Hitchin connections for genus 0 quantum representations

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Preface

This dissertation have been written as part of my PhD studies at the Centre for Quantum Geometry of Moduli Spaces (QGM) and it includes in particular work previously described in my progress report and in a joint paper with Søren Fuglede Jørgensen [26].

First of all, I would like to thank my supervisor Jørgen Ellegaard Andersen for suggesting the topic of the thesis, for his encouragement and for helping me and answering my questions when I have been stuck.

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Resume

Det overordnede tema i denne afhandling går tilbage til Jones, der i 1985 [40] [42] opdagede nogle bestemte repræsentationer af fletningsgrupperne. Disse repræsentationer tillod Jones at definere den første *kvanteinvariant*, nemlig *Jonespolynomiet* $V_L(q) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ af en lænke L. Denne invariant udmærkede sig ved at være meget effektiv til at skelne forskellige lænker og knuder, men også ved dens definition, der udelukkende benytter en projektion til planet samt en række algebraiske operationer. Dette fik Atiyah til i 1987 at efterlyse en rent 3-dimensional definition af Jonespolynomiet. Svaret kom prompte i 1988, da Witten indførte – som et af de første eksempler på en *topologisk kvante-felt teori* – kvante-Chern-Simons teori og på et fysisk grundlag argumenterede for, at Jonespolynomiet kunne bestemmes som forventningsværdierne af Wilson-løkke operatorene. Witten betragtede et principelt G-bundt $P \to M$ over en 3-mangfoldighed M, for en Liegruppe G. Felterne i teorien er givet ved konnektionerne $\mathcal{A}_P$ i P, og stiintegralet er givet i termer af Chern-Simons virkningen $\mathrm{CS} : \mathcal{A}_P/\mathcal{G}_P \to \mathbb{R}/\mathbb{Z}$:

$$\int_{\mathcal{A}/\mathcal{G}} \mathrm{e}^{2\pi \mathrm{i} k\, \mathrm{CS}([A])} \mathcal{D}[A].$$

Hvor $k \in \mathbb{N}$ er *niveauet*, som man kan tænke på som $\hbar^{-1}$. Teoriens obsevable $\mathcal{O}(L, R) : \mathcal{A} \to \mathbb{C}$ er konstrueret ud fra orienterede lænker $L \subset M$, hvor hver komponent L_i er blevet tildelt en endelig dimensional irreducibel repræsentation R_i af G, hvor $\mathcal{O}(L, R)$ er givet ved følgende udtryk:

$$\mathcal{O}(L, R)([A]) = \prod_i \mathrm{Tr}(R_i \circ \mathrm{hol}_{L_i}(A)).$$

Forventingsværdien er så givet ved

$$Z(M, \mathcal{O}(L, R))_k = \int_{\mathcal{A}/\mathcal{G}} \mathcal{O}(L, R) \mathrm{e}^{2\pi \mathrm{i} k\, \mathrm{CS}([A])} \mathcal{D}[A],$$

og det var denne værdi – for $M = S^3$, $\mathrm{G} = \mathrm{SU}(2)$ og R_i standardrepræsentationen – som Witten argumenterede for opfyldte at

$$Z(S^3, \mathcal{O}(L, R))_k = V_L(\mathrm{e}^{\frac{2\pi \mathrm{i}}{k+2}}).$$

I Wittens konstruktion spiller *modulirummet af flade konnektioner* på både 2 og 3 mangfoldigheder en vigtig rolle. Specielt *geometrisk kvantisering* af disse modulirum for flader er fundamentale for hans teori. Geometrisk kvantisering kræver et valg af Kählerstruktur, og modulirummene kommer ikke med en naturlig sådan. Istedet argumentere Witten for at der må eksistere en kanonisk måde at identificere de geometriske kvantiseringer svarende til forskellige Kählerstrukterer, nemlig ved eksistensen af en (projektiv) flad konnektion i bundtet af geometriske kvantiseringer over rummet af Kählerstrukterer. En sådan konnektion – nu kaldes de Hitchin konnektioner – blev konstrueret for modulirummet af flade konnektioner på

en lukket flade af Hitchin i [38] og uafhængigt af Witten, Della-Pietra og Axelrod i [15], og denne konstruktion blev senere genereliserret af Andersen [2] til det generelle tilfælde.

Wittens teori blev defineret matematisk af Reshetikhin og Turaev i [56], men på en fundamental anderledes grundlag. Udover invarianter af lænker giver en topologisk kvantefeltteori også repræsentationer af afbildningsklassegrupper. Andersen og Ueno [8] [10][11][9] viste disse repræsentationer er ækvivalente til repræsentationer konstrueret fra en bestemt konform feltteori. Laszlo [48] viste at repræsentationerne for en lukket flade i denne konforme feltteori igen er ækvivalente til repræsentationer defineret ud fra Hitchin-konnektionen. Hovedresultatet i denne afhandling er en udvidelse af denne ækvivalens til at omfatter flader af genus 0 med mindst 5 markerede punkter. Disse repræsentationer er tæt forbundne med de repræsentationer af fletningsgrupperne som Jones fandt. Derudover indeholder afhandlingen også resultater opnået sammen med Søren Fuglede Jørgensen der generalisere [7] og forbinder Jonesrepræsentationerne med bestemte repræsentationer defineret i termer af homologien af bestemte flader, og bekræfter en formodning fremsagt i [7] for en stor familie af pseudo-Anosov afbildningsklassegruppe elementer. Vi henviser til den engelsksprogede introduktion for flere detaljer samt en oversigt over indholdet i de enkelte kapitler.

Introduction

The themes of this dissertation all owes to the discovery of Jones [40][42] of certain representations of the braid groups, which gave rise to the first *quantum invariant*, namely the Jones polynomial $V_L(q) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ of a link $L \subset S^3$. The invariant was very successful in distinguishing different knots, but it was not clear *why*. Even though its definition is completely elementary in terms of knot diagrams, it remained a mystery why such an invariant should exists and why it was so effective: what was the relation of this invariant with the 3D-topology of knots? This led Atiyah to ask in 1987 for an intrinsically three-dimensional definition of the Jones polynomial; that is, a definition not relying on a knot projection or braid representation. A physical solution was supplied in 1988 by Witten in the celebrated article [65], where he for each compact simple Lie group G describes a quantum field theory of three-manifolds. He consider a principal G-bundle $P \to M$ over a 3-manifold M. The fields of the theory are the connections $\mathcal{A}_P$ in P, acted on by the gauge group $\mathcal{G}_P$ of P. The Lagrangian is given by the *Chern-Simons* functional CS. The partition function is then defined as the path integral

$$\int_{\mathcal{A}/\mathcal{G}} \mathrm{e}^{2\pi \mathrm{i} k\, \mathrm{CS}([A])} \mathcal{D}[A].$$

where $k \in \mathbb{N}$ is the *level* and play the role of $\hbar^{-1}$ in the quantization. The observables of the theory are constructed from oriented links $L \subseteq M$, with each component L_i coloured with a representation R_i of G, as the following function:

$$\mathcal{O}(L,R)([A]) = \prod_i \mathrm{Tr}(R_i \circ \mathrm{hol}_{L_i}(A)).$$

Witten was able to argue that the expectation value of $\mathcal{O}(L,R)$,

$$Z(M, \mathcal{O}(L,R))_k = \int_{\mathcal{A}/\mathcal{G}} \mathcal{O}(L,R)\mathrm{e}^{2\pi \mathrm{i} k\, \mathrm{CS}([A])} \mathcal{D}[A],$$

when $M = S^3$, $\mathrm{G} = \mathrm{SU}(2)$ and R_i the defining representation of SU(2), calculate the Jones polynomial in the following way:

$$Z(S^3, \mathcal{O}(L,R))_k = V_L(\mathrm{e}^{\frac{2\pi \mathrm{i}}{k+2}}),$$

which answered Atiyah's question. Witten's construction is however not satisfying from a mathematical point of view, as the path integral do not have a mathematical rigorous definition and Witten argues only by formal properties such a path integral ought to satisfy, which led to the definition of a *topological quantum field theory*. He argues that the path integral – in an instance of stationary phase approximation – must localize on critical point set of CS, which is exactly the set of *flat* connections $\mathcal{A}_0 \subset \mathcal{A}$. Therefore the moduli spaces of flat connections on three-manifolds and on surfaces (the boundary of three-manifolds) plays an important part in

Witten's theory. Especially the geometric quantization of the moduli space of flat connections on a surface was fundamental to his approach. However such a quantization requires a choice of complex structure on the moduli space, and Witten argued that his construction should be independent of this choice and proposed that the existence of a flat connection on the space of choices, identifying the different quantizations.

This point of view sheds some light on the question why the Jones polynomial, and other quantum invariants, are so strong. The moduli space of flat connections on a manifold with values in a G bundle are in correspondence with the representations of the fundamental group of the manifold into G, modulo conjugations. It is therefore possible to use the quantum invariants to probe the fundamental group, which is a difficult but very strong invariant.

Soon after Reshetikhin and Turaev constructed a theory using categories of representations of quantum groups. Their approach are very different from Witten's. Blanchet, Habegger, Masbaum and Vogel later gave a construction using skein theory in [22]. Andersen and Ueno ([8], [10], [11] and [9]) proved that this skein theoretic description is – as a modular functor – equivalent to a modular functor from a certain conformal field theory. Laszlo have shown that representations of a closed surface $\mathrm{MCG}(\Sigma)$ without marked points from this conformal field theory is equivalent to representations obtained from the geometric quantization of the moduli space of flat $\mathrm{SU}(N)$ connections on a closed surface Σ. One of our main theorems an extension of Laszlo's result from closed surfaces to spheres with marked points.

Let us review the setup of the geometric quantization approach to the TQFT for $\mathrm{G} = \mathrm{SU}(2)$. Let k be a natural number and let Σ be a closed surface with a finite number of marked points $\{p_i\}$, each coloured by a k-admissible irreducible $\mathrm{SU}(2)$-representation. When identifying the $\mathrm{SU}(2)$ representations with $\mathbb{N}_0$ in the standard way, the k-admissible representations corresponds to the numbers between 0 and k. We define the *moduli space of flat connections* $\mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k)$ to be the space of flat connections on $\Sigma \setminus \{p_i\}$ such that the holonomy around p_i are in the conjugacy class corresponding to λ_i, modulo the action of the gauge group $\mathcal{G}$. For $\mathrm{SU}(2)$, this is the conjugacy class with trace $2\cos(\frac{\pi\lambda_i}{k})$. This moduli space have a natural symplectic form ω, and given a complex structure σ on Σ, there exists a complex structure I_σ on $\mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k)$, compatible with ω in the sense that $(\mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k), \omega, I_\sigma)$ forms a Kähler triple. Geometric quantization takes a Kähler manifold equipped with a *prequantum line bundle*, a complex line bundle $\mathcal{L}$ with connection ∇ such that $\mathrm{F}_\nabla = -\mathrm{i}\omega$, and produces a vector space $\mathcal{Q}^{(k)}_\sigma$. The Teichmüller space $\mathcal{T}(\Sigma)$ of Σ parametrizes such complex structures on $\mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k)$, and letting σ vary in $\mathcal{T}(\Sigma)$ give rise to a vector bundle

$$\mathcal{Q}^{(k)} \to \mathcal{T}(\Sigma)$$

of quantum spaces with respect to different Kähler structures on $\mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k)$. When Σ have no marked points, it was shown by Axelrod, Della Pietra and Witten [15] and independently by Hitchin [38] that the vector bundle $\mathcal{Q}^{(k)}$ supports a projectively flat connection, called the Hitchin connection. The mapping class group $\mathrm{MCG}(\Sigma)$ acts on $\mathcal{T}(\Sigma)$ and this action lifts to $\mathcal{Q}^{(k)}(\Sigma)$. The Hitchin connection is natural in the sense that it is invariant under the action of $\mathrm{MCG}(\Sigma)$. We therefore have a projective representation of $\mathrm{MCG}(\Sigma)$ on the space of projectively covariant constant sections of $\mathcal{Q}^{(k)}(\Sigma)$. In [8], [10], [11], [9] Andersen and Ueno proved that the representations from the Reshetikhin-Turaev TQFT are equivalent to representations from a certain conformal field theory. Laszlo proved in [48] that the representations from this TQFT, for a closed surface, are equivalent to the one constructed from the projectively flat sections of the Hitchin connection. In [2] Andersen proved that a family of compatible complex structures on a symplectic manifold satisfying conditions always carries a projectively flat connection, for which he gives an explicit construction, and calls the *Hitchin* connection. Let Σ be S^2 with n marked points, and define a map $\pi : \mathcal{T}(\Sigma) \to \mathrm{Conf}_{n-1}(\mathbb{C})$ as follows. A point in $\mathcal{T}(\Sigma)$ is an equivalence class of diffeomorphisms from Σ to $\mathbb{CP}^1$, such that two diffeomorphisms f, g are considered equivalent if $f^{-1} \circ g\Sigma \to \Sigma$ is isotopic to the identity through diffeomorphisms preserving the marked points. Due to the triple-transitive action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{CP}^1$, any such equivalence class have a representative where the last tree marked points are mapped to $0, 1, \infty$. The map π maps an equivalence class to the image of the first $n-1$ marked point of this special representative. The main result of this thesis is:

Theorem 1. *Let $\Sigma = S^2$ with $n > 4$ marked points, and let $\lambda_i = \lambda_j$ for $1 \leq i, j \leq n$. The Hitchin connection for the family $\mathcal{T}(\Sigma)$ of Kähler structures on $\mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k)$ are projectively equivalent to the pullback of the Knizhnik-Zamolodchikov connection along the map $\pi : \mathcal{T}(\Sigma) \to \mathrm{Conf}_{n-1}(\mathbb{C})$.*

This theorem is joint work with Jørgen Ellegaard Andersen and will soon appear in a joint paper. It have the following consequence:

The Knizhnik-Zamolodchikov (KZ) connection are related to the conformal field theory mentioned above, and is defined in a trivial bundle over $\mathrm{Conf}_n(\mathbb{C})$ with fiber $\hom_{\mathfrak{g}}(\bigotimes_{i=1}^n V_{\lambda_i}, \mathbb{C})$, and is defined as

$$\nabla^{KZ} = \mathrm{d} + \frac{1}{k+\mathrm{h}} \sum_{1 \leq i < j \leq n} \Omega^{ij} \frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j},$$

where Ω^{ij} are certain endomorphisms of the fiber. We will prove theorem 1 in the following steps:

1. Construct a trivial bundle over $\mathcal{T}(\Sigma)$ with fibers the holomorphic sections of a line bundle L_{G} over a space X_{G}, in such a way that Ω^{ij} can be identified with certain second-order differential operators in L_{G}. We will call the KZ connection, when transfered to this bundle, the geometrized KZ connection.

2. Show that the family $\mathcal{T}(\Sigma) \times X_{\mathrm{G}}$ are isomorphic to the family $\mathcal{T}(\Sigma) \times \mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k)$ as families of complex manifolds.

3. This isomorphisms lifts to an isomorphism between the vector bundles $\mathcal{Q}^{(k)}(\Sigma)$ and $\mathrm{H}^0(X_{\mathrm{G}}, L_{\mathrm{G}}) \times \mathcal{T}(\Sigma)$.

4. Using this identification, we can now consider the difference between the Hitchin and the geometrized KZ connection at each point $\sigma \in \mathcal{T}(\Sigma)$, which a priori is a second order differential operator.

5. By explicit calculations, we can show that the degree 2 symbol of this differential operator is 0, so it is in fact at most a first order operator.

6. We can then show that the first order symbol must be 0 as well, leaving us with a 0-order operator. A final consideration then shows that this operator must be constant, which means that the two connections are projectively equivalent.

Remark 1. For simplicity we have stated the theorem only in the case where all labels are equal. The same proof works when we allow different labels, but we need to replace the geometric quantization of the moduli space with the *metaplectic* corrected geometric quantization, and instead of the Hitchin connection from [3] we must use the Hitchin connection for the metaplectic setting, as constructed in [6].

An interesting feature of this approach is that the construction of the geometrized KZ connection is explicit and can be completely computed, which is in contrast of the Hitchin connection which have an explicit construction, but is very difficult to compute explicit in coordinates. For $n = 4$ the space $X_{\mathrm{G}} = \mathbb{CP}^1 \setminus \{0,1,\infty\}$, and $\mathcal{T}(\Sigma)$ is the universal cover of $\mathbb{CP}^1 \setminus \{0,1,\infty\}$, $\mathrm{U}\left(\mathbb{CP}^1 \setminus \{0,1,\infty\}\right)$, and the connection is given as follows:

$$\begin{aligned}\nabla^{\mathrm{KZ}}_{\frac{\partial}{\partial p}} = \nabla^{\mathrm{t}}_{\frac{\partial}{\partial p}} + \frac{1}{8(k+2)}\Bigg(&\frac{\tau(\tau-1)(\tau-p)}{p(p-1)}\nabla_{\frac{\partial}{\partial \tau}}\nabla_{\frac{\partial}{\partial \tau}} + \\ &\left(\frac{4\mathrm{i}}{p}\tau\frac{1-\tau+|\tau-1|}{1+|\tau|+|\tau-1|} + \frac{8}{p-1}\frac{(\tau-1)(|\tau|+\tau)}{1+|\tau|+|\tau-1|}\right)\nabla_{\frac{\partial}{\partial \tau}} \\ &+ \frac{1}{p}\frac{1-3|\tau|+|\tau-1|}{1+|\tau|+|\tau-1|} + \frac{1}{p-1}\frac{1+|\tau|-3|\tau-1|}{1+|\tau|+|\tau-1|}\Bigg)\end{aligned}$$

Due to some irregularities owing to the low dimension (and indeed this is the reason that theorem 1 requires $n > 4$) the fiber of the line bundle is, for $n = 4$, not $\mathrm{H}^0(X_{\mathrm{G}}, L_{\mathrm{G}})$. The reason is that we actually are interested in the sections of a line bundle on a compactification of X_{G}, and for $n > 4$ the compactifying set have codimension ≥ 2, and therefore the holomorphic sections are in bijections with the holomorphic sections of $L_{\mathrm{G}} \to X_{\mathrm{G}}$ by Hartogs theorem. This is however not the case for $n = 4$ where $X_{\mathrm{G}} = \mathbb{CP}^1 \setminus \{0,1,\infty\}$ and the compactifying set $\{0,1,\infty\}$ have codimension 1. The relevant space for $n = 4$ is the subspace of sections in $\mathrm{H}^0(\mathbb{CP}^1 \setminus \{0,1,\infty\}, \mathcal{O})$ that extends to sections of $\mathcal{O}([1])$ over $\mathbb{CP}^1$, that is, the span of the functions 1 and $\frac{1}{\tau}$. In this trivialization we have

$$\nabla = \mathrm{d} + \partial F$$

where

$$F = \log\frac{|\tau||\tau-1|}{(1+|\tau|+|\tau-1|)^3}.$$

We have tried to extend theorem 1 to $n = 4$ by making some of the arguments more explicit, but we are lacking proofs of two claims to make the argument work. We present them here as conjectures. As we will later see, the space X_{G} have a natural Kähler structure.

Conjecture 2. *The families $\mathcal{T}(\Sigma) \times X_{\mathrm{G}}$ and $\mathcal{T}(\Sigma) \times \mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k)$ are isomorphic as families of Kähler manifolds.*

Conjecture 3. *The subbundle $\mathcal{T}(\Sigma) \times \operatorname{span}_{\mathbb{C}}(1, \frac{1}{\tau}) \subseteq \mathcal{T}(\Sigma) \times \mathrm{H}^0(X_{\mathrm{G}}, L_{\mathrm{G}})$ are mapped to a subbundle of $\mathcal{Q}_k(\Sigma)$ that are preserved under the Hitchin connection.*

We further prove that

Theorem 2. *Let $n \in \mathbb{N}$ be odd and greater than 4, Σ a sphere with n punctures and $\rho_k^{p,q}$ the quantum representation of $\mathrm{MCG}(\Sigma)$ where the n points are coloured with pk, at level qk, where $p, q \in \mathbb{N}$ and $\frac{p}{q} < 1$.. Then*

$$\bigoplus_{k=1}^{\infty} \rho_k^{p,q}$$

is a faithful representation of $\mathrm{MCG}(\Sigma)$.

The thesis also contains – in chapter 3 – the results of the paper [26] joint with Søren Fuglede Jørgensen. The results also concerns the Jones representations, but this time viewed from a diagrammatic point of view. We generalize results from the article [7] which deals with the quantum representations for a sphere with four punctures. They find that, when $q \to -1$, the quantum representation approaches the representation of the mapping class group of the 4-punctured sphere on the first homology of $S^1 \times S^1$ through the lift along the branched double cover $S^1 \times S^1 \to S^2$ with 4 branch points. We give a similar result about the Jones representation for a sphere with $n > 4$ points, in terms of the homology of a double cover with n branch points. Whereas [7] prove the result by applying a clever change of basis, we instead find a natural way to interpret the homology the homology of a double cover.

Let Σ_g be a closed surface of genus g and , and let $\omega = \sum_{i=1}^{g} \alpha_i \wedge \beta_i \in \Lambda^2 H_1(\Sigma_g, \mathbb{C})$, where α_i, β_i is a symplectic basis for the intersection pairing. We prove the following

Theorem 3. *The Jones representation $\pi_q^{2n,0}$ specialized to $q = -1$ is equivalent to the action on $\Lambda^g H_1(\Sigma_g, \mathbb{C})/(\omega \wedge \Lambda^{g-2} H_1(\Sigma_g, \mathbb{C}))$, where $g = n - 1$.*

In [7] Andersen, Masbaum and Ueno conjecture that the quantum representation of a pseudo-Anosov has infinite order for all but finitely many levels k, and verify this conjecture the four-holed sphere. In [58], Santharoubane prove the conjecture for the one-holed torus. We use theorem 3 to verify the conjecture for a class of pseudo-Anosovs that for $n = 4$ contains all pseudo-Anosov mapping classes.

The dissertation starts with a short introduction to braids, knots, and mapping class groups in chapter 1, where we will also discuss some quantum invariants of links and braids. In chapter 3 we present the results obtained with S.F. Jørgensen on the Jones representation at $q = -1$. We then give a very short introduction to topological quantum field theory in 2, mostly serving to introduce notation and the quantum representations, and to collect some facts used in the later chapters

In chapter 4 we introduce the concept of *geometric quantization*, which plays an important role in this thesis. First of all, the Hitchin connection – the topic of a later chapter – is a connection constructed to remedy that geometric quantization depends on a non-physical choice, namely a complex structure. Geometric quantization is also involved in a different way in the construction of the *geometrized KZ construction*.

We introduce the moduli spaces that we will work with in chapter 5. The moduli spaces are constructed in terms of data associated to a surface Σ, and are interesting to us because $\mathrm{MCG}(\Sigma)$ acts on them in a highly nontrivial way. This action is then in turns used to construct a representation of $\mathrm{MCG}(\Sigma)$ by – in some sense – approximation the moduli spaces by a vector space. This is done using geometric quantization.

In chapter 6 we introduce the Hitchin connection, crucial for constructing the representations of $\mathrm{MCG}(\Sigma)$ mentioned above, as the choices involved in geometrically quantizing the moduli spaces are not invariant under $\mathrm{MCG}(\Sigma)$. It is therefore necessary to find a $\mathrm{MCG}(\Sigma)$-equivariant way to identify the quantizations of different choices, which is exactly what the Hitchin connection achieves.

In chapter 7 we construct the geometrized KZ connection In chapter 8 we prove that the bundles that the Hitchin and KZ connection lives in can be identified, and in chapter 9 we prove that under this identification, the geometrized KZ connection and the Hitchin connection are equivalent up to a projective factor, and that the representations of the mapping class group that they induce are projectively equivalent.

Chapter 1

Knots, braids and the mapping class group

1.1 The braid group

Let $\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid i \neq j \implies z_i \neq z_j\}$ be the ***configuration space*** of n points in $\mathbb{C}$ and $Y_n = \mathrm{Conf}_n(\mathbb{C})/\mathrm{S}_n$ the quotient by the permutation group acting by permuting the entries: $(z_1, z_2, \ldots, z_n) \cdot \sigma = (z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n)})$. These are manifolds of n complex dimensions, but it is not hard to visualize them: a point in $\mathrm{Conf}_n(\mathbb{C})$ is just n distinct particles in $\mathbb{C}$ and a point in Y_n is represented by n identical particles in $\mathbb{C}$. This makes it easy to imagine a curve inside one of these spaces: we can identify $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ and use the last coordinate as "time", and a curve is just the spacetime path of n particles – and a loop is just a path where each particle end up where a similar particle started. The fundamental group of Y_n is called the ***braid group*** on n strings and denoted by B_n. Artin showed that it has the following presentation:

$$\mathrm{B}_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \rangle / \langle \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1 \rangle$$

where the generator σ_i corresponds to the braid with the i'th string moving over the $i + 1$'th string, as shown in figure 1.1.

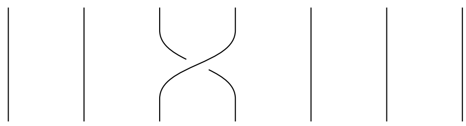


Figure 1.1: The braid $\sigma_3 \in \mathrm{B}_7$.

We notice that the relations are almost the same as for the group S_n, which have a presentation in terms of simple transpositions given as:

$$\mathrm{S}_n = \langle s_1, s_2, \ldots, s_{n-1} \rangle / \langle s_i^2 = 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \text{ for } |i - j| > 1 \rangle.$$

This means that there is a homomorphism $\mathrm{B}_n \to \mathrm{S}_n$ given by mapping $\sigma_i \mapsto s_i$ – the result of applying this map to a braid is to follow the trajectory of each particle and record the

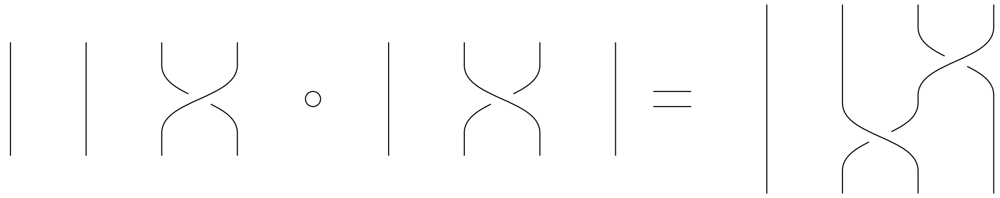


Figure 1.2: The composition in the braid group: the product of σ_3 and σ_2^{-1} .

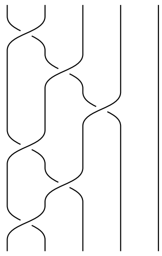


Figure 1.3: A halftwist in B_5 on the first four strings.

end position. The kernel of this map is called the ***pure braid group***, and is isomorphic to $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$.

We collect here the following facts (see i.e. [19]):

Theorem 4. *1. The generators σ_i are all conjugate; σ_{i-1} is conjugate to σ_1 by a half twist on the first i strings.*

2. B_n is torsion free, which give us the very useful property that we can rescale a representation by rescaling the matrices associated to the generators.

3. The center of B_n, $n > 1$, is generated by $(\sigma_1 \cdot \dots \cdot \sigma_{n-1})^n$.

1.2 Mapping class groups

In this section we introduce a central notion in low dimensional topology, namely the group of symmetries of surface, up to isotopy. We consider the group Homeo^+ of orientation-preserving homeomorphisms of a surface Σ and equip it with the compact-open topology. It was shown by Hamstrom in [37] that if Σ have negative Euler characteristic (i.e. not a sphere with 0, 1 or 2 punctures or a torus) then the components of Homeo^+ are contractible and as such does not contain interesting topological information. We will make the following

Definition 4 (Mapping class group). Let Σ be a compact surface with a finite set $\mathrm{D} \subseteq \Sigma$ of marked points. Let $\mathrm{Homeo}^+_D(\Sigma)$ be the group of orientation-preserving homeomorphisms φ of Σ such that $\varphi(D) = D$ and $\varphi_{\restriction \partial\Sigma} = \mathrm{id}_{\partial\Sigma}$. Let $\mathrm{Homeo}^+_{D,0}(\Sigma) \subseteq \mathrm{Homeo}^+_D(\Sigma)$ be the normal subgroup of homeomorphisms isotopic to id_Σ through homeomorphisms in $\mathrm{Homeo}^+_D(\Sigma)$, and define the ***mapping class group*** of Σ as:

$$\mathrm{MCG}(\Sigma) = \mathrm{Homeo}^+_D(\Sigma)/\mathrm{Homeo}^+_{D,0}(\Sigma)$$

Theorem 5 (Alexander). *The mapping class group of a disc is trivial*

An invertible 2×2 matrix with integer coefficients defines a homeomorphism of $\mathbb{R}^2$ preserving the standard lattice $\mathbb{Z}^2$, and therefore descends to the quotients $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. If the inverse matrix also have integer coefficients then the descended map is a homeomorphism. This construction give rise to the isomorphism in the following

Theorem 6. *The mapping class group of $S^1 \times S^1$ are isomorphic to* $\mathrm{SL}(2, \mathbb{Z})$.

The braid group B_n can be identified with the mapping class group of a disc with n punctures $p_1, \dots, p_n$, in the following way: a diffeomorphism φ of the punctured disc can be completed to a diffeomorphism $\bar{\varphi}$ of the disc. By theorem 5, there exists an isotopy $h_t : D^2 \to D^2$ of $\bar{\varphi}$ to the identity, and this isotopy gives a loop in $\mathrm{Conf}_n(D^2)/S_n \cong Y_n$ by $t \mapsto [h_t(p_1), h_t(p_2), \dots, h_t(p_n)]$.

Given a simple, closed curve c on a surface Σ, we can define a homeomorphism of Σ by "cutting Σ along c, twisting a full turn, and glue the two boundaries together". To formalise this we will first consider the annulus $A_{1,3} = \{re^{i\theta} \mid 1 \leq r \leq 3\} \subseteq \mathbb{C}$ and define $\tau(r\mathrm{e}^{\mathrm{i}\theta}) = r\mathrm{e}^{\mathrm{i}(\theta + \pi r - \pi)}$.

Definition 5 (*Dehn twist*). Let c be a simple closed curve on a oriented surface Σ. Then there exists tubular neighbourhood N_c of c such that $N_c \cong A_{1,3}$ by an orientation-preserving homeomorphism φ. We can define a homeomorphism τ_c of Σ by letting τ_c be the identity away from N_c and on N_c equal to $\varphi^{-1} \circ \tau \circ \varphi$. The isotopy class of τ_c depends only on c, and therefore gives a well-defined element in the mapping class group $\tau_c \in \mathrm{MCG}(\Sigma)$, called the *Dehn twist* around c.

Theorem 7. *The mapping class group of an annulus is freely generated by the Dehn twist around the core*

Proposition 6. *Let a, b be simple closed curves on a surface Σ.*

1. *If $a \cap b = 0$ then $\tau_a \tau_b = \tau_b \tau_a$.*
2. *If $a \cap b = \{pt\}$ then $\tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b$.*

A sphere with n punctures can be thought of as a disc with n punctures, glued together with a disc, and this defines a map $p : \mathrm{MCG}(D^2 \setminus \{x_1, \dots, x_n\}) \to \mathrm{MCG}(S^2, \setminus\{x_1, \dots, x_n\})$. This map is surjective and it is shown in [19] that for $n \geq 2$ the kernel is normally generated by $\sigma_1 \sigma_2 \dots \sigma_{n-1}^2 \sigma_{n-2} \dots \sigma_1$ and $(\sigma_1 \sigma_2 \dots \sigma_{n-1})^n$.

Consider surfaces S_g^i of genus g with $i = 0, 1$ boundary components. There is an element $\iota \in \mathrm{MCG}(S_g^i)$ of order two, with $2g + 2 - i$ fixed points. The quotient of S_g^i minus the fixed points of ι with the subgroup $\langle \iota \rangle$ is a sphere with $2g + 2$ punctures if $i = 0$ and a disc with $2g + 1$ puncture if $i = 1$.

Theorem 8 (Birman-Hilden [20]). *Let $g \geq 2$ and $\mathrm{SMCG}(S_g^i)$ be the centralizer of ι in $\mathrm{MCG}(S_g^i)$. Then*

$$\mathrm{SMCG}(S_g^1) \cong \mathrm{MCG}(D^2, x_1, \dots, x_{2n+1}) \cong \mathrm{B}_{2n+1}$$

and

$$\mathrm{SMCG}(S_g)/\langle \iota \rangle \cong \mathrm{MCG}(S^2, x_1, \dots, x_{2g+2})$$

Definition 7. An element in $\mathrm{MCG}(\Sigma)$ is said to be:

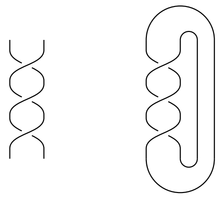


Figure 1.4: The braid $\sigma_1^3 \in \mathrm{B}_2$ and it is closure, the trefoil knot
.

Periodic if it has finite order,

Reducible if it is not of finite order and have a power fixing a simple, essential curve on Σ (a curve with no self-intersections, that does not bound a puncture and is nontrivial in homology),

Pseudo-Anosov if it have a representative φ that preserves two transverse (singular) foliations $\mathcal{F}_1, \mathcal{F}_2$ of Σ equipped with transverse measures μ_1, μ_2 and there exists a real number $\lambda > 1$, called the ***stretch factor***, such that $\varphi(\mathcal{F}_1, \mu_1) = (\mathcal{F}_1, \lambda\mu_1)$ and $\varphi(\mathcal{F}_2, \mu_2) = (\mathcal{F}_2, \lambda^{-1}\mu_2)$. The foliations can be singular in that they can have finitely many leaves of dimension 0, and each of them being in the closure of k dimension-1 leaves; such a singular point is said to have k ***prongs***. The singular points are required to have at least 3 prongs. Marked points on Σ are however allowed to be singular points with just 1 prong.

We can now state the following important theorem (see for instance [27] theorem 13.1):

Theorem 9 (Nielsen–Thurston classification). *A mapping class $f \in \mathrm{MCG}(\Sigma)$ belongs to exactly one of the three types: periodic, reducible or pseudo-Anosov.*

Example 8. In case of $\Sigma = S^1 \times S^1$, let $\varphi \in \mathrm{MCG}(\Sigma) \cong \mathrm{SL}(2, \mathbb{Z})$.

1. if $|\mathrm{Tr}\,\varphi| > 2$ then φ is pseudo-Anosov. The stretch factor is the largest eigenvalue, and the foliations all parallel to the eigenvectors.

2. if $|\mathrm{Tr}\,\varphi| = 2$ and $\varphi \neq \pm\mathrm{id}$ then φ is reducible.

3. if $|\mathrm{Tr}\,\varphi| < 2$ or $\varphi = \pm\mathrm{id}$ then φ is finite order.

1.3 Braids and links

Given a braid $b \in \mathrm{B}_n$ it is possible to close it up to obtain a link $\hat{b}$ by connecting the bottom n points with the top n points by n parallel strands, as shown in Figure 1.4. The resulting link is called the closure of b, and the following two theorems describe the relationship between links and closures of braids.

Theorem 10 (Alexander). *Given a link L there exists a braid $b \in \mathrm{B}_n$ such that $L = \hat{b}$.*

Theorem 11 (Markov). *Two braids have the same closure if and only if they can be connected by a finite sequence of the following two moves, known as Markov moves (note that the second move changes the braid group):*

1. $b \leftrightarrow aba^{-1}$, $a, b \in \mathrm{B}_n$

2. $b \leftrightarrow b'\sigma_n^{\pm 1}$ *for* $b \in \mathrm{B}_n$ *and* b' *denoting the image of* b *in* B_{n+1}.

Therefore the study of functions on $\cup_{n\in\mathbb{N}}\mathrm{B}_n$ that are invariant under the Markov moves is the same as the study of link invariants.

1.3.1 Traces

Definition 9. A trace $\mathrm{Tr} : A \to k$ on a k-algebra A is a linear functional such that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$.

Observe that a trace on $\mathbb{C}[\mathrm{B}_n]$ is invariant under the first Markov move: $\mathrm{Tr}(aba^{-1}) = \mathrm{Tr}(a(ba^{-1})) = \mathrm{Tr}(ba^{-1}a) = \mathrm{Tr}(b)$. Since all σ_i's are conjugate to σ_1, we see that $\mathrm{Tr}(\sigma_i)$ is independent of i. Denote by B_∞ the direct limit of the braid groups, where the inclusion of $\mathrm{B}_n \hookrightarrow \mathrm{B}_{n+m}$ is $\sigma_i \mapsto \sigma_i$ (so it just adds m unbraided strings to the right), and by $\mathrm{B}_{n,m}$ the subgroup generated by $\{\sigma_i \mid n \leq i \leq m-1\}$ (the subset of B_m where the first $n-1$ strings are unbraided). Traces satisfying certain conditions give rise to link invariants.

Definition 10. A trace Tr on $\mathbb{C}[\mathrm{B}_\infty]$ is called a Markov trace if

$$\mathrm{Tr}(\sigma_1 b) = \mathrm{Tr}(\sigma_1)\,\mathrm{Tr}(b)$$

for all $b \in \mathrm{B}_{2,\infty}$

Lemma 11. *A Markov trace satisfies* $\mathrm{Tr}(\sigma_n b) = \mathrm{Tr}(\sigma_n)\,\mathrm{Tr}(b)$ *for* $b \in \mathrm{B}_n$.

Proof. Conjugation by a halftwist Δ_{n+1} of the first $n+1$ strands exchanges σ_i with σ_{n+1-i}, for $i \leq n$. Since we already know that Tr is invariant under the first Markov move, we see that if $a \in \mathrm{B}_n$, then $\mathrm{Tr}(\sigma_n a) = \mathrm{Tr}(\Delta_{n+1}\sigma_n a\Delta_{n+1}^{-1}) = \mathrm{Tr}(\sigma_1(\Delta_{n+1}a\Delta_{n+1}^{-1})) = \mathrm{Tr}(\sigma_1)\,\mathrm{Tr}(a) = \mathrm{Tr}(\Delta_{n+1}\sigma_1\Delta_{n+1}^{-1})\,\mathrm{Tr}(a) = \mathrm{Tr}(\sigma_n)\,\mathrm{Tr}(a)$. □

We say that a Markov trace is ***normalized*** if $\mathrm{Tr}(\sigma_i) = \mathrm{Tr}(\sigma_i^{-1})$ for all i. Given a normalized Markov trace, we can define a link invariant in the following way:

Definition 12. Let Tr be a normalized Markov trace. If $b \in \mathrm{B}_n$, define

$$V_{\mathrm{Tr}}(\hat{b}) = \mathrm{Tr}(\sigma_1)^{1-n}\,\mathrm{Tr}(b).$$

This is a well-defined link invariant because it is invariant under both Markov moves.

Given a Markov trace, we can find an α such that $\alpha^2 = \frac{\mathrm{Tr}(\sigma_i^{-1})}{\mathrm{Tr}(\sigma_i)}$ and define $\widetilde{\mathrm{Tr}}(b) = \alpha^{e(b)}\,\mathrm{Tr}(b)$, where $e(b)$ is the exponent sum of b in the letters σ_i (it is clear from the relations of B_n that this is well-defined). Its easy to check that $\widetilde{\mathrm{Tr}}$ is a normalized Markov trace, and the corresponding link invariant can be written in terms of Tr as:

$$V_{\tilde{\mathrm{Tr}}}(\hat{b}) = (\alpha\,\mathrm{Tr}(\sigma_1))^{1-n}\alpha^{e(b)}\,\mathrm{Tr}(b)$$

We have the following proposition:

Proposition 13. *There is a bijeciton between normalized, multiplicative (i.e. that* $\mathrm{Tr}(ab) = \mathrm{Tr}(a)\,\mathrm{Tr}(b)$ *if* $a \in \mathrm{B}_{1,n}$ *and* $b \in \mathrm{B}_{n+1,\infty}$*) Markov traces with values in the field* k *and link invariants* L *taking values in* k*, such that*

1. *L(unknot) = 1*
2. *There is a $z \in k^*$ such that*

$$L(K_1 \cup K_2) = zL(K_1)L(K_2)$$

for two links K_1, K_2 contained in disjoint balls.

Proof. It is easy to check that a multiplicative, normalized Markov trace gives a link invariant satisfying the two properties, with $z = \mathrm{Tr}(\sigma_1)^{-1}$. On the other hand, we can define a trace from such a link invariant by mapping $b \in \mathrm{B}_n$ to $z^{-n+1}L(\hat{b})$. □

1.4 The Hecke algebras of type A_n

We would like to study representations π of the braid group such that $\pi(\sigma_i)$ is diagonalizable with at most two eigenvalues. Therefore the minimal polynomial of $\pi(\sigma_i)$ is of degree two, i.e. there exist some scalars a and b so that for all i:

$$\pi(\sigma_i)^2 + a\pi(\sigma_i) + b = 0$$

By scaling each $\pi(\sigma_i)$ we can assume that one of the possible eigenvalues is -1. If we call the other possible eigenvalue q, the equation can be written as

$$(g_i + 1)(g_i - q) = g_i^2 + (1 - q)g_i - q = 0,$$

where we have written g_i for $\pi(\sigma_i)$. So representations of the braid group where the generators have at most -1 and q as eigenvalues are the representations of the group algebra $\mathbb{C}\mathrm{B}_n$ that factor through the Hecke algebra $H(n, q)$, which is the algebra over k with the presentation

$$\begin{aligned} H(n, q) = \langle g_1, \ldots, g_{n-1} \mid\ & g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \\ & g_i g_j = g_j g_i \text{ for } |i - j| \geq 2, \\ & g_i^2 = (q - 1)g_i + q\rangle. \end{aligned}$$

where $q \in k$. We will only consider the cases $k = \mathbb{C}$ and $k = \mathbb{C}[q, q^{-1}]$.

Remark 14. We use this generating set only in this section, to agree with the original work of Jones. From the next section we will use the generating set $q^{-\frac{1}{2}}g_i$, which corresponds to the more symmetric choice with $q^{\frac{1}{2}}, -q^{-\frac{1}{2}}$ as eigenvalues.

There is a representation of B_n on $H(q, n)$ given by $\sigma_i \mapsto g_i$, which acts by multiplication. The following theorem gives a family of traces on $H(q, \infty)$:

Theorem 12 (Ocneanu, [30]). *For any $z \in \mathbb{C}$ there is a trace Tr on $H(\infty, q)$, uniquely defined by*

1. $\mathrm{Tr}(1) = 1$
2. $\mathrm{Tr}(xg_n) = z\,\mathrm{Tr}(x)$ *for $x \in H(n, q)$.*

Proof. The idea is to define the trace inductively as

1. $\mathrm{Tr}(1) = 1$
2. $\mathrm{Tr}(xg_ny) = z\,\mathrm{Tr}(xy)$ for $x, y \in H(n, q)$.

Then the two properties are certainly satisfied, but we still need to show that Tr is a trace, i.e. that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$. This is done by induction over n. □

We remark that this is a family of Markov traces, so we obtain knot invariants for any z by normalizing with an α such that $\mathrm{Tr}(\alpha g_i) = \mathrm{Tr}((\alpha g_i)^{-1})$. We can calculate α in the following way: first we find the inverse of g_i:

$$g_i\left(\frac{g_i+1-q}{q}\right) = \frac{g_i^2+(1-q)g_i}{q} = \frac{q+(q-1)g_i+(1-q)g_i}{q} = \frac{q}{q} = 1,$$

so to normalize the trace, we need α to satisfy $\mathrm{Tr}(\alpha g_i) = \mathrm{Tr}(\alpha g_i)^{-1}$ which means that

$$\alpha^2 = \frac{\mathrm{Tr}(g_i^{-1})}{\mathrm{Tr}(g_i)}$$

but

$$\mathrm{Tr}(g_i^{-1}) = \mathrm{Tr}\left(\frac{g_i+1-q}{q}\right) = \frac{1}{q}\left(\mathrm{Tr}(g_i)+\mathrm{Tr}(1-q)\right) = \frac{z+1-q}{q},$$

so we have that

$$\alpha^2 = \frac{z+1-q}{zq}.$$

Thus we get the following link invariant as in Definition 12, where π_α is the representation of B_n in $H(n,q)$ mapping $\sigma_i \mapsto \alpha g_i$, and $b \in \mathrm{B}(n)$ is such that $L = \hat{b}$:

$$X_L(z,q) = \mathrm{Tr}(\sigma_1)^{1-n}\,\mathrm{Tr}_z(\pi_\alpha(b)) = (z\alpha)^{1-n}\alpha^e\,\mathrm{Tr}_z(\pi(b)).$$

This is the HOMFLY polynomial, and we can show that it satisfies a skein relation: given a braid diagram for a link L such that $b = b_1\sigma_i^{-1}b_2$, we will show that the polynomials $X_{L_0}, X_{L_+}, X_{L_-}$ satisfy a linear equality, where $L_0 = \widehat{b_1 b_2}$, $L_+ = L$ and $L_- = \widehat{b_1\sigma_i b_2}$. First notice that we can, by a Markov 1 move, assume that $L_0 = \widehat{c\sigma_i}$, $L_- = \widehat{c}$ and $L_+ = \widehat{c\sigma_i^2}$. Let $e = e(c)$ be the exponent sum. Then we have

$$\begin{aligned}
X_{L_+} &= (z\alpha)^{1-n}\alpha^{e+2}\,\mathrm{Tr}_z(\pi(c\sigma_i^2)),\\
X_{L_0} &= (z\alpha)^{1-n}\alpha^{e+1}\,\mathrm{Tr}_z(\pi(c\sigma_i)),\\
X_{L_+} &= (z\alpha)^{1-n}\alpha^{e}\,\mathrm{Tr}_z(\pi(c)).
\end{aligned}$$

From the relation $g_i^2 = (q-1)g_i + q$, we have that

$$\mathrm{Tr}_z(\pi(c\sigma^2)) = \mathrm{Tr}_z(\pi(c)((q-1)\pi(\sigma_i)+q)) = (q-1)\,\mathrm{Tr}_z(\pi(c\sigma_i)) + q\,\mathrm{Tr}_z(\pi(c)).$$

Multiplying this by $q^{-\frac{1}{2}}$ we see that

$$(q^{\frac{1}{2}} - q^{-\frac{1}{2}})X_{L_0} = q^{-\frac{1}{2}}\alpha^{-1}X_{L_+} - q^{\frac{1}{2}}\alpha X_{L_-}.$$

The polynomial $X_L(z,q)$ has the specialization $V_L(q)$ given by $\alpha = q^{\frac{1}{2}}$ which is the Jones polynomial: since $zq\alpha^2 = z+1-q$, we get that $z = \frac{1-q}{(q-1)(q+1)} = -\frac{1}{q+1}$, so $(z\alpha)^{-1} = -\frac{q+1}{\sqrt{q}} = -q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ and the skein relation becomes

$$(q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_{L_0} = q^{-1}V_{L_+} - qV_{L_-}.$$

The reason for this value of the specialization is that Jones found the Jones polynomial in the context of Temperley–Lieb algebras, which are the following quotients of the Hecke algebras

$$\mathrm{TL}_n = H(n,q)/\langle g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1\rangle,$$

and for the trace Tr_z to factor through the Temperley–Lieb algebra, it must be 0 on this ideal. But if we calculate the trace on the generators:

$$\begin{aligned}
&\mathrm{Tr}_z(g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1)\\
&= \mathrm{Tr}_z(g_i g_{i+1} g_i) + \mathrm{Tr}_z(g_i g_{i+1}) + \mathrm{Tr}_z(g_{i+1} g_i) + \mathrm{Tr}_z(g_i) + \mathrm{Tr}_z(g_{i+1}) + \mathrm{Tr}_z(1)\\
&= z\,\mathrm{Tr}_z(g_i^2) + z\,\mathrm{Tr}_z(g_i) + z\,\mathrm{Tr}(g_{i+1}) + 2z + 1\\
&= z\,\mathrm{Tr}_z((q-1)g_i + q) + 2z^2 + 2z + 1\\
&= (q-1)z^2 + zq + 2z^2 + 2z + 1\\
&= (q+1)z^2 + zq + 2z + 1 = (1+q)z^2 + (2+q)z + 1\\
&= ((1+q)z+1)(z+1)
\end{aligned}$$

which has the roots $z = -1$ and $z = -\frac{1}{q+1}$.

Definition 15. The ***Jones representation*** π_n^J is the representation of the braid group B_n into TL_n given by

$$\mathrm{B}_n \to H(q,n) \twoheadrightarrow \mathrm{TL}_n,$$

and the Jones polynomial of a link obtained as the closure of $b \in \mathrm{B}_n$ is

$$V_{\hat{b}} = (z\alpha)^{1-n}\alpha^e\,\mathrm{Tr}_z(\pi(b)) = \frac{-q^{e(b)/2}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}\,\mathrm{Tr}_{\frac{-1}{1+q}}(b)$$

1.4.1 Representations of Hecke algebras

As remarked in the definition of the Hecke algebra, we will from now on replace the third relation with $g_i^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})g_i + 1$, which just corresponds to scaling the generators with $q^{-\frac{1}{2}}$.

Wenzl defines in [64] some families of representations of Hecke algebras. He defines, in analogy with Young's orthogonal representations of the symmetric group, for each Young diagram λ with n boxes an irreducible representation π_λ of $H(q,n)$, where $q \in \mathbb{C}$ is n***-regular*** ($q \neq 0$ and q is not a l'th root of unity for $2 \leq l \leq n$). It is convenient to define another generating set for $H(n,q)$, given by the spectral projections

$$e_i = \frac{q^{\frac{1}{2}} - g_i}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}.$$

The relations become

1. $e_i^2 = e_i$,
2. $e_i e_{i+1} e_i - \frac{1}{[2]} e_i = e_{i+1} e_i e_{i+1} - \frac{1}{[2]} e_{i+1}$,
3. $e_i e_j = e_j e_i$ for $|i-j| > 1$.

He then defines rational functions of q

$$a_d(q) = \frac{1-q^{d+1}}{(1+q)(1-q^d)} = \frac{[d+1]}{[2][d]},$$

where $[d] = \frac{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$ is the d'th ***quantum integer***. A standard Young tableau of shape λ is a filling of the boxes of λ with the numbers $1, 2, 3\dots, n$, increasing down and to the right. If t is a standard Young taublaux and $1 \leq i, j \leq n$, Wenzl defines $d_{t,i,j} = (c_i - r_i) - (c_j - r_j)$ where c_i, r_i is the column/row where i appear in t. If t is a standard Young tableau and $1 \leq i, j \leq n$ we let $a_{t,i,j}(q) = a_{d_{t,i,j}}(q)$. For a Young diagram λ with n boxes, Wenzl defines a representation on the vector space with basis standard Young tableau of shape λ by:

$$\pi_\lambda(e_i)v_t = a_{t,i,i+1}(q)v_t + (a_{t,i,i+1}(q)a_{t,i+1,i}(q))^{\frac{1}{2}}v_{s_it},$$

where v_t is the basis vector corresponding to the standard Young taubleaux t, and s_it is t with the numbers i and $i+1$ permuted. Note that s_it is not necessarily a standard Young tableau, but Wenzl shows that the coeffecient in front of it vanishes if it is not.

Theorem 13 (Wenzl). *For q generic or $q \in \mathbb{C}$ n-regular, π_λ defines an irreducible representation of $H(n, q)$. If λ, μ are Young diagrams with n boxes, $\pi_\lambda = \pi_\mu$ if and only if $\lambda = \mu$, and*

$$\pi_n = \bigoplus_{\lambda \in \Lambda_n} \pi_\lambda$$

is a faithful representation of $H(n, q)$. The representations π_λ coincide with Youngs orthogonal representations for $q = 1$; in particular, the dimension is given by the hook-length formula.

The automorphism F of $H(q, n)$ that maps $e_i \mapsto 1 - e_i$ has the following property: $\pi_\lambda \circ F$ is equivalent to π_{λ^*}, where λ^* is the Young diagram obtained by switching rows and columns.

Furthermore, we have the following Bratelli property:

$$\pi_\lambda|_{H(n-1,q)} \cong \bigoplus_{\lambda' < \lambda} \pi_{\lambda'},$$

where $\lambda' < \lambda$ means that λ' is obtained from λ by removing a box.

Jones showed in [42] that if λ has $n-1$ rows and n boxes (so of the form ) then π_λ is equivalent to the reduced Burau representation. Jones also showed that the representations that factors through the Temperley–Lieb algebra are exactly the ones corresponding to Young diagrams with at most two rows, and that

$$\mathrm{TL}_n \cong \bigoplus_\lambda \pi_\lambda(\mathrm{TL}_n)$$

where the sum is over Young diagrams with n boxes less than two rows – this is a consequence of the fact that the Temperley–Lieb algebra is a finite dimensional C^*-algebra of dimension $\frac{1}{n+1}\binom{2n}{n}$.

We say that a Young diagram is of type (k, l) if it contains at most k rows, and the number of boxes in the first row, minus the number of boxes in the k'th (perhaps 0) is less than or equal to $l - k$. Wenzl also defines representations $\pi_\lambda^{k,l}$ of $H(n, q)$, q a primitive l'th root of unity, on the space of Young tableaux t of shape λ such that λ is a (k, l)-diagram and t with the box with the highest number removed, also has the shape of a (k, l)-diagram. ($\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ is a

$(2,2)$-tableau, but $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}\,' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ is not). $\pi_\lambda^{k,l}$ is defined by the same formula as π_λ. He then shows that

Theorem 14 (Wenzl). *If q is a primitive root of unity of order at least 4 then $\pi_\lambda^{k,l}$ defines an irreducible representation of $H(n,q)$. If λ is a (k_1,l) and a (k_2,l) diagram with $k_1,k_2 \leq l-1$ then $\pi_\lambda^{k_1,l} = \pi_\lambda^{k_2,l}$, so we can define $\pi_\lambda^l = \pi_\lambda^{k,l}$ if λ is a (k,l)-diagram for some $k \leq l-1$. If $\Lambda_n^{l,k}$ denotes the set of (k,l)-diagrams with $k \leq l-1$, the representation*

$$\pi_n^{k,l} = \bigoplus_{\lambda \in \Lambda_n^{k,l}} \pi_\lambda^{k,l}$$

satisfies that $\pi_n^{k,l}(H(n-1,q))$ is isomorphic to $\pi_{n-1}^{k,l}(H(n-1,q))$.

This allow us to take the direct limit, and we obtain a representation $\pi^{k,l}$ of $H(\infty,q)$.

Theorem 15 (Wenzl). *Let $q = e^{2\pi i/l}$. Then there is a Markov trace with $\operatorname{Tr} e_1 = \eta$ factoring over a C^*-representation of $H(\infty,q)$ if and only if $\eta = a_{-k}(q)$ with $k = 1,2,\ldots,l-1$. The GNS construction applied to this trace is the representation $\pi^{k,l}$.*

We observe that if λ is a Young diagram with at most N rows we have that λ is a $N, k+N$ diagram if $\lambda_1 - \lambda_k \leq k+N-N = k$. This is always satisfied if $k \geq n = |\lambda|$. It is therefore also true if a box is removed from λ, so any standard tableau of shape λ is also a standard $(N,k+N)$-tableau. But then we have that $\pi_\lambda^{N,k+N}$ and π_λ are defined on the same spaces by the same formulas, and therefore that $\pi_\lambda^{N,k+N} = \pi_\lambda$.

1.5 Is the Jones representation faithful?

Jones asked this question in [42], and again as one of his ten problem in [13]. Recall that the Jones representation decomposes as

$$\pi_n^J = \bigoplus_{\lambda \in \Lambda_n^2} \pi_\lambda,$$

where Λ_n^2 is the set of Young diagrams with n boxes and at most two rows. The representation $\pi_{\square\square\cdots\square\square}$ corresponding to a Young diagram with only one row is always one-dimensional since there is only one possible standard Young tableau of this shape. The action of e_i is given by $a_{[1,2,\ldots,n],i,i+1} = a_{-1}(q) = 0$, so we have that $\pi_\lambda(g_i) = \pi_\lambda(-[2]e_1 + q^{\frac{1}{2}}) = 0 + q^{\frac{1}{2}}$. So for a general braid b we get that $\pi_\lambda(b) = q^{\frac{e(b)}{2}}$.

For $n = 2$ the exponent sum defines an isomorphism $\mathrm{B}_2 \cong \mathbb{Z}$, so the representation corresponding to a diagram with one row is clearly faithful for q not a root of unity. We also notice that the sum of infinitely many copies of this representation, at different primitive roots of unity, is faithful.

The reduced Burau representation corresponds to the Young diagram with two columns and $n-1$ rows. For $n = 3$ this diagram has only two rows, and is therefore a summand in the Jones representation. For $n = 3$, it is known (shown for instance in [19]) that the Burau representation is faithful.

For $n \geq 4$, it is unknown if the Jones representation is faithful. It is not known if the Burau representation is faithful for $n = 4$, so it might be possible to show that the $n = 4$ Jones representation is faithful by showing that the Burau representation is. The Burau

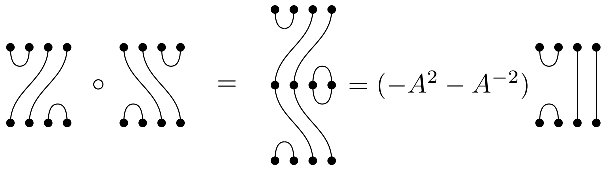


Figure 1.5: The multiplication in $\mathrm{TL}_4(A)$.

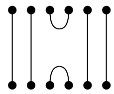


Figure 1.6: The element $e_3 \in \mathrm{TL}_6(A)$.

representation is not faithful for $n > 4$ (this was shown in [52] for $n \geq 9$ in 1991, improved to $n \geq 6$ in [50] and finally in [18] for $n = 5$ in 1999), so another method of proof would be needed.

It is known by [61] that the kernel (if any) of the Burau representation of B_4 is contained in the set of triangular pseudo-Anosov mappings, which consists of maps conjugate to $\Delta_4^{2m} P(\sigma_1^{-1}, \sigma_3\sigma_2)$, where $P(x, y)$ is a word with positive exponents in x and y, and by [49] that – again only for $n = 4$ – that the Burau representation is faithfull if and only if π_q^λ with $\lambda = \boxplus$ is faithfull.

Jones [42] shows that if λ is Young diagram and r is the rank of $\pi_\lambda(e_i)$ and d the dimension of π_λ, then $\pi_\lambda((\sigma_1 \ldots \sigma_{n-1})^n) = q^{\frac{rn(n-1)}{d}}\mathrm{id}$, and $\frac{rn(n-1)}{d}$ is an integer. This implies that π_n^J is never faithful for q a root of unity. However, if we can show that

$$\bigcap_{k=1}^{\infty} \ker \pi^J_{|\restriction q = e^{2\pi i/k+2}} = \{\mathrm{id}\}$$

we have shown that π^J is faithful, for generic q. The Jones representation at these roots of unity have a special geometric interpretation, and indeed the motivation behind this thesis was to develop the geometric picture. Theorem 1 shows that the representation at these roots of unity can be understood in a way that involves the action of B_n on a certain space where all elements in B_n are known to act non-trivially.

1.6 Diagrammatic description of the Jones representations

In chapter 3 we will need a graphical description of the Jones representations of the braid group. It is obtained by giving a graphical description of the Temperley–Lieb algebra, the modules of the Temperley–Lieb algebra, and the map from the braid group to the Temperley–Lieb algebra.

The Temperley–Lieb algebra $\mathrm{TL}_n(A)$ is an algebra over $\mathbb{C}(A)$ and has a basis consisting of noncrossing pairings of $2n$ points, n of them located at the bottom of a square and n of them at the top. The multiplication is given by stacking two squares on top of each other, rescaling the vertical direction to obtain a square, removing all circles, and multiplying by $-[2]_A = -(A^2 + A^{-2})$ for each removed circle, see Figure 1.5. It is generated, as an algebra, by the n elements, $\mathrm{id}, e_1, e_2, \ldots, e_{n-1}$ (Figure 1.6). The map $\mathrm{B}_n \to \mathrm{TL}_n(A)$ is given by

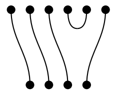


Figure 1.7: A basis element of $V^{6,4}$..

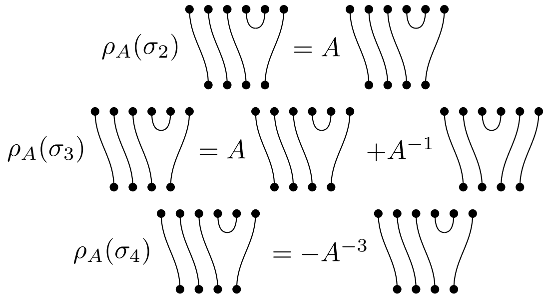


Figure 1.8: The actions of σ_2, σ_3, and σ_4 on the basis element from Figure 1.6..

$\sigma_i \mapsto A\mathrm{id} + A^{-1}e_i$. This map essentially just resolves the braid using the Kauffman skein relations until a non-crossing diagram is obtained. For generic A, the irreducible submodules of this braid group action are in correspondence with two-row Young diagrams with n boxes, which we can parametrize with the *difference* of the lengths of the two rows, $d \in \{0, \dots, n\}$ with $d \equiv n \mod 2$. These modules also admits a natural graphical description:

Definition 16. Let $V^{n,d}$, $0 \leq d \leq n$, denote the complex vector space spanned by non-crossing pairings of $n + d$ points, such that n points are "at the top" and d points "at the bottom", in such a way that all points at the bottom are connected to points at the top; see Figure 1.6. The Temperley–Lieb algebra acts on $V^{n,d}$ in a way similar to the multiplication in TL_n: by stacking diagrams, and removing circles, multiplying by $-[2]_A$ for each removed circle. However, if two bottom points end up connected when doing so, the result is multiplied by 0; see Figure 1.6. We denote by $\eta_A^{n,d}$ the action of $\mathrm{TL}_n(A)$ on $V^{n,d}$.

Proposition 17. *The* $\mathrm{TL}_n(A)$*-module* $V^{n,d}$ *are isomorphic to the representation* $t \otimes \pi_\lambda$ *from section 1.4.1, where* λ *is the* n*-box Two-row Young diagram of shape* $(\frac{n+d}{2} \geq \frac{n-d}{2})$*,* t *is the 1-dimensional representation* $t(b) = A^{-e(b)}$ *and* $A^2 = q^{\frac{1}{2}}$*.*

Proof. We will just descripe the intertwining homomorphism. To a non-crossing diagram we can associate the Young tableau where the second row contains the indices of the top points paired with a top point with a samller index. □

Since $\eta_A^{n,d}$ and the map $\mathrm{B} \to \mathrm{TL}_n(A)$ are defined over $\mathbb{Z}[A, A^{-1}]$, we can specialize them to any complex value of A. We can therefore get any specialize $\pi_q^{n,d}$ to any complex number q by letting $q = A^4$ and taking the tensor product of $\eta_A^{n,d}$ with the 1-dimensional representation $\sigma_i \mapsto A$. We should note that considering the representation in this particular basis is not a new idea: see for instance [57] for a thorough description of the action which moreover puts focus on the case $q = -1$ which is of interest to us.

Chapter 2

TQFT

Let us first make the following

Definition 18. The ***cobordism category*** $\mathbf{Bord}_n$ is the category with:

Objects oriented, closed n-dimensional manifolds M. We consider the empty set a manifold of all dimensions.

Morphisms $M_1 \to M_2$ are equivalence classes of $n+1$ manifolds X with an orientation preserving diffeomorphism $\varphi_X : M_1 \coprod -M_2 \to \partial X$, where $X_1 \sim X_2$ if there exists a diffeomorphism $\psi : X_1 \to X_2$ such that $\psi \circ \varphi_1 = \varphi_2$.

The composition of two such morphisms – called cobordisms – is by gluing the common boundary using the parametrization.

The cobordisms category is useful as it allow one to "chop up" a manifold into smaller pieces and analyze them individually – for instance, consider a closed surface Σ as a cobordism $\Sigma : \emptyset_1 \to \emptyset_1$. We can use a pair of pants decomposition to decompose it as pants (a two-sphere with three boundaries) and annuli. If we want to define invariants of surfaces that can be computed by chopping them up in simpler pieces, compute the invariant for each piece and then fit the computed invariant together, we can conveniently define them as functors from $\mathbf{Bord}_2$. Topological Quantum Field Theories are such an example:

Definition 19. A functor $Z : \mathbf{Bord}_n \to \mathbf{Vect}$ is a TQFT if it satisfies the following axioms:

1. $Z(-M) = Z(M)^*$
2. $Z(M_1 \coprod M_2) = Z(M_1) \otimes Z(M_2)$ for n-manifolds M_1, M_2, and if X_1, X_2 are two cobordisms, then $Z(X_1 \coprod X_2) = Z(X_1) \otimes Z(X_2)$.
3. $Z(\emptyset_n) = \mathbb{C}$ where $\emptyset_n$ is the empty n-manifold.
4. $Z(M \times [0,1]) = \mathrm{id}_{Z(M)}$ where $M \times [0,1]$ is the identity corbordism.

An $n+1$ TQFT provides invariants of $n+1$ manifolds by interpreting a $n+1$-manifold X as a cobordism $X : \emptyset_n \to \emptyset_n$: $Z(X) \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$. They also provide representations of mapping class groups by the following construction:

Definition 20 (Quantum representations from TQFTs). Let Z be a $n+1$ dimensional TQFT. If $\varphi : M \to M$ is a diffeomorphism of a n-manifold M, we can define a cobordism $T_\varphi : M \to M$ by $M \times [0,1]$ by parametrizing the boundaries with $x \mapsto (x, 0)$ and $x \mapsto (\varphi(x), 1)$. The diffeomorphism type of the cobordism depends only on the isotopy type of φ. By the functoriality of Z we have that

$$\varphi \mapsto Z(T_\varphi) \in \mathrm{GL}(Z(M))$$

defines a representation of $\mathrm{MCG}(M)$, which we will call the *quantum representation* of $\mathrm{MCG}(M)$ associated to Z.

The TQFT important to this thesis is the Reshetikhin-Turaev-Witten TQFT, which is of dimension $2+1$. It is actually not a TQFT in the sense we defined above, as it is defined for a cobordism category where the manifolds carry extra structure. The RTW-TQFT depends on a choice of simple Lie group G and a level k. We now define the cobordism category relevant for the RTW-TQFT with $\mathrm{G} = \mathrm{SU}(N)$ and level k, where we consider manifolds with labeled, embedded submanifolds. First we define:

Definition 21. A label set Δ is a finite set equipped with a *trivial* element denoted 0 and an involution $\dagger : \Delta \to \Delta$, such that $0^\dagger = 0$.

The label set for the $\mathrm{SU}(N)$ RTW-TQFT at level k is defined as:

Definition 22. Let $\Lambda_{N,k}$ be the set of Young diagrams with height less than N and length at most k, and the involution $\dagger : \Lambda_{N,k} \to \Lambda_{N,k}$ given by assigning to $\lambda \in \Lambda_{N,k}$ the Young diagram obtained by a 180 degree rotation of the complement of λ in the rectangular diagram with N rows and λ_1 columns. Interpreting λ as an $\mathrm{SU}(N)$ representation, the involution corresponds to the dual representation.

And we can now define the relevant version of the cobordism category:

Definition 23. The cobordism category $\mathbf{Bord^{RTW}}_{N,k}$ is defined to have as:

Objects oriented, closed 2-dimensional manifolds Σ, with a choice of Lagrangian (for the intersection pairing) subspace $L \subseteq H_1(\Sigma, \mathbb{Z})$. Furthermore, Σ can have a finite number of marked points $\mathrm{D} = \{p_1, p_2, \ldots, p_n\}$, each equipped with with a projective tangent vector $v_i \in (T_{p_i}\Sigma \setminus \{0\})/\mathbb{R}_+$, and a label $\lambda_i \in \Lambda_{N,k}$.

Morphisms $\Sigma_1 \to \Sigma_2$ is a pair (X, n) where $n \in \mathbb{Z}$ and X is an oriented, cobordism $X : \Sigma_1 \to \Sigma_2$, which contains finitely many embedded, oriented 1-manifolds L_i, each equipped with a label $\lambda_i \in \Lambda_{N,k}$ and a *framing* of L_i, that is, an embedding $h_i : L_i \times [0,1] \hookrightarrow X$ such that $h_i(s, \frac{1}{2}) = s$. The boundary of L_i, if any, is required to be contained in the marked points of the boundary of M, and each marked point on ∂M must be exactly one of the boundary points of exactly one of the L_i's, and in such a way that the framing of the point match $\frac{\partial}{\partial t} h_i(s,t)_{\restriction t=\frac{1}{2}}$. Furthermore, if L_i is labeled by λ_i it can only meet a point on Σ_1 in a point labeled λ_i, and Σ_2 in points labeled $\lambda_i^\dagger$. Two morphisms $X_i, n_i)$ are composed as $(X_1, n_1) \circ (X-2, n_2) = (X_1 \circ X_2, n_1 + n_2 + \sigma(X_1, X_2))$, where σ is Walls signature cocycle applied to a certain triple of Lagrangian subspaces of the boundary of $X_1 \circ X_2$, see [35, definition 6] for more details.

In particular $\mathbf{Bord^{RTW}}_{N,k}$ contains as morphisms links in S^3 with labeled components.

2.0.1 Construction

Let us review the *universal construction* of [22], however we will only work over $\mathbb{C}$ instead of their more general rings. Let $\langle , \rangle : \hom_{\mathbf{Bord}^{\mathbf{RTW}}}(\emptyset, \emptyset) \to \mathbb{C}$ satisfying:

1. $\langle M_1 \coprod M_2 \rangle = \langle M_1 \rangle \langle M_2 \rangle$.
2. $\langle \emptyset \rangle = 1$.
3. $\langle -M \rangle = \overline{\langle M \rangle}$.

The construction associates to such an invariant a functor $Z : \mathbf{Bord}^{\mathbf{RTW}} \to \mathbf{Vect}$, in the following way. First, for Σ an object in $\mathbf{Bord}^{\mathbf{RTW}}$, an axillary vector space $V(\Sigma)$ is constructed as the vector space freely spanned by $\hom_{\mathbf{Bord}^{\mathbf{RTW}}}(\emptyset, \Sigma)$. We define a bilinear form on $V(\Sigma)$ by

$$\langle M_1, M_2 \rangle_\Sigma = \langle M_1 \coprod_\Sigma -M_2 \rangle.$$

We can now define $Z(\Sigma)$ as $V(\Sigma)/\ker\langle , \rangle$, where $\ker\langle , \rangle = \{x \in V(\Sigma) \mid \langle x, y \rangle = 0 \forall y \in V(\Sigma)\}$. To a cobordism $M : \Sigma_1 \to \Sigma_2$ we associate the map $V(M) : V(\Sigma_1) \to V(\Sigma_2)$ given on the basis as:

$$V(M)(M') = M' \circ M.$$

where $\circ$ denotes the composition on $\mathbf{Bord}^{\mathbf{RTW}}$. If $M' \in \ker\langle , \rangle_{\Sigma_1}$ and $M'' \in V(\Sigma_2)$ then

$$\langle V(M'), M'' \rangle_{\Sigma_2} = \langle M' \circ M \coprod_{\Sigma_2} (-M'') \rangle = \langle M' \coprod_{\Sigma_1} M \circ (-M'') \rangle = \langle M', -M \circ (-M'') \rangle_{\Sigma_1} = 0,$$

so the map $V(M)$ induces a map $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$. Furthermore there are a map $Z(\Sigma_1) \otimes Z(\Sigma_2) \to Z(\Sigma_1 \coprod \Sigma_2)$ induced by the map $M_1 \otimes M_2 \mapsto M_1 \coprod M_2$, and a map $Z(-\Sigma) \to Z(\Sigma)^*$ induced by $M \mapsto \langle \cdot, -M \rangle$. Now Z is a TQFT if these two maps are isomorphisms and $Z(\Sigma)$ is finite dimensional for all Σ, and $Z(\emptyset) = \mathbb{C}$. This happens for the so-called quantum invariants. We will now discuss the 3-manifold invariant which gives rise to the RTW-TQFT. Given a framed link $L \subseteq S^3$ we can define a new three-manifold, the result of the ***integral surgery*** along L, $X_L = S^3 \setminus N_L \coprod_h (\coprod_{i=1}^n D^2 \times S^1)$ where n is the number of components of L, N_L is a tubular neighbourhood of L and $h : \partial N_L \to \partial \coprod_{i=1}^n D^2 \times S^1 = \coprod_{i=1}^n S^1 \times S^1$ is a homeomorphism defined in terms of the framing of L.

Theorem 16 (Lickorish, Wallace). *For any closed, connected 3-manifold Y there exist a framed link $L \subseteq S^3$ such that $X_L \cong Y$.*

Furthermore, there are the theorem:

Theorem 17 (Kirby). *Let L_i be framed link diagrams. Then $X_{L_1} \cong X_{L_2}$ if an only if L_1 can be transformed into L_2 using a combination of the framed Reidermeister moves and the Kirby moves.*

Using this theorem we can try to promote an invariant of framed links to an invariant of three-manifolds by proving that it is invariant under the Kirby moves. Reshetikhin and Turaev found a general way to produce such link invariants from the data of a *modular category*, and the invariants obtained this way always results in a TQFT, see [62]. For the SU(2) RTW TQFT,

$$\Omega_k(L) = \sum_{\vec{\lambda} \in (\Lambda_{2,k})^n} \prod_{i=1}^n [\lambda_i]_{\mathrm{e}^{\frac{2\pi \mathrm{i}}{k+2}}} J(L, \vec{\lambda})(\mathrm{e}^{\frac{2\pi \mathrm{i}}{k+2}}),$$

where $J(L,\vec{\lambda})$ denotes the coloured Jones polynomial. $\Omega_k(L)$ is clearly a knot invariant, but it is not quite invariant under the Kirby moves. However there exists numbers b, c such that $b^n c^\sigma \Omega(L)$ is invariant under the Kirby moves, where n is the number of components and σ the signature of the linking matrix of L. For more details see [56], [46]. Any link $K \subseteq X_L$ can be represented as a link $K' \subseteq S^3 \setminus L$, and we can extend Ω to three-manifolds containing coloured links by

$$\Omega_k(L,K') = \sum_{\vec{\lambda}\in(\Lambda_{2,k})^n} \prod_{i=1}^{n} [\lambda_i]_{\mathrm{e}^{\frac{2\pi\mathrm{i}}{k+2}}} J((L,\vec{\lambda}) \cup K)(\mathrm{e}^{\frac{2\pi\mathrm{i}}{k+2}}).$$

and we obtain an invariant of closed three-manifolds containing coloured, framed links, and this invariant defines via the universal construction to RTW TQFT.

The mapping class group of a surface with labeled, marked points and projective tangent vectors are defined just as the normal mapping class group, but requiring that the involved diffeomorphisms satisfy:

1. If a marked point p_1 is mapped to another marked point p_2, then $\lambda_1 = \lambda_2$.
2. The tangent directions must be preserved.

The effect of the second requirement is effectively that of adding a $\mathbb{Z}$ factor for each marked point, generated by the Dehn twist around that point. The quantum representations provide a representation of the braid groups in the following way: The mapping class group of a sphere with $n+1$ marked points with projective tangent vectors, where the first n of the points have the label μ, and the last point – lets call it ∞ – have a label λ is the (or, if $\lambda = \mu$, contains) the *ribbon braid group*, $\mathbb{Z} \rtimes \mathrm{B}_n$, where B_n acts on $\mathbb{Z}^n$ by the permutation action $\mathrm{B}_n \to \mathrm{S}_n$. The ribbon braid group contains the braid group B_n as the subgroup $\{0\} \times \mathrm{B}_n$. Because of the framings in the definition of the cobordism category for the RTW TQFT, the quantum representation construction does only yield projective representations of the mapping class groups; however, in genus 0 the framing ambiguity are not present, and we get actual representations. Let us write $\widetilde{M}(0,n)_\infty$ for the subgroup of $\widetilde{M}(S^2,(0,1,\dots,n-1,\infty),(\square,\dots,\square,\lambda^\dagger))$ consisting of classes of diffeomorphisms fixing ∞ and its framing. Moreover, we introduce $\widetilde{M}(0,n) = \widetilde{M}(\Sigma,(0,1,\dots,n-1),(\square,\dots,\square))$. Now $\widetilde{M}(0,n)_\infty$ contains B_n as a subgroup as described above. We will denote by $\rho^\lambda_{N,k}$ the restriction of the quantum representation of $\widetilde{M}(0,n)_\infty$ to B_n, and will think of B_n as $\mathrm{MCG}(0,n)_\infty$, the mapping class group of a sphere with n marked points and an extra marked point called ∞ which also carries a preserved projective tangent vector. Finally, we denote by $\rho_{N,k}$ the quantum representation of $\widetilde{M}(0,n)$ for $\mathrm{G} = \mathrm{SU}(N)$ at level k. Like Witten's result about the Jones polynomial, it turns out that the RTW TQFT also computes the Jones representations evaluated at certain roots of unity:

Proposition 24. *There are the following isomorphism of representations of the braid group:*

$$\rho^\lambda_{N,k} \cong \pi^\lambda_{\mathrm{e}^{\frac{2\pi\mathrm{i}}{k+N}}}.$$

Although the quantum representations are very strong, they cannot be faithful for any given level, as a Dehn twist always have finite order. However, in [1] Andersen show that a mapping class of a closed surface cannot be in the kernel of the quantum representations at infinitely many levels, hence proving that these quantum representations are asymptotic faithful. In [7] Andersen, Ueno and Masbaum states the following conjecture:

Conjecture 25 (AMU, [7]). *The quantum representations of a pseudo-Anosov mapping class has infinite order for all but finitely many levels.*

This conjecture was verified in [7] for the sphere with four marked points labeled with $\square$, and also for the torus with one marked point in [58].

Chapter 3

The Jones representation at $q = -1$

In this chapter we present the results of [26], which we will follow rather closely. All results obtained in this chapter are joint with Søren Fuglede Jørgensen. We give an interpretation of the Jones representations at $q = -1$, in terms of the action of the braid group on the homology of certain double covers of punctured disks and spheres.

The study of the Jones representation as contained in this paper was initiated by an attempt to generalize the results of [7] to general punctured spheres. Here, the authors show how to relate the quantum representations of the mapping class group of a sphere with four punctures, obtained from the level k quantum representations to an action on the homology of a torus by considering the limit $k \to \infty$. In [7] they find a clever that change of basis that allow them to obtain the relation to the homology on the torus; in our approach, the connection is more clear as we use a basis for the Jones representation that already consists of curves and show how to map such basis elements to the homology.

3.0.2 The Jones representation and homology

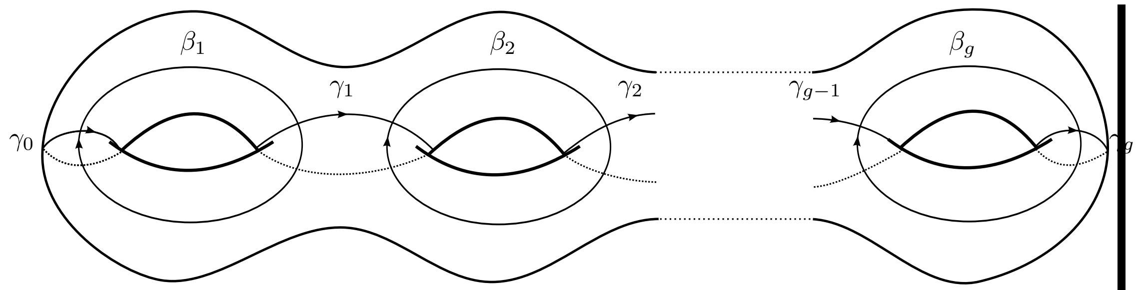
Let $n \in \mathbb{N}$ and let $g = n - 1$ and denote by $\mathrm{MCG}(g, r)$ the mapping class group of a surface of genus g with r boundary components. There is a homomorphism $\Psi : \mathrm{B}_{2n} \to \mathrm{MCG}(g, 0)$ given by mapping the standard braid generators $\sigma_1, \dots, \sigma_{2n-1} \in \mathrm{B}_{2n}$ to the (right) Dehn twists along the curves $\gamma_0, \beta_1, \gamma_1, \dots, \beta_g, \gamma_g$ indicated in Figure 3.1(a) respectively, well-defined due to proposition 6. Similarly, we define $\Psi : \mathrm{B}_{2n-1} \to \mathrm{MCG}(g, 1)$ by mapping $\sigma_1, \dots, \sigma_{2n-2}$ to twists along $\gamma_0, \beta_1, \gamma_1, \dots, \gamma_{g-1}, \beta_g$ respectively (Figure 3.1(b)). Notice also that these homomorphisms are exactly those that appear in theorem 8 .

Now, $\mathrm{MCG}(g, 0)$, respectively $\mathrm{MCG}(g, 1)$, act on the first homology group of a genus g surface, respectively of a genus g surface with one boundary component, by symplectomorphisms with respect to the intersection pairing ω. In particular it preserves the bivector associated to ω, which we by a small abuse of notation also will denote $\omega = \sum_i \alpha_i \wedge \beta_i$. For $m = 0, 1$ and $l \geq 1$, we let

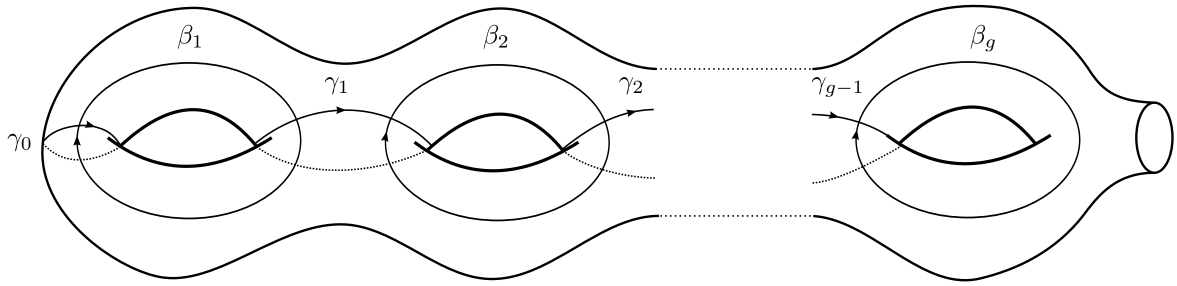
$$\tilde{\rho}^{g,l}_{\mathrm{hom}} : \mathrm{MCG}(g, m) \to \mathrm{GL}(\Lambda^l H_1(\Sigma_g^m, \mathbb{C}) / (\omega \wedge \Lambda^{l-2} H_1(\Sigma_g^m, \mathbb{C})))$$

denote the induced action, i.e. for $[\varphi] \in \mathrm{MCG}(g, m)$ and $v_1, \dots, v_l \in H_1(\Sigma_g^m, \mathbb{C})$,

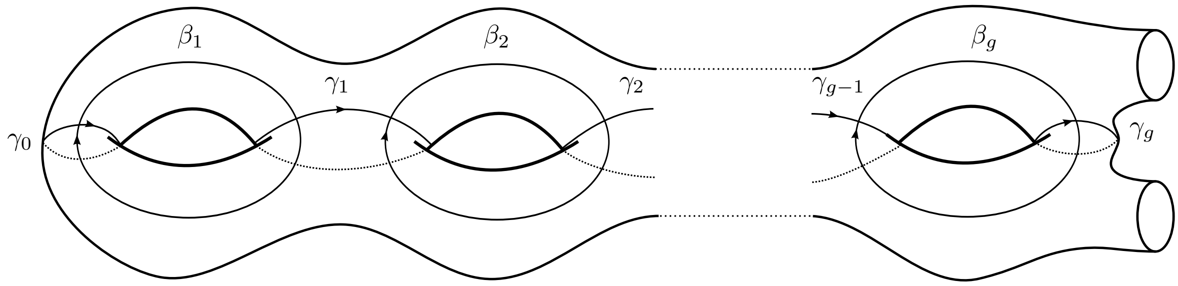
$$\tilde{\rho}^{g,l}_{\mathrm{hom}}([\varphi])[v_1 \wedge \dots \wedge v_l] = [(\varphi_*) v_1 \wedge \dots \wedge (\varphi_*) v_l],$$



(a) The closed surface Σ_g.



(b) The surface Σ_g^1.



(c) The surface Σ_g^2.

Figure 3.1: This figure illustrates how curves and homology cycles on a surface are named..

where we use the conventions that $\Lambda^{-1} H_1(\Sigma_g^m, \mathbb{C}) = \{0\}$ and $\Lambda^0 H_1(\Sigma_g^m, \mathbb{C}) = \mathbb{C}$. Finally, let $\rho_{\mathrm{hom}}^{g,l} = \tilde{\rho}_{\mathrm{hom}}^{g,l} \circ \Psi$ denote the corresponding braid group representations.

In [41, Sect. 10], Jones gave explicit matrices for the representation associated to $\boxplus$ (closely related to $\pi_q^{6,0}$, the representation associated to $\boxplus\!\boxplus$). Moreover, the choice of basis is such that all matrix entries are in $\mathbb{Z}[q, q^{-1}]$, so that one obtains this way representations of B_6 for all non-zero values of q rather than only the 6-regular ones. Kasahara [45, Lem. 2.1] observed in the same vein as above that at $q = -1$, the resulting representation is equivalent to the representation $\rho_{\mathrm{hom}}^{2,2}$. In general, the representation space of $\pi_q^{2n,0}$ has dimension C_{n+1}, the $(n+1)$'th Catalan number.

Theorem 18. *The Jones representation $\pi_q^{2n,0}$ has a natural extension to $q = -1$, for which it is equivalent to $\rho_{\text{hom}}^{g,g}$, where $g = n - 1$.*

By theorem 13 it follows that $\pi_q^{2n-1,1}$ is equivalent to $\pi_q^{2n,0}|_{B_{2n-1}}$, the above Theorem allows us to deduce a similar homological description for $\pi_{q=-1}^{2n-1,1}$.

Theorem 19. *The representation $\pi_q^{2n+1,d}$ has a natural extension to $q = -1$ for which it is equivalent to the action $\rho_{\text{hom}}^{g,l}$, where $g = n$ and $2l = 2n + 1 - d$.*

This leaves us with the two-row diagrams having an even number of boxes. The fact that $\pi_q^{n,d}|_{B_{n-1}} \cong \pi_q^{n-1,d-1} \oplus \pi_q^{n-1,d+1}$ (see e.g. [41]), together with the observation that

$$\binom{2g}{l} - \binom{2g}{l-2} + \binom{2g}{l+1} - \binom{2g}{l-1} = \binom{2g+1}{l+1} - \binom{2g+1}{l-1}$$

for all g and l, leads us to the following: let $\hat{\rho}_{\text{hom}}^{g,l}$ denote the action of B_{2n} on $\Lambda^l H_1(\Sigma_g^2, \mathbb{C})$, where $g = n - 1$, given by mapping $\sigma_1, \dots, \sigma_{2n-1}$ to the action induced by the homological action of the Dehn twists $t_{\gamma_0}, t_{\beta_1}, t_{\gamma_1}, \dots, t_{\beta_g}, t_{\gamma_g}$ respectively (see Figure 3.1(c)).

Theorem 20. *The representation $\pi_q^{2n,d}$ has a natural extension to $q = -1$ for which it is equivalent to a subrepresentation of $\hat{\rho}_{\text{hom}}^{g,l}$, where $g = n - 1$ and $2l = 2n - d$.*

In each of the three cases, the appropriate intertwining operator is constructed as follows: a natural basis for the representation spaces of $\pi_q^{n,d}$ is given in terms of non-intersecting paths in the relevant punctured disc, connecting the punctures. Regarding the punctures as marked points, the disk is realized as the quotient of a surface by the order two element rotating the surface by π along its horizontal axis in Figure 3.1, allowing us to realize the surface as a well-understood branched double cover. Lifting the non-intersecting paths through the double cover defines a collection of loops in the covering surface, and by taking an appropriately ordered and scaled wedge product of the homology classes of these loops, we obtain our desired linear map.

We now prove Theorems 18, 19, and 20. For the remainder of this section, we assume that $A = \exp(-\pi \mathrm{i}/4)$.

3.0.3 Construction of the (iso)morphism

We now construct the intertwining morphism φ as needed in Theorems 19 and 20. For Theorem 18 the construction needs to be tweaked slightly; a concern we defer until it becomes relevant.

For a diagram D in the basis of $V^{n,d}$ we denote by D_0 the set of arcs connecting two top points. We identify the points at the top with the numbers $1, 2, \dots, n$, and for $a \in D_0$ we denote by a_0 and a_1 the left and right end point of a respectively. The set of bottom points will be called ∞.

Definition 26. Let $c_1, \dots, c_{n-1}$ be simple closed curves on the surface Σ_g^m, such that $c_i \cap c_{i+1}$ consists of a single point, and such that c_i and c_j are disjoint for $|i-j| > 1$. We orient the curves such that $t_{c_i}(c_{i+1}) = c_{i+1} + c_i$, where we abuse notation and use c_i also for the element that the oriented curve defines in $H_1(\Sigma_g^m, \mathbb{C})$, and assume that $\operatorname{span}\{c_i \mid i = 1, \dots, n-1\} \subseteq H_1(\Sigma_g^m, \mathbb{C})$ has dimension $n - 1$. We note that $\gamma_0, \beta_1, \gamma_1, \beta_2, \dots, \beta_g$, respectively $\gamma_0, \beta_1, \gamma_1, \beta_2, \dots, \beta_g, \gamma_g$ in fig. 3.1(b) respectively fig. 3.1(c) defines such curves.

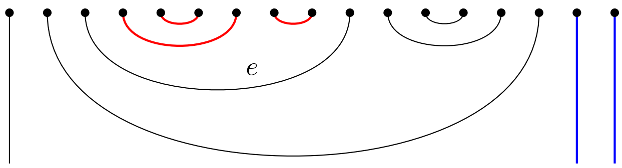


Figure 3.2: The arcs contributing to $w(e) = 3$ have been coloured red, and the arcs contributing to $v(e) = 2$ have been coloured blue. .

We order the arcs in D_0 by their starting points so that $e < e'$ if $e_0 < e'_0$. To each arc e in D_0 we associate the element $X_e = \sum_{i=e_0}^{e_1-1} c_i \in \operatorname{span}\{c_i\} \subseteq H_1(\Sigma_g^m, \mathbb{C})$.

Define a map $\varphi : V^{n,d} \to \Lambda^l H_1(\Sigma_g^m, \mathbb{C})$, by letting, for a basis diagram $D \in V^{n,d}$,

$$\varphi(D) = f(D) \bigwedge_{e \in D_0} X_e, \tag{3.1}$$

where $f(D) = (-\mathrm{i})^{\sum_{e \in D_0} w(e)+v(e)}$; here, $w(e)$ denotes the number of arcs between the starting and ending point of e, $v(e)$ denotes the number of points greater than e_1 that are connected to ∞ (see Figure 3.2), and the wedge product runs through D_0 from the first to the final arc with respect to the ordering on D_0.

3.0.4 Proof of equivariance

Lemma 27. *Denote by T_i the action of e_i on $V^{n,d}$. Then we have, for all $i = 1, 2, \ldots, n-1$ and $D \in V^{n,d}$, that*

$$\varphi(D + iT_i D) = t_{c_i} \varphi(D).$$

Proof. Let $c = c_i$. We first observe that $(t_c)_*$ acts trivially on X_e if e is not connected to exactly one of i or $i+1$. In particular, if the points i, $i+1$ are connected, or if both are connected to ∞, then

$$(t_c)_*(\varphi(D)) = \varphi(D).$$

Since in these cases $T_i D = 0$, we obtain the claim of the Lemma.

Let us therefore consider the case where we have two distinct arcs connecting to i, $i+1$, not both connecting to ∞. Let us first assume that $i+1$ is connected to ∞, and let us denote the arc ending in i by a.

Then $(t_c)_*$ acts trivially on all factors of $\varphi(D)$ except for X_a. If a is the j'th arc in D_0, then

$$(t_c)_*\varphi(D) = f(D) X_1^{j-1} \wedge (X_a - c) \wedge X_{j+1}^k = \varphi(D) + f(D) X_1^{j-1} \wedge (-c) \wedge X_{j+1}^l,$$

where we denote the i'th edge in D_0 by e^i and write

$$X_i^j = \bigwedge_{l=i}^{j} X_{e^i}.$$

On the other hand, we have

$$\begin{aligned}
\varphi(T_i(D)) &= (-i)^{-2w(a)-1} f(D) X_1^{j-1} \wedge X_{j+1}^{j+w(a)} \wedge c \wedge X_{j+w(a)+1}^{l} \\
&= (-i)^{-2w(a)-1}(-1)^{w(a)} f(D) X_1^{j-1} \wedge c \wedge X_{j+1}^{l} \\
&= -i f(D) X_1^{j-1} \wedge (-c) \wedge X_{j+1}^{l}.
\end{aligned}$$

Likewise, if i is connected to ∞ but $i+1$ is not, we denote by a the arc connecting to $i+1$ and find that

$$(t_c)_* \varphi(D) = f(D) X_1^{j-1} \wedge (X_a + c) \wedge X_{j+1}^{l} = \varphi(D) + f(D) X_1^{j-1} \wedge c \wedge X_{j+1}^{l},$$

and

$$\varphi(T_i(D)) = (-i)^{w(a)-w(a)+1} f(D) X_1^{j-1} \wedge c \wedge X_{j+1}^{l} = -i f(D) X_1^{j-1} \wedge c \wedge X_{j+1}^{l}.$$

Now only the cases where two distinct arcs from D_0 connect to i and $i+1$ remain. Denote these arcs by a and b, and assume that they are the j'th and m'th arc respectively, with $j < m$. There are three possibilities, corresponding to whether or not a and b start or end in i and $i+1$ (the case where a starts in i and b ends in $i+1$ is not possible if $a \neq b$). We distinguish these cases by the signs (ξ_a^i, ξ_b^i) of the intersections of the curves with c, i.e. $(t_c)_* X_a = X_a + \xi_a^i c$, and likewise for b. Then

$$\begin{aligned}
(t_c)_* \varphi(D) &= f(D) X_1^{j-1} \wedge (X_a + \xi_a^i c) \wedge X_{j+1}^{m-1} \wedge (X_b + \xi_b^i c) \wedge X_{m+1}^{l} \\
&= \varphi(D) + f(D) X_1^{j-1} \wedge (\xi_b^i X_a - \xi_a^i X_b) \wedge X_{j+1}^{m-1} \wedge c \wedge X_{m+1}^{l} \\
&= \varphi(D) + f(D) X_1^{j-1} \wedge (\xi_b^i X_a \pm c - \xi_a^i X_b) \wedge X_{j+1}^{m-1} \wedge c \wedge X_{m+1}^{l}.
\end{aligned}$$

If now $(\xi_a, \xi_b) = (-, +)$, then we have

$$\begin{aligned}
\varphi(T_i(D)) &= (-i)^{-w(a)-w(b)+w(a)+w(b)+1} f(D) X_1^{j-1} \wedge (a + c + b) \wedge X_{j+1}^{m-1} \wedge c \wedge X_{m+1}^{l} \\
&= -i f(D) X_1^{j-1} \wedge (a + c + b) \wedge X_{j+1}^{m-1} \wedge c \wedge X_{m+1}^{l}.
\end{aligned}$$

If $(\xi_a, \xi_b) = (+, +)$,

$$\begin{aligned}
\varphi(T_i(D)) &= (-i)^{-w(a)-w(b)+w(a)-w(b)-1} f(D) \\
&\quad \cdot X_1^{j-1} \wedge c \wedge X_{j+1}^{m-1} \wedge X_{m+1}^{m+w(b)} \wedge (a - c - b) \wedge X_{m+w(b)+1}^{l} \\
&= i(-i)^{-2w(b)}(-1)^{w(b)} f(D) X_1^{j-1} \wedge c \wedge X_{j+1}^{m-1} \wedge (a - c - b) \wedge X_{m+1}^{l} \\
&= -i f(D) X_1^{j-1} \wedge (a - c - b) \wedge X_{j+1}^{m-1} \wedge c \wedge X_{m+1}^{l}.
\end{aligned}$$

And finally, if $(\xi_a, \xi_b) = (-, -)$,

$$\begin{aligned}
\varphi(T_i(D)) &= (-i)^{-w(a)-w(b)+w(a)-w(b)-1} f(D) \\
&\quad \cdot X_1^{j-1} \wedge (a - c - b) \wedge X_{j+1}^{m-1} \wedge X_{m+1}^{m+w(b)} \wedge c \wedge X_{m+w(b)+1}^{l} \\
&= i(-i)^{-2w(b)}(-1)^{w(b)} f(D) X_1^{j-1} \wedge c \wedge X_{j+1}^{m-1} \wedge (a - c - b) \wedge X_{m+1}^{l} \\
&= -i f(D) X_1^{j-1} \wedge (-a + c + b) \wedge X_{j+1}^{m-1} \wedge c \wedge X_{m+1}^{l}. \qquad \square
\end{aligned}$$

This shows that φ is a homomorphism between the representation $\sigma_i \mapsto A^{-1} \eta_A^{n,d}(\sigma_i)$ and the representation on $\Lambda^l H_1(\Sigma_g^m, \mathbb{C})$ in the cases of Theorems 19, and 20. For Theorem 18, we define

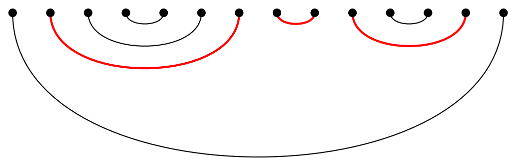


Figure 3.3: Here, the outermost arc is a and the b_k's have been coloured red. .

$\tilde{\varphi}(D)$ for $D \in V^{2n,0}$ in the following way: there is an i such that there is an arc a connecting i to n. We define an element $\tilde{D}$ in $V^{n-1,1}$ by removing a, forgetting n, and connecting i to ∞, and we define $\tilde{\varphi} : V^{2n,0} \to \Lambda^g H_1(\Sigma_g, \mathbb{C})$ by $\tilde{\varphi}(D) = \varphi(\tilde{D})$, identifying $H_1(\Sigma_g, \mathbb{C})$ with $H_1(\Sigma_g^1, \mathbb{C})$. It is clear that the induced map $\tilde{\cdot} : V^{2n,0} \to V^{2n-1,1}$ is an isomorphism of vector spaces.

Lemma 28. *The map $\tilde{\varphi}$ is a homomorphism of representations.*

In order to prove this, we need the following Lemma.

Lemma 29. *If D is a diagram and a is an arc that connects i to j, $i < j$, we have that*

$$X_a \wedge \bigwedge_{e \prec a} X_e = \left(\sum_{\substack{i \le k \le j, \\ k \equiv i \mod 2}} c_k \right) \wedge \bigwedge_{e \prec a} X_e,$$

where we say that $e \prec a$ if $a_0 < e_0 < e_1 < a_1$.

Proof. The proof is by induction on $w(a)$. If $w(a) = 0$, the claim is clear. Otherwise we use the induction hypothesis on the arcs b_k, $k = 1, 2, \ldots, m$, connecting (i_k, j_k), where $i_1 = i + 1$, $j_m = j - 1$, and $j_k + 1 = i_{k+1}$ for $k < m$. See Figure 3.3. □

Up until this point, we have not used the possible description of the curves c_i in terms of the curves β_i and γ_i of Figure 3.1, but for the proof of Lemma 28 below, we will need that

$$c_{n-1} = - \sum_{\substack{1 \le k \le n-3, \\ k \equiv 1 \mod 2}} c_k$$

in $H_1(\Sigma_g, \mathbb{C})$, which will be guaranteed by considering the concrete curves.

Proof (Of lemma 28). If we consider only the action of $B_{n-1} \subseteq B_n$, then $\tilde{\varphi}$ defines a homomorphism by the above, so we only have to check the equivariance of the action of σ_{n-1}. If $n - 1$ and n are connected, then β_g does not appear in $\varphi(\tilde{D})$, and the action of σ_{n-1} on homology is trivial, just as $T_n(D) = 0$.

If n is connected to $i < n - 1$ by the j'th arc, and $n - 1$ is an end point of the m'th arc, which we denote by b, then

$$\begin{aligned} (t_{c_{n-1}})_*(\varphi(\tilde{D})) &= f(\tilde{D}) X_1^{j-1} \wedge X_{j+1}^{m-1} \wedge (X_b - c_{n-1}) \wedge X_{m+1}^l \\ &= \varphi(\tilde{D}) - f(\tilde{D}) X_1^{j-1} \wedge X_{j+1}^{m-1} \wedge c_{n-1} \wedge X_{m+1}^l \end{aligned}$$

$$= \varphi(\tilde{D}) + f(\tilde{D}) X_1^{j-1} \wedge X_{j+1}^{m-1} \wedge \left(\sum_{\substack{1 \le k \le n-3, \\ k \equiv 1 \mod 2}} c_k \right) \wedge X_{m+1}^l$$

$$= \varphi(\tilde{D}) + f(\tilde{D}) X_1^{j-1} \wedge X_{j+1}^{m-1} \wedge \left(\sum_{\substack{i \le k \le b_0, \\ k \equiv 1 \mod 2}} c_k \right) \wedge X_{m+1}^l.$$

Here we used Lemma 29 in last equality, applying it to the arcs left of i that are not contained between the end points of any other arc, and on the arcs between the end points of b. Letting p be the number of arcs to the right of i in $\tilde{D}$, we find, where we denote by d an arc connecting i with b_0, that

$$\begin{aligned}
\varphi(\widetilde{T_{n-1}(D)}) &= (-i)^{p-w(b)+p-w(b)-1} f(\tilde{D}) X_1^{j-1} \wedge X_d \wedge X_{j+1}^{m-1} \wedge X_{m+1}^l \\
&= (-i)^{p-w(b)+p-w(b)-1} f(\tilde{D}) X_1^{j-1} \wedge \left(\sum_{\substack{i \le k \le b_0, \\ k \equiv 1 \mod 2}} c_k \right) \wedge X_{j+1}^{m-1} \wedge X_{m+1}^l \\
&= (-1)^{p-w(b)+1} (-i)^{p-w(b)+p-w(b)-1} f(\tilde{D}) X_1^{j-1} \wedge X_{j+1}^{m-1} \wedge \left(\sum_{\substack{i \le k \le b_0, \\ k \equiv 1 \mod 2}} c_k \right) \wedge X_{m+1}^l \\
&= -i f(\tilde{D}) X_1^{j-1} \wedge X_{j+1}^{m-1} \wedge \left(\sum_{\substack{i \le k \le b_0, \\ k \equiv 1 \mod 2}} c_k \right) \wedge X_{m+1}^l.
\end{aligned}$$

This shows the equivariance of $\tilde{\varphi}$. □

3.0.5 Injectivity

In the case of Theorem 19 we know that the dimension of $V^{n,d}$ agrees with the dimension of $\Lambda^l H_1(\Sigma_g^1, \mathbb{C})/\omega \wedge \Lambda^{l-2} H_1(\Sigma_g^1, \mathbb{C})$, so we can show that $\bar{\varphi}$ – which here denotes φ composed with the map to the quotient – is an isomorphism by showing that it is surjective, which we will do next. This also shows the injectivity claimed in Theorem 18, by construction of $\tilde{\varphi}$.

Proof (Of theorem 18 and theorem 19). Denote by $\bar{\varphi}$ the composition of φ with the projection to the quotient $\Lambda^l H_1(\Sigma_g^1, \mathbb{C})/\omega \wedge \Lambda^{l-2} H_1(\Sigma_g^1, \mathbb{C})$. We just need to show that $\bar{\varphi}$ is injective, as we know that the spaces have the same dimension. Assume that v is in the kernel of $\bar{\varphi}$; then

$$e_i v = T_i v = -i(v + iT_i v) + iv,$$

so

$$\bar{\varphi}(e_i v) = -it_{c_i} \bar{\varphi}(v) + i\bar{\varphi}(v) = 0,$$

which shows that $\ker \bar{\varphi}$ is a $\mathrm{TL}_n(\exp(-\pi i/4))$-subrepresentation of $V^{n,d}$. But it follows from Corollary 4.8 of [57] that the representation on $V^{n,d}$ is irreducible (as noted in the remarks on the case $\beta = 0$ following their Corollary), so $\ker \bar{\varphi}$ is either 0 or $V^{n,d}$. It is easy to check that the latter is not the case, and so we obtain the Theorem. □

Remark 30. We remark that the injectivity above can also be shown by explicitly constructing a basis $a_1, a_2, \dots, a_i, b_1, b_2, \dots, b_j$ of $\Lambda^l H_1(\Sigma, \mathbb{C})$ in such a way that φ surjects onto $\mathrm{span}\{a_k\}$, $\omega \wedge \Lambda^{l-2} H_1(\Sigma, \mathbb{C}) = \mathrm{span}\{b_k\}$, and $\mathrm{span}\{a_k\}$ has the right dimension.

Proof (Of theorem 20). This follows from Theorem 19 in the following way: we can define a map $h_1 : V^{n,d} \to V^{n-1,d-1}$ that maps D to 0 if n is not connected to ∞, and otherwise to the diagram given by removing the arc going from n to ∞; here we let $V^{n-1,-1} = V^{n-1,n+1} = \{0\}$. Likewise, we define a map $h_2 : V^{n,d} \to V^{n-1,d+1}$ that maps a diagram to 0 if n is connected to ∞, and otherwise to the diagram where the point connected to n is now connected to ∞. For a diagram of the latter type, we define $g(D) = \alpha X_e$, where α is a scalar depending on D, and e is an arc such that $\varphi(D) = g(D) \wedge \varphi(h_2(D))$. It is clear that $g(D) = \alpha \sum_{i=k}^{n-1} c_i$ for some k, and by looking at the block decomposition of φ with respect to these two kinds of diagrams and with respect to the two subspaces $c_{n-1} \wedge \Lambda^{l-1} \operatorname{span}\{c_i \mid i = 1, 2, \ldots, n-2\}$ and its orthogonal complement (using wedge products of c_i's as an orthogonal basis), we see that φ is injective, as it has the form

$$\left(\begin{array}{c|c} \varphi^{n-1,d-1} \circ h_1 & * \\ \hline 0 & \alpha c_{n-1} \wedge \varphi^{n-1,d+1} \circ h_2 \end{array}\right),$$

where the diagonal blocks are injective by Theorem 19; here, we interpret $\varphi^{n-1,-1}$ and $\varphi^{n-1,n+1}$ as the maps between two 0-dimensional spaces. □

3.1 The AMU conjecture for homological pA's of a sphere

In the following we let L be the set of Young diagrams with 2 rows and n cells.

Theorem 21. *Let $\varphi \in \mathrm{MCG}(0,n)_\infty$ be a homological pseudo-Anosov, and let $\lambda \in L$. Then $\rho^\lambda_{N,k}(\varphi)$ has infinite order for all but finitely many k.*

As above, we say that $\varphi \in \widetilde{M}(0,n)$ is a *homological pseudo-Anosov* if its image in $\mathrm{MCG}(0,n)$ is a pseudo-Anosov whose invariant foliations have the property that all non-puncture singularities are even-pronged and all punctures have odd-pronged singularities.

Theorem 22. *Assume that n is even. If $\varphi \in \widetilde{M}(0,n)$ is a homological pseudo-Anosov, then $\rho_{2,k}(\varphi)$ has infinite order for all but finitely many k.*

Moreover, we show in Corollary 39 that for the homological pseudo-Anosovs, the quantum representations determine their stretch factors, answering positively [7, Question 1.1 (2)] in this case.

We turn now to the proofs of Theorems 21 and 22. Recall that these are concerned with the genus 0 quantum $\mathrm{SU}(N)$-representations $\rho^\lambda_{N,k}$, depending on a level k and a Young diagram $\lambda \in L$.

Lemma 31. *Let X_n denote the set of primitive n'th roots of unity. Then for every $z \in \mathrm{U}(1)$, there exist $z_n \in X_n$ such that $\lim_{n\to\infty} z_n = z$.*

Remark 32. For $z = -1$ – which in fact is the only case we will need in the proof of Theorem 21 – this is [7, Lem. 5.1], in which the sequence z_n is constructed explicitly.

Proof (Of lemma 31). Write $z = \exp(2\pi i\alpha)$, $\alpha \in [0,1)$, and let

$$Y_n = \{m \mid 0 < m < n,\ \gcd(m,n) = 1\}.$$

The largest gap between two consecutive points of Y_n is bounded from above by $j(n)$, the ordinary Jacobsthal function at n. The result of Iwaniec [39] allows us to bound the size of this gap as $j(n) = O(\log^2(n))$. Thus, since $\log^2(n)/n \to 0$ as $n \to \infty$, we may choose $\alpha_n \in Y_n/n$ such that $\alpha_n - \alpha \to 0$ for $n \to \infty$. Letting $z_n = \exp(2\pi i\alpha_n) \in X_n$, we obtain the result since $z_n z^{-1} \to 1$ as $n \to \infty$. □

For $\varphi \in \mathrm{MCG}(0,n)_\infty$, define $\mathrm{sr}_d(\varphi) : \mathbb{R} \to \mathbb{R}_{>0}$ by

$$\mathrm{sr}_d(\varphi)(x) = \mathrm{sr}(\eta^{n,d}_{\exp(-\pi i x/4)}(\varphi)),$$

for $d \in \{0, \ldots, n\}$ with $d \equiv n \mod 2$, where sr denotes the spectral radius of a linear map.

Lemma 33. *Let $\varphi \in \mathrm{MCG}(0,n)_\infty$. If $\mathrm{sr}_d(\varphi)(x_0) > 1$ for some $x_0 \in [0,1]$, then there exists $k_0 \in \mathbb{N}$ such that the order of $\rho^\lambda_{N,k}(\varphi)$ is infinite for all $k > k_0$, where $\lambda = ((n+d)/2 \geq (n-d)/2)$.*

Proof. It follows from Theorem 24 and the relations described in Section 1.6 that it suffices to show that for some k_0, the order of $\eta^{n,d}_A(\varphi)$ is infinite for $q = A^4 = \exp(2\pi i/(k+N))$ for all k with $k > k_0$. On the other hand, the order of $\eta^{n,d}_A(\varphi)$ at a general primitive $4(k+N)$'th root of unity $A = \exp(\frac{2\pi i l}{4(k+N)})$ is independent of l, as one is the image of the other under the Galois isomorphism permuting the two primitive roots.

Now, apply Lemma 31 to $q = \exp(-\pi i x_0)$ to obtain primitive $4(k+N)$'th roots of unity A_k with $A_k^4 \to q$. Since $\mathrm{sr}_d(\varphi)$ is continuous, $\mathrm{sr}(\eta^{n,d}_{A_k}(\varphi)) > 1$, for all sufficiently large k, and a linear map having an eigenvalue of absolute value greater than 1 necessarily has infinite order. ☐

To show that a given pseudo-Anosov $\varphi \in \mathrm{MCG}(0,n)_\infty$ has infinite order in $\rho^\lambda_{N,k}$ for all but finitely many levels, it thus suffices to find x_0 as above. As we will see below, by Theorems 19 and 20, this is possible for those pseudo-Anosovs whose stretch factors are given by their action on the homology of the double cover. In case $d = n-2$, the claims will follow immediately from Lemma 33 but in general, we will need the following remarks on the actions of surface diffeomorphisms on wedge products of homology.

Remark 34. Let f_* be the action on $H_1(\Sigma, \mathbb{C})$, where Σ is a surface of genus g with 0 or 1 boundary components, induced by a diffeomorphism f. Let $\lambda_1, \ldots . \lambda_{2g}$ be the diagonal entries in the Jordan normal form of the matrix for f_*. Then the action of f on $\Lambda^l H_1(\Sigma, \mathbb{C})$ will have an eigenvector of eigenvalue $\prod_{i\in I} \lambda_i$ for any subset $I \subseteq \{1 : 2g\}$ of size l, given by wedging together the corresponding vectors in the Jordan normal form, in such a way that a non-eigenvector is only included if all the preceding vectors in its block are also included. As all of these eigenvectors are of the form $\bigwedge_{i=1}^l v_i$, they can not be in the subspace $\omega \wedge \Lambda^{l-2} H_1(\Sigma, \mathbb{C})$ if $l \leq g$, and therefore they define eigenvectors of the same eigenvalue in the quotient $\Lambda^l H_1(\Sigma, \mathbb{C})/\omega \wedge \Lambda^{l-2} H_1(\Sigma, \mathbb{C})$.

Assume now that the action on $H_1(\Sigma, \mathbb{C})$ has an eigenvalue with absolute value strictly greater than 1. The action is symplectic, so the eigenvalues come in pairs λ, λ^{-1}, and there must be at least g columns in the Jordan normal form having diagonal entry with absolute value at least 1. Therefore the action on $\Lambda^l H_1(\Sigma, \mathbb{C})$ has an eigenvalue with absolute value greater than 1 for $l \leq g$, and by the above considerations, so does the action on $\Lambda^l H_1(\Sigma, \mathbb{C})/\omega \wedge \Lambda^{l-2} H_1(\Sigma, \mathbb{C})$.

Proposition 35. *Let $\lambda \in L$, let $f \in \mathrm{MCG}(0,n)_\infty$, and assume that the action $\hat{f}_*$ of $\hat{f} = \Psi(f)$ on $H_1(\Sigma^m_g)$ has spectral radius strictly greater than 1. Then the order of $\rho^\lambda_{N,k}(f)$ is infinite for all but finitely many levels k.*

Proof. For general d, we also need to ensure that an appropriate eigenvector – of eigenvalue with absolute value strictly greater than 1 – is actually contained in the image of the morphism of representations. This, on the other hand, is guaranteed by Remark 34 in case n is odd.

Assume now that n is even and that $d < n-2$. Since $\hat{f}$ preserves the boundaries pointwise, it defines a diffeomorphism of Σ^1_{g+1}, obtained by gluing to Σ^2_g a pair of pants. Denote by ι the induced map on (wedge products of) homology. On the level of diagrams, this corresponds to the inclusion $V^{n,d} \hookrightarrow V^{n+1,d+1}$ obtained by adding to a diagram the point $n+1$ and connecting it to ∞. This of course corresponds to the decomposition $V^{n+1,d+1} = V^{n,d} \oplus V^{n,d+2}$ as a vector space, as described in the proof of Theorem 20 (but note that $n+1$ is now odd); that is, $V^{n,d+2}$ is spanned by diagrams where $n+1$ is not connected to ∞. Now, even though the action of f does not preserve the decomposition, it is clearly block triangular.

With these identifications, we have a diagram

$$\begin{tikzcd}
V^{n,d} \arrow[r] \arrow[d, "\varphi^{n,d}"'] & V^{n+1,d+1} \arrow[d, "\varphi^{n+1,d+1}"] & \\
\Lambda^l H_1(\Sigma_g^2, \mathbb{C}) \arrow[r, "\iota"] & \Lambda^l H_1(\Sigma_{g+1}^1, \mathbb{C}) \arrow[r] & \Lambda^l H_1(\Sigma_{g+1}^1, \mathbb{C})/\omega \wedge \Lambda^{l-2} H_1(\Sigma_{g+1}^1, \mathbb{C}),
\end{tikzcd}$$

which is commutative up to a power of $-i$. We wish to show that the action of f on $V^{n,d} \subseteq V^{n+1,d+1}$ contains eigenvectors of the appropriate eigenvalues. Suppose that the eigenvalues of the action of f_* on $H_1(\Sigma_g^2, \mathbb{C})$, counted with algebraic multiplicity, have absolute values

$$(x_1, \dots, x_m, 1, 1, \dots, 1, x_m^{-1}, \dots, x_1^{-1})$$

with $x_i > 1$ for all i so that as before, $m \geq 1$. Consider first the case $l \leq m$. As in Remark 34, the action of $\hat{f}_*$ on $\Lambda^l H_1(\Sigma_g^2, \mathbb{C})$ has an eigenvector $v = v_1 \wedge \dots \wedge v_l$ whose eigenvalue has absolute value $x = x_1 \cdots x_l$. Now, $\iota(v)$ is an eigenvector for the induced action with the same eigenvalue (up to the same root of unity), and it follows from the case of n odd that $\mathrm{Im}(\varphi^{n+1,d+1})$ contains an eigenvector which has the same eigenvalue as $\iota(v)$. Moreover, since the eigenvectors arising from $V^{n,d+2}$ all have absolute value strictly less than x (as we take only the $(l-1)$'st wedge products), we obtain the desired eigenvector of $\eta^{n,d}_{\exp(-\pi i/4)}(f)$. The conclusion now follows as in the case of odd n.

The case $l > m$ is similar but involves also a small combinatorial exercise as in this case, there may also be eigenvectors coming from $V^{n,d+2}$ with eigenvalue of absolute value x. Let $d_x^{n,d}$ be the sum of the algebraic multiplicities of eigenvalues of absolute value x of the action of f on $V^{n,d}$. We claim that for n even,

$$d_x^{n,d} = \binom{2g+1-2m}{l-m} - \binom{2g+1-2m}{l-m-2}. \tag{3.2}$$

To see this, we appeal again to the decomposition used above, as a simple extension of the argument from Remark 34 shows that

$$d_x^{n+1,d+1} = \binom{2g+2-2m}{l-m} - \binom{2g+2-2m}{l-m-2}.$$

Now, since $d_x^{n+1,d+1} = d_x^{n,d} + d_x^{n,d+2}$, equation (3.2) follows by induction on l, starting at $l = m$, by using well-known recursive formulas for binomial coefficients. Since $d_x^{n,d} > 0$, this completes the proof. □

Proof (Of theorem 21). Let $f \in \mathrm{MCG}(0,n)_\infty$ be a homological pseudo-Anosov, let $\tilde{f}$ denote its image in $\mathrm{MCG}(0,n+1)$, and let $\hat{f} \in \mathrm{MCG}(g,m)$ denote the image of f under the Birman–Hilden map used in Theorems 19 and 20 with the appropriate values of g and m. Everything has been set up so that $\hat{f}$ is a pseudo-Anosov of Σ_g^m with the same stretch factor as $\tilde{f}$, and that, moreover, $\hat{f}$ has *orientable* invariant foliations. This follows by the exact same reasoning as in the similar setup in Theorem 5.1 of [16]. In short: the orientability of a foliation is determined by the vanishing of its associated orientation homomorphism, and this on the other hand is ensured by the assumptions on the degrees of the singularities.

Now, the stretch factor of any pseudo-Anosov with orientable invariant foliations is simply the spectral radius of its action on homology which is therefore strictly greater than 1. This is a well-known result and in fact a criterion for having orientable foliations; see e.g. [16, Lemma 4.3] and the discussion preceding it.

The claim then follows directly from Proposition 35.

Proof (Of theorem 22). Consider now the case where $N = 2$, n is even, and λ is the empty diagram. Here, the level k quantum SU(2)-representation, rescaled on each generator by a suitable k-dependent root of unity, defines a representation of $\widetilde{M}(0, n)$, equivalent by construction to $\rho_{2,k}$ (see [41, Sect. 10]). As multiplication by a root of unity does not change whether or not the order of a linear map is finite or infinite, we obtain from Theorem 21 the claimed result. □

Remark 36. More generally, in [41, Sect. 10], Jones finds that his representations may be tweaked by roots of unity to descend to the mapping class groups of spheres whenever the associated Young diagram λ is rectangular. Thus, the same is of course true for the quantum SU(N)-representations $\rho^{\lambda}_{N,k}$. One could therefore proceed as in [7, Sect. 4], define new quantum representations for mapping class groups of punctured spheres, and immediately obtain a version of Theorem 22 for those.

Remark 37.

On a closed torus, the stretch factor of a pseudo-Anosov is always given by its action on homology, and so we recover the main result of [7].

Example 38 (Homological pseudo-Anosovs on the sphere).

As noted in the proof of Theorem 21, [16, Lemma 4.3] tells us that if the spectral radius of the action of a pseudo-Anosov on homology equals its stretch factor, the pseudo-Anosov must necessarily have orientable invariant foliations, and so we can appeal directly to any of the existing homological constructions of pseudo-Anosovs to obtain interesting examples. One such family of examples arises as a special case of the pseudo-Anosovs described in [54] (Penner's construction) on the level of the covering surfaces, which – passing through the Birman–Hilden homomorphism – may be described as follows. Suppose that n is even. Then any word in the generators $\sigma_1, \dots, \sigma_{n-1}$ such that the sign of the exponents for odd-indexed generators all agree, such that exponents of even-indexed generators all have the opposite sign as the odd-indexed ones, and such that each generator appears at least once, is a homological pseudo-Anosov.

3.1.1 Determining stretch factors

In [7, Cor. 5.8], the authors go on to show that the stretch factor of any given pseudo-Anosov of a sphere with four punctures may be obtained as limits of eigenvalues of the quantum representations of φ. The analogous statement in our general case is the following.

Corollary 39. *For any homological pseudo-Anosov $\varphi \in \mathrm{MCG}(0, n)_\infty$ and Young diagram $\lambda \in \Lambda$, there exist eigenvalues $\lambda_k, \tilde{\lambda}_k$ of $\rho^{\lambda}_{N,k}(\varphi)$ such that $\sqrt{|\lambda_k \tilde{\lambda}_k|}$ tends to the stretch factor of φ as $k \to \infty$.*

Proof. The statement follows from the proof of Proposition 35 by continuity in A of the eigenvalues of $\eta_A^{n,d}(\varphi)$, as these include at $q = A^4 = -1$ the values $\tau\mu$ and $\tau\mu^{-1}$, where $|\tau|$ is the stretch factor of φ (and likewise, μ is a product of eigenvalues of the induced action on homology). $\square$

Chapter 4

Geometric quantization

4.1 Symplectic manifolds

A symplectic manifold is a pair (M,ω) where M is a smooth manifold and ω is a closed 2-form on M that is non-degenerate in each fiber, that is, for $p \in M$ and $X \in T_pM$ such that $\omega_p(X,Y) = 0$ for all $Y \in T_pM$, then $X = 0$. Therefore ω induces an isomorphism between the tangent bundle and the cotangent bundle by $X \mapsto i_X\omega$.

Definition 40. Let (M,ω) be a symplectic manifold. For any function $f \in C^\infty(M)$ we can define the ***Hamiltonian vector field*** X_f by

$$\mathrm{d}f(\cdot) = \omega(X_f, \cdot).$$

Definition 41. The ***Poisson bracket*** $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is given by

$$\{f,g\} = -\omega(X_f, X_g) = -\mathrm{d}g(X_f) = -X_f(g)$$

It is a Lie bracket on $C^\infty(M)$ and satisfies the Leibniz rule:

$$\{f,gh\} = \{f,g\}h + g\{f,h\}.$$

The map $C^\infty(M) \to C^\infty(M,TM)$ given by $f \mapsto X_f$ is an homomorphism of Lie algebras:

$$[X_f, X_g] = X_{\{f,g\}}.$$

Given a G-action on (M,ω) by symplectomorphisms, we call it ***Hamiltonian*** if there exist a ***moment map***: a smooth map $\mu : M \to \mathfrak{g}^*$, such that

1. $\underline{\xi} = X_{\mu(\xi)}$ for all $\xi \in \mathfrak{g}$.
2. μ is G-equivariant, with respect to the coadjoint action on $\mathfrak{g}^*$.

From property 2 it is clear that G preserves $\mu^{-1}(0)$.

Theorem 23 (Marsden-Weinstein). *Assume that μ is the moment map for the action of* G *on M and* 0 *is a regular value of μ, and* G *acts freely and properly on $\mu^{-1}(0)$. Then the two-form ω_{G} in $\mu^{-1}(0)/\mathrm{G}$ defined as $\omega_{\mathrm{G}}(\pi_*(A), \pi_*(B)) = \omega(A,B)$ for $A, B \in T_pM$ is well-defined and a symplectic form on $\mu^{-1}(0)/\mathrm{G}$. The symplectic manifold $(\mu^{-1}(0)/\mathrm{G}, \omega_{\mathrm{G}})$ is called the* **symplectic reduction** *of M with respect to the* G*-action, and denoted $M/\!\!/_\omega \mathrm{G}$.*

ω_{G} is the unique form such that $\pi^*(\omega_{\mathrm{G}}) = i^*(\omega)$, where $i : \mu^{-1}(0) \to M$. If furthermore M supports a complex structure compatible with ω and preserved by the G action, then the quotient will also support a complex structure, compatible with ω_{G}. We call the resulting Kähler manifold for the Kähler quotient.

4.2 Geometric quantization

The goal of quantization is to associate to a $2n$-dimensional symplectic manifold (M, ω) and $\hbar \in \mathbb{R}_+$ a complex Hilbert space $\mathcal{H}$ and a map $C^\infty(M) \to \mathrm{Op}(\mathcal{H})$, written as $f \mapsto \hat{f}$, such that

1. The map $f \mapsto \hat{f}$ is linear over $\mathbb{R}$.
2. $\hat{1} = \mathrm{id}$
3. $[\hat{f}_1, \hat{f}_2] = i\hbar\hat{f}$ for $f = \{f_1, f_2\}$.
4. if $(M, \omega) = (T^*\mathbb{R}^n, \mathrm{d}(\sum_{i=1}^n p_i \mathrm{d}q_i))$ then the obtained quantum system is "canonical quantization" of the harmonic oscillator:
   a) The quantum space is
   $$\mathcal{H} = L^2(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{C} \mid \int_{\mathbb{R}^n} |f(q_1, \dots, q_n)|^2 \mathrm{d}q_1 \dots \mathrm{d}q_n < \infty \right\}.$$
   b) $\hat{q}_i f = q_i f$
   c) $\hat{p}_i f = \mathrm{i}\hbar \frac{\partial}{\partial q_i} f$

Unfortunately, such a correspondence is not possible in general, as shown by the Groenewold-Van Hove no-go theorem. It is therefore necessary to relax the conditions in some way. One possibility is to be less ambitious and only aim to quantize a subalgebra of $C^\infty(M)$. Another is to define a family of quantizations, indexed by different values of $\hbar$, but relax the third criterion to $[\hat{f}_1, \hat{f}_2] = i\hbar\hat{f} + \mathrm{O}(\hbar^2)$.

A central notion in geometric quantization is that of a *prequantum line bundle*:

Definition 42. A ***prequantum line bundle*** $\mathcal{L} \to M$ is a complex line bundle with a connection ∇ and a hermitian structure $\langle \cdot, \cdot \rangle$ compatible with the connection, such that the curvature is $\mathrm{F}_\nabla = -i\omega$.

A prequantum line bundle exists if and only if $[\frac{\omega}{2\pi}] \in \mathrm{im}\left(\mathrm{H}^2(M, \mathbb{Z}) \to \mathrm{H}^2(M, \mathbb{R})\right)$. We can define an inner product on the vector space of compactly supported smooth sections of $\mathcal{L}$ by

$$\langle s_1, s_2 \rangle = \int_M \langle s_1, s_2 \rangle \frac{\omega^n}{n!}$$

and our first candidate for a Hilbert space quantizing (M, ω) will be the completion, $L^2(M, \mathcal{L})$. On this space we will let $\hat{f}$ act in the following way on a smooth section s

$$\hat{f}s = i\hbar\nabla_{X_f} s + fs.$$

A constant function f has $X_f = 0$, so it acts by multiplication: $\hat{f}s = fs$. The map $f \mapsto \hat{f}$ is $\mathbb{R}$-linear since $\nabla_{X_{af+bg}} = \nabla_{aX_f + bX_g} = a\nabla_{X_f} + b\nabla_{X_g}$. Similarly it can be shown that $\hat{f}$ is a symmetric operator and that condition 3 holds.

Condition 4 is however not satisfied. To remedy this, we must restrict our quantum space to a subset of $L^2(M,\mathcal{L})$ – right now the prequantum operator depend on too many variables (i.e. the prequantum operator for the momentum mapping could depend on the position as well, something that should not happen in the canonical quantization of the harmonic oscillator). There are different ways to do this. The basic idea is to consider the subset of sections of $\mathcal{L}$ that are constant (with respect to the prequantum connection) along vector fields taking values in a special n-dimensional distribution. There are different kinds of choices for such a distribution that leads to a satisfying theory. We will consider a special case where M is not only symplectic, but also Kähler, in which there always is a good choice, namely $T^{0,1}M$, which we will call the Kähler polarization associated to the Kähler structure. The quantum space of a Kähler manifold with its Kähler polarization is

$$\mathcal{Q}(M,\mathcal{L}) = \{s \in C^\infty(M,\mathcal{L}) \mid \forall X \in T^{0,1}M : \nabla_X s = 0\} = H^0(M,\mathcal{L}),$$

that is, the set of holomorphic sections of the line bundle, with respect to the complex structure defined by ∇ ($\nabla^{0,1}$ defines a complex structure because $\left(\nabla^{0,1}\right)^2 = (\mathrm{F}_\nabla))^{(0,2)} = (-\mathrm{i}\omega)^{(0,2)} = 0$, since ω is a Kähler form and therefore of type $(1,1)$). However, this introduces a new problem: The prequantum operators do not necessarily preserve $Q(M,\mathcal{L})$: let $Y \in T^{0,1}_{\mathbb{C}}M$ and $s \in Q(M,\mathcal{L})$

$$\begin{aligned}\nabla_Y \hat{f}s &= \nabla_Y i\nabla_{X_f}s + \nabla_Y fs = i\nabla_Y\nabla_{X_f}s + df(Y) + f\nabla_Y s\\ &= i(\nabla_{X_f}\nabla_Y - i\omega(Y,X_f) - \nabla_{[Y,X_f]})s + \omega(X_f,Y)s = -i\nabla_{[Y,X_f]}s.\end{aligned}$$

So in general we would have to restrict to functions f such that $[X_f, T^{0,1}_{\mathbb{C}}M] \subseteq T^{0,1}_{\mathbb{C}}M$, but such X_f preserve both the complex structure and the symplectic form, and therefore are Killing vector fields. For M compact this is a finite dimensional vector space corresponding to the Lie algebra of the Kähler isometries of M. However quantum operators for functions whose Hamiltonian vector field is Killing will be very useful for us later, where we will see that they provide a representation of the Lie algebra of the Kähler isomorphisms.

Another approach assumes that M is compact, and lets us quantize all functions. But instead of just one Hilbert space, we instead get a Hilbert space for each natural number k, and we only satisfy our requirements asymptotically in k, where we will think of k^{-1} as $\hbar$. The *level* k quantum space is defined as before, but with $\mathcal{L}$ replaced with $\mathcal{L}^{\otimes k}$ with the naturally induced connection and metric

$$\mathcal{Q}^{(k)}(M) = \{s \in C^\infty(M,\mathcal{L}^{\otimes k}) \mid \forall X \in T^{0,1}M : \nabla_X s = 0\} = H^0(M,\mathcal{L}^{\otimes k}),$$

The quantum space is a L^2-closed subspace of the smooth sections (as the $\bar\partial$ operator is elliptic), so we have a projection operator $\pi^k : \Gamma(M,\mathcal{L}^k) \to \mathcal{Q}(M,\mathcal{L}^k)$, and we simply define the level k quantum operator to be $\hat{f}^k s = \pi^k(\hat{f}s)$. Operators of the form $s \mapsto \pi(Ds)$ where D is a differential operator is called a Toeplitz operator. If D is of degree 0, we will denote by T^k_f the Toeplitz operator on section of $\mathcal{L}^k$ given by $s \mapsto \pi^k(fs)$. There is the following

Theorem 24 (Schlichenmaier).

$$\|[T^k_f, T^k_g] - \frac{i}{k}T^k_{\{f,g\}}\|_k = \mathrm{O}(k^{-2})$$

for $k \to \infty$. Here $\|\cdot\|_k$ is the operator norm on $\mathcal{Q}^{(k)}$.

The quantum operators $\hat{f}^k$ satisfy

Theorem 25 (Tuynman).

$$\hat{f}^k = T^k_{\frac{1}{2k}\Delta f + f}$$

Proof. Let s' be a holomorphic section and s a smooth section. Then

$$\mathcal{L}_{X^{1,0}}(h(s,s')\omega^n) = h(\nabla_{X^{1,0}} s, s')\omega^n + h(s,s')\mathrm{div} X^{1,0}\omega^n,$$

but the integral of the Lie derivative of a top form over a manifold without boundary is 0 (by Cartans formula and Stokes theorem), so

$$\langle \nabla_{X^{1,0}} s, s' \rangle = -\langle \mathrm{div} X^{1,0} s, s' \rangle,$$

therefore we have $T_{-\mathrm{div}X^{1,0}} = T_{\nabla_{X^{1,0}}}$. This shows that the quantum operator is a Toeplitz operator of order 0, and the theorem follows from a formula relating the divergence along the Hamiltonian vector field and the Laplace operator on Kähler manifolds. □

It follows that

$$[\hat{f}_1, \hat{f}_2] = \frac{i}{k}\widehat{\{f_1, f_2\}} + \mathrm{O}(k^{-2})$$

which is the relaxed version of 3.

4.2.1 Quantization and products

Let $(L_i, \nabla^i, \langle\cdot,\cdot\rangle_i)$ be prequantization data for the symplectic manifolds (M_i, ω_i). Let $p_i : \prod_{j=1}^n M_j \to M_i$ denote the projections. We can now form the line bundle $\boxtimes_{i=1}^n L_i = \bigotimes_{i=1}^n p_i^*(L_i)$, that comes with a connection $\otimes p_i^*(\nabla^i)$ and a hermitian structure $\otimes\langle\cdot,\cdot\rangle_i$, and this is prequantum data for $(\prod M_i, \sum p_i^*(\omega_i))$. Likewise, a Kähler structure on each M_i induce a product Kähler structure, and we get a Kähler polarization. As $p_i^*(L_i)$ restricted to $M_1 \times M_2 \ldots M_{i-1} \times \{p\} \times M_{i+1} \times \ldots M_n$ is the trivial bundle, the only holomorphic sections are the constants. Therefore a holomorphic section of $p_i^*(L_i)$ only depends on the projection to M_i. So we have

$$H^0(\prod_i M_i, \boxtimes_i L_i) = \bigotimes_i H^0(M_i, L_i).$$

4.2.2 Quantization and symmetries

If a Lie group G act on M in a Hamiltonian fashion with moment map $\mu : M \to \mathfrak{g}^*$, then the action of G induces an action of $\mathfrak{g}$ on the sections of a prequantum line bundle L by $\xi \mapsto \nabla_{\underline{\xi}} - i\mu(\xi)$: for recall that the pre-quantum operators obtained from geometric quantization satisfy $i\widehat{\{f,g\}} = [\hat{f}, \hat{g}]$, so if we define $\tilde{f} = -i\hat{f}$ we get that

$$\widetilde{\{f,g\}} = -[\hat{f}, \hat{g}] = [-i\hat{f}, -i\hat{g}] = [\tilde{f}, \tilde{g}].$$

Since the map $\mu : \mathfrak{g} \to (C^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ is a Lie algebra homomorphism, we obtain a representation of $\mathfrak{g}$ by

$$\xi \mapsto \widetilde{\mu(\xi)} = \nabla_{\underline{\xi}} - \mathrm{i}\mu(\xi).$$

We say that the prequantum data is G-invariant if this action is induced from an action of G on L. An important example is that of a coadjoint orbit of $\mathcal{O}_\lambda = \mathrm{G}\cdot\lambda \subseteq \mathfrak{g}^*$ a compact Lie group G. The coadjoint orbits are symplectic with the KKS symplectic form $\omega_\mu : T_\mu\mathcal{O}_\lambda \times T_\mu\mathcal{O}_\lambda \to \mathbb{R}$

given by $\omega_\mu(\underline{X}, \underline{Y}) = \mu([X, Y])$. The coadjoint orbits also support a natural complex structure compatible with ω that are preserved under the action of G. Therefore the fundamental vector fields of the action preserves the Kähler polarization, and the prequantum operators acts also on the quantum space of holomorphic sections. If λ is an integral weight, then $[\omega] \in \mathrm{H}^2(\mathcal{O}, \mathbb{R})$ is integral and we can apply geometric quantization to $\mathcal{O}_\lambda$. It follows from the Borel-Weil theorem [59] that if λ is dominant, then the resulting action of $\mathfrak{g}$ integrates to G and is exactly given by the irreducible representation corresponding to λ.

The following important theorem is an instance of a general principle that "reduction commutes with quantization":

Theorem 26 (Guillemin and Sternberg, [36]). *Let L be a* G*-invariant prequantum line bundle for (M, ω), with moment map μ, such that* 0 *is a regular value of μ and* G *acts freely on $\mu^{-1}(0)$. Then the line bundle $L_G := L_{\restriction \mu^{-1}(0)}/\mathrm{G} \to M /\!\!/_\omega G$ has a connection ∇_G such that $i^*(L, \nabla) = \pi^*(L_\mathrm{G}, \nabla_\mathrm{G})$. Furthermore, the curvature satisfies $F_{\nabla_\mathrm{G}} = -\mathrm{i}\omega_\mathrm{G}$, where ω_G the symplectic form of the symplectic reduction. Furthermore, a* G*-invariant Kähler polarization (this is the case if M is Kähler and G acts by biholomorphisms) induces a complex structure on $M /\!\!/_\omega \mathrm{G}$ compatible with ω_G, and the map*

$$H^0(M, L)^\mathrm{G} \to H^0(M /\!\!/_\omega \mathrm{G}, L_\mathrm{G})$$

given by

$$s \mapsto ([x] \mapsto [s(x)])$$

is an isomorphism of vector spaces.

Chapter 5

Moduli spaces

5.1 Moduli spaces of flat connections

Let G be a Lie group and $\pi : P \to \Sigma$ be a principal G-bundle over a manifold Σ.

Definition 43. A connection A in P is a $\mathfrak{g}$-valued 1-form on P satisfying

1. $A(\underline{\xi}) = \xi$,
2. $R_g^*(A) = \mathrm{Ad}_{g^{-1}} A$ for all $g \in \mathrm{G}$.

where $\underline{\xi}$ is the fundamental vector field of $\xi \in \mathfrak{g}$, and $R_g : P \to P$ is the map $p \mapsto pg$.

The kernel of A defines a G-invariant Ehresmann connection, and every G-invariant Ehresmann connection defines a connection in P.

If $\rho : \mathrm{G} \to \mathrm{GL}(V)$ is a representation of G, the associated bundle $(P \times V)/\mathrm{G}$ is a vector bundle over Σ consisting of equivalence classes $[p, v]$ with $p \in P$ and $v \in V$, where the equivalence relation is given by $(p, v) \sim (pg, vg) = (pg, \rho(g^{-1})v)$, and the projection $\pi : P \times V/\mathrm{G} \to \Sigma$ is given by $\pi([p, v]) = \pi(p)$. Let $\mathrm{Ad}_P\, \mathfrak{g} = P \times_{\mathrm{G}} \mathfrak{g}$ be the vector bundle associated to P and the representation Ad of G on $\mathfrak{g}$. The sections of $\mathrm{Ad}\, P$ can be identified with the G-equivariant maps $f : P \to \mathfrak{g}$. A induces an affine connection ∇^A in this bundle:

$$(\nabla^A_X f) = \tilde{X}^A f$$

where $\tilde{X}^A$ denotes the lift of $X_{\pi(p)} \in T_{\pi(p)}\Sigma$ to $T_p P$, using A.

Notice that if given any lift X^0 of X, then $\tilde{X}^A = X^0 - \underline{A(X^0)}$, for this is a lift of X and $A(X^0 - \underline{A(X^0)}) = A(X^0) - A(X^0) = 0$. So $\nabla^A s = ds + [A, s]$.

There is a unique map d_A

$$\Omega^i(\Sigma, \mathrm{Ad}\, P) \xrightarrow{\mathrm{d}_A} \Omega^{i+1}(\Sigma, \mathrm{Ad}\, P) \tag{5.1}$$

satisfying $\mathrm{d}_A(\alpha \otimes \beta) = (\mathrm{d}_A \alpha) \wedge \beta + (-1)^r \alpha \wedge d\beta$, when $\alpha \in \Omega^r(\Sigma, \mathrm{Ad}\, P)$ and $\beta \in \Omega^i(\Sigma)$, and such that $\mathrm{d}_A = \nabla^A$ on $\Omega^0(\Sigma, \mathrm{Ad}\, P)$. It is easy to see that $\omega \mapsto \mathrm{d}\omega + [A, \omega]$ satisfies this for the connection A, and thus $\mathrm{d}_A = d + A$. The ***curvature*** of ∇^A is given by $\mathrm{F}_{\nabla^A} = \nabla^A \circ \nabla^A \in \Omega^2(\Sigma, \mathrm{End}(\mathrm{Ad}\, P))$.

The calculation

$$\mathrm{d}_A \mathrm{d}_A s = \mathrm{d}_A(\mathrm{d}s + [A, s]) = (\mathrm{dd}s + [\mathrm{d}A, s] - [A, \mathrm{d}s] + [A, \mathrm{d}s] + [A, [A, s]])$$

$$= [\mathrm{d}A + \frac{1}{2}[A,A], s] = (\mathrm{d}A + \frac{1}{2}[A,A])s$$

gives us the following formula for the curvature:

$$\mathrm{F}_A = dA + \frac{1}{2}[A,A].$$

Definition 44. A connection A is ***flat*** if $\mathrm{F}_{\nabla^A} = 0$. We denote the space of all connections in P by $\mathcal{A}_P$ and the subset of flat connections by $\mathcal{A}_0$.

Proposition 45. *Let $\gamma_i : [0,1] \to \Sigma$ be smooth curves for $i = 0, 1$. If $\mathrm{F}_{\nabla^A} = 0$ and γ_0 is homotopic to γ_1 relative the endpoints, then $\mathrm{PT}^A_{\gamma_0} = \mathrm{PT}^A_{\gamma_1}$.*

This proposition makes the following definition possible:

Definition 46. If A is flat, we can define a homomorphism $\mathrm{hol}_A : \pi_1(\Sigma, x_0) \to \mathrm{G}$ by

$$\mathrm{hol}_A([\gamma]) = \mathrm{PT}_\gamma(p).$$

Here PT^A_γ denotes parallel transport along γ with respect to A. If γ is a loop on Σ and $\tilde{\gamma}$ is a horizontal lift to P, then we get a horizontal lift $\hat{\gamma}_v$ in $P \times_\rho V$ with $\hat{\gamma}_v(0) = [\tilde{\gamma}(0), v]$ by $t \mapsto [\tilde{\gamma}(t), v]$. Then $\hat{\gamma}_v(1) = [\tilde{\gamma}(1), v] = [\tilde{\gamma}(0)\,\mathrm{PT}_\gamma, v] = [\tilde{\gamma}(0), v\,\mathrm{PT}_\gamma^{-1}] = [\tilde{\gamma}(0), \rho(\mathrm{PT}_\gamma)v]$. This shows that the holonomy of the induced connection is $\rho \circ \mathrm{hol}$.

Given two connections A, B, their difference is a $\mathfrak{g}$-valued 1-form on P satisfying $(A - B)(\underline{\xi}) = \xi - \xi = 0$ and $R_g^*(A - B) = \mathrm{Ad}_{g^{-1}}(A - B)$. On the other hand, if $\dot{A}$ is a $\mathfrak{g}$-valued 1-form such that $\dot{A}(\underline{\xi}) = 0$ and $R_g^*(\dot{A}) = \mathrm{Ad}_{g^{-1}} \dot{A}$ then $A + \dot{A}$ is again a connection. This shows that the space of connections is affine, modelled on the space $\mathrm{Ad}_P\,\mathfrak{g}$-valued 1-forms on Σ.

The automorphism group $\mathcal{G}_P$ of P acts on the space of connections in P, $\mathcal{A}_P$, by pull-back. We have the following formulas, for $g \in \mathrm{Aut}(P)$

$$\begin{aligned} Ag = g^*(A) &= \mathrm{Ad}_{\varphi_g^{-1}} A + \varphi_g^*(\theta), \\ \mathrm{F}_{\nabla^{Ag}} = g^*(\mathrm{F}_{\nabla^A}) &= \mathrm{Ad}_{\varphi_g^{-1}}(\mathrm{F}_{\nabla^A}), \end{aligned}$$

where θ is the Maurer-Cartan one-form on G and $\varphi_g : P \to \mathrm{G}$ is defined by $g(p) = p\varphi_g(p)$. We notice that the flat connections are preserved, and we define the moduli space of flat connections on Σ to be the quotient of the space of flat connections in P by the gauge group:

$$\mathcal{M}^{\mathrm{Flat}}(\Sigma) = \mathcal{A}_0(P)/\mathcal{G}_P.$$

Given a $x_0 \in \Sigma$, we consider the map $\mathrm{hol} : \mathcal{A}_0(P) \to \mathrm{Hom}(\pi_1(\Sigma, x_0), \mathrm{G})$. The effect of a gauge transform on the holonomy is a conjugation:

$$\mathrm{hol}_{Ag}(h) = \varphi_g(x_0)\,\mathrm{hol}_A(h)\varphi_g(x_0)^{-1}.$$

Therefore we have a map

$$\mathrm{hol} : \mathcal{M}^{\mathrm{Flat}}(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma, x_0), \mathrm{G})/\mathrm{G}$$

where G acts on homomorphisms by conjugation. We can forget about the base point x_0, as change of basepoint corresponds to conjugation. We will now assume that Σ is a surface and G is connected and simply-connected. Then there is, up to isomorphism, only one principal G bundle on Σ (as π_2 of a topological group always vanishes, and therefore $B\mathrm{G}$ is 3-connected, which implies that there is unique homotopy class of maps $\Sigma \to B\mathrm{G}$). We define

$$\mathcal{R}(\Sigma, \mathrm{G}) = \mathrm{Hom}(\pi_1(\Sigma, x_0), \mathrm{G})/\mathrm{G}$$

We can give $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{G})$ and $\mathcal{R}(\Sigma, \mathrm{G})$ a topology (and if G is a real/complex matrix group, the structure of a real/complex algebraic variety) by using the identification $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{G}) \subseteq \mathrm{G}^k$, given by fixing k generators $\gamma_1, \ldots, \gamma_k$ of $\pi_1(\varphi)$ and uses the embedding $\rho \mapsto (\rho(\gamma_1), \rho(\gamma_2), \ldots, \rho(\gamma_k))$. The resulting topology does not depend on the chosen system of generators. The structure of a variety come from the fact that any relations on $\gamma_1, \ldots, \gamma_k$ are expressed as polynomials in the coordinates of G.

A morphism $\pi_1(\Sigma) \to \mathrm{G}$ is called reducible if the centralizer of the image is larger than the center of G. The *irreducible* homomorphisms form an open and dense subset of all homomorphisms (if they exists at all; for $\mathrm{G} = \mathrm{SU}(2)$ they do exist precisely when the Euler characteristic of Σ is negative), and by I.2.5 in [14], $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), \mathrm{G})/\mathrm{G}$ is a smooth manifold when G is compact. A flat connection is irreducible if its holonomy is an irreducible representation. We have the following well-known

Theorem 27. *The map* hol *is a homeomorphism. Its restriction to the set of equivalence classes of irreducible connections is a diffeomorphism onto the set of equivalence classes of irreducible representations.*

The inverse to the holonomy map is as follows: given a representation ρ of π_1 in G we can construct a G-bundle by $\tilde{\Sigma} \times G/\pi_1(\Sigma)$ and the trivial connection on $\tilde{\Sigma} \times G$ descend to a flat connection A with holonomy ρ. Since there is only one isomorphism-class of G-bundles on Σ, there is a bundle isomorphism to P, and the image of A is a flat connection with holonomy conjugate to ρ.

Definition 47. We will denote by $\mathcal{M}^{\mathrm{Flat}}(\Sigma)$, respectively $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma)$ the moduli space of flat connections, respectively flat irreducible connection. If Σ have punctures $\mathrm{D} = \{p_i \mid i = 1, \ldots, n\}$ we will consider the moduli space of flat connections with restricted holonomy around the punctures. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let $\lambda_i \in \mathfrak{h}$ for $i = 1, \ldots, n$. Then we denote by $\mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k)$ the moduli space of flat connections on $\Sigma \setminus \{p_i\}$ such that the holonomy around a loop contracting to p_i is in the conjugacy class of $\exp \frac{\lambda_i}{k}$, and by $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$ the subset consisting of irreducible connections.

This parametrisation of conjugacy classes might seem odd at the moment, but it will be convenient later. For SU(2) this means that the holonomy around p_i have trace $2\cos(\frac{2\pi\lambda_i}{k})$. In terms of representations of the fundamental group, we have that if $\Sigma_{g,n}$ is a surface of genus g with n punctures, then $\pi_1(\Sigma_{g,n}) = \langle \gamma_1, \ldots, \gamma_{2g}, c_1, \ldots, c_n \mid \prod_{i=1}^n c_i \prod_{i=1}^g [\gamma_{2i-1}, \gamma_{2i}] = \mathrm{id} \rangle$ where c_i is a curve homotopic to the i'th puncture. Then, if $C_i \in \mathrm{G}/\mathrm{G}$ is the conjugacy class containing $\mathrm{e}^{\frac{\lambda_i}{k}}$, the holonomy gives an homeomorphism

$$\mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}), \mathrm{G}) \mid \rho(c_i) \in C_i\}/\mathrm{G}.$$

that restricts to a diffeomorphism on $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$ to the irreducible representations.

5.2 Complex structures on moduli spaces

In this section we review the construction of complex structures on $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$, induced by a complex structure on Σ. The arguments will be formal, but can be made precise using the theory of infinite dimensional Banach manifolds and Sobolev connections. We want to describe a natural way to obtain complex structures on moduli spaces of irreducible flat connections over a surface, using a complex structure on the surface. First, we recall that the connections in P is an affine space, modeled on $\Omega^1(\Sigma, \mathrm{Ad}_P\,\mathfrak{g})$, so the tangent space at a connection A can be naturally identified with $\Omega^1(\Sigma, \mathrm{Ad}_P\,\mathfrak{g})$. A tangent vector $\dot{A} \in T_A\mathcal{A}$ is in $T_A\mathcal{A}_0$ if (where we will denote ∇^A by d_A)

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}_{t=0}\mathrm{F}_{A+t\dot{A}} = \frac{\mathrm{d}}{\mathrm{d}t}_{t=0}(\mathrm{d}(A+t\dot{A}) + \frac{1}{2}[A+t\dot{A}, A+t\dot{A}]) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}_{t=0}(\mathrm{d}A + \frac{1}{2}[A,A]) + t(\mathrm{d}\dot{A} + \frac{1}{2}[A,\dot{A}]) + \frac{1}{2}t^2[\dot{A},\dot{A}] = \mathrm{d}\dot{A} + [A,\dot{A}] \\ &= \mathrm{d}_A\dot{A}.\end{aligned}$$

The gauge group is identified with $\Omega^0(\Sigma, \mathrm{Ad}_P\, G)$, and its Lie algebra with $\Omega^0(\Sigma, \mathrm{Ad}\,\mathfrak{g})$. In a similar way, we can show that the fundamental vector field of the gauge group is given by

$$\underline{\xi}_A = d_A\xi.$$

This means that we can identify the tangent space at $[A]$ as the quotient of the tangent spaces of the flat connection and the $\mathcal{G}$-orbit: $T_{[A]}\mathcal{M}^{\mathrm{Flat}}(\Sigma) \cong \ker d_A/\operatorname{Im} d_A = H^1(\Sigma, \mathrm{d}_A)$, the first cohomology of the complex (5.1).

Let us first give the construction in the case where Σ have no marked points. The idea is to use a Hodge star $*$ on the Σ to define a map $* : \Omega^1(\Sigma, \mathrm{Ad}_P\,\mathfrak{g}) \to \Omega^1(\Sigma, \mathrm{Ad}_P\,\mathfrak{g})$ that squares to -1: $*^2 = -1$. This defines an almost complex structure on the set of connections $\mathcal{A}$, but we can not just define an almost complex structure on the moduli space by applying this to an arbitrary representative, since $- * \dot{A}$ is not necessarily d_A-closed. Instead we can use a theorem of Hodge theory, that allow us to represent $[\dot{A}]$ by a (unique) harmonic 1-form α, that is, $(\mathrm{d}_A + *\mathrm{d}_A*)\alpha = 0$. But then $*\mathrm{d}_A * \alpha = 0$ so $*\alpha$ is d_A closed. We can now define $I([\dot{A}]) = [- * \alpha]$. As any map $\varphi : \Sigma \to \Sigma$ homotopic to the identity induces the identity on $H^1_A(\Sigma, \mathrm{d}_A)$, this complex structure does not depend of the isotopy class of $*$.

When Σ have marked points, we are working with a non-compact surface and therefore standard Hodge theory does not apply. However, the holonomy condition serves as a boundary condition, and instead we can use Hodge theory with exponential decay towards the punctures, as treated in [**andersen:npotmsattcots**]. First we need another model for the space of connections. Let $\alpha_i \in \mathfrak{su}(2)$ for $i = 1, 2, \ldots, n$. We will consider weighted Sobolev spaces: fix a smooth function d on $\Sigma \setminus \{p_1, \ldots, p_n\}$ such that for each i there is a trivialization of a neighbourhood N_i of p_i to $S^1 \times [0, \infty)$ and $\mathrm{d}(\theta, r) = r$, and d is 0 outside of these neighbourhoods. We let ∇ be some base connection in a trivial SU(2)-bundle P on all of Σ. We can now consider the norms:

$$\|\varphi\|^2_{\varepsilon,k} = \int_\Sigma \sum_{0\leq l\leq k} |\nabla^l e^{\varepsilon d}\varphi|^2,$$

on sections of $\Omega^i(\Sigma \setminus \{p_1, \ldots, p_n\})$ with compact support, and define the weighted Sobolev spaces $L^2_{\varepsilon,k}(E)$ to be the completion with respect to this norm. Here $\varepsilon > 0$ is a small real number, and for the rest of this section we will assume that it is small enough. For more details we again refer to [**andersen:npotmsattcots**]. The moduli space is given as

$$\frac{\{A \in \mathcal{A} \mid \mathrm{F}_A = 0, \mathrm{hol}_A(\partial_i\Sigma) \in C_i\}}{\mathcal{G}},$$

where $\mathrm{hol}_A(\partial_i\Sigma)$ means the holonomy around p_i. We can spend some gauge symmetry to bring the connections into temporal gauge around the punctures

$$\cong \frac{\{A \in \mathcal{A} \mid \mathrm{F}_A = 0, A_{|N_i} = A_i\}}{\{g \in \mathcal{G} \mid g^*_{|N_i} A_i = A_i\}},$$

where $A_i = \alpha_i \mathrm{d}\theta_i$. We can instead of smooth sections use sections of the Sobolev spaces

$$\cong \frac{\{A \in \Omega^1_{k,\varepsilon}(\Sigma, \mathrm{Ad}_P\,\mathfrak{g}) \mid \mathrm{F}_A = 0, \|A_{|N_i} - A_i\|_{k,\varepsilon} < \infty\}}{\{g \in \Omega^0_{k+1,\varepsilon}(\Sigma, \mathrm{Ad}_P\,\mathfrak{g}) \mid \|g^*_{|N_i} A_i - A_i\| < \infty\}}.$$

To find the tangent space, we see that the tangent space of the numerator is $\dot{A} \in \Omega^1_{k-1,\varepsilon}(\Sigma, \mathrm{Ad}_P\,\mathfrak{g})$ and $\mathrm{d}_A\dot{A} = 0$. For the denominator we get $A_i g_{|N_i}(t) = A_i$ so $\mathrm{d}_{A_i}\dot{g} = 0$. Therefore we can identify the tangent space to this subgroup of the gauge group with

$$\Omega^0_{\varepsilon,k+2,\infty}(\Sigma, \mathrm{Ad}_P\,\mathfrak{g}) = \{f \in \Omega_{L^2_{\mathrm{loc}}}(\Sigma, \mathrm{Ad}_P\,\mathfrak{g}) \mid \exists f_\infty \in H^0(\partial\Sigma, \mathrm{d}_{A_i}) : \|f - \rho f_\infty\|_{k+2,\varepsilon} < \infty\},$$

where ρ is a function that takes values in $[0,1]$ and is 1 outside a compact set. We therefore get an identification

$$T_{[A]}\mathcal{M}^{\mathrm{Flat}}(\Sigma) \cong H^1_{k,\varepsilon}(\Sigma, \mathrm{d}_A),$$

where the cohomology is of the complex

$$0 \to \Omega^0_{k+1,\varepsilon,\infty} \xrightarrow{\mathrm{d}_A} \Omega^1_{k,\varepsilon} \xrightarrow{\mathrm{d}_A} \Omega^2_{k-1,\varepsilon,\infty} \to 0.$$

The Hodge theory of this complex was treated in [4], where it was shown that:

Lemma 48 (Andersen). *Any element of $H^1_{\varepsilon,k}(\Sigma, \mathrm{d}_A)$ can be represented by a d_A-closed one form with compact support.*

And

Theorem 28 (Andersen). *Each cohomology class has a unique harmonic representative: there is a natural isomorphism*

$$H^1_{\varepsilon,k}(\Sigma, \mathrm{d}_A) \cong \ker(\mathrm{d}_A + \mathrm{d}_A^*)$$

where d_A^* is the formal adjoint of d_A with respect to the $L^2_{k,\varepsilon}$-inner product.

As before, this allow us to define a complex structure by using $-*$ on the harmonic representative, which give us a complex structure on the moduli space $\mathcal{M}(\Sigma, \vec{\lambda}, k)$, and we will write $\mathcal{M}(\Sigma, \vec{\lambda}, k)_\sigma$ for the moduli space equipped with this complex structure.

5.2.1 The symplectic form

The space of connections on a closed surface Σ have the following natural symplectic structure, where we again identify the tangent space as $T_A\mathcal{A} \cong \Omega^1(\Sigma, \mathrm{ad}\,\mathfrak{g})$:

$$\omega(\dot{A}, \dot{B}) = -\int \mathrm{Tr}(\dot{A} \wedge \dot{B}).$$

It is closed as there are no dependence on the point in $\mathcal{A}$, and it is non-degenerate because $\omega(\dot{A}, -\star\dot{A}) > 0$ for $\dot{A} \neq 0$. Atiyah and Bott noticed that the curvature map $\mathrm{F} : \mathcal{A} \to \Omega^2(\Sigma, \mathrm{ad}\,\mathfrak{g}) \cong \Omega^0(\Sigma, \mathrm{ad}\,\mathfrak{g})^* = (\mathcal{G})^*$ is a moment map for the action of $\mathcal{G}$. The symplectic reduction turns out to also hold in this infinite-dimensional setting, and they introduced the following symplectic form on the moduli space of flat connections:

$$\omega([\dot{A}],[\dot{B}]) = -\int_\Sigma \mathrm{Tr}(\dot{A}\wedge\dot{B}),$$

which is formally the result of the symplectic reduction of $\mathcal{A}$ by $\mathcal{G}$. Using lemma 48 we can define the symplectic form for $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k)$ by the same formula, by choosing representatives with compact support. It is easy to see that the symplectic form is compatible with the induced complex structure from any Hodge star $\star$ on Σ, and therefore $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\lambda,k)$ have a Kähler structure for each $\sigma\in\mathcal{T}(\Sigma)$.

5.2.2 The mapping class group action

The mapping class group $\mathrm{MCG}(\Sigma,\vec{\lambda})\subseteq\mathrm{MCG}(\Sigma)$ of surface with labeled marked points $(\Sigma,\vec{\lambda})$ is the subgroup of $\mathrm{MCG}(\Sigma)$ such that if the marked point p_i is mapped to the marked point p_j, then $\lambda_i=\lambda_j$. $\mathrm{MCG}(\Sigma,\vec{\lambda})$ acts on $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k)$ from the right. In the language of connections the action can be described by the pullback, and in terms of representations, by precomposing with the induced map $\varphi_*:\pi_1(\Sigma)\to\pi_1(\Sigma)$ for $\varphi\in\mathrm{MCG}(\Sigma,\vec{\lambda})$.

Theorem 29. *The action of* $\mathrm{MCG}(\Sigma,\vec{\lambda})$ *on* $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k)$ *preserves the symplectic form, and furthermore, if* $\sigma\in\mathcal{T}$ *and* $\varphi\in\mathrm{MCG}(\Sigma,\vec{\lambda})$*, then:*

$$\varphi:(\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k),\omega_{\vec{\lambda},k},I_\sigma)\to(\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k),\omega_{\vec{\lambda},k},I_{\sigma\varphi})$$

is a Kähler isomorphism.

Theorem 30 ([5]). *If the conjugacy classes corresponding to* $\frac{\vec{\lambda}}{k}$ *are generic, then the only mapping class elements that act trivially on* $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k)$ *are the ones preserving all simple closed curves. Therefore only the central elements of* $\mathrm{MCG}(\Sigma,\vec{\lambda})$ *acts trivially on* $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k)$*.*

5.2.3 Prequantum line bundle

For a closed surface, it was shown in [55] and in [29] how to construct a prequantum line bundle for $(\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma),\omega)$, the so-called ***Chern-Simons prequantum line bundle***, constructed as the quotient line bundle for a cocycle for the gauge group action on $\mathcal{A}_0$, where the cocycle is defined in terms of the Chern-Simons functional. For $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k)$ the following theorem holds, see [12] and the references therein.

Theorem 31. *If* $\vec{\lambda}$ *is in the co-root lattice then there exist a line bundle* $\mathcal{L}^{\mathrm{CS}}{}_{\vec{\lambda},k}\to\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k)$ *with a connection* ∇^{CS} *and hermitian structure, such that* $\mathrm{F}_{\nabla^{\mathrm{CS}}}=-\mathrm{i}k\omega_{\vec{\lambda},k}$*. Furthermore, the Chern-Simons line bundle is natural in the sense that the mapping class group action lifts to* $\mathcal{L}^{\mathrm{CS}}{}_{\vec{\lambda},k}$ *and preserves* ∇^{CS} *and the hermitian structure.*

5.3 Teichmüller space

The complex structures on the moduli space defined in section 5.2 does only depend on the conformal class of the metric. The conformal classes of metrics, up to isotopy, on a surface is parametrized in a very nice way by the Teichmüller space. If we fix a volume form ω on Σ, and I is an almost complex structure, then $\langle v,w\rangle=\omega(v,I(w))$ defines a Riemannian metric on Σ, and this map is surjective onto the set of conformal classes of metrics, since any conformal class defines an almost complex structure by rotation $\frac{\pi}{2}$ in the positive direction determined by ω. Furthermore, any almost complex structure on a surface is a proper complex structure.

Definition 49. The ***Teichmüller space*** of a surface Σ consists of pairs (φ, X) where $\varphi : \Sigma \to X$ is a diffeomorphism, and X is a complex surface, modulo the relation that $(\varphi_1, X_1) \sim (\varphi_2, X_2)$ if there exist a biholomorphism $\Phi : X_1 \to X_2$ such that $\varphi_2^{-1} \circ \Phi \circ \varphi_1$ is isotopic to the identity. If Σ has marked points, we require that $\varphi_2^{-1} \circ \Phi \circ \varphi_1$ is isotopic to the identity through maps that preserve that marked points, and if Σ has marked points with chosen tangent vectors, we require that the isotopy respects the tangent vectors.

We will need the two following important facts about Teichmüller spaces

Theorem 32. *The Teichmüller space of a surface has the structure of a complex manifold, and is homeomorphic to a ball. A diffeomorphism $f : \Sigma_1 \to \Sigma_2$ (preserving marked points and tangent vectors) induces a map $\tilde{f}$ from the Teichmüller space of Σ_2 to the Teichmüller space of Σ_1. This map does not depend on the isotopy class of f, and is in fact a biholomorphism. If $f_3 = f_2 \circ f_1$ then $\tilde{f}_3 = \tilde{f}_2 \circ \tilde{f}_1$.*

Theorem 33 (Douady and Earle, [25]). *The Teichmüller space of a surface is contractible*

Theorem 34 (Bers, Ehrenpreis [17]). *The Teichmüller space of a surface is a Stein manifold*

In fact, there are the following explicit coordinates on the Teichmüller space, known as Fenchel-Nielsen coordinates. Take a pair of pants decomposition of the surface. Then there is exactly one hyperbolic structure on the pair of pants with any set of lengths on the three boundary curves. The first half of the coordinates measures these lengths, around each curve in the pair of pant decomposition. The last half measures the ($\mathbb{R}$-lift of the) angle with which the pants have been glued together.

The genus zero situation $(S^2, p_1, p_2, \ldots, p_n)$ is quite simple. If $\psi : X \to Y$ is a biholomorphism, we always have that $(\varphi, X) \sim (\psi \circ \varphi, Y)$, since ψ gives the required map. Since all complex surfaces diffeomorphic to S^2 are biholomorphic to $\mathbb{CP}^1$, we can represent all equivalence classes with an element of the form $(\varphi, \mathbb{CP}^1)$, and by applying an automorphism to $\mathbb{CP}^1$ we can even assume that $\varphi(p_1) = 0, \varphi(p_2) = 1$ and $\varphi(p_3) = \infty$. But two such pairs $(\varphi_i, \mathbb{CP}^1)$ are equivalent only if $\varphi_1(p_i) = \varphi_2(p_i)$ for all i, and if this is satisfied, the element in the mapping class group given by $\varphi_2^{-1} \circ \varphi_1$ determines if they are equal.

Proposition 50. *The Teichmüller space of a punctured sphere with $n \geq 4$ punctures is the universal cover of a configuration space,* $\mathrm{U}\left(\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0, 1, \infty\})\right)$.

Proof. Choose a base point $\varphi_0 : (S^2, p_1, \ldots, p_n) \to (\mathbb{CP}^1, 2, 3, \ldots, n-2, 0, 1, \infty)$ in $\mathcal{T}(S^2, p_1, \ldots, p_n)$ Given another diffeomorphism $\varphi : S^2 \to \mathbb{CP}^1$ we can find and isotopy h_t from id to $\varphi_0 \circ \varphi^{-1}$, and by applying a family ψ_t of $\mathrm{PSL}(2, \mathbb{C})$ transformations to $\mathbb{CP}^1$, we can assume that this isotopy preserves $0, 1, \infty$. The space $\mathrm{U}\left(\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0, 1, \infty\})\right)$ has as points paths starting at $(2, 3, \ldots, n-2, 0, 1, \infty)$ modulo homotopy preserving end points. The path associated to φ is then given by $t \mapsto (h_t(2), h_t(3), \ldots, h_t(n-2))$. The inverse map is constructed in the same was as the point-pushing map in [27, section 4.6] – notice that $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ has negative Euler characteristic. $\square$

We see that the Teichmüller space of a sphere with $n \geq 4$ punctures has $2(n-3)$ real dimensions.

5.4 Parabolic bundles

Let Σ be a Riemann surface and $p_1, \ldots, p_n \in \Sigma$, $E \to \Sigma$ a complex vector bundle of rank r, and for each p_i, let $(d_i^1, d_i^2, \ldots d_i^{s_i-1})$ be a sequence of positive integers. A ***parabolic vector bundle*** $\mathbb{E}$ associated to this data is a holomorphic structure on E together with, for each p_i, a filtration $0 = E_{p_i}^{s_i} \subset E_{p_i}^{s_i-1} \subset \cdots \subset E_{p_i}^1 = E_{p_i}$, such that $\dim E_{p_i}^j / E_{p_i}^{j+1} = d_i^j$. We now fix numbers $w_i^j \in \mathbb{R}$, $0 \leq j \leq s_i$ such that $w_i^j < w_i^{j+1}$. We define the parabolic degree of a parabolic vector bundle $\mathbb{E}$ to be

$$\operatorname{pdeg} \mathbb{E} = \deg E + \sum_{i,j} w_i^j d_j^i$$

and the slope to be

$$\mu_{\mathrm{p}}(\mathbb{E}) = \frac{\operatorname{pdeg} \mathbb{E}}{\operatorname{rank} \mathbb{E}}.$$

We remark that pdeg depends only on the data (E, d, w), so we will sometimes write $\operatorname{pdeg}(E, d, w)$ for this number. A parabolic morphism $g : \mathbb{E} \to \mathbb{F}$ is a morphism of vector bundles, such that $g_{p_i}(E_{p_i}^j) \subseteq F_{p_i}^{k+1}$ if $w_i^j(\mathbb{E}) > w_i^k(\mathbb{F})$. A holomorphic subbundle $\mathbb{F} \subseteq \mathbb{E}$ is a ***parabolic subbundle*** if the filtration of $\mathbb{F}$ over p_i is the filtration $F_{p_i} \cap E_{p_i}$ with repeated terms deleted, and $w_i^j(\mathbb{F}) = w_i^k(\mathbb{E})$ where k is the largest number such that $F_{p_i}^j \subseteq E_{p_i}^k$. Notice that this also give any subbundle of $\mathbb{E}$ the structure of a parabolic subbundle.

If $f : \mathbb{E} \to \mathbb{F}$ we say that $\mathbb{F}$ is a ***parabolic quotient bundle*** if f is surjective and for all p_i and $1 \leq j \leq s_i(\mathbb{F})$ there exist $1 \leq k \leq s_i(\mathbb{E})$ such that $f_{p_i}(E_{p_i}^k) = F_{p_i}^j$, and, if k is the largest number with this property, $w_i^j(\mathbb{F}) = w_i^k(\mathbb{E})$.

The ***direct sum of two parabolic vector bundles*** $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$ is the direct sum of the vector bundles with weights the union of the weights of $\mathbb{E}_1$ and $\mathbb{E}_2$, and the filtration $E_{p_i}^j = {E_1}_{p_i}^k \oplus {E_2}_{p_i}^l$ where j and k are maximal such that $w_{p_i}^j(\mathbb{E}) \leq w_{p_i}^k(\mathbb{E}_1)$ and $w_{p_i}^j(\mathbb{E}) \leq w_{p_i}^l(\mathbb{E}_2)$.

We say that a parabolic vector bundle $\mathbb{E}$ is ***stable*** with respect to w if for all parabolic subbundles $\mathbb{F} \subset \mathbb{E}$ we have that $\mu_{\mathrm{p}}(\mathbb{F}) < \mu_{\mathrm{p}}(\mathbb{E})$, and ***semistable*** if we for all parabolic subbundles have $\mu_{\mathrm{p}}(\mathbb{F}) \leq \mu_{\mathrm{p}}(\mathbb{E})$, and we call semistable bundles that are not stable for ***strictly semistable***.

A ***parabolic endomorphism*** is an endomorphism of $\mathbb{E}$ preserving the flags, and a ***strongly parabolic endomorphism*** is an endomorphism φ such that $\varphi(E_i) \subseteq E_{i+1}$. We denote trace 0 such by PEnd_0 and SPEnd_0, respectively. The map $(\alpha, \beta) \mapsto \frac{1}{2\pi \mathrm{i}} \operatorname{Tr}(\alpha\beta)$ defines a non-degenerate pairing between $\mathrm{SPEnd}_0(D)$ and PEnd_0 [60], where D is the divisor of parabolic points. Therefore $\mathrm{SPEnd}_0(\mathbb{E})(D)^* \cong \mathrm{PEnd}_0(\mathbb{E})$.

5.5 Moduli space of parabolic bundles

The moduli space $\mathcal{M}^{\mathrm{Par}}(\Sigma, p_1, \ldots, p_n, d_i^j, w_i^j)$ of parabolic bundles over $(\Sigma, p_1, \ldots, p_n)$ with given flag type d_i^j and weights w_i^j, is the space of semi-stable parabolic vector bundles modulo a certain equivalence relation known as s-equivalence. If $\mathbb{E} \sim_s \mathbb{F}$ and $\mathbb{E}$ is stable, then $\mathbb{E}$ must be isomorphic to $\mathbb{F}$, that is, there is a holomorphic section of the bundle of isomorphisms, and this preserves the flags over the marked points $(p_1, \ldots, p_n)$. If $\mathbb{E}$ is a semistable bundle, we can define a new bundle, $\mathrm{Gr}(\mathbb{E})$, well-defined up to isomorphism, in the following way: there exists a series of nested sub-parabolic bundles $0 = \mathbb{F}^n \subset \mathbb{F}^{n-1} \subseteq \cdots \subseteq \mathbb{F}^1 = \mathbb{E}$ such that $\mathbb{F}^i$ is a parabolic subbundle of F^{i-1} with maximal rank such that $\mu_{\mathrm{p}}(\mathbb{F}^i) = \mu_{\mathrm{p}}(\mathbb{F}^{i-1})$. This implies that $\mathbb{F}^{i-1}/\mathbb{F}^i$ is a stable parabolic bundle, and we define $\mathrm{Gr}(\mathbb{E}) = \oplus_{i=1}^{n-1} \mathbb{F}^i/\mathbb{F}^{i+1}$. We say that $\mathbb{E}$ and $\mathbb{F}$ is s-equivalent if $\mathrm{Gr}(\mathbb{E}) \cong \mathrm{Gr}(\mathbb{F})$. We will denote the subset of equivalence classes stable bundles with $\mathcal{M}^{\mathrm{Par}'}(\Sigma, p_1, \ldots, p_n, d_i^j, w_i^j)$. $\mathcal{M}^{\mathrm{Par}'}$ is a smooth manifold, and the tangent space at $[\mathbb{E}]$ is given by $T_{[\mathbb{E}]}\mathcal{M}^{\mathrm{Par}'} = \mathrm{H}^1(\Sigma, \mathrm{PEnd}_0(\mathbb{E}))$ [60]. By Serre duality, the cotangent space is $T^*_{[\mathbb{E}]}\mathcal{M}^{\mathrm{Par}'} = \mathrm{H}^0(\Sigma, \mathrm{SPEnd}_0(\mathbb{E}) \otimes K(D))$.

Theorem 35 (Mehta-Seshadri [51]). *The moduli space of parabolic bundles of type (E, d, w), with $\mathrm{pdeg}(E, d, w) = 0$, is homeomorphic to the moduli space of $\mathrm{U}(r)$ representations of the fundamental group of $X \setminus \{p_i\}$ such that the holonomy around a puncture is conjugate to a diagonal matrix with entries $e^{2\pi i w_i^j}$, each occurring d_i^j times. Furthermore, the restriction to the stable part gives a diffeomorphism to the space of irreducible connections.*

The map going from unitary connections to parabolic bundles can be described in the following way. Given a flat, unitary connection ∇ in $E \to \Sigma \setminus \{p_1, \ldots p_n\}$, we can find a neighborhood around each puncture that is holomorphic to a punctured, open disc, and a unitary frame $g_1, g_2, \ldots, g_r$ such that $\nabla = \mathrm{d} + i\alpha \mathrm{d}\theta$, where α is a diagonal matrix with real entries, increasing along the diagonal. We can get a holomorphic frame for $\bar{\partial}_\nabla$ by setting $f_i = |z|^{\alpha_i} g_i$. We can glue this trivialization, with the standard trivialization of the trivial bundle $D \times \mathbb{C}^r$, and in this way, we obtain a holomorphic bundle over Σ. The parabolic structure is given by the flag $E^i = \mathrm{span}\{e_j \mid \alpha_j \geq \alpha_i\}$ where e_j is the j'th standard basis vector for $\mathbb{C}^r$, and the weights are diagonal entries with duplicates removed.

Remark 51. If $\Sigma = \mathbb{CP}^1$ and the sum of weights, with multiplicities, over each point is zero, the parabolic bundle corresponds to a representation into $\mathrm{SU}(r)$.

Theorem 36 ([24]). *The Mehta-Seshadri map is a biholomorphism from the moduli space of stable parabolic bundles on Σ_σ to the moduli space of irreducible flat connections with prescribed holonomy equipped with the complex structure from σ:*

$$\mathrm{MS} : \mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)_\sigma \to \mathcal{M}^{\mathrm{Par}'}(\Sigma_\sigma, \vec{\lambda}, k).$$

5.6 The case of $\Sigma = \mathbb{CP}^1$

We will consider parabolic bundles on $\mathbb{CP}^1$ with marked points $p_1, \ldots, p_n$, each with flag type $(d_1^i, d_2^i) = (1, 1)$ and parabolic weights $(w_i^0, w_i^1, w_i^2) = (-\frac{1}{k}, \frac{1}{k}, 1)$ for a $k > n$, and $E \to \mathbb{CP}^1$ a rank 2, degree 0 vector bundle. We will write ${\mathcal{M}^{\mathrm{Par}}}_{n,k}$ for the moduli space of such bundles.

Lemma 52. *Any semistable parabolic bundle $\mathbb{E}$ of this type must be a trivializable as a holomorphic vector bundle.*

Proof. Assume that $\mathbb{F} \subseteq \mathbb{E}$ is a holomorphic subbundle. Then we have

$$\mathrm{pdeg}\,\mathbb{F} = \deg \mathbb{F} + -\sum_{i:F_{p_i} = E^2_{p_i}} \frac{1}{k} + \sum_{i:F_{p_i} \neq E^2_{p_i}} \frac{1}{k} = \deg \mathbb{F} + \frac{n}{k} - \frac{2}{k} \sum_{i:F_{p_i} = E^2_{p_i}} 1$$

We first observe that since $\mathrm{pdeg}\,\mathbb{E} = 0$, we must have that $\deg \mathbb{F} \leq \frac{n}{k} < 1$. A theorem of [34] states that any holomorphic vector bundle on $\mathbb{CP}^1$ splits uniquely as a direct sum of holomorphic line bundles, so $\mathbb{E} \cong \mathcal{O}(j) \oplus \mathcal{O}(-j)$ for some j, since the first Chern class is additive under direct sum. But we just showed that E has no subbundles of positive degree, so we must have that $E \cong \mathcal{O}(0) \oplus \mathcal{O}(0)$ – the trivial rank 2 bundle over $\mathbb{CP}^1$. $\square$

To find all the semistable parabolic bundles of this type, we can now assume that $\mathbb{E} = \mathbb{CP}^1 \times \mathbb{C}^2$, and therefore the remaining data just consists of a line in $\mathbb{C}^2$ for each marked point – so the parabolic bundles are parametrized by $(\mathbb{CP}^1)^n$. We want to find the subset corresponding to the semistable and stable bundles, and then consider the quotient.

Remark 53. We first make a small observation: for n odd, there are no strictly semistable bundles. For $\mu_{\mathrm{p}}(\mathbb{F}) = \mu_{\mathrm{p}}(\mathbb{E}) = 0$ is only possible if $\deg \mathbb{F} = 0$ and F agrees with $E^2_{p_i}$ at exactly half of the marked points – impossible if n is odd.

We will now find the precise conditions on $(L_1, L_2, \dots, L_n) \in (\mathbb{CP}^1)^n$ that corresponds to a semistable bundle. It is clear that a subbundle of negative degree has negative parabolic slope, and therefore cannot have larger parabolic slope than $\mathbb{E}$. So we just have to consider subbundles $\mathbb{F}$ that are holomorphically trivial. But then $\mathbb{F} \subseteq \mathbb{E}$ has a holomorphic nowhere vanishing section $s : \mathbb{CP}^1 \to \mathbb{F}$. But this is a holomorphic map from $\mathbb{CP}^1$ to $\mathbb{C}^2$, so it is constant, and $F_p = \mathrm{span}_{\mathbb{C}}(s(p))$, so $F = \mathbb{CP}^1 \times \mathrm{span}_{\mathbb{C}}\{s(p_0)\}$, a trivial subbundle. Thus the set of nonnegative parabolic slopes of subbundles of $(E, (E^2_{p_1}, \dots, E^2_{p_n}))$ is exactly $\{\frac{2j-n}{k} \mid x \in \mathbb{CP}^1, x$ occurs exactly $j > \frac{n}{2}$ times in $(E^2_{p_1}, \dots, E^2_{p_n})\}$. We can now conclude the following

Proposition 54. *The parabolic bundle $(\mathbb{CP}^1 \times \mathbb{C}^2, (E^2_{p_1}, \dots, E^2_{p_n}))$ is semi-stable if and only if at most half of the E_{p_i}'s are equal, and stable if and only strictly less than half of the E_{p_i}'s are equal.*

We are now ready to find the s-equivalence classes of parabolic bundles on $\mathbb{CP}^1$. We first investigate the example of 4 parabolic points.

Example 55. In the case $n = 4$, a bundle is stable if and only if the 4 flags are pairwise different. The isomorphism group of $\mathbb{E}$ is $\mathrm{GL}(2, \mathbb{C})$, and therefore any stable bundle is isomorphic to exactly one where the flags are $(0, 1, \infty, z)$ where $x \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$. There are some strictly semistable bundles – these occur exactly when two of the lines agree. Up to isomorphism there are 6 where $|\{E^2_1, E^2_2, E^2_3, E^2_4\}| = 3$ and three where $|\{E^2_1, E^2_2, E^2_3, E^2_4\}| = 2$. We will show that there are 3 s-equivalence classes of strictly semistable bundles, namely the classes with representatives $F_{\{1,2\}} \oplus F_{\{3,4\}}$, $F_{\{1,3\}} \oplus F_{\{2,4\}}$ and $F_{\{1,4\}} \oplus F_{\{2,3\}}$, where $F_{\{a,b\}}$ is the parabolic bundle $\mathbb{CP}^1 \times \mathbb{C}$ with flags $\mathbb{C}$ over i and j and flag $\{0\}$ over the two other points, with weights $(\frac{1}{k}, 1)$ over i and j and $(-\frac{1}{k}, 1)$ over the other points. For assume that $\mathbb{E} = (\mathbb{CP}^1 \times \mathbb{C}^2, \{E^2_i\}^4_{i=1})$ with weights as above, and that $E_r = E_s$. Then the parabolic subbundle of $\mathbb{CP}^1 \times E_r \subseteq \mathbb{E}$ is isomorphic to $F_{\{r,s\}}$. Since $\mathbb{E}$ is semistable, we know that $E_t \neq E_r$ for $t \neq r, s$, so the quotient parabolic bundle $\mathbb{E}/\left(\mathbb{CP}^1 \times E_s\right)$ is isomorphic to $F_{\{1,2,3,4\} \setminus \{r,s\}}$, and therefore $\mathrm{Gr}(\mathbb{E})$ is isomorphic to one of the 3 bundles listed above. As an example, let

$$\mathbb{E} = (\mathbb{CP}^1 \times \mathbb{C}^2, A, A, B, C)$$

where $A, B, C \in \mathbb{CP}^1$ and $B, C \neq A$ (but we might have $B = C$). Then E is strictly semistable, for it has the same slope as the subbundle $\mathbb{F} = (\mathbb{CP}^1 \times A, A, A, A, A) \cong F_{\{1,2\}}$ where the first two A's have parabolic weight $\frac{1}{k}$ and the last two $-\frac{1}{k}$. There is only one more term in the composition series, and this is

$$\mathbb{E}/\mathbb{F} = (\mathbb{CP}^1 \times (\mathbb{C}^2/A), A/A, A/A, B/A, C/A) \cong F_{\{3,4\}},$$

where the first two lines has weight $-\frac{1}{k}$ and the last two weight $\frac{1}{k}$. The set of s-equivalence classes of strictly semistable points is $\{[0, 1, \infty, 0], [0, 1, \infty, 1], [0, 1, \infty, \infty]\}$. So we have a bijection from $\mathbb{CP}^1$ to $\mathcal{M}^{\mathrm{Par}}{}_{4,k}$ given by $z \mapsto [0, 1, \infty, x]$.

The argument of this example generalized immediately to the following:

Proposition 56. *A strictly semi-stable bundle* $(\mathbb{CP}^1 \times \mathbb{C}^2, (E^2_{p_1}, \dots, E^2_{p_{2n}}))$ *must have* n *equal flags, and their indices partition the set* $\{1, 2, \dots, 2n\}$ *into two sets with* n *elements. Two strictly semistable bundles are* s-*equivalent if and only if their associated partitions are equal.*

Remark 57. We can parametrize an open and dense subset of the stable part of the moduli space by $\mathrm{Conf}_{n-3}\left(\mathbb{CP}^1 \setminus \{0, 1, \infty\}\right)$, by mapping such a tuple $(z_1, z_2, \dots, z_{n-3}) \mapsto [z_1, z_2, \dots, z_{n-3}, 0, 1, \infty]$. This parametrization will be useful for calculations later.

Observe that the only automorphisms of $\mathbb{CP}^1$ that preserve 0 and ∞ are the complex dilations. Therefore we can define a map from the subset of $\mathcal{M}^{\mathrm{Par}}{}_n$ given by $\{[0, \infty, z_1, \dots, z_{n-2}] \mid z_i \neq \infty\}$ to $\mathbb{CP}^{n-3}$ by $[0, \infty, z_1, \dots, z_{n-3}] \mapsto [z_1, \dots, z_{n-2}]$. This map cannot be extended to the points where 2 or more flags coincide. However, if we blow up $\mathbb{CP}^{n-3}$, we can extend to the points with at most two equal flags:

Theorem 37. *There exists a union* X *of submanifolds of codimension* 2 *in* $\mathcal{M}^{\mathrm{Par}}{}_{n,k}$ *and a union* Y *of submanifolds of codimension at least* 2 *in* RM_n*, the blowup of* $\mathbb{CP}^{n-3}$ *in* $n-1$ *points, such that* $\mathcal{M}^{\mathrm{Par}}{}_{n,k} \setminus X$ *is biholomorphic to* $\mathrm{RM}_n \setminus Y$*.*

Proof. Let X be the union of all submanifolds where there exists 4 flags that takes at most 2 distinct values. That is, either there are three flags that agree, or there are two pairs of flags that agree. It is clear that X has codimension 2. We will map the complement of X to $\mathbb{CP}^{n-3}$, blown up at the $n-1$ points $[1, 0, 0, \dots, 0], \dots, [0, 0, \dots, 0, 1]$ and $[1, 1, \dots, 1]$. Let us first define the map on the subset of $\mathcal{M}^{\mathrm{Par}}{}_{n,k}$ where all flags are different. We can assume that the first two flags are $0, \infty$, and we define:

$$f([0, \infty, z_1, \dots, z_{n-2}]) = [z_1, \dots, z_{n-2}].$$

This map is holomorphic, as it is induced from the following $\mathrm{SL}(2, \mathbb{C})$-invariant map from $\{(w_1, \dots, w_n) \in (\mathbb{CP}^1)^n \mid i \neq j \implies w_i \neq w_j\} \to \mathbb{CP}^{n-3}$

$$(w_1, w_2, \dots, w_n) \mapsto \left[1, \frac{(w_4 - w_1)(w_3 - w_2)}{(w_4 - w_2)(w_3 - w_1)}, \frac{(w_5 - w_1)(w_3 - w_2)}{(w_5 - w_2)(w_3 - w_1)}, \dots, \frac{(w_n - w_1)(w_3 - w_2)}{(w_n - w_2)(w_3 - w_1)}\right],$$

which is invariant, as it is can be expressed in terms of the *cross ratio* $\mathrm{CR}(a, b, c, d) = \frac{(b-a)(d-c)}{(b-c)(d-a)}$, which is an $\mathrm{SL}(2, \mathbb{C})$-invariant function on $(\mathbb{CP}^1)^4$. The blow up of $\mathbb{CP}^{n-3}$ at $[1, 0, \dots, 0]$ is the following manifold:

$$(\mathbb{CP}^{n-3} \setminus \{[1, 0, \dots, 0]\}) \coprod \gamma_{n-4} / \sim$$

where $\gamma_{n-4} = \{([z_1, \dots, z_{n-4}], w) \in \mathbb{CP}^{n-4} \times \mathbb{C}^{n-4} \mid w \in [z_1, \dots, z_{n-4}]\}$, and $\sim$ is the identification $[1, w_1, \dots w_{n-3}] \sim ([w_1, \dots w_{n-3}], (1, w_1, \dots w_{n-3}))$. So for $z_1 \neq 0, \infty$ the map f into the blowup is $f([0, \infty, z_1, \dots, z_{n-2}]) = ([\frac{z_2}{z_1}, \dots, \frac{z_{n-2}}{z_1}], (z_1, \dots, z_{n-2}))$. We will extend f to points where $z_1 = \infty$ by setting

$$f([0, \infty, \infty, z_2, \dots, z_{n-2}]) = ([z_2, \dots, z_{n-2}], (0, 0, \dots, 0)).$$

Our goal is now to show that this is holomorphic. To this end, let us pick coordinates on $\mathcal{M}^{\mathrm{Par}}{}_{n,k}$ as

$$(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{n-2}) \mapsto [0, \infty, \frac{1}{z_1}, z_2, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_{n-3}],$$

where $z_1, \dots z_{n-2} \in \mathbb{C}$ and at most one of $z_2, \dots z_{n-2}$ is equal to 0, and $z_1^{-1}, \dots z_{n-2}$ are pairwise distinct, with the exception of at most one pair – that is, the requirements to end up inside the complement of X. Likewise, let us pick coordinates on γ_{n-4} as

$$(w_1, \dots, w_{i-1}, w_{i+1} w_{n-4}, s) \mapsto ([w_1, \dots, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_{n-4}], s(w_1, \dots, 1, \dots, w_{n-4}))\,.$$

We can now calculate f in these coordinates as follows, where the first map is the coordinate map, the second f and the last the inverse coordinate map:

$$\begin{aligned}
&(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{n-2}) \\
&\qquad \mapsto [0, \infty, \frac{1}{z_1}, z_2, \dots, 1, \dots, z_{n-2}] \\
&\qquad = \begin{cases} [0, \infty, 1, z_1 z_2, \dots, z_1 \cdot 1, \dots z_1 z_{n-2}] & \text{if } z_1 \neq 0 \\ [0, \infty, \infty, z_2, \dots, 1, \dots, z_{n-2}] & \text{if } z_1 = 0 \end{cases} \\
&\qquad \mapsto \begin{cases} ([z_1 z_2, \dots, z_1 \cdot 1, \dots, z_1 z_{n-2}], (z_1 z_2, \dots, z_1 \cdot 1, \dots, z_1 z_{n-2})) & \text{if } z_1 \neq 0 \\ ([z_2, \dots, 1, \dots, z_{n-2}], (0, \dots, 0)) & \text{if } z_1 = 0 \end{cases} \\
&\qquad = \begin{cases} ([z_2, \dots, 1, \dots, z_{n-2}], z_1(z_2, \dots, 1, \dots, z_{n-2})) & \text{if } z_1 \neq 0 \\ ([z_2, \dots, 1, \dots, z_{n-2}], (0, \dots, 0)) & \text{if } z_1 = 0 \end{cases} \\
&\qquad = ([z_2, \dots, 1, \dots, z_{n-2}], z_1(z_2, \dots, 1, \dots, z_{n-2})) \\
&\qquad \mapsto (z_2, \dots, z_{i-1}, z_{i+1}, \dots, z_{n-2}, z_1)
\end{aligned}$$

The extension of f to points of the form $[0, \infty, z_1, \dots, \infty, \dots, z_{n-2}]$ (to the blowup at $[0, \dots, 1, \dots, 0]$) and $[0, 0, \infty, z_1, \dots, z_{n-3}$ (to the blowup at $[1, 1, \dots, 1]$) is done in the same way. It is clear from this calculation that all but a codimension-1 subset of the blowup divisor is in the image of f, as well as the image of f contains all but a codimension 2 subsetset of $\mathbb{CP}^{n-3}$. □

Remark 58. Note that in [43] it is shown that the moduli space is birational to $\mathbb{CP}^{n-3}$.

Corollary 59. *For $n > 4$ there are no holomorphic vector fields on ${\mathcal{M}^{\mathrm{Par}}}'_{n,k}$.*

Proof. By Hartog's extension theorem, sections of a vector bundle always extend over subsets of codimension 2. Therefore we just need to conclude that RM_n do not have any holomorphic vector fields. But the holomorphic vector fields on the blowup of a space Z are in correspondence to the holomorphic vector fields on Z with zeroes at the point being blown up. But RM_n is $\mathbb{CP}^{n-3}$ with $n-1$ points blown up. Requiring that a holomorphic vector field vanishes at a point on $\mathbb{CP}^{n-3}$ imposes $n-3$ linear conditions, so requiring that it vanishes at $n-1$ generic points give rise to $(n-3)(n-1) = (n-2)^2 - 1$ linear constraints. Since the group of biholomorphisms of $\mathbb{CP}^{n-3}$ is $\mathrm{PSL}(n-3+1, \mathbb{C})$, we have $\dim H^0(\mathbb{CP}^{n-3}, T^{1,0}) = \dim \mathfrak{sl}(n-2) = (n-2)^2 - 1$. Therefore $\dim H^0(\mathrm{RM}_n, T^{1,0}) = (n-2)^2 - 1 - ((n-2)^2 - 1) = 0$.

A different argument can be given in the following way. Let V be a holomorphic vector field on ${\mathcal{M}^{\mathrm{Par}}}'_{n,k}$ and pull it back to the stable points in $(\mathbb{CP}^1)^n$. As $n > 4$, the vector field must extend holomorphically to all of $(\mathbb{CP}^1)^n$. But it was the pull-back along the quotient map, so it must be $\mathrm{SL}(2, \mathbb{C})$ -invariant, and therefore each of the n components must be invariant. But there are no $\mathrm{SL}(2, \mathbb{C})$ invariant holomorphic vector fields on $\mathbb{CP}^1$. □

Lemma 60. *For $n > 4$ the second cohomology group of ${\mathcal{M}^{\mathrm{Par}}}'_n$ is isomorphic to $\mathbb{Z}^n = \mathrm{H}^2({\mathbb{CP}^1}^n \setminus N, \mathbb{Z})$, and the hmomorphism induced by the quotient map is an isomorphism. In these coordinates on the cohomology the Chern class of ${\mathcal{M}^{\mathrm{Par}}}_n$ is given by $(2, 2, \dots, 2)$.*

Proof. For $n > 4$, N have real codimension ≥ 4 and therefore the inclusion map gives an isomorphism $\mathrm{H}^2((\mathbb{CP}^1)^n \setminus N, \mathbb{Z}) \cong \mathrm{H}^2((\mathbb{CP}^1)^n, \mathbb{Z}) = \mathbb{Z}^n$. Now $(\mathbb{CP}^1)^n \setminus N$ forms a fiber bundle with $\mathrm{SL}_2(\mathbb{C})$ fiber. This follows from the existence of local sections, which we can show using that $\mu^{-1}(0)$ is a SU(2) bundle. Now we can apply Serre's spectral sequence for fiber bundles. As $H^*(\mathrm{SL}_2(\mathbb{C}), \mathbb{Z}) \cong H^*(\mathrm{SU}(2), \mathbb{Z}) \cong H^*(S^3, \mathbb{Z})$, which have all its cohomology in dimension 0 and 3. Therefore the quotient map must define an isomorphism on the second cohomology. The tangent bundle of $(\mathbb{CP}^1)^n \setminus N$ splits as the direct sum of the pullback of the tangent bundle of the quotient and the tangent bundle of the fibers. But the tangent space to the fibers is trivial, as the map from $\mathfrak{sl}(2,\mathbb{C}) \times ((\mathbb{CP}^{1^n}) \setminus N) \to T((\mathbb{CP}^{1^n}) \setminus N)$ mapping $(V, x) \mapsto \underline{V}_x$ defines a global trivialization. The last statement of the theorem now follows from the fact that $\mathrm{c}_1(T\mathbb{CP}^1) = 2 \in \mathbb{Z} \cong \mathrm{H}^2(\mathbb{CP}^1, \mathbb{Z})$. $\square$

Corollary 61. *The second Stiefel-Whitney class* $\mathrm{w}(\mathcal{M}^{\mathrm{Par}}{}_{n,k})$ *of* $\mathcal{M}^{\mathrm{Par}}{}_{n,k}$ *vanishes*

Proof. This follows directly from lemma 60, as w_2 of a vector bundle with complex structure is the mod-2 reduction of the first Chern class. $\square$

Let $D = p_1, \ldots, p_n$ where $p_{n-2} = 0$, $p_{n-1} = 1$ and $p_n = \infty$, and denote by $\mathbb{E}_\tau$ the trivial bundle with parabolic structure at $0, 1, \infty$ be given by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

respectively, and for p_i by

$$\begin{pmatrix} \tau_i \\ 1 \end{pmatrix},$$

where $\tau = (\tau_1, \ldots, \tau_{n-3}) \in \mathrm{Conf}(\mathbb{CP}^1 \setminus \{0, 1, \infty\})$. The map $\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0, 1, \infty\}) \to \mathcal{M}^{\mathrm{Par}'}{}_{n,k}$ given by $\tau \mapsto [\mathbb{E}_\tau]$ is injective and the image is open and dense. The complement of the image inside the stable part consist of all points where some, but less than $\frac{n}{2}$, of the flags coincide.

Cotangent space

The cotangent space at $\mathbb{E}$ is $T^*_{[\mathbb{E}]}\mathcal{M}^{\mathrm{Par}}{}_{n,k} = \mathrm{H}^0(X, \mathrm{SPEnd}_0(\mathbb{E}) \otimes K(D))$, for $\mathbb{E}$ a stable bundle [60]. For $x \in \mathbb{C}$, let $f_x = \frac{\mathrm{d}z}{z-x}$. Then $H^0(X, K(D))$ has the basis $\{f_x \mid x \in D_0\}$, where $D_0 = D \cap \mathbb{C}$. Then

$$\begin{gathered} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes f_{p_1}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes f_{p_{n-1}}, \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes f_{p_1}, \ldots, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes f_{p_{n-1}}, \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes f_{p_1}, \ldots, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes f_{p_{n-1}} \end{gathered} \tag{5.2}$$

is a basis for $\mathrm{H}^0(X, \mathrm{End}_0(\mathbb{E}_\tau) \otimes K(D))$.

The element

$$\sum_{i=1}^{|D_0|} \begin{pmatrix} a_i f_{p_i} & b_i f_{p_i} \\ c_i f_{p_i} & -a_i f_{p_i} \end{pmatrix} \in \mathrm{H}^0(X, \mathrm{End}_0(\mathbb{E}_\tau) \otimes K(D))$$

is in $\mathrm{H}^0(X, \mathrm{SPEnd}_0(\mathbb{E}_\tau) \otimes K(D))$ if:

$$\begin{aligned}
\sum_{i=0}^{|D_0|} a_i &= 0, \\
\sum_{i=0}^{|D_0|} c_i &= 0, \\
\tau_i a_i + b_i &= 0 \quad \text{for } i = 1, \dots, |D_0|, \\
\tau_i c_i - a_i &= 0 \quad \text{for } i = 1, \dots, |D_0|,
\end{aligned}$$

where the first two equations ensure that the properties at ∞ is satisfied (f_p has a simple pole at ∞ of residue -1). Using the basis (5.2), this is the kernel of the matrix that have first row $|D_0|$ 1's and then $2|D_0|$ 0's, second row $2|D_0|$ 0's and then $|D_0|$ 1's, and for each i a row where the i'th column equals τ_i and the $|D_0| + i$'th column is 1, and a row where the i'th column is -1 and the $2|D_0| + i$'th is τ_i.

For $n = 4$ this looks like:

$$\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\tau_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & \tau_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

This matrix has full rank, and it is easy to check that, for $i = 1, 2, \dots, n-3$, the following element is in the kernel:

$$\theta_i = \begin{pmatrix} \frac{-\tau_i}{z-1} + \frac{\tau_i}{z-p_i} & \frac{\tau_i}{z-1} + \frac{-\tau_i^2}{z-p_i} \\ \frac{\tau_i - 1}{z} - \frac{\tau_i}{z-1} + \frac{1}{z-p_i} & \frac{\tau_i}{z-1} - \frac{\tau_i}{z-p_i} \end{pmatrix} \mathrm{d}z.$$

As the matrix is $2|D| \times 3(|D_0|)$ and have full rank, the kernel is $3|D_0| - 2|D| = 3(n-1) - 2n = n-3$ dimensional, and since $\theta_i, i = 1, \dots, n-3$ is linear independent, we have:

$$\mathrm{H}^0(X, \mathrm{SPEnd}_0(\mathbb{E}_\tau) \otimes K(D)) = \mathrm{span}_{\mathbb{C}}\{\theta_i \mid i = 1, 2, \dots, n-3\}.$$

Tangent space

The tangent space at $\mathbb{E}$ is given by $\mathrm{H}^1(X, \mathrm{PEnd}_0(\mathbb{E}))$. We want to find $\frac{\partial}{\partial \tau_i}$ for $i \leq n-3$. Let $\{U_j\}_{j=1}^n$ be an open cover by contractible sets, such that $\bar{U}_j \cap D = p_j$ and $U_j \cap U_j^\infty$ is an annulus, where $U_i^\infty = \cup_{i \in D \setminus \{p_i\}} U_i$. Let $\tau_w = (\tau_1, \dots, \tau_{i-1}, \tau_i + w, \tau_{i+1}, \dots, \tau_{n-3})$ We can make $\mathbb{E}_{\tau_w}$ by gluing together $(\mathbb{E}_\tau)_{U_i}$ and $(\mathbb{E}_\tau)_{U_i^\infty}$, with the transition function $g = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$.

To see this, let $\gamma : (\mathbb{E}_\tau)_{U_i} \sqcup (\mathbb{E}_\tau)_{U_i^\infty} \to \mathbb{E}_{\tau_w}$ be given by $\gamma(z, v) = (z, v)$ for $z \in U_i^\infty$ and $\gamma(z, v) = (z, g(z)v)$ for $z \in U_i$. γ is clearly a morphism of parabolic vector bundles and invariant under the identification of $(z, v) \in U_i^\infty \times \mathbb{C}^2$ with $(z, g(z)v) \in U_i \times \mathbb{C}^2$.

Taking the derivative, we get that $\frac{\partial}{\partial \tau_i}$ – as a one-cycle relative to the cover U_i, U_i^∞ of X – is represented by $\frac{\partial}{\partial \tau_i} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{H}^0(U_i \cap U_i^\infty, \mathrm{PEnd}_0(\mathbb{E}_{\tau_i}))$. Let h_i be a smooth function on

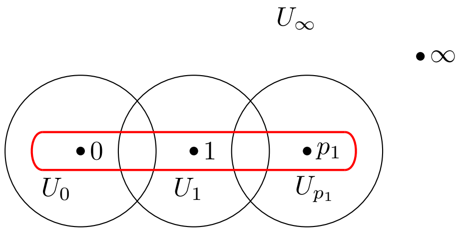


Figure 5.1: An example of an allowed choice of the U_i's. U_i is the interior of the disc centered around p_i, for $p_i \neq \infty$, and U_∞ is the domain outside the red curve..

X that vanish on $U_i^\infty \setminus U_i$ and is 1 on $U_i \setminus U_i^\infty$. Define smooth sections of $\mathrm{PEnd}_0(\mathbb{E})$ over U_i and U_i^∞ by $\psi_i = \begin{pmatrix} 0 & 1-h_i \\ 0 & 0 \end{pmatrix}$ and $\psi_i^\infty = \begin{pmatrix} 0 & -h_i \\ 0 & 0 \end{pmatrix}$. Then

$$\psi_i|_{U_i \cap U_i^\infty} - \psi_i^\infty|_{U_i \cap U_i^\infty} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We conclude:

Proposition 62. *The Dolbeault representative for $\frac{\partial}{\partial \tau_i}$ is given by*

$$\begin{pmatrix} 0 & -\frac{\partial h_i}{\partial \bar{z}} \\ 0 & 0 \end{pmatrix} \otimes \mathrm{d}\bar{z} \in \Omega^{0,1}_{\bar{\partial}}(X, \mathrm{PEnd}_0(\mathbb{E}_{\tau_i})).$$

Let $A \subseteq X$ be the annulus such that h_j only takes the values 0 and 1 outside of $\mathrm{Int}A$.

$$\begin{aligned}
\theta_i(\frac{\partial}{\partial \tau_j}) &= \frac{1}{2\pi \mathrm{i}} \int_X \mathrm{Tr}\,\theta_i \begin{pmatrix} 0 & -\frac{\partial h_j}{\partial \bar{z}} \\ 0 & 0 \end{pmatrix} \mathrm{d}\bar{z} = \int_X -\frac{\partial h_j}{\partial \bar{z}}(\frac{\tau_i - 1}{z} - \frac{\tau_i}{z-1} + \frac{1}{z-p_i})\mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{1}{2\pi \mathrm{i}} \int_A -\frac{\partial h_j}{\partial \bar{z}}(\frac{\tau_i - 1}{z} - \frac{\tau_i}{z-1} + \frac{1}{z-p_i})\mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{1}{2\pi \mathrm{i}} \int_A \bar{\partial}(h_j(\frac{\tau_i - 1}{z} - \frac{\tau_i}{z-1} + \frac{1}{z-p_i}))\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_A \mathrm{d}(h_j(\frac{\tau_i - 1}{z} - \frac{\tau_i}{z-1} + \frac{1}{z-p_i}))\mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\partial A} h_j(\frac{\tau_i - 1}{z} - \frac{\tau_i}{z-1} + \frac{1}{z-p_i})\mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\{x \in \partial A | h_j(x) = 1\}} (\frac{\tau_i - 1}{z} - \frac{\tau_i}{z-1} + \frac{1}{z-p_i})\mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \operatorname*{Res}_{p_j}(\frac{\tau_i - 1}{z} - \frac{\tau_i}{z-1} + \frac{1}{z-p_i})\mathrm{d}z = \delta_{ij}.
\end{aligned} \tag{5.3}$$

∎

We finally see that $\theta_i = \mathrm{d}\tau_i$.

5.7 Moduli space of polygons

Given weights $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+$ we say that a tuple $(\xi_1, \xi_2, \dots, \xi_n) \in \left(\mathbb{R}^3\right)^n$ is a n-gon with side lengths $\vec{\alpha}$ if

$$\sum_{i=1}^{n} \xi_i = 0,$$
$$\|\xi_i\| = \alpha_i \text{ for all } i,$$

where we are thinking of ξ_i as the vector of the i'th edge of the polygon. We say that a polygon is degenerate if $\operatorname{span}\{\xi_i \mid i = 1, \dots, n\}$ is one-dimensional, and non-degenerate otherwise. Let $S^2_r \subseteq \mathbb{R}^3$ be the sphere around 0 of radius r. We define the following map $\mu : \prod_{i=1}^n S^2_{\alpha_i} \to \mathbb{R}^3$ by

$$\mu(\xi_1, \xi_2, \dots, \xi_n) = \sum_{i=1}^{n} \xi_i$$

and we observe that the polygons are exactly the zero set of μ. We define two polygons to be isomorphic if there exists an isometry of $\mathbb{R}^3$ carrying one to the other: that is, $(\xi_1, \dots, \xi_n) \cong (\xi'_1, \dots, \xi'_n)$ if there exists $A \in \mathrm{SO}(3)$ such that $\xi_i = A(\xi'_i)$ for all i. We define the moduli space of polygons of weight $\vec{\alpha}$ to be the space

$$\mathcal{P}_n = \mu^{-1}(0)/\mathrm{SO}(3).$$

It is easy to see that 0 is a regular value of μ away from the non-degenerate polygons, and also that the stabilizer of any non-degenerate polygons is trivial. Therefore the subset of equivalence classes of non-degenerate polygons forms a smooth manifold. Even more is true: identifying $\mathbb{R}^3$ with $\mathfrak{so}(3)^*$ and each $S^2_{\alpha_i}$ with the coadjoint orbit through $(\alpha_i, 0, 0)$, this is just the symplectic quotient of a product of coadjoint orbits, and furthermore, each coadjoint orbit supports an SO(3)-invariant Kähler structure, and $\mathcal{P}_n$ becomes the symplectic reduction of $\prod_i S^2_{\alpha_i}$.

5.7.1 GIT quotient

Let us now assume that $\alpha_i = \frac{\lambda_i}{k}$, for $k >> \sum_i \lambda_i$ and $\lambda_i \in \mathbb{N}$. Then the symplectic form on $\prod_i S^2_{\lambda_i}$ is representing the first Chern class of the line bundle $\boxtimes_i \mathcal{O}(\lambda_i)$ over $\left(\mathbb{CP}^1\right)^{\times n}$. Writing out the conditions for GIT stability we see that they are equivalent to the stability of parabolic bundles, where the weights are $(-\frac{\lambda_i}{k}, \frac{\lambda_i}{k}, 1)$. Note also that an (orbit of) a degenerate polygon corresponds to a partition of $\{1, 2, \dots, n\}$ in two sets N, S such that $\sum_{i \in N} \lambda_i = \sum_{i \in S} \lambda_j$ – orient the polygon so it lies in the x axis an let N be the set of indices whose corresponding vector points in the direction of the north pole of the unit sphere, and S the set of indices pointing south. But we see that this corresponds exactly to a strictly semi-simple parabolic bundle with the given weights.

$$\left(\mathbb{CP}^1\right)^n /\!/_{\vec{\lambda}} \mathrm{SL}(2, \mathbb{C}) \cong \mathcal{M}^{\mathrm{Par}}{}_{\vec{\lambda},k}.$$

It is clear that the subset $\{(\tau_1, \dots, \tau_{n-3}, 0, 1, \infty) \mid \tau_i \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}, i \neq j \implies \tau_i \neq \tau_j\} \subseteq \left(\mathbb{CP}^1\right)^n$ consists only of stable points. It follows from the Kempf-Ness theorem ([53]) that the inclusion $\iota : \mu^{-1}(0) \hookrightarrow (\mathbb{CP}^1)^n$ induces a homeomorphism

$$\mu^{-1}(0)/\mathrm{SO}(3) \cong \left(\mathbb{CP}^1\right)^n /\!/_{\vec{\lambda}} \mathrm{SL}(2, \mathbb{C}) \cong \mathcal{M}^{\mathrm{Par}}{}_{\vec{\lambda},k}$$

such that the restriction to the non-degenerate polygons is a biholomorphism onto $\mathcal{M}^{\mathrm{Par}}{}'_{\vec{\lambda},k}$

5.7.2 Tangent space of $\mu^{-1}(0)/\mathrm{SU}(2)$

If $x \in \mu^{-1}(0)$ then $T_{[x]}(\mu^{-1}(0)/\mathrm{SU}(2)) \cong \{V \in T_x \mid V\mu = 0\}/T_x\mathrm{SU}(2)x$. Let us identify $\mathbb{CP}^1 \cong S^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ using the map

$$z \mapsto \left(\frac{2z}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2} \right).$$

Let $x \in \mu^{-1}(0)$. Then $T_x\mu^{-1}(0) = \{V \in T_x(\mathbb{CP}^1)^n \mid V\mu = 0\}$. For $V = \sum_{i=1}^n a_i \frac{\partial}{\partial z_i} + \bar{a}_i \frac{\partial}{\partial \bar{z}_i}$ this is equivalent to the following 2 equations:

$$0 = \sum_{i=1}^n \frac{a_i \bar{z}_i + \bar{a}_i z_i}{(1+|z_i|^2)^2} \tag{5.4}$$

$$0 = \sum_{i=1}^n \frac{a_i - \bar{a}_i z_i^2}{(1+|z_i|^2)^2} \tag{5.5}$$

It is easy to verify there are the following three solutions:

$$\begin{aligned} a_i &= z_i^2 + 1, \\ a_i &= 2iz_i, \\ a_i &= z_i^2 - 1, \end{aligned}$$

and they corresponds to $T_x(\mathrm{SU}(2)x) \subset T_x\mu^{-1}(0)$. We are looking for solutions outside the span of these. For any subset $S \subseteq \{1, 2, \dots, n\}$ we can deform a polygon by rotating the coordinates corresponding to S around the vector $\sum_{s\in S} \mu_s(x)$. This give us the following vector field:

$$V_S = \mathrm{i} \sum_{s\in S} \left(\left((-b+\mathrm{i}c)z_s^2 - 2az_s + b + \mathrm{i}c\right) \frac{\partial}{\partial z_s} - \overline{\left((-b+\mathrm{i}c)z_s^2 - 2az_s + b + \mathrm{i}c\right)} \frac{\partial}{\partial \bar{z}_s} \right), \tag{5.6}$$

where

$$\begin{aligned} a &= \sum_{s\in S} \frac{1-|z_s|^2}{1+|z_s|^2} \\ b &= \mathrm{Re} \sum_{s\in S} \frac{2z_s}{1+|z_s|^2} \\ c &= \mathrm{Im} \sum_{s\in S} \frac{2z_s}{1+|z_s|^2}. \end{aligned}$$

We check that the equations (5.4) (5.5) are satisfied:

$$\begin{aligned} \sum_{i=1}^n \frac{a_i \bar{z}_i + \bar{a}_i z_i}{(1+|z_i|^2)^2} &= \sum_{s\in S} \frac{\left((-b+\mathrm{i}c)z_s^2 - 2az_s + b + \mathrm{i}c\right)\bar{z}_s - \overline{\left((-b+\mathrm{i}c)z_s^2 - 2az_s + b + \mathrm{i}c\right)} z_s}{(1+|z_s|^2)^2} \\ &= \sum_{s\in S} \frac{(-b+\mathrm{i}c)z_s(|z_s|^2+1) + (b+\mathrm{i}c)\bar{z}_s(1+|z_s|^2)}{(1+|z_s|^2)^2} \\ &= \sum_{s,r\in S} \frac{-2\bar{z}_r}{1+|z_r|^2}\frac{z_s}{1+|z_s|^2} + \frac{2z_r}{1+|z_r|^2}\frac{\bar{z}_s}{1+|z_s|^2} \end{aligned}$$

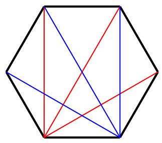


Figure 5.2: The tangent space at the point of $\mathcal{M}_{6,k}$ shown here is spanned by the rotating one part of the hexagon around the coloured lines. For by rotating around the red lines, the length of the blue lines can be adjusted to the corresponding lengths of any nearby hexagon, and afterwards rotations around the blue lines can adjust the angles .

$$= 0.$$

And

$$\begin{aligned}
\sum_{i=1}^{n} \frac{a_i - \bar{a}_i z_i^2}{(1+|z_i|^2)^2} &= \sum_{s\in S} \frac{\big((-b+\mathrm{i}c)z_s^2 - 2az_s + b + \mathrm{i}c\big) + \overline{\big((-b+\mathrm{i}c)z_s^2 - 2az_s + b + \mathrm{i}c\big)} z_i^2}{(1+|z_i|^2)^2} \\
&= \sum_{s\in S} \frac{-2az_s(1+|z_s|^2) + (b+\mathrm{i}c)(1-|z_s|^4)}{(1+|z_s|^2)^2} \\
&= \sum_{s\in S} \frac{-2az_s + (b+\mathrm{i}c)(1-|z_s|^2)}{1+|z_s|^2} \\
&= \sum_{s,r\in S} -2\frac{1-|z_r|^2}{1+|z_r|^2}\frac{z_s}{1+|z_s|^2} + \frac{2z_r}{1+|z_r|^2}\frac{1-|z_s|^2}{1+|z_s|^2} \\
&= 0.
\end{aligned}$$

For generic x, we have that

$$T_x\mu^{-1}(0) = \operatorname{span}\big(\{V_{\{1,2,\dots,i\}} \mid i = 2,\dots,n-2\}\} \cup \{V_{\{n\}\cup\{1,\dots,i\}} \mid i = 1,\dots,n-3\}\big) \oplus T_x\mathrm{SU}(2)x,$$

see figure (5.2)

It is now be possible to use these vectors to calculate the symplectic form of the symplectic quotient.

5.7.3 Computations for $n = 4$

For $n = 4$, the map q is just the cross ratio, and we can write down an explicit section of the quotient $\mu^{-1}(0) \to \mu^{-1}(0)/\mathrm{SO}(3) \cong \mathbb{CP}^1 \setminus \{0,1,\infty\}$. We first observe that the map $h : \mathbb{CP}^1 \to (\mathbb{CP}^1)^4$ given by

$$h(z) = (z^{-1}, \mathrm{i}z, -z^{-1}, -\mathrm{i}z).$$

maps into $\mu^{-1}(0)$, and it is easy to check that $\mathrm{CR}(h(\sqrt{\mathrm{PP}(\sqrt{q})}) = q$, where

$$\mathrm{PP}(x) = \mathrm{i}\frac{x+1}{x-1}.$$

We can therefore for any point $\tau \in \mathbb{CP}^1 \setminus \{0,1,\infty\}$ find an explicit $x \in \mu^{-1}(0) \subseteq (\mathbb{CP}^1)^4$ mapping to τ under the cross ratio. This allow us to do explicit computations for $n = 4$, which we are not able to do for $n > 4$, lacking good explicit coordinates for a slice for the $\mathrm{SU}(2)$ action of $\mu^{-1}(0)$. After some lengthy computations we get the following expressions for the bending vector fields:

$$\begin{aligned} q_*(V_{12}) &= -8\sqrt{\tau}\frac{|\mathrm{PP}(\sqrt{\tau})| + \mathrm{i}\mathrm{PP}(\sqrt{\tau})}{(|\mathrm{PP}(\sqrt{\tau})|+1)(\mathrm{PP}(\sqrt{\tau})+\mathrm{i})}\frac{\partial}{\partial\tau}, \\ q_*(V_{23}) &= -8\tau\frac{|\mathrm{PP}(\sqrt{\tau})| - \mathrm{i}\mathrm{PP}(\sqrt{\tau})}{(|\mathrm{PP}(\sqrt{\tau})|+1)(\mathrm{PP}(\sqrt{\tau})+\mathrm{i})}\frac{\partial}{\partial\tau}. \end{aligned}$$

Even though these expressions seems to depend on the choice of square root of q, the fact that $\mathrm{PP}(-x) = -\mathrm{PP}(x)^{-1}$ shows that the expression for V_S is invariant under $\sqrt{q} \mapsto -\sqrt{q}$. Let us simplify the expression for $q_*(V_{12})$:

$$\begin{aligned} q_*(V_{12}) &= -8\sqrt{\tau}\frac{|\mathrm{PP}(\sqrt{\tau})| + \mathrm{i}\mathrm{PP}(\sqrt{\tau})}{(|\mathrm{PP}(\sqrt{\tau})|+1)(\mathrm{PP}(\sqrt{\tau})+\mathrm{i})}\frac{|\sqrt{\tau}-1|(\sqrt{\tau}-1)}{|\sqrt{\tau}-1|(\sqrt{\tau}-1)}\frac{\partial}{\partial\tau} \\ &= -8\sqrt{\tau}\frac{|(\sqrt{\tau}+1)|(\sqrt{\tau}-1) - (\sqrt{\tau}+1)|\sqrt{\tau}-1|}{(|\sqrt{\tau}+1| + |\sqrt{\tau}-1|)(\mathrm{i}\sqrt{\tau}+1+\mathrm{i}(\sqrt{\tau}-1))}\frac{\partial}{\partial\tau} \\ &= 4\mathrm{i}\frac{|(\sqrt{\tau}+1)|(\sqrt{\tau}-1) - (\sqrt{\tau}+1)|\sqrt{\tau}-1|}{(|\sqrt{\tau}+1| + |\sqrt{\tau}-1|)}\frac{\partial}{\partial\tau} \\ &= 4\mathrm{i}\frac{\sqrt{\tau}(|\sqrt{\tau}+1| - |\sqrt{\tau}-1|) - (|\sqrt{\tau}-1| + |\sqrt{\tau}+1|)}{(|\sqrt{\tau}+1| + |\sqrt{\tau}-1|)}\frac{\partial}{\partial\tau} \\ &= 4\mathrm{i}\frac{\sqrt{\tau}(|\sqrt{\tau}+1|^2 - |\sqrt{\tau}-1|^2) - (|\sqrt{\tau}-1| + |\sqrt{\tau}+1|)^2}{(|\sqrt{\tau}+1| + |\sqrt{\tau}-1|)^2}\frac{\partial}{\partial\tau} \\ &= 4\mathrm{i}\frac{\sqrt{\tau}(4\operatorname{Re}\sqrt{\tau}) - (|\sqrt{\tau}-1| + |\sqrt{\tau}+1|)^2}{(|\sqrt{\tau}+1| + |\sqrt{\tau}-1|)^2}\frac{\partial}{\partial\tau} \\ &= 4\mathrm{i}\frac{2\tau + 2|\tau| - (|\sqrt{\tau}-1| + |\sqrt{\tau}+1|)^2}{(|\sqrt{\tau}+1| + |\sqrt{\tau}-1|)^2}\frac{\partial}{\partial\tau} \\ &= 4\mathrm{i}\frac{2\tau + 2|\tau| - 2(1+|\tau|+|\tau-1|)}{2(1+|\tau|+|\tau-1|)}\frac{\partial}{\partial\tau} \\ &= -4\mathrm{i}\frac{1-\tau+|\tau-1|}{1+|\tau|+|\tau-1|}\frac{\partial}{\partial\tau}. \end{aligned}$$

Similar computations give that

$$q_*(V_{13}) = -4\mathrm{i}\frac{(\tau-1)(|\tau|+\tau)}{1+|\tau|+|\tau-1|}\frac{\partial}{\partial\tau}$$

We can calculate the symplectic form evaluated on the pair (V_{12}, V_{23}) at $h(z)$:

$$\omega(V_{12}, V_{23}) = 8(\operatorname{Re} z^2)\frac{(|z|^4-1)}{(|z|^2+1)^4} = 8\operatorname{Re}\mathrm{PP}(\sqrt{q})\frac{|\mathrm{PP}(\sqrt{q})|^2-1}{(|\mathrm{PP}(\sqrt{q})|+1)^4}$$

G-invariant orthonormal basis

Unfortunately, V_{12} and V_{23} are not orthogonal to $T\mathrm{SO}(3)x$, nor to each other. We can project onto the complement of $T\mathrm{SO}(3)x$ and find a orthonormal basis for the their span. Doing this at the point $h(z)$ we get:

$$\mathrm{b}_1(z) = \frac{\mathrm{i}-1}{\sqrt{8}} \frac{\bar{z}(z^4+1)(|z|^2+1)}{|z||z^4+1|} \begin{pmatrix} \frac{1}{z^2} \\ -\mathrm{i} \\ -\frac{1}{z^2} \\ \mathrm{i} \end{pmatrix}$$

and $\mathrm{b}_2(z) = \mathrm{i}\mathrm{b}_1(z)$. We can calculate the differential:

$$\begin{aligned} q_*(\mathrm{b}_1) &= -i\sqrt{8}(\mathrm{i}+1)\tau\sqrt{|\mathrm{PP}(\sqrt{\tau})|}\frac{|\mathrm{PP}(\sqrt{\tau})+1|}{|\mathrm{PP}(\sqrt{\tau})+1|}\frac{\partial}{\partial\tau} \\ &= -\mathrm{i}\sqrt{8}(\mathrm{i}+1)\frac{\tau\sqrt{|\tau-1|}(|\sqrt{\tau}+1|+|\sqrt{\tau}-1|)}{4|\sqrt{\tau}|}\frac{\partial}{\partial\tau}, \end{aligned}$$

and also the symplectic form between b_1 and b_2 in $(\mathbb{CP}^1)^4$: $\omega(\mathrm{b}_1, \mathrm{b}_2) = -1$. Using this we find that the following expression for the symplectic form ω on the quotient in terms of $\tau \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$:

$$\omega = \frac{\mathrm{i}\mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}}{|\tau||\tau-1|(1+|\tau|+|\tau-1|)}.$$

It is now possible to calculate the curvature of the Kähler quotient, and we find that it is of constant curvature -1. From the expression we see that it has a cone point of angle π at each of the points $0, 1, \infty$. This confirms a theorem in [44] that the moduli spaces have conical complex hyperbolic structure.

Ricci potential

According to a formula derived in [31] we can calculate the Ricci potential of a Kähler quotient as follows: Let $\xi_1, \ldots, \xi_k$ be an orthonormal basis of the Lie algebra of the Lie group, J the complex structure, and let $\xi \in \wedge^k T^{1,0}X$ be $\wedge_{i=1}^k \frac{1}{2}(\underline{\xi} - \mathrm{i}J(\underline{\xi}))$. Then, if $\log F$ is the Ricci potential on X, $\log(\hat{F} + \|\xi\|^2)$ is the Ricci potential on $X/\!\!/G$, where $\hat{F}(p(x)) = F(x)$. For $n = 4$ we can calculate that

$$\|\xi\|^2 = 64\frac{|\mathrm{PP}(\sqrt{q})|(|\mathrm{PP}(\sqrt{q})|^2+1)^2}{|PP(\sqrt{q})+1|^6} = 16\frac{|\tau||\tau-1|}{(1+|\tau|+|\tau-1|)^3},$$

and, as the Ricci potential on $\mathbb{CP}^1$ is 0, this determines the Ricci potential on the quotient.

Calculation of $\langle\xi_i, \xi_j\rangle$

At the point given by $h(z)$ we can calculate the following inner products:

$$\langle\xi_1, \xi_2\rangle = -\frac{1}{2}\left(\frac{|\mathrm{PP}(\sqrt{q})|^2-1}{|\mathrm{PP}(\sqrt{q})|^2+1}\right)^2 = -2\frac{(\mathrm{Re}\sqrt{q})^2}{(|q|+1)^2} = \frac{1-3|\tau|+|\tau-1|}{1+|\tau|+|\tau-1|}$$

and

$$\langle \xi_1, \xi_3 \rangle = \frac{|\mathrm{PP}(\sqrt{q})|^4 - 6|\mathrm{PP}(\sqrt{q})| + 1}{(|\mathrm{PP}(\sqrt{q})|^2 + 1)^2} = \frac{1 + |\tau| - 3|\tau - 1|}{1 + |\tau| + |\tau - 1|}$$

CHAPTER 6

The Hitchin connection

In this section we follow [2] closely. We will see how it is possible, under some restrictions, to make our quantization independent of a choice of Kähler structure. On the quantum side, we are not interested in the actual Hilbert space, but only in $\mathbb{P}(\mathcal{H}) = \mathcal{H}/\mathbb{C}^\times$. We will show the independence on the complex structure by showing the existence of a projectively flat connection in a bundle over the space of complex structures, where the fiber over a point is the quantum space corresponding to this choice of Kähler polarization.

Let (M, ω) be a symplectic manifold and $I : \mathcal{T} \to \Gamma(\mathrm{End}(TM))$ a smooth map (i.e. a smooth section of the vector bundle $\pi_2^*(TM) \to \mathcal{T} \times M$) such that $I(\sigma)^2 = -1$ for all $\sigma \in \mathcal{T}$. $\mathcal{T}$ is a smooth manifold that parametrizes almost complex structures on M. Assume further that (M, ω, I_σ) is a Kähler for all σ.

Let $TM_{\mathbb{C}} = T_\sigma \oplus \bar{T}_\sigma$ be the splitting of the complexified tangent bundle into the eigenspace for i and $-i$ (i.e. into the holomorphic and antiholomorphic tangent spaces). Let $g_\sigma = \omega \cdot I_\sigma$ be the contraction of the symplectic form and the complex structure, so that $g_\sigma(X, Y) = \omega(X, I_\sigma(Y))$ is the metric associated to the Kähler manifold.

Let $V \in \Gamma(T\mathcal{T})$. Since, for $x \in M$ we have that $I(x, \cdot) : \mathcal{T} \to \mathrm{End}(T_x M)$ is a map from $\mathcal{T}$ into a fixed vector space, we can differentiate it along V. We will use the notation $V[I]$ for the result of doing so for each $x \in M$, which is a new map $V[I] : \mathcal{T} \to \Gamma(\mathrm{End}(TM))$. Since the square of I is constant, we get, by the Leibniz rule

$$V[I]I + IV[I] = 0,$$

so $V[I]_\sigma$ anticommutes with I_σ and thus it exchanges types

$$V[I] \in \Gamma((T_\sigma \otimes \bar{T}_\sigma^*) \oplus (\bar{T}_\sigma \otimes T_\sigma^*)).$$

We write $V[I]'$ for the projection to $\bar{T}_\sigma^* \otimes T_\sigma$ and $V[I]''$ for the projection to $T_\sigma^* \otimes \bar{T}_\sigma$. Assume now that $\mathcal{T}$ is a complex manifold and $V'[I] = V[I]'$ and $V''[I] = V[I]''$ (we say that I is holomorphic – indeed, the almost complex structure on $\mathcal{T} \times M$ obtained by combining the complex structure on $\mathcal{T}$ with I is integrable if and only if I is holomorphic in this sense, and each I_σ is integrable), where V' and V'' denote the $(1,0)$ and $(0,1)$ part of V, respectively. Remark that $V[g] = \omega \cdot V[I]$. g is symmetric and ω is $(1,1)$ so we get that

$$V[g]_\sigma \in \Gamma(S^2(T_\sigma^*) \oplus S^2(\bar{T}_\sigma^*))$$

We define $\tilde{G}(V)$ by

$$\tilde{G}(V)\omega = V[I],$$

for all vector fields V on $\mathcal{T}$, and by the types of $V[I]$ and ω, we must have a decomposition as:

$$\tilde{G}(V) = G(V) + \bar{G}(V),$$

where $G(V) \in (T)^{\otimes 2}$, and $\bar{G}(V) \in (\bar{T})^{\otimes 2}$. We have that $V[g] = \omega V[I] = \omega \tilde{G} \omega$, and from this it follows that $\tilde{G}$ – and thus G and $\bar{G}$ – must be symmetric. If $\bar{\partial}_\sigma G(V)_\sigma = 0$ for all vector fields on $\mathcal{T}$, we call the family ***rigid***.

Assume that we have a prequantum line bundle over M, i.e. a complex line bundle $\mathcal{L}$ over M with a hermitian structure and a compatible connection with curvature $-i\omega$. We can define

$$(\bar{\partial}_\sigma)_X s = \nabla_{\frac{1}{2}(1+iI_\sigma)X} s$$

for $s \in \Gamma(\mathcal{L})$ and $X \in TM_{\mathbb{C}}$, and

$$H_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k) = \{s \in \Gamma(\mathcal{L}^k) \mid \bar{\partial}_\sigma s = 0\},$$

which is finite dimensional for each σ (this follows from the theory of elliptic differential operators on compact manifolds). We will assume that these vector spaces fit together as finite dimensional subbundle of the infinite dimensional vector bundle $\mathcal{H}^{(k)} = \mathcal{T} \times \Gamma(\mathcal{L}^k)$ over $\mathcal{T}$. Denote by $\hat{\nabla}^t$ the trivial connection in this bundle, and let $\mathcal{D}(M, \mathcal{L}^k)$ be the set of differential operators acting on $\mathcal{L}^k$. For a $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ we have a connection in $\mathcal{H}^{(k)}$ given by $\hat{\nabla}_V = \hat{\nabla}_V^t - u(V)$.

Definition 63 (Hitchin connection). A ***Hitchin connection*** in $\mathcal{H}^{(k)}$ is a connection of the form $\nabla^t + u$, $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}))$, that preserves the subbundle $\sqcup_\sigma H_\sigma^{(k)}$.

Observe that the existence of such a connection will guarantee that $H_\sigma^{(k)}$ forms a smooth subbundle, and that the Hitchin connection restricts to a connection in this bundle.

6.1 Construction of a Hitchin connection

We will first reformulate the condition of preserving $H^{(k)}$:

Lemma 64. *The connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$ preserves $H^{(k)}$ if and only if*

$$\frac{i}{2} V[I] \nabla^{1,0} s + \nabla^{0,1} u(V) s = 0$$

for all vector fields V on $\mathcal{T}$ and smooth sections s of $H^{(k)}$.

Proof. We differentiate $\nabla^{0,1} s = 0$ and obtain

$$0 = V\left[\nabla^{0,1} s\right] = V\left[\frac{1}{2}(1 + iI)\nabla s\right] = \frac{i}{2} V\left[I\right] \nabla s + \nabla^{0,1} V\left[s\right] = \frac{i}{2} V\left[I\right] \nabla^{1,0} s + \nabla^{0,1} V\left[s\right].$$

We are now ready to check when we get an element of $H^{(k)}$, that is, when the following is 0:

$$\begin{aligned}\nabla_\sigma^{0,1}\left(\left(\hat{\nabla}_V(s)\right)_\sigma\right) &= \nabla_\sigma^{0,1} V\left[s\right] + \nabla_\sigma^{0,1}\left((u(V)s)_\sigma\right) \\ &= -\frac{i}{2}\left(V\left[I\right] \nabla^{1,0} s\right)_\sigma - \nabla_\sigma^{0,1}\left((u(V)s)_\sigma\right)\end{aligned}$$

□

Denote by R_σ the Riemannian curvature tensor on (M,ω,I_σ) and define the Ricci tensor as $\mathrm{ric}_\sigma(X,Y)=\mathrm{Tr}(\zeta\mapsto R_\sigma(\zeta,X)Y)$, and the ***Ricci form*** as $\rho_\sigma(X,Y)=\mathrm{ric}_\sigma(I_\sigma(X),Y)$. We define a second order differential operator $\Delta_G:\Gamma(\mathcal{L}^k)\to\Gamma(\mathcal{L}^k)$ by the composition

$$\begin{aligned}\Gamma(\mathcal{L}^k) &\xrightarrow{\nabla^{1,0}_\sigma} \Gamma(T^*_\sigma\otimes\mathcal{L}^k) \xrightarrow{G\otimes\mathrm{id}}\\ \Gamma(T_\sigma\otimes\mathcal{L}^k) &\xrightarrow{\nabla^{1,0}_\sigma\otimes\mathrm{id}+\mathrm{id}\otimes\nabla^{1,0}_\sigma} \Gamma(T^*_\sigma\otimes T_\sigma\otimes\mathcal{L}^k) \xrightarrow{\mathrm{Tr}} \Gamma(\mathcal{L}^k),\end{aligned}$$

and we note that this operator have symbol GV. We can now define the following second order differential operator for $k,n\in\mathbb{Z}$, $2k+n\neq 0$:

$$u(V)=\frac{1}{2k+n}\left(\frac{1}{2}\Delta_{G(V)}-\nabla_{G(V)dF}-nV'[F]\right)+V'[F],$$

Where F_σ is the Ricci potential of integral 0, that is, a function such that $\mathrm{ric}_\sigma - n\omega = 2\mathrm{i}\partial\bar\partial F_\sigma$. We will use the rest of this section to show that it defines a Hitchin connection.

Lemma 65. *Assume that the first Chern class of (M,ω) is $n[\omega]\in H^2(M,\mathbb{Z})$. For any $\sigma\in T$ and for any $G\in H^0(M_\sigma,S^2(T_\sigma))$ we have the formula:*

$$\nabla^{0,1}_\sigma(\Delta_G(s)-2\nabla_{GdF_\sigma}(s))=-i(2k+n)G\omega\nabla^{1,0}_\sigma(s)-ik\,\mathrm{Tr}(-2G\partial_\sigma F\omega+\nabla^{1,0}_\sigma(G)\omega)s$$

for all $s\in H^0(M_\sigma,\mathcal{L}^k)$

Proof. The idea is to move the $\nabla^{0,1}$ toward the s by commuting it with whatever is in front of it. This produces a lot of curvature terms, both the Riemannian curvature R_σ from the tangent bundle and $-i\omega$ from the prequantum bundle. It is a tensorial calculation, so we will assume that everything is done in a basis of tangent vectors whose Lie bracket vanishes, so that we can replace the commutator of a connection with the curvature. We will also use that the tensor product connection is compatible with Tr. We observe that if $X\in T^{0,1}$ then $\iota_X\rho_\sigma=-i\iota_X\,\mathrm{ric}_\sigma$. We will write $G\nabla^{1,0}s$ instead of $\mathrm{Tr}(G\nabla^{1,0}s)$ and similarly for other contractions.

$$\begin{aligned}
&{\nabla^{0,1}_\sigma}_X(\Delta_G(s))=\mathrm{Tr}\left(\nabla^{0,1}_X\nabla^{1,0}\left(G\nabla^{1,0}s\right)\right)=\mathrm{Tr}\left(\nabla^{1,0}\nabla^{0,1}_X\left(G\nabla^{1,0}s\right)\right)+\mathrm{Tr}\left(\iota_XF_\nabla G\nabla^{1,0}s\right)\\
&=\mathrm{Tr}\left(\nabla^{1,0}\nabla^{0,1}_X\left(G\nabla^{1,0}s\right)\right)+\mathrm{Tr}\left(\mathrm{id}\otimes(-ik\iota_X\omega)\right)G\nabla^{1,0}s+\left(\iota_XR_\sigma\otimes\mathrm{id}\right)\\
&=\mathrm{Tr}\left(\nabla^{1,0}\nabla^{0,1}_X\left(G\nabla^{1,0}s\right)\right)-ik\,\mathrm{Tr}\left(\iota_X\omega G\nabla^{1,0}s\right)+\mathrm{Tr}\left(-\iota_X\,\mathrm{ric}_\sigma G\nabla^{1,0}s\right)\\
&=\mathrm{Tr}\left(\nabla^{1,0}\left(\nabla^{0,1}_XG)\nabla^{1,0}s+G\nabla^{0,1}_X\nabla^{1,0}s\right)\right)-ik\,\mathrm{Tr}\left(\iota_X\omega G\nabla^{1,0}s\right)+\mathrm{Tr}\left(-\iota_X\,\mathrm{ric}_\sigma G\nabla^{1,0}s\right)\\
&=\mathrm{Tr}\left(\nabla^{1,0}G\nabla^{0,1}_X\nabla^{1,0}s\right)-ik\,\mathrm{Tr}\left(\iota_X\omega G\nabla^{1,0}s\right)+\mathrm{Tr}\left(-\iota_X\,\mathrm{ric}_\sigma G\nabla^{1,0}s\right)\\
&=-ik\,\mathrm{Tr}\left(\nabla^{1,0}\left(G\iota_X\omega s\right)\right)-ik\,\mathrm{Tr}\left(\iota_X\omega G\nabla^{1,0}s\right)+\mathrm{Tr}\left(-\iota_X\,\mathrm{ric}_\sigma G\nabla^{1,0}s\right)\\
&=-ik\,\mathrm{Tr}\left(\nabla^{1,0}\left(G\iota_X\omega s\right)\right)-ik\,\mathrm{Tr}\left(\iota_X\omega G\nabla^{1,0}s\right)-i\,\mathrm{Tr}\left(\iota_X\rho_\sigma G\nabla^{1,0}s\right)\\
&=-ik\,\mathrm{Tr}\left(\nabla^{1,0}\left(G\iota_X\omega s\right)\right)-ik\,\mathrm{Tr}\left(\iota_X\omega G\nabla^{1,0}s\right)-i\,\mathrm{Tr}\left(\iota_X\rho_\sigma G\nabla^{1,0}s\right)\\
&=-ik\,\mathrm{Tr}\left(\nabla^{1,0}\left(G\right)\iota_X\omega s+G\nabla^{1,0}(\iota_X\omega)s+G\iota_X\omega\nabla^{1,0}s\right)-ik\,\mathrm{Tr}\left(\iota_X\omega G\nabla^{1,0}s\right)-i\,\mathrm{Tr}\left(\iota_X\rho_\sigma G\nabla^{1,0}s\right)\\
&=-ik\,\mathrm{Tr}\left(\nabla^{1,0}\left(G\right)\iota_X\omega s\right)-2ik\,\mathrm{Tr}\left(\iota_X\omega G\nabla^{1,0}s\right)-i\,\mathrm{Tr}\left(\iota_X\rho_\sigma G\nabla^{1,0}s\right)\\
&=-ik\,\mathrm{Tr}\left(\nabla^{1,0}\left(G\right)\iota_X\omega s\right)-2ik\,\mathrm{Tr}\left(\iota_X\omega G\nabla^{1,0}s\right)-i\,\mathrm{Tr}\left(\iota_Xn\omega G\nabla^{1,0}s\right)-i\,\mathrm{Tr}\left(\iota_X2i\partial\bar\partial FG\nabla^{1,0}s\right),\quad\blacksquare
\end{aligned}$$

where we have used the holomorphicity of G (to see that $\nabla^{0,1}G=0$), that $\nabla\omega=0$ since the connection is induced from a Kähler metric and that mixed covariant derivatives vanish, to conclude that $\nabla^{1,0}\iota_X\omega = \operatorname{Tr}(\nabla^{1,0}X \otimes \omega + X \otimes \nabla^{1,0}\omega) = 0$. We assumed that $c_1(M,\omega) = n[\omega]$, so the curvature in the canonical bundle must belong to the cohomology class of $n\omega$. But the Ricci form is the curvature of the induced connection on this bundle, and since the Kähler form ω is harmonic, we must have that the harmonic part of the Ricci curvature, ρ_σ^H, satisfies $\rho_\sigma^H = n\omega$, since every cohomology class has exactly one harmonic representable. We need the following fact (see e.g. [32], Prop. 1.7): Any exact form $\alpha \in \Omega^{p,q}(M)$ on a compact Kähler manifold can be written as $\alpha = 2i\partial\bar\partial\beta$ for some $\beta \in \Omega^{p-1,q-1}(M)$. This implies that there exists a $F_\sigma \in C^\infty(M)$ such that $\rho_\sigma = n\omega + 2i\partial\bar\partial F_\sigma$. We observe that $\nabla_{GdF}s = \operatorname{Tr}(\partial_\sigma F G \nabla^{1,0}s)$ so we have that

$$\begin{aligned}
\nabla_X^{0,1}\nabla_{GdF}s &= \operatorname{Tr}\left((\iota_X\bar\partial\partial F)G\nabla^{1,0}s + \partial F(\iota_X\bar\partial G)\nabla^{1,0}s + \partial F G \nabla_X^{0,1}\nabla^{1,0}s\right)\\
&= \operatorname{Tr}(\iota_X\bar\partial\partial F G\nabla^{1,0}s - \partial F G i k \iota_X\omega s)\\
&= -\operatorname{Tr}(\iota_X\partial\bar\partial F G\nabla^{1,0}s + ik\partial F G\iota_X\omega s).
\end{aligned}$$

We are now ready to calculate the desired quantity

$$\begin{aligned}
&\nabla_X^{0,1}(\Delta_G(s) + 2\nabla_{GdF_\sigma}(s))\\
&= -ik\operatorname{Tr}\left(\nabla^{1,0}(G)\,\iota_X\omega\right)\otimes s - i\operatorname{Tr}\left(n\iota_X\omega G\otimes\nabla^{1,0}s\right) - i\operatorname{Tr}\left(2i\iota_X\partial\bar\partial F G\otimes\nabla^{1,0}s\right)\\
&- 2ik\operatorname{Tr}\left(\iota_X\omega G\left(\nabla^{1,0}s\right)\right) - 2\operatorname{Tr}(\iota_X\partial\bar\partial F G\nabla^{1,0}s + ik\partial F G\iota_X\omega s)\\
&= -ik\operatorname{Tr}\left(\nabla^{1,0}(G)\,\iota_X\omega\right)\otimes s - i(2k+n)\operatorname{Tr}\left(\iota_X\omega G\otimes\nabla^{1,0}s\right) - 2ik\operatorname{Tr}(\partial F G\iota_X\omega s)
\end{aligned}$$

□

Lemma 66. *If $H^1(M,\mathbb{R}) = 0$ we have the relation*

$$2i\bar\partial_\sigma(V'[F]_\sigma) = \frac{1}{2}\operatorname{Tr}(2G(V)\partial(F)\omega - \nabla^{1,0}(G(V))\omega)_\sigma$$

Proof. Recall that

$$\rho_\sigma = \rho_\sigma^H + 2i\partial\bar\partial_\sigma F_\sigma = n\omega + 2i\mathrm{d}\bar\partial_\sigma F_\sigma$$

So

$$V'[\rho] = 2i(V[\mathrm{d}\bar\partial])F + 2i\mathrm{d}\bar\partial V'[F] = 2i(\frac{i}{2}\mathrm{d}V'[I]dF) + 2i\mathrm{d}\bar\partial V'[F].$$

Observe that the $(0,1)_\sigma$-form

$$\frac{1}{2}\operatorname{Tr}\left(2G(V)\partial F\omega - \nabla^{1,0}G(V)\omega\right)_\sigma - 2i\bar\partial_\sigma V'[F]_\sigma$$

is ∂_σ-closed, by lemma 67. It is also $\bar\partial_\sigma$-closed, since the last term clearly is so, and the first two terms are the last two terms of the equality in Lemma 65 (which also holds locally), in which the first term of the right hand side is $\bar\partial$-closed since $\bar\partial$ of it reduces, after commuting with $\nabla^{1,0}s$ and producing a curvature term, to $G^{ij}\mathrm{d}\bar z_i \wedge \mathrm{d}\bar z_j = 0$ by the symmetry of G. So it is a closed form, and therefore, by hypothesis, an exact form. But this means that it is equal to $\mathrm{d}h$ for some function h, but for this to be a $(0,1)$-form, we must have that $\partial h = 0$. But the only antiholomorphic maps on a compact manifold are the constant maps, so this implies that h is constant, and thus that $\mathrm{d}h = 0$. This gives the desired result. □

Lemma 67. *For any smooth vector field V on $\mathcal{T}$ we have that*

$$(V'[\rho])^{1,1} = -\frac{1}{2}\partial\operatorname{Tr}\left(\nabla^{1,0}(G(V))\omega\right).$$

The 2-form ρ_σ represents the first Chern class of the canonical line bundle – in our case, $n[\omega]$. This implies that the harmonic part is independent of σ: $\rho_\sigma^H = n\omega$.

Theorem 38 (Andersen). *Assume as above that the family $\mathcal{T}$ is holomorphic and rigid, $H^1(M,\mathbb{R}) = 0$, that the first Chern class of $(M,\omega) = n[\omega]$. Then the connection $\nabla^t + u$ is a Hitchin connection.*

Proof. By 64 it is enough to show that $\nabla^{0,1}u(V)s = -\frac{i}{2}V'[I]\nabla^{1,0}s$ for all vector fields V on $\mathcal{T}$ and sections s of $H^{(k)}$.

$$\begin{aligned}
(2k+n)\nabla^{0,1}u(V)s &= \frac{1}{2}\nabla^{0,1}(\Delta_{G(V)} - 2\nabla_{G(V)dF}) + 2k\nabla^{0,1}V'[F]s \\
&= \frac{1}{2}\left(-i(2k+n)G\omega\nabla^{1,0}s - ik\operatorname{Tr}(-2G\partial F\omega + \nabla^{1,0}(G)\omega s)\right) + 2k\nabla^{0,1}V'[F]s \\
&= (2k+n)\frac{-i}{2}G\omega\nabla^{1,0}s + 2iik\bar\partial(V'[F])s + 2k\nabla^{0,1}V'[F]s \\
&= (2k+n)\frac{-i}{2}G\omega\nabla^{1,0}s \\
&= (2k+n)\frac{-i}{2}V'[I]\nabla^{1,0}s,
\end{aligned}$$

Where we use Lemma lemma 65 on page 61 in the first line, Lemma lemma 66 on page 62 in the second and the definition of G, and in the fourth line that $\nabla^{0,1}fs = \bar\partial fs + f\nabla^{0,1}s$. $\square$

6.2 Asymptotic faithfulness

In this section we will describe the main result of [1], which uses the Hitchin connection for the moduli space of flat connections on a closed surface Σ. However, to get a smooth and compact moduli we need to consider a slightly different setup. Let Σ be a surface with one puncture, and γ a loop around the puncture. The fundamental group is given by $2g+1$ generators $A_1,\dots,A_g$ and $B_1,\dots,B_g$ and γ, with the only relations $\prod_i [A_i, B_i] = \gamma$ (so it is also freely generated by the A_i's and B_i's). Let $\operatorname{Hom}_d(\Sigma, \mathrm{SU}(n))$ be the representations π with $\pi(\gamma) = e^{2\pi i d/n}\mathrm{id}_n$. Assume that π is a reducible representation: $\pi = \pi_0 \oplus \pi_1$. Then $\pi_0(\gamma) = e^{2\pi i d/n}\mathrm{id}_{\dim \pi_0}$. But $e^{2\pi i d \dim \pi_0/n} = \det \pi_0(\gamma) = \prod_i \det([A_i, B_i]) = 1$, which implies that $d \dim \pi_0$ divides n. This shows that $\operatorname{Hom}_d(\pi_1(\Sigma), \mathrm{SU}(n))$ only consists of irreducible representations, and therefore the moduli space $\mathcal{M}_g^d$ is a smooth manifold. There is a description similar to the above, that involves taking symplectic reduction at a non-zero coadjoint orbit to obtain all connections with constant, central holonomy, and we can use a description as the one for flat connections, see [38]. It was shown by Axelrod, della Pietra and Witten [15], and Hitchin [38], that geometric quantization applies to these moduli spaces for $g > 2$ with Kähler structure parametrized by Teichmüller space, and they constructed a Hitchin connection. Freed showed [29] that the action of the mapping class group on the moduli space lift to an action of the prequantum line bundle. The Hitchin connection induces a flat connection in the projectivized bundle, and it is clear from Andersen construction that the mapping class group preserves the Hitchin connection. We therefore have an action of the mapping class group on the space of smooth section of $H^{(k)} \to \mathcal{T}$ by $(\varphi \cdot s)(x) = \varphi(s(\varphi^{-1}(x)))$. Since $\mathrm{MCG}(\Sigma)$ preserves the connection, we can restrict the representation to a representation ρ_d^k on the finite dimensional space of covariant constant projective sections. The fiber over a single point can, via parallel transport, be identified with the space of covariant constant sections. Using this description, let σ be the chosen base point and s a covariant constant section. The action on $s(\sigma)$ is then given by:

$$\varphi \cdot (s(\sigma)) = (\varphi \cdot s)(\sigma) = \varphi(s(\varphi^{-1}\sigma)) = \varphi(P_{\sigma,\varphi^{-1}(\sigma)}s(\sigma)) = P_{\varphi(\sigma),\sigma}\varphi(s(\sigma))$$

In [1] it is shown that

Theorem 39 (Andersen). *The Toeplitz operators are asymptotically flat with respect to the connection in* $\mathrm{End}(H^{(k)})$ *induced by the Hitchin connection:*

$$\|P^{(k)}_{\sigma_1,\sigma_2}T^{(k)}_{f,\sigma_1} - T^{(k)}_{f,\sigma_2}\|_{op} = \mathrm{O}(k^{-1})$$

Theorem 40 (Andersen). *For any* $\varphi \in \mathrm{MCG}(\Sigma)$*:*

$$\varphi \in \bigcap_{k=1}^{\infty} \ker \rho_d^k$$

(if and) only if φ *induces the identity on* M*.*

Proof. $\varphi \in \Gamma$ induces an symplectomorphism on M, which we will also denote by φ. Let $f \in C^\infty(M)$. We denote by $T_f, \sigma^{(k)}$ the Toeplitz operator $s \mapsto \pi^k_\sigma(fs)$. We have the following commutative diagram:

$$\begin{tikzcd}
H^0(M_\sigma, \mathcal{L}^k_\sigma) \arrow[r, "\varphi"] \arrow[d, "T^{(k)}_{f,\sigma}"] & H^0(M_{\varphi(\sigma)}, \mathcal{L}^k_{\varphi(\sigma)}) \arrow[r, "P_{\varphi(\sigma),\sigma}"] \arrow[d, "T^{(k)}_{f\circ\varphi^{-1},\varphi(\sigma)}"] & H^0(M_\sigma, \mathcal{L}^k_\sigma) \arrow[d, "P_{\varphi(\sigma),\sigma}T^{(k)}_{f\circ\varphi^{-1},\varphi(\sigma)}"] \\
H^0(M_\sigma, \mathcal{L}^{(k)}_\sigma) \arrow[r, "\varphi"] & H^0(M_{\varphi(\sigma)}, \mathcal{L}^{(k)}_{\varphi(\sigma)}) \arrow[r, "P_{\varphi(\sigma),\sigma}"] & H^0(M_\sigma, \mathcal{L}^{(k)}_\sigma)
\end{tikzcd}$$

where the commutativity of the last square is by definition of the parallel transport in the endomorphism bundle, and in the first because

$$\varphi\pi^k_\sigma(fs) = \pi^k_{\varphi(\sigma)}(\varphi f s) = \pi^k_{\varphi(\sigma)}((f\circ\varphi^{-1})\varphi s)$$

Suppose now that φ is in the kernel for all k. So for each k, the rows are just multiplication by the same nonzero constant – thus, the first and last vertical arrow must be equal

$$T^{(k)}_{f,\sigma}V = P_{\varphi(\sigma),\sigma}T^{(k)}_{f\circ\varphi^{-1},\varphi(\sigma)}.$$

But this now implies that

$$\lim_{k\to\infty}\|T^{(k)}_{f-f\circ\varphi^{-1},\sigma}\| = \lim_{k\to\infty}\|T^{(k)}_{f,\sigma} - T^{(k)}_{f\circ\varphi^{-1},\sigma}\| = \lim_{k\to\infty}\|P_{\varphi(\sigma),\sigma}T_{f\circ\varphi^{-1},\varphi\sigma} - T^{(k)}_{f\circ\varphi^{-1},\sigma}\| = 0$$

where the last equality is (39). By a theorem of Bordemann, Meinrenken and Schlichenmaier [23] $\lim_{k\to\infty}\|T^{(k)}_f\| = \sup|f|$, and therefore we must have that $f = f\circ\varphi^{-1}$. Since $C^\infty(M)$ separates points, φ must act by the identity on M. □

It is known that the only element of $\mathrm{MCG}(\Sigma)$ that acts by the identity on M is the identity or the hyperelliptic involution in genus 2, in the case of $\mathrm{SU}(2)$ and with even holonomy around a marked point (This is in appendix 10 in [33]. The basic idea is using the inclusion of $\mathrm{SU}(2)$ into G to reduce the problem to $\mathrm{SU}(2)$, and then complexify to obtain $\mathrm{Hom}(\pi_1, \mathrm{SL}(2,\mathbb{C}))$. The space has the Teichmüller space as a $\mathrm{MCG}(\Sigma)$-invariant subspace, and the only non-identity mapping class group elements that fix every complex structure are the hyperelliptic involution mentioned).

6.3 Metaplectic correction

There is a way to relax the requirements in theorem 38, as shown in [6]. Here the quantum spaces are replaced by the *metaplectic corrected* quantum spaces. Metaplectic corrected geometric quantization, also known as half-form quantization, is a refinement of geometric quantization as discussed in section 4.2. Standard geometric quantization fails to reproduce the canonical quantization of the harmonic oscillator: the quantization of the Hamiltonian does not have the right spectrum: the physical spectrum of the harmonic oscillator are the *positive* half-integers, but geometric quantization of the Hamiltonian have as spectrum all the *non-negative* half-integers, that is, the physical spectrum shifted down by $\frac{1}{2}$. Metaplectic correction corrects this mismatch by introducing a square root of the canonical bundle of M.

Theorem 41 (Andersen, Gammelgaard, Roed-Lauridsen). *Let (M, ω) be a prequantizable symplectic manifold with vanishing first Stiefel-Whitney class. Let I be a rigid family of M, parametrized by a smooth manifold $\mathcal{T}$, such that $\mathrm{H}^{0,1}(M_\sigma, \mathbb{C}) = 0$ for all $\sigma \in \mathcal{T}$. Then there exists a family of metaplectic structures, also parametrized by $\mathcal{T}$, that is, a line bundle $\delta \to \mathcal{T} \times M$ such that $\delta_\sigma = \delta_{\restriction\{\sigma\}\times M}$ is a square root of the canonical bundle $K_\sigma = \Lambda^{top} T^*_\sigma M$. Let $\mathcal{L}$ be a prequantum line bundle over M. Then there exists a one-form $b \in \Omega^1(\mathcal{T} \times M)$ such that the connection*

$$\nabla^{Hit}_V = \nabla^t_V + \frac{1}{4k}\left(\Delta_{\mathrm{G}(V)} + b(V)\right)$$

in the bundle $\mathcal{T} \times C^\infty(M, \mathcal{L}^{\otimes k} \otimes \delta)$ preserves the subbundle

$$\coprod_{\sigma \in \mathcal{T}} \mathrm{H}^0(M_\sigma, \mathcal{L}_\sigma^{\otimes k} \otimes \delta_\sigma) \to \mathcal{T}.$$

Furthermore,

Theorem 42 (Andersen-Gammelgaard, [32]). *If $\mathrm{H}^0(M_\sigma, \mathrm{T}_\sigma) = 0$ for all $\sigma \in \mathcal{T}$ then the Hitchin connection of theorem 41 is projectively flat.*

Chapter 7

The geometrized KZ connection

7.1 The KZ connection

In this section we introduce the KZ connection. Let G be a compact Lie group, $\mathfrak{g}$ the Lie algebra of G and let $\{I_\nu\}_\nu$ denote an orthonormal basis for $\mathfrak{g}$ with respect to the Killing form.

Definition 68. The ***Knizhnik–Zamolodchikov***, or KZ, connection at level k is a connection in the trivial line bundle with fiber $\left(V_\lambda^{\otimes n}\right)^{\mathfrak{g}}$ over $\mathrm{Conf}_n(\mathbb{C})$, given by

$$\nabla^{KZ} = \mathrm{d} + \frac{1}{k+\mathrm{h}} \sum_{1\leq i<j\leq n} \Omega^{ij} \frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j},$$

where d is the trivial connection, h is the dual Coxeter number of $\mathfrak{g}$ and Ω^{ij} is the action of the quadratic Casimir $\Omega = \sum_\nu I_\nu \otimes I_\nu \in \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$ when $V_\lambda^{\otimes n}$ is given the structure of a $\mathrm{U}(\mathfrak{g})^{\otimes 2}$-module by $(X\otimes Y)(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes v_2 \otimes \cdots \otimes (Xv_i) \otimes \cdots \otimes (Yv_j) \otimes \dots v_n$, that is

$$\Omega^{ij}(v_1 \otimes \cdots \otimes v_n) = \sum_\nu v_1 \otimes v_2 \otimes \cdots \otimes (I_\nu v_i) \otimes \cdots \otimes (I_\nu v_j) \otimes \dots v_n.$$

Let $\Delta : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})^{\otimes 2}$ be the algebra homomorphism given by $X \mapsto X \otimes 1 + 1 \otimes X$, for $X \in \mathfrak{g}$. Then $\Omega = \frac{1}{2}(\Delta(C) - 1 \otimes C - C \otimes 1)$, where $C = \sum_\nu I_\nu \cdot I_\nu \in \mathrm{U}(\mathfrak{g})$ is the ordinary Casimir element. Since $C \otimes 1 - 1 \otimes C$ is in the center, and $[\Delta(C), \Delta(X)] = \Delta([C, X]) = 0$ we have that $[\Omega, \Delta(X)] = 0$. Therefore Ω^{ij} preserves the $\mathfrak{g}$-invariant part of $V_\lambda^{\otimes n}$. The curvature of a connection of the form $\mathrm{d} + A$ is $\mathrm{d}A + \frac{1}{2}[A, A]$. It is easy to check that

$$\mathrm{d}\left(\sum_{1\leq i<j\leq n} \Omega^{ij} \frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j}\right) = 0,$$

by observing that a local choice of logarithm gives us $\frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j} = \mathrm{d}\log(z_i - z_j)$. Using the algebraic properties of Ω^{ij} it can be shown that

$$\left[\sum_{1\leq i<j\leq n} \Omega^{ij} \frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j}, \sum_{1\leq i<j\leq n} \Omega^{ij} \frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j}\right] = 0,$$

and thus the KZ connection is flat for all k. The following theorem allow us to relate the monodromy of the KZ connection to the quantum representations discussed in 2:

Theorem 43 (Drinfeld-Kohno). *The monodromy of the KZ connection for $\mathfrak{g}$ with $\vec{\lambda}$ along a braid b is equal to the RTW-TQFT quantum representation representation of* G *applied to b, seen as an element in the mapping class group of the sphere with $n+1$ marked points, the first n labeled with $\vec{\lambda}$ and the last with* 0.

7.2 Construction of the geometrized KZ connection

The goal of this section is to construct a new connection, the ***geometrized KZ connection***, which interpolates between the KZ and the Hitchin connections. From the construction it will be clear that it is just a "reformulation" of the algebraic KZ connection. However, the actual formulas will rather make us think of the Hitchin connection. Concretely we will find a Kähler manifold X_{G} with a prequantum line bundle $L_{\mathrm{G}} \to X_{\mathrm{G}}$ such that $\mathrm{H}^0(X_{\mathrm{G}}, L_{\mathrm{G}}) \cong \left(V_\lambda^{\otimes n}\right)^{\mathfrak{g}}$ and differential operators Ω^{ij}_{G} acting on sections of L_{G} representing the action of Ω^{ij}. This is the data we need to write the KZ connection in the same setup as the Hitchin connection.

The G representation V_λ can be realized as the geometric quantization of the coadjoint orbit through λ: $V_\lambda \cong \mathrm{H}^0(\mathcal{O}_\lambda, \mathcal{L}_\lambda)$, as we saw in 4.2.2. To get the representation $V_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ we take the exterior tensor product as in section 4.2.1:

$$\bigotimes_i \mathrm{H}^0(\mathcal{O}_{\lambda_i}, \mathcal{L}_\lambda) \cong \mathrm{H}^0(\mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n}, \mathcal{L}_{\lambda_1} \boxtimes \ldots \boxtimes \mathcal{L}_{\lambda_n}),$$

where $\xi \in \mathfrak{g}$ acts by $\nabla_{\underline{\xi}} - \mathrm{i}\mu_{\vec{\lambda}}(\xi)$, $\mu_{\vec{\lambda}} : \mathcal{O}_{\vec{\lambda}} \hookrightarrow (\mathfrak{g}^*)^n \to \mathfrak{g}^*$ is the inclusion followed by the sum of components, and $\underline{\xi}$ is the fundamental vector field generated by ξ under the diagonal action of G. We can also act on a single coordinate by $\nabla_{\underline{\xi}^i} + i(\mu_i(\xi) \circ \pi_i)$, where $\underline{\xi}^i$ is the vector field generated by the action of G on the i'th factor and $\mu_i : \mathcal{O}_{\lambda_i} \to \mathfrak{g}^*$ is the inclusion and π_i is the projection to the i factor. This means that we can realize the operators Ω^{ij} on $V_{\vec{\lambda}}$ by differential operators on $\mathrm{H}^0(\mathcal{O}_{\vec{\lambda}}, \mathcal{L}_{\vec{\lambda}})$ of order 2:

$$\Omega^{jk}_{\mathcal{L}} = \sum_\nu \left(\nabla_{\underline{\nu}^j} - i\mu(\nu) \circ \pi_j\right) \left(\nabla_{\underline{\nu}^k} - i\mu(\nu) \circ \pi_k\right).$$

For the KZ connection we need not the full tensor product, but only the invariant part. From Theorem 26 we realize that the symplectic quotient of $\mathcal{O}_{\vec{\lambda}}$ by the diagonal action of G would be the right space to support a line bundle whose sections are in correspondence with the invariants. The theorem does however not apply directly: 0 is not a regular value of μ and the action of G on $\mu^{-1}(0)$ is not free. Instead, we have the following, when specializing to $\mathrm{G} = \mathrm{SU}(2)$:$\mathfrak{su}(2)$ acts on $\mathfrak{su}(2)^* \cong \mathbb{R}^2$ by rotations, factoring through the map $\mathrm{SU}(2) \to \mathrm{SO}(3)$. This puts us in exactly the situation discussed in section 5.7, from which we can extract the following:

Proposition 69. *The action of* $\mathrm{SU}(2)$ *on $\mathcal{O}_{\vec{\lambda}}$ factors through* $\mathrm{SO}(3)$. *The points in $\mu^{-1}(0)$ with non-trivial* $\mathrm{SO}(3)$ *stabilizer is exactly the non-regular points in $\mu^{-1}(0)$ and they form a* $\mathrm{SO}(3)$ *invariant subset of real dimension 2.*

To be able to apply the symplectic reduction theorem, we want to remove a union of complex submanifolds $N = \cup_i N_i$ of codimension ≥ 2 from $\mathcal{O}_{\vec{\lambda}}$ such that this union is preserved by $\mathrm{SU}(2)$ and the intersection of these submanifolds with $\mu^{-1}(0)$ exactly is the points with non-trivial stabilizer under the $\mathrm{SO}(3)$ action. We choose the submanifolds with at least $\frac{n}{2}$ equal points, which have complex codimension $n - (\lfloor \frac{n}{2} \rfloor + 1)$, which is > 1 for $n > 4$. For $n = 4$ we need to take some special care, which we will discuss later in section 9.2. Hartogs extension theorem guarantees that the restriction map from $\mathrm{H}^0(\mathcal{O}_{\vec{\lambda}}, \mathcal{L}_{\vec{\lambda}})$ to $\mathrm{H}^0(\mathcal{O}_{\vec{\lambda}} \setminus N, \mathcal{L}_{\vec{\lambda}})$ is an isomorphism. We can now take the symplectic reduction for the diagonal action of SU(2) on $X = \mathcal{O}_{\vec{\lambda}} \setminus N$, which has the moment map $\mu_{\vec{\lambda}} : \times_i \mathcal{O}_{\lambda_i} \to \mathfrak{g}^*$ given by $\mu_{\vec{\lambda}}(\xi_1, \dots, \xi_N) = \sum_i \xi_i$. From Theorem 26 and Hartogs extension theorem we now obtain

$$\mathrm{H}^0(X/\!\!/\mathrm{G}, L/\!\!/\mathrm{G}) \cong \mathrm{H}^0(\mathcal{O}_{\vec{\lambda}} \setminus \cup N_i, \mathcal{L}_{\vec{\lambda}})^{\mathrm{G}} \cong \mathrm{H}^0(\mathcal{O}_{\vec{\lambda}}, \mathcal{L}_{\vec{\lambda}})^{\mathrm{G}}.$$

We let $X_{\mathrm{G}} = X/\!\!/\mathrm{G}$ and $L_{\mathrm{G}} = L/\!\!/\mathrm{G}$. We want to show that $\Omega^{ij}_{\mathcal{L}}$ is the pullback of a differential operator Ω^{ij}_{G} in the line bundle $L_{\mathrm{G}} \mapsto X_{\mathrm{G}}$ in the following sense: we want to find another differential operator $\tilde{\Omega_{\mathcal{L}}}^{ij}$ in L such that $\tilde{\Omega_{\mathcal{L}}}^{ij} s = \Omega^{ij}_{\mathcal{L}} s$ for all $s \in \mathrm{H}^0(X, L)^{\mathrm{G}}$ and such that there is a differential operator Ω^{ij}_{G} in L_{G} such that $i^*(\tilde{\Omega}^{ij}) = \pi^*(\Omega^{ij}_{\mathrm{G}})$. We know from 26 that $\pi_0{}^*(\nabla) = \iota^*(\nabla)$. This means that it $\pi_{0*}(X) = Y$ then $\pi_0{}^*(\nabla_Y s) = \nabla_X \pi_0{}^*(s)$ for all $s \in \mathrm{C}^\infty(X_{\mathrm{G}}, L_{\mathrm{G}})$. So we want to write $\Omega^{ij}_{\mathcal{L}}$ in terms of ∇ and invariant vector fields and functions. Let $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{O}_\lambda$. In the following, ν runs over an orthonormal basis (with respect to the Killing form) of $\mathfrak{su}(2)$, and $\underline{\nu}^i$ is the vector field induced from the action of SU(2) on the i'th copy of $\mathbb{CP}^1$.

$$\begin{aligned}
\Omega^{jk}_{\mathcal{L}} &= \sum_\nu \left(\nabla_{\underline{\nu}^j} - i\mu(\nu)\circ\pi_j\right)\left(\nabla_{\underline{\nu}^k} - i\mu(\nu)\circ\pi_k\right) \\
&= \sum_\nu \nabla_{\underline{\nu}^j}\nabla_{\underline{\nu}^k} - (\mathrm{i}\mu(\nu)\circ\pi_j)\,\nabla_{\underline{\nu}^k} - (\mathrm{i}\mu(\nu)\circ\pi_k)\,\nabla_{\underline{\nu}^j} - (\mu(v)\circ\pi_j)(\mu(v)\circ\pi_k) \\
&= \sum_\nu \nabla_{\underline{\nu}^j}\nabla_{\underline{\nu}^k} - \mathrm{i}\nabla_{\underline{\xi_j}^k} - \mathrm{i}\nabla_{\underline{\xi_k}^j} - \langle \xi_k, \xi_j\rangle \\
&= \sum_\nu \nabla_{\underline{\nu}^j}\nabla_{\underline{\nu}^k} - \mathrm{i}\nabla_{\underline{\xi_j+\xi_k}^k + \underline{\xi_j+\xi_k}^j} - \langle \xi_k, \xi_j\rangle \\
&= \sum_\nu \nabla_{(\underline{\nu}^j)^{1,0}}\nabla_{(\underline{\nu}^k)^{1,0}} - \mathrm{i}\nabla_{\underline{\xi_j+\xi_k}^k + \underline{\xi_j+\xi_k}^j} - \langle \xi_k, \xi_j\rangle,
\end{aligned}$$

where the last line uses the fact that we act on holomorphic sections and that the curvature $\mathrm{F}_\nabla\left((\underline{\nu}^j)^{0,1}, (\underline{\nu}^k)^{1,0}\right) = -\mathrm{i}\omega\left((\underline{\nu}^j)^{0,1}, (\underline{\nu}^k)^{1,0}\right) = 0$ and the Lie bracket $[(\underline{\nu}^j)^{0,1}, (\underline{\nu}^k)^{1,0})] = 0$. The function $\langle \xi_k, \xi_j\rangle$ is G invariant, and so are the vector field $\underline{\xi_j + \xi_k}^k + \underline{\xi_j + \xi_k}^j$, which also preserves $\mu^{-1}(0)$ – it is up to a constant just $V_{\{ij\}}$ from section 5.7.2. Therefore the second and third term above both descend to L_{G}, and we can turn our attention towards the first term. Let $s \in \mathrm{H}^0(X, L)^{\mathrm{G}}$, and for tangent vector $V \in T_x X$ let $V'' \in T_x G_{\mathbb{C}} x$ and $V' \in (T_x G_{\mathbb{C}} x)^\perp$ such that $V = V' + V''$. Then

$$\begin{aligned}
\sum_\nu \nabla_{\underline{\nu}^j}\nabla_{\underline{\nu}^k} s = \sum_\nu \nabla_{(\underline{\nu}^j)'+(\underline{\nu}^j)''}\nabla_{(\underline{\nu}^k)'+(\underline{\nu}^k)''} s &= \sum_\nu \nabla_{(\underline{\nu}^j)'+(\underline{\nu}^j)''}\nabla_{(\underline{\nu}^k)'} s \\
&= \sum_\nu \nabla_{(\underline{\nu}^j)'}\nabla_{(\underline{\nu}^k)'} s + \sum_\nu \nabla_{(\underline{\nu}^j)''}\nabla_{(\underline{\nu}^k)'} s \\
&= \sum_\nu \nabla_{(\underline{\nu}^j)'}\nabla_{(\underline{\nu}^k)'} s - i\omega((\underline{\nu}^j)'', (\underline{\nu}^k)')s - \nabla_{[(\underline{\nu}^j)'', (\underline{\nu}^k)']} s \\
&= \sum_\nu \nabla_{(\underline{\nu}^j)'}\nabla_{(\underline{\nu}^k)'} s - \nabla_{[(\underline{\nu}^j)'', (\underline{\nu}^k)']} s
\end{aligned}$$

Since $\omega(I(W), V') = g(W, V') = 0$ for $W \in T_x Gx$, and we can write $W = \underline{a}$ for $a \in \mathfrak{g}$, so $\omega(W, V') = \omega(\underline{a}, V') = \mathrm{d}\mu(a)(V') = 0$, and $\sum_\nu [(\underline{\nu}^j)'', (\underline{\nu}^k)']$ is G-invariant, as we will see later.

Let b_l be an orthonormal, G-invariant frame of $T\mu^{-1}(0)$.

$$\begin{aligned}\sum_\nu \nabla_{(\underline{\nu}^j)'}\nabla_{(\underline{\nu}^k)'} &= \sum_\nu \sum_{l,m} \langle \underline{\nu}^j, \mathrm{b}_l\rangle \nabla_{\mathrm{b}_l} \langle \underline{\nu}^k, \mathrm{b}_m\rangle \nabla_{\mathrm{b}_m} \\ &= \sum_{l,m}\sum_\nu \langle \underline{\nu}^j, \mathrm{b}_l\rangle\langle \underline{\nu}^k, \mathrm{b}_m\rangle \nabla_{\mathrm{b}_l}\nabla_{\mathrm{b}_m} + \langle \underline{\nu}^j, \mathrm{b}_l\rangle \mathrm{b}_l(\langle \underline{\nu}^k, \mathrm{b}_m\rangle)\nabla_{\mathrm{b}_m}\end{aligned}$$

The first term is G-invariant: Let $\mathrm{Ad}_g\, \nu = \sum_\upsilon a_\nu^\upsilon \upsilon$. As the Ad action is by rotations, $\{a_\nu^\upsilon\}$ forms an orthogonal matrix, and therefore $a_\nu^\upsilon a_\nu^\eta = \delta_{\upsilon\eta}$. We can therefore compute

$$\begin{aligned}&\sum_\nu \langle \underline{\nu}^j{}_{g^{-1}x}, (\mathrm{b}_l)_{g^{-1}x}\rangle \langle \underline{\nu}^k{}_{g^{-1}x}, (\mathrm{b}_m)_{g^{-1}x}\rangle \\ &\quad= \sum_\nu \langle (g^{-1})_*(\underline{\mathrm{Ad}_g\,\nu}^j)_x, (g^{-1})_*(\mathrm{b}_l)_x\rangle\langle (g^{-1})_*(\underline{\mathrm{Ad}_g\,\nu}^k)_x, (g^{-1})_*(\mathrm{b}_m)_x\rangle \\ &\quad= \sum_\nu \langle (\underline{\mathrm{Ad}_g\,\nu}^j)_x, (\mathrm{b}_l)_x\rangle\langle (\underline{\mathrm{Ad}_g\,\nu}^k)_x, (\mathrm{b}_m)_x\rangle \\ &\quad= \sum_\nu\sum_\upsilon\sum_\eta \langle (\underline{a_\nu^\upsilon \upsilon}^j)_x, (\mathrm{b}_l)_x\rangle\langle (\underline{a_\nu^\eta \eta}^k)_x, (\mathrm{b}_m)_x\rangle \\ &\quad= \sum_\upsilon\sum_\eta (\sum_\nu a_\nu^\upsilon a_\nu^\eta) \langle (\underline{\upsilon}^j)_x, (\mathrm{b}_l)_x\rangle\langle (\underline{\eta}^k)_x, (\mathrm{b}_m)_x\rangle \\ &\quad= \sum_\upsilon \langle (\underline{\upsilon}^j)_x, (\mathrm{b}_l)_x\rangle\langle (\underline{\upsilon}^k)_x, (\mathrm{b}_m)_x\rangle\end{aligned}$$

Invariance of the second term and of (7.2) are proved in a similar way.

We know that ∇_X descends to $\mu^{-1}(0)/\mathrm{G}$ for X an invariant vector field. Therefore we have shown that there exists a second-order differential operator Ω_G^{jk} such that $(\Omega_\mathcal{L}^{jk} s)|_{\mu^{-1}(0)} = \pi^*(\Omega_\mathrm{G}^{jk}\pi_*(s))$, for $s \in \mathrm{H}^0(X, L)^\mathrm{G}$.

We have thus shown that there exist invariant vector fields $X_k^{ij}, Y_k^{ij}, Z^{ij}$ and an invariant function f^{ij} such that

$$\Omega_\mathcal{L}^{ij} s = \left(\sum_m \nabla_{X_m^{ij}}\nabla_{Y_m^{ij}} + \nabla_{Z^{ij}} + f^{ij}\right) s$$

hold for all $s \in \mathrm{H}^0(X, L)^\mathrm{G}$.

Definition 70. Let

$$\Omega_\mathrm{G}^{ij} = \sum_m \nabla_{\pi_{0*}(X_m^{ij})}\nabla_{\pi_{0*}(Y_m^{ij})} + \nabla_{\pi_{0*}(Z^{ij})} + f^{ij}\circ\pi_0.$$

The connection

$$\nabla^{\mathrm{gKZ}} = \mathrm{d} + \frac{1}{k+2}\sum_{1\le i<j\le n} \Omega_\mathrm{G}^{ij} \frac{\mathrm{d}p_i - \mathrm{d}p_j}{p_i - p_j}$$

in $\mathrm{Conf}_n(\mathbb{C}) \times \mathrm{C}^\infty(X_\mathrm{G}, L_\mathrm{G})$, which preserves $\mathrm{H}^0(X_\mathrm{G}, L_\mathrm{G})$, is called the ***geometrized KZ connection***. Also, we let

$$u^{\mathrm{gKZ}} = \sum_{1\le i<j\le n} \Omega_\mathrm{G}^{ij} \frac{\mathrm{d}p_i - \mathrm{d}p_j}{p_i - p_j}$$

We have shown

Proposition 71. *The pullback of ∇^{gKZ} under the vector bundle isomorphism $(\bigotimes_{i=1}^n V_{\lambda_i})^{\mathrm{G}} \times \mathrm{Conf}_n(\mathbb{C}) \to \mathrm{H}^0(X_{\mathrm{G}}, L_{\mathrm{G}}) \times \mathrm{Conf}_n(\mathbb{C})$ is the ordinary KZ connection.*

7.3 Symbol of the geometrized KZ connection

In this section we compute the symbol of the geometrized KZ connection. The symbol of the second order operator $\tilde{\Omega_{\mathcal{L}}}^{ij}$ is $\sum_\nu (\underline{\nu}^i)^{\perp TG_{\mathbb{C}}} \odot (\underline{\nu}^j)^{\perp TG_{\mathbb{C}}}$ where ν runs over a real basis for $\mathfrak{su}(2)$ that is orthonormal with respect to the Killing form. To find the symbol of Ω^{ij}_{G} we have to compute $p_*(\sum_\nu (\underline{\nu}^i)^{\perp TG_{\mathbb{C}}} \odot (\underline{\nu}^j)^{\perp TG_{\mathbb{C}}})$ where $p : \mu^{-1}(0) \to X/\!\!/G$. This is however a very nontrivial calculation, and we shall instead use the following lemma:

Lemma 72. *If $q : X \to X/\mathrm{G}_{\mathbb{C}}$ is the quotient map, we have that $p_*(\sum_\nu (\underline{\nu}^i)^{\perp TG_{\mathbb{C}}} \odot (\underline{\nu}^j)^{\perp TG_{\mathbb{C}}}) = q_*(\sum_\nu \underline{\nu}^i \odot \underline{\nu}^j)$*

Proof. Both sides are equal to $q_*(\sum_\nu (\underline{\nu}^i)^{\perp TG_{\mathbb{C}}} \odot (\underline{\nu}^j)^{\perp TG_{\mathbb{C}}})$, as q is G-invariant and holomorphic, and therefore invariant under $\mathrm{G}_{\mathbb{C}}$. □

The right hand side is much easier to compute, as we can compute at any point in X without taking care of the $\mu^{-1}(0)$ constraint, and we do not need to compute the projection of the vector fields. We only need to compute the symbol on an dense subset, and we will use the $\mathrm{SL}(2,\mathbb{C})$-invariant map from $\mathrm{Conf}_n(\mathbb{CP}^1) \to \mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0,1,\infty\})$ given by

$$q(z_1, z_2, \ldots, z_n) = (\mathrm{CR}(z_1, z_{n-2}, z_{n-1}, z_n), \mathrm{CR}(z_2, z_{n-2}, z_{n-1}, z_n), \ldots, \mathrm{CR}(z_{n-3}, z_{n-2}, z_{n-1}, z_n)),$$

where we recall that $\mathrm{CR}(a,b,c,d) = \frac{(b-a)(d-c)}{(b-c)(d-a)}$ is the cross ratio in the normalization where $\mathrm{CR}(z,0,1,\infty) = z$. We will denote by $\tau_1, \ldots, \tau_{n-3}$ the coordinates on $\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0,1,\infty\})$

Remark 73. We note that $q(z_1, \ldots, z_n)$ exactly gives the representative in the orbit through $(z_1, \ldots, z_n)$ where $(z_{n-2}, z_{n-1}, z_n) = (0, 1, \infty)$. This is the exactly the same as the τ coordinates on the moduli space of parabolic bundles in section 5.6

The quantity we are interested in is:

$$\sigma\left(u^{\mathrm{gKZ}}(\frac{\partial}{\partial p_i})(\mathrm{d}\tau_j, \mathrm{d}\tau_k)\right) = \sigma\left(\sum_{\substack{l=1\\ l\neq i}}^n \frac{\Omega^{il}_{\mathrm{G}}}{p_i - p_l}\right)(\mathrm{d}\tau_j, \mathrm{d}\tau_k)$$
$$= \sum_{\substack{l=1\\ l\neq i}}^n \frac{1}{2}\frac{\mathrm{d}\tau_j((q_j)_*(\underline{I}^i))\mathrm{d}\tau_k((q_k)_*(\underline{I}^l) + \mathrm{d}\tau_j((q_j)_*(\underline{I}^l))\mathrm{d}\tau_k((q_k)_*(\underline{I}^i)}{p_i - p_j}$$

Before starting to compute, we list the results of some useful calculations:

$$q_i = \frac{(z_{n-2} - z_i)(z_n - z_{n-1})}{(z_{n-2} - z_{n-1})(z_n - z_i)}, \qquad \frac{\partial}{\partial z_i} q_i = \frac{z_{n-2} - z_n}{(z_n - z_i)(z_{n-2} - z_i)} q_i,$$
$$\frac{\partial}{\partial z_{n-2}} q_i = \frac{z_{n-1} - z_i}{(z_{n-2} - z_i)(z_{n-1} - z_{n-2})} q_i, \qquad \frac{\partial}{\partial z_{n-1}} q_i = \frac{z_{n-2} - z_n}{(z_{n-1} - z_{n-2})(z_n - z_{n-1})} q_i,$$
$$\frac{\partial}{\partial z_n} q_i = \frac{z_{n-1} - z_i}{(z_n - z_i)(z_n - z_{n-1})} q_i, \qquad 1 - q_i = \frac{(z_{n-1} - z_i)(z_{n-2} - z_n)}{(z_{n-2} - z_{n-1})(z_n - z_i)}.$$

And we list the vector fields induced by the standard orthonormal basis of $\mathfrak{su}(2)$:

$$\underline{\frac{1}{2\sqrt{2}}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}}^{1,0}_z = \frac{\mathrm{i}}{\sqrt{2}} z \frac{\partial}{\partial z}$$
$$\underline{\frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}^{1,0}_z = \frac{\mathrm{i}}{2\sqrt{2}}(1-z^2)\frac{\partial}{\partial z}$$
$$\underline{\frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}}^{1,0}_z = \frac{1}{2\sqrt{2}}(1+z^2)\frac{\partial}{\partial z}$$

We now observe that

$$\begin{aligned} 8\sum_{\nu}(\underline{\nu}^i q_k)(\underline{\nu}^j q_l) &= (-4z_iz_j - (1-z_i)^2(1-z_j)^2 + (1+z_i)^2(1+z_j)^2)(\frac{\mathrm{d}}{\mathrm{d}z_i}q_k)(\frac{\mathrm{d}}{\mathrm{d}z_j}q_l) \\ &= 2(z_i - z_j)^2(\frac{\mathrm{d}}{\mathrm{d}z_i}q_k)(\frac{\mathrm{d}}{\mathrm{d}z_j}q_l), \end{aligned}$$

We are now ready to compute

$$\begin{aligned} 8\,\sigma\left(u^{\mathrm{KZ}}(\frac{\partial}{\partial p_i})(\mathrm{d}\tau_i, \mathrm{d}\tau_i)\right) &= 8\sum_{j=n-2}^{n-1}\frac{1}{p_i - p_j}\sum_{\nu}\mathrm{d}\tau_i(q_i)_*(\underline{\nu}^i)\mathrm{d}\tau_i(q_i)_*(\underline{\nu}^j) \\ &= 8\sum_{\nu}\frac{(\underline{\nu}^i q_i)(\underline{\nu}^{n-2}q_i)}{p_i} + \frac{(\underline{\nu}^i q_i)(\underline{\nu}^{n-2}q_i)}{p_i - 1} \\ &= q_i^2\Big((1+z_i^2)(1+z_{n-2}^2) - 4z_iz_{n-2} - (1-z_i^2)(1-z_{n-2}^2)) \\ &\cdot\frac{1}{p_i}\frac{z_{n-2}-z_n}{(z_n - z_i)(z_{n-2}-z_i)}\frac{z_{n-1}-z_i}{(z_{n-2}-z_i)(z_{n-1}-z_{n-2})} \\ &+ \frac{1}{p_i - 1}(1+z_i^2)(1+z_{n-1}^2) - 4z_iz_{n-1} - (1-z_i^2)(1-z_{n-1}^2)) \\ &\cdot\frac{z_{n-2}-z_n}{(z_n-z_i)(z_{n-2}-z_i)}\frac{z_{n-2}-z_n}{(z_{n-1}-z_{n-2})(z_n - z_{n-1})}\Big) \\ &= 2q_i^2\Big(\frac{(z_i - z_{n-2})^2}{p_i}\frac{z_{n-2}-z_n}{(z_n-z_i)(z_{n-2}-z_i)}\frac{z_{n-1}-z_i}{(z_{n-2}-z_i)(z_{n-1}-z_{n-2})} \\ &+ \frac{(z_i - z_{n-1})^2}{p_i - 1}\frac{z_{n-2}-z_n}{(z_n-z_i)(z_{n-2}-z_i)}\frac{z_{n-2}-z_n}{(z_{n-1}-z_{n-2})(z_n-z_{n-1})}\Big) \\ &= 2q_i^2\Big(\frac{1}{p_i}\frac{(z_{n-2}-z_n)(z_{n-1}-z_i)}{(z_n-z_i)(z_{n-1}-z_{n-2})} \\ &+ \frac{1}{p_i - 1}\frac{(z_{n-2}-z_n)^2(z_i - z_{n-1})^2}{(z_{n-2}-z_i)(z_{n-1}-z_{n-2})(z_n-z_{n-1})(z_n-z_i)}\Big) \\ &= 2q_i^2\Big(\frac{(q_i - 1)}{p_i} \\ &+ \frac{-1}{p_i - 1}\frac{(z_{n-2}-z_{n-1})(z_n - z_i)}{(z_{n-2}-z_i)(z_n - z_{n-1})}\frac{(z_{n-2}-z_n)^2(z_{n-1}-z_i)^2}{(z_{n-1}-z_{n-2})^2(z_n-z_i)^2}\Big) \\ &= 2q_i^2\Big(\frac{(q_i-1)}{p_i} + \frac{-q_i^{-1}}{p_i - 1}(1-q_i)^2\Big) \end{aligned}$$

$$= 2q_i(q_i-1)(\frac{q_i}{p_i} - \frac{q_i-1}{p_i-1})$$

For $j \neq i$ and $j < n-2$ we have, leaving out the $\frac{1}{p_i - p_j}$ factor, we get similarly:

$$\begin{aligned}
8\,\sigma(\Omega^{ij})(\mathrm{d}\tau_i, \mathrm{d}\tau_j) &= (z_i - z_j)^2 \frac{z_{n-2}-z_n}{(z_n-z_i)(z_{n-2}-z_i)} q_i \frac{z_{n-2}-z_n}{(z_n-z_j)(z_{n-2}-z_j)} q_j \\
&= (z_i - z_j)^2 \frac{z_{n-2}-z_n}{(z_n-z_i)(z_{n-2}-z_i)} \frac{(z_{n-2}-z_i)(z_n-z_{n-1})}{(z_{n-2}-z_{n-1})(z_n-z_i)} \\
&\quad \frac{z_{n-2}-z_n}{(z_n-z_j)(z_{n-2}-z_j)} \frac{(z_{n-2}-z_j)(z_n-z_{n-1})}{(z_{n-2}-z_{n-1})(z_n-z_j)} \\
&= \frac{(z_{n-2}-z_n)^2}{(z_n-z_i)^2} \frac{(z_n-z_{n-1})^2}{(z_{n-2}-z_{n-1})^2} \frac{(z_i-z_j)^2}{(z_n-z_j)^2} \\
&= \left(\frac{z_n-z_{n-1}}{z_{n-2}-z_{n-1}}\right)^2 \left(\frac{z_nz_j + z_iz_{n-2} - z_iz_n - z_jz_{n-2}}{(z_n-z_i)(z_n-z_j)}\right)^2 \\
&= \left(\frac{z_n-z_{n-1}}{z_{n-2}-z_{n-1}}\right)^2 \left(\frac{z_nz_{n-2} + z_iz_j - z_nz_{n-2} - z_iz_j + z_nz_j + z_iz_{n-2} - z_iz_n - z_jz_{n-2}}{(z_n-z_i)(z_n-z_j)}\right)^2 \\
&= \left(\frac{z_n-z_{n-1}}{z_{n-2}-z_{n-1}}\right)^2 \left(\frac{(z_n-z_j)(z_{n-2}-z_i) - (z_n-z_i)(z_{n-2}-z_j)}{(z_n-z_i)(z_n-z_j)}\right)^2 \\
&= \left(\frac{z_n-z_{n-1}}{z_{n-2}-z_{n-1}}\right)^2 \left(\frac{(z_{n-2}-z_i)}{(z_n-z_i)} - \frac{(z_{n-2}-z_j)}{(z_n-z_j)}\right)^2 \\
&= (q_i - q_j)^2
\end{aligned}$$

∎

$$\begin{aligned}
8\,\sigma(\Omega^{i,n-2})(\mathrm{d}\tau_i, \mathrm{d}\tau_j) &= (z_i - z_{n-2})^2 \frac{z_{n-2}-z_n}{(z_n-z_i)(z_{n-2}-z_i)} q_i \frac{z_{n-1}-z_j}{(z_{n-2}-z_j)(z_{n-1}-z_{n-2})} q_j \\
&= (z_i - z_{n-2})^2 \frac{1}{(z_n-z_i)(z_{n-2}-z_i)} q_i \frac{1}{(z_{n-2}-z_j)} q_j (z_n - z_j)(q_j - 1) \\
&= \frac{(z_{n-2}-z_i)(z_n-z_j)}{(z_n-z_i)(z_{n-2}-z_j)} q_i q_j (q_j - 1) \\
&= (q_j^{-1} q_i) q_i q_j (q_j - 1) \\
&= q_i^2 (q_j - 1)
\end{aligned}$$

$$\begin{aligned}
8\,\sigma(\Omega^{i,n-1})(\mathrm{d}\tau_i, \mathrm{d}\tau_j) &= (z_i - z_{n-1})^2 \frac{z_{n-2}-z_n}{(z_n-z_i)(z_{n-2}-z_i)} q_i \frac{z_{n-2}-z_n}{(z_{n-1}-z_{n-2})(z_n-z_{n-1})} q_j \\
&= \frac{(z_{n-2}-z_n)(z_n-j_{n-1})}{(z_{n-2}-z_{n-2})(z_n-z_i)} \frac{(z_i-z_{n-1})^2(z_{n-2}-z_n)^2}{(z_n-z_i)(z_{n-2}-z_i)(z_{n-1}-z_{n-2})(z_n-z_{n-1})} q_j \\
&= -\frac{(z_i-z_{n-1})^2(z_{n-2}-z_n)^2}{(z_n-z_i)^2(z_{n-1}-z_{n-2})^2} q_j \\
&= -(1-q_i)^2 q_j
\end{aligned}$$

∎

Combining it all together we get:

$$8\,\sigma\left(u^{\mathrm{gKZ}}(\frac{\partial}{\partial p_i})\right) = 2q_i(q_i-1)\left(\frac{q_i}{p_i} - \frac{q_i-1}{p_i-1}\right) \frac{\partial}{\partial q_i} \odot \frac{\partial}{\partial q_i}$$

$$+\left(\frac{(q_i-q_j)^2}{p_i-p_j}+\frac{q_i^2(q_j-1)}{p_i}-\frac{q_j(q_i-1)^2}{p_i-1}\right)\frac{\partial}{\partial q_i}\odot\frac{\partial}{\partial q_j}$$

Chapter 8

From the geometrized KZ connection to Hitchin's connection

8.1 The parameter spaces

Let us first identify the base spaces for the KZ and Hitchin connection. The KZ connection is a connection in a bundle over the configuration space of n points in $\mathbb{C}$. We restrict to the subset of configurations where the last 3 points are mapped to $0, 1, \infty$, which is $\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0,1,\infty\})$. We can pull the bundle and the KZ connection back to the universal cover $\mathrm{U}\left(\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0,1,\infty\})\right)$, and the pullback bundle and connection carries an action of $\pi_1(\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0,1,\infty\}))$. The monodromy representation is then equivalent to the representation we get from fixing a point in $x \in \mathrm{U}\left(\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0,1,\infty\})\right)$ and assigning to $\gamma \in \pi_1(\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0,1,\infty\}))$ the representation that first parallel transport to $\gamma \cdot x$ and then act by γ^{-1} to get back to the fiber over x. This is again equivalent to the action on the flat sections. But the Teichmüller space of a sphere with $n > 3$ punctures is exactly the universal cover of $\mathrm{Conf}_{n-3}(\mathbb{CP}^1 \setminus \{0,1,\infty\})$, as we saw in proposition 50. We now have connections in two bundles over the same base manifold, and we can try to identify the bundles under investigation.

8.2 The families

As we saw in section 5.7, we can identify the moduli space of parabolic bundles on $(\mathbb{CP}^1, p_1, \ldots, p_{n-3}, 0, 1, \infty)$, $\mathcal{M}^{\mathrm{Par}}(\mathbb{CP}^1, \vec{\lambda}, k)$, with the GIT quotient $(\mathbb{CP}^1)^{\times n} // \mathrm{SL}(2,\mathbb{C})$. The quotient is exactly with respect to the line bundle with Chern class $[\omega^{\vec{\lambda}}]$. As in 5.7.1, the stable part of the GIT quotient is X_{G}. We get, using the Mehta-Seshadri map and composing it with the map from the parabolic bundles to X_{G}, a map

$$\mathrm{Flags} : \mathcal{T} \times \mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k) \to \mathcal{T} \times X_{\mathrm{G}}.$$

We denote by $\mathcal{T} \times_I \mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$ the left hand side above, endowed with the complex structure $I_{\mathcal{T}} \oplus I_{\sigma}$. We will show that Flags is holomorphic.

Remark 74. In the non-punctured case, Biswas proved in [21] that the left hand side with this complex structure serves as a "universal" moduli space of stable bundles, that is, families of stable bundles where also the complex structure on the surface is allowed to vary in the family. Its possible to prove that the right hand side is such a universal moduli space, and the map Flags would exactly be the classifying map if we showed that the left hand side was a universal moduli space. We will however give a different proof. However, note that in the nonpunctured case, or the punctured of higher genus, it is not true that the family is trivializable, as the individual fibers are not isomorphic.

Let us describe the map Flags_σ. Given $d+A \in \mathcal{M}^{\mathrm{Flat}}$ we construct a parabolic bundle over Σ_σ in the following way. In the smoothly trivial bundle $(\Sigma_\sigma \setminus \mathrm{D}) \times \mathbb{C}^2$ we define a holomorphic structure by the del-bar operator $\bar\partial_A = \pi^{0,1}\mathrm{d}_A = \bar\partial + A^{0,1}$. Around each $p \in \mathrm{D}$ we can find sections e_1, e_2 in a disc $\mathbb{D}^*$ such that $A = \mathrm{d} + i\begin{pmatrix}\alpha & 0\\ 0 & -\alpha\end{pmatrix}\mathrm{d}\theta$, where θ is the angle coordinate on the disc. We have that

$$\mathrm{d}\theta = \mathrm{d}\frac{-\mathrm{i}}{2}\log\left(\frac{z}{\bar z}\right) = \frac{-\mathrm{i}}{2}\frac{\bar z \mathrm{d}z - z\mathrm{d}\bar z}{|z|^2}$$

So $\mathrm{d}\theta^{0,1} = \frac{\mathrm{i}}{2}\frac{\mathrm{d}\bar z}{\bar z}$. Therefore $|z|^\alpha e_1, |z|^{-\alpha}e_2$ defines holomorphic sections:

$$\bar\partial |z|^\alpha e_1 = \frac{\alpha}{2}|z|^{\alpha-1} z \mathrm{d}\bar z e_1 = \frac{\alpha}{2}|z|^\alpha \frac{\mathrm{d}\bar z}{\bar z} e_1 = -\mathrm{i}\begin{pmatrix}\alpha & 0\\ 0 & -\alpha\end{pmatrix}|z|^\alpha e_1 \mathrm{d}\theta^{0,1},$$

and similar for $|z|^{-\alpha}e_2$. We can now glue the bundle together with the holomorphically trivial bundle $\mathbb{D} \times \mathbb{C}^2$ by the transition maps $(z, |z|^\alpha e_1) \to (z, \begin{pmatrix}1\\0\end{pmatrix})$ and $(z, |z|^{-\alpha}e_2) \to (z, \begin{pmatrix}0\\1\end{pmatrix})$. Doing this over each point in D we obtain a holomorphic vector bundle $E_{A,\sigma}$ over Σ_σ and we can at each point $p \in \mathrm{D}$ choose the flag $0 \subseteq \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix} \subseteq \mathbb{C}^2$ to obtain a parabolic bundle. This is the Mehta-Seshadri map. The Chern class of the bundle is $-\sum_{p\in \mathrm{D}}\sum_{i=1}^2 \alpha_i$. If the holonomy around each point in D is in $\mathrm{SU}(2)$, we have that $\alpha_1 = -\alpha_2$, and the degree is 0. This is the case for our specific choice of weights. As we saw in lemma 52, the parabolic bundles are – as holomorphic bundles – trivial, so there exists a holomorphic trivialization $f : E_{A,\sigma} \to \Sigma_\sigma \times \mathbb{C}^2$. The map Flags_σ is then given by

$$\mathrm{Flags}(\sigma, A) = \left[f_{p_1}\left(\begin{pmatrix}1\\0\end{pmatrix}\right), f_{p_2}\left(\begin{pmatrix}1\\0\end{pmatrix}\right), \dots, f_{p_n}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)\right] \in (\mathbb{CP}^1)^{\times n}/\mathrm{SL}(2,\mathbb{C}).$$

which is well defined as the only automorphisms of the trivial vector bundle is the automorphisms that acts as the same $\mathrm{SL}(2,\mathbb{C})$ map in each fiber.

Theorem 44. *The map* Flags *is a biholomorphism.*

Proof. By Hartog's theorem it is enough to show that the map is holomorphic in each variable. For the variables in the $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec\lambda, k)_\sigma$ direction, this follows from theorem 36. Choose a disc $\mathbb{D} \subseteq \mathcal{T}$ centered at σ_0 and a points $[A] \in \mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec\lambda, k)$. We will prove that Flags restricted to $\mathbb{D} \times \{A\}$ is holomorphic. We can define a holomorphic structure $E_\mathbb{D}$ on the vector bundle $(\mathbb{D} \times \Sigma) \times \mathbb{C}^2$ in the following way

1. On $\mathbb{D} \times (\Sigma \setminus D)$, using the $\bar\partial$-operator given by $\bar\partial_\mathbb{D} + \bar\partial_{A_\sigma}$ (σ is the variable of $\mathbb{D}$). This is clearly a $\bar\partial$-operator, and the requirement to define a holomorphic structure is that the square is 0. But it is the $\bar\partial$-operator induced from the prequantum connection on $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec\lambda, k)$ and the trivial connection on $\mathcal{T}$, which have curvature $\pi_2^*(\omega) \in \Omega^{(1,1)}(\mathcal{T} \times_I \mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec\lambda, k))$, and therefore the curvature have $(0,2)$-part equal to 0. But the $(0,2)$ part of the curvature is exactly the square of the induced $\bar\partial$ operator.

2. On $\mathbb{D} \times N(\mathrm{D})$ the bundle $E_{\mathbb{D}}$ have the trivial holomorphic structure.

So the section $(\sigma, z_\sigma) \mapsto \left(\sigma, z_\sigma, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ is holomorphic, where z_σ is a σ-holomorphic coordinate around p_i. In particular the composition m_i of this section with the map $\sigma \mapsto p_i$ is holomorphic. The map Flags is then the composition of $\sigma \mapsto (m_1, \ldots, m_n)$ with a holomorphic trivialization of $E_{\restriction\{\sigma\}\times\Sigma}$, and the quotient from $\mathbb{C}^2 \setminus \{0\} \to \Sigma$. Therefore we just need to show that there exists a holomorphic trivialization of $E_{\restriction\{\sigma\}\times\Sigma}$, varying holomorphically with σ, that is, a holomorphic trivialization $f : E_{\mathbb{D}} \to \mathbb{D} \times \Sigma \times \mathbb{C}^2$. We then get that Flags is the composition of holomorphic maps, and therefore holomorphic. In doing so, we can restrict to smaller discs around 0 in $\mathbb{D}$. As the complex structure on $\mathbb{CP}^1$ is rigid (we are currently not concerned about the marked points), there is a biholomorphism $\mathbb{D} \times \Sigma \cong \mathbb{D} \times \mathbb{CP}^1$ respecting the projection to the first factor. We will denote by $\mathbb{D}_0, \mathbb{D}_\infty$ the two coordinate charts of $\mathbb{CP}^1$.

1. There exists a holomorphic trivialization over $\mathbb{D}\times\mathbb{D}_0$ and $\mathbb{D}\times\mathbb{D}_\infty$ as these are contractible Stein manifolds.

2. The restriction to any $\{\sigma\}\times\mathbb{CP}^1$ is trivializable, so there exists two point-wise independent, holomorphic sections $s_1, s_2 : \{0\} \times \mathbb{CP}^1 \to E_{\mathbb{D}}$.

3. Their restriction to $\{0\} \times \mathbb{D}_0$ extend to holomorphic sections over $\mathbb{D} \times \mathbb{D}_0$, and we can assume these sections are also point-wise independent after a possible restriction to a smaller disc.

4. By applying the transition function, we get independent sections $\mathbb{D} \times \mathbb{D}_\infty \setminus \{\infty\} \to E_{\mathbb{D}}$.

5. By construction the sections extend over $(0, \infty)$. So – after possible another restriction to a smaller disc – we know that they have to be bounded.

6. So by Riemann's removable singularities theorem we get for all $z \in \mathbb{D}$ a holomorphic extension of $s_i|_{\{z\}\times(\mathbb{D}_\infty\setminus\{0\})}$ to $\{z\} \times \mathbb{D}_\infty$.

7. Therefore we get an extension to $\mathbb{D} \times \mathbb{CP}^1$, that might not be holomorphic at the points in $\mathbb{D} \times \{\infty\}$.

8. It now follows from the multi-variable version of Riemanns removeable singularities theorem that this extension is holomorphic, see e.g. [47], theorem 7.3.3.

9. We now have two point-wise independent holomorphic sections, so the bundle is holomorphically trivial. □

The permutation group acts on X_{G} by permuting the coordinates. This defines a action of the mapping class group of Σ by assigning to a mapping class the corresponding permutation of the marked points.

Proposition 75. *The mapping class group of Σ acts on X_{G} by permutations is intertwined with the action of the mapping class group on $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$ under* Flags

Proof. There exists a commutative diagram as below, where the vertical maps are isomorphisms of holomorphic vector bundles covering the identity, and the horizontal maps are isomorphisms of holomorphic vector bundles covering the biholomorphism $\varphi : \Sigma_\sigma \to \Sigma_{\varphi\sigma}$:

$$\begin{tikzcd} E_{A,\sigma} \arrow[r, "\varphi^*"] \arrow[d] & E_{\varphi A,\varphi\sigma} \arrow[d] \\ \Sigma_\sigma \times \mathbb{C}^2 \arrow[r, "\varphi\times\mathrm{id}"] & \Sigma_{\varphi\sigma} \times \mathbb{C}^2 \end{tikzcd}$$

We see that the image under $\mathrm{Flags}_{\varphi\sigma}$ of the parabolic bundle at the lower right corner in X_{G} is just the permutation determined by φ of the image of the bundle at the lower left. □

8.3 The line bundles

In this section we want to show

Proposition 76. *There exists a holomorphic line bundle isomorphism*

$$\psi : \left(\coprod_\sigma \mathcal{L}^{\mathrm{CS}}{}_\sigma \to \mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k) \times_I \mathcal{T} \right) \to (L_{\mathrm{G}} \times \mathcal{T} \to X_{\mathrm{G}} \times \mathcal{T})$$

covering $\mathrm{Flags} : \mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k) \times_I \mathcal{T} \to X_{\mathrm{G}} \times \mathcal{T}$.

Proof. We will prove the existence of an isomorphism by proving the following two claims:

1. Holomorphic line bundles over $X_{\mathrm{G}} \times T$ are classified by their first Chern class.

2. The two line bundles have the same first Chern class, that is, $[\mathrm{Flags}_\sigma^*(\omega_{\mathrm{G}}^{\vec{\lambda}})] = k[\omega_{\vec{\lambda},k}]$.

By theorem 37 X_{G} is – up to subsets of codimension 2 – RM_n, the blowup of $n-1$ points in $\mathbb{CP}^{n-3}$. But this space have $\mathrm{H}^2(\mathrm{RM}_n, \mathbb{Z}) \cong \mathrm{Pic}(\mathrm{RM}_n)$. Therefore – again due to the equivalence up to subsets of codiemsnion 2 – also $\mathrm{H}^2(X_{\mathrm{G}}, \mathbb{Z}) \cong \mathrm{Pic}(X_{\mathrm{G}})$. By Theorem theorem 34, the Teichmüller space $\mathcal{T}$ is a Stein manifold. The theorem of Oka-Grauert – see e.g. [28] – states that each topological vector bundle on a Stein manifolds admits exactly one holomorphic structure, up to isomorphism, in particular the Picard group must be isomorphic to the degree 2 cohomology group with integer coefficients, which for Teichmüller space is the trivial group, as $\mathcal{T}$ is contractible by theorem 33 . To conclude that the Picard group of $\mathcal{T} \times X_{\mathrm{G}}$ is equal to the product of the Picard groups of X_{G} and $\mathcal{T}$, we show that the last map in the following piece of the sheaf exponential sequence is injective:

$$\mathrm{H}^1(\mathcal{T} \times X_{\mathrm{G}}, \mathbb{Z}) \xrightarrow{\iota} \mathrm{H}^1(\mathcal{T} \times X_{\mathrm{G}}, \mathcal{O}) \xrightarrow{\exp} \mathrm{H}^1(\mathcal{T} \times X_{\mathrm{G}}, \mathcal{O}^*) \xrightarrow{c_1} \mathrm{H}^2(\mathcal{T} \times X_{\mathrm{G}}, \mathbb{Z}) \;.$$

This follows from the fact that the kernel is the image of $H^1(\mathcal{T} \times X_{\mathrm{G}}, \mathcal{O})$, and by the Künneth formula:

$$H^1(\mathcal{T} \times X_{\mathrm{G}}, \mathcal{O}) \cong H^0(\mathcal{T}, \mathcal{O}) \otimes H^1(X_{\mathrm{G}}, \mathcal{O}) \bigoplus H^1(\mathcal{T}, \mathcal{O}) \otimes H^0(X_{\mathrm{G}}, \mathcal{O}).$$

It follows from the exponential sequence applied individually to $\mathcal{T}$ and X_{G}, and the fact that both $\mathcal{T}$ and X_{G} have trivial degree 1 cohomology with integer coefficients, that $H^1(\mathcal{T}, \mathcal{O}) \cong 0$ and $H^1(X_{\mathrm{G}}, \mathcal{O}) \cong 0$. Therefore $H^1(\mathcal{T} \times X_{\mathrm{G}}, \mathcal{O}) \cong 0$ and the map is injective.

For the second part, we just need to show that the pullback of the first Chern class of $\mathcal{L}^{\mathrm{CS}}$ along Flags_σ equals the first Chern class of L_{G} restricted to $X_{\mathrm{G}} \times \{\sigma\}$.

This is shown in [24, proposition 6.3] that for a surface with one puncture with genus > 3. The argument uses the calculations used in the proof of [24, Theorem 4.12], which is completely local around the puncture, and therefore extends to multiple punctures. The requirement on the genus is only to ensure high enough co-dimension of the unstable points, which we can guarantee as well by assuming $n > 4$. Their calculation shows that

$$\int_{(\mathrm{Flags}_\sigma^{-1})_*(\iota_i)_*(\mathbb{CP}^1)} \omega_{\vec{\lambda},k} = \frac{\lambda_i}{k}$$

where λ_i is the colour of the i'th flag. However, $\mathrm{c}_1(L_{\mathrm{G}})$ is represented by $\omega_{\mathrm{G}}^{\lambda}$, which is the symplectic form obtained from reduction of the form $\oplus_{i=1}^n \lambda_i \omega_{\mathrm{FS}}$ on $\times_{i=1}^n \mathbb{CP}^1$, and therefore

$$\int_{q_*(\{z_1,z_2,\ldots,z_{i-1}\}\times\mathbb{CP}^1\times\{z_{i+1},\ldots,z_n\})} \omega_{\mathrm{G}}^{\lambda} = \lambda_i \qquad \square$$

But an element in $\mathrm{H}^2(X_{\mathrm{G}},\mathbb{Z})$ is determined exactly by the evaluation at these n cycles. Thus we have $\mathrm{c}_1(\mathcal{L}^{\mathrm{CS}}) = \mathrm{Flags}_\sigma^*(c_1(L_{\mathrm{G}}))$

Remark 77. Such a ψ is unique up to a holomorphic function on $\mathcal{T}$.

8.4 The isomorphism of the vector bundles

ψ gives an isomorphism of holomorphic vector bundles:

$$\psi : \mathcal{V} \to \mathrm{H}^0(X_{\mathrm{G}}, L_{\mathrm{G}}) \times \mathcal{T}$$
$$(\psi(s))(x,\sigma) = \psi(s(\mathrm{Flags}_\sigma^{-1}(x),\sigma))$$

This is compatible with the mapping class group action, at least up to a projective factor:

Proposition 78. *The actions on $\mathrm{H}^0(X_{\mathrm{G}}, L_{\mathrm{G}}) \times \mathcal{T}$ and $\mathcal{V}$ are projectively intertwined by ψ*

Proof. This follows essentially from proposition 75.

$$\begin{aligned}
\varphi\psi(s)(x,\sigma) &= \varphi_{\varphi^{-1}\sigma}\cdot(\psi_{\varphi^{-1}\sigma}s)(\varphi^{-1}x,\varphi^{-1}\sigma) = \varphi_{\varphi^{-1}\sigma}\psi_{\varphi^{-1}\sigma}(s(\mathrm{Flags}_{\varphi^{-1}\sigma}^{-1}\circ\varphi^{-1}(x),\varphi^{-1}\sigma))\\
&= \varphi_{\varphi^{-1}\sigma}\psi_{\varphi^{-1}\sigma}(s(\mathrm{Flags}_{\varphi^{-1}\sigma}^{-1}(x),\varphi^{-1}\sigma))\\
\psi(\varphi s)(x,\sigma) &= \psi_\sigma((\varphi s)(\mathrm{Flags}_\sigma^{-1}(x),\sigma)) = \psi_\sigma(\varphi_{\varphi^{-1}\sigma}s(\varphi^{-1}\circ\mathrm{Flags}_\sigma^{-1}(x),\varphi^{-1}\sigma))\\
&= \psi_\sigma(\varphi_{\varphi^{-1}\sigma}s((\mathrm{Flags}_\sigma^{-1}\circ\mathrm{Flags}_{\varphi^{-1}\sigma})^{-1}\circ\mathrm{Flags}_\sigma^{-1}(x),\varphi^{-1}\sigma))\\
&= \psi_\sigma(\varphi_{\varphi^{-1}\sigma}s(\mathrm{Flags}_{\varphi^{-1}\sigma}^{-1}(x),\varphi^{-1}\sigma))
\end{aligned}$$

so the difference is exactly given by $\varphi_{\varphi^{-1}\sigma}\circ\psi_{\varphi^{-1}\sigma}\circ\varphi_{\varphi^{-1}\sigma}^{-1}\circ\psi_\sigma^{-1}$. This is a holomorphic automorphism of $\mathcal{L}^{\mathrm{CS}}{}_\sigma$, and therefore its multiplication by a non-zero holomorphic function on $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma,\vec{\lambda},k)_\sigma$. However, such a function must be constantly equal to some number α_σ, so $\psi\varphi s = \alpha\varphi\psi s$. $\square$

We can use ψ to push the geometrized KZ connection to a connection in $\mathcal{V}$:

$$\psi(\nabla^{\mathrm{t}} + \frac{1}{k+2}u^{\mathrm{KZ}})\psi^{-1} = \nabla^{\mathrm{t}} + \psi(\mathrm{d}_{\mathcal{T}}\psi^{-1}) + \frac{1}{k+2}\psi(u^{\mathrm{KZ}})\psi^{-1}$$

We see that this is a connection satisfying our definition of a Hitchin connection. The trivial action in the fibers of $\pi^*(L_{\mathrm{G}})$ covering the trivial action on X_{G} and natural action on $\mathcal{T}$, together with ψ defines an action $\bullet$ of MCG on L_{G}. This action preserves the geometrized KZ connection and therefore defines a representation on the flat sections, which is isomorphic to the representation of the KZ connection. The same is true for push-forward, up to a projective factor:

Theorem 45. *The push forward of the geometrized KZ connection is a connection of Hitchin type for the family of complex structures on* $(\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k), \mathcal{L}^{\mathrm{CS}})$ *defined by* $\mathcal{T}$.

Our next goal is to show that the geometrized KZ actually is projectively equal to the Hitchin connection constructed in theorem 38.

Chapter 9

Comparison with the Hitchin connection

Recall that the Hitchin connection was of the form $\nabla^{\mathrm{Hit}} = \nabla^{\mathrm{t}} + u$ where u was a one-form with values in the differential operators on $\mathcal{L}^{\mathrm{CS}}$. For $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$ we will denote this one-form by ${u^{\mathrm{Hit}}}_{\lambda,k}$, and by $\mathrm{G}_{\vec{\lambda},k}(V)$ the symbol of ${u^{\mathrm{Hit}}}_{\vec{\lambda},k}(V)$ – denoted by $\mathrm{G}(V)$ in section 6.1, and was defined by:

$$\mathrm{G}_{\vec{\lambda},k}(V)\omega_{\vec{\lambda},k} = V'[I].$$

We will explicitly calculate $\mathrm{G}_{\vec{\lambda},k}(V)$ for the family of complex structures on ${\mathcal{M}^{\mathrm{Flat,irr}}}_{k,\vec{\lambda}}$ given by $\mathcal{T}(\Sigma)$. The strategy is to first generalize [38, lemma 2.13] from moduli spaces of holomorphic vector bundles to moduli spaces of parabolic vector bundles. This will give us a handle to calculate $\mathrm{G}_{\vec{\lambda},k}(V)$ in coordinates τ_i on $\mathcal{M}^{\mathrm{Par}'}(\Sigma_\sigma, \vec{\lambda}, k)$.

Proposition 79. *Let $[A] \in \mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$, $\sigma \in \mathcal{T}(\Sigma)$ and $\mathbb{E} = \mathrm{MS}([A], \sigma)$. Let $X \in \mathrm{T}^{0,1}_{\mathbb{E}}\mathcal{M}^{\mathrm{Par}'}(\Sigma_\sigma, \vec{\lambda}, k)$ be represented as an harmonic form $\alpha \in \mathcal{H}^{1,0}_{\varepsilon}(\Sigma_\sigma \setminus \mathrm{D}, \mathrm{ad}\,\mathfrak{g}_{\mathbb{C}})$. Then $V[I](X)$ is represented in $\mathrm{H}^1_{\bar\partial_A}(\Sigma_\sigma \setminus \mathrm{D}, \mathrm{ad}\,\mathfrak{g}_{\mathbb{C}})$ by the contraction $-V[\star]\alpha$.*

Proof. This is a direct adaption of Hitchins proof of [38, lemma 2.13], using the Hodge theory with exponential decay from [4]. Let $\star(t)$ be a family of conformal structures on Σ, and $I(t)$ the corresponding family of complex structures on $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$. Then we can write $\alpha = h(t) + \mathrm{d}_A\psi$ where $h(t)$ is harmonic with respect to $\star(t)$. Taking the derivative at $t = 0$ we get

$$0 = \dot{h} + \mathrm{d}_A\dot{\psi}$$

and from the uniqueness of harmonic representatives, that $h(0) = \alpha$. We can now use the definition of $I(t)$ to get:

$$I(t)[\alpha] = [-\star(t)h(t)],$$

and again differentiation at $t = 0$ we get

$$\dot{I}[\alpha] = [-\dot{\star}h(0) - \star\mathrm{d}_A\dot{h}] = [-\dot{\star}\alpha + \star\mathrm{d}_A\dot{\psi}].$$

This class is represented by a harmonic $(0,1)$ form β:

$$\beta = -\star\alpha + \star\mathrm{d}_A\dot{\psi} + \mathrm{d}_A\varphi.$$

By considering the type of the right hand side, we have that $(\star\mathrm{d}_A\dot{\psi} + \mathrm{d}_A\varphi)^{1,0} = 0$, that is $\star\mathrm{d}_A\dot{\psi} + \mathrm{d}_A\varphi = (\star\mathrm{d}_A\dot{\psi} + \mathrm{d}_A\varphi)^{0,1} = -\mathrm{i}\bar\partial_A\dot{\psi} + \bar\partial_A\varphi$. Therefore

$$[\beta] = [-\dot{\star}\alpha]$$

in $\mathrm{H}^1_{\mathrm{d}_A}(\Sigma, \operatorname{ad}\mathfrak{g})$ □

In the same way as Hitchin, we use this to give the following formula for $\mathrm{G}_{\vec{\lambda},k}(V)$ in terms of data associated to the moduli space of holomorphic (parabolic) bundles instead of the moduli space of flat connections:

Proposition 80. *For $\alpha, \beta \in \mathrm{T}^*\mathcal{M}^{\mathrm{Par}'}(\Sigma_\sigma, \vec{\lambda}, k)$ we have*

$$\mathrm{G}_{\vec{\lambda},k}(V)(\alpha,\beta) = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mathrm{Tr}(\alpha\beta)V[\star]$$

Proof. Let α, β be as elements of $\mathrm{H}^0(\Sigma_\sigma, \mathrm{SPEnd}_0(E)\otimes \mathrm{K}(\mathrm{D}))$. Then $\omega^{-1}_{\vec{\lambda},k}$ applied to β is exactly the harmonic representative $h \in \mathrm{T}^{0,1}_{[E]}\mathcal{M}^{\mathrm{Par}}(\Sigma_\sigma, \vec{\lambda}, k)$ such that $\beta = \omega_{\vec{\lambda},k}(h,\cdot)$. So we have:

$$\begin{aligned}
\alpha \mathrm{G}_{\vec{\lambda},k}(V)\beta &= \alpha(V'[I]\omega^{-1}_{\vec{\lambda},k}(\beta)) = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mathrm{Tr}(\alpha\wedge V[I]\omega_k^{-1}\beta)\\
&= \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mathrm{Tr}(\alpha\wedge -V[\star]\omega_k^{-1}\beta) = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mathrm{Tr}(\alpha V[\star]\wedge\omega^{-1}_{\vec{\lambda},k}\beta)\\
&= -\omega_{\vec{\lambda},k}(\alpha V[\star], \omega^{-1}_{\vec{\lambda},k}\beta) = \beta(\alpha V[\star]) = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mathrm{Tr}(\beta\wedge\alpha V[\star]).
\end{aligned}$$ □

9.0.1 Kodaira-Spencer map of Σ

Next we must calculate $\frac{\partial}{\partial p_i}[\star]$. We only need to integrate against it, and it is therefore enough to find a representative in $\mathrm{H}^1(\Sigma_\star, \mathrm{T}^{1,0})$. This is however exactly the Kodaira-Spencer map ρ applied to $\frac{\partial}{\partial p_i}$ of the universal curve over $\mathcal{T}$, at $[\star]$. According to [63, Lemma 5.4.1], $\rho(\frac{\partial}{\partial p_i})$ is as a 1-Čech-cocycle, using again the open cover U_j, given by the sections

$$\rho\left(\frac{\partial}{\partial p_i}\right)_{im} = -\partial_z \in \mathrm{H}^0(U_i\cap U_m, T(-\mathrm{D}))$$

and $\rho\left(\frac{\partial}{\partial p_i}\right)_{mk} = 0$ when $i \notin \{k, m\}$. The computation works the same way as section 5.6. We can find a Dolbeault representative exactly the same way as we did for $\frac{\partial}{\partial \tau_i}$ in proposition 62 and the result is

$$\rho\left(\frac{\partial}{\partial p_i}\right) = \frac{\partial h_i}{\partial \bar z}\frac{\partial}{\partial z}\mathrm{d}\bar z.$$

9.0.2 The symbol $\mathrm{G}_{\vec{\lambda},k}(V)$

The symbol $\mathrm{G}_{\vec{\lambda},k}(V)_\sigma$ is a section of the symmetric tensor square $S^2(T^{1,0}\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)_\sigma)$, which is the same as $S^2((T^{1,0})^*\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)_\sigma)^*$, and as such, it is given by proposition 80 as $\mathrm{G}(V)(\alpha,\beta) = \int_\Sigma \mathrm{Tr}(\alpha\beta)\rho(V)$, where ρ is the Kodaira-Spencer map. For $V = \frac{\partial}{\partial p_i}$, a computation similar to eq. (5.3) shows that

$$G(\frac{\partial}{\partial p_i})(\alpha,\beta) = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mathrm{Tr}(\alpha\beta)\frac{\partial h_i}{\partial \bar z}\mathrm{d}z\wedge\mathrm{d}\bar z = \operatorname*{Res}_{p_i}\mathrm{Tr}\,\alpha\beta.$$

We calculate:

$$\begin{aligned}
G(\frac{\partial}{\partial p_i})(\theta_i,\theta_i) &= \operatorname*{Res}_{p_i} 2(\frac{-\tau_i}{z-1}+\frac{\tau_i}{z-p_i})^2 + 2(\frac{\tau_i}{z-1}+\frac{-\tau_i^2}{z-p_i})(\frac{\tau_i-1}{z}-\frac{\tau_i}{z-1}+\frac{1}{z-p_i})\\
&= 2\left(2\frac{-\tau_i^2}{p_i-1}+\frac{\tau_i}{p_i-1}-\tau_i^2\left(\frac{\tau_i-1}{p_i}-\frac{\tau_i}{p_i-1}\right)\right)\\
&= 2\tau_i(\tau_i-1)\left(\frac{\tau_i-1}{p_i-1}-\frac{\tau_i}{p_i}\right),
\end{aligned}$$

and for $i \neq j$:

$$\begin{aligned}
G(\frac{\partial}{\partial p_i})(\theta_i,\theta_j) = \operatorname*{Res}_{p_i} & 2(\frac{-\tau_i}{z-1}+\frac{\tau_i}{z-p_i})(\frac{-\tau_j}{z-1}+\frac{\tau_j}{z-p_j})\\
&+(\frac{\tau_i}{z-1}+\frac{-\tau_i^2}{z-p_i})(\frac{\tau_j-1}{z}-\frac{\tau_j}{z-1}+\frac{1}{z-p_j})\\
&+(\frac{\tau_j}{z-1}+\frac{-\tau_j^2}{z-p_j})(\frac{\tau_i-1}{z}-\frac{\tau_i}{z-1}+\frac{1}{z-p_i})\\
&= \big(2\tau_i(\frac{-\tau_j}{p_i-1}+\frac{\tau_j}{p_i-p_j})\\
&-\tau_i^2(\frac{\tau_j-1}{p_i}-\frac{\tau_j}{p_i-1}+\frac{1}{p_i-p_j})\\
&+(\frac{\tau_j}{p_i-1}+\frac{-\tau_j^2}{p_i-p_j})\big)\\
&= \left(\frac{-(\tau_i-\tau_j)^2}{p_i-p_j}+\tau_j\frac{(\tau_i-1)^2}{p_i-1}+(1-\tau_j)\frac{\tau_i^2}{p_i}\right)
\end{aligned}$$

and since θ_j does not have a pole at p_i for $i \neq j$, we have $G(\frac{\partial}{\partial p_i})(\theta_j,\theta_k) = 0$ for $i \notin \{j,k\}$.

9.1 Comparison

Comparing the expression for $\mathrm{G}_{\vec{\lambda},k}(V)$ with the symbol calculated in section 7.3, we find that:

$$\sigma\left(\frac{1}{k+2}u^{\mathrm{gKZ}}(V)\right) = -\frac{1}{8(k+2)}\mathrm{G}_{\vec{\lambda},k}(V).$$

The Hitchin connection for $\mathcal{T}\times_I \mathcal{M}^{\mathrm{Flat}}(\Sigma,\vec{\lambda},k)$, on the other hand, have a one-form with the symbol

$$\sigma\left(-{u^{\mathrm{Hit}}}_{k+2}(V)\right) = \sigma\left(\frac{-1}{2(k+2)+n_{k+2}}\frac{1}{2}\delta_{\mathrm{G}_{\vec{\lambda},k}(V)}\right) = -\frac{1}{8(k+2)}\mathrm{G}_{\vec{\lambda},k}(V),$$

using that $n_k = 2k$, by lemma 60 and the second item in the proof of proposition 76. That is, the two operators have the same symbol. The difference between two operators with the same symbol is an operator of one degree lower. We now need the following two lemmas:

Lemma 81. *The differential operator $u^{\mathrm{gKZ}}(V)$ preserves local holomorphic sections.*

Proof. This follows from the description of $u^{\mathrm{gKZ}}(V)$ by reduction. If $U \subseteq X_{\mathrm{G}}$ and $s \in \mathrm{H}^0(U, L_{\mathrm{G}})$, then the action of $u^{\mathrm{gKZ}}(V)$ on s is the same as the action of the pre-reduction operator $\sum_{i<j}\Omega_{\mathcal{L}}^{ij}\frac{\mathrm{d}(p_i-p_j)}{p_i-p_j}(V)$ on the $\mathrm{G}_{\mathbb{C}}$-invariant holomorphic section that lifts s to $\pi^{-1}(U)$. Let us therefore show that the prequantum operator associated to the function $\mu(\xi)$ on X preserves local holomorphic sections, so let $Z \in T^{0,1}$ and let s be a holomorphic section of L over $U \subseteq X$.

$$\nabla_Z(\mathrm{i}\nabla_{X_{\mu(\xi)}} + \mu(\xi)s) = \mathrm{i}(\nabla_{[Z,X_{\mu(\xi)}]} - i\omega(Z, X_{\mu(\xi)}))s + \mathrm{d}f(Z)s = \mathrm{i}\nabla_{[Z,X_{\mu(\xi)}]}s$$

But since G acts by Kähler isometries, $X_{\mu(\xi)} = \underline{\xi}$ preserves both the metric and the symplectic form, and therefore also the complex structure. Therefore

$$[Z, X_{\mu(\xi)}] = -\mathcal{L}_{X_{\mu(\xi)}}Z \in T^{0,1},$$

and we have that $\mathrm{i}\nabla_{[Z,X_{\mu(\xi)}]}s = 0$. Since $\Omega_{\mathcal{L}}^{ij}$ is built from these prequantum operators it must also preserve local holomorphic sections. □

Lemma 82. *The differential operator $u^{\mathrm{Hit}}(V)$ preserves local holomorphic sections.*

Proof. The argument in lemma 64 that $u^{\mathrm{Hit}}(V)$ preserves the global sections works the for local sections as well. □

Theorem 46. *For $n > 4$ there are one-forms a_k on $\mathcal{T}$ such that*

$$\frac{1}{k+2}u^{\mathrm{gKZ}}{}_{\vec{\lambda}}(V) - (\mathrm{Flags}^{-1})^*(u^{\mathrm{Hit}}{}_{\vec{\lambda},k+2}(V)) = a_k(V).$$

Proof. We will consider sections of $\mathrm{H}^0(X_{\mathrm{G}}, L_{\mathrm{G}}) \times \mathcal{T} \to \mathcal{T}$ as sections of $\pi_1^*(L_{\mathrm{G}}) \to X_{\mathrm{G}} \times \mathcal{T}$ that are holomorphic when restricted to $X_{\mathrm{G}} \times \{\sigma\}$ for any $\sigma \in \mathcal{T}$. For a vector field V on $\mathcal{T}$ we can treat ∇_V^{gKZ} and $(\mathrm{Flags}^{-1})^*(\nabla^{\mathrm{Hit}})_V$ as differential operators in $\pi_1^*(L_{\mathrm{G}})$. It follows from lemma 82 and lemma 81 that if either of these differential operators are applied to a local section s defined on an open subset $U \subseteq X_{\mathrm{G}} \times \mathcal{T}$ such that $s_{\restriction U\cap(X_{\mathrm{G}}\times\{\sigma\})}$ is holomorphic, then the result also have this property. Therefore the difference is also holomorphic when restricted to $U \cap (X_{\mathrm{G}} \times \{\sigma\})$. But the difference between $\nabla_V^{\mathrm{gKZ}} - (\mathrm{Flags}^{-1})^*(\nabla^{\mathrm{Hit}})_V$ is a first order operator differentiating only in the X_{G} directions. As it preserves local holomorphic sections, the symbol must be a holomorphic vector field on X_{G}. But by corollary 59 there are no holomorphic vector fields, and we see that it must be a zero order operator. But the only zero order operators that preserves holomorphic sections are the constants. This is exactly the claim. □

Theorem 47. *The representation of the braid group coming from the Hitchin connection is projectively equivalent to representations from the KZ connection*

Proof. Remember that the representations can be realized in the following way. Fix a point $\sigma \in \mathcal{T}$, and let $\mathrm{MCG}(\Sigma)$ act on the fiber over σ in the following way. First choose a path γ from σ to $g \cdot \sigma$, for $g \in \mathrm{MCG}(\Sigma)$. The action of g on the fiber are now given by the parallel transport along γ followed by the action of g^{-1}. For the KZ and the Hitchin connection we have to different parallel transports and two different actions on the line bundles, but both a projectively equivalent: let $\gamma : [0,1] \to \mathcal{T}$ be a curve, and consider the parallel transport along γ. If s is a section along γ, flat with respect to the Hitchin connection, then $\mathrm{e}^{A(t)}s$ is a flat section of the KZ connection along γ, where $A : [0,1] \to \mathbb{C}$ is a solution to the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}A(t) = a(\dot{\gamma}(t))$. Therefore the parallel transport along a curve of the KZ and the Hitchin connection is projectively equivalent. It follows now from proposition 78 that the two mapping class group actions to get back to $\gamma(0)$ are projectively equivalent. Therefore the two representations are projectively equivalent. □

Remark 83. In fact, since the Teichmüller space is contractible, we can even find a function $A : \mathcal{T} \to \mathbb{C}$ such that $\mathrm{d}A = a$. It follows that the map $s \mapsto \mathrm{e}^{A}s$ send the projective covariant sections of the Hitchin connection to the covariant sections of the KZ connection. Therefore the representations consisting of the flat sections of two connections are projectively equivalent.

This theorem opens up for the use of Andersen's theorem on asymptotic faithfulness, as described in section 6.2 , to be applied to the quantum representations on a sphere. To make the argument work, we need to choose $(\Sigma, \vec{\lambda})$ such that $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$ is compact. One way to ensure that is to choose an odd number of marked points, all with the same label, as we saw in remark 53. Next, we need to choose a sequence of levels and labels so that the moduli space associated to each of them is the same space. Recall that only the ratio $\frac{\vec{\lambda}}{k}$ is used in the construction of $\mathcal{M}^{\mathrm{Flat,irr}}(\Sigma, \vec{\lambda}, k)$, and we can therefore achieve this by choosing for each $r \in \mathbb{N}$ the level $k_r = qr$ and the label $\vec{\lambda}_r = r(p, p, \dots, p)$, where $p, q \in \mathbb{N}$ such that $\frac{p}{q} < 1$. The quantum representation for the label $\vec{\lambda}_r$ at level k_r is given by the Hitchin connection for a prequantum line bundle for the symplectic form $k_r\omega_{\vec{\lambda}_r,k_r} = k_r\omega_{\vec{\lambda}_1,k_1}$. Therefore we are exactly in the setup of section 6.2, and the following theorem can be proved by an identical argument:

Theorem 48. *Let $n \in \mathbb{N}$ be odd and greater than 4, Σ a sphere with n punctures and $\rho_k^{p,q}$ the quantum representation of* $\mathrm{MCG}(\Sigma)$ *where the n points are coloured with pk, at level qk, where $p, q \in \mathbb{N}$ and $\frac{p}{q} < 1$.. Then*

$$\bigoplus_{k=1}^{\infty} \rho_k^{p,q}$$

is a faithful representation of $\mathrm{MCG}(\Sigma)$.

9.2 Case of 4 punctures

As mentioned above, the interesting case of 4 marked points are more complicated due to the low codimension of the strictly semistable points. The stable part of the moduli space $\mathcal{M}^{\mathrm{Par}'}(\mathbb{CP}^1, (1,1,1,1), k)$ is parametrized by $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ by the cross ratio, and the three strictly semi-stable points corresponds to $0, 1, \infty$ – that is, a codimension 1 subset, and we therefore cannot apply Hartogs extension theorem. However, we can make some very explicit calculations, which we will present in this section, and although we cannot quite show theorem 1, we can reduce the statement to some reasonable conjectures.

We let $\nabla^{\mathrm{CS}} = \psi^{-1}(\nabla^{\mathrm{CS}})$, and set B to be the one-form $\nabla^{\mathrm{CS}} - \nabla$. We see that B is of type $(1, 0)$ and therefor $B = b\mathrm{d}z$ for some function b.

$$\begin{aligned}
(\psi^{-1})^*(\Delta_G) &= (\psi^{-1})^*(G\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}} + \nabla_{\mathrm{div}(G\frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau}}) \\
&= G\left(\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}} + 2b\nabla_{\frac{\partial}{\partial\tau}} + b\mathrm{d}b(\frac{\partial}{\partial\tau}) + b^2\right) + \nabla_{\mathrm{div}(G\frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau}} + B(\mathrm{div}(G\frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau}) \\
&= G\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}} + \nabla_{G\mathrm{div}(\frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau}} + \nabla_{(\frac{\partial}{\partial\tau}G)\frac{\partial}{\partial\tau}} + G\left(2b\nabla_{\frac{\partial}{\partial\tau}} + b\mathrm{d}b(\frac{\partial}{\partial\tau}) + b^2\right) + b\mathrm{div}(G\frac{\partial}{\partial\tau})
\end{aligned}$$

Lemma 84. *The first 3 terms above match exactly the second and first order terms of the geometrized KZ connection:*

$$G\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}} + \nabla_{G\mathrm{div}(\frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau}} + \nabla_{(\frac{\partial}{\partial\tau}G)\frac{\partial}{\partial\tau}} = \sum_{i=2,3}\sum_{\nu}\nabla_{q_*(\underline{\nu}^1)}\nabla_{q_*(\underline{\nu}^i)} - \mathrm{i}\nabla_{\underline{\xi_1+\xi_i}^1 + \underline{\xi_1+\xi_i}^i}$$

Proof. We start with the second order term on the RHS. We know that, for $x \in \mu^{-1}(0)$, $(T_x \mathrm{G}_{\mathbb{C}} x)^{\perp} = \mathrm{span}_{\mathbb{C}} W$ where W is a tangent vector such that $q_*(W) = \frac{\partial}{\partial \tau}$. We let

$$\underline{\nu}^i = h^i_\nu W + P^i,$$

where $P^i \in T_x(\mathrm{G}_{\mathbb{C}} x)$. We know that

$$G(\frac{\partial}{\partial p}) = q_*(\sum_{\nu, i<j} \Theta_{ij}(p) h^i_\nu h^j_\nu W \otimes W) = \sum_{\nu, i<j} \Theta_{ij}(p) h^i_\nu h^j_\nu \frac{\partial}{\partial \tau} \otimes \frac{\partial}{\partial \tau},$$

where $\Theta_{ij} = \frac{\mathrm{d}p_i - \mathrm{d}p_j}{p_i - p_j}(\frac{\partial}{\partial p_1})$.

$$\begin{aligned} 2\nabla_{\underline{\nu}^i} \nabla_{\underline{\nu}^j} &= \sum_\nu \nabla_{h^i_\nu W} \nabla_{h^j_\nu W} + \sum_\nu \nabla_{h^j_\nu W} \nabla_{h^i_\nu W} \\ &= \sum_\nu 2 h^i_\nu h^j_\nu \nabla_W \nabla_W + \nabla_{h^i_\nu (W h^j_\nu) W} + \nabla_{h^j_\nu (W h^i_\nu) W} \\ &= \sum_\nu 2 h^i_\nu h^j_\nu \nabla_W \nabla_W + \nabla_{(W(h^i_\nu h^j_\nu))W} \end{aligned}$$

so, as Θ only depend on p, we have:

$$\sum_{i<j} \nabla_{\underline{\nu}^i} \nabla_{\underline{\nu}^j} \Theta_{ij} = G \nabla_W \nabla_W + \frac{1}{2} \nabla_{(WG)W},$$

where $G = 2\frac{1}{p(p-1)}(\tau - p)(\tau - 1)\tau$ is the function such that $G \frac{\partial}{\partial \tau} \otimes \frac{\partial}{\partial \tau} = G(\frac{\partial}{\partial p})$. On the quotient this becomes

$$\sum_{i=2,3} \sum_\nu \nabla_{q_*(\underline{\nu}^1)} \nabla_{q_*(\underline{\nu}^i)} = G \nabla_{\frac{\partial}{\partial \tau}} \nabla_{\frac{\partial}{\partial \tau}} + \frac{1}{2} \nabla_{(\frac{\partial}{\partial \tau} G) \frac{\partial}{\partial \tau}}. \tag{9.1}$$

To calculate the second term on the LHS, we first recall that the symplectic form ω_{G} is given by

$$\omega_{\mathrm{G}} = \frac{\mathrm{i} \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}}{|\tau||\tau - 1|(1 + |\tau| + |\tau - 1|)}.$$

Remember that $\mathcal{L}_{\frac{\partial}{\partial \tau}} \omega_{\mathrm{G}} = \mathrm{div}(\frac{\partial}{\partial \tau}) \omega_{\mathrm{G}}$. We use Cartans formula and get

$$\mathcal{L}_{\frac{\partial}{\partial \tau}} \omega_{\mathrm{G}} = \mathrm{d}\omega_{\mathrm{G}}(\frac{\partial}{\partial \tau}, \cdot) = \mathrm{d} \frac{\mathrm{i} \mathrm{d}\bar{\tau}}{|\tau||\tau - 1|(1 + |\tau| + |\tau - 1|)} = \left(\frac{\partial}{\partial \tau} \frac{1}{|\tau||\tau - 1|(1 + |\tau| + |\tau - 1|)} \right) \mathrm{i} \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}$$

And from this we get

$$\begin{aligned} \mathrm{div}(\frac{\partial}{\partial \tau}) &= \left(\frac{\partial}{\partial \tau} \frac{1}{|\tau||\tau - 1|(1 + |\tau| + |\tau - 1|)} \right) |\tau||\tau - 1|(1 + |\tau| + |\tau - 1|) \\ &= -\frac{\partial}{\partial \tau} \log(|\tau||\tau - 1|(1 + |\tau| + |\tau - 1|)) \\ &= -\frac{1}{2\tau} - \frac{1}{2(\tau - 1)} - \frac{\frac{\bar{\tau}}{2|\tau|} + \frac{\bar{\tau} - 1}{2|\tau - 1|}}{1 + |\tau| + |\tau - 1|}. \end{aligned}$$

And we can now us this to calculate the desired quantity:

$$\nabla_{(G\mathrm{div}(\frac{\partial}{\partial\tau})+\frac{\partial}{\partial\tau}G)\frac{\partial}{\partial\tau}}.$$

We compute:

$$\frac{\partial}{\partial\tau}G = \frac{2}{p(p-1)}((\tau-p)(2\tau-1)+\tau(\tau-1)),$$

We already have a contribution of $\frac{1}{2}\nabla_{\frac{\partial}{\partial\tau}G\frac{\partial}{\partial\tau}}$ from eq. (9.1), so we compute:

$$\begin{aligned}
G\left(\mathrm{div}(\frac{\partial}{\partial\tau})\right)+\frac{1}{2}\frac{\partial}{\partial\tau}G &= \frac{(\tau-p)}{p(p-1)}\left(-(\tau-1+\tau)-\frac{(\tau-1)|\tau|+\tau|\tau-1|}{1+|\tau|+|\tau-1|}\right)\\
&+\frac{1}{p(p-1)}((\tau-p)(2\tau-1)+\tau(\tau-1))\\
&=\frac{(\tau-p)}{p(p-1)}\left(-\frac{(\tau-1)|\tau|+\tau|\tau-1|}{1+|\tau|+|\tau-1|}\right)\\
&+\frac{1}{p(p-1)}(\tau(\tau-1)).
\end{aligned}$$

For the KZ connection, we have the following, from calculation in section 5.7.3

$$\begin{aligned}
U_{12} &= -\mathrm{i}q_*(\underline{\xi_1+\xi_2}^1+\underline{\xi_1+\xi_2}^2) = \tau\left(\frac{1-\tau+|\tau-1|}{1+|\tau|+|\tau-1|}\right)\frac{\partial}{\partial\tau}\\
U_{13} &= -\mathrm{i}q_*(\underline{\xi_1+\xi_3}^1+\underline{\xi_1+\xi_3}^3) = \frac{(\tau-1)(|\tau|+\tau)}{1+|\tau|+|\tau-1|}\frac{\partial}{\partial\tau}.
\end{aligned}$$

We also have $\Theta_{12}=\frac{1}{p}$ and $\Theta_{13}=\frac{1}{p-1}$. The relevant vector field for the first-order part of the KZ is

$$\begin{aligned}
\Theta_{12}U_{12}+\Theta_{13}U_{13} &= \frac{1}{p(p-1)}((p-1)U_{12}+pU_{13})\\
&= -\frac{1}{p(p-1)}\left((\tau-p)\frac{(\tau-1)|\tau|+\tau|\tau-1|}{1+|\tau|+|\tau-1|}-\tau(\tau-1)\right)
\end{aligned}$$

□

Remark 85. We see that if $B=(\psi^{-1})^*(\partial F_\sigma)$ then the KZ connection match the pullback of the Hitchin connection up to a zero order operator.

Lemma 86. *$K_{X_\mathrm{G}}^{-\frac{1}{2}}\cong L_\mathrm{G}$ as holomorphic line bundles with connection and hermitian structure.*

Proof. They are isomorphic as holomorphic vector bundles, and the connections have the same curvature (because the Ricci form is $-2\omega_\mathrm{G}$). The obstruction to isomorphism is in $H_1(X_\mathrm{G},\mathrm{U}(1))$, and given by the difference of the monodromy around the punctures. But both bundles (with connection and hermitian structure) are invariant under the permutation action, so the same is true for the monodromy. But the only element $x\in\mathrm{U}(1)$ that satisfies $x=x^2$ is the identity, so the obstruction vanishes. □

Conjecture 87. *The map* $\mathrm{Flags}_\sigma : \mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k) \to X_{\mathrm{G}}$ *is an isometry for all* $\sigma \in \mathcal{T}$

Conjecture 88. *The subset of sections of the quotient line bundle* L_{G} *that satisfy that the pullback to the stable points in* X *extends over the strictly semistable points are mapped, under* ψ*, to a subset of* $\coprod_{\sigma\in\mathcal{T}} \mathrm{H}^0(\mathcal{M}^{\mathrm{Flat}}(\Sigma, \vec{\lambda}, k)_\sigma, \mathcal{L}^{\mathrm{CS}})$ *preserved by the Hitchin connection.*

Assuming the first of these conjectures, we have that B satisfies the requirement presented in remark 85, so the difference is a zero order operator. Assuming the second conjecture, this operator must preserve a finite-dimensional subset of the holomorphic sections. However, multiplication with a non-constant function generates a infinite dimensional subspace, so the zero-order operator is multiplication with a function on $\mathcal{T}$.

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