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J. DEREZIŃSKI AND E. SKIBSTED

Abstract. For a class of negative slowly decaying potentials, including $V(x) := -\gamma|x|^{-\mu}$ with $0 < \mu < 2$, we study the quantum mechanical scattering theory in the low-energy regime. Using modifiers of the Isozaki–Kitada type we show that scattering theory is well behaved on the *whole* continuous spectrum of the Hamiltonian, including the energy 0. We show that the S–matrices are well-defined and strongly continuous down to the zero energy threshold. Similarly, we prove that the wave matrices and generalized eigenfunctions are norm continuous down to the zero energy if we use appropriate weighted spaces. These results are used to derive (oscillatory) asymptotics of the standard short-range and Dollard type S–matrices for the subclasses of potentials where both kinds of S-matrices are defined. For potentials whose leading part is $-\gamma|x|^{-\mu}$ we show that the location of singularities of the kernel of $S(\lambda)$ experiences an abrupt change from passing from positive energies λ to the limiting energy $\lambda = 0$. This change corresponds to the behaviour of the classical orbits. Under stronger conditions we extract the leading term of the asymptotics of the kernel of $S(\lambda)$ at its singularities; this leading term defines a Fourier integral operator in the sense of Hörmander [Hö4].

Contents

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1. Introduction and results

Scattering theory of 2-body systems, both classical and quantum, both short- and long-range, is nowadays a well understood subject [Hö2, II, IK1, IK2, Ya2, DG]. In particular, for large natural classes of potentials we know a lot about the properties of wave and scattering matrices at positive energies. Zero – the only threshold energy – in most works on the subject is avoided, since scattering at zero energy is much more difficult to describe and strongly depends on the choice of the potential.

In this paper we consider a class of potentials that have an especially well behaved, nontrivial and interesting low energy scattering theory. Precise conditions used in our paper are described in Subsection 2. Roughly speaking, the potentials that we consider have a dominant negative radial term $V_1(x)$ similar to $-\gamma|x|^{-\mu}$ with $\gamma > 0$ and $0 < \mu < 2$, plus a faster decaying perturbation.

Similar classes of potentials appeared in the literature already in [Ge]. A systematic study of such 2-body systems at low energies was undertaken in [FS], where a complete expansion of the resolvent at the zero-energy threshold was obtained, and in [DS1], where classical low-energy scattering theory was developed. This paper can be viewed as a continuation of [FS, DS1].

In this paper we show that quantum scattering theory for such potentials is well behaved down to the energy zero. In particular, we study appropriately defined wave matrices and scattering matrices for a fixed energy. We show that they have limits at zero energy. Our results were partly announced in [DS2].

For positive energies most (but probably not all) of our results are contained in the literature, scattered in many sources. Our material about the zero energy case is new.

In the introduction we will first review scattering for positive energies for a rather general class of potentials. Then we will describe a simplified version of the main results of our paper, which concerns scattering at low energies for a more restrictive class of potentials.

1.1. **Classical orbits at positive energies.**

For the presentation of known results about positive energies we assume that the potentials satisfy the following condition:

Condition 1.1. $V = V_1 + V_3$ is a sum of real measurable functions on $\mathbb{R}^d$ such that V_1 is smooth and for some $\mu > 0$

$$\partial_x^\alpha V_1(x) = O\big(|x|^{-\mu-|\alpha|}\big); \quad |\alpha| \geq 0, \tag{1.1}$$

V_3 is compactly supported and $V_3(H_0 + 1)^{-1}$ is a compact operator on the Hilbert space $L^2(\mathbb{R}^d)$. Here $H_0 := 2^{-1}p^2$ with $p := -\mathrm{i}\nabla_x$. The Hamiltonian $H = H_0 + V$ does not have positive eigenvalues.

Let us first consider the classical Hamiltonian $h_1(x, \xi) := \frac{1}{2}\xi^2 + V_1(x)$ on the phase space $\mathbb{R}^d \times \mathbb{R}^d$, using $h_0(x, \xi) := \frac{1}{2}\xi^2$ as the free Hamiltonian. (The analysis of the classical case is needed in the quantum case). One can prove that for any $\xi \in \mathbb{R}^d$, $\xi \neq 0$, and x in an appropriate outgoing/incoming region the following problem admits a solution (strictly speaking, meaning one solution for $t \to +\infty$ and one for $t \to -\infty$):

$$\begin{cases} \ddot{y}(t) = -\nabla V_1(y(t)), \\ y(\pm 1) = x, \\ \xi = \lim_{t\to\pm\infty} \dot{y}(t). \end{cases} \tag{1.2}$$

One obtains a family $y^{\pm}(t,x,\xi)$ of solutions smoothly depending on parameters. All (positive energy) scattering orbits, i.e. orbits satisfying $\lim_{t\to\pm\infty}|y(t)|=\infty$, are of this form (the energy is $\lambda=\frac{1}{2}\xi^2$). Using these solutions, in an appropriate incoming/outgoing region one can construct a solution $\phi^{\pm}(x,\xi)$ to the eikonal equation

$$\tfrac{1}{2}\left(\nabla_x\phi^{\pm}(x,\xi)\right)^2+V_1(x)=\tfrac{1}{2}\xi^2 \tag{1.3}$$

satisfying $\nabla_x\phi^{\pm}(x,\xi)=\dot{y}(\pm 1,x,\xi)$.

1.2. **Wave and scattering matrices at positive energies.** Let us turn to the quantum case. Following Isozaki-Kitada, see [IK1], [IK2], [Ya2] and [RY], one can use the functions $\phi^{\pm}(x,\xi)$ in the quantum case to construct appropriate modifiers, which can be taken to be

$$J^{\pm}f(x):=(2\pi)^{-d}\int \mathrm{e}^{\mathrm{i}\phi^{\pm}(x,\xi)-\mathrm{i}\xi\cdot y}a^{\pm}(x,\xi)f(y)\mathrm{d}x\mathrm{d}\xi. \tag{1.4}$$

Here $a^{\pm}(x,\xi)$ is an appropriate cut-off supported in the domain of the definition of $\phi^{\pm}$, equal to one in the incoming/outgoing region. Then one constructs modified wave operators

$$W^{\pm}f:=\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}J^{\pm}\mathrm{e}^{-\mathrm{i}tH_0}f;\ \hat{f}\in C_c(\mathbb{R}^d\setminus\{0\}), \tag{1.5}$$

and the modified scattering operator

$$S=W^{+*}W^{-}. \tag{1.6}$$

We remark that $W^{\pm}$ are isometric with range $1_{\mathrm{c}}(H)L^2(\mathbb{R}^d)=1_{]0,\infty[}(H)L^2(\mathbb{R}^d)$; whence S is unitary.

The free Hamiltonian H_0 can be diagonalized by the direct integral

$$\mathcal{H}_0=\int_0^{\infty}\oplus L^2(S^{d-1})\,\mathrm{d}\lambda, \tag{1.7}$$

and

$$\mathcal{F}_0(\lambda)f(\omega)=(2\lambda)^{(d-2)/4}\hat{f}(\sqrt{2\lambda}\omega);\quad f\in L^2(\mathbb{R}^d), \tag{1.8}$$

where $\hat{f}$ refers to the d–dimensional Fourier transform. The operator $\mathcal{F}_0(\lambda)$ can be interpreted as a bounded operator from the weighted space $L^{2,s}(\mathbb{R}^d):=\langle x\rangle^{-s}L^2(\mathbb{R}^d)$, $s>\frac{1}{2}$, to $L^2(S^{d-1})$. One can ask whether the wave and scattering operators can be restricted to a fixed energy λ.

This question is conceptually simpler in the case of the scattering operator S. Due to the intertwining property $W^{\pm}H_0=HW^{\pm}$ it satisfies $SH_0=H_0S$, so abstract theory guarantees the existence of a decomposition

$$S\simeq\int_{]0,\infty[}\oplus S(\lambda)\mathrm{d}\lambda,$$

where $S(\lambda)$ are unitary operators on $L^2(S^{d-1})$ defined for almost all λ. One can prove that under Condition 1.1 $S(\lambda)$ can be chosen to be a strongly continuous function (which fixes uniquely $S(\lambda)$ for all $\lambda\in]0,\infty[$). $S(\lambda)$ is called the *scattering matrix at the energy* λ.

The case of wave operators is somewhat more complicated. By the intertwining property it is natural to use the direct integral decomposition (1.7) only from the right and the question is whether we can give a rigorous meaning to $W^{\pm}\mathcal{F}_0(\lambda)^*$. Again, under the condition (1.1) one can show that there exists a unique strongly continuous function $]0,\infty[\ni\lambda\mapsto W^\pm(\lambda)$ with values in the space of bounded operators from $L^2(S^{d-1})$ to $L^{2,-s}(\mathbb{R}^d)$ with $s>\frac12$ such that for $f\in L^{2,s}(\mathbb{R}^d)$

$$W^\pm f=\int_{]0,\infty[}W^\pm(\lambda)\mathcal{F}_0(\lambda)f\mathrm{d}\lambda.$$

The operator $W^\pm(\lambda)$ is called the *wave matrix at energy* λ. One can also extend the domain of $W^\pm(\lambda)$ so that it can act on the delta-function at $\omega\in S^{d-1}$, denoted δ_ω. Now $w^\pm(\omega,\lambda):=W^\pm(\lambda)\delta_\omega$ is an element of $L^{2,-p}(\mathbb{R}^d)$ for $p>\frac{d}{2}$. It satisfies

$$\left(-\tfrac12\Delta+V(x)-\lambda\right)w^\pm(\omega,\lambda)=0. \tag{1.9}$$

It behaves in the outgoing/incoming region as a plane wave. It will be called the *generalized eigenfunction of* H *at energy* λ *and at asymptotic normalized velocity* ω; this terminology is justified in Subsection 1.4.

1.3. **Short-range and Dollard wave and scattering operators.** Let us recall that (1.5) and (1.6) are only one of possible definitions of wave and scattering operators. In the short-range case, that is $\mu>1$, the usual definitions are

$$W_{\rm sr}^\pm f:=\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}f, \tag{1.10}$$

$$S_{\rm sr}:=W_{\rm sr}^{+*}W_{\rm sr}^{-}. \tag{1.11}$$

The operators $W^\pm$ and $W_{\rm sr}^\pm$ differ by a momentum-dependent phase factor:

$$W^\pm=W_{\rm sr}^\pm\mathrm{e}^{\mathrm{i}\psi_{\rm sr}^\pm(p)}, \tag{1.12}$$

$$S=\mathrm{e}^{-\mathrm{i}\psi_{\rm sr}^+(p)}S_{\rm sr}\mathrm{e}^{\mathrm{i}\psi_{\rm sr}^-(p)}. \tag{1.13}$$

Similarly, in the case $\mu>\frac12$ one can use the so-called Dollard construction:

$$W_{\rm dol}^\pm f:=\lim_{t\to\pm\infty}e^{\mathrm{i}tH}U_{\rm dol}(t)f, \tag{1.14}$$

$$U_{\rm dol}(t):=\mathrm{e}^{-\mathrm{i}\int_0^t(p^2/2+V(sp)1_{\{|sp|\geq R_0\}})\,\mathrm{d}s},\ R_0>0, \tag{1.15}$$

$$S_{\rm dol}:=W_{\rm dol}^{+*}W_{\rm dol}^{-}. \tag{1.16}$$

Analogously, we have

$$W^\pm=W_{\rm dol}^\pm\mathrm{e}^{\mathrm{i}\psi_{\rm dol}^\pm(p)}, \tag{1.17}$$

$$S=\mathrm{e}^{-\mathrm{i}\psi_{\rm dol}^+(p)}S_{\rm sr}\mathrm{e}^{\mathrm{i}\psi_{\rm dol}^-(p)}. \tag{1.18}$$

1.4. **Asymptotic normalized velocity operator.** The reader may ask why $W^\pm$, $W_{\rm sr}^\pm$, $W_{\rm dol}^\pm$ are all called wave operators. In fact, it is natural to define a whole family of wave operators associated with a given Schrödinger operator. In this subsection we briefly describe a possible definition of this family, following essentially [De, DG].

Suppose that V satisfies (1.1) (or even much weaker conditions). Then it can be shown that there exists the following operator:

$$v^\pm:=\mathrm{s}-\lim_{t\to\pm\infty}\pm\mathrm{e}^{\mathrm{i}tH}\hat{x}\mathrm{e}^{-\mathrm{i}tH}1_{\rm c}(H);\ \hat{x}=\tfrac{x}{|x|}. \tag{1.19}$$

$v^\pm$ can be called the *asymptotic normalized velocity operator*. It is a vector of commuting self-adjoint operators (on the space $1_{\rm c}(H)L^2(\mathbb{R}^d)$) satisfying

$$(v^\pm)^2=1_{\rm c}(H),\quad [v^\pm,H]=0. \tag{1.20}$$

We say that $\breve{W}^\pm$ is *an outgoing/incoming wave operator associated with* H if it is isometric and satisfies

$$\breve{W}^\pm H_0 = H\breve{W}^\pm, \quad \breve{W}^\pm \hat{p} = v^\pm \breve{W}^\pm, \tag{1.21}$$

where $\hat{p} = \frac{p}{|p|}$.

Note that if $\breve{W}_1^\pm$ and $\breve{W}_2^\pm$ are two wave operators associated with a given H, then there exists a function $\psi^\pm$ such that

$$\breve{W}_1^\pm = \breve{W}_2^\pm \mathrm{e}^{\mathrm{i}\psi^\pm(p)}. \tag{1.22}$$

Therefore, scattering cross-sections, which are usually considered to be the only measurable quantities in scattering theory, are insensitive to the choice of a wave operator.

It is easy to show that $W^\pm$, $W_{\mathrm{sr}}^\pm$, $W_{\mathrm{dol}}^\pm$ are all wave operators in the sense of the above definition. We also note that for the wave operators $W^\pm$ the corresponding generalized eigenfunctions, see (1.9), jointly diagonalize H and $v^\pm$.

1.5. **Low-energy asymptotics of classical orbits.** In the remaining part of the introduction we consider a more restricted class of potentials. To simplify the presentation, in this introduction let us assume that the potential takes the form

$$V(x) = -\gamma|x|^{-\mu} + O(|x|^{-\mu-\epsilon}), \tag{1.23}$$

where $\mu \in]0, 2[$ and $\gamma, \epsilon > 0$. For derivatives, assume that $\partial^\beta \left(V(x) + \gamma|x|^{-\mu}\right) = O(|x|^{-\mu-\epsilon-|\beta|})$. Compactly supported singularities can be included.

For potentials satisfying (1.23) we would like to extend the results described in Subsection 1.1 down to the energy $\lambda = 0$. To this end we change variables to "blow up" the discontinuity at $\lambda = 0$. This amounts to looking at $\xi = \sqrt{2\lambda}\omega$ as depending on two independent variables $\lambda \geq 0$ and $\omega \in S^{d-1}$. It is proven in [DS1] that for any $\omega \in S^{d-1}$, $\lambda \in [0, \infty[$ and x from an appropriate outgoing/incoming region there exists a solution of the problem

$$\begin{cases} \ddot{y}(t) = -\nabla V(y(t)), \\ \lambda = \frac{1}{2}\dot{y}(t)^2 + V(y(t)), \\ y(\pm 1) = x, \\ \omega = \pm \lim_{t\to\pm\infty} y(t)/|y(t)|. \end{cases} \tag{1.24}$$

One obtains a family $y^\pm(t, x, \omega, \lambda)$ of solutions smoothly depending on parameters. All scattering orbits are of this form. Using these solutions one can construct a solution $\phi^\pm(x, \omega, \lambda)$ to the eikonal equation

$$\tfrac{1}{2}\left(\nabla_x \phi^\pm(x, \omega, \lambda)\right)^2 + V(x) = \lambda \tag{1.25}$$

satisfying $\nabla_x \phi^\pm(x, \omega, \lambda) = \dot{y}(\pm 1, x, \omega, \lambda)$.

1.6. **Low-energy asymptotics of wave and scattering matrices.** In the quantum case, we can use the new functions $\phi^\pm(x, \omega, \lambda)$ in the modifiers $J^\pm$, which lead to the definitions of the wave operators $W^\pm$ and the scattering operator S. We can also improve on the choice of the symbols $a^\pm(x, \xi)$ by assuming that in the incoming/outgoing region they satisfy the appropriate transport equations.

The main new result of our paper about wave matrices can be summarized in the following theorem:

Theorem 1.2. *There exists the norm limit of wave matrices at zero energy:*

$$W^{\pm}(0) = \lim_{\lambda \searrow 0} W^{\pm}(\lambda)$$

in the sense of operators in $\mathcal{B}(L^2(S^{d-1}), L^{2,-s}(\mathbb{R}^d))$*, where* $s > \frac{1}{2} + \frac{\mu}{4}$.

The operator $W^{\pm}(0)$ can be called *the wave matrix at zero energy.* We can introduce $w^{\pm}(\omega, 0) := W^{\pm}(0)\delta_\omega$, called the *generalized eigenfunction of* H *at zero energy and fixed asymptotic normalized velocity* ω. It belongs to the weighted space $L^{2,-p}(\mathbb{R}^d)$ where $p > \frac{d}{2} + \frac{\mu}{2} - \frac{d\mu}{4}$. We shall also show weighted L^2– bounds on its ω-derivatives.

It is interesting to note that the behaviour of the generalized eigenfunction $w^{\pm}(\omega, 0)$ depends strongly on the dimension. In dimension 1 it is unbounded, in dimension 2 it is almost bounded and in dimension greater than 2 it decays at infinity (without being square integrable).

The main result of our paper about scattering matrices reads

Theorem 1.3. *There exists the strong limit of scattering matrices at zero energy:*

$$S(0) = \mathrm{s-}\lim_{\lambda \searrow 0} S(\lambda)$$

in the space $\mathcal{B}(L^2(S^{d-1}))$*. This limit* $S(0)$ *is unitary on* $L^2(S^{d-1})$.

We remark that neither $W(\lambda)$ nor $S(\lambda)$ are smooth in $\lambda \geq 0$ at the threshold 0, which can seem somewhat surprising given the fact that the boundary value of the resolvent $R(\lambda + \mathrm{i}0) = (H - \lambda - i0)^{-1}$ (interpreted as acting between appropriate weighted spaces) has this property (see [BGS] for explicit expansions in the purely Coulombic case).

1.7. **Geometric approach to scattering theory.** There exists an alternative approach to scattering theory, based on the study of generalized eigenfunctions. It allows us to characterize scattering matrices by the spatial asymptotics of generalized eigenfunctions. It was used in particular in Vasy [Va1] or [Va2, Remark 19.12]. We shall study this approach, including the case of the zero energy, in Subsection 8.3.

1.8. **Low energy asymptotics of short-range and Dollard operators.** Let us stress that the existence of the limits of wave and scattering matrices at zero energy is made possible not only by appropriate assumptions on the potentials, but also by the use of appropriate modifiers. Wave matrices $W^{\pm}_{\mathrm{sr}}(\lambda)$ defined by the standard short-range procedure, as well as the Dollard modified wave operators $W^{\pm}_{\mathrm{dol}}(\lambda)$, *do not* have this property. They differ from our $W^{\pm}(\lambda)$ by a momentum dependent phase factor that has an oscillatory behaviour as $\lambda \searrow 0$. In particular,

$$W^{\pm}_{\mathrm{sr}}(\lambda) = W^{\pm}(\lambda) \exp\left(\mathrm{i}O(\lambda^{\frac{1}{2}-\frac{1}{\mu}})\right), \qquad 1 < \mu < 2; \tag{1.26a}$$

$$W^{\pm}_{\mathrm{dol}}(\lambda) = W^{\pm}(\lambda) \exp\left(\mathrm{i}O(\lambda^{-\frac{1}{2}} \ln \lambda)\right), \quad \mu = 1; \tag{1.26b}$$

$$W^{\pm}_{\mathrm{dol}}(\lambda) = W^{\pm}(\lambda) \exp\left(\mathrm{i}O(\lambda^{\frac{1}{2}-\frac{1}{\mu}})\right), \qquad \tfrac{1}{2} < \mu < 1. \tag{1.26c}$$

We remark that oscillatory behaviour similar to (1.26a) was proved in [Ya1] in the one-dimensional setting.

1.9. **Location of singularities of the zero energy scattering matrix.** A recurrent idea of scattering theory is the parallel behaviour of classical and quantum systems. One of its manifestations is the relationship between scattering orbits at a given energy and the location of singularities of the scattering matrix.

In the case of positive energies the relationship is simple and well-known. To describe it note that scattering orbits of positive energy have the deflection angle that goes to zero when the distance of the orbit to the center goes to infinity. In the quantum case this corresponds to the fact that the integral kernel of scattering matrices $S(\lambda)(\omega,\omega')$ at positive energies λ are smooth for $\omega\neq\omega'$ and has a singularity at $\omega=\omega'$.

This picture changes at the zero energy. For potentials considered in our paper, the deflection angle of zero-energy orbits does not go to zero for orbits far from the center. The angle of deflection is small for small μ and goes to infinity as μ approaches 2.

For the strictly homogeneous potential, $V(r)=-\gamma r^{-\mu}$, one can solve the equations of motion at zero energy. The (non-collision) zero-energy orbits are given by the implicit equation (in polar coordinates)

$$\sin(1-\frac{\mu}{2})\theta(t)=\Big(\frac{r(t)}{r_{\rm tp}}\Big)^{-1+\frac{\mu}{2}}, \tag{1.27}$$

see [DS1, Example 4.3]. Whence the deflection angle of such trajectories equals $-\frac{\mu\pi}{2-\mu}$. In particular, for attractive Coulomb potentials it equals $-\pi$, which corresponds to the well-known fact that in this case zero-energy orbits are parabolas (see [Ne, p. 126] for example).

One of the main results of our paper is a quantum analogue of this fact:

Theorem 1.4. *The integral kernel of the zero-energy scattering matrix $S(0)(\omega,\omega')$ is smooth away from ω,ω' satisfying $\omega\cdot\omega'=\cos\frac{\mu\pi}{2-\mu}$.*

Note that this fact was known before in the case of Coulomb potentials, at least in dimension $d\geq 3$. In this case $S(0)={\rm e}^{{\rm i}c}P$, where $(P\tau)(\omega)=\tau(-\omega)$. Moreover in this case one can compute (using special functions, see Yafaev [Ya3] for an explicit formula)

$$S_{\rm dol}(\lambda)={\rm e}^{{\rm i}\lambda^{-1/2}\{C_1\ln\lambda+C_2+o(\lambda^0)\}}(P+o(\lambda^0)). \tag{1.28}$$

1.10. **Type of singularity of the scattering matrix.** Let Λ be the operator on $L^2(S^{d-1})$ such that $\Lambda Y_l=(l+d/2-1)Y_l$, where Y_l is a spherical harmonic of order l. Alternatively, it can be introduced as follows

$$\Lambda:=\sqrt{L^2+(d/2-1)^2},$$

where

$$L^2=\sum_{1\leq i<j\leq d}L_{ij}^2;\quad {\rm i}L_{ij}=x_i\partial_{x_j}-x_j\partial_{x_i}.$$

Note that, for any θ, the distributional kernel of ${\rm e}^{{\rm i}\theta\Lambda}$ can be computed explicitly and its singularities appear at $\omega\cdot\omega'=\cos\theta$. This is expressed in the following result:

Proposition 1.5. ${\rm e}^{{\rm i}\theta\Lambda}$ *equals*

(1) *$c_\theta I$, where I is the identity, if $\theta\in\pi 2\mathbb{Z}$;*
(2) *$c_\theta P$, where P is the parity operator, if $\theta\in\pi(2\mathbb{Z}+1)$;*

(3) *the operator whose Schwartz kernel is of the form* $c_\theta(\omega\cdot\omega' - \cos\theta + \mathrm{i}0)^{-\frac{d}{2}}$ *if* $\theta \in]\pi 2k, \pi(2k+1)[$ *for some* $k \in \mathbb{Z}$*;*
(4) *the operator whose Schwartz kernel is of the form* $c_\theta(\omega\cdot\omega' - \cos\theta - \mathrm{i}0)^{-\frac{d}{2}}$ *if* $\theta \in]\pi(2k-1), \pi 2k[$ *for some* $k \in \mathbb{Z}$*.*

Note that for all θ, the operator $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ belongs to the class of Fourier integral operators of order 0 in the sense of Hörmander [Hö2, Hö4].

The operator Λ can be used to describe the leading asymptotics of the scattering matrix at zero energy:

Theorem 1.6. *If* (1.23) *holds with* V *being spherically symmetric (up to a compactly supported possibly singular term) now with the number* ϵ *obeying* $\epsilon > 1 - \frac{\mu}{2}$*, then*

$$S(0) = \mathrm{e}^{\mathrm{i}c}\mathrm{e}^{-\mathrm{i}\frac{\mu\pi}{2-\mu}\Lambda} + K$$

where K *is compact.*

We shall prove Theorem 1.6 by one-dimensional WKB-analysis.

1.11. **Kernel of $S(0)$ as an explicit oscillatory integral.** In the case $V = -\gamma|x|^{-\mu} + O\big(|x|^{-1-\frac{\mu}{2}-\epsilon}\big)$, $\epsilon > 0$, it is possible to represent the distributional kernel of the scattering matrix $S(0)$ (modulo a smoothing term) in terms of a fairly explicit oscillatory integral. This provides an alternative way to prove Theorem 1.4 on the location of singularities of the scattering matrix – given the stronger conditions on the potential (we remark that our proof of Theorem 1.4 is rather abstract, see Subsection 1.13). Moreover, although we shall not elaborate in this paper, it is actually feasible to prove a partial version of Theorem 1.6 (more precisely for the cases $\frac{2}{2-\mu} \notin \mathbb{N}$) using this fairly explicit integral.

1.12. **Generalized eigenfunctions.** A solution of the equation

$$(-\Delta + V(x) - \lambda)u = 0 \tag{1.29}$$

in $\bigcup_s L^{2,-s}(\mathbb{R}^d)$ will be called a *generalized eigenfunction with energy* λ. One of our results says that each generalized eigenfunction with positive or zero energy is of the form $W^{\pm}(\lambda)\tau$, where τ is a distribution on the sphere S^{d-1}.

Such generalized eigenfunctions are never square-integrable. A rough method to describe their behaviour for large x is to use weighted spaces $L^{2,s}(\mathbb{R}^d)$ with appropriate s. A more precise description is provided by the so-called *Besov spaces.* One of our results says that the range of (incoming and outgoing) wave matrices can be described precisely by an appropriate Besov space. One can also describe quite precisely their spatial asymptotics. In the case of zero energy, these results are new.

1.13. **Propagation of singularities for zero-energy generalized eigenfunctions.** It is well-known that some of the properties of solutions of PDE's of the form $P(x,D)u = 0$ can be explained by the behaviour of classical hamiltonian dynamics given by the principal symbol of P. One of the best known expressions of this idea is Hörmander's theorem about propagation of singularities.

Similar ideas are true in the case of Schrödinger operators. This is well understood for positive energies. In the case of zero energy a similar analysis is possible. It has an especially clean formulation if we assume that the potential is $V(x) = -\gamma|x|^{-\mu}$. Under this condition, the set of orbits of the classical system given by $h(x,\xi)$ is invariant with respect to an appropriate scaling. This allows us to reduce the phase space.

In the quantum case, We introduce an appropriate concept of a *wave front set* adapted to the solutions to (1.29), different from Hörmander's. One of our main results describes a possible location of this special wave front set for solutions to (1.29) for $\lambda = 0$ – the statement is very similar to the statement of the original Hörmander's theorem; it is used in a proof of Theorem 1.4.

1.14. **Sommerfeld radiation condition.** Another of our main results is a version of the Sommerfeld radiation condition for zero energies. It says that given v in a certain weighted space a solution u of the equation $(H - \lambda)u = v$ satisfying appropriate outgoing/incoming phase space localization is always of the form $u = R(\lambda \pm \mathrm{i}0)v$.

This somewhat technical result has a number of interesting applications. In particular, we use it in our proof that $S(0)$ can be expressed in terms of an oscillatory integral, and also in the description of the asymptotics of generalized eigenfunctions at large distances.

1.15. **Organization of the paper.** The paper is organized as follows: In Section 2 we impose conditions on the potential. In the case we allow the potential to have a non-spherically symmetric term we shall need certain regularity properties of the leading spherically symmetric term. These properties are stated in Condition 2.2; they are fulfilled for the example (1.23) discussed above.

In Section 3 we describe and extend some of results from our previous papers. In particular, we recall the construction of scattering phases in [DS1] (given there under the same conditions). We describe and to some extend continue the study of the properties of these objects.

In Section 4 we recall various microlocal resolvent estimates from [FS] (slightly extended). We also introduce the concept of the scattering wave front set adapted to energy zero. We give its applications, in particular a result about the Sommerfeld radiation condition at zero energy.

In Section 5 we describe the modifiers used in our paper. They are given by a WKB-type ansatz, which involves solving transport equations.

In Section 6 we introduce wave operators and wave matrices. We describe their low-energy asymptotics.

In Section 7 we introduce scattering operators and matrices. We analyse their low-energy asymptotics.

In Section 8 we study properties of generalized eigenfunctions for non-negative energies.

In Section 9 we restrict our attention to potentials of the form (1.23). We show the classical rule, $\omega \cdot \omega' = \cos \frac{\mu}{2-\mu}\pi$, for the location of zero-energy singularities (cf. Theorem 1.4). We also show a "propagation of scattering singularities result", see Proposition 9.1, on generalized zero-energy eigenfunctions. Under stronger conditions than (1.23) we represent the kernel of $S(0)$ as an explicit oscillatory integral.

In Section 10 we study an explicit Fourier integral operator on the unit sphere – the evolution operator for the wave equation – and we show that it coincides, modulo a compact term, with $S(0)$ (again, under stronger conditions than (1.23)).

In Appendix A we present, in an abstract setting, various elements of stationary scattering theory used in our paper.

2. Conditions

We shall consider a classical Hamiltonian $h = \frac{1}{2}\xi^2 + V$ on $\mathbb{R}^d \times \mathbb{R}^d$ where V satisfies Condition 2.1 (in classical mechanics we can take $V_3 = 0$) and possibly Condition 2.2 (both stated below). We shall throughout the paper use the non-standard notation $\langle x\rangle$ for $x \in \mathbb{R}^d$ to denote a function $\langle x\rangle = f(r)$; $r = |x|$, where here $f \in C^\infty([0,\infty[)$ is taken convex, and obeys $f = \frac{1}{2}$ for $r < \frac{1}{4}$ and $f = r$ for $r > 1$. We shall often use the notation $\hat{x} = x/r$ for vectors $x \in \mathbb{R}^d \setminus \{0\}$. Let $L^{2,s} = L^{2,s}(\mathbb{R}^d_x) = \langle x\rangle^{-s} L^2(\mathbb{R}^d_x)$ for any $s \in \mathbb{R}$ (the corresponding norm will be denoted by $\|\cdot\|_s$). Introduce also $L^{2,-\infty}(= L^{2,-\infty}(\mathbb{R}^d)) = \cup_{s\in\mathbb{R}} L^{2,s}$ and $L^{2,\infty} = \cap_{s\in\mathbb{R}} L^{2,s}$. The notation $F(s > \epsilon)$ denotes a smooth increasing function $= 1$ for $s > \frac{3}{4}\epsilon$ and $= 0$ for $s < \frac{1}{2}\epsilon$; $F(\cdot < \epsilon) := 1 - F(\cdot > \epsilon)$. The notation g will be used extensively; it stands for the function $g(r) = \sqrt{2\lambda - 2V_1(r)}$ (for V_1 obeying Condition 2.1 and $\lambda \in [0,\infty[$).

Condition 2.1. The function V can be written as a sum of three real-valued measurable functions, $V = V_1 + V_2 + V_3$, such that: For some $\mu \in]0,2[$ we have

(1) V_1 is a smooth negative function that only depends on the radial variable r in the region $r \geq 1$ (that is $V_1(x) = V_1(r)$ for $r \geq 1$). There exists $\epsilon_1 > 0$ such that
$$V_1(r) \leq -\epsilon_1 r^{-\mu};\ r \geq 1.$$
(2) For all $\gamma \in (\mathbb{N}\cup\{0\})^d$ there exists $C_\gamma > 0$ such that
$$\langle x\rangle^{\mu+|\gamma|} |\partial^\gamma V_1(x)| \leq C_\gamma.$$
(3) There exists $\tilde{\epsilon}_1 > 0$ such that
$$rV_1'(r) \leq -(2-\tilde{\epsilon}_1)V_1(r);\ r \geq 1. \tag{2.1}$$
(4) $V_2 = V_2(x)$ is smooth and there exists $\epsilon_2 > 0$ such that for all $\gamma \in (\mathbb{N}\cup\{0\})^d$
$$\langle x\rangle^{\mu+\epsilon_2+|\gamma|} |\partial^\gamma V_2(x)| \leq C_\gamma.$$
(5) $V_3 = V_3(x)$ is compactly supported.

The following condition will be needed only in the case $V_2 \neq 0$.

Condition 2.2. Let V_1 be given as in Condition 2.1 and $\alpha := \frac{2}{2+\mu}$. There exists $\bar{\epsilon}_1 > \max(0, 1 - \alpha(\mu + 2\epsilon_2))$ such that
$$\limsup_{r\to\infty} r^{-1}V_1'(r)\Big(\int_1^r (-2V_1(\rho))^{-\frac{1}{2}}\mathrm{d}\rho\Big)^2 < 4^{-1}(1-\bar{\epsilon}_1^2), \tag{2.2}$$
$$\limsup_{r\to\infty} V_1''(r)\Big(\int_1^r (-2V_1(\rho))^{-\frac{1}{2}}\mathrm{d}\rho\Big)^2 < 4^{-1}(1-\bar{\epsilon}_1^2). \tag{2.3}$$

We notice that (2.1) and (2.2) tend to be somewhat strong conditions for $\mu\approx 2$. On the other hand Conditions 2.1 and 2.2 hold for all $\epsilon_2 > 0$ for the particular example $V_1(r) = -\gamma r^{-\mu}$ (with $\epsilon_1 = \gamma$, $\tilde{\epsilon}_1 = 2 - \mu$ and some $\bar{\epsilon}_1 < 1 - \alpha\mu$).

In quantum mechanics we consider $H = H_0 + V$; $H_0 = \frac{1}{2}p^2$, $p = -\mathrm{i}\nabla$, on $\mathcal{H} = L^2(\mathbb{R}^d)$, and we need the following additional condition. Clearly Condition 2.3 (1) assures that H is self-adjoint. For an elaboration of Condition 2.3 (2), see [FS]; it guarantees that zero is not an eigenvalue of H. Condition 2.3 (3) is included here only for convenience of presentation; with the other conditions there are no small positive eigenvalues, cf. [FS].

Condition 2.3. In addition to Condition 2.1

(1) $V_3(H_0+\mathrm{i})^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$.
(2) H satisfies the unique continuation property at infinity.
(3) H does not have positive eigenvalues.

3. Classical orbits

In this section we recall and extend the results of [DS1] about low energy classical orbits that we will need in our paper.

3.1. **Scattering orbits at positive energies.**

We introduce for $R \geq 1$ and $\sigma > 0$

$$\Gamma^+_{R,\sigma}(\omega) = \{y \in \mathbb{R}^d \mid y \cdot \omega \geq (1-\sigma)|y|,\ |y| \geq R\};\ \omega \in S^{d-1},$$
$$\Gamma^+_{R,\sigma} = \{(y,\omega) \in \mathbb{R}^d \times S^{d-1} \mid y \in \Gamma^+_{R,\sigma}(\omega)\}.$$

Lemma 3.1. *Suppose that V_1 satisfies* (1.1). *Let $\sigma \in]0,2[$. Then there exists a decreasing function $]0,\infty[\ni \lambda \mapsto R_0(\lambda)$ such that for all $|\xi| \geq \sqrt{2\lambda}$ and $x \in \Gamma^+_{R_0(\lambda),\sigma}(\hat{\xi})$ there exists a unique solution $y(t) = y^+(t,x,\xi)$ of the problem* (1.2) *such that $y(t) \in \Gamma^+_{R_0(\lambda),\sigma}(\hat{\xi})$ for $t > 1$. If we set*

$$F^+(x,\xi) := \dot{y}^+(1,x,\xi),$$

then $\mathrm{rot}_x F^+(x,\xi) = 0$.

For any $\xi \neq 0$ we let $\lambda = 2^{-1}\xi^2$, $\omega = \hat{\xi}$ and $R = R(\lambda)$. For $(x,\omega) \in \Gamma^+_{R,\sigma}$ we choose a path $[0,1] \ni l \mapsto \gamma(l) \in \Gamma^+_{R,\sigma}(\omega)$ such that $\gamma(0) = R\omega$ and $\gamma(1) = x$. We set

$$\phi^+(x,\xi) := \int_0^1 F^+(\gamma(\tau),\xi) \cdot \frac{\mathrm{d}\gamma(l)}{\mathrm{d}l}\mathrm{d}l + |\xi|R.$$

Note that $\phi^+(x,\xi)$ does not depend on the choice of the path γ. For instance, we can take the interval joining these two points and then

$$\phi^+(x,\xi) = (x - R\omega) \cdot \int_0^1 F^+(l(x-R\hat{\xi}) + R\hat{\xi},\xi)\mathrm{d}l + |\xi|R. \tag{3.1}$$

Another possible choice is the radial interval from $R\omega$ to $|x|\omega$ and then the arc towards x:

$$\phi^+(x,\xi) = \int_R^{|x|} F^+(l\omega,\xi)\cdot\omega\mathrm{d}l + \int_0^{\arccos \omega\cdot\hat{x}} F^+(|x|v_\alpha,\xi)\cdot|x|\frac{\mathrm{d}v_\alpha}{\mathrm{d}\alpha}\mathrm{d}\alpha + |\xi|R, \tag{3.2}$$

where $v_\alpha := \cos\alpha\omega + \sin\alpha\frac{\hat{x}-\omega\ \omega\cdot\hat{x}}{\sqrt{1-(\omega\cdot\hat{x})^2}}$.

The phase function constructed above essentially coincides with the Isozaki Kitada (outgoing) phase function, cf. [Is1], [IK1, Definition 2.3] or [DG, Proposition 2.8.2]. In particular, for any $\xi \neq 0$, there are bounds

$$\partial_\xi^\kappa \partial_x^\gamma(\phi^+(x,\xi) - \xi\cdot x) = O\big(|x|^{\delta-|\gamma|}\big) \text{ for } |x| \to \infty; \tag{3.3}$$
$$\delta > \max(1-\mu,0).$$

These bounds are not uniform in $\xi \neq 0$, they are however uniform on compact subsets of $\mathbb{R}^d \setminus \{0\}$.

3.2. **Scattering orbits at low energies.** Let us now recall some results about scattering orbits taken from [DS1].

We assume Conditions 2.1 and 2.2 (only Condition 2.1 if $V_2 = 0$). The fact that our Condition 2.1 includes a possibly singular potential V_3 is irrelevant for this subsection since by assumption this term is compactly supported. More precisely we just need to make sure that the $R_0 \geq 1$ in Lemma 3.2 stated below is taken so large that $V_3(x) = 0$ for $|x| \geq R_0$, then [DS1] applies.

Lemma 3.2. *There exist $R_0 \geq 1$ and $\sigma_0 > 0$ such that for all $R \geq R_0$ and for all positive $\sigma \leq \sigma_0$ the problem (1.24) is solved for all data $(x,\omega) \in \Gamma^+_{R,\sigma}$ and $\lambda \geq 0$ by a unique function $y^+(t,x,\omega,\lambda)$, $t \geq 1$ such that $y^+(t,x,\omega,\lambda) \in \Gamma^+_{R,\sigma}(\omega)$ for all $t \geq 1$. Define a vector field $F^+(x,\omega,\lambda)$ on $\Gamma^+_{R_0,\sigma_0}(\omega)$ by*

$$F^+(x,\omega,\lambda) = \dot{y}^+(t=1;x,\omega,\lambda); \tag{3.4}$$

Then

$$\mathrm{rot}_x F^+(x,\omega,\lambda) = 0.$$

Note that under the assumptions of Lemma 3.2, we can suppose that $R_0(\lambda)$, introduced in Lemma 3.1, equals R_0 for all $\lambda > 0$. We can define $\phi^+(x,\omega,\lambda)$ on $(x,\omega,\lambda) \in \Gamma^+_{R,\sigma} \times [0,\infty[$. For further reference let us record the analogues of (3.1) and (3.2):

$$\phi^+(x,\omega,\lambda) = (x - R_0\omega) \cdot \int_0^1 F^+(l(x-R_0\omega) + R_0\omega)\mathrm{d}l + \sqrt{2\lambda}R_0,$$

$$\phi^+(x,\omega,\lambda) = \int_{R_0}^{|x|} F^+(l\omega,\omega,\lambda)\cdot\omega\mathrm{d}l + \int_0^{\arccos\omega\cdot\hat{x}} F^+(|x|v_\alpha,\omega,\lambda)\cdot\frac{\mathrm{d}v_\alpha}{\mathrm{d}\alpha}\mathrm{d}\alpha + \sqrt{2\lambda}R_0.$$

We will add the subscript "sph" to all objects where V is replaced by the (spherically symmetric) potential V_1. The following result is proven in [DS1]:

Proposition 3.3. *There exists $\breve{\epsilon} = \breve{\epsilon}(\mu, \bar{\epsilon}_1, \epsilon_2) > 0$ and uniform bounds*

$$F^+(x) - F^+_{\mathrm{sph}}(x) = O\big(|x|^{-\mu/2-\breve{\epsilon}}\big). \tag{3.5a}$$

In particular for constants C, $c > 0$ independent of x, ω and λ

$$\Big|\frac{F^+(x)}{|F^+(x)|} - \frac{F^+_{\mathrm{sph}}(x)}{|F^+_{\mathrm{sph}}(x)|}\Big| \leq C|x|^{-\breve{\epsilon}}, \tag{3.5b}$$

and

$$\frac{F^+(x)}{|F^+(x)|}\cdot\hat{x} \geq 1 - C\big(1 - \hat{x}\cdot\omega\big) - C|x|^{-\breve{\epsilon}}, \tag{3.5c}$$

$$\frac{F^+(x)}{|F^+(x)|}\cdot\hat{x} \leq 1 - c\big(1 - \hat{x}\cdot\omega\big) + C|x|^{-\breve{\epsilon}}, \tag{3.5d}$$

$$\frac{F^+(x)}{|F^+(x)|}\cdot\omega \geq 1 - C\big(1 - \hat{x}\cdot\omega\big) - C|x|^{-\breve{\epsilon}}. \tag{3.5e}$$

More generally (with the same $\breve{\epsilon} > 0$), for all multiindices δ and γ there are uniform bounds

$$\partial^\delta_\omega\partial^\gamma_x F^+(x) = \langle x\rangle^{-|\gamma|}O\left(g(|x|)\right), \tag{3.5f}$$

$$\partial^\delta_\omega\partial^\gamma_x\big(F^+(x) - F^+_{\mathrm{sph}}(x)\big) = \langle x\rangle^{-\breve{\epsilon}-|\gamma|}O\left(g(|x|)\right). \tag{3.5g}$$

The vector field $F^+(x,\omega,\lambda)$ as well as all derivatives $\partial_\omega^\delta \partial_x^\gamma F^+$ are jointly continuous in the variables $(x,\omega) \in \Gamma^+_{R_0,\sigma_0}$ and $\lambda \geq 0$.

The problem (1.24) in the case of $t \to -\infty$ can also be solved. We introduce for $R \geq 1$ and $\sigma > 0$

$$\Gamma^-_{R,\sigma}(\omega) = \{y \in \mathbb{R}^d \mid y \cdot \omega \leq (\sigma - 1)|y|,\ |y| \geq R\};\ \omega \in S^{d-1},$$
$$\Gamma^-_{R,\sigma} = \{(y,\omega) \in \mathbb{R}^d \times S^{d-1} |\ y \in \Gamma^-_{R,\sigma}(\omega)\}.$$

Mimicking the previous procedure, starting from the mixed problem (1.24) in the case of $t \to -\infty$, we can similarly construct a solution $\phi^-(x,\omega,\lambda)$ to the eikonal equation in some $\Gamma^-_{R,\sigma}(\omega)$. This amounts to setting

$$\phi^-(x,\omega,\lambda) = -\phi^+(x,-\omega,\lambda);\ x \in \Gamma^-_{R_0,\sigma_0}(\omega) = \Gamma^+_{R_0,\sigma_0}(-\omega). \tag{3.6}$$

3.3. **Radially symmetric potentials.** In this subsection we assume that $V_2 = 0$, which means that the potential is spherically symmetric. More precisely, we assume that for $r \geq R_0$

$$|\partial_r^n V(r)| \leq c_n r^{-n-\mu},\quad V(r) \leq -cr^{-\mu},\ c > 0,\quad rV'(r) + 2V(r) < 0.$$

Note that motion in such a potential is confined to a 2-dimensional plane. In the case of the trajectory $y^+(t,x,\omega,\lambda)$ it is the plane spanned by ω and $\hat{x}$. It is also convenient to introduce the vectors $x^\perp := \frac{\omega - \cos\theta_1 \hat{x}}{\sin\theta_1}$ and $\omega^\perp := \frac{\hat{x} - \cos\theta_1 \omega}{\sin\theta_1}$, where $\omega \cdot \hat{x} = \cos\theta_1$. Therefore, we can restrict temporarily our attention to a 2-dimensional system. We will use the polar coordinates $(r\cos\theta, r\sin\theta)$. Note that the energy λ and the angular momentum L are preserved quantities. Therefore, the Newton equations (for outgoing orbits) can be reduced to

$$\begin{cases} \dot{\theta} = Lr^{-2}, \\ \dot{r} = \sqrt{2\lambda - 2V(r) - L^2 r^{-2}}. \end{cases} \tag{3.7}$$

Lemma 3.4. *For some $\theta_0 > 0$, for all $r_1 \geq R_0$, $|\theta_1| \leq \theta_0$ and $\lambda \geq 0$ we can find a solution of* (3.7) *satisfying*

$$r(1) = r_1,\ \dot{r}(1) > 0,\ \lim_{t\to\infty} \theta(t) = 0,\ \theta(1) = \theta_1.$$

There exists a function $(r_1,\theta_1,\lambda) \mapsto L(r_1,\theta_1,\lambda) \in \mathbb{R}$ specifying the total angular momentum of the solution $y^+(t,x,\omega,\lambda)$. This function L is an odd function in θ_1. We have the following estimates:

$$\partial_{r_1}^n \partial_{\theta_1^2}^m L^2 = O\left(r_1^{2-n} g(r_1)^2\right),\quad n,m \geq 0; \tag{3.8a}$$

$$\partial_{r_1}^n \partial_{\theta_1^2}^m \frac{L}{\theta_1} = O\left(r_1^{1-n} g(r_1)\right),\quad n,m \geq 0. \tag{3.8b}$$

This allows us to compute the initial velocity of the trajectory:

$$F^+(x,\omega,\lambda) = \sqrt{2\lambda - 2V(r) - L^2/r^2}\,\hat{x} - \frac{L}{r} x^\perp.$$

The function ϕ^+ equals, with $r = |x|$ and $\cos\theta = \hat{x} \cdot \omega$,

$$\phi^+(x,\omega,\lambda) = \sqrt{2\lambda} R_0 + \int_{R_0}^r \sqrt{2\lambda - 2V(r')}\mathrm{d}r' + \int_0^\theta L(r,\theta',\lambda)\mathrm{d}\theta'. \tag{3.9}$$

Therefore, using also that $\nabla_\omega \theta = -\omega^\perp$,

$$\nabla_\omega \phi^+ = -L(r,\theta,\lambda)\omega^\perp. \tag{3.10}$$

This gives the following estimates (in any dimension):

Lemma 3.5. *There exist constants $C, c > 0$ such that*

$$\begin{aligned} |\hat{x}\cdot F^+(x) - g(|x|)| &\le C(1-\hat{x}\cdot\omega)g(|x|), && \text{(3.11a)}\\ |F^+(x) - \hat{x}\ \hat{x}\cdot F^+(x)| &\le C\sqrt{1-\hat{x}\cdot\omega}\, g(|x|), && \text{(3.11b)}\\ |\nabla_\omega \phi^+| &\ge c\sqrt{1-\hat{x}\cdot\omega}\, g(|x|)|x|, && \text{(3.11c)}\\ \partial_\omega^\delta \partial_x^\gamma \phi^+ &= \langle x\rangle^{1-|\gamma|} O(g(|x|)). && \text{(3.11d)} \end{aligned}$$

We calculate for $\lambda > 0$:

$$\begin{aligned} \nabla_\xi F^+ &= (2\lambda)^{-\frac12}\nabla_\omega F^+ + (2\lambda)^{\frac12}\partial_\lambda F^+ \otimes \omega;\\ \nabla_\omega F^+ &= L\partial_\theta L(2\lambda - 2V(r) + L^2 r^{-2})^{-\frac12} r^{-2}\omega^\perp \otimes \hat{x} + \partial_\theta L r^{-1}\omega^\perp \otimes x^\perp - \frac{L}{r}\nabla_\omega x^\perp,\\ \partial_\lambda F^+ &= (2\lambda - 2V(r) - L^2 r^{-2})^{-\frac12}(1 - L\partial_\lambda L r^{-2})\hat{x} - \partial_\lambda L r^{-1} x^\perp. \end{aligned}$$

Specifying to x parallel to ω and noting that $L(x,\hat{x},\lambda) = 0$, we obtain

$$\begin{aligned} \nabla_\xi F^+ &= (2\lambda)^{1/2}\partial_\lambda(2\lambda - 2V(|x|))^{1/2}\hat{x}\otimes\hat{x}\\ &\quad - (2\lambda)^{-1/2}|x|^{-1}\partial_\theta L\ x^\perp \otimes x^\perp\\ &= (2\lambda)^{1/2}(2\lambda - 2V(|x|))^{-1/2}\hat{x}\otimes\hat{x} \qquad\qquad (3.12)\\ &\quad + (2\lambda)^{-1/2}|x|^{-1}\Big(\int_{|x|}^\infty r^{-2}(2\lambda - 2V(r))^{-1/2}\mathrm{d}r\Big)^{-1} x^\perp \otimes x^\perp, \end{aligned}$$

cf. [DS1, (4.5)].

In an arbitrary dimension, the formula is the same except that the second term is repeated $d-1$ times on the diagonal. Therefore,

$$\det\Big(\nabla_\xi \nabla_x \phi^+(x, \sqrt{2\lambda}\hat{x})\Big)^{1/2} = (2\lambda)^{(2-d)/4} g(r)^{-1/2}\Big(r^{-1}h(r)\Big)^{(d-1)/2}, \tag{3.13}$$

where we have introduced the notation

$$h(r) := \Big(\int_r^\infty r'^{-2} g(r')^{-1}\mathrm{d}r'\Big)^{-1}. \tag{3.14}$$

Note the (uniform) bounds

$$crg(r) \le h(r) \le Crg(r). \tag{3.15}$$

Whence, combining (3.13) and (3.15),

$$\begin{aligned} c(2\lambda)^{(2-d)/4} g(r)^{(d-2)/2} &\le \det\big(\nabla_\xi\nabla_x\phi^+(x,\sqrt{2\lambda}\hat{x})\big)^{1/2}\\ &\le C(2\lambda)^{(2-d)/4} g(r)^{(d-2)/2}. \end{aligned} \tag{3.16}$$

4. Boundary values of the resolvent

In this section we impose Conditions 2.1 and 2.3. We shall recall (and extend) some resolvent estimates of [FS]. They are important tools used throughout our paper.

In Subsection 4.2 we will also introduce the notion of the scattering wave front set, which is well adapted to scattering theory at various energies. We will return to this concept in particular in Section 9, where we will prove a theorem about propagation of singularities for potentials with a homogeneous principal part. A somewhat cruder version of this theorem is given already in Subsection 4.2 (valid, however, for a more general class of potentials).

In Subsection 4.4 we prove a version of the Sommerfeld radiation condition for the zero energy.

4.1. **Low energy resolvent estimates.** Let c be a function on the phase space $\mathbb{R}^d \times \mathbb{R}^d$. The left and right Kohn-Nirenberg quantization of the symbol c are the operators $\mathrm{Op}^{\mathrm{l}}(c)$ and $\mathrm{Op}^{\mathrm{r}}(c)$ acting as

$$(\mathrm{Op}^{\mathrm{l}}(c)f)(x) = (2\pi)^{-d/2} \int \mathrm{e}^{\mathrm{i}x\cdot\xi} c(x,\xi) \hat{f}(\xi)\,\mathrm{d}\xi,$$

$$(\mathrm{Op}^{\mathrm{r}}(c)f)(x) = (2\pi)^{-d} \iint \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} c(y,\xi) f(y)\,\mathrm{d}y\mathrm{d}\xi,$$

respectively. Notice that $\mathrm{Op}^{\mathrm{l}}(c)^* = \mathrm{Op}^{\mathrm{r}}(\bar{c})$. In Proposition 4.1 stated below we use for convenience both of these quantizations although they can be used interchangeably. Alternatively one can use Weyl quantization denoted by $\mathrm{Op}^{\mathrm{w}}(c)$, cf. [FS]. We will often use the following (λ-dependent) symbols:

$$a(x,\xi) = \frac{\xi^2}{g(|x|)^2}, \quad b(x,\xi) = \frac{\xi}{g(|x|)} \cdot \frac{x}{\langle x\rangle}. \tag{4.1}$$

It is convenient to introduce the following symbol class: Let $c \in S(m, g_{\mu,\lambda})$ $g_{\mu,\lambda} = \langle x\rangle^{-2}\mathrm{d}x^2 + g^{-2}\mathrm{d}\xi^2$ and $m = m_\lambda = m_\lambda(x,\xi)$ be a uniform weight function [Hö3]. Here $\lambda \in [0,\lambda_0]$ (for an arbitrarily fixed $\lambda_0 > 0$) is considered as a parameter; the function m obeys bounds uniform in this parameter (see [FS, Lemma 4.3 (ii)] for details). For a uniform weight function m the symbol class $S_{\mathrm{unif}}(m, g_{\mu,\lambda})$ is defined to be the set of parameter-dependent smooth symbols $c = c_{\omega,\lambda}$ satisfying

$$|\partial_\omega^\delta \partial_x^\gamma \partial_\xi^\beta c_{\omega,\lambda}(x,\xi)| \le C_{\delta,\gamma,\beta} m_\lambda(x,\xi) \langle x\rangle^{-|\gamma|} g^{-|\beta|}. \tag{4.2}$$

We notice that the "Planck constant" for this class is $\langle x\rangle^{-1} g^{-1}$. The corresponding class of quantizations is denoted by $\Psi_{\mathrm{unif}}(m, g_{\mu,\lambda})$ (it does not depend on whether left or right quantization is used). Finally we remark that the quantizations appearing in Proposition 4.1 stated below belong to $\Psi_{\mathrm{unif}}(1, g_{\mu,\lambda})$ and hence they are bounded uniformly in λ (these symbols are independent of ω).

We can obtain the following estimates by mimicking the proof of [FS, Theorem 4.1] (first for the smooth case $V_3 = 0$, and then the general case by a resolvent equation, see [FS, Subsection 5.1]; here the unique continuation assumption Condition 2.3 (2) comes into play). In particular, Proposition 4.1 (i) follows from [FS, Corollary 3.5] and a resolvent identity (cf. [FS, (5.12)]). Similarly Proposition 4.1 (ii) follows from [FS, Lemma 4.5] and the proof of [FS, Lemma 4.6] (notice that it suffices to show the bounds (4.3b) and (4.3c) for $t = 0$ due to this proof), while Proposition 4.1 (iii) follows from [FS, Lemma 4.9] and the same minor modification of the proof of [FS, Lemma 4.6]. As for the continuity statement at the end of the proposition we refer the reader to the end of this subsection.

The notation $R(\lambda + \mathrm{i}0)$ refers to the limit of the resolvent $R(\lambda + \mathrm{i}\epsilon)$ as $\epsilon \to 0^+$ in the sense of a form on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, cf. Remark 4.2 2).

Proposition 4.1. *Fix any $\lambda_0 > 0$. The following conclusions,* (i)–(v)*, hold uniformly in $\lambda \in [0, \lambda_0]$:*

(i) *For all $\delta > \frac{1}{2}$ there exists $C > 0$ such that*

$$\|\langle x\rangle^{-\delta} g^{\frac{1}{2}} R(\lambda + \mathrm{i}0) g^{\frac{1}{2}} \langle x\rangle^{-\delta}\| \le C. \tag{4.3a}$$

(ii) *There exists $C_0 \ge 1$ such that if $\chi_+ \in C^\infty(\mathbb{R})$, $\operatorname{supp}(\chi_+) \subset]C_0, \infty[$ and $\chi_+' \in C_c^\infty(\mathbb{R})$, then for all $\delta > \frac{1}{2}$ and all $s, t \ge 0$ there exists $C > 0$ such that*

$$\|(\langle x\rangle g)^s \langle x\rangle^{t-\delta} g^{\frac{1}{2}} \mathrm{Op}^{\mathrm{l}}(\chi_+(a)) R(\lambda + \mathrm{i}0) g^{\frac{1}{2}} \langle x\rangle^{-t-\delta} (\langle x\rangle g)^{-s}\| \le C, \tag{4.3b}$$

$$\|(\langle x\rangle g)^{-s} \langle x\rangle^{-t-\delta} g^{\frac{1}{2}} R(\lambda + \mathrm{i}0) \mathrm{Op}^{\mathrm{r}}(\chi_+(a)) g^{\frac{1}{2}} \langle x\rangle^{t-\delta} (\langle x\rangle g)^{s}\| \le C. \tag{4.3c}$$

(iii) *Let $\bar\sigma > 0$ and $\chi_- \in C_c^\infty(\mathbb{R})$. Suppose $\tilde\chi_-, \tilde\chi_+ \in C^\infty(\mathbb{R})$ satisfy*

$$\sup\operatorname{supp}\tilde\chi_- \le 1 - \bar\sigma, \ \inf\operatorname{supp}\tilde\chi_+ \ge \bar\sigma - 1.$$

Then for all $\delta > \frac{1}{2}$ and all $s, t \ge 0$ there exists $C > 0$ such that

$$\|(\langle x\rangle g)^s \langle x\rangle^{t-\delta} g^{\frac{1}{2}} \mathrm{Op}^{\mathrm{l}}(\chi_-(a)\tilde\chi_-(b)) R(\lambda + \mathrm{i}0) g^{\frac{1}{2}} \langle x\rangle^{-t-\delta} (\langle x\rangle g)^{-s}\| \le C, \tag{4.3d}$$

$$\|(\langle x\rangle g)^{-s} \langle x\rangle^{-t-\delta} g^{\frac{1}{2}} R(\lambda + \mathrm{i}0) \mathrm{Op}^{\mathrm{r}}(\chi_-(a)\tilde\chi_+(b)) g^{\frac{1}{2}} \langle x\rangle^{t-\delta} (\langle x\rangle g)^{s}\| \le C. \tag{4.3e}$$

(iv) *Suppose $\chi_-^1, \chi_-^2 \in C_c^\infty(\mathbb{R})$, $\tilde\chi_-$ and $\tilde\chi_+$ satisfy the assumptions from* (iii) *and in addition*

$$\sup\operatorname{supp}\tilde\chi_- < \inf\operatorname{supp}\tilde\chi_+.$$

Then for all $s \ge 0$ there exists $C > 0$ such that

$$\|\langle x\rangle^s \mathrm{Op}^{\mathrm{l}}(\chi_-^1(a)\tilde\chi_-(b)) R(\lambda + \mathrm{i}0) \mathrm{Op}^{\mathrm{r}}(\chi_-^2(a)\tilde\chi_+(b)) \langle x\rangle^s\| \le C. \tag{4.3f}$$

(v) *Suppose χ_+ is given as in* (ii)*, some functions $\tilde\chi_+, \tilde\chi_-, \chi_-$ are given as in* (iii) *and suppose*

$$\operatorname{dist}(\operatorname{supp}\chi_-, \operatorname{supp}\chi_+) > 0.$$

Then for all $s \ge 0$ there exists $C > 0$ such that

$$\|\langle x\rangle^s \mathrm{Op}^{\mathrm{l}}(\chi_+(a)) R(\lambda + \mathrm{i}0) \mathrm{Op}^{\mathrm{r}}(\chi_-(a)\tilde\chi_+(b)) \langle x\rangle^s\| \le C, \tag{4.3g}$$

$$\|\langle x\rangle^s \mathrm{Op}^{\mathrm{l}}(\chi_-(a)\tilde\chi_-(b)) R(\lambda + \mathrm{i}0) \mathrm{Op}^{\mathrm{r}}(\chi_+(a)) \langle x\rangle^s\| \le C. \tag{4.3h}$$

All the forms appearing in (i)–(v) *are continuous in $\lambda \ge 0$. In fact the families of corresponding operators are continuous $\mathcal{B}(L^2(\mathbb{R}^d))$–valued functions.*

Remarks 4.2. 1) Although this will not be needed we have in fact (ii) with $C_0 = 1$; see Corollary 4.4 for a related result.

2) The paper [FS] contains a stronger version of the so-called limiting absorption principle than can be read from Proposition 4.1 (i): For all $\delta > \frac{1}{2}$ there exists $C > 0$ such that

$$\sup_{\lambda + \mathrm{i}\epsilon \in M} \|\langle x\rangle^{-\delta} g^{\frac{1}{2}} R(\lambda + \mathrm{i}\epsilon) g^{\frac{1}{2}} \langle x\rangle^{-\delta}\| \le C; \ M := [0, \lambda_0] \times \mathrm{i}\,]0, 1],$$

and the $\mathcal{B}(L^2(\mathbb{R}^d)$–valued function $\langle x\rangle^{-\delta-\frac{\mu}{4}}R(\zeta)\langle x\rangle^{-\delta-\frac{\mu}{4}}$ is uniformly Hölder continuous in $\zeta \in M$. The (well-known) positive energy analogue of this assertion states that for any positive $\lambda_1 < \lambda_0$ the $\mathcal{B}(L^2(\mathbb{R}^d)$–valued function $\langle x\rangle^{-\delta}R(\zeta)\langle x\rangle^{-\delta}$ is uniformly Hölder continuous in $\zeta \in M \setminus \{\operatorname{Re}\zeta < \lambda_1\}$; see 4) for a related remark.

3) The paper [FS] also contains an extension of Proposition 4.1 to powers of the resolvent, however this will not be useful in the forthcoming sections; see Example 7.5 for a discussion. This is related to the fact that our classical constructions are not smooth in λ at zero energy, cf. [DS1, Remarks 4.7 1)]. The collection of all estimates in Proposition 4.1 (more precisely a collection of similar estimates with a complex spectral parameter) yields similar estimates for powers of the resolvent by a completely algebraic reasoning, cf. [FS, Appendix A].
4) Assume that the potential satisfies Condition 1.1. Then all the bounds of Proposition 4.1 remain true uniformly in $\lambda \in [\lambda_1, \lambda_0]$ for any positive $\lambda_1 < \lambda_0$ provided we replace

$$a \to a := \frac{\xi^2}{2\lambda},\ b \to b := \frac{\xi}{\sqrt{2\lambda}} \cdot \frac{x}{\langle x\rangle} \text{ and } g \to 1. \tag{4.4}$$

(Under the stronger Conditions 2.1 and 2.3 the validity of this modification is a direct consequence of the bounds of Proposition 4.1.) Also in this case the families of associated operators are norm continuous (now in $\lambda > 0$ only).

Proof of continuity statements in Proposition 4.1. Due to Remark 4.2 2) and the calculus of pseudodifferential operators all appearing forms in Proposition 4.1 are continuous in $\lambda \geq 0$.

Norm continuity of the corresponding operator-valued functions also follows from Remark 4.2 2). This can be seen as follows for $B_\delta(\lambda) := \langle x\rangle^{-\delta}g^{\frac{1}{2}}R(\lambda + \mathrm{i}0)g^{\frac{1}{2}}\langle x\rangle^{-\delta}$ (appearing in (i)):

Pick $\delta' \in]\frac{1}{2}, \delta[$, insert for (small) $\kappa > 0$ the identity $I = F(\kappa|x| < 1) + F(\kappa|x| > 1)$ on both sides of $B_\delta(\lambda)$ and expand (into three terms). This yields

$$\|B_\delta(\lambda) - F(\kappa|x| < 1)B_\delta(\lambda)F(\kappa|x| < 1)\| \leq C\kappa^{\delta-\delta'}\|B_{\delta'}(\lambda)\|.$$

Due to Proposition 4.1 (i) the right hand side is $O(\kappa^{\delta-\delta'})$ uniformly $\lambda \geq 0$. On the other hand due to Remark 4.2 2) (and the calculus of pseudodifferential operators) for fixed $\kappa > 0$ the $\mathcal{B}(L^2(\mathbb{R}^d))$–valued function $F(\kappa|x| < 1)B_\delta(\cdot)F(\kappa|x| < 1)$ is continuous. Hence $B_\delta(\cdot)$ is a uniform limit of continuous functions and therefore indeed continuous.

The other operator-valued functions can be dealt with in the same fashion. □

4.2. **Scattering wave front set.** The remaining subsections of Section 4 are devoted to a number of somewhat technical estimates on solutions to the equation $(H - \lambda)u = v$ for a fixed $\lambda \geq 0$. Although they are proved under Conditions 2.1 and 2.3 we remark that there are similar estimates under Condition 1.1 for a fixed $\lambda > 0$. The reader may skip this material on the first reading.

Throughout the remaining part of this section we will use the notation $\langle\xi\rangle_1 = (1 + |\xi|^2)^{1/2}$ and $X = (1 + |x|^2)^{1/2}$ for $\xi, x \in \mathbb{R}^d$.

With reference to the symbol class $S_{\mathrm{unif}}(m, g_{\mu,\lambda})$ from Subsection 4.1 clearly $h_1, h_2 \in S_{\mathrm{unif}}(m, g_{\mu,\lambda})$ with $h_1 := \frac{1}{2}\xi^2 + V_1$, $h_2 := \frac{1}{2}\xi^2 + V_1 + V_2$ and $m = g^2\langle\xi/g\rangle_1^2$.

In the remaining part of Section 4 we shall however only need a reminiscence of this symbol class given by disregarding the uniformity in $\lambda \geq 0$. Whence we shall consider symbols $c \in S(m, g_{\mu,\lambda})$ meaning, by definition, that

$$|\partial_x^\gamma \partial_\xi^\beta c(x,\xi)| \leq C_{\gamma,\beta} m(x,\xi) \langle x \rangle^{-|\gamma|} g^{-|\beta|}. \tag{4.5}$$

The corresponding class of standard Weyl quantizations $\mathrm{Op}^{\mathrm{w}}(c)$ is denoted $\Psi(m, g_{\mu,\lambda})$.

It is convenient to introduce the following constants:

$$s_0 = \begin{cases} (1+\frac{\mu}{2})/2, \\ 1/2, \end{cases} \quad s_1 = \begin{cases} 1-\frac{\mu}{2}, \\ 1, \end{cases} \quad s_2 = \begin{cases} \mu, & \text{for } \lambda = 0, \\ 0, & \text{for } \lambda > 0. \end{cases} \tag{4.6}$$

If $\epsilon > 0$, then $\langle x \rangle^{-s_0-\epsilon}$ will be a typical weight that appears in resolvent estimates. (Notice that in the uniform estimates of Proposition 4.1 the corresponding weight is $g^{\frac{1}{2}} \langle x \rangle^{-\frac{1}{2}-\epsilon}$.) The weight $\langle x \rangle^{-s_1}$ plays the role of the "Planck constant" for the class $\Psi(m, g_{\mu,\lambda})$. Finally, $\langle x \rangle^{-s_2}$ will appear in the "elliptic regularity estimate" of Proposition 4.3. Clearly $s_0 > s_2$ and $s_1 > 0$.

Let us decompose the normalized momentum ξ/g as follows:

$$\frac{\xi}{g} = b\frac{x}{\langle x \rangle} + \bar{c};\ b := \frac{x}{\langle x \rangle} \cdot \frac{\xi}{g} \text{ and } \bar{c} := \Big(I - \big|\frac{x}{\langle x \rangle}\big\rangle\big\langle\frac{x}{\langle x \rangle}\big|\Big)\frac{\xi}{g}. \tag{4.7}$$

Notice that b was already defined in Subsection 4.1, besides for $r = |x| \geq 1$, $b^2 + \bar{c}^2 = a$ with a also defined in Subsection 4.1. Moreover for $r \geq 1$ we have the identification $b = \hat{x} \cdot \frac{\xi}{g} \in \mathbb{R}$ and $\bar{c} = (I - |\hat{x}\rangle\langle\hat{x}|)\frac{\xi}{g} \in T^*_{\hat{x}}(S^{d-1})$ with $\hat{x} = x/r \in S^{d-1}$, which obviously constitute canonical coordinates for "the phase space" $\mathbb{T}^* := T^*(S^{d-1}) \times \mathbb{R} = S^{d-1} \times \mathbb{R}^d$. This partly motivates the following definition:

The wave front set $WF^s_{\mathrm{sc}}(u)$ of a distribution $u \in L^{2,-\infty}$ is the subset of $\mathbb{T}^*$ given by the condition

$$\begin{aligned} &z_1 = (\omega_1, \bar{c}_1, b_1) = (\omega_1, b_1\omega_1 + \bar{c}_1) = (\omega_1, \eta_1) \notin WF^s_{\mathrm{sc}}(u) \\ &\Leftrightarrow \\ &\exists \text{ neighbourhoods } \mathcal{N}_{\omega_1} \ni \omega_1,\ \mathcal{N}_{\eta_1} \ni \eta_1\ \ \forall \chi_{\omega_1} \in C_c^\infty(\mathcal{N}_{\omega_1}),\ \chi_{\eta_1} \in C_c^\infty(\mathcal{N}_{\eta_1}): \\ &\mathrm{Op}^{\mathrm{w}}\big(\chi_{z_1} F(r > 2)\big)u \in L^{2,s} \text{ where } \chi_{z_1}(x,\xi) = \chi_{\omega_1}(\hat{x})\chi_{\eta_1}(b\hat{x} + \bar{c}). \end{aligned} \tag{4.8}$$

Notice that this quantization is defined by the substitution $b\hat{x} + \bar{c} \to \xi/g$, cf. (4.7). Keep in mind that the whole concept depends on the given energy $\lambda \in [0,\infty[$ in consideration (through g, which enters in the definition of b and $\bar{c}$).

The above notion of wave front set is of course adapted to the problem in hand. The classical definition is taylored to measure decay in momentum space; see for example [Hö1, Chapter VIII]. Our definition concerns decay in position space, and thus it is more related to the wave front set introduced in [Me, Section 7] (dubbed there as "the scattering wave front set").

Obviously

$$u \in L^{2,s} \Rightarrow WF^s_{\mathrm{sc}}(u) = \emptyset.$$

Conversely (by a compactness argument), if for some $\chi \in C_c^\infty(\mathbb{R}^d)$

$$u - \mathrm{Op}^{\mathrm{w}}(\chi(\xi/g))u \in L^{2,s}, \tag{4.9}$$

then

$$WF^s_{\mathrm{sc}}(u) = \emptyset \Rightarrow u \in L^{2,s}.$$

Proposition 4.3. *Let $\lambda \geq 0$ and s_2 be defined in* (4.6). *Let $u \in L^{2,-\infty}$, $v \in L^{2,s+s_2}$ and $(H-\lambda)u = v$. Then the estimates* (4.9) *and*

$$WF^s_{\mathrm{sc}}(u) \subseteq \{z \in \mathbb{T}^* | b^2 + \vec{c}^2 = 1\} \tag{4.10}$$

hold.

More generally, suppose $u \in L^{2,-\infty}$, $g^{-1}v \in L^{2,s}$ and $(H-\lambda)u = v$. Then the following estimates hold:

$$\text{For all } \epsilon > 0: \ g\mathrm{Op^w}(F(b^2 + \vec{c}^2 - 1 > \epsilon))u \in L^{2,s}, \tag{4.11a}$$

$$\text{For all } \epsilon > 0, \ g\mathrm{Op^w}\big(\langle \xi/g \rangle_1^2 F(b^2 + \vec{c}^2 - 1 > \epsilon)\big)\, u \in L^{2,s}, \tag{4.11b}$$

$$\text{For all } \epsilon > 0: \ g\mathrm{Op^w}\big(F(1 - b^2 - \vec{c}^2 > \epsilon)\big)u \in L^{2,s}, \tag{4.11c}$$

$$WF^s_{\mathrm{sc}}(gu) \subseteq \{z \in \mathbb{T}^* | b^2 + \vec{c}^2 = 1\}. \tag{4.11d}$$

Proof. Obviously (4.11b) is stronger than (4.11a). Notice also that (4.11a) in some sense is stronger than Proposition 4.1 (ii) (involves weaker weights). It is also obvious that (4.11d) is a consequence of (4.11b) and (4.11c).

The proof of (4.11b) given below is somewhat similar to the proof of the analogue of Proposition 4.1 (ii) given in [FS]. For convenience we have divided the proof into four steps. For the calculus of pseudodifferential operators, used tacitly below, we refer to [Hö1, Theorems 18.5.4, 18.6.3, 18.6.8] (the reader might find it more convenient to consult [FS] for an elaboration).

The bounds (4.11c) can be proved by mimicking Steps III and IV below. We note that the complication due to high energies, cf. Step II below, is absent. For this reason (4.11c) is somewhat easier to establish than (4.11b) and we shall leave the details of proof to the reader.

Step I. At various points in the proof of (4.11b) we need to control the possibly existing local singularities of the potential V_3. This is done in terms of the following elementary bounds.

$$T_1 := \langle x \rangle^{t'} g^{-1} V_3 (H - \mathrm{i})^{-1} g^{-1} \langle x \rangle^{-t} \in \mathcal{B}(L^2); \ t, t' \in \mathbb{R}, \tag{4.12a}$$

$$\widetilde{T}_1 := \langle x \rangle^{t'} g^{-1} V_3 (1 + p^2)^{-1} g^{-1} \langle x \rangle^{-t} \in \mathcal{B}(L^2); \ t, t' \in \mathbb{R}. \tag{4.12b}$$

$$T_2 := \langle x \rangle^{t} (1 + p^2) g (H - \mathrm{i})^{-1} g^{-1} \langle x \rangle^{-t} \in \mathcal{B}(L^2); \ t \in \mathbb{R}. \tag{4.12c}$$

Step II. Suppose $gu \in L^{2,t}$ for some fixed $t \leq s$. We shall prove that then $Agu \in L^{2,t}$ for all $A \in \Psi(\langle \xi/g \rangle_1^2, g_{\mu,\lambda})$, in fact that

$$\text{For all } A \in \Psi(\langle \xi/g \rangle_1^2, g_{\mu,\lambda}): \ \|Agu\|_t \leq C(\|gu\|_t + \|g^{-1}v\|_s). \tag{4.13}$$

For any such operator A and any $m \in \mathbb{R}$ we decompose

$$\langle x \rangle^t A = B_m \langle x \rangle^t \mathrm{Op^w}(\langle \xi/g \rangle_1^2) + R_m, \tag{4.14}$$

where $B_m \in \Psi(1, g_{\mu,\lambda})$ and $R_m \in \Psi(\langle \xi/g \rangle_1^2 \langle x \rangle^{-m}, g_{\mu,\lambda})$.

Now, cf. [FS, proof of Lemma 4.5],

$$\begin{aligned} \mathrm{Op^w}(\langle \xi/g \rangle_1^2) &= g^{-1} p^2 g^{-1} + \mathrm{Op^w}(a_1) \\ &= 2g^{-1}(H - \lambda)g^{-1} + \mathrm{Op^w}(a_2) - 2g^{-2}V_3; \end{aligned} \tag{4.15}$$

$$a_1 = 1 - |\nabla g^{-1}|^2 + 4^{-1}\Delta g^{-2}, \ a_2 = a_1 + 1 - 2g^{-2}V_2 \in S(1, g_{\mu,\lambda}).$$

We substitute (4.15) in (4.14), expand into altogether four terms and apply the resulting sum to the state gu. The contribution from the first term of (4.15) is estimated as

$$\|B_m\langle x\rangle^t 2g^{-1}(H-\lambda)g^{-1}(gu)\| \le C_1\|g^{-1}v\|_t \le C_2\|g^{-1}v\|_s.$$

Similarly, the contribution from the second term of (4.15) is estimated as

$$\|B_m\langle x\rangle^t \mathrm{Op^w}(a_2)gu\| \le C\|gu\|_t.$$

As for the third term of (4.15) we use (4.12a) with $t=t'$ to bound

$$\begin{aligned}2\|B_m\langle x\rangle^t g^{-2}V_3 gu\| &\le 2\|B_m\|\,\|T_1\langle x\rangle^t g(H-\mathrm{i})u\| \\ &\le C_1\big(\|gv\|_t+\|(\lambda-\mathrm{i})gu\|_t\big) \le C_2(\|gu\|_t+\|g^{-1}v\|_s).\end{aligned}$$

To treat the contribution from the second term of (4.14) we note that

$$\Psi(\langle \xi/g\rangle_1^2\langle x\rangle^{-m}, g_{\mu,\lambda}) \subseteq \Psi(\langle \xi\rangle_1^2\langle x\rangle^{2-m}, g_{\mu,\lambda}).$$

Whence, using (4.12c) and choosing $m=2-t$,

$$\|R_m gu\| \le C_1\|T_2\langle x\rangle^t g(H-\mathrm{i})u\| \le C_2(\|gu\|_t+\|g^{-1}v\|_s).$$

We conclude (4.13).

Step III. Suppose $gu\in L^{2,t}$ for some fixed $t<s$. Fix $s'\in]t,t+1-\mu/2]$ with $s'\le s$. We shall show that (4.11a) holds with s replaced by s'. We set $F_\epsilon := F(b^2+\vec{c}^2-1>\epsilon)$.

We need a regularization in x-space given in terms of $\iota_\kappa = X_\kappa^{-\frac{2-\mu}{2}}$ where we for $\kappa\in]0,1]$ let

$$X_\kappa := (1+\kappa|x|^2)^{1/2}. \tag{4.16}$$

Mimicking [FS, proof of Lemma 4.5], clearly for $R>1$ chosen large enough

$$F_\epsilon^2F(r>R)^2 \le \frac{3}{\epsilon}\mathrm{Re}\Big(\frac{2h_2-2\lambda}{g^2}\Big)F_\epsilon^2F(r>R)^2.$$

Let

$$\begin{aligned}&D=\mathrm{Op^w}(d);\ d=\langle \xi/g\rangle_1^{-1}\langle x\rangle^{1-s'}=\hbar^{-1}\langle \xi/g\rangle_1^{-1}g^{-1}\langle x\rangle^{-s'},\\ &P_\kappa=\mathrm{Op^w}(p_\kappa);\ p_\kappa=q_\kappa^2\Big(\frac{6}{\epsilon}\mathrm{Re}\,(h_2-\lambda)-g^2\Big),\ q_\kappa=\langle x\rangle^{s'}F_\epsilon\iota_\kappa F(r>R).\end{aligned}$$

Since $0\le p_\kappa\in S(\hbar^{-2}d^{-2},g_{\mu,\lambda})$

$$D^*P_\kappa D\ge -C$$

uniformly in κ. Since $0<d\in S(d,g_{\mu,\lambda})$ we can for any $m\in\mathbb{R}$ find $e_m\in S(d^{-1},g_{\mu,\lambda})$ such that

$$DE_m-I\in\Psi(\langle x\rangle^{-2m},g_{\mu,\lambda});\ E_m=\mathrm{Op^w}(e_m).$$

Consequently we have the uniform bound

$$P_\kappa\ge -CE_m^*E_m+R_m;\ R_m\in\Psi(\langle \xi/g\rangle_1^2g^2\langle x\rangle^{2s'-2m},g_{\mu,\lambda}),$$

and therefore by choosing $m=s'-t$ and by using (4.13) that the expectation

$$\langle P_\kappa\rangle_u \ge -C((\|gu\|_t+\|g^{-1}v\|_s)^2. \tag{4.17}$$

On the other hand for any $\delta\in]0,1[$

$$\langle P_\kappa\rangle_u \le C((\|gu\|_t+\|g^{-1}v\|_s)^2-(1-\delta)\langle Q_\kappa^*Q_\kappa\rangle_{gu};\ Q_\kappa=\mathrm{Op^w}(q_\kappa), \tag{4.18}$$

here we use that

$$\begin{gathered}\mathrm{Op}^{\mathrm{w}}\big(q_\kappa^2\mathrm{Re}\,(h_2-\lambda)\big)=\mathrm{Re}\,\big((Q_\kappa g)^*Q_\kappa g^{-1}(H-V_3-\lambda)\big)+R_\kappa;\\ R_\kappa\in\Psi(\langle\xi/g\rangle_1^2\hbar^2\langle x\rangle^{2s'}g^2,g_{\mu,\lambda}))\subseteq\Psi(\langle\xi/g\rangle_1^2\langle x\rangle^{2t}g^2),\end{gathered}$$

and the fact that R_κ is bounded in $\kappa\in]0,1]$ in the class $\Psi(\langle\xi/g\rangle_1^2\langle x\rangle^{2t}g^2)$. Notice that

$$\begin{aligned}&\frac{6}{\epsilon}\langle\mathrm{Re}\,\big((Q_\kappa g)^*Q_\kappa g^{-1}(H-\lambda)\big)\rangle_u\\ &\qquad\leq C\|Q_\kappa gu\|\,\|g^{-1}v\|_{s'}\leq\delta\|Q_\kappa gu\|^2+C_\delta\|g^{-1}v\|_s^2,\end{aligned}$$

and that the contributions from V_3 and the term R_κ can be treated by (4.12a) and (4.13), respectively.

Now, combining (4.17) and (4.18) we conclude that

$$\|Q_\kappa gu\|^2\leq C((\|gu\|_t+\|g^{-1}v\|_s)^2$$

uniformly in $\kappa\in]0,1]$. Letting $\kappa\to0$ completes Step III.

Step IV. Note that (4.11b) is equivalent to the following, seemingly stronger statement:

$$\text{For all } \epsilon>0,\ A\in\Psi(\langle\xi/g\rangle_1^2,g_{\mu,\lambda}):\ Ag\mathrm{Op}^{\mathrm{w}}(F_\epsilon)u\in L^{2,s}. \tag{4.19}$$

We will show (4.19) by induction.

By assumption $gu\in L^{2,t}$ for a sufficently small $t\leq s$ and consequently, due to Step II, it follows that $Agu\in L^{2,t}$ for all $A\in\Psi(\langle\xi/g\rangle_1^2,g_{\mu,\lambda})$. Consider for all $k\in\mathbb{N}$ the following claim given in terms of $t_k:=\min(s,t+(1-\mu/2)(k-1))$:

The bound/localization (4.11b) holds for all $\epsilon>0$ and all $A\in\Psi(\langle\xi/g\rangle_1^2,g_{\mu,\lambda})$ provided $u\to u_\epsilon:=\mathrm{Op}^{\mathrm{w}}(F_{\epsilon/2})u$ and s is replaced by t_k. (Notice that this implies in particular that the state $gu_{2\epsilon}\in L^{2,t_k}$ and hence, since $\epsilon>0$ is arbitrary, that $gu_\epsilon\in L^{2,t_k}$.)

We have seen that this claim holds for $k=1$. So suppose $k>1$ and that the claim is true for $k\to k-1$. To show the claim for k we can assume that $t_{k-1}<s$. First, we notice that $v_\epsilon:=(H-\lambda)u_\epsilon$ obeys the condition $g^{-1}v_\epsilon\in L^{2,t_k}$ due to the induction hypothesis, (4.13), (4.12a) and (4.12b). Notice at this point that

$$[H-V_3-\lambda,\mathrm{Op}^{\mathrm{w}}(F_{\epsilon/2})]\in\Psi(g^2\langle\xi/g\rangle_1^2\hbar,g_{\mu,\lambda}),$$

and that in fact (for any $m\in\mathbb{R}$)

$$\begin{gathered}[H-V_3-\lambda,\mathrm{Op}^{\mathrm{w}}(F_{\epsilon/2})]=gAg\mathrm{Op}^{\mathrm{w}}(F_{\epsilon/4})+R_m;\\ A\in\Psi(\langle\xi/g\rangle_1^2\langle x\rangle^{\mu/2-1},g_{\mu,\lambda}),\qquad R_m\in\Psi(\langle\xi/g\rangle_1^2\langle x\rangle^{-m},g_{\mu,\lambda}).\end{gathered}$$

Now, by Step III (4.11a) applies to $u\to u_\epsilon$, $t\to s_{k-1}$ and with s replaced by $s'=t_k$. Next, by applying Step II to the state $u\to\tilde u_\epsilon:=\mathrm{Op}^{\mathrm{w}}(F_\epsilon)u_\epsilon$ (note that as above $g^{-1}(H-\lambda)\tilde u_\epsilon\in L^{2,t_k}$) we conclude that indeed the bound (4.11b) holds with $u\to u_\epsilon$ and s replaced by t_k. The induction is complete.

Finally we obtain, using the above claim, that the bound (4.11b) holds without changing u and with s replaced by t_k. Since clearly $t_k=s$ for k sufficiently large (4.11b) follows. □

The following corollary follows immediately from Proposition 4.3. At a fixed energy, it strengthens Proposition 4.1 (ii).

Corollary 4.4. *Let $\chi \in C_c^\infty(\mathbb{R})$, $\chi = 1$ around 1. Then for any $s > s_0$ we have (with $\lambda \geq 0$, and s_0 and s_2 as given in (4.6))*

$$\|\langle x\rangle^{s-s_2}\mathrm{Op}^{\mathrm{w}}\big((a^2+1)(1-\chi(a)\big)\,R(\lambda \pm \mathrm{i}0)\langle x\rangle^{-s}\| \leq C \tag{4.20}$$

The following proposition is similar to Proposition 9.1 stated later, although the flavour is somewhat "global". These results (as well as their proofs) are modifications of [Hö3, Proposition 3.5.1] (and its proof), see also [Me] and [HMV]. The condition (4.21) is similar to (4.11b); it implies that $WF_{\mathrm{sc}}^s(u) \subseteq \{(b^2+\bar{c}^2 \leq 1\}$ and hence that $WF_{\mathrm{sc}}^s(u)$ is compact.

Proposition 4.5. *Let $\lambda \geq 0$ and s_0 be defined in (4.6). Suppose $u, v \in L^{2,-\infty}$, $(H-\lambda)u = v$, $s \in \mathbb{R}$, $k \in]-1,1[$ and $\{b = k\} \cap WF_{\mathrm{sc}}^s(u) = \emptyset$. Suppose the following condition:*

$$\textit{For all } \delta > 0,\ \mathrm{Op}^{\mathrm{w}}\big(\langle \xi/g\rangle_1^2 F(b^2+\bar{c}^2-1>\delta)\big)\,u \in L^{2,s}. \tag{4.21}$$

Define

$$k^+ = \sup\{\tilde{k} \geq k |\, \{b \in [k,\tilde{k}]\} \cap WF_{\mathrm{sc}}^s(u) = \emptyset\}, \tag{4.22}$$

$$k^- = \inf\{\tilde{k} \leq k |\, \{b \in [\tilde{k},k]\} \cap WF_{\mathrm{sc}}^s(u) = \emptyset\}. \tag{4.23}$$

Then

$$k^+ < 1 \quad \Rightarrow \quad \{b = k^+\} \cap WF_{\mathrm{sc}}^{s+2s_0}(v) \neq \emptyset, \tag{4.24}$$

$$k^- > -1 \quad \Rightarrow \quad \{b = k^-\} \cap WF_{\mathrm{sc}}^{s+2s_0}(v) \neq \emptyset. \tag{4.25}$$

Proof. We shall only deal with the case of superscript "+"; the case of "−"is similar. For convenience we shall assume that $\epsilon_2 \leq 2-\mu$ and divide the proof into two steps. **Step I**. We will first show the following weaker statement: Suppose $u \in L^{2,s-\epsilon_2/2}$, $v \in L^{2,s+2s_0}$ and $(H-\lambda)u = v$ (in this case (4.21) follows from Proposition 4.3). Then

$$k^+ \geq 1. \tag{4.26}$$

Now, suppose on the contrary that $k^+ < 1$. By a compactness argument we can then find a point in $WF_{\mathrm{sc}}^s(u)$ of the form $z_1 = (\omega_1, \bar{c}_1, k^+)$. For $\epsilon > 0$ chosen small enough (less than $(k^+-k)/2$ suffices here)

$$\{b \in]k^+-2\epsilon, k^+[\} \cap WF_{\mathrm{sc}}^s(u) = \emptyset. \tag{4.27}$$

We can assume that $J :=]k^+-2\epsilon, k^++\epsilon[\subseteq]-1,1[$. Pick a non-positive $f \in C_c^\infty(J)$ with $f' \geq 0$ on $[k^+-\epsilon, \infty[$ and $f(k^+) < 0$, and consider for $K > 0$ and $\kappa \in]0,1]$ the symbol

$$b_\kappa = X^{s_0}a_\kappa;\ a_\kappa = X^s X_\kappa^{-\epsilon_2/2}F(r>2)\exp(-Kb)f(b)F(b^2+\bar{c}^2<3); \tag{4.28}$$

here X_κ is defined by (4.16).

We compute the Poisson bracket

$$\begin{aligned}\{h_2, b\} &= \frac{g}{r}\bar{c}^2 + \frac{V_1'(b^2-1)}{g} - \frac{x\cdot\nabla V_2}{g\langle x\rangle}\\ &= \frac{g}{r}\Big(\big(1-rV_1'g^{-2}\big)\bar{c}^2 + rV_1'g^{-2}\big(b^2+\bar{c}^2-1\big) + O\big(r^{-\epsilon_2}\big)\Big) \qquad (4.29)\\ &= \frac{g}{r}\Big(\big(1-rV_1'g^{-2}\big)\big(1-b^2\big) + g^{-2}2(h_2-\lambda) + O\big(r^{-\epsilon_2}\big)\Big). \qquad (4.30)\end{aligned}$$

We expand the right hand side of (4.30) into three terms and notice that due to (2.1) the first term has the following positive lower bound on $\operatorname{supp} b_\kappa$

$$\cdots \geq c\frac{g}{r}; \qquad c = \frac{\tilde{\epsilon}_1}{2}\big(1 - \sup\{t^2 \mid t \in \operatorname{supp} f\}\big).$$

First we fix K: A part of the Poisson bracket with b_κ^2 is

$$\{h_2, X^{2s+2s_0}X_\kappa^{-\epsilon_2}\} = \frac{g}{r}Y_\kappa b X^{2s+2s_0}X_\kappa^{-\epsilon_2}, \tag{4.31}$$

where $Y_\kappa = Y_\kappa(r)$ is uniformly bounded in κ. We pick $K > 0$ such that for all κ

$$2Kc \geq |Y_\kappa| + 2\frac{r}{g}X^{-2s_0} \text{ on } \operatorname{supp} b_\kappa.$$

From (4.30), (4.31) and the properties of K and f, we conclude the following bound at $\{f'(b) \geq 0\}$

$$\{h_2, b_\kappa^2\} \leq -2a_\kappa^2 + g^{-2}(h_2 - \lambda)a_\kappa O\big(r^s\big) + O\big(r^{2s}\big)(F^2)'(b^2 + \bar{c}^2 < 3) + O\big(r^{2s-\epsilon_2}\big).$$

To use this bound effectively we introduce a partition of unity: Let $f_1, f_2 \in C_c^\infty(J)$ be chosen such that $\operatorname{supp} f_1 \subseteq]k^+ - 2\epsilon, k^+[$, $\operatorname{supp} f_2 \subseteq]k^+ - \epsilon, k^+ + \epsilon[$ and $f_1^2 + f_2^2 = 1$ on $\operatorname{supp} f$. We multiply both sides by f_2^2 $(= 1 - f_1^2)$ and obtain after a rearrangement

$$\begin{aligned}\{h_2, b_\kappa^2\} \leq\ & -2a_\kappa^2 + g^{-2}(h_2 - \lambda)a_\kappa d_\kappa \\ & + K_1 f_1^2 F(b^2 + \bar{c}^2 < 3)\langle x\rangle^{2s} \\ & - K_2(F^2)'(b^2 + \bar{c}^2 < 3)\langle x\rangle^{2s} + K_3\langle x\rangle^{2s-\epsilon_2}; \\ d_\kappa \in S(\langle x\rangle^s&, g_{\mu,\lambda});\end{aligned} \tag{4.32}$$

here $K_1, K_2, K_3 > 0$ are independent of κ, and the symbols d_κ are bounded in κ in the indicated class.

We introduce $A_\kappa = \operatorname{Op^w}(a_\kappa)$, $B_\kappa = \operatorname{Op^w}(b_\kappa)$ and regularization $u_R = F(|x|/R < 1)u$ in terms of a parameter $R > 1$. First we compute

$$\langle i[H, B_\kappa^2]\rangle_u = \lim_{R\to\infty}\langle i[H, B_\kappa^2]\rangle_{u_R} = -2\operatorname{Im}\langle v, B_\kappa^2 u\rangle. \tag{4.33}$$

Using (4.33) and the calculus, cf. [Hö1, Theorems 18.5.4, 18.6.3, 18.6.8], we estimate

$$|\langle i[H, B_\kappa^2]\rangle_u| \leq C_1\|v\|_{s+2s_0}\big(\|A_\kappa u\| + \|u\|_{s-\epsilon_2/2}\big) \leq \tfrac{1}{2}\|A_\kappa u\|^2 + C_2. \tag{4.34}$$

On the other hand using (4.21), (4.27) and (4.32) we infer that

$$\begin{aligned}\langle i[H - V_3, B_\kappa^2]\rangle_u &= \lim_{R\to\infty}\langle i[H - V_3, B_\kappa^2]\rangle_{u_R} \\ &\leq -2\|A_\kappa u\|^2 + C_3\|(H - V_3 - \lambda)u\|_{s+\mu}\|A_\kappa u\| + C_4,\end{aligned}$$

and whence using (4.12a) to bound $\|(H - V_3 - \lambda)u\|_{s+\mu} \leq C\big(\|v\|_{s+\mu} + \|u\|_{s-\epsilon_2/2}\big)$ that

$$\langle i[H - V_3, B_\kappa^2]\rangle_u \leq -\tfrac{3}{2}\|A_\kappa u\|^2 + C_5. \tag{4.35}$$

Clearly another application of (4.12a) yields

$$\langle i[V_3, B_\kappa^2]\rangle_u \leq C_6. \tag{4.36}$$

Combining (4.34)–(4.36) yields

$$\|A_\kappa u\|^2 \leq C_7 = C_2 + C_5 + C_6,$$

which in combination with the property that $f(k^+) < 0$ in turn gives a uniform bound

$$\|X_\kappa^{-\epsilon_2/2}\mathrm{Op}^{\mathrm{w}}\big(\chi_{z_1}F(r>2)\big)u\|_s^2 \le C_8; \tag{4.37}$$

here χ_{z_1} signifies any phase-space localization factor of the form entering in (4.8) supported in a sufficiently small neighbourhood of the point $z_1 = (\omega_1, \bar{c}_1, k^+)$.

We let $\kappa \to 0$ in (4.37) and infer that $z_1 \notin WF_{\mathrm{sc}}(u)$ which is a contradiction; whence (4.26) is proven.

Step II. We need to remove the conditions of Step I, $u \in L^{2,s-\epsilon_2/2}$ and $v \in L^{2,s+2s_0}$. This will be accomplished by an iteration and modification of the procedure of Step I.

Pick $t_1 \in \mathbb{R}$ such that $v \in L^{2,t_1}$. Pick $t < s$ such that $u \in L^{2,t}$ and define $s_m = \min(s, t + m\epsilon_2/2)$ for $m \in \mathbb{N}$. Let correspondingly k_m^+ be given by (4.22) with $s \to s_m$. Clearly

$$k_m^+ \le k_{m-1}^+;\ m = 2, 3, \dots \tag{4.38}$$

If $u \in L^{2,s_m-\epsilon_2/2}$ and $v \in L^{2,s_m+2s_0}$ then (4.24) with $k^+ \to k_m^+$ and $s \to s_m$ follows from Step I. Although we shall not verify these conditions we remark that a suitable micro-local modification will come into play in an inductive procedure, see (4.41) and (4.43) below. We shall indeed (inductively) show (4.24) with $k^+ \to k_m^+$ and $s \to s_m$, i.e. that

$$k_m^+ < 1 \Rightarrow \{b = k_m^+\} \cap WF_{\mathrm{sc}}^{s_m+2s_0}(v) \ne \emptyset. \tag{4.39}$$

Notice that (4.24) follows by using (4.39) for an m taken so large that $s_m = s$.

Let us consider the start of induction given by $m = 1$. In this case obviously $u \in L^{2,s_m-\epsilon_2/2}$. Suppose on the contrary that (4.39) is false. Then we consider the following case:

$$k_m^+ < 1 \text{ and } \{b = k_m^+, b^2 + \bar{c}^2 \le 6\} \cap WF_{\mathrm{sc}}^{s_m+2s_0}(v) = \emptyset. \tag{4.40}$$

We let $\epsilon > 0$, J and f be chosen as in Step I with $k^+ \to k_m^+$. Let $\tilde{f} \in C_c^\infty(]k^+ - 3\epsilon, k^+ + 2\epsilon[)$ with $\tilde{f} = 1$ on J. It follows from (4.40), possibly by taking $\epsilon > 0$ smaller than needed in Step I, that

$$I_\epsilon v \in L^{2,s_m+2s_0};\ I_\epsilon = \mathrm{Op}^{\mathrm{w}}\big(\tilde{f}(b)F(b^2 + \bar{c}^2 < 6)\big). \tag{4.41}$$

Next, we introduce the symbol b_κ by (4.28) (with $s \to s_m$) and proceed as in Step I. As for the bounds (4.34) we can replace v by $I_\epsilon v$ up to addition of a term of the form $C\big(\|v\|_{t_1}^2 + \|u\|_{s_m-\epsilon_2/2}^2\big)$. Similarly we can verify (4.35) and (4.36) (using conveniently (4.12b)). So again we obtain (4.37) (with $s \to s_m$) and therefore a contradiction as in Step I. We have shown (4.39) for $m = 1$.

Now suppose $m \ge 2$ and that (4.39) is verified for $m - 1$. We need to show the statement for the given m. Due to (4.38) and the induction hypothesis we can assume that

$$k_m^+ < k_{m-1}^+. \tag{4.42}$$

Again we argue by contradiction assuming (4.40). We proceed as above noticing that it follows from (4.42) that in addition to (4.41) we have

$$I_\epsilon u \in L^{2,s_{m-1}}; \tag{4.43}$$

at this point we possibly need choosing $\epsilon > 0$ even smaller (viz. $\epsilon < (k_{m-1}^+ - k_m^+)/2$). By replacing v by $I_\epsilon v$ and u by $I_\epsilon u$ at various points in the procedure of Step I (using (4.41) and (4.43), respectively) we obtain again a contradiction. Whence (4.39) follows. □

Corollary 4.6. *Let $s \in \mathbb{R}$, $u \in L^{2,-\infty}$, $v \in L^{2,s+2s_0}$, $(H-\lambda)u = v$, $k \in]-1,1[$ and $\{b = k\} \cap WF^s_{\mathrm{sc}}(u) = \emptyset$. Then*

$$WF^s_{\mathrm{sc}}(u) \subseteq \{b = 1\} \cup \{b = -1\}. \tag{4.44}$$

Proof. The condition (4.21) is guaranteed by Proposition 4.3. Then we apply Proposition 4.5. □

4.3. **Wave front set bounds of the boundary value of the resolvent.** Proposition 4.1 implies, that the symbol $R(\lambda \pm \mathrm{i}0)$ in many cases can be treated as an operator although initially it is defined in terms of a quadratic form. Notice that Remark 4.2 2) in one situation gives a slightly different and direct interpretation of $R(\lambda \pm \mathrm{i}0)$ (as a limit of operators and hence avoiding quadratic forms). It will however be convenient to investigate possible other interpretations of states $R(\lambda \pm \mathrm{i}0)v$ (for which in particular Remark 4.2 2) does not apply) and study associated wave front set bounds. The case of $R(\lambda - \mathrm{i}0)$ is similar to that of $R(\lambda + \mathrm{i}0)$ and will not be elaborated regarding proofs.

For sufficiently decaying states v we have (using in (ii) the slightly abused notation $a := b^2 + \bar{c}^2$ for generic points $z = (\omega, \bar{c}, b) = (\omega, b\omega + \bar{c}) \in \mathbb{T}^*$):

Proposition 4.7. *Let $s > s_0$ and $v \in L^{2,s}$. Then the following is true:*

(i) *For any $t > s_0$*

$$R(\lambda \pm \mathrm{i}0)v = \lim_{\epsilon \searrow 0} R(\lambda \pm \mathrm{i}\epsilon)v \text{ exists in } L^{2,-t}.$$

(ii)

$$WF^{s-s_2}_{\mathrm{sc}}(R(\lambda \pm \mathrm{i}0)v) \subseteq \{a = 1\}.$$

(iii) *For any $\epsilon > 0$,*

$$WF^{s-2s_0-\epsilon}_{\mathrm{sc}}(R(\lambda \pm \mathrm{i}0)v) \subseteq \{b = \pm 1\}. \tag{4.45}$$

Proof. **Ad (i).** This statement follows from Remark 4.2 2); notice that the notation for the limit conforms with Proposition 4.1 (i).

Ad (ii). We have $(H-\lambda)u = v$. Therefore (ii) follows from Proposition 4.3 (alternatively by using Corollary 4.4).

Ad (iii). Let $\chi_- \in C_c^\infty(\mathbb{R})$ such that χ_- is zero around 1. Let $\chi \in C_c^\infty(\mathbb{R})$. Then by Proposition 4.1 (iii), for any $\epsilon > 0$

$$\mathrm{Op^w}(\chi(a)\chi_-(b))R(\lambda + \mathrm{i}0)v \in L^{2,s-2s_0-\epsilon}.$$

□

Based completely on Proposition 4.1 one can give a meaning to $R(\lambda \pm \mathrm{i}0)v$ also for some states v with a slower decay provided they have an appropriate phase space localization. (In the statement below $C_0 \geq 1$ is given in agreement with Proposition 4.1 (ii).)

Proposition 4.8. *Let $s \leq s_0$ and $v \in L^{2,s}$. Suppose that for some $t > s_0$ and $k \in]-1,1]$ (or $k \in [-1,1[$)*

$$WF^t_{\mathrm{sc}}(v) \cap \{b < k,\, a < 2C_0\} = \emptyset \ \ (\text{or } WF^t_{\mathrm{sc}}(v) \cap \{b > k,\, a < 2C_0\} = \emptyset). \tag{4.46}$$

(i) *For any* $\epsilon > 0$ *there exists*

$$R(\lambda + \mathrm{i}0)v = \lim_{\kappa \searrow 0} R(\lambda + \mathrm{i}0)v_\kappa$$
$$(R(\lambda - \mathrm{i}0)v := \lim_{\kappa \searrow 0} R(\lambda - \mathrm{i}0)v_\kappa) \text{ in } L^{2,s-2s_0-\epsilon},$$

where $v_\kappa(x) := F(\kappa|x| < 1)v(x)$.

(ii)

$$WF_{\mathrm{sc}}^{s-s_2}(R(\lambda + \mathrm{i}0)v) \subseteq \{a = 1\} \ \ (WF_{\mathrm{sc}}^{s-s_2}(R(\lambda - \mathrm{i}0)v) \subseteq \{a = 1\}).$$

(iii) *For any* $\epsilon > 0$

$$WF_{\mathrm{sc}}^{t-2s_0-\epsilon}(R(\lambda + \mathrm{i}0)v) \cap \{b < k,\, a \le C_0\} = \emptyset \tag{4.47}$$
$$(WF_{\mathrm{sc}}^{t-2s_0-\epsilon}(R(\lambda - \mathrm{i}0)v) \cap \{b > k,\, a \le C_0\} = \emptyset).$$

Proof. **Ad (i).** Let $\chi \in C_c^\infty(]-\infty, 2C_0[)$, $\chi = 1$ around $[0, C_0]$. Let $\chi_- \in C^\infty(\mathbb{R})$ be chosen such that $\chi_- = 1$ around $]-\infty, -1]$ and $\chi_- = 0$ in $[(k-1)/2, \infty[$. Then by the condition (4.46) and the calculus of pseudodifferential operators

$$\mathrm{Op^w}(\chi(a)\chi_-(b))v_\kappa \longrightarrow \mathrm{Op^w}(\chi(a)\chi_-(b))v \text{ in } L^{2,t} \text{ as } \kappa \searrow 0.$$

Whence by Proposition 4.1 (i), for any $\epsilon > 0$

$$u_1 := \lim_{\kappa \searrow 0} R(\lambda + \mathrm{i}0)\mathrm{Op^w}(\chi(a)\chi_-(b))v_\kappa \text{ exists in } L^{2,-s_0-\epsilon}.$$

By Proposition 4.1 (ii) we have

$$u_2 := \lim_{\kappa \searrow 0} R(\lambda + \mathrm{i}0)\mathrm{Op^w}(1 - \chi(a))v_\kappa \text{ exists in } L^{2,s-2s_0-\epsilon}.$$

By Proposition 4.1 (iii) we have

$$u_3 := \lim_{\kappa \searrow 0} R(\lambda + \mathrm{i}0)\mathrm{Op^w}(\chi(a)(1 - \chi_-(b))v_\kappa \text{ exists in } L^{2,s-2s_0-\epsilon}.$$

But $s - 2s_0 \le -s_0$. Hence

$$R(\lambda + \mathrm{i}0)v := \lim_{\kappa \searrow 0} R(\lambda + \mathrm{i}0)v_\kappa = u_1 + u_2 + u_3 \in L^{2,s-2s_0-\epsilon}.$$

Ad (ii). This statement is proven as (ii) of the previous proposition.

Ad (iii). Let χ^1, $\chi^2 \in C_c^\infty(]-\infty, 2C_0[)$, $\chi^2 = 1$ around $[0, \max(\sup\operatorname{supp}\chi^1, C_0)]$. Let $\chi_-^1 \in C_c^\infty(]-\infty, k[)$ and $\chi_-^2 \in C^\infty(\mathbb{R})$ such that $\chi_-^2 = 1$ around $]-\infty, \sup\operatorname{supp}\chi_-^1]$ and $\operatorname{supp}\chi_-^2 \subseteq]-\infty, k[$. Then by the condition (4.46)

$$\mathrm{Op^w}(\chi^2(a)\chi_-^2(b))v \in L^{2,t}.$$

Whence by Proposition 4.1 (i), noting that $t > s_0$, we obtain

$$R(\lambda + \mathrm{i}0)\mathrm{Op^w}(\chi^2(a)\chi_-^2(b))v \in L^{2,-s_0-\epsilon}$$

and

$$WF_{\mathrm{sc}}^{t-2s_0-\epsilon}\left(R(\lambda + \mathrm{i}0)\mathrm{Op^w}(\chi^2(a)\chi_-^2(b))v\right) \subseteq \{b = 1\}. \tag{4.48}$$

By Proposition 4.1 (iv)

$$\mathrm{Op^w}(\chi^1(a)\chi_-^1(b))R(\lambda + \mathrm{i}0)\mathrm{Op^w}(\chi^2(a)(1 - \chi_-^2(b)))v \in L^{2,\infty}, \tag{4.49}$$

and by Proposition 4.1 (v)

$$\mathrm{Op^w}(\chi^1(a)\chi_-^1(b))R(\lambda + \mathrm{i}0)\mathrm{Op^w}(1 - \chi^2(a))v \in L^{2,\infty}. \tag{4.50}$$

Now (4.48)–(4.50) yields

$$\mathrm{Op^w}(\chi^1(a)\chi^1_-(b))R(\lambda+\mathrm{i}0)v \in L^{2,t-2s_0-\epsilon},$$

which implies (4.47). $\square$

We have yet another interpretation very similar to Proposition 4.7 (i):

Proposition 4.9. *Fix real-valued $\chi \in C_c^\infty(\mathbb{R})$ and $\tilde\chi \in C^\infty(\mathbb{R})$ such that $\inf\operatorname{supp}\tilde\chi > -1$ (or $\sup\operatorname{supp}\tilde\chi < 1$). Let $A := \mathrm{Op^w}(\chi(a)\tilde\chi(b))$. Suppose $v \in L^{2,s}$ for some $s \le s_0$.*

For any $\epsilon > 0$ there exists

$$R(\lambda+\mathrm{i}0)Av = \lim_{\kappa\searrow 0} R(\lambda+\mathrm{i}\kappa)Av \text{ in } L^{2,s-2s_0-\epsilon}$$
$$(\text{or } R(\lambda-\mathrm{i}0)Av = \lim_{\kappa\searrow 0} R(\lambda-\mathrm{i}\kappa)Av \text{ in } L^{2,s-2s_0-\epsilon}).$$

Moreover this limit agrees with the interpretation of Proposition 4.8 (i).

Proof. We need to invoke an extended version of the bound (4.3e), see [FS, Lemma 4.10]. First notice that the symbols g and hence also a and b obviously depend on λ. Let $\zeta = \lambda + \mathrm{i}\kappa$ and define g_ζ, a_ζ and b_ζ by replacing λ by $|\zeta|$ in the definition of g in Section 2.1 and of a and b in (4.1), respectively. Now we have the following extension of the bound (4.3e):

For all $\delta > \frac12$ and all $s, t \ge 0$ there exists $C > 0$ such that for all $\kappa \in]0,1]$

$$\|(\langle x\rangle g_\zeta)^{-s}\langle x\rangle^{-t-\delta} g_\zeta^{\frac12} R(\zeta)\mathrm{Op^w}(\chi_-(a_\zeta)\tilde\chi_+(b_\zeta)) g_\zeta^{\frac12}\langle x\rangle^{t-\delta}(\langle x\rangle g_\zeta)^s\| \le C. \tag{4.51}$$

Although this will not be needed, the bound (4.51) is in fact locally uniform in $\lambda \ge 0$.

We pick in (4.51) the functions χ_- and $\tilde\chi_+$ in agreement with Proposition 4.1 (iii) such that in addition we have $\chi_- = 1$ around $[0, \sup\operatorname{supp}\chi]$ and $\tilde\chi_+ = 1$ around $[\min(0, \inf\operatorname{supp}\tilde\chi), \infty[$. Using the bounds $g \le g_\zeta$, $a_\zeta \le a$ and $|b_\zeta| \le |b|$ we then obtain that for any $m \in \mathbb{R}$

$$\big(\mathrm{Op^w}(\chi_-(a_\zeta)\tilde\chi_+(b_\zeta)) - 1\big)A \in \Psi(\langle x\rangle^m, g_{\mu,\lambda}). \tag{4.52}$$

By combining Remark 4.2 2), (4.51) (with $s = 0$, $t = s_0 - s + \frac{\epsilon}{2}$ and $\delta = \frac12 + \frac{\epsilon}{2}$) and (4.52) we obtain the uniform bound: For all $\kappa \in]0,1]$

$$\|\langle x\rangle^{-t-\delta} g^{\frac12} R(\zeta) A g^{\frac12}\langle x\rangle^{t-\delta}\| \le C. \tag{4.53}$$

Obviously we obtain from (4.53) and a density argument that indeed there exists the limit

$$u := \lim_{\kappa\searrow 0} R(\lambda+\mathrm{i}\kappa)Av \text{ in } L^{2,s-2s_0-\epsilon}.$$

Since $u = R(\lambda+\mathrm{i}0)Av$ for $v \in L^{2,\infty}$ we are done (by using density and interchanging limits). $\square$

4.4. **Sommerfeld radiation condition.** In this subsection we describe a version of the Sommerfeld radiation condition close in spirit to [Hö2, Theorem 30.2.7], [Is2] and [Me].

We introduce for $s > 0$ Besov spaces B_s and corresponding duals B_s^* as in [AH] (see [Hö2, Section 14.1] for details about these spaces). They consist of local L^2 functions with a certain (norm) expression being finite.

We shall actually throughout this subsection only use the duals B_s^* for which we can take the norm squared to be

$$\|u\|^2_{B_s^*} := \sup_{R>1} R^{-2s} \int_{|x|<R} |u|^2 \mathrm{d}x.$$

An equivalent norm is given by the square root of the expression

$$\int_{|x|<1} |u|^2 \mathrm{d}x + \sup_{R>1} R^{-2s} \int_{R/2<|x|<R} |u|^2 \mathrm{d}x.$$

In particular we see that for all $s, s' > 0$ the map $X^{s'-s} : B_s^* \to B_{s'}^*$ is bicontinuous.

The subspace $B_{s,0}^* \subseteq B_s^*$ is specified by the additional condition

$$\lim_{R\to\infty} R^{-2s} \int_{|x|<R} |u|^2 \mathrm{d}x = 0,$$

or equivalently,

$$\lim_{R\to\infty} R^{-2s} \int_{R/2<|x|<R} |u|^2 \mathrm{d}x = 0.$$

There are inclusions

$$L^{2,-s} \subseteq B_{s,0}^* \subseteq B_s^* \subseteq \cap_{s'>s} L^{2,-s'}. \tag{4.54}$$

We introduce a notion of scattering wave front set of a distribution $u \in L^{2,-\infty}$ relative to the Besov space $B_{s,0}^*$, $s > 0$, say denoted by $WF(B_{s,0}^*, u)$. It is the complement within $\mathbb{T}^*$ given by replacing $WF_{\mathrm{sc}}^{-s}(u) \to WF(B_{s,0}^*, u)$ and $L^{2,-s} \to B_{s,0}^*$ in (4.8) (here (4.8) is considered with $s \to -s$). Obviously (4.54) implies the inclusions

$$WF_{\mathrm{sc}}^{-s}(u) \supseteq WF(B_{s,0}^*, u) \supseteq WF_{\mathrm{sc}}^{-s'}(u);\ s' > s. \tag{4.55}$$

Proposition 4.10. *Suppose $v \in L^{2,s_0'}$ for some $s_0' > s_0$ (here s_0 is given in (4.6)). Then the equation $(H-\lambda)u = v$ has a unique solution $u \in L^{2,-\infty}$ obeying one of the following conditions*

(i) $WF_{\mathrm{sc}}^{-s_0}(u) \subseteq \{b > -1\}$,

(ii) $WF(B_{s_0,0}^*, u) \subseteq \{b > 0\}$.

This solution is given by $u = R(\lambda + \mathrm{i}0)v \in L^{2,-s}$ for all $s > s_0$ and $WF_{\mathrm{sc}}^{-s_0}(u) \subseteq \{b = 1\}$.

Similarly, under the same condition on v, the equation $(H-\lambda)u = v$ has a unique solution $u \in L^{2,-\infty}$ obeying one of the following conditions

(i)' $WF_{\mathrm{sc}}^{-s_0}(u) \subseteq \{b < 1\}$,

(ii)' $WF(B_{s_0,0}^*, u) \subseteq \{b < 0\}$;

and this solution is given by $u = R(\lambda - \mathrm{i}0)v \in L^{2,-s}$ for all $s > s_0$ and $WF_{\mathrm{sc}}^{-s_0}(u) \subseteq \{b = -1\}$.

Proof. We shall only consider the first mentioned cases (i) or (ii) (they will be treated in parallel); the other cases can be treated similarly. By Proposition 4.7, the function $u = \tilde{u} := R(\lambda + \mathrm{i}0)v$ is a solution to $(H-\lambda)u = v$ enjoying the stated properties (including (i) and (ii)). Suppose in the sequel that $u \in L^{2,-t}$ for some $t > s_0$, $(H-\lambda)u = v$ and $WF_{\mathrm{sc}}^{s_0}(u) \subseteq \{b > -1\}$ or $WF(B_{s,0}^*, u) \subseteq \{b > 0\}$. It remains to be shown that $u = \tilde{u}$.

Step I. We shall show that $u \in L^{2,-s}$ for all $s > s_0$. By Proposition 4.3

$$WF_{\rm sc}^{-s_0}(u) \subseteq \{b^2 + \vec{c}^2 = 1\}, \tag{4.56}$$

$$A\mathrm{Op}^{\rm w}\big(F(b^2 + \vec{c}^2 > 3)\big)u \in L^{2,-s_0} \text{ for all } A \in \Psi(\langle \xi/g\rangle_1^2, g_{\mu,\lambda}). \tag{4.57}$$

It follows from (4.55), Propositions 4.3 and 4.5 and a compactness argument that

$$WF_{\rm sc}^{-s}(u) \subseteq \{b = 1\} \text{ for all } s > s_0. \tag{4.58}$$

Pick a real-valued decreasing $\psi \in C_c^\infty([0,\infty))$ such that $\psi(r) = 1$ in a small neighbourhood of 0 and $\psi'(r) = -1$ if $1/2 \le r \le 1$. Let $\psi_R(x) = \psi\big(|x|/R\big)$; $R > 1$. We also introduce

$$\delta = \max\big(t - s_0, 2t - 2s_0 + \mu - 2\big),$$

and check that

$$\delta + s_0' \ge t,\ s_0 + \delta/2 + 1 - \mu/2 \ge t \text{ and } s_0 + \delta/2 < t.$$

By undoing the commutator we have on one hand that

$$\langle \mathrm{i}[H, X^{-\delta}\psi_R]\rangle_u = -2\,\mathrm{Im}\,\langle v, X^{-\delta}\psi_R u\rangle, \tag{4.59}$$

yielding the estimate

$$|\langle \mathrm{i}[H, X^{-\delta}\psi_R]\rangle_u| \le C_1\|v\|_{s_0'}\|\,\|u\|_{-\delta-s_0'} \le C_2\|v\|_{s_0'}\|\,\|u\|_{-t} = O(R^0). \tag{4.60}$$

On the other hand

$$\mathrm{i}[H, X^{-\delta}\psi_R] = \mathrm{Re}\,\big(g\langle x\rangle h_{\delta,R}\mathrm{Op}^{\rm w}(b)\big);$$
$$h_{\delta,R}(x) = -\delta X^{-2-\delta}\psi_R(x) + X^{-\delta}(|x|R)^{-1}\psi'\big(|x|/R\big),$$

yielding by using (4.57), (4.58) and the calculus (cf. [Hö1, Theorems 18.5.4, 18.6.3, 18.6.8])

$$\langle \mathrm{i}[H, X^{-\delta}\psi_R]\rangle_u = \mathrm{Re}\,\langle g\langle x\rangle h_{\delta,R}\mathrm{Op}^{\rm w}\big(bF(b > 1/2)F(b^2 + \vec{c}^2 < 6)\big)\rangle_u + O(R^0),$$

which in turn (by the same arguments) implies that

$$\langle \mathrm{i}[H, X^{-\delta}\psi_R]\rangle_u \le -\delta 4^{-1}\langle g\langle x\rangle X^{-2-\delta}\psi_R\rangle_u + O(R^0). \tag{4.61}$$

By combining (4.60) and (4.61) we obtain

$$\langle g\langle x\rangle X^{-2-\delta}\psi_R\rangle_u \le C \tag{4.62}$$

for some constant C which is independent of $R > 1$. Whence letting $R \to \infty$ we see that $u \in L^{2,-t_1}$; $t_1 := s_0 + \delta/2$.

More general, we define for $k \in \mathbb{N}$

$$t_k = s_0 + 2^{-1}\max\big(t_{k-1} - s_0, 2t_{k-1} - 2s_0 + \mu - 2\big);\ t_0 := t,$$

iterate the above procedure and conclude that $u \in L^{2,-t_k}$ and hence that indeed $u \in L^{2,-s}$ for all $s > s_0$.

Step II. Due to Step I it suffices to show that $u = 0$ is the only solution to the equation $(H - \lambda)u = 0$ subject to the conditions $u \in L^{2,-s}$ for all $s > s_0$ and either $WF_{\rm sc}^{-s_0}(u) \subseteq \{b > -1\}$ or $WF(B^*_{s_0,0}, u) \subseteq \{b > 0\}$. Consider in the following Steps III and IV this problem.

Step III. We shall show that $u \in B^*_{s_0,0}$. Under Condition (i) the bound (4.58) holds for $s = s_0$ (by Proposition 4.5) which implies that

$$\text{There exists } \epsilon > 0 : WF(B^*_{s,0}, u) \subseteq \{b > \epsilon\}.$$

Under Condition (ii) we have the same conclusion due to (4.56) and a compactness argument. Next, we apply the same scheme as in Step I now with $\delta = 0$ and using a factor of $F(b > \epsilon)$ instead of a factor of $F(b > 1/2)$. This leads to

$$R^{-1}\langle g\langle x\rangle |x|^{-1}\psi'\big(|\cdot|/R^{-1}\big)\rangle_u = o(R^0),$$

and hence $u \in B^*_{s_0,0}$.

Step IV. We shall show that $u = 0$. For convenience we assume that $\epsilon_2 \leq 2 - \mu$. First, letting $s \in]s_0 - \epsilon_2/2, s_0[$ be given arbitrarily, our goal is to show that $u \in L^{2,-s}$. For that consider for $\kappa \in]0, 1/2]$

$$b_\kappa = X^{s_0} a_\kappa;\ a_\kappa = \Big(\tfrac{X}{X_\kappa}\Big)^{-s} X_\kappa^{-s_0} F(-b > 1/2) F(b^2 + \bar{c}^2 < 3); \tag{4.63}$$

here we use the regularization factor of (4.16). We calculate the Poisson bracket

$$\Big\{h_2, \Big(\tfrac{X}{X_\kappa}\Big)^{2s_0-2s}\Big\} = (1-\kappa)(2s_0 - 2s)\langle x\rangle X^{-1} X_\kappa^{-3} \Big(\tfrac{X}{X_\kappa}\Big)^{2s_0-2s-1} gb;$$

obviously this is negative on the support of b_κ, in fact with the (uniform) upper bounds

$$\begin{aligned}\cdots &\leq -8^{-1}(2s_0 - 2s)\Big(\langle x\rangle X^{2s_0-2} g\Big) X_\kappa^{-2}\Big(\big(\tfrac{X}{X_\kappa}\big)^{-s} X_\kappa^{-s_0}\Big)^2\\ &\leq -cX_\kappa^{-2}\Big(\big(\tfrac{X}{X_\kappa}\big)^{-s} X_\kappa^{-s_0}\Big)^2;\ c > 0.\end{aligned}$$

Similarly, by (4.29),

$$\begin{aligned}&\big\{h_2, F^2(-b > 1/2)\big\}\\ &\quad = -\frac{g}{r}(F^2)'(-b > 1/2)\Big(\big(1 - rV_1' g^{-2}\big)\bar{c}^2 + \big(rV_1' g^{-2}\big) g^{-2} 2(h_2 - \lambda) + O\big(r^{-\epsilon_2}\big)\Big);\end{aligned}$$

expanding the right hand side into a sum of three terms the first term is non-positive.

We introduce the quantizations $A_\kappa = \mathrm{Op^w}(a_\kappa)$ and $B_\kappa = \mathrm{Op^w}(b_\kappa)$, and the states $u_R(x) = \psi_R(x)u(x)$; $R > 1$. By Step III

$$\lim_{R\to\infty} \langle \mathrm{i}[H, B_\kappa^2]\rangle_{u_R} = 0. \tag{4.64}$$

On the other hand due to the above considerations the expectation of $\mathrm{i}[H, B_\kappa^2]$ in u_R tends to be negative. Keeping the precise upper bounds in mind we can let $R \to \infty$ (using the calculus, (4.12a) to deal with a contribution from V_3 and (4.64)) obtaining

$$c\|X_\kappa^{-1} A_\kappa u\|^2 \ \big(= \lim_{R\to\infty} c\|X_\kappa^{-1} A_\kappa u_R\|^2\big) \leq C,$$

where the constants c (the one given above) and C are positive and independent of κ. Whence letting $\kappa \to 0$ we conclude that

$$\mathrm{Op^w}\Big(F(-b > 1/2) F(b^2 + \bar{c}^2 < 3)\Big) u \in L^{2,-s}. \tag{4.65}$$

Upon replacing the factor $F(-b > 1/2)$ in (4.63) by $F(b > 1/2)$ we can argue similarly and obtain

$$\mathrm{Op^w}\Big(F(b > 1/2) F(b^2 + \bar{c}^2 < 3)\Big) u \in L^{2,-s}. \tag{4.66}$$

In combination with Proposition 4.5 the bounds (4.65) and (4.66) and the fact that (4.56) holds with s_0 replaced by s (note this is trivial since by assumption now $v = 0$) yield that $u \in L^{2,-s}$.

Next, the above procedure can be iterated: Assuming that $u \in L^{2,-s}$ for all $s > t_k := s_0 - k\epsilon_2/2$ (for some $k \in \mathbb{N}$) the procedure leads to $u \in L^{2,-s}$ for all $s > t_{k+1}$. Consequently $u \in L^{2,s}$ for all $s \in \mathbb{R}$; in particular $u \in L^2$ and therefore $u = 0$. $\square$

5. Fourier integral operators

In this section we construct and study certain modifiers in the form of Fourier integral operators; they will enter in the construction of wave operators in Section 6.

5.1. **The WKB ansatz.** Assume first that Condition 1.1 holds. Fix $\sigma_0 \in]0, 2[$. Recall from Lemma 3.1 that there exists a decreasing function $]0, \infty[\ni \lambda \mapsto R_0(\lambda)$ such that on the set

$$\Big\{(x, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \mid x \in \Gamma^+_{R_0(|\xi|^2/2),\sigma_0}(\hat{\xi})\Big\}$$

we can construct a solution ϕ^+ of the eikonal equation satisfying the (non-uniform in energy) bounds (3.3).

We fix $0 < \sigma < \sigma' < \sigma_0$. Next we introduce smoothed out characteristic functions

$$\chi_1(r) = \begin{cases} 1 & \text{for } r \geq 2 \\ 0 & \text{for } r \leq 1 \end{cases}, \tag{5.1}$$

and

$$\chi_2(l) = \begin{cases} 1 & \text{for } l \geq 1 - \sigma \\ 0 & \text{for } l \leq 1 - \sigma' \end{cases}. \tag{5.2}$$

Define

$$a_0^+(x, \xi) := \chi_2(\hat{x} \cdot \hat{\xi})\chi_1\left(|x|/R_0(|\xi|^2/2)\right).$$

The basic idea of Isozaki-Kitada is to use the modifier given by a Fourier integral operator J_0^+ on $L^2(\mathbb{R}^d)$ of the form

$$(J_0^+ f)(x) = (2\pi)^{-d/2} \int \mathrm{e}^{\mathrm{i}\phi^+(x,\xi)} a_0^+(x, \xi)\hat{f}(\xi)\mathrm{d}\xi, \tag{5.3}$$

where

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x$$

denotes the (unitary) Fourier transform of f.

If we assume that the potentials satisfy Conditions 2.1 and 2.2, then we can assume that the function $R_0(\lambda)$ is the constant R_0 given by Lemma 3.2. Thus in this case the solution $\phi^+(x, \omega, \lambda)$ of the eikonal equation is defined in $\Gamma^+_{R_0,\sigma_0} \times [0, \infty[$ (here σ_0 is also given by Lemma 3.2; possibly it is much smaller than 2), and the amplitude a_0 is simply given by

$$a_0^+(x, \xi) := \chi_2(\hat{x} \cdot \hat{\xi})\chi_1(|x|/R_0).$$

5.2. **The improved WKB ansatz.** The modifier J_0^+ (and its incoming counterpart, say J_0^-) is sufficient only for the most basic purposes, such as the existence of the outgoing (incoming) wave operator. To study finer properties of wave operators it is useful to use a more refined construction suggested by the WKB method.

This more refined construction is possible and useful already under Condition 1.1. However, for simplicity of presentation, in the remaining part of the section we will assume that the potentials satisfy the more restrictive Conditions 2.1 and 2.2. These conditions allow us to extend this and related constructions (see Subsection 5.5) down to (and including) $\lambda = 0$. Therefore, it will be convenient to switch between the two notations $\phi^+(x,\xi)$ and $\phi^+(x,\omega,\lambda)$. This will be done tacitly in the following, and in fact, we shall often slightly abuse notation by writing $(x,\xi) \in \Gamma^+_{R_0,\sigma_0}$ instead of $(x,\omega,\lambda) \in \Gamma^+_{R_0,\sigma_0} \times [0,\infty[$.

The WKB method suggests to approximate the wave operator by a Fourier integral operator J^+ on $L^2(\mathbb{R}^d)$ of the form

$$(J^+f)(x) = (2\pi)^{-d/2} \int \mathrm{e}^{\mathrm{i}\phi^+(x,\xi)} a^+(x,\xi)\hat{f}(\xi)\mathrm{d}\xi, \tag{5.4}$$

where the symbol $a^+(x,\xi)$ is supported in $\Gamma^+_{R_0,\sigma_0}$ and constructed by an iterative procedure to make the difference $T^+ := \mathrm{i}(HJ^+ - J^+H_0)$ small in an outgoing region $\Gamma^+_{R,\sigma}$ for some $R > R_0$, $\sigma < \sigma_0$. We have

$$(T^+f)(x) = (2\pi)^{-d/2} \int \mathrm{e}^{\mathrm{i}\phi^+(x,\xi)} t^+(x,\xi)\hat{f}(\xi)\mathrm{d}\xi, \tag{5.5}$$

where

$$t^+(x,\xi) = \left((\nabla_x\phi^+(x,\xi))\cdot\nabla_x + \tfrac{1}{2}(\triangle_x\phi^+(x,\xi))\right) a^+(x,\xi) - \tfrac{\mathrm{i}}{2}\triangle_x a^+(x,\xi). \tag{5.6}$$

As it is well-known from the WKB method, it is possible to improve on the ansatz by putting (here we need $\xi \neq 0$)

$$a^+(x,\xi) := \left(\det \nabla_\xi\nabla_x\phi^+(x,\xi)\right)^{1/2} b^+(x,\xi), \tag{5.7}$$

$$t^+(x,\xi) := \left(\det \nabla_\xi\nabla_x\phi^+(x,\xi)\right)^{1/2} r^+(x,\xi). \tag{5.8}$$

We have

$$\left((\nabla_x\phi^+(x,\xi))\cdot\nabla_x + \tfrac{1}{2}(\triangle_x\phi^+(x,\xi)\right)\left(\det \nabla_\xi\nabla_x\phi^+(x,\xi)\right)^{1/2} = 0,$$

and therefore

$$\begin{aligned} r^+(x,\xi) &= (\nabla_x\phi^+(x,\xi))\cdot\nabla_x b^+(x,\xi) \\ &\quad - \left(\det \nabla_\xi\nabla_x\phi^+(x,\xi)\right)^{-1/2} \tfrac{\mathrm{i}}{2}\triangle_x \left(\det \nabla_\xi\nabla_x\phi^+(x,\xi)\right)^{1/2} b^+(x,\xi). \end{aligned}$$

It is useful to introduce

$$\zeta^+(x,\xi) = \ln\left(\det \nabla_\xi\nabla_x\phi(x,\xi)\right)^{1/2};\ \xi \neq 0. \tag{5.9}$$

Note that it satisfies the equation

$$(\nabla_x\phi(x,\xi))\cdot\nabla_x\zeta^+(z,\xi) + \tfrac{1}{2}\triangle_x\phi(x,\xi) = 0. \tag{5.10}$$

Proposition 5.1. *For $(x,\xi) \in \Gamma^+_{R,\sigma}$, $\xi \neq 0$,*

$$\zeta^+(x,\xi) = \frac{1}{2}\int_1^\infty \triangle_y\phi^+(y^+(t;x,\xi),\xi)\mathrm{d}t. \tag{5.11}$$

Proof. Both $\zeta^+(x,\xi)$ and the right hand side of (5.11) satisfy the first order equation (5.10). Both go to zero as $|x| \to \infty$. In particular, they go to zero along the characteristics $t \to y^+(t,x,\xi)$. Therefore, they coincide. $\square$

Lemma 5.2. *There exist the uniform limits*

$$\lim_{\lambda \searrow 0} \partial_\omega^\delta \partial_x^\gamma \left(\zeta^+(x,\xi) - \zeta^+_{\mathrm{sph}}(x,\xi)\right).$$

Besides, we have uniform estimates with $\breve{\epsilon}$ given as in Proposition 3.3

$$\partial_\omega^\delta \partial_x^\gamma \left(\zeta^+(x,\xi) - \zeta^+_{\mathrm{sph}}(x,\xi)\right) = O(|x|^{-|\gamma|-\breve{\epsilon}});\ |\delta| + |\gamma| \geq 0.$$

Proof. Below div and ∇ will always involve the derivatives w.r.t. the first argument.

$$\begin{aligned}
\zeta^+(x,\xi) - \zeta^+_{\mathrm{sph}}(x,\xi) &= \int_1^\infty \frac{1}{2}\left(\mathrm{div}F^+(y^+(t),\xi) - \mathrm{div}F^+_{\mathrm{sph}}(y^+_{\mathrm{sph}}(t),\xi)\right)\mathrm{d}t \\
&= \int_1^\infty \mathrm{d}t\, \frac{1}{2}\int_0^1 \nabla\mathrm{div}F^+(y^+_l(t),\xi)\cdot\left(y^+(t) - y^+_{\mathrm{sph}}(t)\right)\mathrm{d}l \\
&\quad + \int_1^\infty \frac{1}{2}\left(\mathrm{div}F^+(y^+_{\mathrm{sph}}(t),\xi) - \mathrm{div}F^+_{\mathrm{sph}}(y^+_{\mathrm{sph}}(t),\xi)\right)\mathrm{d}t \\
&= I + II,
\end{aligned}$$

where $y^+_l(t) = ly^+(t) + (1-l)y^+_{\mathrm{sph}}(t)$.

Now I can be estimated (cf. (3.5f) and [DS1, (6.43)]) by

$$\begin{aligned}
C_1\int_1^\infty |y^+|^{-2}g(|y^+|)t^{\alpha-\epsilon}\mathrm{d}t &\leq C_2\int_{|x|}^\infty |y^+|^{-2}|y^+|^{(\alpha-\epsilon)/\alpha}\mathrm{d}|y^+| \\
&= O(|x|^{-\epsilon/\alpha}) = O(|x|^{-\breve{\epsilon}});
\end{aligned} \tag{5.12}$$

here $\alpha = 2/(2+\mu)$ and $\epsilon > 0$ is specified in [DS1, Subsection 6.1]. We used that

$$\frac{\mathrm{d}|y^+|}{\mathrm{d}t} \geq cg(|y^+|), \quad |y^+| \geq ct^\alpha, \quad c > 0.$$

Splitting the time-integral as $\int_1^{T_0}\mathrm{d}t + \int_{T_0}^\infty \mathrm{d}t$ the argument above yield (uniform) smallness of the second term (provided T_0 is chosen big). As for the contribution from the first term we can apply the dominated convergence theorem; whence we obtain the existence of $\lim_{\lambda\searrow 0} I$.

Next $\partial_\omega^\delta\partial_x^\gamma I$ is a sum integrals of terms of the following form:

$$\partial_\omega^{\delta_1}\partial_x^{\gamma_1}y^+_l \cdots \partial_\omega^{\delta_n}\partial_x^{\gamma_n}y^+_l \partial^n_{y^+_l}\partial^\nu_\omega \nabla\mathrm{div}F(y^+_l,\xi)\cdot\partial_\omega^\alpha\partial_x^\beta\left(y^+(t) - y^+_{\mathrm{sph}}(t)\right).$$

where $\delta_1 + \cdots + \delta_n + \nu + \alpha = \delta$ and $\gamma_1 + \cdots + \gamma_n + \beta = \gamma$. This can be estimated (cf. (3.5f) and [DS1, (4.41) and (6.43)]) by

$$C|x|^{-|\gamma|}|y^+|^{-2}g(|y^+|)t^{\alpha-\epsilon}.$$

We argue as above to obtain uniform bounds on $\partial_\omega^\delta\partial_x^\gamma I$ as well as the existence of $\lim_{\lambda\searrow 0}\partial_\omega^\delta\partial_x^\gamma I$.

Now II is bounded (cf. (3.5g)) by

$$C_1\int_1^\infty |y^+|^{-1-\breve{\epsilon}}g(|y^+|)\mathrm{d}t \leq C_2\int_{|x|}^\infty |y^+|^{-1-\breve{\epsilon}}\mathrm{d}|y^+| = O(|x|^{-\breve{\epsilon}}). \tag{5.13}$$

Then we apply the dominated convergence theorem as above, and we obtain the existence of $\lim_{\lambda\searrow 0} II$.

$\partial_\omega^\delta \partial_x^\gamma II$ is a sum of integrals of terms of the form

$$\partial_\omega^{\delta_1}\partial_x^{\gamma_1} y^+ \cdots \partial_\omega^{\delta_n}\partial_x^{\gamma_n} y^+ \partial_{y^+}^n \partial_\omega^\nu \left(\mathrm{div}F^+(y^+,\xi) - \mathrm{div}F^+_{\mathrm{sph}}(y^+,\xi)\right),$$

where $\delta_1 + \cdots + \delta_n + \nu = \delta$ and $\gamma_1 + \cdots + \gamma_n = \gamma$. This can be estimated (cf. (3.5g) and [DS1, (4.41) and (6.43)]) by

$$C|x|^{-|\gamma|}|y^+|^{-1-\breve{\epsilon}} g(|y^+|).$$

Then we can argue as above. □

Define

$$\tilde{\zeta}^+(x,\omega,\lambda) := \zeta^+(x, \sqrt{2\lambda}\omega) - \ln(2\lambda)^{(2-d)/4}.$$

Proposition 5.3. (i) *There exist (uniform) estimates*

$$|\tilde{\zeta}^+(x,\omega,\lambda) - \ln g(|x|)^{(d-2)/2})| \le C, \tag{5.14a}$$

$$\partial_\omega^\delta \partial_x^\gamma \tilde{\zeta}^+(x,\omega,\lambda) = O\big(|x|^{-|\gamma|}\big), \quad \textit{for } |\delta| + |\gamma| \ge 1 \tag{5.14b}$$

(ii) *There exist (uniform) estimates*

$$\begin{aligned}&(2\lambda)^{(d-2)/4}\partial_\omega^\delta \partial_x^\gamma \left(\det \nabla_\xi \nabla_x \phi^+(x,\xi)\right)^{1/2}, \\ &\quad = g(|x|)^{(d-2)/2} O\big(|x|^{-|\gamma|}\big), \quad \textit{for } |\delta| + |\gamma| \ge 0.\end{aligned} \tag{5.14c}$$

(iii) *There exist the locally uniform limits*

$$\partial_\omega^\delta \partial_x^\gamma \tilde{\zeta}^+(x,\omega,0) := \lim_{\lambda \searrow 0} \partial_\omega^\delta \partial_x^\gamma \tilde{\zeta}^+(x,\omega,\lambda).$$

Proof. Let us first prove the estimates (5.14b) for $|\delta| = 0$, $|\gamma| \ge 1$ in the spherically symmetric case. $\partial_x^\gamma \zeta^+_{\mathrm{sph}}(x,\xi)$ is an integral of terms of the form

$$\partial_x^{\gamma_1} y \cdots \partial_x^{\gamma_n} y \partial_y^n \mathrm{div} F^+_{\mathrm{sph}}(y^+(t),\xi),$$

where $\gamma_1 + \cdots + \gamma_n = \gamma$. Using $\partial_x^\gamma y^+ = O\big(|x|^{1-|\gamma|} g(|x|) g(|y^+|)^{-1}\big)$, cf. [DS1, Proposition 4.9], these integrals are bounded by

$$\begin{aligned}C_1 &\int_1^\infty |x|^{-|\gamma|+n} g(|x|)^n g(|y^+|)^{-n+1} |y^+|^{-n-1} \mathrm{d}t \\ &\le C_2 \int_{|x|}^\infty |x|^{-|\gamma|+n} g(|x|)^n g(|y^+|)^{-n} |y^+|^{-n-1} \mathrm{d}|y^+| = O(|x|^{-|\gamma|}).\end{aligned}$$

Thus

$$\partial_x^\gamma \zeta^+_{\mathrm{sph}}(x,\xi) = O\big(|x|^{-|\gamma|}\big) \text{ for } |\gamma| \ge 1.$$

Clearly we can argue as above for $|\delta| > 0$ as well. If $|\gamma| = 0$ we can use the formula (valid due to spherical symmetry)

$$\zeta^+_{\mathrm{sph}}(x, R_\eta \xi) = \zeta^+_{\mathrm{sph}}(R_\eta^{-1} x, \xi),$$

for any d-dimensional rotation R_η. Clearly this converts ω-derivatives to x-derivatives, and consequently we have shown (5.14b) in the general case.

Taking into account Lemma 5.2 we obtain the estimates (5.14b) in the general case (when V is not necessarily radial).

We have

$$\tilde{\zeta}^+_{\mathrm{sph}}(x,\xi) = \tilde{\zeta}^+_{\mathrm{sph}}(x, \sqrt{2\lambda}\hat{x}) + \int_0^\theta \nabla_\omega \tilde{\zeta}^+_{\mathrm{sph}}(x, \sqrt{2\lambda}\omega(l)) \cdot \omega^\perp(l) \mathrm{d}l,$$

where $[0,\theta] \ni l \mapsto \omega(l)$ is the arc joining $\hat{x}$ and ω and $\omega^\perp(l)$ is the tangent vector. Using (3.16) and (5.14b) with $|\delta| = 1$, $|\gamma| = 0$ and Lemma 5.2 we obtain (5.14a).

The above arguments in conjunction with the proof of Lemma 5.2 can be used to prove that there exist the limits

$$\lim_{\lambda \searrow 0} \partial_\omega^\delta \partial_x^\gamma \tilde{\zeta}^+(x,\omega,\lambda); \quad |\delta| + |\gamma| \geq 1.$$

We know from the explicit formula (3.13) that $\lim_{\lambda\searrow 0} \tilde{\zeta}^+_{\mathrm{sph}}(x,\hat{x},\lambda)$ exists locally uniformly in x. Hence so does $\lim_{\lambda\searrow 0} \tilde{\zeta}^+(x,\omega,\lambda)$ locally uniformly in $(x,\omega) \in \Gamma^+$.

As for the bounds (5.14c) we use (5.14a) and (5.14b). □

5.3. **Solving transport equations.** Introduce the operator

$$\begin{aligned} M &= \left(\det \nabla_\xi \nabla_x \phi^+(x,\xi)\right)^{-1/2} \tfrac{\mathrm{i}}{2} \triangle_x \left(\det \nabla_\xi \nabla_x \phi^+(x,\xi)\right)^{1/2} \\ &= \mathrm{e}^{-\tilde{\zeta}^+(x,\xi)} \tfrac{\mathrm{i}}{2} \triangle_x \mathrm{e}^{\tilde{\zeta}^+(x,\xi)} \\ &= \tfrac{\mathrm{i}}{2} \left(\triangle_x + 2\nabla_x \zeta^+(x,\xi) \cdot \nabla_x + \triangle_x \zeta^+(x,\xi) + \nabla_x \zeta^+(x,\xi)^2\right). \end{aligned}$$

Notice that due to Proposition 5.3 this operator is well-defined at $\lambda = 0$ (more precisely for $(x,\omega,\lambda) \in \Gamma^+_{R_0,\sigma_0} \times \{0\}$).

We define inductively for $(x,\xi) \in \Gamma^+_{R,\sigma_0}$:

$$\begin{aligned} b_0^+(x,\xi) &:= 1; \\ b_{m+1}^+(x,\xi) &:= \int_1^\infty M b_m^+(y(t,x,\xi,t),\xi)\mathrm{d}t. \end{aligned}$$

Proposition 5.4. *There exist the following (uniform) estimates:*

$$\partial_\omega^\delta \partial_x^\gamma b_m^+(x,\xi) = O(|x|^{-m(1-\mu/2)-|\gamma|}), \tag{5.15a}$$

$$\partial_\omega^\delta \partial_x^\gamma M b_m^+(x,\xi) = O(|x|^{-2-m(1-\mu/2)-|\gamma|}). \tag{5.15b}$$

Proof. For a given m, (5.15a) easily implies (5.15b).

Integrating $\partial_\omega^\delta \partial_x^\gamma M b_m(x,\xi)$ we can bound $\partial_\omega^\delta \partial_x^\gamma b_{m+1}(x,\xi)$ by

$$\begin{aligned} &\int_1^\infty |y^+|^{-2-m(1-\mu/2)-|\gamma|}\mathrm{d}t \\ &\quad \leq C_1 \int_{|x|}^\infty |y^+|^{-2-m(1-\mu/2)-|\gamma|} g(|y^+|)^{-1}\mathrm{d}|y^+| \\ &\quad \leq C_2 \int_{|x|}^\infty |y^+|^{-2-m(1-\mu/2)-|\gamma|+\mu/2}\mathrm{d}|y^+| = O(|x|^{-(m+1)(1-\mu/2)-|\gamma|}). \end{aligned}$$

This shows the induction step. □

We set

$$b^+(x,\xi) := \chi_2(\hat{x}\cdot\omega)\breve{b}^+(x,\xi); \qquad \breve{b}^+(x,\xi) = \sum_{m=0}^\infty b_m^+(x,\xi)\chi_1(|x|/R_m)$$

for an appropriately chosen sequence $R_m \to \infty$ (this is an example of the so-called Borel construction, cf. [Hö2, Proposition 18.1.3]). There are (uniform) bounds

$$\partial_\omega^\delta \partial_x^\gamma b^+(x,\xi) = O(|x|^{-|\gamma|}).$$

We introduce

$$\begin{aligned} r^+(x,\xi) &= \big(\nabla_x\phi^+(x,\xi)\cdot\nabla_x + M\big)\, b^+(x,\xi),\\ r^+_{\mathrm{pr}}(x,\xi) &= \chi_2(\hat x\cdot\omega)\,\big(\nabla_x\phi^+(x,\xi)\cdot\nabla_x + M\big)\,\breve b^+(x,\xi),\\ r^+_{\mathrm{bd}}(x,\xi) &= r^+(x,\xi) - r^+_{\mathrm{pr}}(x,\xi). \end{aligned} \tag{5.16}$$

(The subscript pr stands for the *propagation* and bd stands for the *boundary*).

Proposition 5.5. *There exist (uniform) bounds*

$$\partial^\delta_\omega\partial^\gamma_x r^+_{\mathrm{pr}}(x,\xi) = O(|x|^{-\infty}),$$

and $r^+_{\mathrm{bd}}(x,\xi)$ is supported away from $\Gamma^+_{R_0,\sigma}$ and

$$\partial^\delta_\omega\partial^\gamma_x r^+_{\mathrm{bd}}(x,\xi) = O(g(|x|)|x|^{-1-|\gamma|}).$$

5.4. **Constructions in incoming region.** Via the phase function $\phi^- = \phi^-(x,\omega,\lambda)$ given in (3.6) we can construct a symbol $a^- = \mathrm{e}^{\zeta^-}b^-$ with $t^- = \mathrm{e}^{\zeta^-}(r^-_{\mathrm{pr}} + r^-_{\mathrm{bd}})$, $r^-_{\mathrm{pr}} = O(|x|^{-\infty})$ and the symbol $r^-_{\mathrm{bd}} = O\big(g(|x|)|x|^{-1}\big)$ vanishing on a given $\Gamma^-_{R,\sigma} \subseteq \Gamma^-_{R_0,\sigma_0}$ and obeying appropriate analogues of the conditions of the previous subsection.

Similar to (5.4) we consider the Fourier integral operator J^- on $L^2(\mathbb{R}^d)$ given by

$$(J^-f)(x) = (2\pi)^{-d/2}\int \mathrm{e}^{\mathrm{i}\phi^-(x,\xi)}a^-(x,\xi)\hat f(\xi)\mathrm{d}\xi. \tag{5.17}$$

5.5. **Fourier integral operators at fixed energies.** For all $\tau \in L^2(S^{d-1})$ we introduce

$$(J^\pm(\lambda)\tau)(x) := (2\pi)^{-d/2}\int \mathrm{e}^{\mathrm{i}\phi^\pm(x,\omega,\lambda)}\tilde a^\pm(x,\omega,\lambda)\tau(\omega)\mathrm{d}\omega; \tag{5.18}$$

$$(T^\pm(\lambda)\tau)(x) := (2\pi)^{-d/2}\int \mathrm{e}^{\mathrm{i}\phi^\pm(x,\omega,\lambda)}\tilde t^\pm(x,\omega,\lambda)\tau(\omega)\mathrm{d}\omega; \tag{5.19}$$

where

$$\begin{aligned} \tilde a^\pm(x,\omega,\lambda) &:= (2\lambda)^{(d-2)/4}a^\pm(x,\sqrt{2\lambda}\omega),\\ \tilde t^\pm(x,\omega,\lambda) &:= (2\lambda)^{(d-2)/4}t^\pm(x,\sqrt{2\lambda}\omega). \end{aligned}$$

The functions $\tilde a^\pm$ and $\tilde t^\pm$ are continuous in $(x,\omega,\lambda) \in \mathbb{R}^d\times S^{d-1}\times[0,\infty)$. This fact will be very important in the forthcoming sections. Due to these properties we can *define* $J^\pm(\lambda)$ and $T^\pm(\lambda)$ at $\lambda = 0$ by the expressions (5.18) and (5.19), respectively. We can split $T^\pm(\lambda) = T^\pm_{\mathrm{bd}}(\lambda) + T^\pm_{\mathrm{pr}}(\lambda)$ in agreement with the decomposition (5.16) (cf. (5.8)).

Throughout this subsection $\breve\epsilon$ signifies the $\breve\epsilon > 0$ appearing in Proposition 3.3 (it is tacitly assumed that $\breve\epsilon < 1-\mu/2$)). For the problems at hand we can use coordinates for $\omega \in S^{d-1}$ sufficiently close to the d'th standard vector $e_d \in \mathbb{R}^d$ specified as follows (using a partition of unity in the $\hat x$–variable and a rotation of coordinates this is without loss of generality):

$$\omega = \omega_\perp + \omega_d e_d;\ \omega_d = \sqrt{1-\omega_\perp^2},\ \omega_\perp \in \mathbb{R}^{d-1},\ |\omega_\perp| \text{ small}. \tag{5.20}$$

Proposition 5.6. *There exist a (large) $R \ge R_0$ and a (small) $\tilde\sigma \in]0,\sigma_0]$ such that for all $|x| \ge R$ there exists a unique $\omega \in S^{d-1}$ satisfying $\omega\cdot\hat x \ge 1-\tilde\sigma$ (alternatively:*

$x \in \Gamma^+_{R,\tilde\sigma}(\omega)$) and $\partial_\omega \phi^+(x,\omega,\lambda) = 0$. We introduce the notation $\omega^+_{\rm crt} = \omega^+_{\rm crt}(x,\lambda)$ for this vector. It is smooth in x and we have

$$\partial_x^\gamma(\omega^+_{\rm crt} - \hat x) = O(|x|^{-\check\epsilon-|\gamma|}).$$

Let

$$\phi(x,\lambda) = \phi^+(x, \omega^+_{\rm crt}(x,\lambda), \lambda). \tag{5.21}$$

This function solves the eikonal equation

$$(\partial_x\phi(x,\lambda))^2/2 + V(x) = \lambda.$$

In the spherically symmetric case we have $\omega^+_{\rm crt} = \hat x$ and

$$\phi_{\rm sph}(x,\lambda) = \sqrt{2\lambda}R_0 + \int_{R_0}^{|x|}\sqrt{2\lambda - 2V(r)}{\rm d}r. \tag{5.22}$$

The proposition is obvious in the case $V_2 = 0$, cf. (3.9). The general case follows by an application of the fixed point theorem, cf. the proof of the similar statement [II, Lemma 4.1]. At this point one needs some control of the Hessian; we refer the reader to the proof of Theorem 5.7.

Of course, there is an analogue of Proposition 5.6 in the $-$ case; we then need to replace ϕ^+ with ϕ^-, and $\hat x$ with $-\hat x$. We obtain $\omega^-_{\rm crt}(x,\lambda) = -\omega^+_{\rm crt}(x,\lambda)$. Note the identity

$$\phi(x,\lambda) = -\phi^-(x, \omega^-_{\rm crt}(x,\lambda), \lambda).$$

Theorem 5.7. *Let $\tau \in C^\infty(S^{d-1})$. Then*

$$\big(J^\pm(\lambda)\tau\big)(x) = (2\pi)^{-\frac12}{\rm e}^{\mp{\rm i}\pi\frac{d-1}{4}}g^{-\frac12}(r,\lambda)r^{-\frac{d-1}{2}}\big({\rm e}^{\pm{\rm i}\phi(x,\lambda)}\tau(\pm\hat x) + O(r^{-\check\epsilon})\big). \tag{5.23}$$

Moreover (5.23) is uniform in $(\hat x,\lambda) \in S^{d-1}\times[0,\infty[$. The same asymptotics holds for

$$\pm g^{-1}\hat x\cdot pJ^\pm(\lambda)\tau(x).$$

Proof. We invoke the method of stationary phase (with parameter given by the expression $h = h(r)$ of (3.14)), cf. [Hö1, Theorem 7.7.6] or [II, Theorem 4.3]. For simplicity we consider only the $+$ case and we abbreviate $\omega_{\rm crt} = \omega^+_{\rm crt}$. This method yields (up to a minor point that is resolved below) that

$$\begin{aligned}\big(J^+(\lambda)\tau\big)(x) = (2\pi)^{-\frac d2}{\rm e}^{-{\rm i}\pi\frac{d-1}{4}}|\det(\partial^2_\omega\phi^+(x,\omega_{\rm crt},\lambda)/2\pi)|^{-\frac12}\\ \times {\rm e}^{{\rm i}\phi^+(x,\omega_{\rm crt},\lambda)}\big(\tilde a^+(x,\omega_{\rm crt},\lambda)\tau(\omega_{\rm crt}) + g^{\frac{d-2}{2}}O(r^{-\check\epsilon})\big).\end{aligned} \tag{5.24}$$

Let us consider the Hessian. We first compute it in the case $V_2 = 0$ choosing coordinates such that $\hat x = e_d$ and using (5.20):

$$\partial^2_{\omega_\perp}\phi^+_{\rm sph}(\omega = \hat x) = -\partial_{\omega_\perp}\partial_{\hat x}\phi^+_{\rm sph}(\omega = \hat x),$$

and using the fact that

$$\partial_{\omega_\perp}\partial_{\hat x}\phi^+_{\rm sph}(\omega = \hat x) = hI, \tag{5.25}$$

cf. the computation (3.12) (here I refers to the form on $TS^{d-1}_{\hat x=\omega}\times TS^{d-1}_\omega$ given by the Euclidean metric), we obtain that

$$\partial^2_{\omega_\perp}\phi^+_{\rm sph}(\omega = \hat x) = -hI. \tag{5.26}$$

In particular the critical point is non-degenerate in this case.

Since $\omega_{\rm crt}$ is a critical point, the second derivative has an invariant geometric meaning. Therefore, we can drop the reference to the special coordinates $\omega_\perp$ and we can write simply ∂_ω^2 for $\partial_{\omega_\perp}^2$ in the left hand side of (5.26). The formula (5.26) is then valid for all $\hat{x} \in S^{d-1}$.

The general case is similar. In particular, after applying Proposition 3.3 and (5.26), we obtain

$$|\det(\partial_\omega^2 \phi^+(x, \omega_{\rm crt}, \lambda))| = h^{d-1}(1 + O(r^{-\check{\epsilon}})). \tag{5.27}$$

We conclude by combining (3.13), Lemma 5.2, Proposition 5.3, (5.27) and the construction of the symbol $\tilde{a}^+$ that (5.24) and indeed also (5.23) hold.

The second part of the theorem follows similarly. □

6. Wave matrices

In this section we study (modified) wave matrices. We prove that they have a limit at zero energy, in the sense of maps into an appropriate weighted space. This implies asymptotic oscillatory formulas for the standard short-range and Dollard scattering matrices.

6.1. **Wave operators.**

The following theorem is essentially well-known (follows from (3.3)). It describes a construction of modified wave operators similar to that of Isozaki-Kitada [IK1, IK2]. Notice, however, that the original construction involved energies strictly bounded away from zero. Notice also that the construction of $J^\pm$ in Section 5, although given under Conditions 2.1 and 2.2, in fact can be done under Condition 1.1 as well.

Theorem 6.1. *Suppose that V satisfies Condition 1.1. Then*

$$W^\pm f = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} J_0^\pm \mathrm{e}^{-\mathrm{i}tH_0} f = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} J^\pm \mathrm{e}^{-\mathrm{i}tH_0} f;\ \hat{f} \in C_c(\mathbb{R}^d \setminus \{0\}). \tag{6.1}$$

The "wave operator" $W^\pm$ extends to an isometric operator on $L^2(\mathbb{R}^d)$ satisfying $HW^\pm = W^\pm H_0$, and its range is the absolutely continuous spectral subspaces of H. Moreover,

$$0 = \lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}tH} J_0^\pm \mathrm{e}^{-\mathrm{i}tH_0} f = \lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}tH} J^\pm \mathrm{e}^{-\mathrm{i}tH_0} f;\ \hat{f} \in C_c(\mathbb{R}^d \setminus \{0\}). \tag{6.2}$$

Remarks. We know that $J_0^\pm 1_{]\epsilon,\infty[}(H_0)$ and $J^\pm 1_{]\epsilon,\infty[}(H_0)$ are bounded for any $\epsilon > 0$, but we *do not* know if $J_0^\pm$ and $J^\pm$ are bounded (not even under Conditions 2.1 and 2.2). This is the reason for restricting the choice of vectors in (6.1) and (6.2). An alternative, and equivalent, definition of $W^\pm$ as a bounded operator on $L^2(\mathbb{R}^d)$ is the following:

$$W^\pm = \mathrm{s-}\lim_{\epsilon\searrow 0} \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} J^\pm 1_{]\epsilon,\infty[}(H_0) \mathrm{e}^{-\mathrm{i}tH_0}.$$

The following general fact serves as the basic formula in stationary scattering theory, see Appendix A for a derivation.

Lemma 6.2. *Suppose there are densely defined operators $\breve{J}^\pm$ and $\breve{T}^\pm$ on $L^2(\mathbb{R}^d)$ such that $\breve{J}^\pm 1_{]\epsilon,\infty[}(H_0)$ and $\breve{T}^\pm 1_{]\epsilon,\infty[}(H_0)$ are bounded for any $\epsilon > 0$ and that $\breve{T}^\pm f = i(H\breve{J}^\pm - \breve{J}^\pm H_0)f$ for any $f \in L^2(\mathbb{R}^d)$ with $\hat{f} \in C_c(\mathbb{R}^d \setminus \{0\})$. Suppose there exists*

$$\breve{W}^\pm f := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} \breve{J}^\pm \mathrm{e}^{-\mathrm{i}tH_0} f;\ \hat{f} \in C_c(\mathbb{R}^d \setminus \{0\}).$$

Then we have the following formula

$$\breve{W}^{\pm}f \;=\; \lim_{\epsilon\searrow 0}\int(\breve{J}^{\pm}+\mathrm{i}R(\lambda\mp\mathrm{i}\epsilon)\breve{T}^{\pm})\delta_\epsilon(\lambda)f\mathrm{d}\lambda, \tag{6.3}$$

where $\delta_\epsilon(\lambda)=\frac{R_0(\lambda+\mathrm{i}\epsilon)-R_0(\lambda-\mathrm{i}\epsilon)}{2\pi\mathrm{i}}=\frac{\epsilon}{\pi}\left((H_0-\lambda)^2+\epsilon^2\right)^{-1}$.

6.2. **Wave matrices at positive energies.** For any $s\in\mathbb{R}$ we recall the definition of weighted spaces $L^{2,s}(\mathbb{R}^d):=(1+x^2)^{-s/2}L^2(\mathbb{R}^d)$.

Let Δ_ω denote the Laplace-Beltrami operator on the sphere S^{d-1}. For $n\in\mathbb{R}$ we define the Sobolev spaces on the sphere $L^{2,n}(S^{d-1}):=(1-\Delta_\omega)^{-n/2}L^2(S^{d-1})$.

For $\lambda>0$ we introduce $\mathcal{F}_0(\lambda)$ by

$$\mathcal{F}_0(\lambda)f(\omega)=(2\lambda)^{(d-2)/4}\hat{f}(\sqrt{2\lambda}\omega).$$

Let $s>\frac{1}{2}$ and $n\geq 0$. Note that $\mathcal{F}_0(\lambda)$ is a bounded operator in the space $\mathcal{B}(L^{2,s+n}(\mathbb{R}^d),L^{2,n}(S^{d-1}))$ and depends continuously on $\lambda>0$. Likewise, $\mathcal{F}_0(\lambda)^*\in\mathcal{B}(L^{2,-n}(S^{d-1}),L^{2,-s-n}(\mathbb{R}^d))$ and it also depends continuously on $\lambda>0$. Note also that the operator

$$\int\oplus\mathcal{F}_0(\lambda)\,\mathrm{d}\lambda:L^2(\mathbb{R}^d)\to\int_0^\infty\oplus L^2(S^{d-1})\,\mathrm{d}\lambda \tag{6.4}$$

is unitary; consequently the operators $\mathcal{F}_0(\lambda)$ diagonalize the operator H_0. Finally,

$$\mathrm{s}-\lim_{\epsilon\searrow 0}\delta_\epsilon(\lambda) \;=\; \mathcal{F}_0(\lambda)^*\mathcal{F}_0(\lambda)\text{ in }\mathcal{B}(L^{2,s}(\mathbb{R}^d)),L^{2,-s}(\mathbb{R}^d)). \tag{6.5}$$

Due to the limiting absorption principle we have the following partial analogue of (6.5) for the full Hamiltonian, defined under Condition 1.1: Let $s>\frac{1}{2}$ and

$$\delta^V_\epsilon(\lambda):=\frac{R(\lambda+\mathrm{i}\epsilon)-R(\lambda-\mathrm{i}\epsilon)}{2\pi\mathrm{i}}. \tag{6.6}$$

Then there exists

$$\delta^V(\lambda):=\mathrm{s}-\lim_{\epsilon\searrow 0}\delta^V_\epsilon(\lambda)\text{ in }\mathcal{B}(L^{2,s}(\mathbb{R}^d)),L^{2,-s}(\mathbb{R}^d)). \tag{6.7}$$

The operator-valued function $\delta^V(\cdot)$ is a strongly continuous function of $\lambda>0$.

If Conditions 2.1–2.3 are true then we can extend the definition of $\delta^V(\lambda)$ to include $\lambda=0$ if we demand that $s>\frac{1}{2}+\frac{\mu}{4}$, and the corresponding operator-valued function will be a strongly continuous (in fact a norm continuous) function of $\lambda\geq 0$, cf. Remark 4.2 2).

In the remaining part of this section we shall assume that the positive parameter σ' in (5.2) is sufficiently small (this requirement can be fulfilled uniformly in $\lambda\geq 0$). Notice that the condition conforms well with Lemma 3.2; we need it at various points, see for example the proof of Lemma 6.9.

Formally, we have $J^{\pm}(\lambda)=J^{\pm}\mathcal{F}_0(\lambda)^*$ and $T^{\pm}(\lambda)=T^{\pm}\mathcal{F}_0(\lambda)^*$. This suggests that (6.3) can be used to define wave operators at a fixed energy. This idea is used in the following theorem (which is essentially well-known).

Theorem 6.3. *Suppose that the potential satisfies Condition 1.1. Let* $\epsilon>0$, $n\geq 0$ *and* $\lambda>0$. *Then*

$$W^{\pm}(\lambda):=J^{\pm}(\lambda)+\mathrm{i}R(\lambda\mp\mathrm{i}0)T^{\pm}(\lambda) \tag{6.8}$$

defines a bounded operator in $\mathcal{B}(L^{2,-n}(S^{d-1}),L^{2,-\frac{1}{2}-\epsilon-n}(\mathbb{R}^d))$, *which depends continuously on* $\lambda>0$. *It depends only on the splitting of the potential* V *into* V_1 *and* V_3

(but does not depend on the details of the construction of $J^\pm$). For all $f \in L^{2,\frac{1}{2}+\epsilon}(\mathbb{R}^d)$ and $g \in C_c(]0,\infty[)$, we have

$$W^\pm g(H_0) f = \int_0^\infty g(\lambda) W^\pm(\lambda) \mathcal{F}_0(\lambda) f \mathrm{d}\lambda. \tag{6.9}$$

Moreover,

$$W^\pm(\lambda) W^\pm(\lambda)^* = \delta^V(\lambda) \tag{6.10}$$

We set

$$w^\pm(\omega,\lambda) = W^\pm(\lambda)\delta_\omega,$$

where δ_ω denotes the delta-function at $\omega \in S^{d-1}$. Then for all multiindices δ the function

$$S^{d-1} \times]0,\infty[\ni (\omega,\lambda) \mapsto \partial_\omega^\delta w^\pm(\omega,\lambda) \in L^{2,-p}(\mathbb{R}^d);\ p > |\delta| + d/2,$$

is continuous.

Remark. The operator $W^\pm(\lambda) : \mathcal{D}'(S^{d-1}) \to L^{2,-\infty}$ is called the *wave matrix at the energy* λ. Its range consists of generalized eigenfunction at the energy λ. The function $w^\pm(\omega,\lambda)$ (which belongs to $W^\pm(\lambda) L^{2,\frac{1}{2}-p}(S^{d-1})$ for $p > \frac{d}{2}$) is called the *generalized eigenfunction at the energy λ and outgoing (or incoming) asymptotic normalized velocity ω.*

Let us explain the steps of a proof of Theorem 6.3 (in the case of "+"–superscript only); our (main) results contained in Theorems 6.5 and 6.6 will be proved by a parallel procedure.

First one introduces a partition of unity of the form

$$\begin{aligned} I &= \mathrm{Op^r}(\chi_+(a)) + \mathrm{Op^r}(\chi_-(a)\tilde{\chi}_-(b)) + \mathrm{Op^r}(\chi_-(a)\tilde{\chi}_+(b)) \\ &=: \mathrm{Op^r}(\chi_1) + \mathrm{Op^r}(\chi_2) + \mathrm{Op^r}(\chi_3). \end{aligned} \tag{6.11}$$

Here a and b are the symbols introduced in (4.4) (rather than in (4.1) since we do not here impose Conditions 2.1–2.3) and χ_+ is a real-valued function as in Proposition 4.1 (ii) such that $\chi_+(t) = 1$ for $t \geq 2C_0$, and $\chi_- = 1 - \chi_+$. Moreover $\tilde{\chi}_-, \tilde{\chi}_+ \in C^\infty(\mathbb{R})$ are real-valued functions obeying $\tilde{\chi}_- + \tilde{\chi}_+ = 1$ and

$$\operatorname{supp} \tilde{\chi}_- \subseteq (-\infty, 1-\bar{\sigma}], \tag{6.12}$$

$$\operatorname{supp} \tilde{\chi}_+ \subseteq [1-2\bar{\sigma}, \infty[. \tag{6.13}$$

The number $\bar{\sigma}$ needs to be taken (small) positive depending on the parameter σ of Subsection 5.1. (For the proof of Theorems 6.5 and 6.6 to be elaborated on later we refer at this point to (6.38) for the precise requirement.)

The proof of Theorem 6.3 is based on the following lemma:

Lemma 6.4. *Suppose that the potential satisfies Condition 1.1.*

(i) *For all $n \geq 0$ and $\epsilon > 0$, $J^+(\lambda)$ is a continuous function in $\lambda > 0$ with values in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,-\frac{1}{2}-\epsilon-n}(\mathbb{R}^d))$.*
(ii) *For all $n \in \mathbb{R}$ and $\epsilon > 0$, $T^+_{\mathrm{bd}}(\lambda)$ is a continuous function in $\lambda > 0$ with values in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,\frac{1}{2}-\epsilon-n}(\mathbb{R}^d))$.*
(iii) *For all $m, n \in \mathbb{R}$, $\mathrm{Op^r}(\chi_3) T^+_{\mathrm{bd}}(\lambda)$ is a continuous function in $\lambda > 0$ with values in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,m}(\mathbb{R}^d))$.*
(iv) *For all $m, n \in \mathbb{R}$, $T^+_{\mathrm{pr}}(\lambda)$ is a continuous function in $\lambda > 0$ with values in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,m}(\mathbb{R}^d))$.*

More general statements than Lemma 6.4 (i)–(iv) will be given and proven in the context of treating small energies (see Lemma 6.8); these statements are under Conditions 2.1 and 2.2. Let us here use (i)–(iv) in an

Outline of a proof of Theorem 6.3. The expression (6.8) is a well-defined element of $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,-\frac{1}{2}-\epsilon-n}(\mathbb{R}^d))$ due to the positive energy version of Proposition 4.8 and Lemma 6.4; this is for any $\epsilon > 0$ and $n \geq 0$. (Notice that (4.46) holds for any $t \in \mathbb{R}$ by Lemma 6.4.) Effectively this argument is based on the following scheme (to be used below): We insert the right hand side of (6.11) to the right of the resolvent in (6.8) and expand into three terms. Whence by using Remark 4.2 4) and Lemma 6.4 we see that $W^+(\lambda)$ is a sum of four well-defined operators in $\mathcal{B}\big(L^{2,-n}(S^{d-1}), L^{2,-\frac{1}{2}-\epsilon-n}(\mathbb{R}^d)\big)$, hence well-defined.

Next note that $\lambda \mapsto W^+(\lambda)$ is norm continuous due to norm continuity of each of the above mentioned four operators, which in turn may be seen by combining the continuity statements of Remark 4.2 4) and Lemma 6.4.

The statement on the independence of details of construction of $J^\pm$ is based on the positive energy version of Proposition 4.10; the interested reader will realize this by using arguments from the proof of Lemma 6.10 stated later.

The formula (6.9) can be verified by combining (6.3) with arguments used above, see Appendix A for an abstract approach. The identity (6.10) is a consequence of (6.9).

Finally due to the fact that $\partial_\omega^\delta \delta_\omega \in L^{2,\frac{1}{2}-p}(S^{d-1})$ for $p > |\delta| + \frac{d}{2}$ (with continuous dependence of $\omega \in S^{d-1}$) we conclude that indeed $\partial_\omega^\delta w^+(\omega, \lambda) \in L^{2,-p}(\mathbb{R}^d)$ with a continuous dependence of ω and λ.

6.3. **Wave matrices at low energies.** Until the end of this section we assume that Conditions 2.1–2.3 are true. The main new result of this section is expressed in the following two theorems which concern the low-energy behaviour of the wave matrices of Theorem 6.3:

Theorem 6.5. *For $s > \frac{1}{2} + \frac{\mu}{4}$ and $n \geq 0$*

$$W^\pm(0) := J^\pm(0) + \mathrm{i}R(\mp\mathrm{i}0)T^\pm(0) \tag{6.14}$$

defines a bounded operator in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,-s-n(1-\mu/2)}(\mathbb{R}^d))$. It depends only on the splitting of the potential V into $V_1 + V_2$ and V_3 (but does not depend on the details of the construction of $J^\pm$). We have

$$W^\pm(0)W^\pm(0)^* = \delta^V(0). \tag{6.15}$$

If we set

$$w^\pm(\omega, 0) = W^\pm(0)\delta_\omega,$$

then we obtain an element of $L^{2,-p}(\mathbb{R}^d)$ with $p > \frac{d}{2} + \frac{\mu}{2} - \frac{d\mu}{4}$ depending continuously on ω. In fact, more generally, $\partial_\omega^\delta w^\pm(\omega, 0) \in L^{2,-p}(\mathbb{R}^d)$ with $p > (|\delta| + \frac{d}{2})(1 - \frac{\mu}{2}) + \frac{\mu}{2}$ with continuous dependence on ω.

Theorem 6.6. *For all $\epsilon > 0$ and $n \geq 0$*

$$(\langle x \rangle g)^{-n} \langle x \rangle^{-\frac{1}{2}-\epsilon} g^{\frac{1}{2}} W^\pm(\lambda) \tag{6.16}$$

is a continuous $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$–valued function in $\lambda \in [0, \infty[$.

For all $\epsilon > 0$ and all multiindices δ the function

$$S^{d-1} \times [0, \infty[\ni (\omega, \lambda) \mapsto (\langle x \rangle g)^{-|\delta| + \frac{1}{2} - \frac{d}{2}} \langle x \rangle^{-\frac{1}{2} - \epsilon} g^{\frac{1}{2}} \partial_\omega^\delta w^\pm(\omega, \lambda) \in L^2(\mathbb{R}^d)$$

is continuous.

The following corollary interprets Theorem 6.6 in terms of the usual weighted spaces:

Corollary 6.7. *Let $n \geq 0$. We have*

$$W^\pm(0) = \lim_{\lambda \searrow 0} W^\pm(\lambda)$$

in the sense of operators in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,-\tilde{s}_n}(\mathbb{R}^d))$, where $\tilde{s}_n > \frac{1}{2} + n + \max\left(0, \frac{\mu}{4} - n\frac{\mu}{2}\right)$. For all multiindices δ the function

$$S^{d-1} \times [0, \infty[\ni (\omega, \lambda) \mapsto \partial_\omega^\delta w^\pm(\omega, \lambda) \in L^{2,-\tilde{p}}(\mathbb{R}^d)$$

is continuous with $\tilde{p} > \frac{d}{2} + |\delta|$ for $d \geq 2$ and $\tilde{p} > \frac{1}{2} + |\delta| + \max(0, (1 - 2|\delta|)\frac{\mu}{4})$ for $d = 1$.

The proof of Theorems 6.5 and 6.6 is based on the following analogue of Lemma 6.4 (for convenience we focus as before on the case of "+"–superscript only). The symbol χ_3 appearing in the statement (iii) below is specified as before, i.e. by (6.11) and the subsequent discussion.

Lemma 6.8. (i) *For all $n \geq 0$ and $\epsilon > 0$*

$$(\langle x \rangle g)^{-n} \langle x \rangle^{-\frac{1}{2} - \epsilon} g^{\frac{1}{2}} J^+(\lambda) \tag{6.17a}$$

is a continuous $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$–valued function in $\lambda \in [0, \infty[$.

(ii) *For all $n \in \mathbb{R}$ and $\epsilon > 0$*

$$(\langle x \rangle g)^{-n} \langle x \rangle^{\frac{1}{2} - \epsilon} g^{-\frac{1}{2}} T^+_{\mathrm{bd}}(\lambda) \tag{6.17b}$$

is a continuous $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$–valued function in $\lambda \in [0, \infty[$.

(iii) *For all $m, n \in \mathbb{R}$*

$$\langle x \rangle^m \mathrm{Op^r}(\chi_3) T^+_{\mathrm{bd}}(\lambda) \tag{6.17c}$$

is a continuous $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$–valued function in $\lambda \in [0, \infty[$.

(iv) *For all $m, n \in \mathbb{R}$*

$$\langle x \rangle^m T^+_{\mathrm{pr}}(\lambda) \tag{6.17d}$$

is a continuous $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$–valued function in $\lambda \in [0, \infty[$.

Later on we will need a slightly stronger bound than the one of Lemma 6.8 (i) with $n = 0$, which we state below (referring to notation of (4.6) and (4.54)):

Lemma 6.9. *For all $\tau \in L^2(S^{d-1})$, $J^+(\lambda)\tau \in B^*_{s_0}$. In fact with a bounding constant independent of $\lambda \geq 0$*

$$g^{\frac{1}{2}} J^+(\lambda) \in \mathcal{B}(L^2(S^{d-1}), B^*_{\frac{1}{2}}).$$

Proof. We need to bound the operator $P_R := R^{-1} J^+(\lambda)^* g 1_{\{|x| < R\}} J^+(\lambda)$ independently of $R > 1$ and $\lambda \geq 0$. Writing $P_R = R^{-1} \int_0^R \mathrm{d}r \int_{S_r} Q_r \mathrm{d}x$ with $S_r = \{|x| = r\}$ it thus suffices to bound the operator $\int_{S_r} Q_r \mathrm{d}x$ independently of $r > 0$ and $\lambda \geq 0$.

Step I. Analysis of $\int_{S_r} Q_r \mathrm{d}x$. The kernel of Q_r is given by

$$Q_r(\omega, \omega') = \mathrm{e}^{\mathrm{i}(\phi^+(x, \omega', \lambda) - \phi^+(x, \omega, \lambda))} a(x, \omega, \omega', \lambda),$$

where

$$a(x,\omega,\omega',\lambda) = (2\pi)^{-d} g(|x|,\lambda)\overline{\tilde{a}^+(x,\omega,\lambda)}\tilde{a}^+(x,\omega',\lambda).$$

For simplicity we shall henceforth omit the superscript $+$, $r > 0$ and $\lambda \geq 0$ in the notation.

Our goal is to show that $\int_{S_r} Q_r \mathrm{d}x$ is a PsDO on $L^2(S^{d-1})$ with symbol $b(\omega,\omega',z)$ obeying uniform bounds (uniform in $r > 0$ and $\lambda \geq 0$)

$$|\partial_\omega^{\beta_1}\partial_{\omega'}^{\beta_2}\partial_z^\alpha b| \leq C_{\beta_1,\beta_2,\alpha}\langle z\rangle^{-|\alpha|}. \tag{6.18}$$

Clearly this would prove the lemma.

We can use a partition of unity on S^{d-1}, and therefore we can assume that the vectors ω,ω' and $\hat{x}$ are close to the d'th standard vector $e_d \in \mathbb{R}^d$. Consequently we can use coordinates

$$\omega = \omega_\perp + \omega_d e_d; \quad \omega_d = \sqrt{1-\omega_\perp^2}, \tag{6.19}$$

$$x = x_\perp + x_d e_d; \quad x_d = \sqrt{r^2 - x_\perp^2}. \tag{6.20}$$

Next we write

$$\phi(x,\omega') - \phi(x,\omega) = (\omega_\perp - \omega'_\perp)\cdot z;\ z = -\int_0^1 \partial_{\omega_\perp}\phi(x, s(\omega'-\omega)+\omega)\mathrm{d}s.$$

Step II. We shall show that the map

$$S_r \supset \mathcal{U} \ni x \to Tx = z \in \mathbb{R}^{d-1} \text{ is a diffeomorphism onto its range;} \tag{6.21}$$

here $\mathcal{U}$ is an open neighbourhood of e_d containing the supports of $a(\cdot,\omega,\omega')$.

To this end we investigate the bilinear form $\partial_x\partial_\omega\phi(x,\omega)$ on $TS_x^{d-1}\times TS_\omega^{d-1}$. Note that

$$\partial_x\partial_\omega\phi^+_{\mathrm{sph}}(\hat{x}=\omega) = r^{-1}hI, \tag{6.22}$$

cf. (5.25).

In the coordinates (6.19) and (6.20), the identity (6.22) reads for $z_{\mathrm{sph}} = (Tx)_{\mathrm{sph}}$ (here we consider the case where $V_2 = 0$)

$$\partial_{x_j} z_{\mathrm{sph},\,i}(\omega=\omega'=\hat{x}) = -r^{-1}h\big(\delta_{ij} + \omega_d^{-2}\omega_i\omega_j\big);\ i,j \leq d-1. \tag{6.23}$$

Due to (3.15), Proposition 3.3 and (6.23) we obtain the more general result

$$\partial_{x_j} z_i = -r^{-1}h\big(\delta_{ij} + \omega_d^{-2}\omega_i\omega_j + O(\sigma') + O(r^{-\breve{\epsilon}})\big);\ i,j \leq d-1; \tag{6.24}$$

here $O(\sigma')$ refers to a term obeying $|O(\sigma')| \leq C\sigma'$ where $\sigma' > 0$ is given in (5.2) (assumed to be small).

In particular T is a local diffeomorphism with inverse determinant

$$|\partial_{x_j} z_i|^{-1} = (-r^{-1}h)^{1-d}(\omega'_d)^2\big(1 + O(\sigma') + O(r^{-\breve{\epsilon}})\big). \tag{6.25}$$

For a later application we note the uniform bounds

$$\partial_\omega^{\beta_1}\partial_{\omega'}^{\beta_2}\partial_x^\alpha|\partial_{x_j} z_i|^{-1} = g^{1-d}r^{-|\alpha|}O(r^0). \tag{6.26}$$

Also T is injective: Suppose $Tx^1 = Tx^2$, then

$$\begin{aligned} 0 &= \int_0^1 \partial_{x_j} z_i(s(x^1-x^2)+x^2)(x_j^1 - x_j^2)\mathrm{d}s \\ &= -r^{-1}h\big((\delta_{ij} + \omega_d^{-2}\omega_i\omega_j) + O(\sigma') + O(r^{-\breve{\epsilon}})\big)(x_j^1 - x_j^2). \end{aligned}$$

Using the invertibility of the matrix $\delta_{ij} + \omega_d^{-2}\omega_i\omega_j$ it follows that $x^1 = x^2$.

Step III. Analysis of symbol b. Due to Step II we can change coordinates and obtain that $\int_{S_r} Q_r \mathrm{d}x$ is a PsDO with symbol $b = |\partial_{x_j} z_i|^{-1} a$. It remains to show (6.18). For zero indices $\beta_1 = \beta_2 = \alpha = 0$, we obtain the bound by combining Proposition 5.3 and (6.25). For derivatives we note the bounds

$$|\partial_\omega^{\beta_1} \partial_{\omega'}^{\beta_2} \partial_x^\alpha z| \le C_{\beta_1,\beta_2,\alpha} g r^{1-|\alpha|}, \tag{6.27}$$

which by a little bookkeeping yields to

$$|\partial_\omega^{\beta_1} \partial_{\omega'}^{\beta_2} \partial_z^\gamma x| \le C_{\beta_1,\beta_2,\gamma} r \langle z \rangle^{-|\gamma|}. \tag{6.28}$$

Another bookkeeping using Proposition 5.3, (6.25) and (6.28) yields (6.18). □

Proof of Lemma 6.8. We drop the superscript "+" and the parameter λ in the notation. We first prove uniform boundedness on any compact interval $[0, \lambda_1]$.

Re (i). We replace $J = J(\cdot)$ by $J\chi(\omega)$ where $\chi \in C^\infty(S^{d-1})$ with a sufficiently small support. We can assume that n is a non-negative integer. Instead of studying $J(1-\Delta_\omega)^{n/2}$, it then suffices to study $J\partial_\omega^\nu$ for $|\nu| \le n$.

Integrating by parts we observe that the corresponding integral kernel equals

$$C\partial_\omega^\nu \big(\mathrm{e}^{\mathrm{i}\phi(x,\omega)} \tilde{a}(x,\omega) \big) = \mathrm{e}^{\mathrm{i}\phi(x,\omega)} \tilde{a}_\nu(x,\omega),$$

where $\tilde{a}_\nu$ is a linear combinations of terms of the form

$$\partial_\omega^{\nu_1} \phi(x,\omega) \cdots \partial_\omega^{\nu_k} \phi(x,\omega) \partial_\omega^{\nu_0} \tilde{a}(x,\omega),$$

with $\nu_0 + \nu_1 + \cdots + \nu_k = \nu$. Thus, using that $|\partial_\omega^\delta \phi| \le C\langle x \rangle g$ (cf. (3.11d)) and Proposition 5.3, we obtain

$$\partial_\omega^\delta \partial_x^\gamma \tilde{a}_\nu(x,\omega) = O(\langle x \rangle^{n-|\gamma|} g^{n+\frac{d-2}{2}}). \tag{6.29}$$

Then we follow the proof of Lemma 6.9.

Re (ii). Assume first that $n \ge 0$. Then we follow the same scheme as above. The bound on the relevant kernel needs to be replaced by

$$\partial_\omega^\delta \partial_x^\gamma \tilde{t}_\nu(x,\omega) = O(\langle x \rangle^{n-1-|\gamma|} g^{n+\frac{d}{2}}), \tag{6.30}$$

cf. Proposition 5.5. Using (6.30) we can proceed as before.

Assume next that $n < 0$. We can assume that n is a negative integer. For fixed x we decompose $\omega = \omega_\perp + \sqrt{1-\omega_\perp^2}\,\hat{x}$, where $\omega_\perp \cdot x = 0$. By (3.11c) we have the uniform lower bound

$$|\nabla_{\omega_\perp} \phi(x,\omega)| \ge c|x| g \text{ for } \hat{x} \cdot \omega \le 1 - \sigma, \tag{6.31}$$

and by (3.11d) the uniform upper bounds

$$|\partial_{\omega_\perp}^\delta \phi(x,\omega)| \le C|x| g. \tag{6.32}$$

We apply the non-stationary method based on the identity

$$\left(\mathrm{i} \frac{\nabla_{\omega_\perp} \phi}{|\nabla_{\omega_\perp} \phi|^2} \cdot \nabla_{\omega_\perp} \right)^{-n} \mathrm{e}^{\mathrm{i}\phi^+(x,\omega)} = \mathrm{e}^{\mathrm{i}\phi(x,\omega)}.$$

After performing $-n$ integrations by parts, the bounds (6.31) and (6.32) yield

$$T\chi\tau = \sum_{|\nu| \le -n} \int \tilde{t}_\nu(x,\omega) \partial_{\omega_\perp}^\nu \tau(\omega) \mathrm{d}\omega,$$

where the functions $\tilde{t}_\nu$ also satisfy the bounds (6.30). Then we proceed as before.

Re (iii). The kernel of $\mathrm{Op^r}(\chi_3)T_{\mathrm{bd}}(\cdot)$ is given by the integral

$$\int \mathrm{d}\xi \mathrm{e}^{\mathrm{i}x\cdot\xi}\int \mathrm{e}^{\mathrm{i}(\phi(y,\omega)-y\cdot\xi)}k(\omega,y,\xi)\mathrm{d}y;\ k(\omega,y,\xi)=(2\pi)^{-3d/2}\chi_3(y,\xi)\tilde{t}_{\mathrm{bd}}(y,\omega).$$

It suffices to show that

$$|\partial_\xi^\beta\partial_\omega^\delta\int \mathrm{e}^{\mathrm{i}(\phi(y,\omega)-y\cdot\xi)}k(\omega,y,\xi)\mathrm{d}y|\leq C_{\beta,\delta}\text{ uniformly in }\xi,\omega\text{ and }\lambda; \tag{6.33}$$

notice that the symbol k is compactly supported in ξ. First we observe that (using notation of Subsection 4.1)

$$k=k_{\omega,\lambda}\in S_{\mathrm{unif}}(g^{\frac{d}{2}}\langle x\rangle^{-1},g_{\mu,\lambda}).$$

We can substitute $k\to k=F(|y|>2\bar{R})k(\omega,y,\xi)$.

Next we integrate by parts writing first

$$\Big(\mathrm{i}\frac{\xi-\nabla_y\phi}{|\xi-\nabla_y\phi|^2}\cdot\nabla_y\Big)^\ell \mathrm{e}^{\mathrm{i}(\phi(y,\omega)-y\cdot\xi)}=\mathrm{e}^{\mathrm{i}(\phi(y,\omega)-y\cdot\xi)}.$$

We need to argue that $\xi-\partial_y\phi\neq 0$ on the support of the involved symbol. For that we recall the following elementary inequality valid for all $z_1,z_2\in\mathbb{R}^d$ and $\kappa_1,\kappa_2>0$

$$|z_1-z_2|^2\geq\min(\kappa_1^2/2,\kappa_2-\kappa_2^2/2)(|z_1|^2+|z_2|^2), \tag{6.34}$$

provided one of the following three conditions holds

$$|z_2|\leq(1-\kappa_1)|z_1|,\ |z_1|\leq(1-\kappa_1)|z_2|\text{ or }z_1\cdot z_2\leq(1-\kappa_2)|z_1||z_2|.$$

Now, on the support of the symbol k we have $(1-\sigma')|y|\leq y\cdot\omega\leq(1-\sigma)|y|$, cf. (5.2). We use these inequalities in (3.5c) and (3.5d) yielding

$$1-C\sigma'-C|y|^{-\breve{\epsilon}}\leq\frac{\nabla_y\phi(y,\omega)}{|\nabla_y\phi(y,\omega)|}\cdot\hat{y}\leq 1-c\sigma+C|y|^{-\breve{\epsilon}},$$

which in turn (if $\bar{R}$ is taken large enough) implies that

$$1-2C\sigma'\leq\frac{\nabla_y\phi(y,\omega)}{g(|y|)}\cdot\frac{y}{\langle y\rangle}\leq 1-\frac{c}{2}\sigma. \tag{6.35}$$

We claim that there exists a small $c'=c'(\sigma,\sigma')>0$ such that

$$\big|\xi-\nabla_y\phi(y,\omega)\big|\geq c'\Big(|\xi|+\big|\nabla_y\phi(y,\omega)\big|\Big) \tag{6.36}$$

on the support of k (showing in particular that $\xi-\partial_y\phi\neq 0$).

Obviously (6.36) follows from (6.34) with

$$z_1=\tfrac{\xi}{g(|y|)}\text{ and }z_2=\frac{\nabla_y\phi(y,\omega)}{g(|y|)}$$

provided one of the above three conditions hold. If all of those conditions fail, so that intuitively $z_1\approx z_2$, we can replace z_2 in (6.35) by z_1 yielding

$$1-3C\sigma'\leq b(x,\xi)\leq 1-\frac{c}{3}\sigma. \tag{6.37}$$

Here we applied (6.34) for some κ_1 and κ_2 depending on σ and σ'. Now, the second inequality of (6.37) is violated on the support of $\tilde{\chi}_+(b(y,\xi))$ provided the $\bar{\sigma}>0$ of (6.13) is chosen such that

$$2\bar{\sigma}<\frac{c}{3}\sigma. \tag{6.38}$$

We have shown the bound (6.36) on the support of the symbol k and therefore in particular on the support of the relevant symbol after performing the y–integrations by parts. The estimate (6.33) follows.

Re (iv). First we assume that $n \geq 0$. Integrating by parts in ω, as in the proof of (i) and using Proposition 5.5, which says that $t_{\rm pr}$ with all its derivatives is $O(\langle x\rangle^{-\infty})$, we obtain that $\langle x\rangle^m T_{\rm pr}(\lambda)$ is in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$ for any m. The case $n < 0$ then follows trivially.

Let us now prove the continuity. Consider for instance (i). Let $\tau \in C^\infty(S^{d-1})$ and set

$$J_{n,\epsilon}(\lambda) := (\langle x\rangle g)^{-n}\langle x\rangle^{-\frac{1}{2}-\epsilon} g^{\frac{1}{2}} J(\lambda).$$

Clearly for (small) $\kappa > 0$,

$$J_{n,\epsilon}(\lambda)\tau = F(\kappa|x| < 1) J_{n,\epsilon}(\lambda)\tau + F(\kappa|x| > 1)\langle x\rangle^{-\epsilon/2} J_{n,\epsilon/2}(\lambda)\tau. \tag{6.39}$$

We know that $J_{n,\epsilon/2}(\lambda)$ is bounded uniformly in λ. Hence the second term on the right of (6.39) is $O(\kappa^{\epsilon/2})$.

We know that $a(x,\omega,\lambda)$, $\phi(x,\omega,\lambda)$ and $g(x,\lambda)^{\pm 1}$ are continuous down to $\lambda = 0$. The first term on the right of (6.39) involves only variables in a compact set. Therefore it is continuous in λ. Hence $J_{n,\epsilon}(\lambda)\tau$ is continuous as the uniform limit of continuous functions.

By the uniform bound, which we proved before, we conclude that $J_{n,\epsilon}(\lambda)$ is strongly continuous in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$.

Now

$$J_{n,\epsilon}(\lambda) = g^{\epsilon/2} J_{n+\epsilon/2,\epsilon/2}(\lambda)(1-\Delta_\omega)^{-\epsilon/4}(1-\Delta_\omega)^{\epsilon/4},$$

where $g^{\epsilon/2}$ is strongly continuous, $J_{n+\epsilon/2,\epsilon/2}(\lambda)$ is strongly continuous in

$$\mathcal{B}(L^{2,-n-\epsilon/2}(S^{d-1}), L^2(\mathbb{R}^d)),$$

$(1-\Delta_\omega)^{-\epsilon/4}$ is a compact operator on $L^{2,-n-\epsilon/2}(S^{d-1})$ and $(1-\Delta_\omega)^{\epsilon/4}$ is a unitary element of $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,-n-\epsilon/2}(S^{d-1}))$. We invoke the general fact that the product of a strongly continuous operator-valued function and a compact operator is norm continuous. Whence we obtain the norm continuity of $J_{n,\epsilon}(\lambda)$ in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$.

The proof of the norm continuity of the operators in the remaining parts of the lemma is similar. □

Outline of a proof of Theorems 6.5 and 6.6. The proof goes along the lines of the proof of Theorem 6.3. In particular this amount to inserting the right hand side of (6.11) to the right of the resolvent in (6.8) and expand into three terms. Next, using Proposition 4.1 and Lemma 6.8 we conclude that $W^+(\lambda)$ is well-defined as a sum of four operators, say $T_j(\lambda)$. In fact all of the four maps

$$[0,\infty[\ni \lambda \to (\langle x\rangle g)^{-n}\langle x\rangle^{-\frac{1}{2}-\epsilon} g^{\frac{1}{2}} T_j(\lambda) \in \mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$$

are continuous.

For the independence of $W^+(\lambda)$ of cutoffs we use Propositions 4.8 and 4.10 in the same way as in the arguments for deducing (6.40) stated below.

The formula (6.15) follows by combining (6.10), Remark 4.2 2) and the shown continuity properties of $W^+(\lambda)$ and $W^+(\lambda)^*$. □

Lemma 6.10. *For any $\lambda \geq 0$, $R(\lambda\pm i0)T^{\pm}(\lambda)$ is well-defined as a map from $\mathcal{D}'(S^{d-1})$ to $L^{2,-\infty}$ and*

$$0 = J^{\pm}(\lambda) + iR(\lambda\pm i0)T^{\pm}(\lambda). \tag{6.40}$$

Proof. Note that we can extend Lemma 6.8 as follows: Let $\chi_- \in C_c^\infty(\mathbb{R})$ and $\tilde{\chi}_- \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\tilde{\chi}_- \subseteq]-\infty, 2\bar{\sigma}-1[$ for some small $\bar{\sigma} > 0$. Then for all $m, n \in \mathbb{R}$

$$\mathrm{Op^r}(\chi_-(a)\tilde{\chi}_-(b))T^+_{\mathrm{bd}}(\lambda),\ \mathrm{Op^r}(\chi_-(a)\tilde{\chi}_-(b))J^+(\lambda) \in \mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,m}(\mathbb{R}^d)),$$

cf. (6.37) (recall the standing hypothesis of this subsection that the positive parameter σ' in (5.2) is sufficiently small).

Therefore, for all $\tau \in \mathcal{D}'(S^{d-1})$ and $s \in \mathbb{R}$

$$\big(WF^s_{\mathrm{sc}}(T^+(\lambda)\tau) \cup WF^s_{\mathrm{sc}}(J^+(\lambda)\tau)\big) \cap \{b < \bar{\sigma} - 1\} = \emptyset. \tag{6.41}$$

By the definition of $T^+(\lambda)$

$$(H-\lambda)J^+(\lambda)\tau = -iT^+(\lambda)\tau = -i(H-\lambda)R(\lambda+i0)T^+(\lambda)\tau. \tag{6.42}$$

Notice that due to (6.41) and Proposition 4.8 (iii), the vector $u = R(\lambda+i0)T^+(\lambda)\tau$ is in fact well-defined and

$$WF^s_{\mathrm{sc}}(u) \cap \{b < \bar{\sigma} - 1\} = \emptyset. \tag{6.43}$$

Using (6.41)–(6.43) and Proposition 4.10 we conclude that the generalized eigenfunction

$$J^+(\lambda)\tau + iR(\lambda + i0)T^+(\lambda)\tau = 0. \tag{6.44}$$

□

Remark. There exists an alternative time-dependent proof of Lemma 6.10 that avoids the use of Proposition 4.10: Due to (6.2)

$$0 = \lim_{\epsilon\searrow 0}\int (J^{\pm} + iR(\lambda \pm i\epsilon)T^{\pm})\delta_\epsilon(\lambda) f\,d\lambda, \quad \hat{f} \in C_c(\mathbb{R}^d\backslash\{0\}),$$

cf. Lemma 6.2 or Appendix A. The right hand is given by

$$\int (J^{\pm} + iR(\lambda \pm i0)T^{\pm})\delta_0(\lambda) f\,d\lambda,$$

cf. Appendix A. Whence, by a density argument, (6.40) follows. □

We complete this subsection by discussing a certain refined mapping property of $W^{\pm}(\lambda)$. Besides its own interest its application (see Corollary 6.12 stated below) will be needed in Section 8. The result is related to the fact that the continuity in λ of the operators in (6.16) and (6.17a) is proven only for $n \geq 0$ while the continuity in λ of the operator in (6.17b) is valid for all $n \in \mathbb{R}$.

Theorem 6.11. *Fix real-valued $\chi, \tilde{\chi}_- \in C_c^\infty(\mathbb{R})$ and $\chi_+ \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\tilde{\chi}_- \subset]-1,1[$, $\chi'_+ \in C_c^\infty(\mathbb{R})$ and $\operatorname{supp}\chi_+ \subset]C_0, \infty[$. Let $\tilde{A} := \mathrm{Op^w}(\chi(a)\tilde{\chi}_-(b))$ and $A_+ := \mathrm{Op^w}(\chi_+(a))$ for $\lambda \geq 0$. For all $n \in \mathbb{R}$, $\epsilon > 0$ and with $A = \tilde{A}$ or $A = A_+$*

$$W^{\pm}_{n,\epsilon}(\lambda) := (\langle x\rangle g)^{-n}\langle x\rangle^{-\frac{1}{2}-\epsilon} g^{\frac{1}{2}} A W^{\pm}(\lambda) \tag{6.45}$$

is a continuous $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$–valued function in $\lambda \in [0, \infty[$.

Proof. With reference (4.2) (this class of symbols is used extensively in [FS])

$$B(\lambda) := (\langle x\rangle g)^{-n}\langle x\rangle^{-\frac{1}{2}-\epsilon} g^{\frac{1}{2}} A g^{-\frac{1}{2}}\langle x\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}(\langle x\rangle g)^{n} \in \Psi_{\mathrm{unif}}(\langle x\rangle^{-\frac{\epsilon}{2}}, g_{\mu,\lambda}).$$

Whence by the calculus, $B(\lambda) \in \mathcal{B}(L^2(\mathbb{R}^d))$ with a bound locally independent of $\lambda \geq 0$ and in fact $B(\cdot)$ is norm continuous. By using this continuity and Theorem 6.6 we conclude that it suffices to consider the case $n < 0$.

Re $A = \tilde{A}$. Since the construction of $W^+(\lambda)$ is independent of the (small) parameters σ and σ' in (5.2) we can take them smaller (if needed) to assure that

$$\sup\operatorname{supp}\chi_- < 1 - 3C\sigma'; \tag{6.46}$$

here we refer to the left hand side of (6.37).

Now, to show that $W^+_{n,\epsilon}(\lambda)$ is an element of $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$ we consider for $\lambda > 0$ the two terms of (6.8) separately (if $\lambda = 0$ we use instead (6.14)): The contribution from the first term (i.e. from $J^+(\lambda)$) has better mapping properties than specified, cf. Lemma 6.8 (iii). In fact using (6.46) we can mimic the proof of Lemma 6.8 (iii) to handle this contribution. As for the contribution from the second term (i.e. from $\mathrm{i}R(\lambda - \mathrm{i}0)T^+(\lambda)$) we combine Lemma 6.8 (ii) and (iv) and Proposition 4.1 (iii).

By the same arguments continuity in $\lambda \geq 0$ is valid for the contribution from each of the mentioned two terms, hence for $W^+_{n,\epsilon}(\lambda)$.

Re $A = A_+$. Again we consider for $\lambda > 0$ the two terms of (6.8) separately (if $\lambda = 0$ we use instead (6.14)). The contribution from the first term $J^+(\lambda)$ has again better mapping properties than needed, precisely we have the following analogue of Lemma 6.8 (iii):

For all $m \in \mathbb{R}$ the family of operators $\langle x\rangle^m A_+ J^+(\lambda)$ constitutes a continuous $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$–valued function of $\lambda \in [0, \infty[$.

To show this we can again follow the proof of Lemma 6.8 (iii). It suffices to show locally uniform boundedness in the indicated topology and we may replace $A_+ \to \mathrm{Op}^{\mathrm{r}}(\chi_+(a))$. The kernel of $\mathrm{Op}^{\mathrm{r}}(\chi_+(a))J^+(\lambda)$ is given by the integral

$$\int \mathrm{d}\xi \mathrm{e}^{\mathrm{i}x\cdot\xi}\int \mathrm{e}^{\mathrm{i}(\phi(y,\omega)-y\cdot\xi)} k_{\omega,\lambda}(y,\xi)\mathrm{d}y;$$
$$k_{\omega,\lambda}(y,\xi) = (2\pi)^{-3d/2}\chi_+(\xi^2/g(|y|,\lambda)^2)\tilde{a}^+(y,\omega,\lambda).$$

It suffices to show that

$$\Big|\partial^\beta_\xi\partial^\delta_\omega \int \mathrm{e}^{\mathrm{i}(\phi(y,\omega)-y\cdot\xi)} k_{\omega,\lambda}(y,\xi)\mathrm{d}y\Big| \leq C_{\beta,\delta}\langle\xi\rangle^{-d-1} \text{ uniformly in } \xi, \omega \text{ and } \lambda. \tag{6.47}$$

For that we notice that

$$k = k_{\omega,\lambda} \in S_{\mathrm{unif}}(g^{\frac{d}{2}-1}, g_{\mu,\lambda}).$$

It suffices to show (6.47) with $k \to k = F(|y| > 2\bar{R})k_{\omega,\lambda}(y,\xi)$.

Next we integrate by parts writing first

$$\Big(\mathrm{i}\frac{\xi - \nabla_y\phi}{|\xi - \nabla_y\phi|^2}\cdot\nabla_y\Big)^\ell \mathrm{e}^{\mathrm{i}(\phi(y,\omega)-y\cdot\xi)} = \mathrm{e}^{\mathrm{i}(\phi(y,\omega)-y\cdot\xi)},$$

and then we invoking the uniform bounds

$$C|\xi| \geq \big|\xi - \nabla_y\phi(y,\omega)\big| \geq c\big(|\xi| + \big|\nabla_y\phi(y,\omega)\big|\big), \tag{6.48}$$

which are valid on the support of k (provided $\bar{R}$ is chosen sufficiently large). Clearly we obtain (6.47) by this procedure if ℓ (i.e. the number of integrations by parts) is chosen sufficiently large.

As for the contribution from the second term $\mathrm{i}R(\lambda - \mathrm{i}0)T^+(\lambda)$ we combine Lemma 6.8 (ii) and (iv) and Proposition 4.1 (ii). $\square$

We can extend the identities (6.10) and (6.15) (which below corresponds to $s = 0$):

Corollary 6.12. *Let $\chi, \chi_- \in C_c^\infty(\mathbb{R})$ be given as in Theorem 6.11. Fix $\lambda \geq 0$ let again $\tilde{A} := \mathrm{Op}^{\mathrm{w}}(\chi(a)\chi_-(b))$. For all $\delta > \frac{1}{2}$ and $s \leq 0$ there exists the strong limit*

$$\mathrm{s} - \lim_{\epsilon \searrow 0} g^{\frac{1}{2}} \delta_\epsilon^V(\lambda) \tilde{A} g^{\frac{1}{2}} = g^{\frac{1}{2}} \delta^V(\lambda) \tilde{A} g^{\frac{1}{2}} = g^{\frac{1}{2}} W^\pm(\lambda) W^\pm(\lambda)^* \tilde{A} g^{\frac{1}{2}} \tag{6.49}$$

in $\mathcal{B}(L^{2,s+\delta}(\mathbb{R}^d), L^{2,s-\delta}(\mathbb{R}^d))$.

Proof. It follows from Proposition 4.9 that indeed there exists the limit

$$B := \mathrm{s} - \lim_{\epsilon \searrow 0} g^{\frac{1}{2}} \delta_\epsilon^V(\lambda) \tilde{A} g^{\frac{1}{2}} \text{ in } \mathcal{B}(L^{2,s+\delta}(\mathbb{R}^d), L^{2,s-\delta}(\mathbb{R}^d)).$$

Let $n = s/s_1$ where s_1 is given as in (4.6). Due to Theorem 6.11

$$W^\pm(\lambda)^* \tilde{A} g^{\frac{1}{2}} = \big(g^{\frac{1}{2}} \tilde{A} W^\pm(\lambda)\big)^* \in \mathcal{B}(L^{2,s+\delta}(\mathbb{R}^d), L^{2,n}(S^{d-1})),$$

and due to Theorem 6.6

$$g^{\frac{1}{2}} W^\pm(\lambda) \in \mathcal{B}(L^{2,n}(S^{d-1}), L^{2,s-\delta}(\mathbb{R}^d)).$$

We have shown that

$$g^{\frac{1}{2}} W^\pm(\lambda) W^\pm(\lambda)^* \tilde{A} g^{\frac{1}{2}} \in \mathcal{B}(L^{2,s+\delta}(\mathbb{R}^d), L^{2,s-\delta}(\mathbb{R}^d)).$$

Since

$$Bv = g^{\frac{1}{2}} W^\pm(\lambda) W^\pm(\lambda)^* \tilde{A} g^{\frac{1}{2}} v \text{ for } v \in L^{2,\infty},$$

cf. (6.10) and (6.15), we are done by a density argument. $\square$

6.4. **Asymptotics of short-range wave matrices.** Clearly, if $\mu > 1$ there exists

$$W_{\mathrm{sr}}^\pm f = \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} f, \tag{6.50}$$

which is the usual definition of wave operators in the short-range case. In the case $\mu \in]1, 2[$ we can compare our wave matrices with the wave matrices defined by (6.50).

Recall $\hat{p} := p/|p|$.

Theorem 6.13. *For $\mu \in]1, 2[$ the operators*

$$\psi_{\mathrm{sr}}^+(p) := \mathrm{i} \int_{R_0}^\infty (|p| - F^+(l\hat{p}, p, p^2/2) \cdot \hat{p}) \, \mathrm{d}l,$$

$$\psi_{\mathrm{sr}}^-(p) := -\mathrm{i} \int_{R_0}^\infty (|p| + F^+(-l\hat{p}, -p, p^2/2) \cdot \hat{p}) \, \mathrm{d}l$$

are well-defined. If $V_2 = 0$, then $\psi_{\mathrm{sr}}^\pm(p) = \psi_{\mathrm{sr}}^\pm(|p|)$ with

$$\psi_{\mathrm{sr}}^\pm(|p|) = \pm \mathrm{i} \int_{R_0}^\infty \big(|p| - \sqrt{p^2 - 2V_1(r)}\big) \, \mathrm{d}r.$$

We have

$$W_{\mathrm{sr}}^+ = W^+ \mathrm{e}^{\mathrm{i}\psi_{\mathrm{sr}}^+(p)}, \tag{6.51a}$$

$$W_{\mathrm{sr}}^- = W^- \mathrm{e}^{\mathrm{i}\psi_{\mathrm{sr}}^-(p)}. \tag{6.51b}$$

Whence in particular, for all $\lambda > 0$

$$W_{\mathrm{sr}}^{+}(\lambda) = W^{+}(\lambda)\mathrm{e}^{\mathrm{i}\psi_{\mathrm{sr}}^{+}(\sqrt{2\lambda}\cdot)}, \tag{6.52a}$$

$$W_{\mathrm{sr}}^{-}(\lambda) = W^{-}(\lambda)\mathrm{e}^{\mathrm{i}\psi_{\mathrm{sr}}^{-}(\sqrt{2\lambda}\cdot)}. \tag{6.52b}$$

Proof. One can readily show the theorem from well-known properties of the free evolution and the fact that

$$\phi^{+}(x,\omega,\lambda) + \int_{R_0}^{\infty} (\sqrt{2\lambda} - F^{+}(l\omega,\omega,\lambda)\cdot\omega)\,\mathrm{d}l = \sqrt{2\lambda}\omega\cdot x + o(|x|^0), \tag{6.53}$$

which in turn follows from [DS1, (4.50)] and a change a contour of integration. The asymptotics is locally uniform in $(\omega,\lambda) \in S^{d-1}\times]0,\infty[$. $\square$

Remark 6.14. $\psi_{\mathrm{sr}}^{\pm}$ is indeed oscillatory. Notice that for $V_1(r) = -\gamma r^{-\mu}$, as $\lambda \to 0^+$ we have

$$\begin{aligned}
\psi_{\mathrm{sr}}^{+}(\sqrt{2\lambda}) &= \int_{R_0}^{\infty} \left(\sqrt{2\lambda} - \sqrt{2(\lambda + \gamma r^{-\mu})}\right)\mathrm{d}r \\
&= (2\lambda)^{\frac{1}{2}-\frac{1}{\mu}} \int_{R_0(2\lambda)^{\frac{1}{\mu}}}^{\infty} (1 - \sqrt{1+2\gamma s^{-\mu}})\mathrm{d}s \\
&= (2\lambda)^{\frac{1}{2}-\frac{1}{\mu}} \int_{0}^{\infty} (1 - \sqrt{1+2\gamma s^{-\mu}})\mathrm{d}s + O\left(\lambda^0\right),
\end{aligned}$$

cf. [Ya1, (7.11)]. See Remark 6.16 for a similar result.

6.5. **Asymptotics of Dollard-type wave matrices.** For $\mu > \frac{1}{2}$ and $\mu + \epsilon_2 > 1$ the Dollard-type wave operators are given by

$$W_{\mathrm{dol}}^{\pm} f = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} U_{\mathrm{dol}}(t) f,$$

where

$$U_{\mathrm{dol}}(t) = \mathrm{e}^{-\mathrm{i}\int_0^t (p^2/2 + V_1(sp)1_{\{|sp|\geq R_0\}})\,\mathrm{d}s}.$$

We have the following analogue of Theorem 6.13.

Theorem 6.15. *For $\frac{1}{2} < \mu < 2$, $\epsilon_2 < 1$ and $\mu + \epsilon_2 > 1$, the operators*

$$\psi_{\mathrm{dol}}^{+}(p) = \mathrm{i}\int_{R_0}^{\infty} (|p| - F^{+}(l\hat{p},\hat{p},p^2/2)\cdot\hat{p} - |p|^{-1}V_1(l))\,\mathrm{d}l,$$

$$\psi_{\mathrm{dol}}^{-}(p) = -\mathrm{i}\int_{R_0}^{\infty} (|p| + F^{+}(-l\hat{p},-\hat{p},p^2/2)\cdot\hat{p} - |p|^{-1}V_1(l))\,\mathrm{d}l$$

are well-defined. If $V_2 = 0$, then $\psi_{\mathrm{dol}}^{\pm}(p) = \psi_{\mathrm{dol}}^{\pm}(|p|)$ and

$$\psi_{\mathrm{dol}}^{\pm}(|p|) = \pm\mathrm{i}\int_{R_0}^{\infty} (|p| - \sqrt{p^2 - 2V_1(r)} - |p|^{-1}V_1(r))\,\mathrm{d}r.$$

We have

$$W_{\mathrm{dol}}^{+} = W^{+}\mathrm{e}^{\mathrm{i}\psi_{\mathrm{dol}}^{+}(p)}, \tag{6.54a}$$

$$W_{\mathrm{dol}}^{-} = W^{-}\mathrm{e}^{\mathrm{i}\psi_{\mathrm{dol}}^{-}(p)}. \tag{6.54b}$$

Whence in particular, for all $\lambda > 0$

$$W_{\mathrm{dol}}^{+}(\lambda) = W^{+}(\lambda)\mathrm{e}^{\mathrm{i}\psi_{\mathrm{dol}}^{+}(\sqrt{2\lambda}\cdot)}, \tag{6.55a}$$

$$W_{\mathrm{dol}}^{-}(\lambda) = W^{-}(\lambda)\mathrm{e}^{\mathrm{i}\psi_{\mathrm{dol}}^{-}(\sqrt{2\lambda}\cdot)}. \tag{6.55b}$$

Proof. First we notice that $\psi^{\pm}_{\mathrm{dol}}$ are well-defined due to the fact that

$$(F^+ - F^+_{\mathrm{sph}})(l\omega, \omega, \lambda) = O(l^{-\delta})$$

for any $\delta < \min(\mu + \epsilon_2, 2\mu)$, and hence integrable; here F^+_{sph} refers to the F^+ for the case $V_2 = 0$, whence $F^+_{\mathrm{sph}}(l\omega, \omega, \lambda) = g(l, \lambda)\omega$. For this estimate we refer to [DS1, Remarks 6.2 2)] and the proof of [DS1, Lemma 6.4]; it appears stronger at the price of not being uniform in (small) λ. There is an extension of this estimate that allows us to integrate along the line segment joining x and $R\omega$ and taking the limit:

$$\begin{aligned}\int_x^{\infty\omega} (F^+ - F^+_{\mathrm{sph}})(\bar{x}, \omega, \lambda) \cdot d\bar{x} &= \lim_{R\to\infty} \int_x^{R\omega} (F^+ - F^+_{\mathrm{sph}})(\bar{x}, \omega, \lambda) \cdot \mathrm{d}\bar{x} \\ &= o(|x|^0). \end{aligned} \tag{6.56}$$

Introduce the auxillary phases

$$\begin{aligned}\phi^{\pm}_{\mathrm{dol}}(x, \omega, \lambda) &= \sqrt{2\lambda}x \cdot \omega \mp (2\lambda)^{-\frac{1}{2}} \int_{R_0}^{\pm x\cdot\omega} V_1(l)\,\mathrm{d}l, \\ \phi^{\pm}_{\mathrm{aux}}(x, \omega, \lambda) &= \phi^{\pm}_{\mathrm{aux}} = \phi^{\pm}_{\mathrm{dol}} - \int_x^{\pm\infty\omega} (F^+_{\mathrm{sph}} - \nabla_x \phi^{\pm}_{\mathrm{dol}}) \cdot \mathrm{d}\bar{x},\end{aligned}$$

and corresponding modifiers

$$(J^{\pm}_{\sharp} f)(x) = (2\pi)^{-d/2} \int \mathrm{e}^{\mathrm{i}\phi^{\pm}_{\sharp}(x,\xi)} \chi(x, \pm\hat{\xi}) \hat{f}(\xi) \mathrm{d}\xi;\ \xi = \sqrt{2\lambda}\omega.$$

Here we can take the function χ of the form $\chi(x, \omega) = \chi_1(|x|/R)\chi_2(\hat{x} \cdot \omega)$ with χ_1 and χ_2 given as in (5.1) and (5.2), respectively.

By the stationary phase method, [Hö1, Theorem 7.7.6], one derives the following asymptotics in $L^2(\mathbb{R}^d)$ for any state f with $\hat{f} \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$:

$$U_{\mathrm{dol}}(t) f \asymp J^{\pm}_{\mathrm{dol}} \mathrm{e}^{\mathrm{i}tH_0} f \asymp J^{\pm}_{\mathrm{aux}} \mathrm{e}^{\mathrm{i}tH_0} f \text{ as } t \to \pm\infty.$$

Next we notice the following analogue of (6.53), cf. (6.56),

$$\begin{aligned}\phi^{\pm}(x, \omega, \lambda) &+ \int_{R_0}^{\infty} (\nabla\phi^{+}_{\mathrm{dol}} - F^+)(\pm l\omega, \pm\omega, \lambda) \cdot \omega\,\mathrm{d}l \\ &= \phi^{\pm}_{\mathrm{aux}}(x, \omega, \lambda) + o(|x|^0).\end{aligned}$$

Again this asymptotics is locally uniform in $(\omega, \lambda) \in S^{d-1} \times]0, \infty[$. □

Remark 6.16. The first factor on the right hand side of (7.11) is oscillatory. Let us state the following asymptotics for the special case where $V_1(r) = -\gamma r^{-\mu}$ for $r \geq R_0$:

$$\begin{aligned}\psi^{+}_{\mathrm{dol}}(\sqrt{2\lambda}) &= \int_{R_0}^{\infty} \left(\sqrt{2\lambda} - \sqrt{2(\lambda + \gamma r^{-\mu})} + (2\lambda)^{-\frac{1}{2}}\gamma r^{-\mu}\right) \mathrm{d}r \\ &= (2\lambda)^{\frac{1}{2}-\frac{1}{\mu}} \int_{R_0(2\lambda)^{\frac{1}{\mu}}} (1 - \sqrt{1 + 2\gamma s^{-\mu}} + \gamma s^{-\mu}) \mathrm{d}s.\end{aligned}$$

For $\lambda \searrow 0$, this behaves as

$$\begin{aligned} &(2\lambda)^{\frac{1}{2}-\frac{1}{\mu}} C_\mu + O\left(\lambda^{-\frac{1}{2}}\right); && \tfrac{1}{2} < \mu < 1, \\ -\gamma(2\lambda)^{-\frac{1}{2}} \ln 2\lambda + &(2\lambda)^{-\frac{1}{2}} C_1 + O(1); && \mu = 1, \\ &(2\lambda)^{-\frac{1}{2}} \tfrac{R_0^{1-\mu}\gamma}{\mu - 1} + O\left(\lambda^{\frac{1}{2}-\frac{1}{\mu}}\right); && 1 < \mu < 2.\end{aligned}$$

Here

$$C_\mu := \int_0^\infty (1 - \sqrt{1 + 2\gamma s^{-\mu}} + \gamma s^{-\mu})\mathrm{d}s,$$
$$C_1 := \int_1^\infty (1 - \sqrt{1 + 2\gamma s^{-1}} + \gamma s^{-1})\mathrm{d}s + \int_0^1 (1 - \sqrt{1 + 2\gamma s^{-1}})\mathrm{d}s - \gamma \ln R_0.$$

7. Scattering matrices

In this section we study (modified) scattering matrices. We prove that they have a limit at zero energy. This implies low energy oscillatory asymptotics for the standard short-range and Dollard scattering matrices.

7.1. **Scattering matrices at positive energies.** The scattering operator commutes with H_0, which is diagonalized by the direct integral decomposition (6.4). Because of that, the general theory of decomposable operators says that there exists a measurable family $]0,\infty[\ni \lambda \mapsto S(\lambda)$, with $S(\lambda)$ unitary operators on $L^2(S^{d-1})$ defined for almost all λ, such that

$$S \simeq \int_0^\infty \oplus S(\lambda)\, \mathrm{d}\lambda, \tag{7.1}$$

using the decomposition (6.4).

The following theorem is (essentially) well-known:

Theorem 7.1. *Assume Condition 1.1. Then*

$$S(\lambda) = -2\pi J^+(\lambda)^* T^-(\lambda) + 2\pi \mathrm{i} T^+(\lambda)^* R(\lambda + \mathrm{i}0) T^-(\lambda) \tag{7.2a}$$
$$= -2\pi W^+(\lambda)^* T^-(\lambda) \tag{7.2b}$$

defines a unitary operator on $L^2(S^{d-1})$ depending strongly continuously on $\lambda > 0$. Moreover, (7.1) *is true. Furthermore, for all $n \in \mathbb{R}$ and $\epsilon > 0$*

$$S(\lambda) \in \mathcal{B}(L^{2,n}(S^{d-1}), L^{2,n-\epsilon}(S^{d-1}))$$

depending norm continuously on $\lambda > 0$. (Hence in particular $S(\lambda)$ maps $C^\infty(S^{d-1})$ into itself.)

For a derivation of the formula (7.2a) we refer the reader to Appendix A. For the remaining part of the theorem we refer the reader to the proof of Theorem 7.2 stated below (one can use Theorem 6.3 and Lemma 6.4 as substitutes for Theorem 6.6 and Lemma 6.8, respectively).

7.2. **Scattering matrices at low energies.** Until the end of this section we assume that Conditions 2.1–2.3 are true. The main new result of this section is the following theorem:

Theorem 7.2. *The result of Theorem 7.1 is true for all $\lambda \in [0,\infty[$. Specifically, defining*

$$S(0) = -2\pi J^+(0)^* T^-(0) + 2\pi \mathrm{i} T^+(0)^* R(+\mathrm{i}0) T^-(0) \tag{7.3a}$$
$$= -2\pi W^+(0)^* T^-(0), \tag{7.3b}$$

then $S(0)$ is unitary, $\mathrm{s-}\lim_{\lambda\searrow 0} S(\lambda) = S(0)$ *in the sense of $\mathcal{B}(L^2(S^{d-1}))$ and also* $\lim_{\lambda\searrow 0} S(\lambda) = S(0)$ *in the sense of $\mathcal{B}(L^{2,n}(S^{d-1}), L^{2,n-\epsilon}(S^{d-1}))$ for any $n \in \mathbb{R}$ and $\epsilon > 0$.*

Proof. First we notice that the expression

$$S(\lambda) = -2\pi W^+(\lambda)^* T^-(\lambda) \in \mathcal{B}(L^{2,n}(S^{d-1}), L^{2,n-\epsilon}(S^{d-1})) \text{ for } n > 0$$

with a norm continuous dependence of $\lambda \geq 0$. Indeed, fix $n > 0$ and $\epsilon \in]0, n]$ and pick $\epsilon_1, \epsilon_2 \in \mathbb{R}$ such that $\epsilon \frac{\mu}{2} < \epsilon_1 < \epsilon$ and $\epsilon_2 = \frac{1}{2}(\epsilon - \epsilon_1)$. We write

$$\begin{aligned} &W^+(\lambda)^* T^-(\lambda) \qquad (7.4)\\ &= \big(W^+(\lambda)^* g^{\frac{1}{2}} \langle x \rangle^{-\frac{1}{2}-\epsilon_2} (\langle x \rangle g)^{-n+\epsilon}\big)\big(g^{-\epsilon} \langle x \rangle^{-\epsilon_1}\big)\big((\langle x \rangle g)^n \langle x \rangle^{\frac{1}{2}-\epsilon_2} g^{-\frac{1}{2}} T^-(\lambda)\big). \end{aligned}$$

We shall use the analogues of Lemma 6.8 (ii) and (iv) with $T^+(\lambda)$ replaced by $T^-(\lambda)$ (proved in the same way). The third factor on the right of (7.4) is continuous in λ with values in $\mathcal{B}(L^{2,n}(S^{d-1}), L^2(\mathbb{R}^d))$. The second factor is continuous in λ as an operator on $L^2(\mathbb{R}^d)$. The first factor is continuous in λ as an operator in $\mathcal{B}(L^2(\mathbb{R}^d), L^{2,n-\epsilon}(S^{d-1}))$ due to Theorem 6.6. This proves the norm continuity of $S(\lambda)$ in $\mathcal{B}(L^{2,n}(S^{d-1}), L^{2,n-\epsilon}(S^{d-1}))$ for $n > 0$.

Let us prove the same property for $n \leq 0$ using a slight extension of the above scheme: Notice that the positive sign condition above entered only in the condition $n - \epsilon \geq 0$ needed for applying Theorem 6.6. Since $n \leq 0$ we have $n - \epsilon < 0$ and therefore we need a substitute for Theorem 6.6. This is provided by Theorem 6.11 and an analogue of Lemma 6.8 for $T^-(\lambda)$. In fact, choose for (small) $\bar{\sigma} > 0$ real-valued $\tilde{\chi}_- \in C_c^\infty(\mathbb{R})$ and $\chi_+ \in C^\infty(\mathbb{R})$ such that $\operatorname{supp} \tilde{\chi}_- \subset]-1, 1[$, $\tilde{\chi}_- = 1$ in $[\bar{\sigma} - 1, 1 - \bar{\sigma}]$, $\operatorname{supp} \chi_+ \subset]C_0, \infty[$ and $\chi_+ = 1$ in $[2C_0, \infty[$. Let $\chi = 1 - \chi_+$, $\tilde{A} = \mathrm{Op}^{\mathrm{w}}(\chi(a)\tilde{\chi}_-(b))$, $A_+ = \mathrm{Op}^{\mathrm{w}}(\chi_+(a))$ and $\bar{A} = \mathrm{Op}^{\mathrm{w}}(\chi(a)(1 - \tilde{\chi}_-(b)))$. We insert the identity $I = \tilde{A} + A_+ + \bar{A}$

$$W^+(\lambda)^* T^-(\lambda) = ((\tilde{A} + A_+) W^+(\lambda))^* T^-(\lambda) + W^+(\lambda)^* (\bar{A} T^-(\lambda)). \qquad (7.5)$$

Due to Theorem 6.11 the above argument can be repeated for the first term on the right hand side, and if $\bar{\sigma} > 0$ is chosen sufficiently small we have the following analogue of Lemma 6.8 (iii) and (iv) (here stated in combination): For all $m \in \mathbb{R}$ the family of operators $\langle x \rangle^m \bar{A} T^-(\lambda)$ constitutes a continuous $\mathcal{B}(L^{2,-n}(S^{d-1}), L^2(\mathbb{R}^d))$–valued function of $\lambda \in [0, \infty[$. By choosing $m > \frac{1}{2} + \frac{\mu}{4}$ and using Theorem 6.6 we conclude norm continuity of the second term of (7.5).

But from the isometricity of S we see that $S(\lambda)$ is isometric for almost all λ as a map on $L^2(S^{d-1})$. Therefore, it is isometric and strongly continuous as a map on $L^2(S^{d-1})$ for all $\lambda \geq 0$.

By repeating this argument for S^* (not to be elaborated on) we obtain that $S(\lambda)^*$ is isometric and strongly continuous in $\lambda \geq 0$ as a map on $L^2(S^{d-1})$. Whence $S(\lambda)$ is unitary as a map on $L^2(S^{d-1})$. □

Remark. There is an alternative and completely stationary approach to proving the unitarity of the scattering matrices. In fact taking (7.2b) and (7.3b) as definitions the unitarity is a consequence of the formula (8.11), which in turn can be verified directly along the lines of Section 8.

7.3. **Asymptotics of short-range scattering matrices.** In the case $\mu \in]1, 2[$ we can compare $S(\lambda)$ with the S–matrix $S_{\mathrm{sr}}(\lambda)$ defined similarly

$$S_{\mathrm{sr}} = W_{\mathrm{sr}}^{+*} W_{\mathrm{sr}}^- \simeq \int_0^\infty \oplus S_{\mathrm{sr}}(\lambda)\, \mathrm{d}\lambda.$$

Under the condition of radial symmetry Yafaev considered in [Ya1] the component of $S_{\rm sr}(\lambda)$ for each sector of fixed angular momentum. He computed an explicit oscillatory behaviour as $\lambda \to 0$. The following result is a consequence of Theorem 6.13; in combination with Theorem 7.2 it yields oscillatory behaviour in a more general situation than considered in [Ya1].

Theorem 7.3. *For $\mu \in]1,2[$ the operators $S_{\rm sr}$ and S are related by*

$$S_{\rm sr} = \mathrm{e}^{-\mathrm{i}\psi_{\rm sr}^{+}(p)} S \mathrm{e}^{\mathrm{i}\psi_{\rm sr}^{-}(p)}. \tag{7.6}$$

In particular for all $\lambda > 0$

$$S_{\rm sr}(\lambda) = \mathrm{e}^{-\mathrm{i}\psi_{\rm sr}^{+}(\sqrt{2\lambda}\cdot)} S(\lambda) \mathrm{e}^{\mathrm{i}\psi_{\rm sr}^{-}(\sqrt{2\lambda}\cdot)}, \tag{7.7}$$

and if $V_2 = 0$ then

$$S_{\rm sr}(\lambda) = e^{-\mathrm{i}2\int_{R_0}^{\infty}\left(\sqrt{2\lambda}-\sqrt{2(\lambda-V_1(r))}\right)\,\mathrm{d}r} S(\lambda). \tag{7.8}$$

7.4. **Asymptotics of Dollard-type scattering matrices.** For $\mu > \frac{1}{2}$ and $\mu + \epsilon_2 > 1$ the Dollard-type S–matrix is diagonalized as before

$$S_{\rm dol} = W_{\rm dol}^{+*} W_{\rm dol}^{-} \simeq \int_0^\infty \oplus S_{\rm dol}(\lambda)\,\mathrm{d}\lambda.$$

We have the following analogue of Theorem 7.3, cf. Theorem 6.15.

Theorem 7.4. *For $\frac{1}{2} < \mu < 2$, $\epsilon_2 < 1$ and $\mu + \epsilon_2 > 1$ the operators $S_{\rm dol}$ and S are related by*

$$S_{\rm dol} = \mathrm{e}^{-\mathrm{i}\psi_{\rm dol}^{+}(p)} S \mathrm{e}^{\mathrm{i}\psi_{\rm dol}^{-}(p)}. \tag{7.9}$$

In particular for all $\lambda > 0$

$$S_{\rm dol}(\lambda) = \mathrm{e}^{-\mathrm{i}\psi_{\rm dol}^{+}(\sqrt{2\lambda}\cdot)} S(\lambda) \mathrm{e}^{\mathrm{i}\psi_{\rm dol}^{-}(\sqrt{2\lambda}\cdot)}, \tag{7.10}$$

and if $V_2 = 0$ then

$$S_{\rm dol}(\lambda) = \mathrm{e}^{-\mathrm{i}2\int_{R_0}^{\infty}\left(\sqrt{2\lambda}-\sqrt{2(\lambda-V_1(r)}-(2\lambda)^{-1/2}V_1(r)\right)\,\mathrm{d}r} S(\lambda). \tag{7.11}$$

Example 7.5. For the purely Coulombic case $V = -\gamma r^{-1}$ in dimension $d \geq 3$ one can compute

$$S(0) = \mathrm{e}^{\mathrm{i}c} P;\ c \in \mathbb{R}, \tag{7.12}$$

where $(P\tau)(\omega) = \tau(-\omega)$. This formula can be verified using (7.11) and Remark 6.16, the explicit formula [Ya3, (4.3)] for the Coulombic (Dollard) scattering matrix (slightly different from our definition, asymptotics of the gamma function (see for example the reference [3] of [Ya3]) and, for example, the stationary phase formula [Hö1, Theorem 7.7.6] (alternatively one can use the formula [Ya3, (3.4)]).

It follows from (7.12) that the singularities of the kernel $S(0)(\omega,\omega')$ in this particular case are located at $\{(\omega,\omega') \in S^{d-1} \times S^{d-1} | \omega = -\omega'\}$. We devote Section 9 to an extension of this result. In Section 10 we provide a different proof of (7.12) (up to a compact term); this approach yields $c = 4\sqrt{2\gamma R_0} - \pi\frac{d-2}{2}$.

We also note that for the purely Coulombic case there is in fact a complete asymptotic expansion $S(\lambda) \asymp \sum_{j=0}^{\infty} S_j \lambda^{j/2}$. Here (of course) S_0 is given by (7.12), and one can readily check that $S_1 \neq 0$. In particular we see that $S(\lambda)$ is not smooth at $\lambda = 0$, cf. Remark 4.2 3). We refer to [BGS] (and references cited therein) for explicit expansions of the generalized purely Coulombic eigenfunctions at zero energy (for $d = 3$); those are also in $\sqrt{\lambda}$.

8. Generalized eigenfunctions

Throughout this section we impose Conditions 2.1–2.3. For any $\lambda \geq 0$ we define

$$\mathcal{V}^{-\infty}(\lambda) = \{u \in L^{2,-\infty} | (H-\lambda)u = 0\} \subseteq \mathcal{S}'(\mathbb{R}^d).$$

Elements of $\mathcal{V}^{-\infty}(\lambda)$ will be called generalized eigenfunctions of H at energy λ. In this section we study all generalized eigenfunctions of H.

Remark. Note that by Proposition 4.3, for any $u \in V^{-\infty}(\lambda)$ and $s \in \mathbb{R}$

$$WF^s_{\mathrm{sc}}(u) \subseteq \{b^2 + \vec{c}^2 = 1\}. \tag{8.1}$$

8.1. **Representations of generalized eigenfunctions.**

In this subsection we show that all generalized eigenfunctions can be represented by their incoming or outgoing data.

Theorem 8.1. *For any $\lambda \geq 0$ the map*

$$W^{\pm}(\lambda) : \mathcal{D}'(S^{d-1}) \to \mathcal{V}^{-\infty}(\lambda)(\subseteq L^{2,-\infty})$$

is continuous and bijective.

Proof. **Step I**. Clearly $W^{\pm}(\lambda) : \mathcal{D}'(S^{d-1}) \to \mathcal{V}^{-\infty}(\lambda)$ is well-defined and continuous, cf. Theorem 6.6.

Step II. We show that $W^{\pm}(\lambda)$ is onto. Let $u \in \mathcal{V}^{-\infty}(\lambda)$ be given. Let

$$\chi^{\pm} = \chi_-(a)\tilde{\chi}_{\pm}(b) + \tfrac{1}{2}\chi_+(a), \tag{8.2}$$

where $\chi_+ = 1 - \chi_-$ is a real-valued function as in Proposition 4.1 (ii) such that $\chi_+(t) = 1$ for $t \geq 2C_0$, and $\tilde{\chi}_-, \tilde{\chi}_+ \in C^\infty(\mathbb{R})$ are real-valued functions obeying $\tilde{\chi}_- + \tilde{\chi}_+ = 1$ and

$$\operatorname{supp} \tilde{\chi}_- \subseteq]-\infty, 1/2[, \tag{8.3}$$

$$\operatorname{supp} \tilde{\chi}_+ \subseteq]-1/2, \infty[. \tag{8.4}$$

Now

$$\lim_{\epsilon \downarrow 0} R(\lambda \pm \mathrm{i}\epsilon)(H-\lambda)\mathrm{Op^r}(\chi^{\pm})u = \mathrm{Op^r}(\chi^{\pm})u \pm \lim_{\epsilon\downarrow 0} \mathrm{i}\epsilon R(\lambda \pm \mathrm{i}\epsilon)\mathrm{Op^r}(\chi^{\pm})u \tag{8.5}$$

Note that $\lim_{\epsilon\downarrow 0} R(\lambda \pm \mathrm{i}\epsilon)\mathrm{Op^r}(\chi^{\pm})u$ exists, due to Propositions 4.3, 4.7, and 4.9. Therefore the second term on the right of (8.5) is zero. Therefore, we have

$$0 = \mathrm{Op^r}(\chi^{\pm})u - R(\lambda \pm \mathrm{i}0)(H-\lambda)\mathrm{Op^r}(\chi^{\pm})u. \tag{8.6}$$

Adding the two equations of (8.6) yields

$$u = 2\pi\mathrm{i}\delta^V(\lambda)(H-\lambda)\mathrm{Op^r}(\chi^+)u,$$

which in turn in conjunction with Proposition 4.3, (6.10), (6.15) and Corollary 6.12 yields

$$u = W^{\pm}(\lambda)\tau;\ \tau = \pm 2\pi\mathrm{i}W^{\pm}(\lambda)^*[H, \mathrm{Op^r}(\chi^{\pm})]u \in \mathcal{D}'(S^{d-1}). \tag{8.7}$$

Step III. We show that $W^{\pm}(\lambda)$ is injective. For convenience we shall only treat the case of superscript $+$. By (8.7) we need to show that for all $\tau \in \mathcal{D}'(S^{d-1})$

$$\tau = 2\pi\mathrm{i}W^+(\lambda)^*(H-\lambda)\mathrm{Op^r}(\chi^+)W^+(\lambda)\tau. \tag{8.8}$$

By continuity it suffices to verify (8.8) for $\tau \in C^\infty(S^{d-1})$. This can be done as follows. Pick non-negative $f \in C_c^\infty(\mathbb{R})$ with $\int_0^\infty f(s)\mathrm{d}s = 1$, and let $F_R(t) = 1 - \int_0^{t/R} f(s)\mathrm{d}s$; $R > 1$. We write the right hand side of (8.8) as

$$\mathrm{w-}\lim_{R\to\infty} 2\pi \mathrm{i} W^+(\lambda)^* F_R(\langle x\rangle)(H-\lambda)\mathrm{Op^r}(\chi^+)W^+(\lambda)\tau; \tag{8.9}$$

and pull the factor $(H-\lambda)$ to the left. Thus (8.9) equals

$$\mathrm{w-}\lim_{R\to\infty} 2\pi R^{-1} W^+(\lambda)^* f(\langle x\rangle/R) g \mathrm{Op^r}(b\chi^+)W^+(\lambda)\tau.$$

If $\lambda \geq 0$ we insert (6.8) for $W^+(\lambda)$ (if $\lambda = 0$ we use instead (6.14)). By Proposition 4.1 (ii) and (iii) and Lemma 6.8 (ii) and (iv) we can replace each factor of $W^+(\lambda)$ by a factor of $J^+(\lambda)$, cf. the proof of Theorem 6.11. Moreover we can replace the factor $\mathrm{Op^r}(b\chi^+)$ by the operator $g^{-1}\hat{x}\cdot p$. Therefore, (8.9) becomes

$$\mathrm{w-}\lim_{R\to\infty} 2\pi R^{-1} J^+(\lambda)^* f(\langle x\rangle/R)\hat{x}\cdot p J^+(\lambda)\tau. \tag{8.10}$$

By Theorem 5.7, (8.10) equals τ. The identity (8.8) follows. □

Remarks. 1) A somewhat similar representation formula has been derived for representing positive solutions to a PDE, see for example [Mu]. This involves a notion of so-called Martin boundary. In our case, the notion analogous to the "Martin boundary" would be S^{d-1}.

2) For $V_3 = 0$ we have

$$\mathcal{V}^{-\infty}(\lambda) = \{u \in \mathcal{S}'(\mathbb{R}^d) \mid (H-\lambda)u = 0\}$$

and hence the set $\mathcal{V}^{-\infty}(\lambda)$ is closed in $\mathcal{S}'(\mathbb{R}^d)$ (with respect to the weak-$*$ topology of $\mathcal{S}'(\mathbb{R}^d)$). Moreover, in this case $W^\pm(\lambda)$ maps $\mathcal{D}'(S^{d-1})$ bicontinuously onto $\mathcal{V}^{-\infty}(\lambda)$).

In fact, suppose $u \in \mathcal{S}'(\mathbb{R}^d)$ obeys $(H-\lambda)u = 0$. Then for some $m \in \mathbb{N}$ we have $\langle p\rangle^{-2m}u \in L^{2,-\infty}$. But $(H-\lambda+\mathrm{i})^{-m}\langle p\rangle^{2m}$ is bounded on any $L^{2,s}$. Whence, showing that indeed $u \in \mathcal{V}^{-\infty}(\lambda)$,

$$\mathrm{i}^{-m}u = (H-\lambda+\mathrm{i})^{-m}u = (H-\lambda+\mathrm{i})^{-m}\langle p\rangle^{2m}\big(\langle p\rangle^{-2m}u\big) \in L^{2,-\infty}.$$

8.2. **Scattering matrices – an alternative construction.** The construction of scattering matrices given in Subsections 7.1 and 7.2 involved a detailed knowledge of appropriate operators, see the proof of Theorem 7.2. However, given the theory of wave matrices developed in Subsection 8.1 and the basic formulas (6.10) and (6.15) for the spectral resolution we could have constructed the scattering matrix more easily.

Recall from Theorem 8.1 that $W^\pm(\lambda) : \mathcal{D}'(S^{d-1}) \to L^{2,-\infty}$ is injective. Hence, $W^\pm(\lambda)^* : L^{2,\infty} \to C^\infty(S^{d-1})$ has a dense range.

For $\tau \in L^2(S^{d-1})$ of the form $\tau = W^-(\lambda)^*v$ with $v \in L^{2,\infty}$ we define $S(\lambda)\tau := W^+(\lambda)^*v$. By (6.10) and (6.15) we know that

$$\|W^+(\lambda)^*v\|^2 = \|W^-(\lambda)^*v\|^2 = \langle v, \delta^V(\lambda)v\rangle.$$

Hence $S(\lambda)$ is indeed well-defined and isometric. But $W^\pm(\lambda)^*L^{2,\infty}$ is dense in $C^\infty(S^{d-1})$, and therefore also in $L^2(S^{d-1})$. Whence $S(\lambda)$ extends to an isometric operator on $L^2(S^{d-1})$. Reversing the role of $+$ and $-$ we obtain that $S(\lambda)$ is actually unitary. By construction, it satisfies

$$S(\lambda)W^-(\lambda)^* = W^+(\lambda)^*;\ \lambda \geq 0. \tag{8.11}$$

8.3. **Geometric scattering matrices.** The following type of result was proved for a class of constant coefficient Hamiltonians (with no potential) in [AH], and generalized to Schrödinger operators with long-range potentials (for a class including the one given by Condition 1.1) at positive energies by [GY]. It gives a characterization of the space $W^{\pm}(\lambda)L^2(S^{d-1})$ which in turn yields yet another characterization of the scattering matrix $S(\lambda)$.

Let $s_0 = s_0(\lambda)$ be given as in (4.6), and introduce in terms of a dual Besov space

$$\mathcal{V}^{-s_0}(\lambda) := B^*_{s_0} \cap \mathcal{V}^{-\infty}(\lambda)$$

endowed with the topology of $B^*_{s_0}$. The statement (iv) below is given in terms of the phase function $\phi = \phi(x, \lambda)$ of (5.21).

Theorem 8.2. (i) *For all* $\tau \in L^2(S^{d-1})$

$$WF^{-s_0}_{\mathrm{sc}}(W^{\pm}(\lambda)\tau) \subseteq \{b = -1\} \cup \{b = 1\}.$$

(ii) *The operator* $W^{\pm}(\lambda)$ *maps* $L^2(S^{d-1})$ *bijectively and bicontinuously onto* $\mathcal{V}^{-s_0}(\lambda)$.
(iii) *The operator* $W^{\pm}(\lambda)^*$ *(defined a priori on* $B^{**}_{s_0} \supseteq B_{s_0}$*) maps* B_{s_0} *onto* $L^2(S^{d-1})$.
(iv) *For all* $\tau \in L^2(S^{d-1})$

$$W^-(\lambda)\tau(x) - \frac{\mathrm{e}^{\mathrm{i}\pi\frac{d-1}{4}}\mathrm{e}^{-\mathrm{i}\phi(x,\lambda)}\tau(-\hat{x}) + \mathrm{e}^{-\mathrm{i}\pi\frac{d-1}{4}}\mathrm{e}^{\mathrm{i}\phi(x,\lambda)}(S(\lambda)\tau)(\hat{x})}{(2\pi)^{\frac{1}{2}}g^{\frac{1}{2}}(r,\lambda)r^{\frac{d-1}{2}}} \in B^*_{s_0,0}, \tag{8.12}$$

$$W^+(\lambda)\tau(x) - \frac{\mathrm{e}^{-\mathrm{i}\pi\frac{d-1}{4}}\mathrm{e}^{\mathrm{i}\phi(x,\lambda)}\tau(\hat{x}) + \mathrm{e}^{\mathrm{i}\pi\frac{d-1}{4}}\mathrm{e}^{-\mathrm{i}\phi(x,\lambda)}(S(\lambda)^*\tau)(-\hat{x})}{(2\pi)^{\frac{1}{2}}g^{\frac{1}{2}}(r,\lambda)r^{\frac{d-1}{2}}} \in B^*_{s_0,0}, \tag{8.13}$$

$$\|\tau\|^2_{L^2(S^{d-1})} = \lim_{R\to\infty} R^{-1}\int_{r<R} |\sqrt{\pi}g^{\frac{1}{2}}(r,\lambda)W^{\pm}(\lambda)\tau|^2\mathrm{d}x. \tag{8.14}$$

Proof. **Re (i).** Again we concentrate on the case of superscript $+$. Let $\tau \in L^2(S^{d-1})$ be given. We shall use the partition (6.11) as in the proof of Theorems 6.5 and 6.6, so let $\bar{\sigma} > 0$ be given as before, cf. (6.12) and (6.13). As for the partition functions (8.2) we modify (8.3) and (8.4) by replacing here $\tilde{\chi}_{\pm} \to \tilde{\chi}_{\pm,\mathrm{right}}$

$$\operatorname{supp}\tilde{\chi}_{-,\mathrm{right}} \subseteq]-\infty, 1 - \bar{\sigma}/4[, \tag{8.15}$$

$$\operatorname{supp}\tilde{\chi}_{+,\mathrm{right}} \subseteq]1 - \bar{\sigma}/2, \infty[. \tag{8.16}$$

Then it follows from Propositions 4.1 and 4.3 and Lemmas 6.8 and 6.9 that

$$\mathrm{Op^r}(\chi^+_{\mathrm{right}})W^+(\lambda) \in \mathcal{B}(L^2(S^{d-1}), B^*_{s_0}). \tag{8.17}$$

(The fact that this bound holds for $W^+(\lambda) \to J^+(\lambda)$ is indeed a consequence of Lemma 6.9 due to interpolation, cf. [Hö2, Theorem 14.1.4], but it can also be proved concretely along the lines of the proofs of Lemma 6.9 and Theorem 6.11.)

Since $\langle W^+(\lambda)\tau, \mathrm{i}[H, F_R\mathrm{Op^r}(\chi^+_{\mathrm{right}})]W^+(\lambda)\tau\rangle = 0$, we conclude from (4.30) and (8.17) that

$$\sup_{R>1}\mathrm{Re}\,\langle W^+(\lambda)\tau, \mathrm{Op^w}(F_R\chi_-(a)\tilde{\chi}'_{\mathrm{right}}(b)gr^{-1})W^+(\lambda)\tau\rangle \le C\|\tau\|^2; \tag{8.18}$$

here we used the calculus of pseudodifferential operators, cf. [Hö1, Theorem 18.6.8].

In combination with Propositions 4.3 and 4.5 we conclude that

$$\{-1 < b < 1\} \cap WF^{-s_0}_{\mathrm{sc}}(W^+(\lambda)\tau) = \emptyset. \tag{8.19}$$

Re (ii) (Boundedness).

To proceed from here we change (8.15) and (8.16) as follows

$$\operatorname{supp}\tilde{\chi}_{-,\mathrm{left}} \subseteq]-\infty, -1+\bar{\sigma}/2[, \tag{8.20}$$

$$\operatorname{supp}\tilde{\chi}_{+,\mathrm{left}} \subseteq]-1+\bar{\sigma}/4, \infty[. \tag{8.21}$$

With these cutoffs we can show analogously that

$$\mathrm{Op^r}(\chi^-_{\mathrm{left}})W^-(\lambda) \in \mathcal{B}(L^2(S^{d-1}), B^*_{s_0}). \tag{8.22}$$

Using (8.11) this leads to

$$\mathrm{Op^r}(\chi^-_{\mathrm{left}})W^+(\lambda) \in \mathcal{B}(L^2(S^{d-1}), B^*_{s_0}). \tag{8.23}$$

Finally writing (with $\chi_{\mathrm{middle}} := 1 - \chi^+_{\mathrm{right}} - \chi^-_{\mathrm{left}}$)

$$W^+(\lambda) = \mathrm{Op^r}(\chi^+_{\mathrm{right}})W^+(\lambda) + \mathrm{Op^r}(\chi^-_{\mathrm{left}})W^+(\lambda) + \mathrm{Op^r}(\chi_{\mathrm{middle}})W^+(\lambda)$$

we conclude from we (4.54), (8.17), (8.19) and (8.23) that indeed

$$W^+(\lambda) \in \mathcal{B}(L^2(S^{d-1}), B^*_{s_0}). \tag{8.24}$$

Whence $W^+(\lambda)$ maps $L^2(S^{d-1})$ continuously into $\mathcal{V}^{-s_0}(\lambda)$.

Re (ii) (Bijectiveness). We shall show that $W^+(\lambda)$ maps $L^2(S^{d-1})$ onto $\mathcal{V}^{-s_0}(\lambda)$. Using the expression (8.7) for the inverse $\tau \in \mathcal{D}'(S^{d-1})$, mimicking the first part of Step III in the proof of Theorem 8.1 and using the Riesz' representation theorem (see for example [Yo]) in conjunction with (8.24), we obtain that indeed $\tau \in L^2(S^{d-1})$. This argument also shows that

$$W^+(\lambda)^{-1} \in \mathcal{B}(\mathcal{V}^{-s_0}(\lambda), L^2(S^{d-1})). \tag{8.25}$$

Re (iii). The result follows from (ii) by the Banach's closed range theorem, see [Yo].

Re (iv). Let

$$u_{\pm,\tau}(x) = (2\pi)^{-\frac{1}{2}} \mathrm{e}^{\mp \mathrm{i}\pi\frac{d-1}{4}} g^{-\frac{1}{2}}(r,\lambda) r^{-\frac{d-1}{2}} \mathrm{e}^{\pm \mathrm{i}\phi(x,\lambda)} \tau(\pm\hat{x}).$$

Clearly $u_{\pm,\tau} \in B^*_{s_0}$ with a continuous dependence of τ. We claim (with reference to (8.2)) that

$$\mathrm{Op^r}(\chi^\pm)W^\pm(\lambda)\tau - u_{\pm,\tau} \in B^*_{s_0,0}. \tag{8.26}$$

Notice that also the first term is in $B^*_{s_0}$ with a continuous dependence of τ, cf. (8.17) and (8.19), hence it suffices to show (8.26) for $\tau \in C^\infty(S^{d-1})$ in which case the asymptotics follows from Theorem 5.7, cf. Step III of the proof of Theorem 8.1.

Now, combining (8.26) and the identity (8.11) we obtain

$$\mathrm{Op^r}(\chi^+)W^-(\lambda)\tau - u_{+,S(\lambda)\tau},\ \mathrm{Op^r}(\chi^-)W^+(\lambda)\tau - u_{-,S(\lambda)^*\tau} \in B^*_{s_0,0}. \tag{8.27}$$

By (8.26) and (8.27)

$$W^-(\lambda)\tau - (u_{-,\tau} + u_{+,S(\lambda)\tau}),\ W^+(\lambda)\tau - (u_{+,\tau} + u_{-,S(\lambda)^*\tau}) \in B^*_{s_0,0},$$

showing (8.12) and (8.13).

As for (8.14) we use (8.12) and (8.13); notice that the cross terms do not contribute to the limit which can be seen by an integration by parts with respect to the variable $r = |x|$ invoking Proposition 3.3. □

On the basis of Theorem 8.2 we can characterize the scattering matrix $S(\lambda)$ geometrically as follows.

Corollary 8.3. *For all $\tau^- \in L^2(S^{d-1})$ there exist uniquely determined $u \in \mathcal{V}^{-s_0}(\lambda)$ and $\tau^+ \in L^2(S^{d-1})$ such that*

$$u - \frac{\mathrm{e}^{\mathrm{i}\pi\frac{d-1}{4}}\mathrm{e}^{-\mathrm{i}\phi(x,\lambda)}\tau^-(-\hat{x}) + \mathrm{e}^{-\mathrm{i}\pi\frac{d-1}{4}}\mathrm{e}^{\mathrm{i}\phi(x,\lambda)}\tau^+(\hat{x})}{(2\pi)^{\frac{1}{2}}g^{\frac{1}{2}}(r,\lambda)r^{\frac{d-1}{2}}} \in B^*_{s_0,0}. \tag{8.28}$$

We have $\tau^+ = S(\lambda)\tau^-$, $u = W^-(\lambda)\tau^- = W^+(\lambda)\tau^+$.

Proof. The existence part (with $\tau^+ = S(\lambda)\tau^-$) follows from (8.12).

To show the uniqueness, suppose that u_i, τ_i^+, $i = 1, 2$, satisfy the requirements of (8.28) with the same τ_-. Then for the difference, $u = u_1 - u_2$, we have $(H - \lambda)u = 0$ and $WF\left(B^*_{s_0,0}, u\right) \subseteq \{b = 1\}$. Hence by Proposition 4.10, $u = 0$. ☐

Corollary 8.4. *Let $d \geq 2$ and $\lambda \geq 0$. Suppose (in addition to Conditions 2.1 and 2.3) that V_2 and V_3 are spherically symmetric and that $\int_0^\infty r|V_3(r)|\,\mathrm{d}r < \infty$ (Condition 2.2 is not needed since V_2 can be absorbed into V_1). Then there exists a real-valued continuous function $\sigma_l(\cdot)$ such that for all spherical harmonics Y of order l we have $S(\lambda)Y = \mathrm{e}^{\mathrm{i}2\sigma_l(\lambda)}Y$.*

Let $u_l(r)$ denote the regular solution of the reduced Schrödinger equation on the half-line $]0, \infty[$

$$-u'' + V_l u = 0;\ V_l(r) = 2(V(r) - \lambda) + \frac{(l + \frac{d}{2} - 1)^2 - 4^{-1}}{r^2};\ l \geq 0,$$

where, "regular" refers to the asymptotics $u(r) \asymp r^{l+\frac{d-1}{2}}$ as $r \to 0$. Then $\sigma_l(\cdot)$ is uniquely determined $\mod 2\pi$ by the asymptotics

$$\frac{u_l(r)}{r^{\frac{d-1}{2}}} - C\frac{\sin\left(\int_{R_0}^r \sqrt{2(\lambda - V(r'))}\,\mathrm{d}r' + \sqrt{2\lambda R_0} - \frac{d-3+2l}{4}\pi + \sigma_l(\lambda)\right)}{(\lambda - V(r))^{\frac{1}{4}}r^{\frac{d-1}{2}}} \in B^*_{s_0,0},$$

where $C = C(l, \lambda)$ is a (uniquely determined) positive constant.

Proof. Let Y be a spherical harmonic of order l. Note that its parity is $(-1)^l$, i.e. $Y(-\omega) = (-1)^l Y(\omega)$. Besides, $u := r^{-\frac{d-1}{2}}u_l(r)Y(\hat{x})$ solves $(H - \lambda)u = 0$. We apply Corollary 8.3 with this u and with $\tau^- = Y$, so that $\tau^+ = \mathrm{e}^{\mathrm{i}2\sigma_l(\lambda)}Y$. Then

$$\begin{aligned}
&\mathrm{e}^{\mathrm{i}\pi\frac{d-1}{4}}\mathrm{e}^{-\mathrm{i}\phi(x,\lambda)}\tau^-(-\hat{x}) + \mathrm{e}^{-\mathrm{i}\pi\frac{d-1}{4}}\mathrm{e}^{\mathrm{i}\phi(x,\lambda)}\tau^+(\hat{x}) \\
&\quad = \left(\mathrm{e}^{\mathrm{i}\pi\frac{d-1}{4} - \mathrm{i}\phi(x,\lambda) + \mathrm{i}\pi l} + \mathrm{e}^{-\mathrm{i}\pi\frac{d-1}{4} + \mathrm{i}\phi(x,\lambda) + \mathrm{i}2\sigma_l(\lambda)}\right)Y(\hat{x}) \\
&\quad = 2\mathrm{e}^{\mathrm{i}\pi\frac{l}{2} + \mathrm{i}\sigma_l(\lambda)}\sin\left(\phi(x,\lambda) - \tfrac{d-3+2l}{4}\pi + \sigma_l(\lambda)\right)Y(\hat{x}).
\end{aligned}$$

We finish the proof using (5.22). ☐

9. Homogeneous potentials – location of singularities of $S(0)$

In this section we impose Conditions 2.1–2.3 with $d \geq 2$ and the condition $V_1(r) = -\gamma r^{-\mu}$ for $r \geq 1$ and hence $V(r) = -\gamma r^{-\mu} + O(r^{-\mu-\epsilon_2})$, cf. (1.23). Throughout the section $g = g(\lambda = 0) = \sqrt{-2V_1}$.

Our goal is to prove a statement about the localization of the singularities of the (Schwartz) kernel $S(0)(\omega, \omega')$. The purely Coulombic case for which $\mu = 1$ and $d \geq 3$ was treated explicitly in Example 7.5. Under an additional condition we can write down a fairly explicit integral that carries the singularities.

In Section 10 we shall study the nature of these singularities (under the condition of spherical symmetry) using one-dimensional WKB-analysis.

9.1. **Reduced classical equations.** Consider the classical system given by the Hamiltonian $h_1(x,\xi) = \frac{1}{2}\xi^2 - \gamma|x|^{-\mu}$ for $x \neq 0$. The equations of motion for $h_1(x,\xi)$ are invariant with respect to the transformation

$$(x,\xi) \mapsto (\lambda x, \lambda^{-\mu/2}\xi), \quad \lambda \in \mathbb{R}_+, \tag{9.1}$$

upon rescaling of time $t \mapsto t\lambda^{1+\mu/2}$.

Let

$$\mathbb{T}^* := (\mathbb{R}^d \backslash \{0\}) \times \mathbb{R}^d / \sim,$$

where $(x_1,\xi_1) \sim (x_2,\xi_2)$ iff there exists $\lambda > 0$ such that $(x_1,\xi_1) \mapsto (\lambda x_2, \lambda^{-\mu/2}\xi_2)$. Note that $\mathbb{T}^*$ can be conveniently identified with $T^*(S^{d-1}) \times \mathbb{R}$. We shall introduce coordinates of $\mathbb{T}^*$ by setting $b = \hat{x} \cdot \frac{\xi}{g} \in \mathbb{R}$ and $\bar{c} = (I - |\hat{x}\rangle\langle\hat{x}|)\frac{\xi}{g} \in T^*_{\hat{x}}(S^{d-1})$ with $\hat{x} \in S^{d-1}$. (At this point we are slightly abusing the notation of Subsection 4.2, however as noticed there the b and $\bar{c}$ given by (4.7) agree with the above definition for $r \geq 1$.) The equations of motion for the hamiltonian h_1 can be reduced to $\mathbb{T}^*$. Introducing the "new time" τ by $\frac{\mathrm{d}\tau}{\mathrm{d}t} = g/r$ we have the following system of reduced equations of motion:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{x} = \bar{c}, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\bar{c} = -(1-\frac{\mu}{2})b\bar{c} - \bar{c}^2\hat{x}, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}b = (1-\frac{\mu}{2})\bar{c}^2 + \frac{\mu}{2}(b^2+\bar{c}^2-1). \end{cases} \tag{9.2}$$

(Notice that the last equation follows from (4.29)). The maximal solution of (9.2) that passes $z = (\hat{x}, b, \bar{c}) \in \mathbb{T}^*$ at $\tau = 0$ is denoted by $\gamma(\tau, z)$.

Beside (9.2), we shall consider a related dynamics given by the equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{x} = \bar{c}, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\bar{c} = -(1-\frac{\mu}{2})b\bar{c} - \bar{c}^2\hat{x}, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}b = (1-\frac{\mu}{2})\bar{c}^2. \end{cases} \tag{9.3}$$

The (maximal) solution of the system (9.3) that passes $z = (\hat{x}, b, \bar{c}) \in \mathbb{T}^*$ at $\tau = 0$ will be denoted by $\gamma_0(\tau, z)$. Clearly the equation $\bar{c} = 0$ defines the fixed points, and the system is complete.

Notice that the surface $h_1^{-1}(0)$ in the coordinates $(\hat{x}, b, \bar{c})$ corresponds to the condition $b^2 + \bar{c}^2 = 1$. This surface is preserved both by the flow γ and γ_0, and on this surface both flows coincide.

Note that the flow γ_0 is exactly solvable. The variable b is always increasing and $k = b^2 + \bar{c}^2$ is a conserved quantity; of course the relevant value is $k = 1$. For non-fixed points we can compute its dependence on the modified time

$$b(\tau) = \sqrt{k}\tanh\sqrt{k}(1-\tfrac{\mu}{2})(\tau - \tau_0). \tag{9.4}$$

Values $k \neq 1$ correspond in this picture to replacing the coupling constant $\gamma \to k\gamma$. Precisely, if $k = b^2 + \bar{c}^2$ for a solution to (9.3) we can define $r(\tau) = r_0 \exp(\int_0^\tau b d\tau')$, introduce $t = \int_0^\tau \frac{r}{g(r)} d\tau'$ and check that indeed

$$\begin{cases} x(t) = r\hat{x}, \\ \xi(t) = g(r)(b\hat{x} + \bar{c}), \end{cases} \tag{9.5}$$

defines a zero energy solution to Hamilton's equations with $V \to kV$. The equation $b = 0$ corresponds to a turning point (at which $|x(t)|$ is smallest).

Clearly it follows from (9.4) that $\lim_{\tau\to\infty} b = \sqrt{k}$, $\lim_{\tau\to-\infty} b = -\sqrt{k}$. Upon writing $\hat{x}(\tau)\cdot\hat{x}(\infty) = \cos\theta(\tau)$ for some monotone continuous function $\theta(\cdot)$ we obtain from (1.27) that

$$|\theta(\infty) - \theta(-\infty)| = \tfrac{2}{2-\mu}\pi. \tag{9.6}$$

9.2. **Propagation of singularities.** We will use the scattering wave front set at zero energy, introduced in Subsection 4.2. The following proposition is somewhat similar to Hörmander's theorem about propagation of singularities adapted to scattering at the zero energy. It is a "local" version of Proposition 4.5 which takes into account the fact that in the case of a homogeneous potential we can use the dynamics in the reduced phase space. Again the proof is a modification of that of [Hö3, Proposition 3.5.1], see also [Me] and [HMV].

Proposition 9.1. *Suppose $u, v \in L^{2,-\infty}$, $Hu = v$, $s \in \mathbb{R}$, $z \in \mathbb{T}^*$ and $z \notin WF^s_{\mathrm{sc}}(u)$. Define*

$$\tau^+ = \sup\{\tau \ge 0|\ \gamma_0(\tilde\tau, z) \notin WF^s_{\mathrm{sc}}(u) \text{ for all } \tilde\tau \in [0,\tau]\},$$
$$\tau^- = \inf\{\tau \le 0|\ \gamma_0(\tilde\tau, z) \notin WF^s_{\mathrm{sc}}(u) \text{ for all } \tilde\tau \in [\tau,0]\}.$$

If $\tau^+ < \infty$, then $\gamma_0(\tau^+, z) \in WF^{s+2s_0}_{\mathrm{sc}}(v)$. If $\tau^- > -\infty$, then $\gamma_0(\tau^-, z) \in WF^{s+2s_0}_{\mathrm{sc}}(v)$.

Proof. The proof is similar to the one of Proposition 4.5. We shall only deal with the case of forward flow; the case of superscript "$-$"is similar (actually it follows from the case of "$+$"by time reversal invariance). For convenience we shall assume that $\epsilon_2 \le 2 - \mu$.

Step I. We will first show the following weaker statement: Suppose $u \in L^{2,s-\frac{\epsilon_2}{2}}$, $v \in L^{2,s+2s_0}$ and $Hu = v$. Then

$$\gamma_0(\tau, z) \notin WF^s_{\mathrm{sc}}(u) \text{ for all } \tau \ge 0. \tag{9.7}$$

Suppose on the contrary that (9.7) is false. Then we obtain from Proposition 4.3 that the flows of (9.2) and (9.3), starting at z, coincide. Letting $\gamma(\tau) = \gamma(\tau, z)$, it thus needs to be shown that

$$\tau^+ := \sup\{\tau \ge 0|\gamma(\tilde\tau) \notin WF^s_{\mathrm{sc}}(u) \text{ for all } \tilde\tau \in [0,\tau]\} = \infty. \tag{9.8}$$

Suppose on the contrary that τ^+ is finite. Then $\gamma(\tau^+)$ is not a fixed point. Consequently we can pick a slightly smaller $\tilde\tau^+ < \tau^+$ and a transversal $(2d-2)$–dimensional submanifold at $\gamma(\tilde\tau^+)$, say $\mathcal{M}$, such that with $J =]-\epsilon + \tilde\tau^+, \tau^+ + \epsilon[$ for some small $\epsilon > 0$ the map

$$J \times \mathcal{M} \ni (\tau, m) \to \Psi(\tau, m) = \gamma(\tau - \tilde\tau^+, m) \in \mathbb{T}^*,$$

is a diffeomorphism onto its range.

We pick $\chi \in C^\infty_c(\mathcal{M})$ supported in a small neighbourhood of $\gamma(\tilde\tau^+)$ such that $\chi(\gamma(\tilde\tau^+)) = 1$ and

$$\Psi(]-\epsilon + \tilde\tau^+, \tilde\tau^+] \times \operatorname{supp}\chi) \cap WF^s(u) = \emptyset. \tag{9.9}$$

We pick a non-positive function $f \in C^\infty_c(J)$ such that $f' \ge 0$ on a neighbourhood of $[\tilde\tau^+, \tau^+ + \epsilon)$ and $f(\tau^+) < 0$.

Let $f_K(\tau) = \exp(-K\tau)f(\tau)$ for $K > 0$, and $X_\kappa = (1 + \kappa r^2)^{1/2}$ for $\kappa \in]0,1]$. We consider the symbol

$$b_\kappa = g^{-1/2}X^{1/2}a_\kappa;\ a_\kappa = X^s X_\kappa^{-\epsilon_2/2} F(r > 2)(f_K \otimes \chi) \circ \Psi^{-1}. \tag{9.10}$$

First we fix K: A part of the Poisson bracket with b_κ^2 is

$$\{h_2, g^{-1}X^{2s+1}X_\kappa^{-\epsilon_2}\} = r^{-1}Y_\kappa b X^{2s+1}X_\kappa^{-\epsilon_2}, \tag{9.11}$$

where $Y_\kappa = Y_\kappa(r)$ is uniformly bounded in κ. We fix K such that $2K \geq |Y_\kappa b| + 2$ on $\operatorname{supp} b_\kappa$.

We compute

$$\{h_1, (f_K \otimes \chi) \circ \Psi^{-1}\} = \frac{g}{r}\left(\left[\frac{d}{d\tau} f_K\right] \otimes \chi\right) \circ \Psi^{-1}. \tag{9.12}$$

From (9.11) and (9.12), and by the choice of f and K, we conclude that

$$\{h_2, b_\kappa^2\} \leq -2a_\kappa^2 + O\big(r^{2s-\epsilon_2}\big) \text{ at } \mathcal{P} \subseteq \mathbb{T}^* \tag{9.13}$$

given by

$$\mathcal{P} = \Psi(\{\tau \in J | f'(\tau) \geq 0\} \times \operatorname{supp} \chi).$$

Introducing $A_\kappa = \mathrm{Op^w}(a_\kappa)$ and $B_\kappa = \mathrm{Op^w}(b_\kappa)$ we have

$$\langle i[H, B_\kappa^2]\rangle_u = -2\mathrm{Im}\,\langle v, B_\kappa^2 u\rangle, \tag{9.14}$$

and we estimate the right hand side using the calculus of pseudodifferential operators, cf. [Hö1, Theorems 18.5.4, 18.6.3, 18.6.8], to obtain the uniform bound

$$|\langle i[H, B_\kappa^2]\rangle_u| \leq C_1 \|v\|_{s+2s_0} \|A_\kappa u\| + C_2 \leq \|A_\kappa u\|^2 + C_3. \tag{9.15}$$

On the other hand using (9.9) and (9.13) we infer that

$$\langle i[H - V_3, B_\kappa^2]\rangle_u \leq -2\|A_\kappa u\|^2 + C_4. \tag{9.16}$$

An application of (4.12a) yields

$$\langle i[V_3, B_\kappa^2]\rangle_u \leq C_5. \tag{9.17}$$

Combining (9.15)–(9.17) yields

$$\|A_\kappa u\|^2 \leq C_6 = C_3 + C_4 + C_5,$$

which in turn gives a uniform bound

$$\|X_\kappa^{-\epsilon_2/2}\mathrm{Op^w}\big(\chi_{\gamma(\tau^+)}F(r > 2)\big)u\|_s^2 \leq C_7; \tag{9.18}$$

here $\chi_{\gamma(\tau^+)}$ signifies a phase-space localization factor of the form entering in (4.8) supported in a sufficiently small neighbourhood of the point $\gamma(\tau^+)$.

We let $\kappa \to 0$ in (9.18) and infer that $\tau^+ \notin WF_{\mathrm{sc}}^s(u)$, which is a contradiction. We have proved (9.8) and hence (9.7).

Step II. To relax the assumptions on u and v used in Step I we modify the above proof (using localization) in an iterative procedure very similar to Step II of the proof of Proposition 4.5.

Pick $t < s$ such that $u \in L^{2,t}$ and define $s_m = \min(s, t + m\epsilon_2/2)$ for $m \in \mathbb{N}$. Let correspondingly τ_m^+ be given as τ^+ upon replacing $s \to s_m$. Clearly

$$\tau_m^+ \leq \tau_{m-1}^+;\ m = 2, 3, \ldots \tag{9.19}$$

We shall show that

$$\tau_m^+ < \infty \Rightarrow \gamma_0(\tau_m^+, z) \in WF_{\mathrm{sc}}^{s_m+2s_0}(v). \tag{9.20}$$

We are done by using (9.20) for an m taken so large that $s_m = s$.

Let us consider the start of induction given by $m = 1$, in which case obviously $u \in L^{2,s_m-\epsilon_2/2}$. Suppose on the contrary that (9.20) is false. Then we consider the following case:

$$\tau_m^+ < \infty \text{ and } \gamma_0(\tau_m^+, z) \notin WF_{\mathrm{sc}}^{s_m+2s_0}(v). \tag{9.21}$$

It follows from (9.21) and an ellipticity argument that $b^2 + \bar{c}^2 = 1$ at $\gamma_0(\tau_m^+, z)$ (using that $\gamma_0(\tau_m^+, z) \notin WF_{\mathrm{sc}}^{s_m+\mu}(Hu)$). Consequently we can henceforth use the flow of (9.2), $\gamma(\tau) = \gamma(\tau, \cdot)$, exactly as in Step I.

We let $\epsilon > 0$, J, f, f_K, χ and Ψ be chosen as in Step I with $\tau^+ \to \tau_m^+$ and $\tilde{\tau}^+ \to \tilde{\tau}_m^+$. Let $\tilde{f} \in C_c^\infty(]\tilde{\tau}_m^+ - 2\epsilon, \tau_m^+ + 2\epsilon[)$ with $\tilde{f} = 1$ on J. Similarly let $\tilde{\chi} \in C_c^\infty(\mathcal{M})$ be supported in a small neighbourhood of $\gamma(\tilde{\tau}_m^+)$ such that $\tilde{\chi}(\gamma(\tilde{\tau}_m^+)) = 1$ in a neighbourhood of $\operatorname{supp}\chi$.

It follows from (9.21), possibly by shrinking the supports of $\tilde{f}$ and $\tilde{\chi}$, that

$$I_\epsilon v \in L^{2,s_m+2s_0};\ I_\epsilon = \mathrm{Op^w}\big(F(r > 2)(\tilde{f}_K \otimes \tilde{\chi}) \circ \Psi^{-1}\big). \tag{9.22}$$

Next, we introduce the symbol b_κ by (9.10) (with $s \to s_m$) and proceed as in Step I. As for the bounds (9.15) we can replace v by $I_\epsilon v$ up to addition of a term that is bounded uniformly in κ. Clearly we can verify (9.16) and (9.17). So again we obtain (9.18) (with $s \to s_m$) and therefore a contradiction as in Step I. We have shown (9.20) for $m = 1$.

Now suppose $m \geq 2$ and that (9.20) is verified for $m - 1$. We need to show the statement for the given m. Due to (9.19) and the induction hypothesis we can assume that

$$\tau_m^+ < \tau_{m-1}^+. \tag{9.23}$$

Again we argue by contradiction assuming (9.21). We proceed as above noticing that it follows from (9.23) that in addition to (9.22) we have

$$I_\epsilon u \in L^{2,s_{m-1}}; \tag{9.24}$$

at this point we possibly need to shrink the supports of $\tilde{f}$ and $\tilde{\chi}$ even more (viz. taking $\epsilon < (\tau_{m-1}^+ - \tau_m^+)/2$). By replacing v by $I_\epsilon v$ and u by $I_\epsilon u$ at various points in the procedure of Step I (using (9.22) and (9.24), respectively) we obtain again a contradiction. Whence (9.20) follows. □

Remark 9.2. Suppose $u \in L^{2,t_1}$, $v \in L^{2,t_2}$ and $Hu = v$. Suppose $z_0 \notin WF_{\mathrm{sc}}^s(u)$ for some $s > t_1$. Fix $\tilde{\tau}^+ \in]0, \infty[$ and suppose that $\gamma_0(\tau, z_0) \notin WF_{\mathrm{sc}}^{s+2s_0}(v)$ for all $\tau \in [0, \tilde{\tau}^+]$. Write $\gamma_0(\tilde{\tau}^+, z_0) = (\omega_1, \bar{c}_1, b_1) = (\omega_1, \eta_1)$. Then there exist neighbourhoods $\mathcal{N}_{\omega_1} \ni \omega_1$ and $\mathcal{N}_{\eta_1} \ni \eta_1$ such that for all $\chi_{\omega_1} \in C_c^\infty(\mathcal{N}_{\omega_1})$ and $\chi_{\eta_1} \in C_c^\infty(\mathcal{N}_{\eta_1})$: $\mathrm{Op^w}\big(\chi_{z_1} F(r > 2)\big)u \in L^{2,s}$; here $\chi_{z_1}(x, \xi) = \chi_{\omega_1}(\hat{x})\chi_{\eta_1}(\xi/g)$. Notice that this conclusion is already contained in Proposition 9.1; however the above proof yields an additional bound:

First, writing $z_0 = (\omega_0, \eta_0)$, we can pick any similarly defined localization factor, say denoted by χ_{z_0}, with $\chi_{\omega_0} = 1$ and $\chi_{\eta_0} = 1$ around the points ω_0 and η_0, respectively, and such that $\mathrm{Op^w}\big(\chi_{z_0} F(r > 2)\big)u \in L^{2,s}$ (this is by assumption). Next we pick a small neighbourhood U of $\gamma_0([0, \tilde{\tau}^+], z_0) \subset \mathbb{T}^*$ and $\chi \in C_c^\infty(U)$ with $\chi = 1$ around this orbit segment. If U is small enough we have (again by assumption) that $\mathrm{Op^w}\big(\chi_{\gamma_0} F(r > 2)\big)v \in L^{s+2s_0}$, $\chi_{\gamma_0}(x, \xi) := \chi(\hat{x}, \xi/g)$. Now, there are neighbourhoods $\mathcal{N}_{\omega_1} \ni \omega_1$ and $\mathcal{N}_{\eta_1} \ni \eta_1$ depending only on χ_{z_0} and χ_{γ_0} such that for all

$\chi_{\omega_1} \in C_c^\infty(\mathcal{N}_{\omega_1})$ and $\chi_{\eta_1} \in C_c^\infty(\mathcal{N}_{\eta_1})$:

$$\begin{aligned}
&\|\mathrm{Op}^{\mathrm{w}}\big(\chi_{z_1} F(r>2)\big)u\|_s \\
&\quad \le C\big(\|\mathrm{Op}^{\mathrm{w}}\big(\chi_{z_0} F(r>2)\big)u\|_s + \|u\|_{t_1} + \|\mathrm{Op}^{\mathrm{w}}\big(\chi_{\gamma_0} F(r>2)\big)v\|_{s+2s_0} + \|v\|_{t_2}\big);
\end{aligned}$$

the constant C only depends on the various localization factors.

9.3. **Location of singularities of the kernel of the scattering matrix.** In this subsection we describe the location of the singularities of the scattering matrix at zero energy.

Theorem 9.3. *Suppose that $V_1(r) = -\gamma r^{-\mu}$ for $r \ge 1$. Then the kernel $S(0)(\omega,\omega')$ is smooth outside the set $\{(\omega,\omega')|\ \omega\cdot\omega' = \cos\frac{\mu}{2-\mu}\pi\}$.*

To analyse $S(0)(\omega,\omega')$ we shall use the representation (7.3a), which we write (formally) as

$$S(0)(\omega,\omega') = -2\pi\langle j^+(\cdot,\omega), v^-(\cdot,\omega')\rangle + 2\pi\mathrm{i}\langle v^+(\cdot,\omega), R(+\mathrm{i}0)v^-(\cdot,\omega')\rangle,$$

where

$$\begin{aligned}
j^\pm(x,\omega) &= (2\pi)^{-d/2}\big(\mathrm{e}^{\mathrm{i}\phi^\pm}\tilde a^\pm\big)(x,\omega,0),\\
v^\pm(x,\omega) &= (2\pi)^{-d/2}\big(\mathrm{e}^{\mathrm{i}\phi^\pm}\tilde t^\pm\big)(x,\omega,0).
\end{aligned}$$

Let ϕ^+_{sph} denote the solution of the eikonal equation for the potential V_1 at zero energy, cf. (3.9). It is given by

$$\phi^+_{\mathrm{sph}}(x,\omega) = \frac{\sqrt{2\gamma}}{1-\mu/2}\left(r^{1-\mu/2}\cos(1-\mu/2)\theta - R_0^{1-\mu/2}\right); \tag{9.25}$$

here $\cos\theta = \hat x\cdot\omega$. Using $x^\perp = \frac{\omega - \hat x\cos\theta}{\sin\theta}$ and $\nabla_x\theta = -\frac{x^\perp}{r}$, we can also compute

$$\begin{aligned}
F^+_{\mathrm{sph}}(x,\omega) &= \nabla_x\phi^+_{\mathrm{sph}}(x,\omega)\\
&= \sqrt{2\gamma}r^{-\mu/2}\left(\hat x\cos(1-\mu/2)\theta + x^\perp\sin(1-\mu/2)\theta\right)
\end{aligned}$$

Lemma 9.4. *For all $s\in\mathbb{R}$, $\omega\in S^{d-1}$ and multiindices δ*

$$\begin{aligned}
&WF^s_{\mathrm{sc}}(\partial^\delta_\omega v^\pm(\cdot,\omega))\\
&\quad\subseteq \left\{z = (\hat x,\bar c,b)\in\mathbb{T}^*\middle|1-\sigma' \le \pm\hat x\cdot\omega \le 1-\sigma,\ b\hat x + \bar c = \pm\frac{F^+_{\mathrm{sph}}(\hat x,\pm\omega)}{(2\gamma)^{1/2}}\right\},
\end{aligned} \tag{9.26}$$

$$\begin{aligned}
&WF^s_{\mathrm{sc}}(\partial^\delta_\omega j^\pm(\cdot,\omega))\\
&\quad\subseteq \left\{z = (\hat x,\bar c,b)\in\mathbb{T}^*\middle|1-\sigma' \le \pm\hat x\cdot\omega,\ b\hat x + \bar c = \pm\frac{F^+_{\mathrm{sph}}(\hat x,\pm\omega)}{(2\gamma)^{1/2}}\right\}.
\end{aligned} \tag{9.27}$$

Suppose in addition that $\chi_+\in C^\infty(\mathbb{R})$, $\chi'_+\in C_c^\infty(\mathbb{R})$ and $\operatorname{supp}\chi_+ \subset]1,\infty[$. Then

$$\mathrm{Op}^{\mathrm{w}}(\chi_+(a))\partial^\delta_\omega v^\pm(\cdot,\omega),\ \mathrm{Op}^{\mathrm{w}}(\chi_+(a))\partial^\delta_\omega j^\pm(\cdot,\omega)\in L^{2,s}. \tag{9.28}$$

Proof. Only the "+" case needs to be considered (can be seen by complex conjugation). Upon multiplying by a localization operator supported outside of the right hand side of (9.26) we need to demonstrate that the result is in $L^{2,s}$, cf. the definition (4.8). Using right Kohn-Nirenberg quantization (instead of Weyl quantization) this can be done by integrating by parts in the explicit integrals, exactly as in the proofs of Lemma 6.8 (iii) and Theorem 6.11. The arguments for (9.27) and (9.28) are the same, in particular, (9.28) follows from the proof of Theorem 6.11. □

Proof of Theorem 9.3. Due to Proposition 4.8 and Lemma 9.4 we are allowed to act by $R(+\mathrm{i}0)$ on $\partial^{\delta'}_{\omega'} v^-(\cdot,\omega'))$. In fact, for all $\tau \in C^\infty(S^{d-1})$

$$R(+\mathrm{i}0)T^-(0)\tau = \int_{S^{d-1}} R(+\mathrm{i}0)v^-(\cdot,\omega')\tau(\omega')\,\mathrm{d}\omega'. \tag{9.29}$$

Using the representation (7.3a), interpreted as a form on $C^\infty(S^{d-1})$, and (9.29) we have $S_\kappa(0) \to S(0)$ as $\kappa \searrow 0$ where the kernel of $S_\kappa(0)$ is the well-defined smooth expression

$$\begin{aligned} S_\kappa(0)(\omega,\omega') = &-2\pi\langle j^+(\cdot,\omega), F(\kappa|\cdot|<1)v^-(\cdot,\omega')\rangle \\ &+ 2\pi\mathrm{i}\langle v^+(\cdot,\omega), F(\kappa|\cdot|<1)R(+\mathrm{i}0)v^-(\cdot,\omega')\rangle. \end{aligned}$$

It remains to be shown that $S_\kappa(0)(\cdot,\cdot)$ has a limit in $C^\infty\big(\{\omega\cdot\omega' \neq \cos\frac{\mu\pi}{2-\mu}\}\big)$.

By integration by parts it follows that the first term has a limit, in fact in $C^\infty\big(S^{d-1}\times S^{d-1}\big)$, cf. the proof of Lemma 9.4. Whence we only look at the second term.

By Lemma 9.4 and Proposition 3.3, for all s

$$\begin{aligned} WF^s_{\mathrm{sc}}(\partial^\delta_\omega v^+(\cdot,\omega)) \subseteq &\{\bar c \neq 0,\ b^2+\bar c^2 = 1\} \\ &\cap \{z \mid \lim_{\tau\to+\infty} \hat x(\tau) = \omega, \quad \text{where } \gamma_0(\tau,z) = (\hat x(\tau), b(\tau), \bar c(\tau))\}; \end{aligned} \tag{9.30}$$

here $\gamma_0(\tau,z)$ refers to the flow defined by (9.3).

By Propositions 4.8 and 9.1, for all s

$$\begin{aligned} &WF^s_{\mathrm{sc}}(R(+\mathrm{i}0)\partial^{\delta'}_{\omega'} v^-(\cdot,\omega')) \\ &\quad\subseteq \{\gamma_0(\tau,z) \mid \tau \geq 0,\ z \in WF^s_{\mathrm{sc}}(\partial^{\delta'}_{\omega'} v^-(\cdot,\omega'))\} \cup \{\bar c = 0,\ b>0\} \\ &\quad\subseteq \{z \mid \lim_{\tau\to-\infty} \hat x(\tau) = -\omega'\} \cup \{\bar c = 0,\ b > 0\}. \end{aligned} \tag{9.31}$$

By invoking (9.6) we see that the sets on the right hand side of (9.30) and (9.31) are disjoint away from $\{\omega\cdot\omega' \neq \cos\frac{\mu\pi}{2-\mu}\}$. Hence also

$$WF^s_{\mathrm{sc}}(\partial^\delta_\omega v^+(\cdot,\omega)) \cap WF^s_{\mathrm{sc}}(R(+\mathrm{i}0)\partial^{\delta'}_{\omega'} v^-(\cdot,\omega')) = \emptyset,$$

which implies, upon taking $s = 0$ and using (9.28) and a suitable partition of unity, that

$$\langle \partial^\delta_\omega v^+(\cdot,\omega,0), R(+\mathrm{i}0)\partial^{\delta'}_{\omega'} v^-(\cdot,\omega',0)\rangle$$

is well-defined.

By the same arguments

$$\begin{aligned} &\partial^\delta_\omega \partial^{\delta'}_{\omega'} \langle v^+(\cdot,\omega,0), F(\kappa|\cdot|<1)R(+\mathrm{i}0)v^-(\cdot,\omega',0)\rangle \\ &\qquad \to \langle \partial^\delta_\omega v^+(\cdot,\omega,0), R(+\mathrm{i}0)\partial^{\delta'}_{\omega'} v^-(\cdot,\omega',0)\rangle \end{aligned}$$

locally uniformly in $\{\omega\cdot\omega' \neq \cos\frac{\mu\pi}{2-\mu}\}$. Notice that the bound (9.28) is uniform in ω; a similar statement is valid for the bounds underlying (9.26), and we also need at this point to invoke Remark 9.2. □

Remarks 9.5. 1) The somewhat abstract procedure of the proof of Theorem 9.3 does not provide information about the nature of the singularities at the cone $\omega\cdot\omega' = \cos\frac{\mu}{2-\mu}\pi$. In the study of the singularities at the diagonal of the kernel of scattering matrices for positive energies (see [IK2] and [Ya2]) it is important that the eikonal and transport equations can be solved in sufficiently big sectors. In combination with resolvent estimates this allows one to put the singularities in a rather explicit term similar to the first one on the right hand side of (7.2a). A very similar procedure can be used (at least for $V_2 = 0$) for $S(0)(\omega, \omega')$ provided $\mu < 1$, however for $\mu \in [1, 2[$ there is a "glueing problem" due to the fact that in order to apply resolvent estimates in this case the constructed solutions to the eikonal equations $\phi^{\pm}$ need to be extended, viz. as to including some $\theta > \frac{\pi}{2-\mu}$. Therefore, multivalued $\phi^{\pm}$ are needed. We devote Subsection 9.4 to a discussion of this question.

2) Under Condition 1.1 it follows essentially by the same method of proof that the kernel $S(\lambda)(\omega, \omega')$ for $\lambda > 0$ is smooth outside the set $\{(\omega, \omega') |\, \omega = \omega'\}$; for that we use (9.3) with $\mu = 0$. See [Va2, Chapter 19] for a related result and procedure.

3) There is a discrepancy between our results and the main result of [Kv]. The idea of [Kv] is to use a partial wave analysis to obtain an asymptotic expression of the scattering amplitude for $\lambda \to 0$ (with the assumption of radial symmetry and under the short-range condition $\mu > 1$). Unfortunately [Kv, (17)] is incompatible with Theorems 7.2, 7.3 and 9.3.

9.4. **Distributional kernel of $S(0)$ as an oscillatory integral.** In addition to the previous assumption $V_1(r) = -\gamma r^{-\mu}$ for $r \geq 1$ we shall here assume that $V_2 = 0$, see though Remark 9.6 1). We shall explain a procedure which in principle allows us to calculate the singularities of the kernel $S(0)(\omega, \omega')$; a fairly explicit *oscillatory integral* will be specified. Using this integral we derive below the location of the singularities of $S(0)$ by the method of non-stationary phase, which gives an alternative proof of Theorem 9.3 (under the condition that $V_2 = 0$).

We shall improve on the representation (7.3a) for $S(0)$. Notice that the functions $\tilde{a}^+$ and ϕ^+ used up to now are supported near the forward region $\cos\theta = \hat{x} \cdot \omega \approx 1$ only. Now we shall take advantage of the fact that the expression (9.25) defines a solution to the eikonal equation for all values of θ. We shall consider a cut-off at larger values of θ, in fact slightly to the left of the critical angle $\theta = (1 - \mu/2)^{-1}\pi$. The basic idea is similar to the one applied in the study of the kernel of scattering matrices for positive energies, cf. Remark 9.5 1). If we can extend the construction of the phase and amplitude as indicated above then we can apply a "two-sided" resolvent estimate to deal with the second term on the right hand side of (7.3a), i.e. to show that it contributes by a smooth kernel; in our case the appropriate "two-sided" estimate is given by (4.3f).

Now to the problem of extending the phase up to $\theta = (1 - \mu/2)^{-1}\pi$ there is obviously an issue of well-definedness since θ as a function of x is multi-valued; for the case of positive energies this problem does not occur since the cut off in this case occurs before the angle $\theta = \pi$. We have

$$(J^+\tau)(x) = (2\pi)^{-d/2} \int_{S^{d-1}} \big(\mathrm{e}^{\mathrm{i}\phi^+} \tilde{a}^+\big)(x, \omega, 0)\tau(\omega)\mathrm{d}\omega. \tag{9.32}$$

In fact in the present spherically symmetric case the dependence of the variables x and ω is through $r = |x|$ and $\hat{x} \cdot \omega$ only. Writing

$$\omega = \cos\theta\, \hat{x} + \sin\theta\, \widetilde{\omega},$$

where $\widetilde{\omega} \cdot \hat{x} = 0$, (9.32) can be written as

$$(2\pi)^{-d/2} \int_{S^{d-2}} \mathrm{d}\widetilde{\omega} \int_0^{\pi} \big(\mathrm{e}^{\mathrm{i}\phi}\tilde{a}\big)(r, \theta)\tau(\cos\theta\, \hat{x} + \sin\theta\, \widetilde{\omega}) \sin^{d-2}\theta\, \mathrm{d}\theta; \tag{9.33}$$

for convenience we dropped the superscript. The phase ϕ is given by (9.25), and using this expression and the orbit (1.27) we can extend the support of $\tilde{a}$ by solving transport equations as in Subsection 5.3, the cut off is now taken slightly to the left of $\theta = (1-\mu/2)^{-1}\pi$. Precisely the cut off is defined as follows: First pick $L \in \mathbb{N}$ such that $(1-\mu/2)L < 1$ while $(1-\mu/2)(L+1) \geq 1$. We shall assume that the analogue of σ' for the construction of J^-, entering in (5.2) for the construction of J^+, is so small that

$$(1-\mu/2)(L\pi + \cos^{-1}(1-\sigma')) < \pi. \tag{9.34}$$

Next the version of (5.2) that we need is given in terms of the σ of the construction of J^- as follows: Choose angles $\pi L < \theta_0 < \theta_0' < \pi(L+1)$ such that $(1-\mu/2)\theta_0' < \pi$ and $(1-\mu/2)\big(\theta_0 + \cos^{-1}(1-\sigma)\big) > \pi$. Introduce a smoothed out characteristic function

$$\chi_2(s) = \begin{cases} 1 & \text{for } s \leq \theta_0 \\ 0 & \text{for } s \geq \theta_0' \end{cases}, \tag{9.35}$$

and with this choice the new cut off function takes the (essentially same) form $\chi = \chi_1(r)\chi_2(\theta)$.

The extended $\tilde{a}$ has similar properties as before due to the cut off. Whence we are lead to consider the following modification of the expression (9.33):

$$\int_{S^{d-2}} \mathrm{d}\widetilde{\omega} \int_0^\infty f(r,\theta)\tau(\cos\theta\,\hat{x} + \sin\theta\,\widetilde{\omega})\,|\sin^{d-2}\theta|\,\mathrm{d}\theta;\; f = (2\pi)^{-d/2}\mathrm{e}^{\mathrm{i}\phi}\tilde{a},$$

where the θ–integration (due to the cut off) effectively takes place on the interval $[0,(1-\mu/2)^{-1}\pi]$. The next step is to change variable writing for θ in intervals of the form $(2k\pi,(2k+1)\pi]$,

$$\cos\theta\,\hat{x} + \sin\theta\,\widetilde{\omega} = \cos\psi\,\hat{x} + \sin\psi\,\widetilde{\omega};\; \psi = \theta - 2k\pi,$$

while on intervals of the form $(2k+1)\pi,(2k+2)\pi]$,

$$\cos\theta\,\hat{x} + \sin\theta\,\widetilde{\omega} = \cos\psi\,\hat{x} + \sin\psi\,(-\widetilde{\omega});\; \psi = (2k+2)\pi - \theta,$$

respectively; here $k \in \mathbb{N}\cup\{0\}$. Whence we consider the expression

$$\int_{S^{d-1}} F(r,\psi)\tau(\omega)\mathrm{d}\omega,$$

where

$$F(r,\psi) = \sum_{k=0}^{\infty}\big\{f(r,\psi+2k\pi) + f(r,(2k+2)\pi-\psi)\big\},$$

and as above

$$\omega = \cos\psi\,\hat{x} + \sin\psi\,\widetilde{\omega} \text{ with } \widetilde{\omega}\cdot\hat{x} = 0 \text{ and } \psi \in [0,\pi],$$

i.e. $\psi = \cos^{-1}\hat{x}\cdot\omega$.

We claim that $F(r,\psi)$ is smooth in x and ω. Notice that this is not an obvious fact, since although the function $\psi = \cos^{-1}\hat{x}\cdot\omega$ is continuous it has cusp singularity at $\hat{x}\cdot\omega = \pm 1$. However, as can easily verified ψ^2 is smooth at $\hat{x}\cdot\omega = 1$ and $(\pi-\psi)^2$ is smooth at $\hat{x}\cdot\omega = -1$, respectively. Moreover, $f(r,\psi)$ and $f(r,\psi+2(k+1)\pi) + f(r,(2k+2)\pi-\psi)$ are in fact smooth functions of ψ^2 near $\hat{x}\cdot\omega = 1$, and similarly $f(r,\psi+2k\pi)+f(r,(2k+2)\pi-\psi) = f(r,(2k+1)\pi-(\pi-\psi))+f(r,(2k+1)\pi+(\pi-\psi))$ is a smooth function of $(\pi-\psi)^2$ at $\hat{x}\cdot\omega = -1$.

Recall that we have the representation (7.3b)

$$S(0)(\omega,\omega') = -2\pi\langle w^+(\omega,0), \mathrm{e}^{\mathrm{i}\phi^-}\tilde{t}^-(\cdot,\omega',0)\rangle, \tag{9.36}$$

where $w^+(\omega,0)$ is the generalized eigenfunction of Theorem 6.5.

Define $w = w(x,\omega) = F(r,\psi) - R(-\mathrm{i}0)HF$. Due to Proposition 4.10, Proposition 4.1 (iii) and Lemma 6.8 (iii) this w agrees with the eigenfunction $w^+(\omega,0)$, cf. the proof of Lemma 6.10. Therefore, our (extended) version of (7.3a) reads

$$S(0)(\omega,\omega') = -2\pi\langle F, \mathrm{e}^{\mathrm{i}\phi^-}\tilde{t}^-(\cdot,\omega',0)\rangle + 2\pi\langle R(-\mathrm{i}0)HF, \mathrm{e}^{\mathrm{i}\phi^-}\tilde{t}^-(\cdot,\omega',0)\rangle. \tag{9.37}$$

As indicated above the contribution to $S(0)(\omega,\omega')$ from the second term on the right hand side of (9.37) is smooth in ω and ω', if we use a cut off sufficiently close (but to the left of) the critical angle $\theta = (1-\mu/2)^{-1}\pi$; this is indeed accomplished by using (9.35) as cut off function.

We conclude that the singularities of the kernel of $S(0)$ are the same as those of the kernel of the operator $\widetilde{S}(0)$ given by

$$\langle\tau_1, \widetilde{S}(0)\tau_2\rangle = -2\pi\Big\langle \int F(r,\psi)\tau_1(\omega)\mathrm{d}\omega, \int \big(\mathrm{e}^{\mathrm{i}\phi^-}\tilde{t}^-\big)(\cdot,\omega',0)\tau_2(\omega')\mathrm{d}\omega'\Big\rangle.$$

Whence (formally)

$$\widetilde{S}(0)(\omega,\omega') = -2\pi\int \overline{F(r,\psi)}\big(\mathrm{e}^{\mathrm{i}\phi^-}\tilde{t}^-\big)(\cdot,\omega',0)\,\mathrm{d}x. \tag{9.38}$$

Next we introduce the variable $\theta' = \cos^{-1}\hat{x}\cdot(-\omega') \in [0,\pi/2)$; we can represent $\phi^-(x,\omega',0) = -\phi(r,\theta')$, cf. (3.6). The integrand on the right hand side of (9.38) is given as $\sum_{k=0}^{\infty} f_k$ where f_k has the form

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}\big(\phi(r,\psi+2k\pi)+\phi(r,\theta')\big)}g(r,\psi+2k\pi,\theta')\\ &\qquad+\mathrm{e}^{-\mathrm{i}\big(\phi(r,(2k+2)\pi-\psi)+\phi(r,\theta')\big)}g(r,(2k+2)\pi-\psi,\theta').\end{aligned} \tag{9.39}$$

Let us argue that the integral (9.38) is well-defined in $\{\omega\cdot\omega' \neq \cos\frac{\mu}{2-\mu}\pi\}$, in agreement with Theorem 9.3. The argument is based on the method of non-stationary phase. First we notice that the cusp singularities at $\psi=0$ and $\psi=\pi$ correspond to non-stationary points. More precisely we can write

$$x = r(\cos\psi\,\omega + \sin\psi\,\widetilde{\hat{x}}),$$

and perform the x–integration as

$$\int\cdots\mathrm{d}x = \int_0^{\pi}\sin^{d-2}\psi\,\mathrm{d}\psi\int_{S^{d-2}}\mathrm{d}\widetilde{\hat{x}}\int_0^{\infty}\cdots r^{d-1}\mathrm{d}r. \tag{9.40}$$

Now on the support of g the factor $\cos(1-\mu/2)\theta' \geq \cos\theta' \geq 1-\sigma'$ while the factors $\cos(1-\mu/2)(\psi+2k\pi)$ and $\cos(1-\mu/2)((2k+2)\pi-\psi)$ stay sufficiently away from -1 (given that $\psi\approx 0$ or $\psi\approx\pi$) to ensure that the sum of phases does not vanish; here we use (9.34). Thus the phases of f_k are nonzero near the ψ–endpoints of integration and consequently integration by parts with respect to r regularizes the integral (9.38) (upon first substituting (9.40) and localizing near the ψ–endpoints).

By the same reasoning as above, depending on whether L is even or odd (viz. $L=2l$ or $L=2l+1$) only the integral of one term of (9.39) (and only with $k=l$) carries singularities. We first look at the case for which only the first term of

(9.39) $\mathrm{e}^{-\mathrm{i}(\phi(r,\psi+2l\pi)+\phi(r,\theta'))}g(r,\psi+2l\pi,\theta')$ contributes by singularities. Clearly for a stationary point

$$\cos((1-\mu/2)(\psi+2l\pi))+\cos((1-\mu/2)\theta')=0, \tag{9.41}$$

which leads to the condition

$$\cos(\psi+\theta')=\cos(\tfrac{2}{2-\mu}\pi). \tag{9.42}$$

There are three cases to consider.

Case I. $\omega=-\omega'$. In this case $\theta'=\psi$ so that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\psi}\big(\phi(r,\psi+2l\pi)+\phi(r,\theta')\big)\\&=-\sqrt{2\gamma}r^{1-\mu/2}\big(\sin(1-\mu/2)(\psi+2l\pi)+\sin(1-\mu/2)\psi\big)<0.\end{aligned} \tag{9.43}$$

Whence there are no stationary points.

Case II. $\omega=\omega'$. In this case $\theta'=\pi-\psi$ so that (9.42) reads

$$\omega\cdot\omega'=1=-\cos(\tfrac{2}{2-\mu}\pi)=\cos(\tfrac{\mu}{2-\mu}\pi).$$

This agrees with the "rule" of Theorem 9.3.

Case III. $\omega\neq C\omega'$. In dimension $d\geq 3$ the vectors $\widetilde{\hat{x}}=\pm y/|y|$ where $y=\omega'-\omega'\cdot\omega\,\omega$ are the only possible critical points of the map

$$S^{d-2}\ni\widetilde{\hat{x}}\to\theta'=\cos^{-1}(-(\cos\psi\,\omega+\sin\psi\,\widetilde{\hat{x}})\cdot\omega')\in\mathbb{R}.$$

Consequently for any stationary point, $\hat{x}$ must belong to the plane spanned by ω and ω' (like for $d=2$). Let us introduce the angle $\gamma=\cos^{-1}\omega\cdot(-\omega')$. There are three possible relationships to be considered a) $\gamma=|\psi-\theta'|$, b) $\gamma=\psi+\theta'$ and c) $\gamma=2\pi-(\psi+\theta')$. For a) $\theta'=\psi\mp\gamma$ can be substituted into the sum of phases and we compute as in (9.43). Again there will not be any stationary point. For b) we can use (9.42) to compute

$$\omega\cdot\omega'=-\cos\gamma=-\cos(\tfrac{2}{2-\mu}\pi)=\cos(\tfrac{\mu}{2-\mu}\pi),$$

which agrees with the "rule" of Theorem 9.3. Similarly for c) we compute

$$\omega\cdot\omega'=-\cos\gamma=-\cos(\psi+\theta')=-\cos(\tfrac{2}{2-\mu}\pi)=\cos(\tfrac{\mu}{2-\mu}\pi).$$

Next we look at the case for which only $\mathrm{e}^{-\mathrm{i}\left(\phi(r,2(l+1)\pi-\psi)+\phi(r,\theta')\right)}g(r,2(l+1)\pi-\psi,\theta')$ contributes by singularities. For a stationary point

$$\cos((1-\mu/2)(2(l+1)\pi)-\psi)+\cos((1-\mu/2)\theta')=0, \tag{9.44}$$

which leads to the condition

$$\cos(\psi-\theta')=\cos(\tfrac{2}{2-\mu}\pi). \tag{9.45}$$

Again there are three cases to consider.

Case I. $\omega=-\omega'$. In this case $\theta'=\psi$ so that

$$\omega\cdot\omega'=-1=-\cos(\tfrac{2}{2-\mu}\pi)=\cos(\tfrac{\mu}{2-\mu}\pi),$$

which agrees with Theorem 9.3.

Case II. $\omega = \omega'$. We have $\theta' = \pi - \psi$ so that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\psi}\big(\phi(r, 2(l+1)\pi - \psi) + \phi(r, \theta')\big) \qquad (9.46)\\ &= \sqrt{2\gamma} r^{1-\mu/2}\big(\sin(1-\mu/2)(2(l+1)\pi - \psi) + \sin(1-\mu/2)(\pi - \psi)\big) > 0;\end{aligned}$$

whence there are no stationary points.

Case III. $\omega \neq C\omega'$. As in the previous "Case III", for any stationary point the vector $\hat{x}$ must belong to the plane spanned by ω and ω'. Again we define $\gamma = \cos^{-1}\omega\cdot(-\omega')$, and there are three possible relationships to be considered a) $\gamma = |\psi - \theta'|$, b) $\gamma = \psi + \theta'$ and c) $\gamma = 2\pi - (\psi + \theta')$. For a)

$$\omega \cdot \omega' = -\cos\gamma = -\cos(\psi - \theta') = -\cos(\tfrac{2}{2-\mu}\pi) = \cos(\tfrac{\mu}{2-\mu}\pi),$$

which agrees with Theorem 9.3. For b) and c) we compute as in (9.46); there are no stationary points.

Remarks 9.6. 1) For the above considerations (on the location of singularities) it is not strictly needed that $V_2 = 0$. In fact we can include a V_2 as in Condition 2.1 with $\epsilon_2 > 1 - \frac{1}{2}\mu$ and solve transport equations as before using the same phase function (the one determined by V_1 only).

2) Suppose in addition to 1) that V_2 is spherically symmetric. Then the operators $T = S(0)$ as well as $T = \tilde{S}(0)$ obey that $RTR^{-1} = T$ for all d-dimensional rotations R. This means that the kernel $T(\omega, \omega')$ of these operators is a function of $\omega \cdot \omega'$ only. Using the stationary phase method it is feasible for $\frac{\mu}{2-\mu} \notin \mathbb{Z}$ to write (as a possible continuation of the above analysis) the singular part of the kernel of $\tilde{S}(0)$ as a sum of terms of the form $(\omega \cdot \omega' - \nu \pm \mathrm{i}0)^{-\frac{s}{2}} a(\omega \cdot \omega')$ (at least for poly-homogeneous V_2); we shall not elaborate. The next section is devoted to an alternative approach that we find more elementary, and besides, by that method we can extract the singular part in the exceptional cases $\frac{\mu}{2-\mu} \in \mathbb{Z}$ too.

10. Homogeneous potentials – type of singularities of $S(0)$

In this section we shall compute the main contribution of the scattering matrix $S(0)$ for a potential homogeneous of degree μ (plus a lower order term), see Subsection 10.3 for precise conditions. It will turn out to be the evolution operator for the wave equation on the sphere at time $\frac{-\mu}{2-\mu}\pi$. We devote Subsections 10.1–10.3 to a study of this operator. In particular, we will compute explicitly its distributional kernel and determine the location of its singularities. We assume throughout the section that $d \geq 2$.

10.1. **Evolution operator of the wave equation on the sphere.** For any $1 \leq i < j \leq d$, define the corresponding angular momentum operator

$$L_{ij} := -\mathrm{i}(x_i\partial_{x_j} - x_j\partial_{x_i}).$$

Set

$$L^2 := \sum_{1\leq i<j\leq d} L_{ij}^2, \quad \Lambda := \sqrt{L^2 + (d/2-1)^2}.$$

Note that Λ is a self-adjoint operator on $L^2(S^{d-1})$ and its eigenfunctions with eigenvalue $l + d/2 - 1$ are lth order spherical harmonics for $l = 0, 1, \ldots$

For any θ one can compute exactly the integral kernel of $\mathrm{e}^{\mathrm{i}\theta\Lambda}$. Although the result already appears in the literature, see [Ta, Chapter 4, (2.13)], we shall for the readers convenience give a complete derivation (this proof is different from Taylor's). Note that the operator appears naturally when we solve the wave equation on the sphere, therefore we call it the evolution operator of the wave equation on the sphere.

First we need to introduce some notation about distibutions. For any $\epsilon > 0$ and $s \in \mathbb{R}$, the expression

$$\mathbb{R} \in y \mapsto (y \pm \mathrm{i}\epsilon)^{-\frac{s}{2}}$$

defines uniquely a function on a real line, which can be viewed as a distribution in $\mathcal{S}'(\mathbb{R})$. It is well known that for any $\phi \in \mathcal{S}(\mathbb{R})$ there exists a limit

$$\lim_{\epsilon \searrow 0} \int (y \pm \mathrm{i}\epsilon)^{-\frac{s}{2}} \phi(y) \mathrm{d}y =: \int (y \pm \mathrm{i}0)^{-\frac{s}{2}} \phi(y) \mathrm{d}y,$$

which defines a distribution in $\mathcal{S}'(\mathbb{R})$. In the sequel we will treat this distribution as if it were a function denoting it by $(y \pm \mathrm{i}0)^{-\frac{s}{2}}$. Note that for $s, \epsilon > 0$ we have the identity

$$(y \pm \mathrm{i}\epsilon)^{-\frac{s}{2}} = \frac{\mathrm{e}^{\mp \mathrm{i}\pi \frac{s}{4}}}{\Gamma(s/2)} \int_0^\infty \mathrm{e}^{\mathrm{i}t(\pm y + \mathrm{i}\epsilon)} t^{\frac{s-2}{2}} \mathrm{d}t. \tag{10.1}$$

We shall in this section show the following result:

Proposition 10.1. (1) *If $\theta = \pi 2k$, $k \in \mathbb{Z}$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda} = (-1)^{kd}$ times the identity.*

(2) *If $\theta = \pi(2k+1)$, $k \in \mathbb{Z}$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda} = \mathrm{e}^{\mathrm{i}\pi(2k+1)(d/2-1)} P$, where P is the parity operator.*

(3) *If $\theta \in]\pi 2k, \pi(2k+1)[$, $k \in \mathbb{Z}$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ has the distributional kernel*

$$\mathrm{e}^{\mathrm{i}\theta\Lambda}(\omega, \omega') = (2\pi)^{-d/2} \sin\theta\, \Gamma(d/2) \mathrm{e}^{-\mathrm{i}\pi/2} (-\omega \cdot \omega' + \cos\theta - \mathrm{i}0)^{-d/2}.$$

(4) *If $\theta \in]\pi(2k-1), \pi 2k[$, $k \in \mathbb{Z}$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ has the distributional kernel*

$$\mathrm{e}^{\mathrm{i}\theta\Lambda}(\omega, \omega') = (2\pi)^{-d/2} \sin\theta\, \Gamma(d/2) \mathrm{e}^{-\mathrm{i}\pi/2} (-\omega \cdot \omega' + \cos\theta + \mathrm{i}0)^{-d/2}.$$

10.1.1. *Tchebyshev and Gegenbauer polynomials.* Recall that the Tchebyshev polynomials (of the first kind) are defined by the identity

$$T_n(\cos\phi) := \cos n\phi, \quad n = 0, 1, \dots.$$

Let $|t| < 1$. The following generating function of Tchebyshev polynomials follows by an elementary calculation:

$$-\ln(1 - 2wt + t^2) = \sum_{l=1}^\infty \frac{2t^l}{l} T_l(w). \tag{10.2}$$

Gegenbauer polynomials are defined by the generating function [Mü, AAR]

$$\frac{1}{(1 - 2wt + t^2)^{(d-2)/2}} = \sum_{l=0}^\infty t^l C_l^{(d-2)/2}(w). \tag{10.3}$$

The left hand sides of (10.2) and (10.3) look different. But after simple manipulations (involving differentiation of both sides) they become quite similar

$$\frac{-t + t^{-1}}{(t - 2w + t^{-1})^{\frac{d}{2}}} = \begin{cases} T_0(w) + \sum_{l=1}^\infty t^l 2T_l(w), & d = 2; \\ \sum_{l=0}^\infty t^{l+\frac{d}{2}-1} \frac{2l+d-2}{d-2} C_l^{(d-2)/2}(w), & d \geq 3. \end{cases} \tag{10.4}$$

By substituting $t = \mathrm{e}^{\mathrm{i}\theta}$ for $\operatorname{Im}\theta > 0$, we rewrite this as

$$\frac{-\mathrm{i}2\sin\theta}{2^{d/2}(\cos\theta - w)^{\frac{d}{2}}} = \begin{cases} T_0(w) + \sum_{l=1}^{\infty} \mathrm{e}^{\mathrm{i}l\theta} 2T_l(w), & d = 2; \\ \sum_{l=0}^{\infty} \mathrm{e}^{\mathrm{i}(l+\frac{d}{2}-1)\theta} \frac{2l+d-2}{d-2} C_l^{(d-2)/2}(w), & d \geq 3. \end{cases} \tag{10.5}$$

10.1.2. *Projection onto lth sector of spherical harmonics.* It is well-known that the integral kernel of the projection onto lth sector of spherical harmonics in $L^2(S^{d-1})$ can be computed explicitly. This fact is usually presented in the literature as the addition theorem for spherical harmonics, see e.g. Theorem 2, Sect. 2 of [Mü]. In the case $d = 3$ it can also be found in [Vl].

Proposition 10.2. *Let Y be an lth order spherical harmonic in $L^2(S^{d-1})$.*

(1) *In the case $d = 2$,*

$$\int_{S^1} \frac{1}{2\pi} T_0(\hat{x}\cdot\hat{y}) Y(\hat{y}) \mathrm{d}\hat{y} = \delta_{l0} Y(\hat{x}). \tag{10.6}$$

$$\int_{S^1} \frac{1}{\pi} T_n(\hat{x}\cdot\hat{y}) Y(\hat{y}) \mathrm{d}\hat{y} = \delta_{ln} Y(\hat{x}), \quad n = 1, 2, \dots.$$

(2) *In the case $d \geq 3$,*

$$\int_{S^{d-1}} \frac{(d-2+2l)\Gamma(d/2-1)}{4\pi^{d/2}} C_n^{(d-2)/2}(\hat{x}\cdot\hat{y}) Y(\hat{y}) \mathrm{d}\hat{y} = \delta_{ln} Y(\hat{x}). \tag{10.7}$$

Proof. The case (10.6) is elementary. In the proof below we restrict ourselves to $d \geq 3$.

Let us first recall the formula for the Green's function in $\mathbb{R}^d$ for $d \geq 3$:

$$G_d(x) = -\frac{\Gamma(d/2-1)}{4\pi^{d/2}|x|^{d-2}} = -\frac{1}{s_{d-1}(d-2)|x|^{d-2}}, \tag{10.8}$$

where $s_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the area of S^{d-1}. It satisfies

$$\Delta G_d = \delta_0,$$

where δ_0 is Dirac's delta at zero. Recall also the 3rd Green's identity: if $\Delta g = 0$ and Ω is a sufficiently regular domain containing x, then

$$g(x) = \int_{\partial\Omega} g(y) \nabla_y G_d(x-y) \mathrm{d}\vec{s}(y) - \int_{\partial\Omega} (\nabla g)(y) G_d(x-y) \mathrm{d}\vec{s}(y). \tag{10.9}$$

We extend Y to $\mathbb{R}^d$ by setting $g(x) = |x|^l Y(\hat{x})$. Note that

$$\Delta g(x) = 0, \quad \hat{x}\nabla_x g(x) = l g(x).$$

By the (10.3), for $|x| < |y|$

$$G_d(x-y) = -\frac{\Gamma(d/2-1)}{4\pi^{d/2}} \sum_{n=0}^{\infty} C_n^{(d-2)/2}(\hat{x}\hat{y}) |x|^n |y|^{-d+2-n},$$

$$\hat{y}\cdot\nabla_y G_d(x-y) = \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \sum_{n=0}^{\infty} (d-2+n) C_n^{(d-2)/2}(\hat{x}\hat{y}) |x|^n |y|^{-d+1-n}.$$

We apply (10.9) to the unit ball, so that $|y| = 1$ and $|x| < 1$:

$$
\begin{aligned}
|x|^l Y(\hat{x}) &= \int_{S^{d-1}} g(\hat{y})\hat{y} \cdot \nabla G_d(x - \hat{y})\mathrm{d}\hat{y} - \int_{S^{d-1}} (\hat{y} \cdot \nabla g)(\hat{y}) G_d(x - \hat{y})\mathrm{d}\hat{y} \\
&= \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} \sum_{n=0}^{\infty} (d - 2 + n + l) \int_{S^{d-1}} Y(\hat{y}) C_n^{(d-2)/2}(\hat{x}\hat{y})|x|^n \mathrm{d}\hat{y}. \qquad (10.10)
\end{aligned}
$$

Comparing the powers of $|x|$ on both sides of (10.10) we obtain (10.7). $\square$

10.1.3. *Proof of Proposition 10.1.* Let Q_l^{d-1} be the orthogonal projection onto lth order spherical harmonics on S^{d-1}. We multiply (10.5) by $\Gamma(d/2)2^{-1}\pi^{-d/2}$, set $w = \omega \cdot \omega'$ and use Proposition 10.2. We obtain

$$
\begin{aligned}
\frac{-\mathrm{i}\sin\theta\,\Gamma(d/2)}{(2\pi)^{d/2}(\cos\theta - \omega\cdot\omega')^{d/2}} &= \sum_{l=0}^{\infty} Q_l^{d-1}(\omega, \omega')\mathrm{e}^{\mathrm{i}(l+d/2-1)\theta} \\
&= \mathrm{e}^{\mathrm{i}\theta\Lambda}(\omega, \omega').
\end{aligned}
$$

Replace θ with $\theta + \mathrm{i}\epsilon$, where θ is real and ϵ positive. For small ϵ we have

$$
\cos(\theta + \mathrm{i}\epsilon) \approx \cos\theta - \mathrm{i}\sin\theta\epsilon.
$$

Now $\sin\theta > 0$ for $\theta \in]\pi 2k, \pi(2k+1)[$ and $\sin\theta < 0$ for $\theta \in]\pi(2k-1), \pi 2k[$, which ends the proof for the case $\theta \in \mathbb{R} \setminus \pi\mathbb{Z}$.

The case $\theta \in \pi\mathbb{Z}$ is obvious.

10.2. **Evolution operator of the wave equation on the sphere as a FIO.** Let X be a smooth compact manifold of dimension n. Let us recall some basic definitions related to Fourier integral operators on X, cf. [Hö4].

We say that $X \times X \times \mathbb{R}^k \ni (x, x', \theta) \mapsto \phi(x, x', \theta)$ is a non-degenerate phase function if it is a function homogeneous of degree 1 in θ, smooth and satisfying $\nabla\phi \neq 0$ away from $\theta = 0$, and such that

$$
\{(x, x', \theta) \in X \times X \times \mathbb{R}^k \mid \nabla_\theta \phi(x, x', \theta) = 0\}
$$

is a smooth manifold on which $\nabla\nabla_{\theta_1}\phi, \dots, \nabla\nabla_{\theta_k}\phi$ are linearly independent.

Let χ be a smooth and homogeneous transformation on $\mathbb{T}^*X \setminus X\times\{0\}$. We say that it is associated to a non-degenerate phase function ϕ iff two pairs $(x, \xi), (x', \xi') \in \mathbb{T}^*X \setminus \{0\}\times X$ satisfy $\chi(x', \xi') := (x, \xi)$ exactly when

$$
\begin{aligned}
\xi &= \nabla_x \phi(x, x', \theta), \\
\xi' &= -\nabla_{x'} \phi(x, x', \theta), \\
0 &= \nabla_\theta \phi(x, x', \theta). \qquad (10.11)
\end{aligned}
$$

The transformation χ is automatically canonical, that is, it preserves the symplectic form.

We say that a smooth function $X \times X \times \mathbb{R}^k \ni (x, x', \theta) \mapsto u(x, x', \theta)$ is an amplitude of order m iff

$$
\partial_x^\alpha \partial_{x'}^{\alpha'} \partial_\theta^\beta u = O(\langle\theta\rangle^{m-|\beta|}).
$$

Recall from [Hö4] that an operator U from $C^\infty(X)$ to $\mathcal{D}'(X)$ is called a Fourier integral operator of order

$$
m - \frac{n}{2} + \frac{k}{2}.
$$

iff in local coordinate patches its distributional kernel can be written as

$$U(x, x') = \int \mathrm{e}^{\mathrm{i}\phi(x,x'\theta)} u(x, x', \theta)\, \mathrm{d}\theta, \tag{10.12}$$

where $\theta \in \mathbb{R}^k$ are auxiliary variables, the function ϕ is a non-degenerate phase function, and u is an amplitude of order m.

If the phase of U is associated to a canonical transformation χ, we say that U itself is associated to χ. We note that in such a case there are conditions under which we have for all $v \in \mathcal{D}'(X)$ (using here the notion of wave front set of a distribution, cf. [Hö4, Section 2.5])

$$WF(Uv) \subseteq \chi(WF(v));$$

see [Hö4, Proposition 2.5.7 and Theorem 2.5.14] (these conditions are fulfilled for the example $U = U_\theta$ given below).

Theorem 10.3. *The operator* $U_\theta := \mathrm{e}^{\mathrm{i}\theta\Lambda}$ *is a FIO of order* 0.

Proof. If $\theta \in \pi\mathbb{Z}$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ is a so-called point transformation. But point transformations given by diffeomorphisms of the underlying manifold are always FIO of order zero.

Assume that $\theta \notin \pi\mathbb{Z}$. Consider e.g. the case $\theta \in]\pi 2k, \pi(2k+1)[$. By (10.1) and Proposition 10.1 the kernel of U_θ can then be written as

$$U_\theta(\omega, \omega') = C \int_0^\infty \mathrm{e}^{\mathrm{i}t(\omega\cdot\omega' - \cos\theta)} t^{\frac{d-2}{2}} \mathrm{d}t. \tag{10.13}$$

If we compare (10.13) with the definition of a FIO given above, we see that $t(\omega \cdot \omega' - \cos\theta)$ is a non-degenerate phase function. We also have $n = d-1$, $m = \frac{d-2}{2}$ and $k = 1$. Thus U_θ is a FIO of order

$$\frac{d-2}{2} - \frac{d-1}{2} + \frac{1}{2} = 0.$$

□

Let us describe the canonical transformation associated to the FIO U_θ. Let $(\omega, \xi) \in \mathbb{T}^*(S^{d-1})$. It is enough to assume that $|\xi| = 1$. Then the canonical transformation χ_θ associated to U_θ is given by $\chi_\theta(\omega', \xi') = (\omega, \xi)$, where

$$\omega = \omega' \cos\theta - \xi' \sin\theta,$$
$$\xi = \omega' \sin\theta + \xi' \cos\theta.$$

10.3. **Main result.** The main result of this section is

Theorem 10.4. *Suppose Conditions 2.1–2.3 with* $d \geq 2$, *the condition* $V_1(r) = -\gamma r^{-\mu}$ *for* $r \geq 1$, V_2 *is spherically symmetric and that the number* ϵ_2 *of Condition 2.1 obeys* $\epsilon_2 > 1 - \frac{\mu}{2}$ *(viz.* $\frac{\mathrm{d}^k}{\mathrm{d}r^k} V_2(r) = O\big(r^{-1-\frac{\mu}{2}-\epsilon-k}\big)$; $\epsilon > 0$*). Then*

$$S(0) = \mathrm{e}^{\mathrm{i}c_0} \mathrm{e}^{-\mathrm{i}\frac{\mu\pi}{2-\mu}\Lambda} + K,$$

where K *is compact and*

$$c_0 = \tfrac{4\sqrt{2\gamma}}{2-\mu} R_0^{1-\frac{\mu}{2}} + 2\int_{R_0}^\infty \Big(\sqrt{-2V_1(r')} - \sqrt{-2V(r')}\Big)\, \mathrm{d}r'.$$

Let us remark, as a first reduction of the proof of Theorem 10.4, that we can assume that $V_3 = 0$. This can readily be seen by using resolvent equations in the representation formula (7.3a) and Proposition 4.1. Another manifestation is provided by Subsection 9.4 (including Remarks 9.6 1)): Clearly the term $\tilde{S}(0)$ that carries the singularities is independent of $V_3 = 0$.

The analysis will go through under slightly weaker conditions than needed for Theorem 10.4 (given that $V_3 = 0$). Specifically we shall in this subsection impose the following

Condition 10.5. The potential V splits into a sum of three spherically symmetric terms $V = V_1 + V_2 + V_3$, where all terms V_1, V_2, and V_3 are real and continuous functions on $]0,\infty[$, $V_1(r) = -\gamma r^{-\mu}$ for $r \geq 1$, $V_2 = O\big(r^{-1-\frac{\mu}{2}-\epsilon}\big)$ for some $\epsilon > 0$, V_1 and V_2 vanish in a neighbourhood of $r = 0$, V_3 has bounded support and $|V_3(r)| \leq C r^{-2+\kappa}$ for some constants $C, \kappa > 0$.

Under Condition 10.5 we can *define* the phase shift $\sigma_l(0)$ as follows: Fix $l \in \mathbb{N} \cup \{0\}$ and fix $R_0 \geq 0$ so large that $V(r) < 0$ for all $r > R_0$. Then all real solutions zero energy of the reduced Schrödinger equation on the half-line $]0,\infty[$

$$-u'' + V_l u = 0;\ V_l(r) = 2V(r) + \tfrac{(l+\frac{d}{2}-1)^2 - 4^{-1}}{r^2} \tag{10.14}$$

obey

$$\frac{u(r)}{r^{\frac{d-1}{2}}} - C\frac{\sin\big(\int_{R_0}^r \sqrt{-2V_1(r')}\,\mathrm{d}r' + D\big)}{(-2V_1(r))^{\frac{1}{4}} r^{\frac{d-1}{2}}} F(r > 1) \in B^*_{s_0,0}$$

for some $C > 0$ and $D \in \mathbb{R}$ (can be seen from the WKB-analysis given in the bulk of Subsection 10.4). The *regular* solution is characterized by the requirement $\lim_{r\to 0} r^{-l-\frac{d-1}{2}} u(r) = 1$ (existence and uniqueness of the regular solution is usually proven by studying an integral equation of Volterra type, cf. [Ne]). Now we *define* in terms of the constant D for the regular solution

$$\sigma_l(0) = D + \int_{R_0}^{\infty} \Big(\sqrt{-2V_1(r')} - \sqrt{-2V(r')}\Big)\,\mathrm{d}r' + \tfrac{d-3+2l}{4}\pi. \tag{10.15}$$

Note that (10.15) and Corollary 8.4 are consistent; in particular this justifies the joint use of the symbol $\sigma_l(0)$ in (10.15) and Corollary 8.4, see Remark 10.7 for a related discussion.

We shall show the following asymptotics

Proposition 10.6. *Under Condition 10.5 the phase shift obeys*

$$\sigma_l(0) = -\tfrac{\mu\pi}{2(2-\mu)} l + \frac{c}{2} + o(l^0); \tag{10.16}$$

$$\frac{c}{2} = -\tfrac{\pi\mu(d-2)}{4(2-\mu)} + \tfrac{2\sqrt{2\gamma}}{2-\mu} R_0^{1-\frac{\mu}{2}} + \int_{R_0}^{\infty} \Big(\sqrt{-2V_1(r)} - \sqrt{-2V(r)}\Big)\,\mathrm{d}r.$$

Clearly Theorem 10.4 is a consequence of from Proposition 10.6.

Remark 10.7. Note that for $0 < \mu < 2$, $V(x) = -\gamma|x|^{-\mu}$ is an infinitesimal form bounded perturbation of $-\Delta$ in dimension $d \geq 2$. Therefore, $H_\mu := \Delta - \gamma|x|^{-\mu}$ is well-defined and self-adjoint (even though Condition 2.3 (1) may fail). (Actually, H_μ extends to an analytic family of operators for $\operatorname{Re}\mu \in]0,2[$.) It may be tempting to claim that in the cases where operator boundedness fails one can still follow the procedures of Section 9, i.e. use the function ϕ^+_{sph} in (9.25) as the zero energy solution of the eikonal equation to construct and analyse the zero energy scattering matrix. However since the resolvent estimates from [FS] are only derived for operator bounded potentials these estimates would need to be reconsidered. On the other hand since this potential $V(x) = -\gamma|x|^{-\mu}$, $0 < \mu < 2$, indeed fulfills Condition 10.5 we can use (10.15) in this case to the define zero energy scattering matrix (by the formula $S(0)Y = \mathrm{e}^{\mathrm{i}2\sigma_l(0)}Y$ for spherical harmonics Y of order l). Based on this definition and Proposition 10.6 it is natural to conjecture that it equals *exactly* $S(0) = \mathrm{e}^{\mathrm{i}c_0}\mathrm{e}^{-\mathrm{i}\frac{\mu\pi}{2-\mu}\Lambda}$, or alternatively, that the terms $o(l^0)$ in Proposition 10.6 vanishes identically. We leave this as an open problem for the interested reader.

10.4. **One-dimensional WKB-analysis.** This subsection is devoted to the main part of the proof of Proposition 10.6. It is based on detailed 1-dimensional analysis.

For convenience let us note that the effective potential V_l of (10.14) for $V_2 = V_3 = 0$ is given by

$$V_l(r) = 2V_1(r) + \frac{k(k+1)}{r^2} = -2\gamma r^{-\mu} + \frac{k(k+1)}{r^2}; \quad k := l + \frac{d-3}{2}.$$

Abusing slightly notation we shall henceforth denote this expression (whether $V_2 = V_3 = 0$ or not) by V_k and similarly $\sigma_k(0) := \sigma_l(0)$.

In the case $V_2 = V_3 = 0$ there is a unique zero, say denoted r_0, of the effective potential V_k. Explicitly

$$V_k(r_0) = 0 \text{ for } r_0 = \left(\tfrac{k(k+1)}{2\gamma}\right)^{\frac{1}{2-\mu}}. \tag{10.17}$$

For later applications let us notice that

$$V_k'(r_0) = -(2-\mu)\tfrac{k(k+1)}{r_0^3}. \tag{10.18}$$

Clearly V_k is positive to the left of r_0 and negative to the right of r_0.

Proposition 10.8. *Under Conditions 10.5, the regular solution (up to multiplication by a positive constant) satisfies*

$$u(r) = (-V_k)^{-\frac{1}{4}}(r)\Big(\sin\Big(\int_{r_0}^{r} \sqrt{-V_k(r')}\,\mathrm{d}r' + \tfrac{\pi}{4} + o(k^0)\Big) + O(r^{-\epsilon_k})\Big), \tag{10.19}$$

where $o(k^0)$ signifies a vanishing term that is independent of r and $\epsilon_k > 0$.

10.4.1. *Scheme of proof of Proposition 10.8.* We shall first concentrate on the case where $V_2 = V_3 = 0$; the general case will be treated by the same scheme (to be discussed later).

We introduce a partition of $]0,\infty[$ into four subintervals given as follows in terms of $\epsilon_1, \epsilon_2, \epsilon_3 \in (0, 1]$ to be fixed later:

1. $I_1 =]0, r_1]$; $r_1 = r_0 k^{-\frac{\epsilon_1}{2-\mu}}$.
2. $I_2 =]r_1, r_2]$; $r_2 = r_0(1 - k^{-\epsilon_2})$.
3. $I_3 =]r_2, r_3]$; $r_3 = r_0(1 + k^{-\epsilon_3})$.
4. $I_4 =]r_3, \infty[$.

In each of the intervals I_j where $j = 2, 3$ or 4 we shall specify a certain model Schrödinger equation together with its two linearly independent solutions $\phi_j^{\pm}$. In terms of these we can construct exact solutions to the reduced equation

$$-u'' + V_k u = 0 \tag{10.20}$$

by the method of variation of parameters, cf. for example [HS]. Our subject of study is formulas for the regular solution $u = u_k$. Specifically, in the interval I_1 we shall use a comparison argument to get estimates of the regular solution at $r = r_1$. Then we shall use a connection formula to get estimates of the "coefficients" a_2^+ and a_2^- of the ansatz

$$u = a_j^+\phi_j^+ + a_j^-\phi_j^- \tag{10.21}$$

with $j = 2$ at the same point $r = r_1$. Next, using the ODE for a_2^+ and a_2^- we shall derive estimates of these quantities at $r = r_2$. Proceeding similarly we shall consecutively represent u by (10.21) on I_3 and I_4 using connection formulas at r_2 and r_3 and eventually get estimates in the interval I_4 and whence derive the relevant asymptotics of u.

Suppose ϕ^- and ϕ^+ solve the same one-dimensional Schrödinger equation, say

$$-\phi'' + A\phi = 0.$$

The variation of parameter method for the equations (10.20) and (10.21) yields

$$\begin{bmatrix} \phi^+ & \phi^- \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^+ & \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^- \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\tau} \begin{bmatrix} a^+ \\ a^- \end{bmatrix} = (V - A) \begin{bmatrix} 0 & 0 \\ \phi^+ & \phi^- \end{bmatrix} \begin{bmatrix} a^+ \\ a^- \end{bmatrix}. \tag{10.22}$$

(We have omitted the subscript j). We introduce the notation $W(\phi^-, \phi^+)$ for the Wronskian $W(\phi^-, \phi^+) = \phi^-\frac{\mathrm{d}}{\mathrm{d}r}\phi^+ - \phi^+\frac{\mathrm{d}}{\mathrm{d}r}\phi^-$. Then we write $B = V_k - A$ and transform (10.22) into

$$\frac{\mathrm{d}}{\mathrm{d}r}\binom{a^+}{a^-} = N\binom{a^+}{a^-},$$

where

$$N = \frac{B}{W(\phi^-, \phi^+)} \begin{pmatrix} \phi^-\phi^+ & (\phi^-)^2 \\ -(\phi^+)^2 & -\phi^-\phi^+ \end{pmatrix}.$$

For a positive increasing continuous function f on I (to be specified) we introduce the matrix $T = \operatorname{diag}(1, f^{-1})$. We compute

$$TNT^{-1} = \frac{B}{W(\phi^-, \phi^+)} \begin{pmatrix} \phi^-\phi^+ & f(\phi^-)^2 \\ -f^{-1}(\phi^+)^2 & -\phi^-\phi^+ \end{pmatrix}.$$

Introducing the operator $(M_j z)(r) = \int_{r_{j-1}}^r N_j(r')z(r')\,\mathrm{d}r'$, $j \geq 2$, acting on continuous functions $z(\cdot) : I_j \to \mathbb{R}^2$, the above ODE is solved by

$$\binom{a_j^+}{a_j^-}(r) - z_j = \sum_{m=1}^{\infty} M_j^m z_j;\ z_j = \binom{a_j^+}{a_j^-}(r_{j-1}).$$

Whence we have the bound

$$\left\|T_j(r)\left\{\binom{a_j^+}{a_j^-}(r) - z_j\right\}\right\| \leq \sum_{m=1}^{\infty} \left\|\Big((T_j M_j T_j^{-1})^m T_j z_j\Big)(r)\right\|; \tag{10.23}$$

to the right T_j is considered as an operator acting as $(T_j z)(r') = (T_j)(r')z(r')$. Using that f_j is increasing we can estimate

$$\|(T_j M_j T_j^{-1} z)(r)\| \leq \int_{r_{j-1}}^r \left\|(T_j N_j T_j^{-1})(r')\right\| \|z(r')\|\,\mathrm{d}r',$$

which applied repeatedly in (10.23) yields the following bound for $r \in I_j$:

$$\begin{aligned}
&\Big\| T_j(r)\Big\{\binom{a^+}{a^-}(r) - z_j\Big\}\Big\| \\
&\quad \le \Big\{\Big(\exp\int_{r_{j-1}}^{r} \big\|(T_j N_j T_j^{-1})(r')\big\| \, \mathrm{d}r'\Big) - 1\Big\} \sup_{\tilde{r}\in I_j} \|T_j(\tilde{r}) z_j\| \\
&\quad = \Big\{\Big(\exp\int_{r_{j-1}}^{r} \big\|(T_j N_j T_j^{-1})(r')\big\| \, \mathrm{d}r'\Big) - 1\Big\} \|z_j\|. \qquad (10.24)
\end{aligned}$$

We specify in the following $\phi_j^{\pm}$, B_j and f_j for $j = 2, 3$ and 4; in all cases $W(\phi_j^-, \phi_j^+) = 1$:

Ad interval I_2. We define

$$\phi_2^{\pm}(r) = 2^{-\frac{1}{2}} V_k^{-\frac{1}{4}} \mathrm{e}^{\pm \int_{r_1}^{r} \sqrt{V_k}\, \mathrm{d}r'}, \tag{10.25a}$$

compute

$$B_2 = -\big(V_k^{-\frac{1}{4}}\big)'' V_k^{\frac{1}{4}} = -\tfrac{5}{16}\Big(\frac{V_k'}{V_k}\Big)^2 + \tfrac{1}{4}\frac{V_k''}{V_k} \tag{10.25b}$$

and let

$$f_2(r) = \frac{\phi_2^+(r)}{\phi_2^-(r)} = \mathrm{e}^{2\int_{r_1}^{r} \sqrt{V_k}\, \mathrm{d}r'}. \tag{10.25c}$$

Ad interval I_3. We define (in terms of the Airy function, cf. [HS] and [Hö1, Definition 7.6.8])

$$\phi_3^+(r) = \sqrt{\pi}\zeta^{-1}\mathrm{Ai}\big(-\zeta^2(r - r_0)\big);\ \zeta := |V_k'(r_0)|^{\frac{1}{6}}, \tag{10.26a}$$

$$\begin{aligned}
\phi_3^-(r) &= \sqrt{\pi}\mathrm{e}^{\frac{\pi \mathrm{i}}{6}}\zeta^{-1}\mathrm{Ai}\big(-\zeta^2 \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}(r - r_0)\big) \\
&\quad + \sqrt{\pi}\mathrm{e}^{-\frac{\pi \mathrm{i}}{6}}\zeta^{-1}\mathrm{Ai}\big(-\zeta^2 \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}(r - r_0)\big),
\end{aligned} \tag{10.26b}$$

compute

$$B_3(r) = V_k(r) - \big(V_k(r_0) + V_k'(r_0)(r - r_0)\big) = \int_{r_0}^{r} (r - \tilde{r}) V_k''(\tilde{r})\, \mathrm{d}\tilde{r} \tag{10.26c}$$

and let

$$f_3(r) = \begin{cases} \exp\big(-\frac{4}{3}\zeta^3 (r_0 - r)^{\frac{3}{2}}\big) & \text{if } r < r_0 \\ 1 & \text{if } r \ge r_0 \end{cases}. \tag{10.26d}$$

Ad interval I_4. We define

$$\phi_4^+(r) = (-V_k)^{-\frac{1}{4}} \sin\Big(\int_{r_0}^{r} \sqrt{-V_k}\, \mathrm{d}r' + \frac{\pi}{4}\Big), \tag{10.27a}$$

$$\phi_4^-(r) = (-V_k)^{-\frac{1}{4}} \cos\Big(\int_{r_0}^{r} \sqrt{-V_k}\, \mathrm{d}r' + \frac{\pi}{4}\Big), \tag{10.27b}$$

compute

$$B_4 = -\big((-V_k)^{-\frac{1}{4}}\big)''(-V_k)^{\frac{1}{4}} = -\tfrac{5}{16}\Big(\frac{V_k'}{V_k}\Big)^2 + \tfrac{1}{4}\frac{V_k''}{V_k} \tag{10.27c}$$

and let

$$f_4 = 1. \tag{10.27d}$$

10.4.2. *Details of proof of Proposition 10.8.* We start implementing the scheme outlined in Subsubsection 10.4.1.

In the interval I_1 we shall use a standard comparison argument. With V_k replaced by $V = \frac{\tilde{k}(\tilde{k}+1)}{r^2}$ the regular solution is given by the expression $u = r^{\tilde{k}+1}$ and the corresponding Riccati equation

$$\psi' = V - \psi^2 \tag{10.28}$$

is solved by $\psi = \frac{\phi'}{\phi} = \frac{\tilde{k}+1}{r}$.

We fix $\epsilon_1 \in]0, 1]$ (actually $\epsilon_1 > 0$ can be chosen arbitrarily) and notice the following uniform bound in $r \in I_1$

$$V_k(r) = \frac{k(k+1)}{r^2}\big(1 + O(k^{-\epsilon_1})\big). \tag{10.29}$$

Using (10.29) we can find $C > 0$ such that with $k^{\pm} := k(1 \pm Ck^{-\epsilon_1})$ and $V_k^{\pm}(r) := \frac{k^{\pm}(k^{\pm}+1)}{r^2}$ there are estimates

$$V_k(r) \begin{cases} \leq V_k^+(r) \\ \geq V_k^-(r) \end{cases} ;\ r \in I_1.$$

Now, by using [BR, Theorem 1.8] and the Riccati equation it follows that the regular solution u of (10.20) is positive in I_1 and that $v := \frac{u'}{u}$ obeys the bounds

$$v(r) \begin{cases} \leq \frac{k^{+}+1}{r} \\ \geq \frac{k^{-}+1}{r} \end{cases} ;\ r \in I_1.$$

We conclude the uniform bound

$$v(r) = \frac{k+1}{r}\big(1 + O(k^{-\epsilon_1})\big);\ r \in I_1. \tag{10.30}$$

The connection formula at $r = r_1$ reads

$$c_j \begin{pmatrix} 1 \\ v \end{pmatrix}_{r=r_{j-1}} = \begin{pmatrix} a_j^+ \phi_j^+ + a_j^- \phi_j^- \\ a_j^+ (\phi_j^+)' + a_j^- (\phi_j^-)' \end{pmatrix}_{r=r_{j-1}} ;\ j = 2. \tag{10.31}$$

Obviously (10.31) is solved for the coefficients by

$$\begin{pmatrix} a_j^+ \\ a_j^- \end{pmatrix}_{r=r_{j-1}} = \frac{c_j}{W(\phi_j^-, \phi_j^+)} \begin{pmatrix} (-\phi_j^-)' + \phi_j^- v \\ (\phi_j^+)' - \phi_j^+ v \end{pmatrix}_{r=r_{j-1}} ;\ j = 2. \tag{10.32}$$

Next, from (10.25a) we compute

$$(\phi_2^{\pm})' = \Big(\pm\sqrt{V_k} - \frac{1}{4}\frac{V_k'}{V_k}\Big)\phi_2^{\pm}. \tag{10.33}$$

We substitute these expressions and (10.30) in the right hand side of (10.32) and obtain

$$\begin{pmatrix} a_2^+(r_1) \\ a_2^-(r_1) \end{pmatrix} = c_2 \frac{2k}{r_1} \begin{pmatrix} 1 + O\big(k^{-\epsilon_1}\big) \\ O\big(k^{-\epsilon_1}\big) \end{pmatrix}. \tag{10.34}$$

To apply (10.24) we notice that

$$T_2 N_2 T_2^{-1} = B_2 \phi_2^- \phi_2^+ \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = B_2 O\big(V_k^{-\frac{1}{2}}\big).$$

Whence (for the first inequality below we assume that the integral is bounded in k so that the inequality $\exp x - 1 \le Cx$ applies; this will be justified by (10.36))

$$\begin{aligned}
&\Big\| T_2(r_2)\Big\{\Big(\begin{smallmatrix} a_2^+ \\ a_2^- \end{smallmatrix}\Big)(r_2) - \Big(\begin{smallmatrix} a_2^+ \\ a_2^- \end{smallmatrix}\Big)(r_1)\Big\}\Big\| \\
&\quad = \Big\{\Big(\exp \int_{r_1}^{r_2} \Big|\Big(-\tfrac{5}{16}\Big(\frac{V_k'}{V_k}\Big)^2 + \tfrac{1}{4}\frac{V_k''}{V_k}\Big) O\big(V_k^{-\frac{1}{2}}\big)\Big| \, \mathrm{d}r'\Big) - 1\Big\} O\Big(\frac{k}{r_1}\Big) \\
&\quad \le C_1 \tfrac{k}{r_1} r_0 \int_{r_1/r_0}^{r_2/r_0} \left(\frac{\big(V_k'\big)^2}{V_k^{\frac{5}{2}}} + \frac{\big|V_k''\big|}{V_k^{\frac{3}{2}}} \right) \mathrm{d}s \quad \text{(changing variables } r' = r_0 s\text{)} \\
&\quad \le C_2 r_1^{-1} \int_{r_1/r_0}^{r_2/r_0} \left(\frac{s^{-6}}{\big(s^{-2} - s^{-\mu}\big)^{\frac{5}{2}}} + \frac{s^{-4}}{\big(s^{-2} - s^{-\mu}\big)^{\frac{3}{2}}} \right) \mathrm{d}s \\
&\quad = C_2 r_1^{-1} \left(\int_{1/2}^{r_2/r_0} \cdots \, \mathrm{d}s + \int_{r_1/r_0}^{1/2} \cdots \, \mathrm{d}s \right) \\
&\quad \le C_3 r_1^{-1} \max\Big(\int_{1/2}^{r_2/r_0} \big(1 - s^{2-\mu}\big)^{-\frac{5}{2}} \, \mathrm{d}s, \int_{r_1/r_0}^{1/2} s^{-1} \, \mathrm{d}s \Big) \\
&\quad \le C_4 k^{\frac{3}{2}\epsilon_2 - 1} \tfrac{k}{r_1};
\end{aligned} \tag{10.35}$$

we need here

$$\tfrac{3}{2}\epsilon_2 - 1 < 0. \tag{10.36}$$

We conclude by combining (10.34) and (10.35):

$$\begin{pmatrix} a_2^+(r_2) \\ a_2^-(r_2) \end{pmatrix} = c_2 \frac{2k}{r_1} \begin{pmatrix} 1 + O\big(k^{-\epsilon_1}\big) + O\Big(k^{\frac{3}{2}\epsilon_2 - 1}\Big) \\ O\big(k^{-\epsilon_1}\big) + \mathrm{e}^{2\int_{r_1}^{r_2} \sqrt{V_k}\, \mathrm{d}r'} O\Big(k^{\frac{3}{2}\epsilon_2 - 1}\Big) \end{pmatrix}. \tag{10.37}$$

Next we repeat the above procedure passing from the interval I_2 to I_3.

The first issue is the connection formula (10.31) with $j = 2$ replaced by $j = 3$. The left hand side can be estimated using (10.33), (10.37) and the following estimates (where (10.36) is used)

$$\begin{aligned}
\sqrt{V_k(r_2)} &= \frac{\sqrt{k(k+1)}}{r_2} \Big(1 - \big(1 - k^{-\epsilon_2}\big)^{2-\mu}\Big)^{\frac{1}{2}}, \\
&= \frac{k}{r_0} (2-\mu)^{\frac{1}{2}} k^{-\frac{\epsilon_2}{2}} \Big(1 + O\big(k^{-\epsilon_2}\big)\Big),
\end{aligned} \tag{10.38}$$

$$\frac{V_k'(r_2)}{V_k(r_2)} = \frac{O\Big(\frac{k^2}{r_2^3}\Big)}{V_k(r_2)} = r_2^{-1} O\big(k^{\epsilon_2}\big). \tag{10.39}$$

Notice that (10.38) dominates (10.39) (by (10.36) again) so that

$$\Big(\sqrt{V_k} - \frac{1}{4}\frac{V_k'}{V_k}\Big)(r_2) = (2-\mu)^{\frac{1}{2}} \frac{k}{r_0} k^{-\frac{\epsilon_2}{2}} \Big(1 + O\big(k^{-\epsilon_2}\big)\Big).$$

We conclude that

$$
\begin{aligned}
v(r_2) &= \frac{(\phi_2^+)'(r_2)}{\phi_2^+(r_2)}\Big(1+O\big(k^{\frac{3}{2}\epsilon_2-1}\big)\Big) \\
&= (2-\mu)^{\frac12}\frac{k}{r_0}k^{-\frac{\epsilon_2}{2}}\Big(1+O\big(k^{-\epsilon_2}\big)+O\big(k^{\frac32\epsilon_2-1}\big)\Big). \qquad (10.40)
\end{aligned}
$$

By (10.31) and (10.32) with $j=2$ replaced by $j=3$, up to multiplication by a positive constant

$$
\begin{pmatrix} a_3^+ \\ a_3^- \end{pmatrix}_{r=r_2} = \begin{pmatrix} (-\phi_3^-)'+\phi_3^- v \\ (\phi_3^+)'-\phi_3^+ v \end{pmatrix}_{r=r_2}. \qquad (10.41)
$$

It remains to examine the asymptotics of $\phi_3^\pm$ and their derivatives at r_2. For that we notice the asymptotics as $r-r_0\to-\infty$, cf. [HS, Appendix B] and [Hö1, (7.6.20)],

$$
\phi_3^+ = \frac{\exp\big(-\frac23\zeta^3(r_0-r)^{\frac32}\big)}{2\zeta^{\frac32}(r_0-r)^{\frac14}}\Big(1+O\big(\zeta^{-3}(r_0-r)^{-\frac32}\big)\Big), \qquad (10.42a)
$$

$$
(\phi_3^+)' = \zeta^3(r_0-r)^{\frac12}\frac{\exp\big(-\frac23\zeta^3(r_0-r)^{\frac32}\big)}{2\zeta^{\frac32}(r_0-r)^{\frac14}}\Big(1+O\big(\zeta^{-3}(r_0-r)^{-\frac32}\big)\Big), \qquad (10.42b)
$$

$$
\phi_3^- = \frac{\exp\big(\frac23\zeta^3(r_0-r)^{\frac32}\big)}{\zeta^{\frac32}(r_0-r)^{\frac14}}\Big(1+O\big(\zeta^{-3}(r_0-r)^{-\frac32}\big)\Big), \qquad (10.42c)
$$

$$
(\phi_3^-)' = -\zeta^3(r_0-r)^{\frac12}\frac{\exp\big(\frac23\zeta^3(r_0-r)^{\frac32}\big)}{\zeta^{\frac32}(r_0-r)^{\frac14}}\Big(1+O\big(\zeta^{-3}(r_0-r)^{-\frac32}\big)\Big). \qquad (10.42d)
$$

Since $\zeta^3(r_0-r_2)^{\frac32}\asymp\sqrt{2-\mu}\,k^{1-\frac32\frac{\epsilon_2}{2-\mu}}$, cf. (10.18), these asymptotics are applicable. By the same computation (10.40) can be rewritten as

$$
v(r_2) = \zeta^3(r_0-r_2)^{\frac12}\big(1+O(k^{-\epsilon_2})+O\big(k^{\frac32\epsilon_2-1}\big)\big). \qquad (10.43)
$$

Whence in conjunction (10.41) we obtain (up to multiplication by a positive constant)

$$
\begin{pmatrix} a_3^+(r_2) \\ a_3^-(r_2) \end{pmatrix} = \begin{pmatrix} \exp\big(\frac23\zeta^3(r_0-r_2)^{\frac32}\big)\Big(1+O\big(k^{-\epsilon_2}\big)+O\big(k^{\frac32\epsilon_2-1}\big)\Big) \\ \exp\big(-\frac23\zeta^3(r_0-r_2)^{\frac32}\big)\Big(O\big(k^{-\epsilon_2}\big)+O\big(k^{\frac32\epsilon_2-1}\big)\Big) \end{pmatrix}. \qquad (10.44)
$$

Next, to apply (10.24) with $j=3$ we need the following asymptotics of $\phi_3^\pm$ and their derivatives as $r-r_0\to+\infty$, cf. [HS, Appendix B] and [Hö1, (7.6.20) and (7.6.21)],

$$
\phi_3^+ = \zeta^{-\frac32}(r-r_0)^{-\frac14}\Big(\sin\big(\tfrac23\zeta^3(r-r_0)^{\frac32}+\tfrac{\pi}{4}\big)+O\big(\zeta^{-3}(r-r_0)^{-\frac32}\big)\Big), \qquad (10.45a)
$$

$$
(\phi_3^+)' = \zeta^{\frac32}(r-r_0)^{\frac14}\Big(\cos\big(\tfrac23\zeta^3(r-r_0)^{\frac32}+\tfrac{\pi}{4}\big)+O\big(\zeta^{-3}(r-r_0)^{-\frac32}\big)\Big), \qquad (10.45b)
$$

$$
\phi_3^- = \zeta^{-\frac32}(r-r_0)^{-\frac14}\Big(\cos\big(\tfrac23\zeta^3(r-r_0)^{\frac32}+\tfrac{\pi}{4}\big)+O\big(\zeta^{-3}(r-r_0)^{-\frac32}\big)\Big), \qquad (10.45c)
$$

$$
(\phi_3^-)' = -\zeta^{\frac32}(r-r_0)^{\frac14}\Big(\sin\big(\tfrac23\zeta^3(r-r_0)^{\frac32}+\tfrac{\pi}{4}\big)+O\big(\zeta^{-3}(r-r_0)^{-\frac32}\big)\Big). \qquad (10.45d)
$$

In particular

$$
T_3N_3T_3^{-1} = B_3\zeta^{-2}O\big(k^0\big) \text{ uniformly in } r\in I_3.
$$

In conjunction with (10.24), (10.18) and the fact that

$$V_k''(r) = O\Big(k^{-\frac{4+2\mu}{2-\mu}}\Big) \text{ uniformly in } r \in I_3 \tag{10.46}$$

we obtain

$$\begin{aligned}
&\Big\| T_3(r_3)\Big\{\begin{pmatrix} a_3^+ \\ a_3^- \end{pmatrix}(r_3) - \begin{pmatrix} a_3^+ \\ a_3^- \end{pmatrix}(r_2)\Big\}\Big\| \\
&\quad \le C_1\Big((r_3-r_0)^3 + (r_0-r_2)^3\Big)k^{-\frac{4+2\mu}{2-\mu}}\, k^{\frac{2}{3}\frac{1+\mu}{2-\mu}} a_3^+(r_2) \\
&\quad \le C_2 k^{\frac{4}{3}-3\min(\epsilon_2,\epsilon_3)} a_3^+(r_2);
\end{aligned} \tag{10.47}$$

here we need

$$\tfrac{4}{3} - 3\min(\epsilon_2,\epsilon_3) < 0, \tag{10.48}$$

cf. (10.36). At this point let us for convenience take $\epsilon_3 = \epsilon_2$, so that (10.48) simplifies and in conjunction with (10.36) leads to the single requirement

$$\tfrac{2}{3} > \epsilon_2 = \epsilon_3 > \tfrac{4}{9}. \tag{10.49}$$

We conclude that (up to multiplication by the positive constant $a_3^+(r_2)$)

$$\begin{pmatrix} a_3^+(r_3) \\ a_3^-(r_3) \end{pmatrix} = \begin{pmatrix} 1 + O\big(k^{\frac{4}{3}-3\epsilon_2}\big) \\ O\big(k^{\frac{4}{3}-3\epsilon_2}\big) \end{pmatrix}. \tag{10.50}$$

Next we need to study the connection formula passing from I_3 to I_4; a little linear algebra takes it to the form

$$\begin{pmatrix} a_4^+ \\ a_4^- \end{pmatrix} = \begin{pmatrix} W(\phi_4^-,\phi_3^+) & W(\phi_4^-,\phi_3^-) \\ W(\phi_3^+,\phi_4^+) & W(\phi_3^-,\phi_4^+) \end{pmatrix}\begin{pmatrix} a_3^+ \\ a_3^- \end{pmatrix};\ r = r_3.$$

So we need to compute the appearing Wronskians. To this end we note the following uniform asymptotics for $r \in [r_0, r_3]$ which are readily obtained from (10.18) and (10.46) (recall that by now $\epsilon_3 = \epsilon_2$):

$$V_k(r) = V_k'(r_0)(r-r_0)\big(1 + O\big(k^{-\epsilon_2}\big)\big), \tag{10.51a}$$

$$V_k'(r) = V_k'(r_0)\big(1 + O\big(k^{-\epsilon_2}\big)\big), \tag{10.51b}$$

$$\sqrt{-V_k(r)} = \zeta^3(r-r_0)^{\frac{1}{2}}\big(1 + O\big(k^{-\epsilon_2}\big)\big), \tag{10.51c}$$

$$\begin{aligned}
\int_{r_0}^{r}\sqrt{-V_k(r')}\,\mathrm{d}r' &= \frac{2}{3}\zeta^3(r-r_0)^{\frac{3}{2}}\big(1 + O\big(k^{-\epsilon_2}\big)\big) \\
&= \tfrac{2}{3}\zeta^3(r-r_0)^{\frac{3}{2}} + O\big(k^{1-\frac{5}{2}\epsilon_2}\big).
\end{aligned} \tag{10.51d}$$

Due to (10.51c) and (10.51d) the asymptotics (10.45a)–(10.45d) at the point $r = r_3$ can be written in terms of

$$\theta := \int_{r_0}^{r_3}\sqrt{-V_k(r')}\,\mathrm{d}r' + \frac{\pi}{4}$$

as

$$\frac{\phi_3^+(r_3)}{(-V_k(r_3))^{-\frac14}} = \sin\Big(\theta + O\big(k^{1-\frac52\epsilon_2}\big)\Big) + O\big(k^{-\epsilon_2}\big) + O\big(k^{\frac32\epsilon_2-1}\big), \tag{10.52a}$$

$$\frac{(\phi_3^+)'(r_3)}{(-V_k(r_3))^{\frac14}} = \cos\Big(\theta + O\big(k^{1-\frac52\epsilon_2}\big)\Big) + O\big(k^{-\epsilon_2}\big) + O\big(k^{\frac32\epsilon_2-1}\big), \tag{10.52b}$$

$$\frac{\phi_3^-(r_3)}{(-V_k(r_3))^{-\frac14}} = \cos\Big(\theta + O\big(k^{1-\frac52\epsilon_2}\big)\Big) + O\big(k^{-\epsilon_2}\big) + O\big(k^{\frac32\epsilon_2-1}\big), \tag{10.52c}$$

$$\frac{(\phi_3^-)'(r_3)}{(-V_k(r_3))^{\frac14}} = -\sin\Big(\theta + O\big(k^{1-\frac52\epsilon_2}\big)\Big) + O\big(k^{-\epsilon_2}\big) + O\big(k^{\frac32\epsilon_2-1}\big). \tag{10.52d}$$

Next, using that

$$\frac{-V_k'}{(-V_k)^{\frac32}}(r_3) = O\Big(k^{\frac32\epsilon_2-1}\Big),$$

cf. (10.51a) and (10.51b), we obtain for the functions $\phi_4^\pm$

$$\phi_4^+(r_3) = (-V_k(r_3))^{-\frac14}\sin\big(\theta\big), \tag{10.53a}$$

$$(\phi_4^+)'(r_3) = (-V_k(r_3))^{\frac14}\Big(\cos\big(\theta\big) + O\big(k^{\frac32\epsilon_2-1}\big)\Big), \tag{10.53b}$$

$$\phi_4^-(r_3) = (-V_k(r_3))^{-\frac14}\cos\big(\theta\big), \tag{10.53c}$$

$$(\phi_4^-)'(r_3) = -(-V_k(r_3))^{\frac14}\Big(\sin\big(\theta\big) + O\big(k^{\frac32\epsilon_2-1}\big)\Big). \tag{10.53d}$$

The matrix of Wronskians are readily computed using (10.45a)–(10.45d) and (10.52a)– (10.52d), in combination with (10.50) we obtain (using in the second step (10.49))

$$\begin{aligned}\begin{pmatrix} a_4^+(r_3)-1 \\ a_4^-(r_3)\end{pmatrix} &= O\Big(k^{-\epsilon_2}\Big) + O\Big(k^{\frac32\epsilon_2-1}\Big) + O\Big(k^{\frac43-3\epsilon_2}\Big) + O\Big(k^{1-\frac52\epsilon_2}\Big) \\ &= O\Big(k^{-\epsilon_2}\Big) + O\Big(k^{\frac32\epsilon_2-1}\Big) + O\Big(k^{\frac43-3\epsilon_2}\Big).\end{aligned} \tag{10.54}$$

Now we estimate in I_4 using (10.54) (and mimicking partially (10.35))

$$\begin{aligned}&\Big\|\begin{pmatrix} a_4^+ \\ a_4^-\end{pmatrix}(r) - \begin{pmatrix} a_4^+ \\ a_4^-\end{pmatrix}(r_3)\Big\| \\ &\le C_1\Big\{\Big(\exp\int_{r_3}^{r}\Big|\Big(-\tfrac{5}{16}\Big(\frac{V_k'}{V_k}\Big)^2 + \tfrac14\frac{V_k''}{V_k}\Big)O\big((-V_k)^{-\frac12}\big)\Big|\,\mathrm{d}r'\Big) - 1\Big\} \\ &\le C_2 r_0\int_{r_3/r_0}^{r/r_0}\left(\frac{\big(-V_k'\big)^2}{(-V_k)^{\frac52}} + \frac{\big|-V_k''\big|}{(-V_k)^{\frac32}}\right)\mathrm{d}s \quad \text{(changing variables } r' = r_0 s\text{)} \\ &\le C_3 r_0^{\mu/2-1}\int_{r_3/r_0}^{r/r_0} s^{\mu/2-2}\Big(\big(1-s^{\mu-2}\big)^{-\frac52} + \big(1-s^{\mu-2}\big)^{-\frac32}\Big)\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\le C_4 r_0^{\mu/2-1}\Big(\int_2^\infty s^{\mu/2-2}\,\mathrm{d}s + \int_{r_3/r_0}^2 \big(1-s^{\mu-2}\big)^{-\frac{5}{2}}\,\mathrm{d}s\Big)\\
&\le C_5 r_0^{\mu/2-1} k^{\frac{3}{2}\epsilon_2}\\
&= O\big(k^{\frac{3}{2}\epsilon_2-1}\big). \qquad (10.55)
\end{aligned}
$$

By the same type of estimation we also deduce that for fixed k there exist $\epsilon_k > 0$ and $a_4^\pm(\infty) \in \mathbb{R}$ such that

$$a_4^\pm(r) = a_4^\pm(\infty) + O(r^{-\epsilon_k}).$$

By applying (10.55) with $r = \infty$ in combination with (10.54) (and using an elementary trigonometric formula), we conclude that (10.19) is true.

The general case. It remains to prove (10.19) under Condition 10.5 (i.e. without assuming that $V_2 = V_3 = 0$). All previous constructions and estimates carry over, so below we consider only some additional estimates that are needed. Denoting $U = 2V_2 + 2V_3$ the functions $\phi_j^\pm$ and f_j and the potentials A_j are exactly the same while the potentials B_j are given as the old B_j plus U; $j = 2, 3, 4$.

Ad interval I_1. We notice that (10.29) is valid (here with V_k defined upon replacing $2V_1 \to 2V = 2V_1 + U$). Whence we can proceed exactly as before.

Ad interval I_2. In addition to (10.35) we need the following estimation (assuming in the last step that $\frac{\mu}{2} + \epsilon < 1$)

$$
\begin{aligned}
&\int_{r_1}^{r_2} \big|UO\big(V_k^{-\frac{1}{2}}\big)\big|\,\mathrm{d}r' O\Big(\frac{k}{r_1}\Big)\\
&\qquad \le C_1 \tfrac{k}{r_1} r_0 \int_{r_1/r_0}^{r_2/r_0} \frac{r^{-1-\frac{\mu}{2}-\epsilon}}{V_k^{\frac{1}{2}}}\,\mathrm{d}s \quad \text{(changing variables } r = r_0 s)\\
&\qquad \le C_2 \tfrac{k}{r_1} \frac{r_0^{1-\frac{\mu}{2}-\epsilon}}{k} \int_{r_1/r_0}^{r_2/r_0} \frac{s^{-\frac{\mu}{2}-\epsilon}}{\big(1-s^{2-\mu}\big)^{\frac{1}{2}}}\,\mathrm{d}s\\
&\qquad \le C_3 k^{-\frac{2\epsilon}{2-\mu}} \tfrac{k}{r_1}; \qquad (10.56)
\end{aligned}
$$

Ad interval I_3. In addition to (10.47) we need the following estimation

$$
\begin{aligned}
\int_{r_2}^{r_3} \big|U\zeta^{-2}\big|\,\mathrm{d}r' &\le C_1 \int_{r_2}^{r_3} k^{\frac{2}{3}\frac{1+\mu}{2-\mu}} r'^{-1-\frac{\mu}{2}-\epsilon}\,\mathrm{d}r'\\
&\le C_2 k^{\frac{2}{3}\frac{1+\mu}{2-\mu}} r_0^{-\frac{\mu}{2}-\epsilon}\big(k^{-\epsilon_2} + k^{-\epsilon_3}\big)\\
&\le C_3 k^{\frac{1}{3}-\epsilon_2-\frac{2\epsilon}{2-\mu}}. \qquad (10.57)
\end{aligned}
$$

Due to (10.49) the right hand side of (10.57) vanishes.

Ad interval I_4. In addition to (10.55) we need the following estimation

$$\begin{aligned}&\int_{r_3}^{r} \left|UO\big((-V_k)^{-\frac12}\big)\right| \mathrm{d}r' \\ &\le C_1 r_0^{-\epsilon} \int_{r_3/r_0}^{r/r_0} \frac{s^{-1-\epsilon}}{\big(1-s^{\mu-2}\big)^{\frac12}} \,\mathrm{d}s \text{ (changing variables } r' = r_0 s) \\ &\le C_2 k^{-\frac{2\epsilon}{2-\mu}}. \end{aligned} \tag{10.58}$$

This ends the proof of (10.19). □

10.5. **End of proof of Proposition 10.6.** We need the following elementary identity:

Lemma 10.9. *Let $\mu < 2$. Then*

$$\int_1^\infty (\sqrt{r^{-\mu} - r^{-2}} - \sqrt{r^{-\mu}})\mathrm{d}r = \frac{2-\pi}{2-\mu}. \tag{10.59}$$

Proof. We first substitute $r = s^{\frac{1}{\mu-2}}$ and then $s = \sin^2\phi$. Thus the left hand side of (10.59) equals

$$\begin{aligned}\frac{1}{2-\mu}\int_0^1 s^{-\frac32}\left(\sqrt{1-s}-1\right)\mathrm{d}s &= \frac{2}{2-\mu}\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos\phi}{\sin^2\phi}-1\right)\mathrm{d}\phi \\ = \frac{2}{2-\mu}\left(\frac{1-\cos\phi}{\sin\phi}-\phi\right)\Big|_0^{\pi/2} &= \frac{2-\pi}{2-\mu}.\end{aligned}$$

□

Proof of Proposition 10.6. Using Proposition 10.8 we calculate

$$\begin{aligned}\sigma_k(0) &= \lim_{r\to\infty}\Big(\int_{r_0}^{r}\sqrt{-V_k(\tilde r)}\mathrm{d}\tilde r + \frac{\pi}{4} \\ &\quad - \int_{R_0}^{r}\sqrt{-2V(\tilde r)}\mathrm{d}\tilde r + \frac{k\pi}{2}\Big) + o(k^0) \\ &= \int_{r_0}^{\infty}\Big(\sqrt{-V_k(r)} - \sqrt{-2V_1(r)}\Big)\mathrm{d}r \\ &\quad + \int_{R_0}^{\infty}\Big(\sqrt{-2V_1(r)} - \sqrt{-2V(r)}\Big)\mathrm{d}r \\ &\quad - \int_{R_0}^{r_0}\sqrt{-2V_1(r)}\mathrm{d}r + \frac{(k+\frac12)\pi}{2} + o(k^0)\end{aligned}$$

Now (using Lemma 10.9)

$$\begin{aligned}\int_{r_0}^{\infty}(\sqrt{-V_k} - \sqrt{-2V_1(r)})\mathrm{d}r &= \sqrt{k(k+1)}\int_1^\infty(\sqrt{r^{-\mu}-r^{-2}} - \sqrt{r^{-\mu}})\mathrm{d}r \\ &= \sqrt{k(k+1)}\,\frac{2-\pi}{2-\mu}; \\ \int_{r_0}^{R_0}\sqrt{V_1(r)}\mathrm{d}r &= -\tfrac{2}{2-\mu}\sqrt{k(k+1)} + \tfrac{2\sqrt{2\gamma}}{2-\mu}R_0^{1-\frac{\mu}{2}}.\end{aligned}$$

Thus

$$\begin{aligned}
&\sigma_k(0) - \int_{R_0}^{\infty} \left(\sqrt{-2V_1(r)} - \sqrt{-2V(r)}\right) \mathrm{d}r \\
&= -\sqrt{k(k+1)}\frac{\pi}{2-\mu} + \frac{(k+\frac{1}{2})\pi}{2} + \frac{2\sqrt{2\gamma}}{2-\mu}R_0^{1-\frac{\mu}{2}} + o(k^0) \\
&= -\frac{(k+\frac{1}{2})\pi\mu}{2(2-\mu)} + \frac{2\sqrt{2\gamma}}{2-\mu}R_0^{1-\frac{\mu}{2}} + o(k^0).
\end{aligned}$$

□

Appendix A. Elements of abstract scattering theory

Various versions of stationary scattering theory can be found in the literature. In this appendix we give, in an abstract setting, a self-contained presentation of its elements used in our paper. It is a version of the standard approach contained e.g. in [Ya4], adapted to our paper. In our stationary formulas for the scattering operator we use in addition ideas due to Isozaki-Kitada, see the proof of [IK2, Theorem 3.3].

A.1. **Wave operators.** Let H_0 and H be two self-adjoint operators on a Hilbert space $\mathcal{H}$. We assume that H_0 has only continuous spectrum. Let throughout this appendix Λ_n, $n \in \mathbb{N}$, be a sequence of compact subsets of $\sigma(H_0)$ such that Λ_n is a subset of the interior of Λ_{n+1} and such that $\sigma(H_0) \setminus \cup_n \Lambda_n$ has Lebesgue measure zero. Pick a sequence $h_n \in C_c^\infty(\Lambda_{n+1})$ with $h_n = 1$ on Λ_n. Let $\mathcal{D} := \cup_n \mathrm{Ran} 1_{\Lambda_n}(H_0)$; it is dense in $\mathcal{H}$.

We will write $R(z) = (H - \zeta)^{-1}$ and $R_0(\zeta) = (H_0 - \zeta)^{-1}$ for $\zeta \notin \sigma(H_0)$, and

$$\delta_\epsilon(\lambda) = \frac{\epsilon}{\pi((H_0 - \lambda)^2 + \epsilon^2)} = \frac{\epsilon}{\pi} R_0(\lambda - \mathrm{i}\epsilon) R_0(\lambda + \mathrm{i}\epsilon);\ \epsilon > 0.$$

Note that if I is an interval and $f \in \mathcal{H}$, then

$$\| \int_I \frac{\epsilon}{\pi} R_0(\lambda - \mathrm{i}\epsilon) R_0(\lambda + \mathrm{i}\epsilon) f \mathrm{d}\lambda \| \le \|f\|, \tag{A.1}$$

$$\lim_{\epsilon \searrow 0} \int_I \frac{\epsilon}{\pi} R_0(\lambda - \mathrm{i}\epsilon) R_0(\lambda + \mathrm{i}\epsilon) f \mathrm{d}\lambda = E_I(H_0) f. \tag{A.2}$$

Theorem A.1. *Suppose $J^\pm$ is a densely defined operator whose domain contains $\mathcal{D}$ such that $J_n^\pm := J^\pm h_n(H_0)$ is bounded for any n, and*

$$\lim_{t \to \pm\infty} \|J^\pm \mathrm{e}^{\mathrm{i}tH_0} f\|^2 = \|f\|^2, \quad f \in \mathcal{D}.$$

We also suppose that there exists the wave operator

$$W^\pm f := \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} J^\pm \mathrm{e}^{-\mathrm{i}tH_0} f;\ f \in \mathcal{D}. \tag{A.3}$$

Then

(i) *$W^\pm$ extends to an isometric operator and $W^\pm H_0 = H W^\pm$.*
(ii) *For any interval I and $f \in \mathcal{D}$,*

$$W^\pm 1_I(H_0) f = \lim_{\epsilon \searrow 0} \int_I \frac{\epsilon}{\pi} R(\lambda \mp \mathrm{i}\epsilon) J^\pm R_0(\lambda \pm \mathrm{i}\epsilon) f \mathrm{d}\lambda. \tag{A.4}$$

(iii) *For any continuous function $g : \mathbb{R} \to \mathbb{C}$ vanishing at infinity, interval I and $f \in \mathcal{D}$,*

$$W^{\pm} g(H_0) 1_I(H_0) f = \lim_{\epsilon \searrow 0} \int_I \frac{\epsilon}{\pi} g(\lambda) R(\lambda \mp \mathrm{i}\epsilon) J^{\pm} R_0(\lambda \pm \mathrm{i}\epsilon) f \mathrm{d}\lambda. \tag{A.5}$$

(iv) *Suppose in addition that $J^{\pm}$ maps $\mathcal{D}$ into* $\operatorname{Dom} H$. *Suppose that $T^{\pm}$ is a densely defined operator such that $T_n^{\pm} := T^{\pm} h_n(H_0)$ is bounded for any n and that $T^{\pm} f = \mathrm{i}(HJ^{\pm} - J^{\pm} H_0) f$ for any $f \in \mathcal{D}$. Then we have the following modifications of* (A.4) *and* (A.5)*:*

$$W^{\pm} 1_I(H_0) f = \lim_{\epsilon \searrow 0} \int_I (J^{\pm} + \mathrm{i} R(\lambda \mp \mathrm{i}\epsilon) T^{\pm}) \delta_\epsilon(\lambda) f \mathrm{d}\lambda, \tag{A.6}$$

$$W^{\pm} g(H_0) 1_I(H_0) f = \lim_{\epsilon \searrow 0} \int_I g(\lambda)(J^{\pm} + \mathrm{i} R(\lambda \mp \mathrm{i}\epsilon) T^{\pm}) \delta_\epsilon(\lambda) f \mathrm{d}\lambda. \tag{A.7}$$

Proof. (i) is well-known.

Let us prove (ii): By (A.3)

$$W^{\pm} f = \lim_{\epsilon \searrow 0} 2\epsilon \int_0^\infty \mathrm{e}^{-2\epsilon t} \mathrm{e}^{\pm \mathrm{i}tH} J^{\pm} \mathrm{e}^{\mp \mathrm{i}tH_0} f \mathrm{d}t.$$

By the vector-valued Plancherel formula we obtain

$$W^{\pm} f = \lim_{\epsilon \searrow 0} \int \frac{\epsilon}{\pi} R(\lambda \mp \mathrm{i}\epsilon) J^{\pm} R_0(\lambda \pm \mathrm{i}\epsilon) f \mathrm{d}\lambda \tag{A.8}$$

Therefore,

$$\begin{aligned} W^{\pm} 1_I(H_0) f = & \lim_{\epsilon \searrow 0} \int_I \frac{\epsilon}{\pi} R(\lambda \mp \mathrm{i}\epsilon) J^{\pm} R_0(\lambda \pm \mathrm{i}\epsilon) f \mathrm{d}\lambda \\ & - \lim_{\epsilon \searrow 0} \int_I \frac{\epsilon}{\pi} R(\lambda \mp \mathrm{i}\epsilon) J^{\pm} R_0(\lambda \pm \mathrm{i}\epsilon) 1_{\mathbb{R}\setminus I}(H_0) f \mathrm{d}\lambda \\ & + \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus I} \frac{\epsilon}{\pi} R(\lambda \mp \mathrm{i}\epsilon) J^{\pm} R_0(\lambda \pm \mathrm{i}\epsilon) 1_I(H_0) f \mathrm{d}\lambda. \end{aligned}$$

We need to show that the last two terms vanish. The proof for both terms is identical. Consider the last one term. Let $f_1 \in \mathcal{H}$ and pick an n so that $f = 1_{\Lambda_n}(H_0) f$. Then (using (A.1) in the last estimation)

$$\begin{aligned} & \Big| \int_{\mathbb{R}\setminus I} \frac{\epsilon}{\pi} \langle f_1, R(\lambda \mp \mathrm{i}\epsilon) J^{\pm} R_0(\lambda \pm \mathrm{i}\epsilon) 1_I(H_0) f \rangle \mathrm{d}\lambda \Big| \\ & \leq \|J_n^{\pm}\| \left(\int_{\mathbb{R}\setminus I} \frac{\epsilon}{\pi} \|R(\lambda \pm \mathrm{i}\epsilon) f_1\|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}\setminus I} \frac{\epsilon}{\pi} \|R_0(\lambda \pm \mathrm{i}\epsilon) 1_I(H_0) f\|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}} \\ & \leq C_\epsilon \|f_1\|; \; C_\epsilon := \|J_n^{\pm}\| \left(\int_{\mathbb{R}\setminus I} \frac{\epsilon}{\pi} \|R_0(\lambda \pm \mathrm{i}\epsilon) 1_I(H_0) f\|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}}. \end{aligned}$$

Due to (A.2), $C_\epsilon \to 0$ as $\epsilon \to 0$. Whence (ii) follows.

Let us prove (iii): Let $f_1 \in \mathcal{H}$ and pick an n so that $f = 1_{\Lambda_n}(H_0) f$. Any continuous function g vanishing at infinity can be uniformly approximated by g_m, finite linear combinations of characteristic functions of intervals. By (ii) and (A.1),

$$W^{\pm} g_m(H_0) 1_I(H_0) f = \lim_{\epsilon \searrow 0} \int_I \frac{\epsilon}{\pi} g_m(\lambda) R(\lambda \mp \mathrm{i}\epsilon) J^{\pm} R_0(\lambda \pm \mathrm{i}\epsilon) f \mathrm{d}\lambda.$$

Now

$$\left|\int_I \frac{\epsilon}{\pi}\big(g_m(\lambda)-g(\lambda)\big)\langle f_1, R(\lambda\mp \mathrm{i}\epsilon)J^{\pm}R_0(\lambda\pm\mathrm{i}\epsilon)f\rangle \mathrm{d}\lambda\right|$$

$$\leq \|J_n^{\pm}\|\left(\int \frac{\epsilon}{\pi}\left\|R(\lambda\pm\mathrm{i}\epsilon)f_1\right\|^2\mathrm{d}\lambda\right)^{\frac12}\left(\int \frac{\epsilon}{\pi}\left\|R_0(\lambda\pm\mathrm{i}\epsilon)f\right\|^2\mathrm{d}\lambda\right)^{\frac12}\sup|g_m-g|$$

$$\leq C_m\|f_1\|;\ C_m := \|J_n^{\pm}\|\|f\|\sup|g_m-g|.$$

Since $C_m \to\ 0$ we are done.

To prove (iv) we use (iii) and the identity

$$R(\lambda\mp\mathrm{i}\epsilon)J^{\pm} = (J^{\pm}+\mathrm{i}R(\lambda\mp\mathrm{i}\epsilon)T^{\pm})R_0(\lambda\mp\mathrm{i}\epsilon).$$

□

Remark. In the context of our paper, we can take $\Lambda_n = [\frac{1}{n}, n]$.

A.2. **Scattering operator.** Define the scattering operator via $S := W^{+*}W^{-}$. Clearly, $H_0S = SH_0$.

Theorem A.2. *Suppose that the conditions of Theorem A.1 hold. Let the operator J^- satisfy*

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}J^{-}\mathrm{e}^{-\mathrm{i}tH_0}f = 0;\ f\in\mathcal{D}. \tag{A.9}$$

Then for all $f\in\mathcal{D}$

$$Sf = -\lim_{\epsilon\searrow 0} 2\pi\int \delta_\epsilon(\lambda)W^{+*}T^{-}\delta_\epsilon(\lambda)f\mathrm{d}\lambda. \tag{A.10}$$

Proof.

$$\begin{aligned}
W^-f &= -\lim_{t\to+\infty}\left(\mathrm{e}^{\mathrm{i}tH}J^{-}\mathrm{e}^{-\mathrm{i}tH_0}-\mathrm{e}^{-\mathrm{i}tH}J^{-}\mathrm{e}^{\mathrm{i}tH_0}\right)f\\
&= -\lim_{t\to+\infty}\int_{-t}^{t}\mathrm{e}^{\mathrm{i}sH}T^{-}\mathrm{e}^{-\mathrm{i}sH_0}f\mathrm{d}s\\
&= -\lim_{\epsilon\searrow 0}\epsilon\int_0^{\infty}\mathrm{e}^{-\epsilon t}\mathrm{d}t\int_{-t}^{t}\mathrm{e}^{\mathrm{i}sH}T^{-}\mathrm{e}^{-\mathrm{i}sH_0}f\mathrm{d}s\\
&= -\lim_{\epsilon\searrow 0}\int \mathrm{e}^{-\epsilon|s|}\mathrm{e}^{\mathrm{i}sH}T^{-}\mathrm{e}^{-\mathrm{i}sH_0}f\mathrm{d}s.
\end{aligned}$$

Then we use the definition of S and the intertwining property of W^{+*} to obtain

$$Sf = -\lim_{\epsilon\searrow 0}\int \mathrm{e}^{-\epsilon|s|}\mathrm{e}^{\mathrm{i}sH_0}W^{+*}T^{-}\mathrm{e}^{-\mathrm{i}sH_0}f\mathrm{d}s.$$

Finally, we use the vector-valued Plancherel theorem. □

A.3. **Method of rigged Hilbert spaces applied to wave operators.** Consider a family of separable Hilbert spaces $\mathcal{H}$ and $\mathcal{V}_s$, $s>\frac12$, such that $\mathcal{V}_s$ is densely and continuously embedded in $\mathcal{H}$, and similarly, $\mathcal{V}_s$ is densely and continuously embedded in $\mathcal{V}_t$ if $s>t$. Let $\mathcal{V}_s^*$ be the space dual to $\mathcal{V}_s$, so that we have nested Hilbert spaces

$$\mathcal{V}_s\subseteq\mathcal{V}_t\subseteq\mathcal{H}\subseteq\mathcal{V}_t^*\subseteq\mathcal{V}_s^*;\ s>t.$$

We remark that $\mathcal{H}$ equipped with such a structure is sometimes called a *rigged Hilbert space*.

The following theorem allows us to introduce wave matrices:

Theorem A.3. *Fix $s > t > \frac{1}{2}$. Suppose that there exists for almost all λ the limit*

$$\mathrm{s-}\lim_{\epsilon\to 0} \delta_\epsilon(\lambda) =: \delta_0(\lambda) \in \mathcal{B}(\mathcal{V}_t, \mathcal{V}_t^*).$$

Suppose the conditions of Theorem A.1 and that the operators $J_n^\pm$ and $R(\lambda \mp \mathrm{i}\epsilon)T_n^\pm$ with $\lambda \in \Lambda_n$ and $\epsilon > 0$ extend to elements of $\mathcal{B}(\mathcal{V}_t^*, \mathcal{V}_s^*)$. Suppose that for fixed n and almost everywhere in Λ_n there exists

$$R(\lambda \mp \mathrm{i}0)T_n^\pm := \mathrm{s-}\lim_{\epsilon\searrow 0} R(\lambda \mp \mathrm{i}\epsilon)T_n^\pm \in \mathcal{B}(\mathcal{V}_t^*, \mathcal{V}_s^*).$$

Suppose furthermore that for any n there exists $\epsilon_n > 0$ such that

$$\sup_{\lambda\in\Lambda_n}\sup_{\epsilon<\epsilon_n} \|\delta_\epsilon(\lambda)\|_{\mathcal{V}_t\to\mathcal{V}_t^*}, \ \sup_{\lambda\in\Lambda_n}\sup_{\epsilon<\epsilon_0} \left\|R(\lambda \mp \mathrm{i}\epsilon)T_n^\pm\right\|_{\mathcal{V}_t^*\to\mathcal{V}_s^*} < \infty. \tag{A.11}$$

Let I be an interval with $I \subseteq \Lambda_n$ for some n, and let $f \in \mathcal{V}_t$ be given such that $f = h_n(H_0)f$ (in particular this means that $f \in \mathcal{D} \cap \mathcal{V}_t$). Then (in terms of an integral of a $\mathcal{V}_s^*$–valued function) for all $g \in C^\infty(\mathbb{R})$

$$W^\pm g(H_0)1_I(H_0)f = \int_I g(\lambda)\big(J_n^\pm + \mathrm{i}R(\lambda \mp \mathrm{i}0)T_n^\pm\big)\delta_0(\lambda)f\mathrm{d}\lambda. \tag{A.12}$$

Proof. We can replace $T^\pm \to T_n^\pm$ in the integrand of (A.7). Then, by the assumptions, it has a pointwise limit as an element of $\mathcal{V}_s^*$. Due to (A.11) we can apply the dominated convergence theorem. □

Remark. In the context of our paper, we take $\mathcal{V}_s := L^{2,s}$.

A.4. **Method of rigged Hilbert spaces applied to the scattering operator.** The method of rigged Hilbert spaces allows us to introduce scattering matrices:

Theorem A.4. *Suppose that the conditions of Theorem A.3 hold for some $s > t > \frac{1}{2}$ and suppose* (A.9)*. Fix $r > s$. Suppose that for all $n \in \mathbb{R}$ and $\epsilon > 0$ the operators $T_n^-\delta_\epsilon(\lambda) \in \mathcal{B}(\mathcal{V}_r, \mathcal{V}_s)$ with a measurable dependence of $\lambda \in \mathbb{R}$. Suppose that for fixed n and almost everywhere in Λ_n there exists the limit*

$$\mathrm{s-}\lim_{\epsilon\to 0} T_n^-\delta_\epsilon(\lambda) =: T_n^-\delta_0(\lambda) \in \mathcal{B}(\mathcal{V}_r, \mathcal{V}_s).$$

Suppose furthermore that for any n there exists $\epsilon_n > 0$ such that

$$\sup_{\lambda\in\mathbb{R}}\sup_{\epsilon<\epsilon_n} \left\|T_n^-\delta_\epsilon(\lambda)\right\|_{\mathcal{V}_r\to\mathcal{V}_s} < \infty. \tag{A.13}$$

Let I be an interval with $I \subseteq \Lambda_n$ for some n, and let $f_1 \in \mathcal{D}\cap\mathcal{V}_t$ and $f_2 \in \mathcal{D}\cap\mathcal{V}_r$ be given such that $f_1 = 1_I(H_0)f_1$ and $f_2 = h_n(H_0)f_2$. Then

$$\begin{aligned}\langle f_1, Sf_2\rangle =& -2\pi\int_I \langle f_1, \delta_0(\lambda)J_n^{+*}T_n^-\delta_0(\lambda)f_2\rangle\mathrm{d}\lambda \\ &+ 2\pi\mathrm{i}\int_I \langle f_1, \delta_0(\lambda)T_n^{+*}R(\lambda+\mathrm{i}0)T_n^-\delta_0(\lambda)f_2\rangle\mathrm{d}\lambda.\end{aligned}$$

Proof. We insert (A.12) with $g(\lambda) = g(\lambda-\lambda_1,\epsilon) := \frac{\epsilon}{\pi((\lambda-\lambda_1)^2+\epsilon^2)}$ into (A.10):

$$\begin{aligned}\langle f_1, Sf_2\rangle &= -\lim_{\epsilon\searrow 0} 2\pi\int \langle f_1, \delta_\epsilon(\lambda_1)W^{+*}T^-\delta_\epsilon(\lambda_1)f_2\rangle\mathrm{d}\lambda_1 \\ &= -\lim_{\epsilon\searrow 0} 2\pi\int\int_I g(\lambda,\lambda_1,\epsilon)\langle f_1, \delta_0(\lambda)\big(J_n^{+*} - \mathrm{i}T_n^{+*}R(\lambda+\mathrm{i}0)\big)T_n^-\delta_\epsilon(\lambda_1)f_2\rangle \\ &= -\lim_{\epsilon\searrow 0} 2\pi\int_I \langle f_1, \delta_0(\lambda)\big(J_n^{+*} - \mathrm{i}T_n^{+*}R(\lambda+\mathrm{i}0)\big)T_n^-\delta_{2\epsilon}(\lambda)f_2\rangle\mathrm{d}\lambda.\end{aligned}$$

In the last step we interchanged integrals, using (A.13) and the Fubini theorem, and we used that

$$\int \delta_\epsilon(\lambda_1) g(\lambda, \lambda_1, \epsilon) \mathrm{d}\lambda_1 = \delta_{2\epsilon}(\lambda).$$

Then we pass with $\epsilon \to 0$ using (A.13) and the dominated convergence theorem. □

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