Characters of Representations of Algebraic Groups in Small Characteristics

Ph.d. thesis by

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Contents

Abstract

In representation theory, Lusztig's conjecture is one of the important conjectures still standing. It easily extends to a more general character formula, for which several exceptions are already known in small characteristic. The aim of this thesis is to study the exceptions to this generalized character formula in some special cases, and to suggest patterns in these exceptions, and rules which may apply, where the formula fails.

Introduction

This thesis concerns the characters of G-modules for an reductive algebraic group G over an algebraically closed field of finite characteristic p. The primary goal is to investigate Lusztig's character formula, which describes the simple characters $\mathrm{ch}L(\lambda)$ by the characters $\chi(\mu)$ of the Weyl modules:

$$\mathrm{ch}L(w.\lambda_0) = \sum_{x \leq w} P_{w_0 x, w_0 w}(1)\, \chi(x.\lambda_0)$$

where λ_0 is an initial weight, $w \in \mathcal{W}$ is an element of the affine Weyl group and $P_{w_0 x, w_0 w}$ is the Kazhdan-Lusztig-polynomial.

It was first conjectured by G. Lusztig in 1979 that the character formula holds under certain conditions, see Conjecture 3.2.1. The formula is known to hold for large enough p, but for small p there are exceptions. My thesis investigates these exceptions, finding them in for certain affine Weyl groups, and then trying to describe and explain their presence. We focus primarily on the root systems of type A, since the calculations and formulas are somewhat simpler in this case.

This thesis is structured as follows. Chapter 1 introduces the necessary theory of Weyl modules, simple modules, and their characters. In Chapter 2 we develop all the tools needed to actually calculate these characters, most importantly Jantzen's sum formula and a method for finding the dimension of the weight spaces $L(\lambda)_\mu$. Chapter 3 is devoted to Lusztig's conjecture, and the submodule structure of the Weyl modules. In Chapter 4 we use the theory of sheaves on moment graphs to attack the problem from a more combinatorial approach. And in Chapter 5 we study the connections to the representation theory of restricted Lie algebras.

My results represent a thorough investigation of the exceptions to Lusztig's character formula for the root systems A_2, A_3 and A_4 for the primes $p \in \{2, 3\}$. We use computer calculations to find the exceptions, check existing conjectures and on this basis pose new questions and conjectures. Calculation results can be found in the appendices. Our investigation is carried out in three directions (in Chapter 3, 4 and 5, respectively).

First we analyze the submodule structure of the Weyl modules $V(\lambda)$ in the form of diagrams which contain all submodules and the corresponding quotients, expressed in terms of the simple G-modules $L(\mu)$. It is known that, for the regular weights for which Lusztig's character formula holds, the dimension of the

Ext-groups are given by a coefficient of the Kazhdan-Lusztig polynomials (see proposition 3.7.1). This can be generalizes to a dimension formula for Ext-groups for non-regular weights in a natural way. It seems that this generalized formula also holds whenever the weights satisfy Lusztig's character formula. That is; based on our calculations, we observe

> In every calculated case: Each exception to the generalized Ext dimension formula is also an exception to Lusztig's character formula.

The second direction of investigation is more combinatorial, namely we consider the Braden MacPherson sheaf on a moment graph of an affine Weyl group. Firstly, we have a concrete calculation of Braden-MacPherson sheaves for A_2, A_3 and A_4 for the primes $p \in \{2,3\}$. To perform the calculation we use an algorithm developed by T. Braden, and to apply it, it is necessary for the moment graph to be what we call *nice* (certain modules of sections must be graded free). This turns out to hold in all the calculated cases. Thus, a natural question is

> Is every moment graph of an affine Weyl group nice (possibly just for root systems of type A_n)?

The vertex modules of the Braden-MacPherson sheaf are graded free, so they can be written as a sum of shifted copies of the underlying ring. Thus, they define a polynomial using the shifts as exponents. This polynomial is known to coincide with the Kazhdan-Lusztig polynomial $P_{x,w}(q)$ in characteristic zero, and for large p, and thus we call it the modular Kazhdan-Lusztig polynomial, $P^p_{x,w}(q)$. Our first idea was to substitute $P^p_{x,w}(q)$ for $P_{x,w}(q)$ in Lusztig's character formula, thus obtaining a "modular character formula", and compare the results. Sadly, we find more exceptions to the modular character formula; in fact,

> In every calculated case: Each exception to Lusztig's character formula is also an exception to the modular character formula.

It should be noted, though, that in the case of exceptions the two formulas give *different* (wrong) results. This may indicate that the exceptions to Lusztig's character formula and to $P_{x,w}(q) = P^p_{x,w}(q)$ are connected.

The third direction of investigation is concerned with the representation theory of restricted Lie algebras. Here we investigate the composition coefficients of the baby Verma modules $\widehat{Z}(\lambda)$. It is known that these composition coefficients are given by the Kazhdan-Lusztig polynomials

$$\left[\widehat{Z}(x.\lambda_0) : \widehat{L}(w.\lambda_0)\right] = P_{x,w}(1)$$

for p large enough, under certain conditions on x, w and λ_0. One hopes that $p > h$ (the Coxeter number) is large enough (see conjecture 5.2.2).

For small p this does not hold. However, the exceptions seem to align perfectly with the exceptions to $P^p_{x,w} = P_{x,w}$, and the data suggests that

$$\left[\widehat{Z}(x.\lambda_0) : \widehat{L}(w.\lambda_0)\right] = P^p_{x,w}(1)$$

for all initial weights λ_0, and $x, w \in \mathcal{W}$ such that $w.\lambda_0 \in X_p - p\rho$ and $x.\lambda_0 \in -X^+ - \rho$.

To carry out these calculations, we have developed several algorithms. All the algorithms described in this thesis (and several more) have been implemented into the java project ReAlGriDPC (Representations of Algebraic Groups in Defining Prime Characteristic), which can be found on my homepage

`http://home.imf.au.dk/jb`.

Resume på dansk

I 1979 kom G. Lusztig med en formodning som beskrev karaktererne for simple moduler, med en regulær højeste vægt, for en reduktiv algebraisk gruppe G i karakteristik $p > 0$ ved hjælp af Kazhdan-Lusztig polynomier. Den dertilhørende karakterformel kan på naturlig måde generaliseres så den også kan klare vægte som íkke er regulære. Hvis p er stor nok til at der findes regulære vægte ($p \geq h$, hvor h er Coxetertallet for den underliggende affine Weylgruppe), så følger den generaliserede karakterformel fra formlen for regulære vægte.

Lusztigs formodning er blevet bevist hvis karakteristikken af det bagvedliggende legeme p er stor nok (meget stor!), men står stadig ubesvaret for små p, det vil selvfølgelig også sige at man ikke har fundet nogle undtagelser til den. Den generaliserede karakterformel, på den anden side, har man fundet mange undtagelser til.

Målet med denne afhandling er at undersøge disse undtagelser ved at prøve at finde mønstre for hvornår de opstår, og finde andre regler, som så gælder i det generaliserede tilfælde.

Reslutaterne i denne afhandling er gennem undersøgelse af modulerne for de algebraiske grupper $SL_n(k)$ med $n \in \{3, 4, 5\}$ og k et algebraisk lukket legeme med karakteristik $p \in \{2, 3\}$, at finde nogle mulige mønstre (se afsnit 3.7, 4.3.2 og 5.2.2) og finde en mulig mere generel regel (se question 5.2.6).

Meget af det bagvedliggende data kan findes i appendiks A, B og C, og algoritmerne til at udregne data kan findes implementeret i Java på min hjemmeside: `http://home.imf.au.dk/jb`.

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Chapter 1

Preliminaries

1.1 Root systems, affine Weyl groups and actions

Throughout this thesis we will use a lot of repeated notation. In this section we will briefly define this notation. Most of the results of this section can either be found or easily derived from [Hum90].

First we need an algebraic group. Let G be a simple, connected, simply connected algebraic group over an algebraically closed field k of finite characteristic $\mathrm{char}(k) = p > 0$. Let $T \subseteq G$ be a maximal torus, and n be its order, i.e. $T \cong (k^*)^n$.

1.1.1 Roots, weights and the corresponding geometry

Let $\mathbb{R}^n \supseteq \Phi \supseteq \Phi^+ \supseteq \Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the root system corresponding to G, a set of positive roots and a basis.

On $\mathbb{R}^n$ we have the usual euclidean inner product $(\cdot,\cdot)$. This leads to the form $\langle\cdot,\cdot\rangle$ defined by: $\langle\beta,\alpha\rangle = (\beta,\alpha^\vee)$, where $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$. Recall that $\langle\cdot,\cdot\rangle : \Phi\times\Phi \to \mathbb{Z}$.

Let $\Lambda = \sum_{\alpha\in\Phi} \mathbb{Z}\alpha$ be the root lattice, and $\Lambda^+ = \sum_{\alpha\in\Phi^+} \mathbb{N}\alpha$ be the positive part of the root lattice ($\mathbb{N} = \{0, 1, 2, \ldots\}$).

We define the partial relation $\preceq$ on the vectors of $\mathbb{R}^n$ by $x \preceq y \iff y - x \in \Lambda^+$. Let $\tilde{\alpha}$ be the unique greatest (according to the order $\preceq$) of the short (according to $(\cdot,\cdot)$) roots (exists because G is simple, and thus Φ irreducible).

The set $X = X(T) = \mathrm{Hom}_{alg.grp.}(T, k^*)$ of characters for T can be seen as an intermediate set $\Lambda \subseteq X \subseteq \mathbb{R}^n$ with a $\mathbb{Z}$-basis: $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying: $\langle\lambda_i, \alpha_j\rangle = \delta_{i,j}$ (because G is simply connected). Also X has the following property: $X = \{x \in \mathbb{R}^n \mid \langle x, \alpha\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$.

For V a G-module, and $\lambda \in X$ we define the weight space V_λ of V for the weight λ:

$$V_\lambda = \{v \in V \mid tv = \lambda(t)\, v \text{ for all } t \in T\}$$

If $V_\lambda \neq 0$ we say λ is a weight of V, and the vectors of V_λ will be called weight vectors. For modules of algebraic groups we have the following theorem (see [Ste74] section 3.3):

Theorem 1.1.1. *If V is a G-module, then V is a direct sum of its weight spaces:*

$$V = \bigoplus_{\lambda \in X} V_\lambda$$

In general we will refer to X as the weight lattice, and the elements of X as weights. Notice that addition when thinking of X as a lattice becomes multiplication when thinking of X as the set of homomorphisms from T to k^*.

Like we did with the root lattice, we will define the dominant weights: $X^+ = \sum_{i=1}^n \mathbb{N}\lambda_i$. The dominant weights have the following properties:

$$X^+ = \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle \in \mathbb{N}, \text{ for all } \alpha \in \Phi^+\}$$

and for all dominant weights $\lambda \in X^+$ and all positive roots $\alpha \in \Phi^+$: $\langle \lambda, \alpha \rangle \leq \langle \lambda, \tilde{\alpha} \rangle$.

The most important weight $\rho \in X^+$ we define $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. It is easily proved that $\rho = \sum_{i=1}^n \lambda_i$.

We will also be needing some affine hyperplanes. For $\alpha \in \Phi, m \in \mathbb{Z}$ we define

$$\begin{aligned} \tilde{H}_{\alpha,m} &= \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle = mp\} \\ H_{\alpha,m} = \tilde{H}_{\alpha,m} - \rho &= \{x - \rho \in \mathbb{R}^n \mid \langle x, \alpha \rangle = mp\} \\ &= \{x \in \mathbb{R}^n \mid \langle x + \rho, \alpha \rangle = mp\} \end{aligned}$$

We denote the set of these hyperplanes $\tilde{\mathscr{H}} = \left\{ \tilde{H}_{\alpha,m} \,\middle|\, \alpha \in \Phi, m \in \mathbb{Z} \right\}$ and $\mathscr{H} = \{H_{\alpha,m} \mid \alpha \in \Phi, m \in \mathbb{Z}\}$.

Another set of great importance is the set of alcoves $\mathcal{A}$, which is defined to be the set of connected components of $\mathbb{R}^n \setminus \bigcup_{h \in \mathscr{H}} H$. The initial alcove $A_0 \in \mathcal{A}$ defined by:

$$\begin{aligned} A_0 &= \left\{x \in \mathbb{R}^n \,\middle|\, 0 < \langle x + \rho, \alpha \rangle < p \text{ for all } \alpha \in \Phi^+ \right\} \\ &= \{x \in \mathbb{R}^n \mid \langle x + \rho, \alpha_i \rangle > 0, \langle x + \rho, \tilde{\alpha} \rangle < p \text{ for all } 1 \leq i \leq n\} \end{aligned}$$

It is obvious that the alcoves can each be defined as the set of points bounded by some finite set of affine hyperplanes in $\mathscr{H}$ (it will namely always be enough to have 2 hyperplanes $H_{\alpha,m}, H_{\alpha,m+1}$ for each root α). For an alcove $A \in \mathcal{A}$ we define $\mathscr{H}(A) \subseteq \mathscr{H}$ to be the minimal set of bounding hyperplanes defining A. We call $\mathscr{H}(A)$ the walls of A.

1.1.2 The affine Weyl group and its actions

Let $W = \langle s_1, s_2, \ldots, s_n \rangle$ be the Weyl group corresponding to the root system Φ, where $s_i = s_{\alpha_i}$ is the reflection in the hyperplane $\tilde{H}_{\alpha_i} = \tilde{H}_{\alpha_i,0}$ orthogonal to α_i.

In general we define $s_{\alpha,m}$ to be the reflection in $\tilde{H}_{\alpha,m}$. Let $s_0 = s_{\tilde{\alpha},1}$. We now define the affine Weyl group $\mathcal{W} = \langle s_0, s_1, s_2, \ldots s_n \rangle$. Let $S = \{s_0, s_1, \ldots, s_n\}$, then $(\mathcal{W}, S)$ is a Coxeter system.

Both on the affine Weyl group and on the Weyl group we have the standard length map $l : \mathcal{W} \to \mathbb{N}$, $l : W \to \mathbb{N}$ where the latter is just the restriction of the former.

$\mathcal{W}$ acts in a natural way on both $\mathbb{R}^n, X, \Lambda$. We now define the dot-action of $\mathcal{W}$ on $\mathbb{R}^n$. Let $w \in \mathcal{W}, x \in \mathbb{R}^n$, then:

$$w.x = w(x+\rho) - \rho$$

Now we have the following proposition:

Proposition 1.1.2. *The dot-action of $\mathcal{W}$ has the following properties:*

1. *If $\alpha \in \Phi, m \in \mathbb{Z}$ then the dot-action of $s_{\alpha,m}$ on $\mathbb{R}^n$ is the reflection in the affine hyperplan $H_{\alpha,m}$.*
2. *The closure of the initial alcove $\overline{A_0}$ is a fundamental domain for the dot-action of $\mathcal{W}$ on $\mathbb{R}^n$.*
3. *The dot-action permutes the alcoves $\mathcal{A}$ simply transitively.*
4. *The dot-action permutes the affine hyperplanes $\mathscr{H}$.*
5. *If $A = w.A_0$ for an alcove $A \in \mathcal{A}$, then the walls $\mathscr{H}(A)$ of A are $\mathscr{H}(A) = w.\mathscr{H}(A_0)$.*
6. *The dot-action is also an action on the weight lattice.*

1.2 The Weyl modules

1.2.1 Definition

Let $\mathfrak{g}_\mathbb{C}$ be the complex Lie algebra with the root system isomorphic to Φ. Then clearly the weights of the maximal toral subalgebra of $\mathfrak{g}_\mathbb{C}$ (corresponding to the root system of $\mathfrak{g}_\mathbb{C}$) correspond to X by the same isomorphism.

Choose a Chevalley basis $\{e_\alpha \mid \alpha \in \Phi\} \cup \{h_\alpha \mid \alpha \in \Delta\}$ for $\mathfrak{g}_\mathbb{C}$. Let $\mathcal{U}_\mathbb{C} = \mathcal{U}(\mathfrak{g}_\mathbb{C})$ be the enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Define $e_\alpha^{(r)} = \frac{1}{r!}e_\alpha^r$ and let $\mathcal{U}_\mathbb{Z}$ be the $\mathbb{Z}$-subalgebra of $\mathcal{U}_\mathbb{C}$ generated by $\left\{e_\alpha^{(r)} \,\middle|\, \alpha \in \Phi, r \in \mathbb{N}\right\}$. $\mathcal{U}_\mathbb{Z}$ is called the Kostant form of $\mathcal{U}_\mathbb{C}$.

For $\lambda \in X$, let $V(\lambda)_\mathbb{C}$ be the unique simple $\mathfrak{g}_\mathbb{C}$-module with the highest weight λ, called the complex Weyl module. Recall that $V(\lambda)_\mathbb{C}$ is generated by any non-zero weight vector $v_\lambda \in V(\lambda)_{\mathbb{C},\lambda}$ in the one dimensional weight space $V(\lambda)_{\mathbb{C},\lambda}$.

Let $v_\lambda \in V(\lambda)_{\mathbb{C},\lambda}$ be a non-zero weight vector in $V(\lambda)_\mathbb{C}$ of weight λ. Now define $V(\lambda)_\mathbb{Z} = \mathcal{U}_\mathbb{Z} v_\lambda \subseteq V(\lambda)_\mathbb{C}$. Notice that $V(\lambda)_\mathbb{Z}$ is up to isomorphism independent of the choice of v_λ.

$V(\lambda)_\mathbb{Z}$ is a lattice in $V(\lambda)_\mathbb{C}$, and more generally the weight spaces $V(\lambda)_{\mathbb{Z},\mu} = V(\lambda)_\mathbb{Z} \cap V(\lambda)_{\mathbb{C},\mu}$ are lattices in $V(\lambda)_{\mathbb{C},\mu}$, and thus we have $V(\lambda)_\mathbb{Z} = \bigoplus_\mu V(\lambda)_{\mathbb{Z},\mu}$ by thm. 27.1 [Hum72].

Now let $V(\lambda) = V(\lambda)_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$, this we will call a Weyl module. Likewise let $\mathcal{U}_k = \mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$.

The Weyl module $V(\lambda)$ is clearly a finite dimensional k-vector space, seeing that the $\mathbb{Z}$-basis of $V(\lambda)_{\mathbb{Z}}$ becomes a basis for $V(\lambda)$ when you tensor it with $1 \in k$.

We now have that $V(\lambda) = \bigoplus_{\mu} V(\lambda)_{\mu}$ where $V(\lambda)_{\mu} = V(\lambda)_{\mathbb{Z},\mu} \otimes_{\mathbb{Z}} k$. $V(\lambda)_{\mu}$ is the weight spaces of $V(\lambda)$ for the weight $\mu \in X$.

With a slight abuse of notation we define $v_\lambda = v_\lambda \otimes 1 \in V(\lambda)_{\lambda}$, and we define the action of $\mathcal{U}_k$ on $V(\lambda)$, by letting $u \otimes x \in \mathcal{U}_k$ act by letting $u \in \mathcal{U}_{\mathbb{Z}}$ act on the $V(\lambda)_{\mathbb{Z}}$-part, and $x \in k$ act (multiplicatively) on the k-part of $V(\lambda)_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$.

Notice that $e_\alpha^{(r)}$ sends $V(\lambda)_{\mu}$ to $V(\lambda)_{\mu+r\alpha}$, and if $E = e_{\beta_1}^{(r_1)} \cdots e_{\beta_l}^{(r_l)}$ then E maps $V(\lambda)_{\mu}$ to $V(\lambda)_{\mu+\gamma(E)}$ with $\gamma(E) = \sum_{i=1}^{l} r_i \beta_i \in \Lambda$, we call $\gamma(E)$ the total weight of E.

Inherited from $\mathcal{U}_{\mathbb{C}}$ we have the following proposition about the relations in $\mathcal{U}_k$ and its action on $V(\lambda)$:

Proposition 1.2.1 (The Standard Relations)**.**

$$e_\alpha e_\beta = \begin{cases} e_\beta e_\alpha + N_{\alpha,\beta} e_{\alpha+\beta} & \text{for } \alpha \neq -\beta \\ e_\beta e_\alpha + h_\alpha & \text{for } \alpha = -\beta \end{cases}$$

$$h_\alpha e_\beta = e_\beta h_\alpha + \langle \beta, \alpha \rangle e_\beta$$

$$h_\alpha v_\mu = \langle \mu, \alpha \rangle v_\mu$$

where $N_{\alpha,\beta}$ is some integer satisfying $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Phi$, and $v_\mu \in V(\lambda)_{\mu}$ is any weight vector for the weight μ. Furthermore,

$$e_\alpha v_\lambda = 0$$

if $\alpha \in \Phi^+$ is a positive root.

1.2.2 The Weyl module as a G-module

The reason for calling $V(\lambda)$ the Weyl module is of course that it is a G-module:

Theorem 1.2.2. *Let $\lambda \in X^+$, then $V(\lambda)$ is a G-module. Furthermore a subspace $M \subseteq V(\lambda)$ is a G-submodule of $V(\lambda)$ if and only if M is an $\mathcal{U}_k$-submodule.*

We will sketch the proof of this theorem:

Sketch of proof. To shorten the notation define $V = V(\lambda)$. For $\alpha \in \Phi$ and $t \in k$ we define:

$$x_{\alpha,V}(t) = \sum_{r \geq 0} t^r e_\alpha^{(r)} \in \mathrm{End}(V)$$

Notice that the sum is finite (and thus makes sense) because there are only finitely many weights of V, and $e_\alpha^{(r)}, e_\alpha^{(s)}$ map a weight vector to different weight spaces for $r \neq s$ (recall V is a direct sum of its weight spaces).

For a given $\alpha \in \Phi$, and $t, u \in k$ we observe:

$$\begin{aligned} x_{\alpha,V}(t)\, x_{\alpha,V}(u) &= \sum_{r\geq 0} t^r e_\alpha^{(r)} \sum_{s\geq 0} u^s e_\alpha^{(r)} \\ &= \sum_{r,s\geq 0} t^r u^s \binom{r+s}{r} e_\alpha^{(r+s)} \\ &= \sum_{l\geq 0} e_\alpha^{(l)} \sum_{r=0}^{l} t^r u^{l-r} \binom{l}{r} \\ &= \sum_{l\geq 0} e_\alpha^{(l)} (t+u)^l = x_{\alpha,V}(t+u) \end{aligned}$$

which follows directly from the binomial formula.

Because $x_{\alpha,V}(0)$ is the identity on V we get that $x_{\alpha,V}(t) \in GL(V)$, and the set $\{x_{\alpha,V}(t) \,|\, t \in k\}$ (for constant $\alpha \in \Phi$) is a group isomorphic to $(k, +)$.

Now let $G_V = \langle x_{\alpha,V}(t) \mid \alpha \in \Phi, t \in k\rangle \subseteq GL(V)$. It can now be proved that G_V is a connected and simple algebraic group with root system Φ, which by the classification theorem 32.1 of [Hum95], gives us a representation $G \to G_V \hookrightarrow GL(V(\lambda))$, which makes $V(\lambda)$ a G-module.

To see the second part of the theorem, we start by noticing that the above shows that M is a G-module if and only if M is a G_V-module. This is the same as $x_{\alpha,V}(t)\, M = M$ for all $\alpha \in \Phi$ and all $t \in k$. So what it all boils down to is to prove that for each $\alpha \in \Phi$ we have:

$$\forall t \in k : x_{\alpha,V}(t)\, M = M \iff \forall r \geq 0 : e_\alpha^{(r)} M \subseteq M$$

The direction $\Leftarrow$ is trivial since $e_\alpha^{(r)} M \subseteq M$ for all $r \geq 0$ gives us both that $x_{\alpha,V}(t)\, M \subseteq M$ and $x_{\alpha,V}(t)^{-1} M = x_{\alpha,V}(-t)\, M \subseteq M$ which together gives us the left side.

$\Rightarrow$) Let $N \in \mathbb{N}$ be a value such that $e_\alpha^{(r)} = 0 \in \operatorname{End}(V)$ for all $r \geq N$. Now choose N different values $t_1, \ldots, t_N \in k$ (this is possible, since k is algebraically closed, and thus infinite). Since the t_i's are all different, we get that the Vandermonde matrix $A = \left(t_i^{j-1}\right)_{1\leq i,j\leq N}$ is invertible, so let $B = A^{-1} \in GL\left(k^N\right)$ be the inverse.

If we look at the vector $u = \left(e_\alpha^{(0)}, e_\alpha^{(1)}, \ldots, e_\alpha^{(N-1)}\right)^T$ we immediately see that $(Au)_i = x_{\alpha,V}(t_i)$, and thus (for $0 \leq r < N$):

$$e_\alpha^{(r)} M = \sum_{j=1}^{N} B_{r+1,j}\, x_{\alpha,V}(t_j) M \subseteq M$$

This concludes the sketch of the proof. □

Because of the above theorem, we will now refer to both G-submodules and $\mathcal{U}_k$-submodules of $V(\lambda)$ simply as submodules of $V(\lambda)$.

Notice that the $V(\lambda)$ we have constructed here coincides with the $V(\lambda)$ defined (differently) in [Jan03] (see II.8.3(3) in [Jan03]).

Furthermore we notice that in the first part of the proof the image of T in G_V is a torus T_V and that the "weight space" for $\mu \in X$ of $V(\lambda)$ from the definition of $V(\lambda)$ (derived from the weight space of $V(\lambda)_{\mathbb{C}}$) corresponds to the weight space of $\mu_V \in X_V = \mathrm{Hom}_{\text{alg.grp}}(T_V, k^*)$, from the definition of weight space, where μ_V comes from $\mu \in X$ under the isomorphism $X \cong X_V$. Thus we have that the two definitions of weight spaces are equivalent. And our abuse of notation is justified.

1.2.3 The case A

In this thesis we will especially focus on the case where the root system Φ is of type A, that is to say that $G = SL_{n+1}(k)$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_{n+1}(\mathbb{C})$.

So let V be k^{n+1} with the standard basis $\epsilon_1, \ldots, \epsilon_{n+1}$, and identify $\Phi \subseteq V$ by $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Notice that then $\Delta = \{\epsilon_i - \epsilon_{i+1} \,|\, 1 \leq i \leq n\}$, $\Phi^+ = \{\epsilon_i - \epsilon_j \,|\, i < j\}$ and $\Phi = \{\epsilon_i - \epsilon_j \,|\, i \neq j\}$.

Now we identify $e_{\epsilon_i - \epsilon_j}$ with the matrix $E_{i,j} \in \mathfrak{sl}_{n+1}(\mathbb{C})$ with 1 on the i,j'th entry and 0 on all other entries.

It is easily proved that $SL_n(k)$ is generated by transvections (matrices of the type $I + tE_{i,j}$ with $t \in k$), and thus it is enough to define the action of these on the module $V(\lambda)$. So a matrix $I + tE_{i,j} \in G = SL(V)$ acts by $x_{\epsilon_i - \epsilon_j, V(\lambda)}(t) = \sum_{n \geq 0} t^n e^{(n)}_{\epsilon_i - \epsilon_j}$ on $V(\lambda)$.

One of the reasons for working in the case A is that the values $N_{\alpha,\beta}$ (from the standard relations) are very easy to calculate in this case:

Lemma 1.2.3. *Let $\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l \in \Phi$ be different roots, then $N_{\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l} = \delta_{j,k} - \delta_{i,l}$.*

Proof. The lemma follows trivially from the following calculation:

$$[E_{i,j}, E_{k,l}] = E_{i,j}E_{k,l} - E_{k,l}E_{i,j} = \delta_{j,k}E_{i,l} - \delta_{i,l}E_{k,j}$$

□

1.2.4 More and less complex calculations in $V(\lambda)$

We would like to do calculations in $V(\lambda)_{\mathbb{Z}}$ and thus in $V(\lambda)$ without always consulting $V(\lambda)_{\mathbb{C}}$. To do that we set up a series of rules for calculation in $\mathcal{U}_{\mathbb{Z}}, \mathcal{U}_k$ and $V(\lambda)_{\mathbb{Z}}, V(\lambda)$:

Theorem 1.2.4. *We have the following rules*

$$e_\alpha^{(r)} e_\alpha^{(s)} = \binom{r+s}{r} e_\alpha^{(r+s)} \tag{1.1}$$
$$h_\alpha e_\beta^{(r)} = e_\beta^{(r)} h_\alpha + r \langle \beta, \alpha \rangle e_\beta^{(r)} \tag{1.2}$$
$$h_\alpha E v_\lambda = \langle \lambda + \gamma(E), \alpha \rangle E v_\lambda \tag{1.3}$$
$$e_\alpha e_{-\alpha}^{(r)} = e_{-\alpha}^{(r)} e_\alpha + e_{-\alpha}^{(r-1)} h_\alpha - (r-1) e_{-\alpha}^{(r-1)} \tag{1.4}$$
$$e_\alpha^{(r)} e_{-\alpha}^{(s)} E v_\lambda = \sum_{i=0}^{\min(r,s)} \binom{\langle \lambda + \gamma(E), \alpha \rangle + r - s}{i} e_{-\alpha}^{(s-i)} e_\alpha^{(r-i)} E v_\lambda \tag{1.5}$$

where E is a product of elements of $\mathcal{U}_\mathbb{Z}$ of the type $e_\alpha^{(r)}$ (the generating elements of $\mathcal{U}_\mathbb{Z}$).

Proof. We will prove that the equalities holds in $V(\lambda)_\mathbb{C}$ and $\mathcal{U}_\mathbb{C}$ and thus holds in $V(\lambda)_\mathbb{Z}, V(\lambda)$ and $\mathcal{U}_\mathbb{Z}, \mathcal{U}_k$.

1. We prove this part by straight forward calculation:

$$e_\alpha^{(r)} e_\alpha^{(s)} = \frac{1}{r!s!} e_\alpha^{r+s} = \frac{(r+s)!}{r!s!} e_\alpha^{(r+s)} = \binom{r+s}{r} e_\alpha^{(r+s)}$$

   done!

2. We prove this by induction. Notice that the induction start $(r = 1)$ follows directly from the standard relation. Assume the statement to be true for $r - 1$:

$$\begin{aligned}
h_\alpha e_\beta^{(r)} &= \frac{1}{r} h_\alpha e_\beta e_\beta^{(r-1)} \\
&= \frac{1}{r} \left(e_\beta h_\alpha e_\beta^{(r-1)} + \langle \beta, \alpha \rangle e_\beta e_\beta^{(r-1)} \right) \\
&= \langle \beta, \alpha \rangle e_\beta^{(r)} + \frac{1}{r} e_\beta \left(h_\alpha e_\beta^{(r-1)} \right) \\
&= \langle \beta, \alpha \rangle e_\beta^{(r)} + \frac{1}{r} e_\beta \left(e_\beta^{(r-1)} h_\alpha + (r-1) \langle \beta, \alpha \rangle e_\beta^{(r-1)} \right) \\
&= \langle \beta, \alpha \rangle e_\beta^{(r)} + e_\beta^{(r)} h_\alpha + (r-1) \langle \beta, \alpha \rangle e_\beta^{(r)} \\
&= e_\beta^{(r)} h_\alpha + r \langle \beta, \alpha \rangle e_\beta^{(r)}
\end{aligned}$$

   Which was what we wanted.

3. Let $E = e_{\beta_1}^{(r_1)} \cdots e_{\beta_l}^{(r_l)}$ be any product of the generators of $\mathcal{U}_\mathbb{Z}$. We do induction in l. For $l = 0$ it follows from the standard relation. Assume it to be ok for

$l-1$.

$$\begin{aligned}
&h_\alpha e_{\beta_1}^{(r_1)} e_{\beta_2}^{(r_2)} \dots e_{\beta_l}^{(r_l)} v_\lambda \\
&= e_{\beta_1}^{(r_1)} h_\alpha e_{\beta_2}^{(r_2)} \dots e_{\beta_l}^{(r_l)} v_\lambda + r_1 \langle \beta_1, \alpha \rangle e_{\beta_1}^{(r_1)} e_{\beta_2}^{(r_2)} \dots e_{\beta_l}^{(r_l)} v_\lambda \\
&= \left\langle \lambda + \sum_{i=2}^{l} r_i \beta_i, \alpha \right\rangle e_{\beta_1}^{(r_1)} e_{\beta_2}^{(r_2)} \dots e_{\beta_l}^{(r_l)} v_\lambda + \langle r_1 \beta_1, \alpha \rangle e_{\beta_1}^{(r_1)} e_{\beta_2}^{(r_2)} \dots e_{\beta_l}^{(r_l)} v_\lambda \\
&= \left\langle \lambda + \sum_{i=1}^{l} r_i \beta_i, \alpha \right\rangle e_{\beta_1}^{(r_1)} e_{\beta_2}^{(r_2)} \dots e_{\beta_l}^{(r_l)} v_\lambda \\
&= \langle \lambda + \gamma(E), \alpha \rangle E v_\lambda
\end{aligned}$$

4. Again we go forward by induction. And again we notice that the induction start follows directly from the standard relation. Assume the statement to be true for $r-1$:

$$\begin{aligned}
e_\alpha e_{-\alpha}^{(r)} &= \frac{1}{r} e_\alpha e_{-\alpha} e_{-\alpha}^{(r-1)} \\
&= \frac{1}{r} \left(e_{-\alpha} e_\alpha e_{-\alpha}^{(r-1)} + h_\alpha e_{-\alpha}^{(r-1)} \right)
\end{aligned}$$

We now use both the induction hypothesis and part 2 of the theorem:

$$\begin{aligned}
e_\alpha e_{-\alpha}^{(r)} &= \frac{1}{r} \left(e_{-\alpha} e_\alpha e_{-\alpha}^{(r-1)} + h_\alpha e_{-\alpha}^{(r-1)} \right) \\
&= \frac{1}{r} e_{-\alpha} \left(e_{-\alpha}^{(r-1)} e_\alpha + e_{-\alpha}^{(r-2)} h_\alpha - (r-2) e_{-\alpha}^{(r-2)} \right) \\
&\quad + \frac{1}{r} \left(e_{-\alpha}^{(r-1)} h_\alpha + (r-1) \langle -\alpha, \alpha \rangle e_{-\alpha}^{(r-1)} \right) \\
&= e_{-\alpha}^{(r)} e_\alpha + \frac{r-1}{r} e_{-\alpha}^{(r-1)} h_\alpha - \frac{(r-1)(r-2)}{r} e_{-\alpha}^{(r-1)} \\
&\quad + \frac{1}{r} e_{-\alpha}^{(r-1)} h_\alpha - \frac{2(r-1)}{r} e_{-\alpha}^{(r-1)} \\
&= e_{-\alpha}^{(r)} e_\alpha + e_{-\alpha}^{(r-1)} h_\alpha - (r-1) e_{-\alpha}^{(r-1)}
\end{aligned}$$

which again was what we wanted.

5. We use induction in r. For $r=0$ it is trivial, so assume it to be true for

$r-1$. For a shorter notation let $<>=\langle\lambda+\gamma(E),\alpha\rangle$:

$$
\begin{aligned}
&e_{\alpha}^{(r)}e_{-\alpha}^{(s)}Ev_{\lambda}\\
&=\frac{1}{r}e_{\alpha}\sum_{i=0}^{\min(r-1,s)}\binom{<>+r-1-s}{i}e_{-\alpha}^{(s-i)}e_{\alpha}^{(r-1-i)}Ev_{\lambda}\\
&=\sum_{i=0}^{\min(r-1,s)}\frac{r-i}{r}\binom{<>+r-1-s}{i}e_{-\alpha}^{(s-i)}e_{\alpha}^{(r-i)}Ev_{\lambda}\\
&+\sum_{i=0}^{\min(r-1,s)}\frac{1}{r}\binom{<>+r-1-s}{i}e_{-\alpha}^{(s-1-i)}h_{\alpha}e_{\alpha}^{(r-1-i)}Ev_{\lambda}\\
&+\sum_{i=0}^{\min(r-1,s)}\frac{1+i-s}{r}\binom{<>+r-1-s}{i}e_{-\alpha}^{(s-1-i)}e_{\alpha}^{(r-1-i)}Ev_{\lambda}\\
&=\sum_{i=0}^{\min(r-1,s)}\frac{r-i}{r}\binom{<>+r-1-s}{i}e_{-\alpha}^{(s-i)}e_{\alpha}^{(r-i)}Ev_{\lambda}\\
&+\sum_{j=1}^{\min(r-1,s)+1}\frac{\langle\lambda+\gamma(E)+(r-j)\alpha,\alpha\rangle}{r}\binom{<>+r-1-s}{j-1}e_{-\alpha}^{(s-j)}e_{\alpha}^{(r-j)}Ev_{\lambda}\\
&+\sum_{j=1}^{\min(r-1,s)+1}\frac{j-s}{r}\binom{<>+r-1-s}{j-1}e_{-\alpha}^{(s-j)}e_{\alpha}^{(r-j)}Ev_{\lambda}
\end{aligned}
$$

using the convention that $e_{\alpha}^{(-1)}=0$. We now calculate the values of the coefficient c_i of $e_{-\alpha}^{(s-i)}e_{\alpha}^{(r-i)}Ev_{\lambda}$ for $1\leq i\leq\min(r-1,s)$ (using the notation

$[x]_i = x(x-1)\cdots(x-i+1)$):

$$
\begin{aligned}
c_i &= \frac{(r-i)\,[<> +r-1-s]_i}{ri!} \\
&+ \frac{(\langle \lambda + \gamma(E) + (r-i)\,\alpha, \alpha\rangle + i - s)\,[<> +r-1-s]_{i-1}}{r\,(i-1)!} \\
&= \frac{(<> +r-s-i)\,(r-i)\,[<> +r-1-s]_{i-1}}{ri!} \\
&+ \frac{i\,(<> +2r-s-i)\,[<> +r-1-s]_{i-1}}{ri!} \\
&= \frac{((<> +r-s-i)\,(r-i) + i\,(<> +2r-s-i))\,[<> +r-1-s]_{i-1}}{ri!} \\
&= \frac{(r <> +r^2 - sr - ir + ir)\,[<> +r-1-s]_{i-1}}{ri!} \\
&= \frac{r\,(<> +r-s)\,[<> +r-1-s]_{i-1}}{ri!} \\
&= \frac{[<> +r-s]_i}{i!} \\
&= \binom{<> +r-s}{i}
\end{aligned}
$$

So we have the correct coefficients for $1 \leq i \leq \min(r-1, s)$. We now look at the coefficient c_0 for $e_{-\alpha}^{(s)} e_{\alpha}^{(r)} E v_\lambda$

$$c_0 = \frac{r-0}{r}\binom{<> +r-1-s}{0} = 1 = \binom{<> +r-s}{0}$$

So also here we have the correct coefficient. Now we have to divide into two different possibilities: 1) $min(r,s) \neq r$ and 2) $min(r,s) = r$. Luckily we have already handled case 1) if we just notice that this case leads to $\min(r-1,s) = min(r,s) = s$, which means that the last summand for $j = \min(r-1,s)+1 = s+1$ yields 0 because $e_{-\alpha}^{(s-(n+1))} = 0$. So we only need to calculate $c_{\min(r,s)}$ in case 2. In this case we have that $r = \min(r,s)$ and $r-1 = \min(r-1,s)$.

$$
\begin{aligned}
c_{\min(r,s)} &= c_r \\
&= \frac{(\langle \lambda + \gamma(E) + (r-r)\,\alpha, \alpha\rangle) + (r-s)}{r}\binom{<> +r-1-s}{r-1} \\
&= \frac{<> +r-s}{r}\binom{<> +r-1-s}{r-1} = \binom{<> +r-s}{r}
\end{aligned}
$$

which was exactly what we wanted, so we have obtained the correct formula.

□

1.3 An algorithm for the cases A, D and E

In the preceding section we described the Weyl modules and some rules for calculating with their elements. Now we want to be able to put any element of $V(\lambda)$ into a standard form. To do this the rules of the last section are not enough, we also need some rules involving $e_\alpha^{(r)}, e_\beta^{(s)}$ where $\alpha \neq \pm\beta$. These rules will involve $N_{\alpha,\beta}$ which are very dependent on which type of root system we are working in.

In this section we will make a concrete algorithm for converting an element of $V(\lambda)$ to a standard form when working in root systems of type A, D or E.

1.3.1 A few extra rules of calculation

We need some more rules of calculation:

Theorem 1.3.1. *If our root system is of type A, D or E and $\alpha \neq \pm\beta$ we have:*

$$e_\alpha e_\beta^{(s)} = e_\beta^{(s)} e_\alpha + N_{\alpha,\beta} e_\beta^{(s-1)} e_{\alpha+\beta} \tag{1.6}$$

$$e_\alpha^{(r)} e_\beta^{(s)} = \sum_{i=0}^{\min(r,s)} N_{\alpha,\beta}^i e_\beta^{(s-i)} e_{\alpha+\beta}^{(i)} e_\alpha^{(r-i)} \tag{1.7}$$

Proof. Let us start by assuming that $\langle \alpha, \beta \rangle < 0$, since otherwise $\alpha + \beta \notin \Phi$ and then e_α and e_β commutes, making the two statements trivial.

Now when in type A, D or E this only leaves one possibility, namely that $\langle \alpha, \beta \rangle = -1$. By [Hum72] Proposition 8.4(e) this eliminates the possibility that $\alpha + 2\beta$ or $2\alpha + \beta$ is a root, and thus we have that both $e_\alpha^{(r)}$ and $e_\beta^{(s)}$ commutes with $e_{\alpha+\beta}^{(t)}$. This is the fact we will be using to prove the two identities.

In this proof we will, like in the proof of Theorem 1.2.4, do the calculations in $\mathcal{U}_\mathbb{C}$ and then transfer them into $\mathcal{U}_\mathbb{Z}$ and $\mathcal{U}_k$.

6. When $s = 1$ the statement follows from the standard relations. Notice that if we use the convention $e_\alpha^{(-1)} = 0$ it also holds for $s = 0$. We proceed by induction in s, so assume it to hold for $s - 1$, then:

$$\begin{aligned}
e_\alpha e_\beta^{(s)} &= \frac{1}{s} e_\alpha e_\beta e_\beta^{(s-1)} \\
&= \frac{1}{s} \left(e_\beta e_\alpha e_\beta^{(s-1)} + N_{\alpha,\beta} e_{\alpha+\beta} e_\beta^{(s-1)} \right) \\
&= \frac{1}{s} \left(e_\beta e_\beta^{(s-1)} e_\alpha + N_{\alpha,\beta} e_\beta e_\beta^{(s-2)} e_{\alpha+\beta} + N_{\alpha,\beta} e_{\alpha+\beta} e_\beta^{(s-1)} \right) \\
&= e_\beta^{(s)} e_\alpha + \left(\frac{s-1}{s} + \frac{1}{s} \right) N_{\alpha,\beta} e_\beta^{(s-1)} e_{\alpha+\beta} \\
&= e_\beta^{(s)} e_\alpha + N_{\alpha,\beta} e_\beta^{(s-1)} e_{\alpha+\beta}
\end{aligned}$$

Which is what we wanted.

7. When $r = 0$ the statement is trivial. So we proceed by induction and assume that it is true for $r - 1$:

$$\begin{aligned}
e_\alpha^{(r)} e_\beta^{(s)} &= \frac{1}{r} e_\alpha^{(r-1)} e_\alpha e_\beta^{(s)} \\
&= \frac{1}{r} e_\alpha^{(r-1)} \left(e_\beta^{(s)} e_\alpha + N_{\alpha,\beta} e_\beta^{(s-1)} e_{\alpha+\beta} \right) \\
&= \frac{1}{r} \sum_{i=0}^{\min(r-1,s)} N_{\alpha,\beta}^i e_\beta^{(s-i)} e_{\alpha+\beta}^{(i)} e_\alpha^{(r-1-i)} e_\alpha \\
&\quad + \sum_{i=0}^{\min(r-1,s-1)} N_{\alpha,\beta}^{i+1} e_\beta^{(s-1-i)} e_{\alpha+\beta}^{(i)} e_\alpha^{(r-1-i)} e_{\alpha+\beta} \\
&= \sum_{i=0}^{\min(r-1,s)} \frac{r-i}{r} N_{\alpha,\beta}^i e_\beta^{(s-i)} e_{\alpha+\beta}^{(i)} e_\alpha^{(r-i)} \\
&\quad + \sum_{j=1}^{\min(r,s)} \frac{j}{r} N_{\alpha,\beta}^j e_\beta^{(s-j)} e_{\alpha+\beta}^{(j)} e_\alpha^{(r-j)}
\end{aligned}$$

It is now clear that all the coefficients c_i of $e_{\alpha+\beta}^{(i)} e_\alpha^{(r-i)}$ are the correct values $c_i = N_{\alpha,\beta}^i$ for $0 \le i \le \min(r-1, s) \le \min(r, s)$, so now we only need to take care of the coefficient $c_{\min(r,s)}$ in the case where $\min(r, s) \neq \min(r-1, s)$. In this case we must have that $\min(r, s) = r$ and thus $c_{\min(r,s)} = c_r = \frac{r}{r} N_{\alpha,\beta}^r = N_{\alpha,\beta}^r$, which indeed is the value desired.

We hereby conclude the proof. □

1.3.2 The algorithm

Firstly we will have to decide on a standard form. This is done by choosing an ordering $\beta_1 \lneqq \beta_2 \lneqq \cdots \lneqq \beta_m$ of the negative roots $\Phi^- = \{\beta_1, \ldots, \beta_m\}$. Now our standard form of writing the elements of $V(\lambda)$ is as a linear combination of terms of the form $e_{\beta_1}^{(r_1)} \cdots e_{\beta_m}^{(r_m)} v_\lambda$, where $r_1, \ldots, r_m \in \mathbb{N}$.

We can now easily describe our algorithm. The elements of $V(\lambda)$ are represented by a linear combination of monomials of $e_\alpha^{(r)}$'s and $h_\alpha^{(r)}$'s, applied to v_λ.

The algorithm then works through each monomial individually (possibly making more monomials in the process), using the rules (1.1), (1.3), (1.5) and (1.7) plus the fact that $e_\alpha^{(r)} v_\lambda = 0$ for any positive root $\alpha \in \Phi^+$ and any positive integer $r > 0$.

More concretely we can start by eliminating the h_α's from the right by using (1.3), having done that we can eliminate the $e_\alpha^{(r)}$ with $\alpha \in \Phi^+$ positive by using (1.5) and (1.7) to move it further to the right and finally eliminating them if they reach the right end of the string, see the standard relations proposition 1.2.1.

Having done that, what remains is a linear combination of strings of $e_\alpha^{(r)}$ with $\alpha \in \Phi^-$ negative, so all we need to do is get the correct order of the elements.

This is done easily by using by using (1.1) and (1.7) on any pair not in the correct order. Here, however it is not so obvious that this process will terminate seeing that (1.7) might add an extra element to some of the strings. This problem is solved by Lemma 26.3C in [Hum72], which makes sure that the terms with the extra elements do not give any problems.

It is possible to make some more complicated but roughly similar rules for the rest of the types of root systems, and also make a similar algorithm, observe for example Kostant's theorem (Theorem 26.4 in [Hum72]). Notice that this yields that there must be some finite subset of the monomials on the standard form that forms a basis for $V(\lambda)$. Sadly the question of which subsets makes a basis is difficult. This question we will return to later.

1.4 The simple modules

The Weyl modules are, though they derive from simple modules of $\mathfrak{g}_{\mathbb{C}}$, in general not simple. But they are however related to some simple modules.

Proposition 1.4.1. *$V(\lambda)$ has a unique maximal proper submodule $V(\lambda)_{\max}$.*

Proof. Let $M \subsetneq V(\lambda)$ be a proper submodule of $V(\lambda)$. By Theorem 1.1.1 have that M is the direct sum of its weight spaces $M_\mu = M \cap V(\lambda)_\mu$.

Since $M \neq V(\lambda)$, $\dim(V(\lambda)_\lambda) = 1$ and $V(\lambda) = \mathcal{U}_k v_\lambda$, we get that $M_\lambda = 0$. That is to say:

$$M \subseteq \bigoplus_{\mu \prec \lambda} V(\lambda)_\mu$$

Now let:

$$V(\lambda)_{\max} = \sum_{M \subsetneq V(\lambda) \text{ submodule}} M$$

Then clearly $V(\lambda)_{\max} \subseteq V(\lambda)$ is a submodule of $V(\lambda)$ and it is proper because $V(\lambda)_{\max,\lambda} = V(\lambda)_{\max} \cap V(\lambda)_\lambda = 0$. Furthermore $V(\lambda)_{\max}$ contains all proper submodules of $V(\lambda)$, and thus it is a unique maximal proper submodule of $V(\lambda)$. □

Since $V(\lambda)_{\max} \subsetneq V(\lambda)$ is the unique maximal submodule we can define the simple module $L(\lambda)$:

$$L(\lambda) = V(\lambda) / V(\lambda)_{\max}$$

We have that $L(\lambda) = \bigoplus_{\mu \preceq \lambda} L(\lambda)_\mu$ is a direct sum of weight spaces $L(\lambda)_\mu = V(\lambda)_\mu \big/ V(\lambda)_{\max,\mu}$. This also gives us that $\dim(L(\lambda)_\lambda) = 1$. And now we can conclude from Theorem 31.3 of [Hum95]:

Theorem 1.4.2. *Let L be a simple G-module, then there exists a $\lambda \in X^+$ such that $L \cong L(\lambda)$.*

1.5 Characters

Since the characters of the modules are very central in this thesis we start by defining them. Let $\{e(\lambda) \mid \lambda \in X\} \subseteq \mathbb{Z}[X]$ be a basis for the group ring $\mathbb{Z}[X]$ where $e(\lambda)e(\mu) = e(\lambda+\mu)$.

Definition 1.5.1. The *formal character* (or just the *character*) of a G-module M (and thus a representation) is the element $\mathrm{ch}(M) \in \mathbb{Z}[X]$ given by:

$$\mathrm{ch}(M) = \sum_{\mu \in X} \dim(M_\mu)\, e(\mu)$$

Some well known facts about characters are:

Proposition 1.5.2. *Let M, N, V be G-modules then:*
If $0 \to M \to V \to N \to 0$ is exact, then $\mathrm{ch}(V) = \mathrm{ch}(M) + \mathrm{ch}(N)$. In particular $\mathrm{ch}(M \oplus N) = \mathrm{ch}(M) + \mathrm{ch}(N)$. Also $\mathrm{ch}(M \otimes N) = \mathrm{ch}(M)\,\mathrm{ch}(N)$.

Proposition 1.5.3. *Let M be a finite dimensional G-module, then the character of M is a sum of simple characters (= characters of simple G-modules). Furthermore the simple characters are linearly independent.*

Proof. Let $M = M^0 \supsetneq M^1 \supsetneq \cdots \supsetneq M^m = 0$ be a composition series of M with $M^{i+1} \subsetneq M^i$ a maximal submodule.

If $m = 1$ then M must already be simple, so in that case we are done. Proceed by induction in m, and assume it to be true for $m-1$. We start by using the induction on M^1 to write $\mathrm{ch}(M^1)$ as a sum of characters of simple modules.

Since $M^1 \subseteq M$ is a submodule we have a short exact sequence: $0 \to M^1 \to M \to M/M^1 \to 0$. Now by Proposition 1.5.2 $\mathrm{ch}(M) = \mathrm{ch}(M^1) + \mathrm{ch}(M/M^1)$. This proves the statement, since M^1 is maximal in $M^0 = M$ and thus M/M^1 must be simple.

Now to prove that the simple characters are linearly independent, we observe that by Theorem 1.4.2 the simple characters are just $\mathrm{ch}(L(\lambda))$ for $\lambda \in X^+$. So let $(c_\lambda)_{\lambda \in X^+} \subseteq \mathbb{Z}$ (we could just as well take $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) with almost all $c_\lambda = 0$, such that:

$$\sum_{\lambda \in X^+} c_\lambda \mathrm{ch}(L(\lambda)) = 0$$

assume that not all c_λ are 0, and choose $\lambda \in X^+$ a maximal element in the $\preceq$-order such that $c_\lambda \neq 0$. Since λ is the highest weight of $L(\lambda)$ we get that the coefficient of $e(\lambda)$ on the left side of the above identity is $c_\lambda \neq 0$ where on the right side it is clearly 0. So clearly all $c_\lambda = 0$ and we have linear independence. □

The above proposition tells us every character is a unique sum of simple characters, thus we can define:

Definition 1.5.4. Let M be a G-module. We define the *composition coefficients* $[M : L(\lambda)] \in \mathbb{N}$ for $\lambda \in X^+$ such that:

$$\operatorname{ch}(M) = \sum_{\lambda \in X^+} [M : L(\lambda)] \operatorname{ch}(L(\lambda))$$

That is to say if $M = M^0 \supseteq M^1 \supseteq \dots$ is a composition series with M^{i+1} maximal in M^i, then

$$[M : L(\lambda)] = \left|\left\{ i \,\middle|\, M^i / M^{i+1} \cong L(\lambda) \right\}\right|$$

If $[M : L(\lambda)] \neq 0$ we say that $L(\lambda)$ is a *composition factor* of M.

The composition length of M is the sum:

$$\sum_{\lambda \in X^+} [M : L(\lambda)]$$

If $M = M^0 \supsetneq M^1 \supsetneq \dots \supsetneq M^m = 0$ is a composition series like the one in the above proposition, then the composition length is m.

1.5.1 The characters of the Weyl modules and $\chi(\lambda)$

For $\lambda \in X$ we define:

$$\chi(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e(w(\lambda + \rho))}{\sum_{w \in W} (-1)^{l(w)} e(w\rho)}$$

This makes sense because $\mathbb{Z}[X]$ is an integral domain, so we can think of $\chi(\lambda)$ as an element of the fraction field – notice that the denominator is different from 0 since all the $w\rho$ are different (ρ is in the interior of the Weyl chamber). Now according to Weyl's formula, Theorem 24.3 in [Hum72], we have for $\lambda \in X^+$ that $\chi(\lambda) = \operatorname{ch}(V(\lambda))$.

Lemma 1.5.5. *Let $\lambda \in X$ and $u \in W$. Then:*

$$\chi(\lambda) = (-1)^{l(u)} \chi(u.\lambda)$$

Proof.

$$\begin{aligned}
(-1)^{l(u)} \chi(u.\lambda) &= \frac{\sum_{w \in W} (-1)^{l(w)+l(u)} e(w(u.\lambda + \rho))}{\sum_{w \in W} (-1)^{l(w)} e(w\rho)} \\
&= \frac{\sum_{w \in W} (-1)^{l(w)+l(u)} e(w(u(\lambda + \rho) - \rho + \rho))}{\sum_{w \in W} (-1)^{l(w)} e(w\rho)} \\
&= \frac{\sum_{w \in W} (-1)^{l(wu)} e(wu(\lambda + \rho))}{\sum_{w \in W} (-1)^{l(w)} e(w\rho)} \\
&= \frac{\sum_{w \in W} (-1)^{l(w)} e(w(\lambda + \rho))}{\sum_{w \in W} (-1)^{l(w)} e(w\rho)} = \chi(\lambda)
\end{aligned}$$

where the second-to-last equality comes from the fact that $Wu = W$ (W being a group). $\square$

Lemma 1.5.6. *Let* $\lambda \in (X^+ - \rho) \setminus X^+ = X \cap \left(\left(\overline{C} - \rho\right) \setminus \overline{C}\right)$, *where*

$$\overline{C} = \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle \geq 0 \text{ for } \alpha \in \Delta\}$$

is the closure of the Weyl chamber. Then $\chi(\lambda) = 0$.

Proof. Let $\lambda \in (X^+ - \rho) \setminus X^+$, and let $\Delta_\lambda = \{\alpha \in \Delta \mid \langle \lambda + \rho, \alpha \rangle = 0\}$. Our first aim is to prove that Δ_λ is non-empty. Assume otherwise, then since

$$\begin{aligned}\overline{C} - \rho &= \{x - \rho \in \mathbb{R}^n \mid \langle x, \alpha \rangle \geq 0 \text{ for } \alpha \in \Delta\} \\ &= \{x \in \mathbb{R}^n \mid \langle x + \rho, \alpha \rangle \geq 0 \text{ for } \alpha \in \Delta\}\end{aligned}$$

we must have $\lambda \in \{x \in \mathbb{R}^n \mid \langle x + \rho, \alpha \rangle \geq 1 \text{ for } \alpha \in \Delta\}$ since $\langle \lambda + \rho, \alpha \rangle \in \mathbb{Z}$ is an integer. Now write $\lambda = \sum_{i=1}^n a_i \lambda_i$ we now get:

$$a_i = \langle \lambda, \alpha_i \rangle = \langle \lambda + \rho, \alpha_i \rangle - 1 \geq 0$$

which yields $\lambda \in X^+$ which is a contradiction. Thus we can conclude that there is an $\alpha_i \in \Delta_\lambda$, and thus $\lambda = s_i.\lambda$. This gives, by Lemma 1.5.5, that $\chi(\lambda) = -\chi(\lambda)$, so $\chi(\lambda) = 0$. □

Lemma 1.5.7. *Let* $\mu \in X$. *Then there exists a dominant weight* $\lambda \in X^+$ *and a constant* $c \in \{-1, 0, 1\}$ *such that* $\chi(\mu) = c \cdot \chi(\lambda) = c \cdot \mathrm{ch}(V(\lambda))$.

Proof. We start by noticing that $\overline{C} - \rho$ is a fundamental domain of the dot-action of W on $\mathbb{R}^n$. This follows from the fact that $\overline{C}$ is a fundamental domain for the action of W on $\mathbb{R}^n$, in particular if $x \in \mathbb{R}^n$ then there exists a $y \in \overline{C}$ and a $w \in W$ such that $x + \rho = wy$. Then $y - \rho \in \overline{C} - \rho$ and $x = wy - \rho = w.(y - \rho)$.

Now let $\lambda \in X^+ - \rho$ be a weight in the same dot-orbit as μ. The lemma then follows from Lemma 1.5.5 if $\lambda \in X^+$ and from Lemma 1.5.6 with $\chi(\mu) = 0 = 0 \cdot \chi(0)$ if $\lambda \notin X^+$. □

Notice that this last lemma also proves that $\chi(\lambda) \in \mathbb{Z}[X]$ for every $\lambda \in X$.

Chapter 2

Calculating characters

2.1 The approach

Now to find the characters for $L(\lambda)$ we start out by the now well known characters $\chi(\lambda) = \mathrm{ch}V(\lambda)$.

Our first aim is to write $\mathrm{ch}V(\lambda)$ as a sum of characters of some of the simple modules (this is possible by Proposition 1.5.3).

We know that the weights μ of $V(\lambda)$ (with non-zero weight space $V(\lambda)_\mu$) all satisfy $\mu \preceq \lambda$, and thus we get:

$$\chi(\lambda) = \sum_{\mu \preceq \lambda} [V(\lambda) : L(\mu)]\, \mathrm{ch}L(\mu) \tag{2.1}$$

where $[V(\lambda) : L(\mu)] \in \mathbb{N}$ are the non-negative composition coefficients, $\mu \in X^+$ are dominant weights. Notice that we get from the construction of $L(\lambda)$ (section 1.4) that $[V(\lambda) : L(\lambda)] = 1$.

Seeing that the set of dominant weights is bounded below, we always get a finite sum and, we also see that $([V(\mu) : L(\xi)])_{\xi,\mu \preceq \lambda}$ forms an upper triangular matrix with 1 on the diagonal. This matrix we can now easily invert over the integers to get the matrix $([L(\mu) : V(\xi)])_{\xi,\mu \preceq \lambda}$ (again upper triangular with ones in the diagonal) which will finally give us a formula for the character $\mathrm{ch}L(\lambda)$:

$$\mathrm{ch}L(\lambda) = \chi(\lambda) + \sum_{\mu \prec \lambda} [L(\lambda) : V(\mu)]\, \chi(\mu) \tag{2.2}$$

where we have $[L(\lambda) : V(\mu)] \in \mathbb{Z}$ (but not necessarily non-negative).

Thus we have now reduced our problem to finding the $[V(\lambda) : L(\mu)]$.

First we want to cut down the number of $\mu \in X$ for which $[V(\lambda) : L(\mu)]$ might be non-zero:

Proposition 2.1.1. *If $\mu \in X$ for which $[V(\lambda) : L(\mu)] \neq 0$ then $\mu \preceq \lambda$ and there exists a $w \in \mathcal{W}$ such that $\mu = w.\lambda$.*

This proposition is a consequence of the linkage principle (see section II.6 in [Jan03]).

2.2 Steinberg's tensor product theorem

So far calculating the characters of all the simple modules seems like an impossible task, since there is a different one for each $\lambda \in X^+$. This problem we will solve in this section.

2.2.1 Do the Frobenius twist

Let $\varphi : k \to k$ be the Frobenius map $\varphi(x) = x^p$. For $r \in \mathbb{N}$ let $\varphi_r : k \to k$ be the composition of r Frobenius maps $\varphi_p(x) = x^{p^r}$. By freshman's dream φ_r is a field homomorphism, which is clearly injective, and since k is algebraically closed $x^{p^r} = a$ has a solution for all $a \in k$, and thus φ_r is also surjective. All in all we conclude that φ_r is a field automorphism and denote the inverse: $x^{p^{-r}} := \varphi_r^{-1}(x)$.

Let M be a finite dimensional G-module. For $r \in \mathbb{N}$ we define $M^{[r]}$ to be M as an abelian group, but as a vector space scalar multiplication $\cdot$ in $M^{[r]}$ is defined by: $a \cdot v = a^{p^{-r}} v$ for all $a \in k$ and $v \in M$. The G action on $M^{[r]}$ is defined to be the same as in M: $g \cdot v = gv$ for all $g \in G$ and $v \in M$.

Proposition 2.2.1. *$M^{[r]}$ is a G-module with the actions described above.*

Proof. Since φ_r and thus also φ_r^{-1} is a field automorphism, $M^{[r]}$ is a k-vector space. Let $v_1, \ldots, v_m$ be a basis of M. Since M is a G-module there exists functions $f_{i,j} : G \to k$, such that $f_{i,j} \in k[G]$ and for $g \in G$:

$$gv_i = \sum_{j=1}^{m} f_{j,i}(g)\, v_j$$

with this we get:

$$g \cdot v_i = gv_i = \sum_{j=1}^{m} f_{j,i}(g)\, v_j = \sum_{j=1}^{m} (f_{j,i}(g))^{p^r} \cdot v_j$$

And since $f_{i,j} \in k[G]$ then also $f_{i,j}^{p^r} \in k[G]$. Thus $M^{[r]}$ is a G-module. □

Definition 2.2.2. If M is a G-module we call $M^{[r]}$ the r'th *Frobenius twist* of M.

Proposition 2.2.3. *Let M be a finite dimensional G-module, let $r \in \mathbb{N}$. Then:*

$$ch\left(M^{[r]}\right) = \sum_{\mu \in X} \dim(M_\mu)\, e(p^r \mu) \in \mathbb{Z}[X]$$

Proof. We prove that for $\lambda \in X$ we have $M_\lambda = \left(M^{[r]}\right)_{p^r\lambda}$. Let $v \in M_\lambda$ and $t \in T$ then:

$$t \cdot v = tv = \lambda(t)\, v = \lambda(t)^{p^r} \cdot v = (p^r\lambda)(t) \cdot v$$

since the addition in X is defined to be the multiplication of the functions the elements represent. Thus we have $M_\lambda \subseteq \left(M^{[r]}\right)_{p^r\lambda}$.

Now since $M^{[r]}$ is a G-module, it is a direct sum of its weight spaces. So as a vector space,

$$\bigoplus_\lambda \left(M^{[r]}\right)_\lambda = M^{[r]} = M = \bigoplus_\mu M_\mu$$

which gives us that $M_\lambda = \left(M^{[r]}\right)_{p^r\lambda}$ (since we already have the one inclusion). Also we clearly get that if $\lambda \notin p^r X$ then $\left(M^{[r]}\right)_\lambda = 0$. Now by the definition of a formal character we get the proposition. □

2.2.2 Steinberg's tensor product theorem and characters

Any natural number $m \in \mathbb{N}$ can be written in base p (called the p-adic expansion of m), that is to say, there are unique values $m_{(0)}, m_{(1)}, \ldots, m_{(r)} \in \{0, 1, \ldots, p-1\}$, with $r \in \mathbb{N}$ the maximal number such that $p^r \leq m$, such that $m = \sum_{i=0}^r m_{(i)} p^i$. This procedure can also be done for a vector $v = \sum_{i=1}^l a_i x_i, a_i \in \mathbb{N}$ in a grid, $\sum_{i=1}^l \mathrm{span}_\mathbb{Z} \{x_1, \ldots, x_l\}$ by letting $v_{(j)} = \sum_{i=1}^l a_{i,(j)} x_i$.

Definition 2.2.4. We define the *fundamental box* X_p:

$$\begin{aligned} X_p &= \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha_i \rangle < p \text{ for } 1 \leq i \leq n\} \\ &= \left\{ \sum_{i=1}^n a_i \lambda_i \,\middle|\, 0 \leq a_i < p \text{ for } 1 \leq i \leq n \right\} \end{aligned}$$

Now we see that if $\lambda \in X^+$ there exists unique $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(r)} \in X_p$ (for some $r \in \mathbb{N}$), such that $\lambda = \sum_{i=0}^r \lambda_{(i)} p^i$.

We can now state Steinberg's tensor product theorem (for a proof see [Jan03] subsections II.3.16-II.3.17).

Theorem 2.2.5 (Steinberg's tensor product theorem)**.** *Let $\lambda \in X^+$ be a dominant weight and let $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(r)} \in X_p$ such that $\lambda = \sum_{i=0}^r p^i \lambda_{(i)}$. Then:*

$$L(\lambda) \cong L\left(\lambda_{(0)}\right) \otimes L\left(\lambda_{(1)}\right)^{[1]} \otimes \cdots \otimes L\left(\lambda_{(r)}\right)^{[r]}$$

Now from Proposition 2.2.3 we know how to calculate $\mathrm{ch}\left(L(\lambda)^{[r]}\right)$ just knowing how $\mathrm{ch}(L(\lambda))$ looks in $\mathbb{Z}[X]$. So now we can (using Proposition 1.5.2) calculate the character $\mathrm{ch}(L(\lambda))$ of $L(\lambda)$ for any $\lambda \in X^+$, just knowing the characters of $L(\lambda)$ for $\lambda \in X_p$ which is a finite set! Thus we have greatly reduced the problem of finding all characters of the simple finite G-modules.

2.3 Translation functors

This section mostly follows section II.7 of [Jan03].

2.3.1 Facets and stabilizers

In this subsection we will introduce a lot of geometry, that we will use to facilitate the process of finding the $[V(\lambda) : L(\mu)]$.

Definition 2.3.1. A *facet* is defined by a division of the positive roots Φ^+ into a disjoint union $\Phi^+ = \Phi_0^+ \dot\cup \Phi_1^+$, and a tuple of integers $(n_\alpha)_{\alpha\in\Phi^+}$ indexed by the positive roots. The *facet* $F \subseteq \mathbb{R}^n$ corresponding to these parameters is defined to be:

$$\begin{aligned} F = \big\{ x \in \mathbb{R}^n \big| \, &\langle x+\rho, \alpha\rangle = n_\alpha p \text{ for } \alpha \in \Phi_0^+, \\ &(n_\alpha - 1)\, p < \langle x+\rho, \alpha\rangle < n_\alpha p \text{ for } \alpha \in \Phi_1^+ \big\} \end{aligned}$$

The name facet is natural because they are geometrically speaking facets of the alcoves $\mathcal{A}$.

It is easily seen, that the entire space $\mathbb{R}^n$ is a disjoint union of all the facets, and thus we can define for $\lambda \in \mathbb{R}^n$ the facet F_λ the unique facet containing λ.

Notice also that just because we are given the some values n_α, and some partition of the positive roots $\Phi^+ = \Phi_0^+ \dot\cup \Phi_1^+$, then it does not mean that the corresponding facet is non-empty. As an example if $\Phi_1^+ = \Phi^+$, and $n_{\alpha_i} = 1$ for all $1 \leq i \leq n$, and $n_{\tilde\alpha} = 1$ then clearly $F \subseteq A_0$, this leaves only one possible value n_β, for the each of the rest of the roots β, for which $F \neq \emptyset$.

One may also notice that the extreme cases $\Phi^+ = \Phi_0^+$ and $\Phi^+ = \Phi_1^+$ correspond to (if F is non-empty) a point and an alcove.

If F is a facet (with parameters $n_\alpha, \Phi_0^+, \Phi_1^+$) we also want to define the upper closure $\widehat{F}$ of F:

$$\begin{aligned} \widehat{F} = \big\{ x \in \mathbb{R}^n \big| \, &\langle x+\rho, \alpha\rangle = n_\alpha p \text{ for } \alpha \in \Phi_0^+, \\ &(n_\alpha - 1)\, p < \langle x+\rho, \alpha\rangle \leq n_\alpha p \text{ for } \alpha \in \Phi_1^+ \big\} \end{aligned}$$

In the same way that $\mathbb{R}^n$ is a disjoint union of all the facets, it is also a disjoint union of the upper closure of the alcoves:

$$\mathbb{R}^n = \bigcup_{A\in\mathcal{A}} \widehat{A}$$

For $\lambda \in \mathbb{R}^n$ we define the set $\mathscr{H}_\lambda = \{H \in \mathscr{H} \,|\, \lambda \in H\}$ of all the affine hyperplanes containing λ. Then $\alpha \in \Phi_0^+$ for F_λ if and only if $H_{\alpha,n_\alpha} \in \mathscr{H}_\lambda$ if and only if $s_{\alpha,n_\alpha}.\lambda = \lambda$. Thus there is a clear connection between $F_\lambda, \mathscr{H}_\lambda$ and $\mathcal{W}_\lambda = \{w \in \mathcal{W} \,|\, w.\lambda = \lambda\}$.

Definition 2.3.2. Let $\mu \in X$, and let $\lambda \in \overline{A_0}$ be the unique element in $\overline{A_0}$ also in the dot-orbit of μ. Then we call λ the *initial representative* of μ.

Define $\mathrm{stab}_\mu = \{s \in S \,|\, s.\lambda = \lambda\}$ the *stabilizer* of μ (or of the dot-orbit of μ), where λ is the initial representative of μ.

If $\mathrm{stab}_\mu = \emptyset$ for a $\mu \in X$ we say that μ is a *regular weight*.

Notice that a weight μ is regular if and only if it is in (the interior of) an alcove.

Notice furthermore that for $\lambda, \mu \in \overline{A_0}$ we have $\mu \in \overline{F_\lambda}$ if and only if $\mathrm{stab}_\lambda \subseteq \mathrm{stab}_\mu$.

2.3.2 The translation functors

So what is the use of defining all this extra geometry in the last subsection? The simple answer is the translation functor.

Let $\mathcal{C}$ be the category of all finite dimensional G-modules. For $\lambda \in \overline{A_0} \cap X$ let $\mathcal{C}_\lambda$ be the subcategory of $\mathcal{C}$ of all finite dimensional G-modules with all its composition factors on the form $L(w.\lambda)$ for $w \in \mathcal{W}$.

Since $\overline{A_0}$ is a fundamental domain for the dot-action of $\mathcal{W}$ we get by Proposition 2.1.1 that for $\mu \in X^+$ with $\lambda \in \overline{A_0} \cap X$ the initial representative of μ, $V(\mu) \in \mathcal{C}_\lambda$.

It can be proved that $\mathcal{C} = \bigoplus_{\lambda \in \overline{A_0} \cap X} \mathcal{C}_\lambda$ (see sections II.7.1-II.7.2 in [Jan03]). That is to say that if $M \in \mathcal{C}$ then $M = \bigoplus_{\lambda \in \overline{A_0} \cap X} M_{(\lambda)}$ with $M_{(\lambda)} \in \mathcal{C}_\lambda$. And we have a projection functor $P_\lambda : \mathcal{C} \to \mathcal{C}_\lambda$ defined by $P_\lambda(M) = M_{(\lambda)}$.

Definition 2.3.3. Let $\lambda, \lambda' \in \overline{A_0} \cap X$, and let $w \in W$ be the element such that $w(\lambda' - \lambda) \in X^+$. We now define the *translation functor* $T_\lambda^{\lambda'}$ from λ to λ' by:

$$T_\lambda^{\lambda'} : \mathcal{C}_\lambda \to \mathcal{C}_{\lambda'}$$
$$T_\lambda^{\lambda'}(M) = P_{\lambda'}(M \otimes L(w(\lambda' - \lambda)))$$

By Lemma II.7.6 in [Jan03] the translation functor is exact. That is to say if $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence of modules in $\mathcal{C}_\lambda$, then $0 \to T_\lambda^{\lambda'}(M_1) \to T_\lambda^{\lambda'}(M) \to T_\lambda^{\lambda'}(M_2) \to 0$ is a short exact sequence of modules in $\mathcal{C}_{\lambda'}$. This gives us that if $\mathrm{ch}(M) = \sum_i c_i \mathrm{ch}(M_i)$ then $\mathrm{ch}\left(T_\lambda^{\lambda'}(M)\right) = \sum_i c_i \mathrm{ch}\left(T_\lambda^{\lambda'}(M_i)\right)$.

However as long as we do not know how $T_\lambda^{\lambda'}(V(\lambda))$ and $T_\lambda^{\lambda'}(L(\lambda))$ look, it will not help us. Luckily we have:

Theorem 2.3.4. *Let $\lambda, \lambda' \in \overline{A_0} \cap X$, with $\lambda' \in \overline{F}_\lambda$, let $w \in \mathcal{W}$ then we have:*

$$T_\lambda^{\lambda'}(V(w.\lambda)) \cong \begin{cases} V(w.\lambda') & \text{if } w.\lambda' \in X^+ \\ 0 & \text{otherwise} \end{cases}$$
$$T_\lambda^{\lambda'}(L(w.\lambda)) \cong \begin{cases} L(w.\lambda') & \text{if } w.\lambda' \in X^+ \cap \widehat{F}_{w.\lambda} \\ 0 & \text{otherwise} \end{cases}$$

For a proof see [Jan03] Proposition II.7.11 (combined with II.4.2(10)) for the first identity and Proposition II.7.15 for the second. Now some easy corollaries follow:

Corollary 2.3.5. *Let $\lambda \in A_0 \cap X$ be a regular weight, let $\mu \in \overline{A_0} \cap X, w \in \mathcal{W}$ such that $w.\mu \in \widehat{w.A_0}$ then we have, that if*

$$\mathrm{ch}(L(w.\lambda)) = \sum_{w'} a_{w,w'} ch(V(w'.\lambda))$$

then

$$\mathrm{ch}(L(w.\lambda')) = \sum_{w'} a_{w,w'} ch(V(w'.\lambda'))$$

Notice that some of the $V(w'.\lambda')$ may be equal for different values of $w' \in \mathcal{W}$ while all the $V(w'.\lambda)$ are different, since λ is regular.

Corollary 2.3.6. *Let $\lambda, \lambda' \in \overline{A_0} \cap X$, with $\lambda' \in F_\lambda$, let $w, w' \in \mathcal{W}$ with $w.\lambda, w'.\lambda \in X^+$ then we have:*

$$[V(w.\lambda) : L(w'.\lambda)] = [V(w.\lambda') : L(w'.\lambda')]$$

Seeing that the proofs of the two statements are very similar, we only prove the latter.

Proof. Since λ, λ' are in the same facet, we have $u.\lambda \in X^+$ if and only if $u.\lambda' \in X^+$, thus the equality makes sense. We have

$$\mathrm{ch}(V(w.\lambda)) = \sum_{u.\lambda \preceq w.\lambda} [V(w.\lambda) : L(u.\lambda)] \, \mathrm{ch}(L(u.\lambda))$$

We now get by the theorem (since if $\lambda' \in F_\lambda$ then $u.\lambda' \in F_{u.\lambda} \subseteq \widehat{F}_{u.\lambda}$, for any $u \in \mathcal{W}$) that

$$\begin{aligned} \mathrm{ch}(V(w.\lambda')) &= \mathrm{ch}\left(T_\lambda^{\lambda'}(V(w.\lambda))\right) \\ &= \sum_{u.\lambda \preceq w.\lambda} [V(w.\lambda) : L(u.\lambda)] \, \mathrm{ch}\left(T_\lambda^{\lambda'}(L(u.\lambda))\right) \\ &= \sum_{u.\lambda \preceq w.\lambda} [V(w.\lambda) : L(u.\lambda)] \, \mathrm{ch}(L(u.\lambda')) \end{aligned}$$

This gives us the equality. □

The latter corollary implies that when calculating the $[V(\lambda) : L(\mu)]$ we can get them all by calculating the composition coefficients for the dot-orbit of just one λ for each facet F_λ in $\overline{A_0}$. In other words, one for each stabilizer $\mathrm{stab}_\lambda \subseteq S$.

Still this does not actually calculate the values $[V(\lambda) : L(\mu)]$, it only pairs them in groups. To calculate the values we need a lot of tools.

2.4 The sum formula and dimension arguments

2.4.1 Messing with the sum formula

The first tool we will be using is the sum formula. This will enable us to put some bounds on the coefficients.

Theorem 2.4.1 (Jantzen's Sum Formula)**.** *Let $V = V(\lambda)$ be a Weyl module. Then there exists a filtration of submodules: $V = V^0 \supseteq V^1 \supseteq \ldots$ such that, $V/V^0 \cong L(\lambda)$ and:*

$$\sum_{i>0} \mathrm{ch}\left(V^i\right) = \sum_{\alpha \in \Phi^+} \sum_{m=1}^{\left\lfloor \frac{\langle \lambda+\rho, \alpha \rangle - 1}{p} \right\rfloor} \nu_p(mp) \, \chi(s_{\alpha,m}.\lambda)$$

Where $\nu_p(m) = \max\{i \in \mathbb{N} \mid p^i | m\}$.

For a proof see [Jan03] Proposition II 8.19. The filtration from the sum formula is called the Jantzen filtration.

So how can we use this to bound the coefficients? The point is that we are dealing with a finite partially ordered set $M_\lambda = \{\mu \in X^+ \mid \mu \preceq \lambda, \mu \in \mathcal{W}.\lambda\}$ of weights μ, for which $[V(\lambda) : L(\mu)]$ potentially is different from 0. And the smaller (in the $\preceq$-order) the weight λ, the smaller the set M_λ, that is the fewer coefficients to calculate.

Seeing that if $M_\lambda = \{\lambda\}$ then $V(\lambda) = L(\lambda)$, we can start from the bottom, and then work our way inductively upwards.

So from now on we assume we have found the value of $[V(\mu) : L(\xi)]$ for all $\mu \in M_\lambda \setminus \{\lambda\}$ and all $\xi \in M_\mu$. Having found these we can now replace the $\chi(\mu)$ by a linear combination of $\operatorname{ch}(L(\xi))$ (possibly using Lemma 1.5.7) and thus we get:

$$\sum_{i>0} \operatorname{ch}\left(V^i\right) = \sum_{\mu \prec \lambda} c_{\lambda,\mu} \operatorname{ch}(L(\mu))$$

Because we know that $V/V^1 \cong L(\lambda)$, we have:

$$\operatorname{ch}\left(V^1\right) = \sum_{\mu \prec \lambda} [V(\lambda) : L(\mu)] \operatorname{ch}(L(\mu)) = \chi(\lambda) - \operatorname{ch}(L(\lambda))$$

we must clearly have that $[V(\lambda) : L(\mu)] > 0$ if and only if $c_{\lambda,\mu} > 0$. Furthermore we have that $[V(\lambda) : L(\mu)] \leq c_{\lambda,\mu}$. And thus we have our initial bounds on the coefficients.

In some simple cases these bounds may be enough, namely if all $c_{\lambda,\mu} \in \{0, 1\}$ then we know that $[V(\lambda) : L(\mu)] = c_{\lambda,\mu}$.

Example 2.4.2. In our examples $[a_1, \ldots, a_n] = a_1\lambda_1 + \cdots + a_n\lambda_n$ will represent a weight in X, and $(a_1, \ldots, a_n) = a_1\alpha_1 + \cdots + a_n\alpha_n$ will represent an element in the root lattice (typically a root).

Let Φ be of type A_3 and $p = 3$. We want to find the $[V(\lambda) : L(\mu)]$ for $\lambda = [1, 2, 1]$, and $\mu \in M_\lambda = \{[1, 2, 1], [0, 2, 0], [0, 0, 0]\}$.

We assume that we have already calculated that

$$\begin{aligned} \chi([0, 0, 0]) &= \operatorname{ch}(L([0, 0, 0])) \\ \chi([0, 2, 0]) &= \operatorname{ch}(L([0, 2, 0])) + \operatorname{ch}(L([0, 0, 0])) \end{aligned}$$

We now use the sum formula on $[1, 2, 1]$. Notice that $\lambda + \rho = [2, 3, 2]$, and that there are only 3 roots $\alpha \in \{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ such that $\langle [2, 3, 2], \alpha\rangle > p$, namely we get $\langle [2, 3, 2], (1, 1, 0)\rangle = \langle [2, 3, 2], (0, 1, 1)\rangle = 5$, and $\langle [2, 3, 2], (1, 1, 1)\rangle = 7$. Thus only these roots contribute to the sum formula.

Now

$$\begin{aligned} s_{(1,1,0),1}.\, [1, 2, 1] &= [-1, 0, 3] \in \left(X^+ - \rho\right) \setminus X^+ \\ s_{(0,1,1),1}.\, [1, 2, 1] &= [3, 0, -1] \in \left(X^+ - \rho\right) \setminus X^+ \end{aligned}$$

and thus by Lemma 1.5.6, we get that they do not contribute to the sum formula.

So we look at the next summand, with the weight $s_{(1,1,1),1}.\lambda = [-3, 2, -3]$, and get $[0, 0, 0] = (s_2 s_3 s_1).[-3, 2, -3]$, and thus this summand by Lemma 1.5.5 yields $-\chi([0, 0, 0])$.

The final summand is the one with the weight $s_{(1,1,1),2}.[1, 2, 1] = [0, 2, 0]$ so here we get the right weight at first try, and we can conclude that:

$$\sum_{i>0} \operatorname{ch}\left(V^i\right) = \chi([0, 2, 0]) - \chi([0, 0, 0])$$

For $V = V([1, 2, 1])$. We plug in the (as assumed) already known values of the χ:

$$\sum_{i>0} \operatorname{ch}\left(V^i\right) = \operatorname{ch}(L([0, 2, 0])) = 1 \cdot \operatorname{ch}(L([0, 2, 0])) + 0 \cdot \operatorname{ch}(L([0, 0, 0]))$$

so we see that the $c_{[1,2,1],[0,2,0]} = 1, c_{[1,2,1],[0,0,0]} = 0 \in \{0, 1\}$, and thus by the discussion of the sum formula we get:

$$[V([1, 2, 1]) : L([0, 2, 0])] = 1$$
$$[V([1, 2, 1]) : L([0, 0, 0])] = 0$$

Thus we have used the sum formula to find all the coefficients for character for $V([1, 2, 1])$.

2.4.2 Dimensional arguments

It is a fairly trivial task calculating the dimension of the modules $V(\lambda)$ using Weyl's character formula (cor 24.3 [Hum72]). This knowledge can be used to find the precise values of $[V(\lambda) : L(\mu)]$. Again let us assume that we have already dealt with the cases $\prec \lambda$.

We are now able to calculate the dimensions of the $L(\mu)$ by simply plugging the identity into our formula (2.2):

$$\dim(L(\mu)) = \sum_{\xi \preceq \mu} [L(\mu) : V(\xi)] \dim(V(\xi)) \tag{2.3}$$

We can of course also plug the identity into our formula (2.1) getting:

$$\dim(V(\lambda)) = \sum_{\mu \preceq \lambda} [V(\lambda) : L(\mu)] \dim(L(\mu)) \tag{2.4}$$

If we can somehow also calculate the dimension of $L(\lambda)$, then this yields an equation with $[V(\lambda) : L(\mu)]$ as unknown variables. Combining this equation with the bounds from the sum formula often leaves only one possible solution, thus giving us all the rest of the coefficients $[V(\lambda) : L(\mu)]$.

So when can we calculate $\dim(L(\lambda))$ without knowing the coefficients for $\operatorname{ch}(L(\lambda))$? One situation is when $\lambda \notin X_p$ then we can use the Steinberg's tensor product theorem 2.2.5.

Example 2.4.3. Let Φ be of type A_4, and let $p = 3$. Let $\lambda = [2,3,0,0] = [2,0,0,0] + p\,[0,1,0,0]$. For the weights $\mu \prec \lambda$ in the dot-orbit of λ we assume that we already know the coefficients, so we can use (2.3) to find $\dim(L(\mu))$ (first using Weyl's character formula to calculate $\dim(V(\mu))$).

We apply the sum formula to get:

$$\sum_{i>0} \mathrm{ch}\left(V^i\right) = \mathrm{ch}\left(L\left([3,1,1,0]\right)\right) + 2\mathrm{ch}\left(L\left([1,2,1,0]\right)\right)$$

For $V = V([2,3,0,0])$.

We notice that both $V([2,0,0,0])$ and $V([0,1,0,0])$ are simple (so we don't have to calculate for other weights in those dot-orbits). We gather all the necessary calculated values in a table:

μ	$\dim(V(\mu))$	$\dim(L(\mu))$
$[2,3,0,0]$	1260	?
$[3,1,1,0]$	1440	255
$[1,2,1,0]$	1050	855
$[2,0,0,0]$	15	15
$[0,1,0,0]$	10	10

Now we use Steinbergs tensor product theorem to calculate $\dim(L([2,3,0,0]))$:

$$\begin{aligned}\dim\left(L\left([2,3,0,0]\right)\right) &= \dim\left(L\left([2,0,0,0]\right)\right)\dim\left(L\left([0,1,0,0]\right)^{[1]}\right)\\ &= \dim\left(L\left([2,0,0,0]\right)\right)\dim\left(L\left([0,1,0,0]\right)\right)\\ &= 15\cdot 10 = 150.\end{aligned}$$

Now let $x = [V([2,3,0,0]) : L([1,2,1,0])]$ be the unknown coefficient. From (2.4) we get following equation:

$$1260 = 1\cdot 150 + 1\cdot 255 + x855$$

We now see that $x = 1$ so in this case we don't even need the information that $x \leq 2$, from the sum formula.

Sadly we are not able to use this method very often. We may however refine this approach to achieve a fool-proof way of finding the coefficients.

2.5 The foolproof method

This section builds on [GS88], in which it is stated that $\dim\left(L(\lambda)_\mu\right) = \mathrm{rank}\left(M_{\lambda,\mu}\right)$ for a certain matrix $M_{\lambda,\mu}$.

2.5.1 The approach

By definition of the formal characters the coefficient of $e(\mu) \in \mathbb{Z}[X]$ in $\mathrm{ch}(L(\lambda))$ and $\mathrm{ch}(V(\lambda))$ are respectively $\dim\left(\mathrm{ch}(L(\lambda))_\mu\right)$ and $\dim\left(\mathrm{ch}(V(\lambda))_\mu\right)$. Seeing that $\{e(\lambda) \mid \lambda \in X\}$ is a basis of $\mathbb{Z}[X]$, we get that the coefficient of $e(\xi)$ is the same on either side of (2.1), giving us:

$$\dim\left(V(\lambda)_\xi\right) = \sum_{\mu \preceq \lambda} [V(\lambda) : L(\mu)] \dim\left(L(\mu)_\xi\right)$$

Now we can use the fact that μ is the highest weight of $L(\mu)$ and thus $L(\mu)_\xi = 0$ for $\xi \npreceq \mu$. Furthermore we use $\dim\left(L(\xi)_\xi\right) = 1$, and thus rewrite the identity above:

$$[V(\lambda) : L(\xi)] = \dim\left(V(\lambda)_\xi\right) - \sum_{\xi \prec \mu \preceq \lambda} [V(\lambda) : L(\mu)] \dim\left(L(\mu)_\xi\right) \tag{2.5}$$

If we choose ξ to be the highest weight for which $[V(\lambda) : L(\xi)]$ is still unknown, we can calculate $[V(\lambda) : L(\xi)]$ by calculating $\dim\left(V(\lambda)_\xi\right)$ and all $\dim\left(L(\mu)_\xi\right)$ for $\xi \prec \mu \prec \lambda$.

As usual the Weyl module case is the easiest, since they derive from the complex Weyl module $V(\lambda)_\mathbb{C}$, and thus have $\dim\left(V(\lambda)_\mu\right) = \dim\left(V(\lambda)_{\mathbb{C},\mu}\right)$.

For finding the value of $\dim\left(V(\lambda)_{\mathbb{C},\mu}\right)$ we have several possibilities, for example using Freudenthal's formula (Theorem 22.3 [Hum72]) or Kostant's multiplicity formula (Theorem 24.2 [Hum72]).

So the only part missing before we can find the coefficient $[V(\lambda) : L(\xi)]$, is to find $\dim\left(L(\mu)_\xi\right)$, which is not an easy task. That is why the whole of the rest of this section is about just this task.

2.5.2 The Verma modules $M(\lambda)$

Let $\lambda \in X^+$ be a dominant weight. We define the $\mathcal{U}_\mathbb{C}$-module $M(\lambda)_\mathbb{C}$ by

$$M(\lambda)_\mathbb{C} = \mathcal{U}_\mathbb{C} \Big/ \left(\sum_{\alpha \in \Phi^+} \mathcal{U}_\mathbb{C} e_\alpha + \sum_{h \in \mathfrak{h}} \mathcal{U}_\mathbb{C}(h - \lambda(h))\right)$$

where $\mathfrak{h}$ is the sub-k-algebra generated by $\{h_\alpha \mid \alpha \in \Phi^+\}$ (the part that comes from the maximal toral subalgebra of $\mathfrak{g}_\mathbb{C}$). This module we call the complex Verma module.

If we define $v_\lambda = [1] \in M(\lambda)_\mathbb{C}$ to be the class containing 1, then clearly the standard relations (Proposition 1.2.1), and the rules of calculation (Theorem 1.2.4 and for type A, D or E Theorem 1.3.1) hold in $M(\lambda)_\mathbb{C}$ also.

$M(\lambda)_{\mathbb{C}}$ however is not the same as $V(\lambda)_{\mathbb{C}}$, because $M(\lambda)_{\mathbb{C}}$ is infinite dimensional. Still one can prove (using many of the arguments from subsection 1.3.2) that the set of elements on the standard form, for some total ordering, forms a basis of $M(\lambda)_{\mathbb{C}}$.

As usual we also have $Ev_\lambda \in M(\lambda)_{\mathbb{C},\lambda+\gamma(E)}$, when E is a monomial in $\mathcal{U}_{\mathbb{C}}$, because $M(\lambda)_{\mathbb{C},\mu}$ is defined to be exactly the elements of $v \in M(\lambda)_{\mathbb{C}}$ satisfying that $hv = \mu(h)v$ for all $h \in \mathfrak{h}$.

As in the Weyl module case we define: $M(\lambda)_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}}v_\lambda$ and $M(\lambda) = M(\lambda)_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. The last one we call the Verma module.

Clearly the dimension of $M(\lambda)_\mu = M(\lambda)_{\mathbb{Z},\mu} \otimes_{\mathbb{Z}} k$ is the number of ways to write $\lambda - \mu \in \Lambda^+$ as a sum of positive roots, seeing that each way yields exactly one element on the standard form. Thereby we get

$$M(\lambda)_\mu = \operatorname{span}_k \{Ev_\lambda \in M(\lambda) \mid \gamma(E) = \mu - \lambda\}$$

We have $M(\lambda) = \bigoplus_{\mu \preceq \lambda} M(\lambda)_\mu$.

We get a natural surjective $\mathcal{U}_{\mathbb{C}}$-module-homomorphism from $M(\lambda)_{\mathbb{C}}$ to $V(\lambda)_{\mathbb{C}}$, sending $Ev_\lambda \in M(\lambda)_{\mathbb{C}}$ to $Ev_\lambda \in V(\lambda)_{\mathbb{C}}$. This homomorphism restricts to a $\mathcal{U}_{\mathbb{Z}}$-homomorphisms $M(\lambda)_{\mathbb{Z}} \to V(\lambda)_{\mathbb{Z}}$, and further induces a $\mathcal{U}_k$-homomorphism $M(\lambda) \to V(\lambda)$. The last of these homomorphisms we want to compose with the projection of $V(\lambda)$ onto $L(\lambda)$ to get the surjective homomorphism

$$\varphi_\lambda : M(\lambda) \twoheadrightarrow V(\lambda) \twoheadrightarrow L(\lambda)$$

This homomorphism we will return to in a while.

2.5.3 A bilinear form on $M(\lambda)$

According to [Hum08] subsections 3.14, 3.15, there exists a symmetric bilinear form $(\cdot,\cdot)$ on $M(\lambda)_{\mathbb{C}}$ satisfying:

$$\begin{aligned}&(v_\lambda, v_\lambda) = 1\\ &(e_\alpha v, v') = (v, e_{-\alpha}v') \text{ for all } v, v' \in M(\lambda)_{\mathbb{C}}, \text{ and } \alpha \in \Phi\end{aligned}$$

We have following result:

Lemma 2.5.1. *When restricted to $M(\lambda)_{\mathbb{Z}}$, the form maps into the integers:*

$$(\cdot,\cdot)_{|M(\lambda)_{\mathbb{Z}} \times M(\lambda)_{\mathbb{Z}}} : M(\lambda)_{\mathbb{Z}} \times M(\lambda)_{\mathbb{Z}} \to \mathbb{Z}$$

Proof. It is enough to prove it for that every pair of basis vectors of $M(\lambda)_{\mathbb{Z}}$ is mapped into $\mathbb{Z}$, that is $(Ev_\lambda, E'v_\lambda) \in \mathbb{Z}$, where $E, E' \in \mathcal{U}_{\mathbb{Z}}$ are strings on the standard form.

Seeing that E has standard form, we may write E as $E = e_{\beta_1}^{(r_1)} \cdots e_{\beta_m}^{(r_m)} \in \mathcal{U}_{\mathbb{Z}}$. We now define: $\overline{E} = e_{-\beta_m}^{(r_m)} \cdots e_{-\beta_1}^{(r_1)} \in \mathcal{U}_{\mathbb{Z}}$. We clearly get: $(Ev_\lambda, E'v_\lambda) = \left(v_\lambda, \overline{E}E'v_\lambda\right)$. Now we can rewrite $\overline{E}E'v_\lambda$ as a $\mathbb{Z}$-linear combination of strings on the standard form, which proves that it is enough to prove for that $(v_\lambda, Fv_\lambda) \in \mathbb{Z}$ for any string $F \in \mathcal{U}_\mathbb{Z}$ on the standard form.

But now if F is different from the empty string we can just say:

$$(v_\lambda, Fv_\lambda) = \left(\overline{F}v_\lambda, v_\lambda\right) = (0, v_\lambda) = 0 \in \mathbb{Z}$$

where the second to last $=$ comes from F being a string $F = e_{\beta_1}^{(s_1)} \cdots e_{\beta_m}^{(s_m)}$ where the β_i's are negative roots (and not all $s_i = 0$), and thus $\overline{F} = e_{-\beta_m}^{(s_m)} \cdots e_{-\beta_1}^{(s_1)}$ where the $-\beta_i$'s are positive roots. The case where $F = 1$ we just get $(v_\lambda, v_\lambda) = 1 \in \mathbb{Z}$. This concludes the proof. □

In the proof you may notice that $(Ev_\lambda, E'v_\lambda) \neq 0$ only if $\gamma(E) = \gamma(E')$, from this we can directly conclude:

Lemma 2.5.2. *The weight spaces $M(\lambda)_{\mathbb{Z},\mu}, M(\lambda)_{\mathbb{Z},\xi}$ are orthogonal in the form $(\cdot,\cdot)$ when $\mu \neq \xi$.*

Now we want to induce the form to a symmetric bilinear form $(\cdot,\cdot) : M(\lambda) \times M(\lambda) \to k$; this is simply done by setting

$$(v \otimes 1, v' \otimes 1) = (v, v')$$

and extending linearly. This form also have the properties of the former form:

$$\begin{aligned} &(v_\lambda, v_\lambda) = 1 \\ &(e_\alpha v, v') = (v, e_{-\alpha} v') \text{ for all } v, v' \in M(\lambda), \text{ and } \alpha \in \Phi \\ &M(\lambda)_\mu \perp M(\lambda)_\xi \text{ for } \mu \neq \xi \end{aligned}$$

2.5.4 The kernel of the homomorphism φ_λ

Recall from subsection 2.5.2 that we have a natural surjective homomorphism $\varphi_\lambda : M(\lambda) \twoheadrightarrow L(\lambda)$. This we will use to describe $L(\lambda)$.

We start by noticing that φ_λ, being a homomorphism, maps weight spaces to weight spaces of the same weight.

Proposition 2.5.3. *The kernel of φ_λ can be described by the form $(\cdot,\cdot)$ as follows:*

$$\ker(\varphi_\lambda) = \{v \in M(\lambda) \mid (v, M(\lambda)) = 0\}$$

Proof. Notice first that both $M(\lambda)$ and $L(\lambda)$ is a direct sum of their weight spaces, φ_λ maps weight spaces to weight spaces of the same weight, and $M(\lambda)_\mu \perp M(\lambda)_\xi$. These three properties imply that it is enough to prove that for any $v \in M(\lambda)_\mu$:

$$v \in \ker(\varphi_\lambda) \iff (v, M(\lambda)) = 0$$

So let $v \in M(\lambda)_\mu$ for some weight μ. Since $L(\lambda)$ is simple we must have that $\varphi_\lambda(\mathcal{U}_k v)$ is either 0 or all of $L(\lambda)$.

We define the $\mathcal{U}_k$-submodule $N = \mathcal{U}_k v$ of $M(\lambda)$. Using that $\varphi_\lambda(v_\lambda) = v_\lambda \neq 0$, we get that $v \in \ker(\varphi_\lambda)$ if and only if $v_\lambda \notin N$. Now N being a k-vector space, and $M(\lambda)_\lambda$ being 1-dimensional, this ($v_\lambda \notin N$) is the same as $N \cap M(\lambda)_\lambda = 0$.

Now because $v \in M(\lambda)_\mu$ we get that N is generated by Ev for all terms $E \in \mathcal{U}_k$ on the standard form. The Ev are then all weight vectors (in N_ξ for $\xi = \mu + \gamma(E)$) and thus $N = \bigoplus_\xi N_\xi$ where $N_\xi = N \cap M(\lambda)_\xi$. Keeping that in mind we get that $N \cap M(\lambda)_\lambda = 0$ if and only if $N \subseteq \bigoplus_{\xi \prec \lambda} M(\lambda)_\xi$ which by the orthogonality of the weight spaces of $M(\lambda)$ is clearly the same as (the first "="):

$$0 = (v_\lambda, N) = (v_\lambda, \mathcal{U}_k v) = (\mathcal{U}_k v_\lambda, v) = (v, M(\lambda))$$

Which concludes our proof. □

2.5.5 The matrix $M_{\lambda,\mu}$

We are now ready to define the matrix $M_{\lambda,\mu}$ and prove the main theorem of this section. Let $E_1, \ldots, E_l \in \mathcal{U}_\mathbb{Z}$ be all the different monomials on the standard form, with $\gamma(E_i) = \mu - \lambda$, that is to say $E_1 v_\lambda, \ldots, E_l v_\lambda$ forms a basis for $M(\lambda)_\mu$. Now define:

$$M_{\lambda,\mu} = ((E_i v_\lambda, E_j v_\lambda))_{1 \leq i,j \leq l}$$

And now to the main theorem of this section.

Theorem 2.5.4. *Let $\lambda \in X^+$ and $\mu \in X$ such that $\mu \preceq \lambda$. Then*

$$\dim\left(L(\lambda)_\mu\right) = \operatorname{rank}(M_{\lambda,\mu})$$

Proof. Define $\psi : M(\lambda)_\mu \to \left(M(\lambda)_\mu\right)^*$, by $\psi(v)(x) = (v, x)$. Let $v_1, \ldots, v_l$ be a basis for $M(\lambda)_\mu$ such that $v_{r+1}, \ldots, v_l$ is a basis of $\ker(\psi)$ and $\psi(v_1), \ldots, \psi(v_r)$ is a basis of $\psi\left(M(\lambda)_\mu\right)$.

Now extend the basis $\psi(v_1), \ldots, \psi(v_r)$ of $\psi\left(M(\lambda)_\mu\right)$ to a basis $u_1, \ldots, u_l$ of $\left(M(\lambda)_\mu\right)^*$, such that $u_1 = \psi(v_1), \ldots, u_r = \psi(v_r)$. Let $w_1, \ldots, w_l$ be a dual basis of $u_1, \ldots, u_l$. That is to say $u_i(w_j) = \delta_{i,j}$, which gives us:

$$(v_i, w_j) = \begin{cases} \delta_{i,j} & \text{, for } i \leq r \\ 0 & \text{, for } i > r \end{cases}$$

Seeing that $v_{r+1}, \ldots, v_l$ is a basis for $\ker(\psi)$ we get:

$$\begin{aligned} \operatorname{span}_k\{v_i \mid i > r\} &= \left\{v \in M(\lambda)_\mu \,\middle|\, \left(v, M(\lambda)_\mu\right) = 0\right\} \\ &= \left\{v \in M(\lambda)_\mu \,\middle|\, (v, M(\lambda)) = 0\right\} = \ker\left(\varphi_{\lambda|M(\lambda)_\mu}\right) \end{aligned}$$

Here the second equality follows from the orthogonality of the weight spaces, and the last equality follows from Proposition 2.5.3. From the above results and the rank-nullity theorem we get that:

$$\begin{aligned}\dim\left(L(\lambda)_\mu\right) &= \dim\left(\varphi_\lambda\left(M(\lambda)_\mu\right)\right)\\ &= l - \dim\left(\ker\left(\varphi_{\lambda|M(\lambda)_\mu}\right)\right)\\ &= r\\ &= \operatorname{rank}\left(((v_i, w_j))_{1\le i,j\le l}\right)\end{aligned}$$

Now in general if $b_1, \ldots, b_m$, $c_1, \ldots, c_m$, $d_1, \ldots, d_m$ and $e_1, \ldots, e_m$ are bases of a vector space with a bilinear symmetric form $(\cdot,\cdot)$, then the matrices $((b_i, c_j))_{1\le i,j\le l}$ and $((d_i, e_j))_{1\le i,j\le l}$ have the same rank. This is easily seen by using change of basis matrices (which are invertible), and the fact that multiplication by invertible matrices do not change the rank. Thus we get:

$$\begin{aligned}\operatorname{rank}(M_{\lambda,\mu}) &= \operatorname{rank}\left(((E_i v_\lambda, E_j v_\lambda))_{1\le i,j\le l}\right)\\ &= \operatorname{rank}\left(((v_i, w_j))_{1\le i,j\le l}\right)\\ &= \dim\left(L(\lambda)_\mu\right)\end{aligned}$$

This concludes the proof. □

2.5.6 The algorithm

We can now easily make an algorithm that finds $\dim\left(L(\lambda)_\mu\right)$. The algorithm starts by finding all the strings $E_1, \ldots, E_l$ on standard form with $\gamma(E_i) = \mu - \lambda$. Next it uses an algorithm like the one in subsection 1.3.2 to calculate $\overline{E_i} E_j v_\lambda \in M(\lambda)_{\mathbb{Z},\lambda}$ which is an integer $c_{i,j}$ times v_λ ($M(\lambda)_\lambda$ is 1-dimensional). Now use simple Gaussian reduction to calculate the rank of the $M_{\lambda,\mu}$ which has the entries $[c_{i,j}]_p \in \mathbb{F}_p \subseteq k$, and now by Theorem 2.5.4 we have found the dimension of the weight space $L(\lambda)_\mu$. By this algorithm we now have a way of finding the characters of $L(\lambda)$.

Example 2.5.5. Let Φ be of type A_4 and let $p = 3$. Let $\lambda = [2, 1, 1, 2] \in X^+$ and $\xi = [1, 1, 1, 1] \in X^+$.

We will try to find $[V(\lambda) : L(\xi)]$ using the method described in this section and in subsection 2.4.2. So let us assume that we have already calculated $[V(\lambda) : L(\mu)]$ (they are all 1) for all $\xi \prec \mu \preceq \lambda$ with μ in the dot-orbit of λ (and of ξ).

Now we need to calculate $\dim\left(V(\lambda)_\xi\right)$ (which we will not go into details with), and $\dim\left(L(\mu)_\xi\right)$ for all the before mentioned μ's. In general this can take some time if you do it by hand. But in this case the only matrix we get that is bigger that 1×1 is the 8×8-matrix $M_{\lambda,\xi}$. So let us look at that one.

First we need to see in how many ways we can write $\lambda - \mu = [1, 0, 0, 1] = (1, 1, 1, 1)$ as a sum of positive roots. This is where the number 8 enters, for that is exactly how many ways it can be done:

$$\begin{aligned}
&1 : (1, 1, 1, 1)\,,\\
&2 : (0, 0, 1, 1) + (1, 1, 0, 0)\,,\\
&3 : (1, 0, 0, 0) + (0, 1, 1, 1)\,,\\
&4 : (0, 1, 0, 0) + (1, 0, 0, 0) + (0, 0, 1, 1)\,,\\
&5 : (0, 0, 0, 1) + (1, 1, 1, 0)\,,\\
&6 : (0, 0, 0, 1) + (1, 0, 0, 0) + (0, 1, 1, 0)\,,\\
&7 : (0, 0, 0, 1) + (0, 0, 1, 0) + (1, 1, 0, 0)\,,\\
&8 : (0, 0, 0, 1) + (0, 0, 1, 0) + (0, 1, 0, 0) + (1, 0, 0, 0)
\end{aligned}$$

Now each of these ways represents a monomial E_i (for $1 \le i \le 8$) on the standard form for example: $3 : (1, 0, 0, 0) + (0, 1, 1, 1)$ represents $E_3 = e_{-(1,0,0,0)}e_{-(0,1,1,1)}$, and $6 : (0, 0, 0, 1) + (1, 0, 0, 0) + (0, 1, 1, 0)$ represents $E_6 = e_{-(0,0,0,1)}e_{-(1,0,0,0)}e_{-(0,1,1,0)}$.

Now the next step is to calculate $\overline{E}_i E_j v_\lambda$ for all pairs $1 \le i \le j \le 8$. we will demonstrate this using the algorithm in subsection 1.3.2 in the fairly simple case $\overline{E}_1 E_3 v_\lambda$:

$$\begin{aligned}
&e_{(1,1,1,1)}e_{-(1,0,0,0)}e_{-(0,1,1,1)}v_\lambda\\
&= e_{-(1,0,0,0)}e_{(1,1,1,1)}e_{-(0,1,1,1)}v_\lambda - e_{(0,1,1,1)}e_{-(0,1,1,1)}v_\lambda\\
&= e_{-(1,0,0,0)}e_{-(0,1,1,1)}e_{(1,1,1,1)}v_\lambda + e_{-(1,0,0,0)}e_{(1,0,0,0)}v_\lambda\\
&\quad - e_{-(0,1,1,1)}e_{(0,1,1,1)}v_\lambda - \langle\lambda, (0, 1, 1, 1)\rangle\, v_\lambda\\
&= -4v_\lambda
\end{aligned}$$

This means that entry $1, 3$ (and $3, 1$) in $M_{\lambda,\xi}$ is $-4 = 2$ in k. The whole of the matrix looks like this:

$$\begin{pmatrix}
0 & 0 & 2 & 2 & 1 & 0 & 2 & 1\\
0 & 0 & 0 & 0 & 2 & 0 & 1 & 1\\
2 & 0 & 1 & 2 & 0 & 2 & 0 & 1\\
2 & 0 & 2 & 0 & 0 & 0 & 2 & 0\\
1 & 2 & 0 & 0 & 1 & 1 & 2 & 1\\
0 & 0 & 2 & 0 & 1 & 1 & 0 & 1\\
2 & 1 & 0 & 2 & 2 & 0 & 1 & 1\\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 2 & 1 & 0 & 1 & 0 & 2\\
0 & 1 & 2 & 1 & 2 & 0 & 1 & 1\\
0 & 0 & 1 & 1 & 2 & 0 & 1 & 2\\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 2\\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

So we see that $\operatorname{rank}(M_{\lambda,\xi}) = 6$. Data needed can be seen in the following table:

μ	$[2, 1, 1, 2]$	$[3, 0, 0, 3]$	$[3, 0, 1, 1]$	$[1, 1, 0, 3]$
$\operatorname{rank}(M_{\mu,\xi})$	6	0	0	0

Furthermore $\dim (V(\lambda))_\xi = 8$. And now we can find the coefficient using (2.5):

$$\begin{aligned}[V(\lambda) : L(\xi)] &= \dim \left(V(\lambda)_\xi \right) - \sum_{\xi \prec \mu \prec \lambda} [V(\lambda) : L(\mu)] \dim \left(L(\mu)_\xi \right) \\ &= 8 - 1 \cdot 0 - 1 \cdot 0 - 1 \cdot 0 - 1 \cdot 6 = 2\end{aligned}$$

And thus we have calculated the coefficient $[V(\lambda) : L(\xi)]$.

Chapter 3

Lusztig's conjecture

For a mathematician it is of course not enough to be able to calculate something; we want to find a pattern. Such a pattern for the characters for the simple modules $L(\lambda)$ is described in Lusztig's conjecture.

3.1 Prerequisite theory

Before we are able to understand Lusztig's conjecture, we need some general theory about Hecke algebras and Kazhdan-Lusztig polynomials, and about the Coxeter number.

3.1.1 Kazhdan-Lusztig polynomials

The constructions in this subsection will depend only on the affine Weyl group $\mathcal{W}$ (actually it can be done for any Coxeter group).

Definition 3.1.1. The *Hecke algebra* $\mathcal{H}$ is a $\mathbb{Z}[q, q^{-1}]$-algebra spanned by the set $\{T_w \mid w \in \mathcal{W}\}$, with product defined by:

$$T_{s_i} T_w = \begin{cases} T_{s_i w} & \text{for } l(s_i w) > l(w) \\ (q-1) T_w + q T_{s_i w} & \text{for } l(s_i w) < l(w) \end{cases}$$

where $w \in \mathcal{W}$ is any element of $\mathcal{W}$, and s_i is one of the simple generators.

To see that this actually is an algebra, see [Hum90]. Notice that the product of any two generators T_x, T_y (and thus any two elements of $\mathcal{H}$) can be calculated by the above identity since if $x = s_{i_1} s_{i_2} \cdots s_{i_l}$ is a reduced expression for x, then $T_x = T_{s_{i_1}} T_{s_{i_2}} \cdots T_{s_{i_l}}$. Since T_1 acts neutrally on all of $\mathcal{H}$, so it makes sense to define $1 = T_1 \in \mathcal{H}$. And seeing that we now have an identity element we can also notice that all the generators have an inverse element in $\mathcal{H}$, in particular:

$$T_{s_i}^{-1} = q^{-1} T_{s_i} - \left(1 - q^{-1}\right) T_1$$

Writing the inverse element of T_w in the T_x-basis we get:

$$T_w^{-1} = (-1)^{l(w)} q^{(l(w))} \sum_{x \leq w} (-1)^{l(x)} R_{x,w}(q) T_x$$

where $\leq$ is the Bruhat ordering, and $R_{x,w} \in \mathbb{Z}[q]$ is a polynomial (called the R-polynomial) of degree $l(w) - l(x)$ with $R_{w,w}(q) = 1$, see Proposition 7.4 in [Hum90]. These R-polynomials play an important role in calculating the Kazhdan-Lusztig polynomials. On how to calculate the R-polynomials, see [Hum90] subsection 7.5.

To define the Kazhdan-Lusztig polynomials we first need to introduce an involution $\bar{\cdot} : \mathcal{H} \to \mathcal{H}$ on $\mathcal{H}$:

$$\begin{aligned} \overline{x} &= x && \text{for } x \in \mathbb{Z} \\ \overline{q} &= q^{-1} \\ \overline{T}_w &= T_{w^{-1}}^{-1} && \text{for } w \in \mathcal{W} \end{aligned}$$

Now it can be proved (Theorem 7.9 in [Hum90]) that there exists a unique basis $\{C_w \mid w \in \mathcal{W}\}$ of $\mathcal{H}$ (if we view $\mathcal{H}$ as an $\mathbb{Z}\left[q^{\frac{1}{2}}, q^{-\frac{1}{2}}\right]$-algebra) such that $\overline{C}_w = C_w$ and $C_w - T_w \in \operatorname{span}\{T_x \mid x < w\}$. We can now define the Kazhdan-Lusztig polynomials:

Definition 3.1.2. When we write C_w in the T_x-basis we get:

$$C_w = (-1)^{l(w)} q^{\frac{1}{2}l(w)} \sum_{x \leq w} (-1)^{l(x)} q^{-l(x)} P_{x,w}(q) T_x$$

and the polynomials $P_{x,w}(q)$ are called the *Kazhdan-Lusztig polynomials.*

The Kazhdan-Lusztig polynomials are polynomials in q, of degree less that or equal to $\frac{1}{2}(l(w) - l(x) - 1)$ if $x < w$, $P_{w,w} = 1$.

To get a method of calculating the Kazhdan-Lusztig polynomials, see [Hum90] 7.10.

3.1.2 The Coxeter number

In this subsection we will be working in the Weyl group W. (W, S') is a Coxeter system with $S' = \{s_1, s_2, \dots, s_n\}$. We define:

Definition 3.1.3. A *Coxeter element* $c \in W$ is a product of one of each of the generators S' in any order. The order $h \in \mathbb{N}$ of a Coxeter element is called the *Coxeter number.*

It is not obvious that the Coxeter number is a well defined number, but one can prove (Proposition 3.16 in [Hum90]) that all Coxeter elements are conjugate in W, and thus have the same order.

When Φ is of type A_n, the Weyl group $W = S_{n+1}$ is the symmetric group where the simple generators $s_i = (i \ \ i+1)$ are the simple transpositions. We easily see that a Coxeter element here is a $(n+1)$-cycle, and thus the Coxeter number is $h = n+1$.

Some properties of the Coxeter number are:

Proposition 3.1.4. *The Coxeter number h satisfies:*

- $h = \frac{|\Phi|}{n}$
- $h = \langle \rho, \tilde{\alpha} \rangle + 1$

See Proposition 3.18 in [Hum90] for the first part, and Proposition 31 in [Bou68] chap. VI, §1 for the second part. Because of the last property one can easily prove that $A_0 \cap X$ is non-empty if and only if $0 \in A_0 \cap X$ if and only if $p \geq h$.

3.2 The conjecture and questions

We start by defining the Jantzen region J:

$$\begin{aligned} J &= \left\{ \lambda \in X^+ \,\middle|\, \langle \lambda + \rho, \alpha \rangle \leq p\,(p - h + 2) \text{ for all } \alpha \in \Phi^+ \right\} \\ &= \left\{ \lambda \in X^+ \,\middle|\, \langle \lambda + \rho, \tilde{\alpha} \rangle \leq p\,(p - h + 2) \right\} \end{aligned}$$

We are now ready to state Lusztig's conjecture:

Conjecture 3.2.1 (Lusztig's conjecture)**.** Let $\lambda \in A_0 \cap X$, and $w \in \mathcal{W}$ such that $w.\lambda \in J$, then:

$$\mathrm{ch} L\,(w.\lambda) = \sum_{\substack{y \leq w \\ y.\lambda \in X^+}} (-1)^{l(yw)}\, P_{w_0 y, w_0 w}\,(1)\, \chi\,(y.\lambda)$$

Here $w_0 \in W \subseteq \mathcal{W}$ is the unique longest element in W, and $P_{w_0 y, w_0 w}$ is the Kazhdan-Lusztig polynomial, and $y \leq w$ is in the Bruhat order.

The identity in the conjecture we will call *Lusztig's character formula.*

In [AJS94] it was proved that the conjecture holds at least for big enough values of p. More precisely:

Theorem 3.2.2. *There exists a constant $m\,(\Phi) \in \mathbb{N}$ dependent only on the root system Φ of G, such that if char$\,(k) = p \geq m\,(\Phi)$ then Lusztig's conjecture holds.*

One might notice that λ is required to be in A_0, that is to say be a regular weight. This problem can however be solved by the translation functor. Seeing that $\mathbb{R}^n$ is the disjoint union of the upper closure of the alcoves, we know that any $\lambda' \in X^+$ is in the upper closure of some alcove A. Now since the dot-action of $\mathcal{W}$ permutes the alcoves simply transitively, there is a unique $w \in \mathcal{W}$ such that $w.A_0 = A$. Then we can apply corollary 2.3.5 to any regular weight $\lambda \in A_0 \cap X$, and w and λ', to get $\mathrm{ch}L\,(w.\lambda')$.

Because the right side in Lusztig's character formula doesn't really depend on p, it is necessary with the bound that $w.\lambda \in J$ which depends on p. To overcome this we can use Steinberg's tensor product Theorem 2.2.5, if $X_p \subseteq J$.

These two observations lead to another similar question:

Question 3.2.3. Let $\lambda \in \overline{A_0} \cap X$, and $w \in \mathcal{W}$ such that $w.\lambda \in X_p$ and $w.\lambda \in \widehat{w.A_0}$. Does Lusztig's character formula

$$\mathrm{ch} L(w.\lambda) = \sum_{\substack{y \leq w \\ y.\lambda \in X^+}} (-1)^{l(yw)} P_{w_0 y, w_0 w}(1) \chi(y.\lambda)$$

hold in this case?

The hope is that the answer is yes if $p \geq h$ (that is if there exists regular weights), even if $X_p \nsubseteq J$ ($X_p \subseteq J \iff p \geq 2h-3$).

In the general case however, the answer is no. In the data in appendix A one can find two places (but only these two places) where the actual characters do not match the ones from Lusztig's character formula:

In type A_4 with $p = 2$ we have that

$$\mathrm{ch}(L([0,1,1,0])) = \chi([0,1,1,0]) - \chi([0,0,0,0])$$

whereas Lusztig's character formula claims that it is

$$\mathrm{ch}(L([0,1,1,0])) = \chi([0,1,1,0])$$

In type A_4 with $p = 3$ we have that

$$\mathrm{ch}(L([1,2,2,1])) = \chi([1,2,2,1]) - \chi([0,2,2,0])$$

whereas Lusztig's character formula claims that it is

$$\mathrm{ch}(L([1,2,2,1])) = \chi([1,2,2,1]) - \chi([0,2,2,0]) + \chi([0,1,1,0]) - \chi([1,0,0,1])$$

We are still left with the following two natural questions:

1. How big need p be for Lusztig's character formula to be correct?

2. Can we find another pattern that holds more generally, or at least in the small p cases ($p < h$)?

To the first question some progress has been made in [Fie09], in which a concrete way of calculating an upper bound for $m(\Phi)$ is described. These upper bounds, however are far greater than the value hoped for (namely h).

It is question number two that we will be looking into primarily. One way to attack this problem is to actually do some calculations for some small values of p, and compare the results to Lusztig's character formula.

In appendix A can be seen the inverse composition factors $[L(\lambda) : V(\mu)]$ for each $\lambda \in X_p$ and $\mu \in X^+$ with $\mu \preceq \lambda$ and $\mu = w.\lambda$ for some $w \in \mathcal{W}$. These numbers are then compared to what the numbers would be according to Lusztig's character formula. These calculations are made for root systems Φ of type A_2, A_3 and A_4 for $p \in \{2,3\}$ (Except the case A_2, $p = 2$ where $V(\lambda) = L(\lambda)$ for all $\lambda \in X_p$).

3.3 Homomorphisms between Weyl modules

To better be able to describe the simple modules, we try to describe the Weyl modules. One way of describing $V(\lambda)$ is to find homomorphisms from $V(\mu)$ to $V(\lambda)$ for some μ. Now since $V(\mu)$ is generated (over $\mathcal{U}_k$) by v_μ, it is enough to know the image of v_μ, to know the whole homomorphism. And since a homomorphism must send weight spaces to weight spaces of the same weight we get the following result:

Proposition 3.3.1. *Let $\varphi : V(\mu) \to V(\lambda)$ be a non-zero homomorphism, then $\mu \preceq \lambda$, and $\varphi(v_\mu) = \sum_{i=1}^{l} a_i E_i v_\lambda$ where $a_i \in k$ and the E_i's are monomials on the standard form in $\mathcal{U}_k$ with $\gamma(E_i) = \mu - \lambda$.*

Notice that we use that φ is a G-homomorphism between Weyl modules if and only if φ is a $\mathcal{U}_k$-homomorphism between Weyl modules – a fact that is easily proved (see the closely related last part of Theorem 1.2.2).

This however in most cases leaves a lot of choices for φ, not all of which are actual homomorphisms. This problem we solve by the following theorem:

Theorem 3.3.2. *Let $\mu \preceq \lambda$, and let $E = \sum_{i=1}^{l} a_i E_i$, with $a_i \in k$ and the E_i's monomials on the standard form in $\mathcal{U}_k$ with $\gamma(E_i) = \mu - \lambda = \sum_{i=1}^{n} r_i \alpha_i$. Then there exists a homomorphism $\varphi : V(\mu) \to V(\lambda)$ sending v_μ to $E v_\lambda$, if and only if for all $1 \leq i \leq n$ and for all $1 \leq s \leq r_i$, we have $e_{\alpha_i}^{(s)} E v_\lambda = 0 \in V(\lambda)$.*

The theorem follows from Lemma II.2.13.a in [Jan03].

Now to actually find the homomorphism we need be able to write any element of $V(\lambda)$ on a unique standard form. More precisely we need a basis of $V(\lambda)$.

3.4 The structure of $V(\lambda)$ in type A

3.4.1 The basis of $V(\lambda)$

Finding a basis for $V(\lambda)$ in general in not an easy task. First we need some extra structure on $V(\lambda)$. This can be found in [Hum72] chapter 21. In particular Theorem 21.4:

Theorem 3.4.1. *Let $\lambda \in X^+$ with $\lambda = \sum_{i=1}^{n} m_i \lambda_i$. Then*

$$V(\lambda)_{\mathbb{C}} \cong M(\lambda)_{\mathbb{C}} / J(\lambda)$$

where $J(\lambda)$ is the $M(\lambda)_{\mathbb{C}}$-submodule generated by $e_{-\alpha_i}^{(m_i+1)}$ for $1 \leq i \leq n$.

It may be outside the scope of this thesis to find a general basis for the Weyl module, but we need one to find the homomorphisms mentioned in the previous subsection.

In [Jan73] is a description of a basis of $V(\lambda)$, when Φ is of type A_n. To explain it as a subset of the monomials on the standard form, we use the ordering $\trianglelefteq$ of the negative roots given by (for $1 \le j < i \le n+1, 1 \le l < k \le n+1$):

$$\epsilon_i - \epsilon_j \trianglelefteq \epsilon_k - \epsilon_l \iff j > l \text{ or } (j = l \text{ and } i \ge k)$$

now it can be concluded from page 85 (just below formula (9) in the bottom half of the page) of [Jan73] that:

Theorem 3.4.2. *Let Φ be a root system of type A_n (basis $\epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n$). Let $\lambda \in X^+$ with $\lambda = \sum_{i=1}^n m_i \lambda_i$. The set of monomials on the standard form according to the ordering $\trianglelefteq$, where the exponent $r_{i,j} \in \mathbb{N}$ of $e_{\epsilon_i - \epsilon_j}$ for $1 \le j < i \le n+1$ satisfies*

$$0 \le r_{i,j} \le m_{i-1} + \sum_{l=1}^{j-1} (r_{i-1,l} - r_{i,l})$$

forms a basis of $V(\lambda)$. That is to say the basis vectors are on the form:

$$e^{(r_{n+1,n})}_{\epsilon_{n+1}-\epsilon_n} e^{(r_{n+1,n-1})}_{\epsilon_{n+1}-\epsilon_{n-1}} e^{(r_{n,n-1})}_{\epsilon_n-\epsilon_{n-1}} e^{(r_{n+1,n-2})}_{\epsilon_{n+1}-\epsilon_{n-2}} \cdots e^{(r_{3,2})}_{\epsilon_3-\epsilon_2} e^{(r_{n+1,1})}_{\epsilon_{n+1}-\epsilon_1} e^{(r_{n,1})}_{\epsilon_n-\epsilon_1} \cdots e^{(r_{2,1})}_{\epsilon_2-\epsilon_1} v_\lambda$$

where the $r_{i,j}$ satisfy the above constraints.

3.4.2 Yet another rule of calculation

Now to transform an element in $V(\lambda)$ to a linear combination of the basis vectors described in Theorem 3.4.2, we need one more rule of calculation:

Proposition 3.4.3. *Let Φ be of type A, D or E. If $\alpha, \beta, \alpha+\beta \in \Phi$, and $t, r, s \in \mathbb{N}$ with $t + r \ge s$, then:*

$$\begin{aligned} e^{(t)}_\alpha e^{(r)}_{\alpha+\beta} e^{(s)}_\beta =& N^{-r}_{\beta,\alpha} \sum_{j=0}^{s} N^j_{\alpha,\beta} \binom{s+r-j}{r} e^{(s+r-j)}_\beta e^{(j)}_{\alpha+\beta} e^{(t+r-j)}_\alpha \\ & - \sum_{i=1}^{r} N^{-i}_{\beta,\alpha} \binom{s+i}{i} e^{(t+i)}_\alpha e^{(r-i)}_{\alpha+\beta} e^{(s+i)}_\beta \end{aligned} \tag{3.1}$$

Proof. We calculate in $\mathcal{U}_\mathbb{C}$.

As usual for these rules we use induction, and this time in s. To see the induction start ($s = 0$) we use (1.7):

$$e^{(r)}_\beta e^{(r+t)}_\alpha = \sum_{i=0}^{r} N^i_{\beta,\alpha} e^{(r+t-i)}_\alpha e^{(i)}_{\alpha+\beta} e^{(r-i)}_\beta$$

We now extract the last summand (for $i = r$):

$$\begin{aligned}
e_\alpha^{(t)} e_{\alpha+\beta}^{(r)} &= N_{\beta,\alpha}^{-r} e_\beta^{(r)} e_\alpha^{(r+t)} - \sum_{i=0}^{r-1} N_{\beta,\alpha}^{i-r} e_\alpha^{(t+r-i)} e_{\alpha+\beta}^{(i)} e_\beta^{(r-i)} \\
&= N_{\beta,\alpha}^{-r} e_\beta^{(r)} e_\alpha^{(r+t)} - \sum_{j=1}^{r} N_{\beta,\alpha}^{-j} e_\alpha^{(t+j)} e_{\alpha+\beta}^{(r-j)} e_\beta^{(j)}
\end{aligned}$$

Which is clearly what it should be. So now assume the proposition to be ok for $s-1$, and let $t + r \geq s > s - 1$.

$$\begin{aligned}
& e_\alpha^{(t)} e_{\alpha+\beta}^{(r)} e_\beta^{(s)} \\
&= \frac{1}{s} e_\alpha^{(t)} e_{\alpha+\beta}^{(r)} e_\beta^{(s-1)} e_\beta \\
&= \frac{N_{\beta,\alpha}^{-r}}{s} \sum_{j=0}^{s-1} N_{\alpha,\beta}^{j} \binom{s-1+r-j}{r} e_\beta^{(s-1+r-j)} e_{\alpha+\beta}^{(j)} e_\alpha^{(r+t-j)} e_\beta \\
&\quad - \frac{1}{s} \sum_{i=1}^{r} N_{\beta,\alpha}^{-i} \binom{s-1+i}{i} e_\alpha^{(t+i)} e_{\alpha+\beta}^{(r-i)} e_\beta^{(s-1+i)} e_\beta \\
&= \frac{N_{\beta,\alpha}^{-r}}{s} \sum_{j=0}^{s-1} (s+r-j) N_{\alpha,\beta}^{j} \binom{s-1+r-j}{r} e_\beta^{(s+r-j)} e_{\alpha+\beta}^{(j)} e_\alpha^{(r+t-j)} \\
&\quad + \frac{N_{\beta,\alpha}^{-r}}{s} \sum_{j=0}^{s-1} (j+1) N_{\alpha,\beta}^{j+1} \binom{s-1+r-j}{r} e_\beta^{(s-1+r-j)} e_{\alpha+\beta}^{(j+1)} e_\alpha^{(r+t-j-1)} \\
&\quad - \sum_{i=1}^{r} N_{\beta,\alpha}^{-i} \frac{s+i}{s} \binom{s-1+i}{s-1} e_\alpha^{(t+i)} e_{\alpha+\beta}^{(r-i)} e_\beta^{(s+i)}
\end{aligned}$$

Here the second equality is by the induction hypothesis, and the third equality is just (1.7) used on the terms in the first sum and (1.1) used to merge factors with the same root. We now substitute $k = j + 1$ in the second sum:

$$\begin{aligned}
&= \frac{N_{\beta,\alpha}^{-r}}{s} \sum_{j=0}^{s-1} (s+r-j) N_{\alpha,\beta}^{j} \binom{s-1+r-j}{r} e_\beta^{(s+r-j)} e_{\alpha+\beta}^{(j)} e_\alpha^{(r+t-j)} \\
&\quad + \frac{N_{\beta,\alpha}^{-r}}{s} \sum_{k=1}^{s} k N_{\alpha,\beta}^{k} \binom{s+r-k}{r} e_\beta^{(s+r-k)} e_{\alpha+\beta}^{(k)} e_\alpha^{(r+t-k)} \\
&\quad - \sum_{i=1}^{r} N_{\beta,\alpha}^{-i} \binom{s+i}{s} e_\alpha^{(t+i)} e_{\alpha+\beta}^{(r-i)} e_\beta^{(s+i)}
\end{aligned}$$

We notice that the coefficients of $e_\alpha^{(t+i)} e_{\alpha+\beta}^{(r-i)} e_\beta^{(s+i)}$ are the correct ones for all i, so we now need to examine the coefficients c_j of $e_\beta^{(s+r-j)} e_{\alpha+\beta}^{(j)} e_\alpha^{(r+t-j)}$. First let $0 < j < s$:

$$\begin{aligned}
c_j &= \frac{N_{\beta,\alpha}^{-r} N_{\alpha,\beta}^{j}}{s} \left((s+r-j) \binom{s-1+r-j}{r} + j \binom{s+r-j}{r} \right) \\
&= N_{\beta,\alpha}^{-r} N_{\alpha,\beta}^{j} \left(\frac{(s+r-j)(s-1+r-j)!}{sr!\,(s-1-j)!} + \frac{j\,(s+r-j)!}{sr!\,(s-j)!} \right) \\
&= N_{\beta,\alpha}^{-r} N_{\alpha,\beta}^{j} \frac{(s-j+j)(s+r-j)!}{sr!\,(s-j)!} \\
&= N_{\beta,\alpha}^{-r} N_{\alpha,\beta}^{j} \binom{s+r-j}{r}
\end{aligned}$$

which fits the theorem. Now lets look at c_0:

$$\begin{aligned}
c_0 &= \frac{N_{\beta,\alpha}^{-r} N_{\alpha,\beta}^{0}}{s} (s+r-0) \binom{s-1+r-0}{r} \\
&= N_{\beta,\alpha}^{-r} \frac{s+r}{s} \binom{s+r-1}{s-1} = N_{\beta,\alpha}^{-r} \binom{s+r}{s} = N_{\beta,\alpha}^{-r} \binom{s+r}{r}
\end{aligned}$$

which is the correct value. Now we just need the last coefficient c_s:

$$c_s = \frac{N_{\beta,\alpha}^{-r}}{s} s N_{\alpha,\beta}^{s} \binom{s+r-s}{r} = N_{\beta,\alpha}^{-r} N_{\alpha,\beta}^{s} \binom{s+r-s}{r}$$

And we hereby conclude the proof. □

Notice that if Φ is of type A we get from Lemma 1.2.3, that the $N_{\alpha,\beta}, N_{\beta,\alpha} \in \{-1, 1\}$ so, it does not matter that we raise them to a negative power. That is to say that in case A the proposition holds in $\mathcal{U}_\mathbb{Z}$, and thus in $\mathcal{U}_k$.

3.4.3 An algorithm in type A

We are now ready to sketch how to write the elements of $V(\lambda)$ as linear combinations of the basis vectors described in Theorem 3.4.2.

Theorem 3.4.4. *Let Φ be of type A_n, let $\lambda = \sum_{i=1}^n m_i \lambda_i \in X^+$, and let $E \in \mathcal{U}_k$ be a monomial on the standard form (according to the $\trianglelefteq$ order) – that is to say $E = E_n E_{n-1} \cdots E_1$ with*

$$E_i = e_{\epsilon_{n+1}-\epsilon_i}^{(r_{n+1,i})} e_{\epsilon_n-\epsilon_i}^{(r_{n,i})} \cdots e_{\epsilon_{i+1}-\epsilon_i}^{(r_{i+1,i})}$$

Then E can be written as a sum of a linear combination of monomials on the standard form satisfying the constraints of 3.4.2 and an element belonging to $\sum_{i=1}^n \sum_{s>m_i} \mathcal{U}_\mathbb{Z}^- e_{\epsilon_{i+1}-\epsilon_i}^{(s)}$, where $\mathcal{U}_\mathbb{Z}^- \subseteq \mathcal{U}_\mathbb{Z}$ is the subalgebra generated by $e_\alpha^{(r)}$ for $\alpha \in \Phi^-$, $r \in \mathbb{N}$.

Proof. We will prove this by induction both in n and in the total weight $\gamma(E_1)$ of E_1.

Notice first that $E_n E_{n-1} \cdots E_2$ can be seen as an element of $\mathcal{U}_{\mathbb{Z}}$ with Φ of type A_{n-1} (since if you remove all the roots not orthogonal to ϵ_1 in Φ you are left with a root system of type A_{n-1}). Thus if $\gamma(E_1) = 0$ then $E_1 = 1$ and $E = E_n E_{n-1} \cdots E_2$ and we are done by induction in n. So assume $E_1 \neq 1$ (that is $\gamma(E_1) \prec 0$), and assume the theorem to holds for all $E' = E'_n \cdots E'_1$ with $\gamma(E'_1) \succ \gamma(E_1)$, and that the theorem holds for all E when Φ is of type A_r for $r < n$.

First we see what we can do if there is an $2 \leq i \leq n+1$ such that $r_{i,1} > m_{i-1}$. If $i = 2$ then E ends with $\ldots e_{\epsilon_2-\epsilon_1}^{(r_{2,1})}$ with $r_{2,1} > m_1$, and we are already done. So assume that $i > 2$.

Now we use (1.7) on $e_{\epsilon_{i-1}-\epsilon_1}^{(r_{i,1}+r_{i-1,1})} e_{\epsilon_i-\epsilon_{i-1}}^{(r_{i,1})}$ to get:

$$\begin{aligned} & e_{\epsilon_i-\epsilon_1}^{(r_{i,1})} e_{\epsilon_{i-1}-\epsilon_1}^{(r_{i-1,1})} \\ & = (-1)^{-r_{i,1}} \left(e_{\epsilon_{i-1}-\epsilon_1}^{(r_{i,1}+r_{i-1,1})} e_{\epsilon_i-\epsilon_{i-1}}^{(r_{i,1})} - \sum_{m=0}^{r_{i,1}-1} (-1)^m \, e_{\epsilon_i-\epsilon_{i-1}}^{(r_{i,j}-m)} e_{\epsilon_i-\epsilon_1}^{(m)} e_{\epsilon_{i-1}-\epsilon_1}^{(r_{i,1}+r_{i-1,1}-m)} \right) \end{aligned}$$

Where the (-1) comes from $N_{\epsilon_{i-1}-\epsilon_1,\epsilon_i-\epsilon_{i-1}} = -1$ by Lemma 1.2.3. This expression we use inside E_1 to rewrite E as a linear combination of new monomials. In the first monomial in this linear combination (the one coming from $e_{\epsilon_{i-1}-\epsilon_1}^{(r_{i,1}+r_{i-1,1})} e_{\epsilon_i-\epsilon_{i-1}}^{(r_{i,1})}$) we see that every factor to the right of $e_{\epsilon_i-\epsilon_{i-1}}^{(r_{i,1})}$ commutes with $e_{\epsilon_i-\epsilon_{i-1}}^{(r_{i,1})}$, so we can move $e_{\epsilon_i-\epsilon_{i-1}}^{(r_{i,1})}$ all the way to the right end, thus getting something on the form $\mathcal{U}_{\mathbb{Z}}^{-} e_{\epsilon_i-\epsilon_{i-1}}^{(r_{i,1})}$ with $r_{i,1} > m_{i-1}$, so this term is on one of the correct forms. For the rest of the monomials we can commute $e_{\epsilon_i-\epsilon_{i-1}}^{(r_{i,j}-m)}$ to the left into its rightful place in $E_{i-1} \neq E_1$ to achieve a monomial $E' = E'_n \cdots E'_1$ with

$$\begin{aligned} \gamma(E'_1) &= \gamma(E_1) - (r_{i,j} - m)(\epsilon_i - \epsilon_{i-1}) \\ &= \gamma(E_1) + (r_{i,j} - m)(\epsilon_{i-1} - \epsilon_i) \succ \gamma(E_1) \end{aligned}$$

which by induction can be put on the right form.

Now we are left with the case where all the $r_{i,1}$ satisfy $r_{i,1} \leq m_{i-1}$. We use the induction hypothesis on $E_n \cdots E_2$ and the weight

$$\lambda' = \sum_{i=2}^{n-1} (m_i + r_{i,1} - r_{i+1,1}) \lambda_i$$

to rewrite E as a sum of monomials on the standard form, with the correct exponents, plus a linear combination of monomials on the form

$$E' = E'_n \cdots E'_2 e_{\epsilon_{i+1}-\epsilon_i}^{(r)} E_1$$

with $r > m_i + r_{i,1} - r_{i+1,1}$.

We notice that $e_{\epsilon_{i+1}-\epsilon_i}^{(r)}$ commutes with all the factors of E_1 except $e_{\epsilon_i-\epsilon_1}^{(r_{i,1})}$. So with the monomials of the last type we rewrite

$$E' = E_n' \cdots E_2' e_{\epsilon_n+1-\epsilon_1}^{(r_{n+1,1})} \cdots \underline{e_{\epsilon_{i+1}-\epsilon_i}^{(r)} e_{\epsilon_{i+1}-\epsilon_1}^{(r_{i+1,1})} e_{\epsilon_i-\epsilon_1}^{(r_{i,1})}} \cdots e_{\epsilon_2-\epsilon_1}^{(r_{2,1})}$$

Here we can apply (3.1) to the underlined factors above, since $r > m_i + r_{i,1} - r_{i+1,1}$ and thus $r + r_{i+1,1} \geq r_{i,1}$.

Now the first sum on the right hand side of (3.1) (when the underlined is plugged into the left hand side) consists of monomials ending on $e_{\epsilon_{i+1}-\epsilon_i}^{(r+r_{i+1,1}-j)}$ for $0 \leq j \leq r_{i,1}$. In the monomials that comes from these terms we can commute $e_{\epsilon_{i+1}-\epsilon_i}^{(r+r_{i+1,1}-j)}$ to the very right, and since $r+r_{i+1,1}-j > m_i+r_{i,1}-r_{i+1,1}+r_{i+1,1}-r_{i,1} = m_i$ we get monomials on the correct form.

In the second sum on the right hand side of (3.1) we get monomials on the form $e_{\epsilon_{i+1}-\epsilon_i}^{(r+j)} e_{\epsilon_{i+1}-\epsilon_1}^{(r_{i+1,1}-j)} e_{\epsilon_i-\epsilon_1}^{(r_{i,1}-j)}$ for $j > 0$. The leftmost factor here can now be commuted out of E_1 to the left giving us a new monomial E'' on the standard form $E'' = E_n'' \ldots E_1''$ with:

$$\begin{aligned} \gamma\left(E_1''\right) &= \gamma\left(E_1\right) - j\left(\epsilon_{i+1} - \epsilon_i\right) \\ &= \gamma\left(E_1\right) + j\left(\epsilon_i - \epsilon_{i+1}\right) \succ \gamma\left(E_1\right) \end{aligned}$$

We use induction on these terms, and have now transformed the monomial to a linear combination of the monomials on the desired form. $\square$

This proof together with the algorithm for putting elements on the standard form described in subsection 1.3.2 and Theorem 3.4.1 gives us an algorithm for how to write an element of $V(\lambda)$ as a linear combination of the basis vectors from Theorem 3.4.2 when Φ is of type A_n.

Notice also that (by Weyl's character formula) if one proves that there are

$$\sum_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$$

different vectors on the form described in Theorem 3.4.2 then the above theorem can be extended to a proof of Theorem 3.4.2.

3.5 Radicals and Ext groups

To further explore the Weyl modules and the simple modules we need some more tools.

3.5.1 The radical of a module

Definition 3.5.1. The *radical* $\operatorname{rad}(V)$ of a module V is the intersection of all the maximal proper submodules:

$$\operatorname{rad}(V) = \bigcap_{M \subsetneq V \text{ maximal submodule}} M$$

The *i'th radical* $\mathrm{rad}^i(V)$ of a module is defined inductively by:

$$\begin{aligned}\mathrm{rad}^1(V) &= \mathrm{rad}(V)\\ \mathrm{rad}^{i+1}(V) &= \mathrm{rad}\left(\mathrm{rad}^i(V)\right)\end{aligned}$$

The filtration $M \supsetneq \mathrm{rad}^1(M) \supsetneq \mathrm{rad}^2(M) \supsetneq \dots$ of M is called the *radical filtration*.

A module M is said to be semisimple if it is a direct sum of simple submodules.

Notice that for any G-module M of finite composition length, we have that $M/\mathrm{rad}(M)$ is semisimple. Furthermore M is semisimple if and only if $\mathrm{rad}(M) = 0$. That is $\mathrm{rad}(M)$ is the smallest submodule such that $M/\mathrm{rad}(M)$ is semisimple.

3.5.2 The Ext group

For later use we now define the Ext groups.

Definition 3.5.2. Let N, M be G-modules. On the class of short exact sequences of G-module homomorphisms of the form

$$0 \to N \to L \to M \to 0$$

we have an equivalence relation $\sim$ given by:

$$\begin{aligned} &0 \to N \to L \to M \to 0\\ \sim\ \ &0 \to N \to L' \to M \to 0\end{aligned}$$

if and only if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0\\ & & \downarrow \mathrm{id} & & \downarrow & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & N & \longrightarrow & L' & \longrightarrow & M & \longrightarrow & 0\end{array}$$

commutes for some map for $L \to L'$.

Let $\mathrm{Ext}^1_G(M, N)$ be the set of equivalence classes of $\sim$. $\mathrm{Ext}^1_G(M, N)$ is equipped with an addition operator (called the Baer sum) making $\mathrm{Ext}^1_G(M, N)$ an abelian group with 0-element

$$0 \to N \to N \oplus M \to M \to 0$$

where the two middle maps are $x \mapsto (x, 0)$ and $(x, y) \mapsto y$. $\mathrm{Ext}^1_G(N, M)$ is also equipped with a scalar multiplication, making $\mathrm{Ext}^1_G(N, M)$ a k-vector space. $\mathrm{Ext}^1_G(N, M)$ is called the *Ext group*.

If a short exact sequence is 0 in the Ext group we say that it is a *split* short exact sequence.

Notice that by the 5-lemma gives us that if

$$\begin{aligned} & 0 \to N \to L \to M \to 0 \\ \sim\ & 0 \to N \to L' \to M \to 0 \end{aligned}$$

for two G-modules L, L', then $L \cong L'$.

Now we want to find the Ext group $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$ between two simple modules $L(\lambda), L(\mu)$. We have the following nice results:

Proposition 3.5.3. *Let $\lambda, \mu \in X^+$, then*

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\mu), L(\lambda))$$

Furthermore if $\lambda \not\prec \mu$, then

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(\mathrm{rad}(V(\lambda)), L(\mu))$$

For a proof see II.2.12(4) and Proposition II.2.14 in [Jan03].

So to investigate the Ext groups we want to study $\mathrm{Hom}_G(\mathrm{rad}(V(\lambda)), L(\mu))$. Notice first that if $\varphi : \mathrm{rad}(V(\lambda)) \to L(\mu)$ is a non-zero G-homomorphism then φ must be surjective and $\ker(\varphi)$ is a maximal proper submodule of $\mathrm{rad}(V(\lambda))$. Thereby $\mathrm{rad}^2(V(\lambda)) = \mathrm{rad}(\mathrm{rad}(V(\lambda))) \supseteq \ker(\varphi)$. Which gives us that $\varphi = \varphi' \circ \pi$ for some G-homomorphism $\varphi' : \mathrm{rad}(V(\lambda)) / \mathrm{rad}^2(V(\lambda)) \to L(\mu)$, where $\pi : \mathrm{rad}(V(\lambda)) \to \mathrm{rad}(V(\lambda)) / \mathrm{rad}^2(V(\lambda))$ is the standard projection map. We can conclude

$$\mathrm{Hom}_G(\mathrm{rad}(V(\lambda)), L(\mu)) \cong \mathrm{Hom}_G\left(\mathrm{rad}(V(\lambda)) / \mathrm{rad}^2(V(\lambda)), L(\mu)\right)$$

Now since $\mathrm{rad}(V(\lambda)) / \mathrm{rad}^2(V(\lambda))$ is semisimple we have

$$\mathrm{rad}(V(\lambda)) / \mathrm{rad}^2(V(\lambda)) \cong \bigoplus_{\nu \prec \lambda} L(\nu)^{m(\nu)}$$

and thus by Schur's lemma we get the following result (using the convention $m(\nu) = 0$ for $\nu \not\prec \lambda$):

$$\begin{aligned} m(\nu) &= \dim\left(\mathrm{Hom}_G\left(\mathrm{rad}(V(\lambda)) / \mathrm{rad}^2(V(\lambda)), L(\nu)\right)\right) \\ &= \dim\left(\mathrm{Ext}^1_G(L(\nu), L(\mu))\right) \end{aligned} \tag{3.2}$$

This gives us the following result (notice that for any $\lambda, \mu \in X$ then either $\lambda \not\prec \mu$ or $\mu \not\prec \lambda$):

Proposition 3.5.4. *Let $\lambda, \mu \in X^+$ such that $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$ then either $\lambda \prec \mu$ or $\mu \prec \lambda$. Furthermore if $\mu \prec \lambda$, then:*

$$\left[\mathrm{rad}(V(\lambda)) / \mathrm{rad}^2(V(\lambda)) : L(\mu)\right] = \dim\left(\mathrm{Ext}^1_G(L(\lambda), L(\mu))\right)$$

3.6 The submodule structure of $V(\lambda)$

By the previous section we see that to find the dimension of the Ext group between two simple modules, we just need to find the composition coefficients $m(\nu)$ of $\mathrm{rad}(V(\lambda))/\mathrm{rad}^2(V(\lambda))$. These composition coefficients are of course closely related to the composition coefficients of $V(\lambda)$, namely $m(\nu) \leq [V(\lambda) : L(\nu)]$. But also to the submodule structure of $V(\lambda)$. This subsection builds mostly on [Alp80].

3.6.1 Preliminary lemmas

We want to describe the submodule structure of $V(\lambda)$ by the composition factors. This can be done very elegantly using diagrams. First we need some lemmas:

Lemma 3.6.1. *Let M be a G-module, and let E a simple G-module satisfying $[M : E] = 1$. If $M_1, M_2 \subseteq M$ are submodules with E as a composition factor, then $[M_1 \cap M_2 : E] = 1$.*

Proof. Assume otherwise that $[M_1 \cap M_2 : E] = 0$. Then we must have that $[M_1/(M_1 \cap M_2) : E] > 0$, since E is a composition factor of M_1 but not of $M_1 \cap M_2$. Now since $(M_1 + M_2)/M_2 \cong M_1/(M_1 \cap M_2)$ we get that $[(M_1 + M_2)/M_2 : E] > 0$. Since E is a composition factor of M_2 this gives us that $[M_1 + M_2 : E] > 1$, but that contradicts the fact that $[M : E] = 1$, since $M_1 + M_2 \subseteq M$ is a submodule of M. □

Lemma 3.6.2. *Let M be a finite dimensional G-module, and let E a simple G-module with $[M : E] = 1$. Define the submodule $t_E(M) \subseteq M$ to be*

$$t_E(M) = \bigcap_{\substack{M' \subseteq M \text{ submodule} \\ [M':E] \neq 0}} M'$$

Then $E \cong t_E(M)/\mathrm{rad}(t_E(M))$.

Proof. We start by noticing that $[t_E(M) : E] = 1$. This follows from the above lemma, together with the fact that since M is finite dimensional we can see the intersection defining $t_E(M)$ as a finite intersection. This observation also gives us that $t_E(M) \neq 0$.

Let $N \subsetneq t_E(M)$ be a maximal proper submodule of $t_E(M)$, then $[N : E] = 0$ since otherwise we would have $t_E(M) \subseteq N$ by the definition of $t_E(M)$. Thereby we get that $t_E(M)/N \cong E$. Now since $t_E(M)/\mathrm{rad}(t_E(M)) \cong E_1 \oplus \cdots \oplus E_m$ for some simple modules $E_1, \ldots, E_m$, we get by the above (since N was any maximal submodule of $t_E(M)$)

$$E_i \cong t_E(M)/N_i \cong E$$

for some maximal proper submodule N_i of $t_E(M)$. So $t_E(M)/\mathrm{rad}(t_E(M)) \cong E^m$, with $0 < m \leq [t_E(M) : E] = 1$, which concludes our proof. □

We need some properties of this $t_E(M)$.

Lemma 3.6.3. *Let M be a finite dimensional G-module, and let E be a simple G-module with $[M : E] = 1$. Let $U \subseteq M$ be a submodule such that $U/\mathrm{rad}(U) \cong E$, then $U = t_E(M)$.*

Proof. Since $[U : E] > 0$ we have $t_E(M) \subseteq U$. Assume now that $t_E(M) \neq U$, that is $t_E(M) \subsetneq U$. Since $U/\mathrm{rad}(U)$ is simple we must have that $\mathrm{rad}(U) \subsetneq U$ is a unique maximal submodule of U. Every proper submodule is contained in a maximal submodule, thus $t_E(M) \subseteq \mathrm{rad}(U)$. Thereby $[U/t_E(M) : E] \geq [U/\mathrm{rad}(U) : E] = 1$, but this gives us $[U : E] > 1$, which contradicts that $[M : E] = 1$. □

Since the only simple G-modules are the $L(\lambda)$ we will use the notation $t_\lambda(M) = t_{L(\lambda)}(M)$ for any G-module M with $[M : L(\lambda)] = 1$.

We now apply the above lemma on the Weyl modules.

Proposition 3.6.4. *Let $\lambda \in X^+$, then $t_\lambda(V(\lambda)) = V(\lambda)$. Furthermore if M is a G-module with $[M : L(\lambda)] = 1$, and there exists a non-zero homomorphism $\varphi : V(\lambda) \to M$, then $t_\lambda(M) = \varphi(V(\lambda))$.*

Proof. That $t_\lambda(V(\lambda)) = V(\lambda)$ follows directly from the above lemma, since $V(\lambda)/\mathrm{rad}(V(\lambda)) = L(\lambda)$ by Proposition 1.4.1.

And now to the second part of the proposition. Claim: $\mathrm{rad}(\varphi(V(\lambda))) = \varphi(\mathrm{rad}(V(\lambda)))$. We prove the more general statement: if M, N are G-modules with finite composition length, and there is a surjective map $\varphi : M \to N$, then $\mathrm{rad}(N) = \varphi(\mathrm{rad}(M))$.

$\supseteq$ We start by noticing that a proper submodule of N is maximal if and only if it is the kernel of a (non-zero) homomorphism from N to a simple module. That gives us:

$$\mathrm{rad}(N) = \bigcap_{\substack{f \in \mathrm{Hom}_G(N,S) \\ S \text{ simple}}} \ker(f)$$

Now clearly if $f \in \mathrm{Hom}_G(N, S)$ where S is simple, then $f \circ \varphi \in \mathrm{Hom}_G(M, S)$. This gives us

$$\mathrm{rad}(M) \subseteq \ker(f \circ \varphi)$$

which implies that

$$\varphi(\mathrm{rad}(M)) \subseteq \ker(f)$$

And since this holds for any $f \in \mathrm{Hom}_G(N, S)$ where S is simple, we get

$$\varphi(\mathrm{rad}(M)) \subseteq \mathrm{rad}(N)$$

which was what we wanted.

$\subseteq$ We start by considering the surjective homomorphism

$$M \overset{\varphi}{\twoheadrightarrow} N \twoheadrightarrow N/\varphi(\mathrm{rad}(M))$$

Then the kernel of the above homomorphism is $\mathrm{rad}(M) + \ker(\varphi)$. We recall that since M is of finite composition length, $M/\mathrm{rad}(M)$ is semisimple. We also have a homomorphism

$$M/\mathrm{rad}(M) \twoheadrightarrow M/(\mathrm{rad}(M) + \ker(\varphi)) \overset{\sim}{\to} N/\varphi(\mathrm{rad}(M))$$

Since the image of a homomorphism from a semisimple module is semisimple we get that $N/\varphi(\mathrm{rad}(M))$ is semisimple. And thus we get the claim by the fact that the radical of N is the smallest submodule such that $N/\mathrm{rad}(N)$ is semisimple.

And now back to the proof. Observe the surjective homomorphism

$$\psi : V(\lambda) \overset{\varphi}{\twoheadrightarrow} \varphi(V(\lambda)) \overset{\pi}{\twoheadrightarrow} \varphi(V(\lambda))/\mathrm{rad}(\varphi(V(\lambda)))$$

where π is the projection map. We now calculate the kernel of ψ:

$$V(\lambda) \supsetneq \ker(\psi) = \varphi^{-1}(\mathrm{rad}(\varphi(V(\lambda)))) = \varphi^{-1}(\varphi(\mathrm{rad}(V))) \supseteq \mathrm{rad}(V(\lambda))$$

Now by the two inclusions, the latter must be an equality, that is $\ker(\psi) = \mathrm{rad}(V(\lambda))$. And thus by the isomorphism theorem (on ψ) we get:

$$\varphi(V(\lambda))/\mathrm{rad}(\varphi(V(\lambda))) \cong V(\lambda)/\mathrm{rad}(V(\lambda)) \cong L(\lambda)$$

This gives us that $t_\lambda(M) = \varphi(V)$ by the lemma. $\square$

3.6.2 Diagrams for multiplicity free modules

Definition 3.6.5. A G-module M is called *multiplicity free* if $[M : L(\lambda)] \leq 1$ for all $\lambda \in X^+$. Let M be a multiplicity free G-module, we define:

$$\mathrm{comp}(M) = \left\{\lambda \in X^+ \,\middle|\, [M : L(\lambda)] \neq 0\right\}$$

Proposition 3.6.6. *Let M be a multiplicity free G-module. Let $\lambda, \mu \in \mathrm{comp}(M)$, then the following statements are equivalent:*

1. $\left[\mathrm{rad}(t_\lambda(M))/\mathrm{rad}^2(t_\lambda(M)) : L(\mu)\right] \neq 0$.
2. $\mathrm{Hom}_G(\mathrm{rad}(t_\lambda(M)), L(\mu)) \neq 0$.
3. *There exists submodules $V \subseteq U \subseteq M$ such that*

$$0 \to L(\mu) \to U/V \to L(\lambda) \to 0$$

is a non-split exact sequence.

Proof. In all of this proof we use the notation $t_\lambda = t_\lambda(M)$.

1 $\Rightarrow$ *2* : Assume $\left[\mathrm{rad}(t_\lambda)/\mathrm{rad}^2(t_\lambda) : L(\mu)\right] \neq 0$. We recall that for any G-module N: $\mathrm{rad}(N)/\mathrm{rad}^2(N)$ is semisimple. Thus $\mathrm{rad}(t_\lambda)/\mathrm{rad}^2(t_\lambda) \cong L(\mu) \oplus N$, for some submodule N. Thus we have a non-zero homomorphism

$$\varphi : \mathrm{rad}(t_\lambda) \to \left(\mathrm{rad}(t_\lambda)/\mathrm{rad}^2(t_\lambda)\right)/N \cong L(\mu)$$

2 $\Rightarrow$ *3* : Let $\varphi : \mathrm{rad}(t_\lambda) \to L(\mu)$ be a non-zero homomorphism. Let $U = t_\lambda$ and let $V = \ker(\varphi) \subsetneq \mathrm{rad}(t_\lambda) \subsetneq t_\lambda = U \subseteq M$. Now by the isomorphism theorem ($L(\mu)$ is simple so φ must be surjective) we get $L(\mu) \cong \mathrm{rad}(t_\lambda)/V$ which can be included in U/V. U/V on the other hand projects onto $U/\mathrm{rad}(U) \cong L(\lambda)$, with kernel $\mathrm{rad}(t_\lambda)/V$ by Lemma 3.6.2. We now have a short exact sequence:

$$0 \to L(\mu) \to U/V \to L(\lambda) \to 0$$

which is clearly non-split since $V \subsetneq \mathrm{rad}(U) = \mathrm{rad}(t_\lambda)$.

3 $\Rightarrow$ *1* : Let $V \subseteq U \subseteq M$ be submodules such that

$$0 \to L(\mu) \xrightarrow{d_1} U/V \xrightarrow{d_2} L(\lambda) \to 0$$

is exact and non-split. Let W be the kernel of the map $U \twoheadrightarrow U/V \twoheadrightarrow L(\lambda)$. Then $V \subseteq W \subseteq U$, and $U/W \cong L(\lambda)$. Furthermore

$$L(\mu) \cong d_1(L(\mu)) = \ker(d_2) \cong W/V$$

Since $\mathrm{comp}(U/V) = \{\lambda, \mu\}$ and the exact sequence is non-split we get that W is the only strictly intermediate module of V, U (only module satisfying $V \subsetneq W \subsetneq U$).

Clearly $[U : L(\lambda)] \neq 0$, so $t_\lambda \subseteq U$. And clearly $[W : L(\lambda)] = 0$, so $t_\lambda \nsubseteq W$. These give us $V \subsetneq V + t_\lambda \subseteq U$ with $V + t_\lambda \neq W$. Thereby $U = V + t_\lambda$ and thus:

$$U/V = (V + t_\lambda)/V \cong t_\lambda/(t_\lambda \cap V)$$

Now $\mathrm{comp}(t_\lambda/(t_\lambda \cap V)) = \{\lambda, \mu\}$. Since t_λ has a unique maximal submodule we have $t_\lambda \cap V \subseteq \mathrm{rad}(t_\lambda)$. We have $\mathrm{rad}(t_\lambda)/(t_\lambda \cap V) \cong L(\mu)$, making $t_\lambda \cap V$ a maximal submodule of $\mathrm{rad}(t_\lambda)$, which gives us that $\mathrm{rad}^2(t_\lambda) \subseteq t_\lambda \cap V$, and thereby

$$\left[\mathrm{rad}(t_\lambda)/\mathrm{rad}^2(t_\lambda) : L(\mu)\right] \geq [\mathrm{rad}(t_\lambda)/(t_\lambda \cap V) : L(\mu)] = 1$$

This concludes our proof. $\square$

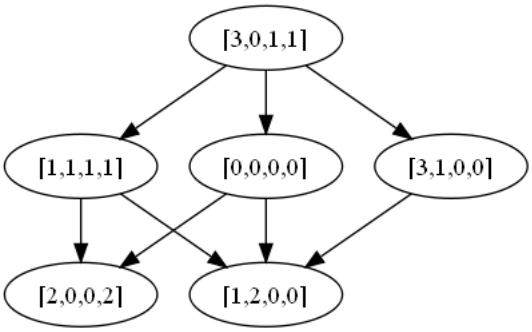
We are now ready to define the diagram of a finite multiplicity free G-module.

Definition 3.6.7. Let M be a finite dimensional multiplicity free G-module. We define the *diagram* $\mathcal{D}(M)$ of M to be the directed graph with vertices $\mathrm{comp}(M)$, and with an arc from $\lambda \in \mathrm{comp}(M)$ to $\mu \in \mathrm{comp}(M)$ if and only if λ, μ satisfy the equivalent conditions *1*, *2*, *3* of the above proposition.

Let $C \subseteq \mathrm{comp}(M)$, then we call C *arrow closed* or a *bottom diagram* if it satisfies that if $\lambda \in C$ and there is an arc from λ to $\mu \in \mathrm{comp}(M)$, then also $\mu \in C$.

Let $O \subseteq \mathrm{comp}(M)$ then we call O *arrow open* or a *top diagram* if it satisfies that if $\lambda \in C$ and there is an arc from $\mu \in \mathrm{comp}(M)$ to λ, then also $\mu \in C$.

To give an idea of what a diagram looks like, here is an example of a typical diagram, namely of $V([3,0,1,1])$, which we will later calculate in detail.



Notice that the set of arrow closed subsets and arrow open subsets of $\mathrm{comp}(M)$ makes $\mathrm{comp}(M)$ a topological space. That is they are closed under $\cap$ and $\cup$, all of the diagram and the empty diagram are both both arrow open and arrow closed, and the arrow open subdiagrams are exactly those with arrow closed complement.

And now for the great correspondence (see Theorem 1 of [Alp80]):

Theorem 3.6.8. *Let M be a finite dimensional multiplicity free G-module. Then the map* comp *from the set of G-submodules of M to the set of subsets of* $\mathrm{comp}(M)$ *is injective and has image equal to the set of arrow closed subsets of* $\mathrm{comp}(M)$. *That is* comp *describes a one-to-one correspondence between the G-submodules of M and the arrow closed subsets of* $\mathrm{comp}(M)$.

3.6.3 Finding the diagrams

Now we want to calculate how exactly the diagrams should look. To do this we have some efficient tools:

Proposition 3.6.9. *Let M be a finite dimensional multiplicity free G-module and let $N \subseteq M$ be a G-submodule. Then the diagram of N is just the arrow closed subdiagram of M of all the vertices* $\mathrm{comp}(N)$. *And the diagram of M/N is just the arrow open subdiagram of M of all the vertices* $\mathrm{comp}(M/N) = \mathrm{comp}(M) \setminus \mathrm{comp}(N)$.

The first statement follows immediately from Theorem 3.6.8, and the second follows from Theorem 3.(4) of [Alp80].

To get the next rule, we need some more insight into the sum formula.

Lemma 3.6.10. *Let $\lambda \in X^+$ such that $V(\lambda)$ is multiplicity free. Let $V(\lambda) = V^0 \supseteq V^1 \supseteq V^2 \supseteq \cdots$ be the Jantzen filtration. For each i there exists a non-degenerate symmetric bilinear form*

$$(\cdot,\cdot)_i : \left(V^i/V^{i+1}\right) \times \left(V^i/V^{i+1}\right) \to k$$

satisfying:

$$\begin{aligned} (e_\alpha v, v')_i &= (v, e_{-\alpha} v')_i \text{ for all } v, v' \in V^i/V^{i+1}, \text{ and } \alpha \in \Phi \\ \left(V^i/V^{i+1}\right)_\mu &\perp \left(V^i/V^{i+1}\right)_\xi \text{ for } \mu \neq \xi \end{aligned}$$

This can be proved by following the proof of a similar result for Verma modules in [Jan79], Satz 5.3.

Lemma 3.6.11. *Assume M is a multiplicity free finite dimensional G-module (and thus $\mathcal{U}_k$-module) with a non-degenerate symmetric bilinear form*

$$(\cdot,\cdot) : M \times M \to k$$

satisfying:

$$\begin{aligned} (e_\alpha v, v') = (v, e_{-\alpha} v') \text{ for all } v, v' \in M, \text{ and } \alpha \in \Phi \\ M_\mu \perp M_\xi \text{ for } \mu \neq \xi \end{aligned}$$

Then M is semisimple.

Proof. We prove this by induction in $\dim(M)$. If M is simple we are done. Otherwise let $E \subsetneq M$ be a simple submodule.

Claim: $E^\perp \subsetneq M$ is a non-trivial submodule. This follows easily from (for $x \in E^\perp$):

$$(\mathcal{U}_k x, E) = (x, \mathcal{U}_k E) = (x, E) = 0$$

and non-triviality follows from non-degeneracy. By induction $E^\perp$ is now semisimple, so all we need to prove is $M = E \oplus E^\perp$.

Claim: $\dim(M_\lambda) = \dim(E_\lambda) + \dim\left(\left(E^\perp\right)_\lambda\right)$ for any weight $\lambda \in X$. To prove this claim we notice that when $(\cdot,\cdot)$ is non-degenerate we have that the map $\varphi : M \to M^*$ given by $\varphi(x) = (x,\cdot)$ in an injective linear map, and thus a vector space isomorphism (since $\dim(M) = \dim(M^*)$). Now since the weight spaces of M are orthogonal we get that the restriction of φ to M_λ maps isomorphically to M_λ^*. By the injectivity of φ we get $\dim(\varphi(E_\lambda)) = \dim(E_\lambda) = \dim(E_\lambda^*)$. And now we have

$$\left(\left(E^\perp\right)_\lambda\right)^* \cong M_\lambda^*/\varphi(E_\lambda) \cong M_\lambda^*/E_\lambda^*$$

And thus we have the claim.

The claim now gives us that $\operatorname{ch}(M) = \operatorname{ch}(E) + \operatorname{ch}\left(E^\perp\right)$, and thereby that $\operatorname{ch}\left(M/E^\perp\right) = \operatorname{ch}(E)$. Since we assumed M to be multiplicity free, this gives us that $\left[E^\perp : E\right] = 0$, and thus $E \cap E^\perp = 0$ (since $E \cap E^\perp \subseteq E$ is a submodule of the simple module E). By the claim $\dim(M) = \dim(E) + \dim\left(E^\perp\right)$ so since E and $E^\perp$ intersects trivially we get $M = E \oplus E^\perp$ and we conclude our proof. $\square$

Lemma 3.6.12. *Let M be a finite dimensional multiplicity free G-module. Then M is semisimple if and only if there are no arcs in $\mathcal{D}(M)$.*

Proof. We prove each direction separately.

- $\Rightarrow$ Let M be semisimple, and let $\lambda, \mu \in \operatorname{comp}(M)$ with $\lambda \neq \mu$. We prove that there is no arc from λ to μ. Since $\lambda \in \operatorname{comp}(M)$ and M is semisimple, then $L(\lambda) \subseteq M$ is a submodule. And thus $t_\lambda(M) = L(\lambda)$ so:

$$[t_\lambda(M)/\operatorname{rad}(t_\lambda(M)) : L(\mu)] = [L(\lambda) : L(\mu)] = 0$$

  since $\mu \neq \lambda$, and thus there is no arc from λ to μ.

- $\Leftarrow$ Let M be such that there are no arcs in $\mathcal{D}(M)$. We prove the result by induction in the composition length of M ($|\operatorname{comp}(M)|$). Let $N \subsetneq M$ be a maximal submodule. By induction it is enough to prove that $M \cong N \oplus M/N$, since M/N is simple, and $\operatorname{comp}(N) \subsetneq \operatorname{comp}(M)$. Since M/N is simple there is a $\lambda \in \operatorname{comp}(M)$, such that $\operatorname{comp}(M/N) = \{\lambda\}$. And since there are no arcs in $\mathcal{D}(M)$ the set $\{\lambda\}$ is arrow closed, and thus we can see $L(\lambda)$ as a subset of M. Now when M is multiplicity free and $\{\lambda\} = \operatorname{comp}(M/N) = \operatorname{comp}(M) \setminus \operatorname{comp}(N)$, we must have that $N \cap L(\lambda) = 0$, so $M = N \oplus L(\lambda)$, since $\operatorname{comp}(M) = \operatorname{comp}(N) \cup \{\lambda\}$.

This concludes our proof. □

Proposition 3.6.13. *Let $\lambda \in X^+$ such that $V(\lambda)$ is multiplicity free. Let $V(\lambda) = V^0 \supseteq V^1 \supseteq V^2 \supseteq \cdots$ be the Jantzen filtration. Then in the diagram $\mathcal{D}(V(\lambda))$ of $V(\lambda)$ there is no arc from $\mu \in \operatorname{comp}(V^i)$ to $\xi \in \operatorname{comp}(V(\lambda)/V^{i+1})$. Moreover there is an arc from λ to μ for all $\mu \in \operatorname{comp}(V^1/V^2)$.*

Proof. By the three just proved lemmas we clearly get that there are no arcs between $\mu \in \operatorname{comp}(V^i/V^{i+1})$ and $\xi \in \operatorname{comp}(V^i/V^{i+1})$. Let $\mu \in \operatorname{comp}(V^i)$ and $\xi \in \operatorname{comp}(V(\lambda)/V^{i+1})$, by the above argument we may assume that $\mu \notin \operatorname{comp}(V^i/V^{i+1})$ or $\xi \notin \operatorname{comp}(V^i/V^{i+1})$. Assume without loss of generality that

$$\begin{aligned}
\xi &\in \operatorname{comp}\left(V(\lambda)/V^{i+1}\right) \setminus \operatorname{comp}\left(V^i/V^{i+1}\right) \\
&= \operatorname{comp}\left(\left(V(\lambda)/V^{i+1}\right) / \left(V^i/V^{i+1}\right)\right) \\
&= \operatorname{comp}\left(V(\lambda)/V^i\right) \\
&= \operatorname{comp}(V(\lambda)) \setminus \operatorname{comp}\left(V^i\right)
\end{aligned}$$

in particular $\xi \notin \operatorname{comp}(V^i)$. Since V^i is a submodule $\operatorname{comp}(V^i)$ must be arrow closed, and thus there can be no arc from $\lambda \in V^i$ to $\xi \notin V^i$.

And now for the second part of the proof. We know from Proposition 1.4.1 that $V^1 = \operatorname{rad}(V(\lambda))$ since $V(\lambda)/V^1 \cong L(\lambda)$. Now since V^1/V^2 is semisimple we must have that $\operatorname{rad}^2(V(\lambda)) = \operatorname{rad}(V^1) \subseteq V^2$. Thereby $\operatorname{comp}(V^1/V^2) \subseteq \operatorname{comp}\left(\operatorname{rad}(V(\lambda))/\operatorname{rad}^2(V(\lambda))\right)$. This proves that there is an arc from λ to all of $\operatorname{comp}(V^1/V^2)$ by *2* of the definition of the arcs (Proposition 3.6.6) and Proposition 3.6.4. □

To better understand the process of finding the diagram for $V(\lambda)$ we do an example.

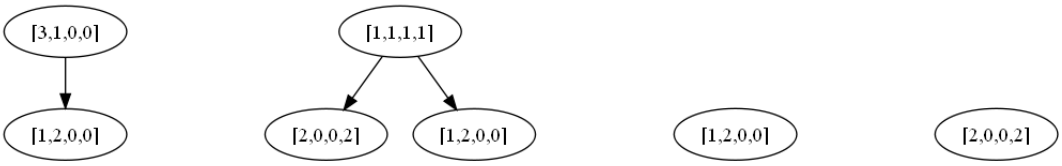
Example 3.6.14. As an example we will construct the diagram for the module $V([3,0,1,1])$, when Φ is of type A_4 and $p = 3$. Much of the needed data can be found in the appendices. We start by noticing that $V([3,0,1,1])$ is multiplicity free, and that $\operatorname{comp}(V([3,0,1,1]))$ is the following set:

$$\{[3,0,1,1], [3,1,0,0], [1,1,1,1], [1,2,0,0], [2,0,0,2], [0,0,0,0]\}$$

Using Theorem 3.3.2, we find out for which $\mu \in \operatorname{comp}(V(\lambda)) \setminus \{\lambda\}$ there is a non-zero homomorphism $\varphi : V(\mu) \to V(\lambda)$. This is the case for $[3,1,0,0]$, $[1,1,1,1]$, $[1,2,0,0]$ and $[2,0,0,2]$. Now assume we have found the diagram for all Weyl modules of each of these weights (see appendix B). Knowing a basis for each of the Weyl modules (Theorem 3.4.2), we can convert the homomorphisms to matrices, and calculate the rank, that is the dimension of the image. The needed data can be seen in the following table, where $\varphi : V(\mu) \to V(\lambda)$ is a (non-zero if possible) homomorphism:

μ	$\dim(V(\mu))$	$\dim(L(\mu))$	$\dim(\varphi(V(\mu)))$
$[3,0,1,1]$	1050	150	1050
$[3,1,0,0]$	224	50	224
$[1,1,1,1]$	1024	476	849
$[1,2,0,0]$	175	174	174
$[2,0,0,2]$	200	199	199
$[0,0,0,0]$	1	1	0

We now want to find the arrow open subdiagrams of $\mathcal{D}(V(\mu))$ corresponding to $V(\mu)/\ker(\varphi) \cong \varphi(V(\mu))$, corresponding to an arrow closed subdiagram of $\mathcal{D}(V(\lambda))$. By the data in the above table we calculate that there is only one possible arrow open subdiagram $\mathcal{D}(V(\mu))$ giving the correct dimension for each of the μ. The subdiagrams are the following:



Now we calculate the sum formula for $V(\lambda)$:

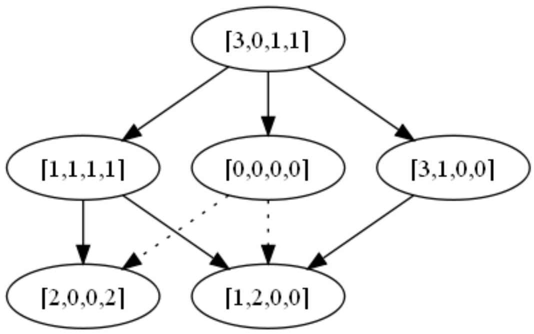
$$\sum_{i>0} \operatorname{ch}(V^i) = 2\operatorname{ch}L([1,2,0,0]) + 2\operatorname{ch}L([2,0,0,2])$$
$$+ \operatorname{ch}L([3,1,0,0]) + \operatorname{ch}L([1,1,1,1]) + \operatorname{ch}L([0,0,0,0])$$

Since $V(\lambda)$ is multiplicity free this gives us:

$$\operatorname{comp}(V^1/V^2) = \{[3,1,0,0], [1,1,1,1], [0,0,0,0]\}$$
$$\operatorname{comp}(V^2/V^3) = \{[1,2,0,0], [2,0,0,2]\}$$

and $V^3 = 0$.

We now merge the known subdiagrams, and add the information given by the V^i's (Proposition 3.6.13) putting dotted arrows where we still don't know if there is an arc:



Now this leaves four possible diagrams, but since there is no non-zero homomorphism from $V([0,0,0,0])$ to $V([3,0,1,1])$, we immediately rule out the possibility that none of the arcs are there.

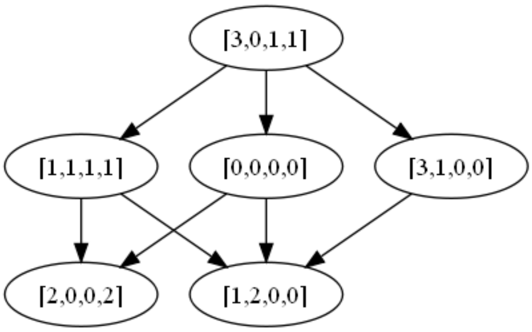
The theorem used for finding homomorphisms from $V(\mu)$ to $V(\lambda)$ (Theorem 3.3.2), can actually be generalized so it can be used to find homomorphisms from $V(\mu)$ to M for any G-module M (see Lemma II.2.13.a in [Jan03]). This can be used to find out which arcs are there.

If there is a non-zero homomorphism from $V([0,0,0,0])$ to the quotient module $V([3,0,1,1])/L([1,2,0,0])$, then the set $\{[0,0,0,0]\}$ corresponds to a submodule in $V([3,0,1,1])/L([1,2,0,0])$ (since $V([0,0,0,0]) = L([0,0,0,0])$), and thus $\{[0,0,0,0]\}$ is arrow closed in $\mathcal{D}(V([3,0,1,1])/L([1,2,0,0]))$. Thereby there cannot be an arc from $[0,0,0,0]$ to $[2,0,0,2]$.

Likewise if there is a non-zero homomorphism from $V([0,0,0,0])$ to the quotient module $V([3,0,1,1])/L([2,0,0,2])$, then the set $\{[0,0,0,0]\}$ corresponds to a submodule in $V([3,0,1,1])/L([2,0,0,2])$, and thus $\{[0,0,0,0]\}$ is arrow closed in $\mathcal{D}(V([3,0,1,1])/L([2,0,0,2]))$. Thus there cannot be an arc from $[0,0,0,0]$ to $[2,0,0,2]$.

If however 0 is the only homomorphism from $V([0,0,0,0])$ to any of the two quotient modules, then $\{[0,0,0,0]\}$ isn't arrow closed in any of the two aforementioned subdiagrams, that is both arcs must be there.

Doing the calculations we find out that it is the latter case which happens, so we finally can conclude that the diagram for $V([3,0,1,1])$ looks like:



In appendix B can be seen the diagrams all the Weyl modules for which we calculated the simple characters in appendix A, except those $V(\lambda)$ with composition length $|\mathrm{comp}(V(\lambda))| \leq 2$, in which cases there are only one possible diagram ($\{\lambda\}$ is always the only arrow open singleton set by Proposition 1.4.1). Notice that this excludes all of A_2 for $p = 3$ and A_3 for $p = 2$.

3.6.4 The diagram of a module

It is of course not enough to be able to describe the submodule structure of the multiplicity free modules. Luckily it is possible generalize the concept of the diagram of a finite dimensional G-module.

Definition 3.6.15. Let D be directed graph with no directed loops and no multiple arcs between the same pair of vertices. If for any chain of at least three connected vertices $x_1 \to x_2 \to \ldots \to x_m$ for $m \geq 3$ ($x_i \to x_j$ denotes that there is an arc from x_i to x_j), there is no arc from x_1 to x_m ($x_1 \nrightarrow x_m$), then we call D a *module diagram*.

For a module diagram we can define arrow closed and arrow open just like in the previous subsections.

The radical of a module diagram is the intersection of all the maximal (proper) arrow closed subdiagrams.

Let M be a finite dimensional G-module, and let D be a module diagram. Then we call D a *diagram of* M if the number of vertices of D is equal to the composition length of M, and there is a function δ from the set of arrow closed subdiagrams of D to the submodules of M, such that δ sends unions to sums, and preserves both intersection and radicals.

Notice that the diagrams defined for multiplicity free modules are also diagrams of the modules in this sense.

In Theorem 3 of [Alp80] is a list of properties of the diagrams of modules, summarized in the following proposition.

Proposition 3.6.16. *If M is a G-module and D is a diagram of G with δ the associated map, and with vertex set V, then:*

- $\delta(\emptyset) = 0$.

- *δ preserves inclusions.*
- *δ is injective.*
- *If $U \subseteq V$ is an arrow closed subset of the vertices, then the induced subdiagram with vertices U and map δ restricted to the arrow closed subsets of U is a diagram of the module $\delta(U) \subseteq M$.*
- *If $U \subseteq V$ is an arrow closed subset of the vertices, then the induced subdiagram with vertices V/U and map δ restricted to the arrow closed subsets of V/U is a diagram of the module $M/\delta(U)$.*
- *There exists a labeling $\lambda : V \to X^+$ of the vertices, such that if $v \in V$ is a vertex, and $v \in U \subseteq V$ is an arrow closed subset of the vertices such that $U \setminus \{v\}$ also is arrow closed, then $\delta(U)/\delta(U \setminus \{v\}) \cong L(\lambda(v))$.*

It is easily seen that the labeling λ must satisfy

$$[M : L(\mu)] = |\{v \in V \mid \lambda(v) = \mu\}|$$

Furthermore we see that we can use a method similar to the one from the previous subsection to find a diagram of any G-module M, as long as we are extra careful around the $\lambda \in X^+$ with $[M : L(\lambda)] > 1$.

In my calculations there is only one Weyl module $V([2,1,1,2])$ (in type A_4, $p = 3$) that is not multiplicity free. The diagram of this module can also be found in appendix B.

3.7 Ext groups and Kazhdan-Lusztig polynomials

We recall from Proposition 3.5.4 that if $\mu \prec \lambda$, then

$$\dim\left(\operatorname{Ext}^1_G(L(\lambda), L(\mu))\right) = \left[\operatorname{rad}(V(\lambda))/\operatorname{rad}^2(V(\lambda)) : L(\mu)\right]$$

and $\dim\left(\operatorname{Ext}^1_G(L(\lambda), L(\mu))\right) = 0$ in all other cases. If we now combine Proposition 3.6.4 with *2* of Proposition 3.6.6, we see that, in the case where $\mu \prec \lambda$ and $V(\lambda)$ is multiplicity free, the dimension of $\operatorname{Ext}^1_G(L(\lambda), L(\mu))$ is 1 if and only if there is an arc from λ to μ in $\mathcal{D}(V(\lambda))$.

In general it can easily be seen that if $\mu \prec \lambda$ then $\dim\left(\operatorname{Ext}^1_G(L(\lambda), L(\mu))\right)$ is equal to the number of arcs from the vertex labeled λ to vertices labeled μ i the diagram of $V(\lambda)$.

So why are we interested in the dimension of these Ext groups? For one thing they have a connection to Lusztigs conjecture. Namely Proposition II.C.2 of [Jan03] (combined with Proposition II.C.10 also from [Jan03]) gives us the following result:

Proposition 3.7.1. *Let $\lambda \in A_0 \cap X$ be a regular weight, and let $w, w' \in \mathcal{W}$ such that $w.\lambda, w'.\lambda \in X^+$ with $w'.\lambda \prec w.\lambda$. If the character of $L(x.\lambda)$ is given by Lusztig's character formula for all $x \preceq w$. Then:*

- *If* $l(w) - l(w')$ *is even then* $\operatorname{Ext}^1_G(L(w.\lambda), L(w'.\lambda)) = 0$.
- *If* $l(w) - l(w')$ *is odd then we get*

$$\dim\left(\operatorname{Ext}^1_G(L(w.\lambda), L(w'.\lambda))\right) = [P_{w_0w', w_0w}(q)]_{\left(\frac{l(w)-l(w')-1}{2}\right)}$$

where $P_{w_0w',w_0w}(q)$ *is the Kazhdan-Lusztig polynomial,* $w_0 \in W$ *is the longest element, and* $[P(q)]_m$ *means the coefficient of* q^m *in the polynomial* $P(q)$.

It is now natural to see if we can generalize this to any weight $\lambda \in \overline{A_0} \cap X$ and not just the regular weights, just like we asked for Lusztig's conjecture.

Question 3.7.2. Let $\lambda \in \overline{A_0} \cap X$, and let $w, w' \in \mathcal{W}$ such that $w.\lambda, w'.\lambda \in X^+$, $w.\lambda \in \widehat{w.A_0}$, $w'.\lambda \in \widehat{w'.A_0}$, $w'.\lambda \prec w.\lambda$.

When is there the above connection between the parity of $l(w) - l(w')$, the Kazhdan-Lusztig-polynomials and the dimension of $\operatorname{Ext}^1_G(L(w.\lambda), L(w'.\lambda))$?

To answer the question we have (as usual) calculated the actual values in the usual cases (A_n, $n \in \{2,3,4\}$, $p \in \{2,3\}$). And the data suggests that there is a clear connection.

The result we got was that when $L(w.\lambda)$ was given by Lusztig's character formula, then the connection from Proposition 3.7.1 did indeed hold. While in the two exceptions the connections did not hold at all:

Let Φ be of type A_4 with $p = 2$, and $\lambda = [0, -1, -1, 0] \in \overline{A_0} \cap X$. For $w \in \mathcal{W}$ such that $w.\lambda = [0, 1, 1, 0]$, and $w.\lambda \in \widehat{w.A_0}$, we have that

$$\dim\left(\operatorname{Ext}^1_G(L([0,1,1,0]), L([0,0,0,0]))\right) = 1$$

even though $l(w) - l(w') = 6$ is even, when $w' \in \mathcal{W}$ such that $w'.\lambda = [0,0,0,0]$ and $w'.\lambda \in \widehat{w'.A_0}$.

Let Φ be of type A_4 with $p = 3$, and $\lambda = [0, -1, -1, 0] \in \overline{A_0} \cap X$. For $w \in \mathcal{W}$ such that $w.\lambda = [1, 2, 2, 1]$, and $w.\lambda \in \widehat{w.A_0}$, we have that

$$\dim\left(\operatorname{Ext}^1_G(L([1,2,2,1]), L([0,2,2,0]))\right) = 0$$

even though $l(w) - l(w') = 1$ is odd and $[P_{w_0w',w_0w}(q)]_{\left(\frac{l(w)-l(w')-1}{2}\right)} = 1$, when $w' \in \mathcal{W}$ such that $w'.\lambda = [0,0,0,0]$ and $w'.\lambda \in \widehat{w'.A_0}$. Furthermore we have that

$$\dim\left(\operatorname{Ext}^1_G(L([1,2,2,1]), L([0,1,1,0]))\right) = 1$$

even though $l(w) - l(w') = 6$ is even, when $w' \in \mathcal{W}$ such that $w'.\lambda = [0,1,1,0]$ and $w'.\lambda \in \widehat{w'.A_0}$. Finally we have that

$$\dim\left(\operatorname{Ext}^1_G(L([1,2,2,1]), L([1,0,0,1]))\right) = 0$$

which fits the fact that $l(w) - l(w') = 7$ is odd and $[P_{w_0 w', w_0 w}(q)]_{\left(\frac{l(w)-l(w')-1}{2}\right)} = 0$, when $w' \in \mathcal{W}$ such that $w'.\lambda = [1, 0, 0, 1]$ and $w'.\lambda \in \widehat{w'.A_0}$.

A closer inspection of these exceptions shows that if the characters of $L(x.\lambda)$ were all given by Lusztig's character formula (for all $x.\lambda \in X^+$ such that $x.\lambda \preceq w.\lambda$), then we would indeed have that the dimensions of the Ext groups were given by the parity of $l(w) - l(w')$, and the Kazhdan-Lusztig polynomials like in Proposition 3.7.1.

So it seems that there is a strong connection between when Lusztig's character formula holds, and when we have the aforementioned connection between the dimension of the Ext groups, and the parity of $l(w) - l(w')$ and the coefficients of the Kazhdan-Lustig Polynomials.

Chapter 4

Braden MacPherson sheaves

4.1 Moment graphs and sheaves

The representation theory of algebraic groups is closely tied to the theory of sheaves on moment graphs. In this section we will primarily give the definitions needed to use this approach.

Definition 4.1.1. Let V be a vector space. A V*-moment graph* (or simple *moment graph*) G is a 4-tuple: $G = (\mathcal{V}, \mathcal{E}, \leq, \tau)$, where:

- $(\mathcal{V}, \mathcal{E})$ is a simple graph, with vertices $\mathcal{V}$ and edges $\mathcal{E}$.
- $\leq$ is a partial ordering of $\mathcal{V}$, satisfying that $u, v \in \mathcal{V}$ are comparable by $\leq$ if $\{u, v\} \in \mathcal{E}$.
- $\tau : \mathcal{E} \to \mathcal{P}(V)$ is a labeling of the edges by 1-dimensional subspaces of V.

For now let's fix a finite dimensional vector space V, later we will worry about which vector space to choose.

Let $S = S(V)$ denote the symmetric algebra of V. We equip S with the usual grading, that is to say the elements of $V \subseteq S$ are the elements of degree 1.

It is in general not required of a moment graph to be a finite graph, however the ones we will be needing are all finite, so to make things easier we from now on require moment graphs to be finite.

An important type of moment graphs is the Goresky-Kottwitz-MacPherson-graph or the GKM-graph:

Definition 4.1.2. Let $G = (\mathcal{V}, \mathcal{E}, \leq, \tau)$ be a moment graph. We say that G is a *GKM-graph* if for any pair of edges $E, F \in \mathcal{E}$ meeting in exactly one vertex we have: $\tau(E) \neq \tau(F)$.

We can now define a sheaf on a moment graph.

Definition 4.1.3. Let $G = (\mathcal{V}, \mathcal{E}, \leq, \tau)$ be a moment graph. A 3-tuple $\mathcal{M} = \left((\mathcal{M}_x)_{x\in\mathcal{V}}, (\mathcal{M}_E)_{E\in\mathcal{E}}, (\rho_{x,E})_{x\in E\in\mathcal{V}}\right)$ is called a *G-sheaf*, if:

- $\mathcal{M}_x$ are graded S-modules for each $x \in \mathcal{V}$,
- $\mathcal{M}_E$ are graded S-modules satisfying $\tau(E)\mathcal{M}_E = 0$ (under the natural inclusion $\tau(E) \subseteq V \subseteq S$) for each $E \in \mathcal{E}$,
- $\rho_{x,E} : \mathcal{M}_x \to \mathcal{M}_E$ are graded S-module homomorphisms, for each $x \in E \in \mathcal{E}$.

The basic example of a sheaf on a moment graph is that of the structure sheaf:

Example 4.1.4. Let $G = (\mathcal{V}, \mathcal{E}, \leq, \tau)$ be a moment graph. The structure sheaf $\mathcal{A} = \mathcal{A}(G)$ is the G-sheaf with

- $\mathcal{A}_x = S$ for all $x \in \mathcal{V}$,
- $\mathcal{A}_E = S/\tau(E)S$ for all $E \in \mathcal{E}$,
- $\rho_{x,E} : S \to S/\tau(E)S$ are the quotient homomorphisms for all $x \in E \in \mathcal{E}$.

When we have a G-sheaf we can define the modules of sections:

Definition 4.1.5. Let $G = (\mathcal{V}, \mathcal{E}, \leq, \tau)$ be a moment graph, and let $\mathcal{M}$ be a G-sheaf. For $\mathcal{U} \subseteq \mathcal{V}$ we define the module of *local sections* $\mathcal{M}(\mathcal{U})$:

$$\mathcal{M}(\mathcal{U}) = \left\{ (m_x)_{x\in\mathcal{U}} \in \bigoplus_{x\in\mathcal{U}} \mathcal{M}_x \,\middle|\, \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \text{ for each } \{x,y\} = E \in \mathcal{E}_{|\mathcal{U}} \right\}$$

where $\mathcal{E}_{|\mathcal{U}} = \{\{x,y\} \in \mathcal{E} \mid x, y \in \mathcal{U}\}$.

Furthermore if $\mathcal{U} = \mathcal{V}$, we call $\Gamma(\mathcal{M}) = \mathcal{M}(\mathcal{V})$ the set of global sections.

Notice these are called modules since they are S-modules.

For $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \mathcal{V}$ we define the projection:

$$\pi_{\mathcal{U}_1}^{\mathcal{U}_2} : \bigoplus_{v\in\mathcal{U}_2} \mathcal{M}_v \to \bigoplus_{v\in\mathcal{U}_1} \mathcal{M}_v$$

Lemma 4.1.6. *Let G be a moment graph, and let $\mathcal{M}$ be a sheaf on G. Let $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_3 \subseteq \mathcal{V}$ be sets of vertices. Then we have*

$$\pi_{\mathcal{U}_1}^{\mathcal{U}_3}(\mathcal{M}(\mathcal{U}_3)) \subseteq \pi_{\mathcal{U}_1}^{\mathcal{U}_2}(\mathcal{M}(\mathcal{U}_2))$$

In particular we have if $\mathcal{U}_1 = \mathcal{U}_2$, that the restriction of the map

$$\pi_{\mathcal{U}_1}^{\mathcal{U}_3} : \mathcal{M}(\mathcal{U}_3) \to \mathcal{M}(\mathcal{U}_1)$$

is well defined (that is, maps into $\mathcal{M}(\mathcal{U}_1)$).

The proof follows trivially from the fact that:

$$\{E \in \mathcal{E} \mid E \subseteq \mathcal{U}_3\} \supseteq \{E \in \mathcal{E} \mid E \subseteq \mathcal{U}_2\}$$

and thus the coordinates of an element of $\mathcal{M}(\mathcal{U}_3)$ has at least as many constraints as the coordinates of an element of $\mathcal{M}(\mathcal{U}_2)$.

From now on (if nothing else is said), π_A^B, with $A \subseteq B \subseteq \mathcal{V}$ will be assumed to be the projection

$$\pi_A^B : \mathcal{M}(B) \to \mathcal{M}(A)$$

for whatever sheaf $\mathcal{M}$ we are working with.

4.2 Braden MacPherson sheaves

In this section we will define a very important sheaf, namely the Braden MacPherson sheaf.

4.2.1 Constructing Braden MacPherson sheaves

In this subsection fix a moment graph $G = (\mathcal{V}, \mathcal{E}, \leq, \tau)$.

Before we can construct a Braden MacPherson sheaf, we need look at the ordering $\leq$ of the vertices $\mathcal{V}$ of our moment graph. We start by giving the edges a direction. If $E = \{x, y\} \in \mathcal{E}$ is an edge in G, then by the definition of a moment graph, either $x \leq y$ of $y \leq x$. Assume $x \leq y$, then we call x the lower vertex of E and y the upper vertex of E. Likewise we say that E is an upper edge (or up-edge) of x (since it points to a vertex greater than x) and E is a lower edge (or down-edge) of y (since it points to a vertex smaller than y). We define the set U_x of up-edges of x, and D_x of down-edges of x, that is:

$$U_x = \{\{x, y\} \in \mathcal{E} \mid x \leq y\}$$
$$D_x = \{\{x, y\} \in \mathcal{E} \mid y \leq x\}$$

We now need to look at restrictions of G. Let $x \in \mathcal{V}$. We define the moment graph $G_{>x} = (\mathcal{V}_{>x}, \mathcal{E}_{>x}, \leq_{>x}, \tau_{>x})$, where:

- $\mathcal{V}_{>x} = \{v \in \mathcal{V} \mid x < v\}$,
- $\mathcal{E}_{>x} = \{\{a, b\} \in \mathcal{E} \mid x < a, x < b\}$,
- $\leq_{>x} = \leq_{|\mathcal{V}_{>x} \times \mathcal{V}_{>x}}$,
- $\tau_{>x} = \tau_{|\mathcal{E}_{>x}}$.

The Braden MacPherson sheaf $\mathcal{B}(z)$ is parametrized by a vertex $z \in \mathcal{V}$, the construction goes as follows:

Start by setting $\mathcal{B}(z)_x = 0$ for all $x \not\leq z$, and setting $\mathcal{B}(z)_z = S$. The rest of the sheaf is constructed inductively. Let $x < z$, and assume that we have already constructed $\mathcal{B}(z)_y$, $\mathcal{B}(z)_E$ and $\rho_{y,E}$ for all $y \in E \in \mathcal{E}_{>x}$. Now for each $y \in U_x$ we define $\mathcal{B}(z)_{\{x,y\}} = \mathcal{B}(z)_y / \tau(\{x, y\}) \mathcal{B}(z)_y$, and $\rho_{y,\{x,y\}} : \mathcal{B}(z)_y \to \mathcal{B}(z)_y / \tau(\{x, y\}) \mathcal{B}(z)_y$ as the quotient homomorphism.

Write $\Gamma\left(\mathcal{B}(z)_{>x}\right) = \mathcal{B}(z)(\mathcal{V}_{>x})$. Notice that this module of sections can be calculated from what we already know.

Definition 4.2.1. The map $\rho_x^U : \Gamma\left(\mathcal{B}(z)_{>x}\right) \to \bigoplus_{E \in U_x} \mathcal{B}(z)_E$ is given by

$$\rho_x^U : \Gamma\left(\mathcal{B}(z)_{>x}\right) \xrightarrow{\pi} \bigoplus_{\{x,y\} \in U_x} \mathcal{B}(z)_y \xrightarrow{\oplus\rho} \bigoplus_{E \in U_x} \mathcal{B}(z)_E$$

where the first map is the projection, and the second map is the direct sum of the maps $\rho_{y,\{x,y\}}$ for all $\{x, y\} \in U_x$. The image of ρ_x^U we denote by $\mathcal{B}(z)_{\partial x} = \operatorname{im}\left(\rho_x^U\right)$.

With this definition, we now set $\mathcal{B}(z)_x$ to be the projective (free) cover of $\mathcal{B}(z)_{\partial x}$.

Finally we can define $\rho_{x,\{x,y\}}$ for $\{x,y\} \in U_x$:

$$\rho_{x,\{x,y\}} = \pi_{\{x,y\}} \circ \varphi_x, \tag{4.1}$$

where $\varphi_x : \mathcal{B}(z)_x \to \mathcal{B}(z)_{\partial x}$ is the cover map, and $\pi_{\{x,y\}}$ is the projection of $\mathcal{B}(z)_{\partial x} \subseteq \bigoplus_{E \in U_x} \mathcal{B}(z)_E$ on $\mathcal{B}(z)_{\{x,y\}}$.

Now we have constructed the Braden MacPherson sheaf.

4.2.2 Towards calculating the Braden MacPherson sheaf

Now we have a construction of the Braden MacPherson sheaf, we can in theory calculate it. However the calculation is very(!) time consuming, namely the step of calculating the module of sections $\Gamma\left(\mathcal{B}(z)_{>x}\right)$ seems to make the calculation impossible. To avoid this problem we want to make a total ordering out of the ordering of the vertices, and then add one vertex at the time - that is one coordinate at the time - to the module of sections, and in that way calculate $\Gamma\left(\mathcal{B}(z)_{>x}\right)$ in small (or at least smaller) inductive steps from what we have already calculated.

It is now practical to change the notation slightly. So fix a moment graph $G = (\mathcal{V}, \mathcal{E}, \preceq, \tau)$ and a vertex $z \in \mathcal{V}$. Since the Braden MacPherson sheaf $\mathcal{B}(z)$ does not depend on elements $y \in \mathcal{V}$ with $y \npreceq z$, we assume that $\mathcal{V}$ satisfies that $y \preceq z$ for all $y \in \mathcal{V}$. We now rename the vertices so that $\mathcal{V} = \{0, 1, \ldots, n-1\}$ (with n the number of vertices), such that if there is a relation $i \prec j$ for two vertices, then $i > j$ as integers. That way we must have $z = 0$, and $\mathcal{V}_{\succ x} \subseteq \{0, 1, \ldots, x-1\}$. Because of this we now short the notation down to $\mathcal{B} = \mathcal{B}(0) = \mathcal{B}(z)$. This way we will start the construction at 0 and work our way one step at the time up to $n-1$. Because we will be working inductively like that, we introduce some more projections.

Definition 4.2.2. For $0 \leq j \leq i < n$ we define the projection π_j^i

$$\pi_j^i : \mathcal{B}(\{0, 1, \ldots, i\}) \to \mathcal{B}_j$$

Notice that $\pi_j^i = \pi_{\{j\}}^{\{0,1,\ldots,i\}}$.

Now, sadly, we need some extra assumptions for our moment graph:

Definition 4.2.3. A moment graph $G = (\mathcal{V} = \{0, 1, \ldots, n-1\}, \mathcal{E}, \preceq, \tau)$ is said to be *nice* if the module of sections $\mathcal{B}(\{0, 1, \ldots, i\})$ is a graded free S-module for all $0 \leq i < n$.

We will later return to this condition, to see when it applies.

We are now ready to prove an important technical proposition:

Proposition 4.2.4. *Let* $G = (\mathcal{V} = \{0, 1, \ldots, n-1\}, \mathcal{E}, \preceq, \tau)$ *be a nice moment graph. Then for any* $0 < i < n$ *we have:*

1. *There exists a homomorphism* $\xi_i : \mathcal{B}(\{0,1,\ldots,i-1\}) \to \mathcal{B}_i$*, such that if* $x \in \mathcal{B}(\{0,1,\ldots,i-1\})$*, then* $(x, \xi_i(x)) \in \mathcal{B}(\{0,1,\ldots,i\})$.

2. *We have an isomorphism:*

$$\begin{aligned}&\psi_i : \mathcal{B}(\{0,1,\ldots,i-1\}) \oplus \ker(\varphi_i) \tilde{\to} \mathcal{B}(\{0,1,\ldots,i\})\\ &\psi_i(x,y) = (x, \xi_i(x) + y)\end{aligned}$$

3. *The kernels of the cover maps* $\ker(\varphi_i)$ *are graded free* S*-modules.*

4. *The projections* $\pi^i_j : \mathcal{B}(\{0,1,2,\ldots,i\}) \to \mathcal{B}_j$ *for* $j \leq i$ *are under the identification* ψ_i *given by:*

$$\begin{aligned}&\pi^i_i(x,y) = \xi_i(x) + y,\\ &\pi^i_j(x,y) = \pi^{i-1}_j(x)\,, \text{for } j < i\end{aligned}$$

where $x \in \mathcal{B}(\{0,1,\ldots,i-1\})$ *and* $y \in \ker(\varphi_i)$.

Proof. We do the proof by induction in i. So assume 1-4 to be true for all values smaller than a given i.

1. To construct this homomorphism ξ_i we start by noticing that the homomorphisms ρ^U_i and φ_i by definition have the same images $\mathcal{B}_{\partial i}$. By niceness of G we have that $\mathcal{B}(\{0,1,\ldots,i-1\})$ is graded free, so we can choose a homogeneous basis $e_1,\ldots,e_m$ of $\mathcal{B}(\{0,1,\ldots,i-1\})$. We can now choose $f_k \in \varphi_i^{-1}\left(\rho^U_i(\pi(e_k))\right) \subseteq \mathcal{B}_i$ homogeneous an of same degree as e_k, where π is the projection:

$$\pi = \pi^{\{0,1,\ldots,i-1\}}_{\mathcal{V}_{\succ i}} : \mathcal{B}(\{0,1,\ldots,i-1\}) \to \mathcal{B}(\mathcal{V}_{\succ i})$$

(see lemma 4.1.6). We now use the universal property of a graded free module to construct the unique homomorphism:

$$\xi_i : \mathcal{B}(\{0,1,\ldots,i-1\}) \to \mathcal{B}_i$$

satisfying $\xi_i(e_k) = f_k$ for all $1 \leq k \leq m$. Notice that since the e_k's form a basis, we get:

$$\varphi_i \circ \xi_i = \rho^U_i \circ \pi \tag{4.2}$$

Let $(m_k)_{k<i} \in \mathcal{B}(\{0,1,\ldots,i-1\})$, and set $m_i = \xi_i\left((m_k)_{k<i}\right)$. We now want to prove that $(m_k)_{k\leq i} \in \mathcal{B}(\{0,1,\ldots,i\})$. To prove this we need to prove that for any edge $\{j,k\} \in \mathcal{E}$ with $j,k \leq i$ we have $\rho_{j,\{j,k\}}(m_j) = \rho_{k,\{j,k\}}(m_k)$. Now since $(m_k)_{k<i} \in \mathcal{B}(\{0,1,\ldots,i-1\})$, this is clear whenever $j,k < i$. So let $\{i,j\} = E \in \mathcal{E}$ be an edge. Notice that since $j < i$ we have that $E \in U_i$. We now calculate:

$$\begin{aligned}\rho_{i,E}(m_i) &= \pi_E \circ \varphi_i(m_i) \\ &= \pi_E \circ \varphi_i \circ \xi_i\left((m_k)_{k<i}\right) \\ &= \pi_E \circ \rho_i^U \circ \pi\left((m_k)_{k<i}\right) \\ &= \rho_{j,E} \circ \pi_j^i\left((m_k)_{k<i}\right) \\ &= \rho_{j,E}(m_j)\end{aligned}$$

where the equal signs follow from (in that order) (4.1), the definition of m_i, (4.2), the definition of ρ_i^U and finally the definition of the projection π_j^i. Thus ξ_i satisfies our requirements.

2. Now to prove that

$$\begin{aligned}&\psi_i : \mathcal{B}(\{0,1,\ldots,i-1\}) \oplus \ker(\varphi_i) \tilde{\to} \mathcal{B}(\{0,1,\ldots,i\}) \\ &\psi_i(x,y) = (x, \xi_i(x) + y)\end{aligned}$$

is a well defined isomorphism. To show that it is well defined we have to prove that it actually maps into $\mathcal{B}(\{0,1,\ldots,i\})$. Notice first that since $\rho_{i,E} = \pi_E \circ \varphi_i$ for any $E \in U_i$, we have

$$\ker(\varphi_i) = \bigcap_{E \in U_i} \ker(\rho_{i,E})$$

So an element $(0,0,\ldots,0,m) \in \mathcal{B}(\{0,1,\ldots,i\})$ if and only if $\rho_{i,E}(m) = 0$ for all $E \in U_i$. This is the case if and only if $m \in \bigcap_{E \in U_i} \ker(\rho_{i,E}) = \ker(\varphi_i)$. If $x \in \mathcal{B}(\{0,1,\ldots,i-1\})$, and $y \in \ker(\varphi_i)$, then $(x, \xi_i((x))) \in \mathcal{B}(\{0,1,\ldots,i\})$ and $(0,y) \in \mathcal{B}(\{0,1,\ldots,i\})$ and thereby $(x, \xi_i(x) + y) \in \mathcal{B}(\{0,1,\ldots,i\})$, since $\mathcal{B}(\{0,1,\ldots,i\})$ is a module. So the homomorphism is well defined. Now to prove that it is an isomorphism.

Surjective Let $(l_j)_{j\le i} \in \mathcal{B}(\{0,1,\ldots,i\})$. Let $x = (l_j)_{j<i} \in \mathcal{B}(\{0,1,\ldots,i-1\})$, then we have $(x, \xi_i(x)) \in \mathcal{B}(\{0,1,\ldots,i\})$, since $\mathcal{B}(\{0,1,\ldots,i\})$ is a module we have $(0, l_i - \xi_i(x)) \in \mathcal{B}(\{0,1,\ldots,i\})$, and thereby $y = l_i - \xi_i(x) \in \ker(\varphi_i)$, so $\psi_i(x,y) = (l_j)_{j\le i}$, and we have surjectivity.

Injective Let $x \in \mathcal{B}(\{0,1,\ldots,i-1\})$ and $y \in \ker(\varphi_i)$, such that $\psi_i(x,y) = 0$. By the definition of ψ_i we must then have $x = 0$, and $y = y + \xi_i(0) = y + \xi_i(x) = 0$, so we have injectivity.

Thus ψ_i is an isomorphism.

3. Now we wish to prove that $\ker(\varphi_i)$ is a graded free S-module. By 2 we have that $\ker(\varphi_i)$ is a direct summand of the graded free module $\mathcal{B}(\{0,1,\ldots,i\})$, and thereby it is a graded projective S-module. Now by corollary 4.7 of [Lam78] we have that $\ker(\varphi_i)$ is a graded free S-module.

4. The identification of the projection maps π_i^j is clear from part 2.

□

Notice that the property of $\ker(\varphi_i)$ being a graded free S-module for all $0 < i < n$ is equivalent to G being nice, since $\mathcal{B}(\{0\}) = S$ is always graded free, and the isomorphisms $\psi_i : \mathcal{B}(\{0, 1, \ldots, i-1\}) \oplus \ker(\varphi_i) \tilde{\rightarrow} \mathcal{B}(\{0, 1, \ldots, i\})$ then inductively yields the niceness. Later on we will use this definition of a nice moment graph in the algorithm to calculate the Braden-MacPherson sheaf.

The final obstacle before devising an algorithm for calculating the Braden-MacPherson sheaf, is that we want to define $\mathcal{B}_i$ from $\Gamma_{<i} = \mathcal{B}(\{0, \ldots, i-1\})$, whereas $\mathcal{B}_i$ is in the construction defined from $\Gamma_{\succ i} = \mathcal{B}(\{j \in \{0, \ldots, n-1\} \mid j \succ i\})$. This obstacle is overcome by the following lemma:

Lemma 4.2.5. *Let G be nice. Let $0 < i < n$, and let $\mathcal{U} \subseteq \{0, \ldots, i-1\}$ satisfy that if $j \in \mathcal{U}$ and $k \in \{0, \ldots, i-1\}$ with $k \succ j$, then $k \in \mathcal{U}$. Then the projection $\pi = \pi_{\mathcal{U}}^{\{0,\ldots,i-1\}} : \mathcal{B}(\{0, \ldots, i-1\}) \to \mathcal{B}(\mathcal{U})$ is surjective.*

Proof. All the projections we will be working with in this proof will be well defined by lemma 4.1.6. We proceed to prove the lemma by induction in the size of $\mathcal{D} = \{0, \ldots, i-1\} \setminus \mathcal{U}$. Naturally if $\mathcal{D} = \emptyset$, then π is the identity on $\Gamma_{<i}$ and thus surjective.

Assume the lemma to hold for all $\mathcal{D}$ of size $m-1$, and let $\mathcal{U} \subseteq \{0, \ldots, i-1\}$ satisfy the conditions in the lemma, and that $\mathcal{D} = \{0, \ldots, i-1\} \setminus \mathcal{U}$ has size m. Choose $j \in \mathcal{D}$ minimal in the usual ordering ($<$) of the integers. That is $\{0, \ldots, j-1\} \subseteq \mathcal{U}$, but $j \notin \mathcal{U}$. Let $\mathcal{U}' = \mathcal{U} \cup \{j\}$, then $\mathcal{U}'$ satisfies the conditions of the lemma (since if $k \succ j$ the $k < j$ and then $k \in \mathcal{U}'$). So by induction the projection $\pi' : \mathcal{B}(\{0, \ldots, i-1\}) \to \mathcal{B}(\mathcal{U}')$ is surjective, since $|\{0, \ldots, i-1\} \setminus \mathcal{U}'| = m-1$. Thus all we need to prove is that the projection $\pi_{\mathcal{U}}^{\mathcal{U}'}$ is surjective. Let $m_{\mathcal{U}} = (m_x)_{x \in \mathcal{U}} \in \mathcal{B}(\mathcal{U})$. Define

$$m_j = \xi_j\left((m_x)_{x<j}\right) = \xi_j\left(\pi_{\{0,\ldots,j-1\}}^{\mathcal{U}}(m_{\mathcal{U}})\right)$$

where ξ_j is the homomorphism defined in proposition 4.2.4 part 1. We have now defined $m_{\mathcal{U}'} = (m_x)_{x \in \mathcal{U}'}$, and want to prove that it is an element of $\mathcal{B}(\mathcal{U}')$. Let $x, y \in \mathcal{U}'$, such that $\{x, y\} \in \mathcal{E}$. If $x, y \neq j$, then $x, y \in \mathcal{U}$, and thus $\rho_{x,\{x,y\}}(m_x) = \rho_{y,\{x,y\}}(m_y)$. If however one of the vertices is j, let's say $y = j$, then $x \in \mathcal{U}$ and thus we cannot have $j \succ x$ (since that would mean $j \in \mathcal{U}$), so $x \succ j$, and thus $x < j$, and we have by proposition 4.2.4 part 1, that $\rho_{x,\{x,j\}}(m_x) = \rho_{j,\{x,j\}}(m_j)$, and thereby we have $m_{\mathcal{U}'} \in \mathcal{B}(\mathcal{U}')$, making $\pi_{\mathcal{U}}^{\mathcal{U}'}$ surjective, and thus concluding the proof. □

4.2.3 An algorithm for calculating the Braden MacPherson sheaf

With the proofs of proposition 4.2.4 and lemma 4.2.5, we can now attack the problem of calculating the Braden MacPherson sheaf of a nice moment graph. The following algorithm is based on the algorithm found on Tom Bradens homepage: `http://www.math.umass.edu/~braden/MG/index.html`

Proposition 4.2.6. *Let $G = (\mathcal{V} = \{0, 1, \dots, n-1\}, \mathcal{E}, \preceq, \tau)$ be a moment graph (with vertices ordered such that if $i \preceq j$ then $i \geq j$). Let S be the symmetric algebra over the vector space of G. If the algorithm below does not err, then G is nice, and the final datum $\mathcal{B}_i$, Γ, π_i (for $0 \leq i < n$) describes the vertex-modules, the module of global sections and the projection maps $\pi_i : \Gamma \to \mathcal{B}_i$ for the Braden MacPherson sheaf of G.*

$\mathcal{B}_0 = S$
$\Gamma = S$
$\pi_0 = id_S$
for $i = 1$ *to* $n-1$ ***do***
 $upVertex = \{j \in \{0, \dots, i-1\} \mid \{j, i\} \in \mathcal{E}\}$
 $\mathcal{B}_i^U = \bigoplus_{j \in upVertex} \mathcal{B}_j / \tau(\{j, i\}) \mathcal{B}_j$
 $\omega : \Gamma \to \mathcal{B}_i^U, \omega = \bigoplus_{j \in upVertex} \rho_{j,\{j,i\}} \circ \pi_j$ ▷ $\rho_{j,\{j,i\}}$ *is the quotient map*
 $generators\partial = \emptyset$ ▷ *This set will contain the generators of* $\mathcal{B}_{\partial i}$
 for $e \in \Gamma.generators$ ***do***
  $generators\partial = generators\partial \cup \{\omega(e)\}$
 end for
 $minGen =$ ***GetMinimalGenerators*** $(generators\partial)$
 $\mathcal{B}_i = 0$
 for $gen \in minGen$ ***do***
  $\mathcal{B}_i = \mathcal{B}_i \oplus S \langle \deg(gen) \rangle$
 end for
 $\varphi_i : \mathcal{B}_i \to \mathcal{B}_i^U$, $\varphi_i(e_k) = minGen_k$ *for* $k \in \{1, \dots, \operatorname{rank}(\mathcal{B}_i)\}$
 for $k = 1$ *to* $\operatorname{rank}(\Gamma)$ ***do***
  $f_k =$ ***ChooseElementInPreimage*** $(\varphi_i, \omega(e_k))$ ▷ $f_k \in \varphi_i^{-1}(\omega(e_k))$
 end for
 $\xi_i : \Gamma \to \mathcal{B}_i$, $\xi_i(e_k) = f_k$ *for* $k \in \{1, \dots, \operatorname{rank}(\Gamma)\}$
 $ker = \ker(\varphi_i)$
 if *not* ***IsFree***(ker) ***then***
  error: "G is not nice"
 end if
 $\Gamma = \Gamma \oplus ker$
 $\pi_i : \Gamma \to \mathcal{B}_i$, $\pi_i = \xi_i \oplus \mathrm{id}_{ker}$
 for $j = 0$ *to* $i-1$ ***do***
  $\pi_j = \pi_j \oplus 0_{ker}$
 end for
end for

The proof follows from proposition 4.2.4 and the definition of the Braden-MacPherson sheaf, if you also notice that since the map ω in the algorithm is $\rho_i^U \circ \pi_{\{0,\dots,i-1\}}^{\mathcal{V}_{\succ i}}$, and since $\pi_{\{0,\dots,i-1\}}^{\mathcal{V}_{\succ i}}$ is surjective by lemma 4.2.5, we have that the images of ω and ρ_i^U are the same $\mathcal{B}_{\partial i}$.

Still a lot of the methods are not explained properly. The most important thing to notice before handling these methods is that S is really a polynomial F-algebra, that is $S \cong F\left[x_1, x_2, \ldots, x_{\dim(V)}\right]$ (the i'th basis vector of V is identified with x_i). Since we are working over a polynomial ring, all the unexplained methods can be translated into some problems regarding Gröbner bases. For details on how to solve these problems regarding Gröbner bases see [KR00], or see my implementation of the above algorithm in the ReAlGriDPC-project found on my web page `http://home.imf.au.dk/jb`.

4.3 The connection

We are now finally ready to connect the theory of sheaves on moment graphs with the representation theory of algebraic groups.

4.3.1 The moment graph of an affine Weyl group

In this section we want to tie the concept of a moment graph to the main theme of representations of algebraic groups. To do this we construct the moment graph for the affine Weyl group. Recall that $(\mathcal{W}, \{s_0, \ldots, s_n\})$ is our affine Weyl group (to avoid confusion, in this section S will only denote the symmetric algebra $S(V)$ and NOT the set of simple reflections $\{s_0, \ldots, s_n\}$).

Definition 4.3.1. Let $(\mathcal{W}, \{s_0, \ldots, s_n\})$ be an affine Weyl group, let F be a field and let $V = F^{n+1}$ an F-vector space. The V-moment graph $G_{\mathcal{W}} = (\mathcal{W}, \mathcal{E}_{\mathcal{W}}, \leq, \tau_{\mathcal{W}})$ for the affine Weyl group is defined by:

- The vertices $\mathcal{W}$ are the elements of the affine Weyl group,
- The edges are $\mathcal{E}_{\mathcal{W}} = \{\{x, y\} \subseteq \mathcal{W} \mid xy^{-1} \text{ is a reflection}\}$,
- The ordering $\leq$ is the Bruhat order,
- The labeling $\tau_{\mathcal{W}} : \mathcal{E}_{\mathcal{W}} \to F^{n+1}$ is given by $\tau_{\mathcal{W}}(\{x, y\}) = (a_1, a_2, \ldots a_n, m) \in F^{n+1}$, when $xy^{-1} = s_{\alpha,m}$, and $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$.

Notice that this is not a finite moment graph, however whenever we choose an element $z \in \mathcal{W}$, we get that $G_{\mathcal{W}_{\leq z}}$ is finite, and it is this moment graph, we will be using to make Braden-MacPherson-sheaves. Notice further that since $\mathcal{B}(w)_x$ only depends on the vertices in the Bruhat interval $[x; w]$, the actual graphs we will be doing our calculations over are usually much smaller than $G_{\mathcal{W}_{\leq w}}$. So from now on when we $\mathcal{B}(w)$ will denote the Braden-MacPherson-sheaf of $G_{\mathcal{W}_{\leq w}}$.

4.3.2 Moment graphs and Kazhdan-Lusztig polynomials

By definition of the Braden MacPherson sheaf, the vertex modules are graded free modules; that is for any pair of vertices $x, w \in \mathcal{V}$ in a moment graph, we can write

$$\mathcal{B}(w)_x = \bigoplus_{i=0}^{r} S[d_i]$$

where $S[d_i]$ is the symmetric algebra shifted to the degree d_i, that is the 1-element of $S[d_i]$ has degree d_i.

Definition 4.3.2. Let G be a moment graph, and let w, x be vertices of $\mathcal{M}$. We define:

$$P(G)_{x,w} = \sum_{i=0}^{r} q^{d_i} \in \mathbb{Z}[q]$$

where

$$\mathcal{B}(w)_x = \bigoplus_{i=0}^{r} S[d_i]$$

and for $G = G_{\mathcal{W}}$ we set

$$P^p_{x,w} = P(G_{\mathcal{W}})_{x,w}$$

where p is the characteristic of F.

Form [Fie10] theorem 4.5 we get the following result (tying the theories together):

Theorem 4.3.3. *Let $x, w \in \mathcal{W}$ satisfy $x \leq w$ then*

$$P^0_{x,w} = P_{x,w}$$

where $P_{x,w}$ is the Kazhdan-Lusztig-polynomial. Further more if p is big enough (depending on $\mathcal{W}$), then also

$$P^p_{x,w} = P_{x,w}$$

Because of the above theorem, it seems fair to call the polynomials $P^p_{x,w}$, with $p > 0$ the modular Kazhdan-Lusztig-polynomials.

In figure 4.1 can be seen the moment graph $G_{w_0 \leq \mathcal{W} \leq w_0 s_0 s_1 s_3 s_2}$ for $\mathcal{W}$ of type A_3 with $p = 2$, in the vertices can be seen the polynomials corresponding to the vertex modules of the Braden-MacPherson sheaf of this graph.

Having defined these modular Kazhdan-Lusztig-polynomials, an obvious idea would be to check if the generalized Lusztig's Conjecture (Question 3.2.3) may hold if the Kazhdan-Lusztig-polynomials are replaced by these. Recall from section 3.2

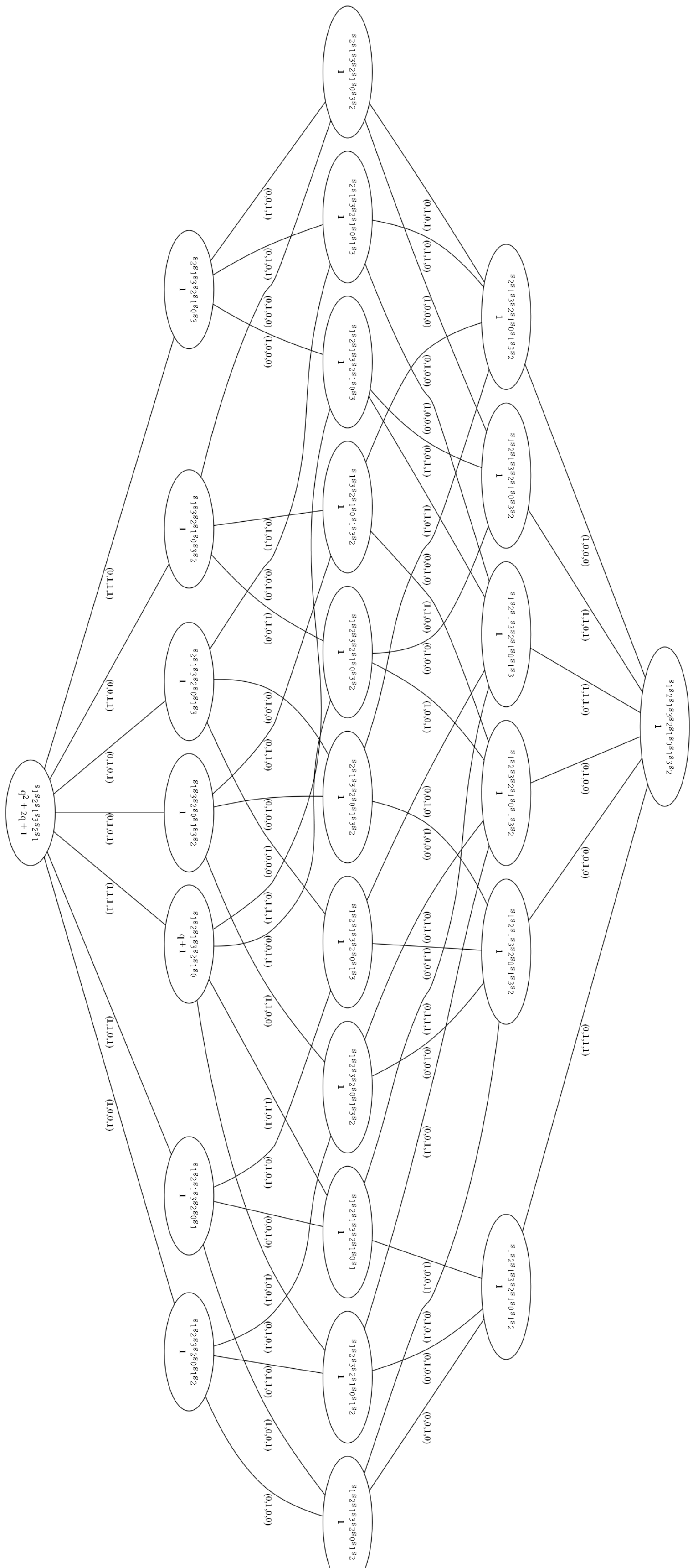


Figure 4.1: The Braden-MacPherson sheaf of $G_{w_0 \le W \le w_0 s_0 s_1 s_3 s_2}$ in type A_3 with $p = 2$.

the cases where the Lusztig's conjecture predicts a wrong answer. In the case A_4 with $p = 2$ we have that

$$\mathrm{ch}\left(L\left([0,1,1,0]\right)\right) = \chi\left([0,1,1,0]\right) - \chi\left([0,0,0,0]\right)$$

whereas Lusztig's character formula predicted it to be

$$\mathrm{ch}\left(L\left([0,1,1,0]\right)\right) = \chi\left([0,1,1,0]\right)$$

If we take the polynomials $P^2_{x,w}$ instead, then we would get,

$$\mathrm{ch}\left(L\left([0,1,1,0]\right)\right) = \chi\left([0,1,1,0]\right) + \chi\left([0,0,0,0]\right)$$

which again is not correct. And in the case A_4 with $p = 3$ we have that

$$\mathrm{ch}\left(L\left([1,2,2,1]\right)\right) = \chi\left([1,2,2,1]\right) - \chi\left([0,2,2,0]\right)$$

whereas Lusztig's character formula claims that it is

$$\mathrm{ch}\left(L\left([1,2,2,1]\right)\right) = \chi\left([1,2,2,1]\right) - \chi\left([0,2,2,0]\right) + \chi\left([0,1,1,0]\right) - \chi\left([1,0,0,1]\right)$$

If we take the polynomials $P^3_{x,w}$ instead, then we would get,

$$\mathrm{ch}\left(L\left([1,2,2,1]\right)\right) = \chi\left([1,2,2,1]\right) - \chi\left([0,2,2,0]\right) + 2\chi\left([0,1,1,0]\right) - 4\chi\left([1,0,0,1]\right)$$

which like in the case $p = 2$ again is not correct.

In appendix A we have compared the values of the simple characters $\mathrm{ch}L\left(\lambda\right)$ to characters given by Lusztig's character formula, both with the Kazhdan-Lusztig-polynomials, and with the modular Kazhdan-Lusztig-polynomials for all $\lambda \in X_p$. From these values, it can be seen that there are a few cases (namely the inverse composition coefficients $\left[L\left([1,2,1,2]\right) : V\left([1,0,0,0]\right)\right]$, $\left[L\left([2,1,1,2]\right) : V\left([0,0,0,0]\right)\right]$, $\left[L\left([2,1,2,1]\right) : V\left([0,0,0,1]\right)\right]$, all in type A_4 with $p = 3$) where Lusztig's character formula predicts the correct characters but the formula using the modular Kazhdan-Lusztig polynomials predicts a wrong result. However it can also be seen that whenever the formula using the modular Kazhdan-Lusztig polynomials predicts the correct result, then so does Lusztig's character formula. This indicates that the failure of Lusztig's conjecture may be related to when $P \neq P^p$, but sadly not in a simple way.

4.3.3 When is $G_{\mathcal{W}}$ nice?

When we look at the Braden-MacPherson sheaves and the modular Kazhdan-Lusztig polynomials, then the values that we are interested in are of the type $P^p_{w_0 x, w_0 w}$, for that reason we have calculated the Braden-MacPherson-sheaf of the moment graph $G_{w_0 \leq \mathcal{W} \leq w_0 w}$ with the Bruhat interval $[w_0; w_0 w]$ as vertices, or more precisely we are interested in the pairs:

$$Calc^{Lusztig}_{\mathcal{W}} = \left\{ (w_0 x, w_0 w) \in \mathcal{W} \times \mathcal{W} \,\middle|\, \exists \lambda_0 \in \overline{A_0} : w.\lambda_0 \in X_p \cap \widehat{w.A_0}, x \leq w, \right\} x.\lambda_0 \in X^+$$

For these pairs, in type A_2, A_3, A_4 with F of characteristic 2 and 3, we have used the algorithm described in the above subsection and we have found the corresponding moment graph $G_{w_0x\leq\mathcal{W}\leq w_0w}$ to be nice. That is except in the graph where the upper bound on the vertices was $w_0s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0 \in \mathcal{W}$, with $\mathcal{W}$ of type A_4 and $p = 3$; in that graph, I have only checked the niceness until I got to the elements of length $l(w_0) + 2$, a which point the algorithm (running on a state of the art private computer) was trying to find out whether the kernel of one of the maps φ_i was free, and crashed the computer after about 20 days of calculating. So sadly we have to leave the question, whether the graphs $G_{w_0x\leq\mathcal{W}\leq w_0w}$ are nice for all $(w_0x, w_0w) \in Calc_{\mathcal{W}}^{Lusztig}$ open. So, for the rest of this thesis we will assume that the graphs $G_{w_0x\leq\mathcal{W}w_0w}$ are nice for $(w_0x, w_0w) \in Calc_{\mathcal{W}}^{Lusztig}$ and hope that it is so.

The interesting question to ask is however:

Question 4.3.4. Is $G_{x\leq\mathcal{W}\leq w}$ nice for any $x \leq w \in \mathcal{W}$ in any affine Weyl group $\mathcal{W}$?

It seems fair to conjecture it to be true, at least in type A_n when $(x, w) \in Calc_{\mathcal{W}}^{Lusztig}$, since all the places where I actually got to determining whether $\ker(\varphi_i)$ was free, it was. A proof of the above question would do wonders for the algorithm, since the checking whether $\ker(\varphi_i)$ is free could be removed, and the running time could be dramatically decreased.

Chapter 5

Representations of restricted Lie algebras

5.1 Restricted Lie algebras

In this section we will introduce restricted Lie algebras, and their representations, so that we in the next section can relate it to the problems at hand.

5.1.1 The basic definitions

Definition 5.1.1. Let $\mathfrak{g}$ be a k-Lie algebra. We say $\mathfrak{g}$ is a *restricted Lie algebra* if there exists a p-power map $\mathfrak{g} \to \mathfrak{g}$, $x \mapsto x^{[p]}$, satisfying:

- $x^p - x^{[p]}$ is central in $\mathcal{U}(\mathfrak{g})$ for any $x \in \mathfrak{g}$,
- $(x+y)^p - (x+y)^{[p]} = x^p - x^{[p]} + y^p - y^{[p]}$ for any pair $x, y \in \mathfrak{g}$ and
- $(ax)^{[p]} = a^p x^{[p]}$ for any $a \in k$ and $x \in \mathfrak{g}$.

From now on let $\mathfrak{g}$ denote $\mathfrak{g} = \operatorname{Lie}(G)$ the Lie algebra of our algebraic group G, furthermore let $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.

Proposition 5.1.2. *The Lie algebra $\mathfrak{g}$ over any algebraic group of finite characteristic is an restricted Lie algebra. And the p-power map sends e_α to $e_\alpha^{[p]} = 0$ for all $\alpha \in \Phi$.*

For the first claim see [Jan03] I.7.10, for the second claim see [Jan98] 6.1.

Let $I \subseteq \mathcal{U}$ be the two-sided ideal generated by $x^p - x^{[p]}, x \in \mathfrak{g}$. We call $\mathcal{U}^{\text{res}} = \mathcal{U}/I$ the restricted universal enveloping algebra. Let $\mathfrak{h} = \operatorname{Lie}(T)$, then $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$. Since $\mathcal{U}$ is generated by $\{e_\alpha \mid \alpha \in \Phi\} \cup \{h_\alpha \mid \alpha \in \Delta\}$ we get that $\mathcal{U}^{\text{res}}$ is generated by the image under the quotient map. The generators of $\mathcal{U}^{\text{res}}$ we will also denote $\{e_\alpha \mid \alpha \in \Phi\} \cup \{h_\alpha \mid \alpha \in \Delta\}$ (abusing the notation slightly). The differential gives us a map $\overline{\cdot} : X = \operatorname{Hom}(T, k^*) \to \overline{X} = \operatorname{Hom}(\mathfrak{h}, k)$. We are interested in a certain category $\widehat{\mathcal{C}}$ of $\mathcal{U}^{\text{res}}$-modules.

Definition 5.1.3. We say that an $\mathcal{U}^{\mathrm{res}}$-module is in $\widehat{\mathcal{C}}$ if:

- M is finitely generated,
- $M = \bigoplus_{\lambda \in X} M_\lambda$ for some k-vector spaces M_λ,
- $e_\alpha M_\lambda \subseteq M_{\lambda+\alpha}$ for all roots α and all weights λ,
- $hx = \overline{\lambda}(h)\, x$, whenever $h \in \mathfrak{h}$ and $x \in M_\lambda$ for some $\lambda \in X$.

Notice that the category $\widehat{\mathcal{C}}$ can be identified to the category of G_1T-modules of [Jan03].

Definition 5.1.4. Let $M \in \widehat{\mathcal{C}}$, the formal character $\mathrm{ch}(M)$ of M is defined as follows:

$$\mathrm{ch}(M) = \sum_{\lambda \in X} \dim(M_\lambda)\, e(\lambda)$$

5.1.2 The modules and characters of $\widehat{\mathcal{C}}$

There are two types of modules in $\widehat{\mathcal{C}}$ that we are interested in, the first type is the Baby Verma Modules:

Definition 5.1.5. For $\lambda \in X$ we define the baby Verma module $\widehat{Z}(\lambda)$ as follows:

$$\widehat{Z}(\lambda) = \mathcal{U}^{\mathrm{res}} \Big/ \left(\sum_{\alpha \in \Phi^+} \mathcal{U}^{\mathrm{res}} e_\alpha + \sum_{h \in \mathfrak{h}} \mathcal{U}^{\mathrm{res}} \left(h - \overline{\lambda}(h)\right) \right)$$

Notice the similarities with the definition of the Verma module (subsection 2.5.2). The analogue to the PBW theorem for $\mathcal{U}^{\mathrm{res}}$ (see Prop 2.8 in [Jan03]) implies that we have a basis of $\widehat{Z}(\lambda)$ consisting of

$$\left\{ \prod_{\alpha \in \Phi^+} e_{-\alpha}^{r(\alpha)} \,\middle|\, 0 \le r(\alpha) < p \right\}$$

One may now see that $\widehat{Z}(\lambda)$ is in $\widehat{\mathcal{C}}$ if we equip it with the X-grading such that

$$\widehat{Z}(\lambda)_\mu = \mathrm{span}_k \left\{ \prod_{\alpha \in \Phi^+} e_{-\alpha}^{r(\alpha)} \,\middle|\, \sum_{\alpha \in \Phi^+} r(\alpha)\, \alpha = \lambda - \mu \right\}$$

We now conclude the following proposition:

Proposition 5.1.6. *Let* $\mathrm{Par}_p : X \to \mathbb{N}$ *be the partition function given by:*

$$\mathrm{Par}_p(\lambda) = \left| \left\{ (r(\alpha))_{\alpha \in \Phi^+} \,\middle|\, 0 \le r(\alpha) < p, \sum_{\alpha \in \Phi^+} r(\alpha)\, \alpha = \lambda \right\} \right|$$

Then the character $\mathrm{ch}\left(\widehat{Z}(\lambda)\right)$ *of* $\widehat{Z}(\lambda)$ *is given by:*

$$\mathrm{ch}\left(\widehat{Z}(\lambda)\right) = \sum_{\mu \in X} \mathrm{Par}_p(\lambda - \mu)\, e(\mu)$$

Notice that $\mathrm{Par}_p(\lambda) = 0$ if $\lambda \notin \mathbb{N}\Phi^+$ or if any of the coordinates of λ, when written in the Δ-basis, are greater than the corresponding coordinate of $\sum_{\alpha \in \Phi^+} (p-1)\alpha$ in the Δ-basis.

Notice further that

$$\begin{aligned}
\mathrm{ch}\left(\widehat{Z}(\lambda)\right) &= \sum_{\mu \in X} \mathrm{Par}_p(\lambda - \mu)\, e(\mu) \\
&= \sum_{\xi \in X} \mathrm{Par}_p(\xi)\, e(\lambda - \xi) \\
&= e(\lambda) \sum_{\xi \in X} \mathrm{Par}_p(\xi)\, e(-\xi) \\
&= e(\lambda)\, \mathrm{ch}\left(\widehat{Z}(0)\right)
\end{aligned}$$

for any $\lambda \in X$. This gives us that for any pair $\lambda, \mu \in X$ we have

$$\mathrm{ch}\left(\widehat{Z}(\lambda)\right) = e(\lambda - \mu)\, \mathrm{ch}\left(\widehat{Z}(\mu)\right) \tag{5.1}$$

The other type of modules in $\widehat{\mathcal{C}}$ that have our interest are of course the simple modules. They can be constructed from the baby Verma modules much like we constructed the simple G-modules from the Weyl-modules. By fitting the proof of proposition 1.4.1 to this setting we can prove that $\widehat{Z}(\lambda)$ has a unique maximal submodule $\widehat{Z}(\lambda)_{\max}$, and define the simple modules $\widehat{L}(\lambda) = \widehat{Z}(\lambda) / \widehat{Z}(\lambda)_{\max}$ for all $\lambda \in X$ (not just the ones in X^+).

We have the following proposition (see proposition II.9.6 of [Jan03]):

Proposition 5.1.7. *If* $\lambda \in X$ *and let* $\lambda_0 \in X_p$ *and* $\lambda_p \in X$ *such that* $\lambda = \lambda_0 + p\lambda_p$, *then*

$$\mathrm{ch}\left(\widehat{L}(\lambda)\right) = e(p\lambda_p)\, \mathrm{ch}(L(\lambda_0))$$

5.1.3 The composition coefficients of the baby Verma modules

As with the Weyl modules and the simple G-modules, we would like to investigate the composition coefficients of the baby Verma modules. One reason for wanting to investigate that is that if we know the composition coefficients of the baby Verma modules, then we can find the characters of the simple G-modules: We have (like in (2.1)) that:

$$\mathrm{ch}\widehat{Z}(\lambda) = \sum_{\mu \preceq \lambda} \left[\widehat{Z}(\lambda) : \widehat{L}(\mu)\right] \mathrm{ch}\widehat{L}(\mu)$$

and thus

$$\dim \widehat{Z}(\lambda)_\xi = \sum_{\xi \preceq \mu \preceq \lambda} \left[\widehat{Z}(\lambda) : \widehat{L}(\mu)\right] \dim \widehat{L}(\mu)_\xi$$

and thereby

$$\dim \widehat{L}(\lambda)_\xi = \dim \widehat{Z}(\lambda)_\xi - \sum_{\xi \preceq \mu \prec \lambda} \left[\widehat{Z}(\lambda) : \widehat{L}(\mu)\right] \dim \widehat{L}(\mu)_\xi$$

It is easy to calculate $\dim \widehat{Z}(\lambda)_\xi$ using $\mathrm{Par}_p(\lambda - \xi)$, but for the other $\dim \widehat{L}(\mu)_\xi$ are a bit more difficult. We can however use the above equality recursively, and notice that for each step, the new sum will have will have fewer and fewer summands since there by $\xi \preceq \mu \prec \lambda$ clearly are fewer weights in the $\preceq$-interval $[\xi; \mu]$ than in $[\xi; \lambda]$, and there by, if we keep on replacing the $\dim \widehat{L}(\mu)_\xi$ by new sums using the above formula, we will at some point have no $\dim \widehat{L}(\mu)_\xi$'s left, and we can calculate $\dim \widehat{L}(\lambda)_\xi$ just from the composition coefficients of the baby Verma modules, and the characters of the baby Verma modules. To calculate the character of the simple $\mathcal{U}^{\mathrm{res}}$-modules, we now just need to notice that $\widehat{L}(\lambda)$ is a homomorphic image of $\widehat{Z}(\lambda)$, and thus $\dim \widehat{L}(\lambda)_\mu \leq \dim \widehat{Z}(\lambda)_\mu$, and we now have only finitely many weights to check. Using proposition 5.1.7, we can use this knowledge to find all the characters of the simple G-modules $\mathrm{ch} L(\lambda)$ with $\lambda \in X_p$, and from these characters we can calculate all the characters of the simple G-modules using Steinberg's tensor product theorem 2.2.5.

To simplify the task of finding the composition coefficients of the baby Verma modules, we need the following proposition

Proposition 5.1.8. *Let $\lambda, \mu, \xi \in X$, then the composition coefficients satisfy:*

$$\left[\widehat{Z}(\mu + p\xi) : \widehat{L}(\lambda + p\xi)\right] = \left[\widehat{Z}(\mu) : \widehat{L}(\lambda)\right]$$

The proposition follows from (5.1) and proposition 5.1.7.

Another nice result about the composition coefficients of the Verma modules is from [Jan03] II.9.16(4) (with $r = 1$ and $\mu_1 = -\rho$):

Proposition 5.1.9. *Let $\lambda_0 \in \overline{A_0}$ and $w, x \in \mathcal{W}$, such that $w.\lambda_0 \in X_p - p\rho$. Let $v \in W$ (the Weyl group), then*

$$\left[\widehat{Z}(x.\lambda_0) : \widehat{L}(w.\lambda_0)\right] = \left[\widehat{Z}(vx.\lambda_0) : \widehat{L}(w.\lambda_0)\right]$$

Notice that the above two propositions give us that to know all the composition

$$\left[\widehat{Z}(\mu) : \widehat{L}(\lambda)\right]$$

it is enough to know them for $\lambda \in X_p - p\rho$ and $\mu \in -X^+ - \rho$, since $-X^+ - \rho$ is a fundamental domain for the dot-orbit of the Weyl group W on the weights X.

We still have no smart way of finding the composition coefficients (we will return to that later), we do, however, have a not very smart way of finding the composition coefficients. What we can do is calculate the characters of the simple G-modules, use these to calculate the characters of the simple modules in $\widehat{\mathcal{C}}$, then we can use the function Par_p to calculate the characters of the baby Verma modules, and then we can finally decompose these characters in a sum of the characters of the simple modules in $\widehat{\mathcal{C}}$, and like that we can calculate the composition coefficients of the baby Verma modules.

We have actually used the above method to find the all the composition coefficients $\left[\widehat{Z}(\mu) : \widehat{L}(\lambda)\right]$ with $\lambda \in X_p - p\rho$ and $\mu \in -X^+ - \rho$ for root systems of type A_n with $n \in \{2,3,4\}$ in characteristic 2 and 3. In appendix C can be found a comprehensive list of all these composition coefficients.

5.2 Connected conjectures

5.2.1 The conjectures

In the article [Fie10] three conjectures that are very relevant to this thesis are mentioned. In the terminology of this thesis they say:

Conjecture 5.2.1. Lusztig's Conjecture+ (Conjecture 2.4 of [Fie10]) Let $p = \mathrm{char}(k) > h$ (the Coxeter number). Let $w \in \mathcal{W}$, such that $w.0 \in X_p$ then

$$\mathrm{ch}(L(w.0)) = \sum_{x \leq w} (-1)^{l(w)-l(x)} P_{w_0x,w_0w}(1)\,\chi(x.\lambda)$$

Conjecture 5.2.2. The generic multiplicity conjecture (Conjecture 3.4 of [Fie10]) Let $p = \mathrm{char}(k) > h$. For $x, w \in \mathcal{W}$, such that $w.0 \in X_p - p\rho$

$$\left[\widehat{Z}(x.0) : \widehat{L}(w.0)\right] = P_{x,w}(1)$$

Conjecture 5.2.3. (Conjecture 4.4 of [Fie10]) Let $p = \mathrm{char}(k) \neq 2$, let $w \in \mathcal{W}$, such that $G_{\mathcal{W}_{\leq w}}$ is a GKM-graph. Then

$$P_{x,w} = P^p_{x,w}$$

for all $x \leq w$.

Notice that the requirements of the conjectures are related; more precisely, if $p \geq h$, then $G_{\mathcal{W}_{\leq w_0 w}}$ is GKM for any $w \in \mathcal{W}$ such that $w.0 \in X_p$.

And now for the connection between the conjectures (see theorem 3.5 and theorem 5.2 of [Fie10]):

Theorem 5.2.4. *The conjectures 5.2.1 and 5.2.2 are equivalent, and follow from conjecture 5.2.3.*

5.2.2 The modified conjectures

It is very striking that the Kazhdan-Lusztig polynomials appear both in a formula for the characters of a simple G-module and as composition coefficients of baby Verma modules. The above theorem goes part of the way of explaining the relation. It says nothing, however, about small primes p, and it is not at all clear that we should have any such correspondence in the small p case.

We have already seen in subsection 4.3.2 that the modular Kazhdan-Lusztig polynomials do not fit into conjecture 5.2.1 (which corresponds to question 3.2.3 with $p > h$). Since the modular Kazhdan-Lusztig polynomials coincide with the Kazhdan-Lusztig polynomials for p large enough, but differ for small p, and conjectures 5.2.1 and 5.2.2 are equivalent, the obvious question is now whether the modular Kazhdan-Lusztig polynomials fit into conjecture 5.2.2 regarding the composition coefficients of baby Verma modules, for small p.

First, however, we need to generalize conjecture 5.2.2 a bit. In section 3.2 we saw that we can use corollary 2.3.5 to generalize Lusztig's character formula from any regular dominant weight – like 0 when $p \geq h$ – to any dominant weight. A very similar trick can be used in the restricted Lie algebra case; from II.9.22 of [Jan03] we get (recall the definition of the facets F_λ in subsection 2.3.1)

Proposition 5.2.5. *Let $\lambda, \mu \in X$ with $\mu \in \overline{F_\lambda}$, and let $x, w \in \mathcal{W}$ such that $w.\mu \in \widehat{F_{w.\lambda}}$, then*

$$\left[\widehat{Z}\left(x.\mu\right) : \widehat{L}\left(w.\mu\right)\right] = \left[\widehat{Z}\left(x.\lambda\right) : \widehat{L}\left(w.\lambda\right)\right]$$

This proposition gives us a natural way of generalizing conjecture 5.2.2, namely using that if $p \geq h$, $\lambda_0 \in \overline{A_0}$ and $w.\lambda_0 \in \widehat{w.A_0}$, then

$$\left[\widehat{Z}\left(x.\lambda_0\right) : \widehat{L}\left(w.\lambda_0\right)\right] = \left[\widehat{Z}\left(x.0\right) : \widehat{L}\left(w.0\right)\right]$$

Our aim is to substitute $P^p_{x,w}$ for $P_{x,w}$ in this generalized version of conjecture 5.2.2, so we need to extend our area of calculation by the set of (x, w) relevant for this conjecture, which is

$$Calc^{\widehat{\mathcal{C}}}_{\mathcal{W}} = \left\{(x,w) \in \mathcal{W} \times \mathcal{W} \,\middle|\, w \geq x \geq w_0, \exists \lambda_0 \in \overline{A_0} : w.\lambda_0 \in \widehat{w.A_0} \cap X_p - p\rho, x.\lambda_0 \in -X^+ - \rho\right\}$$

and the extended area of calculation becomes

$$Calc_{\mathcal{W}} = Calc^{Lusztig}_{\mathcal{W}} \cup Calc^{\widehat{\mathcal{C}}}_{\mathcal{W}}$$

We have calculated $P^p_{x,w}$ for all $(x, w) \in Calc_{\mathcal{W}}$ with $\mathcal{W}$ of type A_n for $n \in \{2, 3, 4\}$ and $p \in \{2, 3\}$, with the possible exception of the cases from $Calc^{Lusztig}_{\mathcal{W}}$ where we have not checked if the moment graph is nice. It turns out that all the moment graphs $G_{x \leq \mathcal{W} \leq w}$ with $(x, w) \in Calc^{\widehat{\mathcal{C}}}_{\mathcal{W}}$ are nice.

Table 5.1 is a list of all exceptions to $P_{x,w} = P^p_{x,w}$ within the calculated area. An extremely interesting entry in table 5.1 is the last entry, because $G_{\mathcal{W} \leq w_0 s_0 s_1 s_2 s_4 s_3 s_2}$

is actually a GKM-graph, so this particular example actually disproves conjecture 5.2.3. So the GKM property of the moment graph is no guarantee for $P_{x,w} = P^p_{x,w}$.

We now make a precise modified version of conjecture 5.2.2 with the modular Kazhdan-Lusztig polynomials:

Question 5.2.6. Let $\lambda \in \overline{A_0} \cap X$, and let $w \in \mathcal{W}$ such that $w.\lambda \in \widehat{w.A_0} \cap (X_p - p\rho)$, and let $x \in \mathcal{W}$ such that $x.\lambda \in -X^+ - \rho$. Are the composition coefficients then given by

$$\left[\widehat{Z}\left(x.\lambda\right) : \widehat{L}\left(w.\lambda\right)\right] = P^p_{x,w}\left(1\right)?$$

Note that this question is asked for all primes p not only the primes $p > h$. If $p > h$, then the answer to the question is "yes" assuming conjecture 5.2.3 holds (in these cases); this follows from theorem 5.2.4 and II.9.22 of [Jan03].

We will answer this question when $\mathcal{W}$ is of type A_n with $n \in \{2, 3, 4\}$ and with $p \in \{2, 3\}$, by calculating the composition coefficients. We have calculated $\left[\widehat{Z}\left(x.\lambda_0\right) : \widehat{L}\left(w.\lambda_0\right)\right]$, and found it to be equal to $P^p_{x,w}\left(1\right)$, for each $(x, w) \in Calc^{\widehat{\mathcal{C}}}_{\mathcal{W}}$ and for each initial weight $\lambda_0 \in \overline{A_0}$ fitting the definition of $Calc^{\widehat{\mathcal{C}}}_{\mathcal{W}}$.

Moreover in type A_4 with $p = 2$, for $(x, w) = (w_0 s_0, w_0 s_0 s_1 s_2 s_4 s_3 s_2 s_0) \in Calc^{\widehat{\mathcal{C}}}_{\mathcal{W}}$, for initial weight $[0, -1, -1, 0]$ we have

$$\left[\widehat{Z}\left([-2, -1, -1, -2]\right) : \widehat{L}\left([-2, -2, -2, -2]\right)\right] = 6 = P^2_{x,w}\left(1\right) \neq 5 = P_{x,w}\left(1\right)$$

and for $(x, w) = (w_0, w_0 s_0 s_1 s_2 s_4 s_3 s_2 s_0) \in Calc^{\widehat{\mathcal{C}}}_{\mathcal{W}}$ for initial weight $[0, -1, -1, 0]$ we have

$$\left[\widehat{Z}\left([-2, -1, -1, -2]\right) : \widehat{L}\left([-2, -2, -2, -2]\right)\right] = 6 = P^2_{x,w}\left(1\right) \neq 5 = P_{x,w}\left(1\right)$$

and in type A_4 with $p = 3$, for $(x, w) = (w_0, w_0 s_0 s_1 s_2 s_4 s_3 s_2) \in Calc^{\widehat{\mathcal{C}}}_{\mathcal{W}}$ for initial weight $[0, -1, -1, 0]$ we have

$$\left[\widehat{Z}\left([-2, -1, -1, -2]\right) : \widehat{L}\left([-3, -2, -2, -3]\right)\right] = 5 = P^3_{x,w}\left(1\right) \neq 4 = P_{x,w}\left(1\right)$$

For all the rest of the pairs $(x, w) \in Calc^{\widehat{\mathcal{C}}}_{\mathcal{W}}$ we have $P_{x,w} = P^p_{x,w}$.

We see that we answer question 5.2.6 in the affirmative in every calculated case. Moreover, there are several exceptions in the calculated area to the corresponding formula with the usual Kazhdan-Lusztig polynomials from conjecture 5.2.2,

$$\left[\widehat{Z}\left(x.\lambda_0\right) : \widehat{L}\left(w.\lambda_0\right)\right] = P_{x,w}\left(1\right)$$

Consequently, there is reason to believe that the correct version of the formula should include the modular Kazhdan-Lusztig polynomials, and we conjecture that the answer to question 5.2.6 is yes.

Type	p	x	w	$P_{w_0x,w_0w}(q)$	$P^p_{w_0x,w_0w}(q)$	Difference
A_4	2	$s_0s_1s_4$	$s_0s_1s_2s_4s_3s_2s_0s_1s_4$	$3q^2+4q+1$	q^3+5q^2+4q+1	q^3+2q^2
A_4	2	$s_0s_1s_2s_4$	$s_0s_1s_2s_4s_3s_2s_0s_1s_4$	q^2+3q+1	$2q^2+3q+1$	q^2
A_4	2	$s_0s_1s_4s_3$	$s_0s_1s_2s_4s_3s_2s_0s_1s_4$	q^2+3q+1	$2q^2+3q+1$	q^2
A_4	2	s_0	$s_0s_1s_2s_4s_3s_2s_0$	$2q^2+2q+1$	q^3+2q^2+2q+1	q^3
A_4	2	1	$s_0s_1s_2s_4s_3s_2s_0$	$2q^2+2q+1$	q^3+2q^2+2q+1	q^3
A_4	3	s_0s_1	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$2q^3+5q^2+5q+1$	$3q^5+2q^4+3q^3+5q^2+5q+1$	$3q^5+2q^4+q^3$
A_4	3	$s_0s_1s_4$	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$3q^2+5q+1$	$2q^4+q^3+3q^2+5q+1$	$2q^4+q^3$
A_4	3	$s_0s_1s_4$	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$3q^2+5q+1$	$2q^4+q^3+3q^2+5q+1$	$2q^4+q^3$
A_4	3	$s_0s_1s_4s_0$	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$3q^2+5q+1$	q^3+3q^2+5q+1	q^3
A_4	3	s_0	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$2q^4+6q^3+7q^2+5q+1$	$q^6+6q^5+4q^4+7q^3+7q^2+5q+1$	$q^6+6q^5+2q^4+q^3$
A_4	3	s_0s_1	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$2q^3+5q^2+5q+1$	$3q^5+2q^4+3q^3+5q^2+5q+1$	$3q^5+2q^4+q^3$
A_4	3	s_0s_4	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$2q^3+5q^2+5q+1$	$3q^5+2q^4+3q^3+5q^2+5q+1$	$3q^5+2q^4+q^3$
A_4	3	$s_0s_1s_4$	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$3q^2+5q+1$	$2q^4+q^3+3q^2+5q+1$	$2q^4+q^3$
A_4	3	s_0s_4	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$2q^3+5q^2+5q+1$	$3q^5+2q^4+3q^3+5q^2+5q+1$	$3q^5+2q^4+q^3$
A_4	3	$s_0s_1s_4$	$s_0s_1s_2s_4s_3s_2s_0s_1s_4s_0$	$3q^2+5q+1$	$2q^4+q^3+3q^2+5q+1$	$2q^4+q^3$
A_4	3	1	$s_0s_1s_2s_4s_3s_2$	$2q^2+q+1$	q^3+2q^2+q+1	q^3

Table 5.1: The exceptions to $P_{x,w} = P^p_{x,w}$ within $Calc_{\mathcal{W}}$

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Appendix A

Calculated characters

Here is a list of all the values of $[L(\lambda) : V(\mu)]$ for all the weights $\lambda \in X_p$, these values are compared to the values given by Lusztig's character formula, and the values given by Lusztig's character formula using the modular Kazhdan-Lusztig-polynomials in stead of the usual Kazhdan-Lusztig-polynomials. The dot-orbits with no more than one element in the fundamental box X_p are omitted.

Note that the weights are represented as:

$$[a_1, a_2, \ldots, a_n] = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n$$

In the matrices we have underlined the coefficients where the actual value differs from that predicted by Lusztig's conjecture.

A.1 Type A_2, $p = 3$

Stabilizer $\emptyset$, initial weight $[0, 0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,1]) \\ \mathrm{ch}L([0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1]) \\ \chi([0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1]) \\ \mathrm{ch}L([0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1]) \\ \chi([0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1]) \\ \mathrm{ch}L([0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1]) \\ \chi([0,0]) \end{pmatrix}$$

A.2 Type A_3, $p = 2$

Stabilizer $\{s_1, s_3\}$, initial weight $[-1, 0, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L\left([1,0,1]\right) \\ \mathrm{ch}L\left([0,0,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([1,0,1]\right) \\ \chi\left([0,0,0]\right) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([1,0,1]\right) \\ \mathrm{ch}L\left([0,0,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([1,0,1]\right) \\ \chi\left([0,0,0]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([1,0,1]\right) \\ \mathrm{ch}L\left([0,0,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([1,0,1]\right) \\ \chi\left([0,0,0]\right) \end{pmatrix}$$

A.3 Type A_3, $p = 3$

Stabilizer $\{s_0\}$, initial weight $[0,0,0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L\left([1,2,1]\right) \\ \mathrm{ch}L\left([0,2,0]\right) \\ \mathrm{ch}L\left([0,0,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([1,2,1]\right) \\ \chi\left([0,2,0]\right) \\ \chi\left([0,0,0]\right) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([1,2,1]\right) \\ \mathrm{ch}L\left([0,2,0]\right) \\ \mathrm{ch}L\left([0,0,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([1,2,1]\right) \\ \chi\left([0,2,0]\right) \\ \chi\left([0,0,0]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([1,2,1]\right) \\ \mathrm{ch}L\left([0,2,0]\right) \\ \mathrm{ch}L\left([0,0,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([1,2,1]\right) \\ \chi\left([0,2,0]\right) \\ \chi\left([0,0,0]\right) \end{pmatrix}$$

Stabilizer $\{s_1\}$, initial weight $[-1,0,0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,1]\right) \\ \mathrm{ch}L\left([1,1,0]\right) \\ \mathrm{ch}L\left([0,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,1]\right) \\ \chi\left([3,0,0]\right) \\ \chi\left([1,1,0]\right) \\ \chi\left([0,0,1]\right) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,1]\right) \\ \mathrm{ch}L\left([1,1,0]\right) \\ \mathrm{ch}L\left([0,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,1]\right) \\ \chi\left([3,0,0]\right) \\ \chi\left([1,1,0]\right) \\ \chi\left([0,0,1]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,1]\right) \\ \mathrm{ch}L\left([1,1,0]\right) \\ \mathrm{ch}L\left([0,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,1]\right) \\ \chi\left([3,0,0]\right) \\ \chi\left([1,1,0]\right) \\ \chi\left([0,0,1]\right) \end{pmatrix}$$

Stabilizer $\{s_2\}$, initial weight $[0, -1, 0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,1,1]) \\ \mathrm{ch}L([2,0,0]) \\ \mathrm{ch}L([0,0,2]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,1]) \\ \chi([2,0,0]) \\ \chi([0,0,2]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,1]) \\ \mathrm{ch}L([2,0,0]) \\ \mathrm{ch}L([0,0,2]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,1]) \\ \chi([2,0,0]) \\ \chi([0,0,2]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,1]) \\ \mathrm{ch}L([2,0,0]) \\ \mathrm{ch}L([0,0,2]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,1]) \\ \chi([2,0,0]) \\ \chi([0,0,2]) \end{pmatrix}$$

Stabilizer $\{s_3\}$, initial weight $[0, 0, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,1,2]) \\ \mathrm{ch}L([0,1,1]) \\ \mathrm{ch}L([1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,2]) \\ \chi([0,0,3]) \\ \chi([0,1,1]) \\ \chi([1,0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,2]) \\ \mathrm{ch}L([0,1,1]) \\ \mathrm{ch}L([1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,2]) \\ \chi([0,0,3]) \\ \chi([0,1,1]) \\ \chi([1,0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,2]) \\ \mathrm{ch}L([0,1,1]) \\ \mathrm{ch}L([1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,2]) \\ \chi([0,0,3]) \\ \chi([0,1,1]) \\ \chi([1,0,0]) \end{pmatrix}$$

Stabilizer $\{s_1, s_3\}$, initial weight $[-1, 0, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([2,0,2]) \\ \mathrm{ch}L([1,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,0,2]) \\ \chi([1,0,1]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,0,2]\right) \\ \mathrm{ch}L\left([1,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,0,2]\right) \\ \chi\left([1,0,1]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,0,2]\right) \\ \mathrm{ch}L\left([1,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,0,2]\right) \\ \chi\left([1,0,1]\right) \end{pmatrix}$$

Stabilizer $\{s_1, s_3\}$, initial weight $[-1, 1, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2]\right) \\ \mathrm{ch}L\left([0,1,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2]\right) \\ \chi\left([0,1,0]\right) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2]\right) \\ \mathrm{ch}L\left([0,1,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2]\right) \\ \chi\left([0,1,0]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2]\right) \\ \mathrm{ch}L\left([0,1,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2]\right) \\ \chi\left([0,1,0]\right) \end{pmatrix}$$

A.4 Type A_4, $p = 2$

Stabilizer $\{s_0, s_1, s_3\}$, initial weight $[-1, 0, -1, 0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([0,1,0,1]) \\ \mathrm{ch}L([1,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,1,0,1]) \\ \chi([1,0,0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([0,1,0,1]) \\ \mathrm{ch}L([1,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,1,0,1]) \\ \chi([1,0,0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([0,1,0,1]) \\ \mathrm{ch}L([1,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,1,0,1]) \\ \chi([1,0,0,0]) \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_3\}$, initial weight $[0, -1, -1, 0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([0,1,1,0]) \\ \mathrm{ch}L([0,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,1,1,0]) \\ \chi([0,0,0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([0,1,1,0]) \\ \mathrm{ch}L([0,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & \underline{0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,1,1,0]) \\ \chi([0,0,0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([0,1,1,0]) \\ \mathrm{ch}L([0,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & \underline{1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,1,1,0]) \\ \chi([0,0,0,0]) \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_4\}$, initial weight $[0, -1, 0, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,0,1,0]) \\ \mathrm{ch}L([0,0,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,0,1,0]) \\ \chi([0,0,0,1]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,0,1,0]) \\ \mathrm{ch}L([0,0,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,0,1,0]) \\ \chi([0,0,0,1]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,0,1,0]) \\ \mathrm{ch}L([0,0,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,0,1,0]) \\ \chi([0,0,0,1]) \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_4\}$, initial weight $[-1, -1, 0, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,0,1,1]) \\ \mathrm{ch}L([0,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,0,1,1]) \\ \chi([0,0,0,2]) \\ \chi([0,0,1,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,0,1,1]) \\ \mathrm{ch}L([0,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,0,1,1]) \\ \chi([0,0,0,2]) \\ \chi([0,0,1,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,0,1,1]) \\ \mathrm{ch}L([0,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,0,1,1]) \\ \chi([0,0,0,2]) \\ \chi([0,0,1,0]) \end{pmatrix}$$

Stabilizer $\{s_1, s_3, s_4\}$, initial weight $[-1, 0, -1, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,1,0,1]) \\ \mathrm{ch}L([0,1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,0,1]) \\ \chi([2,0,0,0]) \\ \chi([0,1,0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,0,1]) \\ \mathrm{ch}L([0,1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,0,1]) \\ \chi([2,0,0,0]) \\ \chi([0,1,0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,0,1]) \\ \mathrm{ch}L([0,1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,0,1]) \\ \chi([2,0,0,0]) \\ \chi([0,1,0,0]) \end{pmatrix}$$

A.5 Type A_4, $p = 3$

Stabilizer $\{s_0, s_1\}$, initial weight $[-1, 0, 0, 0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,1,1,2]) \\ \mathrm{ch}L([0,0,2,2]) \\ \mathrm{ch}L([2,0,1,1]) \\ \mathrm{ch}L([2,1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,1,2]) \\ \chi([2,0,0,3]) \\ \chi([0,0,2,2]) \\ \chi([2,0,1,1]) \\ \chi([2,1,0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,1,2]) \\ \mathrm{ch}L([0,0,2,2]) \\ \mathrm{ch}L([2,0,1,1]) \\ \mathrm{ch}L([2,1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,1,2]) \\ \chi([2,0,0,3]) \\ \chi([0,0,2,2]) \\ \chi([2,0,1,1]) \\ \chi([2,1,0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,1,2]) \\ \mathrm{ch}L([0,0,2,2]) \\ \mathrm{ch}L([2,0,1,1]) \\ \mathrm{ch}L([2,1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,1,2]) \\ \chi([2,0,0,3]) \\ \chi([0,0,2,2]) \\ \chi([2,0,1,1]) \\ \chi([2,1,0,0]) \end{pmatrix}$$

Stabilizer $\{s_0, s_2\}$, initial weight $[0, -1, 0, 0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,1,2,1]) \\ \mathrm{ch}L([2,0,2,0]) \\ \mathrm{ch}L([0,1,1,2]) \\ \mathrm{ch}L([1,0,1,1]) \\ \mathrm{ch}L([1,1,0,0]) \\ \mathrm{ch}L([0,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,2,1]) \\ \chi([0,0,3,1]) \\ \chi([2,0,2,0]) \\ \chi([0,1,1,2]) \\ \chi([1,0,0,3]) \\ \chi([1,0,1,1]) \\ \chi([3,0,0,0]) \\ \chi([1,1,0,0]) \\ \chi([0,0,1,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,2,1]) \\ \mathrm{ch}L([2,0,2,0]) \\ \mathrm{ch}L([0,1,1,2]) \\ \mathrm{ch}L([1,0,1,1]) \\ \mathrm{ch}L([1,1,0,0]) \\ \mathrm{ch}L([0,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,2,1]) \\ \chi([0,0,3,1]) \\ \chi([2,0,2,0]) \\ \chi([0,1,1,2]) \\ \chi([1,0,0,3]) \\ \chi([1,0,1,1]) \\ \chi([3,0,0,0]) \\ \chi([1,1,0,0]) \\ \chi([0,0,1,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,2,1]) \\ \mathrm{ch}L([2,0,2,0]) \\ \mathrm{ch}L([0,1,1,2]) \\ \mathrm{ch}L([1,0,1,1]) \\ \mathrm{ch}L([1,1,0,0]) \\ \mathrm{ch}L([0,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,2,1]) \\ \chi([0,0,3,1]) \\ \chi([2,0,2,0]) \\ \chi([0,1,1,2]) \\ \chi([1,0,0,3]) \\ \chi([1,0,1,1]) \\ \chi([3,0,0,0]) \\ \chi([1,1,0,0]) \\ \chi([0,0,1,0]) \end{pmatrix}$$

Stabilizer $\{s_0, s_3\}$, initial weight $[0,0,-1,0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,2,1,1]) \\ \mathrm{ch}L([2,1,1,0]) \\ \mathrm{ch}L([0,2,0,2]) \\ \mathrm{ch}L([1,1,0,1]) \\ \mathrm{ch}L([0,0,1,1]) \\ \mathrm{ch}L([0,1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,1,1]) \\ \chi([1,3,0,0]) \\ \chi([2,1,1,0]) \\ \chi([0,2,0,2]) \\ \chi([3,0,0,1]) \\ \chi([1,1,0,1]) \\ \chi([0,0,0,3]) \\ \chi([0,0,1,1]) \\ \chi([0,1,0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,2,1,1]) \\ \mathrm{ch}L([2,1,1,0]) \\ \mathrm{ch}L([0,2,0,2]) \\ \mathrm{ch}L([1,1,0,1]) \\ \mathrm{ch}L([0,0,1,1]) \\ \mathrm{ch}L([0,1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,1,1]) \\ \chi([1,3,0,0]) \\ \chi([2,1,1,0]) \\ \chi([0,2,0,2]) \\ \chi([3,0,0,1]) \\ \chi([1,1,0,1]) \\ \chi([0,0,0,3]) \\ \chi([0,0,1,1]) \\ \chi([0,1,0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,2,1,1]) \\ \mathrm{ch}L([2,1,1,0]) \\ \mathrm{ch}L([0,2,0,2]) \\ \mathrm{ch}L([1,1,0,1]) \\ \mathrm{ch}L([0,0,1,1]) \\ \mathrm{ch}L([0,1,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,1,1]) \\ \chi([1,3,0,0]) \\ \chi([2,1,1,0]) \\ \chi([0,2,0,2]) \\ \chi([3,0,0,1]) \\ \chi([1,1,0,1]) \\ \chi([0,0,0,3]) \\ \chi([0,0,1,1]) \\ \chi([0,1,0,0]) \end{pmatrix}$$

Stabilizer $\{s_0, s_4\}$, initial weight $[0,0,0,-1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([2,1,1,1]) \\ \mathrm{ch}L([2,2,0,0]) \\ \mathrm{ch}L([1,1,0,2]) \\ \mathrm{ch}L([0,0,1,2]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,1,1,1]) \\ \chi([2,2,0,0]) \\ \chi([3,0,0,2]) \\ \chi([1,1,0,2]) \\ \chi([0,0,1,2]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,1,1,1]) \\ \mathrm{ch}L([2,2,0,0]) \\ \mathrm{ch}L([1,1,0,2]) \\ \mathrm{ch}L([0,0,1,2]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,1,1,1]) \\ \chi([2,2,0,0]) \\ \chi([3,0,0,2]) \\ \chi([1,1,0,2]) \\ \chi([0,0,1,2]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,1,1,1]) \\ \mathrm{ch}L([2,2,0,0]) \\ \mathrm{ch}L([1,1,0,2]) \\ \mathrm{ch}L([0,0,1,2]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,1,1,1]) \\ \chi([2,2,0,0]) \\ \chi([3,0,0,2]) \\ \chi([1,1,0,2]) \\ \chi([0,0,1,2]) \end{pmatrix}$$

Stabilizer $\{s_1, s_2\}$, initial weight $[-1,-1,0,0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,1,2,2]) \\ \mathrm{ch}L([0,1,2,1]) \\ \mathrm{ch}L([1,0,2,0]) \\ \mathrm{ch}L([2,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,2,2]) \\ \chi([0,0,3,2]) \\ \chi([0,1,2,1]) \\ \chi([1,0,0,4]) \\ \chi([1,0,2,0]) \\ \chi([2,0,0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,2,2]) \\ \mathrm{ch}L([0,1,2,1]) \\ \mathrm{ch}L([1,0,2,0]) \\ \mathrm{ch}L([2,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,2,2]) \\ \chi([0,0,3,2]) \\ \chi([0,1,2,1]) \\ \chi([1,0,0,4]) \\ \chi([1,0,2,0]) \\ \chi([2,0,0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,1,2,2]) \\ \mathrm{ch}L([0,1,2,1]) \\ \mathrm{ch}L([1,0,2,0]) \\ \mathrm{ch}L([2,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,1,2,2]) \\ \chi([0,0,3,2]) \\ \chi([0,1,2,1]) \\ \chi([1,0,0,4]) \\ \chi([1,0,2,0]) \\ \chi([2,0,0,0]) \end{pmatrix}$$

Stabilizer $\{s_1, s_3\}$, initial weight $[-1,0,-1,0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,2,1,2]) \\ \mathrm{ch}L([0,2,1,1]) \\ \mathrm{ch}L([1,1,1,0]) \\ \mathrm{ch}L([2,0,0,1]) \\ \mathrm{ch}L([0,0,2,0]) \\ \mathrm{ch}L([1,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 1 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,1,2]) \\ \chi([0,2,0,3]) \\ \chi([0,2,1,1]) \\ \chi([3,0,1,0]) \\ \chi([0,3,0,0]) \\ \chi([1,1,1,0]) \\ \chi([0,0,0,4]) \\ \chi([2,0,0,1]) \\ \chi([0,0,2,0]) \\ \chi([1,0,0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,2,1,2]) \\ \mathrm{ch}L([0,2,1,1]) \\ \mathrm{ch}L([1,1,1,0]) \\ \mathrm{ch}L([2,0,0,1]) \\ \mathrm{ch}L([0,0,2,0]) \\ \mathrm{ch}L([1,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 1 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,1,2]) \\ \chi([0,2,0,3]) \\ \chi([0,2,1,1]) \\ \chi([3,0,1,0]) \\ \chi([0,3,0,0]) \\ \chi([1,1,1,0]) \\ \chi([0,0,0,4]) \\ \chi([2,0,0,1]) \\ \chi([0,0,2,0]) \\ \chi([1,0,0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,2,1,2]) \\ \mathrm{ch}L([0,2,1,1]) \\ \mathrm{ch}L([1,1,1,0]) \\ \mathrm{ch}L([2,0,0,1]) \\ \mathrm{ch}L([0,0,2,0]) \\ \mathrm{ch}L([1,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 1 & 1 & 0 & -2 & \underline{4} \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,1,2]) \\ \chi([0,2,0,3]) \\ \chi([0,2,1,1]) \\ \chi([3,0,1,0]) \\ \chi([0,3,0,0]) \\ \chi([1,1,1,0]) \\ \chi([0,0,0,4]) \\ \chi([2,0,0,1]) \\ \chi([0,0,2,0]) \\ \chi([1,0,0,0]) \end{pmatrix}$$

Stabilizer $\{s_1, s_4\}$, initial weight $[-1, 0, 0, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([2,1,1,2]) \\ \mathrm{ch}L([1,1,1,1]) \\ \mathrm{ch}L([1,2,0,0]) \\ \mathrm{ch}L([2,0,0,2]) \\ \mathrm{ch}L([0,0,2,1]) \\ \mathrm{ch}L([0,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,1,1,2]) \\ \chi([3,0,0,3]) \\ \chi([1,1,1,1]) \\ \chi([3,1,0,0]) \\ \chi([0,0,1,3]) \\ \chi([1,2,0,0]) \\ \chi([2,0,0,2]) \\ \chi([0,0,2,1]) \\ \chi([0,0,0,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,1,1,2]) \\ \mathrm{ch}L([1,1,1,1]) \\ \mathrm{ch}L([1,2,0,0]) \\ \mathrm{ch}L([2,0,0,2]) \\ \mathrm{ch}L([0,0,2,1]) \\ \mathrm{ch}L([0,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,1,1,2]) \\ \chi([3,0,0,3]) \\ \chi([1,1,1,1]) \\ \chi([3,1,0,0]) \\ \chi([0,0,1,3]) \\ \chi([1,2,0,0]) \\ \chi([2,0,0,2]) \\ \chi([0,0,2,1]) \\ \chi([0,0,0,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,1,1,2]) \\ \mathrm{ch}L([1,1,1,1]) \\ \mathrm{ch}L([1,2,0,0]) \\ \mathrm{ch}L([2,0,0,2]) \\ \mathrm{ch}L([0,0,2,1]) \\ \mathrm{ch}L([0,0,0,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & \underline{-5} \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,1,1,2]) \\ \chi([3,0,0,3]) \\ \chi([1,1,1,1]) \\ \chi([3,1,0,0]) \\ \chi([0,0,1,3]) \\ \chi([1,2,0,0]) \\ \chi([2,0,0,2]) \\ \chi([0,0,2,1]) \\ \chi([0,0,0,0]) \end{pmatrix}$$

Stabilizer $\{s_2, s_3\}$, initial weight $[0, -1, -1, 0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,2,2,1]) \\ \mathrm{ch}L([0,2,2,0]) \\ \mathrm{ch}L([0,1,1,0]) \\ \mathrm{ch}L([1,0,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,2,1]) \\ \chi([0,2,2,0]) \\ \chi([0,1,1,0]) \\ \chi([1,0,0,1]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,2,2,1]) \\ \mathrm{ch}L([0,2,2,0]) \\ \mathrm{ch}L([0,1,1,0]) \\ \mathrm{ch}L([1,0,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & \underline{1} & \underline{-1} \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,2,1]) \\ \chi([0,2,2,0]) \\ \chi([0,1,1,0]) \\ \chi([1,0,0,1]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([1,2,2,1]\right) \\ \mathrm{ch}L\left([0,2,2,0]\right) \\ \mathrm{ch}L\left([0,1,1,0]\right) \\ \mathrm{ch}L\left([1,0,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & \underline{2} & \underline{-4} \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([1,2,2,1]\right) \\ \chi\left([0,2,2,0]\right) \\ \chi\left([0,1,1,0]\right) \\ \chi\left([1,0,0,1]\right) \end{pmatrix}$$

Stabilizer $\{s_2, s_4\}$, initial weight $[0,-1,0,-1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2,1]\right) \\ \mathrm{ch}L\left([1,1,2,0]\right) \\ \mathrm{ch}L\left([0,1,1,1]\right) \\ \mathrm{ch}L\left([0,2,0,0]\right) \\ \mathrm{ch}L\left([1,0,0,2]\right) \\ \mathrm{ch}L\left([0,0,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2,1]\right) \\ \chi\left([3,0,2,0]\right) \\ \chi\left([1,1,2,0]\right) \\ \chi\left([0,0,3,0]\right) \\ \chi\left([0,1,0,3]\right) \\ \chi\left([4,0,0,0]\right) \\ \chi\left([0,1,1,1]\right) \\ \chi\left([0,2,0,0]\right) \\ \chi\left([1,0,0,2]\right) \\ \chi\left([0,0,0,1]\right) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2,1]\right) \\ \mathrm{ch}L\left([1,1,2,0]\right) \\ \mathrm{ch}L\left([0,1,1,1]\right) \\ \mathrm{ch}L\left([0,2,0,0]\right) \\ \mathrm{ch}L\left([1,0,0,2]\right) \\ \mathrm{ch}L\left([0,0,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2,1]\right) \\ \chi\left([3,0,2,0]\right) \\ \chi\left([1,1,2,0]\right) \\ \chi\left([0,0,3,0]\right) \\ \chi\left([0,1,0,3]\right) \\ \chi\left([4,0,0,0]\right) \\ \chi\left([0,1,1,1]\right) \\ \chi\left([0,2,0,0]\right) \\ \chi\left([1,0,0,2]\right) \\ \chi\left([0,0,0,1]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2,1]\right) \\ \mathrm{ch}L\left([1,1,2,0]\right) \\ \mathrm{ch}L\left([0,1,1,1]\right) \\ \mathrm{ch}L\left([0,2,0,0]\right) \\ \mathrm{ch}L\left([1,0,0,2]\right) \\ \mathrm{ch}L\left([0,0,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 1 & 1 & -2 & 0 & \underline{4} \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2,1]\right) \\ \chi\left([3,0,2,0]\right) \\ \chi\left([1,1,2,0]\right) \\ \chi\left([0,0,3,0]\right) \\ \chi\left([0,1,0,3]\right) \\ \chi\left([4,0,0,0]\right) \\ \chi\left([0,1,1,1]\right) \\ \chi\left([0,2,0,0]\right) \\ \chi\left([1,0,0,2]\right) \\ \chi\left([0,0,0,1]\right) \end{pmatrix}$$

Stabilizer $\{s_3, s_4\}$, initial weight $[0, 0, -1, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([2,2,1,1]) \\ \mathrm{ch}L([1,2,1,0]) \\ \mathrm{ch}L([0,2,0,1]) \\ \mathrm{ch}L([0,0,0,2]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,2,1,1]) \\ \chi([2,3,0,0]) \\ \chi([1,2,1,0]) \\ \chi([4,0,0,1]) \\ \chi([0,2,0,1]) \\ \chi([0,0,0,2]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,2,1,1]) \\ \mathrm{ch}L([1,2,1,0]) \\ \mathrm{ch}L([0,2,0,1]) \\ \mathrm{ch}L([0,0,0,2]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,2,1,1]) \\ \chi([2,3,0,0]) \\ \chi([1,2,1,0]) \\ \chi([4,0,0,1]) \\ \chi([0,2,0,1]) \\ \chi([0,0,0,2]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,2,1,1]) \\ \mathrm{ch}L([1,2,1,0]) \\ \mathrm{ch}L([0,2,0,1]) \\ \mathrm{ch}L([0,0,0,2]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,2,1,1]) \\ \chi([2,3,0,0]) \\ \chi([1,2,1,0]) \\ \chi([4,0,0,1]) \\ \chi([0,2,0,1]) \\ \chi([0,0,0,2]) \end{pmatrix}$$

Stabilizer $\{s_0, s_1, s_3\}$, initial weight $[-1, 0, -1, 1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([0,2,1,2]) \\ \mathrm{ch}L([2,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,2,1,2]) \\ \chi([2,0,1,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([0,2,1,2]) \\ \mathrm{ch}L([2,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,2,1,2]) \\ \chi([2,0,1,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([0,2,1,2]) \\ \mathrm{ch}L([2,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,2,1,2]) \\ \chi([2,0,1,0]) \end{pmatrix}$$

Stabilizer $\{s_0, s_1, s_3\}$, initial weight $[-1, 1, -1, 0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,2,0,2]) \\ \mathrm{ch}L([2,1,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,0,2]) \\ \chi([2,1,0,1]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,2,0,2]) \\ \mathrm{ch}L([2,1,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,0,2]) \\ \chi([2,1,0,1]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,2,0,2]) \\ \mathrm{ch}L([2,1,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,0,2]) \\ \chi([2,1,0,1]) \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_3\}$, initial weight $[0, -1, -1, 1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([0,2,2,1]) \\ \mathrm{ch}L([1,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,2,2,1]) \\ \chi([1,0,1,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([0,2,2,1]) \\ \mathrm{ch}L([1,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,2,2,1]) \\ \chi([1,0,1,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([0,2,2,1]) \\ \mathrm{ch}L([1,0,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([0,2,2,1]) \\ \chi([1,0,1,0]) \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_3\}$, initial weight $[1, -1, -1, 0]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([1,2,2,0]) \\ \mathrm{ch}L([0,1,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,2,0]) \\ \chi([0,1,0,1]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,2,2,0]) \\ \mathrm{ch}L([0,1,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,2,0]) \\ \chi([0,1,0,1]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([1,2,2,0]) \\ \mathrm{ch}L([0,1,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([1,2,2,0]) \\ \chi([0,1,0,1]) \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_4\}$, initial weight $[1, -1, 0, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2,0]\right) \\ \mathrm{ch}L\left([0,1,0,2]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2,0]\right) \\ \chi\left([0,1,0,2]\right) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2,0]\right) \\ \mathrm{ch}L\left([0,1,0,2]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2,0]\right) \\ \chi\left([0,1,0,2]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2,0]\right) \\ \mathrm{ch}L\left([0,1,0,2]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2,0]\right) \\ \chi\left([0,1,0,2]\right) \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_4\}$, initial weight $[0, -1, 1, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L\left([2,0,2,1]\right) \\ \mathrm{ch}L\left([1,0,1,2]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,0,2,1]\right) \\ \chi\left([1,0,1,2]\right) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,0,2,1]\right) \\ \mathrm{ch}L\left([1,0,1,2]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,0,2,1]\right) \\ \chi\left([1,0,1,2]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,0,2,1]\right) \\ \mathrm{ch}L\left([1,0,1,2]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,0,2,1]\right) \\ \chi\left([1,0,1,2]\right) \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_4\}$, initial weight $[-1, -1, 0, -1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2,2]\right) \\ \mathrm{ch}L\left([0,1,2,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2,2]\right) \\ \chi\left([0,1,0,4]\right) \\ \chi\left([0,1,2,0]\right) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2,2]\right) \\ \mathrm{ch}L\left([0,1,2,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2,2]\right) \\ \chi\left([0,1,0,4]\right) \\ \chi\left([0,1,2,0]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,1,2,2]\right) \\ \mathrm{ch}L\left([0,1,2,0]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,1,2,2]\right) \\ \chi\left([0,1,0,4]\right) \\ \chi\left([0,1,2,0]\right) \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_4\}$, initial weight $[-1,-1,1,-1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([2,0,2,2]) \\ \mathrm{ch}L([1,0,2,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,0,2,2]) \\ \chi([1,0,1,3]) \\ \chi([1,0,2,1]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,0,2,2]) \\ \mathrm{ch}L([1,0,2,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,0,2,2]) \\ \chi([1,0,1,3]) \\ \chi([1,0,2,1]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,0,2,2]) \\ \mathrm{ch}L([1,0,2,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,0,2,2]) \\ \chi([1,0,1,3]) \\ \chi([1,0,2,1]) \end{pmatrix}$$

Stabilizer $\{s_1, s_3, s_4\}$, initial weight $[-1,0,-1,-1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([2,2,1,2]) \\ \mathrm{ch}L([0,2,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,2,1,2]) \\ \chi([4,0,1,0]) \\ \chi([0,2,1,0]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,2,1,2]) \\ \mathrm{ch}L([0,2,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,2,1,2]) \\ \chi([4,0,1,0]) \\ \chi([0,2,1,0]) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L([2,2,1,2]) \\ \mathrm{ch}L([0,2,1,0]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,2,1,2]) \\ \chi([4,0,1,0]) \\ \chi([0,2,1,0]) \end{pmatrix}$$

Stabilizer $\{s_1, s_3, s_4\}$, initial weight $[-1,1,-1,-1]$

The simple characters are:

$$\begin{pmatrix} \mathrm{ch}L([2,2,0,2]) \\ \mathrm{ch}L([1,2,0,1]) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi([2,2,0,2]) \\ \chi([3,1,0,1]) \\ \chi([1,2,0,1]) \end{pmatrix}$$

Lusztig's conjecture predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,2,0,2]\right) \\ \mathrm{ch}L\left([1,2,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,2,0,2]\right) \\ \chi\left([3,1,0,1]\right) \\ \chi\left([1,2,0,1]\right) \end{pmatrix}$$

The character formula using the modular Kazhdan-Lusztig polynomials predicts:

$$\begin{pmatrix} \mathrm{ch}L\left([2,2,0,2]\right) \\ \mathrm{ch}L\left([1,2,0,1]\right) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\left([2,2,0,2]\right) \\ \chi\left([3,1,0,1]\right) \\ \chi\left([1,2,0,1]\right) \end{pmatrix}$$

Appendix B

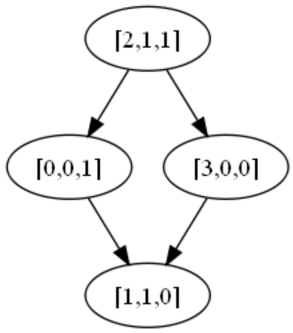
Submodule diagrams

In this appendix we list the module diagrams for those of the Weyl modules seen in appendix A with composition length ≥ 3.

B.1 Diagrams for A_3, $p = 3$:

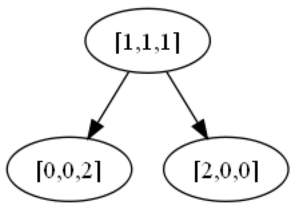
Stabilizer $\{s_1\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[-1, 0, 0]$.

Diagram for $V([2, 1, 1])$:



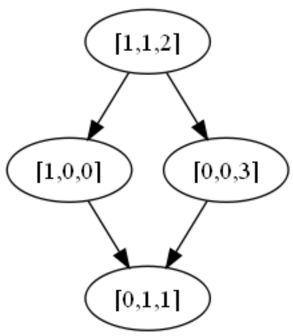
Stabilizer $\{s_2\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[0, -1, 0]$.

Diagram for $V([1, 1, 1])$:



Stabilizer $\{s_3\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[0, 0, -1]$.

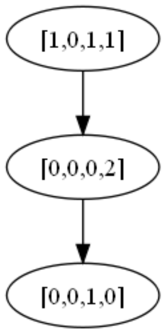
Diagram for $V([1, 1, 2])$:



B.2 Diagrams for A_4, $p = 2$:

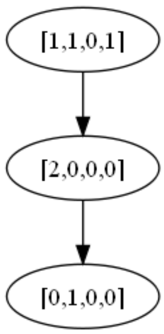
Stabilizer $\{s_1, s_2, s_4\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[-1, -1, 0, -1]$.

Diagram for $V([1, 0, 1, 1])$:



Stabilizer $\{s_1, s_3, s_4\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[-1, 0, -1, -1]$.

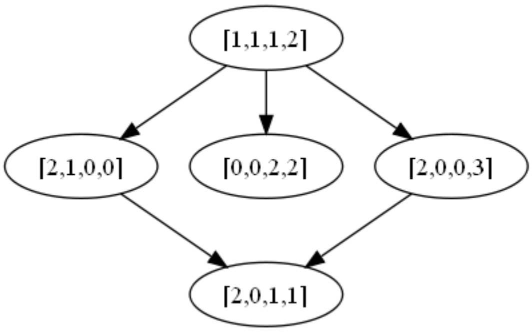
Diagram for $V([1, 1, 0, 1])$:



B.3 Diagrams for A_4, $p = 3$:

Stabilizer $\{s_0, s_1\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[-1, 0, 0, 0]$.

Diagram for $V([1, 1, 1, 2])$:



Stabilizer $\{s_0, s_2\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[0, -1, 0, 0]$.

Diagram for $V([1, 0, 1, 1])$:

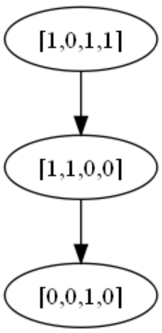


Diagram for $V([0, 1, 1, 2])$:

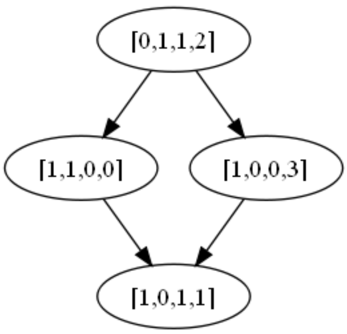


Diagram for $V([2, 0, 2, 0])$:

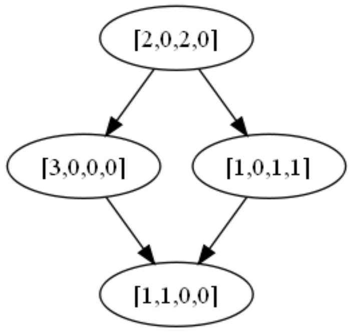
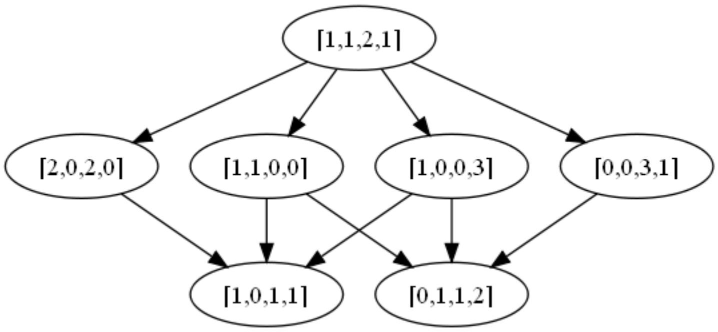


Diagram for $V([1,1,2,1])$:



Stabilizer $\{s_0, s_3\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[0,0,-1,0]$.

Diagram for $V([1,1,0,1])$:

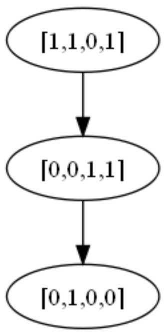


Diagram for $V([0,2,0,2])$:

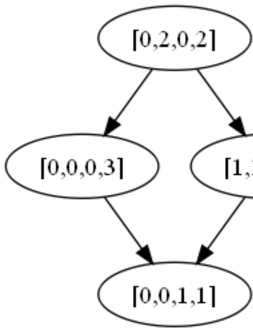


Diagram for $V([2,1,1,0])$:

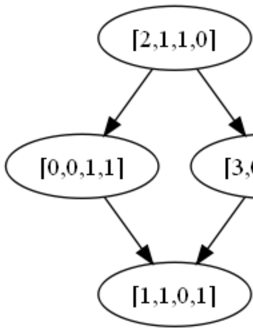
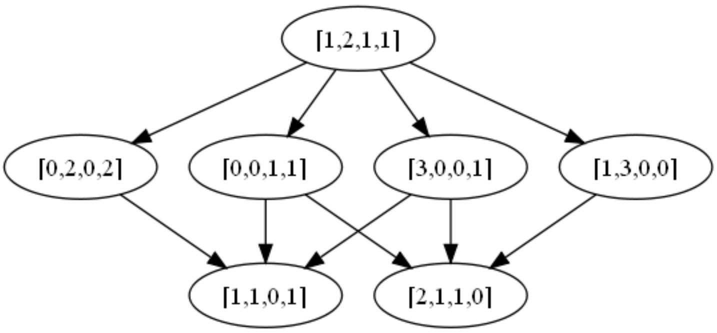
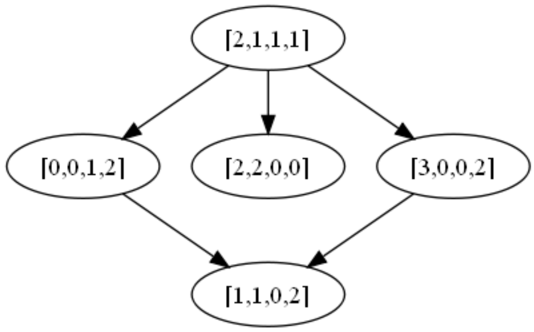


Diagram for $V([1,2,1,1])$:



Stabilizer $\{s_0, s_4\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[0,0,0,-1]$.

Diagram for $V([2,1,1,1])$:



Stabilizer $\{s_1, s_2\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[-1,-1,0,0]$.

Diagram for $V([0,1,1,3])$:

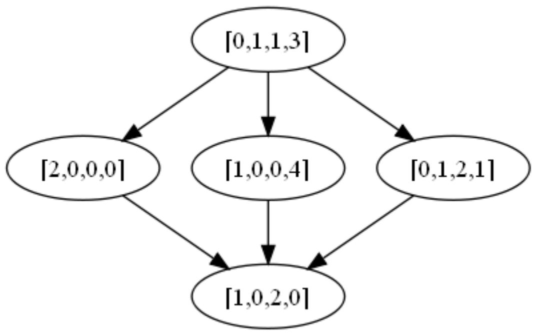


Diagram for $V([0,0,3,2])$:

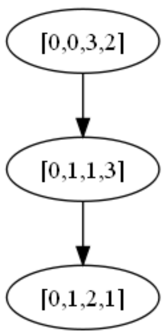
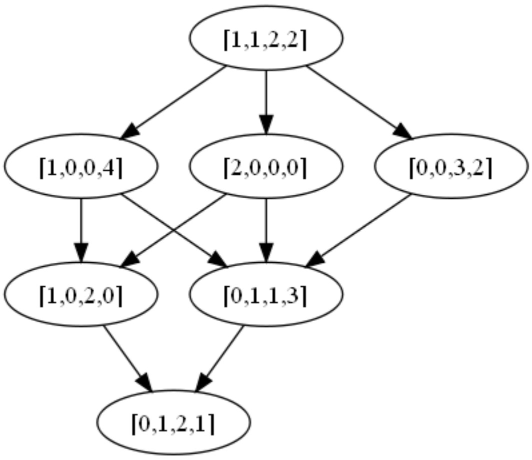


Diagram for $V([1,1,2,2])$:



Stabilizer $\{s_1, s_3\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[-1, 0, -1, 0]$.

Diagram for $V\,([1, 1, 1, 0])$:

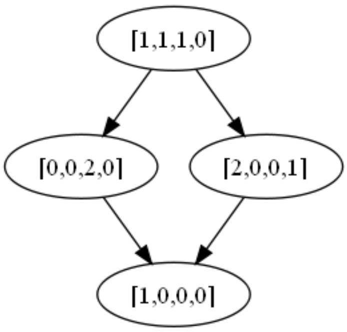


Diagram for $V\,([3, 0, 1, 0])$:

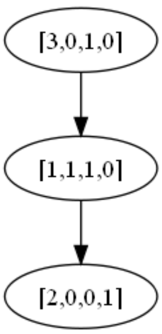


Diagram for $V\,([0, 2, 1, 1])$:

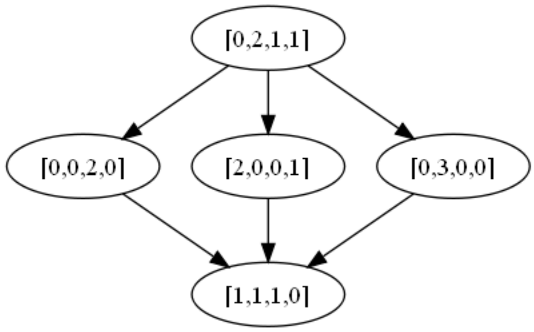


Diagram for $V([0,2,0,3])$:

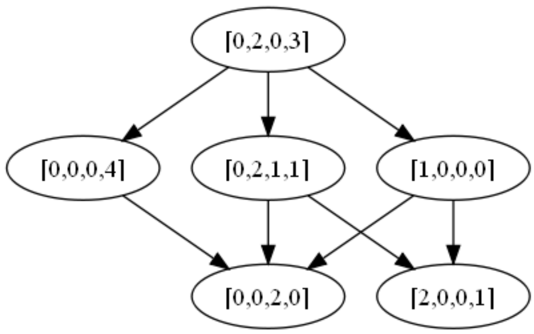
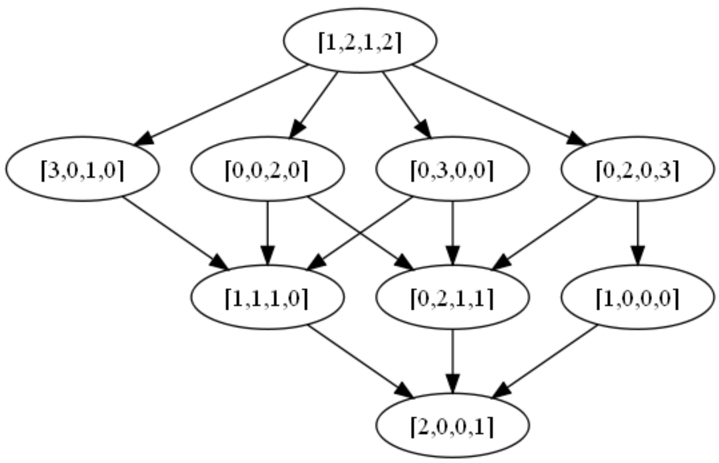


Diagram for $V([1,2,1,2])$:



Stabilizer $\{s_1, s_4\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[-1,0,0,-1]$.

Diagram for $V([1,1,1,1])$:

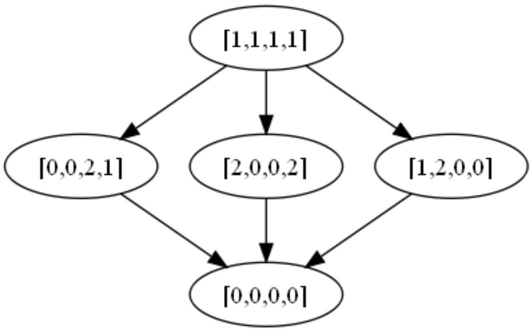


Diagram for $V([1,1,0,3])$:

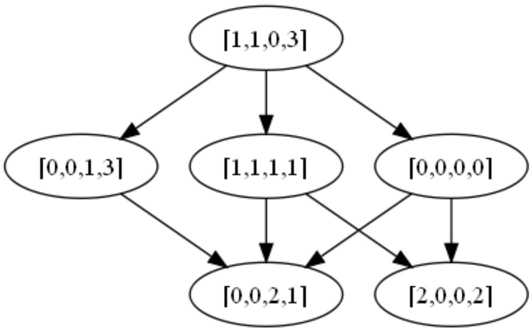


Diagram for $V([3,0,1,1])$:

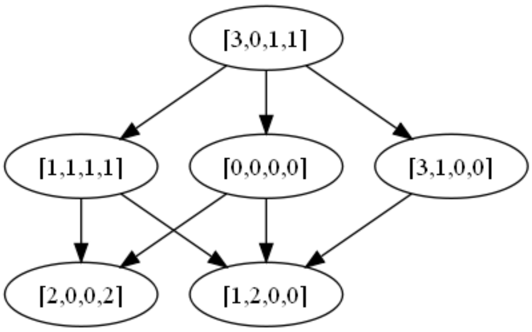


Diagram for $V([3,0,0,3])$:

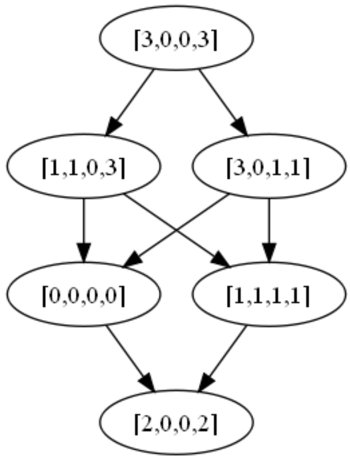
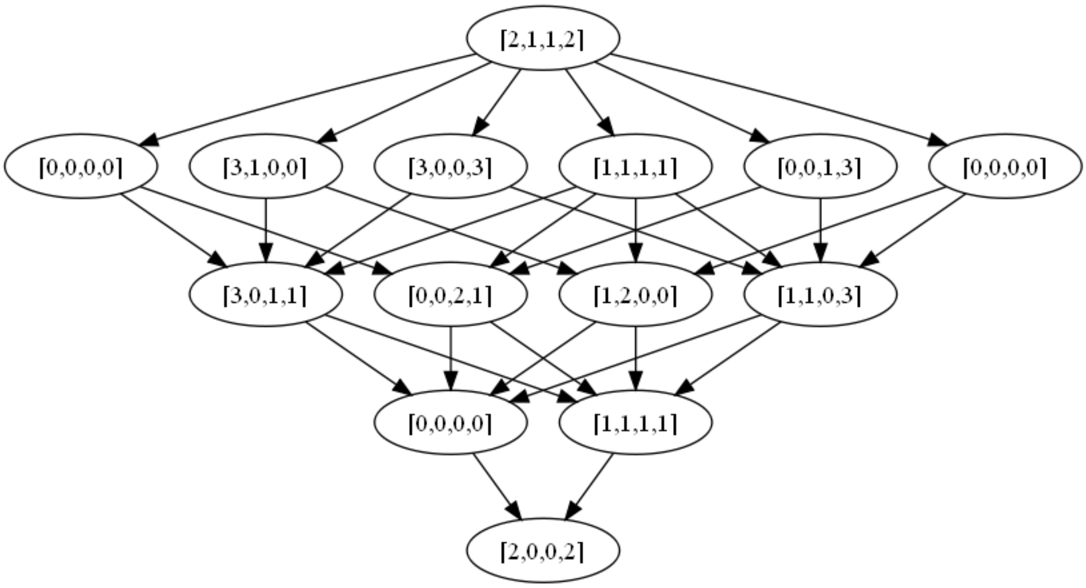
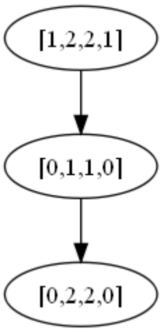


Diagram for $V\left([2,1,1,2]\right)$:



Stabilizer $\{s_2, s_3\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[0,-1,-1,0]$.

Diagram for $V\left([1,2,2,1]\right)$:



Stabilizer $\{s_2, s_4\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[0, -1, 0, -1]$.

Diagram for $V([0, 1, 1, 1])$:

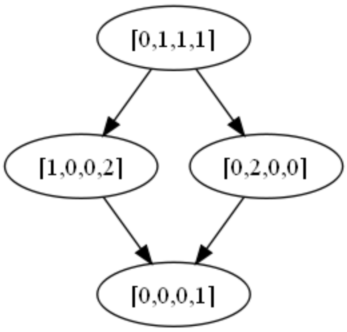


Diagram for $V([0, 1, 0, 3])$:

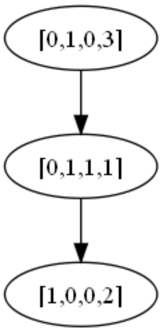


Diagram for $V([1, 1, 2, 0])$:

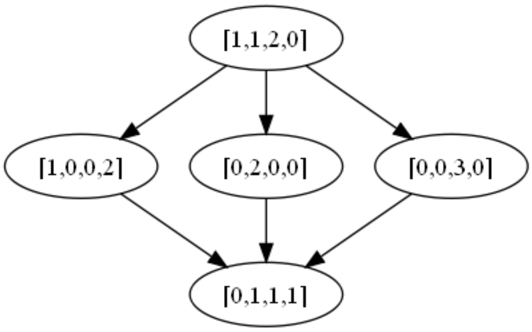


Diagram for $V([3,0,2,0])$:

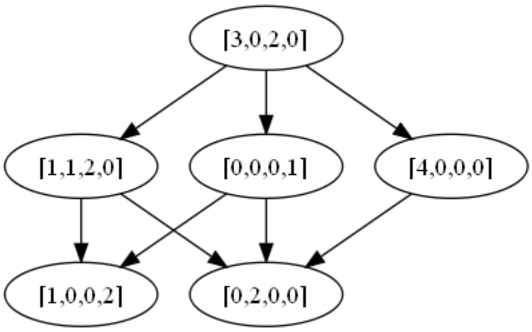
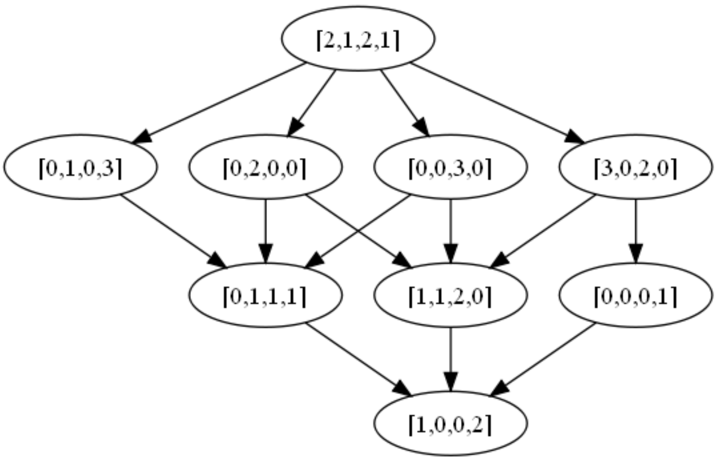


Diagram for $V([2,1,2,1])$:



Stabilizer $\{s_3, s_4\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[0,0,-1,-1]$.

Diagram for $V([3,1,1,0])$:

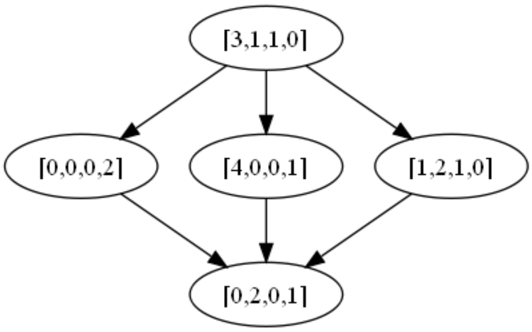


Diagram for $V([2,3,0,0])$:

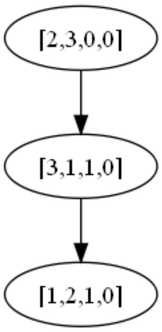
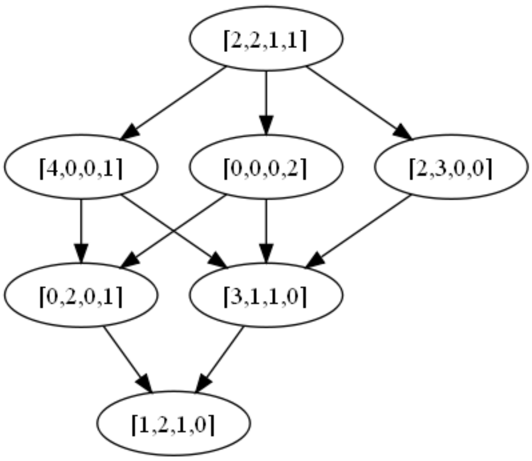
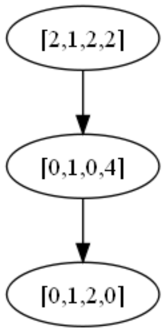


Diagram for $V([2,2,1,1])$:



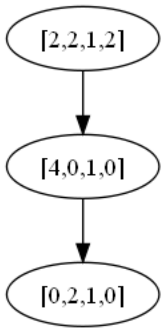
Stabilizer $\{s_1, s_2, s_4\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[-1,-1,0,-1]$.

Diagram for $V([2,1,2,2])$:



Stabilizer $\{s_1, s_3, s_4\}$ As representative for this stabilizer we choose the dot-orbit that has initial representative $[-1, 0, -1, -1]$.

Diagram for $V\left([2, 2, 1, 2]\right)$:



Appendix C

Composition coefficients of baby Verma modules

In this appendix we list all the composition coefficients $\left[\widehat{Z}(\lambda) : \widehat{L}(\mu)\right]$ of the baby Verma modules with $\mu \in X_p - p\rho$ and $\lambda \in -X^+ - \rho$. The rows in the matrices corresponds to the baby Verma modules $\widehat{Z}(\lambda)$ and the columns correspond to the simple $\mathcal{U}^{\mathrm{res}}$-modules $\widehat{L}(\mu)$.

C.1 Type A_2, $p = 2$

Stabilizer $\{s_0\}$, initial weight $[0, 0]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-2,-2]\right)\ \right) : \left(\ \widehat{L}\left([-2,-2]\right)\ \right)\right] = \left(\ 1\ \right)$$

Stabilizer $\{s_1\}$, initial weight $[-1, 0]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-2,-1]\right)\ \right) : \left(\ \widehat{L}\left([-2,-1]\right)\ \right)\right] = \left(\ 1\ \right)$$

Stabilizer $\{s_2\}$, initial weight $[0, -1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-1,-2]\right)\ \right) : \left(\ \widehat{L}\left([-1,-2]\right)\ \right)\right] = \left(\ 1\ \right)$$

Stabilizer $\{s_1, s_2\}$, initial weight $[-1, -1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-1,-1]\right)\ \right) : \left(\ \widehat{L}\left([-1,-1]\right)\ \right)\right] = \left(\ 1\ \right)$$

C.2 Type A_2, $p = 3$

Stabilizer $\emptyset$, initial weight $[0, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-2,-2]) \\ \widehat{Z}\,([-3,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-2,-2]) \\ \widehat{L}\,([-3,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0\}$, initial weight $[0, 1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-3,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-3,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_0\}$, initial weight $[1, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-2,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-2,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_1\}$, initial weight $[-1, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-2,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-2,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_1\}$, initial weight $[-1, 1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-3,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-3,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_2\}$, initial weight $[0, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-1,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-1,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_2\}$, initial weight $[1, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-1,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-1,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_2\}$, initial weight $[-1,-1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\,([-1,-1])\ \right) : \left(\ \widehat{L}\,([-1,-1])\ \right)\right] = \left(\ 1\ \right)$$

C.3 Type A_3, $p = 2$

Stabilizer $\{s_0, s_1\}$, initial weight $[-1, 0, 0]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-2,-2,-1]\right)\ \right) : \left(\ \widehat{L}\left([-2,-2,-1]\right)\ \right)\right] = \left(\ 1\ \right)$$

Stabilizer $\{s_0, s_2\}$, initial weight $[0, -1, 0]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-2,-1,-2]\right)\ \right) : \left(\ \widehat{L}\left([-2,-1,-2]\right)\ \right)\right] = \left(\ 1\ \right)$$

Stabilizer $\{s_0, s_3\}$, initial weight $[0, 0, -1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-1,-2,-2]\right)\ \right) : \left(\ \widehat{L}\left([-1,-2,-2]\right)\ \right)\right] = \left(\ 1\ \right)$$

Stabilizer $\{s_1, s_2\}$, initial weight $[-1, -1, 0]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-2,-1,-1]\right)\ \right) : \left(\ \widehat{L}\left([-2,-1,-1]\right)\ \right)\right] = \left(\ 1\ \right)$$

Stabilizer $\{s_1, s_3\}$, initial weight $[-1, 0, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\left([-1,-2,-1]\right) \\ \widehat{Z}\left([-2,-2,-2]\right) \end{pmatrix} : \begin{pmatrix} \widehat{L}\left([-1,-2,-1]\right) \\ \widehat{L}\left([-2,-2,-2]\right) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_2, s_3\}$, initial weight $[0, -1, -1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-1,-1,-2]\right)\ \right) : \left(\ \widehat{L}\left([-1,-1,-2]\right)\ \right)\right] = \left(\ 1\ \right)$$

Stabilizer $\{s_1, s_2, s_3\}$, initial weight $[-1, -1, -1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-1,-1,-1]\right)\ \right) : \left(\ \widehat{L}\left([-1,-1,-1]\right)\ \right)\right] = \left(\ 1\ \right)$$

C.4 Type A_3, $p = 3$

Stabilizer $\{s_0\}$, initial weight $[0,0,0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-2,-2]) \\ \widehat{Z}([-1,-3,-3]) \\ \widehat{Z}([-3,-3,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-2,-2]) \\ \widehat{L}([-1,-3,-3]) \\ \widehat{L}([-3,-3,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_1\}$, initial weight $[-1,0,0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-2,-1]) \\ \widehat{Z}([-3,-2,-2]) \\ \widehat{Z}([-2,-3,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-2,-1]) \\ \widehat{L}([-3,-2,-2]) \\ \widehat{L}([-2,-3,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_2\}$, initial weight $[0,-1,0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-1,-2]) \\ \widehat{Z}([-3,-1,-3]) \\ \widehat{Z}([-2,-2,-4]) \\ \widehat{Z}([-4,-2,-2]) \\ \widehat{Z}([-3,-3,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-1,-2]) \\ \widehat{L}([-3,-1,-3]) \\ \widehat{L}([-3,-3,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_3\}$, initial weight $[0,0,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-2,-2]) \\ \widehat{Z}([-2,-2,-3]) \\ \widehat{Z}([-3,-3,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-2,-2]) \\ \widehat{L}([-2,-2,-3]) \\ \widehat{L}([-3,-3,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_1\}$, initial weight $[-1,0,1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-3,-2,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-3,-2,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_1\}$, initial weight $[-1,1,0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-3,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-3,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_2\}$, initial weight $[0, -1, 1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-3,-1,-2]\right)\ \right):\left(\ \widehat{L}\left([-3,-1,-2]\right)\ \right)\right]=\left(\ 1\ \right)$$

Stabilizer $\{s_0, s_2\}$, initial weight $[1, -1, 0]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-2,-1,-3]\right)\ \right):\left(\ \widehat{L}\left([-2,-1,-3]\right)\ \right)\right]=\left(\ 1\ \right)$$

Stabilizer $\{s_0, s_3\}$, initial weight $[1, 0, -1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-1,-2,-3]\right)\ \right):\left(\ \widehat{L}\left([-1,-2,-3]\right)\ \right)\right]=\left(\ 1\ \right)$$

Stabilizer $\{s_0, s_3\}$, initial weight $[0, 1, -1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-1,-3,-2]\right)\ \right):\left(\ \widehat{L}\left([-1,-3,-2]\right)\ \right)\right]=\left(\ 1\ \right)$$

Stabilizer $\{s_1, s_2\}$, initial weight $[-1, -1, 0]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-2,-1,-1]\right)\ \right):\left(\ \widehat{L}\left([-2,-1,-1]\right)\ \right)\right]=\left(\ 1\ \right)$$

Stabilizer $\{s_1, s_2\}$, initial weight $[-1, -1, 1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-3,-1,-1]\right)\ \right):\left(\ \widehat{L}\left([-3,-1,-1]\right)\ \right)\right]=\left(\ 1\ \right)$$

Stabilizer $\{s_1, s_3\}$, initial weight $[-1, 0, -1]$

The composition coefficients are:

$$\left[\left(\begin{array}{c}\widehat{Z}\left([-1,-2,-1]\right)\\ \widehat{Z}\left([-3,-2,-3]\right)\end{array}\right):\left(\begin{array}{c}\widehat{L}\left([-1,-2,-1]\right)\\ \widehat{L}\left([-3,-2,-3]\right)\end{array}\right)\right]=\left(\begin{array}{cc}1&2\\0&1\end{array}\right)$$

Stabilizer $\{s_1, s_3\}$, initial weight $[-1, 1, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-1,-3,-1]) \\ \widehat{Z}\,([-2,-3,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-1,-3,-1]) \\ \widehat{L}\,([-2,-3,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_2, s_3\}$, initial weight $[0, -1, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-1,-1,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-1,-1,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_2, s_3\}$, initial weight $[1, -1, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-1,-1,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-1,-1,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_3\}$, initial weight $[-1, -1, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\,([-1,-1,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}\,([-1,-1,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

C.5 Type A_4, $p = 2$

Stabilizer $\{s_0, s_1, s_2\}$, initial weight $[-1, -1, 0, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-2,-1,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-2,-1,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_1, s_3\}$, initial weight $[-1, 0, -1, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-1,-2,-1]) \\ \widehat{Z}([-1,-2,-2,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-1,-2,-1]) \\ \widehat{L}([-1,-2,-2,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_1, s_4\}$, initial weight $[-1, 0, 0, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-2,-2,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-2,-2,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_3\}$, initial weight $[0, -1, -1, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-1,-1,-2]) \\ \widehat{Z}([-1,-2,-1,-3]) \\ \widehat{Z}([-3,-1,-2,-1]) \\ \widehat{Z}([-2,-2,-2,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-1,-1,-2]) \\ \widehat{L}([-2,-2,-2,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 6 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_4\}$, initial weight $[0, -1, 0, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-2,-1,-2]) \\ \widehat{Z}([-2,-2,-2,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-2,-1,-2]) \\ \widehat{L}([-2,-2,-2,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_3, s_4\}$, initial weight $[0, 0, -1, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-1,-2,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-1,-2,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_3\}$, initial weight $[-1,-1,-1,0]$

The composition coefficients are:

$$\left[\left(\begin{array}{c}\widehat{Z}\left([-2,-1,-1,-1]\right)\end{array}\right) : \left(\begin{array}{c}\widehat{L}\left([-2,-1,-1,-1]\right)\end{array}\right)\right] = \left(\begin{array}{c}1\end{array}\right)$$

Stabilizer $\{s_1, s_2, s_4\}$, initial weight $[-1,-1,0,-1]$

The composition coefficients are:

$$\left[\left(\begin{array}{c}\widehat{Z}\left([-1,-2,-1,-1]\right)\\ \widehat{Z}\left([-2,-2,-1,-2]\right)\end{array}\right) : \left(\begin{array}{c}\widehat{L}\left([-1,-2,-1,-1]\right)\\ \widehat{L}\left([-2,-2,-1,-2]\right)\end{array}\right)\right] = \left(\begin{array}{cc}1 & 2\\ 0 & 1\end{array}\right)$$

Stabilizer $\{s_1, s_3, s_4\}$, initial weight $[-1,0,-1,-1]$

The composition coefficients are:

$$\left[\left(\begin{array}{c}\widehat{Z}\left([-1,-1,-2,-1]\right)\\ \widehat{Z}\left([-2,-1,-2,-2]\right)\end{array}\right) : \left(\begin{array}{c}\widehat{L}\left([-1,-1,-2,-1]\right)\\ \widehat{L}\left([-2,-1,-2,-2]\right)\end{array}\right)\right] = \left(\begin{array}{cc}1 & 2\\ 0 & 1\end{array}\right)$$

Stabilizer $\{s_2, s_3, s_4\}$, initial weight $[0,-1,-1,-1]$

The composition coefficients are:

$$\left[\left(\begin{array}{c}\widehat{Z}\left([-1,-1,-1,-2]\right)\end{array}\right) : \left(\begin{array}{c}\widehat{L}\left([-1,-1,-1,-2]\right)\end{array}\right)\right] = \left(\begin{array}{c}1\end{array}\right)$$

Stabilizer $\{s_1, s_2, s_3, s_4\}$, initial weight $[-1,-1,-1,-1]$

The composition coefficients are:

$$\left[\left(\begin{array}{c}\widehat{Z}\left([-1,-1,-1,-1]\right)\end{array}\right) : \left(\begin{array}{c}\widehat{L}\left([-1,-1,-1,-1]\right)\end{array}\right)\right] = \left(\begin{array}{c}1\end{array}\right)$$

C.6 Type A_4, $p = 3$

Stabilizer $\{s_0, s_1\}$, initial weight $[-1, 0, 0, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-2,-2,-1]) \\ \widehat{Z}([-3,-3,-1,-1]) \\ \widehat{Z}([-1,-3,-2,-2]) \\ \widehat{Z}([-1,-2,-3,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-2,-2,-1]) \\ \widehat{L}([-3,-3,-1,-1]) \\ \widehat{L}([-1,-3,-2,-2]) \\ \widehat{L}([-1,-2,-3,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_2\}$, initial weight $[0, -1, 0, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-2,-1,-2]) \\ \widehat{Z}([-1,-3,-1,-3]) \\ \widehat{Z}([-3,-2,-2,-1]) \\ \widehat{Z}([-5,-1,-2,-1]) \\ \widehat{Z}([-1,-2,-2,-4]) \\ \widehat{Z}([-2,-3,-2,-2]) \\ \widehat{Z}([-2,-2,-3,-3]) \\ \widehat{Z}([-3,-3,-2,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-2,-1,-2]) \\ \widehat{L}([-1,-3,-1,-3]) \\ \widehat{L}([-3,-2,-2,-1]) \\ \widehat{L}([-2,-3,-2,-2]) \\ \widehat{L}([-2,-2,-3,-3]) \\ \widehat{L}([-3,-3,-2,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 1 & 2 & 4 & 5 \\ 0 & 1 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_3\}$, initial weight $[0, 0, -1, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-1,-2,-2]) \\ \widehat{Z}([-1,-2,-2,-3]) \\ \widehat{Z}([-3,-1,-3,-1]) \\ \widehat{Z}([-1,-2,-1,-5]) \\ \widehat{Z}([-4,-2,-2,-1]) \\ \widehat{Z}([-2,-2,-3,-2]) \\ \widehat{Z}([-3,-3,-2,-2]) \\ \widehat{Z}([-3,-2,-3,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-1,-2,-2]) \\ \widehat{L}([-1,-2,-2,-3]) \\ \widehat{L}([-3,-1,-3,-1]) \\ \widehat{L}([-2,-2,-3,-2]) \\ \widehat{L}([-3,-3,-2,-2]) \\ \widehat{L}([-3,-2,-3,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 1 & 2 & 4 & 5 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_4\}$, initial weight $[0, 0, 0, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-2,-2,-2]) \\ \widehat{Z}([-1,-1,-3,-3]) \\ \widehat{Z}([-2,-2,-3,-1]) \\ \widehat{Z}([-3,-3,-2,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-2,-2,-2]) \\ \widehat{L}([-1,-1,-3,-3]) \\ \widehat{L}([-2,-2,-3,-1]) \\ \widehat{L}([-3,-3,-2,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_2\}$, initial weight $[-1,-1,0,0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-2,-1,-1]) \\ \widehat{Z}([-3,-2,-1,-2]) \\ \widehat{Z}([-2,-3,-1,-3]) \\ \widehat{Z}([-2,-2,-2,-4]) \\ \widehat{Z}([-1,-4,-2,-2]) \\ \widehat{Z}([-1,-3,-3,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-2,-1,-1]) \\ \widehat{L}([-3,-2,-1,-2]) \\ \widehat{L}([-2,-3,-1,-3]) \\ \widehat{L}([-1,-3,-3,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_3\}$, initial weight $[-1,0,-1,0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-1,-2,-1]) \\ \widehat{Z}([-3,-1,-2,-2]) \\ \widehat{Z}([-4,-2,-1,-2]) \\ \widehat{Z}([-2,-2,-2,-3]) \\ \widehat{Z}([-2,-2,-1,-5]) \\ \widehat{Z}([-1,-3,-3,-2]) \\ \widehat{Z}([-3,-3,-1,-3]) \\ \widehat{Z}([-3,-2,-2,-4]) \\ \widehat{Z}([-2,-4,-2,-2]) \\ \widehat{Z}([-2,-3,-3,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-1,-2,-1]) \\ \widehat{L}([-3,-1,-2,-2]) \\ \widehat{L}([-2,-2,-2,-3]) \\ \widehat{L}([-1,-3,-3,-2]) \\ \widehat{L}([-3,-3,-1,-3]) \\ \widehat{L}([-2,-3,-3,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 2 & 2 & 4 & 9 \\ 0 & 1 & 1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_4\}$, initial weight $[-1,0,0,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-2,-2,-1]) \\ \widehat{Z}([-2,-2,-2,-2]) \\ \widehat{Z}([-2,-1,-3,-3]) \\ \widehat{Z}([-1,-3,-3,-1]) \\ \widehat{Z}([-3,-3,-1,-2]) \\ \widehat{Z}([-2,-1,-2,-5]) \\ \widehat{Z}([-5,-2,-1,-2]) \\ \widehat{Z}([-1,-2,-4,-2]) \\ \widehat{Z}([-2,-4,-2,-1]) \\ \widehat{Z}([-3,-1,-3,-4]) \\ \widehat{Z}([-4,-3,-1,-3]) \\ \widehat{Z}([-2,-2,-4,-3]) \\ \widehat{Z}([-4,-2,-2,-4]) \\ \widehat{Z}([-3,-4,-2,-2]) \\ \widehat{Z}([-3,-3,-3,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-2,-2,-1]) \\ \widehat{L}([-2,-2,-2,-2]) \\ \widehat{L}([-2,-1,-3,-3]) \\ \widehat{L}([-1,-3,-3,-1]) \\ \widehat{L}([-3,-3,-1,-2]) \\ \widehat{L}([-3,-3,-3,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 & 2 & 1 & 2 & 16 \\ 0 & 1 & 1 & 1 & 1 & 8 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_2, s_3\}$, initial weight $[0,-1,-1,0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-1,-1,-2]) \\ \widehat{Z}([-3,-1,-1,-3]) \\ \widehat{Z}([-2,-2,-1,-4]) \\ \widehat{Z}([-4,-1,-2,-2]) \\ \widehat{Z}([-3,-2,-2,-3]) \\ \widehat{Z}([-2,-3,-3,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-1,-1,-2]) \\ \widehat{L}([-3,-1,-1,-3]) \\ \widehat{L}([-3,-2,-2,-3]) \\ \widehat{L}([-2,-3,-3,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 5 & 5 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_2, s_4\}$, initial weight $[0,-1,0,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-2,-1,-2]) \\ \widehat{Z}([-2,-2,-1,-3]) \\ \widehat{Z}([-2,-1,-2,-4]) \\ \widehat{Z}([-3,-2,-2,-2]) \\ \widehat{Z}([-5,-1,-2,-2]) \\ \widehat{Z}([-3,-1,-3,-3]) \\ \widehat{Z}([-2,-3,-3,-1]) \\ \widehat{Z}([-2,-2,-4,-2]) \\ \widehat{Z}([-4,-2,-2,-3]) \\ \widehat{Z}([-3,-3,-3,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-2,-1,-2]) \\ \widehat{L}([-2,-2,-1,-3]) \\ \widehat{L}([-3,-2,-2,-2]) \\ \widehat{L}([-3,-1,-3,-3]) \\ \widehat{L}([-2,-3,-3,-1]) \\ \widehat{L}([-3,-3,-3,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 2 & 4 & 2 & 9 \\ 0 & 1 & 1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_3, s_4\}$, initial weight $[0,0,-1,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-1,-2,-2]) \\ \widehat{Z}([-2,-1,-2,-3]) \\ \widehat{Z}([-3,-1,-3,-2]) \\ \widehat{Z}([-2,-2,-4,-1]) \\ \widehat{Z}([-4,-2,-2,-2]) \\ \widehat{Z}([-3,-3,-3,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-1,-2,-2]) \\ \widehat{L}([-2,-1,-2,-3]) \\ \widehat{L}([-3,-1,-3,-2]) \\ \widehat{L}([-3,-3,-3,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_1, s_2\}$, initial weight $[-1,-1,0,1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-3,-2,-1,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-3,-2,-1,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_1, s_2\}$, initial weight $[-1,-1,1,0]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-2,-3,-1,-1]\right)\ \right):\left(\ \widehat{L}\left([-2,-3,-1,-1]\right)\ \right)\right]=\left(\ 1\ \right)$$

Stabilizer $\{s_0, s_1, s_3\}$, initial weight $[-1,0,-1,1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\left([-3,-1,-2,-1]\right) \\ \widehat{Z}\left([-1,-3,-2,-3]\right) \end{pmatrix}:\begin{pmatrix} \widehat{L}\left([-3,-1,-2,-1]\right) \\ \widehat{L}\left([-1,-3,-2,-3]\right) \end{pmatrix}\right]=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_1, s_3\}$, initial weight $[-1,1,-1,0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\left([-2,-1,-3,-1]\right) \\ \widehat{Z}\left([-1,-2,-3,-2]\right) \end{pmatrix}:\begin{pmatrix} \widehat{L}\left([-2,-1,-3,-1]\right) \\ \widehat{L}\left([-1,-2,-3,-2]\right) \end{pmatrix}\right]=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_1, s_4\}$, initial weight $[-1,0,1,-1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-1,-3,-2,-1]\right)\ \right):\left(\ \widehat{L}\left([-1,-3,-2,-1]\right)\ \right)\right]=\left(\ 1\ \right)$$

Stabilizer $\{s_0, s_1, s_4\}$, initial weight $[-1,1,0,-1]$

The composition coefficients are:

$$\left[\left(\ \widehat{Z}\left([-1,-2,-3,-1]\right)\ \right):\left(\ \widehat{L}\left([-1,-2,-3,-1]\right)\ \right)\right]=\left(\ 1\ \right)$$

Stabilizer $\{s_0, s_2, s_3\}$, initial weight $[0,-1,-1,1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}\left([-3,-1,-1,-2]\right) \\ \widehat{Z}\left([-4,-1,-2,-1]\right) \\ \widehat{Z}\left([-1,-3,-1,-4]\right) \\ \widehat{Z}\left([-2,-3,-2,-3]\right) \end{pmatrix}:\begin{pmatrix} \widehat{L}\left([-3,-1,-1,-2]\right) \\ \widehat{L}\left([-2,-3,-2,-3]\right) \end{pmatrix}\right]=\begin{pmatrix} 1 & 5 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_3\}$, initial weight $[1, -1, -1, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-1,-1,-3]) \\ \widehat{Z}([-1,-2,-1,-4]) \\ \widehat{Z}([-4,-1,-3,-1]) \\ \widehat{Z}([-3,-2,-3,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-1,-1,-3]) \\ \widehat{L}([-3,-2,-3,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 5 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_4\}$, initial weight $[1, -1, 0, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-2,-1,-3]) \\ \widehat{Z}([-3,-2,-3,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-2,-1,-3]) \\ \widehat{L}([-3,-2,-3,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_2, s_4\}$, initial weight $[0, -1, 1, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-3,-1,-2]) \\ \widehat{Z}([-2,-3,-2,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-3,-1,-2]) \\ \widehat{L}([-2,-3,-2,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_3, s_4\}$, initial weight $[1, 0, -1, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-1,-2,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-1,-2,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_0, s_3, s_4\}$, initial weight $[0, 1, -1, -1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-1,-3,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-1,-3,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_3\}$, initial weight $[-1, -1, -1, 0]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-2,-1,-1,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-2,-1,-1,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_3\}$, initial weight $[-1, -1, -1, 1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-3,-1,-1,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-3,-1,-1,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_4\}$, initial weight $[-1,-1,0,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-2,-1,-1]) \\ \widehat{Z}([-3,-2,-1,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-2,-1,-1]) \\ \widehat{L}([-3,-2,-1,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_4\}$, initial weight $[-1,-1,1,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-3,-1,-1]) \\ \widehat{Z}([-2,-3,-1,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-3,-1,-1]) \\ \widehat{L}([-2,-3,-1,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_3, s_4\}$, initial weight $[-1,0,-1,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-1,-2,-1]) \\ \widehat{Z}([-3,-1,-2,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-1,-2,-1]) \\ \widehat{L}([-3,-1,-2,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_3, s_4\}$, initial weight $[-1,1,-1,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-1,-3,-1]) \\ \widehat{Z}([-2,-1,-3,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-1,-3,-1]) \\ \widehat{L}([-2,-1,-3,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Stabilizer $\{s_2, s_3, s_4\}$, initial weight $[0,-1,-1,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-1,-1,-2]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-1,-1,-2]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_2, s_3, s_4\}$, initial weight $[1,-1,-1,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-1,-1,-3]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-1,-1,-3]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$

Stabilizer $\{s_1, s_2, s_3, s_4\}$, initial weight $[-1,-1,-1,-1]$

The composition coefficients are:

$$\left[\begin{pmatrix} \widehat{Z}([-1,-1,-1,-1]) \end{pmatrix} : \begin{pmatrix} \widehat{L}([-1,-1,-1,-1]) \end{pmatrix}\right] = \begin{pmatrix} 1 \end{pmatrix}$$