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Abstract

We present two algorithms for reconstruction of the shape of convex bodies in the two-dimensional Euclidean space. The first reconstruction algorithm requires knowledge of the exact surface tensors of a convex body up to rank s for some natural number s. The second algorithm uses harmonic intrinsic volumes which are certain values of the surface tensors and allows for noisy measurements. From a generalized version of Wirtinger's inequality, we derive stability results that are utilized to ensure consistency of both reconstruction procedures. Consistency of the reconstruction procedure based on measurements subject to noise is established under certain assumptions on the noise variables.

Keywords: Convex body, shape, reconstruction algorithm, surface tensor, harmonic intrinsic volume, generalized Wirtinger's inequality

1 Introduction

The problem of determining and reconstructing an unknown geometric object from indirect measurements is treated in a number of papers, see, e.g., Gardner (2006). In Prince and Willsky (1990), a convex body is reconstructed from measurements of its support function. Measurements of the brightness function are used in Gardner and Milanfar (2003), and in Campi et al. (2012) it is shown that a convex body can be uniquely determined up to translation from measurements of its lightness function. Milanfar et al. (1995) developed a reconstruction algorithm for planar polygons and quadrature domains from moments of the Lebesgue measure restricted to these sets. In particular, they showed that a non-degenerate convex polygon in $\mathbb{R}^2$ with k vertices is uniquely determined by its moments up to order $2k-3$. The reconstruction algorithm and the uniqueness result were generalized to convex polytopes in $\mathbb{R}^n$ in Gravin et al. (2012).

In continuation of the work in this area, we discuss reconstruction of convex bodies from a certain type of Minkowski tensors. In recent years, Minkowski tensors have been studied intensively. On the applied side, Minkowski tensors have been

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established as robust and versatile descriptors of shape and morphology of spatial patterns of physical systems, see e.g., Beisbart et al. (2002); Schröder-Turk et al. (2010, 2013). The importance of Minkowski tensors is further indicated by Alesker's characterization theorem, see Alesker (1999), that states that products of Minkowski tensors and powers of the metric tensor span the space of tensor-valued valuations on convex bodies satisfying some natural conditions.

In the present work, we consider translation invariant Minkowski tensors, $\Phi_j^s(K)$ of rank s, which are tensors derived from the j'th area measure $S_j(K,\cdot)$ of a convex body $K \subseteq \mathbb{R}^n$, $j = 0, \dots, n-1$. For details, see Section 2. For a given $j = 1, \dots, n-1$, the set $\{\Phi_j^s(K) \mid s \in \mathbb{N}_0\}$ of all Minkowski tensors determines K up to translation. Calling the equivalence class of all translations of K the *shape* of K, we can say that $\{\Phi_j^s(K) \mid s \in \mathbb{N}_0\}$ determines the shape of K. When only Minkowski tensors $\Phi_j^s(K)$, $s \leq s_o$ up to a certain rank s_o are given, this is, in general, no longer true. We establish a stability result (Theorem 3.8) stating that the shapes of two convex bodies are close to one another when the two convex bodies have coinciding Minkowski tensors $\Phi_1^s(K)$ of rank $s \leq s_o$. The proof uses a generalization of Wirtinger's inequality (Corollary 3.7), which is different from existing generalizations in the literature (e.g. Cheng and Zhang (2009); Giova and Ricciardi (2010)) as it involves a higher order spherical harmonic expansion. We also show (Theorem 3.1) that there always exists a convex polytope P with the same surface tensors Φ_{n-1}^s of rank $s \leq s_o$ as a given convex body. The number of facets of P can be bounded by a polynomial of s_o of degree $n-1$. Using this result, we conclude (Corollary 3.2) that a convex body K is a polytope if the shape of K is uniquely determined by a finite number of surface tensors. In fact, the shape of a convex body K is uniquely determined by a finite number of its surface tensors if and only if K is a polytope (Theorem 3.3).

For actual reconstructions, we restrict considerations to the planar case. We consider two cases. Firstly, the case when the exact tensors are given, and secondly, the case when certain values of the tensors are measured with noise. *Algorithm Surface Tensor* in Section 4 allows to reconstruct an unknown convex body K_0 in $\mathbb{R}^2$ based on surface tensors $\Phi_1^s(K_0)$ up to rank s_o. The output of the reconstruction procedure is a polygon P with surface tensors identical to the surface tensors of K_0 up to rank s_o. Theorem 3.1 yields the existence of a polygon with the described property. Due to the bound on the number of facets of P and to the simple structure of surface tensors of polygons, the reconstruction problem can be solved by first finding the surface area measure of P using a least squares optimization, and then constructing P with the help of Algorithm MinkData in Gardner (2006). The consistency of the reconstruction procedure is established using the mentioned stability result.

Reconstruction algorithms for dimensions $n \geq 2$ could be developed along the same lines when surface tensors $\Phi_{n-1}^s(K_0)$, $s \leq s_o$ are used as input. However, the methods in this paper yield a stability result for $\Phi_1^s(K_0)$, $s \leq s_o$, and this is why we only consider the case $n = 2$. The higher dimensional situation will be discussed in future work.

Algorithm Harmonic Intrinsic Volumes LSQ reconstructs an unknown convex body K_0 based on measurements of harmonic intrinsic volumes up to degree s_o, where the measurements are subject to noise. The harmonic intrinsic volumes of a convex body in $\mathbb{R}^2$ are certain values of the surface tensors, and the harmonic intrinsic volumes up to degree s_o determine the surface tensors up to rank s_o. The output of the reconstruction is a polygon with surface tensors best fitting the measurements of the harmonic intrinsic volumes of K_0 in a least squares sense. As for the procedure for reconstruction of convex bodies from exact surface tensors, this reconstruction procedure is based on Theorem 3.1 and Algorithm MinkData. The consistency of the reconstruction algorithm is established using the stability result and requires that the variances of all measurements converge to zero sufficiently fast. It is the structure of the stability result that suggests that we should consider reconstruction based on harmonic intrinsic volumes when the measurements are subject to noise.

The paper is organized as follows: After introducing notations and preliminaries in Section 2, we present the main theoretical results in Section 3 in $\mathbb{R}^n$, $n \geq 2$: The existence of a polytope with finitely many surface tensors coinciding with those of a given convex body, the uniqueness result for shapes of polytopes, the generalized Wirtinger's inequality, and the derived stability result. In Section 4 Algorithm Surface Tensor and its properties are discussed, and Section 5 is devoted to the reconstruction from noisy measurements of harmonic intrinsic volumes.

2 Notation and preliminaries

We work in the n-dimensional Euclidean vector space $\mathbb{R}^n$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|\cdot|$. As usual, S^{n-1} is the unit sphere in $\mathbb{R}^n$, and κ_n and ω_n denote the volume and the surface area of the unit ball B^n, respectively. The Borel σ-algebra of a topological space X is denoted by $\mathcal{B}(X)$. Further, let λ denote the Lebesgue measure on $\mathbb{R}^n$. The set $L^2(S^{n-1})$ of square integrable functions on S^{n-1} with respect to the spherical Lebesgue measure σ is equipped with the usual inner product $\langle \cdot, \cdot \rangle_2$ and the associated norm $\| \cdot \|$.

For a function F on the unit sphere S^{n-1}, we let $\check{F}$ denote the radial extension of F to $\mathbb{R}^n \setminus \{o\}$, that is,

$$\check{F}(x) = F\left(\frac{x}{|x|}\right)$$

for $x \in \mathbb{R}^n \setminus \{o\}$. Let $\nabla_{\mathrm{S}} F$ denote the restriction of the gradient $\nabla \check{F}$ of $\check{F}$ to S^{n-1}, when the partial derivatives of $\check{F}$ exist. If further, $\check{F}$ has partial derivatives of second order, the Laplace-Beltrami operator $\Delta_{\mathrm{S}} F$ of F is defined as the restriction of $\Delta \check{F}$ to S^{n-1}, where Δ denotes the Laplace operator on functions on $\mathbb{R}^n$.

In the proofs of Lemma 3.6 and Theorem 3.8, spherical harmonics are a key ingredient. We use Groemer (1996) as a general reference on the theory of spherical harmonics. A polynomial p on $\mathbb{R}^n$ is said to be harmonic if it is homogeneous and $\Delta p = 0$. A spherical harmonic of degree m is the restriction to S^{n-1} of a harmonic polynomial of degree m. Let $\mathcal{H}_m^n$ be the vector space of spherical harmonics of degree m on S^{n-1}, and let $N(n, m)$ denote the dimension of $\mathcal{H}_m^n$. For $m \in \mathbb{N}_0$, let $H_{m1}, \ldots, H_{mN(n,m)}$ be an orthogonal basis for $\mathcal{H}_m^n$. Then the condensed harmonic expansion of a function $F \in L^2(S^{n-1})$ is $\sum_{m=0}^{\infty} F_m$, where $F_m = \sum_{j=1}^{N(n,m)} \alpha_{mj} H_{mj}$ with

$$\alpha_{mj} = \frac{\langle F, H_{mj} \rangle_2}{\|H_{mj}\|^2}.$$

We write $F \sim \sum_{m=0}^{\infty} F_m$, when $\sum_{n=0}^{\infty} F_m$ is the condensed harmonic expansion of F. The condensed harmonic expansion of F is independent of the choice of bases of spherical harmonic used to derive it. The spherical harmonics are eigenfunctions of the Laplace-Beltrami operator as

$$\Delta_{\mathrm{S}} H_m = -m(m+n-2)H_m$$

for $H_m \in \mathcal{H}_m^n$. We let γ_m denote the absolute value of the eigenvalues of Δ_{S}, that is $\gamma_m = m(m+n-2)$ for $m \in \mathbb{N}_0$.

As in Campi (1998), the Sobolev space W^α for $\alpha \geq 0$ is defined as the space of square integrable functions $F \sim \sum_{m=0}^{\infty} F_m$ on the sphere, for which

$$\sum_{m=0}^{\infty} \gamma_m^\alpha \|F_m\|^2 < \infty.$$

By definition $W^\alpha \subseteq L^2(S^{n-1})$ for $\alpha \geq 0$, and $W^0 = L^2(S^{n-1})$. For $F \in W^\alpha$, the sum

$$\sum_{m=0}^{\infty} (\gamma_m)^{\frac{\alpha}{2}} F_m$$

converges in the L^2-sense. The limit is denoted by $(-\Delta_{\mathrm{S}})^{\frac{\alpha}{2}} F$, and thus

$$\|(-\Delta_{\mathrm{S}})^{\frac{\alpha}{2}} F\|^2 = \sum_{m=0}^{\infty} \gamma_m^\alpha \|F_m\|^2. \tag{2.1}$$

The notation is explained by the fact that

$$\Delta_{\mathrm{S}} F \sim -\sum_{m=0}^{\infty} \gamma_m F_m$$

for any $F \sim \sum_{m=0}^{\infty} F_m$ that is twice continuously differentiable.

In the two-dimensional setting we have $N(2,0) = 1$ and $N(2,m) = 2$ for $m \in \mathbb{N}$, and the spherical harmonic expansion is closely related to classical Fourier expansion. We obtain an orthonormal sequence of spherical harmonics constituting a basis of $L^2(S^1)$ by letting $H_{01}(u_1, u_2) = (2\pi)^{-\frac{1}{2}}$,

$$H_{m1}(u_1, u_2) = \pi^{-\frac{1}{2}} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \binom{m}{2i} u_1^{m-2i} u_2^{2i} \tag{2.2}$$

and

$$H_{m2}(u_1, u_2) = \pi^{-\frac{1}{2}} \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m}{2i+1} u_1^{m-2i-1} u_2^{2i+1}, \tag{2.3}$$

for $(u_1, u_2) \in S^1$ and $m \in \mathbb{N}$, where $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{R}$. If the polynomials in (2.2) and (2.3) are considered as polynomials on $\mathbb{R}^2$, then due to homogeneity, the polynomials can be decomposed into linear factors. More precisely,

$$H_{m1}(u_1, u_2) = \pi^{-\frac{1}{2}} (u_1 - \zeta_1 u_2) \cdot \dots \cdot (u_1 - \zeta_m u_2), \tag{2.4}$$

where

$$\zeta_j = \frac{\cos(\frac{(2j-1)\pi}{2m})}{\sin(\frac{(2j-1)\pi}{2m})} \qquad \text{for } j = 1, \dots, m.$$

The lines where H_{m1} vanishes (and herewith ζ_j, $j = 1, \dots, m$) are determined using the fact that

$$H_{m1}(\cos(\omega), \sin(\omega)) = \sqrt{\pi} \cos(m\omega)$$

for $\omega \in [0, 2\pi)$. Similarly, we can factorize H_{m2}. In this case, however, the factorization depends on the parity of m. This is due to the fact that a term involving u_1^m does not appear in H_{m2} and that the term involving u_2^m only appears, when m is odd. We get that

$$H_{m2}(u_1, u_2) = \begin{cases} \pi^{-\frac{1}{2}} m \, u_2 \, (u_1 - \lambda_1 u_2) \cdot \dots \cdot (u_1 - \lambda_{m-1} u_2) & \text{if m is even} \\ \pi^{-\frac{1}{2}} (-1)^{\frac{m-1}{2}} (u_2 - \rho_1 u_1) \cdot \dots \cdot (u_2 - \rho_m u_1) & \text{if m is odd,} \end{cases} \tag{2.5}$$

where

$$\lambda_j = \frac{\cos(\frac{j\pi}{m})}{\sin(\frac{j\pi}{m})} \qquad \text{for } j = 1, \dots, m-1$$

and

$$\rho_j = \frac{\sin(\frac{(j-1)\pi}{m})}{\cos(\frac{(j-1)\pi}{m})} \qquad \text{for } j = 1, \dots, m.$$

Here, we have used that

$$H_{m2}(\cos(\omega), \sin(\omega)) = \sqrt{\pi} \sin(m\omega)$$

for $\omega \in [0, 2\pi)$ in order to determine the lines where H_{m2} vanishes.

As general reference on convex geometry and Minkowski tensors, we use Schneider (2014). Let $\mathcal{K}^n$ denote the set of convex bodies (that is, compact, convex, non-empty sets) in $\mathbb{R}^n$, and let $\mathcal{K}^n_n$ denote the set of convex bodies with non-empty interior. We refer to convex polytopes and convex polygons by 'polytopes' and 'polygons', and let $\mathcal{P}^n_m$ denote the set of non-empty polytopes in $\mathbb{R}^n$ with at most m facets, $m \in \{n+1, n+2, \dots\}$. The support function (restricted to S^{n-1}) of a convex body K is denoted by h_K. The set of support functions $\{h_K \mid K \in \mathcal{K}^n, K \subseteq RB^n\}$ for $R > 0$ is bounded in W^α for $0 < \alpha < \frac{3}{2}$, see (Kiderlen, 2008, Prop. 2.1). The set $\mathcal{K}^n$ of convex bodies is equipped with the Hausdorff metric δ, which can be expressed as the distance of support functions with respect to the supremum norm on S^{n-1}, i.e.

$$\delta(K, L) = \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

In addition to the Hausdorff metric, we use the L^2-metric on $\mathcal{K}^n$. The L^2-distance between two convex bodies K and L is defined as the L^2-distance of their support functions, i.e.

$$\delta_2(K, L) = \|h_K - h_L\|.$$

The Hausdorff metric and the L^2-metric are equivalent and related by inequalities, see (Groemer, 1996, Prop. 2.3.1). This is used in Theorem 3.9 to transfer bounds on the L^2-distance to bounds on the Hausdorff distance between convex bodies satisfying certain conditions.

In the present work, two convex bodies are said to have the same shape if and only if they are translates. The position of a convex body has major influence on the above described distances, and as a measure of difference in shape only, we consider the translation invariant versions

$$\delta^t(K,L) = \inf_{x\in\mathbb{R}^n} \delta(K, L+x)$$

and

$$\delta_2^t(K,L) = \inf_{x\in\mathbb{R}^n} \delta_2(K, L+x).$$

If the support function h_K of a convex body K has condensed harmonic expansion $\sum_{m=0}^{\infty}(h_K)_m$, then $(h_K)_1 = \langle s(K), \cdot\rangle$, where $s(K)$ is the Steiner point of K,

$$s(K) = \frac{1}{\kappa_n}\int_{S^{n-1}} h_K(u)\, u\, \sigma(du).$$

For convex bodies K and L, this implies that $\delta_2^t(K,L) = \delta_2(K,L)$ if and only if K and L have coinciding Steiner points, see (Groemer, 1996, Prop. 5.1.2).

Let $\mathbb{T}^p$ be the vector space of symmetric tensors of rank p over $\mathbb{R}^n$, that is, the space of symmetric multilinear functions of p variables in $\mathbb{R}^n$. Due to linearity, a tensor $T \in \mathbb{T}^p$ can be identified with the array $\{T(e_{i_1}, \dots, e_{i_p})\}_{i_1,\dots,i_p=1}^n$, where $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$. We refer to the entries of the array as the components of T. For symmetric tensors $a \in \mathbb{T}^{p_1}$ and $b \in \mathbb{T}^{p_2}$, let $ab \in \mathbb{T}^{p_1+p_2}$ denote the symmetric tensor product of a and b. Identifying $x \in \mathbb{R}^n$ with the rank 1 tensor $z \mapsto \langle z, x\rangle$, we write $x^p \in \mathbb{T}^p$ for the p-fold symmetric tensor product of x. The metric tensor $Q \in \mathbb{T}^2$ is defined by $Q(x,y) = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$.

Let $p(K,x)$ be the metric projection of $x \in \mathbb{R}^n$ on a convex body K, and define $u(K,x) := \frac{x-p(K,x)}{|x-p(K,x)|}$ for $x \notin K$. For $\epsilon > 0$ and a Borel set $A \in \mathcal{B}(\mathbb{R}^n \times S^{n-1})$, the Lebesgue measure of the local parallel set

$$M_\epsilon(K,A) := \{x \in (K+\epsilon B^n)\setminus K \mid (p(K,x), u(K,x)) \in A\}$$

of K is a polynomial in $\epsilon \geq 0$, hence

$$\lambda(M_\epsilon(K,A)) = \sum_{k=0}^{n-1} \epsilon^{n-k}\kappa_{n-k}\Lambda_k(K,A).$$

This local version of the Steiner formula defines the support measures $\Lambda_0(K,\cdot)$, $\dots, \Lambda_{n-1}(K,\cdot)$ of a convex body $K \in \mathcal{K}^n$. The intrinsic volumes of K appear as total masses of the support measures, $V_j(K) = \Lambda_j(K, \mathbb{R}^n \times S^{n-1})$ for $0 = 1, \dots, n-1$. The area measures $S_0(K,\cdot), \dots, S_{n-1}(K,\cdot)$ of K are rescaled projections of the corresponding support measures on the second component. More explicitly, they are given by

$$\binom{n}{j} S_j(K,\omega) = n\kappa_{n-j}\Lambda_j(K, \mathbb{R}^n \times \omega)$$

for $\omega \in \mathcal{B}(S^{n-1})$ and $j = 0, \dots, n-1$. The area measure of order $n-1$ is called the surface area measure, and for $K \in \mathcal{K}^n_n$ the surface area measure is the $(n-1)$-dimensional Hausdorff measure of the reverse spherical image of K. That is,

$$S_{n-1}(K, \omega) = \mathcal{H}^{n-1}(\tau(K, \omega)),$$

for $\omega \in \mathcal{B}(S^{n-1})$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and $\tau(K, \omega)$ is the set of all boundary points of K at which there exists an outer normal vector of K belonging to ω.

For a convex body $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$, and $j \in \{0, 1, \dots, n-1\}$, we define the *Minkowski tensors* of K as

$$\Phi_j^{r,s}(K) := \frac{\omega_{n-j}}{r!s!\omega_{n-j+s}} \int_{\mathbb{R}^n \times S^{n-1}} x^r u^s \, \Lambda_j(K, d(x, u))$$

and supplement this definition by

$$\Phi_n^{r,0}(K) := \frac{1}{r!} \int_K x^r \, \lambda(dx).$$

The tensor functionals $\Phi_j^{r,s}$ and $\Phi_n^{r,0}$ are motion covariant valuations on $\mathcal{K}^n$ and continuous with respect to the Hausdorff metric. In Hug et al. (2008) the tensor functionals $Q^m \Phi_j^{r,s}$ with $m, r, s \in \mathbb{N}_0$ and either $j \in \{0, \dots, n-1\}$ or $(j, s) = (n, 0)$ are called the basic tensor valuations. Due to Alesker's characterization theorem, every motion covariant, continuous tensor-valued valuation is a linear combination of the basic tensor valuations.

In the present work, we only consider translation invariant Minkowski tensors, which are obtained by letting $r = 0$. We use the notation

$$\Phi_j^s(K) = \Phi_j^{0,s}(K) = \frac{\binom{n-1}{j}}{s! \, \omega_{n-j+s}} \int_{S^{n-1}} u^s S_j(K, du)$$

for $j \in \{0, \dots, n-1\}$ and $s \in \mathbb{N}_0$. For $s \in \mathbb{N}_0$, the tensors $\Phi_{n-1}^s(K)$ derived from the surface area measure of a convex body K are called *surface tensors* of K. For later use, we mention that

$$\Phi_j^1(K) = 0 \tag{2.6}$$

for $j = 0, \dots, n-1$ and any $K \in \mathcal{K}^n$, which is a special case of (Schneider, 2014, eq. (5.30)). For $s_o \in \mathbb{N}_0$, $j \in \{0, \dots, n-1\}$ and $K \in \mathcal{K}^n$, we let

$$\mathcal{M}_j^{s_o}(K) = \{L \in \mathcal{K}^n \ \mid \Phi_j^s(L) = \Phi_j^s(K), \, 0 \le s \le s_o\}.$$

As $S_0(K, \cdot) = \sigma$ independently of $K \in \mathcal{K}^n$, we have trivially $\mathcal{M}_0^{s_o}(K) = \mathcal{K}^n$. In the following, we will only consider these classes for $j = 1$ and $j = n-1$.

Remark 2.1. Let $K \in \mathcal{K}^n$ be given. By computing the trace of the tensor $\Phi_j^s(K)$, $j \in \{0, \dots, n-1\}$, $s \ge 2$, the rank of the tensor is reduced by 2, and the tensor $\frac{n-j+s-2}{2\pi s(s-1)} \Phi_j^{s-2}(K)$ is obtained. This follows from the identity

$$\sum_{k=1}^n \int_{S^{n-1}} u_{i_1} \cdots u_{i_{s-2}} \, u_k^2 \, S_j(K, du) = \int_{S^{n-1}} u_{i_1} \cdots u_{i_{s-2}} \, S_j(K, du).$$

Therefore, the tensors $\Phi_j^s(K)$ and $\Phi_j^{s-1}(K)$ determine all tensors $\Phi_j^{s'}(K)$ of rank $s' \leq s$. More generally, the moments of order at most s of a measure μ on S^{n-1} are determined by the moments of μ of order $s-1$ and s.

For $s \in \mathbb{N}_0$ and a convex body K in $\mathbb{R}^2$, we let $\phi_{sj}(K)$ denote the different components of the surface tensor $\Phi_1^s(K)$ of rank s. That is,

$$\phi_{sj}(K) = \frac{1}{s!\omega_{s+1}} \int_{S^1} u_1^j u_2^{s-j} S_1(K, du)$$

for $j = 0, \dots, s$. For $s_o \in \mathbb{N}$, Remark 2.1 implies that it is sufficient to require knowledge of the $2s_o + 1$ components of $\Phi_1^{s_o-1}(K)$ and $\Phi_1^{s_o}(K)$ in a reconstruction algorithm of shape based on surface tensors up to rank s_o as these components determine the surface tensors $\Phi_1^0(K), \dots, \Phi_1^{s_o}(K)$. This will be used in Section 4.

Instead of using only values of the surface tensors of rank $s_o - 1$ and s_o for the reconstruction, another option is to use the value of $\Phi_1^0(K)$ and two values of each surface tensor $\Phi_1^s(K)$ for $1 \leq s \leq s_o$. That this information is equivalent to the knowledge of $\Phi_1^s(K)$, $0 \leq s \leq s_o$, can be seen as follows. Due to the factorization into linear factors of the spherical harmonics in (2.4) and (2.5), there are vectors $(v_{s1}^i)_{i=1}^s, (v_{s2}^i)_{i=1}^s \subseteq (\mathbb{R}^2)$ for $s \in \mathbb{N}$ such that for $j = 1, 2$ is

$$\psi_{sj}(K) := \Phi_1^s(K)(v_{sj}^1, \dots, v_{sj}^s) = \int_{S^1} H_{sj}(u)\, S_1(K, du), \tag{2.7}$$

where (H_{sj}) is the orthonormal sequence of spherical harmonics given by (2.2) and (2.3). Further, we have that

$$\psi_{01}(K) := \sqrt{\frac{2}{\pi}}\Phi_1^0(K) = \int_{S^1} H_{01}(u)\, S_1(K, du). \tag{2.8}$$

Equations (2.7) and (2.8) show that $\psi_{sj}(K)$ is a value of $\Phi_1^s(K)$ when $s \geq 1$ and that $\psi_{01}(K)$ is the value of $\Phi_1^0(K)$ up to a known constant. Thus trivially, the vector $(\psi_{01}(K), \psi_{11}(K), \psi_{12}(K), \dots, \psi_{s_o1}(K), \psi_{s_o2}(K))$ is determined by $(\Phi_1^0(K), \dots, \Phi_1^{s_o}(K))$. The converse is also true, as polynomials on S^1 of degree at most s_o are linear combinations of the spherical harmonics of degree at most s_o (see (Groemer, 1996, Cor. 3.2.6)). It follows that the knowledge of the $2s_o+1$ values $\psi_{01}(K)$, $\psi_{s1}(K)$, $\psi_{s2}(K)$ for $1 \leq s \leq s_o$ is sufficient for a reconstruction algorithm based on surface tensors up to rank s_o.

The described values $(\psi_{sj}(K))$ are moments of the surface area measure of $K \in \mathcal{K}^2$ with respect to an orthonormal sequence of spherical harmonics. In Hörrmann (2014) such moments are called *harmonic intrinsic volumes*. In general, the harmonic intrinsic volumes associated to a convex body K in $\mathbb{R}^n$ are defined as

$$\psi_{jmk}(K) = \int_{S^{n-1}} H_{mk}(u)\, S_j(K, du)$$

for $j = 0, \dots, n-1$, $m \in \mathbb{N}_0$ and $k = 1, \dots, N(n, m)$. The harmonic intrinsic volume $\psi_{jmk} \colon \mathcal{K}^n \to \mathbb{R}$ is positively homogeneous of degree m, and we, therefore, refer to m as the degree of ψ_{jmk}. The harmonic intrinsic volumes depend on the choice of orthonormal bases for $\mathcal{H}^n_m$ for $m \in \mathbb{N}_0$. For $n = 2$, we use the bases given by (2.2) and (2.3). We remark however that

$$\sum_{k=1}^{N(n,m)} \psi_{jmk}(K)^2 \text{ and } \sum_{k=1}^{N(n,m)} \psi_{jmk}(K)\psi_{jkm}(M),$$

$K, M \in \mathcal{K}^n$, do not depend on the chosen basis of $\mathcal{H}^n_m$ due to the addition theorem for spherical harmonics (Groemer, 1996, Theorem 3.3.3). In particluar condition (3.5) in Theorem 3.8 does not depend on the basis chosen.

As we mainly consider harmonic intrinsic volumes derived from the surface area measure, we refer to those as harmonic intrinsic volumes. When referring to harmonic intrinsic volumes derived from area measures of lower order, this is explicitly stated. For $n = 2$ and $j = 1$, we write $\psi_{mk} = \psi_{1mk}$. The notation is consistent with (2.7) and (2.8).

As described above, the surface tensors and the harmonic intrinsic volumes of a convex body K are closely related. For $s_o \in \mathbb{N}_0$, the surface tensors $\Phi^0_{n-1}(K), \dots, \Phi^{s_o}_{n-1}(K)$ are uniquely determined by $\psi_{(n-1)mk}(K)$ for $m = 0, \dots, s_o$ and $k = 1, \dots, N(n,m)$, see (Groemer, 1996, Cor. 3.2.6), and vice versa. Due to the nice properties of spherical harmonics, the harmonic intrinsic volumes are beneficial in the establishment of stability results for surface tensors.

3 Uniqueness and stability results

The components of the Minkowski tensors $\Phi^s_j(K)$, $s \in \mathbb{N}_0$ are coinciding with the moments of $S_j(K, \cdot)$ up to known constants. As S^{n-1} is compact, an application of Stone-Weiserstrass's theorem implies that $\{\Phi^s_j(K) \mid s \in \mathbb{N}_0\}$ determine $S_j(K, \cdot)$. Hence, these tensors determine $K \in \mathcal{K}^n_n$ up to translation when $1 \le j \le n-1$ by the Aleksandrov-Fenchel-Jessen theorem (Schneider, 2014, Thm. 8.1.1). Hence, the shape (as defined in Section 2) of a convex body $K \in \mathcal{K}^n_n$ is uniquely determined by $\{\Phi^s_j(K) \mid s \in \mathbb{N}_0\}$. For $n = 2$, the tensors $\{\Phi^s_1(K) \mid s \in \mathbb{N}_0\}$ even determine the shape of K when K is lower-dimensional.

In order to investigate how different the shape of two convex bodies with identical surface tensors up to a certain rank can be, we discuss properties of the sets $\mathcal{M}^{s_o}_1(K)$ and $\mathcal{M}^{s_o}_{n-1}(K)$ for $s_o \in \mathbb{N}_0$ and $K \in \mathcal{K}^n$. In Theorem 3.1, it is shown that $\mathcal{M}^{s_o}_{n-1}(K)$ contains a polytope, and in Theorem 3.3 a uniqueness result is established stating that $\mathcal{M}^{2s_o}_{n-1}$ is the class of translates of K if K is a polytope with non-empty interior and at most s_o facets. In Theorem 3.8, we show that for large s_o the set $\mathcal{M}^{s_o}_1(K)$ contains only translations of convex bodies close to K in Hausdorff distance.

In the following, we let m_s denote the number of different components of the tensors u^{s-1} and u^s for $s \in \mathbb{N}$ and $u \in S^{n-1}$. Then

$$m_s = \binom{s+n-2}{n-1} + \binom{s+n-1}{n-1} = \left(2 + \frac{n-1}{s}\right)\binom{s+n-2}{n-1} = \mathcal{O}(s^{n-1})$$

for fixed $n \in \mathbb{N}$ as $s \to \infty$. For instance, $m_s = 2s + 1$ for $n = 2$, and $m_s = (s+1)^2$ for $n = 3$. The number of different components of u^{s-1} and u^s is identical to the dimension of $\mathcal{H}^n_0 \oplus \mathcal{H}^n_1 \oplus \cdots \oplus \mathcal{H}^n_s$, that is, $m_s = \sum_{m=0}^{s} N(n,m)$.

Theorem 3.1. *Let $K \in \mathcal{K}^n$ and $s_o \in \mathbb{N}$. Then there exists a $P \in \mathcal{P}^n_{m_{s_o}}$, such that*

$$\Phi^s_{n-1}(K) = \Phi^s_{n-1}(P) \tag{3.1}$$

for $0 \leq s \leq s_o$.

The proof of Theorem 3.1 follows the lines of the proof of Lemma 6.9 in Campi et al. (2012) (see also Skouborg (2012)). For the readers convenience, the proof of Theorem 3.1 is included.

Proof. If the interior of K is empty, then is either $S_{n-1}(K, \cdot) = 0$ or $S_{n-1}(K, \cdot) = \alpha(\delta_u + \delta_{-u})$ for some $u \in S^{n-1}$ and $\alpha > 0$. In the first case, let $P = \{o\}$. In the latter case, let P be a polytope contained in the orthogonal complement $u^\perp$ of u with surface area α.

We may from now on assume that $K \in \mathcal{K}^n_n$. If $s_o = 1$, we let P be a polytope with at most $m_1 = n + 1$ facets with the same surface area as K. Then (3.1) is satisfied due to (2.6). Now assume $s_o \geq 2$. To prove the claim in this case, we construct a Borel measure μ on S^{n-1} with support containing at most m_{s_o} points, satisfying the assumptions of Minkowski's existence theorem, see (Schneider, 2014, Thm. 8.2.2), and such that μ has the same moments as $S_{n-1}(K, \cdot)$ up to order s_o. Due to homogeneity of the surface area measure (and herewith of the surface tensors), we may assume that $S_{n-1}(K, S^{n-1}) = 1$.

Let $f_1, \dots, f_{m_{s_o}}$ denote the different components of the tensors u^{s_o-1} and u^{s_o}. For a Borel probability measure ν on S^{n-1}, let

$$\Gamma(\nu) = \left(\int_{S^{n-1}} f_1(u)\, \nu(du), \dots, \int_{S^{n-1}} f_{m_{s_o}}(u)\, \nu(du) \right).$$

Put

$$M := \{\Gamma(\nu) \mid \nu \text{ is a Borel probability measure on } S^{n-1}\}$$

and

$$N := \{\Gamma(\delta_u) \mid u \in S^{n-1}\} = \{(f_1(u), \dots, f_{m_{s_o}}(u)) \mid u \in S^{n-1}\},$$

where δ_u denotes the Dirac measure at $u \in S^{n-1}$. As $f_1, \dots, f_{m_{s_o}}$ are continuous, the set N is compact in $\mathbb{R}^{m_{s_o}}$, so the convex hull $\operatorname{conv} N$ of N is compact and, in particular, closed. The convex hull $\operatorname{conv} N$ of N is the image of the set of Borel probability measures on S^{n-1} with finite support under Γ. Hence, $M = \operatorname{conv} N$ as every Borel probability measure on S^{n-1} can be weakly approximated by such measures, see e.g., (Bauer, 2001, Cor. 30.5). This implies that $\Gamma(S(K, \cdot)) \in \operatorname{conv} N$. As S^{n-1} is connected and $f_1, \dots, f_{m_{s_o}}$ are continuous, the set N is connected. Then a version of Caratheodory's theorem due to Fenchel (see Hanner and Rådström (1951) and references given there) yields the existence of unit vectors $v_1, \dots, v_{m_{s_o}} \in S^{n-1}$ and $\alpha_1, \dots, \alpha_{m_{s_o}} \geq 0$ with $\sum_{i=1}^{m_{s_o}} \alpha_j = 1$ such that

$$\Gamma(S(K, \cdot)) = \sum_{i=1}^{m_{s_o}} \alpha_i \Gamma(\delta_{v_i}) = \Gamma(\mu), \tag{3.2}$$

where $\mu := \sum_{i=1}^{m_{s_o}} \alpha_i \delta_{v_i}$ is a probability measure with support containing at most m_{s_o} points. Remark 2.1, (2.6) and (3.2) yield that

$$\int_{S^{n-1}} u_i \, \mu(du) = \int_{S^{n-1}} u_i \, S_{n-1}(K, du) = 0$$

for $i = 1, \ldots, n$, hence the centroid of μ is at the origin.

If the support of μ was concentrated on a great subsphere $v^\perp \cap S^{n-1}$ of S^{n-1} for some $v \in S^{n-1}$, then

$$\int_{S^{n-1}} \langle u, v\rangle^2 S_{n-1}(K, du) = \int_{S^{n-1}} \langle u, v\rangle^2 \mu(du) = 0$$

by Remark 2.1 and (3.2) as $s_o \geq 2$. This would imply that $S_{n-1}(K, \cdot)$ is concentrated on $v^\perp \cap S^{n-1}$, which is a contradiction as K has interior points. Hence, the measure μ has full-dimensional support.

Herewith, μ satisfies the assumptions in Minkowski's existence theorem, and there is a polytope P with interior points such that $S_{n-1}(P, \cdot) = \mu$. As the support of $S_{n-1}(P, \cdot)$ contains at most m_{s_o} points, the polytope P has at most m_{s_o} facets. Due to (3.2) and Remark 2.1, the measures $S_{n-1}(K, \cdot)$ and $S_{n-1}(P, \cdot)$ have identical moments up to order s_o, which ensures that equation (3.1) is satisfied. $\square$

Corollary 3.2. *If K is determined up to translation among all convex bodies in $\mathbb{R}^n$ by its surface tensors up to rank s_o then $K \in \mathcal{P}^n_{m_{s_o}}$.*

On the other hand, a polytope is determined up to translation by finitely many surface tensors.

Theorem 3.3. *Let $m \geq n+1$ be a natural number. The shape of any $P \in \mathcal{P}^n_m$ with non-empty interior is uniquely determined in $\mathcal{K}^n$ by its surface tensors up to rank $2m$. If $n = 2$ then the result holds for any $P \in \mathcal{P}^n_m$.*

Proof. Let $P \in \mathcal{P}^n_m$ be given. We may assume without loss of generality that P has m facets. The surface area measure of P is of the form

$$S_{n-1}(P, \cdot) = \sum_{i=1}^{m} \alpha_i \delta_{u_i}$$

with $\alpha_1, \ldots, \alpha_m > 0$ and pairwise different $u_1, \ldots, u_m \in S^{n-1}$.

Let $K \in \mathcal{K}^n$ be a convex body such that $\Phi^s_{n-1}(K) = \Phi^{0,s}_{n-1}(P)$ for all $s \leq 2m$. We first show that $\operatorname{supp} S_{n-1}(K, \cdot) \subseteq \{\pm u_1, \ldots, \pm u_m\}$. Assume that $w \notin \{\pm u_1, \ldots, \pm u_m\}$. Then there exists $v_j \in u_j^\perp \setminus w^\perp$, $j = 1, \ldots, m$. Hence, the polynomial

$$q_1(u) = \prod_{j=1}^{m} \langle v_j, u\rangle^2,$$

$u \in S^{n-1}$, vanishes at $\pm u_1, \ldots, \pm u_m$ but not at w. By assumption on coinciding tensors and as q_1 has degree $2m$, we have

$$\int_{S^{n-1}} q_1(u) S_{n-1}(K, du) = \int_{S^{n-1}} q_1(u) S_{n-1}(P, du) = \sum_{i=1}^{m} \alpha_i q_1(u_i) = 0.$$

As $q_1 \geq 0$, this shows that q_1 is zero for $S_{n-1}(K, \cdot)$-almost all u. As q_1 is continuous,

$$\operatorname{supp} S_{n-1}(K, \cdot) \subseteq \{u \in S^{n-1} | q_1(u) = 0\} \subseteq S^{n-1} \setminus \{w\}.$$

Hence $w \notin \operatorname{supp} S_{n-1}(K, \cdot)$ and then $\operatorname{supp} S_{n-1}(K, \cdot) \subseteq \{\pm u_1, \dots, \pm u_m\}$. In particular, K is a polytope. Its surface area measure is of the form

$$S_{n-1}(K, \cdot) = \sum_{i=1}^{m} \left(\beta_i^+ \delta_{u_i} + \beta_i^- \delta_{-u_i}\right)$$

with $\beta_1^+, \beta_1^-, \dots, \beta_m^+, \beta_m^- \geq 0$, where we may assume $\beta_i^- = 0$ whenever $-u_i \in \{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_m\}$.

Consider now two cases. If $-u_1 \notin \{u_2, \dots, u_m\}$, we can find $v_j \in u_j^\perp \setminus u_1^\perp$, $j = 2, \dots, m$, and thus we have $q_2(u_1) \neq 0 \neq q_3(u_1)$ for

$$q_2(u) = \prod_{j=2}^{m} \langle v_j, u\rangle^2, \qquad q_3(u) = \Big(\prod_{j=2}^{m-1} \langle v_j, u\rangle^2\Big) \langle v_m, u\rangle.$$

By assumption on coinciding tensors, q_2 gives the same value when integrated with respect to $S_{n-1}(K, \cdot)$ and $S_{n-1}(P, \cdot)$. The same is true for q_3. This gives

$$\beta_1^+ + \beta_1^- = \alpha_1, \qquad \beta_1^+ - \beta_1^- = \alpha_1,$$

so $\beta_1^+ = \alpha_1$ and $\beta_1^- = 0$. If $-u_1 \in \{u_2, \dots, u_m\}$ we may without loss of generality assume $-u_1 = u_2 \notin \{\pm u_3, \dots, \pm u_m\}$. In this case, we have $\beta_1^- = \beta_2^- = 0$, and the remaining two parameters β_1^+ and β_2^+ can be determined with arguments similar to the ones above using

$$q_2(u) = \prod_{j=3}^{m} \langle v_j, u\rangle^2, \qquad q_3(u) = \Big(\prod_{j=3}^{m-1} \langle v_j, u\rangle^2\Big) \langle v_m, u\rangle.$$

These arguments can be applied to any index i showing that $S_{n-1}(K, \cdot) = S_{n-1}(P, \cdot)$. If P has non-empty interior or if $n = 2$, this implies that P and K are translates. $\square$

Theorem 3.4, below, is a version of Theorem 3.1 for centrally symmetric convex bodies. If $K \in \mathcal{K}^n$ is centrally symmetric its surface area measure is even on S^{n-1}, and hence $\Phi_{n-1}^s(K) = 0$ for all odd s. This simplifies the arguments in the proof of Theorem 3.1 as outlined in the following. Let $s_o \in \mathbb{N}$ be even. Let l_{s_o} denote the number of components of u^{s_o}, that is,

$$l_{s_o} = \binom{s_o + n - 1}{n - 1}.$$

In particular, $l_{s_o} = s_o + 1$ for $n = 2$. Let $h_1, \dots, h_{l_{s_o}}$ denote the different components of u^{s_o}. Following the proof of Theorem 3.1 with Γ, M and N replaced by

$$\Gamma_s(\nu) = \left(\int_{S^{n-1}} h_1(u)\nu(du), \dots, \int_{S^{n-1}} h_{l_{s_o}}(u)\nu(du)\right),$$

$$M_s = \left\{\Gamma(\nu) \mid \nu \text{ is a symmetric Borel probability measure on } S^{n-1}\right\},$$

and

$$N_s = \left\{\tfrac{1}{2}(\Gamma(\delta_u) + \Gamma(\delta_{-u})) \mid u \in S^{n-1}\right\},$$

we obtain an even probability measure $\mu_s = \sum_{j=1}^{l_{s_o}} \alpha_j(\delta_{u_j} + \delta_{-u_j})$ on S^{n-1}, such that

$$\Gamma_s(S_{n-1}(K,\cdot)) = \Gamma_s(\mu_s). \tag{3.3}$$

As μ_s and $S_{n-1}(K,\dot{)}$ are even, equation (3.3) implies that $\Gamma(\mu_s) = \Gamma(S_{n-1}(K,\cdot))$ with the notation from the proof of Theorem 3.1, and the result of Theorem 3.4 follows.

Theorem 3.4. *Let $K \in \mathcal{K}^n$ be centrally symmetric and $s_o \in \mathbb{N}$ be even. Then there exists an origin-symmetric polytope $P \in \mathcal{P}^n_{2l_{s_o}}$, such that*

$$\Phi^s_{n-1}(K) = \Phi^s_{n-1}(P)$$

for $0 \le s \le s_o$.

Remark 3.5. For later use, we note that the polytope P and the convex body K in Theorems 3.1 and 3.4 have identical harmonic intrinsic volumes up to degree s_o, as they have identical surface tensors up to rank s_o.

The following lemma gives a generalized version of Wirtinger's inequality, which is used in Theorem 3.8 to establish stability estimates for harmonic intrinsic volumes derived from the area measure of order 1.

Recall that $F \sim \sum_{m=0}^{\infty} F_m$ is the condensed harmonic expansion of $F \in L^2(S^{n-1})$.

Lemma 3.6. *Let $n \ge 2$, $s \in \mathbb{N}$ and $F \sim \sum_{m=0}^{\infty} F_m \in W^{\alpha}$ be given for some $\alpha > 0$. For $\gamma_m = m(m+n-2)$ we have*

$$\|F\|^2 \le \gamma_s^{-\alpha} \|(-\Delta_{\mathrm{S}})^{\frac{\alpha}{2}} F\|^2 + \sum_{m=0}^{s-1} \left(1 - (\gamma_m \gamma_s^{-1})^{\alpha}\right) \|F_m\|^2$$

with equality if and only if $F \in \bigoplus_{m=0}^{s} \mathcal{H}^n_m$.

Proof. It follows from (2.1) that

$$\begin{aligned}
\|F\|^2 &= \sum_{m=0}^{s-1} \|F_m\|^2 + \sum_{m=s}^{\infty} \gamma_m^{-\alpha} \gamma_m^{\alpha} \|F_m\|^2 \\
&\le \sum_{m=0}^{s-1} \|F_m\|^2 + \gamma_s^{-\alpha} \sum_{m=s}^{\infty} \gamma_m^{\alpha} \|F_m\|^2 \\
&= \gamma_s^{-\alpha} \|(-\Delta_{\mathrm{S}})^{\frac{\alpha}{2}} F\|^2 + \sum_{m=0}^{s-1} (1 - (\gamma_m \gamma_s^{-1})^{\alpha}) \|F_m\|^2.
\end{aligned}$$

Equality holds in the above calculations if and only if $F = \sum_{m=0}^{s} F_m$. $\square$

Lemma 3.6 immediately yields Corollary 3.7, where the second statement is a generalized version of Wirtinger's inequality.

Corollary 3.7 (Generalized Wirtinger's inequality). *Let $n \geq 2$, $s \in \mathbb{N}$ and $F \sim \sum_{m=0}^{\infty} F_m \in W^{\alpha}$ be given for some $\alpha \geq 0$. Then*

(i) $\|F\|^2 \leq \gamma_s^{-\alpha} \|(-\Delta_{\mathrm{S}})^{\frac{\alpha}{2}} F\|^2 + \sum_{m=0}^{s-1} \|F_m\|^2$

(ii) if $F_0 = \cdots = F_{s-1} = 0$, *then* $\|F\|^2 \leq \gamma_s^{-\alpha} \|(-\Delta_{\mathrm{S}})^{\frac{\alpha}{2}} F\|^2$

Equality holds in (i) and/or (ii) if and only if F is a spherical harmonic of degree s.

If F is twice continuously differentiable, then $F \in W^1$ by (Groemer, 1996, Cor. 3.2.12). Hence, Corollary 3.7 (ii) with $\alpha = 1$ can be applied to F if $F_0 = \cdots = F_{s-1} = 0$. For $s \in \{1, 2\}$, this yields the usual versions of Wirtinger's inequality of functions on S^{n-1}, see, e.g., (Groemer, 1996, Thm. 5.4.1).

A convex body $K \in \mathcal{K}_n^n$ is said to be of class C_+^2 if the boundary of K is a regular submanifold of $\mathbb{R}^n$ of class C^2 with positive Gauss curvature at each point. If $n \geq 2$ and K is of class C_+^2, then the support function h_K is twice continuously differentiable (see (Schneider, 2014, Sec. 2.5)), and the area measure $S_1(K, \cdot)$ of order 1 has density

$$s_1 = h_K + \frac{1}{n-1} \Delta_{\mathrm{S}} h_K \tag{3.4}$$

with respect to the spherical Lebesgue measure on S^{n-1}, see (Schneider, 2014, (2.56) and (4.26)). This establishes a connection between the support function of K and the harmonic intrinsic volumes of K derived from the area measure of order one. In combination with the generalized version of Wirtinger's inequality, this connection can be used to show the stability results in Theorems 3.8 and 3.9.

Theorem 3.8. *Let $n \geq 2$, $s_o \in \mathbb{N}_0$ and $\rho \geq 0$. Let $K, L \in \mathcal{K}^n$ such that $K, L \subseteq RB^n$ for some $R > 0$. Assume that*

$$\frac{1}{(m \vee 1)^{3-\varepsilon}} \sum_{k=1}^{N(n,m)} \big(\psi_{1mk}(K) - \psi_{1mk}(L)\big)^2 \leq \rho \tag{3.5}$$

for $m = 0, \dots, s_o$ and some $\varepsilon > 0$. Then

$$\delta_2^t(K, L)^2 \leq c_1 \big((s_o + 1)(n + s_o - 1)\big)^{-\alpha} + \rho M(n, \varepsilon) \tag{3.6}$$

for $0 < \alpha < \frac{3}{2}$, where $c_1 = c_1(\alpha, n, R)$ is a constant depending only on n, α and R, and M is a constant depending only on n and ε.

Proof. By (Schneider, 2014, Thm. 3.4.1 and subsequent remarks) there exists a sequence $(K_j)_{j \in \mathbb{N}}$ of convex bodies of class C_+^2 converging to K in the Hausdorff metric. For each $j \in \mathbb{N}$, the support function h_{K_j} is twice continuously differentiable, as K_j is of class C_+^2. Then an application of Green's formula (see, e.g., (Groemer, 1996, (1.2.7))), implies that

$$\langle H_m, \Delta_{\mathrm{S}} h_{K_j} \rangle_2 = \langle \Delta_{\mathrm{S}} H_m, h_{K_j} \rangle_2 = -\gamma_m \langle H_m, h_{K_j} \rangle_2$$

for $H_m \in \mathcal{H}^n_m$ as spherical harmonics are eigenfunctions of the Laplace-Beltrami operator. Thus, (3.4) yields that

$$\int_{S^{n-1}} H_m(u)\, S_1(K_j, du) = \alpha_{nm} \langle H_m, h_{K_j} \rangle_2 \tag{3.7}$$

for $H_m \in \mathcal{H}^n_m$, where $\alpha_{nm} = 1-(n-1)^{-1}\gamma_m$. Note that $\alpha_{nm} = 0$ if and only if $m = 1$. As $S_1(K_j, \cdot)$ converges weakly to $S_1(K, \cdot)$ (see (Schneider, 2014, Thm. 4.2.1)), and h_{K_j} converges uniformly to h_K, equation (3.7) implies that

$$\int_{S^{n-1}} H_m(u)\, S_1(K, du) = \alpha_{n,m} \langle H_m, h_K \rangle_2. \tag{3.8}$$

By the same arguments, equation (3.8) holds with K replaced by L.

Now let $F = h_K - h_L + \langle x, \cdot \rangle$, where $x = s(L) - s(K)$. Then $F_1 = 0$, and by equation (3.8), inequality (3.5), and the fact that $\langle x, \cdot \rangle \in \mathcal{H}^n_1$ we obtain that

$$\begin{aligned} \sum_{m=0}^{s_o} \|F_m\|^2 &= \sum_{\substack{m=0\\ m\neq 1}}^{s_o} \sum_{k=1}^{N(n,m)} \left(\int_{S^{n-1}} H_{mk}(u) F(u)\, \sigma(du) \right)^2 \\ &= \sum_{\substack{m=0\\ m\neq 1}}^{s_o} \alpha_{nm}^{-2} \sum_{k=1}^{N(n,m)} (\psi_{1mk}(K) - \psi_{1mk}(L))^2 \leq \rho\, M(n, \varepsilon), \end{aligned}$$

where $M(n,\varepsilon) = \sum_{m=2}^{\infty} \frac{m^{3-\varepsilon}}{\alpha_{n,m}^2} + 1 < \infty$. For $0 < \alpha < \frac{3}{2}$, we have that

$$\|(-\Delta_{\mathrm{S}})^{\frac{\alpha}{2}} F\| \leq \|(-\Delta_{\mathrm{S}})^{\frac{\alpha}{2}} h_{K-s(K)}\| + \|(-\Delta_{\mathrm{S}})^{\frac{\alpha}{2}} h_{L-s(L)}\| \leq c_1(\alpha, n, R)$$

due to (Kiderlen, 2008, (2.12)). This implies that $F \in W^\alpha$ for $0 < \alpha < \frac{3}{2}$. Then Corollary 3.7 (i) with s replaced by $s_o + 1$ can be applied to F, which yields that

$$\|F\|^2 \leq \big((s_o+1)(s_o+n-1)\big)^{-\alpha} c_1(\alpha, n, R) + \rho M(n, \varepsilon)$$

for $0 < \alpha < \frac{3}{2}$. Then inequality (3.6) follows, since $\delta_2^t(K,L)^2 = \|F\|^2$. □

The result of Theorem 3.8 can be transferred to a stability result for the Minkowski tensors Φ_1^s (which are the surface tensors in the two-dimensional setting).

Theorem 3.9. *Let $n \geq 2$, $s_o \in \mathbb{N}_0$ and let $K, L \in \mathcal{K}^n$ such that $K, L \subseteq RB^n$ for some $R > 0$. If $\Phi_1^s(K) = \Phi_1^s(L)$ for $s \in \{s_o - 1 \vee 0, s_o\}$, then*

$$\delta_2^t(K, L) \leq c_1\, s_o^{-\alpha} \tag{3.9}$$

and

$$\delta^t(K, L) \leq c_2\, s_o^{-\frac{2\alpha}{n+1}} \tag{3.10}$$

for $0 < \alpha < \frac{3}{2}$, where $c_1 = c_1(\alpha, n, R)$ and $c_2 = c_2(\alpha, n, R)$ are constants depending only on α, n and R.

Proof. Inequality (3.9) follows from Theorem 3.8, since equation (3.5) is satisfied with $\rho = 0$, as $\Phi_1^s(K) = \Phi_1^s(L)$ for $0 \leq s \leq s_o$, see Remark 2.1. Inequality (3.9) in combination with a known connection between the L^2-distance and the Hausdorff distance (see, (Groemer, 1996, Prop. 2.3.1)) yields inequality (3.10). □

4 Reconstruction of shape from surface tensors

We assume throughout this section that $n = 2$. In arbitrary dimension n, the surface tensors determine the shape of a convex body with interior points. In the two-dimensional case, however, the assumption on interior points is redundant, see (Schneider, 2014, Thm. 8.3.6). In the attempt to reconstruct shape from surface tensors in $\mathbb{R}^2$, it is therefore natural to consider $K_0 \in \mathcal{K}^2$. We suppose that the convex body K_0 is unknown and that the surface tensors $\Phi_1^0(K_0), \dots, \Phi_1^{s_o}(K_0)$ are known for some $s_o \in \mathbb{N}_0$. By Remark 2.1, this is equivalent to assuming that the components $\phi_{sj}(K_0)$ for $j = 0, \dots, s$ of $\Phi_1^s(K_0)$ are known for $s = s_o - 1, s_o$ (If $s_o = 0$, only the value of $\Phi_1^0(K_0)$ is assumed to be known).

Section 4.1 presents a reconstruction procedure of the shape of K_0 based on the components of the surface tensors of rank $s_o - 1$ and s_o. The output of the reconstruction procedure is a polygon P, where the surface tensors of P are identical to the surface tensors K_0 up to rank s_o. In Section 4.2 we use results from Section 3 to show consistency of the reconstruction algorithm developed in Section 4.1.

As described in Section 2, the harmonic intrinsic volumes of K_0 up to degree s_o constitute a set of values of surface tensors that contains the same shape information as the components of $\Phi_1^{s_o-1}(K_0)$ and $\Phi_1^{s_o}(K_0)$. It only requires minor adjustments of the reconstruction algorithm to obtain an algorithm based on the harmonic intrinsic volumes.

4.1 Reconstruction

Assume that $s_o \geq 1$, and define $D_{s_o} \colon \mathcal{K}^2 \to [0, \infty)$ as the sum of squared deviations of the components of the surface tensors of K to the components of the surface tensors of K_0 of rank $s_o - 1$ and s_o. That is

$$D_{s_o}(K) = \sum_{s=s_o-1}^{s_o} \sum_{j=0}^{s} \big(\phi_{sj}(K_0) - \phi_{sj}(K)\big)^2.$$

By Remark 2.1, the surface tensors of a convex body K and the surface tensors of K_0 are identical up to rank s_o if and only if $D_{s_o}(K) = 0$. In order to reconstruct the shape of K_0 from the surface tensors, it therefore suffices to find a convex body that minimizes D_{s_o}. Due to Theorem 3.1, there exists a $P \in \mathcal{P}^2_{2s_o+1}$ satisfying this condition.

Let δ_u denote the Dirac measure at $u \in S^1$, and let

$$M = \{(\alpha, u) \in \mathbb{R}^{2s_o+1} \times (S^1)^{2s_o+1} \mid \alpha_i \geq 0, \sum_{i=1}^{2s_o+1} \alpha_i u_i = o\}.$$

Then the surface area measure of a $P \in \mathcal{P}^2_{2s_o+1}$ is of the form

$$S_1(P, \cdot) = \sum_{i=1}^{2s_o+1} \alpha_i \delta_{u_i},$$

where $(\alpha, u) \in M$. The vectors $u_1, \ldots, u_{2s_o+1}$ are the facet normals of P, and $\alpha_1, \ldots, \alpha_{2s_o+1}$ are the corresponding facet lengths, see (Schneider, 2014, (4.24) and (8.15)). Conversely, if a Borel measure φ on S^1 is of the form

$$\varphi = \sum_{i=1}^{2s_o+1} \alpha_i \delta_{u_i}$$

for some $(\alpha, u) \in M$, then by Minkowski's existence theorem there is a $P \in \mathcal{P}^2_{2s_o+1}$, such that φ is the surface area measure of P, see (Schneider, 2014, Thm. 8.2.1). Notice that the assumption on the dimension of φ in Minkowski's existence theorem can be omitted as $n = 2$, see (Schneider, 2014, Thm. 8.3.1). The minimization of D_{s_o} can now be reduced to its minimization on $\mathcal{P}^2_{2s_o+1}$, and hence to the finite dimensional minimization problem

$$\min_{(\alpha,u)\in M} \sum_{s=s_o-1}^{s_o} \sum_{i=0}^{s} \Big(\phi_{sj}(K_0) - \frac{1}{s!\,\omega_{s+1}} \sum_{i=1}^{2s_o+1} \alpha_j u_{i1}^j u_{i2}^{s-j}\Big)^2. \tag{4.1}$$

This can solved numerically.

A solution to the minimization problem (4.1) is a vector $(\alpha, u) \in M$, which describes the surface area measure of a polygon. The reconstruction of the polygon from the surface area measure can be executed by means of Algorithm MinkData, see (Gardner, 2006, Sec. A.4). For $n = 2$, the reconstruction algorithm is simple. The vectors $\alpha_1 u_1, \ldots, \alpha_{2s_o+1} u_{2s_o+1}$ are sorted such that the polar angles are increasing, and hereafter, the vectors are positioned successively such that they form the boundary of a polygon $\tilde{P}$ with facets of length α_j parallel to u_j for $j = 1, \ldots, 2s_o+1$. The output polygon $\hat{K}_{s_o}$ of the algorithm is $\tilde{P}$ rotated $\frac{\pi}{2}$ about the origin. Then $\hat{K}_{s_o}$ minimizes D_{s_o}, and it follows that the convex bodies $\hat{K}_{s_o}$ and K_0 have identical surface tensors up to rank s_o.

If $s_o = 0$, let $\hat{K}_{s_o}$ be the line segment $[0, \phi_{00}(K_0)e_1]$, where e_1 is the first standard basis vector in $\mathbb{R}^2$. Then $\hat{K}_{s_o}$ is a polygon with 1 facet, and $\Phi_1^0(K_0) = \Phi_1^0(\hat{K}_{s_o})$.

The reconstruction algorithm can be summarized as follows.

Algorithm Surface Tensor

Input: A natural number $s_o \in \mathbb{N}_0$ and the components of the surface tensors $\Phi_1^{s_o}(K_0)$ and $\Phi_1^{s_o-1\vee 0}(K_0)$ of an unknown convex body $K_0 \in \mathcal{K}^2$.

Task: Construct a polygon $\hat{K}_{s_o}$ in $\mathbb{R}^2$ with at most $2s_o + 1$ facets such that $\hat{K}_{s_o}$ and K_0 have identical surface tensors up to rank s_o.

Action: If $s_o = 0$, let $\hat{K}_{s_o}$ be the line segment $[0, \phi_{00}(K_0)e_1]$. Otherwise,

Phase I: Find a vector $(\alpha, u) \in M$ that minimizes

$$\sum_{s=s_o-1}^{s_o} \sum_{j=0}^{s} \Big(\phi_{sj}(K_0) - \frac{1}{s!\,\omega_{s+1}} \sum_{i=1}^{2s_o+1} \alpha_i u_{i1}^j u_{i2}^{s-j}\Big)^2,$$

where $\phi_{s0}(K_0), \ldots, \phi_{ss}(K_0)$ denote the components of $\Phi_1^s(K_0)$.

Phase II: The vector (α, u) describes a polygon $\hat{K}_{s_o}$ in $\mathbb{R}^2$ with at most $2s_o+1$ facets. Reconstruct $\hat{K}_{s_o}$ from (α, u) using Algorithm MinkData.

It is worth mentioning that certain a priori information on $K_0 \in \mathcal{K}^n$ can be included in the reconstruction algorithm by modifying the set M in (4.1). We give two examples.

Example 4.1. If K_0 is known to be centrally symmetric, M can be replaced by

$$\big\{(\alpha, u) \in \mathbb{R}^{2s_o+2} \times (S^1)^{2s_o+2} \mid \alpha_j = \alpha_{(s_o+1)+j} \geq 0,\, u_j = -u_{(s_o+1)+j}\big\},$$

due to Theorem 3.4. This ensures central symmetry of the output polygon $\hat{K}_{s_o}$ of the reconstruction algorithm.

Example 4.2. If K_0 is known to be a polygon with at most m facets, M can be replaced by

$$\tilde{M} = \Big\{(\alpha, u) \in \mathbb{R}^m \times (S^1)^m \mid \alpha_j \geq 0, \sum_{j=1}^m \alpha_j u_j = 0\Big\}.$$

The assumption on K_0 implies that the optimization of (4.1) with M replaced by $\tilde{M}$ still has a solution with objective function value zero. The uniqueness statement in Theorem 3.3 even implies that the output $\hat{K}_{s_o}$ of this modified Algorithm Surface Tensor is unique and has the same shape as K_0 if $s_o \geq 2m$.

Remark 4.3. If K_0 is a polygon with at most $m \in \mathbb{N}$ facets and known surface tensors of rank $2m-1$ and $2m-2$, then an alternative reconstruction procedure similar to methods for reconstruction of planar polygons from complex moments described in Milanfar et al. (1995) and Golub et al. (1999) can be applied. We let $k \leq m$ denote the number of facets of K_0, let $u_1, \dots, u_k$ denote the facet normals and $\alpha_1, \dots, \alpha_k$ denote the corresponding facet lengths. The facet normals are identified with complex numbers in the natural way (in particular, u^s denotes complex multiplication and not tensor multiplication in this remark). For $s = 0, \dots, 2m-1$, we let

$$\tau_s = \sum_{j=1}^k \alpha_j u_j^s = s!\omega_{s+1} \sum_{j=0}^s \binom{s}{j} i^{s-j} \phi_{sj}(K_0)$$

and define the Hankel matrix

$$H = \begin{pmatrix} \tau_0 & \cdots & \tau_{m-1} \\ \vdots & \ddots & \vdots \\ \tau_{m-1} & \cdots & \tau_{2m-2} \end{pmatrix}.$$

As

$$H = V \operatorname{diag}(\alpha_1, \dots, \alpha_k) V^\top$$

where V is the Vandermonde matrix

$$V = \begin{pmatrix} 1 & \cdots & 1 \\ u_1 & \cdot & u_k \\ \vdots & & \vdots \\ u_1^{2m-1} & \cdots & u_k^{2m-1} \end{pmatrix} \in \mathbb{C}^{2m \times k},$$

the rank of H is the number k of facets of K_0. The facet normals and facet lengths of K_0 can be restored from H (or a submatrix of H, if $k < m$) using Prony's method, see Milanfar et al. (1995) or Hildebrand (1956). The shape of the polygon K_0 can then be reconstructed from the facet normals and facet lengths by means of Algorithm MinkData. The facet normals and facet lengths can also be obtained by solving the generalized eigenvalue problem $Hx = \lambda H_1 x$ where H_1 is defined as H but its entries start with τ_1 and end with τ_{2m-1}, see Golub et al. (1999).

4.2 Consistency of the reconstruction algorithm

Algorithm Surface Tensor described in Section 4.1 is consistent. This follows from Theorem 4.4.

Theorem 4.4. *Let $K_0 \in \mathcal{K}^2$ and $s_o \in \mathbb{N}_0$. If $K_{s_o} \in \mathcal{K}^2$ and K_0 have identical surface tensors up to rank s_o then*

$$\delta^t(K_0, K_{s_o}) = \mathcal{O}(s_o^{-1+\varepsilon})$$

for any $\varepsilon > 0$. Hence, if $K_{s_o}, s_o = 0, 1, 2, \ldots,$ is a sequence of such bodies then the shape of K_{s_o} converges to the shape of K_0.

Proof. As K_0 is compact, there is an $R > 0$ such that $K_0 \subseteq RB^2$. Let $s_o \in \mathbb{N}_0$, and let $x, y \in K_{s_o}$. Then

$$|x - y| = V_1([x, y]) \leq V_1(K_{s_o}) = V_1(K_0) \leq \pi R$$

by monotonicity of the intrinsic volumes on $\mathcal{K}^2$, see, e.g., Schneider and Weil (2008). It follows that there is a translate $K_{s_o} + x_{s_o}$ of K_{s_o} which is a subset of πRB^2. For each $s_o \in \mathbb{N}_0$, Theorem 3.9 with R replaced by πR can now be applied to K_0 and $K_{s_o} + x_{s_o}$, and we obtain that

$$\delta^t(K_0, K_{s_o}) \leq c_2(\alpha, 2, \pi R)(s_o + 1)^{-\frac{2\alpha}{3}}$$

for $0 < \alpha < \frac{3}{2}$. This yields the result. $\square$

4.3 Examples of reconstructions

This section consists of two examples where Algorithm Surface Tensor is used to reconstruct a polytope (see Figure 1) and a half disc (see Figure 2). For each two of the convex bodies, the reconstruction is executed for $s_o = 2, 4, 6$. The minimization (4.1) is performed by use of the procedure *fmincon* provided by MatLab. As initial values for this procedure, we use regular polytopes with $2s_o + 1$ facets. The reconstructions are illustrated in Figure 3 and Figure 4.

The reconstructions with $s_o = 2$ and the corresponding underlying convex bodies have identical surface tensors up to rank 2, so the reconstructions have, in particular, the same boundary length as the corresponding underlying bodies. Further, the reconstructions (in particular, the reconstruction of the polytope) seem to have the same orientation and degree of anisotropy as the corresponding underlying convex bodies. This is due to the influence of the surface tensor of rank 2. As expected, the reconstructions with $s_o = 4$ are more accurate than the reconstructions with $s_o = 2$. In the current two examples, the Algorithm Surface Tensor provides very precise approximations of the polytope and the half disc already for $s_o = 6$.

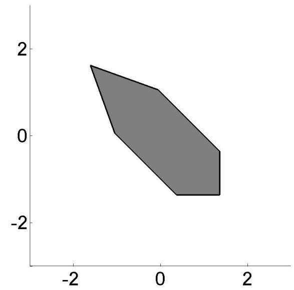

Figure 1: Polytope with six facets

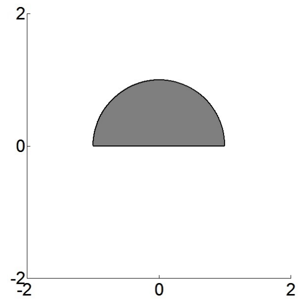

Figure 2: Half Disc

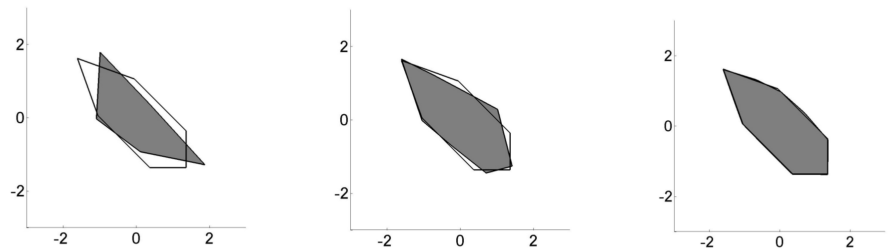

Figure 3: Reconstructions of polytope based on surface tensors up to rank $s_o = 2, 4, 6$.

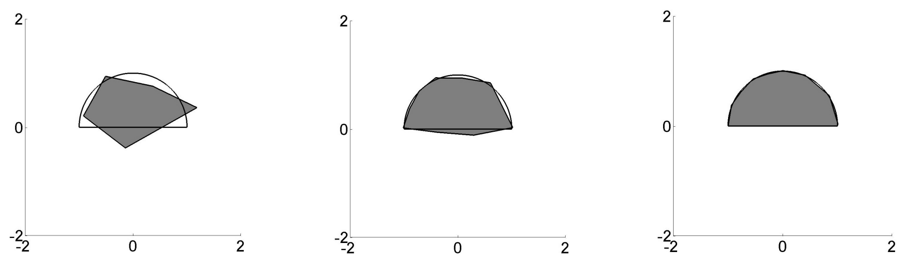

Figure 4: Reconstructions of half disc based on surface tensors up to rank $s_o = 2, 4, 6$.

5 Reconstruction of shape from measurements of harmonic intrinsic volumes

In Section 4, the reconstruction of shape from surface tensors was treated. In this section, we consider the problem of reconstructing shape from *noisy* measurements of surface tensors. As in Section 4, we assume that $n = 2$. As described in Section 2, the harmonic intrinsic volumes up to degree s contain the same shape information of a convex body as all surface tensors up to rank s. When only noisy measurements of the surface tensors are available, the structure of the stability result Theorem 3.8 proposes to use the harmonic intrinsic volumes for the reconstruction in order to obtain consistency of the reconstruction algorithm.

Let $s_o \in \mathbb{N}_0$, and suppose that $K_0 \in \mathcal{K}^2$ is an unknown convex body, where measurements of the harmonic intrinsic volumes up to degree s_o are known. To include noise, the measurements are assumed to be of the form

$$\lambda_{sj}(K_0) = \psi_{sj}(K_0) + \epsilon_{sj} \tag{5.1}$$

for $j = 1, \dots, N(2, s)$ and $s = 0, \dots, s_o$, where (ϵ_{sj}) are independent random variables with zero mean and finite variance. In the following, let

$$\psi_s(K) = (\psi_{01}(K), \psi_{11}(K), \psi_{12}(K), \dots, \psi_{s2}(K))$$

and similarly

$$\lambda_s(K) = (\lambda_{01}(K), \lambda_{11}(K), \lambda_{12}(K), \dots, \lambda_{s2}(K))$$

for $s \in \mathbb{N}_0$ and $K \in \mathcal{K}^2$.

Section 5.1 presents a reconstruction algorithm for the shape of K_0 based on the measurements (5.1). The output of the reconstruction procedure is a polygon, which fits the measurements (5.1) in a least squares sense. It is natural to consider least squares estimation as this is equivalent to maximum likelihood estimation when the noise terms (ϵ_{sj}) are independent, identically distributed normal random variables. The consistency of the least squares estimator is discussed in Section 5.2.

5.1 Reconstruction

Assume that $s_o \geq 1$, and define $D^H_{s_o} \colon \mathcal{K}^2 \to [0, \infty)$ as the sum of squared deviations of the harmonic intrinsic volumes of a convex body K to the measurements (5.1). That is

$$D^H_{s_o}(K) = \sum_{s=0}^{s_o} \sum_{j=1}^{n_s} \big(\lambda_{sj}(K_0) - \psi_{sj}(K)\big)^2 = |\lambda_{s_o}(K_0) - \psi_{s_o}(K)|^2,$$

where $n_s = N(2, s)$ for $s = 0, \dots, s_o$, ($n_0 = 1$ and $n_s = 2$ for $s \geq 1$). In order to obtain a least squares estimator, the infimum of $D^H_{s_o}$ has to be attained. In contrast to the situation in Section 4.1, the convex body K_0 does not necessarily minimize $D^H_{s_o}$. However, Lemma 5.1 ensures the existence a polygon that minimizes $D^H_{s_o}$.

Lemma 5.1. *There exists a $P \in \mathcal{P}^2_{2s_o+1}$ such that*

$$D^H_{s_o}(P) = \inf_{K \in \mathcal{K}^2} D^H_{s_o}(K). \tag{5.2}$$

Furthermore, if $K', K'' \in \mathcal{K}^2$ both are solutions of (5.2) *then $\psi_{s_o}(K') = \psi_{s_o}(K'')$, i.e. K' and K'' have the same surface tensors of rank at most s_o.*

Proof. Let $\mathcal{M}_{s_o} = \{\psi_{s_o}(K) \mid K \in \mathcal{K}^2\} \subseteq \mathbb{R}^{2s_o+1}$. Due to Minkowski linearity of the area measure of order one, see (Schneider, 2014, eq. (8.23)), $\mathcal{M}_{s_o}$ is convex.

We first show that $\mathcal{M}_{s_o}$ is closed in $\mathbb{R}^{2s_o+1}$. Let $(\psi_{s_o}(K_n))_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_{s_o}$, such that $\psi_{s_o}(K_n) \to \xi$ for some $\xi \in \mathbb{R}^{2s_o+1}$. For sufficiently large n we have

$$\begin{aligned}\sqrt{\frac{2}{\pi}} V_1(K_n) = \psi_{01}(K_n) &\leq |\xi_1 - \psi_{01}(K_n)| + |\xi_1| \\ &\leq |\xi - \psi_{s_o}(K_n)| + |\xi| \leq 1 + |\xi|.\end{aligned}$$

By monotonicity of the intrinsic volumes on $\mathcal{K}^2$ (see, e.g, Schneider and Weil (2008)), we have

$$|x - y| = V_1([x, y]) \leq V_1(K_n) \leq \sqrt{\frac{\pi}{2}}(1 + |\xi|)$$

for $x, y \in K_n$. This implies that a translate of K_n is a subset of $\sqrt{\frac{\pi}{2}}(1 + |\xi|)B^2$ for n sufficiently large. By continuity of $K \mapsto \psi_{s_o}(K)$ (with respect to the Hausdorff metric), an application of Blaschke's selection theorem (see, e.g., (Schneider, 2014, Thm. 1.8.7)), yields the existence of a subsequence $(n_l)_{l\in\mathbb{N}}$ and a convex body $K \in \mathcal{K}^2$ satisfying $\psi_{s_o}(K_{n_l}) \to \psi_{s_o}(K)$ for $l \to \infty$. Hence, $\xi = \psi_{s_o}(K) \in \mathcal{M}_{s_o}$, so $\mathcal{M}_{s_o}$ is closed. The optimization problem

$$\inf_{K \in \mathcal{K}^2} D^H_{s_o}(K) = \inf_{\psi \in \mathcal{M}_{s_o}} |\lambda_{s_o}(K_0) - \psi|^2$$

corresponds to finding the metric projection of $\lambda_{s_o}(K_0)$ to the non-empty closed and convex set $\mathcal{M}_{s_o}$. This metric projection $\psi_{s_o}(K') \in \mathcal{M}_{s_o}$ always exists and is unique; see, e.g., (Schneider, 2014, Section 1.2). Note that $K' \in \mathcal{K}^2$ is not uniquely determined here, but any two sets $K', K'' \in \mathcal{K}^2$ minimizing (5.2) must satisfy $\psi_{s_o}(K') = \psi_{s_o}(K'')$. By Theorem 3.1 (and Remark 3.5), this ensures the existence of a polygon P with at most $2s_o + 1$ facets satisfying (5.2). □

Remark 5.2. It follows from Lemma 5.1 that the measurements (5.1) are the exact harmonic intrinsic volumes of a convex body if and only if $\inf_{K\in\mathcal{K}^2} D^H_{s_o}(K) = 0$.

By Lemma 5.1 and considerations similar to those in Section 4.1, the minimization of $D^H_{s_o}$ can be reduced to the finite dimensional minimization problem

$$\min_{(\alpha,u)\in M} \sum_{s=0}^{s_o} \sum_{j=1}^{n_s} \Big(\lambda_{sj}(K_0) - \sum_{i=1}^{2s_o+1} \alpha_i H_{sj}(u_i)\Big)^2, \tag{5.3}$$

where M is defined as in Section 4.1. This finite minimization problem can be solved numerically. The solution to the minimization problem (5.3) is a vector (α, u) in M, that describes the surface area measure of a polygon. As described in Section 4.1, the MinkData Algorithm can be applied for the reconstruction of this polygon.

The least squares estimator $\hat{K}^H_{s_o}$ of the shape of K_0 is defined to be the output polygon of this algorithm. Then $\hat{K}^H_{s_o}$ minimizes $D^H_{s_o}$, so the harmonic intrinsic volumes of $\hat{K}^H_{s_o}$ fit the measurements (5.1) in a least squares sense. For $s_o = 0$, the estimator $\hat{K}^H_{s_o}$ is defined as the line segment $[0, \lambda_{00}(K_0)e_1]$ if $\lambda_{01}(K_0) \geq 0$. Otherwise, $\hat{K}^H_{s_o}$ is defined as the singleton $\{0\}$.

The reconstruction algorithm can be summarized as follows.

Algorithm Harmonic Intrinsic Volume LSQ

Input: A natural number $s_o \in \mathbb{N}_0$ and measurements $\lambda_{sj}(K_0)$, $j = 0, \ldots, N(2, s)$, $s = 0, \ldots, s_o$ of the harmonic intrinsic volumes up to degree s_o of an unknown convex body $K_0 \in \mathcal{K}^2$.

Task: Construct a polygon $\hat{K}^H_{s_o}$ in $\mathbb{R}^2$ with at most $2s_o + 1$ facets such that the harmonic intrinsic volumes of $\hat{K}^H_{s_o}$ fit the measurements of the harmonic intrinsic volumes of K_0 in a least squares sense.

Action: If $s_o = 0$, let $\hat{K}^H_{s_o}$ be the line segment (or singleton) $[0, (\lambda_{01}(K_0) \vee 0)e_1]$. Otherwise,

Phase I: Find a vector $(\alpha, u) \in M$ that minimizes

$$\sum_{s=0}^{s_o} \sum_{j=1}^{n_s} \Big(\lambda_{sj}(K_0) - \sum_{i=1}^{2s_o+1} \alpha_i H_{sj}(u_i) \Big)^2.$$

Phase II: The vector (α, u) describes a polygon $\hat{K}^H_{s_o}$ in $\mathbb{R}^2$ with at most $2s_o + 1$ facets. Reconstruct $\hat{K}^H_{s_o}$ from (α, u) using the MinkData Algorithm.

As described in Examples 4.1 and 4.2, additional information on the unknown convex body K_0 can be included in the reconstruction algorithm by modifying the set M in a suitable way.

5.2 Consistency of the least squares estimator

So far, we have oppressed the dependence of the noise term in the notation of $D^H_{s_o}$. In the following, for $s_o \in \mathbb{N}$, we write

$$D^H_{s_o}(K, x) = |\psi_{s_o}(K_0) + x - \psi_{s_o}(K)|^2$$

where $K \in \mathcal{K}^2$ and $x \in \mathbb{R}^{2s_o+1}$. Further, we let

$$\mathbb{K}_{s_o}(x) = \{K \in \mathcal{K}^2 \mid D^H_{s_o}(K, x) = \inf_{L \in \mathcal{K}^2} D^H_{s_o}(L, x)\}.$$

If $\epsilon_{s_o} = (\epsilon_{01}, \epsilon_{11}, \epsilon_{12}, \ldots, \epsilon_{s_o 2})$ denotes the random vector of noise variables in the measurements (5.1), then $\mathbb{K}_{s_0}(\epsilon_{s_o})$ is the random set of solutions to the minimization (5.3). Due to Lemma 5.1, the set $\mathbb{K}_{s_o}(\epsilon_{s_o})$ is non-empty for all $s_o \in \mathbb{N}$. We can without loss of generality assume that the noise variables are defined on a complete probability space.

In the following, we show that $\sup_{K\in\mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K)$ is measurable. To this end, we use the notion of permissible sets, see (Pollard, 1984, App. C). For $K \in \mathcal{K}^2$ and $x \in \mathbb{R}^{2s_o+1}$, define

$$f(K, x) = \delta^t(K_0, K)\, \mathbf{1}_{\{0\}}(g(K, x))$$

where $g(K, x) = \inf_{L\in\mathcal{K}^2} D^H_{s_o}(L, x) - D^H_{s_o}(K, x)$, and let $\mathcal{F} = \{f(K, \cdot) \mid K \in \mathcal{K}^2\}$. Then

$$\sup_{K\in\mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K) = \sup_{K\in\mathcal{K}^2} f(K, \epsilon_{s_o}).$$

As $D^H_{s_o}$ is continuous in the first variable and is measurable as a function of two variables, the mapping g is measurable as $\mathcal{K}^2$ is separable. As $\delta^t(K_0, \cdot)$ is continuous, this implies that f is measurable.

Let $\mathcal{F}_2$ denote the family of closed subsets of $\mathbb{R}^2$ equipped with the Fell topology, see, e.g., (Schneider and Weil, 2008, Chapter 12.2). Then, $\mathcal{F}_2$ is compact and metrizable, and the set of convex bodies $\mathcal{K}^2$ is an analytic subset of $\mathcal{F}_2$ as $\mathcal{K}^2 \in \mathcal{B}(\mathcal{F}_2)$, see, e.g., (Schneider and Weil, 2008, Thm. 12.2.1, the subsequent remark and Thm 2.4.2). Further, the topology on the separable set $\mathcal{K}^2$ induced by the Fell topology and the topology on $\mathcal{K}^2$ induced by the Hausdorff metric coincide, see, e.g, (Schneider and Weil, 2008, Thm. 12.3.4), so the set $\mathcal{F}$ is permissible. Due to (Pollard, 1984, App. C, p. 197), this implies that $\sup_{K\in\mathcal{K}^2} f(K, \epsilon_{s_o})$ is measurable.

For $s_o \in \mathbb{N}$, the noise variables $\epsilon_{01}, \epsilon_{11}, \dots, \epsilon_{s_o 2}$ are assumed to be independent with zero mean and finite variance bounded by a constant $\sigma^2_{s_o} < \infty$.

Theorem 5.3. *If $\sigma^2_{s_o} = \mathcal{O}(\frac{1}{{s_o}^{1+\varepsilon}})$ for some $\varepsilon > 0$, then*

$$\sup_{K\in\mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K) \to 0$$

in probability as $s_o \to \infty$. If $\sigma^2_{s_o} = \mathcal{O}(\frac{1}{{s_o}^{2+\varepsilon}})$, then the convergence is almost surely.

Proof. Let $\delta > 0$, and let $\rho < \frac{\delta}{2M} \wedge 1$ where $M = M(2, 3)$ is defined in Theorem 3.8. Let $s_o \in \mathbb{N}$, $K \in \mathbb{K}_{s_o}(\epsilon_{s_o})$, and assume first that $D^H_{s_o}(K_0, \epsilon_{s_o}) < \frac{\rho}{8}$. Then,

$$\begin{aligned}
&\max_{s=0,\dots,s_o} \sum_{j=1}^{n_s} \big(\psi_{sj}(K_0) - \psi_{sj}(K)\big)^2 \\
&\qquad \le 4 \max_{s=0,\dots,s_o} \sum_{j=1}^{n_s} \Big(\epsilon^2_{sj} + \big(\lambda_{sj}(K_0) - \psi_{sj}(K)\big)^2\Big) \\
&\qquad \le 8D^H_{s_o}(K_0, \epsilon_{s_o}) < \rho.
\end{aligned}$$

In particular, $(\psi_{01}(K_0) - \psi_{01}(K))^2 < \rho$ which implies that

$$V_1(K) < \tfrac{\pi}{2} + V_1(K_0) =: R(K_0).$$

By arguments similar to those in the proof of Theorem 4.4, this implies that there are translates of K and K_0 contained in RB^2. As R is independent of s_o and K, we obtain by Theorem 3.8 that

$$\sup_{K\in\mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t_2(K_0, K) \le c_1(1, 2, R)(s_o + 1)^{-2} + \rho M < \delta$$

for s_o sufficiently large. Due to the connection between the Hausdorff metric and L_2-metric, see, e.g., (Groemer, 1996, Prop. 2.3.1), we obtain

$$\sup_{K \in \mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K) < (3R\delta^2)^{\frac{1}{3}}. \tag{5.4}$$

As $D^H_{s_o}(K_0, \epsilon_{s_o}) = \sum_{s=0}^{s_o} \sum_{j=1}^{n_s} \epsilon_{sj}^2$, the assumption on the convergence rate, $\sigma^2_{s_o} = \mathcal{O}(\frac{1}{s_o{}^{1+\varepsilon}})$ for some $\varepsilon > 0$, implies that $D^H_{s_o}(K_0, \epsilon_{s_o})$ convergences to zero in mean and then in probability, when s_o increases. If $\sigma^2_{s_o} = \mathcal{O}(\frac{1}{s_o{}^{2+\varepsilon}})$, then $\sum_{s_o=1}^{\infty} \mathbb{E}\, D^H_{s_o}(K_0, \epsilon_{s_o}) < \infty$, which ensures that $D^H_{s_o}(K_0)$ convergences to zero almost surely. In combination with inequality (5.4), this yields the convergence results. □

As $\hat{K}^H_{s_o} \in \mathbb{K}_{s_o}(\epsilon_{s_o})$ for $s_o \in \mathbb{N}$, Theorem 5.3 yields consistency of Algorithm Harmonic Intrinsic Volume LSQ.

5.3 Example on reconstruction from harmonic intrinsic volumes

This section is an example where Algorithm Harmonic Intrinsic Volume LSQ is used to reconstruct a half disc K_0 from noisy measurements of the harmonic intrinsic volumes. The reconstruction of the half disc is executed for $s_o = 2, 4, \dots, 12$. The noise terms (ϵ_{sj}) are independent and normally distributed with zero mean. For the reconstruction based on harmonic intrinsic volumes up to degree s_o, the variance of the noise terms is $\sigma^2_{s_o} = \frac{1}{s_o{}^{2.1}}$. Due to Theorem 5.3 this ensures that $\delta^t(K_0, \hat{K}_{s_o}) \to 0$ almost surely for $s_o \to \infty$. The minimization (5.3) is carried out by use of the procedure *fmincon* provided by MatLab. As initial values for the minimization procedure, we use regular polytopes with $2s_o + 1$ facets. The reconstructions are plotted in Figure 5.

For the reconstruction based on exact surface tensors, the values of $D_{s_o}(K_0)$ and $D_{s_o}(\hat{K}_{s_o})$ are always zero. This is not the case when the reconstruction is based on measurements subject to noise. In Figure 6, the values of $D^H_{s_o}(K_0)$ and $D^H_{s_o}(\hat{K}^H_{s_o})$ are plotted for $s_o = 2, 4, \dots, 12$. As $\hat{K}^H_{s_o}$ minimizes $D^H_{s_o}$, the value of $D^H_{s_o}(\hat{K}_{s_o})$ is smaller than the value of $D^H_{s_o}(K_0)$ for each s_o. As the variance of the noise terms converges to zero sufficiently fast, the values of $D^H_{s_o}(K_0)$ and hence also the values of $D^H_{s_o}(\hat{K}^H_{s_o})$ tend to zero, when s_o increases.

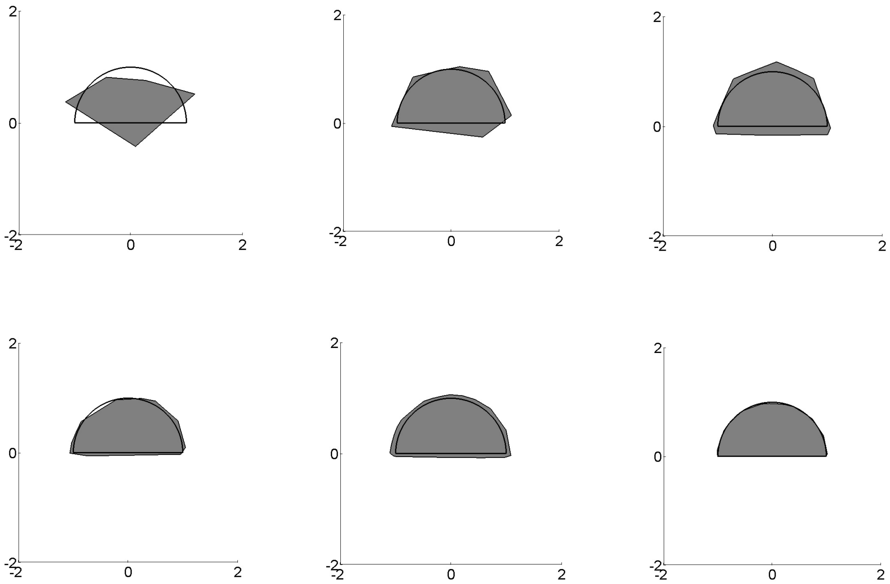

Figure 5: Reconstruction of a half disc based on measurements of harmonic intrinsic volumes up to degree $s_o = 2, 4, 6, 8, 10, 12$. The noise variables are normally distributed with zero mean and variance $\frac{1}{s_o^{2.1}}$.

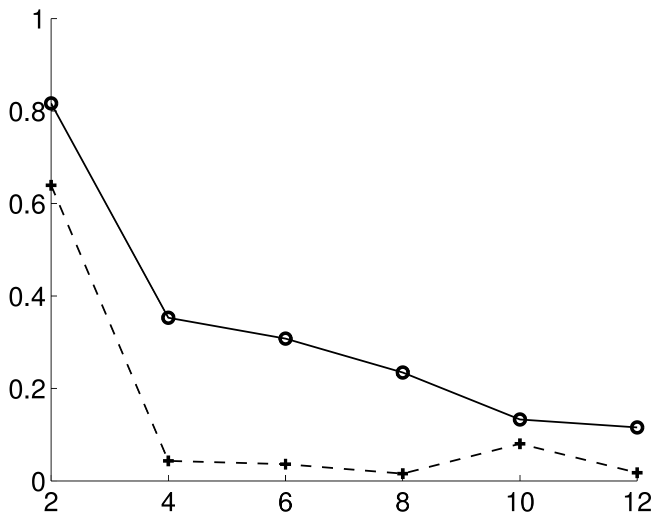

Figure 6: $D^H_{s_o}(K_0)$ ('o') and $D^H_{s_o}(\hat{K}^H_{s_o})$ ('+') plotted for $s_o = 2, 4, \ldots, 12$.

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References

S. Alesker. Description of continuous isometry covariant valuations on convex sets. *Geom. Dedicata*, 74:241–248, 1999.

H. Bauer. *Measure and integration theory.* Walter de Gruyter, Berlin, New York, 2001.

C. Beisbart, R. Dahlke, K. Mecke, and H. Wagner. Vector- and tensor-valued descriptors for spatial patterns. In K. Mecke and D. Stoyan, editors, *Morphology of Condensed Matter*. Springer, Heidelberg, 2002.

S. Campi. Stability estimates for star bodies in terms of their intersection bodies. *Mathematika*, 45:287–303, 1998.

S. Campi, R. J. Gardner, P. Gronchi, and M. Kiderlen. Lightness functions. *Adv. Math.*, 231:3118–3146, 2012.

R. Cheng and D. Zhang. A generalized Wirtinger's inequality with applications to a class of ordinary differential equations. *J. Inequal. Appl.*, 2009:710475, 2009.

R. J. Gardner. *Geometric Tomography.* Cambridge University Press, New York, second edition, 2006.

R. J. Gardner and P. Milanfar. Reconstruction of convex bodies from brightness functions. *Discrete Comput Geom*, 29:279–303, 2003.

R. Giova and T. Ricciardi. A sharp weighted Wirtinger inequality and some related functional spaces. *Bull. Belg. Math. Soc. Simon Stevin*, 17:209–218, 2010.

G.H. Golub, P. Milanfar, and J. Varah. A stable numerical method for inverting shape from moments. *Siam J. Sci. Comput.*, 21(4):1222–1243, 1999.

N. Gravin, J. Lasserre, D. V. Pasechnik, and S. Robins. The inverse moment problem for convex polytopes. *Discrete Comput Geom*, 48:596–621, 2012.

H. Groemer. *Geometric applications of Fourier series and spherical harmonics.* Cambridge University Press, Cambridge, 1996.

O. Hanner and H. Rådström. A generalization of a theorem of Fenchel. *Proc. Amer. Math. Soc.*, 2:589–593, 1951.

A. Hildebrand. *Introduction to numerical analysis.* McGraw-Hill, New York, 1956.

J. Hörrmann. *The method of densities for non-isotropic Boolean Models.* PhD thesis, Karlsruhe Institute of Technology, Karlsruhe, 2014.

D. Hug, R. Schneider, and R. Schuster. Integral geometry of tensor valuations. *Adv. Appl. Math.*, 41(4):482–509, 2008.

M. Kiderlen. Stability results for convex bodies in geometric tomography. *Indiana University Mathematics Journal*, 57:2000–2037, 2008.

P. Milanfar, G. C. Verghese, W. C. Karl, and A. S. Willsky. Reconstructing polygons from moments with connections to array processing. *IEEE Transactions on signal processing*, 43:432–443, 1995.

D. Pollard. *Convergence of stochastic processes.* Springer-Verlag, New York, 1984.

J. J. Prince and A. S. Willsky. Reconstructing convex sets from support line measurements. *IEEE Trans. Pattern Anal. Mach. Intell.*, 12(4):377–389, 1990.

R. Schneider. *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge University Press, Cambridge, second edition, 2014.

R. Schneider and W. Weil. *Stochastic and Integral Geometry.* Springer, Heidelberg, 2008.

G. E. Schröder-Turk, S. Kapfer, B. Breidenbach, C. Beisbart, and K. Mecke. Tensorial Minkowski functionals and anisotropy measures for planar patterns. *J. Microsc.*, 238 (1):57–74, 2010.

G. E. Schröder-Turk, W. Mickel, S. C. Kapfer, F. M. Schaller, B. Breidenbach, D. Hug, and K. Mecke. Minkowski tensors of anisotropic spatial structure. *New J. Phys.*, 15: 083028, 2013.

Casper Skouborg. Rekonstruktion af konvekse mængder fra overfladetensorer. Master's thesis, Aarhus University, 2012.