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**PhD Dissertation**

# Harish-Chandra characters of some minimal representations

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# Abstract

Let G be a simple Lie group of Hermitian type. Let $\mathrm{K} \subset \mathrm{G}$ be a maximal compact subgroup and let $\mathrm{H} \subset \mathrm{K}$ be a Cartan subgroup. Then H is also a Cartan subgroup of G. We denote their Lie algebras by fraktur script i.e. $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$. Let $\Lambda$ be a weight of $\mathfrak{h}$ then $\Lambda$ can be decomposed into two parameters $\Lambda_0$ and $\lambda$ such that $\Lambda_0$ is a weight of $[\mathfrak{k}, \mathfrak{k}] \cap \mathfrak{h}$ and $\lambda \in \mathbb{R}$. Let $\Delta^+(\mathfrak{g}, \mathfrak{h})$ be a positive system such that any positive non-compact root is greater than any compact root, let $\Delta_n^+$ be the set of positive non-compact roots. Then set $\mathfrak{n} = \oplus_{\alpha \in \Delta_n^+} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space of $\alpha$ in $\mathfrak{g}_{\mathbb{C}}$.

Let $\gamma$ be a dominant integral weight and let $L^{\mathfrak{g}_{\mathbb{C}}}(\gamma)$ be the unique simple quotient of $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{k}_{\mathbb{C}}+\mathfrak{n}} F^{\mathfrak{k}_{\mathbb{C}}}(\gamma)$. Where $F^{\mathfrak{k}_{\mathbb{C}}}(\gamma)$ is the finite dimensional highest-weight representation of $\mathfrak{k}_{\mathbb{C}}$ with highest-weight $\gamma$. The set of $(\Lambda_0, \lambda)$ such that $L^{\mathfrak{g}_{\mathbb{C}}}((\Lambda_0, \lambda))$ is unitarizable has been classified, take $\lambda$ minimal such that $L^{\mathfrak{g}_{\mathbb{C}}}((0, \lambda))$ is unitarizable. Then we call $L^{\mathfrak{g}_{\mathbb{C}}}((0, \lambda))$ the minimal holomorphic representation of G and denote it by $\pi_{\mathrm{Min}}$. This representation integrates to a representation of G. There exists $\alpha, \beta$ such that

$$\pi_{\mathrm{Min}}|_{\mathrm{K}} = \hat{\oplus}_{n=0}^{\infty} F^{\mathrm{K}}(\alpha + n\beta).$$

Let $Z \in Z(\mathfrak{k})$ such that $\alpha(Z) < 0$ for all $\alpha \in \Delta_n^+$. Then $C_{\mathrm{Min}} = \overline{\mathrm{Cone}(\mathrm{Ad}(\mathrm{G})Z)}$ is a non-trivial, Ad(G)-invariant, proper convex cone in $\mathfrak{g}$ such that $C_{\mathrm{Min}} \neq \{0\}, \mathfrak{g}$. Furthermore $C^o_{\mathrm{Min}} \neq \emptyset$. Then there exists a semigroup $\Gamma_{\mathrm{G}}(C_{\mathrm{Min}})$ such that G is a subgroup and $\Gamma_{\mathrm{G}}(C_{\mathrm{Min}})$ is homeomorphic to $\mathrm{G} \times C_{\mathrm{Min}}$. Inside this semigroup there is an ideal $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ which is homeomorphic to $\mathrm{G} \times C^o_{\mathrm{Min}}$. The ideal $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ has a complex manifold structure. There exists a representation of $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ which we will also denote by $\pi_{\mathrm{Min}}$ such that:

1. For $s \in \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ the operator $\pi_{\mathrm{Min}}(s)$ is trace-class and the function $s \mapsto \operatorname{tr} \pi_{\mathrm{Min}}(s)$ is holomorphic.

2. There exists a sequence $s_n \in \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ such that $s_n$ converges to the identity in $\Gamma_{\mathrm{G}}(C_{\mathrm{Min}})$. Furthermore the functions $\mathrm{G} \ni g \mapsto \operatorname{tr} \pi_{\mathrm{Min}}(g s_n)$ converge to the Harish-Chandra character of $\pi_{\mathrm{Min}}$ in the sense of distributions.

In this thesis we calculate $\operatorname{tr} \pi_{\mathrm{Min}}(h \operatorname{Exp}(iX))$ for $h \in \mathrm{H}$ and $X \in \mathfrak{h} \cap C^o_{\mathrm{Min}}$. Then for $\mathfrak{g} = \mathfrak{su}(p, q), \mathfrak{sp}(n, \mathbb{R}), \mathfrak{so}^*(2n)$ we use this to calculate the Harish-Chandra character of $\pi_{\mathrm{Min}}$. Furthermore we also calculate the Harish-Chandra character of the odd part of the metaplectic representation in this way. For $\mathfrak{g} = \mathfrak{so}(2, n)$ we get a reduced expression for $\operatorname{tr} \pi_{\mathrm{Min}}(h \operatorname{Exp}(iX))$ which we use to conjecture a character formula for $\pi_{\mathrm{Min}}$ in this case.



## Dansk resumé

Lad G være en simpel Lie gruppe af Hermitisk type. Lad $\mathrm{K} \subset \mathrm{G}$ være en maksiml kompakt undergruppe og lad $\mathrm{H} \subset \mathrm{K}$ være en Cartanundergruppe. Så er H også en

Cartanundergruppe af G. Vi skriver Liealgebraer med frakturskrift(også kaldet gotiske bogstaver) så $\mathfrak{g} = \mathrm{Lie(G)}$, $\mathfrak{k} = \mathrm{Lie(K)}$, etc. Lad $\Lambda$ være en vægt for $\mathfrak{h} \subset \mathfrak{k}$ så kan $\Lambda$ dekomponeres i to parametre $\Lambda_0$ og $\lambda$ så $\Lambda_0$ er en vægt for $[\mathfrak{k}, \mathfrak{k}] \cap \mathfrak{h} \subset [\mathfrak{k}, \mathfrak{k}]$ og $\lambda \in \mathbb{R}$. Lad $\Delta^+(\mathfrak{g}, \mathfrak{h})$ være et positivt system som opfylder at enhver positiv ikke-kompakt rod er større end enhver kompakt rod. Lad $\Delta_n^+$ være mængden af positive ikke-kompakte rødder. Så lader vi $\mathfrak{n} = \oplus_{\alpha \in \Delta_n^+} \mathfrak{g}_\alpha$ hvor $\mathfrak{g}_\alpha$ er rodrummet i $\mathfrak{g}_\mathbb{C}$ tilhørende $\alpha$.

Lad $\gamma$ være en dominant heltallig vægt og lad $L^{\mathfrak{g}_\mathbb{C}}(\gamma)$ være den entydige simple kvotient af modulet $\mathfrak{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathfrak{k}_\mathbb{C} + \mathfrak{n}} F^{\mathfrak{k}_\mathbb{C}}(\gamma)$. Hvor $F^{\mathfrak{k}_\mathbb{C}}(\gamma)$ er den endeligt dimensionelle højestevægtrepræsentation af $\mathfrak{k}_\mathbb{C}$ med højeste vægt $\gamma$. Mængden af par $(\Lambda_0, \lambda)$ så $L^{\mathfrak{g}_\mathbb{C}}((\Lambda_0, \lambda))$ kan gøres til en unitær repræsentation er tidligere blevet bestemt. Lad $\lambda$ være minimal således at $L^{\mathfrak{g}_\mathbb{C}}((0, \lambda))$ kan gøres unitær. Vi betegner $L^{\mathfrak{g}_\mathbb{C}}((0, \lambda))$ med dette $\lambda$ som den minimale holomorfe repræsentation og skriver $\pi_{\mathrm{Min}}$. Så giver $\pi_{\mathrm{Min}}$ anledning til en repræsentation af G. Der findes $\alpha$ og $\beta$ således at

$$\pi_{\mathrm{Min}}|_{\mathrm{K}} = \hat{\oplus}_{n=0}^{\infty} F^{\mathrm{K}}(\alpha + n\beta).$$

Lad $Z \in Z(\mathfrak{k})$ og vælg $Z$ så det opfylder $\alpha(Z) < 0$ for alle $\alpha \in \Delta_n^+$. Sæt $C_{\mathrm{Min}} := \overline{\mathrm{Cone}(\mathrm{Ad(G)}Z)}$ så er $C_{\mathrm{Min}}$ en Ad(G)-invariant konveks kegle i $\mathfrak{g}$, som opfylder at $C_{\mathrm{Min}} \neq \{0\}, \mathfrak{g}$. Ydermore da $\mathfrak{g}$ er simpel er $C^o_{\mathrm{Min}} \neq \emptyset$. Så eksisterer der en semigruppe $\Gamma_{\mathrm{G}}(C_{\mathrm{Min}})$ så G er en undergruppe og $\Gamma_{\mathrm{G}}(C_{\mathrm{Min}})$ er homøomorf til $\mathrm{G} \times C_{\mathrm{Min}}$. I denne semigruppe findes et semigruppeideal $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$, som under den samme homøomorfi som før bliver sendt til $\mathrm{G} \times C^o_{\mathrm{Min}}$. Idealet $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ er en kompleksmangfoldighed. Der findes en repræsentation af $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ som vi også kalder $\pi_{\mathrm{Min}}$ som opfylder

1. Lad $s \in \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ så er operatoren $\pi_{\mathrm{Min}}(s)$ sporklasse og funktionen $s \mapsto \mathrm{tr}\, \pi_{\mathrm{Min}}(s)$ er holomorf.

2. Der findes en følge $s_n \in \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$, så $s_n$ konvergerer til identiteten i $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$. Ydermere konvergerer funktionerne $\mathrm{G} \ni g \mapsto \mathrm{tr}\, \pi_{\mathrm{Min}}(g s_n)$ til Harish-Chandra karakteren af $\pi_{\mathrm{Min}}$. Denne konvergens er i distributionsforstand.

I denne afhandling beregner vi $\mathrm{tr}\, \pi_{\mathrm{Min}}(h \,\mathrm{Exp}(iX))$ når $h \in \mathrm{H}$ og $X \in \mathfrak{h} \cap C^o_{\mathrm{Min}}$. I tilfældene $\mathfrak{g} = \mathfrak{su}(p, q), \mathfrak{sp}(n, \mathbb{R})$ og $\mathfrak{so}^*(2n)$ bruger vi dette til at beregne en formel for Harish-Chandra karakteren af $\pi_{\mathrm{Min}}$. Vi beregner også Harish-Chandra karakteren af den ulige metaplektiske repræsentation med disse metoder. I tilfældet $\mathfrak{so}(2, 2n)$ beregner vi et reduceret udtryk for $\mathrm{tr}\, \pi_{\mathrm{Min}}(h \,\mathrm{Exp}(iX))$ som vi bruger til at opstille en formodning om Harish-Chandra karakteren.

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# Contents

# 1 Introduction

The aim of this thesis is to calculate the Harish-Chandra characters for minimal representations of simple Lie groups of Hermitian type. In the thesis we introduce some general results that we hope should be applicable to the calculation of characters of highest-weight representations of Lie groups of Hermitian type. However they rely on an assumption of continuous extension of the character on the Ol'shanskiĭ semigroup to the group. The author is unaware of any results about when to expect such extensions and thus we rely on case by case analysis to show that it holds for the character of the minimal representations of $\mathfrak{sp}(n, \mathbb{R}), \mathfrak{su}(p, q)$ and $\mathfrak{so}^*(2n)$. In general the structure of minimal representations is exceptionally simple from our perspective and thus the natural test cases for our method. However for the mentioned cases the Ol'shanskiĭ semigroups also possess very explicit descriptions and thus we can show the extension property directly. In conclusion we get explicit formulas for the Harish-Chandra characters of minimal representations of $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{su}(p, q)$ and $\mathfrak{so}^*(2n)$.

## 1.1 Harish-Chandra characters

In the theory of representations of finite and compact Lie groups a central idea is the notion of a character of a representation. Let $\pi$ be a finite dimensional representation of a group $\mathrm{G}$ on a complex vector space then the character $\chi$ of $\pi$ is the function $\chi : \mathrm{G} \to \mathbb{C}$ given by

$$\chi(g) = \operatorname{tr} \pi(g).$$

Characters of finite dimensional representations of finite or compact groups satisfy the Schur orthogonality relations, which state that inequivalent irreducible representations are orthogonal. Thus in principle the character contains all information necessary to uniquely identify the isomorphism class of a representation. Let $\chi_1, \ldots$ be a maximal set of characters of pairwise inequivalent irreducible representations. Then we can determine the decomposition of $\pi$ into its irreducible constituents by considering $\langle \chi, \chi_j \rangle$ for all $j$.

In the study of unitary representations of non-compact semisimple Lie groups we would like to do something similar. However all non-trivial unitary representations of non-compact semisimple Lie groups are infinite dimensional, and furthermore since $\pi(g)$ is unitary it is not trace-class. Let $\pi$ be an irreducible unitary representation of $\mathrm{G}$ a semisimple Lie group. For $f \in C_c^\infty(\mathrm{G})$ that is $f$ is a compactly supported smooth function on $\mathrm{G}$ we can define $\pi(f)$ by

$$\pi(f)v = \int_{\mathrm{G}} f(g)\pi(g)v \, dg. \tag{1.1}$$

Then it turns out that $\pi(f)$ is trace-class and in fact the mapping $f \mapsto \operatorname{tr} \pi(f)$ is a continuous linear functional on $C_c^\infty(\mathrm{G})$ and we call this Schwartz distribution the character of $\pi$, we denote it by $\Theta$. Harish-Chandra has shown that on the set of regular elements $\mathrm{G}'$ the character $\Theta$ is given by a real analytic function. In fact Harish-Chandra showed that there exists a locally integrable function $\theta : \mathrm{G} \to \mathbb{C}$ on G such that

$$\Theta(f) = \int_{\mathrm{G}} \theta(g) f(g)\, dg$$

and $\theta|_{\mathrm{G}'}$ is real analytic. When we say that we want to calculate the character of a representation we usually mean that we want to calculate $\theta|_{\mathrm{G}'}$. Furthermore if $\pi_1, \dots, \pi_n$ are irreducible unitary inequivalent representations then their characters $\Theta_1, \dots, \Theta_n$ are linearly independent. Specifically the Harish-Chandra character determines the equivalence class of an irreducible representation.

## 1.2 Minimal representations

Minimal representations have received much study in the last 30 years. They are interesting from various different angles and for various different reasons. The most well-known and well-studied minimal representation is the metaplectic representation also known as the Segal-Shale-Weil representation or the oscillator representation. Much of its prominence is due to the work of Howe with Howe dual pairs and the Howe correspondence. This has inspired much work in finding realizations of minimal representations for other groups and the study of their branching laws and the special functions arising in this theory.

Another reason to study minimal representations comes from the study of unipotent representations and the orbit philosophy. In the Kirillov orbit method there is a correspondence between irreducible unitary representations and coadjoint orbits. Kirillov showed that this holds for nilpotent groups and it was extended to solvable groups. However there is no such correspondence in general for simple Lie groups but it is still used as a guiding philosophy in the attempt to classify the unitary dual of simple Lie groups. In this sense the minimal representations should be the representations corresponding to the minimal nilpotent orbit.

To be more explicit let G be a simple real Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathcal{J} \subset \mathfrak{U}(\mathfrak{g}_\mathbb{C})$ be a two-sided ideal in the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Then by the PBW theorem there exists a filtration of $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ such that $\operatorname{gr} \mathfrak{U}(\mathfrak{g}_\mathbb{C}) \cong S(\mathfrak{g}_\mathbb{C})$ where $S(\mathfrak{g}_\mathbb{C})$ is the symmetric algebra. Then $\operatorname{gr} \mathcal{J}$ is an ideal in $S(\mathfrak{g}_\mathbb{C})$ and thus the zero set of $\operatorname{gr} \mathcal{J}$ defines a variety in $\mathfrak{g}_\mathbb{C}^*$. We call this the associated variety of $\mathcal{J}$ and denote it by $\operatorname{Ass}(\mathcal{J})$.

Then Joseph[Jos76] has shown that if $\mathfrak{g}_\mathbb{C}$ is not of type $A$ there exists a unique completely-prime ideal $\mathcal{J} \subset \mathfrak{U}(\mathfrak{g}_\mathbb{C})$ such that $\operatorname{Ass}(\mathcal{J}) = \overline{\mathcal{O}_{\mathrm{Min}}}$ where $\overline{\mathcal{O}_{\mathrm{Min}}}$ is the closure of the minimal nilpotent orbit. That is the associated variety of $\mathcal{J}$ is the closure of the minimal orbit in $\mathfrak{g}_\mathbb{C}^*$. This ideal is called the Joseph ideal. In type $A_n$ there exists a $\mathbb{C}$-parametrized family of ideals such that the associated variety is the closure of the minimal orbit[BJ98, p. 7.5; Bor77].

Let $\pi$ be a unitary representation of G. Then to $\pi$ corresponds a $(\mathfrak{g}, \mathrm{K})$ module and thus a $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ representation. We let $\mathrm{Ann}(\pi) \subset \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ denote the kernel of this representation.

**Definition 1.2.1 (4.10 in [GS05])** Let $\pi$ be a unitary irreducible representation of G.

- We call $\pi$ *weakly minimal* if $\mathrm{Ann}(\pi)$ is completely prime and has $\mathrm{Ass}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}_{\mathrm{Min}}}$.
- We call $\pi$ *minimal* if $\mathrm{Ann}(\pi)$ is the Joseph ideal.

Since we restrict ourselves to simple Lie groups of Hermitian type there is an alternative notion of minimal representations which is also interesting to us. Let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of a maximal compact subgroup of G. Let $\mathfrak{h} \subset \mathfrak{k}$ be a Cartan subalgebra, then $\mathfrak{h}$ is also a Cartan sublagebra of $\mathfrak{g}$. Let $\Delta_n^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the set of positive non-compact roots of $\mathfrak{g}_{\mathbb{C}}$ where the ordering is a *good ordering* in the sense that any positive non-compact root is greater than any compact root. Then set $\mathfrak{n} = \oplus_{\alpha \in \Delta_n^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the rootspace of $\mathfrak{g}_{\mathbb{C}}$ corresponding to the root $\alpha$. A weight of $\mathfrak{h}$ can be decomposed $\Lambda = (\Lambda_0, \lambda)$ such that $\Lambda_0$ is a weight of $[\mathfrak{k}, \mathfrak{k}]$ and $\lambda \in \mathbb{R}$. Assume that $\Lambda_0$ is dominant integral and let $F^{\mathfrak{k}_{\mathbb{C}}}(\Lambda)$ denote the simple $\mathfrak{k}_{\mathbb{C}}$ module with highest weight $\Lambda$. Let $L^{\mathfrak{g}_{\mathbb{C}}}(\Lambda)$ be the unique simple quotient of $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{U}(\mathfrak{k}_{\mathbb{C}} + \mathfrak{n})} F^{\mathfrak{k}_{\mathbb{C}}}(\Lambda)$. Then Jakobsen and Enright, Howe and Wallach determined the $\Lambda$ such that $L^{\mathfrak{g}_{\mathbb{C}}}(\Lambda)$ are unitarizable, the set of such $\Lambda$ is usually called the Wallach set. Then the minimal holomorphic representation $\pi_{\mathrm{Min}}$ is the representation such that $\Lambda = (0, \lambda)$ is the element in the Wallach set with $\lambda$ minimal and $\Lambda_0 = 0$. For all $\mathfrak{g}$ such that $\mathfrak{g}_{\mathbb{C}}$ is not of type $A$ the minimal holomorphic representation is minimal and almost all minimal representations are either a minimal holomorphic representation or the contragredient representation of a minimal holomorphic representation[MO14]. Observe that $\pi_{\mathrm{Min}}$ is a lowest weight representation.

All minimal holomorphic representations have a pencil of K-types.

**Proposition 1.2.2** *Let* G *be a simple Lie group of Hermitian type and let* $\mathrm{K} \subset \mathrm{G}$ *be a maximal compact subgroup. Let* $\pi_{\mathrm{Min}}$ *be the minimal holomorphic representation. Then* $\pi_{\mathrm{Min}}$ *has a* pencil of K-types *that is there exists weights* $\alpha, \beta \in \mathfrak{h}_{\mathbb{C}}^*$ *such that*

$$\pi_{\mathrm{Min}}|_{\mathrm{K}} = \widehat{\bigoplus_{n=0}^{\infty}} F^{\mathrm{K}}(\alpha + n\beta).$$

Actually this seems to be the case for most minimal representations: It is at least the case for the representations constructed in [HKM14].

## 1.3 Ol'shanskiĭ semigroups

Let G be a simple Lie group of Hermitian type and let $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ be its Lie algebra. Kostant[Seg76] and Vinberg[Vin80] has shown that then there exist a non-trivial, proper convex cone $C \subset \mathfrak{g}$ such that $\mathrm{Ad}(\mathrm{G})C = C$. Furthermore since G is simple the interior of $C$ has to be non-empty. Then a special case of Lawsons theorem gives us a semigroup.

**Theorem 2.4.1 (Lawson's theorem)** *Let* $\mathrm{G}$ *be a connected simple Lie group of Hermitian type with a complexification* $\mathrm{G}_{\mathbb{C}}$. *Let* $C \subset \mathfrak{g}$ *be a non-trivial, closed,* $\mathrm{Ad}(\mathrm{G})$*-invariant, proper convex cone. Then* $\Gamma_{\mathrm{G}}(C) := \mathrm{G}\exp(iC)$ *is a closed subsemigroup of* $\mathrm{G}_{\mathbb{C}}$ *such that*

$$\mathrm{G} \times C \to \Gamma_{\mathrm{G}}(C) \qquad\qquad (g, X) \mapsto g\exp(iX)$$

*is a homeomorphism.*

This gives a complex manifold structure on $\mathrm{G}\exp(iC^o)$ since it is the interior of $\Gamma_{\mathrm{G}}(C)$. As $C$ is a convex cone it is contractible. Thus $\Gamma_{\mathrm{G}}(C)$ contracts to $\mathrm{G}$. Then we see that the coverings of $\Gamma_{\mathrm{G}}(C)$ exactly correspond to the coverings of $\mathrm{G}$. Furthermore if $\mathrm{A} \subset \mathrm{G}$ is some discrete central subgroup in $\mathrm{G}$ then it acts properly on $\Gamma_{\mathrm{G}}(C)$ and thus we can consider $\Gamma_{\mathrm{G}}(C)/\mathrm{A}$ which is homeomorphic to $(\mathrm{G}/\mathrm{A}) \times C$. In this way for any connected Lie group $\hat{\mathrm{G}}$ with Lie algebra $\mathfrak{g}$ we construct a semigroup $\Gamma_{\hat{\mathrm{G}}}(C)$ which is homeomorphic to $\hat{\mathrm{G}} \times C$.

Ol'shanskiĭ has shown that any highest-weight representation of a simple Lie group of Hermitian type with a complexification extend to a representation of the interior of the associated Ol'shanskiĭ semigroup. This result can be generalized to all complex open Ol'shanskiĭ semigroups. Specifically all minimal holomorphic representations extend to an open complex Ol'shanskiĭ semigroup. We will also denote these extensions by $\pi_{\mathrm{Min}}$. Let $s$ be an element in the interior of the Ol'shanskiĭ semigroup then since $\pi_{\mathrm{Min}}$ is irreducible there is a theorem which tells us that $\pi_{\mathrm{Min}}(s)$ is a trace-class operator. Furthermore the function $s \mapsto \operatorname{tr}\pi_{\mathrm{Min}}(s)$ is a holomorphic function on the interior of the Ol'shanskiĭ semigroup. Let $s_n$ be a sequence in the open Ol'shanskiĭ semigroup such that $\lim_{n\to\infty} s_n = e$ i.e. the identity. Then the Harish-Chandra character of $\pi_{\mathrm{Min}}$ is the limit of the sequence of functions $\mathrm{G} \ni g \mapsto \operatorname{tr}\pi_{\mathrm{Min}}(gs_n)$ in the sense of distributions. Thus $\operatorname{tr}\pi_{\mathrm{Min}}$ on the interior of the Ol'shanskiĭ semigroup determines the Harish-Chandra character of $\pi_{\mathrm{Min}}$.

## 1.4 Results

The idea in the thesis is now to use the K-type decomposition of $\pi_{\mathrm{Min}}$ and Weyl's character formula to calculate the function $\operatorname{tr}\pi_{\mathrm{Min}}|_{\mathrm{H}\,\mathrm{Exp}(i\mathfrak{h}\cap C)}$. In fact we calculate the character on the Ol'shanskiĭ semigroup for any unitary irreducible highest-weight representation with a pencil of K-types. We usually denote $c_m^o = \mathfrak{h} \cap C_{\mathrm{Min}}^o$ where $C_{\mathrm{Min}}^o$ is the minimal non-trivial, open, $\mathrm{Ad}(\mathrm{G})$-invariant, proper convex cone in $\mathfrak{g}$.

**Proposition 2.6.3** *Let* $\mathrm{G}$ *be a connected, simple Lie group of Hermitian type. Let* $\pi$ *be a unitary, irreducible, highest-weight representation of* $\mathrm{G}$ *with a pencil of* $\mathrm{K}$*-types, with* $\alpha$ *and* $\beta$ *as in definition 2.6.1. Let* $h \in \mathrm{H}$ *and* $X \in c_m^o$ *then*

$$\operatorname{tr}\pi(h\,\mathrm{Exp}(iX)) = \sum_{w \in W_{\mathrm{K}}} \frac{\epsilon(w)\xi_{w(\alpha+\delta_k)-\delta_k}(h)e^{i(w(\alpha+\delta_k)-\delta_k)(X)}}{(1-\xi_{w\beta}(h)e^{iw\beta(X)})\prod_{\gamma\in\Delta_k^+}(1-\xi_{-\gamma}(h)e^{-i\gamma(X)})}. \tag{2.7}$$

Then we exploit that the trace is conjugation invariant hence $\operatorname{tr}\pi_{\mathrm{Min}}$ is a conjugation invariant and holomorphic function for which we have a formula on $\operatorname{Exp}(\mathfrak{h} \cap C)$. Then by analytic continuation this is enough to determine $\operatorname{tr}\pi_{\mathrm{Min}}$ as a holomorphic function on the entire open Ol'shanskiĭ semigroup. However we want to get more precise information on how $\operatorname{tr}\pi_{\mathrm{Min}}$ looks away from the fundamental Cartan subgroup. We do this by constructing curves in the Ol'shanskiĭ semigroup connecting the various Cartan subgroups to the fundamental Cartan subgroup. These curves are constructed such that along them $\operatorname{tr}\pi_{\mathrm{Min}}$ is constant.

**Proposition 2.8.9** *Let* $\mathrm{G}$ *be a connected, simple Lie group of Hermitian type. Let* $\theta : \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}}) \to \mathbb{C}$ *be a holomorphic, conjugation invariant function which extends continuously to* $\mathrm{G}'$. *Let* $S = \{\nu_1, \dots, \nu_k\} \subset \Delta_n^+$ *be pairwise strongly orthogonal roots. Assume that* $\mathfrak{g} \in \{\mathfrak{so}(2,n), \mathfrak{sp}(n,\mathbb{R}), \mathfrak{su}(p,q), \mathfrak{so}^*(2n)\}$.

*Let* $n_1, \dots, n_k \in \mathbb{Z}$, $s_1, \dots, s_k \in \mathbb{R} \setminus \{0\}$ *and* $X \in \mathfrak{h}'_S$ *such that the element on the left-hand side in eq.* (2.15) *is regular. Then*

$$\begin{aligned}\theta\Big(\Big(\prod_{j=1}^{k} \exp(n_j\pi i H'_{\nu_j}) \exp(s_j(E_{\nu_j} + \overline{E_{\nu_j}}))\Big) \exp(X)\Big) \\ = \lim_{s\to 0^+} \theta\Big(\Big(\prod_{j=1}^{k} \exp(n_j\pi i H'_{\nu_j}) \operatorname{Exp}(|s_j| H'_{\nu_j})\Big) \exp(X) \operatorname{Exp}(siZ')\Big). \quad (2.15)\end{aligned}$$

*Where* $Z'$ *is an element in* $\mathfrak{h}'_S$ *independent of* $X, n_1, \dots, n_k, s_1, \dots, s_k$ *but such that the right hand side of eq.* (2.15) *is contained in* $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ *for* $s > 0$ *sufficiently small.*

For $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R}), \mathfrak{su}(p,q), \mathfrak{so}^*(2n)$ the Ol'shanskiĭ semigroup can be realized as a subsemigroup of the contractive operators with respect to some non-degenerate indefinite Hermitian form. As done in [Ols95] we use this to show that for $s$ an element in the Ol'shanskiĭ semigroup we can split the eigenvalues of $s$ into two sets, those with norm strictly greater than 1 and those with norm strictly less than 1. This allows us to define holomorphic functions on the interior of the semigroup in terms of eigenvalues in one of these sets and we can express the weights and the roots on the fundamental Cartan subgroup in terms of these eigenvalues. In this way we can recognize the character as a function of eigenvalues on the interior of the semigroup and use this to show that it extends to the regular elements of $\mathrm{G}$. Thus we show that the Harish-Chandra character of $\pi_{\mathrm{Min}}$ is uniformly given in the sense of proposition 2.8.9.

**Proposition 1.4.1** *For* $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R}), \mathfrak{su}(p,q), \mathfrak{so}^*(2n)$ *the Harish-Chandra character associated to the minimal holomorphic representation* $\pi_{\mathrm{Min}}$ *is uniformly given in the sense of proposition 2.8.9. Explicit formulas are given in theorems 3.4.11, 3.5.11 and 3.3.21.*

In the case of $\mathfrak{so}(2,n)$ we have not been able to show the continuous extension of the character formula, however based on the results in chapter 2 we propose an explicit character formula for the minimal holomorphic representation of $\mathfrak{so}(2,2n)$.

## 1.5 Structure of this thesis

In chapter 2 we introduce the theory and notation of Ol'shanskiĭ semigroups necessary to treat characters of holomorphic representations. Sections 2.1 to 2.5 are meant to quickly set up the theory. As such they do not contain any essentially new results but are mostly results from the literature quoted in the hope that they will ease the process for the reader. A few of the results are proven because the author does not know any places in the literature in which they occur in exactly the form we use them, but there should be no surprises to a reader familiar with the field. The result farthest from anything which the author has seen before is lemma 2.4.13 which is used to show that a product of $\mathfrak{sl}_2$ semigroup homomorphisms as in lemma 2.8.6 is holomorphic.

The rest of chapter 2 is concerned with applying this theory to our case. Most results in these sections are new as far as the author is aware. Particularly the approach of realizing a Cayley transform as a continuous curve along which conjugation invariant functions are constant seems new. This is roughly what we do in section 2.7 and section 2.8.1.

The calculation of a character of a representation with a pencil of K-types by summing Weyl characters in a geometric series as we do in section 2.6 is not new. This is a commonly used back-of-the-envelope type calculation. However using the Ol'shanskiĭ semigroup to get an absolutely convergent series and exploiting the character theory of Ol'shanskiĭ semigroups is as far as the author is aware new.

Chapter 3 treats the minimal holomorphic representations of the classical simple Lie algebras of Hermitian type on a case by case basis. Except for the case $\mathfrak{so}(2,n)$ they all follow the same pattern:

1. Introduce a realization of the group and Lie algebra. This can be found in any book on non-compact simple Lie groups.

2. Use this to get a description of the minimal non-trivial Ad-invariant proper convex cone. This can also be found along with much more detailed information about such cones in [Pan81].

3. Using this we can show that the Ol'shanskiĭ semigroup consists of linear operators which are contractive with respect to some Hermitian form. Hence they have no eigenvalues of norm 1. This allows us to recognize functions on $\mathrm{H}\,\mathrm{Exp}(i(\mathfrak{h} \cap C^o_{\mathrm{Min}}))$ invariant under the compact Weyl group as functions on the interior of the semigroup given in terms of eigenvalues. This is essentially what Ol'shanskiĭ does in [Ols95].

4. Given the explicit expression for the character in proposition 2.6.3 as expressed in terms of these eigenvalues we can see that the character actually extends continuously to the regular elements of the Lie group.

5. Then finally we can apply proposition 2.8.9 to get explicit formulas for the Harish-Chandra character on a maximal set of non-conjugate Cartan subgroups.

For $\mathfrak{so}(2,n)$ we are at the moment unable to show whether the character extends continuously to the group. Hence we can not complete the above procedure in this case. We do however assume that this is the case and use this to conjecture a character formula for the minimal holomorphic representation.

## 1.6 Deficiencies and outlook

- The most glaring deficiency of this thesis is the lack of a character formula for the minimal representations of $\mathfrak{so}(2,n)$. With such a formula we could claim to have found the characters of all minimal representations of the classical types. The theory in chapter 2 does cover $\mathfrak{so}(2,n)$ but as can be seen in section 3.6 and section 3.7 the methods that we use to show that the character function on the Ol'shanskiĭ semigroup extends continuously to the regular elements of the Lie group for $\mathfrak{su}(p,q), \mathfrak{sp}(n,\mathbb{R})$ and $\mathfrak{so}^*(2n)$ do not trivially apply in this case. This is possibly related to the fact that realizing the Hermitian symmetric space associated to $\mathrm{SO}(2,n)$ is also more complicated than the three other cases. As can be seen in section 3.6.4 the major difficulty seems to be that the weight associated with the $\mathfrak{so}(2)$ component of the compact Cartan algebra enters the character formula in a significantly different way than the other weight basis vectors. So one way to prove the continuous extension property might be to prove that the eigenvalue associated with $\mathrm{SO}(2)$ globalizes to a holomorphic function on the open Ol'shanskiĭ semigroup.

- The next natural step to consider would be the exceptional simple Lie algebras of Hermitian type. In this case one should notice that in lemma 2.8.4 we have not considered these cases, but if one could show this lemma, then the rest of section 2.8.1 applies. However it seems natural to expect that the question of the continuous extension of the character on the Ol'shanskiĭ semigroup only becomes more complicated in the exceptional cases.

- Assuming that it is possible to prove something like lemma 2.8.4 in the exceptional cases, possibly a modified version like lemma 2.8.5. Then it would be interesting to see if these lemmas could be formulated and proved in a uniform way without resorting to case by case analysis in such a way that it is still possible to prove a version of proposition 2.8.9.

- Overall the question of which characters on the Ol'shanskiĭ semigroups extend continuously to the regular elements of the Lie group seems to be an interesting question. From our perspective it seems like it might easily be a very complicated question since we seem to be asking for a characterization of which holomorphic functions on a wedge that have a continuous extension to specific open dense subset of the boundary. However we do seem to be in a more advantageous position since we know that the character on the Lie group is a distributional limit of the character on the Ol'shanskiĭ semigroup, furthermore we know that the character on the Lie group is a real analytic function on the regular elements. Considering this setup it shares certain similarities with the reflection principle of the Edge of the Wedge theorem of holomorphic function theory in flat space.

- In proposition 2.8.9 we show that if the character on the Ol'shanskiĭ semigroup extends continuously to the regular elements of G then it is uniformly given in the same sense as the character formula in [Hec76]. However there are a lot of subtleties related to connected components of the regular elements of Cartan subgroups, especially when G is not complexifiable. It is not clear when the formulas in proposition 2.8.9 are surjective on the Cartan subgroups. It would be interesting to try and extend the proposition such that the surjectivity is guaranteed.

## 1.7 Notation

We try as much as possible to adhere to standard notation, but since notation tends to differ slightly from author to author we have tried to include a comprehensive list of notation below. For most of the notation we also try to introduce it during the main text.

| | |
|---|---|
| $\lvert(\alpha_1,\dots,\alpha_n)\rvert$ | The degree of the tuple: $\sum_{i=1}^n \alpha_i$ for $\alpha \in \mathbb{N}^n$. |
| $B_r(c)$ | The open ball of radius $r$ with center $c$: $\{v \mid \lVert v-c\rVert < r\}$. |
| $C_{\mathrm{Min}}$ | The minimal closed cone in $\mathfrak{g}$. |
| $c_m$ | The intersection of $C_{\mathrm{Min}}$ and $\mathfrak{h}$. |
| $\delta$ | Half the sum of the positive roots: $\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. |
| $\delta_k$ | Half the sum of the positive compact roots: $\frac{1}{2}\sum_{\alpha\in\Delta_k^+}\alpha$. |
| $\Delta(\mathfrak{g},\mathfrak{h})$ | The roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$. |
| $\epsilon(w)$ | The sign of the Weyl group element $w$: $(-1)^{l(w)}$. |
| $l(w)$ | Length of the Weyl group element $w$. |
| $\mathbf{e}_j$ | The $j$'th standard weight vector. |
| $\vec{e}_j$ | The $j$'th standard basis vector. |
| $\mathrm{G}'$ | The regular elements of G. |
| $N_{\mathrm{G}}(X)$ | The normalizer in G of the set $X$: $\{g \in \mathrm{G} \mid g.X \subset X\}$ |
| $\mathbb{N}$ | $= \{0, 1, \dots\}$ |
| $Z_{\mathrm{G}}(X)$ | The centralizer of $X$ in G: $\{g \in \mathrm{G} \mid \forall x \in X : g.x = x\}$. |
| $W(\mathrm{G},\mathfrak{h})$ | The analytic Weyl group of $\mathfrak{h}$ in G: $N_G(\mathfrak{h})/Z_G(\mathfrak{h})$. |
| $W(\Delta)$ | The algebraic Weyl group associated to the root system $\Delta$. |
| $W$ | $W(\Delta)$. |
| $W_{\mathrm{K}}$ | $W(\mathrm{K},\mathfrak{h})$. |
| $X^o$ | The interior of $X$. |
| $\overline{X}$ | The closure of $X$. |

# 2 Characters on Ol'shanskiĭ semigroups

In the following chapter we will introduce some theory for Ol'shanskiĭ semigroups. Then we use it to calculate the characters of minimal representations on the Ol'shanskiĭ semigroups and show that if they extend continuously to the regular elements of the Lie groups then the Harish-Chandra characters are uniformly given on all Cartan subgroups. The central results are proposition 2.6.3 and proposition 2.8.9. Since we want to extend our representations to Ol'shanskiĭ semigroups we restrict ourselves to simple Lie groups of Hermitian type and their highest-weight representations.

The idea of calculating Harish-Chandra characters by extending the representations to the corresponding Ol'shanskiĭ semigroups can be seen as a natural extension of the ideas in [Hec76]. In [Hec76] Hecht calculates the Harish-Chandra characters of the analytic continuation of the holomorphic discrete series representations by extending the representations to $\mathrm{G} \times \mathbb{R}_+ iZ$ where $Z \in Z(\mathfrak{k})$. Then he shows that for $0 < t < 1$ the extended representation $\pi_t(g)$ is a trace class operator and the Harish-Chandra character of the original representation $\pi$ is the distributional limit of $\operatorname{tr} \pi_t$ for $t \to 1^-$. Then he shows $\operatorname{tr} \pi_t$ converges uniformly on compact sets for $t \to 1^-$ hence the Harish-Chandra character is the pointwise limit. He shows that the trace of $\pi_t(g)$ can be computed by computing the fixed points of the action of $g \exp(tiZ)$ on $\mathrm{G}_\mathbb{C}/\mathrm{K}_\mathbb{C} P^+$ and thus he can calculate the Harish-Chandra character directly and show that it is uniformly given on all Cartan subgroups. That is Hecht calculates the Harish-Chandra character of the entire holomorphically induced module, whereas we are interested in minimal representations which are irreducible submodules of some holomorphically induced module.

In this chapter we will exploit that by the semigroup theory developed in the 90s we know that we can extend highest-weight representations to the entire semigroup. We furthermore know that on the interior of the semigroup the image of the representation is contained in the trace class operators and $\operatorname{tr} \pi$ is a holomorphic function on the interior of the semigroup. We exploit this information to construct curves in $\Gamma_\mathrm{G}(C^o_{\mathrm{Min}})$ between a fundamental Cartan subgroup of G and Cayley transforms of the fundamental Cartan subgroup along which $\operatorname{tr} \pi$ is constant. Thus indicating that if we know the character on one Cartan subgroup(actually on the "Cartan semigroup" in the Ol'shanskiĭ semigroup) then we know it on all other Cartan subgroups. This is roughly the contents of proposition 2.8.9.

## 2.1 Setup

Since non-trivial Ad(G)-invariant proper convex cones in simple Lie algebras exist if and only if the Lie algebra is of Hermitian type we restrict ourselves to Lie groups of Hermitian type.

We use G, K, H to denote Lie groups. Usually G denotes a connected non-compact real simple Lie group of Hermitian type, K will denote a maximal compact subgroup of G and H a Cartan subgroup of K. The subscript $_\mathbb{C}$ denotes the complexification of a vector space thus also a Lie algebra. Often we will consider groups G such that they have a complexification and then we will usually denotes this complexification by $\mathrm{G}_\mathbb{C}$. We will also use fraktur script to denote Lie algebras so $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ and $\mathfrak{g}_\mathbb{C}$ its complexification.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote a Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Then since G is of Hermitian type we know that H is also a Cartan subgroup of G[Kna01, page 504].

If $\alpha \in \mathfrak{h}_\mathbb{C}^*$ then we let $\mathfrak{g}_\alpha$ denote the root space of $\alpha$ in $\mathfrak{g}_\mathbb{C}$, that is

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}_\mathbb{C} \,|\, \forall H \in \mathfrak{h}_\mathbb{C} : \mathrm{ad}(H)(X) = \alpha(H)X\}.$$

Then we let $\Delta(\mathfrak{g}, \mathfrak{h})$ denote the roots

$$\Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}_\mathbb{C}^* \setminus \{0\} \,|\, \mathfrak{g}_\alpha \neq 0\}.$$

Then $\mathfrak{g}_0 = \mathfrak{h}_\mathbb{C}$ and we get the decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha.$$

Usually the Lie algebra $\mathfrak{g}$ and the Cartan subalgebra $\mathfrak{h}$ will be clear from context and we will just write $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ to denote the set of roots.

Since $\mathfrak{h}$ is invariant under the Cartan involution(it is a subset of $\mathfrak{k}$), a root space is either contained in $\mathfrak{k}_\mathbb{C}$ or $\mathfrak{p}_\mathbb{C}$. We call the roots $\alpha$ such that $\mathfrak{g}_\alpha \subset \mathfrak{k}_\mathbb{C}$ compact and the ones such that $\mathfrak{g}_\alpha \subset \mathfrak{p}_\mathbb{C}$ non-compact. We denote the set of compact roots $\Delta_k$ and the non-compact roots $\Delta_n$

$$\Delta_k := \{\alpha \in \Delta \,|\, \mathfrak{g}_\alpha \subset \mathfrak{k}_\mathbb{C}\} \qquad \Delta_n := \{\alpha \in \Delta \,|\, \mathfrak{g}_\alpha \subset \mathfrak{p}_\mathbb{C}\}.$$

and we get decompositions:

$$\mathfrak{k}_\mathbb{C} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_k} \mathfrak{g}_\alpha \qquad \mathfrak{p}_\mathbb{C} = \bigoplus_{\alpha \in \Delta_n} \mathfrak{g}_\alpha$$

Since $\mathfrak{g}$ is of Hermitian we fix a good ordering. Thus we get a positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ such that every compact root is smaller than every positive non-compact root. We let $\Delta_k^+$ and $\Delta_n^+$ denote the positive compact and non-compact roots respectively. We set

$$\delta = \sum_{\alpha \in \Delta^+} \alpha \qquad \delta_k = \sum_{\alpha \in \Delta_k^+} \alpha.$$

For a root $\alpha \in \Delta$ we let $H_\alpha \in \mathfrak{h}_\mathbb{C}$ denote the element such that $\alpha(H) = B(H_\alpha, H)$ where $B(,)$ is the Killing form. We let $W(\Delta)$ denote the algebraic Weyl group of the root system $\Delta$. Most of the time the root system will be clear from context and we will just denote the algebraic Weyl group with $W$. Similarly we let $W(\mathrm{G}, \mathfrak{b})$ denote the analytic Weyl group associated to the Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ that is

$$W(\mathrm{G}, \mathfrak{b}) = N_{\mathrm{G}}(\mathfrak{b})/Z_{\mathrm{G}}(\mathfrak{b}).$$

We will only be working with Cartan subalgebras which are stable under the Cartan involution hence

$$W(\mathrm{G}, \mathfrak{b}) = N_{\mathrm{K}}(\mathfrak{b})/Z_{\mathrm{K}}(\mathfrak{b}).$$

For $\mathfrak{h} \subset \mathfrak{k}$ it follows that

$$W(\mathrm{G}, \mathfrak{h}) = W(\mathrm{K}, \mathfrak{h}) = W(\Delta_k)$$

thus we denote this Weyl group $W_{\mathrm{K}}$ and also call it the compact Weyl group.

## 2.2 Complex Lie groups

The way we usually work with Ol'shanskiĭ semigroups is by first considering G such that G has simply-connected complexification $\mathrm{G}_{\mathbb{C}}$. Then the closed respectively open Ol'shanskiĭ semigroups are closed respectively open subsemigroups of $\mathrm{G}_{\mathbb{C}}$. Thus we introduce a few technical results which will be useful when working with Ol'shanskiĭ semigroups.

**Definition 2.2.1** Let G be a Lie group. We say that G has a complexification if there exists a complex Lie group $\mathrm{G}_{\mathbb{C}}$ such that $\mathrm{Lie}(\mathrm{G}_{\mathbb{C}}) = \mathrm{Lie}(\mathrm{G})_{\mathbb{C}}$ and G is a closed subgroup of $\mathrm{G}_{\mathbb{C}}$.

We first note that connected complex groups are complexifications of the analytic subgroups associated to real forms of their Lie algebra.

**Lemma 2.2.2** *Let $\mathrm{G}_{\mathbb{C}}$ be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and assume that $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ is a real subalgebra such that $\dim_{\mathbb{R}} \mathfrak{g} = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g} + i\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$. Then the analytic subgroup associated to $\mathfrak{g}$ in $\mathrm{G}_{\mathbb{C}}$ is a closed subgroup of $\mathrm{G}_{\mathbb{C}}$.*

**Proof.** This follows from [Kna02, corollary 7.6 and proposition 7.9]. □

The following coordinates will be very useful later as coordinates on Ol'shanskiĭ semigroups.

**Proposition 2.2.3** *Let G be a Lie group with complexification $\mathrm{G}_{\mathbb{C}}$.*

*There exists $W \subset \mathfrak{g}$ such that $W$ is an open convex neighborhood of $0$ and $\Phi : W \times W \to \mathrm{G}_{\mathbb{C}}$, $\Phi(v, w) = \exp(v)\exp(iw)$ is a diffeomorphism onto an open neighborhood of the identity. Furthermore $\Phi^{-1}(\mathrm{G}) = W \times \{0\}$.*

**Proof.** Since G is a Lie subgroup and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ it follows from [Hel01, lemma 2.5] that by taking $W$ to be a small enough open neighborhood of 0 in $\mathfrak{g}$, we can ensure that $\exp : W \oplus iW \to \mathrm{G}_{\mathbb{C}}$ is a diffeomorphism onto an open neighborhood $V \subset \mathrm{G}_{\mathbb{C}}$ of the identity and $\exp(W) = \exp(W \oplus iW) \cap \mathrm{G}$. Then it follows from [Hel01, lemma 2.4] that by taking a smaller $W$ we can also make sure that $\Phi$ is a diffeomorphism onto an open neighborhood of the identity. By taking $W$ possibly even smaller we can assume that $\Phi(W, W) \subset V$.

Assume that there exists $X, Y \in W$ such that $\exp(X)\exp(iY) \in \mathrm{G}$ then $\exp(X)\exp(iY) = \exp(Z)$ for $Z \in \mathfrak{g}$ but then $\exp(iY) = \exp(-X)\exp(Z) \in \mathrm{G}$ but we choose $W$ such that $\exp|_{W \oplus iW}$ is a diffeomorphism onto its image and $\exp(iW) \cap \mathrm{G} = \{e\}$. Hence we get that $Y = 0$.

Now take $W' \subset W$ such that $W'$ is a convex open neighborhood of 0 in $\mathfrak{g}$. $\square$

## 2.3 $\mathrm{Ad}(G)$-invariant cones

The basis of the study of Ol'shanskiĭ semigroups is the realization that certain simple Lie algebras contain non-trivial Ad(G)-invariant proper convex cones. It was shown independently by Kostant[Seg76] and Vinberg[Vin80] that $\mathfrak{g}$ contains such a cone if and only if $\mathfrak{g}$ is of Hermitian type. Furthermore Vinberg and Paneitz[Pan81; Pan83] studied the structure of such cones, they showed that there exists unique minimal and maximal cones. We will not give an extensive introduction to the subject but will formulate some results that we will use repeatedly in the thesis.

**Proposition 2.3.1** *Let* $\mathrm{G}$ *be a connected simple Lie group with* $\mathrm{K} \subset \mathrm{G}$ *a maximal compact subgroup. Assume* $C \subset \mathfrak{g}$ *is a non-trivial, closed,* $\mathrm{Ad}(\mathrm{G})$*-invariant, proper convex cone.*

*Then* $\mathrm{G}$ *is of Hermitian type and there exists non-zero* $Z \in Z(\mathfrak{k})$ *such that* $Z \in C$.

**Proof.** [Vin80, Theorem 1] $\square$

**Proposition 2.3.2** *Let* $\mathrm{G}$ *be a simple connected Lie group of Hermitian type, let* $\mathrm{K} \subset \mathrm{G}$ *denote a maximal compact subgroup and let* $\mathrm{H} \subset \mathrm{K}$ *be a maximal torus, hence a Cartan subgroup of* $\mathrm{G}$.

1. *There exists a unique(up to multiplication by* $-1$*) minimal non-trivial, closed, pointed,* $\mathrm{Ad}(\mathrm{G})$*-invariant, proper convex cone* $C_{\mathrm{Min}} \subset \mathfrak{g}$. *Let* $Z$ *be a non-zero element of the center of* $\mathfrak{k}$ *such that* $\alpha(iZ) > 0$ *for all* $\alpha \in \Delta_n^+$ *then*
$$C_{\mathrm{Min}} := \overline{\mathrm{Cone}(\mathrm{Ad}(\mathrm{G})Z)} \tag{2.1}$$
*and it is minimal in the sense that for any other closed,* $\mathrm{Ad}(\mathrm{G})$*-invariant, proper convex cone* $C$ *either* $C_{\mathrm{Min}} \subset C$ *or* $-C_{\mathrm{Min}} \subset C$.

2. *Recall* $\Delta^+$ *comes from a good ordering thus*
$$c_m := C_{\mathrm{Min}} \cap \mathfrak{h} = \{\sum_{\alpha \in \Delta_n^+} (-i) t_\alpha H_\alpha \,|\, \forall \alpha \in \Delta_n^+ : t_\alpha \geq 0\}. \tag{2.2}$$

3. *There exists a unique(up to multiplication by* $-1$*) maximal, non-trivial, closed,* $\mathrm{Ad}(\mathrm{G})$*-invariant, proper convex cone* $C_{\mathrm{Max}} \subset \mathfrak{g}$ *such that any other closed,* $\mathrm{Ad}(\mathrm{G})$*-invariant, proper convex cone* $C$ *satisfies* $C \subset C_{\mathrm{Max}}$ *or* $C \subset -C_{\mathrm{Max}}$.

4. *There is a bijective correspondence between open(closed),* $\mathrm{Ad}(\mathrm{G})$*-invariant, proper convex cones and open(closed),* $W_{\mathrm{K}}$*-invariant, proper convex cones in* $\mathfrak{h}$ *given by*
$$C \mapsto C \cap \mathfrak{h}.$$
*Moreover any orbit in* $C_{\mathrm{Max}}$ *intersects* $\mathfrak{h}$*.*

5. *Let* $C \subset \mathfrak{g}$ *be an* $\mathrm{Ad}(\mathrm{G})$*-invariant convex cone then*
$$C^o \cap \mathfrak{h} = (C \cap \mathfrak{h})^o. \tag{2.3}$$

**Proof.** This is in large parts the contents of [Pan83]. □

**Corollary 2.3.3** *Any* $X \in C_{\mathrm{Max}}$ *is* weakly elliptic *i.e. all eigenvalues of* $\mathrm{ad}(X)$ *are purely imaginary.*

**Proof.** There exists $g \in \mathrm{G}$ such that $\mathrm{Ad}(g)X \in \mathfrak{h}$ then the spectrum of $\mathrm{ad}(\mathrm{Ad}(g)X))$ is purely imaginary but $\mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g) \circ \mathrm{ad}(X) \circ \mathrm{Ad}(g)^{-1}$. □

## 2.4 Ol'shanskiĭ semigroups

We do not define and introduce Ol'shanskiĭ semigroups in the greatest generality possible, rather we are only concerned with the case where G is a simple Lie group of Hermitian type. The first step in the construction of Ol'shanskiĭ semigroups is Lawsons theorem. We include a special case:

**Theorem 2.4.1 (Lawson's theorem)** *Let* G *be a connected simple Lie group of Hermitian type with a complexification* $\mathrm{G}_{\mathbb{C}}$*. Let* $C \subset \mathfrak{g}$ *be a non-trivial, closed,* $\mathrm{Ad}(\mathrm{G})$*-invariant, proper convex cone. Then* $\Gamma_{\mathrm{G}}(C) := \mathrm{G}\exp(iC)$ *is a closed subsemigroup of* $\mathrm{G}_{\mathbb{C}}$ *such that*
$$\mathrm{G} \times C \to \Gamma_{\mathrm{G}}(C) \qquad\qquad (g, X) \mapsto g\exp(iX)$$
*is a homeomorphism.*

**Proof.** [Nee00, Theorem XI.1.10]. □

Then $\Gamma_{\mathrm{G}}(C)$ is a closed Ol'shanskiĭ semigroup associated to G. As can be seen from theorem 2.4.1 $\Gamma_{\mathrm{G}}(C)$ contracts onto G since $C$ contracts to 0. Then it can be shown that Lie semigroups has a covering theory very similar to Lie groups. Much information about coverings of Lie semigroups can be found in [HN93, section 3.4] and [Nee92].

**Proposition 2.4.2** *Let* G *and* $C$ *be as in theorem 2.4.1. Let* $\tilde{\Gamma}$ *be the universal covering space of* $\Gamma_{\mathrm{G}}(C)$*.*

*Then* $\tilde{\Gamma}$ *has a semigroup structure such that the covering map is a semigroup homomorphism and the universal covering group* $\tilde{\mathrm{G}}$ *of* G *is a subgroup of* $\tilde{\Gamma}$ *such that*
$$\tilde{\mathrm{G}} \times C \to \tilde{\Gamma} \qquad\qquad (g, X) \mapsto g\,\mathrm{Exp}(iX)$$
*is a homeomorphism. Here* Exp *is the lift of* $\exp|_{iC}$ *to* $\tilde{\Gamma}$ *such that* $\mathrm{Exp}(0)$ *is the identity element.*

**Proof.** [HN93, chapter 3]. □

Let $\mathrm{A} \subset \tilde{\mathrm{G}}$ be a discrete central subgroup of $\tilde{\mathrm{G}}$. Then any connected Lie group with Lie algebra $\mathfrak{g}$ can be realized as $\tilde{\mathrm{G}}/\mathrm{A}$ for some such A. We construct the associated Ol'shanskiĭ semigroup similarly.

**Proposition 2.4.3** *Let* G*,* $C$ *and* $\tilde{\Gamma}$ *be as in the previous proposition. Let* $\mathrm{A} \subset \tilde{\mathrm{G}}$ *be a discrete central subgroup.*

*Then* A *acts properly on* $\tilde{\Gamma}$ *and* $S := \tilde{\Gamma}/\mathrm{A}$ *is a Lie semigroup such that the quotient mapping* $q : \tilde{\Gamma} \to S$ *is a covering homomorphism and*

$$(\tilde{\mathrm{G}}/\mathrm{A}) \times C \to S \qquad\qquad (g, X) \mapsto gq(\mathrm{Exp}(iX))$$

*is a homeomorphism.*

**Proof.** [HN93, Theorem 3.20]. □

Now we are finally ready to define a closed Ol'shanskiĭ semigroup.

**Definition 2.4.4** Let G be a connected simple Lie group of Hermitian type and $C \subset \mathfrak{g}$ a closed, non-trivial, Ad(G)-invariant, proper convex cone. Let $\tilde{\mathrm{G}}$ be the universal covering group of G and let $\mathrm{A} \subset \tilde{\mathrm{G}}$ be a discrete central subgroup such that $\mathrm{G} = \tilde{\mathrm{G}}/\mathrm{A}$. Then we denote the quotient $\tilde{\Gamma}/A$ from proposition 2.4.3 by $\Gamma_{\mathrm{G}}(C)$ and call it the *closed complex Ol'shanskiĭ semigroup* associated to G and $C$.

**Remark 2.4.5** We will usually let $\mathrm{Exp} : C \to \Gamma_{\mathrm{G}}(C)$ denote the composition $q \circ \widetilde{\mathrm{Exp}}$ where $\widetilde{\mathrm{Exp}}$ is the exponential function on $\tilde{\Gamma} = \Gamma_{\tilde{\mathrm{G}}}(C)$.

Most often we will however not work on the entire closed semigroup but only on the interior of the semigroup.

**Proposition 2.4.6** *Let* $\Gamma_{\mathrm{G}}(C)$ *be a closed complex Ol'shanskiĭ semigroup. Then* $\mathrm{G}\,\mathrm{Exp}(iC^o)$ *is an open dense semigroup ideal and it has a complex manifold structure such that multiplication is holomorphic.*

*When* G *has a complexification, then* $\mathrm{G}\,\mathrm{Exp}(iC^o)$ *is the interior of* $\mathrm{G}\,\mathrm{Exp}(iC)$ *as a subset of* $\mathrm{G}_{\mathbb{C}}$*.*

**Proof.** [Nee00, Theorem XI.1.12]. □

**Definition 2.4.7** Let G be a connected simple Lie group of Hermitian type and let $C \subset \mathfrak{g}$ be an open, non-trivial, Ad(G)-invariant, proper convex cone. Then we write $\Gamma_{\mathrm{G}}(C) := \mathrm{G}\,\mathrm{Exp}(iC)$ for the open semigroup ideal in $\Gamma_{\mathrm{G}}(\overline{C})$. We call $\Gamma_{\mathrm{G}}(C)$ the *open complex Ol'shanskiĭ semigroup* associated to G and $C$.

**Remark 2.4.8** We will generally abuse notation and for H a subgroup of G and $W \subset C$ let $\Gamma_{\mathrm{H}}(W) = \mathrm{H}\,\mathrm{Exp}(iW)$. We will mostly use this for H a Cartan subgroup of K and $W = C \cap \mathfrak{h}$ or for $W = C \cup \{0\}$.

**Lemma 2.4.9** *Let $\Gamma_{\mathrm{G}}(C)$ be a closed complex Ol'shanskiĭ semigroup. Let $g \in \mathrm{G}$ and $X \in C$ then*

$$\operatorname{Exp}(i \operatorname{Ad}(g)X) = g \operatorname{Exp}(iX)g^{-1}.$$

**Proof.** Let $\hat{\mathrm{G}}$ be such that $\operatorname{Lie}\hat{\mathrm{G}} = \operatorname{Lie}\mathrm{G}$ and $\hat{\mathrm{G}}$ has a simply-connected complexification $\hat{\mathrm{G}}_{\mathbb{C}}$. Then the lemma is clear since $\operatorname{Exp} = \exp|_{\mathfrak{g}\oplus iC}$. Let $\Gamma_{\tilde{\mathrm{G}}}(C)$ be the simply-connected covering semigroup of $\Gamma_{\mathrm{G}}(C)$ and let $\alpha(t) = g\operatorname{Exp}(itX)g^{-1}$ and $\beta(t) = \operatorname{Exp}(it\operatorname{Ad}(g)X)$ for $t \in [0,1]$. If $p : \Gamma_{\tilde{\mathrm{G}}}(C) \to \Gamma_{\hat{\mathrm{G}}}(C)$ is the covering homomorphism then $p\circ\alpha = p\circ\beta$ since $\operatorname{Ad}(p(g)) = \operatorname{Ad}(g)$ since $p$ restricts to a covering homomorphism of the groups $\tilde{\mathrm{G}} \to \hat{\mathrm{G}}$. But $\beta(0) = e = \alpha(0)$ thus $\alpha$ and $\beta$ are lifts of the same curve with the same initial point hence they are equal and thus $\alpha(1) = \beta(1)$.

In the general case we let $q : \Gamma_{\tilde{\mathrm{G}}}(C) \to \Gamma_{\mathrm{G}}(C)$ denote the covering homomorphism. Let $g \in \mathrm{G}$ and take $\tilde{g} \in q^{-1}(g)$ then

$$\begin{aligned} g\operatorname{Exp}(iX)g^{-1} = q(\tilde{g})\operatorname{Exp}(iX)q(\tilde{g})^{-1} &= q(\tilde{g}\widetilde{\operatorname{Exp}}(iX)\tilde{g}^{-1}) \\ &= q(\widetilde{\operatorname{Exp}}(i\operatorname{Ad}(\tilde{g})X)) = \operatorname{Exp}(i\operatorname{Ad}(\tilde{g})X). \end{aligned}$$

But again $q$ restricts to a group covering homomorphism and hence $\operatorname{Ad}(\tilde{g}) = \operatorname{Ad}(g)$. □

**Lemma 2.4.10** *Let $\Gamma_{\mathrm{G}}(C)$ be an open complex Ol'shanskiĭ semigroup. Let $\gamma \in \Gamma_{\mathrm{G}}(C)$ then there exists a small open ball $V$ with center $0$ in $\mathfrak{g}_{\mathbb{C}}$ such that*

$$f_\gamma : V \cap iC \to \Gamma_{\mathrm{G}}(C) \qquad\qquad f_\gamma(X) = \gamma\operatorname{Exp}(X)$$

*extends to $V$ and these maps give a holomorphic atlas of $\Gamma_{\mathrm{G}}(C)$.*

**Proof.** Note first of all that by analytic continuation since we know Exp on $iC$ which is an open subset of $i\mathfrak{g}$ if such an extension exists it is unique.

Assume that G has a simply-connected complexification $\mathrm{G}_{\mathbb{C}}$ then $\Gamma_{\mathrm{G}}(C)$ is an open subset of $\mathrm{G}_{\mathbb{C}}$ and the complex structure on the semigroup is the one inherited from $\mathrm{G}_{\mathbb{C}}$. In this case Exp = exp and by taking $V$ small enough we can make sure that exp is a biholomorphism. The set of maps $X \mapsto \gamma\exp(X)$ constitute a holomorphic atlas of $\mathrm{G}_{\mathbb{C}}$ thus also of $\Gamma_{\mathrm{G}}(C)$.

If $\tilde{\mathrm{G}}$ is the universal covering group of G then Exp is the lift of exp. Let $p : \Gamma_{\tilde{\mathrm{G}}}(C) \to \Gamma_{\mathrm{G}}(C)$ be the covering map and let $\gamma' = p(\gamma)$. Then we use the $V \subset \mathfrak{g}_{\mathbb{C}}$ that we get from before for $\gamma'$ and consider the map $V \ni X \mapsto \gamma'\exp(X) \in \Gamma_{\mathrm{G}}(C)$. Since $V$ is simply-connected there exists a lift $f_\gamma$ of this map such that $f_\gamma(0) = \gamma$. Furthermore since $f_\gamma$ is a lift of a biholomorphism it must be a biholomorphism. Since $V \cap C$ is simply-connected and for $X \in C$ we have $p(f_\gamma(iX)) = \gamma'\exp(iX) = p(\gamma\operatorname{Exp}(iX))$ we must have $f_\gamma(iX) = \gamma\operatorname{Exp}(iX)$.

In the general case let $p : \Gamma_{\tilde{\mathrm{G}}}(C) \to \Gamma_{\mathrm{G}}(C)$ be a covering map and let $\tilde{\gamma}$ be such that $p(\tilde{\gamma}) = \gamma$. Then by the previous there exists $V$ and $f_{\tilde{\gamma}}$ such that $f_{\tilde{\gamma}}$ is a biholomorphism. Then we put $f_\gamma = p\circ f_{\tilde{\gamma}}$. Since $p$ is a covering map we know it is a local biholomorphism hence $f_\gamma$ must be a local biholomorphism. By making $V$ smaller we make sure that $f_\gamma$ is a biholomorphism. □

**Lemma 2.4.11** *Let $X_1, X_2$ be path-connected, locally path-connected, simply-connected topological spaces. Let for $j = 1, 2$: $p_j : \tilde{Z}_j \to Z_j$ be a covering space and $f_j : X_j \to Z$ a continuous map and $\tilde{f}_j : X_j \to \tilde{Z}_j$ a lift of $f_j$. Then $p_1 \times p_2$ is a covering map of $Z_1 \times Z_2$ and $\tilde{f}_1 \times \tilde{f}_2$ is a lift of $f_1 \times f_2$.*

**Lemma 2.4.12** *Let $\Gamma_{\mathrm{G}}(C), \Gamma_{\mathrm{H}}(D)$ be two closed complex Ol'shanskiĭ semigroups. Let $\tau : \mathrm{G} \to \mathrm{H}$ be a Lie group homomorphism such that $d\tau(C) \subset D$.*

*Then $\tau$ extends to a homomorphism of Ol'shanskiĭ semigroups $f$ such that*

$$f(g\,\mathrm{Exp}(iX)) = \tau(g)\,\mathrm{Exp}(id\tau(X)).$$

**Proof.** By Lawsons theorem if for $g \in \mathrm{G}$ and $X \in C$ we let $f$ be given by

$$f(g\,\mathrm{Exp}(iX)) = \tau(g)\,\mathrm{Exp}(id\tau(X))$$

then $f$ is a well-defined continuous map. Thus we want to show that $f$ is a homomorphism.

Let $\tilde{\mathrm{G}}$ be the connected simply-connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathrm{G}'$ be some Lie group with Lie algebra $\mathfrak{g}$ then we let $p_{\mathrm{G}'}$ denote a covering morphism $\tilde{\mathrm{G}} \to \mathrm{G}'$ and also for semigroups.

Then $f \circ p_{\mathrm{G}}$ is a continuous map from the simply-connected space $\Gamma_{\tilde{\mathrm{G}}}(C)$ to $\Gamma_{\mathrm{H}}(D)$ hence by lemma 2.4.11 it lifts to the continuous map

$$\tilde{f}(g\,\mathrm{Exp}(iX)) = \tilde{\tau}(g)\,\mathrm{Exp}(id\tau(X)).$$

Such that $\tilde{f}$ sends the identity to the identity and $\tilde{\tau}$ is the lift of $\tau \circ p_{\mathrm{G}}$ to $\tilde{\mathrm{H}}$ i.e. the unique group homomorphism between $\tilde{\mathrm{G}}$ and $\tilde{\mathrm{H}}$ such that $d\tilde{\tau} = d\tau$.

We want to show that $\tilde{f}$ is a homomorphism because then for $s_1, s_2 \in \Gamma_{\mathrm{G}}(C)$ if we have $\tilde{s}_j \in p_{\mathrm{G}}^{-1}(s_j)$ we get

$$\begin{aligned} f(s_1 s_2) &= f(p_{\mathrm{G}}(\tilde{s}_1) p_{\mathrm{G}}(\tilde{s}_2)) = f(p_{\mathrm{G}}(\tilde{s}_1 \tilde{s}_2)) = p_{\mathrm{H}}(\tilde{f}(\tilde{s}_1 \tilde{s}_2)) \\ &= p_{\mathrm{H}}(\tilde{f}(\tilde{s}_1)\tilde{f}(\tilde{s}_2)) = f(s_1) f(s_2). \end{aligned}$$

Let $\hat{\mathrm{G}}_{\mathbb{C}}$ and $\hat{\mathrm{H}}_{\mathbb{C}}$ be the simply-connected complex Lie groups with Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ respectively. Then $d\tau$ integrates to a complex Lie group homomorphism $\beta : \hat{\mathrm{G}}_{\mathbb{C}} \to \hat{\mathrm{H}}_{\mathbb{C}}$ and for $g \in \hat{\mathrm{G}}$ and $X \in C \subset \mathfrak{g}$ we get

$$\beta(g \exp(iX)) = \beta(g) \exp(id\tau(X)).$$

Where $\beta(g) \in \hat{\mathrm{H}}$ since $d\beta(\mathfrak{g}) = d\tau(\mathfrak{g}) \subset \mathfrak{h}$ thus we see that $\beta(\Gamma_{\hat{\mathrm{G}}}(C)) \subset \Gamma_{\hat{\mathrm{H}}}(D)$. Then $\beta \circ p_{\hat{\mathrm{G}}}$ is a continuous semigroup homomorphism from $\Gamma_{\tilde{\mathrm{G}}}(C)$ to $\Gamma_{\hat{\mathrm{H}}}(D)$. So $\beta \circ p_{\hat{\mathrm{G}}}$ lifts to a semigroup homomorphism $\tilde{\beta} : \Gamma_{\tilde{\mathrm{G}}}(C) \to \Gamma_{\tilde{\mathrm{H}}}(D)$. However $\tilde{\beta}$ restricts to a group homomorphism on $\tilde{\mathrm{G}}$ and $d\tilde{\beta}|_{\tilde{\mathrm{G}}} = d\tau$ hence $\tilde{\beta}|_{\tilde{\mathrm{G}}} = \tilde{\tau}$ and we get

$$\tilde{\beta}(g\,\mathrm{Exp}(iX)) = \tilde{\beta}(g)\tilde{\beta}(\mathrm{Exp}(iX)) = \tilde{\tau}(g)\,\mathrm{Exp}(id\tau X) = \tilde{f}(g\,\mathrm{Exp}(iX)).$$

Which shows that $\tilde{f}$ is a semigroup homomorphism hence that $f$ is a semigroup homomorphism. $\square$

**Lemma 2.4.13** *Continue the setup from lemma 2.4.12. If* $s \in \Gamma_{\mathrm{G}}(C^o)$ *then there exists an open ball* $V \subset \mathfrak{g}_{\mathbb{C}}$ *centered in* 0 *such that for* $X \in V$

$$f(s\,\mathrm{Exp}(X)) = f(s)\,\mathrm{Exp}(d\tau(X)).$$

*Where* Exp *on the left hand side is the extension from lemma 2.4.10 and* Exp *on the right-hand side is an extension to* $d\tau(V) \cup D$ *by following* $\exp|_{d\tau(V)\cup D}$ *through the construction of* $\Gamma_{\mathrm{H}}(D)$*.*

**Proof.** If G and H have simply-connected complexifications then the lemma follows from the fact that $f$ is a restriction of a complex Lie group homomorphism.

Now assume that G and H are simply connected and let $\hat{\mathrm{G}}$ and $\hat{\mathrm{H}}$ be quotients with simply-connected complexifications. Let $\hat{f}$ be the semigroup homomorphism between the semigroups associated to $\hat{\mathrm{G}}$ and $\hat{\mathrm{H}}$ and let us abuse notation a bit and let $p$ denote the covering map in all cases. Then for $X \in V$ we get

$$p(f(s\,\mathrm{Exp}(X))) = \hat{f}(p(s\,\mathrm{Exp}(X))) = \hat{f}(p(s)\exp(X)) = p(f(s))\exp(d\tau(X))$$

where the second equality follows from the construction of the extension of Exp in the proof of lemma 2.4.10. We thus see that $f(s\,\mathrm{Exp}(X))$ is a lift of the map $X \mapsto p(f(s))\exp(d\tau(X))$. Hence $f(s\,\mathrm{Exp}(X)) = f(s)\,\mathrm{Exp}(d\tau(X))$ where Exp is an appropriate extension of $X \mapsto f(s)\,\mathrm{Exp}(X)$ to $d\tau(V) \cup C$ since $d\tau(V) \cup D$ star-shaped around 0 thus simply-connected.

Now the general case follows by taking quotients of simply-connected case. □

**Lemma 2.4.14** *Assume that* G *is a simple Lie group of Hermitian type and that* G *has a complexification* $\mathrm{G}_{\mathbb{C}}$*. Let* $\Gamma_{\mathrm{G}}(C)$ *be an open complex Ol'shanskiĭ semigroup.*

*Then* $\Gamma_{\mathrm{G}}(C) \cap \mathrm{G}'_{\mathbb{C}}$ *is a connected open dense subset of* $\Gamma_{\mathrm{G}}(C)$*.*

**Proof.** Consider the function $D : \mathrm{G}_{\mathbb{C}} \times \mathbb{C} \to \mathbb{C}$ given by

$$D(x,\lambda) = \det_{\mathbb{C}}((\lambda+1)I - \mathrm{Ad}(x)) = \lambda^{\dim \mathrm{G}} + \sum_{j=0}^{\dim \mathrm{G}-1} D_j(x)\lambda^j.$$

Let $n$ denote the complex rank of $\mathrm{G}_{\mathbb{C}}$ then $D_n$ is a holomorphic function not identically zero on $\mathrm{G}_{\mathbb{C}}$. Then since $\Gamma_{\mathrm{G}}(C)$ is an open subset of $\mathrm{G}_{\mathbb{C}}$ the function $D_n|_{\Gamma_{\mathrm{G}}(C)}$ is a holomorphic non-zero function, and hence by [Kna02, lemma 7.96] the complement of the zero set is open, connected and dense. Now an element $x \in \mathrm{G}_{\mathbb{C}}$ is regular if and only if $D_n(x) \neq 0$. □

**Lemma 2.4.15** *Let* $\Gamma_{\mathrm{G}}(C)$ *be an open complex Ol'shanskiĭ semigroup such that* G *has a complexification* $\mathrm{G}_{\mathbb{C}}$*. Let* $f : \Gamma_{\mathrm{G}}(C) \to \mathbb{C}$ *be a continuous function such that* $f|_{\Gamma_{\mathrm{G}}(C)\cap \mathrm{G}'_{\mathbb{C}}}$ *is holomorphic.*

*Then* $f$ *is holomorphic.*

**Proof.** This follows from the Riemann removable singularities theorem in several variables [Kra01, theorem 7.3.3]. $\square$

**Lemma 2.4.16** *Let $\Gamma_{\mathrm{G}}(C)$ be an open complex Ol'shanskiĭ semigroup. Let $X \in C$ then there exists an open neighborhood $V$ of $iX$ in $\mathfrak{g}_{\mathbb{C}}$ such that* Exp *extends to $V$ and* Exp *is holomorphic on $V$ and regular at $iX$.*

**Proof.** Assume that $\hat{\mathrm{G}}$ has a simply-connected complexification $\hat{\mathrm{G}}_{\mathbb{C}}$. According to [Kna02, proposition 1.110] for a complex Lie group exp is a holomorphic mapping. Furthermore since $\Gamma_{\hat{\mathrm{G}}}(C)$ is an open subset of $\hat{\mathrm{G}}_{\mathbb{C}}$ and $\exp(iX) \in \Gamma_{\hat{\mathrm{G}}}(C)$ there exists an open convex neighborhood $V$ of $iX$ in $\mathfrak{g}_{\mathbb{C}}$ such that $\exp(V) \subset \Gamma_{\hat{\mathrm{G}}}(C)$. By corollary 2.3.3 $iC$ consists of weakly hyperbolic elements i.e. $\mathrm{ad}(iX)$ has real spectrum on $\mathfrak{g}_{\mathbb{C}}$. It then follows from [Hel01, theorem 1.7] that exp is regular at $iX$.

Let $\Gamma_{\tilde{\mathrm{G}}}(C)$ be the simply-connected open complex Ol'shanskiĭ semigroup and let $p : \Gamma_{\tilde{\mathrm{G}}}(C) \to \Gamma_{\hat{\mathrm{G}}}(C)$ be the covering morphism. Let $f = \exp|_V : V \to \Gamma_{\hat{\mathrm{G}}}(C)$ then since $V$ is simply-connected there exists a lift $\tilde{f}$ of $f$ to $\Gamma_{\tilde{\mathrm{G}}}(C)$ such that $\tilde{f}(iX) = \widetilde{\mathrm{Exp}}(iX)$. Since $C$ and $V$ are convex $iC \cap V$ is convex and thus simply-connected and hence $\tilde{f}|_{V\cap iC} = \widetilde{\mathrm{Exp}}|_{V\cap iC}$. Since $iC$ is an open subset of the real form $i\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ we get that $\tilde{f}$ is the unique holomorphic extension of $\widetilde{\mathrm{Exp}}$ to a neighborhood of $iX$. Then $\tilde{f}$ must also be regular at $iX$ since $\exp = p \circ \widetilde{\mathrm{Exp}}$ and $dp$ is an isomorphism.

Then $\Gamma_{\mathrm{G}}(C)$ is a quotient of $\Gamma_{\tilde{\mathrm{G}}}(C)$ such that $q : \Gamma_{\tilde{\mathrm{G}}}(C) \to \Gamma_{\mathrm{G}}(C)$ is a covering morphism. By definition $q \circ \widetilde{\mathrm{Exp}} = \mathrm{Exp}$ hence we must have that $q \circ \tilde{f}|_{V\cap iC} = \mathrm{Exp}\,|_{V\cap iC}$. Furthermore $q$ is a local biholomorphism thus $q \circ \tilde{f}$ is holomorphic. And since $dq$ is an isomorphism we get that $q \circ \tilde{f}$ is regular at $iX$. $\square$

**Lemma 2.4.17** *Let $\Gamma_{\mathrm{G}}(C)$ be an open complex Ol'shanskiĭ semigroup and let $U \subset \Gamma_{\mathrm{G}}(C)$ be an open connected subset. Let $f, g : U \to \mathbb{C}$ be holomorphic maps and assume that there exists an open subset $V \subset C$ such that $\mathrm{Exp}(iV) \cap U \neq \emptyset$ and*

$$f|_{\mathrm{Exp}(iV)\cap U} = g|_{\mathrm{Exp}(iV)\cap U}.$$

*Then $f = g$.*

**Proof.** Let $X \in V$ such that $\mathrm{Exp}(iX) \in U$. Let $W$ be an open connected neighborhood in $\mathfrak{g}_{\mathbb{C}}$ of $iX$ such that Exp extends to a holomorphic function on$W$ as in lemma 2.4.16. Then $V \cap W$ is a open subset of the real form $i\mathfrak{g} \cap W$ and $f \circ \mathrm{Exp}\,|_{iV\cap W} = g \circ \mathrm{Exp}\,|_{iV\cap W}$. Then analytic continuation implies that $f \circ \mathrm{Exp} = g \circ \mathrm{Exp}$ on $W$. But Exp is regular at $iX$ hence $\mathrm{Exp}(W)$ is a neighborhood of $\mathrm{Exp}(iX)$ and hence $f$ and $g$ agree on an open set, thus since $U$ is connected analytic continuation implies that $f = g$. $\square$

**Proposition 2.4.18** *Let $\Gamma_{\mathrm{G}}(C)$ be an open complex Ol'shanskiĭ semigroup. Let* H *be a Cartan subgroup of* K *a maximal compact subgroup of* G*. Let $V \subset \Gamma_{\mathrm{G}}(C)$ be an open connected subset such that $V \cap \mathrm{Exp}(i(\mathfrak{h} \cap C)) \neq \emptyset$. Let $f, g : V \to \mathbb{C}$ be* G*-conjugation* *invariant*[1] *holomorphic maps that satisfy*

$$\forall X \in \mathfrak{h} \cap C : f(\operatorname{Exp}(iX)) = g(\operatorname{Exp}(iX)).$$

*Then* $f = g$.

**Proof.** It follows from proposition 2.3.2 that $\mathfrak{h} \cap C$ is a non-trivial, open, proper convex cone in $\mathfrak{h}$. Let $Y \in \mathfrak{h} \cap C$ such that $\operatorname{Exp}(iY) \in V$ then since Exp is continuous $\operatorname{Exp}^{-1}(V)$ is an open neighborhood of $iY$ in $iC$ but then $W = \mathfrak{h} \cap (-i) \operatorname{Exp}^{-1}(V)$ must be a neighborhood of $Y$. Thus there exists $X \in W$ such that $X$ is regular since the regular elements are dense. Consider $\Phi : \mathrm{G} \times (\mathfrak{h} \cap C) \to C$ such that

$$\Phi(a, X) = \operatorname{Ad}(a)X.$$

Then we get $T\Phi_{(e,X)} : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g}$ given by

$$T\Phi_{(e,X)}(Y, Z) = [Y, X] + Z = -\operatorname{ad}(X)(Y) + Z. \tag{2.4}$$

Now consider $T\Phi_{(e,X)} : \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ where we extend $T\Phi_{(e,X)}$ complex linearly. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\gamma \in \Delta} (\mathfrak{g}_{\mathbb{C}})_\gamma$ be the root space decomposition of $\mathfrak{g}_{\mathbb{C}}$. Since $X$ is regular $\operatorname{ad}(X)$ is non-zero on every root space; therefore the image contains every root space, and it is clear from eq. (2.4) that the image contains the complexified Cartan subalgebra. Hence we conclude that the complexification of $T\Phi_{(e,X)}$ is surjective.

Thus we conclude that $\dim_{\mathbb{C}} \operatorname{Ker}_{\mathbb{C}} T\Phi_{(e,X)} = \dim_{\mathbb{C}} \mathfrak{h}_{\mathbb{C}}$ but the kernel of the complexification of $T\Phi_{(e,X)}$ is just $\operatorname{Ker}_{\mathbb{R}} T\Phi_{(e,X)} \oplus i \operatorname{Ker}_{\mathbb{R}} T\Phi_{(e,X)}$, hence $\dim_{\mathbb{R}} \operatorname{Ker}_{\mathbb{R}} T\Phi_{(e,X)} = \dim_{\mathbb{C}} \operatorname{Ker}_{\mathbb{C}} T\Phi_{(e,X)}$. Then it follows by dimensional considerations that $T\Phi(\mathfrak{g} \oplus \mathfrak{h}) = \mathfrak{g}$.

The image of $\Phi$ thus contains a neighborhood $N$ of $X$ in $C$. Furthermore since $\operatorname{Exp}(iX) \in V$ we can by shrinking $N$ assume that $\operatorname{Exp}(iN) \subset V$. Now

$$\begin{aligned} f(\operatorname{Exp}(i \operatorname{Ad}(a)X)) = f(a \operatorname{Exp}(iX)a^{-1}) &= f(\operatorname{Exp}(iX)) \\ &= g(\operatorname{Exp}(iX)) = g(\operatorname{Exp}(i \operatorname{Ad}(a)X)). \end{aligned}$$

Thus $f \circ \operatorname{Exp}$ and $g \circ \operatorname{Exp}$ agree on an open subset of $iC$. Then it follows from lemma 2.4.17 that $f$ and $g$ must agree. □

## 2.5 Representation theory of Ol'shanskiĭ semigroups

Let $\mathcal{H}$ be a Hilbert space then we give $B(\mathcal{H})$ the Banach space structure coming from the operator norm. Then if $M$ is a complex manifold it makes sense to talk of holomorphic functions between $M$ and $B(\mathcal{H})$ and by [Rud91, p. 3.31] this is the same as weakly holomorphic functions.

[1] Where by G-conjugation invariant we mean that if $h \in \mathrm{G}$, $\gamma \in V$ such that $h\gamma h^{-1} \in V$ then $f(h\gamma h^{-1}) = f(\gamma)$.

**Definition 2.5.1** Let $\Gamma_{\mathrm{G}}(C)$ be an open complex Ol'shanskiĭ semigroup. A representation $\pi$ of $\Gamma_{\mathrm{G}}(C)$ on a Hilbert space $\mathcal{H}$ is a holomorphic semigroup homomorphism $\pi : \Gamma_{\mathrm{G}}(C) \to B(\mathcal{H})$ such that for $g \in \mathrm{G}$ and $X \in C$

$$\pi(\mathrm{Exp}(iX)g^{-1}) = \pi(g\,\mathrm{Exp}(iX))^*.$$

Let $\tau$ be a unitary representation of G then we say that $\pi$ is an extension of $\tau$ if

$$\pi(g\,\mathrm{Exp}(iX)) = \tau(g)e^{id\tau(X)}$$

and $\pi(gh) = \tau(g)\pi(h)$ for all $g \in \mathrm{G}$ and $h \in \Gamma_{\mathrm{G}}(C)$.

Note that the terminology in definition 2.5.1 differs from the terminology in [Nee00]. What we call a representation of $\Gamma_{\mathrm{G}}(C)$ Neeb calls a holomorphic representation of a complex involutive semigroup. But since these are the representations we get from unitary highest-weight representations of simple Lie groups of Hermitian type and thus the only ones we are interested in we leave out some adjectives.

**Remark 2.5.2** Let G be a simple, connected Hermitian Lie group and let $\pi$ be a unitary, highest-weight or lowest-weight representation with respect to some positive system. Then $\pi$ is also a highest-weight representation with respect to a positive system coming from a good ordering, hence satisfying the conditions of proposition 2.5.3.

**Proof.** Since G is of Hermitian type it is admissible in the sense of [Nee96] and then this follows from [Nee96, corollary 4.13]. □

**Proposition 2.5.3** *Let* G *be a simple, connected Lie group of Hermitian type and let* $\pi$ *be a unitary, highest-weight representation with respect to the positive system* $\Delta^+$*. Then* $\pi$ *extends to the open Ol'shanskiĭ-semigroup* $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$*. Where* $C^o_{\mathrm{Min}}$ *is defined as in proposition 2.3.2.*

$\pi$ *is irreducible as a representation of* G *if and only if it is irreducible as a representation of* $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$*.*

**Proof.** This follows from [Nee00, theorem XI.4.5]. □

**Corollary 2.5.4** *Consider the setup from proposition 2.5.3 but assume that* $\pi$ *is a unitary lowest-weight representation with respect to* $\Delta^+$*. Then* $\pi$ *extends to the open Ol'shanskiĭ semigroup* $\Gamma_{\mathrm{G}}(-C^o_{\mathrm{Min}})$*.*

**Proof.** Since $\Delta^+$ comes from a good ordering, we can assume that they are associated to an ordered basis $Z, \vec{e}_1, \ldots, \vec{e}_{n-1} \in \mathfrak{h}$ where $Z \in Z(\mathfrak{k})$. We consider the ordering coming from $-Z, \vec{e}_1, \ldots, \vec{e}_{n-1}$ and denote the associated positive system by $\Phi \subset \Delta$. This corresponds to making all non-compact positive roots into negative roots, and vice versa all negative non-compact roots become positive. But the ordering of the compact roots stay as they are. Thus $\pi$ is a highest-weight representation with respect to $\Phi$ but $C^o_{\mathrm{Min},\Phi} = -C^o_{\mathrm{Min},\Delta^+}$. □

**Proposition 2.5.5** *Let $\pi$ be a unitary irreducible highest-weight representation of a simple connected Lie group* G *of Hermitian type.*

*Then $\pi(\gamma)$ is trace class for all $\gamma \in \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ and $\operatorname{tr} \circ \pi$ is a holomorphic map on the open semigroup.*

*Let $s_n \in \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ be a sequence such that $\lim_{n\to\infty} s_n = 1$ and $|\pi(s_n)|$ is bounded. Let $\theta_n(g) = \operatorname{tr} \pi(g s_n)$ for $g \in \mathrm{G}$. Then $\theta_n$ converges to the Harish-Chandra character of $\pi$ in the sense of distributions.*

**Proof.** [Nee00, Theorem XI.6.1] and [Nee00, proposition XI.6.7]. □

## 2.6 Pencil of K-types

**Definition 2.6.1** Let $\pi$ be a unitary representation of G. Take the decomposition into K-types

$$\pi|_{\mathrm{K}} = \hat{\oplus}_{\tau \in \hat{\mathrm{K}}} n_\tau \tau.$$

Where we have identified $\hat{\mathrm{K}}$ with the dominant, analytically integral weights in $\mathfrak{h}^*_{\mathbb{C}}$.

If there exists weights $\alpha, \beta$ such that $n_\tau = 0$ for all $\tau \notin \alpha + \mathbb{N}\beta$ and $n_{\alpha + k\beta} = 1$ for all $k \in \mathbb{N}$ then we say that $\pi$ has a *pencil of* K*-types.*

The terminology in definition 2.6.1 was introduced by Vogan in [Vog81] where he also shows that many minimal representations have a pencil of K-types. Given a representation with a pencil of K-types an alluring idea is to look at the Weyl character formula for compact groups, observe that the highest-weight of the K-type only enters in the numerator of the fraction and hence if we sum the characters of the K-types we get a geometric series. This way by a back-of-the-envelope calculation one can get a formula that we expect to be related to the character of the minimal representations. The central observation in this section is that by extending the representation to an open complex Ol'shanskiĭ semigroup this geometric series becomes absolutely convergent. Since the character on the open semigroup is holomorphic it follows by proposition 2.4.18 that this uniquely determines the character. Thus the work left for later sections is to use this information to calculate the Harish-Chandra character on the group.

Let $\mathrm{H} \subset \mathrm{K}$ be a maximal torus in the connected compact group K. Then to an analytically integral $\Delta_k^+$-dominant weight $\lambda \in \mathfrak{h}^*$ we denote by $\xi_\lambda : \mathrm{H} \to \mathbb{C}$ the associated character such that $d\xi_\lambda = \lambda$. Note that a highest-weight of an irreducible representation of K is analytically integral and $\Delta_k^+$-dominant. First we need a slight reformulation of Weyl's character formula. Note that here we work only with a compact group K and thus $\Delta$ and $\delta$ are specified with respect to $(\mathfrak{k}, \mathfrak{h})$.

**Lemma 2.6.2** *Let $\pi$ be an irreducible representation of the compact connected group* K*. Let $\mathrm{H} \subset \mathrm{K}$ be a maximal torus and let $\pi$ have highest weight $\lambda$. Let $h \in \mathrm{H}$ and $X \in \mathfrak{h}$ then*

$$\operatorname{tr}(\pi(h) e^{id\pi(X)}) = \frac{\sum_{w \in W} \epsilon(w) \xi_{w(\lambda+\delta)-\delta}(h) e^{i(w(\lambda+\delta)-\delta)(X)}}{\prod_{\alpha \in \Delta^+(\mathfrak{k},\mathfrak{h})} (1 - \xi_{-\alpha}(h) e^{-i\alpha(X)})} \tag{2.5}$$

**Proof.** Since K is compact $\mathfrak{k}$ is reductive and hence $[\mathfrak{k}, \mathfrak{k}]$ is semisimple. Thus from the Weyl character formula for complex semisimple Lie algebras [Kna02, theorem 5.75] we get that as formal characters

$$\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) \operatorname{Char}(\pi) = \sum_{w \in W} \epsilon(w) e^{w(\lambda+\delta)-\delta}. \tag{2.6}$$

Specifically we know that $d\pi(\mathfrak{h}_{\mathbb{C}})$ is a commuting family of simultaneously diagonalizable operators. From [Kna02, lemma 5.106] we know that all weights of $\pi$ are analytically integral hence it follows that they are imaginary. This implies that $d\pi(X)$ is diagonalizable with purely imaginary eigenvalues, hence it must be skew-Hermitian and thus $id\pi(X)$ is Hermitian. It follows from the spectral theorem that $e^{id\pi(X)}$ is a well-defined operator with eigenvalue $e^{i\mu(X)}$ on the weight space $\pi_\mu$.

We know from the proof of the Weyl character formula for compact groups[Kna02, theorem 5.113] that $\pi(k)$ is diagonalizable with eigenvalue $\xi_\mu(k)$ on weight space $\pi_\mu$. Thus we know that on each weight space $\pi_\mu$ the action of $\pi(k)e^{id\pi(X)}$ is given by $\xi_\mu(k)e^{i\mu(X)}$. We know from [Kna02, lemma 5.112] that all weights in eq. (2.6) are analytically integral and hence we can evaluate both sides on $\mathrm{H} \times \mathfrak{h}$ and get eq. (2.5). □

Then we can apply the lemma to calculate the character on the Ol'shanskiĭ semigroup. It is worth noting that in most cases when $\pi$ is a minimal representation the parameter $\alpha$ is invariant under $W_{\mathrm{K}}$.

**Proposition 2.6.3** *Let* G *be a connected, simple Lie group of Hermitian type. Let $\pi$ be a unitary, irreducible, highest-weight representation of* G *with a pencil of* K*-types, with $\alpha$ and $\beta$ as in definition 2.6.1. Let $h \in \mathrm{H}$ and $X \in c_m^o$ then*

$$\operatorname{tr} \pi(h \operatorname{Exp}(iX)) = \sum_{w \in W_{\mathrm{K}}} \frac{\epsilon(w)\xi_{w(\alpha+\delta_k)-\delta_k}(h) e^{i(w(\alpha+\delta_k)-\delta_k)(X)}}{(1 - \xi_{w\beta}(h)e^{iw\beta(X)}) \prod_{\gamma \in \Delta_k^+}(1 - \xi_{-\gamma}(h)e^{-i\gamma(X)})}. \tag{2.7}$$

**Proof.** Choose an orthonormal basis $\{v_n\}$ of $\pi$ consisting of weight-vectors from each K-type. Then since $\pi(\operatorname{Exp}(iX))$ is trace class for each $X \in c_m^o$ we know that $\sum_{n=1}^{\infty} \langle e^{id\pi(X)} v_n, v_n \rangle$ is absolutely convergent. Thus it must also be absolutely convergent if we choose a subset of the orthonormal basis, so choose $\{w_n\} \subset \{v_n\}$ such that $\pi(h)w_n = \xi_{\alpha+n\beta}(h)w_n$. Thus we get

$$\sum_{n=0}^{\infty} \langle e^{id\pi(X)} w_n, w_n \rangle = \sum_{n=0}^{\infty} e^{i(\alpha+n\beta)(X)} = e^{i\alpha(X)} \sum_{n=0}^{\infty} \left(e^{i\beta(X)}\right)^n$$

and this sum is absolutely convergent. Since $\pi$ is unitary we know that $\alpha(X)$ is imaginary and $(\alpha + \beta)(X)$ is imaginary hence $\beta(X)$ is imaginary. We conclude that $i\beta(c_m^o) < 0$.

Let $\tau_n$ be the component of the K-type decomposition of $\pi$ with highest-weight $\alpha + n\beta$ and let $V_n = \{v_k \mid v_k \in \tau_n\}$. Since $\pi(h \operatorname{Exp}(iX))$ is trace-class the sum

$$\operatorname{tr} \pi(h \operatorname{Exp}(iX)) = \sum_{n=0}^{\infty} \langle \pi(h \operatorname{Exp}(iX)) v_n, v_n \rangle$$

converges absolutely. Thus we can rearrange it to be a sum over K-types and then a trace for each K-type which we can compute using lemma 2.6.2

$$\begin{aligned}
\operatorname{tr}\pi(h\operatorname{Exp}(iX)) &= \sum_{n=0}^{\infty}\sum_{v\in V_n}\langle \pi(h\operatorname{Exp}(iX))v,v\rangle \\
&= \sum_{n=0}^{\infty}\operatorname{tr}(\pi(h\operatorname{Exp}(iX))|_{\operatorname{span}(V_n)}) \\
&= \sum_{n=0}^{\infty}\frac{\sum_{w\in W_{\mathrm{K}}}\epsilon(w)\xi_{w(\alpha+n\beta+\delta)-\delta}(h)e^{i(w(\alpha+n\beta+\delta)-\delta)(X)}}{\prod_{\gamma\in\Delta^+(\mathfrak{k},\mathfrak{h})}(1-\xi_{-\gamma}(h)e^{-i\gamma(X)})} \\
&= \frac{\sum_{n=0}^{\infty}\sum_{w\in W_{\mathrm{K}}}\epsilon(w)\xi_{w(\alpha+n\beta+\delta)-\delta}(h)e^{i(w(\alpha+n\beta+\delta)-\delta)(X)}}{\prod_{\gamma\in\Delta^+(\mathfrak{k},\mathfrak{h})}(1-\xi_{-\gamma}(h)e^{-i\gamma(X)})}. \qquad (2.8)
\end{aligned}$$

However we want to interchange the sum over K-types and the sum over $W_{\mathrm{K}}$ thus we check that the sum in eq. (2.8) is absolutely convergent. First we see that since $W_{\mathrm{K}}.c_m^o \subset c_m^o$ we know $iw\beta(c_m^o) < 0$. Thus we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{w\in W_{\mathrm{K}}}|\epsilon(w)\xi_{w(\alpha+n\beta+\delta_k)-\delta_k}(h)e^{i(w(\alpha+n\beta+\delta_k)-\delta_k)(X)}| \\
&\qquad = \sum_{n=0}^{\infty}\sum_{w\in W_{\mathrm{K}}}|e^{i(w(\alpha+\delta_k)-\delta_k)(X)}||e^{iw\beta(X)}|^n \\
&\qquad\qquad = \sum_{w\in W_{\mathrm{K}}}|e^{i(w(\alpha+\delta_k)-\delta_k)(X)}|\sum_{n=0}^{\infty}|e^{iw\beta(X)}|^n < \infty.
\end{aligned}$$

Note that we get $|\xi_{w\beta}(h)e^{iw\beta(X)}| < 1$.

We know that $\alpha$ and $\alpha+\beta$ are analytically integral, hence $\beta$ must be analytically integral and thus $w\beta$ is analytically integral for all $w \in W_{\mathrm{K}}$. Furthermore let $D_{\mathrm{K}}$ denote the compact Weyl denominator, i.e.

$$D_{\mathrm{K}}(h\operatorname{Exp}(iX)) := \prod_{\gamma\in\Delta_k^+}(1-\xi_{-\gamma}(h)e^{-i\gamma(X)}).$$

Now we can compute the trace

$$\begin{aligned}
&\operatorname{tr}(\pi(h\operatorname{Exp}(iX))) \\
&= \frac{1}{D_{\mathrm{K}}(h\operatorname{Exp}(iX))}\sum_{w\in W_{\mathrm{K}}}\sum_{n=0}^{\infty}\epsilon(w)\xi_{w(\alpha+\delta_k)-\delta_k}(h)e^{i(w(\alpha+\delta_k)-\delta_k)(X)}(\xi_{w\beta}(h)e^{iw\beta(X)})^n \\
&\qquad = \frac{1}{D_{\mathrm{K}}(h\operatorname{Exp}(iX))}\sum_{w\in W_{\mathrm{K}}}\frac{\epsilon(w)\xi_{w(\alpha+\delta_k)-\delta_k}(h)e^{i(w(\alpha+\delta_k)-\delta_k)(X)}}{1-\xi_{w\beta}(h)e^{iw\beta(X)}}. \qquad \square
\end{aligned}$$

Since tr is conjugation-invariant it follows from analytic continuation that proposition 2.6.3 determines the character of the representation on the open Ol'shanskiĭ semigroup and thus also the Harish-Chandra character. However determining the behavior of a Harish-Chandra character on one Cartan subgroup from the behavior on another Cartan subgroup is a highly non-trivial task. For example the Harish-Chandra characters of the unitary principal series representations of $\mathrm{SL}(2,\mathbb{R})$ are 0 on all regular elliptic elements. The rest of this chapter we try to indicate why this should not happen for irreducible highest-weight representations. Unfortunately we can not prove this but we do get a partial result that we can apply in chapter 3 to some minimal representations to calculate their Harish-Chandra characters. Our central idea is that we can realize a Cayley transform as a continuous curve in the Ol'shanskiĭ semigroup in such a way that the holomorphic character function is constant along this curve. Thereby relating the behavior of the character on the various Cartan subgroups.

## 2.7 $\mathrm{SL}(2,\mathbb{R})$ considerations

We realize $\mathrm{SL}(2,\mathbb{R})$ as the group of 2 by 2 real matrices of determinant 1 with $\mathfrak{sl}_2$ the Lie algebra of 2 by 2 real matrices of trace 0. We fix $\mathrm{K} = \{\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix} \mid \theta \in \mathbb{R}\}$ a maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$. Then $\mathfrak{k} = \mathbb{R}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ is its Lie algebra. Let $\alpha$ denote the root $\alpha(\phi\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}) = 2i\phi$, then we fix as positive root system $\Delta^+ = \{-\alpha\}$. According to proposition 2.3.2 there are exactly two minimal open, non-trivial, $\mathrm{Ad}(\mathrm{SL}(2,\mathbb{R}))$-invariant, proper convex cones in $\mathfrak{sl}_2$. Then $C^o_{\mathrm{Min}} \subset \mathfrak{sl}_2$ denotes the open, convex, $\mathrm{Ad}(\mathrm{SL}(2,\mathbb{R}))$ invariant cone containing $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. See section 3.3.1 for details.

**Corollary 2.7.1** *There exists a unique holomorphic function* $\lambda : \Gamma_{\mathrm{SL}(2,\mathbb{R})}(C^o_{\mathrm{Min}}) \to \mathbb{C}$ *such that* $|\lambda(g)| < 1$ *and* $\lambda(g)$ *is an eigenvalue of* $g$. *Furthermore* $\lambda$ *is invariant under conjugation by* $\mathrm{SL}(2,\mathbb{R})$.

**Proof.** Note that $\mathrm{SL}(2,\mathbb{R}) = \mathrm{Sp}(1,\mathbb{R})$, hence it follows from proposition 3.3.9 that $\lambda$ is a well-defined holomorphic function.

The spectrum of a linear operator is invariant under conjugation hence $\lambda$ must be conjugation-invariant since an element in $\mathrm{SL}(2,\mathbb{C})$ has at most one eigenvalue of norm less than 1. □

**Lemma 2.7.2** *Consider* $\lambda$ *from corollary 2.7.1.*
$\lambda$ *extends continuously to* $\mathrm{SL}(2,\mathbb{R})'$.

**Proof.** This follows from proposition 3.3.10. □

### 2.7.1 A curve in $\Gamma_{\mathrm{SL}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$

Consider the function $\phi : \mathbb{R}_{\geq 0} \times [-1,1] \to \mathrm{SL}(2,\mathbb{C})$ given by

$$\phi(s,t) = \begin{pmatrix} \cosh(s) + t\sinh(s) & -i\sqrt{1-t^2}\sinh(s) \\ i\sqrt{1-t^2}\sinh(s) & \cosh(s) - t\sinh(s) \end{pmatrix}.$$

First of all observe that $\det(\phi(s,t)) = 1$ so $\phi$ actually maps into $\mathrm{SL}(2,\mathbb{C})$. Then observe that $\phi$ is continuous.

**Lemma 2.7.3** *For $s > 0$, $-1 < t < 1$ we have $\phi(s,t) \in \Gamma_{\mathrm{SL}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$.*
*Furthermore for $s \geq 0$, $-1 \leq t \leq 1$ we get $\phi(s,t) \in \Gamma_{\mathrm{SL}(2,\mathbb{R})}(\{0\} \cup C^o_{\mathrm{Min}})$.*

**Proof.** Put

$$\begin{aligned} r &= \sinh^{-1}(\sinh(s)\sqrt{1-t^2}) > 0 \\ a &= \frac{\cosh(s) + t\sinh(s)}{\sqrt{\cosh^2(s) - t^2\sinh^2(s)}} > 0. \end{aligned}$$

Then since $r > 0$ we get that $\exp(ir\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \in \Gamma_{\mathrm{SL}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ and hence

$$A(r,a) = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix} \exp(ir\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix} \in \Gamma_{\mathrm{SL}(2,\mathbb{R})}(C^o_{\mathrm{Min}}).$$

Calculating we get

$$\begin{aligned} A(r,a) &= \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix} \begin{pmatrix} \cosh(r) & -i\sinh(r) \\ i\sinh(r) & \cosh(r) \end{pmatrix} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a\cosh(r) & -i\sinh(r) \\ i\sinh(r) & a^{-1}\cosh(r) \end{pmatrix}. \end{aligned} \tag{2.9}$$

Now consider

$$\begin{aligned} \cosh(r) = \sqrt{1+\sinh(r)^2} &= \sqrt{1 + (\sinh(s)\sqrt{1-t^2})^2} \\ &= \sqrt{1 + (1-t^2)\sinh(s)^2} = \sqrt{\cosh(s)^2 - t^2\sinh(s)^2}. \end{aligned} \tag{2.10}$$

Furthermore we get

$$\frac{\cosh(s)^2 - t^2\sinh(s)^2}{\cosh(s) + t\sinh(s)} = \cosh(s) - t\sinh(s). \tag{2.11}$$

Combining eq. (2.9), eq. (2.10) and eq. (2.11) we get

$$A(r,a) = \begin{pmatrix} \cosh(s) + t\sinh(s) & -i\sqrt{1-t^2}\sinh(s) \\ i\sqrt{1-t^2}\sinh(s) & \cosh(s) - t\sinh(s) \end{pmatrix} = \phi(s,t).$$

This proves the first assertion.

The second assertion follows by observing that

$$\phi(0,t) = I \in \mathrm{SL}(2,\mathbb{R})$$
$$\phi(s,1) = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$
$$\phi(s,-1) = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \qquad \square$$

**Proposition 2.7.4** *Fix $s > 0$ then $\lambda(\phi(s,\cdot))$ is constant on $]-1,1[$. Similarly $\lambda(-\phi(s,\cdot))$ is constant on $]-1,1[$.*

**Proof.** Let $x \in \mathrm{SL}(2,\mathbb{C})$ then $x$ has the eigenvalues $\mu$ and $\mu^{-1}$ and $\operatorname{tr} x = \mu + \mu^{-1}$. The map

$$\{z \in \mathbb{C} \,|\, 0 < |z| < 1\} \ni z \mapsto z + z^{-1}$$

is injective. Thus for $x \in \Gamma_{\mathrm{SL}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ we must get that $\lambda(x)$ can be expressed as a function of $\operatorname{tr} x$.

We see that $\operatorname{tr} \phi(s,t) = 2\cosh(s)$ thus constant in $t$ hence $\lambda(\phi(s,t))$ must be constant in $t$. Similarly $\operatorname{tr} -\phi(s,t) = -2\cosh(s)$ and thus constant in $t$. $\square$

### 2.7.2 Lifts to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$

Let $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ denote the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ with covering morphism $p$.

**Lemma 2.7.5** *$\widetilde{\mathrm{SL}}(2,\mathbb{R})' \cup \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ is simply-connected.*

**Proof.** Let $\gamma : S^1 \to \widetilde{\mathrm{SL}}(2,\mathbb{R})' \cup \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ be a closed curve. Then consider the homotopy $h : S^1 \times [0,1] \to \widetilde{\mathrm{SL}}(2,\mathbb{R})' \cup \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ given by

$$h(s,t) = \gamma(s)\,\mathrm{Exp}(it \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}).$$

Then $h(\cdot,1)$ is a curve in $\Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ but this an open simply-connected semigroup hence $h(\cdot,1)$ is homotopic to the trivial curve. By transitivity this shows that $\gamma$ is homotopic to a constant curve. $\square$

We abuse notation a bit and also let $p$ denote the covering morphism of semigroups: $\Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}}) \to \Gamma_{\mathrm{SL}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$.

**Lemma 2.7.6** *Let $\lambda$ be the function from lemma 2.7.2. There exists a continuous logarithm of $\lambda \circ p$ on $\widetilde{\mathrm{SL}}(2,\mathbb{R})' \cup \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ such that*

$$e^{\log \lambda(g)} = \lambda(p(g))$$
$$\log \lambda(\mathrm{Exp}(i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})) = -1.$$

*Furthermore* $\log\lambda$ *is holomorphic on* $\Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ *and invariant under conjugation by elements of* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

**Proof.** The existence follows since $\widetilde{\mathrm{SL}}(2,\mathbb{R})' \cup \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ is connected and simply-connected. The holomorphicity comes from the fact that $\lambda$ is holomorphic on the open semigroup. Let $g \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ and let $\alpha : [0,1] \to \widetilde{\mathrm{SL}}(2,\mathbb{R})$ be a continuous curve such that $\alpha(0)$ is the identity and $\alpha(1) = g$. Let $h \in \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$ then

$$e^{\log\lambda(\alpha(t)h\alpha(t)^{-1})} = \lambda(p(\alpha(t)h\alpha(t)^{-1})) = \lambda(p(\alpha(t))p(h)p(\alpha(t))^{-1}) = \lambda(p(h)).$$

Thus $\log\lambda(\alpha(t)h\alpha(t)^{-1})$ must be constant in $t$ and hence $\log\lambda$ is conjugation invariant.$\square$

Define the function $\gamma$ by

$$\begin{aligned} \gamma : \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\} \times \mathbb{R} &\to \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(\{0\} \cup C^o_{\mathrm{Min}}) \\ \gamma(0,0) &= e \\ p(\gamma(\theta,t)) &= \exp(\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \end{aligned}$$

Note that there exists a unique function satisfying the given properties, since it is a lift of a function from the simply-connected domain $\mathbb{C}_{\mathrm{Im}\geq} \times \mathbb{R}$ with $\gamma(0,0)$ the identity in $\Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(\{0\} \cup C^o_{\mathrm{Min}})$. We see that when $\mathrm{Im}(\theta) > 0$ then $\gamma(\theta,t) \in \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})$. Furthermore we observe that when $\theta \notin \pi\mathbb{Z}$ then $\gamma(\theta,0) \in \widetilde{\mathrm{SL}}(2,\mathbb{R})'$ and when $t \neq 0$ then $\gamma(\pi n,t) \in \widetilde{\mathrm{SL}}(2,\mathbb{R})'$.

Let $\phi$ denote the curve introduced in section 2.7.1, and let $\tilde\phi_{2n} : \mathbb{R}_{\leq} \times [-1,1] \to \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(\{0\} \cup C^o_{\mathrm{Min}})$ denote the lift of $\phi$ such that $\tilde\phi_{2n}(0,0) = \gamma(2n\pi,0)$ and $p \circ \tilde\phi_{2n} = \phi$. Similarly let $\tilde\phi_{2n-1}$ denote lift of $-\phi$ such that $p \circ \tilde\phi_{2n-1} = -\phi$ and $\tilde\phi_{2n-1}(0,0) = \gamma((2n-1)\pi,0)$.

**Lemma 2.7.7** $\tilde\phi_n$ *is continuous and* $\log\lambda(\tilde\phi_n(s,t))$ *is constant in* $t$*. For* $s \in \mathbb{R}$

$$\begin{aligned} \tilde\phi_n(|s|,0) &= \gamma(n\pi + i|s|,0) \\ \tilde\phi_n(|s|,\operatorname{sgn}(s)) &= \gamma(n\pi,s). \end{aligned}$$

**Proof.** $\tilde\phi_n$ is a lift of a continuous function hence continuous. Proposition 2.7.4 gives that $\lambda(p(\tilde\phi_n(s,t))) = \lambda(\phi(s,t))$ is constant in $t$. Hence $\log\lambda(\tilde\phi_n(s,t))$ is constant in $t$.

Observe that $\phi(0,t)$ is constant in $t$, hence $\tilde\phi_n(0,\pm 1) = \gamma(n\pi,0)$. Let $s > 0$ then $\phi(s,1) = p(\gamma(2n\pi,s))$ hence $\gamma(2n\pi,\cdot)$ is a lift of $\phi(\cdot,1)$ and we thus get that $\gamma(2n\pi,s) = \tilde\phi_{2n}(s,1)$ since they agree for $s = 0$ and are lifts of the same curve. Let $s < 0$ then $\phi(-s,-1) = p(\gamma(2n\pi,s))$ and $\tilde\phi_{2n}(0,-1) = \tilde\phi_{2n}(0,0) = \gamma(2n\pi,0)$ so $\gamma(2n\pi,s) = \tilde\phi_{2n}(-s,-1)$. Similar arguments apply to $-\phi$ and $\gamma((2n-1)\pi,\cdot)$. $\square$

**Proposition 2.7.8** *Let* $\theta \in \mathbb{C}_{\mathrm{Im}>0} \cup (\mathbb{R} \setminus \pi\mathbb{Z})$ *then* $\log \lambda(\gamma(\theta, 0)) = i\theta$.

*Let* $t \in \mathbb{R} \setminus \{0\}$ *then* $\log \lambda(\gamma(n\pi, t)) = -|t| + in\pi$.

**Proof.** First of all we observe that for $\theta \in \mathbb{C}_{\mathrm{Im}>0}$ we have $\gamma(\theta, 0) = \mathrm{Exp}(\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$. Hence the map $\mathbb{C}_{\mathrm{Im}>0} \ni \theta \mapsto \gamma(\theta, 0)$ is holomorphic. Furthermore we know that $p(\gamma(\theta, 0)) = \exp(\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ which has eigenvalues $e^{\pm i\theta}$ hence if $\mathrm{Im}\, \theta > 0$ we know that $|e^{i\theta}| < 1$ and thus that $\lambda(p(\gamma(\theta, 0))) = e^{i\theta}$.

Now consider the map $f(\theta) = i\theta$ which is a holomorphic map on $\mathbb{C}$ which satisfies $e^{f(\theta)} = \lambda(p(\gamma(\theta, 0)))$ i.e. $f$ is a holomorphic logarithm of $\lambda(p(\gamma(\cdot, 0)))$ which satisfies that $-1 = f(i)$. By the uniqueness of holomorphic logarithms we must have that $f(\theta) = \log \lambda(\gamma(\theta, 0))$ for $\mathrm{Im}\, \theta > 0$. Furthermore we know that $\log \lambda$ extends continuously to $\widetilde{\mathrm{SL}}(2, \mathbb{R})'$ so by taking a limit we get

$$\log \lambda(\gamma(\theta_{\mathrm{re}}, 0)) = \lim_{\theta_{\mathrm{im}} \to 0^+} \log \lambda(\gamma(\theta_{\mathrm{re}} + i\theta_{\mathrm{im}}, 0)) = \lim_{\theta_{\mathrm{im}} \to 0^+} i\theta_{\mathrm{re}} - \theta_{\mathrm{im}} = i\theta_{\mathrm{re}}.$$

This proves the first assertion.

Since $\tilde{\phi}_n(s, \pm 1)$ is regular for $s \neq 0$ we get by continuity and lemma 2.7.7 that

$$\begin{aligned} \log \lambda(\gamma(n\pi, s)) = \log \lambda(\tilde{\phi}_n(|s|, \mathrm{sgn}(s))) &= \log \lambda(\tilde{\phi}_n(|s|, 0)) \\ &= \log \lambda(\gamma(n\pi + i|s|, 0)) = -|s| + n\pi i. \end{aligned}$$

$\square$

**Proposition 2.7.9** *Let* $h : \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}}) \to \mathbb{C}$ *be a holomorphic, conjugation invariant function.*

*Then* $h(g) = h(\mathrm{Exp}(-i \log \lambda(g) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}))$ *and* $h \circ \tilde{\phi}_n(s, t)$ *is constant for* $s > 0$ *and* $-1 < t < 1$.

**Proof.** Let $f(g) = h(\mathrm{Exp}(-i \log \lambda(g) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}))$ then it follows from proposition 2.7.8 that $f$ and $h$ agree on $\mathrm{Exp}(i\mathbb{R}_+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$. It follows from lemma 2.7.6 that $f$ is conjugation invariant. Then it follows from proposition 2.4.18 that $f = h$ and it follows from lemma 2.7.7 that $f \circ \tilde{\phi}_n(s, t)$ is constant in $t$. $\square$

## 2.8 Characters on the Ol'shanskiĭ semigroups

**Lemma 2.8.1** *Let* $\Gamma_{\mathrm{G}}(C)$ *be an open complex Ol'shanskiĭ semigroup. Let* $\pi$ *be an irreducible representation of* $\Gamma_{\mathrm{G}}(C)$.

*Then for every* $X \in C$ *we have that* $\|\pi(\mathrm{Exp}((0, 1]iX))\|$ *is bounded.*

**Proof.** According to [Nee00, proposition XI.3.1] we have $X \in B(I_{\pi_{\mathrm{G}}})$ that is there exists $a > 0$ such that $\inf\langle I_{\pi_{\mathrm{G}}}, X\rangle \geq -a$. From [Nee00, proposition X.1.5] we get $\|\pi(\mathrm{Exp}(iX))\| = e^{-\inf\langle I_{\pi_{\mathrm{G}}}, X\rangle} \leq e^{a}$ but this gives that $\|\pi(\mathrm{Exp}(itX))\| \leq e^{ta}$ for $t \geq 0$. This gives that $\|\pi(\mathrm{Exp}(i[0,1]X))\|$ is bounded. □

**Lemma 2.8.2** *Let $\Gamma_{\mathrm{G}}(C)$ be an open complex Ol'shanskiĭ semigroup. Let $\pi$ be an irreducible representation of $\Gamma_{\mathrm{G}}(C)$. Let $\Theta_\pi : \Gamma_{\mathrm{G}}(C) \to \mathbb{C}$ be given by $\Theta_\pi(g) = \mathrm{tr}(\pi(g))$. Let $V \subset \mathrm{G}$ be an open subset of $\mathrm{G}$. Let $f : V \to \mathbb{C}$ such that $\tilde{f} : \Gamma_{\mathrm{G}}(C) \cup V \to \mathbb{C}$ given by*

$$\tilde{f}(g) = \begin{cases} \Theta_\pi(g) & \text{for } g \in \Gamma_{\mathrm{G}}(C) \\ f(g) & \text{for } g \in V \end{cases}$$

*is continuous, where $\Gamma_{\mathrm{G}}(C) \cup V$ is given the subspace topology in $\Gamma_{\mathrm{G}}(\overline{C})$.*

*Then $f|_{V\cap \mathrm{G}'} = \theta|_{V \cap \mathrm{G}'}$ where $\theta$ is the analytic function on $\mathrm{G}'$ associated to the Harish-Chandra character of $\pi$.*

**Proof.** Let $X \in C$ and consider $s_n = \mathrm{Exp}(i\frac{1}{n}X)$ then by lemma 2.8.1 we have $\|\pi(s_n)\|$ is bounded. Then it follows from proposition 2.5.5 that $\Theta_\pi(gs_n)$ converges to $\theta$ in the sense of distributions. Hence we want to show that $\Theta_\pi(gs_n)$ also converges to $f$ in the sense of distributions on $V \cap \mathrm{G}'$.

We show that $\Theta_\pi(gs_n)$ converges uniformly on compact subsets of $V \cap \mathrm{G}'$ to $f$. Let $A$ be a compact subset of $V \cap \mathrm{G}'$. Let $g \in A$ then there exists a small neighborhood $U_g$ in $\Gamma_{\mathrm{G}}(C) \cup (V \cap \mathrm{G}')$ of $g$ such that for $y \in U_g$ we have $|f(g) - \tilde{f}(y)| \leq \frac{\varepsilon}{2}$. Observe furthermore that for $x, y \in U_g$ we have $|\tilde{f}(x) - \tilde{f}(y)| \leq \varepsilon$. By Lawson's theorem we can take $U_g$ smaller and assume that $U_g = V_g \,\mathrm{Exp}(iW_g \cap (C \cup \{0\}))$ such that $V_g \subset \mathrm{G}$ is an open neighborhood of $g$ and $W_g \subset \overline{C}$ an open neighborhood of 0. Now since $A$ is compact there exists $g_1, \dots, g_n$ such that $A \subset \cup_{j=1}^n V_{g_j}$ define $W = \cap_{j=1}^n W_{g_j}$. Then $W$ is an open neighborhood of 0 in $\overline{C}$. Therefore there exists $N$ such that for $n \geq N$ we have $s_n \in \mathrm{Exp}(iW)$. But this gives that $|f(y) - \Theta_\pi(ys_n)| \leq \varepsilon$ for all $y \in A$.

Thus $\Theta_\pi(gs_n)$ converges uniformly on compact subsets of $V \cap \mathrm{G}'$ to $f$. Hence $\Theta_\pi(gs_n)$ converges in the sense of distributions on $V \cap \mathrm{G}'$ to $f$, but this implies that $f|_{V\cap\mathrm{G}'} = \theta|_{V\cap\mathrm{G}'}$ almost everywhere. But since both $\theta$ and $f$ are continuous on $V \cap \mathrm{G}'$ they agree. □

### 2.8.1 Cayley transforms

We quickly review the definition of Cayley transforms and their application to Cartan subalgebras. Details can be found in [Kna02, section VI.7] and [Sch75]. Let $\alpha \in \Delta_n^+$ and let $E_\alpha \in \mathfrak{g}_\alpha$ be a nonzero root vector of $\alpha$. Then $\overline{E_\alpha} \in \mathfrak{g}_{-\alpha}$ since $\alpha$ is imaginary. Furthermore since $\alpha$ is non-compact we have $0 < B(E_\alpha, \overline{E_\alpha})$. We can normalize $E_\alpha$ so that $B(E_\alpha, \overline{E_\alpha}) = \frac{2}{|a|^2}$ then let $H'_\alpha = [E_\alpha, \overline{E_\alpha}]$. Then $\{iH'_\alpha, E_\alpha + \overline{E_\alpha}, i(E_\alpha - \overline{E_\alpha})\} \subset \mathfrak{g}$ span a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. We denote this subalgebra $\mathfrak{sl}_2(\alpha) \subset \mathfrak{g}$ and its complexification by $\mathfrak{sl}_2(\alpha, \mathbb{C})$. We denote the associated Cayley transform by $\mathbf{c}_\alpha = \mathrm{Ad}(\exp \frac{\pi}{4}(\overline{E_\alpha} - E_\alpha))$.

Let $\alpha, \beta \in \Delta$. We call $\alpha$ and $\beta$ strongly orthogonal if $\beta \notin \mathbb{R}\alpha$ and neither $\alpha + \beta$ nor $\alpha - \beta$ are roots. Let $\alpha, \beta \in \Delta_n^+$ be strongly orthogonal roots; then $\mathfrak{sl}_2(\alpha, \mathbb{C})$ and $\mathfrak{sl}_2(\beta, \mathbb{C})$ commute. Specifically the associated Cayley transforms commute. Let $S \subset \Delta_n^+$ be a set of pairwise strongly orthogonal roots. Then to $S$ we associate a Cayley transform $\mathbf{c}_S = \prod_{\alpha \in S} \mathbf{c}_\alpha$ and define $\mathfrak{h}_S = \mathfrak{g} \cap \mathbf{c}_S(\mathfrak{h}_\mathbb{C})$. We define $\mathfrak{h}'_S = \cap_{\alpha \in S} \operatorname{Ker}(\alpha) = \mathfrak{h} \cap \mathfrak{h}_S$ such that $\mathfrak{h}_S = \mathfrak{h}'_S \oplus_{j=1}^k \mathbb{R}(E_\alpha + \overline{E_\alpha})$ and $\mathfrak{h} = \mathfrak{h}'_S \oplus_{j=1}^k \mathbb{R}iH'_\alpha$. Every Cartan subalgebra of $\mathfrak{g}$ is conjugate to some $\mathfrak{h}_S$.

Our first step is to show that each such $\mathfrak{sl}_2(\alpha)$ subalgebra gives us a homomorphism of Ol'shanskiĭ semigroups.

**Lemma 2.8.3** *Let* $\mathrm{G}$ *be a connected, simple Lie group of Hermitian type and let* $\mathrm{H} \subset \mathrm{G}$ *be a Cartan subgroup of a maximal compact subgroup, and let* $\Delta^+(\mathfrak{g}, \mathfrak{h})$ *be a positive system corresponding to a good ordering.*

*Let* $\alpha \in \Delta_n^+$ *then there exists a semigroup homomorphism*

$$f_\alpha : \Gamma_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}(C_{\mathrm{Min}}) \to \Gamma_{\mathrm{G}}(C_{\mathrm{Min}})$$

$$\begin{aligned} f_\alpha(\exp(r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + s \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix})) \\ = \exp(riH'_\alpha + s(E_\alpha + \overline{E_\alpha}) + ti(E_\alpha - \overline{E_\alpha})) \end{aligned}$$

$$\begin{aligned} f_\alpha(\operatorname{Exp}(i(r \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + s \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}))) \\ = \operatorname{Exp}(i(r(-i)H'_\alpha + s(E_\alpha + \overline{E_\alpha}) + ti(E_\alpha - \overline{E_\alpha}))). \end{aligned}$$

*Where in the last equality we assume that* $r, s, t \in \mathbb{R}$ *are chosen such that the argument to* $\operatorname{Exp}(i\cdot)$ *is contained in* $C_{\mathrm{Min}}(\mathfrak{sl}_2)$. *Furthermore let* $\alpha, \beta \in \Delta_n^+$ *be strongly orthogonal then the images of* $f_\alpha$ *and* $f_\beta$ *commute.*

**Proof.** According to [Kna02, page 391] the map $df_\alpha : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}_\mathbb{C}$ given below defines an injective Lie algebra homomorphism

$$\begin{aligned} df_\alpha(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}) &= H'_\alpha \\ df_\alpha(\frac{1}{2} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}) &= E_\alpha \\ df_\alpha(\frac{1}{2} \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}) &= \overline{E_\alpha}. \end{aligned}$$

Furthermore $df_\alpha(\mathfrak{sl}_2(\mathbb{R})) \subset \mathfrak{g}$ thus since $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is simply-connected and connected it induces a Lie group homomorphism $\hat{f}_\alpha : \widetilde{\mathrm{SL}}(2, \mathbb{R}) \to \mathrm{G}$. According to proposition 2.3.2

$-iH'_\alpha \in C_{\mathrm{Min}}$ thus

$$df_\alpha(C_{\mathrm{Min}}(\mathfrak{sl}_2)) = df_\alpha(\overline{\mathrm{Cone}(\mathrm{Ad}(\mathrm{SL}(2,\mathbb{R}))\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix})}) \subset \overline{\mathrm{Cone}(\mathrm{Ad}(\hat{f}_\alpha(\widetilde{\mathrm{SL}}(2,\mathbb{R})))(-i)H'_\alpha)}$$
$$\subset \overline{\mathrm{Cone}(\mathrm{Ad}(\mathrm{G})(-i)H'_\alpha)} \subset C_{\mathrm{Min}}(\mathfrak{g}).$$

By lemma 2.4.12 this implies that $df_\alpha$ induces a homomorphism $f_\alpha$ of Ol'shanskiĭ semigroups such that

$$f_\alpha(g\,\mathrm{Exp}(iX)) = \hat{f}_\alpha(g)\,\mathrm{Exp}(idf_\alpha(X)).$$

Let $\alpha, \beta \in \Delta_n^+$ be strongly orthogonal then the Lie subalgebras $\mathfrak{sl}_2(\alpha, \mathbb{C})$ and $\mathfrak{sl}_2(\beta, \mathbb{C})$ commute. As $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is connected if $x \in f_\alpha(\widetilde{\mathrm{SL}}(2,\mathbb{R}))$ and $y \in f_\beta(\widetilde{\mathrm{SL}}(2,\mathbb{R}))$ then $x$ and $y$ commute. Furthermore $\mathrm{Ad}(f_\alpha(\widetilde{\mathrm{SL}}(2,\mathbb{R})))|_{\mathfrak{sl}_2(\beta,\mathbb{C})} = I$ and vice versa. Thus $\mathrm{Exp}(idf_\alpha(X))f_\beta(b) = f_\beta(b)\,\mathrm{Exp}(idf_\alpha(X))$ and hence

$$f_\alpha(a\,\mathrm{Exp}(iA))f_\beta(b\,\mathrm{Exp}(iB)) = f_\alpha(a)\,\mathrm{Exp}(idf_\alpha(A))f_\beta(b)\,\mathrm{Exp}(idf_\beta(B))$$
$$= f_\beta(b\,\mathrm{Exp}(iB))f_\alpha(a\,\mathrm{Exp}(iA)). \qquad \square$$

Thus to a set $S$ of pairwise strongly orthogonal positive non-compact roots we now get a commuting family of semigroup homomorphisms. However their image is in the closed complex Ol'shanskiĭ semigroup $\Gamma_{\mathrm{G}}(C_{\mathrm{Min}})$ but we mostly want to work on the open semigroup $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$. For example we only know that our representations extend to $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ and even if they do extend to the closed semigroup it is on the open semigroup that we get trace class operators. So we want to push the image of a product of such homomorphisms into the open semigroup. Doing this is at first glance trivial since $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ is a semigroup ideal so we just multiply by some element from the open semigroup. In many cases we can however do a lot better and instead multiply with an element which commutes with the images of all the homomorphisms.

**Lemma 2.8.4** *Let $\mathfrak{g} \in \{\mathfrak{sp}(n,\mathbb{R}), \mathfrak{su}(p,q), \mathfrak{so}(2,2n), \mathfrak{so}(2,2n+1)\}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra consisting of fixpoints of a Cartan involution. Let $S = \{\gamma_1, \ldots, \gamma_k\} \subset \Delta_n^+(\mathfrak{g}, \mathfrak{h})$ be a set of pairwise strongly orthogonal non-compact positive roots.*

*Then there exists $Z' \in \mathfrak{h}'_S = \cap_{j=1}^k \mathrm{Ker}(\gamma_j)$ such that for any $s, s_1, \ldots, s_k > 0$*

$$sZ' + s_1(-i)H_{\gamma_1} + \cdots + s_k(-i)H_{\gamma_k} \in c^o_m. \tag{2.12}$$

**Proof.** We proceed by case by case analysis, the notation for roots in each case is introduced in their respective structure sections in chapter 3. But the notation should be consistent with most sources e.g. [Kna02, Appendix C]. Note that the lemma is formulated in terms of $H_\alpha$ and not $H'_\alpha$. This makes no difference since the lemma holds for all positive scalars $s, s_1, \ldots, s_k$ and $H'_\alpha$ is just a positive scalar times $H_\alpha$.

$\mathfrak{sp}(n,\mathbb{R})$ In this case $\Delta_n^+ = \{\mathbf{e}_j + \mathbf{e}_k \,|\, 1 \le j,k \le n\}$ thus by possibly reordering the basis of $\mathfrak{h}$ we can assume that

$$S = \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4, \ldots, \mathbf{e}_{2l-1} + \mathbf{e}_{2l}, 2\mathbf{e}_{2l+1}, 2\mathbf{e}_{2l+2}, 2\mathbf{e}_{l+k}\}.$$

It follows from lemma 3.3.5 that $c_m^o = \{t_j(-i)H_{\mathbf{e}_j} \,|\, t_j > 0\}$. We set $Z' = \sum_{j=2(l+k)+1}^n (-i)H_{\mathbf{e}_j}$ then it follows from lemma 3.3.4 that $Z' \in \cap_{j=1}^k \operatorname{Ker}(\gamma_j)$. Then

$$\begin{aligned} &sZ' + \sum_{j=1}^k s_j(-i)H_{\gamma_j} \\ &= \sum_{j=1}^l s_j(-i)H_{\mathbf{e}_{2j-1}} + s_j(-i)H_{\mathbf{e}_{2j}} + \sum_{j=1}^{k-l} s_{j+l}(-i)2H_{\mathbf{e}_{j+2l}} + \sum_{j=2(l+k)+1}^{n} s(-i)H_{\mathbf{e}_j} \end{aligned}$$

and we see that all the basis vectors $(-i)H_{\mathbf{e}_j}$ appear with a positive coefficient in the above sum thus the sum is in $c_m^o$.

$\mathfrak{so}(2,2n)$ We have $\Delta_n^+ = \{\mathbf{e}_0 \pm \mathbf{e}_j \,|\, 1 \le j \le n\}$ and according to lemma 3.6.5 $c_m^o = \{\sum_{j=0}^n t_j(-i)H_{\mathbf{e}_0} \,|\, t_0 > \sum_{j=1}^n |t_j|\}$. We can by acting by $W_{\mathrm{K}}$ assume that we are in one of the three cases $S_+ = \{\mathbf{e}_0 + \mathbf{e}_1\}$, $S_- = \{\mathbf{e}_0 - \mathbf{e}_1\}$ or $S_2 = S_+ \cup S_-$. The cases $S_+$ and $S_-$ are symmetric so we only consider $S_+$ and $S_2$. If $S = \{\mathbf{e}_0 + \mathbf{e}_1\}$ we set $Z' = (-i)H_{\mathbf{e}_0 - \mathbf{e}_1}$. Then it follows from lemma 3.6.4 that $Z' \in \operatorname{Ker}(\mathbf{e}_0 + \mathbf{e}_1)$ and

$$sZ' + s_1(-i)H_{\mathbf{e}_0+\mathbf{e}_1} = (s+s_1)(-i)H_{\mathbf{e}_0} + (s-s_1)(-i)H_{\mathbf{e}_1}$$

and $|s - s_1| < s + s_1$ and thus $sZ' + s_1(-i)H_{\gamma_1} \in c_m^o$. If $S = S_2$ we put $Z' = 0$ and then the above calculation shows that eq. (2.12) is satisfied.

$\mathfrak{so}(2,2n+1)$ We have $\Delta_n^+ = \{\mathbf{e}_0 \pm \mathbf{e}_j \,|\, 1 \le j \le n\} \cup \{\mathbf{e}_0\}$. The minimal cone is given as for $\mathfrak{so}(2,2n)$. Thus we only need to consider the new case $S = \{\mathbf{e}_0\}$ but here $(-i)H_{\mathbf{e}_0} \in c_m^o$ so we can just put $Z' = 0$.

$\mathfrak{su}(p,q)$ We realize $\mathfrak{su}(p,q)$ as in section 3.4.1. Then $\Delta_n^+ = \{\mathbf{e}_j - \mathbf{e}_k \,|\, 1 \le j \le p < k \le p+q\}$ and lemma 3.4.4 gives

$$c_m^o = \{A(t) \,|\, \forall 1 \le j \le p+q : t_j > 0 \text{ and } \sum_{j=1}^p t_j = \sum_{j=p+1}^{p+q} t_j\}.$$

We can by acting with an element from $W_{\mathrm{K}}$ ensure that $S = \{\mathbf{e}_1 - \mathbf{e}_{p+1}, \ldots, \mathbf{e}_k - \mathbf{e}_{p+k}\}$. Let $Z' = A(\phi)$ such that $\phi_j = \phi_{p+j} = 0$ for all $1 \le j \le k$ and for $k < j \le p$ set $\phi_j = 1$ and for $k < j \le q$ set $\phi_{p+j} = \frac{p-k}{q-k}$. Then $Z' \in \mathfrak{h}$ and $(\mathbf{e}_j - \mathbf{e}_{p+j})(Z') = 0$ for $1 \le j \le k$. Furthermore it follows from lemma 3.4.3 and lemma 3.4.4 that

$$sZ' + s_1 H_{\mathbf{e}_1 - \mathbf{e}_{p+1}} + \cdots + s_k H_{\mathbf{e}_k - \mathbf{e}_{p+k}} \in c_m^o$$

for any $s, s_1, \ldots, s_k > 0$. □

In the case of $\mathfrak{so}^*(2n)$ lemma 2.8.4 is a bit more complicated. For $n$ odd such a $Z'$ does not exist for a maximally split Cartan subalgebra. To see this let $n = 2k+1$, let $\mathfrak{h}$ be the diagonal Cartan subalgebra and let $S = \{\mathbf{e}_1 + \mathbf{e}_2, \dots, \mathbf{e}_{2k-1} + \mathbf{e}_{2k}\}$. For details on the notation and realization see section 3.5.1. We see that

$$\mathfrak{h}'_S = \cap_{j=1}^k \operatorname{Ker}(\mathbf{e}_{2j-1} + \mathbf{e}_{2j}) = \{\begin{pmatrix}\phi & 0 \\ 0 & -\phi\end{pmatrix} \mid \phi = (z_1, -z_1, z_2, -z_2, \dots, z_k, -z_k, z)\}. \quad (2.13)$$

So if we wanted $Z' \in \mathfrak{h}'_S$ to satisfy the same properties as in lemma 2.8.4 then $Z'$ would have to be in $c_m$(take the limit $s_1, \dots, s_k \to 0$). Let $Z' = \begin{pmatrix} i\phi & 0 \\ 0 & -i\phi \end{pmatrix}$ then since $Z' \in c_m$ it follows from lemma 3.5.5 that $\phi_j \leq 0$ for all $j$. Combining this with eq. (2.13) we get that $\phi_j = 0$ for $1 \leq j \leq 2k$. Thus $\phi = (0, \dots, 0, z)$ and then lemma 3.5.5 implies that $z = 0$. But $(-i)H_{\mathbf{e}_1+\mathbf{e}_2} + \cdots + (-i)H_{\mathbf{e}_{2k-1}+\mathbf{e}_{2k}} \notin c_m^o$.

**Lemma 2.8.5** *Let $n \geq 3$ and let $\mathfrak{h} \subset \mathfrak{so}^*(2n)$ be the diagonal Cartan subalgebra. Let $S = \{\gamma_1, \dots, \gamma_k\} \subset \{\gamma_1, \dots, \gamma_k\} \subset \Delta_n^+$ be a set of pairwise strongly orthogonal positive non-compact roots. Assume that $n$ is even or that $k < \lfloor \frac{n}{2} \rfloor$.*

*Then there exists $Z' \in \mathfrak{h}'_S$ such that for any $s, s_1, \dots, s_k > 0$ we have*

$$sZ' + s_1(-i)H_{\gamma_1} + \cdots + s_k(-i)H_{\gamma_k} \in c_m^o.$$

**Proof.** We can by the Weyl group action transform $S$ such that it satisfies $S = \{\mathbf{e}_1 + \mathbf{e}_2, \dots, \mathbf{e}_{2k-1} + \mathbf{e}_{2k}\}$. Let us first assume that $k < \lfloor \frac{n}{2} \rfloor$ then we can put $Z' = \begin{pmatrix} i\phi & 0 \\ 0 & -i\phi \end{pmatrix}$ such that $\phi_j = 0$ for $1 \leq j \leq 2k$ and $\phi_j = -1$ for $2k+1 \leq j \leq n$. Then $Z' \in \mathfrak{h}'_S$ and since $n - 2k \geq 2$ it follows from lemma 3.5.5 that $Z' \in c_m$. If we choose $\psi$ such that

$$sZ' + s_1(-i)H_{\mathbf{e}_1+\mathbf{e}_2} + \cdots + s_k(-i)H_{\mathbf{e}_{2k-1}+\mathbf{e}_{2k}} = \begin{pmatrix} i\psi & 0 \\ 0 & -i\psi \end{pmatrix}.$$

Then $\psi_{2j-1} = \psi_{2j} = -s_j$ for $1 \leq j \leq k$ and $\psi_j = -s$ for $2k+1 \leq j \leq n$. It now follows from lemma 3.5.5 that $\begin{pmatrix} i\psi & 0 \\ 0 & -i\psi \end{pmatrix} \in c_m^o$. When $n = 2k$ we see by the above calculation that we can simply put $Z' = 0$. $\square$

**Lemma 2.8.6** *Let $\mathrm{G}$ be a connected, simple Lie group of Hermitian type and let $S = \{\gamma_1, \dots, \gamma_k\} \subset \Delta_n^+$ be a set of pairwise strongly orthogonal roots. Assume that $\mathfrak{g}$ and $S$ satisfy the assumptions in lemma 2.8.4 or lemma 2.8.5. Let $g_1, \dots, g_k \in \Gamma_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}(C^o_{\mathrm{Min}})$, $X \in \mathfrak{h}'_S$ and $z \in \mathbb{C}_{\mathrm{Im}>0}$. Then*

$$(\prod_{j=1}^k f_{\gamma_j}(g_j)) \exp(X) \operatorname{Exp}(zZ') \in \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}}). \quad (2.14)$$

**Proof.** For each $j$ we write $g_j = h_j \operatorname{Exp}(iY_j)$ such that $h_j \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ and $Y_j \in C^o_{\mathrm{Min}}$. Then put $h = (\prod_{j=1}^k f_{\gamma_j}(h_j)) \exp(X) \exp(\mathrm{Re}(z)Z) \in \mathrm{G}$. Since the images of the $f_{\gamma_j}$ commute and commute with $\exp(\mathfrak{h}'_S)$ we can write eq. (2.14) as

$$h(\prod_{j=1}^k f_{\gamma_j}(\operatorname{Exp}(iY_j))) \operatorname{Exp}(i \operatorname{Im}(z)Z').$$

But $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ is an ideal in $\Gamma_{\mathrm{G}}(C_{\mathrm{Min}})$ thus we can ignore $h$. Each element in $C^o_{\mathrm{Min}}$ is conjugate to an element in $c^o_m$. Thus we can write $Y_j = \mathrm{Ad}(a_j)Y'_j$ with $Y'_j \in c^o_m$ then $\operatorname{Exp}(iY_j) = a_j \operatorname{Exp}(iY'_j)a_j^{-1}$ and we can commute the $f_{\gamma_j}(a_j)$'s to the left into $h$ and the $f_{\gamma_j}(a_j^{-1})$'s to the right. Then using again that $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ is an ideal we will ignore these. $Y'_j = s_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for some $s_j > 0$ and we just have to show that

$$\prod_{j=1}^k f_{\gamma_j}(\operatorname{Exp}(s_j i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \operatorname{Exp}(\mathrm{Im}(z)iZ') \in \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}}).$$

But then the formulas in lemma 2.8.3 gives

$$\prod_{j=1}^k f_{\gamma_j}(\operatorname{Exp}(t_j i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \operatorname{Exp}(\mathrm{Im}(z)iZ') = \operatorname{Exp}(i(\mathrm{Im}(z)Z' + \sum_{j=1}^k s_j(-i)H'_{\gamma_j}))$$

and the result follows from lemma 2.8.4 or lemma 2.8.5. $\square$

We want to show that not only does the product of semigroup homomorphisms land in $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$, the product of homomorphisms becomes a holomorphic map. Thus we need a version of lemma 2.8.3 that is easy to relate to the complex structure coming from the atlas from lemma 2.4.10.

**Lemma 2.8.7** *Continue the notation and setup from lemma 2.8.3. Let $\gamma \in \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{C})}(C^o_{\mathrm{Min}})$ then for $a, b, c \in \mathbb{C}$ sufficiently close to 0*

$$f_\alpha(\gamma \operatorname{Exp}(a \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{b}{2} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} + \frac{c}{2} \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix})) = f_\alpha(\gamma) \operatorname{Exp}(aH'_\alpha + bE_\alpha + c\overline{E_\alpha}).$$

**Proof.** This follows from lemma 2.4.12 and lemma 2.4.13. $\square$

**Lemma 2.8.8** *Let the setup be as in lemma 2.8.6 and let $X \in \mathfrak{h}$ then $h : \Gamma_{\widetilde{\mathrm{SL}}(2,\mathbb{R})}(C^o_{\mathrm{Min}})^k \times \mathbb{C}_{\mathrm{Im}>0} \to \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ given by*

$$h(\rho_1, \ldots, \rho_k, z) = (\prod_{j=1}^k f_{\gamma_j}(\rho_j)) \exp(X) \operatorname{Exp}(zZ')$$

*is holomorphic.*

**Proof.** Let $\rho_j \in \Gamma_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}(C^o_{\mathrm{Min}})$ and let $\tau_j : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$ corresponding to the root $\gamma_j$ such that it matches up with $f_{\gamma_j}$. Note that $\tau_j$ is complex linear. Then it follows from lemma 2.8.7 that

$$h(\rho_1 \operatorname{Exp}(X_1), \dots, \rho_k \operatorname{Exp}(X_k), z + w) = (\prod_{j=1}^{k} f_{\gamma_j}(\rho_j) \exp(X) \operatorname{Exp}(zZ')) \operatorname{Exp}(wZ' + \sum_{j=1}^{k} \tau_j(X_j))$$

which is holomorphic in $X_1, \dots, X_k, w$ by lemma 2.4.10. □

## 2.8.2 Uniformity of characters

Now we are ready to prove the main proposition of this chapter.

**Proposition 2.8.9** *Let* G *be a connected, simple Lie group of Hermitian type. Let* $\theta : \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}}) \to \mathbb{C}$ *be a holomorphic, conjugation invariant function which extends continuously to* $\mathrm{G}'$*. Let* $S = \{\nu_1, \dots, \nu_k\} \subset \Delta_n^+$ *be pairwise strongly orthogonal roots. Assume that* $\mathfrak{g} \in \{\mathfrak{so}(2,n), \mathfrak{sp}(n,\mathbb{R}), \mathfrak{su}(p,q), \mathfrak{so}^*(2n)\}$*.*

*Let* $n_1, \dots, n_k \in \mathbb{Z}$*,* $s_1, \dots, s_k \in \mathbb{R} \setminus \{0\}$ *and* $X \in \mathfrak{h}'_S$ *such that the element on the left-hand side in eq.* (2.15) *is regular. Then*

$$\theta((\prod_{j=1}^{k} \exp(n_j \pi i H'_{\nu_j}) \exp(s_j(E_{\nu_j} + \overline{E_{\nu_j}}))) \exp(X)) = \lim_{s \to 0^+} \theta((\prod_{j=1}^{k} \exp(n_j \pi i H'_{\nu_j}) \operatorname{Exp}(|s_j| H'_{\nu_j})) \exp(X) \operatorname{Exp}(siZ')). \quad (2.15)$$

*Where* $Z'$ *is an element in* $\mathfrak{h}'_S$ *independent of* $X, n_1, \dots, n_k, s_1, \dots, s_k$ *but such that the right hand side of eq.* (2.15) *is contained in* $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ *for* $s > 0$ *sufficiently small.*

**Proof.** If $\mathfrak{g}$ satisfies the assumptions in lemma 2.8.4 or $\mathfrak{g} = \mathfrak{so}^*(2n)$ and $S$ satisfies the assumptions of lemma 2.8.5 then it follows from lemma 2.8.6 that for any $s > 0$ the argument on the right hand side of eq. (2.15) is in $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$. Thus we can actually apply $\theta$ to it. However that the limit exists is not so clear, since we do not take a limit towards $\mathrm{G}'$. Rather we take the limit towards an element in $\Gamma_{\mathrm{G}}(C_{\mathrm{Min}})$.

Let $f_{\nu_j}$ denote the maps from lemma 2.8.3. Let $\gamma$ and $\tilde{\phi}$ denote the curves from lemma 2.7.7. Let $h$ be as in lemma 2.8.8. Then $\theta \circ h$ is $\widetilde{\mathrm{SL}}(2,\mathbb{R})^k$ conjugation invariant since the images of the $f_{\gamma_j}$ commute and $\theta$ is G conjugation invariant. Then it follows from proposition 2.7.9 that $\theta(h(\tilde{\phi}_{n_1}(|s_1|, t_1), \dots, \tilde{\phi}_{n_k}(|s_k|, t_k), z))$ is constant for all $t_1, \dots, t_k \in (-1, 1)$. Now for $0 \le t < 1$ let

$$\eta(t) = h(\tilde{\phi}_{n_1}(|s_1|, \operatorname{sgn}(s_1)t), \dots, \tilde{\phi}_{n_k}(|s_k|, \operatorname{sgn}(s_k)t), i(1-t)).$$

Then it follows from lemma 2.8.3 that

$$\lim_{t\to 1}\eta(t) = (\prod_{j=1}^{k} \exp(n_j\pi i H'_{\nu_j}) \exp(s_j(E_{\nu_j} + \overline{E_{\nu_j}}))) \exp(X) \in \mathrm{G}.$$

If this element is regular then since $\theta$ extends continuously we have $\lim_{t\to 1}\theta(\eta(t)) = \theta(\lim_{t\to 1}\eta(t))$. But we know from earlier that for all $t < 1$

$$\begin{aligned}\theta(\eta(t)) &= \theta(h(\tilde{\phi}_{n_1}(|s_1|, 0), \ldots, \tilde{\phi}_{n_k}(|s_k|, 0), i(1-t))) \\ &= \theta((\prod_{j=1}^{k} \exp(n_j\pi i H'_{\nu_j}) \operatorname{Exp}(|s_j| H'_{\nu_j})) \exp(X) \operatorname{Exp}((1-t)iZ')).\end{aligned}$$

This finishes the proof in most cases.

In the case $\mathfrak{g} = \mathfrak{so}^*(4k+2)$ we need to be a bit more careful since as we saw in the paragraph before lemma 2.8.5 we cannot have a $Z'$ that satisfies all the properties of lemma 2.8.4. By an action of the compact Weyl group we can assume that $\nu_j = H_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}$. Then we put $Z' = \begin{pmatrix} i\phi & 0 \\ 0 & -i\phi \end{pmatrix}$ with $\phi_j = 0$ for $1 \le j \le 2k$ and $\phi_{2k+1} = -1$. Then

$$s_1(-i)H'_{\mathbf{e}_1+\mathbf{e}_2} + \cdots + s_k(-i)H'_{\mathbf{e}_{2k-1}+\mathbf{e}_{2k}} + sZ' \in C^o_{\mathrm{Min}}$$

for $s < \min(s_1, \ldots, s_k)$. By [Pan83, lemma 6] any element in $C^o_{\mathrm{Min}}(\mathfrak{sl}_2)$ is conjugate to an element in $c^o_m(\mathfrak{sl}_2)$ and thus for $\rho_1, \ldots, \rho_k \in \Gamma_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}(C^o_{\mathrm{Min}})$ and for $0 < s$ sufficiently small depending on $\rho_1, \ldots, \rho_k$ we get

$$(\prod_{j=1}^{k} f_{\nu_j}(\rho_j)) \operatorname{Exp}(isZ') \in \Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}}).$$

Then by a proof identical to the ones for lemma 2.8.6 and lemma 2.8.8 we get that there exists some open set $V$ satisfying

$$\Gamma_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}(C^o_{\mathrm{Min}}) \times \mathbb{R} \subset V \subset \Gamma_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}(C^o_{\mathrm{Min}})^k \times \mathbb{C}_{\mathrm{Im}\ge 0}$$

such that

$$h : V \to \Gamma_{\mathrm{G}}(C_{\mathrm{Min}}) \qquad h(\rho_1, \ldots, \rho_k, z) = (\prod_{j=1}^{k} f_{\nu_j}(\rho_j)) \exp(X) \operatorname{Exp}(zZ')$$

is a well-defined continuous map such that on $V \cap (\Gamma_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}(C^o_{\mathrm{Min}})^k \times \mathbb{C}_{i>0})$ it lands in $\Gamma_{\mathrm{G}}(C^o_{\mathrm{Min}})$ and is holomorphic. That is if we fix $\rho_1, \ldots, \rho_k$ the map $z \mapsto h(\rho_1, \ldots, \rho_k, z)$ is defined for $0 < \operatorname{Im} z < \varepsilon$ for some sufficiently small $\varepsilon$ depending on $\rho_1, \ldots, \rho_k$.

It follows from the proof of lemma 2.7.3 and lemma 2.8.3 that

$$\begin{aligned}f_{\nu_j}(\tilde{\phi}_n(s,t)) &= f_{\nu_j}(g(s,t)h(s,t) \operatorname{Exp}(ir(s,t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) h(s,t)^{-1}) \\ &= f_{\nu_j}(g(s,t)) \operatorname{Exp}(ir(s,t)(-i) \operatorname{Ad}(f_{\nu_j}(h(s,t))) H'_{\nu_j})\end{aligned}$$

where $g, h : \mathbb{R}_+ \times [-1,1] \to \widetilde{\mathrm{SL}}(2,\mathbb{R})$ are continuous and $r : \mathbb{R}_+ \times [-1,1] \to \mathbb{R}_\geq$ is the continuous function

$$r(s,t) = \sinh^{-1}(\sinh(s\sqrt{1-t^2})).$$

Thus if we consider the map

$$(-1,1)^k \times \mathbb{C}_{\mathrm{Im}>0} \ni (t_1,\dots,t_k,z) \mapsto h(\tilde{\phi}_{n_1}(|s_1|,t_1),\dots,\tilde{\phi}_{n_k}(|s_k|,t_k),z)$$

it is defined for $\operatorname{Im} z < \min(r(|s_1|,t_1),\dots,r(|s_k|,t_k))$ and as before it is constant in $t_j$. Thus we put

$$\alpha(t) = \frac{1}{2}\sinh^{-1}(\sinh(\min(|s_1|,\dots,|s_k|)\sqrt{1-t^2})).$$

Then $\alpha$ is a continuous function $\alpha : [0,1] \to \mathbb{R}_\geq$ such that $\alpha(t) > 0$ for $t < 1$ and $\alpha(1) = 0$ and we can define

$$\eta(t) = h(\tilde{\phi}_{n_1}(|s_1|,\operatorname{sgn}(s_1)t),\dots,\tilde{\phi}_{n_k}(|s_k|,\operatorname{sgn}(s_k)t),i\alpha(t))$$

and finish the argument as previously. □

**Corollary 2.8.10** *Consider the setup from proposition 2.8.9 but assume instead that $\theta$ is a function on $\Gamma_{\mathrm{G}}(-C^o_{\mathrm{Min}})$ then*

$$\theta((\prod_{j=1}^{k}\exp(n_j\pi i H'_{\nu_j})\exp(s_j(E_{\nu_j}+\overline{E_{\nu_j}})))\exp(X))$$
$$= \lim_{s\to 0^-}\theta((\prod_{j=1}^{k}\exp(n_j\pi i H'_{\nu_j})\operatorname{Exp}(-|s_j|H'_{\nu_j}))\exp(X)\operatorname{Exp}(siZ')).$$

**Proof.** As we have noted earlier we can pass from $C^o_{\mathrm{Min}}$ to $-C^o_{\mathrm{Min}}$ by making positive non-compact roots into negative non-compact roots and vice versa, but keeping the relative order of functionals which agree on $Z(\mathfrak{k})$. Thus if $S \subset \Delta_n^+$ we will instead have to consider $-S$ and then the result follows by observing $H'_{-\alpha} = -H'_\alpha$. □

# 3 Minimal Representations

## 3.1 Minimal holomorphic representations

Minimal holomorphic representations are unitary highest-weight representations of a simple Lie group of Hermitian type corresponding to the minimal non-trivial parameter in the Wallach set. They all have a pencil of K-types. Thus they are natural candidates on which to attempt to apply the results of chapter 2. Furthermore most minimal representations of simple Lie groups of Hermitian type are minimal holomorphic or its contragredient. For groups of tube type the odd part of the metaplectic representation and its contragredient are the only minimal representations not arising in this way.

Let G be a real simple Lie group of Hermitian type with K the fixpoint group of a Cartan involution and $\mathfrak{h} \subset \mathfrak{k}$ a maximal torus. Let $\Delta^+(\mathfrak{g}_\mathbb{C})$ be a positive system coming from a good ordering. Let $\mathfrak{p}_- = \sum_{\alpha \in \Delta_n^+} \mathfrak{g}_{-\alpha}$ denote the sum of negative non-compact root-spaces. For a dominant integral weight $\lambda \in \mathfrak{h}_\mathbb{C}^*$ let $F^{\mathfrak{k}_\mathbb{C}}(\lambda)$ denote the irreducible, finite-dimensional representation of $\mathfrak{k}_\mathbb{C}$ with highest-weight $\lambda$. We thus let $L^{\mathfrak{g}_\mathbb{C}}(\lambda)$ denote the unique simple quotient of the associated module $\mathfrak{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathfrak{k}_\mathbb{C} \oplus \mathfrak{n}} F^{\mathfrak{k}_\mathbb{C}}(\lambda)$ where $\mathfrak{p}_-$ acts by 0 on $F^{\mathfrak{k}_\mathbb{C}}(\lambda)$. Let $W$ denote the set of weights $\lambda$ such that $L^{\mathfrak{g}_\mathbb{C}}(\lambda)$ is a unitarizable $(\mathfrak{g}, \mathrm{K})$-module. Then $W$ is known as the Wallach set and was determined independently by Jakobsen[Jak83] and Enright, Howe and Wallach[EHW83]. Let $Z \in Z(\mathfrak{k}_\mathbb{C})$ such that $\alpha(Z) > 0$ for $\alpha \in \Delta_n^+$ and let $\zeta \in i\mathfrak{h}^*$ such that $\zeta(Z) > 0$ and $\zeta([\mathfrak{k}, \mathfrak{k}] \cap \mathfrak{h}) = 0$. Then there exists a smallest non-zero real number $c$ such that $c\zeta \in W$ and this $c$ is positive.

**Definition 3.1.1** Let $\mathfrak{g}$ be a simple Lie algebra of Hermitian type and real rank greater than 1. Then we call $L^{\mathfrak{g}_\mathbb{C}}(c\zeta)$ *the minimal holomorphic representation* and we denote it by $\pi_{\mathrm{Min}}$.

Inspired by Howe's study of the metaplectic representation the branching laws of minimal holomorphic representations have been studied in many different cases [Zha01; Sep07b; Sep07a; Sek13; PZ04; MS10; BZ94; MO14]. Hilgert, Kobayashi and Möllers in [HKM14, definition 2.29, theorem 2.30] characterize the minimal covering groups to which $\pi_{\mathrm{Min}}$ integrates. For $\mathfrak{g} = \mathfrak{so}(2, 2n), \mathfrak{so}^*(2n)$ or $\mathfrak{su}(p, q)$ $\pi_{\mathrm{Min}}$ integrates to the corresponding classical group $\mathrm{SO}(2, 2n), \mathrm{SO}^*(2n)$ and $\mathrm{SU}(p, q)$ respectively. In the cases $\mathfrak{so}(2, 2n+1)$ and $\mathfrak{sp}(n, \mathbb{R})$ one needs to consider an appropriate double covering group.

In the interest of applying the theory from chapter 2 we need the K-type decomposition of the minimal holomorphic representations.

**Lemma 3.1.2** *Let* G *be a simple Lie group of Hermitian type and let* K *be the fixpoint group of a Cartan involution. Assume that the holomorphic minimal representation* $\pi_{\mathrm{Min}}$

*of $\mathfrak{g}$ integrates to a representation of* G*. Then for $\alpha = c\zeta$ and $\beta$ the highest root in $\Delta_n^+$ we have*

$$\pi_{\mathrm{Min}}|_{\mathrm{K}} = \bigoplus_{n=0}^{\infty} F^{\mathrm{K}}(\alpha + n\beta).$$

*For the classical cases $\alpha$ and $\beta$ are given by the following list:*

$\mathfrak{sp}(n,\mathbb{R})$*: Then $\alpha = \frac{1}{2}\sum_{j=1}^{n} \mathbf{e}_j$ and $\beta = 2\mathbf{e}_1$.*

$\mathfrak{su}(p,q)$*: Then $\alpha = \frac{q}{p+q}\sum_{j=1}^{p} \mathbf{e}_j - \frac{p}{p+q}\sum_{j=1}^{q} \mathbf{e}_{p+j} = \sum_{j=1}^{p} \mathbf{e}_j$ and $\beta = \mathbf{e}_1 - \mathbf{e}_{p+q}$.*

$\mathfrak{so}^*(2n)$*: Then $\alpha = \sum_{j=1}^{n} \mathbf{e}_j$ and $\beta = \mathbf{e}_1 + \mathbf{e}_2$.*

$\mathfrak{so}(2,2n)$*: Then $\alpha = (n-1)\mathbf{e}_0$ and $\beta = \mathbf{e}_0 + \mathbf{e}_1$.*

$\mathfrak{so}(2,2n+1)$*: Then $\alpha = (n-\frac{1}{2})\mathbf{e}_0$ and $\beta = \mathbf{e}_0 + \mathbf{e}_1$.*

**Proof.** [MO14, page 2, page 4] ☐

**Lemma 3.1.3** *Let the setup be as in lemma 3.1.2 and let $C_{\mathrm{Min}}$ be the minimal cone as in proposition 2.3.2. Then $\pi_{\mathrm{Min}}$ extends to a holomorphic representation of $\Gamma_{\mathrm{G}}(-C^o_{\mathrm{Min}})$ and the character on the interior of the Ol'shanskiĭ semigroup is determined by*

$$\operatorname{tr}\pi(h\operatorname{Exp}(iX)) = \sum_{w\in W_{\mathrm{K}}} \frac{\epsilon(w)\xi_{w(\alpha+\delta_k)-\delta_k}(h)e^{i(w(\alpha+\delta_k)-\delta_k)(X)}}{(1-\xi_{w\beta}(h)e^{iw\beta(X)})\prod_{\gamma\in\Delta_k^+}(1-\xi_{-\gamma}(h)e^{-i\gamma(X)})}.$$

**Proof.** $\pi_{\mathrm{Min}}$ is an irreducible lowest-weight representation of G then it follows from corollary 2.5.4 that it extends to $\Gamma_{\mathrm{G}}(-C^o_{\mathrm{Min}})$. Then the character formula is exactly as in eq. (2.7) since $\pi_{\mathrm{Min}}$ satisfies all the assumptions in proposition 2.6.3. ☐

Now the rest of the work is in attempting to show that the character on the Ol'shanskiĭ semigroup $\Gamma_{\mathrm{G}}(-C^o_{\mathrm{Min}})$ extends to G′ such that we can apply proposition 2.8.9. We do this for $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R}), \mathfrak{su}(p,q), \mathfrak{so}^*(2n)$ since in these cases we can realize the Ol'shanskiĭ semigroup as the semigroup of contractions with respect to some non-degenerate indefinite Hermitian inner product. This was for example observed in [KØ97, proposition 2.2]. Then we can show that there exists $k, m$ such that every element $\gamma \in \Gamma_{\mathrm{G}}(-C^o_{\mathrm{Min}})$ has $k$ eigenvalues $\lambda_1, \dots, \lambda_k$ such that $|\lambda_k| < 1$ and $m$ eigenvalues $\mu_1, \dots, \mu_m$ such that $|\mu_m| > 1$. Thus we can define holomorphic global functions on $\Gamma_{\mathrm{G}}(-C^o_{\mathrm{Min}})$ in terms of the eigenvalues $\lambda_1, \dots, \lambda_k$ and $\mu_1, \dots, \mu_m$ as long as these functions are invariant under permutations of the $\lambda_j$ and permutations of the $\mu_j$. But this turns out to exactly be the symmetry Weyl group $W_{\mathrm{K}} = W(\mathrm{K}, \mathfrak{h})$ of G in these three cases. Thus the character on the Ol'shanskiĭ semigroup has the right symmetry properties and we can extend it to all of $\Gamma_{\mathrm{G}}(-C^o_{\mathrm{Min}})$.

Furthermore from the explicit formula of the character that we get from proposition 2.6.3 we can actually show that it extends continuously to G′. The method of identifying

G-conjugation invariant functions on $\Gamma_{\mathrm{G}}(-C^{o}_{\mathrm{Min}})$ to with functions of eigenvalues to get a global description was used by Ol'shanskiĭ in [Ols95]. Unfortunately the author has not been able to find a sufficiently general description of this method to avoid case by case analysis. Therefore the sections 3.3 to 3.5 are largely similar and unless the reader is extremely interested it is probably beneficial to just read the section on their favorite group. The other cases are almost exact copies in their structure and only a few arguments are different.

We should however note that in the case of $\mathfrak{sp}(n,\mathbb{R})$ we consider not only the minimal holomorphic representation which corresponds to the even part of the metaplectic representation, but also the "other" minimal representation; the odd part of the metaplectic representation. In this case we can furthermore simplify the formula in proposition 2.6.3 quite a lot to arrive at the character formulas in [Tor80; Ada97].

For $\mathfrak{so}(2,n)$ the author is unfortunately not able to show that the character on the Ol'shanskiĭ semigroup extends to the regular elements. However we can show a simplification of the general character formula in this case like for the metaplectic representation and we include it here in the hope that it might be useful to someone.

## 3.2 Technicalities

We include a couple of well known results on finite groups and basic linear algebra. They are included here for comprehensiveness since we use them in many of the following calculations.

**Lemma 3.2.1** *Let* G *be a finite group, let* $V$ *be a representation of* G*. Assume that we have* $v \in V$ *and a subgroup* $\mathrm{H} \subset \mathrm{G}$ *such that*

$$\sum_{h\in\mathrm{H}} h.v = 0$$

*then*

$$\sum_{g\in\mathrm{G}} g.v = 0.$$

**Proof.** Let $g_1 = e, \dots, g_k$ be representatives of the left cosets G/H. Then $\mathrm{G} = \sqcup_{j=1}^{k} g_j\mathrm{H}$ and we can rearrange the sum

$$\sum_{g\in\mathrm{G}} g.v = \sum_{j=1}^{k}\sum_{h\in\mathrm{H}} g_j.(h.v) = \sum_{j=1}^{k} g.0 = 0. \qquad \square$$

**Corollary 3.2.2** *Let the setup be as in lemma 3.2.1, assume that* $v, w \in V$ *such that*

$$\sum_{h\in\mathrm{H}} h.v = \sum_{h\in\mathrm{H}} h.w$$

*then*

$$\sum_{g\in \mathrm{G}} g.v = \sum_{g\in \mathrm{G}} g.w.$$

**Lemma 3.2.3** *Let* G *be a finite group, let* $V$ *be a representation of* G *and assume that* $V$ *is a vector space over a field not of characteristic* 2*. Assume we have* $g \in \mathrm{G}$*,* $v \in V$ *such that* $g.v = -v$ *then*

$$\sum_{s\in \mathrm{G}} s.v = 0.$$

**Proof.**

$$2\sum_{s\in \mathrm{G}} s.v = \sum_{s\in \mathrm{G}} s.v + \sum_{s\in \mathrm{G}} s.(g.v) = \sum_{s\in \mathrm{G}} s.v - \sum_{s\in \mathrm{G}} s.v = 0.$$ □

**Lemma 3.2.4** *Let* $V$ *be a complex vector space of dimension* $n$*. Let* $\langle , \rangle$ *be a non-degenerate* $\mathbb{C}$*-bilinear form on* $V$*. Let* $g \in \mathrm{GL}(V)$ *such that* $\langle gv, gw\rangle = \langle v, w\rangle$ *for all* $v, w \in V$*. Let* $V = \oplus_\lambda V_\lambda$ *be the decomposition of* $V$ *into generalized eigenspaces*[1] *of* $g$*.*

*Then* $\dim V_\lambda = \dim V_{\lambda^{-1}}$*.*

**Proof.** Let $x$ be an eigenvector of $\lambda$ and $y$ an eigenvector of $\mu$ with $\mu \neq \lambda^{-1}$, then

$$\langle x, y\rangle = \langle gx, gy\rangle = \lambda\mu\langle x, y\rangle.$$

Thus $\langle x, y\rangle = 0$, by induction we see that if $(g-\mu)^k y = 0$ then $\langle x, y\rangle = 0$ since $gy = \mu y + y'$ where $(g-\mu)^{k-1}y' = 0$. Thus

$$\langle x, y\rangle = \lambda\mu\langle x, y\rangle + \lambda\langle x, y'\rangle = \lambda\mu\langle x, y\rangle$$

and the previous argument applies. By a further induction we see that if $x$, $y$ are generalized eigenvectors of eigenvalues $\lambda$ and $\mu$ respectively, then $\langle x, y\rangle = 0$.

Let $x \in V_\lambda$ with $x \neq 0$. Then since $\langle , \rangle$ is non-degenerate there exists $y \in V$ such that $\langle x, y\rangle \neq 0$. Let $y = \sum_\mu y_\mu$ be the decomposition of $y$ into generalized eigenvectors such that $y_\mu \in V_\mu$. Now for all $\mu \neq \lambda^{-1}$ we have seen that $\langle x, y_\mu\rangle = 0$. Specifically we must have that $y_{\lambda^{-1}} \neq 0$ and thus $V_{\lambda^{-1}} \neq 0$. This argument also shows that $\langle , \rangle|_{V_\lambda \oplus V_{\lambda^{-1}}}$ is non-degenerate. Thus $\langle , \rangle$ establishes an injective linear map $V_\lambda \to V^*_{\lambda^{-1}}$ and $V_{\lambda^{-1}} \to V^*_\lambda$ and thus we must have

$$\dim V_\lambda \leq \dim V_{\lambda^{-1}} \leq V_\lambda.$$ □

**Definition 3.2.5** Let $\langle , \rangle$ be a Hermitian form. We call an element $g \in \mathrm{GL}(n, \mathbb{C})$ *weakly expansive* with respect to $\langle , \rangle$ if $\langle gv, gv\rangle \geq \langle v, v\rangle$ for all $v \in \mathbb{C}^{2n}$. We call $g$ *strictly expansive* if $\langle gv, gv\rangle > \langle v, v\rangle$. We similarly define *weakly contractive* if $\langle gv, gv\rangle \leq \langle v, v\rangle$ and *strictly contractive* if $\langle gv, gv\rangle < \langle v, v\rangle$.

[1] In the sense that $V_\lambda = \cup_{k=1}^{\infty} \mathrm{Ker}((g-\lambda)^k))$

**Lemma 3.2.6** *Let $\langle , \rangle$ be a non-degenerate Hermitian form. Let $g \in \mathrm{GL}(n, \mathbb{C})$ be a strictly contractive or strictly expansive linear automorphism and $\lambda$ an eigenvalue of $g$.*
*Then $|\lambda| \neq 1$.*

**Proof.** Let $v$ be an eigenvector with eigenvalue $\lambda$ then

$$|\lambda|^2 \langle v, v\rangle = \langle gv, gv\rangle < \langle v, v\rangle.$$

Which implies that $|\lambda| \neq 1$. □

**Lemma 3.2.7** *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}[X]$ be the function*

$$f(a_0, \dots, a_n) = \sum_{j=0}^{n} a_j X^j.$$

*Let $g : \mathbb{C}^n \times \mathbb{C}^* \to \mathbb{C}^n/\mathrm{S}_n$ be the function which to $a \in \mathbb{C}^{n+1}$ assigns the roots of $f(a)$ occurring with multiplicity. Give $\mathbb{C}^n/\mathrm{S}_n$ the quotient topology, then $g$ is a continuous function.*

**Proof.** We let $q : \mathbb{C}^n \to \mathbb{C}^n/\mathrm{S}_n$ denote the quotient map. Let $a \in \mathbb{C}^n \times \mathbb{C}^*$ and let $V_a$ be an open neighborhood of $g(a)$ in $\mathbb{C}^n/\mathrm{S}_n$. Let $\lambda_1, \dots, \lambda_n$ be the roots of $f(a)$. Then $q^{-1}(V_a)$ is an open $\mathrm{S}_n$-invariant neighborhood of $(\lambda_1, \dots, \lambda_n)$. Thus there exists $\epsilon > 0$ such that if $|\mu_j - \lambda_j| < \epsilon$ for all $j$ then $q(\mu) \in V_a$. Then it follows from [Mar66, theorem (1,4)] that there exists $\delta > 0$ such that if $|b_j - a_j| < \delta$ then $g(b) \in V_a$. □

**Lemma 3.2.8** *Let $f$ and $g$ be given as in lemma 3.2.7. Let $V$ be a connected non-empty subset of $\mathbb{C}^n \times \mathbb{C}^*$ and assume that there exists $W_1, \dots, W_k$ open pairwise disjoint subsets of $\mathbb{C}$ such that for all $a \in V$ all roots of $f(a)$ are in $\cup_{j=1}^k W_j$. Let $b$ be some element in $V$ and let $n_j$ be the number of roots counted with multiplicity that $f(b)$ has in $W_j$.*
*Then there exists continuous maps*

$$g_j : V \to W_j^{n_j}/\mathrm{S}_{n_j}$$

*such that for any $a \in V$ $f(a)$ has exactly $n_j$ roots in $W_j$ and they are given by $g_j(a)$.*

**Proof.** Let us for $\alpha \in \mathbb{N}^k$ have $W^\alpha = \times_{j=1}^k W_j^{\alpha_j}$. Note that the resemblance of the usual multiindex notation can be deceptive since the Cartesian product is not commutative. However for $\sum_{j=1}^k \alpha_j = n$ and any $\gamma \in \{1, \dots, k\}^n$ such that

$$\#\{i \,|\, \gamma_i = j\} = \alpha_j.$$

Then

$$(\times_{j=1}^n W_{\gamma_j})/\mathrm{S}_n = W^\alpha/\mathrm{S}_n.$$

We also see that $W^\alpha/\mathrm{S}_n$ is open since

$$q^{-1}(W^\alpha/\mathrm{S}_n) = \cup_\gamma \times_{j=1}^n W_{\gamma_j},$$

that is the inverse image under the quotient map is a union of open sets thus open.

Observe that for any $a \in V$ $g(a) \in (\cup_{j=1}^k W_j)^n/\mathrm{S}_n$. Then by the observation in the first paragraph we get

$$(\cup_{j=1}^k W_j)^n/\mathrm{S}_n = (\cup_{\gamma\in\{1,\dots,k\}^n} \times_{j=1}^n W_{\gamma_j})/\mathrm{S}_n = \cup_{\substack{\alpha\in\mathbb{N}^k\\ \sum_{j=1}^k \alpha_j = n}} W^\alpha/\mathrm{S}_n.$$

Let $\alpha, \beta \in \mathbb{N}^k$ be as above and assume that $v \in \mathbb{C}^n$ such that $q(v) \in (W^\alpha/\mathrm{S}_n) \cap (W^\beta/\mathrm{S}_n)$. Then reordering the entries of does not change $q(v)$ and we can thus assume that

$$v_{\sum_{l=1}^{j-1} n_l+1}, \dots, v_{\sum_{l=1}^{j} n_l} \in W_j.$$

But since the $W_j$ are disjoint this implies that $\alpha = \beta$.

We conclude that the $W^\alpha/\mathrm{S}_n$ give a covering of disjoint open connected subsets. Thus since $V$ is connected it follows from lemma 3.2.7 that $g(V) \subset W^\alpha/\mathrm{S}_n$ for some $\alpha$. This implies that the $g_j$ are well-defined functions and then continuity follows by an argument similar to the proof of lemma 3.2.7. □

**Lemma 3.2.9** *Let $\mathfrak{g}$ be a real, semisimple Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Let $S \subset \Delta(\mathfrak{g}, \mathfrak{h})$ be a set of pairwise strongly orthogonal roots. Such that either all roots in $S$ are non-compact imaginary, or all roots in $S$ are real. Let $\mathbf{c}_S$ be the associated Cayley transform and let $\mathfrak{h}_S$ be the Cayley transformed Cartan subalgebra. Let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}_S)$ if $H_{S,\alpha}$ denotes the element in $\mathfrak{h}_{S,\mathbb{C}}$ such that $\alpha = B(H_{S,\alpha}, \cdot)$ then*

$$\begin{aligned} H_{S,\alpha} &= \mathbf{c}_S(H_{\mathbf{c}_S^{-1}(\alpha)}) \\ H'_{S,\alpha} &= \mathbf{c}_S(H'_{\mathbf{c}_S^{-1}(\alpha)}). \end{aligned}$$

**Proof.**

$$\begin{aligned} B(H_{S,\alpha}, H) = \alpha(H) &= (\mathbf{c}_S^{-1}(\alpha))(\mathbf{c}_S^{-1}(H)) \\ &= B(H_{\mathbf{c}_S^{-1}(\alpha)}, \mathbf{c}_S^{-1}(H)) = B(\mathbf{c}_S(H_{\mathbf{c}_S^{-1}(\alpha)}), H). \end{aligned}$$

Where the last equality follows since the Killing form is invariant under inner automorphisms of $\mathfrak{g}_\mathbb{C}$ and Cayley transforms are inner automorphisms of the complexified Lie algebra. We see furthermore that

$$\begin{aligned} |\alpha|^2 &= B(H_{S,\alpha}, H_{S,\alpha}) = B(\mathbf{c}_S(H_{\mathbf{c}_S^{-1}(\alpha)}), \mathbf{c}_S(H_{\mathbf{c}_S^{-1}(\alpha)})) \\ &= B(H_{\mathbf{c}_S^{-1}(\alpha)}, H_{\mathbf{c}_S^{-1}(\alpha)}) = |\mathbf{c}_S^{-1}(\alpha)|^2. \end{aligned}$$ □

## 3.3 $\mathfrak{sp}(n,\mathbb{R})$

The probably most well-studied example of a minimal representation is the metaplectic representation $\pi_{\mathrm{Met}}$ of the metaplectic group $\mathrm{Mp}(n,\mathbb{R})$. We will not provide a comprehensive introduction to the study of the metaplectic representation, this can be found in [Fol89, chapter 4]. We will simply review the necessary characteristics of the metaplectic representation to calculate its character using the methods introduced previously.

**Remark 3.3.1** Consider a function $f:\mathbb{C}\to\mathbb{C}$ we will abuse notation and write

$$\begin{aligned} f &: \mathbb{C}^n \to \mathrm{Mat}_{n\times n}(\mathbb{C}) \\ f(\phi_1,\dots,\phi_n) &= \mathrm{diag}(f(\phi_1),\dots,f(\phi_n)) \end{aligned}$$

where appropriate.

### 3.3.1 Some structure theory

First we will quickly introduce the notation for this section and review the basic structure of $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{Mp}(n,\mathbb{R})$. We give a symplectic structure on $\mathbb{R}^{2n}$ by considering

$$\begin{aligned} J &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \\ \omega(v,w) &= \langle v, Jw\rangle = \sum_{i=1}^{n} v_{n+i}w_i - v_i w_{n+i} \\ \mathrm{Sp}(n,\mathbb{R}) &= \{A\in \mathrm{Mat}_{2n\times 2n}(\mathbb{R}) \,|\, \omega(Av,Aw)=\omega(v,w)\}. \end{aligned}$$

**Lemma 3.3.2** $\mathrm{Sp}(n,\mathbb{R})$ *is a connected, simple Lie group of Hermitian type with maximal compact subgroup* $\mathrm{K}\cong\mathrm{U}(n)$*. A compact Cartan subalgebra of* $\mathrm{Sp}(n,\mathbb{R})$ *is given by*

$$\begin{aligned} \mathfrak{h} &= \{\begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \,|\, \phi\in\mathbb{R}^n\} \\ \mathrm{H} &= \{\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \,|\, \phi\in\mathbb{R}^n\}. \end{aligned}$$

*We get specifically that* $\pi_1(\mathrm{Sp}(n,\mathbb{R}))=\mathbb{Z}$ *and hence* $\mathrm{Mp}(n,\mathbb{R})$ *is the unique double-covering group of* $\mathrm{Sp}(n,\mathbb{R})$*. Let* $p:\mathrm{Mp}(n,\mathbb{R})\to\mathrm{Sp}(n,\mathbb{R})$ *then* $p^{-1}(\mathrm{U}(n))$ *is a maximal compact subgroup of* $\mathrm{Mp}(n,\mathbb{R})$ *and* $p^{-1}(\mathrm{H})$ *a compact Cartan subgroup.*

**Proof.** [Kna02, page 513], [Tor80, section I.1 and I.2], [Fol89, chapter 4]. ☐

**Lemma 3.3.3** *Let* $\mathbf{e}_k \in \mathfrak{h}_{\mathbb{C}}^*$ *be given by* $\mathbf{e}_k(\begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}) = i\phi_k$. *Then for* $\mathrm{Sp}(n, \mathbb{R})$ *with Cartan subgroup* H *we have*

$$\begin{aligned}
\Delta &= \{\pm\mathbf{e}_i \pm \mathbf{e}_j, \pm 2\mathbf{e}_i \,|\, i \neq j\} \\
\Delta^+ &= \{\mathbf{e}_i \pm \mathbf{e}_j, 2\mathbf{e}_i \,|\, i < j\} \\
\Delta_k &= \{\mathbf{e}_i - \mathbf{e}_j \,|\, i \neq j\} \\
\Delta_k^+ &= \{\mathbf{e}_i - \mathbf{e}_j \,|\, i < j\} \\
\Delta_n^+ &= \{\mathbf{e}_i + \mathbf{e}_j, 2\mathbf{e}_i\} \\
W_{\mathrm{K}} &= \{\textit{all permutations of the } \mathbf{e}_i\} \cong \mathrm{S}_n \\
\delta_k &= \frac{1}{2}\sum_{i=1}^{n}(n+1-2i)\mathbf{e}_i
\end{aligned}$$

**Proof.** [Kna02, page 150, 155, 164] □

**Lemma 3.3.4** *Let* $H, H' \in \mathfrak{h}_{\mathbb{C}} \subset \mathfrak{sp}(n, \mathbb{C})$ *then*

$$B(H, H') = (8 + (4n-1)) \sum_{j=1}^{n} \mathbf{e}_j(H)\mathbf{e}_j(H')$$

*and if*

$$H_{\mathbf{e}_j} = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \tag{3.1}$$

*then* $\phi_k = -i\frac{1}{8+4(n-1)}\delta_{j,k}$.

**Proof.** We calculate

$$\begin{aligned}
B(H, H') &= \mathrm{tr}(\mathrm{ad}(H)\,\mathrm{ad}(H')) = \sum_{\alpha\in\Delta} \alpha(H)\alpha(H') \\
&= 8\sum_{j=1}^{n} \mathbf{e}_j(H)\mathbf{e}_j(H') + 2\sum_{j<k}(\mathbf{e}_j \pm \mathbf{e}_k)(H)(\mathbf{e}_j \pm \mathbf{e}_k)(H') \\
&= 8\sum_{j=1}^{n} \mathbf{e}_j(H)\mathbf{e}_j(H') + 2\sum_{j<k} 2\mathbf{e}_j(H)\mathbf{e}_j(H') + 2\mathbf{e}_k(H)\mathbf{e}_k(H') \\
&= 8\sum_{j=1}^{n} \mathbf{e}_j(H)\mathbf{e}_j(H') + 4\sum_{j=1}^{n}(n-j)\mathbf{e}_j(H)\mathbf{e}_j(H') + 4\sum_{j=1}^{n}(j-1)\mathbf{e}_j(H)\mathbf{e}_j(H') \\
&= (8 + 4(n-1))\sum_{j=1}^{n} \mathbf{e}_j(H)\mathbf{e}_j(H').
\end{aligned}$$

Hence if we let $H_{\mathbf{e}_j} \in \mathfrak{h}_{\mathbb{C}}$ be such that $\mathbf{e}_j = B(H_{\mathbf{e}_j}, \cdot)$ and $H_{\mathbf{e}_j} = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}$ then

$$\mathbf{e}_j(H) = B(H_{\mathbf{e}_j}, H) = (8 + 4(n-1)) \sum_{k=1}^{n} \mathbf{e}_k(H_{\mathbf{e}_j}) \mathbf{e}_k(H).$$

But $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are a basis of the dual of the Cartan subalgebra, specifically they are linearly independent and thus we get $i\phi_k = \mathbf{e}_k(H_{\mathbf{e}_j}) = \frac{1}{8+4(n-1)} \delta_{j,k}$. This gives $\phi_k = \frac{-i}{8+4(n-1)} \delta_{j,k}$. □

### The semigroup

**Lemma 3.3.5** *Let $C_{\mathrm{Min}}$ be the minimal cone according to the positive system defined in lemma 3.3.3, let $c_m = C_{\mathrm{Min}} \cap \mathfrak{h}$ then*

$$\begin{aligned} c_m &= \{\begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \mid \phi \in \mathbb{R}^n, \forall j : \phi_j \leq 0\} \\ c_m^o &= \{\begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \mid \phi \in \mathbb{R}^n, \forall j : \phi_j < 0\} \end{aligned} \tag{3.2}$$

**Proof.** Many more details about the cone can be found [Pan81, section II.6] but we include a proof for completeness.

It follows from lemma 3.3.4 and proposition 2.3.2 that

$$c_m = -i \operatorname{span}_{\mathbb{R}_{\geq}}(\{H_{\mathbf{e}_j+\mathbf{e}_k}, H_{2\mathbf{e}_j}\}) \subseteq (-i)^2 \{\begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \mid \phi_k \geq 0\} = \{\begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \mid \phi_k \leq 0\}.$$

Conversely let $X = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}$ and $\phi_j \leq 0$ for all j. Then let $a_j = -\frac{8+4(n-1)}{2}\phi_j \geq 0$ and set $Y = (-i)\sum_{j=1}^{n} a_j H_{2\mathbf{e}_j}$ and let $\psi \in \mathbb{C}^n$ such that $Y = \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix}$ then it follows from lemma 3.3.4 that

$$\psi_j = (-i) \sum_{k=1}^{n} a_k \frac{(-i)}{8 + 4(n-1)} 2\delta_{j,k} = \phi_j.$$

Thus we get $X = Y$. But it follows from proposition 2.3.2 that $Y \in c_m$. □

In order to recognize the character in lemma 3.3.13 as a global function we want some prototype holomorphic functions on $\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(c_m^o)$. Therefore we realize $\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C_{\mathrm{Min}}^o)$ as contractive elements of $\mathrm{Sp}(n, \mathbb{C})$ as done in for example [Ols95].

We extend $\omega(,)$ to a complex bilinear form on $\mathbb{C}^{2n}$. We furthermore introduce the indefinite Hermitian form given by $\Omega\langle v, w\rangle = -i\langle v, Jw\rangle$ where $\langle, \rangle$ is the usual Hermitian form on $\mathbb{C}^{2n}$. Then

$$
\begin{aligned}
\Omega\langle v, w\rangle &= -i\sum_{i=1}^{n} \overline{v_{n+i}}w_i - \overline{v_i}w_{n+i} \\
\mathrm{Sp}(n,\mathbb{C}) &= \{A \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \,|\, \omega(Av, Aw) = \omega(v,w)\} \\
\mathrm{U}(\Omega\langle,\rangle) &= \{A \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \,|\, \Omega\langle Av, Aw\rangle = \Omega\langle v,w\rangle\}.
\end{aligned}
$$

**Lemma 3.3.6** *$\Omega\langle,\rangle$ is a non-degenerate, Hermitian form of signature $(n,n)$ and*

$$\mathrm{Sp}(n,\mathbb{R}) \subset \mathrm{U}(\Omega\langle,\rangle).$$

**Proof.** Consider the basis of $\mathbb{C}^{2n}$ given by $\{\frac{1}{\sqrt{2}}(\vec{e}_j \pm i\vec{e}_{j+n})\}_{j=1}^{n}$ then in this basis $\Omega\langle,\rangle$ is just the standard Hermitian form of signature $(n,n)$

$$
\begin{aligned}
&\Omega\langle \vec{e}_j + i\vec{e}_{j+n}, \vec{e}_j + i\vec{e}_{j+n}\rangle = -2 \\
&\Omega\langle \vec{e}_j - i\vec{e}_{j+n}, \vec{e}_j - i\vec{e}_{j+n}\rangle = 2 \\
&\Omega\langle \vec{e}_j - i\vec{e}_{j+n}, \vec{e}_j + i\vec{e}_{j+n}\rangle = 0 \\
&\Omega\langle \vec{e}_j \pm i\vec{e}_{j+n}, \vec{e}_k \pm i\vec{e}_{k+n}\rangle = 0 \quad \text{for } j \neq k.
\end{aligned}
$$

Let $g \in \mathrm{Sp}(n,\mathbb{R})$ then

$$\Omega\langle gv, gw\rangle = (-i)(gv)^H Jgw = (-i)v^H g^T Jgw = (-i)v^H Jw = \Omega\langle v, w\rangle. \qquad \square$$

**Lemma 3.3.7** *Let $g \in \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ then $g$ is strictly expansive with respect to $\Omega\langle,\rangle$. The eigenvalues of $g$ are $\lambda_1,\ldots,\lambda_n,\lambda_1^{-1},\ldots,\lambda_n^{-1}$ with $|\lambda_1| < 1,\ldots,|\lambda_n| < 1$.*

**Proof.** It follows from Lawsons theorem that $\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}}) = \mathrm{Sp}(n,\mathbb{R})\,\mathrm{Exp}(iC^o_{\mathrm{Min}})$. To see that all elements in the open complex Ol'shanskiĭ semigroup are strictly expansive, it is thus enough by lemma 3.3.6 to observe that the elements of $\mathrm{Exp}(iC^o_{\mathrm{Min}})$ are strictly expansive. Furthermore observe that from lemma 2.4.9 we have $\mathrm{Exp}(i\,\mathrm{Ad}(g)X) = g\,\mathrm{Exp}(iX)g^{-1}$. Then [Pan83, lemma 6] says that any orbit in $C^o_{\mathrm{Min}}$ intersects $\mathfrak{h}$ and hence it is enough to consider $X \in c_m^o = \mathfrak{h} \cap C_{\mathrm{Min}}^o$. Consider $X = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \in c_m^o$ then

$$\begin{aligned}
&\Omega\langle \mathrm{Exp}(iX)v, \mathrm{Exp}(iX)v\rangle - \Omega\langle v, v\rangle \\
&= (-i)v^H \left( \begin{pmatrix} \cos(i\phi) & -\sin(i\phi) \\ \sin(i\phi) & \cos(i\phi) \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \cos(i\phi) & -\sin(i\phi) \\ \sin(i\phi) & \cos(i\phi) \end{pmatrix} - \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \right) v \\
&= (-i)v^H \begin{pmatrix} -2\cos(i\phi)\sin(i\phi) & \sin^2(i\phi) - \cos^2(i\phi) + 1 \\ \cos^2(i\phi) - \sin^2(i\phi) - 1 & -2\cos(i\phi)\sin(i\phi) \end{pmatrix} v \\
&= (-i)v^H \begin{pmatrix} -2\cos(i\phi)\sin(i\phi) & 2\sin^2(i\phi) \\ -2\sin^2(i\phi) & -2\cos(i\phi)\sin(i\phi) \end{pmatrix} v \\
&= v^H \begin{pmatrix} -2\cosh(\phi)\sinh(\phi) & 2i\sinh^2(\phi) \\ -2i\sinh^2(\phi) & -2\cosh(\phi)\sinh(\phi) \end{pmatrix} v \\
&= \sum_{j=1}^{n} -2\cosh(\phi_j)\sinh(\phi_j)(|v_j|^2 + |v_{n+j}|^2) - 2\sinh^2(\phi_j) 2\,\mathrm{Im}(\overline{v_j} v_{n+j}) > 0.
\end{aligned}$$

We get the last equality since $\phi_j < 0$ we have $\sinh(\phi_j) < 0$ and thus the first term in the sum is positive, but then

$$|\sinh^2(\phi_j) 2\,\mathrm{Im}(\overline{v_j} v_{n+j})| < \cosh(\phi_j)|\sinh(\phi_j)|(|v_j|^2 + |v_{n+j}|^2)$$

gives that the sum is positive.

Let $\lambda$ be an eigenvalue of $g \in \mathrm{Sp}(n, \mathbb{C})$ such that $g$ is strictly expansive. Then it follows from lemma 3.2.6 that $|\lambda| \neq 1$. It then follows from lemma 3.2.4 that there are exactly $n$ eigenvalues such that $|\lambda| > 1$ and $n$ such that $|\lambda| < 1$. □

**Proposition 3.3.8** *Let $f : (B_1(0) \setminus \{0\})^n \to \mathbb{C}$ be a continuous function such that for any $\sigma \in \mathrm{S}_n$ we have*

$$f(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) = f(z_1, \ldots, z_n).$$

*For each $g \in \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C_{\mathrm{Min}}^o)$ put $\tilde{f}(g) = f(\lambda_1, \ldots, \lambda_n)$ where $\{\lambda_j\}_{j=1}^n$ are the eigenvalues of $g$ such that $|\lambda_j| < 1$.*

*Then $\tilde{f} : \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C_{\mathrm{Min}}^o) \to \mathbb{C}$ is a continuous function.*

**Proof.** This follows from lemma 3.3.7 and lemma 3.2.8. □

**Proposition 3.3.9** *Let the setup be as in proposition 3.3.8. Assume $f$ is holomorphic.*

*Then $\tilde{f}$ is holomorphic on $\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C_{\mathrm{Min}}^o)$.*

**Proof.** From lemma 2.4.15 it is enough to check holomorphicity on the regular elements. Let $g \in \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C_{\mathrm{Min}}^o)$ and assume $g \in \mathrm{Sp}(n, \mathbb{C})'$. Then $g$ must have all eigenvalues distinct, since $g$ is conjugate in $\mathrm{Sp}(n, \mathbb{C})$ to a regular element in the diagonal Cartan subgroup. Then $\frac{d}{d\lambda} \det(g - \lambda)|_{\lambda=\lambda_i} \neq 0$ for $\lambda_i$ an eigenvalue of $g$ since it has multiplicity 1. Then since $\det(g - \lambda)$ is a holomorphic function on $\mathrm{Sp}(n, \mathbb{C})$ specifically on $\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C_{\mathrm{Min}}^o)$ the holomorphic implicit function theorem[Hör90, theorem 2.1.2] gives that there exists a small neighborhood of $g$ and holomorphic functions $\lambda_1(\cdot), \ldots, \lambda_n(\cdot)$ such that they are the eigenvalues of norm less than 1. Then $\tilde{f}$ in this neighborhood is just given by $f(\lambda_1(\cdot), \ldots, \lambda_n(\cdot))$ which is a composition of holomorphic functions hence holomorphic.□

**Proposition 3.3.10** *Let $f$ be as in proposition 3.3.8 and assume that $f$ extends continuously to $(\overline{B}_1(0) \setminus \{0, \pm 1\})^n$.*

*Then $\tilde{f}$ extends continuously to* $\mathrm{Sp}(n, \mathbb{R})'$.

**Proof.** First of all observe that if there exists a continuous extension of $\tilde{f}$ it is unique since for each element in $\mathrm{Sp}(n, \mathbb{R})'$ there exists a sequence in $\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ converging to it.

Let $g \in \mathrm{Sp}(n, \mathbb{R})'$ then $g$ is a regular element of $\mathrm{Sp}(n, \mathbb{C})$. Hence it is conjugate to a regular element in the diagonal Cartan subgroup. Thus $e^{2\mathbf{e}_i}(g) \neq 1$ for all $i$ which shows $g$ does not have eigenvalue $\pm 1$ thus all eigenvalues must be distinct. Then by the implicit function theorem applied to $(g, \lambda) \mapsto \det(g - \lambda)$ there exists a neighborhood $V_g \subset \mathrm{Sp}(n, \mathbb{C})'$ and continuous functions $\alpha_1, \ldots, \alpha_n$ such that for all $h \in V_g$ the eigenvalues of $h$ are $\alpha_1(h), \alpha_1(h)^{-1}, \ldots, \alpha_n(h), \alpha_n(h)^{-1}$. We can furthermore take $V_g$ smaller so that $\alpha_i(V_g) \cap \alpha_j(V_g) = \emptyset$ for $i \neq j$ and $\alpha_i(V_g) \cap \alpha_j(V_g)^{-1} = \emptyset$ for all $i, j$. Furthermore by proposition 2.2.3 we can take $V_g$ smaller so that $(v, w) \mapsto g \exp(v) \exp(iw)$ is a diffeomorphism onto $V_g$ where $v, w \in W$ for $W \subset \mathfrak{sp}(n, \mathbb{R})$ a convex open neighborhood of 0. By Lawsons theorem the map $\mathrm{Sp}(n, \mathbb{R}) \times C_{\mathrm{Min}} \ni (h, X) \mapsto h \exp(iX)$ is a homeomorphism onto the closed semigroup, hence we can take a potentially even smaller $W$ such that $g \exp(W) \exp(iW) \cap \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C_{\mathrm{Min}}) = g \exp(W) \exp(iW \cap C_{\mathrm{Min}})$.

Let $\gamma \in V_g \cap \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})'$ and name the eigenvalue functions such that

$$|\alpha_1(\gamma)|, \ldots, |\alpha_n(\gamma)| < 1$$

and let $\gamma = g \exp(v_\gamma) \exp(iw_\gamma)$. Assume that there exists $\rho \in V_g$ and $1 \leq i \leq n$ such that $|\alpha_i(\rho)| > 1$ and let $\rho = g \exp(v_\rho) \exp(iw_\rho)$. Then we must have $w_\rho, w_\gamma \in W \cap C^o_{\mathrm{Min}}$. But $W$ and $C^o_{\mathrm{Min}}$ are convex and thus their intersection is convex. Set $h$ to be the curve from $\gamma$ to $\rho$ given by

$$h(t) = g \exp(tv_\rho + (1 - t)v_\gamma) \exp(i(tw_\rho + (1 - t)w_\gamma)).$$

Then $h(t) \in \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ for all $t$ and $|\alpha_i(h(t))|$ is a continuous function into $\mathbb{R}_+$ different from 1 such that $|\alpha_i(h(0))| < 1$ and $|\alpha_i(h(1))| > 1$ which is impossible by the intermediate value theorem. This implies that $|\alpha_i(\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}}))| < 1$ for all $i$. By the continuity of $\alpha_i$ we get $\alpha_i(g) \in \overline{B}_1(0) \setminus \{\pm 1\}$.

Furthermore we conclude that

$$\tilde{f}(\gamma) = f(\alpha_1(\gamma), \ldots, \alpha_n(\gamma)) \text{ for all } \gamma \in V_g \cap \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}}).$$

Thus set $\hat{f}(h) = f(\alpha_1(h), \ldots, \alpha_n(h))$ for $h \in V_g \cap (\mathrm{Sp}(n, \mathbb{R})' \cup \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}}))$. This is a composition of continuous functions hence $\hat{f}$ is continuous and thus a continuous extension of $\tilde{f}$. □

### 3.3.2 The metaplectic representation

**Proposition 3.3.11** *Let $\pi_{\text{Met}}$ be the metaplectic representation of $\mathrm{Mp}(n,\mathbb{R})$. Then*

1. *$\pi_{\text{Met}}$ is a unitary representation, and it decomposes into two irreducible subrepresentations:*
$$\pi_{\text{Met}} = \pi_{\text{Met,Even}} \oplus \pi_{\text{Met,Odd}}.$$
2. *$\pi_{\text{Met,Even}}$ and $\pi_{\text{Met,Odd}}$ are highest-weight representations with respect to the positive system given in lemma 3.3.3.*
3. *Let $\alpha = -\frac{1}{2}\sum_{i=1}^{n} \mathbf{e}_i$ and $\beta = -\mathbf{e}_n$ then the $\mathrm{K}$-type decompositions are given by:*
$$\pi_{\text{Met}}|_{\mathrm{U}(n)} = \widehat{\bigoplus}_{n=0}^{\infty} V_{\alpha+n\beta}$$
$$\pi_{\text{Met,Even}}|_{\mathrm{U}(n)} = \widehat{\bigoplus}_{n=0}^{\infty} V_{\alpha+n2\beta}$$
$$\pi_{\text{Met,Odd}}|_{\mathrm{U}(n)} = \widehat{\bigoplus}_{n=0}^{\infty} V_{\alpha+\beta+n2\beta}.$$
*Where $V_\mu$ denotes the irreducible representation of $\mathrm{U}(n)$ with highest-weight $\mu$.*
4. *We can realize $V_{\alpha+n\beta}$ as the space of homogeneous polynomials of degree $n$ with the action $\mathrm{H}$ given by*
$$\left(\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}.f\right)(z) = e^{-\frac{1}{2}\sum_{i=1}^{n}\phi_i} f(e^{-\phi_1}z_1, \ldots, e^{-\phi_n}z_n). \tag{3.3}$$
*In this realization the set monomials are an orthonormal basis of $\pi_{\text{Met}}$.*
*Note that the square root is well-defined on $\mathrm{Mp}(n,\mathbb{R})$. Since topologically $\mathrm{Sp}(n,\mathbb{R}) \cong \mathrm{U}(n)\times\mathbb{R}^k$. Then $e^{-\frac{1}{2}\sum_{i=1}^{n}\phi_i}$ is just $\frac{1}{\sqrt{\det}}$ on $\mathrm{U}(n)$. But $\det$ is a well-defined, non-zero continuous function on $\mathrm{U}(n)$ hence taking its square root gives a double cover of $\mathrm{U}(n)$, and since all double-coverings of $\mathrm{U}(n)$ are homeomorphic this must be a maximal compact subgroup $p^{-1}(\mathrm{U}(n))$ of $\mathrm{Mp}(n,\mathbb{R})$ and thus we get a well-defined, continuous square root there.*

**Proof.** [Fol89, proposition 4.39, theorem 4.56]. □

**Lemma 3.3.12** *$\pi_{\text{Met}}$, $\pi_{\text{Met,Even}}$ and $\pi_{\text{Met,Odd}}$ extend to $\Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\text{Min}})$.*

**Proof.** Let $X \in c^o_m$ with $X = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}$. Then $i\mathbf{e}_j(X) = i^2\phi_j = -\phi_j > 0$. Thus $i\alpha(c^o_m) > 0$ for all $\alpha \in \Delta_n^+$ and hence $C^o_{\text{Min}}$ is $\Delta_n^+$ adapted in the sense of [Nee00, definition VII.3.6, theorem XI.4.5]. Then since $\pi_{\text{Met,Even}}$ and $\pi_{\text{Met,Odd}}$ are highest-weight representations with respect to this positive system it follows from proposition 3.3.11 and [Nee00, theorem XI.4.5] that both $\pi_{\text{Met,Even}}$ and $\pi_{\text{Met,Odd}}$ extend to $\Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\text{Min}})$. Thus $\pi_{\text{Met}}$ extends to a representation of $\Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\text{Min}})$. □

### 3.3.3 The character on the compact Cartan subgroup

**Lemma 3.3.13** *Let $h \in \mathrm{H}$ and $X \in c_m^o$ and let*

$$\theta_\pm(h\,\mathrm{Exp}(iX)) = \frac{\xi_{-\frac{1}{2}\sum_{j=1}^n \mathbf{e}_j}(h) e^{-i\frac{1}{2}\sum_{j=1}^n \mathbf{e}_j(X)}}{\prod_{j=1}^n 1 \pm \xi_{-\mathbf{e}_j}(h) e^{-i\mathbf{e}_j(X)}}.$$

*Then*

$$\mathrm{tr}(\pi_{\mathrm{Met}}(h\,\mathrm{Exp}(iX))) = \theta_-(h\,\mathrm{Exp}(iX)) \tag{3.4}$$

$$\mathrm{tr}(\pi_{\mathrm{Met,Even}}(h\,\mathrm{Exp}(iX))) = \frac{1}{2}(\theta_-(h\,\mathrm{Exp}(iX)) + \theta_+(h\,\mathrm{Exp}(iX))) \tag{3.5}$$

$$\mathrm{tr}(\pi_{\mathrm{Met,Odd}}(h\,\mathrm{Exp}(iX))) = \frac{1}{2}(\theta_-(h\,\mathrm{Exp}(iX)) - \theta_+(h\,\mathrm{Exp}(iX))). \tag{3.6}$$

We give two different proofs of this lemma, the first straightforward and using the orthonormal basis of monomials. For the second proof we use the formula from proposition 2.6.3 to derive the formula in eq. (3.4). In both cases we observe that using proposition 2.4.18 it is enough to do the computation for $h = e$. This simplifies the computations significantly in the second proof, since we can then ignore the question of whether the weights are analytically integral while computing. Thus observe first that all the weights in the right-hand side of eq. (3.4) are analytically integral, hence the right-hand side is a well-defined holomorphic function on $\Gamma_{\mathrm{H}}(c_m^o)$.

**Proof (Direct).** For $\alpha \in \mathbb{N}^n$ let $f_\alpha(z) = z^\alpha$. Let $X \in c_m^o$ with $X = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}$. It follows from eq. (3.3) that

$$\begin{aligned}
(\pi_{\mathrm{Met}}(e^{tX})f_\alpha)(z) &= e^{-i\frac{1}{2}\sum_{j=1}^n t\phi_j} f_\alpha(e^{-it\phi_1}z_1, \ldots, e^{-it\phi_n}z_n) \\
&= e^{-ti\sum_{j=1}^n(\alpha_j + \frac{1}{2})\phi_j} f_\alpha(z) \\
d\pi_{\mathrm{Met}}(X)f_\alpha &= -i\sum_{j=1}^n (\frac{1}{2} + \alpha_j)\phi_j \\
e^{id\pi_{\mathrm{Met}}(X)} f_\alpha &= (e^{\frac{1}{2}\sum_{j=1}^n \phi_j} \prod_{j=1}^n e^{\alpha_j\phi_j}) f_\alpha.
\end{aligned}$$

We know from proposition 3.3.11 that the monomials are an orthonormal basis hence

$$\mathrm{tr}(e^{id\pi_{\mathrm{Met}}(X)}) = e^{\frac{1}{2}\sum_{j=1}^n \phi_j} \sum_{\alpha\in\mathbb{N}^n} \prod_{j=1}^n e^{\alpha_j\phi_j} = e^{-i\frac{1}{2}\sum_{j=1}^n \mathbf{e}_j(X)} \frac{1}{\prod_{j=1}^n 1 - e^{-i\mathbf{e}_j(X)}}. \tag{3.7}$$

The last equality follows since it is just a product of geometric series, and since $\phi_j < 0$ for all $j$ all the respective geometric series converge absolutely.

An orthonormal basis of the even part of the metaplectic representation is given by all the monomials of even degree. The monomials of odd degree give an orthonormal basis of the odd part. Thus we are interested in the sums

$$a(z) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \text{ even}}} z^\alpha$$
$$b(z) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \text{ odd}}} z^\alpha$$

for $\|z\| < 1$, we know they are absolutely convergent since the multi-index geometric series is absolutely convergent. Furthermore we see that $a$ as a function in $z$ is even and $b$ is odd and $a(z) + b(z) = \prod_{i=1}^n \frac{1}{1-z_i}$. Hence we must have

$$a(z) = \frac{1}{2}\left(\prod_{i=1}^n \frac{1}{1-z_i} + \prod_{i=1}^n \frac{1}{1+z_i}\right)$$
$$b(z) = \frac{1}{2}\left(\prod_{i=1}^n \frac{1}{1-z_i} - \prod_{i=1}^n \frac{1}{1+z_i}\right)$$

and applying this to eq. (3.7) proves the lemma. □

In order to simplify the notation a bit we suppress elements of $\mathfrak{h}$ and for $\alpha \in \mathfrak{h}_\mathbb{C}^*$ let $e^\alpha$ denote the function

$$e^\alpha : \mathfrak{h}_\mathbb{C} \to \mathbb{C} \qquad\qquad e^\alpha(X) = e^{\alpha(X)}.$$

The Weyl group acts on $\mathfrak{h}_\mathbb{C}$ thus it also induces an action on functions on $\mathfrak{h}_\mathbb{C}$, we observe

$$(w.e^\alpha)(X) = e^\alpha(w^{-1}X) = e^{\alpha(w^{-1}X)} = e^{(w.\alpha)(X)} = e^{w.\alpha}(X).$$

**Lemma 3.3.14** *For $n \geq 1$ and $\alpha, \beta \in \mathbb{C}$*

$$\sum_{\sigma \in \mathrm{S}_n} \epsilon(\sigma) e^{\sum_{j=1}^n (-j)\mathbf{e}_{\sigma(j)}} \prod_{j=1}^n (\alpha + \beta e^{-\mathbf{e}_{\sigma(j)}}) = \sum_{k=0}^n \alpha^{n-k}\beta^k \sum_{\sigma \in \mathrm{S}_n} \epsilon(\sigma) e^{\sigma.(\sum_{j=1}^n (-j)\mathbf{e}_j - \sum_{j=0}^{k-1} \mathbf{e}_{n-j})}. \quad (3.8)$$

**Proof.** We proceed by induction, observe that we get the equality for $n = 1$ simply by expanding the product. Let $n \geq 2$ then

$$\begin{aligned}
&\sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{\sum_{j=1}^n(-j)\mathbf{e}_{\sigma(j)}} \prod_{j=1}^{n} (\alpha + \beta e^{-\mathbf{e}_{\sigma(j)}}) \\
=&\alpha \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{\sum_{j=1}^n(-j)\mathbf{e}_{\sigma(j)}} \prod_{j=1}^{n-1} (\alpha + \beta e^{-\mathbf{e}_{\sigma(j)}}) \\
&+ \beta \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{\sum_{j=1}^n(-j)\mathbf{e}_{\sigma(j)}-\mathbf{e}_{\sigma(n)}} \prod_{j=1}^{n-1} (\alpha + \beta e^{-\mathbf{e}_{\sigma(j)}}) \\
=&\alpha \sum_{k=0}^{n-1} \alpha^{n-1-k}\beta^k \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{\sum_{j=1}^n(-j)\mathbf{e}_{\sigma(j)}} e^{\sigma.(\sum_{j=1}^n(-j)\mathbf{e}_j - \sum_{j=0}^{k-1}\mathbf{e}_{n-1-j})} \\
&+ \beta \sum_{k=0}^{n-1} \alpha^{n-1-k}\beta^k \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{\sigma.(\sum_{j=1}^n(-j)\mathbf{e}_j - \sum_{j=0}^{k-1}\mathbf{e}_{n-1-j} - \mathbf{e}_n)}.
\end{aligned}$$

The first equality is simply the expansion of the parenthesis in the $j = n$ part of the product. For the second equality we let $\mathrm{S}_{n-1} \subset \mathrm{S}_n$ such that $\mathrm{S}_{n-1}$ permutes $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ then apply the induction hypothesis for $\mathrm{S}_{n-1}$ and corollary 3.2.2 to both terms. Then we observe that in the first term on the right-hand side of the last equality; for $k \geq 1$ all expressions in the exponential are invariant under $(n-1; n) \in \mathrm{S}_n$. Thus by lemma 3.2.3 they do not contribute. Then collecting the terms we get the right-hand side of eq. (3.8).□

**Corollary 3.3.15** *For $n \geq 1$ and $\alpha, \beta \in \mathbb{C}$*

$$\sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{\sum_{j=1}^n(-j)\mathbf{e}_{\sigma(j)}} \prod_{j=1}^{n-1} (\alpha + \beta e^{-\mathbf{e}_{\sigma(j)}}) = \alpha^{n-1} \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{\sum_{j=1}^n(-j)\mathbf{e}_{\sigma(j)}}.$$

**Proof.** Let $\mathrm{S}_{n-1} \subset \mathrm{S}_n$ such that $\mathrm{S}_{n-1}$ acts on $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ then we apply corollary 3.2.2 and lemma 3.3.14 and get

$$\begin{aligned}
&\sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{\sum_{j=1}^n(-j)\mathbf{e}_{\sigma(j)}} \prod_{j=1}^{n-1} (\alpha + \beta e^{-\mathbf{e}_{\sigma(j)}}) \\
&\qquad = \sum_{k=0}^{n-1} \alpha^{n-1-k}\beta^k \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{\sigma.(\sum_{j=1}^n(-j)\mathbf{e}_j \sum_{j=0}^{k-1}\mathbf{e}_{n-1-j})}.
\end{aligned}$$

We observe that for all terms with $k \geq 1$ the expression in the exponential is invariant under $(n-1; n)$ thus by lemma 3.2.3 they are zero. □

**Lemma 3.3.16**

$$\sum_{\sigma\in S_n}\epsilon(\sigma)e^{\frac{1}{2}\sum_{j=1}^{n}(n+1-2j)\mathbf{e}_{\sigma(j)}}\prod_{j=1}^{n-1}(1-e^{-2\mathbf{e}_{\sigma(j)}})$$
$$=\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k e^{\frac{1}{2}\left(\sum_{j=1}^{n}(n+1-2j)\mathbf{e}_{\sigma(j)}-4\sum_{j=1}^{k}\mathbf{e}_{\sigma(n-(2j-1))}\right)}. \tag{3.9}$$

**Proof.** Let $l=\lceil\frac{n}{2}\rceil$ and $m=\lfloor\frac{n}{2}\rfloor$ and let $S_m$ act on $\mathbf{e}_1,\dots,\mathbf{e}_n$ as permutations of the elements $\mathbf{e}_{n-1},\mathbf{e}_{n-3},\dots$ and $S_l$ as permutations of the elements $\mathbf{e}_n,\mathbf{e}_{n-2},\dots$. Then $S_l\times S_m\subset S_n$ and we can factor the right-hand side of eq. (3.9)

$$\begin{aligned}
&\sum_{\sigma\in S_l\times S_m}\epsilon(\sigma)e^{\frac{1}{2}\sum_{j=1}^{n}(n+1-2j)\mathbf{e}_{\sigma(j)}}\prod_{j=1}^{n-1}(1-e^{-2\mathbf{e}_{\sigma(j)}})\\
=&e^{\frac{1}{2}\sum_{j=1}^{n}(n+1)\mathbf{e}_j+\sum_{j=1}^{l}\mathbf{e}_{2j-1}}\left(\sum_{\sigma\in S_l}\epsilon(\sigma)e^{\sum_{j=0}^{l-1}-2(l-j)\mathbf{e}_{\sigma(n-2j)}}\prod_{j=1}^{l-1}(1-e^{-2\mathbf{e}_{\sigma(n-2j)}})\right)\\
&\cdot\left(\sum_{\sigma\in S_m}\epsilon(\sigma)e^{\sum_{j=1}^{m}-2(m-j)\mathbf{e}_{\sigma(n-(2j-1))}}\prod_{j=1}^{m}(1-e^{-2\mathbf{e}_{\sigma(n-(2j-1))}})\right)\\
=&e^{\frac{1}{2}\sum_{j=1}^{n}(n+1)\mathbf{e}_j+\sum_{j=1}^{l}\mathbf{e}_{2j-1}}\left(\sum_{\sigma\in S_l}\epsilon(\sigma)e^{\sum_{j=0}^{l-1}-2(l-j)\mathbf{e}_{\sigma(n-2j)}}\right)\\
&\cdot\left(\sum_{k=0}^{m}(-1)^k\sum_{\sigma\in S_m}\epsilon(\sigma)e^{\sum_{j=1}^{m}-2(m-j)\mathbf{e}_{\sigma(n-(2j-1))}-\sum_{j=1}^{k}2\mathbf{e}_{\sigma(n-(2j-1)}}\right).
\end{aligned}$$

The first equality is simply rewriting the sum into the product of two sums. For the second equality we apply corollary 3.3.15 to the sum over $S_l$ and lemma 3.3.14 to the sum over $S_m$. The eq. (3.9) follows by multiplying the two expressions together and applying corollary 3.2.2. □

For the proof of eq. (3.6) we need a slight modification of lemma 3.3.16.

**Lemma 3.3.17**

$$\sum_{\sigma\in S_n}\epsilon(\sigma)e^{\frac{1}{2}\sum_{j=1}^{n}(n+1-2j)\mathbf{e}_{\sigma(j)}-\mathbf{e}_{\sigma(n)}}\prod_{j=1}^{n-1}(1-e^{-2\mathbf{e}_{\sigma(j)}})$$
$$=\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k e^{\frac{1}{2}\left(\sum_{j=1}^{n}(n+1-2j)\mathbf{e}_{\sigma(j)}-2\mathbf{e}_{\sigma(n)}-4\sum_{j=1}^{k}\mathbf{e}_{\sigma(n-2j)}\right)}. \tag{3.10}$$

**Proof.**

$$\sum_{\sigma\in \mathrm{S}_n}\epsilon(\sigma)e^{\frac{1}{2}\sum_{j=1}^n(n+1-2j)\mathbf{e}_{\sigma(j)}-\mathbf{e}_{\sigma(n)}}\prod_{j=1}^{n-1}(1-e^{-2\mathbf{e}_{\sigma(j)}})$$

$$=\sum_{\sigma\in \mathrm{S}_n}\epsilon(\sigma)e^{\frac{1}{2}\sum_{j=1}^n(n+1-2j)\mathbf{e}_{\sigma(j)}-\mathbf{e}_{\sigma(n)}}\prod_{j=1}^{n-2}(1-e^{-2\mathbf{e}_{\sigma(j)}})$$
$$-\sum_{\sigma\in \mathrm{S}_n}\epsilon(\sigma)e^{\sigma\cdot\frac{1}{2}\left(\sum_{j=1}^{n-2}(n+1-2j)\mathbf{e}_j-(n+1)\mathbf{e}_{n-1}-(n+1)\mathbf{e}_n\right)}\prod_{j=1}^{n-2}(1-e^{-2\mathbf{e}_{\sigma(j)}}).$$

To the first term on the right-hand side we let $\mathrm{S}_{n-1}\subset \mathrm{S}_n$ such that $\mathrm{S}_{n-1}$ permutes $\mathbf{e}_1,\dots,\mathbf{e}_{n-1}$ then we can apply lemma 3.3.16 to give the right hand side of eq. (3.10). Thus we only need to argue that the second term is zero. But this follows by observing that the exponential is invariant under the cycle $(n-1;n)$. □

**Proof (Even part of lemma 3.3.13).** We attempt to calculate the trace using proposition 2.6.3 and the K-type decomposition, since this approach should generalize to other representations where we do not necessarily know an explicit eigenbasis for the H-action.

Let $X\in c_m^o$ and $\alpha,\beta$ as in proposition 3.3.11 then according to proposition 2.6.3

$$\begin{aligned}\mathrm{tr}(e^{id\pi_{\mathrm{Met,Even}}(X)})&=\sum_{w\in W_{\mathrm{K}}}\frac{\epsilon(w)e^{i(w(\alpha+\delta_k)-\delta_k)(X)}}{(1-e^{i2w\beta(X)})\prod_{\gamma\in\Delta_k^+}(1-e^{-i\gamma(X)})}\\&=\left(\frac{e^{-i\frac{1}{2}\sum_{j=1}^n\mathbf{e}_j(X)}}{e^{i\delta_k(X)}\prod_{\gamma\in\Delta_k^+}(1-e^{-i\gamma(X)})\prod_{j=1}^n(1-e^{-2i\mathbf{e}_j(X)})}\right)\\&\quad\cdot\left(\sum_{\sigma\in \mathrm{S}_n}\epsilon(\sigma)e^{i\frac{1}{2}\sum_{j=1}^n(n+1-2j)\mathbf{e}_{\sigma(j)}(X)}\prod_{j=1}^{n-1}(1-e^{-i2\mathbf{e}_{\sigma(j)}(X)})\right).\end{aligned}\tag{3.11}$$

Where we have used that $w\alpha=\alpha$ for all $w\in W_{\mathrm{K}}$. Then we can apply lemma 3.3.16 to the right-hand side. Let $0\le k\le\lfloor\frac{n}{2}\rfloor$, let $\equiv$ denote equivalence modulo 2 and rewrite

$$\begin{aligned}\sum_{j=1}^n(n+1-2j)\mathbf{e}_j-4\sum_{j=1}^k\mathbf{e}_{n-(2j-1)}=&\sum_{j=1}^{n-2k}(n+1-2j)\mathbf{e}_j+\sum_{\substack{j=n-2k+1\\ n\equiv j+1}}^{n}(n+1-2(j+1))\mathbf{e}_j\\&+\sum_{\substack{j=n-2k+1\\ n\equiv j}}^{n}(n+1-2(j-1))\mathbf{e}_j-\sum_{j=n-2k+1}^{n}2\mathbf{e}_j\end{aligned}\tag{3.12}$$

put $\omega_k=(n;n-1)(n-2;n-3)\cdots(n-(2k-2);n-(2k-1))$ then we get

$$\sum_{j=1}^n(n+1-2j)\mathbf{e}_j-4\sum_{j=1}^k\mathbf{e}_{n-(2j-1)}=\sum_{j=1}^n(n+1-2j)\mathbf{e}_{\omega_k(j)}-2\sum_{j=n-2k+1}^{n}\mathbf{e}_{\omega_k(j)}\tag{3.13}$$

We see that $\epsilon(\omega_k) = (-1)^k$ hence by reordering terms we get

$$\begin{aligned}
&\sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} (-1)^k e^{i\frac{1}{2}\left(\sum_{j=1}^n (n+1-2j)\mathbf{e}_{\sigma(j)} - 4\sum_{j=1}^k \mathbf{e}_{\sigma(n-(2j-1))}\right)(X)} \\
&\qquad = \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \epsilon(\omega_k) e^{i\frac{1}{2}\left(\sum_{j=1}^n (n+1-2j)\mathbf{e}_{\sigma(\omega_k(j))}(X) - 2\sum_{j=0}^{2k-1} \mathbf{e}_{\sigma(\omega_k(n-j))}\right)(X)} \\
&\qquad\qquad = \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} e^{i\frac{1}{2}\left(\sum_{j=1}^n (n+1-2j)\mathbf{e}_{\sigma(j)}(X) - 2\sum_{j=0}^{2k-1} \mathbf{e}_{\sigma(n-j)}\right)(X)} \\
&= \frac{1}{2}\left(\prod_{j=1}^n (1+e^{-i\mathbf{e}_j(X)}) + \prod_{j=1}^n (1-e^{-i\mathbf{e}_j(X)})\right) \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{i\frac{1}{2}\sum_{j=1}^n (n+1-2j)\mathbf{e}_{\sigma(j)}(X)}. \qquad (3.14)
\end{aligned}$$

Where the last equality comes from two applications of lemma 3.3.14. Then combining eq. (3.11) and eq. (3.14) we get

$$\begin{aligned}
\mathrm{tr}(\pi_{\mathrm{Met,Even}}(\mathrm{Exp}(iX))) &= \frac{e^{i\alpha(X)}(\prod_{j=1}^n (1-e^{-i\mathbf{e}_j(X)}) + \prod_{j=1}^n (1+e^{-i\mathbf{e}_j(X)}))}{\prod_{j=1}^n 1-(e^{-i\mathbf{e}_j(X)})^2} \\
&= \frac{1}{2}(\theta_+(\mathrm{Exp}(iX)) + \theta_-(\mathrm{Exp}(iX))). \qquad \square
\end{aligned}$$

**Proof (Odd part of lemma 3.3.13).** This follows by very similar computations as in the proof of eq. (3.5).

Let $X \in c_m^o$, $\alpha$ and $\beta$ as in proposition 3.3.11 then it follows from proposition 2.6.3 that

$$\begin{aligned}
\mathrm{tr}(\pi_{\mathrm{Met,Odd}}(\mathrm{Exp}(iX))) = &\left(\frac{e^{-i\frac{1}{2}\sum_{j=1}^n \mathbf{e}_j(X)}}{e^{i\delta_k(X)}\prod_{\gamma\in\Delta_k^+}(1-e^{-i\gamma(X)})\prod_{j=1}^n (1-e^{-2i\mathbf{e}_j(X)})}\right) \\
&\cdot\left(\sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{i\frac{1}{2}\sum_{j=1}^n (n+1-2j)\mathbf{e}_{\sigma(j)}(X) - i\mathbf{e}_{\sigma(n)}(X)} \prod_{j=1}^{n-1}(1-e^{-i2\mathbf{e}_{\sigma(j)}(X)})\right). \qquad (3.15)
\end{aligned}$$

For $0 \le k \le \lfloor \frac{n}{2} \rfloor$ put $\omega_k = (n-1;n-2)(n-3;n-4)\cdots(n-2k+1;n-2k)$. Then by a calculation very similar to eq. (3.12) and eq. (3.13) we get

$$\sum_{j=1}^n (n+1-2j)\mathbf{e}_j - 2\mathbf{e}_n - 4\sum_{j=1}^k \mathbf{e}_{n-2j} = \sum_{j=1}^n (n+1-2j)\mathbf{e}_{\omega_k(j)} - 2\sum_{j=n-2k}^n \mathbf{e}_{\omega_k(j)}.$$

Note that $\epsilon(\omega_k) = (-1)^k$ and hence from lemma 3.3.17 and lemma 3.3.14 we get

$$\begin{aligned}
&\sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{i\frac{1}{2}\sum_{j=1}^n (n+1-2j)\mathbf{e}_{\sigma(j)}(X) - i\mathbf{e}_{\sigma(n)}(X)} \prod_{j=1}^{n-1}(1-e^{-i2\mathbf{e}_{\sigma(j)}(X)}) \\
&\qquad = \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \epsilon(\omega_k) e^{i\frac{1}{2}\left(\sum_{j=1}^n (n+1-2j)\mathbf{e}_{\sigma(\omega_k(j))} - 2\sum_{j=n-2k}^n \mathbf{e}_{\sigma(\omega_k(j))}\right)(X)}. \\
&= \frac{1}{2}\left(\prod_{j=1}^n (1+e^{-i\mathbf{e}_j(X)}) - \prod_{j=1}^n (1-e^{-i\mathbf{e}_j(X)})\right) \sum_{\sigma\in \mathrm{S}_n} \epsilon(\sigma) e^{i\frac{1}{2}\sum_{j=1}^n (n+1-2j)\mathbf{e}_{\sigma(j)}(X)}. \qquad \square
\end{aligned}$$

Then we get eq. (3.4) simply by addition of eq. (3.5) and eq. (3.6).

### 3.3.4 The character function

**Lemma 3.3.18** *Let $p : \Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}}) \to \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ denote the covering morphism. For $g \in \Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ let $\lambda_1(g), \dots, \lambda_n(g)$ denote the eigenvalues of $p(g)$ with $|\lambda_j| < 1$.*

*Then there exists a continuous function $f$ on $\mathrm{Mp}(n,\mathbb{R})' \cup \Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ such that for $g \in \Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ and $X \in c^o_m$ it satisfies*

$$\begin{aligned}
f(g)^2 &= \prod_{j=1}^n \lambda_j(g) \\
f(\mathrm{Exp}(iX)) &= e^{-i\frac{1}{2}\sum_{j=1}^n \mathbf{e}_j(X)}.
\end{aligned}$$

*Furthermore $f$ is holomorphic on $\Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$.*

**Proof.** Let $h(z_1, \dots, z_n) = \prod_{j=1}^n z_j$ then $h$ is a holomorphic function invariant under $\mathrm{S}_n$ and according to propositions 3.3.9 and 3.3.10 $\tilde{h}$ is a continuous function on $\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ holomorphic on the set of regular elements. Since $\pi_1(\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})) = \mathbb{Z}$ the semigroup has a unique double cover. Thus since $\tilde{h}$ is a non-zero continuous function it must have a continuous square root on the double cover $\Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$. The square root is holomorphic on the set of regular elements since $\tilde{h}$ is. Let $f$ denote the square root such that for $X \in c^o_m$ $f(\mathrm{Exp}(iX)) = e^{-i\frac{1}{2}\sum_{j=1}^n \mathbf{e}_j(X)}$. We want to show that $f$ extends continuously to $\mathrm{Mp}(n,\mathbb{R})'$. First of all observe that there can be at most one continuous extension of $f$ since for every element $g \in \mathrm{Mp}(n,\mathbb{R})'$ we can find a sequence in $\Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ converging to $g$. Choose a convex open neighborhood $0 \in W \subset \mathfrak{sp}(n,\mathbb{R})$ such that

$$W \times (\{0\} \cup (W \cap C^o_{\mathrm{Min}})) \ni (X, Y) \mapsto g\exp(X)\,\mathrm{Exp}(iY)$$

is a homeomorphism and has image in $\mathrm{Mp}(n,\mathbb{R})' \cup \Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$. Let

$$\begin{aligned}
B &= g\exp(W)\,\mathrm{Exp}(i(\{0\} \cup (W \cap C^o_{\mathrm{Min}}))) \\
C &= B \cap \Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}}) = g\exp(W)\,\mathrm{Exp}(i(W \cap C^o_{\mathrm{Min}})).
\end{aligned}$$

Then $B$ and $C$ are simply-connected. By proposition 3.3.10 $\tilde{h}\circ p$ is a continuous non-zero function on $B$. Thus there exists a continuous function $\alpha$ on $B$ such that $\alpha^2 = \tilde{h}\circ p$ we can furthermore assume that $\alpha$ agrees with $f$ on a single element of $C$. But then $\alpha|_C$ and $f|_C$ are two continuous square roots of $\tilde{h}\circ p$ which agree on a single point of $C$ hence they must agree on all of $C$ since $C$ is connected. Then $\alpha$ is a continuous extension of $f$. $\square$

**Proposition 3.3.19** *Let* $p : \Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}}) \to \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ *denote the covering morphism. For* $g \in \Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ *let* $\lambda_1(g),\dots,\lambda_n(g)$ *denote the eigenvalues of* $p(g)$ *with* $|\lambda_j| < 1$ *then*

$$\mathrm{tr}(\pi_{\mathrm{Met}}(g)) = \frac{\sqrt{\prod_{j=1}^n \lambda_j(g)}}{\prod_{j=1}^n 1-\lambda_j(g)} \tag{3.16}$$

$$\mathrm{tr}(\pi_{\mathrm{Met,Even}}(g)) = \frac{\sqrt{\prod_{j=1}^n \lambda_j(g)}}{2}\left(\frac{1}{\prod_{j=1}^n 1-\lambda_j(g)} + \frac{1}{\prod_{j=1}^n 1+\lambda_j(g)}\right) \tag{3.17}$$

$$\mathrm{tr}(\pi_{\mathrm{Met,Odd}}(g)) = \frac{\sqrt{\prod_{j=1}^n \lambda_j(g)}}{2}\left(\frac{1}{\prod_{j=1}^n 1-\lambda_j(g)} - \frac{1}{\prod_{j=1}^n 1+\lambda_j(g)}\right). \tag{3.18}$$

*Furthermore* $\mathrm{tr}\circ\pi_{\mathrm{Met}}$, $\mathrm{tr}\circ\pi_{\mathrm{Met,Even}}$ *and* $\mathrm{tr}\circ\pi_{\mathrm{Met,Odd}}$ *extend continuously to* $\mathrm{Mp}(n,\mathbb{R})'$.

**Proof.** Let $f$ denote the function from lemma 3.3.18. Let

$$h_\pm(z_1,\dots,z_n) = \frac{1}{\prod_{j=1}^n (1 \pm z_i)}.$$

Then $h_\pm$ are holomorphic functions on $(\mathbb{C}\setminus\{\pm 1\})^n$ invariant under permutations. Thus by propositions 3.3.8, 3.3.8 and 3.3.10 we get continuous functions $\tilde{h}_\pm$ on $\mathrm{Sp}(n,\mathbb{R})' \cup \Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$ which are holomorphic on $\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$. Then it follows from lemma 3.3.13 that for $X \in c^o_m$

$$\begin{aligned}
\mathrm{tr}(\pi_{\mathrm{Met}}(\mathrm{Exp}(iX))) &= \frac{f(\mathrm{Exp}(ix))}{\tilde{h}_-(\mathrm{Exp}(iX))}\\
\mathrm{tr}(\pi_{\mathrm{Met,Even}}(g)) &= \frac{f(\mathrm{Exp}(ix))}{2}\left(\frac{1}{\tilde{h}_-(\mathrm{Exp}(iX))} + \frac{1}{\tilde{h}_+(\mathrm{Exp}(iX))}\right)\\
\mathrm{tr}(\pi_{\mathrm{Met,Odd}}(g)) &= \frac{f(\mathrm{Exp}(ix))}{2}\left(\frac{1}{\tilde{h}_-(\mathrm{Exp}(iX))} - \frac{1}{\tilde{h}_+(\mathrm{Exp}(iX))}\right).
\end{aligned}$$

Then since both the right- and left-hand sides are holomorphic functions on regular elements it follows from proposition 2.4.18 that they agree on $\Gamma_{\mathrm{Mp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$. $\square$

Let $f_n : \mathbb{C} \to \mathbb{C}$ denote the function

$$f_n(z) = \begin{cases} \sinh(z) & n \text{ even} \\ \cosh(z) & n \text{ odd} \end{cases}.$$

**Remark 3.3.20** [Tor80, section I.2] describes a maximal family of non-conjugate Cartan subgroups of $\mathrm{Sp}(n,\mathbb{R})$ invariant under the Cartan involution. Let $a, b, s_j, t_j, v_j, u_j$ as in theorem 3.3.21 and let $g$ by the element of Mp corresponding to these. Then this corresponds to the element in the parametrization of [Tor80] given by

$$\begin{aligned} &m = a && p = n - 2a - b && n - 2m - p = b \\ &t_j = s_j && \nu_j = v_j && c_j = u_j + in_j \\ &\mu_j = t_j. \end{aligned}$$

Where the left hand side is the notation of [Tor80] and the right hand side is the one used in theorem 3.3.21. Then [Tor80, proposition 1] says that this parametrization is surjective for $n \neq 2a$. To see this observe first that the parametrization is surjective on $\mathrm{Sp}(n,\mathbb{R})$, this is immediate from the explicit description in [Tor80]. Then the only complication will be Cartan subgroups $H \subset \mathrm{Sp}(n,\mathbb{R})$ such that $p^{-1}(H)_e$ does not contain $e_-$ where $e_-$ is the element of $p^{-1}(e_{\mathrm{Sp}})$ such that $e_- \neq e_{\mathrm{Mp}}$. That is the identity component of the inverse image of H under the projection from $\mathrm{Mp}(n,\mathbb{R})$ to $\mathrm{Sp}(n,\mathbb{R})$ does not contain the "negative" identity. Observe that in $\mathrm{Mp}(n,\mathbb{R})$ we have $\exp(2n_j\pi i H'_{2\mathbf{e}_j}) = e_-$ thus if $b > 0$ or $n - 2a - b > 0$ we have that $e_-$ is in the identity component. Thus the only case left is $n = 2a$. But this is exactly why we consider the case separately in theorem 3.3.21 where we get the non-identity component by putting $k = 1$.

**Theorem 3.3.21** *Let $\theta_{\mathrm{Met}}$, $\theta_{\mathrm{Met,Even}}$ and $\theta_{\mathrm{Met,Odd}}$ denote the character functions of the metaplectic representation, the even part and the odd part respectively. Let $\psi = \theta_{\mathrm{Met,Even}} - \theta_{\mathrm{Met,Odd}}$ denote the character of the virtual representation $\pi_{\mathrm{Met,Even}} - \pi_{\mathrm{Met,Odd}}$.*

*Let $0 \leq a < \frac{n}{2}$, $0 \leq b \leq n$ such that*

$$\begin{gathered} 2a + b \leq n \\ s_1, t_1, \ldots, s_a, t_a, v_1, \ldots, v_{n-2a-b}, u_1, \ldots, u_b \in \mathbb{R} \\ n_1, \ldots, n_b \in \mathbb{Z} \end{gathered}$$

*and*

$$\begin{aligned} g = &\prod_{j=1}^{a} \exp(s_j(E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}} + \overline{E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}}) + t_j i H'_{\mathbf{e}_{2j-1}-\mathbf{e}_{2j}}) \\ &\prod_{j=1}^{b} \exp(n_j \pi i H'_{2\mathbf{e}_{2a+j}}) \exp(u_j(E_{2\mathbf{e}_{2a+j}} + \overline{E_{2\mathbf{e}_{2a+j}}})) \\ &\prod_{j=1}^{n-2a-b} \exp(v_j i H'_{2\mathbf{e}_{2a+b+j}}) \end{aligned} \tag{3.19}$$

*then*

$$\begin{aligned}\theta_{\mathrm{Met}}(g) =&2^{-(n-a)}\prod_{j=1}^{a}(\cosh(s_j)-\cos(t_j))^{-1}\cdot\prod_{j=1}^{b}(-i)^{n_j}|f_{n_j}(\frac{u_j}{2})|^{-1}\\ &\cdot(-i)^{n-2a-b}\prod_{j=1}^{n-2a-b}\sin(\frac{v_j}{2})^{-1}\\ \psi(g) =&2^{-(n-a)}\prod_{j=1}^{a}(\cosh(s_j)+\cos(t_j))^{-1}\cdot\prod_{j=1}^{b}(-i)^{n_j}|f_{1+n_j}(\frac{u_j}{2})|^{-1}\\ &\cdot\prod_{j=1}^{n-2a-b}\cos(\frac{v_j}{2})^{-1}.\end{aligned}\tag{3.20}$$

*If $n=2a$, $s_1,t_1,\ldots,s_a,t_a\in\mathbb{R}$ and $k\in\mathbb{Z}$ and we set*

$$g=\exp(2k\pi iH'_{2\mathbf{e}_1})\prod_{j=1}^{a}\exp(s_j(E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}+\overline{E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}})+t_jiH'_{\mathbf{e}_{2j-1}-\mathbf{e}_{2j}})$$

*then*

$$\begin{aligned}\theta_{\mathrm{Met}}(g)&=(-1)^k2^{-a}\prod_{j=1}^{a}(\cosh(s_j)-\cos(t_j))^{-1}\\ \psi(g)&=(-1)^k2^{-a}\prod_{j=1}^{a}(\cosh(s_j)+\cos(t_j))^{-1}.\end{aligned}\tag{3.21}$$

**Proof.** For $\phi\in\mathbb{C}^n$ the eigenvalues of

$$\mathrm{Sp}(n,\mathbb{C})\ni g_\phi=\exp(\begin{pmatrix}0&-\phi\\ \phi&0\end{pmatrix})=\begin{pmatrix}\cos(\phi)&-\sin(\phi)\\ \sin(\phi)&\cos(\phi)\end{pmatrix}$$

are $\{e^{i\phi_1},e^{-i\phi_1},\ldots,e^{i\phi_n},e^{-i\phi_n}\}$ and it follows from lemma 3.3.5 that if $\operatorname{Im}\phi_j<0$ for all $j$ we get $g_\phi\in\Gamma_{\mathrm{Sp}(n,\mathbb{R})}(C^o_{\mathrm{Min}})$. Then $|e^{i\phi_j}|>1$ and thus we can let $\lambda_j(g_\phi)=e^{-i\phi_j}$. We thus get from lemma 3.3.18 and proposition 3.3.19 that

$$\begin{aligned}\theta_{\mathrm{Met}}(\exp(\begin{pmatrix}0&-\phi\\ \phi&0\end{pmatrix}))&=\frac{e^{-i\frac{1}{2}\sum_{j=1}^n\phi_j}}{\prod_{j=1}^n 1-e^{-i\phi_j}}=(\prod_{j=1}^{n}e^{\frac{1}{2}i\phi_j}-e^{-\frac{1}{2}i\phi_j})^{-1}\\ \psi(\exp(\begin{pmatrix}0&-\phi\\ \phi&0\end{pmatrix}))&=\frac{e^{-i\frac{1}{2}\sum_{j=1}^n\phi_j}}{\prod_{j=1}^n 1+e^{-i\phi_j}}=(\prod_{j=1}^{n}e^{\frac{1}{2}i\phi_j}+e^{-\frac{1}{2}i\phi_j})^{-1}.\end{aligned}\tag{3.22}$$

Note that the formula hold for all $\phi\in\mathbb{C}^n_{\mathrm{Im}<0}$ specifically by continuous extension it also holds for $\phi\in\mathbb{R}^n$ such that $\exp(\begin{pmatrix}0&-\phi\\ \phi&0\end{pmatrix})$ is regular. Let $S\subset\Delta_n^+$ be a set of pairwise strongly orthogonal roots. Then possibly after acting by an element from the compact Weyl group we can assume that $S = \{\mathbf{e}_1+\mathbf{e}_2, \dots, \mathbf{e}_{2k-1}+\mathbf{e}_{2k}, 2\mathbf{e}_{2k+1}, \dots, 2\mathbf{e}_{2k+l}\}$. Consider $H'_{\mathbf{e}_j\pm\mathbf{e}_k} = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}$ then it follows from lemma 3.3.4 that $\phi_l = -i(\delta_{l,j} \pm \delta_{l,k})$. Similarly if $H_{2\mathbf{e}_j} = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}$ then $\phi_k = -i\delta_{j,k}$. Then eq. (3.20) follows from eq. (3.22) by applying proposition 2.8.9.

For eq. (3.21) note that $H'_{2\mathbf{e}_1} = \frac{1}{2}(H'_{\mathbf{e}_1+\mathbf{e}_2} + H'_{\mathbf{e}_1-\mathbf{e}_2})$ and then eq. (3.21) follows from eq. (3.22) and proposition 2.8.9. □

**Remark 3.3.22** A comparison of the formulas in theorem 3.3.21 with the literature gives

- The formula for $\theta_{\mathrm{Met}}$ in theorem 3.3.21 agrees with the formula for $\theta(\psi_-)$ in [Ada97, proposition 4.7].

- The formula for $\theta_{\mathrm{Met}}$ is the conjugation of the formula in [Tor80, théorème 2] for the character of the metaplectic representation.

But the character function of a contragredient representation is just the complex conjugate of the character function. Thus the above formula agrees with the litteratue and the only disagreement is whether the metaplectic should be a highest-weight or a lowest-weight representation in the positive system from lemma 3.3.3.

Theorem 3.3.21 specifies the characters of all minimal representations of $\mathfrak{sp}(n, \mathbb{R})$: The even part of the metaplectic representation, the odd part of the metaplectic and their contragredients(by conjugation).

## 3.4 $\mathfrak{su}(p, q)$

### 3.4.1 Structure and realization

We fix a realization of $\mathrm{SU}(p, q)$ by

$$\begin{aligned} I_k &= \mathrm{diag}(1, 1, \dots, 1) \in \mathrm{Mat}_{k\times k} \\ I_{p,q} &= \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \in \mathrm{Mat}_{(p+q)\times(p+q)} \\ s(v, w) &= v^T I_{p,q} w \\ S\langle v, w\rangle &= v^H I_{p,q} w \\ \mathrm{SU}(p, q) &= \{A \in \mathrm{SL}(p+q, \mathbb{C}) \mid \forall v, w \in \mathbb{C}^{p+q} : S\langle Av, Aw\rangle = S\langle v, w\rangle\}. \end{aligned}$$

Let $A : \mathbb{C}^{p+q} \to \mathrm{Mat}_{(p+q)\times(p+q)}(\mathbb{C})$ be given by

$$A(\phi) = \begin{pmatrix} -i\phi_1 & & & & & \\ & \ddots & & & & \\ & & -i\phi_p & & & \\ & & & i\phi_{p+1} & & \\ & & & & \ddots & \\ & & & & & i\phi_{p+q} \end{pmatrix} \tag{3.23}$$

**Proposition 3.4.1** *Let* K *denote* $S(\mathrm{U}(p)\times\mathrm{U}(q))$ *embedded as block-diagonal matrices in* $\mathrm{SU}(p,q)$ *then* K *is a maximal compact subgroup. Let*

$$\mathfrak{h} = \{\begin{pmatrix} i\theta_1 & & & \\ & i\theta_2 & & \\ & & \ddots & \\ & & & i\theta_{p+q} \end{pmatrix} \,|\, \theta_1,\dots,\theta_{p+q} \in \mathbb{R}, \sum_{j=1}^{p+q}\theta_j = 0\}$$

$$\mathrm{H} = \{\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_{p+q}} \end{pmatrix} \,|\, \theta_1,\dots,\theta_{p+q} \in \mathbb{R}, 1 = \prod_{j=1}^{p+q} e^{i\theta_j}\}.$$

*Then* H *is a Cartan subgroup of* $\mathrm{SU}(p,q)$ *and a maximal torus in* $S(\mathrm{U}(p)\times\mathrm{U}(q))$*. Furthermore* $\mathrm{SL}(p+q,\mathbb{C})$ *is a complexification of* $\mathrm{SU}(p,q)$*.*

*Let* $\mathbf{e}_j \in \mathfrak{h}^*_{\mathbb{C}}$ *be given by* $\mathbf{e}_j(\begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_{p+q} \end{pmatrix}) = \theta_j$*. Then*

$$\begin{aligned}
\Delta &= \{\mathbf{e}_j - \mathbf{e}_k \,|\, 1 \le j \ne k \le n\}\\
\Delta^+ &= \{\mathbf{e}_j - \mathbf{e}_k \,|\, 1 \le j < k \le n\}\\
\Delta_k &= \{\mathbf{e}_j - \mathbf{e}_k \,|\, 1 \le j \ne k \le p \text{ or } p+1 \le j \ne k \le p+q\}\\
\Delta_k^+ &= \{\mathbf{e}_j - \mathbf{e}_k \,|\, 1 \le j < k \le p \text{ or } p+1 \le j < k \le p+q\}\\
\Delta_n^+ &= \{\mathbf{e}_j - \mathbf{e}_k \,|\, 1 \le j \le p < k \le p+q\}\\
W_{\mathrm{K}} &\cong \mathrm{S}_p \times \mathrm{S}_q\\
\delta_k &= \frac{1}{2}\sum_{j=1}^{p}(p+1-2j)\mathbf{e}_j + \frac{1}{2}\sum_{j=1}^{q}(q+1-2j)\mathbf{e}_{p+j}.
\end{aligned}$$

**Remark 3.4.2** $W_{\mathrm{K}}$ acts on $\mathfrak{h}^*_{\mathbb{C}}$ such that $\mathrm{S}_p$ permutes $\mathbf{e}_1,\dots,\mathbf{e}_p$ and $\mathrm{S}_q$ permutes $\mathbf{e}_{p+1},\dots,\mathbf{e}_{p+q}$.

**Proof.** [Kna02, page 150, 155, 164, 513]. $\square$

**Lemma 3.4.3** *Let $H, H' \in \mathfrak{h}_{\mathbb{C}} \subset \mathfrak{su}(p,q)_{\mathbb{C}}$ then*

$$B(H,H') = 2(p+q)\sum_{j=1}^{p+q} \mathbf{e}_j(H)\mathbf{e}_j(H')$$

*and let* $\begin{pmatrix} \phi_1 & & \\ & \ddots & \\ & & \phi_{p+q} \end{pmatrix} = H_{\mathbf{e}_j - \mathbf{e}_k} \in \mathfrak{h}_{\mathbb{C}}$ *be the element such that* $\mathbf{e}_j - \mathbf{e}_k = B(H_{\mathbf{e}_j-\mathbf{e}_k}, \cdot)$ *then*

$$\phi_l = \frac{1}{2(p+q)}(\delta_{l,j} - \delta_{l,k}).$$

**Proof.** We calculate

$$\begin{aligned}
B(H,H') &= 2\sum_{1\le j<k\le p+q} (\mathbf{e}_j - \mathbf{e}_k)(H)(\mathbf{e}_j - \mathbf{e}_k)(H') \\
&= 2(\sum_{1\le j<k\le p+q} \mathbf{e}_j(H)\mathbf{e}_j(H') + \mathbf{e}_k(H)\mathbf{e}_k(H') - \mathbf{e}_j(H)\mathbf{e}_k(H') - \mathbf{e}_k(H)\mathbf{e}_j(H')) \\
&= 2(p+q-1)\sum_{j=1}^{p+q} \mathbf{e}_j(H)\mathbf{e}_j(H') - 2\sum_{j=1}^{p+q}\mathbf{e}_j(H)(\sum_{\substack{1\le k\le p+q \\ k\neq j}} \mathbf{e}_k(H')) \\
&= 2(p+q)\sum_{j=1}^{p+q} \mathbf{e}_j(H)\mathbf{e}_j(H') - 2\sum_{j=1}^{p+q}\mathbf{e}_j(H)(\sum k = 1^{p+q}\mathbf{e}_k(H')) \\
&= 2(p+q)\sum_{j=1}^{p+q} \mathbf{e}_j(H)\mathbf{e}_j(H') - 2(\sum_{j=1}^{p+q}\mathbf{e}_j)(H)(\sum_{j=1}^{p+q}\mathbf{e}_j)(H') \\
&= 2(p+q)\sum_{j=1}^{p+q} \mathbf{e}_j(H)\mathbf{e}_j(H').
\end{aligned}$$

Where the last equality comes since $\mathfrak{su}(p,q)_{\mathbb{C}} = \mathfrak{sl}(p+q,\mathbb{C})$ thus all matrices are trace-less and $\sum_{j=1}^n \mathbf{e}_j = \mathrm{tr}$. Thus the last term is just a product of traces hence zero.

Let $\begin{pmatrix} \phi_1 & & \\ & \ddots & \\ & & \phi_{p+q} \end{pmatrix} = H_{\mathbf{e}_j - \mathbf{e}_k} \in \mathfrak{h}_{\mathbb{C}}$ and assume that $j,k < p+q$ then

$$\mathbf{e}_j(H) - \mathbf{e}_k(H) = B(H_{\mathbf{e}_j-\mathbf{e}_k}, H) = 2(p+q)\sum_{l=1}^{p+q}\phi_l\mathbf{e}_l(H) = 2(p+q)\sum_{l=1}^{p+q-1}(\phi_l - \phi_{p+q})\mathbf{e}_l(H).$$

Since $\mathfrak{sl}(p+q,\mathbb{C})$ has rank $p+q$ the linear functionals $\mathbf{e}_1, \ldots, \mathbf{e}_{p+q-1}$ are linearly independent. This implies that $\phi_l - \phi_{p+q} = \frac{1}{2(p+q)}(\delta_{l,j} - \delta_{l,k})$. Furthermore since $H_{\mathbf{e}_j-\mathbf{e}_k} \in \mathfrak{h}_{\mathbb{C}}$

we must have

$$0 = \sum_{l=1}^{p+q} \phi_j = (p+q)\phi_{p+q} + \sum_{l=1}^{p+q-1} (\phi_l - \phi_{p+q}) = (p+q)\phi_{p+q} + \frac{1}{2(p+q)}(1-1) = (p+q)\phi_{p+q}.$$

Which implies $\phi_{p+q} = 0$ and thus lemma 3.4.3. The assumption that $j, k < p+q$ is not a restriction on the argument when $p+q \geq 3$ since we can simply reorder the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{p+q}$ to ensure that this is satisfied. Thus we only need to consider the case $p+q = 2$. Then we see

$$2\mathbf{e}_1(H) = \mathbf{e}_1(H) - \mathbf{e}_2(H) = B(H_{\mathbf{e}_1-\mathbf{e}_2}, H) = 4(\phi_1 - \phi_2)\mathbf{e}_1(H).$$

Which implies that $\phi_1 - \phi_2 = \frac{1}{2}$ furthermore since $H_{\mathbf{e}_1-\mathbf{e}_2} \in \mathfrak{h}_{\mathbb{C}}$ we know $\phi_1 + \phi_2 = 0$. Thus $-\phi_2 = \phi_1 = \frac{1}{4}$ and $\phi_j = \frac{1}{2(p+q)}(\delta_{j,1} - \delta_{j,2})$. And the argument is symmetric for $H_{\mathbf{e}_2-\mathbf{e}_1}$. □

**Lemma 3.4.4** *Let $C_{\mathrm{Min}}$ denote the minimal cone associated to the positive system in proposition 3.4.1 and let $c_m = C_{\mathrm{Min}} \cap \mathfrak{h}$ then*

$$c_m = \{A(\phi) \,|\, \forall 1 \leq j \leq p+q : \phi_j \geq 0 \text{ and } \sum_{j=1}^{p} \phi_j = \sum_{j=1}^{q} \phi_{p+j}\}$$

$$c_m^o = \{A(\phi) \,|\, \forall 1 \leq j \leq p+q : \phi_j > 0 \text{ and } \sum_{j=1}^{p} \phi_j = \sum_{j=1}^{q} \phi_{p+j}\}.$$

**Proof.** More details about the invariant convex cones in $\mathfrak{su}(p,q)$ can be found in [Pan81, section III.8].

Denote

$$R = \{A(\phi) \,|\, \forall 1 \leq j \leq p+q : \phi_j \geq 0 \text{ and } \sum_{j=1}^{p} \phi_j = \sum_{j=1}^{q} \phi_{p+j}\}$$

and observe that by lemma 3.4.3 we have for $1 \leq j \leq p$ and $p+1 \leq k \leq q$ that $(-i)H_{\mathbf{e}_j-\mathbf{e}_k} \in R$. Then proposition 2.3.2 implies

$$c_m = (-i)\,\mathrm{span}_{\mathbb{R}_{\geq 0}}(\Delta_n^+) \subseteq R.$$

Conversely let $A(\phi) = X \in R$. For $1 \leq j \leq p$ and $1 \leq k \leq q$ define $a_{j,k}$ inductively by

$$a_{j,k} = \min(\phi_j - \sum_{l=1}^{k-1} a_{j,l}, \phi_{p+k} - \sum_{l=1}^{j-1} a_{l,k}).$$

It follows directly from the definition that

$$\sum_{l=1}^{k} a_{j,l} \leq \phi_j \qquad\qquad \sum_{l=1}^{j} a_{l,k} \leq \phi_{p+k}.$$

This implies that for all $j,k \geq 2$ we have $a_{j,k} \geq 0$ since it is the minimum of two non-negative numbers. Furthermore by assumptiont we have $\phi_j \geq 0$ and $\phi_{p+k} \geq 0$ which implies that $a_{j,1} \geq 0$ and $a_{1,k} \geq 0$. Thus $a_{j,k} \geq 0$ for all $j,k$. Assume that there exists $1 \leq j \leq p$ such that

$$\sum_{l=1}^{q} a_{j,l} < \phi_j.$$

Then for all $1 \leq k \leq q$ we must have

$$a_{j,k} = \phi_{p+k} - \sum_{l=1}^{j-1} a_{l,k}.$$

This implies that $\phi_{p+k} = \sum_{l=1}^{p} a_{l,k}$ and thus

$$\sum_{k=1}^{q} \phi_{p+k} = \sum_{k=1}^{q} \sum_{l=1}^{p} a_{l,k} < \sum_{l=1}^{p} \phi_l.$$

But this contradicts the assumption that $X \in R$. Thus we must have $\sum_{l=1}^{q} a_{j,l} = \phi_j$ for all $j$. With a symmetric argument we see that $\sum_{l=1}^{p} a_{l,k} = \phi_{p+k}$. But then it follows from lemma 3.4.3 that

$$X = \sum_{j=1}^{p} \sum_{k=1}^{q} a_{j,k}(-i)2(p+q)H_{\mathbf{e}_j - \mathbf{e}_k} \in c_m. \qquad \square$$

### The Semigroup

**Lemma 3.4.5** *Let $g \in \Gamma_{\mathrm{SU}(p,q)}(-C^o_{\mathrm{Min}})$ then $g$ is strictly contractive with respect to $S\langle,\rangle$.*

**Proof.** As in the proof of lemma 3.3.7 we see that it is enough to prove strict expansiveness for $\mathrm{Exp}(iX)$ for $X \in -c^o_m$. It follows from lemma 3.4.4 that $X = A(-\phi)$ such that all $\phi_j$ are positive. Then

$$\mathrm{Exp}(iX) = \exp(iX) = \mathrm{diag}(e^{-\phi_1}, \ldots, e^{-\phi_p}, e^{\phi_{p+1}}, \ldots, e^{\phi_{p+q}}).$$

Then we calculate and see

$$\begin{aligned} S\langle \mathrm{Exp}(iX)v, \mathrm{Exp}(iX)v\rangle &= \sum_{j=1}^{p} e^{-2\phi_j}|v_j|^2 - \sum_{j=1}^{q} e^{2\phi_{p+j}}|v_{p+j}|^2 \\ &< \sum_{j=1}^{p}|v_j|^2 - \sum_{j=1}^{q}|v_{p+j}|^2 \\ &= S\langle v, v\rangle. \end{aligned} \qquad \square$$

**Lemma 3.4.6** *For* $g \in \Gamma_{\mathrm{SU}(p,q)}(-C^o_{\mathrm{Min}})$ *let* $\lambda_1, \dots, \lambda_{p+q}$ *be the eigenvalues of* $g$ *counted with multiplicity. Then they can be ordered such that* $|\lambda_j| < 1$ *for* $1 \leq j \leq p$ *and* $|\lambda_j| > 1$ *for* $p+1 \leq j \leq p+q$*. Let* $f : (B_1(0) \setminus \{0\})^p \times \{z \in \mathbb{C} \,|\, |z| > 1\}^q \to \mathbb{C}$ *be a continuous* $\mathrm{S}_p \times \mathrm{S}_q$*-invariant function.*

*Then* $\hat{f} : g \mapsto f(\lambda_1, \dots, \lambda_{p+q})$ *is a well-defined continuous map on the Ol'shanskiĭ semigroup.*

**Proof.** It follows from lemma 3.4.5 and lemma 3.2.6 that for all $g \in \Gamma_{\mathrm{SU}(p,q)}(-C^o_{\mathrm{Min}})$ all eigenvalues $\lambda$ satisfy that $|\lambda| \neq 1$. Thus we can put $W_1 = B_1(0) \setminus \{0\}$ and $W_2 = \overline{B}_1(0)^c$. Furthermore we see that for $X \in -c^o_m$ $\mathrm{Exp}(iX)$ is diagonal, hence we can read off the eigenvalues from the diagonal and they are ordered such that $|\lambda_j| < 1$ for $1 \leq j \leq p$ and $|\lambda_j| > 1$ for $p+1 \leq j \leq p+q$. That is $\lambda_1, \dots, \lambda_p \in W_1$ and $\lambda_{p+1}, \dots, \lambda_{p+q} \in W_2$. Then it follows from lemma 3.2.8 that this is true for the eigenvalues of all $g \in \Gamma_{\mathrm{SU}(p,q)}(C^o_{\mathrm{Min}})$. Since $f$ is $\mathrm{S}_p \times \mathrm{S}_q$-invariant it factors through quotienting out by $\mathrm{S}_p \times \mathrm{S}_q$ and then the continuity of $\hat{f}$ follows from lemma 3.2.8. □

**Lemma 3.4.7** *Let the assumptions be as in lemma 3.4.6 and assume that* $f$ *is holomorphic. Then* $\hat{f}$ *is holomorphic on* $\Gamma_{\mathrm{SU}(p,q)}(-C^o_{\mathrm{Min}})$.

**Proof.** Let $g \in \Gamma_{\mathrm{SU}(p,q)}(-C^o_{\mathrm{Min}}) \cap \mathrm{SL}(p+q, \mathbb{C})'$ then since $g \in \mathrm{SL}(p+q, \mathbb{C})'$ it is conjugate to a regular element in the diagonal Cartan subgroup. But all regular elements in the diagonal Cartan subgroup of $\mathrm{SL}(p+q, \mathbb{C})$ have distint eigenvalues. Then the rest of the argument follows from the holomorphic implicit function theorem as in proposition 3.3.9.□

We want to prove an extension proposition similar to proposition 3.3.10 however in the formulation of proposition 3.3.10 we deliberately had stronger assumptions than we used in the proof; since it was much easier to formulate and was satisfied by the character functions in proposition 3.3.19. The exact requirement should be that $f$ should extend to all tuples of eigenvalues of regular elements, but we are satisfied if we can show that the character function in eq. (2.7) satisfies it. So let

$$A = \{z \in \{w \in \mathbb{C} \,|\, 0 < |w| \leq 1\}^p \times \{w \in \mathbb{C} \,|\, |z| \geq 1\}^q \,|\, \forall j \neq k : z_j \neq z_k\}.$$

Then $\hat{f}$ extends if $f$ extends to $A$.

**Proposition 3.4.8** *Let the setup be as in lemma 3.4.6 and assume that* $f$ *extends continuously to* $A$*. Then* $\hat{f}$ *extends continuously to* $\mathrm{SU}(p, q)'$.

**Proof.** The argument is very similar to the argument in the proof of proposition 3.3.10 so it will not be as detailed.

Let $g \in \mathrm{SU}(p, q)'$ then $g$ is regular when considered as an element of $\mathrm{SL}(p+q, \mathbb{C})$. Thus $g$ is conjugate to a regular element of the diagonal Cartan subgroup. But on the diagonal Cartan subgroup of $\mathrm{SL}(p+q, \mathbb{C})$ we have $e^{\mathbf{e}_j - \mathbf{e}_k}(g) = g_{j,j} g^{-1}_{k,k}$. Thus we get that any product of one eigenvalue and the inverse of another eigenvalue is not equal to 1 and hence all eigenvalues of $g$ must be distinct. Let $\lambda_1, \dots, \lambda_{p+q}$ be the eigenvalues of $g$ then we get that $(\lambda_1, \dots, \lambda_{p+q}) \notin A$. The idea is now to extend $\hat{f}$ by setting $\hat{f}(g) = f(\lambda_1, \dots, \lambda_{p+q})$. It is however not clear in what way to order the eigenvalues.

As in the proof of proposition 3.3.10 we can find a small open convex neighborhood of 0 in $\mathfrak{su}(p,q)$ such that $W \ni X, Y \mapsto g\exp(X)\exp(iY)$ is a homeomorphism onto the open neighborhood of $g$ $V_g = g\exp(W)\exp(iW)$ and

$$g\exp(W)\exp(iW) \cap \Gamma_{\mathrm{SU}(p,q)}(C_{\mathrm{Min}}) = g\exp(W)\exp(i(W \cap C_{\mathrm{Min}})),$$

and there exists $\alpha_1, \dots, \alpha_{p+q} : V_g \to \mathbb{C}$ such that $\alpha_1(h), \dots, \alpha_{p+q}(h)$ are the eigenvalues of $h$. Let $\gamma \in V_g \cap \Gamma_{\mathrm{SU}(p,q)}(C^o_{\mathrm{Min}})$ be some element, then we can reorder the functions $\alpha_1, \dots, \alpha_{p+q}$ such that $|\alpha_j(\gamma)| < 1$ for $1 \leq j \leq p$ and $|\alpha_j(\gamma)| > 1$ for $p+1 \leq j \leq p+q$. we now want to show that for any other $\rho \in V_g \cap \Gamma_{\mathrm{SU}(p,q)}(C^o_{\mathrm{Min}})$ with the same ordering we also get $|\alpha_1(\rho)|, \dots, |\alpha_p(\rho)| < 1$ and $|\alpha_{p+1}(\rho)|, \dots, |\alpha_{p+q}(\rho)| > 1$.

Assume that there exists some $\rho \in V_g \cap \Gamma_{\mathrm{SU}(p,q)}(C^o_{\mathrm{Min}})$ and $1 \leq j \leq p$ such that $|\alpha_j(\rho)| > 1$. Let $\gamma = g\exp(v_\gamma)\exp(iw_\gamma)$ and $\rho = g\exp(v_\rho)\exp(iw_\rho)$ then

$$h(t) = g\exp((1-t)v_\gamma + tv_\rho)\exp(i((1-t)w_\gamma + tw_\rho))$$

is a curve between $\gamma$ and $\rho$ such that $t \mapsto |\alpha_j(h(t))|$ contradicts the intermediate value theorem. This shows that on $V_g \cap \Gamma_{\mathrm{SU}(p,q)}(C^o_{\mathrm{Min}})$ with the previously desribed ordering $\tilde{f} = f(\alpha_1(\cdot), \dots, \alpha_{p+q}(\cdot))$ and this function extends continuosly to $\mathrm{SU}(p,q)' \cap V_g$. $\square$

### 3.4.2 The minimal representation

We are now in a position to show that the character of $\pi_{\mathrm{Min}}$ is uniform in the sense of proposition 2.8.9. To make the character formula more explicit we parametrize a maximal set of non-conjugate Cartan subgroups.

**Lemma 3.4.9** *Let $0 \leq k \leq p \leq q$ and assume that $k < q$ then $f_k : (S^1)^k \times \mathbb{R}^k \times (S^1)^{p+q-k} \to \mathrm{SU}(p,q)$ parametrizes a Cartan subgroup of $\mathrm{SU}(p,q)$.*

$$f_k(w_1, \dots, w_k, t_1, \dots, t_k, z_1, \dots, z_{p+q-2k-1}) = \begin{pmatrix} \alpha & & \beta & \\ & \gamma & & \\ \beta & & \alpha & \\ & & & \delta \end{pmatrix}$$

*where*

$$\begin{aligned}
\alpha &= \mathrm{diag}(w_1\cosh(t_1), \dots, w_k\cosh(t_k)) \\
\beta &= \mathrm{diag}(w_1\sinh(t_1), \dots, w_k\sinh(t_k)) \\
\gamma &= \mathrm{diag}(z_1, \dots, z_{p-k}) \\
\delta &= \mathrm{diag}(z_{p-k+1}, \dots, z_{p+q-2k-1}, (w_1^2 \cdots w_k^2 z_1 \cdots z_{p+q-2k-1})^{-1}).
\end{aligned}$$

*For $k = p = q$ we let $f_k : \{\pm\} \times (S^1)^{k-1} \times \mathbb{R}^k \to \mathrm{SU}(k,k)$ be given as below*

$$f_k(\pm, w_1, \dots, w_{k-1}, t_1, \dots, t_k) = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

*where*

$$
\begin{aligned}
z &= (w_1 \cdots w_{k-1})^{-2}\\
\alpha &= \operatorname{diag}(w_1\cosh(t_1),\ldots,w_{k-1}\cosh(t_{k-1}),\pm z\cosh(t_k))\\
\beta &= \operatorname{diag}(w_1\sinh(t_1),\ldots,w_{k-1}\sinh(t_{k-1}),\pm z\sin(t_k)).
\end{aligned}
$$

*Then* $f_k$ *parametrizes a maximally split Cartan subgroup of* $\mathrm{SU}(k,k)$.

*Furthermore the collection of maps* $f_0,\ldots,f_p$ *parametrize a maximal set of non-conjugate Cartan subgroups of* $\mathrm{SU}(p,q)$.

**Remark 3.4.10** If we let $S_k = \{\mathbf{e}_1 - \mathbf{e}_{p+1},\ldots,\mathbf{e}_k - \mathbf{e}_{p+k}\}$ then $f_k$ parametrizes the Cartan subgroup associated to $\mathfrak{h}_{S_k}$. Furthermore

$$
\begin{aligned}
&f_k(e^{i\theta_1},\ldots,e^{i\theta_k},t_1,\ldots,t_k,e^{i\phi_1},\ldots,e^{i\phi_{p+q-2k-1}})\\
&\qquad\qquad = \prod_{j=1}^{k}\exp(t_1(E_{\mathbf{e}_j-\mathbf{e}_{p+j}} + \overline{E_{\mathbf{e}_j-\mathbf{e}_{p+j}}}))\\
&\cdot\exp(i\operatorname{diag}(\theta_1,\ldots,\theta_k,\phi_1,\ldots,\phi_{p-k},\theta_1,\ldots,\theta_k,\phi_{p-k+1},\ldots,\phi_{p+q-2k-1},-\sum_{j=1}^{k}2\theta_j-\sum_{j=1}^{p+q-2k-1}\phi_j)).
\end{aligned}
$$

For $k=p=q$ the map $f_k$ is given by

$$
\begin{aligned}
&f_k((-1)^k,e^{i\theta_1},\ldots,e^{i\theta_{k-1}},t_1\ldots,t_k) =\\
&\qquad \exp(\pi k i H'_{\mathbf{e}_p-\mathbf{e}_{2p}})\prod_{j=1}^{k}\exp(t_1(E_{\mathbf{e}_j-\mathbf{e}_{p+j}} + \overline{E_{\mathbf{e}_j-\mathbf{e}_{p+j}}}))\\
&\qquad\qquad \cdot\exp(i\operatorname{diag}(\theta_1,\ldots,\theta_{k-1},-\sum_{j=1}^{k-1}\theta_j,\theta_1,\ldots,\theta_{k-1},-\sum_{j=1}^{k-1}\theta_j)).
\end{aligned}
$$

**Theorem 3.4.11** *Let* $\theta_{\mathrm{Min}}$ *be the character function associated to* $\pi_{\mathrm{Min}}$. *Let* $k \le p \le q$ *and assume that* $k < q$ *and let* $\lambda_1,\ldots,\lambda_{p+q} \in \mathbb{C}$ *and let*

$$
g = f_k(w_1,\ldots,w_k,t_1,\ldots,t_k,z_1,\ldots,z_{p+q-2k-1})
$$

*for* $1 \le j \le k$

$$
\begin{aligned}
\lambda_j &= w_j e^{-|t_j|}\\
\lambda_{p+j} &= w_j e^{|t_j|}
\end{aligned}
$$

*for* $j \le p-k$ *in the first case and* $j \le q-k-1$ *in the second*

$$
\begin{aligned}
\lambda_{k+j} &= z_j\\
\lambda_{p+k+j} &= z_{p-k+j}\\
\lambda_{p+q} &= (w_1^2\cdots w_k^2 z_1\ldots z_{p+q-2k-1})^{-1}.
\end{aligned}
$$

*In the case $k = p = q$ we let $1 \leq j \leq k-1$*

$$\begin{aligned} g &= f_k(\pm, w_1, \ldots, w_{k-1}, t_1, \ldots, t_k) \\ \lambda_j &= w_j e^{-|t_j|} \\ \lambda_{p+j} &= w_j e^{|t_j|} \\ \lambda_k &= \pm(w_1 \ldots w_{k-1})^{-2} e^{-|t_k|} \\ \lambda_{p+k} &= \pm(w_1 \ldots w_{k-1})^{-2} e^{|t_k|} \\ &. \end{aligned}$$

*Then the Harish-Chandra character of $\pi_{\mathrm{Min}}$ is given by*

$$\theta_{\mathrm{Min}}(g) = \frac{\prod_{j=1}^{p} \lambda_j}{(\prod_{1 \leq j < k \leq p} 1 - \lambda_j \lambda_k^{-1})(\prod_{1 \leq j < k \leq q} 1 - \lambda_{p+j} \lambda_{p+k}^{-1})} \cdot \sum_{(\sigma,\omega) \in \mathrm{S}_p \times \mathrm{S}_q} \frac{\epsilon(\sigma)\epsilon(\omega) \prod_{j=1}^{p} \lambda_j^{j-\sigma^{-1}(j)} \prod_{j=1}^{q} \lambda_{p+j}^{j-\omega^{-1}(j)}}{1 - \lambda_{\sigma(1)} \lambda_{p+\omega(q)}^{-1}}. \quad (3.24)$$

**Proof.** Since $\pi_{\mathrm{Min}}$ is a unitary irreducible lowest-weight representation it follows from corollary 2.5.4 that it extends to $\Gamma_{\mathrm{SU}(p,q)}(-C^o_{\mathrm{Min}})$. Let $g \in \mathrm{H}\,\mathrm{Exp}(-ic^o_m)$ and let $\lambda_1, \ldots, \lambda_{p+q}$ be the eigenvalues of $g$ ordered such that $\lambda_1, \ldots, \lambda_p$ have norm strictly less than 1 and $\lambda_{p+1}, \ldots, \lambda_{p+q}$ have norm strictly greater than 1. Then it follows from proposition 2.6.3 that $\mathrm{tr}\,\pi_{\mathrm{Min}}(g)$ is given by the formula in eq. (3.24). Let $a(\lambda_1, \ldots, \lambda_{p+q})$ be the meromorphic function on $\mathbb{C}^{p+q}$ given by the expression on the right hand side of eq. (3.24). Since $a$ is a formula for the character on $\mathrm{H}\,\mathrm{Exp}(-ic^o_m)$ and the analytic Weyl group of $\mathfrak{h}$ is isomorphic to $\mathrm{S}_p \times \mathrm{S}_q$ we get that $a$ is invariant under this action. Let $b : \mathbb{C}^{p+q} \to \mathbb{C}$ be given by

$$b(z_1, \ldots, z_{p+q}) = (\prod_{1 \leq j \neq k \leq p} 1 - z_j z_k^{-1})(\prod_{1 \leq j \neq k \leq q} 1 - z_{p+j} z_{p+k}^{-1}).$$

Then $b$ satisfies all the assumptions of proposition 3.4.8. Furthermore we see that $ba$ also satisfies the assumptions of proposition 3.4.8. Let the function they induce on the semigroup be denote by a hat $\hat{}$ that is $\hat{b}$ and $\widehat{ab}$. Then we get that $\mathrm{tr}\,\pi_{\mathrm{Min}}(s) = \frac{\widehat{ab}}{\hat{b}}$ for $s \in \mathrm{H}\,\mathrm{Exp}(-ic^o_m)$ then by proposition 2.4.18 this identity holds on all of $\Gamma_{\mathrm{SU}(p,q)}(-C^o_{\mathrm{Min}})$. Then by writing out the expressions of $\widehat{ab}$ and $\hat{b}$ we see that $\frac{\widehat{ab}}{\hat{b}}$ is non-zero on the regular elements of $\mathrm{SU}(p,q)$. Thus we get that $\mathrm{tr}\,\pi_{\mathrm{Min}}$ extends continuously to $\mathrm{SU}(p,q)'$ and hence it satisfies the assumptions of corollary 2.8.10 which finishes the proof. $\square$

## 3.5 $\mathfrak{so}^*(2n)$

Like the symplectic groups and the indefinite special unitary groups we can realize the Ol'shanskiĭ semigroups of $\mathrm{SO}^*(2n)$ as contractive elements of $\mathrm{SO}(2n, \mathbb{C})$ with respect to some indefinite hermitian form. And similarly this will allow us prove that the formula in proposition 2.6.3 extends continuously to $\mathrm{SO}^*(2n)'$ so that we can apply proposition 2.8.9. Note that we only consider $\mathfrak{so}^*(2n)$ for $n \geq 3$.

### 3.5.1 Structure and realization

We once again start with describing an explicit realization of $\mathfrak{so}^*(2n)$ and $\mathrm{SO}^*(2n)$. We introduce some notation, note that $J$ and $\omega(,)$ are not as in section 3.3.1 specifically they here define a symmetric bilinear form

$$\begin{aligned}
I_n &= \mathrm{diag}(1,1,\dots,1) \in \mathrm{Mat}_{n\times n} \\
I_{n,n} &= \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \\
J &= \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \\
S\langle v,w\rangle &= v^H I_{n,n} w \\
\omega(v,w) &= v^T J w \\
\mathrm{SU}(n,n) &= \{A \in \mathrm{SL}(2n,\mathbb{C}) \,|\, \forall v,w \in \mathbb{C}^{2n} : S\langle Av,Aw\rangle = S\langle v,w\rangle\} \\
\mathrm{SO}^*(2n) &= \{A \in \mathrm{SU}(n,n) \,|\, \forall v,w \in \mathbb{C}^{2n} : \omega(Av,Aw) = \omega(v,w)\} \\
\mathrm{SO}(2n,\mathbb{C}) &= \{A \in \mathrm{SL}(2n,\mathbb{C}) \,|\, \forall v,w \in \mathbb{C}^{2n} : \omega(Av,Aw) = \omega(v,w)\}.
\end{aligned}$$

Then as usual we define a parametrization of the Lie algebra of a maximal torus

$$\begin{aligned}
A :&\mathbb{R}^n \to \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \\
A(\phi) =& \begin{pmatrix} i\phi & 0 \\ 0 & -i\phi \end{pmatrix}.
\end{aligned}$$

**Lemma 3.5.1** *Let* $\mathrm{K} \cong \mathrm{U}(n)$ *be embedded diagonally by* $M \mapsto \begin{pmatrix} M & 0 \\ 0 & \overline{M} \end{pmatrix}$. *Then* $\mathrm{K}$ *is a maximal compact subgroup of* $\mathrm{SO}^*(2n)$. *Let*

$$\begin{aligned}
\mathfrak{h} &= \{A(\phi) \,|\, \phi \in \mathbb{R}^n\} \\
\mathrm{H} &= \{\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \,|\, \phi \in \mathbb{R}^n\}
\end{aligned}$$

*then* $\mathfrak{h}$ *is a Cartan subalgebra of* $\mathfrak{so}^*(2n)$ *and* $\mathrm{H}$ *is a maximal Torus in* $\mathrm{K}$ *and thus a Cartan subgroup in* $\mathrm{SO}^*(2n)$. *Furthermore* $\mathrm{SO}(2n,\mathbb{C})$ *is a complexification of* $\mathrm{SO}^*(2n)$. *Let* $\mathbf{e}_j \in \mathfrak{h}_{\mathbb{C}}^*$ *be given by* $\mathbf{e}_j(\begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}) = \phi_j$. *Then a positive system coming from a good* *ordering is*

$$\begin{aligned}
\Delta &= \{\pm\mathbf{e}_j \pm \mathbf{e}_k \,|\, 1 \le j,k \le n\}\\
\Delta^+ &= \{\mathbf{e}_j \pm \mathbf{e}_k \,|\, 1 \le j < k \le n\}\\
\Delta_k &= \{\mathbf{e}_j - \mathbf{e}_k \,|\, 1 \le j,k \le n\}\\
\Delta_k^+ &= \{\mathbf{e}_j - \mathbf{e}_k \,|\, 1 \le j < k \le n\}\\
\Delta_n^+ &= \{\mathbf{e}_j + \mathbf{e}_k \,|\, 1 \le j,k \le n\}\\
W_{\mathrm{K}} &\cong \mathrm{S}_n\\
\delta_k &= \frac{1}{2}\sum_{j=1}^{n}(n-1-2j)\mathbf{e}_j.
\end{aligned}$$

**Remark 3.5.2** $W_{\mathrm{K}}$ acts as $\mathrm{S}_n$ on $\mathfrak{h}_{\mathbb{C}}^*$ by permuting the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$.

**Lemma 3.5.3** *Let $H, H' \in \mathfrak{h}_{\mathbb{C}} \subset \mathfrak{so}(2n, \mathbb{C})$ then*

$$B(H,H') = 4(n-1)\sum_{j=1}^{n}\mathbf{e}_j(H)\mathbf{e}_j(H').$$

**Proof.** We calculate

$$\begin{aligned}
B(H,H') &= \operatorname{tr}\operatorname{ad}(H)\operatorname{ad}(H') = \sum_{\alpha\in\Delta}\alpha(H)\alpha(H')\\
&= \sum_{1\le j<k\le n}(\pm\mathbf{e}_j \pm \mathbf{e}_k)(H)(\pm\mathbf{e}_j \pm \mathbf{e}_k)(H')\\
&= \sum_{1\le j<k\le n}4\mathbf{e}_j(H)\mathbf{e}_j(H') + 4\mathbf{e}_k(H)\mathbf{e}_k(H')\\
&= 4\sum_{j=1}^{n}(n-j)\mathbf{e}_j(H)\mathbf{e}_j(H') + 4\sum_{j=1}^{n}(j-1)\mathbf{e}_j(H)\mathbf{e}_j(H')\\
&= 4(n-1)\sum_{j=1}^{n}\mathbf{e}_j(H)\mathbf{e}_j(H').
\end{aligned}$$

□

**Lemma 3.5.4** *Let $H_{\mathbf{e}_j} \in \mathfrak{h}_{\mathbb{C}}$ such that $\mathbf{e}_j = B(H_{\mathbf{e}_j}, \cdot)$ and let $\phi \in \mathbb{C}$ such that $H_{\mathbf{e}_j} = \begin{pmatrix}\phi & 0\\ 0 & -\phi\end{pmatrix}$ then*

$$\phi_k = \frac{1}{4(n-1)}\delta_{j,k}.$$

**Proof.** This follows from lemma 3.5.3 since

$$\mathbf{e}_j(H) = B(H_{\mathbf{e}_j}, H) = 4(n-1)\sum_{k=1}^{n} \mathbf{e}_k(H_{\mathbf{e}_j})\mathbf{e}_k(H).$$

Since the $\mathbf{e}_k \in \mathfrak{h}_{\mathbb{C}}^*$ are linearly independent we get

$$\phi_k = \mathbf{e}_k(H_{\mathbf{e}_j}) = \frac{1}{4(n-1)}\delta_{j,k}. \qquad \square$$

**Lemma 3.5.5** *Let $C_{\mathrm{Min}}$ denote the minimal, non-trivial, closed,* Ad*-invariant, proper convex cone in $\mathfrak{so}^*(2n)$ associated to the positive system in lemma 3.5.1 and let as usual $c_m = \mathfrak{h} \cap C_{\mathrm{Min}}$ then*

$$c_m = \{A(\phi) \,|\, \forall j : \sum_{\substack{k=1\\k\neq j}}^{n} \phi_k \le \phi_j \le 0\}$$

$$c_m^o = \{A(\phi) \,|\, \forall j : \sum_{\substack{k=1\\k\neq j}}^{n} \phi_k < \phi_j < 0\}.$$

**Proof.** Let $R = \{A(\phi) \,|\, \forall j : \sum_{\substack{k=1\\k\neq j}}{}^{n}{}_{\phi_k} \le \phi_j \le 0\}$ then it follows from lemma 3.5.4 that $(-i)H_{\mathbf{e}_j+\mathbf{e}_k} \in R$. Since $R$ is convex and $\mathbb{R}_+ R \subset R$ we get $c_m \subset R$. Conversely let $X \in R$ and $\phi \in \mathbb{R}^n$ such that $X = A(\phi)$, we can reorder the entries and assume that

$$\sum_{j=2}^{n} \phi_j \le \phi_1 \le \phi_2 \le \cdots \le \phi_n \le 0. \tag{3.25}$$

Let $k$ be such that $\phi_k < 0$ and $\phi_{k+1} = 0$ and if there is no $j$ such that $\phi_j = 0$ then we put $k = n$. We want to show by induction in $k$ that $X \in c_m$. First observe that $k = 1$ gives $0 \le \phi_1 < 0$ and thus a contradiction. If $k = 2$ we have $\phi_2 \le \phi_1 \le \phi_2$ and thus

$$X = (-\phi_1)(-i)H_{\mathbf{e}_1+\mathbf{e}_2} \in c_m.$$

If $k = 3$ we have that $(-i)H_{\mathbf{e}_1+\mathbf{e}_2}, (-i)H_{\mathbf{e}_1+\mathbf{e}_3}$ and $(-i)H_{\mathbf{e}_2+\mathbf{e}_3}$ are a basis of $A(\mathbb{R}^3 \times \{0\}^{n-3})$ and thus there exists unique elements $b_{1,2}, b_{1,3}$ and $b_{2,3}$ such that

$$X = b_{1,2}(-i)H_{\mathbf{e}_1+\mathbf{e}_2} + b_{1,3}(-i)H_{\mathbf{e}_1+\mathbf{e}_3} + b_{2,3}(-i)H_{\mathbf{e}_2+\mathbf{e}_3}.$$

If $b_{1,2} < 0$ then $b_{1,3} = -\phi_1 - b_{1,2}$ and $b_{2,3} = -\phi_2 - b_{1,2}$ but then

$$\phi_3 = -b_{1,3} - b_{2,3} = \phi_1 + \phi_2 + 2b_{1,2} < \phi_1 + \phi_2.$$

Which contradicts that $X \in R$, similarly for any other $b_{1,3}$ and $b_{2,3}$.

For $k \geq 4$ we put $Y = X - (-\phi_k)(-i)H_{\mathbf{e}_1+\mathbf{e}_n} = A(\psi)$ then $\psi_j = \phi_j$ for $j \neq 1, n$ and thus

$$\psi_1 = \phi_1 - \phi_n \geq (\sum_{j=2}^{n} \phi_j) - \phi_n = \sum_{j=2}^{n} \psi_j$$

$$\psi_2 = \phi_2 \geq \phi_1 \geq \phi_1 - \phi_k + \phi_3 \geq \sum_{\substack{j=1 \\ j\neq 2}}^{n} \psi_j.$$

Thus whether $\psi_1 \leq \psi_2$ or $\psi_2 < \psi_1$ we can reorder the entries in $\psi$ such that it satisfies eq. (3.25). Thus $Y \in R$ and by induction $Y \in c_m$ and hence $X \in c_m$. $\square$

**Lemma 3.5.6** *Let $g \in \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$ then $g$ is a strictly contractive operator with respect to $S\langle , \rangle$.*

**Proof.** As in the proof of lemma 3.3.7 it is enough to prove strict contractiveness for $\mathrm{Exp}(iX)$ with $X \in -c^o_m$. Let $\phi \in \mathbb{R}^n$ such that $X = A(\phi)$ then

$$\mathrm{Exp}(iX) = \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix}$$

and $\phi_j > 0$. Then

$$S\langle \mathrm{Exp}(iX)v, \mathrm{Exp}(iX)v \rangle = \sum_{j=1}^{n} e^{-2\phi_j}|v_j|^2 - e^{2\phi_j}|v_{n+j}|^2 < \sum_{j=1}^{n} |v_j|^2 - |v_{n+j}|^2 = S\langle v, v \rangle \square$$

**Lemma 3.5.7** *Let $g \in \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$ and let $\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}$ be the eigenvalues of $g$ counted with multiplicity. Then they can be ordered such that $|\lambda_1|, \ldots, |\lambda_n| < 1$.*

*Let $f : \{z \in \mathbb{C} \,|\, 0 < |c| < 1\}^n \to \mathbb{C}$ be a continuous function invariant under permutations. Then the map $\hat{f} : \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}}) \to \mathbb{C}$ given by $\hat{f} : g \mapsto f(\lambda_1, \ldots, \lambda_n)$ is continuous.*

**Proof.** It follows from lemma 3.5.6 and lemma 3.2.6 that all eigenvalues of $g$ have $|\lambda| \neq 1$ and then it follows from lemma 3.2.4 that if $\lambda$ is an eigenvalue of $g$ so is $\lambda^{-1}$ with the same algebraic multiplicity.

Then it follows from lemma 3.2.8 that $\hat{f}$ is a well-defined, continuous function on $\Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$. $\square$

**Lemma 3.5.8** *Let $f$ be as in lemma 3.5.7 and assume furthermore that $f$ is holomorphic. Then $\hat{f}$ is holomorphic.*

**Proof.** It is enough to check for $g \in \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$ such that $g$ is a regular element of $\mathrm{SO}(2n, \mathbb{C})$. But then $g$ is conjugate to a regular element in the diagonal Cartan subgroup. Let $g = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ for $z \in \mathbb{C}^n$. Then $z_i \neq z_j$ and $z_i \neq \frac{1}{z_j}$ for $i \neq j$ since $g$ is regular. So if $\lambda_1, \ldots \lambda_n$ denote the eigenvalues with $|\lambda| < 1$ then $\lambda_i \neq \lambda_j$ for all $i \neq j$. Thus we conclude that all eigenvalues of $g$ are distinct and holomorphicity of $\hat{f}$ follows as in the proof of proposition 3.3.9. $\square$

We want to find a condition on $f$ such that when it is satisfied we can show $\hat{f}$ extends to $\mathrm{SO}^*(2n)'$ however we want this condition to be tight enough that we can show it is satisfied by the character for the minimal representation of $\mathrm{SO}^*(2n)$ on $\Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$. To write this down explicitly we have to define a few extra sets

$$\begin{aligned} B &= \{z \in \{w \in \mathbb{C} \,|\, 0 < |w| \leq 1\}^n \,|\, \forall i \neq j : z_i \neq z_j \text{ and } z_i \neq z_j^{-1}\} \\ \mathrm{SO}^*(2n)_{\mathrm{good}} &= \{g \in \mathrm{SO}^*(2n)' \,|\, \det(g-1) \neq 0 \neq \det(g+1)\}. \end{aligned}$$

That is $\mathrm{SO}^*(2n)_{\mathrm{good}}$ consists of all regular elements of $\mathrm{SO}^*(2n)$ which do not have 1 or $-1$ as eigenvalues. We restrict to this smaller open set since all elements $g \in \mathrm{SO}^*(2n)_{\mathrm{good}}$ have all eigenvalues distinct and we can argue as in proposition 3.3.10 and proposition 3.4.8.

**Proposition 3.5.9** *Let $f$ be as in lemma 3.5.7 and assume furthermore that $f$ extends continuously to $B$. Then $\hat{f}$ extends continuously to* $\mathrm{SO}^*(2n)_{\mathrm{good}}$.

**Proof.** Once again the argument follows the same structure as proposition 3.3.10 and proposition 3.4.8. First of all observe that since $\mathrm{SO}^*(2n)$ sits on the boundary of $\Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$ the value at $\mathrm{SO}^*(2n)'$ is uniquely given by $\hat{f}$, thus the question of the existence of a continuous extension is purely local.

Let $g \in \mathrm{SO}^*(2n)_{\mathrm{good}}$ then $g$ is regular as an element of $\mathrm{SO}(2n,\mathbb{C})$ and hence it is conjugate in $\mathrm{SO}(2n,\mathbb{C})$ to a regular element of the diagonal Cartan subgroup. This implies that there exists $z \in \mathbb{C}^n$ such that $g = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$. Then it follows from the regularity of $g$ that $z_j \neq z_k$ and $z_j \neq z_k^{-1}$ for $j \neq k$. Thus the only way that $g$ can have an eigenvalue of multiplicity $\geq 2$ is if $z_j = z_j^{-1}$ which implies that $z_j = \pm 1$. But for $g \in \mathrm{SO}^*(2n)_{\mathrm{good}}$ we have excluded this possibility and then as usual it follows from the implicit function theorem that there exists a small open neighborhood $g \in V \subset \mathrm{SO}(2n,\mathbb{C})$ and continuous[2] functions $\lambda_1, \ldots, \lambda_n$ such that the eigenvalues of $h \in V$ are given by $\lambda_1(h), \ldots, \lambda_n(h), \lambda_1(h)^{-1}, \ldots, \lambda_n(h)^{-1}$. By taking $V$ small enough we can assume that there exists $W \subset \mathfrak{so}^*(2n)$ open convex neighborhood of 0 such that $V = g\exp(W)\exp(iW)$ and $V \cap \Gamma_{\mathrm{G}}(-C^o_{\mathrm{Min}}) = g\exp(W)\exp(iW \cap -C^o_{\mathrm{Min}})$.

Let $h_0 \in V \cap \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$ be some fixed element, we can then assume that $|\lambda_1(h_0)|, \ldots, |\lambda_n(h_0)|$. We now want to show that $\hat{f}|_{V \cap \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})} = f(\lambda_1, \ldots, \lambda_n)$ since this would then give a continuous extension of $\hat{f}$ to all of $V$, specifically to $V \cap \mathrm{SO}^*(2n)$. By the construction of $\hat{f}$ the only thing we have to show is that for any other $h \in V \cap \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$ we have $|\lambda_1(h)|, \ldots, |\lambda_n(h)| < 1$.

Thus assume that there exists $h_1 \in \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$ and $j$ such that $|\lambda_j(h_1)| > 1$. Let $h_j = g\exp(X_j)\exp(iY_j)$ for $j = 0, 1$ with $X_j \in W$ and $Y_j \in W \cap -C^o_{\mathrm{Min}}$. Then since $W$ and $-C^o_{\mathrm{Min}}$ are convex $\alpha(t) = g\exp(tX_0 + (1-t)X_1)\exp(i(tY_0 + (1-t)Y_1))$ is a curve between $h_0$ and $h_1$ inside $V \cap \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$. Then $|\lambda_j(\alpha(t))|$ is a continuous function from $[0,1]$ to $\mathbb{R}_+$ such that $|\lambda_j(\alpha(0))| < 1$ and $|\lambda_j(\alpha(1))| > 1$, thus by the intermediate value theorem there exists $t$ such that $|\lambda_j(\alpha(1))| = 1$, but this contradicts lemma 3.5.6 and lemma 3.2.6. □

[2]They are even analytic.

### 3.5.2 Minimal representation

According to lemma 3.1.2 the K-type decomposition of the minimal holomorphic representation is given by

$$\pi_{\mathrm{Min}}|_{\mathrm{K}} = \bigoplus_{n=0}^{\infty} F^{\mathrm{K}}(\sum_{j=1}^{n} \mathbf{e}_j + n(\mathbf{e}_1 + \mathbf{e}_2)).$$

Furthermore the minimal holomorphic representation is an irreducible lowest weight representation with respect to $\Delta^+$ thus we can apply the theory from chapter 2 to it. However before we take a short detour to describe a family of non-conjugate Cartan subgroups. We do this in order to attempt to simplify the expression in proposition 2.8.9.

It follows from the usual theory of Cayley transforms and Cartan subgroups that we can get every conjugacy class of Cartan subalgebra by Cayley transforming $\mathfrak{h}$ using the cayley transforma associated to a subset $S \subset \Delta_n^+$ of pairwise strongly orthogonal roots. As we have seen in e.g. the proof of lemma 2.8.5 we can by the action of the Weyl group $W_{\mathrm{K}}$ assume that $S$ is of the form $S_k = \{\mathbf{e}_1 + \mathbf{e}_2, \ldots, \mathbf{e}_{2k-1} + \mathbf{e}_{2k}\}$. Thus we index the Cartan subgroups by $k$.

**Lemma 3.5.10** *Let $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and let $f : \mathbb{R}^k \times (S^1)^{n-k} \to \mathrm{SO}^*(2n)$ be the map*

$$f(t_1, \ldots, t_k, w_1, \ldots w_k, z_1, \ldots, z_{n-2k}) = \begin{pmatrix} \alpha & & \beta & \\ & \gamma & & \\ \beta^T & & \delta & \\ & & & \gamma^{-1} \end{pmatrix}.$$

*Where the block are given:*

$$\alpha = \begin{pmatrix} w_1 \cosh(t_1) & & & & \\ & w_1^{-1} \cosh(t_1) & & & \\ & & \ddots & & \\ & & & w_k \cosh(t_k) & \\ & & & & w_k^{-1} \cosh(t_k) \end{pmatrix}$$

$$\beta = \begin{pmatrix} & w_1 \sinh(t_1) & & & \\ -w_1^{-1} \sinh(t_1) & & & & \\ & & \ddots & & \\ & & & & w_k \sinh(t_k) \\ & & & -w_k^{-1} \sinh(t_k) & \end{pmatrix}$$

$$\delta = \begin{pmatrix} w_1^{-1} \cosh(t_1) & & & & \\ & w_1 \cosh(t_1) & & & \\ & & \ddots & & \\ & & & w_k^{-1} \cosh(t_k) & \\ & & & & w_k \cosh(t_k) \end{pmatrix}$$

$$\gamma = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_{n-2k} \end{pmatrix}.$$

*Then $f(\mathbb{R}^k \times (S^1)^{n-2k})$ is a Cartan subgroup of $\mathrm{SO}^*(2n)$ and $f$ is a group isomorphism onto this Cartan subgroup. Furthermore*

$$f(t_1, \ldots, t_k, e^{i\phi_1}, \ldots, e^{i\phi_k}, e^{i\psi_1}, \ldots, e^{i\psi_{n-2k}})$$
$$= \prod_{j=1}^{k} \exp(t_j(E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}} + \overline{E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}}) + i\phi_1 H'_{\mathbf{e}_{2j-1}-\mathbf{e}_{2j}}) \exp(A(0, \ldots, 0, \psi_1, \ldots, \psi_{n-2k})).$$

**Proof.** In the following we will work both with the Cartan subalgebra $\mathfrak{h}$ and the Cayley transformed Cartan subalgebra $\mathfrak{h}_{S_k} = \mathbf{c}_{S_k}(\mathfrak{h})$ but $H_\alpha$ will always denote the dual with respect to the Killing form of $\alpha \in \mathfrak{h}^*$ in $\mathfrak{h}$. To denote dual elements in $\mathfrak{h}_{S_k}$ we write $H_{S_k,\alpha}$.

If we put $E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}} = E_{2j-1,n+2j} - E_{2j,n+2j-1}$ where $E_{a,b}$ denotes the matrix with 0's everywhere except the entry $(a, b)$ where it is 1, that is the $a$'th row and $b$'th column. Then

$$\begin{aligned} [A(\phi), E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}] &= (\phi_{2j-1} + \phi_{2j})E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}} \\ \overline{E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}} &= E_{n+2j,2j-1} - E_{n+2j-1,2j} \\ [E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}, \overline{E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}}] &= E_{2j-1,2j-1} + E_{2j,2j} - E_{n+2j-1,n+2j-1} - E_{n+2j,n+2j} \\ &= H'_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}. \end{aligned}$$

Thus $E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}$ is normalized as in section 2.8.1 and we see that $E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}} = \frac{\partial f}{\partial t_j}(e)$ where $e$ is the identity element in $\mathbb{R}^k \times (S^1)^{n-k}$ furthermore we see that $iH'_{\mathbf{e}_{2j-1}-\mathbf{e}_{2j}} = \frac{\partial f}{\partial w_j}(e)$ and $A(\mathbf{e}_{k+j}) = \frac{\partial f}{\partial z_j}(e)$. We thus see that $\mathfrak{h}_{S_k} = T_e f \mathbb{R}^n$ that is we have a parametrization of the Cayley transformation of the Lie algebra but we still need to show that the image of $f$ is the centralizer of the Lie algebra.

The image of $f$ is the analytic subgroup corresponding $\mathfrak{h}_{S_k}$, according to [Kna02, proposition 7.110] the associated Cartan subgroup is generated by the image of $f$ and $\exp(\pi i H'_{S_k,\alpha})$ for $\alpha \in \Delta(\mathfrak{so}^*(2n), \mathfrak{h}_{S_k})$ with $\alpha$ real. If we let $a > 2k$ we have $Tf\vec{e}_a = A(\vec{e}_a) \in \cap_{j=1}^k \mathrm{Ker}(\mathbf{e}_{2j-1} + \mathbf{e}_{2j})$ and thus $\mathbf{c}_{S_k}^{-1}(Tf\vec{e}_a) = Tf\vec{e}_a$. If $1 \leq j \leq k$ we have $Tf\vec{e}_{k+j} = iH'_{\mathbf{e}_{2j-1}-\mathbf{e}_{2j}}$ which is strongly orthogonal to all elements of $S_k$ thus $\mathbf{c}_{S_k}^{-1}(iH'_{\mathbf{e}_{2j-1}-\mathbf{e}_{2j}}) = iH'_{\mathbf{e}_{2j-1}-\mathbf{e}_{2j}}$. If $1 \leq j \leq k$ we have $Tf\vec{e}_j = E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}} + \overline{E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}}$ thus $\mathbf{c}_{S_k}^{-1}(Tf\vec{e}_{k+j}) = H'_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}$.

If we put $\alpha_j = \mathbf{c}_{S_k}(\mathbf{e}_j)$ and $v = (t_1, \ldots, t_k, s_1, \ldots, s_k, u_1, \ldots, u_{n-2k})$ then it follows from the previous that

$$\alpha_j(Tfv) = \begin{cases} t_l + is_l & \text{if } 1 \leq j \leq 2k \text{ and } j = 2l - 1 \\ t_l - is_l & \text{if } 1 \leq j \leq 2k \text{ and } j = 2l \\ iu_l & \text{if } 2k < j \text{ and } j = 2k + l \end{cases}.$$

Since all roots are of the form $\pm\alpha_l \pm \alpha_j$ for $l \neq j$ we see that the only real roots are $\alpha_{2j-1} + \alpha_{2j}$ for $1 \leq j \leq k$. That is the real roots are exactly $\mathbf{c}_{S_k}(S_k)$ and it follows from lemma 3.2.9 that

$$H'_{S_k,\alpha_{2j-1}+\alpha_{2j}} = \mathbf{c}_{S_k}(H'_{\mathbf{c}_{S_k}^{-1}(\alpha_{2j-1}+\alpha_{2j})}) = \mathbf{c}_{S_k}(H'_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}) = E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}} + \overline{E_{\mathbf{e}_{2j-1}+\mathbf{e}_{2j}}}.$$

Thus to see that $\exp(\pi i H'_{S_k,\alpha_{2j-1}+\alpha_{2j}})$ is in the image of $f$ it is enough to observe that

$$\exp\begin{pmatrix} & & & \pi i \\ & & -\pi i & \\ & -\pi i & & \\ \pi i & & & \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Which is clearly in the image of $f$ by setting $t_j = 0$ and $w_j = -1$ and all other parameters 0. ☐

**Theorem 3.5.11** *Let $\theta$ be the function on $\mathrm{SO}^*(2n)'$ asssociated to the character of $\pi_{\mathrm{Min}}$. Furthermore for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $t_1, \ldots, t_k \in \mathbb{R}$ and $w_1, \ldots, w_k, z_1, \ldots, z_{n-2k} \in S^1$ set*

$$\lambda_j = \begin{cases} w_l e^{-|t_l|} & \text{if } 1 \leq j \leq 2k \text{ and } j = 2l-1 \\ w_l^{-1} e^{-|t_l|} & \text{if } 1 \leq j \leq 2k \text{ and } j = 2l \\ z_{j-2k} & \text{if } 2k < j \end{cases} \tag{3.26}$$
$$g = f(t_1, \ldots, t_k, w_1, \ldots, w_k, z_1, \ldots, z_{n-2k}).$$

*Then the character function is given by*

$$\theta(g) = \frac{(\prod_{j=1}^n \lambda_j)}{\prod_{1\leq j<k\leq n} 1 - \lambda_k \lambda_j^{-1}} \sum_{\sigma \in \mathrm{S}_n} \frac{\epsilon(\sigma) \prod_{j=1}^n \lambda_j^{j-\sigma^{-1}(j)}}{(1 - \lambda_{\sigma(1)}\lambda_{\sigma(2)})}. \tag{3.27}$$

**Proof.** Since $\pi_{\mathrm{Min}}$ is a lowest-weight representation with respect to $\Delta^+$ it follows from corollary 2.5.4 that $\pi_{\mathrm{Min}}$ extends to $\Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$. Now let $X \in -c^o_m$ and let $\gamma = \mathrm{Exp}(iX) \in \Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\gamma$ such that $|\lambda_j| < 1$. Let $X = A(\phi)$ then it follows from lemma 3.5.5 that $\phi_j > 0$ and $\mathbf{e}_j(X) = i\phi_j$ and thus by a reordering we can assume $\lambda_j = e^{-\phi_j} = e^{i\mathbf{e}_j(X)}$. Then it follows from proposition 2.6.3 that

$$\operatorname{tr}\pi(\gamma) = (\prod_{j=1}^n \lambda_j) \sum_{\sigma\in\mathrm{S}_n} \frac{\epsilon(\sigma)\prod_{j=1}^n \lambda_j^{j-\sigma^{-1}(j)}}{(1-\lambda_{\sigma(1)}\lambda_{\sigma(2)}) \prod_{1\leq j<k\leq n} 1 - \lambda_k\lambda_j^{-1}} =: f(\lambda_1, \ldots, \lambda_n). \tag{3.28}$$

From eq. (3.28) it is not clear that $f$ is invariant under permutations, nor that $f$ does not have singularities in $\{z \in \mathbb{C} \mid 0 < |z| < 1\}^n$. The permutation invariance follows since $\operatorname{tr}\pi$ is invariant under conjugations and the analytic Weyl group $W(\mathrm{G}, \mathfrak{h}_{\mathbb{C}})$ is isomorphic to $\mathrm{S}_n$ and acts by permutation of the standard weight vectors $\mathbf{e}_j$ thus by permuting the $\lambda_j$.

To see that $f$ does not have singularities in the polydisc we introduce the function $a : (\mathbb{C} \setminus \{0\})^n \to \mathbb{C}$ given by

$$a(z_1, \ldots, z_n) = \prod_{1 \le j \ne k \le n} 1 - z_j z_k^{-1}.$$

This is a holomorphic function not identically zero which satisfies the assumptions of lemma 3.5.8 and proposition 3.5.9 thus let $\hat{a}$ denote the associated function on $\Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$. Then the function $\hat{a} \operatorname{tr} \pi$ is a holomorphic function on $\Gamma_{\mathrm{SO}^*(2n)}(-C^o_{\mathrm{Min}})$ which for $\gamma = \operatorname{Exp}(iX)$ is given by

$$\hat{a}(\gamma) \operatorname{tr} \pi(\gamma) = \prod_{1 \le j < k \le n} (1 - \lambda_j \lambda_k^{-1}) \prod_{j=1}^{n} \lambda_j \sum_{\sigma \in \mathrm{S}_n} \frac{\epsilon(\sigma) \prod_{j=1}^{n} \lambda_j^{j - \sigma^{-1}(j)}}{1 - \lambda_{\sigma(1)} \lambda_{\sigma(2)}} =: b(\lambda_1, \ldots, \lambda_n).$$

However it is clear that $b$ does not have singularities when all $\lambda_j$ satisfy $|\lambda_j| < 1$. Thus we can apply proposition 3.5.9 to $b$ and get $\hat{b}$, then $\operatorname{tr} \pi|_{\operatorname{Exp}(ic^o_m)} = \frac{\hat{a}}{\hat{b}}|_{\operatorname{Exp}(ic^o_m)}$ and then it follows from proposition 2.4.18 that this holds on the entire open Ol'shanskiĭ semigroup.

Specifically we get that $\operatorname{tr} \pi$ satisfies the assumptions of proposition 2.8.9. Let $w_j = e^{i\phi_j}$ and $z_j = e^{i\psi_j}$ then we get from proposition 2.8.9 and lemma 3.5.10 that

$$\begin{aligned} \theta(g) = \lim_{s \to 0^+} \theta(\prod_{j=1}^{k} \operatorname{Exp}(-|t_j| H'_{\mathbf{e}_{2j-1} + \mathbf{e}_{2j}} + i\phi_j H'_{\mathbf{e}_{2j-1} - \mathbf{e}_{2j}}) \\ \cdot \exp(A(0, \ldots, 0, u_1 + is, \ldots, u_{n-2k} + is))). \end{aligned} \tag{3.29}$$

If we let $\gamma_s$ denote the argument to $\theta$ in the right-hand side of eq. (3.29) then we see that if we let $\lambda_j(s)$ be given by

$$\lambda_j(s) = \begin{cases} w_l e^{-|t_l|} & \text{if } 1 \le j \le 2k \text{ and } j = 2l - 1 \\ w_l^{-1} e^{-|t_l|} & \text{if } 1 \le j \le 2k \text{ and } j = 2l \\ z_{j-2k} e^{-s} & \text{if } 2k < j \end{cases}.$$

Then $|\lambda_j(s)| < 1$ and $\lambda_1(s), \ldots, \lambda_n(s)$ are eigenvalues of $\gamma_s$, furthermore we see that $\lambda_j(s)$ converges to eq. (3.26). Thus the right hand side of eq. (3.29) is equal to the right hand side of eq. (3.27). □

## 3.6 $\mathfrak{so}(2,2n)$

Though the minimal representation $\varpi^{p,q}$ for $\mathrm{O}(p,q)$ was constructed much later than the metaplectic representation, in the last twenty years it has received much study. In [BZ91] Binegar and Zierau constructs a minimal representation of $\mathrm{SO}_e(p,q)$ for $p+q$ even and $p+q$ greater than or equal to 6. This representation is in general irreducible and not a highest weight representation, but for $p = 2$, $q = 2n$ it decomposes into two irreducible highest weight representations. Their construction was generalized in

[HT93] and [ZH97] in different directions. In a series of papers [KØ03a; KØ03b; KØ03c] Kobayashi and Ørsted give different realisations of this representation for $\mathrm{O}(p,q)$ from primarily and an analytic and geometric point of view, and investigate different properties of the representation including branching laws. Kobayashi and Mano make further study of the analytic properties of this representation including the unitary inversion operator [KM07; KM11].

We will not attempt to provide a complete construction of the minimal representation of $\mathrm{SO}_e(2,2n)$. Rather we will use that based on the theory of chapter 2 we need only to know that a representation is an irreducible, unitary highest-weight representation and its K-type decomposition to calculate its character.

### 3.6.1 Structure and realization

We introduce some notation and fix a realisation of $\mathrm{SO}(2n+2,\mathbb{C})$ and $\mathrm{SO}_e(2,2n)$. By $\mathrm{SO}_e(2,2n)$ we denote the identity component of $\mathrm{SO}(2,2n)$.

$$\begin{aligned} I_k &= \mathrm{diag}(1,1,\dots,1) \in \mathrm{Mat}_{k\times k} \\ I_{p,q} &= \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \in \mathrm{Mat}_{(p+q)\times(p+q)} \\ s(v,w) &= v^T I_{2,2n} w \\ \mathrm{SO}(p,q,\mathbb{C}) &= \{A \in \mathrm{SL}(p+q,\mathbb{C}) \,|\, \forall v,w \in \mathbb{C}^{p+q} : s(Av,Aw) = s(v,w)\} \\ \mathrm{SO}(p,q) &= \mathrm{SO}(p,q,\mathbb{C}) \cap \mathrm{GL}(p+q,\mathbb{R}) \\ &. \end{aligned}$$

**Remark 3.6.1** For any $p,q \in \mathbb{Z}_{\geq 0}$ we get by the basis change

$$\vec{e}_j \mapsto \begin{cases} \vec{e}_j & 1 \leq j \leq p \\ i\vec{e}_j & p+1 \leq j \leq p+q \end{cases}$$

that $\mathrm{SO}(p,q,\mathbb{C}) \cong \mathrm{SO}(p+q,0,\mathbb{C})$.

Let $A : \mathbb{C}^{n+1} \to \mathrm{Mat}_{(2n+2)\times(2n+2)}(\mathbb{C})$ denote the map

$$A(\theta_0,\dots,\theta_n) = \begin{pmatrix} 0 & -\theta_0 & & & \\ \theta_0 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\theta_n \\ & & & \theta_n & 0 \end{pmatrix}.$$

**Proposition 3.6.2** *Let* K *denote* $\mathrm{SO}(2) \times \mathrm{SO}(2n)$ *embedded as block-diagonal matrices in* $\mathrm{SO}_e(2,2n)$ *then* K *is a maximal compact subgroup. Let*

$$\mathfrak{h} = \{A(\theta_0,\dots,\theta_n) \,|\, \theta_0,\dots,\theta_n \in \mathbb{R}\}$$

$$\mathrm{H} = \{\begin{pmatrix} \cos(\theta_0) & -\sin(\theta_0) & & & \\ \sin(\theta_0) & \cos(\theta_0) & & & \\ & & \ddots & & \\ & & & \cos(\theta_n) & -\sin(\theta_n) \\ & & & \sin(\theta_n) & \cos(\theta_n) \end{pmatrix} \,|\, \theta_0,\dots,\theta_n \in \mathbb{R}\}.$$

*Then* H *is a Cartan subgroup of* $\mathrm{SO}_e(2,2n)$ *and a maximal torus in* $\mathrm{SO}(2) \times \mathrm{SO}(2n)$. *Furthermore* $\mathrm{SO}(2n+2,\mathbb{C})$ *is a complexification of* $\mathrm{SO}_e(2,2n)$.

*Let* $\mathbf{e}_j \in \mathfrak{h}_{\mathbb{C}}^*$ *be given by* $\mathbf{e}_j(A(\theta)) = i\theta_j$. *Then*

$$\begin{aligned} \Delta &= \{\pm\mathbf{e}_j \pm \mathbf{e}_k \,|\, 0 \le j \ne k \le n\} \\ \Delta^+ &= \{\mathbf{e}_j \pm \mathbf{e}_k \,|\, 0 \le j < k \le n\} \\ \Delta_k &= \{\pm\mathbf{e}_j \pm \mathbf{e}_k \,|\, 1 \le j \ne k \le n\} \\ \Delta_k^+ &= \{\mathbf{e}_j \pm \mathbf{e}_k \,|\, 1 \le j < k \le n\} \\ \Delta_n^+ &= \{\mathbf{e}_0 \pm \mathbf{e}_k \,|\, 1 \le k \le n\} \\ W_{\mathrm{K}} &\cong \mathrm{S}_n \ltimes \{\pm\}^{n-1} \\ \delta_k &= \sum_{j=1}^{n} (n-j)\mathbf{e}_j. \end{aligned}$$

**Remark 3.6.3** $W_{\mathrm{K}}$ acts on $\mathfrak{h}_{\mathbb{C}}^*$ by permuting the $\mathbf{e}_j$ for $j \ge 1$ and by changing an even number of signs on the same basis vectors. If $\sigma \in \{\pm\}^{n-1}$ then written as a matrix that acts on $\mathfrak{h}_{\mathbb{C}}^*$ in the basis $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$ it is

$$\sigma = \begin{pmatrix} 1 & & & \\ & \epsilon_1 & & \\ & & \ddots & \\ & & & \epsilon_n \end{pmatrix}$$

with $\epsilon_j \in \{\pm 1\}$. Thus $\epsilon(\sigma) = \det(\sigma) = \prod_{j=1}^n \epsilon_j = 1$ since it is an even sign change.

**Proof.** [Kna02, page 150, 155, 164, 513]. □

**Lemma 3.6.4** *Let* $H, H' \in \mathfrak{h}_{\mathbb{C}} \subset \mathfrak{so}(2,2n)_{\mathbb{C}}$ *then*

$$B(H,H') = 4n \sum_{j=0}^{n} \mathbf{e}_j(H)\mathbf{e}_j(H')$$

*and let* $H_{\mathbf{e}_k} = A(\phi)$ *then*

$$\phi_j = \frac{-i}{4n}\delta_{j,k}.$$

**Proof.** The first equality follows from lemma 3.5.3 since $\mathfrak{so}(2,2n)_{\mathbb{C}}$ is isomorphic to $\mathfrak{so}(2(n+1),\mathbb{C})$. Let $H_{\mathbf{e}_k} \in \mathfrak{h}_{\mathbb{C}}$ denote the element such that $\mathbf{e}_k = B(H_{\mathbf{e}_k},\cdot)$ and let $H_{\mathbf{e}_k} = A(\phi)$ for $\phi \in \mathbb{C}^{n+1}$. Then

$$\mathbf{e}_k(H) = B(H_{\mathbf{e}_k}, H) = 4n\sum_{j=0}^{n}\mathbf{e}_j(H_{\mathbf{e}_k})\mathbf{e}_j(H)$$

which gives since the $\mathbf{e}_0,\dots,\mathbf{e}_n$ are linearly independent that $i\phi_j = \mathbf{e}_j(H_{\mathbf{e}_k}) = \frac{1}{4n}\delta_{j,k}$. □

**Lemma 3.6.5** *Let* $C_{\mathrm{Min}}$ *denote the minimal cone associated to the positive system in proposition 3.6.2, and let as usual* $c_m = C_{\mathrm{Min}} \cap \mathfrak{h}$ *then*

$$c_m = A(\{\phi \in \mathbb{R}^{n+1} \,|\, \phi_0 \le -\sum_{j=1}^{n}|\phi_j|\})$$
$$c_m^o = A(\{\phi \in \mathbb{R}^{n+1} \,|\, \phi_0 < -\sum_{j=1}^{n}|\phi_j|\}).$$

**Proof.** More details about the cone is contained [Pan81, section II.12] but we include a proof for completeness.

Let $X \in c_m$ then it follows from proposition 2.3.2 that $X = (-i)\sum_{j=1}^{n} a_j H_{\mathbf{e}_0+\mathbf{e}_j} + b_j H_{\mathbf{e}_0-\mathbf{e}_j}$ for $a_j, b_j \ge 0$. Then

$$X = (-i)\sum_{j=1}^{n} a_j H_{\mathbf{e}_0+\mathbf{e}_j} + b_j H_{\mathbf{e}_0-\mathbf{e}_j} = -i(\sum_{j=1}^{n} a_j + b_j)H_{\mathbf{e}_0} - i\sum_{j=1}^{n}(a_j - b_j)H_{\mathbf{e}_j}.$$

Let $\phi \in \mathbb{C}^{n+1}$ such that $A(\phi) = X$ then it follows from lemma 3.6.4 that

$$\phi_0 = \frac{-1}{4n}\sum_{j=1}^{n} a_j + b_j$$
$$\phi_j = \frac{1}{4n}(b_j - a_j).$$

This implies that $-\phi_0 \ge \sum_{j=1}^{n}|\phi_j|$.

Conversely if $-\phi_0 \geq \sum_{j=1}^{n}|\phi_j|$ then put

$$\begin{aligned}
a_1 &= 4n((\phi_1)_- + \frac{1}{2}(-\phi_0 - \sum_{j=1}^{n}|\phi_j|)) \\
b_1 &= 4n((\phi_1)_+ + \frac{1}{2}(-\phi_0 - \sum_{j=1}^{n}|\phi_j|)) \\
a_j &= 4n(\phi_j)_- \\
b_j &= 4n(\phi_j)_+.
\end{aligned}$$

Then we see that $a_1, b_1, \dots, a_n, b_n \geq 0$ and let $\psi \in \mathbb{C}^{n+1}$ such that

$$A(\psi) = (-i)\sum_{j=1}^{n} a_j H_{\mathbf{e}_0+\mathbf{e}_j} + b_j H_{\mathbf{e}_0-\mathbf{e}_j} \in c_m.$$

Then it follows from our previous calculations that

$$\begin{aligned}
\psi_0 &= \frac{-1}{4n}\sum_{j=1}^{n} a_j + b_j = -(-\phi_0 - \sum_{j=1}^{n}|\phi_j| + \sum_{j=1}^{n}(\phi_j)_- + (\phi_j)_+) = \phi_0 \\
\psi_j &= \frac{1}{4n}(b_j - a_j) = (\phi_j)_+ - (\phi_j)_i = \phi_j.
\end{aligned}$$

□

### 3.6.2 Minimal representations

**Proposition 3.6.6** *Using notation from proposition 3.6.2 for the roots of* $\mathrm{SO}(p)$*. For* $p \geq 3$ *and* $k \geq 0$ *we denote by* $\mathcal{H}^k(\mathbb{C}^p)$ *the representation of* $\mathrm{SO}(p)$ *on spherical harmonics of degree* $k$*.* $\mathcal{H}^k(\mathbb{C}^p)$ *is an irreducible representation. For* $p \neq 4$ *the highest weight of* $\mathcal{H}^k(\mathbb{C}^p)$ *is* $k\mathbf{e}_1$*. For* $p = 4$ *the highest weight of* $\mathcal{H}^k(\mathbb{C}^p)$ *is* $k(\mathbf{e}_1 + \mathbf{e}_2)$*.*

**Proof.** [Vil68, section IX.2] shows that $\mathcal{H}^k(\mathbb{C}^p)$ is an irreducible representation of $\mathrm{SO}(p)$. [GW09, theorem 5.6.11] gives the highest weights. □

For $k \in \mathbb{Z}$ we let $\chi_k : \mathrm{SO}(2) \to \mathbb{C}$ denote the character $\chi_k(\begin{pmatrix}\cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta)\end{pmatrix}) = e^{ik\theta}$.

**Proposition 3.6.7** *Let* $\varpi^{2,2n}$ *denote the representation of* $\mathrm{SO}_e(2,2n)$ *constructed in [BZ91] and let* $\varpi^{2,2n} = \varpi_+^{2,2n} \oplus \varpi_-^{2,2n}$ *denote the decomposition described in [BZ91, proposition IV].*

*Then* $\varpi_+^{2,2n}$ *and* $\varpi_-^{2,2n}$ *are irreducible unitary highest weight representations of* $\mathrm{SO}_e(2,2n)$*.*

$$\begin{aligned}
\varpi_+^{2,2n}|_{\mathrm{SO}(2)\times\mathrm{SO}(2n)} &= \widehat{\bigoplus}_{k=0}^{\infty}\chi_{k+n-1} \boxtimes \mathcal{H}^k(\mathbb{C}^{2n}) \\
\varpi_+^{2,2n}|_{\mathrm{SO}(2)\times\mathrm{SO}(2n)} &= \widehat{\bigoplus}_{k=0}^{\infty}\chi_{-(k+n-1)} \boxtimes \mathcal{H}^k(\mathbb{C}^{2n}).
\end{aligned}$$

**Proof.** [BZ91]. □

**Lemma 3.6.8** *Let $C_{\mathrm{Min}}$ be the minimal cone in $\mathfrak{so}(2,2n)$ associated to the positive system from proposition 3.6.2. Then $\varpi_+^{2,2n}$ extends to a representation of $\Gamma_{\mathrm{SO}(2,2n)}(-C^o_{\mathrm{Min}})$ and $\varpi_-^{2,2n}$ extends to a representation of $\Gamma_{\mathrm{SO}(2,2n)}(C^o_{\mathrm{Min}})$.*

**Proof.** $\varpi_-^{2,2n}$ is a highest-weight representation with respect to $\Delta^+$ hence by proposition 2.5.3 it extends to $\Gamma_{\mathrm{SO}(2,2n)}(C^o_{\mathrm{Min}})$. Similarly it follows from corollary 2.5.4 that $\varpi_+^{2,2n}$ extends to $\Gamma_{\mathrm{SO}(2,2n)}(-C^o_{\mathrm{Min}})$. □

### 3.6.3 Combinatorics

We intend to calculate the characters of $\varpi_+^{2,2n}$ and $\varpi_-^{2,2n}$ using similar methods to the second proof of lemma 3.3.13. In this case we will also need to rewrite some alternating sums over the analytic Weyl group. In order to simplify the notation a bit we suppress elements of $\mathfrak{h}$ and for $\alpha \in \mathfrak{h}^*_{\mathbb{C}}$ let $e^\alpha$ denote the function

$$e^\alpha : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C} \qquad\qquad e^\alpha(X) = e^{\alpha(X)}.$$

Sometimes it will be easier to use the notation $(a_1, \dots, a_n) = \sum_{j=1}^n a_j \mathbf{e}_j \in \mathfrak{h}^*_{\mathbb{C}}$ thus

$$e^{(a_1,\dots,a_n)} = e^{\sum_{j=1}^n a_j \mathbf{e}_j}.$$

The Weyl group acts on $\mathfrak{h}_{\mathbb{C}}$ thus it also induces an action on functions on $\mathfrak{h}_{\mathbb{C}}$ we observe

$$(w.e^\alpha)(X) = e^\alpha(w^{-1}X) = e^{\alpha(w^{-1}X)} = e^{(w.\alpha)(X)} = e^{w.\alpha}(X).$$

We will show many of the results in this section by induction over thus let $W_n$ denote the analytic Weyl group $W(\mathrm{SO}(2n), \mathrm{H})$.

**Lemma 3.6.9** *For $n \geq 1$ and $\alpha, \beta \in \mathbb{C}$*

$$\begin{aligned}\sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^n (n-j)\mathbf{e}_j)} \prod_{j=1}^n (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)}) \\ = \sum_{k=0}^n (\alpha^{2(n-k)} + \beta^{2(n-k)} - \delta_{k,n})\alpha^k \beta^k \sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^k \mathbf{e}_j + \sum_{j=1}^n (n-j)\mathbf{e}_j)} \\ + \alpha^n \beta^n \sum_{w \in W_n} \epsilon(w) e^{w.(-\mathbf{e}_n + \sum_{j=1}^{n-1} \mathbf{e}_j + \sum_{j=1}^n (n-j)\mathbf{e}_j)}. \end{aligned} \tag{3.30}$$

**Proof.** For $n = 1$ we just get

$$\sum_{w \in W_n} \epsilon(w)(\alpha^2 + \beta^2 + \alpha\beta e^{w.\mathbf{e}_1} + \alpha e^{w.(-\mathbf{e}_1)})$$

for both sides of the equation.

For $n \geq 2$ we get

$$
\begin{aligned}
&\sum_{w\in W_n} \epsilon(w) e^{w.(\sum_{j=1}^n (n-j)\mathbf{e}_j)} \prod_{j=1}^n (\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
&= (\alpha^2+\beta^2) \sum_{w\in W_n} \epsilon(w) e^{w.(\sum_{j=1}^n (n-j)\mathbf{e}_j)} \prod_{j=2}^n (\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
&\quad + \alpha\beta \sum_{w\in W_n} \epsilon(w) e^{w.(\mathbf{e}_1+\sum_{j=1}^n (n-j)\mathbf{e}_j)} \prod_{j=2}^n (\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
&\quad + \alpha\beta \sum_{w\in W_n} \epsilon(w) e^{w.(-\mathbf{e}_1+\sum_{j=1}^n (n-j)\mathbf{e}_j-\mathbf{e}_1)} \prod_{j=1}^n (\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
&= (\alpha^2+\beta^2)\sum_{k=0}^{n-1}(\alpha^{2(n-1-k)}+\beta^{2(n-1-k)}-\delta_{k,n-1})\alpha^k\beta^k \\
&\qquad\qquad \cdot\left(\sum_{w\in W_n}\epsilon(w)e^{w.(\sum_{j=2}^{k+1}\mathbf{e}_j+\sum_{j=1}^n(n-j)\mathbf{e}_j)}\right) && (3.31)\\
&\quad + (\alpha^2+\beta^2)\alpha^{n-1}\beta^{n-1}\sum_{w\in W_n}\epsilon(w)e^{w.(-\mathbf{e}_n+\sum_{j=2}^{n-1}\mathbf{e}_j+\sum_{j=1}^n(n-j)\mathbf{e}_j)} && (3.32)\\
&\quad + \alpha\beta\sum_{k=0}^{n-1}(\alpha^{2(n-1-k)}+\beta^{2(n-1-k)}-\delta_{k,n-1})\alpha^k\beta^k \\
&\qquad\qquad \cdot\left(\sum_{w\in W_n}\epsilon(w)e^{w.(\sum_{j=1}^{k+1}\mathbf{e}_j+\sum_{j=1}^n(n-j)\mathbf{e}_j)}\right) \\
&\quad + \alpha\beta\alpha^{n-1}\beta^{n-1}\sum_{w\in W_n}\epsilon(w)e^{w.(-\mathbf{e}_n+\sum_{j=1}^{n-1}\mathbf{e}_j+\sum_{j=1}^n(n-j)\mathbf{e}_j)} \\
&\quad + \alpha\beta\sum_{k=0}^{n-1}(\alpha^{2(n-1-k)}+\beta^{2(n-1-k)}-\delta_{k,n-1})\alpha^k\beta^k \\
&\qquad\qquad \cdot\left(\sum_{w\in W_n}\epsilon(w)e^{w.(-\mathbf{e}_1+\sum_{j=2}^{k+1}\mathbf{e}_j+\sum_{j=1}^n(n-j)\mathbf{e}_j)}\right) && (3.33)\\
&\quad + \alpha\beta\alpha^{n-1}\beta^{n-1}\sum_{w\in W_n}\epsilon(w)e^{w.(-\mathbf{e}_n-\mathbf{e}_1+\sum_{j=2}^{n-1}\mathbf{e}_j+\sum_{j=1}^n(n-j)\mathbf{e}_j)}. && (3.34)
\end{aligned}
$$

The first equality is just expanding the factor with $j=1$. Then we let $W_{n-1}\subset W_n$ such that $W_{n-1}$ acts on $\mathbf{e}_2,\ldots,\mathbf{e}_n$ and apply corollary 3.2.2 and the induction hypothesis to all terms. In eq. (3.31) we observe that in all terms with $k\geq 1$ the expression in the exponential is invariant when we act by (1 2) hence by lemma 3.2.3 they are all zero. For eq. (3.33) observe that for $k=0$ the expression in the exponential is invariant under (1 2) and for $n\geq 3$ and $k\geq 2$ it is invariant under (1 3) hence by lemma 3.2.3 they are all zero.

For $k = 1$ we apply (1 2) and reorder the sum to get

$$-\alpha^2\beta^2(\alpha^{2(n-2)} + \beta^{2(n-2-k)}).$$

For $n \geq 3$ we see that in eq. (3.32) the expression in the exponential is invariant under (1 2), for $n = 2$ the expression in the exponential is $(1, -1)$ which is invariant under $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. In both cases lemma 3.2.3 gives that the sum is zero. In eq. (3.34) with $n \geq 4$ the expression in the exponential is invariant under (1 3) for $n = 3$ it is $(1, 2, -1)$ hence it is invariant under $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. In both cases lemma 3.2.3 gives that the sum is zero. For $n = 2$ the expression in the exponential is $(0, -1)$ we thus act by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and reorder the sum and add it to the $k = 0$ term from eq. (3.31) and the $k = 1$ term from eq. (3.33) to get

$$((\alpha^2 + \beta^2)(\alpha^2 + \beta^2) - 2\alpha^2\beta^2) \sum_{w \in W_2} \epsilon(w) e^{w.(1,0)} = (\alpha^4 + \beta^4) \sum_{w \in W_2} \epsilon(w) e^{w.(1,0)}.$$

For $n = 3$ we add the $k = 0$ term from eq. (3.31) and the $k = 1$ term from eq. (3.33) and get

$$\begin{aligned}(\alpha^2 + \beta^2)&(\alpha^{2(n-1)} + \beta^{2(n-1)}) - \alpha^2\beta^2(\alpha^{2(n-2)} + \beta^{2(n-2)}) \\ &= \alpha^{2n} + \beta^{2n} + \alpha^2\beta^{2(n-1)} + \alpha^{2(n-1)}\beta^2 - \alpha^2\beta^{2(n-1)} - \alpha^{2(n-1)}\beta^2 = \alpha^{2n} + \beta^{2n}.\end{aligned}$$

Then collecting terms we get the right hand side of eq. (3.30). □

**Corollary 3.6.10** *For $n \geq 1$ and $\alpha, \beta \in \mathbb{C}$*

$$\begin{aligned}\sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^{n}(n-j)\mathbf{e}_j)} &\prod_{j=2}^{n} (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)}) \\ &= (\alpha^{2(n-1)} + \beta^{2(n-1)} - \delta_{n,1}) \sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^{n}(n-j)\mathbf{e}_j)}.\end{aligned}$$

**Proof.** Observe first that statement is trivial for $n = 1$. The $n \geq 2$ case we argued in the proof of lemma 3.6.9 when rewriting eqs. (3.31) and (3.32). □

**Corollary 3.6.11** *For $n \geq 2$ and $\alpha, \beta \in \mathbb{C}$*

$$\begin{aligned}\sum_{w \in W_n} \epsilon(w) e^{w.(-\mathbf{e}_1 + \sum_{j=1}^{n}(n-j)\mathbf{e}_j)} &\prod_{j=2}^{n} (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)}) \\ &= -(\alpha^{2(n-2)} + \beta^{2(n-2)})\alpha\beta \sum_{w \in W_n} \epsilon(w) e^{w.\sum_{j=1}^{n}(n-j)\mathbf{e}_j}.\end{aligned}$$

**Proof.** This is also argued during the proof of lemma 3.6.9. $\square$

**Corollary 3.6.12** *For $n \geq 2$ and $\alpha, \beta \in \mathbb{C}$*

$$\sum_{w \in W_n} \epsilon(w) e^{w.\sum_{j=1}^n (n-j)\mathbf{e}_j} (\alpha + \beta e^{w.(-\mathbf{e}_1)}) \prod_{j=2}^n (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)})$$
$$= \alpha^{2n-3}(\alpha^2 - \beta^2) \sum_{w \in W_n} \epsilon(w) e^{w.\sum_{j=1}^n (n-j)\mathbf{e}_j}.$$

**Proof.** This follows from corollary 3.6.10 and corollary 3.6.11 since

$$\alpha(\alpha^{2n-2} + \beta^{2n-2}) - \alpha\beta^2(\alpha^{2n-4} + \beta^{2n-4})$$
$$= \alpha^{2n-1} + \alpha\beta^{2n-2} - \alpha^{2n-3}\beta^2 - \alpha\beta^{2n-2} = \alpha^{2n-3}(\alpha^2 - \beta^2). \quad \square$$

### 3.6.4 The character

**Lemma 3.6.13** *Let $n \geq 3$ and $X \in c_m^o$ then*

$$\operatorname{tr}(\varpi_+^{2,2n}(\operatorname{Exp}(-iX))) = \frac{e^{-i(n-1)\mathbf{e}_0(X)} - e^{-i(n+1)\mathbf{e}_0(X)}}{\prod_{j=1}^n (1 - e^{-i(\mathbf{e}_0 + \mathbf{e}_j)(X)})(1 - e^{-i(\mathbf{e}_0 - \mathbf{e}_j)(X)})}$$
$$\operatorname{tr}(\varpi_-^{2,2n}(\operatorname{Exp}(iX))) = \frac{e^{-i(n-1)\mathbf{e}_0(X)} - e^{-i(n+1)\mathbf{e}_0(X)}}{\prod_{j=1}^n (1 - e^{-i(\mathbf{e}_0 + \mathbf{e}_j)(X)})(1 - e^{-i(\mathbf{e}_0 - \mathbf{e}_j)(X)})}$$

**Proof.** Let us first consider $\varpi_+^{2,2n}$: Then it follows from proposition 3.6.7 that in the notation of proposition 2.6.3 $\alpha = (n-1)\mathbf{e}_0$ and $\beta = \mathbf{e}_0 + \mathbf{e}_1$ using corollary 3.6.12 we get

$$\begin{aligned}
&\operatorname{tr} \varpi_+^{2,2n}(\operatorname{Exp}(-iX)) \\
=&\frac{e^{-i(n-1)\mathbf{e}_0(X)}}{(\prod_{j=1}^n (1 - e^{-i(\mathbf{e}_0 + \mathbf{e}_j)(X)})(1 - e^{-i(\mathbf{e}_0 - \mathbf{e}_j)(X)}))(\sum_{w \in W_{\mathrm{K}}} \epsilon(w) e^{-i(w.\delta_k)(X)})} \\
&\cdot \sum_{w \in W_{\mathrm{K}}} \Big[\epsilon(w) e^{-i(w.\delta_k)(X)} (1 + e^{-i(w.(-\mathbf{e}_1))(X)}) \\
&\qquad \cdot \prod_{j=2}^n (1 - e^{-i(\mathbf{e}_0 + w.\mathbf{e}_j)(X)})(1 - e^{-i(\mathbf{e}_0 - w.\mathbf{e}_j)(X)}))\Big] \\
=&\frac{e^{-i(n-1)\mathbf{e}_0(X)}(1 - e^{-2i\mathbf{e}_0(X)})}{\prod_{j=1}^n (1 - e^{-i(\mathbf{e}_0 + \mathbf{e}_j)(X)})(1 - e^{-i(\mathbf{e}_0 + \mathbf{e}_j)(X)})}.
\end{aligned}$$

And the calculation for $\varpi_-^{2,2n}$ follows similarly. $\square$

**Lemma 3.6.14** *Let $n \geq 3$ and consider the maps:*

$$\begin{aligned}
f_0 &: (S^1)^{n+1} \to \mathrm{SO}_e(2,2n) \\
f_0(w_0,\dots,w_n) &= \begin{pmatrix} r(w_0) & & \\ & \ddots & \\ & & r(w_n) \end{pmatrix} \\
f_1 &: \mathbb{R} \times (S^1)^n \to \mathrm{SO}_e(2,2n) \\
f_1(t,w_1,\dots,w_n) &= \begin{pmatrix} \alpha(t,w_1) & & & \\ & r(w_2) & & \\ & & \ddots & \\ & & & r(w_n) \end{pmatrix} \\
f_2 &: \{\pm\} \times \mathbb{R}^2 \times (S^1)^{n-1} \to \mathrm{SO}_e(2,2n) \\
f_2(\pm,s,t,w_2,\dots,w_n) &= \begin{pmatrix} \pm\beta(s,t) & & & \\ & r(w_2) & & \\ & & \ddots & \\ & & & r(w_n) \end{pmatrix}
\end{aligned}$$

*where*

$$\begin{aligned}
r(e^{i\phi}) &= \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \\
\alpha(t,e^{i\phi}) &= \begin{pmatrix} r(\phi) & \\ & r(\phi) \end{pmatrix} \begin{pmatrix} \cosh(t)I_2 & \sinh(t)I_2 \\ \sinh(t)I_2 & \cosh(t)I_2 \end{pmatrix} \\
&= \begin{pmatrix} \cosh(t)\cos(\phi) & -\cosh(t)\sin(\phi) & \sinh(t)\cos(\phi) & -\sinh(t)\sin(\phi) \\ \cosh(t)\sin(\phi) & \cosh(t)\cos(\phi) & \sinh(t)\sin(\phi) & \sinh(t)\cos(\phi) \\ \sinh(t)\cos(\phi) & -\sinh(t)\sin(\phi) & \cosh(t)\cos(\phi) & -\cosh(t)\sin(\phi) \\ \sinh(t)\sin(\phi) & \sinh(t)\cos(\phi) & \cosh(t)\sin(\phi) & \cosh(t)\cos(\phi) \end{pmatrix} \\
\beta(s,t) &= \begin{pmatrix} \cosh(s) & & \sinh(s) & \\ & \cosh(t) & & \sinh(t) \\ \sinh(s) & & \cosh(s) & \\ & \sinh(t) & & \cosh(t) \end{pmatrix}.
\end{aligned}$$

*Then $f_0, f_1$ and $f_2$ parametrize a maximal set of non-conjugate Cartan subgroups of* $\mathrm{SO}_e(2,2n)$.

**Proof.** Let $S \subset \Delta_n^+$ be a set of pairwise strongly orthogonal roots, for $n \geq 3$ there is up to an action by $W_{\mathrm{K}}$ three possibilities: $S = \emptyset, \{\mathbf{e}_0 - \mathbf{e}_1\}, \{\mathbf{e}_0 + \mathbf{e}_1, \mathbf{e}_0 - \mathbf{e}_1\}$. We want to show that $f_1$ parametrizes the Cartan subgroup corresponding to $\mathfrak{h}_{\mathbf{e}_0 - \mathbf{e}_1}$ and $f_2$ the one corresponding to $\mathfrak{h}_{\{\mathbf{e}_0 \pm \mathbf{e}_1\}}$.

First of all let

$$E_{\mathbf{e}_0-\mathbf{e}_1} = \frac{1}{2}\begin{pmatrix} & & 1 & i & \\ & & -i & 1 & \\ 1 & -i & & & \\ i & 1 & & & \\ & & & & 0 \end{pmatrix}$$

$$E_{\mathbf{e}_0+\mathbf{e}_1} = \frac{1}{2}\begin{pmatrix} & & 1 & -i & \\ & & -i & -1 & \\ 1 & -i & & & \\ -i & -1 & & & \\ & & & & 0 \end{pmatrix}$$

then

$$[E_{\mathbf{e}_0-\mathbf{e}_1}, \overline{E_{\mathbf{e}_0-\mathbf{e}_1}}] = \begin{pmatrix} 0 & i & & & \\ -i & 0 & & & \\ & & 0 & -i & \\ & & i & 0 & \\ & & & & 0 \end{pmatrix} = H'_{\mathbf{e}_0-\mathbf{e}_1}$$

$$[E_{\mathbf{e}_0+\mathbf{e}_1}, \overline{E_{\mathbf{e}_0+\mathbf{e}_1}}] = \begin{pmatrix} 0 & i & & & \\ -i & 0 & & & \\ & & 0 & i & \\ & & -i & 0 & \\ & & & & 0 \end{pmatrix} = H'_{\mathbf{e}_0+\mathbf{e}_1}.$$

Hence we let $\mathbf{c}_{\mathbf{e}_0\pm\mathbf{e}_0}$ be the Cayley transforms defined from $E_{\mathbf{e}_0\pm\mathbf{e}_1}$. Furthermore we see that

$$\begin{aligned}
\frac{\partial f_1}{\partial t}(e) &= E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}} \\
\frac{\partial f_1}{\partial w_1}(e) &= iH'_{\mathbf{e}_0+\mathbf{e}_1} \\
\frac{\partial f_2}{\partial s}(e) &= \frac{1}{2}(E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}} + E_{\mathbf{e}_0+\mathbf{e}_1} + \overline{E_{\mathbf{e}_0+\mathbf{e}_1}}) \\
\frac{\partial f_2}{\partial t}(e) &= \frac{1}{2}(E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}} - E_{\mathbf{e}_0+\mathbf{e}_1} - \overline{E_{\mathbf{e}_0+\mathbf{e}_1}})
\end{aligned}$$

thus $Tf_1\mathbb{R}^{n+1} = \mathfrak{h}_{\mathbf{e}_0-\mathbf{e}_1}$ and $Tf_2\mathbb{R}^{n+1} = \mathfrak{h}_{\{\mathbf{e}_0\pm\mathbf{e}_1\}}$. Thus the $f_1$ and $f_2(+,\cdot)$ parametrize the analytic subgroups associated to the Cayley transformed Cartan subalgebras. Let $H_{1,\gamma}$ denote the dual element to $\gamma \in \mathfrak{h}^*_{\mathbf{e}_0-\mathbf{e}_1,\mathbb{C}}$ and $H_{2,\gamma}$ similarly for $\gamma \in \mathfrak{h}^*_{\{\mathbf{e}_0\pm\mathbf{e}_1\},\mathbb{C}}$. Since $\mathrm{SO}_e(2,2n)$ has a complexification; to see that $f_1$ and $f_2$ parametrize the Cartan subgroups, it is enough by [Kna02, proposition 7.110] to check that $\exp(\pi i H'_{1,\gamma})$ and $\exp(\pi i H'_{2,\gamma})$ are in their respective images for all real roots $\gamma$.

Now let $\alpha_j = \mathbf{c}_{\mathbf{e}_0-\mathbf{e}_1}(\mathbf{e}_j)$ and $\beta_j = \mathbf{c}_{\{\mathbf{e}_0\pm\mathbf{e}_1\}}(\mathbf{e}_j)$ be the Cayley transformed standard weight vectors. We observe that for $\phi \in \mathbb{R}^{n-1}$ we get $A(0,0,\phi) \in \mathrm{Ker}(\mathbf{e}_0), \mathrm{Ker}(\mathbf{e}_1)$ hence $A(0,0,\phi)$ is fixed by both Cayley transforms. Since $\mathbf{e}_0 - \mathbf{e}_1$ and $\mathbf{e}_0 + \mathbf{e}_1$ are strongly orthogonal we get that $H'_{\mathbf{e}_0+\mathbf{e}_1}$ is fixed by $\mathbf{c}_{\mathbf{e}_0-\mathbf{e}_1}$. Thus if $s, t, \phi_1, \dots, \phi_n \in \mathbb{R}^n$ and we set $v = (t, \phi_1, \dots, \phi_n)$ and $w = (s, t, \phi_2, \dots, \phi_n)$ then we get

$$\begin{aligned}
(\alpha_0 - \alpha_1)(Tf_1 v) &= (\alpha_0 - \alpha_1)(t(E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}}) + \phi_1 i H'_{\mathbf{e}_0+\mathbf{e}_1} + A(0,0,\phi_2,\dots,\phi_n))\\
&= (\mathbf{e}_0 - \mathbf{e}_1)(tH'_{\mathbf{e}_0-\mathbf{e}_1} + \phi_1 i H'_{\mathbf{e}_0+\mathbf{e}_1}) = 2t\\
(\alpha_0 + \alpha_1)(Tf_1 v) &= (\mathbf{e}_0 + \mathbf{e}_1)(tH'_{\mathbf{e}_0-\mathbf{e}_1} + \phi_1 i H'_{\mathbf{e}_0+\mathbf{e}_1}) = 2i\phi_1\\
(\beta_0 - \beta_1)(Tf_2 w) &= \frac{1}{2}(\beta_0 - \beta_1)(s(E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}} + E_{\mathbf{e}_0+\mathbf{e}_1} + \overline{E_{\mathbf{e}_0+\mathbf{e}_1}})\\
&\qquad + t(E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}} - E_{\mathbf{e}_0+\mathbf{e}_1} + \overline{E_{\mathbf{e}_0+\mathbf{e}_1}}) + A(0,0,\phi_2,\dots,\phi_n))\\
&= \frac{1}{2}(\mathbf{e}_0 - \mathbf{e}_1)((s+t)H'_{\mathbf{e}_0-\mathbf{e}_1} + (s-t)H'_{\mathbf{e}_0+\mathbf{e}_1}) = \frac{1}{2}(s+t)\\
(\beta_0 + \beta_1)(Tf_2 w) &= \frac{1}{2}(\mathbf{e}_0 + \mathbf{e}_1)((s+t)H'_{\mathbf{e}_0-\mathbf{e}_1} + (s-t)H'_{\mathbf{e}_0+\mathbf{e}_1}) = \frac{1}{2}(s-t).
\end{aligned}$$

Which gives

$$\begin{aligned}
\alpha_0(Tf_1 v) &= t + i\phi_1 & \alpha_1(Tf_1 v) &= -t + i\phi_1\\
\alpha_j(Tf_1 v) &= i\phi_j & &\\
\beta_0(Tf_2 w) &= \frac{1}{2}s & \beta_1(Tf_2 w) &= -\frac{1}{2}t\\
\beta_j(Tf_2 w) &= i\phi_j. & &
\end{aligned}$$

Thus we see that the real roots are respectively $\{\pm(\alpha_0 - \alpha_1)\}$ and $\{\pm\beta_0 \pm \beta_1\}$. We get from lemma 3.2.9 that

$$\begin{aligned}
H'_{\mathbf{e}_0-\mathbf{e}_1,\alpha 1-\alpha 2} &= \mathbf{c}_{\mathbf{e}_0-\mathbf{e}_1}(H'_{\mathbf{e}_0-\mathbf{e}_1}) = E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}}\\
H'_{\{\mathbf{e}_0\pm\mathbf{e}_1\},\alpha 1\pm\alpha 2} &= \mathbf{c}_{\{\mathbf{e}_0\pm\mathbf{e}_1\}}(H'_{\mathbf{e}_0\pm\mathbf{e}_1}) = E_{\mathbf{e}_0\pm\mathbf{e}_1} + \overline{E_{\mathbf{e}_0\pm\mathbf{e}_1}}.
\end{aligned}$$

Then to see that $\exp(\pi i H'_{\mathbf{e}_0-\mathbf{e}_1,\alpha 1-\alpha 2}) \in f_1(\mathbb{R} \times (S^1)^n)$ and $\exp(\pi i H'_{\{\mathbf{e}_0\pm\mathbf{e}_1\},\alpha 1\pm\alpha 2}) \in f_2(-, \mathbb{R}^2 \times (S^1)^{n-1})$ it is enough to see that

$$\exp(\pi i E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}}) = \begin{pmatrix} -I_2 & & \\ & -I_2 & \\ & & I_{2(n-1)} \end{pmatrix} = \exp(\pi i E_{\mathbf{e}_0+\mathbf{e}_1} + \overline{E_{\mathbf{e}_0+\mathbf{e}_1}}).$$

Which is also $f_1(0, -1, 1, \dots, 1)$ and $f_2(-, 0, 0, 1, \dots, 1)$. □

**Remark 3.6.15** From the calculations and realizations in the proof of lemma 3.6.14 we also get

$$\begin{aligned}
f_0(e^{i\phi_0}, \dots, e^{i\phi_n}) &= \exp(A(\phi))\\
f_1(t, e^{i\phi_1}, \dots, e^{i\phi_n}) &= \exp(t(E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}}))\exp(A(\phi_1, \phi_1, \phi_2, \dots, \phi_n))\\
f_2((-1)^k, s, t, \phi_2, \dots, \phi_n) &= \exp(\frac{s+t}{2}(E_{\mathbf{e}_0-\mathbf{e}_1} + \overline{E_{\mathbf{e}_0-\mathbf{e}_1}}))\exp(\frac{s-t}{2}(E_{\mathbf{e}_0+\mathbf{e}_1} + \overline{E_{\mathbf{e}_0+\mathbf{e}_1}}))\\
&\quad \cdot \exp(A(0,0,\phi_2,\dots,\phi_n))\exp(k\pi i H'_{\mathbf{e}_0+\mathbf{e}_1}).
\end{aligned}$$

**Hypothesis 3.6.16** The holomorphic function $\operatorname{tr} \varpi_-^{2,2n}(\cdot)$ on $\Gamma_{\mathrm{SO}(2,2n)}(C^o_{\mathrm{Min}})$ extends continuously to $\mathrm{SO}(2,2n)'$.

**Consequence 3.6.17** *Assume that hypothesis 3.6.16 is true. Let $\theta_-$ be the function associated to the distribution character of $\varpi_-^{2,2n}$ then*

$$\theta_-(f_0(e^{i\phi_0},\ldots,e^{i\phi_n})) = \frac{i\sin(\phi_0)}{2^{n-1}\prod_{j=1}^n \cos(\phi_0)-\cos(\phi_j)} \tag{3.35}$$

$$\begin{aligned}\theta_-(f_1(t,e^{i\phi_1},\ldots,e^{i\phi_n})) =& \frac{-i\sinh(|t|+i\phi_1)}{2^n\sinh(|t|)\sin(\phi_1)} \\ &\cdot \frac{1}{\prod_{j=2}^n \cosh(|t|+i\phi_1)-\cos(\phi_j)}\end{aligned} \tag{3.36}$$

$$\begin{aligned}\theta_-(f_2(\pm,s,t,e^{i\phi_2},\ldots,e^{i\phi_n})) =& \frac{(-1)^{n-1}\sinh(\frac{1}{2}(|s+t|+|s-t|))}{2^n\sinh(\frac{1}{2}|s+t|)\sinh(\frac{1}{2}|s-t|)} \\ &\cdot \frac{1}{\prod_{j=2}^n \cos(\phi_j)\mp\cosh(\frac{1}{2}(|s+t|+|s-t|))}.\end{aligned} \tag{3.37}$$

**Proof.** Consider the semigroup homomorphisms $e^{\mathbf{e}_j} : \mathrm{H}\operatorname{Exp}(ic_m^o) \to \mathbb{C}$ given by

$$e^{\mathbf{e}_j}(h\operatorname{Exp}(iX)) = \xi_{\mathbf{e}_j}(h)e^{i\mathbf{e}_j(X)}.$$

Then $e^{\mathbf{e}_j}$ is a holomorphic function and thus by holomorphic continuation and lemma 3.6.13 we get

$$\operatorname{tr}\varpi_-^{2,2n}|_{\Gamma_{\mathrm{H}}(c_m^o)} = \frac{e^{\mathbf{e}_0}-e^{-\mathbf{e}_0}}{\prod_{j=1}^n (e^{\mathbf{e}_0}+e^{-\mathbf{e}_0}-e^{\mathbf{e}_j}-e^{-\mathbf{e}_j})}.$$

Then by applying hypothesis 3.6.16, lemma 2.8.2 and remark 3.6.15 we get eq. (3.35). Noting that

$$\begin{aligned} e^{\mathbf{e}_0}(\exp(|t|H'_{\mathbf{e}_0-\mathbf{e}_1})) &= e^{|t|} \\ e^{\mathbf{e}_1}(\exp(|t|H'_{\mathbf{e}_0-\mathbf{e}_1})) &= e^{-|t|} \\ e^{\mathbf{e}_0}(\exp(A(\phi_1,\phi_1,\phi_2,\ldots,\phi_n))) &= e^{i\phi_1} \\ e^{\mathbf{e}_1}(\exp(A(\phi_1,\phi_1,\phi_2,\ldots,\phi_n))) &= e^{i\phi_1} \end{aligned}$$

we get eq. (3.36) by applying proposition 2.8.9. For eq. (3.37) note

$$\begin{aligned} e^{\mathbf{e}_0}(\exp(|b|H'_{\mathbf{e}_0+\mathbf{e}_1})) &= e^{|b|} \\ e^{\mathbf{e}_1}(\exp(|b|H'_{\mathbf{e}_0+\mathbf{e}_1})) &= e^{|b|} \\ e^{\mathbf{e}_0}(\exp(\pi i H'_{\mathbf{e}_0+\mathbf{e}_1})) &= -1 \\ e^{\mathbf{e}_1}(\exp(\pi i H'_{\mathbf{e}_0+\mathbf{e}_1})) &= -1 \end{aligned}$$

. □

**Hypothesis 3.6.18** The holomorphic function $\operatorname{tr}\varpi_+^{2,2n}(\cdot)$ on $\Gamma_{\mathrm{SO}(2,2n)}(-C^o_{\mathrm{Min}})$ extends continuously to $\mathrm{SO}(2,2n)'$.

**Consequence 3.6.19** *Assume that hypothesis 3.6.18 is true. Let $\theta_+$ be the function associated to the distribution character of $\varpi_+^{2,2n}$ then*

$$\begin{aligned}
\theta_+(f_0(e^{i\phi_0},\dots,e^{i\phi_n})) =& \frac{-i\sin(\phi_0)}{2^{n-1}\prod_{j=1}^n \cos(\phi_0)-\cos(\phi_j)}\\
\theta_+(f_1(t,e^{i\phi_1},\dots,e^{i\phi_n})) =& \frac{i\sinh(|t|-i\phi_1)}{2^n\sinh(|t|)\sin(\phi_1)}\\
&\cdot\frac{1}{\prod_{j=2}^n \cosh(|t|-i\phi_1)-\cos(\phi_j)}\\
\theta_+(f_2(\pm,s,t,e^{i\phi_2},\dots,e^{i\phi_n})) =& \frac{(-1)^{n-1}\sinh(\frac{1}{2}(|s+t|+|s-t|))}{2^n\sinh(\frac{1}{2}|s+t|)\sinh(\frac{1}{2}|s-t|)}\\
&\cdot\frac{1}{\prod_{j=2}^n \cos(\phi_j)\mp\cosh(\frac{1}{2}(|s+t|+|s-t|))}.
\end{aligned}$$

**Proof.** This follows almost exactly as consequence 3.6.17 except that the associated Ol'shanskiĭ semigroup is $\Gamma_{\mathrm{SO}_e(2,2n)}(-C^o_{\mathrm{Min}})$ and hence we need to make a sign change as in corollary 2.8.10. $\square$

**Remark 3.6.20** Note that $\theta_+=-\theta_-$ on the fundamental Cartan subgroup, hence the representations defined by Binegar-Zierau and Kobayashi-Ørsted are not elliptic representations for $n\geq 3$. But the minimal representations $\varpi_-^{2,2n}$ and $\varpi_+^{2,2n}$ are.

## 3.7 $\mathfrak{so}(2,2n+1)$

### 3.7.1 Structure and realization

We use notation as in section 3.6.1. However in this case we let $A$ denote the map

$$A:\mathbb{C}^{n+1}\to \mathrm{Mat}_{(2n+3)\times(2n+3)}(\mathbb{C})$$
$$A(\theta_0,\dots,\theta_n)=\begin{pmatrix} 0 & -\theta_0 & & & & \\ \theta_0 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -\theta_n & \\ & & & \theta_n & 0 & \\ & & & & & 0\end{pmatrix}.$$

**Proposition 3.7.1** *Let* K *denote* $\mathrm{SO}(2)\times\mathrm{SO}(2n+1)$ *embedded as block-diagonal matrices in* $\mathrm{SO}_e(2, 2n+1)$ *then* K *is a maximal compact subgroup. Let*

$$\mathfrak{h} = \{\big(A(\theta_0, \ldots, \theta_n)\big) \mid \theta_0, \ldots, \theta_n \in \mathbb{R}\}$$

$$\mathrm{H} = \{\begin{pmatrix} \cos(\theta_0) & -\sin(\theta_0) & & & & \\ \sin(\theta_0) & \cos(\theta_0) & & & & \\ & & \ddots & & & \\ & & & \cos(\theta_n) & -\sin(\theta_n) & \\ & & & \sin(\theta_n) & \cos(\theta_n) & \\ & & & & & 1 \end{pmatrix} \mid \theta_0, \ldots, \theta_n \in \mathbb{R}\}.$$

*Then* H *is a Cartan subgroup of* $\mathrm{SO}_e(2, 2n+1)$ *and a maximal torus in* $\mathrm{SO}(2)\times\mathrm{SO}(2n+1)$. *Furthermore* $\mathrm{SO}(2n+3, \mathbb{C})$ *is a complexification of* $\mathrm{SO}_e(2, 2n+1)$.

*Let* $\mathbf{e}_j \in \mathfrak{h}_{\mathbb{C}}^*$ *be given by* $\mathbf{e}_j(A(\theta)) = i\theta_j$. *Then*

$$\begin{aligned}
\Delta &= \{\pm\mathbf{e}_j \pm \mathbf{e}_k \mid 0 \le j \ne k \le n\} \cup \{\pm\mathbf{e}_j \mid 0 \le j \le n\}\\
\Delta^+ &= \{\mathbf{e}_j \pm \mathbf{e}_k \mid 0 \le j < k \le n\} \cup \{\mathbf{e}_j \mid 0 \le j \le n\}\\
\Delta_k &= \{\pm\mathbf{e}_j \pm \mathbf{e}_k \mid 1 \le j \ne k \le n\} \cup \{\pm\mathbf{e}_j \mid 1 \le j \le n\}\\
\Delta_k^+ &= \{\mathbf{e}_j \pm \mathbf{e}_k \mid 1 \le j < k \le n\} \cup \{\mathbf{e}_j \mid 1 \le j \le n\}\\
\Delta_n^+ &= \{\mathbf{e}_0 \pm \mathbf{e}_k \mid 1 \le k \le n\} \cup \{\mathbf{e}_0\}\\
W_{\mathrm{K}} &\cong \mathrm{S}_n \ltimes \{\pm\}^n\\
\delta_k &= \sum_{j=1}^{n}(n - j + \frac{1}{2})\mathbf{e}_k.
\end{aligned}$$

**Remark 3.7.2** $W_{\mathrm{K}}$ acts on $\mathfrak{h}_{\mathbb{C}}^*$ by permuting the $\mathbf{e}_j$ for $j \ge 1$ and by sign changes on the same basis vectors. If $\sigma \in \{\pm\}^n$ then written as a matrix that acts on $\mathfrak{h}_{\mathbb{C}}^*$

$$\sigma = \begin{pmatrix} 1 & & & \\ & \epsilon_1 & & \\ & & \ddots & \\ & & & \epsilon_n \end{pmatrix}$$

with $\epsilon_j \in \{\pm 1\}$. Thus $\epsilon(\sigma) = \det(\sigma) = \prod_{j=1}^n \epsilon_j = \pm 1$ depending on whether it changes an even or an odd number of signs.

**Proof.** [Kna02, page 150, 155, 164, 513]. □

**Lemma 3.7.3** *Let* $H, H' \in \mathfrak{h}_{\mathbb{C}} \subset \mathfrak{so}(2, 2n+1)_{\mathbb{C}}$ *then*

$$B(H, H') = (4n+2)\sum_{j=0}^{n} \mathbf{e}_j(H)\mathbf{e}_j(H')$$

*and let* $H_{\mathbf{e}_k} = A(\phi)$ *then*

$$\phi_j = \frac{-i}{4n+2}\delta_{j,k}.$$

**Proof.** We calculate

$$\begin{aligned}
B(H,H') = \operatorname{tr}\operatorname{ad}(H)\operatorname{ad}(H') &= \sum_{\alpha\in\Delta}\alpha(H)\alpha(H') \\
&= \sum_{0\le j<k\le n}(\pm\mathbf{e}_j\pm\mathbf{e}_k)(H)(\pm\mathbf{e}_j\pm\mathbf{e}_k)(H') + \sum_{j=0}^{n}(\pm\mathbf{e}_j)(H)(\pm\mathbf{e}_j)(H') \\
&= \sum_{0\le j<k\le n} 4\mathbf{e}_j(H)\mathbf{e}_j(H') + 4\mathbf{e}_k(H)\mathbf{e}_k(H') + 2\sum_{j=0}^{n}\mathbf{e}_j(H)\mathbf{e}_j(H') \\
&= 4\sum_{j=0}^{n}(n-j)\mathbf{e}_j(H)\mathbf{e}_j(H') + 4\sum_{j=0}^{n} j\mathbf{e}_j(H)\mathbf{e}_j(H') + 2\sum_{j=0}^{n}\mathbf{e}_j(H)\mathbf{e}_j(H') \\
&= (4n+2)\sum_{j=0}^{n}\mathbf{e}_j(H)\mathbf{e}_j(H').
\end{aligned}$$

Let $A(\phi) = H_{\mathbf{e}_k} \in \mathfrak{h}_{\mathbb{C}}$ denote the element such that $\mathbf{e}_k = B(H_{\mathbf{e}_k},\cdot)$. Then

$$\mathbf{e}_k(H) = B(H_{\mathbf{e}_k},H) = (4n+2)\sum_{j=1}^{n}\mathbf{e}_j(H_{\mathbf{e}_k})\mathbf{e}_j(H)$$

but $\mathbf{e}_1,\ldots,\mathbf{e}_n$ are linearly independent hence $i\phi_j = \mathbf{e}_j(H_{\mathbf{e}_k}) = \frac{1}{4n+2}\delta_{j,k}$. □

**Lemma 3.7.4** *Let* $C_{\mathrm{Min}}$ *denote the minimal cone associated to the positive system in proposition 3.7.1, and let as usual* $c_m = C_{\mathrm{Min}} \cap \mathfrak{h}$ *then*

$$\begin{aligned}
c_m &= A(\{\phi\in\mathbb{R}^{n+1} \mid \phi_0 \le -\sum_{j=1}^{n}|\phi_j|\}) \\
c_m^o &= A(\{\phi\in\mathbb{R}^{n+1} \mid \phi_0 < -\sum_{j=1}^{n}|\phi_j|\}).
\end{aligned}$$

**Proof.** More details about the cone can be found in [Pan81, section II.12].

Consider the notation as in the proof of lemma 3.6.5. We can identify the Cartan algebra from proposition 3.6.2 with the Cartan algebra from proposition 3.7.1 by embedding it as the upper left $(2n+2)\times(2n+2)$ block of a $(2n+3)\times(2n+3)$ matrix and zeros elsewhere. Let $H_\alpha \in \mathfrak{h}_{\mathbb{C}}$ denote the element such that $\alpha = B(H_\alpha,\cdot)_{\mathfrak{so}(2,2n+1)}$ and $\tilde{H}_\alpha \in \mathfrak{h}_{\mathbb{C}}$

such that $\alpha = B(\tilde{H}_\alpha, \cdot)_{\mathfrak{so}(2,2n)}$. Then it follows from lemma 3.6.4 and lemma 3.7.3 that $\tilde{H}_\alpha \in \mathbb{R}_{>0} H_\alpha$. Hence we conclude from lemma 3.6.5 that

$$A(\{\phi \in \mathbb{R}^{n+1} \mid \phi_0 \leq -\sum_{j=1}^{n} |\phi_j|\}) = (-i) \operatorname{span}_{\mathbb{R}_\geq}(H_{\mathbf{e}_0 \pm \mathbf{e}_1}, \ldots, H_{\mathbf{e}_0 \pm \mathbf{e}_n}).$$

But we also see that

$$H_{\mathbf{e}_0} = \frac{1}{2}(H_{\mathbf{e}_0+\mathbf{e}_1} + H_{\mathbf{e}_0-\mathbf{e}_1}) \in A(\{\phi \in \mathbb{R}^{n+1} \mid \phi_0 \leq -\sum_{j=1}^{n} |\phi_j|\}).$$

Thus

$$A(\{\phi \in \mathbb{R}^{n+1} \mid \phi_0 \leq -\sum_{j=1}^{n} |\phi_j|\}) = (-i) \operatorname{span}_{\mathbb{R}_{\geq 0}}(H_{\mathbf{e}_0}, H_{\mathbf{e}_0 \pm \mathbf{e}_1}, \ldots, H_{\mathbf{e}_0 \pm \mathbf{e}_n}) = c_m$$

where the last equality follows from proposition 2.3.2. □

### 3.7.2 Combinatorics

**Lemma 3.7.5** *Let $n \geq 1$ and $\alpha, \beta \in \mathbb{C}$ then*

$$\begin{aligned} &\sum_{w \in W_n} \epsilon(w) e^{w.(\frac{1}{2}\sum_{j=1}^{n}(2(n-j)+1)\mathbf{e}_j)} \prod_{j=1}^{n} (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)}) \\ &= \sum_{k=0}^{n} \alpha^k \beta^k (\sum_{j=0}^{2(n-k)} (-1)^j \alpha^{2(n-k)-j} \beta^j) \sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^{k} \mathbf{e}_j + \sum_{j=1}^{n}(2(n-j)+1)\mathbf{e}_j)} \end{aligned} \tag{3.38}$$

**Proof.** We prove this by induction, the basis case $n = 1$ is just expanding the product

$$\begin{aligned} &\sum_{w \in W_n} \epsilon(w) e^{w.\frac{1}{2}\mathbf{e}_1} (\alpha^2 + \beta^2 + \alpha\beta e^{w.\mathbf{e}_1} + \alpha\beta e^{w.(-\mathbf{e}_1)}) \\ &\qquad = \sum_{w \in W_n} \epsilon(w)((\alpha^2 - \alpha\beta + \beta^2) e^{w.\frac{1}{2}\mathbf{e}_1} + \alpha\beta e^{w.\frac{3}{2}\mathbf{e}_1}). \end{aligned}$$

Where we applied a single sign change which has signature $-1$.

Let $\rho = \frac{1}{2}\sum_{j=1}^{n}(2(n-j)+1)\mathbf{e}_j$ then for $n \geq 2$ we expand

$$\begin{aligned}
&\sum_{w\in W_n} \epsilon(w)e^{w.\rho}\prod_{j=1}^{n}(\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
=&(\alpha^2+\beta^2)\sum_{w\in W_n} \epsilon(w)e^{w.\rho}\prod_{j=2}^{n}(\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
&+\alpha\beta\sum_{w\in W_n} \epsilon(w)e^{w.(\mathbf{e}_1+\rho)}\prod_{j=2}^{n}(\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
&+\alpha\beta\sum_{w\in W_n} \epsilon(w)e^{w.(-\mathbf{e}_1+\rho)}\prod_{j=2}^{n}(\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)})
\end{aligned}$$

and consider each of the terms on the right hand side by itself, first applying lemma 3.2.1 and the induction hypothesis to the first term we get

$$\begin{aligned}
&(\alpha^2+\beta^2)\sum_{w\in W_n} \epsilon(w)e^{w.\rho}\prod_{j=2}^{n}(\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
&=\sum_{k=0}^{n-1}(\alpha^2+\beta^2)\alpha^k\beta^k(\sum_{j=0}^{2(n-1-k)}(-1)^j\alpha^{2(n-1-k)-j}\beta^j)\sum_{w\in W_n}\epsilon(w)e^{w.(\sum_{j=1}^{k}\mathbf{e}_{j+1}+\rho)}. \quad (3.39)
\end{aligned}$$

In eq. (3.39) the terms with $k \geq 1$ are zero by lemma 3.2.3 since the expressions in the exponentials are invariant under (1 2). Thus we get

$$\begin{aligned}
&(\alpha^2+\beta^2)\sum_{w\in W_n} \epsilon(w)e^{w.\rho}\prod_{j=2}^{n}(\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
&\qquad\qquad = (\alpha^2+\beta^2)(\sum_{j=0}^{2(n-1)}(-1)^j\alpha^{2(n-1)-j}\beta^j)\sum_{w\in W_n}e^{w.\rho}. \quad (3.40)
\end{aligned}$$

This also proves corollary 3.7.6.

Then applying the induction hypothesis to the next term we get

$$\begin{aligned}
&\alpha\beta\sum_{w\in W_n} \epsilon(w)e^{w.(\mathbf{e}_1+\rho)}\prod_{j=2}^{n}(\alpha+\beta e^{w.\mathbf{e}_j})(\alpha+\beta e^{w.(-\mathbf{e}_j)}) \\
&\qquad\qquad =\sum_{k=1}^{n}\alpha^k\beta^k(\sum_{j=0}^{n-k}(-1)^j\alpha^{2(n-k)-j}\beta^j)\sum_{w\in W_n}\epsilon(w)e^{w.(\sum_{j=1}^{k}\mathbf{e}_j+\rho)}.
\end{aligned}$$

Which are most of the terms on the right hand side of eq. (3.38). Applying the induction hypothesis to the last term we get

$$\alpha\beta \sum_{w\in W_n} \epsilon(w) e^{w.(-\mathbf{e}_1+\rho)} \prod_{j=2}^{n} (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)})$$

$$= \sum_{k=1}^{n} \alpha^k \beta^k ( \sum_{j=0}^{2(n-k)} (-1)^j \alpha^{2(n-k)} \beta^j ) \sum_{w\in W_n} \epsilon(w) e^{w.(-\mathbf{e}_1 + \sum_{j=2}^{k} \mathbf{e}_j + \rho)}. \quad (3.41)$$

Let us first consider $n \geq 3$ then for $k \geq 3$ the exponential on the right hand side of eq. (3.41)is invariant under (1 3) hence zero. For $k = 1$ it is invariant under (1 2). Thus for $n \geq 3$ we get

$$\alpha\beta \sum_{w\in W_n} \epsilon(w) e^{w.(-\mathbf{e}_1+\rho)} \prod_{j=2}^{n} (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)})$$

$$= -\alpha^2\beta^2 ( \sum_{j=0}^{2(n-2)} (-1)^j \alpha^{2(n-2)-j} \beta^j ) \sum_{w\in W_n} \epsilon(w) e^{w.(\rho)}$$

then combining this with eq. (3.40) and the following equation we are finished for $n \geq 3$

$$(\alpha^2 + \beta^2) \sum_{j=0}^{2(n-1)} (-1)^j \alpha^{2(n-1)-j} \beta^j - \alpha^2\beta^2 \sum_{j=0}^{2(n-2)} (-1)^j \alpha^{2(n-2)-j} \beta^j = \sum_{j=0}^{2n} (-1)^j \alpha^{2n-j} \beta^j.$$

For $n = 2$ we get

$$\alpha\beta \sum_{w\in W_2} \epsilon(w) e^{w.(\frac{1}{2},\frac{1}{2})} (\alpha^2 + \beta^2 + \alpha\beta e^{w.\mathbf{e}_2} + e^{w.(-\mathbf{e}_2)})$$

$$= \alpha^2\beta^2 \sum_{w\in W_2} \epsilon(w) e^{w.(\frac{1}{2},\frac{3}{2})} + \alpha^2\beta^2 \sum_{w\in W_2} \epsilon(w) e^{w.(\frac{1}{2},-\frac{1}{2})}$$

$$= -\alpha^2\beta^2 \sum_{w\in W_2} \epsilon(w) e^{w.(\frac{3}{2},\frac{1}{2})}.$$

Where we get the last equality since the second term is invariant under $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. But then we simply see that

$$(\alpha^2 + \beta^2)(\alpha^2 - \alpha\beta + \beta^2) - (\alpha\beta)^2 = \sum_{j=0}^{4} (-1)^j \alpha^{4-j} \beta^j$$

and when we gather terms with eq. (3.40) we are finished. $\square$

**Corollary 3.7.6** *For $n \geq 1$ and $\alpha, \beta \in \mathbb{C}$*

$$\sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^n \frac{1}{2}(2n+1-2j)\mathbf{e}_j)} \prod_{j=2}^n (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)})$$
$$= (\sum_{j=0}^{2(n-1)} (-1)^j \alpha^{2(n-1)-j} \beta^j) \sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^n \frac{1}{2}(2n+1-2j)\mathbf{e}_j)}. \quad (3.42)$$

**Proof.** This is proved during the proof of lemma 3.7.5 □

**Corollary 3.7.7** *For $n \geq 2$ and $\alpha, \beta \in \mathbb{C}$*

$$\sum_{w \in W_n} \epsilon(w) e^{w.(-\mathbf{e}_1 + \sum_{j=1}^n \frac{1}{2}(2n+1-2j)\mathbf{e}_j)} \prod_{j=2}^n (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)})$$
$$= -\alpha\beta(\sum_{j=0}^{2(n-2)} (-1)^j \alpha^{2(n-2)-j} \beta^j) \sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^n \frac{1}{2}(2n+1-2j)\mathbf{e}_j)}. \quad (3.43)$$

**Proof.** This is proved during the proof of lemma 3.7.5. □

**Corollary 3.7.8** *Let $n \geq 2$ and $\alpha, \beta \in \mathbb{C}$ then*

$$\sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^n \frac{1}{2}(2n+1-2j)\mathbf{e}_j)} (\alpha + \beta e^{w.(-\mathbf{e}_1)}) \prod_{j=2}^n (\alpha + \beta e^{w.\mathbf{e}_j})(\alpha + \beta e^{w.(-\mathbf{e}_j)})$$
$$= \alpha^{2n-2}(\alpha - \beta) \sum_{w \in W_n} \epsilon(w) e^{w.(\sum_{j=1}^n \frac{1}{2}(2n+1-2j)\mathbf{e}_j)}.$$

**Proof.** Add the right-hand sides of eqs. (3.42) and (3.43). Then the following calculation gives the desired conclusion

$$\sum_{j=0}^{2(n-1)} (-1)^j \alpha^{2(n-1)-j+1} \beta^j - \sum_{j=0}^{2(n-2)} (-1)^j \alpha^{2(n-2)-j+1} \beta^{j+2}$$
$$= \sum_{j=0}^{2(n-1)} (-1)^j \alpha^{2n-1-j} \beta^j - \sum_{j=1}^{2(n-1)} (-1)^j \alpha^{2n-1-j} \beta^j = \alpha^{2n-1} - \alpha^{2n-2}\beta. \quad \square$$

**Corollary 3.7.9** *Let $n \geq 2$ and $X \in -c_m^o$ then*

$$\mathrm{tr}(\pi_{\mathrm{Min}}(\mathrm{Exp}(iX))) = \frac{e^{i(n-\frac{1}{2})\mathbf{e}_0(X)} - e^{i(n+\frac{1}{2})\mathbf{e}_0(X)}}{\prod_{j=1}^n (1 - e^{i(\mathbf{e}_0+\mathbf{e}_j)(X)})(1 - e^{i(\mathbf{e}_0-\mathbf{e}_j)(X)})}$$

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