KMS weights on groupoid C^*-algebras

with an emphasis on graph C^*-algebras

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Resumé

Denne afhandling er opdelt i to dele. I den første del introduceres læseren først til groupoider, den fulde groupoid C^*-algebra og definitionen af KMS vægte på C^*-algebraer. Derefter præsenteres og bevises en udvidelse af Neshveyev's klassiske Sætning fra [20] til KMS vægte på groupoid C^*-algebraer. Ligesom den klassiske Sætning af Neshveyev deler vores udvidelse beskrivelsen af KMS vægte op i beskrivelsen af visse quasi-invariante mål på enhedsrummet af groupoiden og visse målelige legemer af tilstande. Vi viser, at disse quasi-invariante mål er extremale, præcist når de er ergodiske, og for en stor klasse af groupoider viser vi desmere, at de målelige legemer af sportilstande kan beskrives som tilstande på en tilhørende gruppe C^*-algebra.

Del to af afhandlingen er en samling af artikler, hvortil forfatteren har bidraget, og den skal betragtes som den primære del af denne afhandling. I [A] og [B] påbegyndes et studie af KMS vægtene for generaliserede gauge-virkninger på graf C^*–algebraer. I [A] beskrives alle disse, når grafen er endelig. I [B] opnår vi en delvis beskrivelse for Cayley-grafer for grupper, og vi viser, at beskrivelsen er fuldstændig, når gruppen er nilpotent. I artikel [C] og [D] gives der en fuldstændig beskrivelse af KMS tilstandene for gauge-virkningen på både Toeplitz og Cuntz-Krieger algebraen for alle endelige grafer af højere rank. Afslutningsvis beskæftiger vi os i [E] med at beskrive, hvordan der for en stor klasse af groupoider gælder, at KMS vægte givet ved mål på enhedsrummet kun kan forekomme for 1-parameter grupper givet ved kontinuerte groupoid homomorfier.

Abstract

This dissertation consists of two parts. In the first part the reader is introduced to groupoids, the full groupoid C^* algebra and the definition of KMS weights on C^*-algebras. We then present and prove an extension of a classical theorem of Neshveyev [20] to KMS weights on groupoid C^*-algebras. As in the original theorem in [20] this extended version splits the description of KMS weights into the description of certain quasi-invariant measures on the unit space of the groupoid and certain measurable fields of states. We show that these quasi-invariant measures are extremal if and only if they are ergodic, and for a large class of groupoid C^*-algebras we prove that the measurable fields of states can be described by states on an associated group C^*-algebra.

The second part of this dissertation is a collection of papers which the author has made contributions to, and this part is to be considered the main contribution of this dissertation. In [A] and [B] we investigate KMS weights for generalised gauge-actions on the C^*–algebra of directed graphs. In [A] we give a complete description when the graph is finite. In [B] we obtain a partial description for the graphs arising as Cayley graphs for groups, and we prove that this description is complete when the group is nilpotent. In article [C] and [D] we give a complete description of the KMS states for the gauge-action on both the Toeplitz and the Cuntz-Krieger algebras for all finite higher-rank graphs. Finally we describe in [E] how for a large class of groupoids KMS weights given by measures on the unit space can only occur for 1-parameter groups that are given by continuous groupoid homomorphism.

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Contents

Part II: Articles

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Introduction

This dissertation is divided into two parts. Part II is a collection of articles that the author has made contributions to. In Part I we give a general introduction to the study of KMS weights on groupoid C^*-algebras, and we give an overview of the main results obtained in the articles in Part II. Although Part I is merely an introduction to the subject, it also contains some new ideas and proofs. These are included to make the introduction as general as possible. The new work presented in Part I was done during the last few months of the authors Ph.D., so it is not as polished as the ideas presented in Part II, and the authors contributions to the articles in Part II are to be considered the main results of this dissertation.

When speaking of a KMS state, there are always two underlying objects involved: The C^*-algebra $\mathcal{A}$ the state is defined on, and a continuous 1-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$. The pair $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$ is called a C^*-dynamical system over $\mathbb{R}$ and a state ω on $\mathcal{A}$ is called a β-KMS state for α when:

$$\omega(a\alpha_{i\beta}(b)) = \omega(ba) \tag{1.1}$$

for all a, b in a norm dense, α-invariant $*$-subalgebra of the entire analytic elements for α. The formula (1.1) is often called the KMS condition, and it was originally considered in quantum statistical mechanics. One approach to building a model of a system in quantum statistical mechanics is to describe the observables of the system by a C^*-algebra $\mathcal{A}$. The dynamics of the system, which is how the observables changes over time, can then be modelled by a 1-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$. In this theoretical picture one can interpret the set of β-KMS states for α as the set of *equilibrium states* at inverse temperature β for the system modelled by $(\mathcal{A}, \alpha)$. This theoretical connection to physics was originally the reason why some mathematicians became interested in KMS states, yet over time the subject has also become very relevant for mathematicians like the author, who has little interest in (and knowledge about) the possible applications in physics. This is first and foremost due to the emergence of Tomita-Takesaki theory which associates a canonical continuous 1-parameter group to any normal faithful state on a von Neumann algebra, and this 1-parameter group can best be described as the unique 1-parameter group that makes the state a -1 KMS state on the algebra. This relation was an essential ingredient in the work of Connes and Haagerup that classify the hyperfinite type III factors, and it is therefore evident that the KMS condition has been a keen subject for operator-algebraist to ponder about, and it validates that KMS states are a natural class of objects to study. Over the years this had lead to the discovery of several deep relations between KMS states on C^*-algebras and different fields of mathematics. A prominent relation is the one between number theory and KMS states on C^*-algebras that has been established by work of Boost and Connes, which has lead to a flurry of activity in the study of KMS states on C^*-algebras. Another prominent relation, which is perhaps the most relevant one for the subjects studied in this dissertation, is the discovery that KMS states on C^*-algebras can be related to the study of dynamical systems. If none of the above sound reasons for studying KMS states can spark the readers enthusiasm, then the author can provide one last motivation: The theory of KMS states is both rich and beautiful, and faced with the task of describing KMS states one often gets exposed to elegant ideas and problems from many different fields of mathematics. No doubt this makes the study of KMS states a challenging task, but since the results are often thought-provoking and deep it also makes it an extremely rewarding task.

A natural class of C^*-algebras to study KMS states on is the class of *étale groupoid C^*-algebras*, by which we mean the family of C^*-algebras that can be realised as the C^*-algebra of a locally compact second countable Hausdorff étale groupoid. Most if not all C^*-algebras where a study of KMS states has been instigated can be described, priori of a priori, as an étale groupoid C^*-algebra, so the class is general enough to encompass most known examples. On the other hand, étale groupoid C^*-algebras have enough structure that it is possible to prove general and interesting results about KMS states that are valid for all étale groupoid C^*-algebras and a very large class of continuous 1-parameter groups. The continuous 1-parameter groups we have in mind here are the ones arising from a continuous groupoid homomorphism from the underlying groupoid to the reals (which is sometimes referred to as a 1-cocycle), and we will throughout call the family of these 1-parameter groups *diagonal*. Since the C^*-algebras we consider are not necessarily unital, we will instead of KMS states study their generalisation called KMS weights. This sets up the scope of this dissertation: We are going to study KMS weights for diagonal continuous 1-parameter groups on étale groupoid C^*-algebras, and we are going to provide a lot of concrete descriptions of KMS weights on C^*-dynamical systems arising from both directed graphs and higher-rank graphs.

The theory of groupoid C^*-algebras was pioneered by Jean Renault in his dissertation [26], which also contains the first great general insight into the structure of KMS weights on étale groupoid C^*-algebras. To describe this result, let $\mathcal{G}$ be an étale groupoid with unitspace $\mathcal{G}^{(0)}$ and let $C^*(\mathcal{G})$ be its full C^*-algebra. For a continuous groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ we then obtain a diagonal continuous 1-parameter group α^c on $C^*(\mathcal{G})$. The insight in [26] is that any quasi-invariant regular Borel measure on $\mathcal{G}^{(0)}$ with Radon-Nikodym derivative $e^{-\beta c}$ will give rise to a KMS weight on $C^*(\mathcal{G})$[1]. Furthermore Renault proved that for principal étale groupoids this becomes a bijective correspondence, and hence the study of KMS weights reduces to a study of measures on $\mathcal{G}^{(0)}$. This result builds a bridge between KMS weights and the field of dynamical systems, and it often greatly reduces the problem of describing the KMS weights. There is however a lot of interesting groupoids that are not principal, for example the ones associated to graphs and higher-rank graphs, and in the non-principal case the map between measures and KMS weights fails to be a bijection. There is a very nice description of KMS *states* on general étale groupoid C^*-algebras which are not principal, and this description was established in [20]. The insight obtained in [20] is that in the non-principal case there is a bijection between β-KMS states for α^c on $C^*(\mathcal{G})$ and pairs $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ consisting of a quasi-invariant Borel probability measure on $\mathcal{G}^{(0)}$ with Radon-Nikodym derivative $e^{-\beta c}$ and a μ-measurable field of states $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$.

We use this description of the KMS states as the starting point of our exposition. In chapter 2 we give an introduction to topological groupoids and the full groupoid C^*-algebra, focusing especially on the results and terminology we will need in the subsequent chapters. In chapter 3 we will briefly review some theory on continuous 1-parameter groups, KMS weights and their extension to the von Neumann algebra of their GNS representation. In chapter 4 we will extend Neshveyev's description of KMS states to a description of KMS weights, and we will then prove how this theorem regarding weights follows from the original theorem in [20]. Neshveyev's theorem translates the study of KMS weights into the study of two other objects, the quasi-invariant regular Borel measures and the measurable fields of states. This decomposition sets the theme for the next two chapters.

In chapter 5 we will study the quasi-invariant regular Borel measures. We will do this by proving that the extremal measures are exactly the

[1] In the terminology of this dissertation this is not entirely true, since we define a weight differently than in [26].

ergodic measures. Chapter 6 contains our investigation of measurable fields of states. Unlike in the other chapters, we have to restrict our analysis to a particular class of étale groupoids. This class contains the groupoids of higher-rank graphs and directed graphs. For this class of groupoids we will obtain a description of the measurable fields of states that is much in the spirit of the one obtained for traces on crossed products by abelian groups in Corollary 2.4 in [20], but which encompasses many more cases.

This dissertation contains an introduction to groupoid C^*-algebras and KMS weights, but its introduction to the theory of directed graphs and higher-rank graphs is fairly superficial. For more background on higher-rank graphs we refer to [24, 25, 18] and for their groupoid picture we refer to Appendix B in [10] and [8, 37, 18]. For more background on directed graphs we refer to [35, 19, 1] and for information on their groupoid picture we refer to [19, 22] and section 5 in [A]. There is a natural way of identifying the higher-rank graphs of rank 1 with the directed graphs, but in this dissertation we will keep the two terminologies separate. This is so because when we associate a C^*-algebra to a higher-rank graph we follow the standard higher-rank graph convention of letting a partial isometry S_e corresponding to an edge e have source projection $S_e^* S_e$ equal to the projection $p_{s(e)}$, while for directed graphs we follow the usual[2] convention of setting $S_e^* S_e = p_{r(e)}$.

In accordance with the rules of the Graduate School of Science and Technology (GSST) at Aarhus University chapter 7 contains a summary of the papers contained in Part II and relates them to current trends within the field. The papers have been divided into three themes. The first theme is the description of the KMS states for generalised gauge actions on directed graphs, where a description has been obtained for finite graphs [A] in collaboration with Klaus Thomsen, and a partial description has been obtained for infinite graphs structured as a Cayley graph [B] in collaboration with Klaus Thomsen. The authors contributions to [A, B] are proportional. The second theme concerns the description of the KMS states for the gauge-actions on the Toeplitz and Cuntz-Krieger algebra of a finite higher-rank graph and is a summary of the papers [C, D]. The last paper [E] is an investigation of the relationship between KMS weights given by measures on the unit space of the groupoids and diagonal actions. This paper is written jointly with Klaus Thomsen, and the authors contributions to this paper is proportional as well. In accordance with the GSST rules, parts of this dissertation were also used in the progress report for the qualifying examination.

[2] This convention is at least usual in the northern hemisphere, but some papers, e.g. [12, 15], use the higher-rank graph convention for directed graphs

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Groupoid C^*-algebras

In this chapter we will introduce groupoids and the full C^*-algebra of a groupoid and we will collect a few results from the literature, which we will use in the chapters to come. We will follow [26] for consistency.

2.1 Topological groupoids

Definition 2.1.1. A groupoid is a set $\mathcal{G}$ with an inverse map $\mathcal{G} \ni g \to g^{-1} \in \mathcal{G}$ and a partially defined product map $\mathcal{G}^2 \ni (g,h) \to gh \in \mathcal{G}$ defined on a set $\mathcal{G}^2 \subseteq \mathcal{G} \times \mathcal{G}$, that satisfies the following relations:

1. $(g^{-1})^{-1} = g$ for all $g \in \mathcal{G}$.

2. If $(g,h),(h,f) \in \mathcal{G}^2$ then $(gh,f),(g,hf) \in \mathcal{G}^2$ and $(gh)f = g(hf)$.

3. $(g^{-1},g) \in \mathcal{G}^2$ for all $g \in \mathcal{G}$, and $g^{-1}(gh) = h$ for $(g,h) \in \mathcal{G}^2$.

4. $(g,g^{-1}) \in \mathcal{G}^2$ for all $g \in \mathcal{G}$ and $(hg)g^{-1} = h$ for $(h,g) \in \mathcal{G}^2$.

We call $g \to g^{-1}g$ the *source map* and denote it by s and $g \to gg^{-1}$ the *range map* and denote it by r. We denote the image of these two maps by $\mathcal{G}^{(0)}$. The set $\mathcal{G}^2$ is called the set of *composable pairs*.

Groupoids are a generalisation of groups where the requirement that there is a fully defined product has been relaxed. One can prove that two elements g,h in a groupoid $\mathcal{G}$ can be composed into gh exactly when $s(g) = r(h)$. Since there is not a fully defined product there can not be a unit as for groups, but since $gs(g) = g$ and $r(g)g = g$ the elements in $\mathcal{G}^{(0)}$ act as units where composition makes sense, which is why $\mathcal{G}^{(0)}$ is called the *unit space*.

We follow the standard conventions in the literature on groupoids and set $\mathcal{G}_x = s^{-1}(x)$, $\mathcal{G}^x = r^{-1}(x)$ and $\mathcal{G}_x^x := \mathcal{G}_x \cap \mathcal{G}^x$ whenever $x \in \mathcal{G}^{(0)}$. The set $\mathcal{G}_x^x$ is called the *isotropy group at $x \in \mathcal{G}^{(0)}$*, and it is in fact a group with unit x when it inherits the groupoid operations from $\mathcal{G}$.

Definition 2.1.2. A topological groupoid is a groupoid $\mathcal{G}$ with a topology such that:

- The inverse map $\mathcal{G} \ni g \to g^{-1} \in \mathcal{G}$ is continuous.
- The product map $\mathcal{G}^2 \ni (g,h) \to gh \in \mathcal{G}$ is continuous when $\mathcal{G}^2$ has the relative product topology from $\mathcal{G} \times \mathcal{G}$.

To make sure that the topology on a topological groupoid is well behaved we follow [26] and always assume that it is locally compact, Hausdorff and second countable. With these restrictions on the topology one can use the continuity of the groupoid operations to prove:

Lemma 2.1.3. *Let $\mathcal{G}$ be a topological groupoid with a locally compact second countable Hausdorff topology. Then the inverse map $\mathcal{G} \ni g \to g^{-1} \in \mathcal{G}$ is a homeomorphism, $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ and $r, s : \mathcal{G} \to \mathcal{G}$ are continuous maps.*

For the scope of this dissertation we will require that the topology satisfies one more thing, namely that the groupoid is *étale*.

Definition 2.1.4. Let $\mathcal{G}$ be a topological groupoid. We call $\mathcal{G}$ a locally compact second countable Hausdorff étale groupoid when the topology is locally compact, second countable and Hausdorff and the maps $r, s : \mathcal{G} \to \mathcal{G}$ are local homeomorphisms.

Assumption 2.1.5. From this point and onwards all groupoids are locally compact second countable Hausdorff étale groupoids.

We remind the reader that r is a local homeomorphism when for every $g \in \mathcal{G}$ there is an open neighbourhood U of g with $r(U)$ open and $r|_U : U \to r(U)$ a homeomorphism when U and $r(U)$ are equipped with the relative topology from $\mathcal{G}$. If W is open with $r(W)$ and $s(W)$ open and $r|_W : W \to r(W)$ and $s|_W : W \to s(W)$ are homeomorphisms we call W a *bisection*. For many of our arguments we will need to choose sufficiently small bisections. To ease notation throughout we introduce the following:

Notation 2.1.6. We call a subset W of $\mathcal{G}$ a *small bisection* if W is an open bisection with compact closure and there exists another open bisection U with $W \subseteq \overline{W} \subseteq U$.

We can now prove the following regarding the topology on our groupoids:

Lemma 2.1.7. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid. Then:*

1. *$\mathcal{G}^{(0)}$ is a clopen subset of $\mathcal{G}$.*

2. *$\mathcal{G}^{(0)}$ and $\mathcal{G}$ are σ-compact*

3. *There exists a countable basis for the topology on $\mathcal{G}$ consisting of small bisections.*

4. *r and s maps Borel sets to Borel sets.*

Proof. Since $r : \mathcal{G} \to \mathcal{G}$ is a local homeomorphism with image $\mathcal{G}^{(0)}$ we get 1. 2 follows since $\mathcal{G}$ is locally compact and second countable, and 3 follows since $\mathcal{G}$ is second countable and since an open subset of a bisection is a bisection. If $\{W_i\}_{i\in\mathbb{N}}$ is a countable basis of small bisections and B is Borel, we have:

$$r(B) = \bigcup_{i=1}^{\infty} r(B \cap W_i)$$

Since $r(B \cap W_i)$ is Borel this proves 4. $\square$

We will need one more fact about étale groupoids for our exposition.

Lemma 2.1.8. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $V \subseteq \mathcal{G}^{(0)}$ be open. The reduction:*

$$\mathcal{G}|_V := \{g \in \mathcal{G} \ : \ r(g) \in V \text{ and } s(g) \in V\}$$

is an open subgroupoid of $\mathcal{G}$ and it is a locally compact second countable Hausdorff étale groupoid in the relative topology.

Proof. Checking that $\mathcal{G}|_V$ is a subgroupoid of $\mathcal{G}$ is straightforward. For $g \in \mathcal{G}|_V$ we can use that $\mathcal{G}$ is étale and V is open to find an open bisection W of g with $r(W) \subseteq V$ and $s(W) \subseteq V$. Hence $g \in W \subseteq \mathcal{G}|_V$, so $\mathcal{G}|_V$ is an open subset of $\mathcal{G}$. Knowing this, it is straightforward to prove that $\mathcal{G}|_V$ is a topological groupoid in the relative topology, and that it is locally compact second countable and Hausdorff. Since the open set W is a bisection in $\mathcal{G}|_V$, it follows that $\mathcal{G}|_V$ is also étale. $\square$

2.2 Groupoid C^*-algebras

We will fix a locally compact second countable Hausdorff étale groupoid $\mathcal{G}$ throughout this section, and we will describe how to introduce the full groupoid C^*-algebra $C^*(\mathcal{G})$ of $\mathcal{G}$. The first step in this construction is to consider the set $C_c(\mathcal{G})$ of continuous compactly supported complex functions, i.e.:

$$C_c(\mathcal{G}) := \{f : \mathcal{G} \to \mathbb{C} \mid f \text{ is continuous and } \operatorname{supp}(f) \text{ is compact}\}$$

$C_c(\mathcal{G})$ is a complex vector space under point wise addition and scalar multiplication, and one of the great benefits of working with $C_c(\mathcal{G})$ is that one can often use the compactness of the support of an element $f \in C_c(\mathcal{G})$ to write it as the finite sum of functions in $C_c(\mathcal{G})$ with well behaved support. As a key example of this, one can obtain the following Lemma by using 3 in Lemma 2.1.7 and a partition of unity.

Lemma 2.2.1. *Let $f \in C_c(\mathcal{G})$. There exists $n \in \mathbb{N}$ and small bisections $\{W_i\}_{i=1}^n$ such that $f = \sum_{i=1}^n f_i$ where $f_i \in C_c(\mathcal{G})$ satisfies $\mathrm{supp}(f_i) \subseteq W_i$ for each i.*

To build a C^*-algebra that uses the groupoid features of $\mathcal{G}$ we need to define an involution and a product of elements that incorporate the groupoid operations. For $f_1, f_2 \in C_c(\mathcal{G})$ we define the product $f_1 * f_2$ by:

$$(f_1 * f_2)(g) = \sum_{h \in \mathcal{G}^{r(g)}} f_1(h) f_2(h^{-1}g) \quad \text{for all } g \in \mathcal{G}. \tag{2.1}$$

To define the involution of an element $f \in C_c(\mathcal{G})$ we set:

$$f^*(g) = \overline{f(g^{-1})} \qquad \text{for all } g \in \mathcal{G}. \tag{2.2}$$

Proposition 2.2.2. *For every $f_1, f_2 \in C_c(\mathcal{G})$ we have $f_1 * f_2 \in C_c(\mathcal{G})$ and $f_1^* \in C_c(\mathcal{G})$. With this product and involution $C_c(\mathcal{G})$ becomes a $*$-algebra.*

The next step in constructing $C^*(\mathcal{G})$ is to define a norm on $C_c(\mathcal{G})$, and since we will construct the full C^*-algebra we will do this by taking the supremum of the norm under all bounded representations. To do this, let us introduce what we mean by a representation:

Definition 2.2.3. Let H be a Hilbert space and let $\mathcal{B}(H)$ denote the bounded operators on H. A representation π of $C_c(\mathcal{G})$ on H is a $*$-homomorphism $\pi : C_c(\mathcal{G}) \to \mathcal{B}(H)$ which is continuous when $C_c(\mathcal{G})$ has the inductive limit topology and $\mathcal{B}(H)$ has the weak operator topology and which is non-degenerate, i.e. $\mathrm{span}\{\pi(f)\xi \mid f \in C_c(\mathcal{G}),\ \xi \in H\}$ is dense in H.

We define *the I-norm* on $C_c(\mathcal{G})$ via the formula:

$$\|f\|_I = \max\Big\{ \sup_{x \in \mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}^x} |f(g)| \ , \ \sup_{x \in \mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x} |f(g)| \Big\} \tag{2.3}$$

A $*$-homomorphism $\pi : C_c(\mathcal{G}) \to \mathcal{B}(H)$ on a Hilbert space H is then *bounded* when $\|\pi(f)\| \leq \|f\|_I$ for all $f \in C_c(\mathcal{G})$. Any non-degenerate $*$-homomorphism bounded in the I-norm is continuous in the inductive limit topology and hence is a representation. One of the important contributions of [26] is that the converse is also true, i.e.:

Theorem 2.2.4. *$\|\cdot\|_I$ is a norm on $C_c(\mathcal{G})$ that defines a topology coarser than the inductive limit topology on $C_c(\mathcal{G})$, and every representation of $C_c(\mathcal{G})$ on a separable Hilbert space is automatically bounded in the norm $\|\cdot\|_I$.*

Proof. The content of this theorem corresponds to Corollary II.1.22 and (i) in Proposition II.1.4 in [26]. Since we are working with étale groupoids these statements can be proved without the heavy machinery of the Disintegration Theorem, see f.x. Lemma 3.2.3 in [27]. $\square$

C.f. Proposition II.1.11 in [26] we can now define a C^*-norm on $C_c(\mathcal{G})$ by setting:

$$\|f\| := \sup\{\|\pi(f)\| \mid \pi \text{ is a bounded representation of } C_c(\mathcal{G})\} \tag{2.4}$$

for all $f \in C_c(\mathcal{G})$.

Definition 2.2.5. The full C^*-algebra of $\mathcal{G}$ is the C^*-algebra $C^*(\mathcal{G})$ obtained by completing $C_c(\mathcal{G})$ in the full norm defined in (2.4).

Properties of $C^*(\mathcal{G})$

To get a understanding of the full norm on $C_c(\mathcal{G})$, let us first relate it to the natural occurring sup-norm $\|\cdot\|_\infty$ on $C_c(\mathcal{G})$:

Lemma 2.2.6. *Let $f \in C_c(\mathcal{G})$, then $\|f\|_\infty \leq \|f\|$ for all $f \in C_c(\mathcal{G})$ and we have equality when f is supported on an open bisection.*

Proof. The first statement follows from Proposition II.4.1 in [26]. If f is supported in a bisection, then by definition of the I-norm in (2.3) we have $\|f\|_I \leq \|f\|_\infty$, so by definition of the full norm $\|f\| \leq \|f\|_\infty$. $\square$

For functions $f_1, f_2 \in C_c(\mathcal{G}^{(0)})$ it follows from (2.1) and (2.2) that:

$$f_1 * f_2(x) = f_1(x) \cdot f_2(x)$$

and $f_1^*(x) = \overline{f(x)}$, i.e. the product and involution of $C_c(\mathcal{G})$ on $C_c(\mathcal{G}^{(0)})$ reduces to the natural point wise product and involution, so since $\mathcal{G}^{(0)}$ is a bisection Lemma 2.2.6 will imply:

Lemma 2.2.7. *Completing $C_c(\mathcal{G}^{(0)})$ in the full norm gives $C_0(\mathcal{G}^{(0)})$, so $C_0(\mathcal{G}^{(0)}) \subseteq C^*(\mathcal{G})$. The restriction map:*

$$C_c(\mathcal{G}) \ni f \to f|_{\mathcal{G}^{(0)}} \in C_c(\mathcal{G}^{(0)}) \tag{2.5}$$

extends to a conditional expectation $P : C^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$.

Proof. Since $\|f\|_\infty = \|f\|$ for $f \in C_c(\mathcal{G}^{(0)})$ the first statement follows. Since $\|f|_{\mathcal{G}^{(0)}}\|_\infty \leq \|f\|_\infty \leq \|f\|$ the map in (2.5) is a linear contraction, and hence it extends to a linear contraction $P : C^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$. Since $P(a) = a$ for $a \in C_0(\mathcal{G}^{(0)})$ then P is a contractive projection, and by Tomiyama's theorem, see e.g. Theorem 1.5.10 in [3], it is a conditional expectation. □

The following Lemma follows straightforward from standard calculations and the definition of the full norm.

Lemma 2.2.8. *If $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$ the map $\iota : C_c(\mathcal{H}) \to C_c(\mathcal{G})$ defined by extending functions by 0 is an injective $*$-homomorphism that extends to a $*$-homomorphism $\iota : C^*(\mathcal{H}) \to C^*(\mathcal{G})$.*

Finally let us recall that there is a natural way to construct *C^*-dynamical systems* on $C^*(\mathcal{G})$, see Definition 2.7.1 in [2] for a definition of a C^*-dynamical systems.

Definition 2.2.9. Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let G be a locally compact group. A map $\Phi : \mathcal{G} \to G$ is called a *continuous groupoid homomorphism*[1] if it is continuous and satisfies

$$\Phi(g^{-1}) = \Phi(g)^{-1} \quad \text{and} \quad \Phi(gh) = \Phi(g)\Phi(h)$$

for all $g, h \in \mathcal{G}$ with $s(g) = r(h)$.

When Φ is a continuous groupoid homomorphism into a locally compact abelian group A there is a canonical way to construct a C^*-dynamical system $(C^*(\mathcal{G}), \widehat{A}, \alpha)$ where $\widehat{A}$ denotes the Pontryagin dual group of A.

Proposition 2.2.10 (Proposition II.5.1 in [26])**.** *Let A be a locally compact abelian group and $\Phi : \mathcal{G} \to A$ be a continuous groupoid homomorphism. Then:*

1. *For each $\xi \in \widehat{A}$ and $f \in C_c(\mathcal{G})$ we set:*
$$\alpha_\xi(f)(g) := \xi(\Phi(g))f(g) \quad \text{for all } g \in \mathcal{G} \tag{2.6}$$
Equation (2.6) defines an automorphism α_ξ on $C_c(\mathcal{G})$.
2. *α_ξ extends to an automorphism on $C^*(\mathcal{G})$ that fixes $C_0(\mathcal{G}^{(0)})$ point wise.*
3. *$(C^*(\mathcal{G}), \widehat{A}, \alpha)$ is a C^*-dynamical system.*

[1] In the literature these are also called (continuous) 1-cocycles

More on representations of $C_c(\mathcal{G})$

Representations of $C_c(\mathcal{G})$ have the very nice property that they can be extended to a slightly bigger algebra $b_c(\mathcal{G})$. This fact will be crucial to this exposition, since the algebra $b_c(\mathcal{G})$, unlike $C_c(\mathcal{G})$, always contains a lot of projections. Let us formalise this:

Proposition 2.2.11. *Let $b_c(\mathcal{G})$ denote the set of bounded Borel functions from $\mathcal{G}$ to $\mathbb{C}$ with compact support. Defining a product and involution using the same formulas (2.1) and (2.2) as for $C_c(\mathcal{G})$, $b_c(\mathcal{G})$ becomes a $*$-algebra.*

We say that a sequence $\{f_n\}_{n=1}^{\infty} \subseteq b_c(\mathcal{G})$ converges to $f \in b_c(\mathcal{G})$ if $\lim_n f_n(g) = f(g)$ for all $g \in \mathcal{G}$ and there exists $h \in b_c(\mathcal{G})$ with $|f| \leq h$ and $|f_n| \leq h$ for all n. With this notion of convergence we can define what we mean by a representation of $b_c(\mathcal{G})$:

Definition 2.2.12. Let H be a Hilbert space. A representation L of $b_c(\mathcal{G})$ on H is a $*$-homomorphism $L : b_c(\mathcal{G}) \to \mathcal{B}(H)$ which is non-degenerate and satisfies that if $f_n \to f$ in $b_c(\mathcal{G})$ then $L(f_n)$ converges in the weak operator topology to $L(f)$.

Lemma 2.2.13 (Lemma II.1.17 in [26])**.** *Any representation π of $C_c(\mathcal{G})$ on a Hilbert space H extends to a representation L of $b_c(\mathcal{G})$ on H.*

— 3 —

KMS weights on C^*-algebras

In this chapter we will introduce KMS weights. First we will summarise a few facts concerning continuous 1-parameter groups, then we will turn to the definition of a KMS weight on a C^*-algebra. Our main source of information on these subjects are [16] and [17].

3.1 On continuous 1-parameter groups

Definition 3.1.1. Let $\mathcal{A}$ be a C^*-algebra. We call α a *continuous* 1-*parameter group* when $(\mathcal{A}, \mathbb{R}, \alpha)$ is a C^*-dynamical system, i.e. when $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ satisfies:

- $\alpha_{t+s} = \alpha_t \alpha_s$ and $\alpha_0 = \mathrm{Id}_{\mathcal{A}}$ for every $s, t \in \mathbb{R}$.
- The mapping $\mathbb{R} \ni t \to \alpha_t(a) \in \mathcal{A}$ is norm-continuous for all $a \in \mathcal{A}$.

Example 3.1.2. Assume $\mathcal{G}$ is a locally compact second countable Hausdorff étale groupoid and $c : \mathcal{G} \to \mathbb{R}$ is a continuous groupoid homomorphism. By Proposition 2.2.10 then c gives rise to a continuous 1-parameter group which we will denote by α^c. We will call these continuous 1-parameter groups *diagonal*.

We will extend a continuous 1-parameter group α from a map on $\mathbb{R}$ to a map on $\mathbb{C}$. To do this let $z \in \mathbb{C}$, then we let $S(z)$ denote the set of complex numbers with imaginary part between 0 and $\mathrm{Im}(z)$, i.e. if $\mathrm{Im}(z) \geq 0$ then:

$$S(z) = \{y \in \mathbb{C} \ : \ \mathrm{Im}(y) \in [0, \mathrm{Im}(z)]\}$$

and if $\mathrm{Im}(z) \leq 0$ then $S(z)$ is defined by requiring $\mathrm{Im}(y) \in [\mathrm{Im}(z), 0]$. We let $S(z)^0$ denote the interior of $S(z)$.

Definition 3.1.3. Let $z \in \mathbb{C}$ and let α be a continuous 1-parameter group on a C^*-algebra $\mathcal{A}$. Define $\mathrm{D}(\alpha_z)$ to be the $a \in \mathcal{A}$ such that:

- There exists a continuous function $f : S(z) \to \mathcal{A}$ that is analytic on $S(z)^0$ and satisfies $f(t) = \alpha_t(a)$ for all $t \in \mathbb{R}$.

and for such an a define $\alpha_z(a) = f(z)$.

Remark 3.1.4. When saying f is analytic in $z \in \mathbb{C}$ we mean that:

$$\frac{f(z+h) - f(z)}{h}$$

exists in norm for $h \to 0$. Remark that:

- We have in fact defined an extension in the sense that using the definition on a $z \in \mathbb{R}$ gives back the original continuous 1-parameter group α.
- $\mathrm{D}(\alpha_z)$ is a linear subspace of $\mathcal{A}$ and α_z is linear on $\mathrm{D}(\alpha_z)$.

We call an element $a \in \mathcal{A}$ *analytic for* α if there exists an analytic function $f : \mathbb{C} \to \mathcal{A}$ with $f(t) = \alpha_t(a)$ for all $t \in \mathbb{R}$, and it then follows that $a \in D(\alpha_z)$ for each $z \in \mathbb{C}$. To argue that $\mathrm{D}(\alpha_z)$ is dense it suffices to argue that the analytic elements are dense.

Lemma 3.1.5. *Let α be a continuous 1-parameter group on a C^*-algebra $\mathcal{A}$. For each $a \in \mathcal{A}$ and $n \in \mathbb{N}$ define:*

$$a(n) := \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t(a) e^{-nt^2} \; dt$$

Then $a(n)$ is analytic for α, $\|a(n)\| \leq \|a\|$ and $\|a(n) - a\| \to 0$ for $n \to \infty$.

Proof. See e.g. the proof of Proposition 2.5.22 in [2]. □

That we can find a core for α_z is going to be crucial.

Lemma 3.1.6. *Let α be a continuous 1-parameter group on a C^*-algebra $\mathcal{A}$ and $z \in \mathbb{C}$. Assume that $\mathcal{B}$ is a dense linear subspace of $\mathcal{A}$ and $b(n) \in \mathcal{B}$ for all $b \in \mathcal{B}$ and $n \in \mathbb{N}$. Then $\mathcal{B}$ is a core for α_z.*

Proof. As in the proof of 3 in Proposition 6.1 in [36]. □

Regarding diagonal continuous 1-parameter groups on $C^*(\mathcal{G})$ we can now conclude the following:

Proposition 3.1.7. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism. Then $C_c(\mathcal{G})$ is a core for $D(\alpha^c_z)$ for any $z \in \mathbb{C}$.*

Proof. For any $f \in C_c(\mathcal{G})$ supported in a small bisection W we have:

$$f(n) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t^c(f) e^{-nt^2} \, dt \tag{3.1}$$

By Lemma 3.1.6 and Lemma 2.2.1 it follows that $C_c(\mathcal{G})$ is a core if we can prove that $f(n) \in C_c(\mathcal{G})$. The expression in (3.1) can be approximated in norm by elements on the form $\sum_{i=1}^m \lambda_i \alpha_{t_i}^c(f)$ with $m \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$ and $t_i \in \mathbb{R}$. By definition of α^c each $\alpha_{t_i}^c(f)$ has support in W, so $\sum_{i=1}^m \lambda_i \alpha_{t_i}^c(f)$ has support in W. Hence it suffices to prove that if $\{f_n\}_{n\in\mathbb{N}} \subseteq C_c(\mathcal{G})$ is a sequence with $\|f_n - a\| \to 0$ for some $a \in C^*(\mathcal{G})$ and $\mathrm{supp}(f_n) \subseteq W$ for each n, then $a \in C_c(\mathcal{G})$. For such a sequence Lemma 2.2.6 implies that f_n is cauchy in the norm $\|\cdot\|_\infty$, so there is a function $f \in C_0(\mathcal{G})$ with $\|f_n - f\|_\infty \to 0$. Letting U be an open bisection such that $\overline{W} \subseteq U$, then $\mathrm{supp}(f) \subseteq \overline{W} \subseteq U$, implying that $f \in C_c(\mathcal{G})$ and $\|f_n - f\|_\infty = \|f_n - f\|$ for each n by another use of Lemma 2.2.6. In conclusion $a = f \in C_c(\mathcal{G})$. □

3.2 KMS weights

For a C^*-algebra $\mathcal{A}$ we let $\mathcal{A}_+$ denote the convex cone of positive elements of $\mathcal{A}$. A weight on $\mathcal{A}$ is a map $\psi : \mathcal{A}_+ \to [0, \infty]$ such that $\psi(a+b) = \psi(a)+\psi(b)$ and $\psi(\lambda a) = \lambda\psi(a)$ for all $a, b \in \mathcal{A}_+$ and $\lambda \geq 0$. We call a weight ψ:

- *densely defined* if $\{a \in \mathcal{A}_+ \;:\; \psi(a) < \infty\}$ is dense in $\mathcal{A}_+$.
- *lower semi-continuous* if $\{a \in \mathcal{A}_+ \;:\; \psi(a) \leq \lambda\}$ is closed for all $\lambda \geq 0$.
- *proper* if it is non-zero, densely defined and lower semi-continuous.

For any proper weight ψ we define a left ideal $\mathcal{N}_\psi := \{a \in \mathcal{A} \;:\; \psi(a^*a) < \infty\}$ and we set $\mathcal{M}_\psi^+ := \{a \in \mathcal{A}_+ \mid \psi(a) < \infty\}$. Setting:

$$\mathcal{M}_\psi := \mathrm{span}\{a^*b \;:\; a, b \in \mathcal{N}_\psi\}$$

then $\mathcal{M}_\psi = \mathrm{span}\ \mathcal{M}_\psi^+$ and it is a dense $*$-subalgebra of $\mathcal{A}$. There is a unique linear extension $\mathcal{M}_\psi \to \mathbb{C}$ of ψ on $\mathcal{M}_\psi \cap \mathcal{A}_+$ which we also denote by ψ.

Originally the theory of KMS weights on C^*-algebras goes back to Combes [5], but we will follow [17] and [16] for our information on KMS weights.

Definition 3.2.1. Let $\mathcal{A}$ be a C^*-algebra, $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ a continuous 1-parameter group and let $\beta \in \mathbb{R}$. We call a weight ψ on $\mathcal{A}$ a β-KMS weight for α if it is a proper weight satisfying:

1. $\psi \circ \alpha_t = \psi$ for all $t \in \mathbb{R}$.
2. For every $a \in D(\alpha_{-\beta i/2})$ we have
$$\psi(a^*a) = \psi(\alpha_{-\beta i/2}(a)\alpha_{-\beta i/2}(a)^*).$$

We call ψ a β-KMS state for α if $\sup\{\psi(a) \ : \ 0 \leq a \leq 1\} = 1$.

The KMS weights for α defined in [16, 17] are exactly the -1 KMS weights for α in Definition 3.2.1. For $\beta \neq 0$ we can translate the results of [16, 17] into our setting by noticing that in Definition 3.2.1 a weight is a β-KMS weight for $\{\alpha_t\}_{t\in\mathbb{R}}$ if and only if it is a -1 KMS weight for $\{\alpha_{-\beta t}\}_{t\in\mathbb{R}}$. For $\beta = 0$ we can translate the results by using that a 0-KMS weight for α is precisely a -1 KMS weight for $\{\mathrm{Id}_{\mathcal{A}}\}_{t\in\mathbb{R}}$ invariant under α.

With this in mind it follows from Proposition 1.11 in [17] that for $\beta \neq 0$ condition 2. could equivalently have been exchanged with condition $2'$.:

$2'$. For every $a, b \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$ there is a continuous and bounded function $F : S(i\beta) \to \mathbb{C}$ holomorphic on $S(i\beta)^0$ satisfying:
$$F(t) = \psi(a\alpha_t(b)) \quad , \quad F(t + i\beta) = \psi(\alpha_t(b)a)$$

Comparing this with the [5] it follows that our definition of a KMS weight agrees with the one in [5] up to a change in orientation.

Remark 3.2.2. By Proposition 5.3.3 and Proposition 5.3.7 in [2] the notion of a β-KMS state when $\beta \neq 0$ given in Definition 3.2.1 is identical to the classical one given in equation (1.1) in the introduction. The 0-KMS states for α defined by (1.1) are exactly the tracial state, while 0-KMS states for α in Definition 3.2.1 are tracial states invariant under α. The invariance under α has been chosen in this exposition to ensure that the definition is in accordance with the definition of a KMS state in [20]. In general there is no consistency in the literature on which definition on 0-KMS states to work with, and the author honours this tradition by being inconsistent himself, i.e. in [A, B, C, D] 0-KMS states are tracial states while in [E] they are invariant tracial states.

To get a better idea of how the notion of KMS weights in Definition 3.2.1 and the notion of KMS states in (1.1) are related, let us present Lemma 3.1 in [E]:

Lemma 3.2.3 (Lemma 3.1 in [E]). *Let $\mathcal{A}$ be a C^*-algebra and α a continuous 1-parameter group of automorphisms on $\mathcal{A}$ and ψ a KMS weight for α. Let $p \in \mathcal{A}$ be a projection in the fixed point algebra of α. Then $\psi(p) < \infty$.*

When $\mathcal{A}$ contains a unit $I_{\mathcal{A}}$ this Lemma implies that $\psi(I_{\mathcal{A}}) < \infty$ for any KMS weight ψ, and it follows that the KMS weights on $\mathcal{A}$ are just positive scalars of KMS states, and hence in the unital case it suffices to describe the KMS states. For non-unital C^*-algebras the KMS weights will in general not be scalars of KMS states, and for many non-unital C^*-algebras the structure of KMS states is poor compared to the structure of KMS weights. Therefore the key argument for studying weights is to ensure that intriguing information regarding the C^*-dynamical system is reflected by the set of KMS weights.

Most textbooks that introduce the notion of a weight will inform the reader that the abstraction of studying weights instead of states on a C^*-algebra can be compared to, or is indeed the non-commutative version of, the abstraction that takes place when one study general measures instead of probability measures on a measure space. As we shall see later on, c.f. Theorem 4.2.1, this point of view is strikingly correct when studying KMS weights on groupoid C^*-algebras, and it is worth remembering as another argument for studying KMS weights instead of KMS states.

3.3 The GNS construction

A KMS states for a C^*-dynamical system can be extended to a KMS state for a W^*-dynamical system using the GNS construction, c.f. Corollary 5.3.4 in [2]. A similar extension can be done for KMS weights, as we will see in the following.

Definition 3.3.1. Let ψ be a proper weight on a C^*-algebra $\mathcal{A}$. A *GNS construction* for ψ is a triple $(H_\psi, \pi_\psi, \Lambda_\psi)$ where H_ψ is a Hilbertspace, $\Lambda_\psi : \mathcal{N}_\psi \to H_\psi$ is a linear map with dense image such that:

$$\langle \Lambda_\psi(a), \Lambda_\psi(b) \rangle = \psi(b^* a) \qquad \text{for all } a, b \in \mathcal{N}_\psi$$

and $\pi_\psi : \mathcal{A} \to \mathcal{B}(H_\psi)$ is a representation with $\pi_\psi(a)\Lambda_\psi(b) = \Lambda_\psi(ab)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{N}_\psi$.

In a similar way that it is done for states one can prove that a GNS construction for weights exists and is unique up to unitary transformations. To extend a KMS weight via a GNS representation let us recall that a strongly continuous 1-parameter group α on a von Neumann algebra M is a 1-parameter group in Definition 3.1.1 with the exception that the map $\mathbb{R} \ni t \to \alpha_t(a)$ should be continuous in the σ-weak topology instead of in the norm-topology, or equivalently in the strong or weak operator topology c.f. 7.4.2 in [23], and similarly one can extend α by defining a linear map α_z on $D(\alpha_z)$ for every $z \in \mathbb{C}$. We call a weight ϕ on a von Neumann algebra M *semi-finite* when $\{a \in M_+ \ : \ \phi(a) < \infty\}$ is σ-weak dense i M_+ and we call ϕ *normal* when $\{a \in M_+ \ : \ \phi(a) \leq \lambda\}$ is σ-weak closed for all $\lambda \geq 0$.

Theorem 3.3.2. *Let ψ be a β-KMS weight for a continuous 1-parameter group α on a C^*-algebra $\mathcal{A}$ and let $(H_\psi, \pi_\psi, \Lambda_\psi)$ be a GNS-construction for ψ. Then*

1. *$\pi_\psi : \mathcal{A} \to \mathcal{B}(H_\psi)$ is non-degenerate.*
2. *There exists a normal semi-finite weight $\tilde{\psi}$ on $\pi_\psi(\mathcal{A})''$ such that $\psi = \tilde{\psi} \circ \pi_\psi$.*
3. *There exists a strongly continuous 1-parameter group $\tilde{\alpha}$ on $\pi_\psi(\mathcal{A})''$ that is implemented by unitaries such that $\pi_\psi \circ \alpha_t = \tilde{\alpha}_t \circ \pi_\psi$ and $\tilde{\psi} \circ \tilde{\alpha}_t = \tilde{\psi}$ for all $t \in \mathbb{R}$.*
4. *$\tilde{\psi}(a^*a) = \tilde{\psi}(\tilde{\alpha}_{-i\beta/2}(a)\tilde{\alpha}_{-i\beta/2}(a)^*)$ for all elements $a \in D(\tilde{\alpha}_{-i\beta/2})$.*

Proof of Theorem 3.3.2. The first statement is contained in Proposition 1.7 in [17]. The second statement is Definition 2.10 in [17] and the third is Corollary 2.20 in [17]. When $\beta = -1$ the last statement follows from combining the conclusion after Proposition 2.22 in [17] with Proposition 2.7 in [17]. For $\beta \neq 0$ the conclusion then follows from scaling the action, and for $\beta = 0$ it follows by noticing again that since ψ is a -1 weight for $\{\mathrm{Id}_{\mathcal{A}}\}_{t\in\mathbb{R}}$ then $\tilde{\psi}(a^*a) = \tilde{\psi}(aa^*)$ for all $a \in \pi_\psi(A)''$, which is the same as the statement for $\tilde{\alpha}_0$. □

— 4 —

Neshveyev's Theorem

In this chapter we will present Neshveyev's theorem on KMS states for diagonal actions on the C^*-algebras of locally compact second countable Hausdorff étale groupoids. Before getting to the theorem we will summarise some facts about measure theory and quasi-invariant measures, which albeit being fairly fundamental, has found its way into this dissertation to provide a rigorous background. Instead of presenting Neshveyev's original theorem we will present a generalisation to weights, and we will prove that this generalisation follows from the original theorem.

4.1 Background

Measure theory

We will throughout use [4] as our source on measure theory. A measure space is a triple (X, Σ, μ) where X is a set, Σ is a σ-algebra on X and μ is a measure on (X, Σ). In this dissertation all measures will be non-negative. If X is a topological space the Borel σ-algebra is the σ-algebra generated by the open sets, and we will denote this $\mathcal{B}(X)$. If μ is a measure on $(X, \mathcal{B}(X))$ we call μ a *Borel measure*. A Borel measure μ on a locally compact Hausdorff space X is *regular* when it is non-negative and satisfies:

- $\mu(K) < \infty$ for all compact sets $K \subseteq X$.
- $\mu(E) = \inf\{\mu(V) \ : \ E \subseteq V, \ V \text{ is open}\}$ for all Borel sets E.
- $\mu(E) = \sup\{\mu(K) \ : \ K \subseteq E, \ K \text{ is compact}\}$ for all open sets E and all Borel sets E with $\mu(E) < \infty$.

The locally compact Hausdorff spaces X we encounter in this dissertation are going to be second countable, and in that case most Borel measures are going to be regular as explained in the following Lemma.

Lemma 4.1.1. *Let X be a locally compact second countable Hausdorff space and let μ be a Borel measure on X. If $\mu(K) < \infty$ for all compact sets K*

in X then μ is regular. Furthermore any regular Borel measure μ on X satisfies:

$$\mu(E) = \sup\{\mu(K) \ : \ K \subseteq E, \ K \textit{ is compact}\}$$

for all Borel sets E.

Proof. Since X is σ-compact this follows from Proposition 7.2.3 and Proposition 7.2.6 in [4]. □

A measure space (X, Σ, μ) is called complete if $Y \subseteq Z \subseteq X$ and $Z \in \Sigma$ with $\mu(Z) = 0$ implies that $Y \in \Sigma$. A regular Borel measure μ on a locally compact Hausdorff space X is not necessarily complete, but there is a canonical way to complete it. Consider the set $\mathcal{B}(X)_\mu$ consisting of subsets $B \subseteq X$ satisfying that there are $A, C \in \mathcal{B}(X)$ with $A \subseteq B \subseteq C$ and:

$$\mu(C \setminus A) = 0$$

It can then be proven that setting $\overline{\mu}(B) := \mu(C) = \mu(A)$ gives a well defined measure with $(X, \mathcal{B}(X)_\mu, \overline{\mu})$ complete, $\mathcal{B}(X) \subseteq \mathcal{B}(X)_\mu$ and $\overline{\mu}$ agreeing with μ on Borel sets, see e.g. Proposition 1.5.1 in [4]. A function $f : X \to \mathbb{C}$ is called *Borel* when it is $(\mathcal{B}(X), \mathcal{B}(\mathbb{C}))$-measurable, and if μ is a Borel measure on a locally compact Hausdorff space X we say that $f : X \to \mathbb{C}$ is μ-measurable if it is $(\mathcal{B}(X)_\mu, \mathcal{B}(\mathbb{C}))$ -measurable.

Lemma 4.1.2. *Let X be a locally compact Hausdorff space and μ a Borel measure on X. If $f : X \to \mathbb{C}$ is μ-measurable then there exists a Borel function $f' : X \to \mathbb{C}$ and a $N \in \mathcal{B}(X)$ such that $\mu(N) = 0$ and $f(x) = f'(x)$ for all $x \in X \setminus N$.*

Proof. When $f : X \to \mathbb{C}$ is μ-measurable then $\mathrm{Re}(f), \mathrm{Im}(f) : X \to \mathbb{R}$ are $(\mathcal{B}(X)_\mu, \mathcal{B}(\mathbb{R}))$ measurable, so the Lemma follows from Proposition 2.2.3 in [4]. □

Quasi-invariant measures and fields of states

To state Neshveyev's theorem we will introduce the notion of μ-measurable fields of states and quasi-invariant measures. First let us introduce μ-measurable fields.

Definition 4.1.3. Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid, and let μ be a regular Borel measure on $\mathcal{G}^{(0)}$. For each $x \in \mathcal{G}^{(0)}$ we let u_g, $g \in \mathcal{G}_x^x$ denote the canonical unitary generators of $C^*(\mathcal{G}_x^x)$.

We call a collection $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ a *μ-measurable field of states* if each φ_x is a state on $C^*(\mathcal{G}_x^x)$ and the function:

$$\mathcal{G}^{(0)} \ni x \to \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g) \tag{4.1}$$

is μ-measurable for each $f \in C_c(\mathcal{G})$.

We identify two μ-measurable fields $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ and $\{\varphi'_x\}_{x\in\mathcal{G}^{(0)}}$ when $\varphi'_x = \varphi_x$ for μ-a.e. x.

Remark 4.1.4. The function in (4.1) has support in $r(\mathrm{supp}(f))$, which is compact since r is continuous. There is a $n \geq 1$ such that $\mathrm{supp}(f)$ is contained in n bisections, and hence $\sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)$ has at most n non-zero terms, implying that:

$$\left|\sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)\right| \leq n\|f\|_\infty$$

so the function in (4.1) is also bounded.

To introduce quasi-invariant measures let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid, and let μ be a regular Borel measure on $\mathcal{G}^{(0)}$. Arguing as in the above remark Riesz representation theorem implies that there exist unique regular Borel measures μ_r and μ_s on $\mathcal{G}$ with:

$$\int_{\mathcal{G}} f\ d\mu_r = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}^x} f(g)\ d\mu(x)\ ,\ \int_{\mathcal{G}} f\ d\mu_s = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x} f(g)\ d\mu(x)$$

for all $f \in C_c(\mathcal{G})$.

Definition 4.1.5. Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid, let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. Let μ be a regular Borel measure on $\mathcal{G}^{(0)}$. We call μ quasi-invariant with Radon-Nikodym cocycle $e^{-\beta c}$ if μ_r and μ_s are equivalent and $d\mu_r/d\mu_s = e^{-\beta c}$.

Although this is the standard definition of quasi-invariant measures with Radon-Nikodym cocycle $e^{-\beta c}$, we will give other characterisations of these measures which are easier to work with.

Proposition 4.1.6. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid, let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. If μ is a regular Borel measure on $\mathcal{G}^{(0)}$, the following are equivalent:*

1. *μ is quasi-invariant with Radon-Nikodym cocycle $e^{-\beta c}$.*

2. *For all small bisections $W \subseteq \mathcal{G}$ we have:*

$$\mu(s(W)) = \int_{r(W)} e^{\beta c(r_W^{-1}(x))}\, d\mu(x)$$

where r_W^{-1} is the inverse of $r_W : W \to r(W)$.

3. *Whenever $B \in \mathcal{B}(\mathcal{G})$ is contained in some small bisection W:*

$$\mu(s(B)) = \int_{r(B)} e^{\beta c(r_W^{-1}(x))}\, d\mu(x).$$

Proof. Let W be a small bisection and let μ be a regular Borel measure on $\mathcal{G}^{(0)}$. There exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq C_c(\mathcal{G})$ such that $0 \leq f_n \leq 1$ and f_n point wise increases to 1_W, and then it follows that $x \to \sum_{g\in\mathcal{G}_x} f_n(g)$ is an increasing sequence of Borel functions converging point wise to $1_{s(W)}$. This implies:

$$\mu_s(W) = \lim_n \int_{\mathcal{G}} f_n\, d\mu_s = \lim_n \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x} f_n(g)\, d\mu(x) = \mu(s(W))$$

Defining $f_n'(g) = f_n(g)e^{\beta c(g)}$ and using monotone convergence for f_n' gives:

$$\begin{aligned}\int_W e^{\beta c(g)} d\mu_r(g) &= \lim_n \int_{\mathcal{G}} f_n'(g) d\mu_r(g) = \lim_n \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}^x} f_n'(g) d\mu(x)\\ &= \int_{r(W)} e^{\beta c(r_W^{-1}(x))}\, d\mu(x)\end{aligned}$$

Any open subset $U \subseteq W$ is a small bisection, so the two Borel measures:

$$\mathcal{B}(W) \ni B \to \mu_s(B) \quad , \quad \mathcal{B}(W) \ni B \to \mu(s(B))$$

on W agree on open sets. Since W is a small bisection they are finite, so by Lemma 4.1.1 they agree on all Borel sets B contained in W. A similar argument gives that:

$$\int_B e^{\beta c(g)} d\mu_r(g) = \int_{r(B)} e^{\beta c(r_W^{-1}(x))}\, d\mu(x)$$

for all Borel sets $B \subseteq W$.

1 $\Rightarrow$ 2. Assume that μ is quasi-invariant with Radon-Nikodym cocycle $e^{-\beta c}$ and let $W \subseteq \mathcal{G}$ be a small bisection. Since $d\mu_s/d\mu_r = e^{\beta c}$ we get:

$$\mu(s(W)) = \mu_s(W) = \int_W e^{\beta c(g)} d\mu_r(g) = \int_{r(W)} e^{\beta c(r_W^{-1}(x))}\, d\mu(x)$$

2 $\Rightarrow$ 3. Let B' be a Borel set with $B' \subseteq W$ for some small bisection W. The two finite Borel measures on W given by:

$$\mathcal{B}(W) \ni B \to \mu(s(B)) \quad , \quad \mathcal{B}(W) \ni B \to \int_{r(B)} e^{\beta c(r_W^{-1}(x))}\, d\mu(x)$$

agree on open sets by assumption, so by regularity they agree on B'.

3 $\Rightarrow$ 1. Let $B \in \mathcal{B}(\mathcal{G})$, then by 3 in Lemma 2.1.7 we can write B as a countable disjoint union $\bigsqcup_i B_i$ with each B_i Borel and contained in a small bisection. Fix i and suppose $B_i \subseteq W$ for a small bisection W, the observation in the beginning of the proof then implies that:

$$\mu_s(B_i) = \mu(s(B_i)) = \int_{r(B_i)} e^{\beta c(r_W^{-1}(x))}\, d\mu(x) = \int_{B_i} e^{\beta c(g)} d\mu_r(g).$$

Since $g \to e^{\beta c(g)}$ is a strictly positive function this proves 3 $\Rightarrow$ 1. $\square$

Lemma 4.1.7. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid, let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. If $N \subseteq \mathcal{G}^{(0)}$ is a Borel set, then*

$$s(r^{-1}(N)) = r(s^{-1}(N))$$

and this set is Borel. If μ is quasi-invariant with Radon-Nikodym cocycle $e^{-\beta c}$ and $\mu(N) = 0$ then $\mu(s(r^{-1}(N))) = 0$.

Proof. Consider $g \in \mathcal{G}$ with $r(g) \in N$, then $s(g^{-1}) = r(g) \in N$ and $r(g^{-1}) = s(g)$, proving that $s(r^{-1}(N)) \subseteq r(s^{-1}(N))$. The other inclusion follows in the same way. Choose a basis $\{W_i\}_{i=1}^{\infty}$ of small bisections as in Lemma 2.1.7, then we have:

$$s(r^{-1}(N)) = \bigcup_{i=1}^{\infty} s(r_{W_i}^{-1}(N \cap r(W_i)))$$

Since $s \circ r_{W_i}^{-1}$ is a homeomorphism from $r(W_i)$ to $s(W_i)$ this proves that the set is Borel. If $\mu(N) = 0$ Proposition 4.1.6 implies:

$$\mu(s(r_{W_i}^{-1}(N \cap r(W_i)))) = \int_{N \cap r(W_i)} e^{\beta c(r_{W_i}^{-1}(x))}\, d\mu(x) = 0$$

which proves $\mu(s(r^{-1}(N))) = 0$. $\square$

4.2 Neshveyev's Theorem

We now have the necessary background to state Neshveyev's Theorem for weights:

Theorem 4.2.1 (Neshveyev)**.** *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid, let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and let $\beta \in \mathbb{R}$.*

There is a bijective correspondence between the β-KMS weights for α^c on $C^*(\mathcal{G})$ and the pairs $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$, where μ is a non-zero regular Borel measure on $\mathcal{G}^{(0)}$ and $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field of states φ_x on $C^*(\mathcal{G}_x^x)$ such that:

1. *μ is quasi-invariant with Radon-Nikodym cocycle $e^{-\beta c}$.*
2. *$\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}})$ for μ-a.e. $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x$ and $h \in \mathcal{G}_x$.*
3. *$\varphi_x(u_g) = 0$ for μ-a.e. $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x \setminus c^{-1}(0)$.*

The β-KMS weight ψ corresponding to the pair $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ has the property that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\psi$ and it is the unique β-KMS weight satisfying:

$$\psi(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)\, d\mu(x) \tag{4.2}$$

for all $f \in C_c(\mathcal{G})$.

Remark 4.2.2. By choosing an approximation of the identity $\{f_n\}_{n=1}^\infty \subseteq C_c(\mathcal{G}^{(0)})$ one can prove that ψ is a state if and only if $\mu(\mathcal{G}^{(0)}) = 1$. Hence we get a bijection between KMS states and the pairs $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$, where μ is a regular Borel probability measure on $\mathcal{G}^{(0)}$ and $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field of states satisfying 1.-3. This is the content of the original theorem by Neshveyev presented in [20].

Before we give a proof of Theorem 4.2.1 we will prove a Proposition of independent interest.

Proposition 4.2.3. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. Let ψ be a β-KMS weight for α^c and let $(H_\psi, \pi_\psi, \Lambda_\psi)$ be its GNS triple, with $\tilde{\psi}$ the extension of ψ to $\pi_\psi(C^*(\mathcal{G}))''$ and $\tilde{\alpha}^c$ the extension of α^c as in Theorem 3.3.2. Then we have:*

1. *Restricting π_ψ to $C_c(\mathcal{G})$ gives a bounded representation of $C_c(\mathcal{G})$ on H_ψ as in Definition 2.2.3.*

2. *Let L_ψ be the extension of $\pi_\psi|_{C_c(\mathcal{G})}$ to $b_c(\mathcal{G})$, then for any compact set $K \subseteq \mathcal{G}^{(0)}$ we have $L_\psi(1_K) \in \pi_\psi(C^*(\mathcal{G}))''$ and:*

$$\tilde{\psi}(L_\psi(1_K)) < \infty$$

Proof. Since $\pi_\psi(C_c(\mathcal{G}))$ is norm-dense in $\pi_\psi(C^*(\mathcal{G}))$ it follows that $\pi_\psi|_{C_c(\mathcal{G})}$ is a non-degenerate $*$-homomorphism. Since π_ψ is bounded then $\|\pi_\psi(f)\| \leq \|f\| \leq \|f\|_I$, so $\pi_\psi|_{C_c(\mathcal{G})}$ is bounded in the I-norm, and Theorem 2.2.4 then implies that it is a representation.

To prove 2 notice that 1_K is a projection in $b_c(\mathcal{G})$, and by the shrinking Lemma and second countability we can find a sequence of functions $\{f_n\} \subseteq C_c(\mathcal{G}^{(0)})$ with $f_n \to 1_K$ in $b_c(\mathcal{G})$. Lemma 2.2.13 and Definition 2.2.12 implies that $L_\psi(f_n) \to L_\psi(1_K)$ in the weak operator topology, so $L_\psi(1_K) \in \pi_\psi(C^*(\mathcal{G}))''$. Since $\tilde{\alpha}^c_t$ is implemented by unitaries it is weak operator continuous, so for any $t \in \mathbb{R}$ and $\xi, \eta \in H_\psi$:

$$\begin{aligned}\langle \tilde{\alpha}^c_t(L_\psi(1_K))\xi, \eta\rangle &= \lim_n \langle \tilde{\alpha}^c_t(L_\psi(f_n))\xi, \eta\rangle = \lim_n \langle \tilde{\alpha}^c_t(\pi_\psi(f_n))\xi, \eta\rangle \\ &= \lim_n \langle \pi_\psi(\alpha^c_t(f_n))\xi, \eta\rangle = \lim_n \langle \pi_\psi(f_n)\xi, \eta\rangle = \langle L_\psi(1_K)\xi, \eta\rangle\end{aligned}$$

So $L_\psi(1_K)$ is a projection in the fixed point algebra of $\tilde{\alpha}^c$.

Let $f \in C_c(\mathcal{G}^{(0)})$ satisfy that $f(x) = 1$ for $x \in K$ and $0 \leq f \leq 1$. Since ψ is proper and $f \in C^*(\mathcal{G})_+$ there is a sequence of positive elements $\{A^k\}_{k=1}^\infty \subseteq C^*(\mathcal{G})_+$ such that $\psi(A^k) < \infty$ for each k and $\|A^k - f\| \to 0$ for $k \to \infty$. Setting:

$$A^k(n) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha^c_t(A^k) e^{-nt^2} \, dt$$

then $A^k(n)$ is analytic for α^c for each n, k, $A^k(n) \to A^k$ for $n \to \infty$ in norm and by e.g. Lemma 2.12 in [16] then:

$$\psi((A^k(n))^2) \leq \|A^k(n)\| \psi(A^k(n)) < \infty.$$

Since

$$\lim_k \lim_n (A^k(n))^2 = f^2$$

there is an analytic element $A \in C^*(\mathcal{G})_+$ for α^c with $\psi(A^2) < \infty$ such that $\|A^2 - f^2\| < 1/2$. By definition of the product in $b_c(\mathcal{G})$, $1_K = 1_K * f^2 * 1_K$, so:

$$\begin{aligned}&\|L_\psi(1_K)\pi_\psi(A^2)L_\psi(1_K) - L_\psi(1_K)\|_{\mathcal{B}(H_\psi)} \\ &\quad = \|L_\psi(1_K)\left(\pi_\psi(A^2) - L_\psi(f^2)\right)L_\psi(1_K)\|_{\mathcal{B}(H_\psi)} \\ &\quad \leq \|\pi_\psi(A^2) - \pi_\psi(f^2)\|_{\mathcal{B}(H_\psi)} \leq \|A^2 - f^2\| < 1/2\end{aligned}$$

where $\|\cdot\|_{\mathcal{B}(H_\psi)}$ is the operator norm on $\mathcal{B}(H_\psi)$. Spectral theory now implies that $L_\psi(1_K)\pi_\psi(A^2)L_\psi(1_K) \geq (1/2)L_\psi(1_K)$. Since A is analytic for α^c there is an analytic function $F : \mathbb{C} \to C^*(\mathcal{G})$ with $F(t) = \alpha^c_t(A)$ for all $t \in \mathbb{R}$. Defining $G : \mathbb{C} \to \pi_\psi(C^*(\mathcal{G}))''$ by $G(z) = \pi_\psi(F(z))L_\psi(1_K)$ then G is analytic in norm and:

$$G(t) = \pi_\psi(\alpha^c_t(A))L_\psi(1_K) = \tilde{\alpha}^c_t(\pi_\psi(A)L_\psi(1_K)) \qquad \forall t \in \mathbb{R}.$$

In conclusion $\pi_\psi(A)L_\psi(1_K)$, and by a similar argument $\pi_\psi(A)$, is analytic for $\tilde{\alpha}^c$ and hence:

$$\begin{aligned}
\tilde{\psi}(L_\psi(1_K)) &\leq 2\tilde{\psi}(L_\psi(1_K)\pi_\psi(A^2)L_\psi(1_K)) \\
&= 2\tilde{\psi}(\tilde{\alpha}^c_{-i\beta/2}(\pi_\psi(A))L_\psi(1_K)\tilde{\alpha}^c_{-i\beta/2}(\pi_\psi(A))^*) \\
&\leq 2\tilde{\psi}(\tilde{\alpha}^c_{-i\beta/2}(\pi_\psi(A))\tilde{\alpha}^c_{-i\beta/2}(\pi_\psi(A))^*) \\
&= 2\tilde{\psi}(\pi_\psi(A^2)) = 2\psi(A^2) < \infty
\end{aligned}$$

which proves 2. □

We can now prove Theorem 4.2.1. This proof have a predecessor in Theorem 3.2 in [E].

Proof of Theorem 4.2.1. Since this proof is quite long we have divided it into four steps.

Step 1: Associating a pair $(\mu, \{\varphi_x\})$ to a weight ψ

Let ψ be a β-KMS weight for α^c, and borrow the notation used in Proposition 4.2.3. Using Lemma 2.1.7 we can find a countable sequence $\{V_i\}_{i=1}^\infty$ of open sets in $\mathcal{G}^{(0)}$ with compact closure $K_i := \overline{V_i}$, such that $K_i \subseteq V_{i+1}$ for each i and:

$$\mathcal{G}^{(0)} = \bigcup_{i=1}^\infty V_i$$

Since $L_\psi(1_{K_{i-1}}) \leq L_\psi(1_{V_i}) \leq L_\psi(1_{K_i})$ it follows from Proposition 4.2.3 that $L_\psi(1_{V_1}) \leq L_\psi(1_{V_2}) \leq \cdots$ is a sequence of projections with $\tilde{\psi}(L_\psi(1_{V_i})) < \infty$ for each i, and by approximating 1_{V_i} by an increasing sequence of functions in $C_c(\mathcal{G})$ it follows that $L_\psi(1_{V_i}) \in \pi_\psi(C^*(\mathcal{G}))''$. So $L_\psi(1_{V_i})$ converges to a projection $P \in \pi_\psi(C^*(\mathcal{G}))''$ in the strong topology and by definition of P then $P\pi_\psi(f) = \pi_\psi(f)$ for all $f \in C_c(\mathcal{G})$, implying since $\pi_\psi|_{C_c(\mathcal{G})}$ is non-degenerate that $P = I_{B(H_\psi)}$. Since $\{L_\psi(1_{V_i})\}_{i=1}^\infty$ is a bounded sequence it converges to $I_{B(H_\psi)}$ in the σ-weak topology, so since $\tilde{\psi}(I_{B(H_\psi)}) > 0$ and $\tilde{\psi}$ is normal we can then assume that:

$$0 < \tilde{\psi}(L_\psi(1_{V_i})) < \infty \qquad \text{for each } i.$$

For $f \in C_c(\mathcal{G})$ we can use compactness to find a i such that:

$$r(\operatorname{supp}(f)) \subseteq V_i \text{ and } s(\operatorname{supp}(f)) \subseteq V_i$$

By definition of the product in $b_c(\mathcal{G})$ we have $f = 1_{V_i} * f * 1_{V_i}$. Considering f as an element of the C^*-algebra $C^*(\mathcal{G})$ there exists $a_j \in C^*(\mathcal{G})_+$ and $\lambda_j \in \mathbb{C}$ for $j = 1, 2, 3, 4$ with $f = \sum_{j=1}^4 \lambda_j a_j$. Since:

$$\tilde{\psi}(L_\psi(1_{V_i})\pi_\psi(a_j)L_\psi(1_{V_i})) \leq \|\pi_\psi(a_j)\|\tilde{\psi}(L_\psi(1_{V_i})) < \infty$$

and:

$$\pi_\psi(f) = L_\psi(1_{V_i})\pi_\psi(f)L_\psi(1_{V_i}) = \sum_{j=1}^{4} \lambda_j L_\psi(1_{V_i})\pi_\psi(a_j)L_\psi(1_{V_i})$$

we get that $|\psi(f)| = |\tilde{\psi}(\pi_\psi(f))| < \infty$, proving that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\psi$. In particular ψ is a positive linear functional on $C_c(\mathcal{G}^{(0)})$, so by the Riesz Representation Theorem there is a unique regular Borel measure μ on $\mathcal{G}^{(0)}$ such that:

$$\psi(f) = \int_{\mathcal{G}^{(0)}} f(x)\, d\mu(x) \quad \text{for all } f \in C_c(\mathcal{G}^{(0)})$$

By Lemma 2.1.8 we have an open subgroupoid $\mathcal{G}_n := \mathcal{G}|_{V_n}$ and by Lemma 2.2.8 inclusion gives a $*$-homomorphism $\iota_n : C^*(\mathcal{G}_n) \to C^*(\mathcal{G})$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we want to define a function ω_n on $C^*(\mathcal{G}_n)$ by:

$$\omega_n(a) = \mu(V_n)^{-1}\psi(\iota_n(a))$$

Since $\tilde{\psi}(L_\psi(1_{V_n})) < \infty$ then $\mu(V_n) > 0$. Choosing $f \in C_c(\mathcal{G}^{(0)})$ with $f = 1$ on K_n we get $f * \iota_n(a) = \iota_n(a)$ for all $a \in C^*(\mathcal{G}_n)$, and hence $\psi(\iota_n(a)) = \psi(f\iota_n(a)f) < \infty$ for all $a \in C^*(\mathcal{G}_n)_+$, proving that ω_n is a positive linear functional. Using e.g. Proposition 2.3.11 in [2] the definition of μ implies that ω_n is a state. Since $c_n = c|_{\mathcal{G}_n}$ is a continuous groupoid homomorphism on $\mathcal{G}_n$ and $\iota_n \circ \alpha^{c_n} = \alpha^c \circ \iota_n$, we get that ω_n is a β-KMS state for α^{c_n} on $C^*(\mathcal{G}_n)$. Using Theorem 4.2.1 for states, we get a regular Borel probability measure μ_n on $V_n = \mathcal{G}_n^{(0)}$ and a μ_n-measurable field of states $\{\varphi_x^n\}_{x \in V_n}$ such that:

$a_n)$ μ_n is quasi-invariant on $\mathcal{G}_n$ with Radon-Nikodym cocycle $e^{-\beta c_n}$.

$b_n)$ $\varphi_x^n(u_g) = \varphi_{r(h)}^n(u_{hgh^{-1}})$ for μ_n-a.e. $x \in V_n$ and all $g \in \mathcal{G}_x^x$, $h \in (\mathcal{G}_n)_x$.

$c_n)$ $\varphi_x^n(u_g) = 0$ for μ_n-a.e. $x \in V_n$ and all $g \in \mathcal{G}_x^x \setminus c_n^{-1}(0)$

and that for $f \in C_c(\mathcal{G}_n)$ we have:

$$\mu(V_n)^{-1}\psi(\iota_n(f)) = \int_{V_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^n(u_g)\ d\mu_n(x)$$

For every function $f \in C_c(V_n)$ we have:

$$\begin{aligned} \mu(V_n)^{-1} \int_{V_n} f(x)\ d\mu(x) &= \mu(V_n)^{-1} \int_{\mathcal{G}^{(0)}} \iota_n(f)(x)\ d\mu(x) = \mu(V_n)^{-1}\psi(\iota_n(f)) \\ &= \int_{V_n} f(x)\ d\mu_n(x) \end{aligned}$$

So $\mu|_{V_n} = \mu(V_n)\mu_n$, which implies that $\mu_n = \mu(V_{n+1})/\mu(V_n)\mu_{n+1}|_{V_n}$. Any small bisection W satisfies $W \subseteq \mathcal{G}_n$ for all sufficiently large n, so by Proposition 4.1.6 μ satisfies (1) since μ_n satisfies a_n) for each n. For $f \in C_c(\mathcal{G}_n)$ we have:

$$\begin{aligned} \int_{V_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^n(u_g)\ d\mu_n(x) &= \mu(V_n)^{-1}\psi(\iota_n(f)) \\ &= \frac{\mu(V_{n+1})}{\mu(V_n)} \int_{V_{n+1}} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^{n+1}(u_g)\ d\mu_{n+1}(x) \\ &= \int_{V_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^{n+1}(u_g)\ d\mu_n(x) \end{aligned}$$

Since $\mu(N) = 0$ for $N \subseteq V_n$ iff $\mu_n(N) = 0$ then $\{\varphi_x^{n+1}\}_{x \in V_n}$ satisfies b_n) and c_n), so injectivity of the map in Theorem 1.3 in [20] implies that $\varphi_x^{n+1} = \varphi_x^n$ for μ_n-almost all $x \in V_n$. Let $N_n \subseteq V_n$ be a Borel set with $\mu_n(N_n) = 0$ and $\varphi_x^{n+1} = \varphi_x^n$ for $x \in V_n \setminus N_n$, and set $N = s(r^{-1}(\bigcup_n N_n))$. Then N is Borel and $\mu(N) = 0$ by Lemma 4.1.7. Let Tr_x be the canonical trace on $C^*(\mathcal{G}_x^x)$ and set:

$$\varphi_x = \begin{cases} \varphi_x^n & \text{if } x \in V_n \setminus N \text{ for some } n \\ Tr_x & \text{if } x \in N \end{cases}$$

This is well defined by choice of N, and it is straightforward to check that φ_x satisfies (2) and (3). Since any $f \in C_c(\mathcal{G})$ satisfies $\text{supp}(f) \subseteq \mathcal{G}_n$ for large n then $\{\varphi_x\}$ is μ-measurable since each $\{\varphi_x^n\}$ is μ_n-measurable. For $f \in C_c(\mathcal{G})$ there exists a n such that $f = \iota_n(f|_{\mathcal{G}_n})$, and hence:

$$\begin{aligned} \psi(f) = \psi(\iota_n(f|_{\mathcal{G}_n})) &= \mu(V_n) \int_{V_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^n(u_g)\ d\mu_n(x) \\ &= \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)\ d\mu(x) \end{aligned}$$

which proves that to any KMS weight ψ corresponds a pair $(\mu, \{\varphi_x\}_{x \in \mathcal{G}^{(0)}})$.

Step 2: Every pair $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ gives rise to a KMS weight

Assume now $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ satisfies (1) – (3), then every $x \in \mathcal{G}^{(0)}$ gives rise to a state ψ_x on $C^*(\mathcal{G})$ such that:

$$\psi_x(f) = \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)$$

for all $f \in C_c(\mathcal{G})$, c.f. the proof of Theorem 1.1 in [20]. Since $x \to \psi_x(f)$ is μ-measurable, so is $x \to \psi_x(a)$ for all $a \in C^*(\mathcal{G})$. For $a \geq 0$ we can define:

$$\psi(a) = \int_{\mathcal{G}^{(0)}} \psi_x(a)\ d\mu(x)$$

Then ψ is a non-zero weight on $C^*(\mathcal{G})$. Fatou's lemma implies that ψ is lower semi-continuous and for $f \in C_c(\mathcal{G})$ then $\psi_x(f) = 0$ for $x \notin s(\mathrm{supp}(f))$, so:

$$|\psi(f)| \leq \int_{\mathcal{G}^{(0)}} |\psi_x(f)|\ d\mu(x) \leq \|f\|\mu(s(\mathrm{supp}(f))) < \infty$$

In conclusion ψ is a proper weight with $C_c(\mathcal{G}) \subseteq \mathcal{M}_\psi$. Choose a sequence $\{V_n\}_{n=1}^\infty$ as in step 1, since μ is regular we can assume $0 < \mu(V_n) < \infty$ for all n. Then $(\mu(V_n)^{-1}\mu, \{\varphi_x\}_{x\in V_n})$ gives a β-KMS state ω_n on $C^*(\mathcal{G}_n)$ for α^{c_n} by Theorem 1.3 in [20]. For any $f \in C_c(\mathcal{G})$, there is a n such that $f = \iota_n(f|_{\mathcal{G}_n})$, and we get:

$$\begin{aligned}
\psi(f^* * f) &= \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} (f^* * f)(g)\varphi_x(u_g)\ d\mu(x) \\
&= \mu(V_n) \int_{V_n} \sum_{g\in\mathcal{G}_x^x} (f^* * f)(g)\varphi_x(u_g)\ d(\mu(V_n)^{-1}\mu)(x) \\
&= \mu(V_n)\omega_n((f|_{\mathcal{G}_n})^*(f|_{\mathcal{G}_n})) = \mu(V_n)\omega_n(\alpha^{c_n}_{-i\beta/2}(f|_{\mathcal{G}_n})\alpha^{c_n}_{-i\beta/2}(f|_{\mathcal{G}_n})^*) \\
&= \int_{V_n} \sum_{g\in\mathcal{G}_x^x} (\alpha^{c_n}_{-i\beta/2}(f|_{\mathcal{G}_n}) * \alpha^{c_n}_{-i\beta/2}(f|_{\mathcal{G}_n})^*)(g)\varphi_x(u_g)\ d\mu(x) \\
&= \int_{V_n} \sum_{g\in\mathcal{G}_x^x} (\alpha^{c}_{-i\beta/2}(f) * \alpha^{c}_{-i\beta/2}(f)^*)(g)\varphi_x(u_g)\ d\mu(x) \\
&= \psi(\alpha^c_{-i\beta/2}(f)\alpha^c_{-i\beta/2}(f)^*)
\end{aligned}$$

So we have proved that ψ satisfies the KMS condition on $C_c(\mathcal{G})$, yet to prove that it is a KMS weight we need to prove this equality for all $a \in D(\alpha^c_{-i\beta/2})$. $C_c(\mathcal{G})$ is a core for $\alpha^c_{-i\beta/2}$ by Proposition 3.1.7, so for $a \in D(\alpha^c_{-i\beta/2})$ we can find a sequence $\{f_m\}_{m\in\mathbb{N}} \subseteq C_c(\mathcal{G})$ such that $f_m \to a$ and $\alpha^c_{-i\beta/2}(f_m) \to \alpha^c_{-i\beta/2}(a)$ in norm. Now choose a sequence of functions $\{g_n\}_{n=1}^\infty \subseteq C_c(\mathcal{G}^{(0)})$ with the property that $g_n(x) = 1$ for $x \in \overline{V_n}$ and $\mathrm{supp}(g_n) \subseteq V_{n+1}$ for all n

and with $0 \leq g_n \leq 1$. Then $\{g_n\}_{n\in\mathbb{N}}$ is an approximation of the identity on $C^*(\mathcal{G})$ and since $\alpha_t^c(g_n) = g_n$ for all t, we get that:

$$\psi(g_l f_m^* g_n^2 f_m g_l) = \psi((g_n f_m g_l)^*(g_n f_m g_l)) = \psi(g_n \alpha_{-i\beta/2}^c(f_m) g_l^2 \alpha_{-i\beta/2}^c(f_m)^* g_n)$$

By definition of ψ the functions $C^*(\mathcal{G}) \ni x \to \psi(g_l x g_l)$ and $C^*(\mathcal{G}) \ni x \to \psi(g_n x g_n)$ are continuous, so letting $m \to \infty$ we get:

$$\psi(g_l a^* g_n g_n a g_l) = \psi(g_n \alpha_{-i\beta/2}^c(a) g_l^2 \alpha_{-i\beta/2}^c(a)^* g_n) \tag{4.3}$$

Since $\psi_x(g_n f g_n) = \psi_x(f) g_n(x)^2$ for $f \in C_c(\mathcal{G})$ the same holds for all $b \in C^*(\mathcal{G})$ by continuity and:

$$\int_{V_n} \psi_x(\alpha_{-i\beta/2}^c(a) g_l^2 \alpha_{-i\beta/2}^c(a)^*) d\mu(x) \leq \psi(g_n \alpha_{-i\beta/2}^c(a) g_l^2 \alpha_{-i\beta/2}^c(a)^* g_n)$$
$$\leq \int_{V_{n+1}} \psi_x(\alpha_{-i\beta/2}^c(a) g_l^2 \alpha_{-i\beta/2}^c(a)^*) d\mu(x)$$

so we have:

$$\psi(g_n \alpha_{-i\beta/2}^c(a) g_l^2 \alpha_{-i\beta/2}^c(a)^* g_n) \to \psi(\alpha_{-i\beta/2}^c(a) g_l^2 \alpha_{-i\beta/2}^c(a)^*) \qquad \text{for } n \to \infty.$$

Considering (4.3) for $n \to \infty$ and using lower semi-continuity we get:

$$\psi(g_l a^* a g_l) = \psi(\alpha_{-i\beta/2}^c(a) g_l^2 \alpha_{-i\beta/2}^c(a)^*).$$

When $l \to \infty$ a similar argument to the one just given implies that:

$$\psi(a^* a) = \psi(\alpha_{-i\beta/2}^c(a) \alpha_{-i\beta/2}^c(a)^*).$$

To see that $\psi \circ \alpha^c = \psi$ fix $t \in \mathbb{R}$. By condition (3) we get that $\psi_x(\alpha_t^c(f)) = \psi_x(f)$ for $f \in C_c(\mathcal{G})$ and almost all x, hence by continuity of ψ_x we have $\psi_x = \psi_x \circ \alpha_t^c$ for μ-a.e. x, proving $\psi \circ \alpha_t^c = \psi$. So the formula (4.2) defines a β-KMS weight ψ for α^c.

Step 3: The formula (4.2) defines a unique β-KMS weight.

This follows if we can prove ψ constructed in step 2 is the only β-KMS weight for α^c satisfying (4.2), so let ψ' be a β-KMS weight for α^c that agrees with ψ on $C_c(\mathcal{G})$. Let $\{g_n\}_{n=1}^\infty \subseteq C_c(\mathcal{G}^{(0)})$ be the approximation of the identity defined in step 2. For each n the maps $C^*(\mathcal{G}) \ni a \to \psi(g_n a g_n)$ and $C^*(\mathcal{G}) \ni a \to \psi'(g_n a g_n)$ are continuous and they agree on $C_c(\mathcal{G})$, so they agree on all elements. Choose $a \in C_c(\mathcal{G})_+$, for each $x \in \mathcal{G}^{(0)}$ we have $\psi_x(g_n a g_n) = g_n(x)^2 \psi_x(a)$ so:

$$\psi'(g_n a g_n) = \psi(g_n a g_n) = \int_{\mathcal{G}^{(0)}} \psi_x(a) g_n(x)^2 \, d\mu(x)$$

Since $\psi'(g_n a g_n) \leq \psi(a)$ for each n lower semi-continuity implies $\psi'(a) \leq \psi(a)$. Assume now that $a \in C_c(\mathcal{G})_+$ satisfies $\psi(a) < \infty$, then we also have $\psi'(a) < \infty$ and hence for any n we have $\sqrt{a}g_n, g_n\sqrt{a} \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$ and $\sqrt{a}g_n, g_n\sqrt{a} \in \mathcal{N}_{\psi'} \cap \mathcal{N}_{\psi'}^*$. If $\beta = 0$ we then have:

$$\psi(\sqrt{a}g_n^2\sqrt{a}) = \psi(g_n a g_n) = \psi'(g_n a g_n) = \psi'(\sqrt{a}g_n^2\sqrt{a})$$

Since $\sqrt{a}g_n^2\sqrt{a}$ increases to a lower semi-continuity implies that $\psi(a) = \psi'(a)$. To prove the same is true when $\beta \neq 0$ we can assume $\beta > 0$ by symmetry. By definition there exist bounded and continuous functions $F, F' : S(i\beta) \to \mathbb{C}$ analytic on $S(i\beta)^0$ satisfying:

$$F(t) = \psi(g_n\sqrt{a}\alpha_t^c(\sqrt{a})g_n) \quad , \quad F(t+i\beta) = \psi(\alpha_t^c(\sqrt{a})g_n g_n\sqrt{a})$$

and:

$$F'(t) = \psi'(g_n\sqrt{a}\alpha_t^c(\sqrt{a})g_n) \quad , \quad F'(t+i\beta) = \psi'(\alpha_t^c(\sqrt{a})g_n g_n\sqrt{a})$$

The function $F - F'$ is then 0 on $\mathbb{R}$ and analytic on $S(i\beta)^0$, so the edge of the wedge theorem, see e.g. Proposition 5.3.6 in [2], implies that $F - F'$ is 0 on $S(i\beta)^0$, and hence by continuity $F(i\beta) = F'(i\beta)$, i.e:

$$\psi'(\sqrt{a}g_n g_n\sqrt{a}) = \psi(\sqrt{a}g_n g_n\sqrt{a})$$

Since this is true for all n we get $\psi(a) = \psi'(a)$. It now follows that $\psi = \psi'$ by Corollary 1.15 in [17].

Step 4: The map is a bijection.

By step 2 and step 3 the map is well defined, and by step 1 it is surjective. So we only need to prove that it is injective. For this assume that $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ and $(\mu', \{\varphi'_x\}_{x\in\mathcal{G}^{(0)}})$ represent the same β-KMS weight for α^c via the formula (4.2). Riesz Representation Theorem implies immediately that $\mu = \mu'$. Choose a sequence $\{f_n\}_{n=1}^\infty \subseteq C_c(\mathcal{G})$ where each f_n is supported in a bisection, and such that for each $g \in \mathcal{G}$ there is a n with $f_n(g) = 1$. For any $f \in C_c(\mathcal{G})$ and $k \in C_c(\mathcal{G}^{(0)})$ we get by definition:

$$\int_{\mathcal{G}^{(0)}} k(x) \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)\, d\mu(x) = \int_{\mathcal{G}^{(0)}} k(x) \sum_{g\in\mathcal{G}_x^x} f(g)\varphi'_x(u_g)\, d\mu(x)$$

From this it follows that:

$$\sum_{g\in\mathcal{G}_x^x} f_n(g)\varphi_x(u_g) = \sum_{g\in\mathcal{G}_x^x} f_n(g)\varphi'_x(u_g)$$

for μ-a.e. $x \in \mathcal{G}^{(0)}$ and all n. This implies that $\varphi_x = \varphi'_x$ for μ-a.e. x. $\square$

Neshveyev's theorem allows us to divide the study of KMS weights into two different questions - one regarding measures on $\mathcal{G}^{(0)}$ and one regarding measurable fields of states on the C^*-algebras of the isotropy groups. We will focus on these two questions in the two chapters to come. For clarity let us make a remark that was also made in [20].

Remark 4.2.4. Condition 3 in Theorem 4.2.1 is automatically satisfied when $\beta \neq 0$, since a quasi-invariant measure μ with Radon-Nikodym cocycle $e^{-\beta c}$ will automatically be concentrated on the $x \in \mathcal{G}^{(0)}$ with $\mathcal{G}_x^x \subseteq c^{-1}(0)$. The set:

$$M = \{g \in \mathcal{G} \ : \ c(g) > 0 \text{ and } r(g) = s(g)\}$$

is Borel since c, r and s are continuous. Since s maps Borel sets to Borel sets by Lemma 2.1.7, we have that:

$$\{x \in \mathcal{G}^{(0)} \ : \ \mathcal{G}_x^x \nsubseteq c^{-1}(0)\} = s(M)$$

is Borel. To prove that $\mu(s(M)) = 0$ it follows from Lemma 2.1.7 that it suffices to show that $\mu(s(M \cap W)) = 0$ for any small bisection W. Proposition 4.1.6 now implies that:

$$\mu(s(M \cap W)) = \int_{r(M \cap W)} e^{\beta c(r_W^{-1}(x))} \, d\mu(x)$$

By choice of M the function $r(M \cap W) \ni x \to c(r_W^{-1}(x))$ is positive and $r(M \cap W) = s(M \cap W)$, so when $\beta \neq 0$ this can only be the case when $\mu(s(M \cap W)) = 0$.

Corollary 4.2.5. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid, let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and let $\beta \in \mathbb{R}$. If ψ is a β-KMS weight for α^c given by $(\mu, \{\varphi_x\}_{x \in \mathcal{G}^{(0)}})$ then $\psi = \psi \circ P$ if and only if $\varphi_x = Tr_x$ for μ-a.e. $x \in \mathcal{G}^{(0)}$.*

Proof. Theorem 4.2.1 implies that if two KMS weights agree on $C_c(\mathcal{G})$ then they are equal, but if $\psi \circ P = \psi$ then ψ agrees with the KMS weight given by $(\mu, \{\mathrm{Tr}_x\}_{x \in \mathcal{G}^{(0)}})$, proving one direction. Assume now that ψ is given by $(\mu, \{\mathrm{Tr}_x\}_{x \in \mathcal{G}^{(0)}})$ and consider $a \in C^*(\mathcal{G})$. Clearly $\psi(f) = \psi(P(f))$ for all $f \in C_c(\mathcal{G})$. Let $\{f_m\}_{m=1}^{\infty} \subseteq C_c(\mathcal{G}^{(0)})_+$ be an approximation of the identify with $0 \leq f_m \leq f_{m+1} \leq 1$ for all m and assume that $\{h_n\}_{n=1}^{\infty} \subseteq C_c(\mathcal{G})$ is a sequence with $h_n \to a \in C^*(\mathcal{G})$. Then:

$$\psi(f_m a^* a f_m) = \lim_n \psi(f_m h_n^* h_n f_m) = \lim_n \psi(f_m P(h_n^* h_n) f_m) = \psi(f_m P(a^* a) f_m)$$

By definition of ψ:

$$\psi(a^* a) = \lim_m \psi(f_m a^* a f_m) = \lim_m \psi(f_m P(a^* a) f_m) = \psi(P(a^* a))$$

proving the corollary. □

— 5 —

Ergodicity

In this chapter we will first prove that the extremal quasi-invariant measures with Radon-Nikodym cocycle $e^{-\beta c}$ exactly are the ones that are ergodic. This result has a predecessor in Theorem 4.15 in [30]. After having established this, we will analyse the relationship between extremal measures and extremal weights.

5.1 Ergodicity

Definition 5.1.1. Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid. For $c : \mathcal{G} \to \mathbb{R}$ a continuous groupoid homomorphism and $\beta \in \mathbb{R}$ we let $\Delta(\beta, c)$ denote the set of non-negative regular Borel measures on $\mathcal{G}^{(0)}$ that are quasi-invariant with Radon-Nikodym cocycle $e^{-\beta c}$.

Remark 5.1.2. Notice that the zero-measure is an element of $\Delta(\beta, c)$, yet it does not give rise to any weights by Theorem 4.2.1. Also notice that $\Delta(\beta, c)$ is a convex set and by Lemma 2.1.7 every measure in $\Delta(\beta, c)$ is σ-finite.

Before proving that the extremal measures in $\Delta(\beta, c)$ are the ergodic measures in $\Delta(\beta, c)$, let us define what we mean by extremal and ergodic.

Definition 5.1.3. We call a measure $\mu \in \Delta(\beta, c)$ *extremal* when any $\mu_1, \mu_2 \in \Delta(\beta, c)$ with $\mu = \mu_1 + \mu_2$ satisfies $\mu_1, \mu_2 \in \mathbb{R}_+\mu := \{\lambda\mu \ : \ \lambda \geq 0\}$.

Definition 5.1.4. Following [26] we call a set $B \subseteq \mathcal{G}^{(0)}$ invariant if for all $g \in \mathcal{G}$, $r(g) \in B$ if and only if $s(g) \in B$, or equivalently if

$$B = r(s^{-1}(B)) \text{ or } B = s(r^{-1}(B))$$

We say that a Borel measure μ on $\mathcal{G}^{(0)}$ is ergodic if for all invariant Borel sets B we either have $\mu(B) = 0$ or $\mu(B^C) = 0$.

Remark 5.1.5. Notice that if B is invariant then $B^C = \mathcal{G}^{(0)} \setminus B$ is also invariant.

The aim of this section is to prove the following Theorem, which links ergodicity with being extremal:

Theorem 5.1.6. *A measure in $\Delta(\beta, c)$ is extremal if and only if it is ergodic.*

Before proving Theorem 5.1.6 we will prove the following Proposition which is of independent interest.

Proposition 5.1.7. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid, let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. If $\mu \in \Delta(\beta, c)$ and B is an invariant Borel set, then the restriction μ_B of μ to B is again an element of $\Delta(\beta, c)$.*

Proof. Let W be a small bisection in $\mathcal{G}$. Since B is invariant we get that:

$$s(W) \cap B = s(r_W^{-1}(r(W) \cap B))$$

Since $\mu \in \Delta(\beta, c)$ we can use Proposition 4.1.6 to see that:

$$\begin{aligned}\mu_B(s(W)) = \mu(s(r_W^{-1}(r(W) \cap B))) &= \int_{r(W)\cap B} e^{\beta c(r_W^{-1}(x))}\ d\mu(x) \\ &= \int_{r(W)} e^{\beta c(r_W^{-1}(x))}\ d\mu_B(x)\end{aligned}$$

Since μ_B is Borel and finite on compact sets it is regular, so since W was an arbitrary small bisection Proposition 4.1.6 implies that $\mu_B \in \Delta(\beta, c)$. $\square$

Proof of Theorem 5.1.6. If $\mu = 0$ it is both extremal and ergodic, so we can assume $\mu \neq 0$. Let μ be extremal and $B \subseteq \mathcal{G}^{(0)}$ invariant and Borel with $\mu(B) > 0$, then B^C is invariant and Borel as well and $\mu = \mu_B + \mu_{B^C}$. Proposition 5.1.7 then implies that $\mu_B = \lambda\mu$, and evaluating in B gives $\lambda > 0$. Since $\lambda\mu(B^C) = \mu_B(B^C) = 0$ we then get $\mu(B^C) = 0$, proving that μ is ergodic.

Assume now that μ is ergodic. Let $\mu_1, \mu_2 \in \Delta(\beta, c) \setminus \{0\}$ satisfy that:

$$\mu = \mu_1 + \mu_2$$

Since $\mathcal{G}^{(0)}$ is σ-compact both μ,μ_1 and μ_2 are σ-finite and clearly μ_1 and μ_2 are absolutely continuous with respect to μ. Using the Radon-Nikodym theorem, see e.g. Theorem 4.2.2 in [4], we then get Borel functions $f_i : \mathcal{G}^{(0)} \to [0, \infty[$, $i = 1, 2$, such that:

$$\mu_i(B) = \int_B f_i(x)\ d\mu(x)$$

for all Borel sets $B \subseteq \mathcal{G}^{(0)}$. Since $\mu = \mu_1 + \mu_2$ we have that $f_1(x) + f_2(x) = 1$ for μ-a.e. x. If f_1 is constant μ-almost everywhere it follows that μ_1 and μ_2

are scalar multiples of μ, completing the proof, so assume for contradiction this is not the case. Since f_1 is not constant μ-almost everywhere, we can use that the function $[0,1] \ni k \to \mu(f_1^{-1}([0,k[))$ is non-decreasing and lower semi-continuous to find a $k \in]0,1[$ such that $\mu(f_1^{-1}([0,k[)) > 0$ and $\mu(f_1^{-1}(]k,1])) > 0$.

Choose a Borel set $B_0 \subseteq f_1^{-1}([0,k[)$ such that $\mu(B_0) \in]0,\infty[$. Lemma 4.1.7 implies that $B = s(r^{-1}(B_0))$ is Borel and invariant, and since $B_0 \subseteq B$ we have $\mu(B) \geq \mu(B_0) > 0$, so ergodicity of μ implies $\mu(B^C) = 0$. Since $\mu(f_1^{-1}(]k,1])) > 0$ and $f_1(x) + f_2(x) = 1$ for μ-a.e. x we must have that:

$$\mu(f_2^{-1}([0,1-k[)) > 0.$$

So we can choose $C_0 \subseteq f_2^{-1}([0,1-k[)$ with $\mu(C_0) \in]0,\infty[$, and setting $C = s(r^{-1}(C_0))$ this is again an invariant Borel set with $\mu(C^C) = 0$.

Claim: $\mu(\{x \in B \mid f_1(x) > k\}) = 0$ and $\mu(\{x \in C \mid f_2(x) > 1-k\}) = 0$.

If this claim is true the Proposition follows: Notice that $\mu(B^C \cup C^C) = 0$, and that for μ-a.e. $x \in B \cap C$ we have $f_1(x) \leq k$ and $f_2(x) \leq 1-k$. Since $f_1(x) + f_2(x) = 1$ for μ-a.e. x this implies that $f_1(x) = k$ for almost all $x \in B \cap C$, i.e. $f_1(x) = k$ almost everywhere, contradicting that f_1 was not constant. In conclusion $\mu_1, \mu_2 \in \mathbb{R}_+\mu$, proving that μ is extremal.

To prove the claim fix an $i \in \{1,2\}$, it suffices to prove that if $D \subseteq f_i^{-1}([0,l[)$ for some $l \in [0,1]$ with $\mu(D) \in]0,\infty[$ then

$$\mu(\{x \in s(r^{-1}(D)) \mid f_i(x) > l\}) = 0.$$

Assume for a contradiction that this is not true, and let $\{W_j\}_{j=1}^{\infty}$ be a countable basis of small bisections, then:

$$s(r^{-1}(D)) = \bigcup_{j=1}^{\infty} s(r_{W_j}^{-1}(D \cap r(W_j))).$$

Since $\mu(s(W_j)) < \infty$ for each j, there must then be some j such that:

$$0 < \mu(\{x \in s(r_{W_j}^{-1}(D \cap r(W_j))) \ : \ f_i(x) > l\}) < \infty.$$

Set $H = \{x \in s(r_{W_j}^{-1}(D \cap r(W_j))) \ : \ f_i(x) > l\}$, then:

$$\mu_i(H) = \int_H f_i(x) \ d\mu(x) > l\mu(H)$$

on the other hand, since $r(s_{W_j}^{-1}(H)) \subseteq D \subseteq f_i^{-1}([0,l[)$ we get:

$$\mu_i(H) = \mu_i\left(s\left(s_{W_j}^{-1}(H)\right)\right) = \int_{r(s_{W_j}^{-1}(H))} e^{\beta c(r_{W_j}^{-1}(x))}\ d\mu_i(x)$$
$$\leq l \int_{r(s_{W_j}^{-1}(H))} e^{\beta c(r_{W_j}^{-1}(x))}\ d\mu(x) = l\mu\left(s\left(s_{W_j}^{-1}(H)\right)\right) = l\mu(H)$$

a contradiction. Hence the claim is true, which proves that ergodic measures are extremal. □

For the last part of our analysis of $\Delta(\beta, c)$ we conclude what the above theorem says when $\mathcal{G}^{(0)}$ is compact. For this, define for any $f \in C_c(\mathcal{G}^{(0)})_+$:

$$\Delta_f(\beta, c) := \left\{\mu \in \Delta(\beta, c)\ :\ \mu(f) := \int f\ d\mu = 1\right\}$$

When $\mathcal{G}^{(0)}$ is compact (which is equivalent to $C^*(\mathcal{G})$ being unital) it is sufficient to describe the KMS states, and in this case we are therefore only interested in the set $\Delta_{\mathcal{G}^{(0)}}(\beta, c) := \Delta_{1_{\mathcal{G}^{(0)}}}(\beta, c)$ of Borel probability measures with Radon-Nikodym cocycle $e^{-\beta c}$. Notice for the statement of the next result that we use the notation ∂K for any convex set K to denote the set of extreme points, i.e. the elements $x \in K$ satisfying that $x = \lambda x_1 + (1-\lambda)x_2$ implies $x_1 = x_2 = x$ for any $\lambda \in]0,1[$ and $x_1, x_2 \in K$.

Corollary 5.1.8. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. Assume $f \in C_c(\mathcal{G}^{(0)})_+$ satisfies $\mu(f) > 0$ for all $\mu \in \Delta(\beta, c) \setminus \{0\}$. Then $\Delta_f(\beta, c)$ is a convex set, and:*

$$\partial\Delta_f(\beta, c) = \{\mu \in \Delta_f(\beta, c)\ :\ \mu \textit{ is ergodic}\}$$

Proof. For this it suffices to notice that with the assumption on f:

$$\partial\Delta_f(\beta, c) = \{\mu \in \Delta_f(\beta, c)\ :\ \mu \text{ is extremal in } \Delta(\beta, c)\}$$

□

5.2 Extremal KMS weights

It is possible to define an *extremal KMS weight* in a similar fashion as extremality was defined for measures. In this section we will briefly analyse the relationship between extremal measures and extremal weights. Let us first introduce some notation.

Definition 5.2.1. Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid. For $c : \mathcal{G} \to \mathbb{R}$ a continuous groupoid homomorphism and $\beta \in \mathbb{R}$ we let $\mathcal{W}(\beta, c)$ denote the set of β-KMS weights for α^c *including the zero weight.*

Our reason for including 0 in $\mathcal{W}(\beta, c)$ is that the zero measure was included in the definition of $\Delta(\beta, c)$.

Definition 5.2.2. We call $\psi \in \mathcal{W}(\beta, c)$ *extremal* if $\psi = \psi_1 + \psi_2$ with $\psi_1, \psi_2 \in \mathcal{W}(\beta, c)$ implies that $\psi_1, \psi_2 \in \mathbb{R}_+\psi$.

Notice that this definition is weaker than the definition of extremal weights in e.g. [33, 30], but the two definitions coincide when every $\psi \in \mathcal{W}(\beta, c)$ satisfies $\psi = \psi \circ P$. It is still unclear if they always coincide.

For each $\mu \in \Delta(\beta, c)$ let $\mathcal{W}(\beta, c)_\mu$ denote the set of KMS weights ω that restricts to the integral over μ on $\mathcal{G}^{(0)}$, or in different terms the KMS weights that in Theorem 4.2.1 are given by a pair $(\mu, \{\varphi_x\}_{x \in \mathcal{G}^{(0)}})$ for some μ-measurable field of states $\{\varphi_x\}_{x \in \mathcal{G}^{(0)}}$.

Lemma 5.2.3. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. Let $\psi \in \mathcal{W}(\beta, c)$ be given by the pair $(\mu, \{\varphi_x\}_{x \in \mathcal{G}^{(0)}})$ with $\mu \in \Delta(\beta, c)$. Then ψ is extremal in $\mathcal{W}(\beta, c)$ if and only if μ is extremal in $\Delta(\beta, c)$ and $\psi \in \partial(\mathcal{W}(\beta, c)_\mu)$.*

Proof. The statement is trivial if $\psi = 0$, so assume $\psi \neq 0$. Assume ψ is extremal in $\mathcal{W}(\beta, c)$ and $\mu = \mu_1 + \mu_2$. Since μ nullsets must be μ_1 and μ_2 nullsets $\{\varphi_x\}_{x \in \mathcal{G}^{(0)}}$ is both μ_1- and μ_2-measurable, and by Theorem 4.2.1 $(\mu_1, \{\varphi_x\}_{x \in \mathcal{G}^{(0)}})$ and $(\mu_2, \{\varphi_x\}_{x \in \mathcal{G}^{(0)}})$ represent elements $\psi_1, \psi_2 \in \mathcal{W}(\beta, c)$. The map $\psi_1 + \psi_2$ is lower semi-continuous (since ψ_1 and ψ_2 are), it is densely defined since $C_c(\mathcal{G}) \subseteq \mathcal{M}_{\psi_1} \cap \mathcal{M}_{\psi_2}$ and it is straightforward to check that it is a KMS weight. Since $\psi_1 + \psi_2$ agrees with ψ on $C_c(\mathcal{G})$ then $\psi = \psi_1 + \psi_2$, and hence $\psi_1 = \lambda\psi$ for some $\lambda \geq 0$. In particular this implies that $\mu_1 = \lambda\mu$, so $\mu_1, \mu_2 \in \mathbb{R}_+\mu$. If $\psi = t\psi_1 + (1 - t)\psi_2$ with $t \in]0, 1[$ and $\psi_1, \psi_2 \in \mathcal{W}(\beta, c)_\mu$ then $t\psi_1 = \lambda\psi$ for some $\Lambda > 0$. Since ψ_1 and ψ agrees on $C_c(\mathcal{G}^{(0)})$ then $t = \lambda$, giving $\psi_1 = \psi$. This proves one direction.

For the other direction assume μ is extremal in $\Delta(\beta, c)$ and $\psi \in \partial(\mathcal{W}(\beta, c)_\mu)$ and $\psi = \psi_1 + \psi_2$ with $\psi_1, \psi_2 \in \mathcal{W}(\beta, c) \setminus \{0\}$. Letting μ_1 and μ_2 be the measures associated to ψ_1, ψ_2, then evaluating in $C_c(\mathcal{G}^{(0)})$ gives $\mu = \mu_1 + \mu_2$, so there is $t_1, t_2 > 0$ with $\mu_1 = t_1\mu$ and $\mu_2 = t_2\mu$. This implies $t_1 + t_2 = 1$, so since $t_1^{-1}\psi_1, t_2^{-1}\psi_2 \in \mathcal{W}(\beta, c)_\mu$ then $\psi = t_1^{-1}\psi_1 = t_2^{-1}\psi_2$, proving the other direction. □

— 6 —

Symmetries of the KMS simplex

With the results on quasi-invariant measures obtained in last chapter we are now ready to study the measurable fields of states. Unfortunately we can not say anything clever about the measurable fields of states for general locally compact second countable Hausdorff étale groupoids. However we can do a thorough analysis for a large class of groupoids, the ones that are *injectively graded by an abelian group*.

6.1 Measurable fields of states

Let us first define what we mean by an injectively graded groupoid.

Definition 6.1.1. Let G be a discrete countable group and $\mathcal{G}$ a locally compact second countable Hausdorff étale groupoid. We say that $\mathcal{G}$ is *injectively graded by* G if there is a continuous groupoid homomorphism $\Phi : \mathcal{G} \to G$ satisfying:

$$\ker(\Phi) \cap \mathcal{G}_x^x = \{x\} \qquad \text{for all } x \in \mathcal{G}^{(0)}$$

Remark 6.1.2. Restricting Φ to $\mathcal{G}_x^x$ gives an injective group homomorphism from $\mathcal{G}_x^x$ into G. In [C] we consider groupoids that satisfy Definition 6.1.1 for an abelian group and that furthermore has compact unit space, and we refer to them as groupoids *admitting an abelian valued homomorphism*. Since writing [C] the author has been introduced to the notion of *graded groupoids*: A groupoid is graded by a discrete group G if there exists a continuous groupoid homomorphism $\Phi : \mathcal{G} \to G$. Due to this the terminology *injectively graded* seems more appropriate, and we will use this term in Part I of this dissertation.

Example 6.1.3. Let (Λ, d) be a compactly aligned topological k-graph for some $k \in \mathbb{N}$. We will refer to [37] for the definition of this and for the results we summarise in the following. Using (Λ, d) one can define a space of paths X_Λ and for each $m \in \mathbb{N}^k$ a map σ^m on $\{x \in X_\Lambda \mid d(x) \geq m\}$ and obtain a groupoid:

$$\mathcal{G}_\Lambda = \{(x,m,y) \in X_\Lambda \times \mathbb{Z}^k \times X_\Lambda \mid \exists p,q \in \mathbb{N}^k \text{ with } p \le d(x), \\ q \le d(y),\ p-q=m \text{ and } \sigma^p(x) = \sigma^q(y)\}$$

with composition $(x,m,y)(y,n,z) = (x,m+n,z)$. We can equip $\mathcal{G}_\Lambda$ with a topology such that the homomorphism $\mathcal{G}_\Lambda \ni (x,m,y) \to m \in \mathbb{Z}^k$ becomes continuous and $\mathcal{G}_\Lambda$ is a locally compact second countable Hausdorff étale groupoid, and hence $\mathcal{G}_\Lambda$ is injectively graded by $\mathbb{Z}^k$. So the groupoid for the Toeplitz algebra of a compactly aligned topological k-graph is injectively graded by $\mathbb{Z}^k$. Since the groupoid for the Cuntz-Krieger algebra for a compactly aligned topological k-graph is a reduction of $\mathcal{G}_\Lambda$, it is also injectively graded by $\mathbb{Z}^k$. This provides us with a lot of examples, see e.g. the ones listed in Example 7.1 in [37], and most importantly for this dissertation it implies that the groupoids of the Cuntz-Krieger and the Toeplitz algebra of a finitely aligned higher-rank graph are injectively graded by an abelian group.

Theorem 6.1.4 below is a generalisation of Theorem 5.2 in [C], which contains a similar result but only for unital groupoid C^*-algebras. However, the technique for proving Theorem 5.2 in [C] is completely different and a lot more complicated than the one used in Theorem 6.1.4 below.

Theorem 6.1.4. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid injectively graded by a discrete countable abelian group A via a map $\Phi : \mathcal{G} \to A$. Assume $\mu \in \Delta(\beta, c) \setminus \{0\}$ is ergodic. The following is then true:*

1. *The subset:*
$$X(C) := \{x \in \mathcal{G}^{(0)} \ : \ \Phi(\mathcal{G}_x^x) = C\}$$
is Borel and invariant for each subgroup $C \subseteq A$.

2. *There exists a unique subgroup B of A with $\mu(X(B)^C) = 0$.*

3. *For $x \in X(B)$ let $\Phi_x : C^*(\mathcal{G}_x^x) \to C^*(B)$ be the isomorphism induced by the restriction of Φ. If $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field of states satisfying*
$$\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}}) \text{ for } \mu\text{-a.e. } x \in \mathcal{G}^{(0)} \text{ and all } g \in \mathcal{G}_x^x \text{ and } h \in \mathcal{G}_x$$
then there exists a state φ on $C^*(B)$ such that $\varphi \circ \Phi_x = \varphi_x$ for μ-a.e. $x \in X(B)$.

Proof. For each $a \in A$ the set $X(a) := \{x \in \mathcal{G}^{(0)} \ : \ a \in \Phi(\mathcal{G}_x^x)\}$ can be realised as:

$$X(a) = r(\{g \in \Phi^{-1}(\{a\}) \ : \ r(g) = s(g)\})$$

Since Φ is continuous Lemma 2.1.7 implies $X(a)$ is a Borel set. For any subgroup $D \subseteq A$ we have:

$$X(D) = \Big(\bigcap_{d \in D} X(d)\Big) \cap \Big(\bigcap_{a \in A \setminus D} X(a)^C\Big) \tag{6.1}$$

which is Borel. If $h \in \mathcal{G}_y^x$ and $g \in \mathcal{G}_x^x$ then:

$$\Phi(h^{-1}gh) = \Phi(h^{-1})\Phi(g)\Phi(h) = \Phi(g).$$

Since $h^{-1}\mathcal{G}_x^x h = \mathcal{G}_y^y$ this implies $\Phi(\mathcal{G}_x^x) = \Phi(\mathcal{G}_y^y)$, proving 1.

Using the same argument as above, we see that $X(a)$ is also an invariant set for each $a \in A$, so since μ is ergodic then either $\mu(X(a)) = 0$ or $\mu(X(a)^C) = 0$. Since $X(a) \cap X(b) \subseteq X(ab)$ and $X(a) = X(a^{-1})$ we get that:

$$B := \{a \in A \ : \ \mu(X(a)^C) = 0\}$$

is a subgroup of A, and by comparing with (6.1) we see that $\mu(X(B)^C) = 0$. If $\mu(X(C)^C) = 0$ for some subgroup C of A then $\mu(X(B)^C \cup X(C)^C) = 0$, so since $\mu \neq 0$ we must have $\mu(X(B) \cap X(C)) > 0$, but $X(B) \cap X(C) = \emptyset$ when $C \neq B$. So B is unique, proving 2.

For 3 let N_1 be a Borel set with $\mu(N_1) = 0$ such that $\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}})$ for all $x \in \mathcal{G}^{(0)} \setminus N_1$ and all $g \in \mathcal{G}_x^x$ and $h \in \mathcal{G}_x$. For each $a \in A$ we can realise the Borel function $1_{\Phi^{-1}(a)}$ as the point wise limit of functions in $C_c(\mathcal{G})$, implying that the map:

$$\mathcal{G}^{(0)} \ni x \to \sum_{g \in \mathcal{G}_x^x} 1_{\Phi^{-1}(a)}(g)\varphi_x(u_g)$$

is μ-measurable. Using Lemma 4.1.2 we find a Borel set N_a such that $\mu(N_a) = 0$ and:

$$\mathcal{G}^{(0)} \ni x \to 1_{N_a^C}(x) \sum_{g \in \mathcal{G}_x^x} 1_{\Phi^{-1}(a)}(g)\varphi_x(u_g)$$

is Borel. Set

$$N = s\Big(r^{-1}\Big(N_1 \cup \bigcup_{a \in A} N_a\Big)\Big)$$

which is then a Borel set with $\mu(N) = 0$. Set $\varphi'_x = \varphi_x$ for $x \notin N$ and $\varphi'_x = \mathrm{Tr}_x$ for $x \in N$, where Tr_x denotes the canonical trace on $C^*(\mathcal{G}_x^x)$. Since

N is invariant $\{\varphi'_x\}_{x\in\mathcal{G}^{(0)}}$ satisfies the criterion in 3 for all $x \in \mathcal{G}^{(0)}$, and for any $f \in C_c(\mathcal{G})$:

$$\mathcal{G}^{(0)} \ni x \to \sum_{g\in\mathcal{G}_x^x} f(g)\varphi'_x(u_g) = \begin{cases} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g) & \text{if } x \notin N \\ f(x) & \text{if } x \in N \end{cases}$$

is μ-measurable. Notice that the function:

$$\mathcal{G}^{(0)} \ni x \to \sum_{g\in\mathcal{G}_x^x} 1_{\Phi^{-1}(a)}(g)\varphi'_x(u_g)$$

is Borel for every $a \in A$. Let E denote the weak* compact set of states on $C^*(B)$ and let $\mathrm{Tr} \in E$ denote the canonical tracial state on $C^*(B)$. We can now define a map $F : \mathcal{G}^{(0)} \to E$ by:

$$\mathcal{G}^{(0)} \ni x \xrightarrow{F} \begin{cases} \varphi'_x \circ \Phi_x^{-1} & \text{if } x \in X(B) \\ \mathrm{Tr} & \text{if } x \notin X(B) \end{cases}$$

The sets:

$$\{\varphi \in E \ : \ |\varphi(u_b) - \omega(u_b)| < \varepsilon\} \quad \text{for } \omega \in E,\ \varepsilon > 0 \text{ and } b \in B$$

is a subbasis for the weak* topology on E. It follows from this that F is Borel if $x \to F(x)(u_b)$ is Borel for each $b \in B$, however:

$$X(B) \ni x \to \sum_{g\in\mathcal{G}_x^x} 1_{\Phi^{-1}(b)}(g)\varphi'_x(u_g) = \varphi'_x(\Phi_x^{-1}(u_b)).$$

So $x \to F(x)(u_b)$ is the restriction of two Borel functions to two disjoint Borel sets, and hence it is Borel. In conclusion $F : \mathcal{G}^{(0)} \to E$ is Borel.

Assume $M \subseteq E$ is Borel and $x \in F^{-1}(M)$ and $h \in \mathcal{G}_x^y$. If $x \notin X(B)$ then $y \notin X(B)$ and $F(y) = \mathrm{Tr} = F(x) \in M$, so $y \in F^{-1}(M)$. If $x \in X(B)$ then $y \in X(B)$ and for $b \in B$ we pick $g \in \mathcal{G}_x^x$ with $\Phi(g) = b$ to get:

$$F(x)(u_b) = \varphi'_x(u_g) = \varphi'_y(u_{hgh^{-1}}) = F(y)(u_b).$$

It follows from this that $F(y) = F(x)$, so $y \in F^{-1}(M)$. So $F^{-1}(M)$ is an invariant Borel set for any Borel set M in E. Since $C^*(B)$ is separable and unital the weak* topology on E is metrizable for some metric d. So for each $n \in \mathbb{N}$ we can use that E is compact to cover it by finitely many disjoint Borel sets $M_1, \dots, M_{k_n}$ with diameter at most $1/n$. The sets $F^{-1}(M_i)$, $i = 1, \dots, k_n$ is then a disjoint partition of $\mathcal{G}^{(0)}$ into invariant Borel sets, hence there is a unique i with $\mu(F^{-1}(M_i)^C) = 0$. Denote this M_i by S_n, and find such a set S_n for each $n \in \mathbb{N}$. So $\{S_n\}_{n=1}^{\infty}$ is a sequence of Borel sets where S_n has diameter at most $1/n$ and satisfies $\mu(F^{-1}(S_n)^C) = 0$ for each n. Set:

$$S := \bigcap_{n=1}^{\infty} S_n$$

it then follows that:

$$\mu(F^{-1}(S^C)) = \mu\Big(\bigcup_{n=1}^{\infty} F^{-1}(S_n)^C\Big) = 0$$

implying in particular since $\mu \neq 0$ that $S \neq \emptyset$. However by choice of the diameter on S_n then S can at most contain one point, i.e. $S = \{\varphi\}$ for some $\varphi \in E$. For all $x \in F^{-1}(S) \cap X(B)$ we now have:

$$\varphi = F(x) = \varphi'_x \circ \Phi_x^{-1} \Rightarrow \varphi'_x = \varphi \circ \Phi_x$$

Since $\varphi_x = \varphi'_x$ for μ-a.e. x and:

$$\mu((F^{-1}(S) \cap X(B))^C) = \mu(F^{-1}(S^C) \cup X(B)^C) = 0$$

the Theorem now follows. □

6.2 Symmetries of the KMS simplex

We will now use Theorem 6.1.4 to describe the KMS weights on the groupoid C^*-algebra of a groupoid injectively graded by a discrete countable abelian group. Notice that Theorem 6.2.1 only works for $\beta \neq 0$ by remark 4.2.4.

Theorem 6.2.1. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid that is injectively graded by an abelian countable discrete group A via a map $\Phi : \mathcal{G} \to A$. Let $\beta \in \mathbb{R} \setminus \{0\}$, $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\mu \in \Delta(\beta, c) \setminus \{0\}$ be ergodic. Denote by B the subgroup of A associated to μ given by (2) in Theorem 6.1.4. There exists an affine bijection from the state-space on $C^*(B)$ to $\mathcal{W}(\beta, c)_\mu$ that maps a state φ into the β-KMS weight ω_φ given by:*

$$\omega_\varphi(f) = \int_{X(B)} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi(u_{\Phi(g)})\, d\mu(x) \qquad \text{for all } f \in C_c(\mathcal{G}) \tag{6.2}$$

Proof. Assume φ is a state on $C^*(B)$, we claim that:

$$\varphi_x = \begin{cases} \varphi \circ \Phi_x & \text{for } x \in X(B) \\ Tr_x & \text{for } x \notin X(B) \end{cases}$$

is a μ-measurable field of states satisfying the conditions in Theorem 4.2.1. For any $f \in C_c(\mathcal{G})$ and $x \in X(B)$ we have:

$$\sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g) = \sum_{g \in \mathcal{G}_x^x} f(g)\varphi(u_{\Phi(g)})$$

so, using e.g. Lemma 2.2.1, it follows that $\{\varphi_x\}_{x \in \mathcal{G}^{(0)}}$ is a μ-measurable field of states. For $x \in X(B)$, $g \in \mathcal{G}_x^x$ and $h \in \mathcal{G}_x$ we have $r(h) \in X(B)$, so:

$$\varphi_{r(h)}(u_{hgh^{-1}}) = \varphi(u_{\Phi(hgh^{-1})}) = \varphi(u_{\Phi(g)}) = \varphi_x(u_g)$$

Since $\mu(X(B)^C) = 0$ remark 4.2.4 implies that the formula (6.2) defines a KMS weight, and hence the map $\varphi \to \omega_\varphi$ is well defined. It is surjective by 3 in Theorem 6.1.4. For injectivity notice that if $\omega_\varphi = \omega_\psi$ for some states φ, ψ on $C^*(B)$ then $\varphi \circ \Phi_x = \psi \circ \Phi_x$ for μ-almost all $x \in X(B)$, so since Φ_x is an isomorphism this implies that $\varphi = \psi$. We leave it to the reader to check that the map $\varphi \to \omega_\varphi$ is affine. □

Remark 6.2.2. Since B is a discrete abelian group then $C^*(B) \simeq C(\widehat{B})$, so the state-space of $C^*(B)$ is homeomorphic to the space of regular Borel probability measures on the Pontryagin dual $\widehat{B}$ of B. Notice that Theorem 6.2.1 also gives a description of the proper tracial weights on $C^*(\mathcal{G})$ by taking the groupoid homomorphism c to be the zero function.

Corollary 6.2.3. *In the setting of Theorem 6.2.1 there is a bijection between pairs (μ, ξ) and extremal KMS weights ψ in $\mathcal{W}(\beta, c) \setminus \{0\}$, where (μ, ξ) consists of an ergodic measure $\mu \in \Delta(\beta, c) \setminus \{0\}$ and a character $\xi \in \widehat{B}$ where $B \subseteq A$ is the subgroup corresponding to μ via Theorem 6.1.4. The extremal weight $\psi_{\mu,\xi}$ corresponding to (μ, ξ) is given as:*

$$\psi_{\mu,\xi}(f) = \int_{X(B)} \sum_{g \in \mathcal{G}_x^x} f(g)\xi(\Phi(g)) \, d\mu(x) \qquad \textit{for all } f \in C_c(\mathcal{G}) \tag{6.3}$$

Proof. The extremal state on $C^*(B)$ are the elements of $\widehat{B}$ under the map $u_b \to \xi(b)$ for $b \in B$ and $\xi \in \widehat{B}$. Hence the Corollary follows from combining Theorem 6.2.1, Theorem 5.1.6 and Lemma 5.2.3. □

The set of β-KMS states for some continuous 1-parameter group α on a unital C^*-algebra is a simplex, implying that any KMS state can be decomposed as extremal KMS states. In the light of Corollary 6.2.3 it would be very interesting if one could obtain a similar decomposition for weights. The author is currently working on answering this question, and likewise looking into the question of decomposing measures in $\Delta(\beta, c)$ into ergodic ones. Since this is still work in progress, and since there is a restriction on the length of a dissertation, we will not pursue it further here, but only state the following Corollary which gives a unique decomposition in a setting which suffices for the results obtained in [C, D] in Part II.

Corollary 6.2.4. *In the setting of Theorem 6.2.1 assume that $\mathcal{G}^{(0)}$ is compact and $\Delta_{\mathcal{G}^{(0)}}(\beta, c)$ has finitely many extremal points $\{\mu_i\}_{i=1}^n$ with corresponding subgroups $\{B_i\}_{i=1}^n$. Assume there exists disjoint Borel sets $\{X_i\}_{i=1}^n$ in $\mathcal{G}^{(0)}$ with $\mu_i(X_i) = 1$ for each i. Then the set of β-KMS states is a Bauer simplex and any β-KMS state ω for α^c is given uniquely as:*

$$\omega(f) = \sum_{i=1}^n \lambda_i \int_{X(B_i)} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_i(u_{\Phi(g)}) \, d\mu_i(x) \qquad \text{for all } f \in C_c(\mathcal{G}) \tag{6.4}$$

where $\lambda_i \geq 0$ for each i, $\sum_{i=1}^n \lambda_i = 1$ and φ_i is a unique state on $C^*(B_i)$ when $\lambda_i \neq 0$. This correspondence is a bijection.

Proof. Let Δ denote the set of β-KMS states for α^c, since $\mathcal{G}^{(0)}$ is compact $C^*(\mathcal{G})$ is unital and Δ is a simplex, c.f. Theorem 5.3.30 in [2]. Assume that $\xi_n \to \xi$ in $\widehat{B_i}$ and $f \in C_c(\mathcal{G})$ with $\operatorname{supp}(f) \subseteq \Phi^{-1}(b)$ for some $b \in B$, then by (6.3):

$$\psi_{\mu_i,\xi_n}(f) = \int_{X(B_i)} \sum_{g \in \mathcal{G}_x^x} f(g)\xi_n(\Phi(g)) \, d\mu_i(x) = \xi_n(b) \int_{X(B_i)} \sum_{g \in \mathcal{G}_x^x} f(g) \, d\mu_i(x)$$

which clearly converges to $\psi_{\mu_i,\xi}(f)$. Since $\widehat{B_i}$ is compact it follows from this that the map $F_i : \widehat{B_i} \to \Delta$ given by $F_i(\xi) = \psi_{\mu_i,\xi}$ is a homeomorphism from $\widehat{B_i}$ onto its image $F_i(\widehat{B_i})$ in Δ, which is then compact. Since

$$\partial\Delta = \bigsqcup_{i=1}^n F_i(\widehat{B_i})$$

it follows that $\partial\Delta$ is a closed set in Δ, and hence Δ is a Bauer simplex. Any KMS state is now given by a unique probability measure m on $\partial\Delta$, but this measure can be decomposed as $\sum_{i=1}^n \lambda_i m_i$ where $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$ and each m_i is a probability measure supported on $F_i(\widehat{B_i})$. Since $F_i(\widehat{B_i}) \simeq \widehat{B_i}$ each m_i defines a state φ_i on $C^*(B_i)$, giving the existence statement. Clearly (6.4) defines a KMS state ω for any family $\{\lambda_i\}_{i=1}^n$ and $\{\varphi_i\}_{i=1}^n$. If ω is also given by $\{\lambda_i'\}_{i=1}^n$ and $\{\varphi_i'\}_{i=1}^n$ evaluating in $C_c(\mathcal{G}^{(0)})$ gives $\sum_i \lambda_i \mu_i = \sum_i \lambda_i' \mu_i$, so evaluating in X_j gives $\lambda_j' = \lambda_j$ for each j. Now:

$$0 = \omega(hf) - \omega(hf) = \sum_{i=1}^n \lambda_i \int_{X(B_i)} h(x) \sum_{g \in \mathcal{G}_x^x} f(g)(\varphi_i(u_{\Phi(g)}) - \varphi_i'(u_{\Phi(g)})) \, d\mu_i(x)$$

for all $h \in C_c(\mathcal{G}^{(0)})$ and $f \in C_c(\mathcal{G})$. For any j find $K \subseteq X_j$ with $\mu_j(K) > 0$, letting $h_n \to 1_K$ then gives:

$$0 = \lambda_j \int_{X(B_j)\cap K} \sum_{g \in \mathcal{G}_x^x} f(g)(\varphi_j(u_{\Phi(g)}) - \varphi_j'(u_{\Phi(g)}))\, d\mu_j(x)$$

for any $f \in C_c(\mathcal{G})$, which must imply that $\varphi_j = \varphi_j'$ when $\lambda_j > 0$. □

Theorem 6.2.1 was proved in [C] for states when the groupoid C^*-algebra was unital. The technique in [C] was to use the C^*-dynamical system $(C^*(\mathcal{G}), \hat{A}, \gamma)$ defined in Proposition 2.2.10 by Φ to prove that there where symmetries in the KMS simplex (explaining the name of this chapter) and using this to derive the theorem. We will not pursue this further since it is explained in [C], but we will remark the following useful fact, which is a generalisation of Lemma 3.1 in [C].

Proposition 6.2.5. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid that is injectively graded by an abelian countable discrete group A via a map $\Phi : \mathcal{G} \to A$ and let $(C^*(\mathcal{G}), \widehat{A}, \gamma)$ be the C^*-dynamical system defined by Φ. Assume $c : \mathcal{G} \to \mathbb{R}$ is a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. If $\psi \in \mathcal{W}(\beta, c)$ then ψ is invariant under γ if and only if $\psi = \psi \circ P$.*

Proof. Since $\gamma_\xi(P(a)) = P(\gamma_\xi(a)) = P(a)$ for all $\xi \in \widehat{A}$ and $a \in C^*(\mathcal{G})$ the equation $\psi = \psi \circ P$ implies that ψ is invariant under γ. Assume ψ is invariant under γ and that ψ is given by the pair $(\mu, \{\varphi_x\}_{x \in \mathcal{G}^{(0)}})$. For any $a \in A \setminus \{e\}$ we can pick $\xi^a \in \widehat{A}$ such that $\xi^a(a) \neq 1$. For any $x \in \mathcal{G}^{(0)}$ the map $u_g \to u_g \xi^a(\Phi(g))$ defines an automorphism $\gamma_{\xi^a}^x$ of $C^*(\mathcal{G}_x^x)$, and it is straightforward to check that defining $\varphi_x^a = \varphi_x \circ \gamma_{\xi^a}^x$ then $\{\varphi_x^a\}_{x \in \mathcal{G}^{(0)}}$ is a μ-measurable field satisfying 2 and 3 of Theorem 4.2.1, and since:

$$\psi(\gamma_{\xi^a}(f)) = \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x^x} f(g)\xi^a(\Phi(g))\varphi_x(u_g)\, d\mu(x) = \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^a(u_g)\, d\mu(x)$$

there is a μ-null set $N_a \subseteq \mathcal{G}^{(0)}$ with $\varphi_x \circ \gamma_{\xi^a}^x = \varphi_x$ for all $x \notin N^a$. Since A is countable then $\varphi_x \circ \gamma_{\xi^a}^x = \varphi_x$ for μ-a.e x and all $a \in A \setminus \{e_0\}$ and hence for $g \in \mathcal{G}_x^x \setminus \{x\}$ then $\varphi_x(u_g) = \varphi_x(u_g)\xi^{\Phi(g)}(\Phi(g))$, implying that $\varphi_x(u_g) = 0$. In conclusion $\varphi_x = \mathrm{Tr}_x$ for μ-a.e. x, so $\psi = \psi \circ P$ by Corollary 4.2.5. □

— 7 —

KMS weights on graph C^*-algebras

In this chapter we will review the main results of the articles from Part II. These results are the main contribution of this dissertation, and we will present them as follows: First we will review article [A] and [B] that concern KMS states on directed graphs. We will then turn to our results on higher–rank graphs, reviewing article [C] and [D]. After this we will review the results of [E]. The article [E] is not concerned with describing KMS weights for specific C^*-algebras and 1-parameter groups but instead with a more philosophical result concerning the connection between KMS weights arising from measures on the unit space and 1-parameter groups given by continuous groupoid homomorphisms.

Let us introduce some notation for this chapter. If S is a finite set, we let $M_S(\mathbb{F})$ denote the set of matrices over S with entries in $\mathbb{F}$, and for any subset $D \subseteq S$ and any matrix $M \in M_S(\mathbb{F})$ we let $M^D \in M_D(\mathbb{F})$ denote the restriction of M from $S \times S$ to $D \times D$. For any matrix $M \in M_S(\mathbb{F})$ we denote by $\rho(M)$ the spectral radius of M.

7.1 KMS weights on the Cuntz-Krieger algebra of directed graphs

In this section we will first give a brief introduction to directed graphs, then we will give a summary of our results in [A] and then we will give a summary of our results in [B].

Directed graphs

A *directed graph* $G = (V, E, r, s)$ consists of a countable set of vertexes V, a countable set of edges E and maps $r, s : E \to V$ called the *range* and the *source* map. If $v \in V$ satisfies $s^{-1}(v) = \emptyset$ we call v a *sink*, and if $|s^{-1}(v)| = \infty$ we call v an *infinite emitter*. We say our graph is *row-finite* if there is no infinite emitters, and call it *finite* if V and E are finite. A *finite path* δ of length n in G is a concatenation $\delta = e_1 \cdots e_n$ of elements

$e_i \in E$ with $r(e_i) = s(e_{i+1})$ for each i, and we define $s(\delta) = s(e_1)$ and $r(\delta) = r(e_n)$. For a graph G we can introduce the graph C^*-algebra $C^*(G)$ as the universal C^*-algebra generated by a family $\{S_e : e \in E\}$ of partial isometries with mutually orthogonal ranges and a family $\{p_v : v \in V\}$ of mutually orthogonal projections subject to the conditions:

- $S_e^* S_e = p_{r(e)}$ for all $e \in E$.
- $S_e S_e^* \leq p_{s(e)}$ for all $e \in E$.
- $p_v = \sum_{e \in s^{-1}(v)} S_e S_e^*$ for all $v \in V$ with $0 < |s^{-1}(v)| < \infty$.

For any finite path $\delta = e_1 \cdots e_n$ we set $S_\delta := S_{e_1} S_{e_2} \cdots S_{e_n}$. When originally introduced for row-finite graphs without sinks in [19] graph C^*-algebras were defined as the C^*-algebra of an étale groupoid constructed using the graph, and it was then proven that this C^*-algebra was universal with the properties above. For general graphs this is also true [22], i.e. there is a locally compact second countable Hausdorff étale groupoid $\mathcal{G}$ associated to G such that $C^*(\mathcal{G}) \simeq C^*(G)$. We will not introduce the groupoid and its topology here and refer to section 5 in [A] for an introduction to it, but we will note that, as in Example 6.1.3, one can define a continuous groupoid homomorphism $\Phi : \mathcal{G} \to \mathbb{Z}$ that makes $\mathcal{G}$ injectively graded. The C^*-dynamical system $(C^*(G), \mathbb{T}, \gamma)$ constructed by Φ via Proposition 2.2.10 is the *gauge-action*, i.e. the action guaranteed by universality satisfying $\gamma_z(S_e) = zS_e$ and $\gamma_z(p_v) = p_v$ for all $e \in E$, $v \in V$ and $z \in \mathbb{T}$.

Generalised gauge-actions on finite graphs

In this section we will summarize our results from [A]. The study of KMS weights on C^*-algebras that can be realised as the graph C^*-algebra of a directed graph goes back to work of Olesen and Pedersen [21] and Enomoto, Fujii and Watatani [7][1], and during the last 40 years several contributions have been made to the subject. The inspiration for our analysis in [A] was recent work, both recent results in [12, 15] and more general results on the structure of KMS states on graphs and groupoids by Klaus Thomsen in [29, 31, 32, 33]. In [12, 15] the KMS states on the graph C^*-algebra of a general finite graph were described for the 1-parameter group defined via the map $\mathbb{R} \ni t \to \gamma_{e^{it}}$. We extend this result in [A].

For any function $F : E \to \mathbb{R}$ we can use the universal property of $C^*(G)$ to obtain a continuous 1-parameter group α^F such that $\alpha_t^F(p_v) = p_v$ for all $v \in V$ and $\alpha_t^F(S_e) = e^{itF(e)} S_e$ for all $e \in E$, and in the groupoid picture

[1] This work was done in the setting of Cuntz-Krieger algebras

this 1-parameter group can also be described via a continuous groupoid homomorphism c_F (see e.g. section 5 in [A] again). We call such an action a *generalised gauge-action*, and the main objective of [A] is to describe the KMS states for such actions on the graph C^*-algebras of finite graphs.

Our general approach to describing the β-KMS states for α^F on $C^*(G)$ for a graph $G = (V, E, r, s)$ is a recurrent theme in this dissertation, and it roughly consists of the following three steps:

1. We first describe an affine bijection between the gauge-invariant KMS states and certain vectors in $[0, \infty[^V$.

2. We then describe certain subsets of the graph G that can be used to construct such vectors, and we argue that the vectors we construct are exactly the extremal vectors. This implies that we have a description of all these vectors, and hence we have a description of the gauge-invariant KMS states.

3. Finally we describe the non gauge-invariant KMS states.

To describe the non gauge-invariant KMS states in [A] we use results from [29], but we would like to emphasise that the theory in chapter 6 has been developed to be used in the third step. When we have a description of the gauge-invariant KMS states Proposition 6.2.5 implies that we have a complete description of the quasi-invariant measures with Radon-Nikodym cocycle $e^{-\beta c_F}$, and hence we can obtain a description of the non-gauge invariant KMS states using the ideas from chapter 6.

To describe the vectors corresponding to the gauge-invariant KMS states define for each $\beta \in \mathbb{R}$ a matrix $A(\beta) = (A(\beta)_{v,w})$ over V by:

$$A(\beta)_{v,w} = \sum_{e \in s^{-1}(v) \cap r^{-1}(w)} e^{-\beta F(e)}$$

We call a vector $\psi \in [0, \infty[^V$ *almost $A(\beta)$-harmonic* if $(A(\beta)\psi)_v = \psi_v$ for all $v \in V$ that is not a sink, and we say a vector has *unit 1-norm* when $\sum_{v \in V} |\psi_v| = 1$.

Lemma 7.1.1 (Lemma 2.1 in [A], [31]). *Let $G = (V, E, r, s)$ be a finite directed graph and $\beta \in \mathbb{R}$. There is an affine bijection $\psi \to \omega_\psi$ between almost $A(\beta)$-harmonic vectors of unit 1-norm and gauge-invariant β-KMS states for α^F. The state ω_ψ is given by:*

$$\omega_\psi(S_\mu S_\nu^*) = \delta_{\mu,\nu} e^{-\beta F(\mu)} \psi_{r(\mu)}$$

for all finite paths μ, ν.

To describe the subsets of V that give rise to the extremal almost $A(\beta)$-harmonic vectors we need to describe the non-circular KMS components, the circular KMS-components and the KMS sinks. To do this we define a relation $\leq$ on V by:

$$v \leq w \iff \text{there exists a finite path } \mu \text{ with } s(\mu) = v,\ r(\mu) = w$$

We write $v \sim w$ if $v \leq w$ and $w \leq v$, and we call an equivalence class C in $\sim$ a component if it is not on the form $C = \{v\}$ with $r^{-1}(v) \cap s^{-1}(v) = \emptyset$. We define the closure $\overline{S}$ for any $S \subseteq V$ as:

$$\overline{S} := \{v \in V \mid v \leq w \text{ for some } w \in S\}$$

A finite path μ that is not a vertex is a loop if $r(\mu) = s(\mu)$, and a component C is a *circular component* if it only contains one loop. Extending F to any finite path $\mu = \mu_1 \dots \mu_n$ via:

$$F(\mu) := F(\mu_1) + \cdots + F(\mu_n)$$

it follows from Lemma 4.2 and Lemma 4.3 in [A] that if a component C satisfies $F(\mu) > 0$ for every loop μ contained in C, or $F(\mu) < 0$ for every loop μ in C, then there is a unique number $\beta_C \in \mathbb{R}$, such that $\rho(A(\beta_C)^C) = 1$, where $A(\beta_C)^C$ is the restriction of $A(\beta_C)$ to C. Knowing this we can make the following definitions.

Definition 7.1.2. Let C be a non-circular component. We say that C is a:

- *KMS component of positive type* if it satisfies $F(\mu) > 0$ for all loops μ contained in $\overline{C}$ and $\beta_{C'} < \beta_C$ for all components C' contained in $\overline{C} \setminus C$.

- *KMS component of negative type* if it satisfies $F(\mu) < 0$ for all loops μ contained in $\overline{C}$ and $\beta_{C'} > \beta_C$ for all components C' contained in $\overline{C} \setminus C$.

For a $\beta \in \mathbb{R} \setminus \{0\}$ we let $\mathcal{C}(\beta)$ be the set of non-circular positive and negative KMS-components C such that $\beta_C = \beta$.

Definition 7.1.3. For a circular component D, we say that D is a:

- *KMS component of positive type* when $F(\nu) = 0$ for the loop ν in D, and $F(\mu) > 0$ for all loops μ in $\overline{D} \setminus D$.

- *KMS component of negative type* when $F(\nu) = 0$ for the loop ν in D, and $F(\mu) < 0$ for all loops μ in $\overline{D} \setminus D$.

To each circular KMS-component D we associate an interval I_D. If D is a positive (or negative) circular KMS component with no loops in $\overline{D} \setminus D$, we set $I_D = \mathbb{R}$. If D is a positive circular KMS component with components in $\overline{D} \setminus D$, we set $I_D =]\beta_D, \infty[$ where

$$\beta_D = \max\{\beta_{D'} \mid D' \text{ is a component in } \overline{D} \setminus D\}$$

and if D is a negative circular KMS component with components in $\overline{D} \setminus D$, we set $I_D =]-\infty, \beta_D[$ where

$$\beta_D = \min\{\beta_{D'} \mid D' \text{ is a component in } \overline{D} \setminus D\}$$

For a $\beta \in \mathbb{R} \setminus \{0\}$ we let $\mathcal{Z}(\beta)$ be the set of circular positive and negative KMS components D such that $\beta \in I_D$.

Definition 7.1.4. We say that a sink $s \in V$ is a

- *KMS sink of positive type* when $F(\mu) > 0$ for every loop μ in $\overline{\{s\}}$.
- *KMS sink of negative type* when $F(\mu) < 0$ for every loop μ in $\overline{\{s\}}$.

We associate to each KMS sink s an interval I_s as well. If $\overline{\{s\}}$ contains no components, we set $I_s = \mathbb{R}$. If s is of positive type and $\overline{\{s\}}$ contains a component we set $I_s =]\beta_s, \infty[$ where:

$$\beta_s = \max\{\beta_{C'} \mid C' \text{ is a component in } \overline{\{s\}}\}$$

and if s is of negative type and $\overline{\{s\}}$ contains a component we set $I_s =]-\infty, \beta_s[$ where:

$$\beta_s = \min\{\beta_{C'} \mid C' \text{ is a component in } \overline{\{s\}}\}$$

For each $\beta \in \mathbb{R} \setminus \{0\}$ we let $\mathcal{S}(\beta)$ be the set of KMS sinks s such that $\beta \in I_s$.

When $\beta \neq 0$ it turns out that the elements of $\mathcal{S}(\beta)$, $\mathcal{Z}(\beta)$ and $\mathcal{C}(\beta)$ give rise to the extremal almost $A(\beta)$-harmonic vectors, c.f. Theorem 4.10 in [A], and hence we have completed the second step, i.e. we have described the gauge-invariant KMS states. Adding the non-gauge invariant KMS states does not require the theory developed in Theorem 6.2.1 because there are only countably many units with non-trivial isotropy in the groupoid $\mathcal{G}$ with $C^*(\mathcal{G}) \simeq C^*(G)$, and hence we can describe the non gauge-invariant states by results in [29]. It turns out that the circular KMS components are the only ones giving rise to non-gauge invariant KMS states. For a fixed $\beta \in \mathbb{R} \setminus \{0\}$ we associate to each $C \in \mathcal{C}(\beta)$ the unique β-KMS state φ_C obtained by combining Lemma 4.5, Lemma 3.4 and Lemma 2.1 in [A], we associate to each $s \in \mathcal{S}(\beta)$ the unique β-KMS state φ_s obtained by combining Lemma 4.9, Lemma 3.2 and Lemma 2.1 in [A], and we associate to each $D \in \mathcal{Z}(\beta)$ and Borel probability measure m_D on $\mathbb{T}$ the unique β-KMS state $\omega_D^{m_D}$ described in section 5 in [A].

Theorem 7.1.5 (Theorem 5.2 in [A])**.** *For a $\beta \in \mathbb{R} \setminus \{0\}$, define $\mathcal{C}(\beta)$, $\mathcal{Z}(\beta)$ and $\mathcal{S}(\beta)$ as above.*

For every β-KMS state φ for α^F there are numbers $\alpha_C \in [0,1]$, $C \in \mathcal{C}(\beta)$, $\alpha_s \in [0,1]$, $s \in \mathcal{S}(\beta)$ and $\alpha_D \in [0,1]$, $D \in \mathcal{Z}(\beta)$ and Borel probability measures m_D, $D \in \mathcal{Z}(\beta)$, on $\mathbb{T}$, such that:

$$\sum_{C \in \mathcal{C}(\beta)} \alpha_C + \sum_{s \in \mathcal{S}(\beta)} \alpha_s + \sum_{D \in \mathcal{Z}(\beta)} \alpha_D = 1$$

and:

$$\varphi = \sum_{C \in \mathcal{C}(\beta)} \alpha_C \varphi_C + \sum_{s \in \mathcal{S}(\beta)} \alpha_s \varphi_s + \sum_{D \in \mathcal{Z}(\beta)} \alpha_D \omega_D^{m_D}$$

where the numbers α_C, α_s and α_D and the measures m_D where $\alpha_D > 0$ are uniquely determined by φ.

In [A] we use the convention that the 0-KMS states are tracial states, and we use a different approach to describe these, c.f. section 5.1 in [A].

Generalised gauge-actions on Cayley graphs

In this section we will summarize our results from [B]. The results in [A] do not imply that we in full understand the KMS states for diagonal actions on graph C^*-algebras of directed graphs. First and foremost there are a lot of 1-parameter groups that can not be described as a generalised gauge-action, and we have no description of the KMS states for these actions, see e.g. [28] for results in this direction. Furthermore the results in [A] only cover finite graphs, and in general we have no description of the KMS weights of these actions on infinite graphs. In the special cases where we do have concrete descriptions the structure of the set of KMS weights is surprisingly rich [31, 30], and these results indicate that a description for all graphs is currently out of reach. In [B] we give an answer to the problem when the infinite graph is the Cayley graph of a nilpotent group and the generalised gauge-action is sufficiently nice. We will now introduce these Cayley graphs and generalised gauge-actions.

Consider a group G and a finite set Y in G with $|Y| \geq 2$ such that Y generates G as a semi-group. One can then define a directed graph $\Gamma(G, Y)$ by considering G as the set of vertexes and drawing an edge from $g \in G$ to

$h \in G$ exactly when $g^{-1}h \in Y$, and with our assumption on Y it follows that $C^*(\Gamma(G,Y))$ is a simple C^*-algebra. Letting $e_0 \in G$ denote the unit then p_{e_0} is a full projection in $C^*(\Gamma(G,Y))$, and this implies that the KMS weights on $C^*(\Gamma(G,Y))$ are in a natural correspondence with the KMS states on $p_{e_0}C^*(\Gamma(G,Y))p_{e_0}$, c.f Theorem 2.4 in [31]. To introduce the 1-parameter group let $F : Y \to \mathbb{R}$ be a function. If e is the edge from g to h we set $F(e) = F(g^{-1}h)$, so setting $\gamma_t^F(S_e) = e^{itF(e)}S_e$ we get a 1-parameter group γ^F that leaves $p_{e_0}C^*(\Gamma(G,Y))p_{e_0}$ globally invariant. The aim in [B] is to give a description of the KMS states for $(p_{e_0}C^*(\Gamma(G,Y))p_{e_0}, \gamma^F)$.

First we identify $p_{e_0}C^*(\Gamma(G,Y))p_{e_0}$ with a C^*-subalgebra of the Cuntz-algebra $O_{|Y|}$. We identify $O_{|Y|}$ with O_Y, i.e. we assume the Cuntz-algebra is generated by isometries V_y, $y \in Y$. For $t = (t_1, \dots, t_n) \in Y^n$ we set $\overline{t} = t_1t_2\cdots t_n \in G$ and let $V_t \in O_Y$ be the isometry $V_{t_1}V_{t_2}\cdots V_{t_n}$, and use the convention $Y^0 = \emptyset$, $\overline{\emptyset} = e_0$ and $V_\emptyset = 1 \in O_Y$. It then turns out that:

$$O_Y(G) := \overline{\operatorname{span}}\{V_tV_u^* \mid \overline{t} = \overline{u}\}$$

is a C^*-subalgebra of O_Y. Using the universal properties of O_Y we can define a 1-parameter group α^F on O_Y satisfying $\alpha_t^F(V_y) = e^{itF(y)}V_y$, and α^F then leaves $O_Y(G)$ globally invariant.

Proposition 7.1.6 (Proposition 2.2 in [B])**.** *There is a $*$-isomorphism $\pi : p_{e_0}C^*(\Gamma(G,Y))p_{e_0} \to O_Y(G)$ such that $\pi \circ \gamma_t^F = \alpha_t^F \circ \pi$ for all $t \in \mathbb{R}$.*

This Proposition implies that it suffices to describe the KMS states of $(O_Y(G), \alpha^F)$. The C^*-subalgebra:

$$\overline{\operatorname{span}}\Big\{V_tV_t^* \mid t \in \bigcup_n Y^n\Big\}$$

generates a copy of $C(Y^{\mathbb{N}})$ inside $O_Y(G)$, and there is a conditional expectation $P : O_Y(G) \to C(Y^{\mathbb{N}})$ with:

$$P(V_tV_u^*) = \begin{cases} V_tV_t^* & \text{if } t = u \\ 0 & \text{if } t \neq u \end{cases}$$

We now proceed as in [A], but in this case all KMS states for α^F on $O_Y(G)$ are gauge-invariant, so we can now formulate a bijective correspondence between KMS states, certain measures on $Y^{\mathbb{N}}$ and certain vectors. To describe this correspondence define for each $t = (t_1, \dots, t_n) \in Y^n$ the cylinder:

$$tY^{\mathbb{N}} = \{(y_i)_{i=1}^\infty \in Y^{\mathbb{N}} \mid t_i = y_i,\ i \leq n\}$$

then the measures and vectors in question are the following:

Definition 7.1.7 (Definition 3.2 and Lemma 3.3 in [B])**.** We call a Borel probability measure m on $Y^{\mathbb{N}}$ a β-KMS measure for α^F when:

$$e^{\beta F(t)} m(tY^{\mathbb{N}}) = e^{\beta F(u)} m(uY^{\mathbb{N}})$$

whenever $t, u \in \bigcup_n Y^n$ satisfy that $\overline{t} = \overline{u}$.

Definition 7.1.8. A vector (or function) $\psi : G \to [0, \infty[$ is called *β-harmonic* when

$$\sum_{y \in Y} e^{-\beta F(y)} \psi_{gy} = \psi_g$$

for all $g \in G$ and *normalized* when $\psi_{e_0} = 1$

We then get:

Theorem 7.1.9 (Section 3 in [A])**.** *Let $\beta \in \mathbb{R}$. The formula:*

$$\omega(a) = \int_{Y^{\mathbb{N}}} P(a) \; dm \qquad \forall a \in O_Y(G)$$

defines an affine bijection between the β-KMS states ω for α^F and the β-KMS measures m for α^F.

The formula:

$$m(tY^{\mathbb{N}}) = e^{-\beta F(t)} \psi_{\overline{t}} \qquad t \in \bigcup_n Y^n$$

defines an affine bijection between the β-KMS measures m for α^F and the normalised β-harmonic vectors.

Equipping these three sets with suitable topologies we can conclude from this theorem that there are affine homeomorphism between the following three sets:

- The β-KMS states on $O_Y(G)$ for α^F.
- The β-KMS measures on $Y^{\mathbb{N}}$ for α^F.
- The normalised β-harmonic vectors on G for α^F.

For general groups we can not describe any of these sets. However, we can describe a subset of them. Therefore we will in the following restrict attention to the set Δ of *abelian* normalised β-harmonic vectors on G, i.e. normalised β-harmonic vectors ψ satisfying:

$$\psi_{hgk} = \psi_{hkg} \quad \text{for all } h, g, k \in G.$$

Δ is a closed convex subset of the normalised β-harmonic vectors on G, and in general it is a proper subset. We call the states and measures corresponding to Δ under our affine bijection abelian as well. We will in the following give a description of the abelian normalised β-harmonic vectors, and by Theorem 4.1 in [B] we are able to conclude that all normalised β-harmonic vectors ψ are abelian when G is nilpotent, so for a large class of groups we will describe all β-KMS states. We obtain the following description of the extremal points of Δ.

Lemma 7.1.10 (Lemma 4.2 and Lemma 4.3 in [B])**.** *An element $\psi \in \Delta$ is extremal in Δ if and only if there is a homomorphism $c : G \to \mathbb{R}$ such that:*

$$\psi_g = e^{c(g)} \qquad \forall g \in G$$

An abelian β-KMS measure is extremal in the set of abelian β-KMS measures if and only if it is a Bernoulli measure*, i.e. there is a probability vector $p : Y \to [0, 1]$ with $\sum_{y \in Y} p(y) = 1$ such that*

$$m(tY^{\mathbb{N}}) = \prod_{i=1}^{n} p(t_i) \qquad \text{for any } n \text{ and } t = (t_1, \cdots, t_n) \in Y^n$$

We can now describe the abelian β-KMS states for α^F. Any extremal abelian β-harmonic vector is given by a homomorphism c on G. Since:

$$c(ghg^{-1}h^{-1}) = 0 \qquad \text{for all } g, h \in G$$

we must have $[G, G] \subseteq \ker(c)$, where $[G, G]$ is the commutator subgroup of G. This explains the main idea in obtaining the next results:

Proposition 7.1.11 (Proposition 4.6 in [B])**.** *When the abelianisation $G/[G, G]$ of G is trivial or a finite group there is an abelian β-KMS state for α^F if and only if:*

$$\sum_{y \in Y} e^{-\beta F(y)} = 1$$

and in that case it is unique.

Theorem 7.1.12 (Theorem 4.13 in [B])**.** *Assume the abelianisation $G/[G, G]$ of G has rank $n \geq 1$ and that $F(y) > 0$ for all $y \in Y$. It follows that there is a $\beta_0 > 0$ such that:*

- *There are no abelian β-KMS states for α^F when $\beta < \beta_0$.*
- *There is a unique abelian β_0-KMS states for α^F.*

- *For $\beta > \beta_0$ the simplex of abelian β-KMS states for α^F is affinely homeomorphic to the simplex of Borel probability measures on the $(n-1)$ sphere S^{n-1}.*

As pointed out, this describes *all* KMS states for α^F when G is nilpotent. In light of the third conclusion in Theorem 7.1.12 it is relevant to ask if we can conclude anything about the *KMS_∞ states*. If ω_n is a β_n-KMS state for each n, and $\{\beta_n\}_{n=1}^\infty$ is a sequence with $\beta_n \to \infty$ and ω_n converge to some state ω in the weak* topology then we call ω a KMS$_\infty$ state [6]. If each ω_n can be chosen abelian then we call ω an abelian KMS$_\infty$ state.

Theorem 7.1.13 (Corollary 5.4 in [B]). *Assume $G/[G,G]$ has rank $n \geq 1$ and $F(y) > 0$ for all $y \in Y$. The abelian KMS_∞ states for α^F on $O_Y(G)$ constitute a compact convex set affinely homeomorphic to the set of Borel probability measures on the $(n-1)$ sphere S^{n-1}.*

For G nilpotent, this theorem describes the entire set of KMS$_\infty$ states.

7.2 KMS weights on the C^*-algebras of finite higher-rank graphs

In this section we will first give an introduction to higher-rank graphs, and then we will give a summary of our results in [C, D].

Higher-rank graphs

Let $\mathbb{N}$ be the natural numbers including 0 and for $k \geq 1$ write $e_1, \dots, e_k$ for the canonical generators of $\mathbb{N}^k$. A *higher-rank graph* of rank $k \geq 1$ is a pair (Λ, d) consisting of a countable small category Λ and a functor $d : \Lambda \to \mathbb{N}^k$ that has the unique factorisation property, i.e. if $d(\lambda) = n + m$ there exist unique $\mu, \eta \in \Lambda$ with $d(\mu) = n$, $d(\eta) = m$ and $\lambda = \mu\eta$. The category Λ is equipped with a range map r and source map s, and any two morphisms $\lambda, \mu \in \Lambda$ can be composed to $\lambda\mu$ exactly when $s(\lambda) = r(\mu)$. Notice that this is the opposite convention than the one used when composing paths in directed graphs. For $n \in \mathbb{N}^k$ we write $\Lambda^n := d^{-1}(n)$ and identify the objects of the category with Λ^0. We write:

$$v\Lambda = \{\lambda \in \Lambda \mid r(\lambda) = v\} \quad , \quad \Lambda v = \{\lambda \in \Lambda \mid s(\lambda) = v\}$$

and for any $n \in \mathbb{N}^k$ we set $v\Lambda^n = v\Lambda \cap \Lambda^n$, $\Lambda^n v = \Lambda^n \cap \Lambda v$ and $v\Lambda^n w = v\Lambda^n \cap \Lambda^n w$ for any $v, w \in \Lambda^0$. The *vertex matrices* $A_1, \dots, A_k \in M_{\Lambda^0}(\mathbb{N})$ of

Λ are the matrices with entries $A_i(v,w) = |v\Lambda^{e_i}w|$ for $v,w \in \Lambda^0$. We will only consider *finite* k-graphs, i.e. we always assume that Λ^n is finite for all $n \in \mathbb{N}^k$. For any paths $\lambda, \mu \in \Lambda$ we define:

$$\Lambda^{\min}(\mu,\lambda) := \{(\kappa,\eta) \in \Lambda \times \Lambda \ : \mu\kappa = \lambda\eta,\ d(\mu\kappa) = d(\mu) \vee d(\lambda)\}.$$

where $\vee$ denotes the point wise maximum. For $v \in \Lambda^0$ we denote by $v\mathcal{FE}(\Lambda)$ the collection of finite sets $E \subseteq v\Lambda$ satisfying that for any $\mu \in v\Lambda$ there is a $\lambda \in E$ with $\Lambda^{\min}(\mu,\lambda) \neq \emptyset$.

The Toeplitz C^*-algebra $\mathcal{T}C^*(\Lambda)$ of a finite higher-rank graph Λ is then the universal C^*-algebra generated by a family of partial isometries $\{S_\lambda :\ \lambda \in \Lambda\}$ subject to the conditions:

1. $\{p_v := S_v \ : v \in \Lambda^0\}$ are mutually orthogonal projections.
2. When $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$ we have $S_{\lambda\mu} = S_\lambda S_\mu$.
3. $S_\mu^* S_\lambda = \sum_{(\kappa,\eta)\in\Lambda^{\min}(\mu,\lambda)} S_\kappa S_\eta^*$ for all $\mu, \lambda \in \Lambda$.

The Cuntz-Krieger algebra $C^*(\Lambda)$ is the universal C^*-algebra generated by a family of partial isometries $\{S_\lambda :\ \lambda \in \Lambda\}$ satisfying 1.-3. above and:

$$\prod_{\lambda\in E}(p_v - S_\lambda S_\lambda^*) = 0 \text{ for all } v \in \Lambda^0 \text{ and } E \in v\mathcal{FE}(\Lambda)$$

c.f. [8]. There is then an ideal J in $\mathcal{T}C^*(\Lambda)$ such that $\mathcal{T}C^*(\Lambda)/J \simeq C^*(\Lambda)$.

Higher-rank graphs for finite graphs without sources[2] were introduced in [18][3] as a generalisation of directed graphs, and their Cuntz-Krieger C^*-algebras were introduced both as a universal C^*-algebra and as the C^*-algebra of a locally compact second countable Hausdorff étale groupoid. Analogous to directed graphs, the definitions was extended to finite graphs with sources and these were also described as groupoid C^*-algebras [8], so as for directed graphs we can consider $C^*(\Lambda)$ and $\mathcal{T}C^*(\Lambda)$ as universal C^*-algebras introduced above or we can consider them as the full C^*-algebras of locally compact second countable Hausdorff étale groupoids. For the definitions of these groupoids we refer to [8, 37], and remark only that if we denote by $\mathcal{G}_\Lambda$ the groupoid with $C^*(\mathcal{G}_\Lambda) \simeq \mathcal{T}C^*(\Lambda)$ then there is a continuous groupoid homomorphism $\Phi : \mathcal{G}_\Lambda \to \mathbb{Z}^k$ that makes $\mathcal{G}_\Lambda$ injectively graded and satisfies that the C^*-dynamical system $(\mathcal{T}C^*(\Lambda), \mathbb{T}^k, \gamma)$ arising

[2] This means $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

[3] Instead of considering finite graphs they consider *row-finite graphs*, falling beyond the scope of this dissertation.

from Φ is once again the *gauge-action*, i.e. the action on $\mathcal{T}C^*(\Lambda)$ guaranteed by setting:

$$\gamma_z(S_\lambda) = z^{d(\lambda)} S_\lambda = \left(\prod_{i=1}^{k} z_i^{d(\lambda)_i} \right) S_\lambda \quad \text{for all } z \in \mathbb{T}^k \text{ and } \lambda \in \Lambda$$

For any $r \in \mathbb{R}^k$ we can compose the map $\mathbb{R} \ni t \to (e^{itr_1}, e^{itr_2}, \cdots, e^{itr_k}) \in \mathbb{T}^k$ with the gauge-action γ to obtain a continuous 1-parameter group α^r. Equivalently the map $\phi_r : \mathbb{Z}^k \ni z \to r \cdot z \in \mathbb{R}$ is a homomorphism and letting $c_r := \phi_r \circ \Phi$ the continuous 1-parameter group α^r is the same as the one α^{c_r} obtained by using Proposition 2.2.10 on c_r, c.f. appendix B in [10]. This 1-parameter group factor through a 1-parameter group on the Cuntz-Krieger algebra.

KMS states on higher-rank graphs

We will now describe the main results from [C, D]. The aim of [C, D] is to describe the KMS states for the actions α^r with $r \in \mathbb{R}^k$ on $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$. When an investigation of the KMS states for these actions on the Toeplitz and Cuntz-Krieger algebra of a higher-rank graph was started in [13] the analysis was carried out using the universal picture of these C^*-algebras. This was also the dominant picture in the subsequent work on the subject that appeared in the following years, cf [13, 14, 10, 11, 9]. In this body of work the KMS states for the Cuntz-Krieger algebra of a strongly connected graph[4] were successfully described [14], but unlike for directed graphs [12, 15] it was only possible to give a description of the KMS states for finite higher-rank graphs when numerous restrictions were imposed on the graph and the 1-parameter groups [9].

In [C] and [D] the problem of describing the KMS states is attacked using the theory developed in chapter 6, and this approach gives a complete description of all the KMS states for all the 1-parameter groups α^r for the Toeplitz and Cuntz-Krieger algebras of all finite higher-rank graphs. Our general approach is completely similar to the one used for directed graphs. First we find a bijection between gauge-invariant KMS states and certain vectors over Λ^0, then we get a complete description of these vectors by analysing certain components in the graph, and as the last step we describe the subgroups of $\mathbb{Z}^k$ corresponding to the extremal vectors, giving us a complete description of the KMS states by e.g. Corollary 6.2.4. The first part of [C] deals with the development of Theorem 5.2 in [C], which has been generalised in our chapter 6, and the second part deals with the description

[4] A graph is strongly connected if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$

of KMS states on the Cuntz-Krieger algebra of a finite higher-rank graph without sources, which has been generalised in [D]. Therefore the results of [C] play only a minor role in the following presentation, yet it should be emphasised that the results of [C] are absolutely essential for the work carried out in [D].

The gauge-invariant KMS states

First let us describe the vectors corresponding to gauge-invariant states.

Definition 7.2.1. Let Λ be a finite k-graph with vertex matrices $A_1, \dots, A_k$, let $r \in \mathbb{R}^k$ and let $\beta \in \mathbb{R}$. Let $1_{\Lambda^0} \in M_{\Lambda^0}(\mathbb{R})$ be the identity matrix. We say a vector $\psi \in [0, \infty[^{\Lambda^0}$ is sub-invariant for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$ if:

$$\prod_{i \in I} (1_{\Lambda^0} - e^{-\beta r_i} A_i)\psi \geq 0 \qquad \text{for each subset } I \subseteq \{1, \dots, k\}.$$

Proposition 7.2.2 (Lemma 4.1 and Proposition 4.3 in [D])**.** *Let Λ be a finite k-graph and let $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. The map:*

$$\omega \to \{\omega(p_v)\}_{v \in \Lambda^0}$$

is an affine bijection between the set of gauge-invariant β-KMS states for α^r on $\mathcal{T}C^*(\Lambda)$ and the set of sub-invariant vectors for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$ of unit 1-norm.

For any finite k-graph Λ with vertex matrices $A_1, \dots, A_k$ and groupoid $\mathcal{G}_\Lambda$, and any $\beta \in \mathbb{R}$ and $r \in \mathbb{R}^k$ Proposition 7.2.2 and Proposition 6.2.5 implies that we have affine bijections between:

- The set of sub-invariant vectors for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$ of unit 1-norm.
- The set of quasi-invariant probability measures on $\mathcal{G}_\Lambda^{(0)}$ with Radon-Nikodym cocycle $e^{-\beta c_r}$.
- The set of gauge-invariant β-KMS states on $\mathcal{T}C^*(\Lambda)$ for α^r.

To study these convex sets further we will analyse certain faces. For this consider the following linear algebraic result concerning sub-invariant vectors:

Proposition 7.2.3 (Proposition 3.2 in [D])**.** *Let Λ be a finite k-graph, let $\beta \in \mathbb{R}$, $r \in \mathbb{R}^k$ and let $\psi \in [0, \infty[^{\Lambda^0}$ be sub-invariant for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$. For each $I \subseteq \{1, \dots, k\}$ there exists a vector h^I that is sub-invariant for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$ such that:*

1. $e^{-\beta r_i} A_i h^I = h^I$ *for all* $i \in I$.

2. $\lim_{n\to\infty}(e^{-\beta r_j} A_j)^n h^I = 0$ *for* $j \in \{1, \dots, k\} \setminus I$.

3. $\psi = \sum_{I\subseteq\{1,\dots,k\}} h^I$.

Furthermore this decomposition is unique in the sense that there is only one family of sub-invariant vectors satisfying 1.-3.

It follows that the convex set of sub-invariant vectors for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$ of unit 1-norm has a face for each $I \subseteq \{1, \dots, k\}$ consisting of vectors h_I satisfying 1 and 2 in Proposition 7.2.3 above. Proposition 7.2.3 implies that it suffices to describe this face for each I, and it turns out that the vectors in these faces arise from certain subsets of the graph. To describe these sets we need to introduce different equivalences on Λ^0. For each $I \subseteq \{1, \dots, k\}$ we define a relation $\leq_I$ on Λ^0 by:

$$v \leq_I w \iff \exists \lambda \in \Lambda \text{ with } r(\lambda) = v,\ s(\lambda) = w \text{ and } d(\lambda)_j = 0 \text{ for } j \notin I.$$

for $v, w \in \Lambda^0$. We can then define an equivalence relation on Λ^0 by:

$$v \sim_I w \iff v \leq_I w \text{ and } w \leq_I v$$

We furthermore define the I-closure $\overline{S}^I$ of a set $S \subseteq \Lambda^0$ by:

$$\overline{S}^I = \{w \in \Lambda^0 \mid \exists v \in S\ w \leq_I v\}$$

and we let $\overline{S} := \overline{S}^{\{1,\dots,k\}}$. Whenever we have a k-graph Λ with vertex matrices $A_1, \dots, A_k$ and some $S \subseteq \Lambda^0$ we set:

$$\rho(A^S) := (\rho(A_1^S), \rho(A_2^S), \dots, \rho(A_k^S)) \in \mathbb{R}^k$$

Definition 7.2.4. Let Λ be a finite k-graph, $r \in \mathbb{R}^k$, $\beta \in \mathbb{R}$ and let $I \subseteq \{1, \dots, k\}$. An equivalence class C for $\sim_I$ is called a (I, β, r)-subharmonic component, if it satisfies:

1. All equivalence classes D in $\sim_I$ with $D \neq C$ and $D \subseteq \overline{C}^I$ satisfies:
$$\rho(A^D)_I \lneq \rho(A^C)_I$$

2. $\rho(A_i^C) = e^{\beta r_i}$ for $i \in I$.

3. $\rho(A_j^{\overline{C}}) < e^{\beta r_j}$ for $j \in J := \{1, \dots, k\} \setminus I$.

We can use these components to complete our description of the gauge-invariant KMS states as follows:

Theorem 7.2.5 (Proposition 5.4, Definition 5.5 and Proposition 5.8 in [D])**.** *Let Λ be a finite k-graph, $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$, and let $I \sqcup J = \{1, \dots, k\}$ be a partition. For any (I, β, r)-subharmonic component C we can associate a canonical vector y^C. If $\psi \in [0, \infty[^{\Lambda^0}$ is a sub-invariant vector for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$ and $e^{-\beta r_i} A_i \psi = \psi$ for $i \in I$ and $\lim_{n\to\infty} (e^{-\beta r_j} A_j)^n \psi = 0$ for $j \in J$, then there exist a unique collection of (I, β, r)-subharmonic components $\mathcal{C}$ and numbers $t_C > 0$, $C \in \mathcal{C}$, such that:*

$$\psi = \sum_{C \in \mathcal{C}} t_C y^C.$$

This theorem implies that we completely understand the face of the convex set of sub-invariant vectors ψ for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$ that satisfies $e^{-\beta r_i} A_i \psi = \psi$ for $i \in I$ and $\lim_{n\to\infty} (e^{-\beta r_j} A_j)^n \psi = 0$ for $j \in J$. So combining Theorem 7.2.5, Proposition 7.2.3 and Proposition 7.2.2 give enough insight into the set of gauge-invariant KMS states to prove:

Theorem 7.2.6 (Theorem 5.9 in [D])**.** *Let Λ be a finite k-graph, $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. For $I \subseteq \{1, \dots, k\}$ let $\mathcal{C}_r^I(\beta)$ be the set of (I, β, r)-subharmonic components and set:*

$$\mathcal{C}_r(\beta) := \bigsqcup_{I \subseteq \{1, \dots, k\}} \mathcal{C}_r^I(\beta).$$

There is an affine bijective correspondence between functions $f : \mathcal{C}_r(\beta) \to [0, 1]$ with $\sum_{C \in \mathcal{C}_r(\beta)} f(C) = 1$ and the gauge-invariant β-KMS states for α^r on $\mathcal{T}C^*(\Lambda)$. A KMS state ω corresponding to a function f is given by:

$$\omega(S_\lambda S_\mu^*) = \delta_{\lambda,\mu} e^{-\beta r \cdot d(\lambda)} \psi_{s(\lambda)}$$

where:

$$\psi = \sum_{C \in \mathcal{C}_r(\beta)} f(C) y^C.$$

Adding the non gauge-invariant KMS states

If Λ is a finite k-graph, $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$ we can interpret Theorem 7.2.6 as a description of the quasi-invariant measures with Radon-Nikodym cocycle $e^{-\beta c_r}$, i.e. we have a bijective correspondence between extremal measures and components C where $C \in \mathcal{C}_r^I(\beta)$ for some $I \subseteq \{1, \dots, k\}$. Let m_C denote the ergodic probability measure on $\mathcal{G}_\Lambda^{(0)}$ corresponding to the component $C \in \mathcal{C}_r^I(\beta)$ for a $I \subseteq \{1, \dots, k\}$, we then associate a subgroup of $\mathbb{Z}^k$ to C

we denote by $\mathrm{Per}_I(C)$. If $I = \emptyset$ we set $\mathrm{Per}_I(C) = \{0\}$. If $I \neq \emptyset$ we will not explain how to define $\mathrm{Per}_I(C)$ but refer the reader to Section 6 in [D]. We will however remark that defining:

$$C\Lambda_I C := \{\lambda \in \Lambda \mid r(\lambda), s(\lambda) \in C \text{ and } d(\lambda)_j = 0 \text{ for } j \notin I\}$$

then this set can be considered a strongly connected $|I|$-graph in a natural way, and $\mathrm{Per}_I(C)$ will be isomorphic to the periodicity group of the $|I|$-graph $C\Lambda_I C$ as it was originally defined in [14]. The group $\mathrm{Per}_I(C)$ then turns out to be the group corresponding to the measure m_C as in Theorem 6.1.4, and this gives the following complete description of the KMS states on $\mathcal{T}C^*(\Lambda)$:

Theorem 7.2.7 (Theorem 6.3 in [D])**.** *Let Λ be a finite k-graph and fix $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R} \setminus \{0\}$. There is a bijection between pairs (C, ξ), where $C \in \mathcal{C}_r^I(\beta)$ for some $I \subseteq \{1, \dots, k\}$ and ξ lies in the dual $\widehat{Per_I(C)}$ of $Per_I(C)$, to the set of extremal β-KMS states for α^r on $\mathcal{T}C^*(\Lambda)$:*

$$(C, \xi) \to \omega_{C,\xi}$$

where:

$$\omega_{C,\xi}(f) = \int_{X(Per_I(C))} \sum_{g \in \mathcal{G}_x^x} f(g)\xi(\Phi(g)) \, dm_C(x) \qquad \text{for all } f \in C_c(\mathcal{G}_\Lambda).$$

The next natural question to ask is which KMS states factor through a KMS state of the Cuntz-Krieger algebra $C^*(\Lambda)$ of Λ? It turns out that this can be described using only the properties of the component C giving rise to the extremal KMS state.

Corollary 7.2.8 (Corollary 6.8 in [D])**.** *In the setting of Theorem 7.2.7 a state $\omega_{D,\xi}$ for a $D \in \mathcal{C}_r^I(\beta)$ and $\xi \in \widehat{Per_I(D)}$ factors through a state of $C^*(\Lambda)$ if and only if $D \subseteq \overline{C}^I$ for a component C in $\sim_I$ satisfying:*

$$C\Lambda^{e_j} := \bigcup_{v \in C} v\Lambda^{e_j} = \emptyset \qquad \forall j \in \{1, \dots, k\} \setminus I$$

7.3 Diagonality of actions and KMS weights

The results in this dissertation are only concerned with 1-parameter groups arising from some continuous groupoid homomorphisms c on the groupoid $\mathcal{G}$, and for these we have seen that the quasi-invariant measures with Radon-Nikodym cocycle $e^{-\beta c}$ give rise to β-KMS weights. The last result we will introduce is from [E], written jointly with Klaus Thomsen, and it is a result that ties the existence of KMS weights given by measures on the unit space to 1-parameter groups given by continuous groupoid homomorphisms.

The results in [E] are for the reduced C^*-algebra $C_r^*(\mathcal{G})$ of a locally compact second countable Hausdorff étale groupoid. We will not give a thorough introduction to this C^*-algebra here, but only remark that it is also defined as the completion of $C_c(\mathcal{G})$, and that any continuous groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ gives rise to a continuous 1-parameter group α^c, and we denote such 1-parameter groups as *diagonal* as well. As for the full groupoid C^*-algebra $C_0(\mathcal{G}^{(0)})$ embeds as a sub-C^*-algebra in $C_r^*(\mathcal{G})$ and there is a conditional expectation $P : C_r^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$. We call a weight ω on $C_r^*(\mathcal{G})$ diagonal when $\omega = \omega \circ P$. The main result in [E] is that these two concepts are closely related. Due to Theorem 4.2.1 we can present the result for more general groupoids than it was done in [E]:

Theorem 7.3.1 (An extension of Theorem 2.1 in [E])**.** *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid satisfying that for at least one $x \in \mathcal{G}^{(0)}$ the isotropy group $\mathcal{G}_x^x$ is trivial and that $\mathcal{G}$ is minimal in the sense that $s(r^{-1}(y))$ is dense in $\mathcal{G}^{(0)}$ for all $y \in \mathcal{G}^{(0)}$.*

Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a continuous 1-parameter group on $C_r^*(\mathcal{G})$ and assume that there is a β_0-KMS weight for α for some $\beta_0 \neq 0$. Then the following are equivalent:

1. *There is a $\beta_1 \neq 0$ and a diagonal β_1-KMS weight for α.*
2. *Whenever $\beta \neq 0$ and there is a β-KMS weight for α, there is also a diagonal β-KMS weight for α.*
3. *$\alpha_t(f) = f$ for all $t \in \mathbb{R}$ and all $f \in C_0(\mathcal{G}^{(0)})$.*
4. *α is diagonal.*

Proof. Theorem 2.1 in [E] has the extra requirement that $\mathcal{G}^{(0)}$ is totally disconnected, but for the proof of this theorem we only use this when invoking Corollary 3.4 in [E]. Since Theorem 3.2 in [E] is generalised in Theorem 4.2.1 Corollary 3.4 remains true without the assumption that $\mathcal{G}^{(0)}$ is totally disconnected, which proves the Theorem. □

Notice that under the assumption that there is an element in the unit space with trivial isotropy, minimality of $\mathcal{G}$ is equivalent to simplicity of $C_r^*(\mathcal{G})$, c.f Corollary 2.18 in [34], so the C^*-algebras in the theorem are all simple.

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Finite digraphs and KMS states

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ABSTRACT

The paper contains a description of the KMS states of a generalized gauge action on the C^*-algebra of a finite graph.

1. Introduction

In a recent paper by an Huef, Laca, Raeburn and Sims, [6], the authors describe an algorithm by which it is possible to determine all the KMS states of the gauge action on the C^*-algebra of a finite graph. Their results cover also the gauge action on the Toeplitz extension of the algebra and extend the result of Enomoto, Fujii and Watatani, [4], which deals with strongly connected graphs. Almost simultaneously with this work, Carlsen and Larsen obtained an abstract description of the KMS states for some of the generalized gauge actions on the C^*-algebra of a finite graph as well as its Toeplitz extension. Their work builds on and extends methods and results obtained by Exel and Laca in [5] and brings our knowledge about the KMS states of the actions they consider to the point where the work on the gauge action begins in [6]. It is the purpose of the present paper to take the steps from the abstract to the concrete which were taken by an Huef, Laca, Raeburn and Sims, but now for all the generalized gauge actions.

The point of departure for our work are results of the second author from [14] from which it follows that the relevant results of Carlsen and Larsen from [3] remain valid for all generalized gauge actions, provided attention is restricted to the KMS states that are gauge invariant; a condition which is automatically satisfied for the actions considered by Carlsen and Larsen. What we do first is to develop the approach from [6] to make it applicable to generalized gauge actions. In this way we obtain a description of the gauge invariant KMS states for all generalized gauge actions. The main input for this is a generalization of the Perron–Frobenius theory for positive matrices which was obtained by Victory, [16]. See also [11]. The theory

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handles arbitrary finite non-negative matrices and can also be used to simplify some of the steps in [6]. We give here a new proof of the relevant results from [16] and [11] by using ideas from [6].

In order to identify the KMS states that are not gauge invariant we use results by Neshveyev, [8], in a form presented in [12]. By combining the result with our study of the gauge invariant KMS states we obtain in Theorem 5.2 our main result which describes the β-KMS states for all $\beta \in \mathbb{R}\backslash\{0\}$ and for an arbitrary generalized gauge action on the C^*-algebra of a finite graph. As with the gauge action, [6], it is a sub-collection of the components and the sinks in the graph that parametrize the extremal KMS states, although in general some of the components, corresponding to a loop without exits, may contribute a family of extremal KMS states parametrized by a circle. Which components and sinks play a role depends very much on the action, as we illustrate by examples.

It is intrinsic to our approach that the case $\beta = 0$, where the KMS states are the trace states, must be handled separately as we do in Section 5.1. For completeness we describe also in a final section the ground states for the same actions. While there are no ground states for the gauge action unless the graph has sinks, this is no longer the case for generalized gauge actions and even for strongly connected graphs their structure can be very rich.

2. Preparations

Let G be a finite directed graph with vertex set V and edge set E. The maps $r, s : E \to V$ associate to an edge $e \in E$ its source vertex $s(e) \in V$ and range vertex $r(e) \in V$. Thus the set of edges emitted from a vertex v is the set $s^{-1}(v)$ while $r^{-1}(v)$ is the set of edges terminating at v. A *sink* in G is a vertex v that does not emit an edge, i.e. $s^{-1}(v) = \emptyset$.

Formulated in terms of generators and relations the C^*-algebra $C^*(G)$ of G is the universal C^*-algebra generated by a set S_e, $e \in E$, of partial isometries and a set P_v, $v \in V$, of mutually orthogonal projections such that

$$\begin{aligned} &1)\quad S_e^* S_e = P_{r(e)} \ \forall e \in E, \ \text{and} \\ &2)\quad P_v = \sum_{e \in s^{-1}(v)} S_e S_e^* \ \text{for every vertex } v \in V \text{ which is not a sink.} \end{aligned} \tag{2.1}$$

A finite path μ in G is an element $\mu = e_1 e_2 \cdots e_n \in E^n$ for some $n \in \mathbb{N}$ such that $r(e_i) = s(e_{i+1})$, $i = 1, 2, \cdots, n-1$. For such a path we set

$$S_\mu = S_{e_1} S_{e_2} \cdots S_{e_{n-1}} S_{e_n}.$$

The number $|\mu| = n$ is the length of the path. We consider a vertex v as a path ν of length 0, and set $S_\nu = P_v$ in this case. Let $P_f(G)$ denote the set of finite paths in G. Then

$$\mathcal{A} = \operatorname{Span}\{S_\mu S_\nu^* : \ \mu, \nu \in P_f(G)\} \tag{2.2}$$

is a dense $*$-subalgebra of $C^*(G)$.

Let $F : E \to \mathbb{R}$ be a function. The universal property of $C^*(G)$ guarantees the existence of a one-parameter group α_t^F, $t \in \mathbb{R}$, on $C^*(G)$ such that

$$\alpha_t^F(P_v) = P_v \ \forall v \in V, \quad \text{and} \quad \alpha_t^F(S_e) = e^{iF(e)t} S_e \ \forall e \in E.$$

For $\beta \in \mathbb{R}$ a *β-KMS state* for α^F is a state φ on $C^*(G)$ such that

$$\varphi(ab) = \varphi\left(b \alpha_{i\beta}^F(a)\right)$$

for all $a, b \in \mathcal{A}$, cf. Definition 5.3.1 in [2]. When F is constant 1 the automorphism group $\{\alpha_t^1\}$ is the so-called *gauge action* and we study first the gauge invariant KMS states for α^F, i.e. the KMS states φ for α^F with the additional property that $\varphi \circ \alpha_t^1 = \varphi$ for all $t \in \mathbb{R}$. For this purpose we use the following description of the gauge invariant KMS states. It was obtained by Carlsen and Larsen in [3] when F is strictly positive (in which case all KMS states for α^F are gauge-invariant). The general case follows from Theorem 2.8 in [14].

Let B be a non-negative matrix over V with the property that $B_{vw} > 0$ iff there is an edge in G from v to w. A vector $\psi \in [0, \infty)^V$ is *almost harmonic* for B (or *almost B-harmonic*) when

$$\sum_{w \in V} B_{vw} \psi_w = \psi_v \tag{2.3}$$

for every vertex $v \in V$ which is not a sink, and *normalized* when $\sum_{v \in V} \psi_v = 1$. When the identity (2.3) holds for all $v \in V$ we say that ψ is *harmonic* for B (or *B-harmonic*). Thus an almost B-harmonic vector ψ is B-harmonic if and only if $\psi_s = 0$ for every sink $s \in V$. For $\beta \in \mathbb{R}$, consider the matrix $A(\beta) = (A(\beta)_{vw})$ over V defined such that

$$A(\beta)_{vw} \;=\; \sum_{e \in vEw} e^{-\beta F(e)},$$

where $vEw = s^{-1}(v) \cap r^{-1}(w)$. For a finite path $\mu = e_1 e_2 \cdots e_n$ in G, set

$$F(\mu) = F(e_1) + F(e_2) + \cdots + F(e_n).$$

Lemma 2.1. *(See [3,14].) For every normalized $A(\beta)$-almost harmonic vector ψ there is a unique gauge invariant β-KMS state ω_ψ for α^F such that*

$$\omega_\psi\left(S_\mu S_\nu^*\right) = \delta_{\mu,\nu} e^{-\beta F(\mu)} \psi_{r(\mu)} \tag{2.4}$$

for every pair μ, ν of finite paths in G. Furthermore, every gauge invariant β-KMS state for α^F arises from a normalized $A(\beta)$-almost harmonic vector in this way.

By Lemma 2.1 the study of the gauge invariant KMS states becomes a study of normalized almost harmonic vectors for the family $A(\beta)$, $\beta \in \mathbb{R}$, of non-negative matrices over V.

3. Almost harmonic vectors for a non-negative matrix

Let B be a non-negative matrix over V with the property that $B_{vw} > 0$ iff there is an edge in G from v to w. We seek to obtain a description of the B-almost harmonic vectors.

We shall need the following well-known lemma, cf. e.g. 6.43 in [17].

Lemma 3.1 *(Riesz decomposition). Let $\psi = (\psi_v)_{v \in V} \in [0, \infty[^V$ be a non-negative vector such that*

$$\sum_{w \in V} B_{vw} \psi_w \leq \psi_v$$

for all $v \in V$. It follows that there are unique non-negative vectors $h, k \in [0, \infty[^V$ such that h is B-harmonic and

$$\psi_v = h_v + \sum_{w \in V} \sum_{n=0}^{\infty} B^n_{vw} k_w \tag{3.1}$$

for all $v \in V$. The vector k is given by

$$k_v = \psi_v - \sum_{w \in V} B_{vw} \psi_w,$$

while

$$h_v = \lim_{n \to \infty} \sum_{w \in V} B^n_{vw} \psi_w.$$

We say that a sink $s \in V$ is B-*summable* when

$$\sum_{n=0}^{\infty} B^n_{vs} < \infty$$

for all $v \in V$. For such a sink we define a vector $\phi^s \in [0, \infty)^V$ by

$$\phi^s_v = \frac{\sum_{n=0}^{\infty} B^n_{vs}}{\sum_{w \in V} \sum_{n=0}^{\infty} B^n_{ws}}.$$

Lemma 3.2. *ϕ^s in an extremal normalized B-almost harmonic vector.*

Proof. The only assertion which may not be straightforward to verify is that ϕ^s is extremal in the set of normalized B-almost harmonic vectors. To show this, consider a B-almost harmonic vector φ with the property that $\varphi \leq \phi^s$. Since

$$B^m_{vw} \varphi_w \leq B^m_{vw} \phi^s_w \ \leq \ \frac{\sum_{n=m}^{\infty} B^n_{vs}}{\sum_{w \in V} \sum_{n=0}^{\infty} B^n_{ws}} \ \to \ 0 \tag{3.2}$$

as $m \to \infty$, it follows that the harmonic part from the Riesz decomposition of φ is zero. Thus

$$\varphi_v = \sum_{w \in V} \sum_{n=0}^{\infty} B^n_{vw} k_w$$

where $k_v = \varphi_v - \sum_{w \in V} B_{vw} \varphi_w$. Note that $k_v = 0$ when v is not a sink since φ is B-almost harmonic, and that $k_{s'} = \varphi_{s'}$ for every sink s'. Note also that $\phi^s_{s'} = 0$ for every sink s' in G other than s. Since $\varphi \leq \phi^s$ it follows that the same is true for φ. Hence

$$\varphi_v = \sum_{n=0}^{\infty} B^n_{vs} \varphi_s = t \phi^s_v,$$

where

$$t = \varphi_s \sum_{w \in V} \sum_{n=0}^{\infty} B^n_{ws}. \qquad \square$$

By combining Lemma 3.1 and Lemma 3.2 we obtain the following

Proposition 3.3. *Let ψ be a normalized B-almost harmonic vector. There are a unique (possibly empty) set $\mathcal{S}$ of summable sinks in G, unique positive numbers $t_s \in \,]0, 1]$, $s \in \mathcal{S}$, and a unique B-harmonic vector h such that*

$$\psi = h + \sum_{s \in \mathcal{S}} t_s \phi^s.$$

We turn to a study of the B-harmonic vectors. For any pair of subsets $E, D \subseteq V$ we let $B^{E,D}$ denote the $E \times D$-matrix obtained by restricting B to $E \times D$, and we set $B^E = B^{E,E}$ for any subset $E \subseteq V$.

Write $v \rightsquigarrow w$ between two vertexes v, w when there is a finite path $\mu = e_1 \cdots e_n$ in G such that $s(e_1) = v$ and $r(e_n) = w$, and $v \sim w$ when $v \rightsquigarrow w$ and $w \rightsquigarrow v$. Then $\sim$ is an equivalence relation since we consider a vertex v as a finite path (of length 0) from v to v. A *component* C in G is an equivalence class in $V/\sim$ such that $B^C \neq 0$. For any collection F of vertexes in G we define the *closure* of F to be the set of vertexes that 'talk' to an element of F, i.e. $v \in \overline{F}$ if and only if there is a vertex $w \in F$ such that $v \rightsquigarrow w$. In contrast the *hereditary closure* of a set F consists of the vertexes $w \in V$ such that $v \rightsquigarrow w$ for some $v \in F$. The hereditary closure will be denoted by $\widehat{F}$.

In the following we denote the spectral radius of a finite matrix A by $\rho(A)$. A component C in G is *B-harmonic* when

a) $\rho\left(B^C\right) = 1$ and
b) $\rho\left(B^{\overline{C}\backslash C}\right) < 1$ if $\overline{C}\backslash C \neq \emptyset$.

This definition, as well as the proof of the following lemma, is inspired by Theorem 4.3 in [6].

Lemma 3.4. *Let C be a B-harmonic component in G. There is a unique normalized B-harmonic vector ϕ^C such that $B^C\phi^C|_C = \phi^C|_C$ and $\phi^C_v \neq 0 \ \Leftrightarrow \ v \in \overline{C}$.*

Proof. Existence: Since $\rho\left(B^C\right) = 1$ it follows from Perron–Frobenius theory that there is a strictly positive vector $x^C \in [0,\infty)^C$ such that $B^C x^C = x^C$. Since $\rho\left(B^{\overline{C}\backslash C}\right) < 1$, the matrix $1^{\overline{C}\backslash C} - B^{\overline{C}\backslash C}$ is invertible and we set

$$\phi^C = \left(1^{\overline{C}\backslash C} - B^{\overline{C}\backslash C}\right)^{-1} B^{\overline{C}\backslash C, C} x^C \ + \ x^C,$$

which is a strictly positive vector in $[0,\infty)^{\overline{C}}$. For any pair of vertexes $v, w \in \overline{C}\backslash C$,

$$\limsup_n \left(B^n_{v,w}\right)^{\frac{1}{n}} \leq \rho\left(B^{\overline{C}\backslash C}\right) < 1,$$

and hence

$$\left(1^{\overline{C}\backslash C} - B^{\overline{C}\backslash C}\right)^{-1} = \sum_{n=0}^{\infty} \left(B^{\overline{C}\backslash C}\right)^n.$$

Using this and that no vertex in C talks to a vertex in $\overline{C}\backslash C$, we find that

$$\begin{aligned}
B^{\overline{C}}\phi^C &= B^{\overline{C}\backslash C}\left(1^{\overline{C}\backslash C} - B^{\overline{C}\backslash C}\right)^{-1} B^{\overline{C}\backslash C, C}x^C + B^{\overline{C}\backslash C, C}x^C + B^C x^C \\
&= B^{\overline{C}\backslash C}\sum_{n=0}^{\infty}\left(B^{\overline{C}\backslash C}\right)^n B^{\overline{C}\backslash C, C}x^C + B^{\overline{C}\backslash C, C}x^C + x^C \\
&= \sum_{n=1}^{\infty}\left(B^{\overline{C}\backslash C}\right)^n B^{\overline{C}\backslash C, C}x^C + B^{\overline{C}\backslash C, C}x^C + x^C \\
&= \sum_{n=0}^{\infty}\left(B^{\overline{C}\backslash C}\right)^n B^{\overline{C}\backslash C, C}x^C + x^C \\
&= \phi^C.
\end{aligned} \tag{3.3}$$

Set $\phi^C_v = 0$ when $v \notin \overline{C}$ and normalize the resulting vector in $[0,\infty)^V$. It follows from (3.3) that ϕ^C is B-harmonic. Since $\phi^C|_C$ is multiple of x^C by construction, it follows that $B^C\phi^C|_C = \phi^C|_C$.

Uniqueness: If ψ is a normalized B-harmonic vector such that $B^C\psi|_C = \psi|_C$ and $\psi_v \neq 0 \Leftrightarrow v \in \overline{C}$, it follows from Perron–Frobenius theory that there is a $\lambda > 0$ such that $\psi_v = \lambda\phi^C_v \ \forall v \in C$. Then $\psi - \lambda\phi^C$ is vector supported in $\overline{C}\backslash C$ such that $B^{\overline{C}\backslash C}(\psi - \lambda\phi^C) = \psi - \lambda\phi^C$. Since $\rho\left(B^{\overline{C}\backslash C}\right) < 1$, it follows first that $\psi = \lambda\phi^C$ and then that $\psi = \phi^C$ because both vectors are normalized. $\square$

The following theorem is equivalent to the Frobenius–Victory theorem stated as Theorem 2.7 in [11].

Theorem 3.5. *Let $\psi \in [0,1]^V$ be a normalized B-harmonic vector. There is a unique collection $\mathcal{C}$ of B-harmonic components in G and positive numbers $t_C \in]0,1]$, $C \in \mathcal{C}$, such that*

$$\psi = \sum_{C\in\mathcal{C}} t_C\phi^C. \tag{3.4}$$

Proof. Set $W = \{v \in V : \ \psi_v > 0\}$. Let $v \in W$. Since $B^n_{vv}\psi_v \leq \psi_v$ for all n, it follows that

$$\limsup_n \left(B^n_{vv}\right)^{\frac{1}{n}} \leq 1.$$

Hence

$$\rho\left(B^W\right) = \sup_{v\in W}\limsup_n \left(B^n_{vv}\right)^{\frac{1}{n}} \leq 1.$$

On the other hand, the fact that $B^W\psi|_W = \psi|_W$ implies that $\rho\left(B^W\right) \geq 1$, and we conclude that

$$\rho\left(B^W\right) = 1. \tag{3.5}$$

Since

$$\rho\left(B^W\right) = \sup_C \rho\left(B^C\right),$$

where we take the supremum over the components of G contained in W, the collection $\mathcal{C}'$ of components C from G such that $C \subseteq W$ and $\rho(B^C) = 1$ is not empty. Order the elements of $\mathcal{C}'$ such that $C \leq C'$ when the elements in C talk to the elements of C'. Let $\mathcal{C}$ be the minimal elements of $\mathcal{C}'$ with respect to this order. Let $D \in \mathcal{C}$. We claim that D is a B-harmonic component, i.e. we assert that

$$\rho\left(B^{\overline{D}\backslash D}\right) < 1.$$

Since $\overline{D} \subseteq W$ it follows from (3.5) that $\rho\left(B^{\overline{D}\backslash D}\right) \leq 1$. If $\rho\left(B^{\overline{D}\backslash D}\right) = 1$, there must be one of G's components, say D', contained in $\overline{D}\backslash D$ such that $\rho\left(B^{D'}\right) = 1$. But then $D' \in \mathcal{C}'$, $D' \neq D$ and $D' \leq D$, contradicting the minimality of D. Hence D is B-harmonic as claimed, and we conclude that $\mathcal{C}$ consists of B-harmonic components.

Let $D \in \mathcal{C}$. Then $B^D\psi|_D \leq \psi|_D$ so it follows from Perron–Frobenius theory that there is $t_D \geq 0$ such that $\psi|_D = t_D\psi^D|_D$. Since $\psi|_D$ and $\psi^D|_D$ are strictly positive, t_D is positive too. Set

$$\eta = \psi - \sum_{D\in\mathcal{C}} t_D\psi^D.$$

We claim that $\eta = 0$. To show this, set $K = \bigcup_{D \in \mathcal{C}} D$, and note that $\eta|_K = 0$. Let H be the hereditary closure of K, i.e. $H = \widehat{K}$. Consider a $D \in \mathcal{C}$. When $v \in (H \backslash K) \cap \overline{D}$, there is a path from (some element of) $D' \subseteq K$ to v and a path from v to (some element of) D. Note that $D' \neq D$ since otherwise v would have to be an element of $D \subseteq K$. But $D' \neq D$ is impossible since D is minimal for the order on $\mathcal{C}'$. Hence $(H \backslash K) \cap \overline{D} = \emptyset$, showing that $\psi^D|_{H \backslash K} = 0$. It follows that $\eta|_{H \backslash K} = \psi_{H \backslash K}$, and hence that $\eta|_H \geq 0$. Let $w \in H$. There is an $l \in \mathbb{N}$ and $v \in K$ such that $B^l_{vw} \neq 0$. Since $B^l \eta = \eta$ we find that $0 = \eta_v = \sum_{u \in V} B^l_{vu} \eta_u \geq B^l_{vw} \eta_w \geq 0$, implying that $\eta_w = 0$. Hence $\eta|_H = 0$. Now note that

$$\rho\left(B^{W \backslash H}\right) < 1 \tag{3.6}$$

since all components D in W with $\rho\left(B^D\right) = 1$ are contained in H. Since

$$\left(B^{W \backslash H} \eta\right)_v = \sum_{w \in W \backslash H} B_{vw} \eta_w = \sum_{w \in V} B_{vw} \eta_w = \eta_v$$

for all $v \in W \backslash H$, it follows from (3.6) that $\eta|_{W \backslash H} = 0$. Thus $\eta = 0$ as claimed and (3.4) follows.

To prove the uniqueness part of the statement let $\mathcal{D}$ be a collection of B-harmonic components in G and s_C, $C \in \mathcal{D}$, positive numbers such that

$$\psi = \sum_{C \in \mathcal{D}} s_C \phi^C .$$

Then $W = \bigcup_{C \in \mathcal{C}} \overline{C} = \bigcup_{C \in \mathcal{D}} \overline{C}$, so when $C \in \mathcal{D}$ there is a $C' \in \mathcal{C}$ such that $C \cap \overline{C'} \neq \emptyset$. It follows that $C \subseteq \overline{C'}$ and that either $C' = C$ or $C \subseteq \overline{C'} \backslash C'$. However, $\rho(B^C) = 1$ while $\rho\left(B^{\overline{C'} \backslash C'}\right) < 1$, and it follows therefore that $C = C'$. In this way we conclude that $\mathcal{D} = \mathcal{C}$. Since the preceding argument shows that $C \cap \overline{C'} = \emptyset$ when C and C' are distinct elements from $\mathcal{C}$, we find that

$$s_C \phi^C|_C = \psi|_C = t_C \phi^C|_C ,$$

and hence that $s_C = t_C$ for all $C \in \mathcal{C}$. $\square$

Corollary 3.6. *The normalized B-harmonic vectors constitute a finite dimensional simplex whose set of extreme points is*

$$\left\{\phi^C : \ C \text{ a } B\text{-harmonic component in } G\right\} .$$

Combining Theorem 3.5 with Proposition 3.3 we obtain the following

Corollary 3.7. *The set of normalized B-almost harmonic vectors constitutes a finite dimensional simplex whose set of extreme points is*

$$\left\{\phi^C : \ C \text{ a } B\text{-harmonic component in } G\right\} \cup \left\{\phi^s : \ s \text{ a } B\text{-summable sink in } G\right\} .$$

4. Gauge invariant KMS states

It follows from Lemma 2.1 and Corollary 3.7 that the gauge invariant β-KMS states for α^F are determined by the $A(\beta)$-harmonic components and the $A(\beta)$-summable sinks. In this section we complete the description of the gauge invariant KMS states for $\beta \neq 0$ by finding the $A(\beta)$-harmonic components and the $A(\beta)$-summable sinks for each $\beta \in \mathbb{R}\backslash\{0\}$.[1]

4.1. $A(\beta)$-harmonic components

A *loop* in G is a finite path $\mu = e_1 e_2 \cdots e_n$ (of positive length, i.e. $n \geq 1$) such that $s(e_1) = r(e_n)$. If a component C only contains a single loop, we call it *circular*.

Lemma 4.1. *Let $C \subseteq V$ be a component. The function*

$$\mathbb{R} \ni \beta \mapsto \rho\left(A(\beta)^C\right)$$

is log-convex and continuous.

Proof. Since C is a component there is a loop in C, of length p, say. Let v be a vertex on this loop. It follows that $\log \rho\left(A(\beta)^C\right) \geq \frac{1}{p}\log\left(A(\beta)^C\right)^p_{vv}$, showing that the logarithm of the function we consider takes finite values for all β. Its continuity follows therefore from its log-convexity which is established as follows. Let $v \in C$ and $\beta, \beta' \in \mathbb{R}$, $t \in [0,1]$. For each $n \in \mathbb{N}$ let $vE^n v$ denote the set of paths of length n from v back to itself. Then

$$\left(A(t\beta + (1-t)\beta')^C\right)^n_{vv} = \sum_{\mu \in vE^n v} e^{-(t\beta+(1-t)\beta')F(\mu)} = \sum_{\mu \in vE^n v} \left(e^{-\beta F(\mu)}\right)^t \left(e^{-\beta' F(\mu)}\right)^{1-t}.$$

Then Hölder's inequality shows that

$$\left(A(t\beta + (1-t)\beta')^C\right)^n_{vv} \leq \left(\left(A(\beta)^C\right)^n_{vv}\right)^t \left(\left(A(\beta')^C\right)^n_{vv}\right)^{1-t}.$$

It follows that

$$\rho\left(A(t\beta + (1-t)\beta')^C\right) = \limsup_n \left(\left(A(t\beta + (1-t)\beta')^C\right)^n_{vv}\right)^{\frac{1}{n}}$$

is dominated by the product

$$\rho\left(A(\beta)^C\right)^t \rho\left(A(\beta')^C\right)^{1-t},$$

which is what we needed to prove. □

Lemma 4.2. *Let C be a component in G which is not circular.*

i) *If $F(\mu) > 0$ for all loops μ in C, there is a unique $\beta_0 \in \mathbb{R}$ such that $\rho(A(\beta_0)^C) = 1$. This β_0 is positive and $\rho\left(A(\beta)^C\right) < 1$ if and only if $\beta > \beta_0$.*
ii) *If $F(\mu) < 0$ for all loops μ in C, there is a unique $\beta_0 \in \mathbb{R}$ such that $\rho(A(\beta_0)^C) = 1$. This β_0 is negative and $\rho\left(A(\beta)^C\right) < 1$ if and only if $\beta < \beta_0$.*
iii) *In all other cases, i.e. if $F(\mu) = 0$ for some loop in C or there are loops μ_1, μ_2 in C such that $F(\mu_1) < 0 < F(\mu_2)$, it follows that $\rho\left(A(\beta)^C\right) > 1$ for all $\beta \in \mathbb{R}$.*

[1] We could have handled the case $\beta = 0$ here also, but it does simplify things a little when $\beta \neq 0$, and we will have to consider the $\beta = 0$ case separately for other reasons anyway.

Proof. Some of the following arguments have appeared in [14]. i): We claim that $\beta \mapsto \rho(A(\beta)^C)$ is strictly decreasing. To see this, set

$$a = \min \{F(\mu) : \ \mu \text{ is a loop in } C \text{ of length } |\mu| \leq \#C\} .$$

Consider $\beta' < \beta$ and a loop μ in C of length n. Then $\mu = \mu_1\mu_2 \cdots \mu_m$, where each μ_i is a loop in C of length $\leq \#C$, and

$$e^{-\beta' F(\mu)} e^{\beta F(\mu)} = \prod_j e^{(\beta-\beta')F(\mu_j)} \geq e^{m(\beta-\beta')a} \geq e^{\frac{n}{\#C}(\beta-\beta')a}.$$

Summing over all loops of length n starting and ending at the same vertex v in C, it follows first that

$$\left(A(\beta')^C\right)^n_{vv} \ \geq \ e^{\frac{n}{\#C}(\beta-\beta')a} \left(A(\beta)^C\right)^n_{vv},$$

and then that

$$\rho\left(A(\beta')^C\right) = \limsup_n \left(\left(A(\beta')^C\right)^n_{vv}\right)^{\frac{1}{n}} \geq \rho\left(A(\beta)^C\right) e^{\frac{1}{\#C}(\beta-\beta')a} > \rho\left(A(\beta)^C\right).$$

This proves the claim. Note that $A(0)^C$ is the adjacency matrix of the subgraph H of G whose vertex set is C. This is a finite strongly connected graph and it is well-known, and easy to show, that $\rho(A(0)^C) > 1$ because H by assumption consists of more than a single loop. In view of Lemma 4.1 it suffices now to show that $\lim_{\beta\to\infty} \rho(A(\beta)^C) = 0$. To this end note that any path in H of length $\geq \#C$ must visit at least one vertex twice. It follows that for any path $\mu \in P_f(H)$ of length n with $r(\mu) = s(\mu)$ there is a finite collection

$$\{\nu_1, \nu_2, \cdots, \nu_N\} \subseteq \{\mu \in P_f(H) : \ 1 \leq |\mu| \leq \#C, \ s(\mu) = r(\mu)\}$$

such that $N \geq \frac{n}{\#C}$ and

$$F(\mu) = \sum_{j=1}^{N} F(\nu_j) \geq Na \geq \frac{na}{\#C}.$$

Let $\beta > 0$ and $v \in C$. Then

$$A(\beta)^n_{vv} = \sum_{\mu\in vE^n v} e^{-\beta F(\mu)} \leq A(0)^n_{vv} e^{-\frac{\beta a n}{\#C}}.$$

Hence

$$\rho\left(A(\beta)^C\right) = \limsup_n \left(\left(A(\beta)^C\right)^n_{vv}\right)^{\frac{1}{n}} \leq \rho\left(A(0)^C\right) e^{-\frac{\beta a}{\#C}}.$$

Since $a > 0$, it follows that $\lim_{\beta\to\infty} \rho(A(\beta)^C) = 0$.

The proof of ii) is analogous to that of i).

iii): Assume first that $F(\mu) = 0$ for some loop in C. Since we assume that C is not circular, there is a path ν such that $|\nu| = m|\mu|$ for some $m \in \mathbb{N}$, $s(\nu) = r(\nu) = s(\mu)$ and ν is not the composition of m copies of μ. It follows that, with $v = s(\mu)$,

$$\left(A(\beta)^C\right)^{nm|\mu|}_{vv} \geq \left(\left(A(\beta)^C\right)^{m|\mu|}_{vv}\right)^n \geq (e^{-\beta m F(\mu)} + e^{-\beta F(\nu)})^n = (1 + e^{-\beta F(\nu)})^n$$

for all $n \in \mathbb{N}$, showing that

$$\rho\left(A(\beta)^C\right) \geq \left(1+e^{-\beta F(\nu)}\right)^{\frac{1}{m|\mu|}} > 1$$

for all $\beta \in \mathbb{R}$. Assume then that there are loops μ_1, μ_2 in C such that $F(\mu_1) < 0 < F(\mu_2)$. We may assume that μ_1 and μ_2 start at the same vertex v, if necessary after a modification of μ_1 or μ_2. Then

$$\left(A(\beta)^C\right)^{n|\mu_1||\mu_2|}_{vv} \geq \max\left\{e^{-\beta n|\mu_2|F(\mu_1)}, e^{-\beta n|\mu_1|F(\mu_2)}\right\}$$

for all $n \in \mathbb{N}$, proving that

$$\rho\left(A(\beta)^C\right) \geq \max\left\{e^{-\beta\frac{F(\mu_1)}{|\mu_1|}}, e^{-\beta\frac{F(\mu_2)}{|\mu_2|}}\right\} > 1$$

for all $\beta \neq 0$. This completes the proof because $\rho\left(A(0)^C\right) > 1$ since C is not circular. $\square$

Lemma 4.3. *Let C be a circular component consisting of the vertexes in the loop μ. Then*

$$\rho\left(A(\beta)^C\right) = e^{-\beta\frac{F(\mu)}{|\mu|}}$$

for all $\beta \in \mathbb{R}$.

Proof. Left to the reader. $\square$

Let C be a component. It follows from Lemma 4.2 and Lemma 4.3 that when $F(\mu) > 0$ for every loop μ in C, or $F(\mu) < 0$ for every loop in C, there is a unique number $\beta_C \in \mathbb{R}$ such that

$$\rho\left(A(\beta_C)^C\right) = 1.$$

Definition 4.4. A non-circular component C in G is a *KMS component of positive type* when

i) $F(\mu) > 0$ for every loop μ in $\overline{C}$, and
ii) $\beta_{C'} < \beta_C$ for every component C' in $\overline{C}\backslash C$, if any.

Similarly, a non-circular component C in G is a *KMS component of negative type* when

i) $F(\mu) < 0$ for every loop μ in $\overline{C}$, and
ii) $\beta_C < \beta_{C'}$ for every component C' in $\overline{C}\backslash C$, if any.

Lemma 4.5.

i) *Let $\beta > 0$. A non-circular component C is $A(\beta)$-harmonic if and only if C is a KMS component of positive type and $\beta_C = \beta$.*
ii) *Let $\beta < 0$. A non-circular component C is $A(\beta)$-harmonic if and only if C is a KMS component of negative type and $\beta_C = \beta$.*

Proof. The proofs of the two cases are identical and we consider here only case i): By definition, C is $A(\beta)$-harmonic if and only if $\rho\left(A(\beta)^C\right) = 1$ and $\rho\left(A(\beta)^{\overline{C}\backslash C}\right) < 1$. In view of Lemma 4.2 the first condition is equivalent to $F(\mu)$ being strictly positive for every loop μ in C and that $\beta_C = \beta$. Note that $\rho\left(A(\beta_C)^{\overline{C}\backslash C}\right) = 0$ when $\overline{C}\backslash C$ is non-empty, but does not contain any components, while

$$\rho\left(A(\beta_C)^{\overline{C}\backslash C}\right) = \max\left\{\rho\left(A(\beta_C)^{C'}\right) : \ C' \text{ a component in } \overline{C}\backslash C\right\}$$

otherwise. In view of i) in Lemma 4.2 and Lemma 4.3 this shows that the second condition,

$$\rho\left(A(\beta_C)^{\overline{C}\backslash C}\right) < 1,$$

holds if and only if $F(\mu) > 0$ for every loop μ in $\overline{C}\backslash C$ and $\beta_{C'} < \beta_C$ for every component in $\overline{C}\backslash C$. □

We consider then the circular components.

Definition 4.6. A circular component C in G is a *KMS component of positive type* when

i) $F(\nu) = 0$ for the loop ν in C,
ii) $F(\mu) > 0$ for all loops μ in $\overline{C} \setminus C$, if any.

Similarly, a circular component C in G is a *KMS component of negative type* when

i) $F(\nu) = 0$ for the loop ν in C, and
ii) $F(\mu) < 0$ for all loops μ in $\overline{C} \setminus C$, if any.

Unlike non-circular components, a circular component C can be a KMS component of both positive and negative types. This occurs when there are no loops in $\overline{C}\backslash C$.

Let C be a circular component. Assume that C is a KMS component of positive type. If there are no components in $\overline{C}\backslash C$, it follows $\rho\left(A(\beta)^{\overline{C}\backslash C}\right) = 0$ for all $\beta \in \mathbb{R}$ and we set $I_C = \mathbb{R}$ in this case. Otherwise, set $I_C =]\beta_C, \infty[$, where

$$\beta_C = \max\left\{\beta_{C'} : \ C' \text{ a component in } \overline{C}\backslash C\right\}.$$

Assume then that C is a KMS component of negative type. If there are no components in $\overline{C}\backslash C$, we set $I_C = \mathbb{R}$. Otherwise, set $I_C =]-\infty, \beta_C[$, where

$$\beta_C = \min\left\{\beta_{C'} : \ C' \text{ a component in } \overline{C}\backslash C\right\}.$$

In analogy with Lemma 4.5 we have the following.

Lemma 4.7.

i) *Let $\beta > 0$. A circular component C is $A(\beta)$-harmonic if and only if C is a KMS component of positive type and $\beta \in I_C$.*
ii) *Let $\beta < 0$. A circular component C is $A(\beta)$-harmonic if and only if C is a KMS component of negative type and $\beta \in I_C$.*

Proof. Basically the same as for Lemma 4.5. □

4.2. $A(\beta)$-summable sinks

Definition 4.8. A sink s in G is a *KMS sink of positive type* when $F(\mu) > 0$ for every loop μ in $\overline{\{s\}}$, if any, and a *KMS sink of negative type* when $F(\mu) < 0$ for every loop μ in $\overline{\{s\}}$, if any.

When there are no loops in $\overline{\{s\}}$ we set $I_s = \mathbb{R}$. When s is a KMS sink of positive type with components in $\overline{\{s\}}$, we set $I_s =]\beta_s, \infty[$ where

$$\beta_s = \max\left\{\beta_{C'} : \ C' \text{ a component in } \overline{\{s\}}\right\}.$$

Similarly, when s is a KMS sink of negative type with components in $\overline{\{s\}}$, we set $I_s =]-\infty, \beta_s[$ where

$$\beta_s = \min\left\{\beta_{C'} : \ C' \text{ a component in } \overline{\{s\}}\right\}.$$

Lemma 4.9.

i) *Let $\beta > 0$. A sink s in G is $A(\beta)$-summable if and only if s is a KMS sink of positive type and $\beta \in I_s$.*
ii) *Let $\beta < 0$. A sink s in G is $A(\beta)$-summable if and only if s is a KMS sink of negative type and $\beta \in I_s$.*

Proof. Left to the reader. □

4.3. The gauge invariant β-KMS states, $\beta \neq 0$

For $\beta \in \mathbb{R}\backslash\{0\}$, let $\mathcal{C}(\beta)$ be the set of non-circular KMS components C such that $\beta_C = \beta$, and $\mathcal{Z}(\beta)$ the set of circular KMS components D such that $\beta \in I_D$. Let $\mathcal{S}(\beta)$ be the set of KMS sinks s with $\beta \in I_s$. We can then summarize our findings with regard to the gauge invariant KMS states as follows.

Theorem 4.10. *Let $\beta \in \mathbb{R}\backslash\{0\}$. For every gauge invariant β-KMS state φ for α^F there are unique functions $f : \mathcal{C}(\beta) \to [0, 1]$, $g : \mathcal{Z}(\beta) \to [0, 1]$ and $h : \mathcal{S}(\beta) \to [0, 1]$ such that $\sum_C f(C) + \sum_D g(D) + \sum_s h(s) = 1$ and*

$$\varphi(S_\mu S_\nu^*) = \delta_{\mu,\nu} e^{-\beta F(\mu)} \phi_{r(\mu)}$$

for all finite paths μ, ν, where $\phi \in [0, \infty)^V$ is the vector

$$\phi_v = \sum_{C \in \mathcal{C}(\beta)} f(C)\phi_v^C + \sum_{D \in \mathcal{Z}(\beta)} g(D)\phi_v^D + \sum_{s \in \mathcal{S}(\beta)} h(s)\phi_v^s.$$

5. Including the KMS states that are not gauge invariant

To handle KMS states that are not gauge invariant we draw on the results of Neshveyev, [8]. For this it is necessary to introduce the groupoid picture of $C^*(G)$.

Originally graph C^*-algebras were introduced using groupoids, [7], but only for row-finite graphs without sinks. For general graphs the realization as a groupoid C^*-algebra was obtained by A. Paterson in [9]. To describe the groupoid for a general graph, possibly infinite but countable, let $P_f(G)$ and $P(G)$ denote the set of finite and infinite paths in G, respectively. The range and source maps, r and s on edges, extend in the natural way to $P_f(G)$; the source map also to $P(G)$. A vertex $v \in V$ will be considered as a finite path of length 0 and we set $r(v) = s(v) = v$ when v is considered as an element of $P_f(G)$. Let V_∞ be the set of vertexes v that are either sinks, or infinite emitters in the sense that $s^{-1}(v)$ is infinite. The unit space Ω_G of $\mathcal{G}$ is the union $\Omega_G = P(G) \cup Q(G)$, where

$$Q(G) = \{p \in P_f(G) : \ r(p) \in V_\infty\}$$

is the set of finite paths that terminate at a vertex in V_∞. In particular, $V_\infty \subseteq Q(G)$ because vertexes are considered to be finite paths of length 0. For any $p \in P_f(G)$, let $|p|$ denote the length of p. When $|p| \geq 1$, set

$$Z(p) = \{q \in \Omega_G : \ |q| \geq |p|, \ q_i = p_i, \ i = 1, 2, \cdots, |p|\},$$

and

$$Z(v) = \{q \in \Omega_G : \ s(q) = v\}$$

when $v \in V$. When $\nu \in P_f(G)$ and F is a finite subset of $P_f(G)$, set

$$Z_F(\nu) = Z(\nu) \backslash \left(\bigcup_{\mu \in F} Z(\mu) \right). \tag{5.1}$$

The sets $Z_F(\nu)$ form a basis of compact and open subsets for a locally compact Hausdorff topology on Ω_G.[2] When $\mu \in P_f(G)$ and $x \in \Omega_G$, we can define the concatenation $\mu x \in \Omega_G$ in the obvious way when $r(\mu) = s(x)$. The groupoid $\mathcal{G}$ consists of the elements in $\Omega_G \times \mathbb{Z} \times \Omega_G$ of the form

$$(\mu x, |\mu| - |\mu'|, \mu' x),$$

for some $x \in \Omega_G$ and some $\mu, \mu' \in P_f(G)$. The product in $\mathcal{G}$ is defined by

$$(\mu x, |\mu| - |\mu'|, \mu' x)(\nu y, |\nu| - |\nu'|, \nu' y) = (\mu x, \ |\mu| + |\nu| - |\mu'| - |\nu'|, \nu' y),$$

when $\mu' x = \nu y$, and the involution by $(\mu x, |\mu| - |\mu'|, \mu' x)^{-1} = (\mu' x, |\mu'| - |\mu|, \mu x)$. To describe the topology on $\mathcal{G}$, let $Z_F(\mu)$ and $Z_{F'}(\mu')$ be two sets of the form (5.1) with $r(\mu) = r(\mu')$. The topology we shall consider has as a basis the sets of the form

$$\{(\mu x, |\mu| - |\mu'|, \mu' x) : \ \mu x \in Z_F(\mu), \ \mu' x \in Z_{F'}(\mu')\}. \tag{5.2}$$

With this topology $\mathcal{G}$ becomes an étale second countable locally compact Hausdorff groupoid and we can consider the reduced C^*-algebra $C_r^*(\mathcal{G})$ as in [10]. As shown by Paterson in [9] there is an isomorphism $C^*(G) \to C_r^*(\mathcal{G})$ which sends S_e to 1_e, where 1_e is the characteristic function of the compact and open set

$$\{(ex, 1, r(e)x) : \ x \in \Omega_G\} \ \subseteq \ \mathcal{G},$$

and P_v to 1_v, where 1_v is the characteristic function of the compact and open set

$$\{(vx, 0, vx) \ : \ x \in \Omega_G\} \ \subseteq \ \mathcal{G}.$$

In the following we use the identification $C^*(G) = C_r^*(\mathcal{G})$ and identify Ω_G with the unit space of $\mathcal{G}$ via the embedding $\Omega_G \ni x \mapsto (x, 0, x)$. In this way we get a canonical embedding $C(\Omega_G) \subseteq C^*(G)$ and there is a conditional expectation $P : C^*(G) \to C(\Omega_G)$ defined such that

$$P(f)(x) = f(x, 0, x)$$

when $f \in C_c(\mathcal{G})$, cf. [10]. This conditional expectation can be used to characterize the gauge invariant KMS states because it follows from Theorem 2.2 in [13] that a KMS state for α^F is gauge invariant if and only if it factorizes through P.

[2] Since we here deal with finite graphs where there are no infinite emitters, the topology has as an alternative basis the sets $Z(\nu)$, corresponding to $Z_F(\nu)$ with $F = \emptyset$.

To describe the automorphism group α^F in the groupoid picture we define a continuous homomorphism $c_F : \mathcal{G} \to \mathbb{R}$ by

$$c_F(ux, |u| - |u'|, u'x) = F(u) - F(u').$$

The automorphism group α^F on $C_r^*(\mathcal{G})$ is then defined such that

$$\alpha_t^F(f)(\gamma) = e^{itc_F(\gamma)} f(\gamma)$$

when $f \in C_c(\mathcal{G})$, cf. [10].

Thanks to this picture of $C^*(G)$ and α^F, and because we consider finite graphs in this paper, we can draw on the results of Neshveyev, [8], to obtain a decomposition of the KMS states into those that are gauge invariant and those that are not. Since the groupoid $\mathcal{G}$ has the additional properties required in Section 2 of [12] we can use the description obtained in Theorem 2.4 of [12] when $\beta \neq 0$. Of the β-KMS states considered in Theorem 2.4 in [12], it is only those of the form $\omega_{\mathcal{O}}^{\varphi}$ which may not factor through P. Here $\mathcal{O}$ is an orbit in Ω_G under the canonical action of the groupoid $\mathcal{G}$ on its unit space, and $\mathcal{O}$ must be consistent and β-summable for $\omega_{\mathcal{O}}^{\varphi}$ to be defined. Furthermore, the formula for $\omega_{\mathcal{O}}^{\varphi}$ shows that it is only if the points in $\mathcal{O}$ have non-trivial isotropy group in $\mathcal{G}$ that $\omega_{\mathcal{O}}^{\varphi}$ does not factor through P.

Note that the isotropy group $\mathcal{G}_x^x \subseteq \mathcal{G}$ of an element $x \in \Omega_G$ is trivial unless x is an infinite path in G which is pre-period. Its orbit under $\mathcal{G}$ is then the orbit of an infinite periodic path. We may therefore assume that there is a loop δ in G such that $x = \delta^\infty \in P(G)$. Then

$$\mathcal{G}_x^x = \{(x, kp, x) : \ k \in \mathbb{Z}\},$$

where p is the period of δ^∞. We may assume that $p = |\delta|$ and find then that $c_F(x, kp, x) = kF(\delta)$. It follows that the $\mathcal{G}$-orbit $\mathcal{G}x$ is consistent in the sense used in [12] if and only if $F(\delta) = 0$. If the component of G containing δ contains a second loop, there will be another loop δ' in G starting and ending at the same vertex as δ. Then

$$x_n = \delta^n \delta' \delta^\infty, \ \ n \in \mathbb{N},$$

are distinct elements in $\mathcal{G}x$, and when we use the notation from [12], we have that

$$l_x(x_n) = e^{-F(\delta')}.$$

This shows that

$$\sum_{z \in \mathcal{G}x} l_x(z)^\beta = \infty$$

for all $\beta \in \mathbb{R}$, and we conclude therefore that $\mathcal{G}x$ is not β-summable for any $\beta \in \mathbb{R}$. It follows that the only $\mathcal{G}$-orbits of elements with non-trivial isotropy groups which can be both consistent and β-summable in the sense of [12], are the $\mathcal{G}$-orbits of a periodic infinite path lying in a circular component consisting of a loop δ with $F(\delta) = 0$. On the other hand, for such an infinite path x the corresponding $\mathcal{G}$-orbit will be β-summable if and only if

$$\sum_{\mu \in E_\delta^* s(x)} e^{-\beta F(\mu)} < \infty, \tag{5.3}$$

where $E_\delta^* s(x)$ denotes the set of finite paths μ in G that terminate at $s(x) \in V$ and do not contain δ. Note that (5.3) will hold if and only if C is a circular KMS component with $\beta \in I_C$. In this case the β-KMS state

$\omega^{\varphi}_{\mathcal{O}}$ is defined for every state φ on $C^*(\mathcal{G}^x_x)$, but it will only be extremal when φ is a pure state. By using the identification $C^*(\mathcal{G}^x_x) = C(\mathbb{T})$ this means that the extremal β-KMS states occurring in Theorem 2.4 in [12] that are not gauge invariant arise from a number $\lambda \in \mathbb{T}$, considered as a pure state on $C(\mathbb{T})$, and a component C of zero type with $\beta \in I_C$. We will denote this extremal β-KMS state by ω^{λ}_C. The formula for this state, as it was given in [12], becomes

$$\omega^{\lambda}_C(f) = \left(\sum_{\nu \in E^*_\delta s(x)} e^{-\beta F(\nu)} \right)^{-1} \sum_{k \in \mathbb{Z}} \sum_{\mu \in E^*_\delta s(x)} \lambda^k e^{-\beta F(\mu)} f(\mu x, kp, \mu x) \tag{5.4}$$

when $f \in C_c(\mathcal{G})$. A general state φ on $C(\mathbb{T})$ is given by integration against a Borel probability measure μ on $\mathbb{T}$ and the corresponding β-KMS state $\omega^{\varphi}_{\mathcal{O}}$ from [12], which we in the present setting will denote by ω^{μ}_C, is then given as an integral

$$\omega^{\mu}_C(a) = \int_{\mathbb{T}} \omega^{\lambda}_C(a)\ d\mu(\lambda). \tag{5.5}$$

The conclusions we need here can then be summarized in the following way.

Lemma 5.1. *Let $\beta \in \mathbb{R}\backslash\{0\}$. For every β-KMS state φ for α^F there is a Borel probability measure ν on Ω_G, Borel probability measures μ_D, $D \in \mathcal{Z}(\beta)$, on $\mathbb{T}$ and numbers t and t_D, $D \in \mathcal{Z}(\beta)$, in $[0,1]$ such that $t + \sum_{D \in \mathcal{Z}(\beta)} t_D = 1$ and*

$$\varphi(a) = t \int_{\Omega_G} P(a)\ d\nu + \sum_{D \in \mathcal{Z}(\beta)} t_D \omega^{\mu_D}_D(a). \tag{5.6}$$

The numbers t and t_D are uniquely determined by φ, as are the Borel probability measures μ_D with $t_D > 0$.

The measure ν in Lemma 5.1 has certain properties which reflect that φ is a KMS state, and they can be found in [12], but what matters here is only that

$$a \mapsto \int_{\Omega_G} P(a)\ d\nu$$

is β-KMS state which is gauge invariant. It is therefore a convex combination of the states φ_C, φ_s, φ_D given by the formula (2.4) when the vector ψ occurring there is substituted by the $A(\beta)$-almost harmonic vectors ϕ^C, $C \in \mathcal{C}(\beta)$, ϕ^s, $s \in \mathcal{S}(\beta)$, and ϕ^D, $D \in \mathcal{Z}(\beta)$, respectively. Note that the state φ_D corresponding to a component $D \in \mathcal{Z}(\beta)$ is the same as the state ω^m_D from (5.5) when m is the normalized Lebesgue measure on $\mathbb{T}$. We can therefore now use Theorem 2.4 in [12] and combine Lemma 5.1 with Theorem 4.10 to obtain the following description of the β-KMS states when $\beta \neq 0$.

Theorem 5.2. *For $\beta \in \mathbb{R}\backslash\{0\}$,*

- *let $\mathcal{C}(\beta)$ be the set of non-circular KMS components C in G with $\beta_C = \beta$,*
- *let $\mathcal{S}(\beta)$ be the set of KMS sinks s in G with $\beta \in I_s$, and*
- *let $\mathcal{Z}(\beta)$ be the set of circular KMS components D with $\beta \in I_D$.*

For every β-KMS state φ for α^F there are numbers $\alpha_C \in [0,1]$, $C \in \mathcal{C}(\beta)$, $\alpha_s \in [0,1]$, $s \in \mathcal{S}(\beta)$, and $\alpha_D \in [0,1]$, $D \in \mathcal{Z}(\beta)$, and Borel probability measures μ_D, $D \in \mathcal{Z}(\beta)$, on $\mathbb{T}$, such that $\sum_C \alpha_C + \sum_s \alpha_s + \sum_D \alpha_D = 1$, and

$$\varphi = \sum_{C \in \mathcal{C}(\beta)} \alpha_C \varphi_C + \sum_{s \in \mathcal{S}(\beta)} \alpha_s \varphi_s + \sum_{D \in \mathcal{Z}(\beta)} \alpha_D \omega_D^{\mu_D}.$$

The numbers α_C, α_s, α_D are uniquely determined by φ, as are the Borel probability measures μ_D for the components $D \in \mathcal{Z}(\beta)$ with $\alpha_D > 0$.

5.1. Trace states

We need a different approach when $\beta = 0$. Since the 0-KMS states are the trace states of $C^*(G)$ we must determine these.

Let $\mathcal{Z}(0)$ denote the set of circular components C in G with the property that $\overline{C} \backslash C$ does not contain any components, and similarly $\mathcal{S}(0)$ the set of sinks s in G such that $\overline{\{s\}} \backslash \{s\}$ does not contain a component. For every $C \in \mathcal{Z}(0)$ the set $V \backslash \overline{C}$ is hereditary and saturated, and there is a surjective $*$-homomorphism $\pi_C : C^*(G) \to C^*(\overline{C})$, where $\overline{C}$ is considered as a directed graph with vertex set $\overline{C} \subseteq V$ and the edge set $\left\{e \in E : \ s(e), r(e) \in \overline{C}\right\}$, cf. Theorem 4.1 in [1]. Similarly, when $s \in \mathcal{S}(0)$ there is also a surjective $*$-homomorphism $\pi_s : C^*(G) \to C^*(\overline{\{s\}})$, where $\overline{\{s\}}$ is considered as a directed graph with vertex set $\overline{\{s\}} \subseteq V$ and the edge set $\left\{e \in E : \ s(e), r(e) \in \overline{\{s\}}\right\}$.

When $s \in \mathcal{S}(0)$ we let n_s be the number of paths in G terminating at s. When $C \in \mathcal{Z}(0)$ we choose a vertex $v_C \in C$ and set

$$n_C = \# \left\{\mu \in P_f(G) : \ r(\mu) = v_C, \ s(\mu_i) \neq v_C, \ \text{for } i \leq |\mu|\right\},$$

where the condition that $s(\mu_i) \neq v_C$ is negligible when $|\mu| = 0$.

Theorem 5.3. *For every $s \in \mathcal{S}(0)$,*

$$C^*(\overline{\{s\}}) \simeq M_{n_s}(\mathbb{C}),$$

and for every $C \in \mathcal{Z}(0)$,

$$C^*(\overline{C}) \simeq M_{n_C}\left(C(\mathbb{T})\right), \ C \in \mathcal{Z}(0).$$

For every trace state ω on $C^*(G)$ there are unique numbers $\alpha_s \in [0,1]$ and $\alpha_C \in [0,1]$, and trace states ω_s on $C^*(\overline{\{s\}})$ and ω_C on $C^*(\overline{C})$, $s \in \mathcal{S}(0)$, $C \in \mathcal{Z}(0)$, such that

$$\sum_{s \in \mathcal{S}(0)} \alpha_s + \sum_{C \in \mathcal{Z}(0)} \alpha_C = 1$$

and

$$\omega = \sum_{s \in \mathcal{S}(0)} \alpha_s \omega_s \circ \pi_s + \sum_{C \in \mathcal{Z}(0)} \alpha_C \omega_C \circ \pi_C.$$

For the proof of Theorem 5.3 set

$$N = V \backslash \left(\bigcup_{C \in \mathcal{Z}(0)} \overline{C} \cup \bigcup_{s \in \mathcal{S}(0)} \overline{\{s\}} \right).$$

Then N is hereditary and saturated, and the set $\{P_v : \ v \in N\}$ generates an ideal I_N in $C^*(G)$ such that $C^*(G)/I_N \simeq C^*(\tilde{G})$ where $\tilde{G}$ is the graph with vertex set

$$\tilde{V} = \bigcup_{C \in \mathcal{Z}(0)} \overline{C} \cup \bigcup_{s \in \mathcal{S}(0)} \overline{\{s\}}$$

and edge set $\tilde{E} = \{e \in E : \ r(e) \notin N\}$, cf. Theorem 4.1 in [1].

Lemma 5.4. *Let ω be a trace state on $C^*(G)$. Then $\omega(I_N) = 0$.*

Proof. It suffices to show that $\omega(P_v) = 0$ when $v \in N$. To this end consider a loop μ in G with vertexes $v_1, \dots, v_n, v_1$. The Cuntz–Krieger relations (2.1) imply:

$$\begin{aligned}\omega(P_{v_1}) &= \omega\big(\sum_{e \in s^{-1}(v_1)} S_e S_e^*\big) = \sum_{e \in s^{-1}(v_1)} \omega(S_e^* S_e) = \sum_{e \in s^{-1}(v_1)} \omega(P_{r(e)}) \geq \omega(P_{v_2}) \\ &= \sum_{e \in s^{-1}(v_2)} \omega(P_{r(e)}) \geq \omega(P_{v_3}) = \cdots \geq \omega(P_{v_n}) = \sum_{e \in s^{-1}(v_n)} \omega(P_{r(e)}) \geq \omega(P_{v_1}).\end{aligned}$$

Hence we must have equality everywhere, which implies that $\omega(P_{r(e)}) = 0$ if $e \in s^{-1}(v_i)$ for some i, but $e \notin \mu$. It follows from this that $\omega(P_w) = 0$ when

$$w \in \bigcup_{C \in \mathcal{C}} \widehat{C} \setminus \bigcup_{C' \in \mathcal{Z}(0)} C'$$

where $\mathcal{C}$ is the set of components. Hence if s is a sink in G it follows that $\omega(P_s) = 0$ unless $s \in \mathcal{S}(0)$. Consider a vertex $v \in N$. If v is sink, $\omega(P_v) = 0$ and we are done. Otherwise, if $\omega(P_v) > 0$, the Cuntz–Krieger relations (2.1) imply that there is an edge $e_1 \in s^{-1}(v)$ such that $\omega(P_{r(e_1)}) > 0$. Then $r(e_1)$ cannot be a sink and we can find an edge e_2 such that $s(e_2) = r(e_1)$ and $\omega(P_{r(e_2)}) > 0$. We can continue this construction of edges e_i indefinitely so there are $i < i'$ such that $s(e_i) = r(e_{i'})$, and the path $e_i e_{i+1} \cdots e_{i'}$ is contained in a component C. Since $\omega(P_{r(e_i)}) > 0$ this component must be circular and without components in $\overline{C} \backslash C$, which contradicts that $v \in N$. It follows that $\omega(P_v) = 0$. $\square$

For each $C \in \mathcal{Z}(0)$, fix a vertex $v_C \in C$, and set $v_s = s$ for $s \in \mathcal{S}(0)$. For all $v \in \tilde{V}$ and $a \in \mathcal{Z}(0) \cup \mathcal{S}(0)$, we define:

$$N_v^a = \{\mu \in P_f(\tilde{G}) \mid s(\mu) = v \ , \ r(\mu) = v_a \ , \ s(\mu_i) \neq v_a \text{ for } i \leq |\mu|\}$$

where the condition that $s(\mu_i) \neq v_a$ is negligible when $|\mu| = 0$. We define $N^a = \bigcup_{v \in \tilde{V}} N_v^a$ for $a \in \mathcal{Z}(0) \cup \mathcal{S}(0)$.

Lemma 5.5.

$$C^*(\tilde{G}) \simeq \Big(\bigoplus_{s \in \mathcal{S}(0)} M_{\# N^s}(\mathbb{C}) \Big) \oplus \Big(\bigoplus_{C \in \mathcal{Z}(0)} M_{\# N^C}(C(\mathbb{T})) \Big)$$

Proof. For $a \in \mathcal{Z}(0) \cup \mathcal{S}(0)$, let $e_{\alpha,\beta}$, $\alpha, \beta \in N^a$ be the standard matrix units in $M_{N^a}(\mathbb{C}) \simeq M_{\# N^a}(\mathbb{C})$. For $v \in \tilde{V}$, set

$$\tilde{P}_v = \sum_{a \in \mathcal{Z}(0) \cup \mathcal{S}(0)} \sum_{\alpha \in N_v^a} e_{\alpha,\alpha}.$$

Then $\tilde{P}_v$, $v \in \tilde{V}$, are mutually orthogonal projections. For each $f \in \tilde{E}$ such that $s(f) \notin \{v_a : \ a \in \mathcal{Z}(0) \cup \mathcal{S}(0)\}$, set

$$\tilde{S}_f = \sum_{a \in \mathcal{Z}(0) \cup \mathcal{S}(0)} \sum_{\alpha \in N^a_{r(f)}} e_{f\alpha,\alpha}.$$

If $s(f) \in \{v_a : a \in \mathcal{Z}(0) \cup \mathcal{S}(0)\}$, then $s(f) = v_C$ for some $C \in \mathcal{Z}(0)$, and we let μ^C denote the unique shortest path in $\tilde{G}$ with $s(\mu^C) = r(f)$ and $r\left(\mu^C\right) = v_C$. We define an element

$$\tilde{S}_f \in C\left(\mathbb{T}, M_{N^C}(\mathbb{C})\right)$$

such that

$$\tilde{S}_f(z) = z e_{v_C,\mu_C}.$$

It is straightforward to verify that $\tilde{P}_v$, $v \in \tilde{V}$, and $\tilde{S}_f$, $f \in \tilde{E}$, is a Cuntz–Krieger family, i.e. they satisfy (2.1) relative to $\tilde{G}$. Since

$$\tilde{P}_v, \tilde{S}_f \in \Big(\bigoplus_{s \in \mathcal{S}(0)} M_{\#N^s}(\mathbb{C}) \Big) \oplus \Big(\bigoplus_{C \in \mathcal{Z}(0)} M_{\#N^C}(C(\mathbb{T})) \Big)$$

for all $v \in \tilde{V}$ and all $f \in \tilde{E}$, the universal property of $C^*(\tilde{G})$ gives us a canonical $*$-homomorphism

$$C^*(\tilde{G}) \ \to \ \Big(\bigoplus_{s \in \mathcal{S}(0)} M_{\#N^s}(\mathbb{C}) \Big) \oplus \Big(\bigoplus_{C \in \mathcal{Z}(0)} M_{\#N^C}(C(\mathbb{T})) \Big).$$

To show that this is an isomorphism, note first that it is surjective because the target algebra is generated as a C^*-algebra by $\tilde{P}_v$, $v \in \tilde{V}$, and $\tilde{S}_f$, $f \in \tilde{E}$. For the injectivity we shall appeal to the gauge-invariant uniqueness theorem, Theorem 2.1 in [1]. For an $a \in \mathcal{S}(0) \cup \mathcal{Z}(0)$, define for each $\omega \in \mathbb{T}$ the unitary:

$$U^a_\omega = \sum_{\alpha \in N^a} \omega^{|\alpha|} e_{\alpha,\alpha}$$

For $s \in \mathcal{S}(0)$ we define an automorphism ψ^s_ω on $M_{\#N^s}(\mathbb{C})$ by $\psi^s_\omega(A) = U^s_\omega A U^s_{\overline{\omega}}$, and for $C \in \mathcal{Z}(0)$ we define an automorphism on $M_{\#N^C}(C(\mathbb{T}))$ by $\psi^C_\omega(f)(z) = U^C_\omega f(\omega^{\#C} z) U^C_{\overline{\omega}}$. It is straightforward to check that:

$$\mathbb{T} \ni \omega \to \psi_\omega := (\bigoplus_{s \in \mathcal{S}} \psi^s_\omega) \oplus (\bigoplus_{C \in \mathcal{Z}(0)} \psi^C_\omega)$$

is an action, and that we for $f \in \tilde{E}$ and $v \in \tilde{V}$ have:

$$\psi_\omega(\tilde{S}_f) = \omega \tilde{S}_f \qquad\qquad \psi_\omega(\tilde{P}_v) = \tilde{P}_v$$

for all $\omega \in \mathbb{T}$. It follows therefore from Theorem 2.1 in [1] that the homomorphism under consideration is injective. □

Proof of Theorem 5.3. Consider $C \in \mathcal{Z}(0)$ and let $C^*(G) \to M_{\#N^C}(C(\mathbb{T}))$ be the surjective $*$-homomorphism obtained by composing the quotient map $C^*(G) \to C^*(\tilde{G})$ with the projection $C^*(\tilde{G}) \to M_{\#N^C}(C(\mathbb{T}))$ obtained from Lemma 5.5. The kernel of this $*$-homomorphism is the same as the kernel of $\pi_C : C^*(G) \to C^*(\overline{C})$, namely the ideal generated by

$$\left\{P_v : \ v \notin \overline{C}\right\}.$$

It follows that $C^*(\overline{C}) \simeq M_{\#N^C}(C(\mathbb{T}))$. In the same way we see that $C^*(\overline{\{s\}}) \simeq M_{\#N^s}(\mathbb{C})$ when $s \in \mathcal{S}(0)$. The statements regarding a trace state ω follow from Lemma 5.4 and Lemma 5.5. □

6. Ground states

To describe the ground states we use again the groupoid picture described in Section 5 in order to adapt the approach from Section 5 in [15] to the present setting. The fixed point algebra of α^F is the C^*-algebra of the open sub-groupoid

$$\mathcal{F} = \{(\mu x, |\mu| - |\mu'|, \mu' x) : \ x \in \Omega_G, \ F(\mu) = F(\mu')\}$$

of $\mathcal{G}$. The conditional expectation

$$Q : C^*(G) \to C_r^*(\mathcal{F})$$

extending the restriction map $C_c(\mathcal{G}) \to C_c(\mathcal{F})$ can be described as a limit:

$$Q(a) = \lim_{R \to \infty} \frac{1}{R} \int_0^R \alpha_t^F(a) \ dt, \tag{6.1}$$

cf. the proof of Theorem 2.2 in [14].

When $x \in \Omega_G$, $z \in P_f(G)$, write $z \subseteq x$ when $1 \leq |z|$ and $x|_{[1,|z|]} = z$ or $|z| = 0$ and $z = s(x)$. An element $x \in \Omega_G$ *has minimal F-weight* when the following holds:

$$z, z' \in P_f(G), \ z \subseteq x, \ r(z') = r(z) \ \Rightarrow \ F(z') \geq F(z) \, .$$

We denote the set of elements in Ω_G with minimal F-weight by $\mathrm{Min}(F, G)$. Then $\mathrm{Min}(F, G)$ is closed in Ω_G and $\mathcal{F}$-invariant in the sense that

$$(x, k, y) \in \mathcal{F}, \ x \in \mathrm{Min}(F, G) \ \Rightarrow \ y \in \mathrm{Min}(F, G).$$

It follows that the reduction $\mathcal{F}|_{\mathrm{Min}(F,G)}$ of $\mathcal{F}$ to $\mathrm{Min}(F, G)$, defined by

$$\mathcal{F}|_{\mathrm{Min}(F,G)} = \{(\mu x, |\mu| - |\mu'|, \mu' x) : \ x \in \Omega_G, \ F(\mu) = F(\mu'), \ \mu x \in \mathrm{Min}(F, G)\} \, ,$$

is a locally compact étale groupoid. Furthermore, there is a surjective $*$-homomorphism

$$R : C_r^*(\mathcal{F}) \to C_r^* \left(\mathcal{F}|_{\mathrm{Min}(F,G)} \right)$$

extending the restriction map $C_c(\mathcal{F}) \to C_c \left(\mathcal{F}|_{\mathrm{Min}(F,G)} \right)$. Now the proof of Theorem 5.3 in [15] can be repeated almost ad verbatim to yield the following.

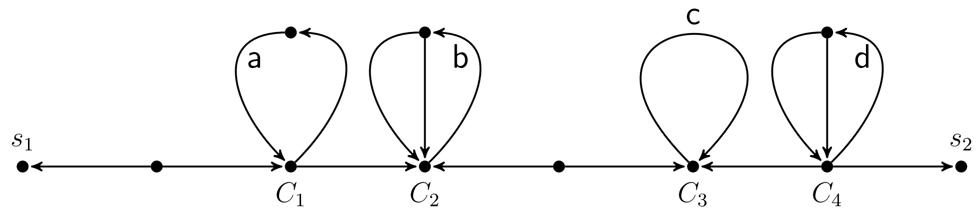
Theorem 6.1. *The map $\omega \mapsto \omega \circ R \circ Q$ is an affine homeomorphism from the state space of $C_r^* \left(\mathcal{F}|_{Min(F,G)} \right)$ onto the ground states of α^F.*

The structure of the C^*-algebra $C_r^* \left(\mathcal{F}|_{\mathrm{Min}(F,G)} \right)$ varies a lot with the choice of F. When F is constant zero, it is equal to $C^*(G)$, and when F is strictly positive it is isomorphic to $\mathbb{C}^n$, where n is the number of sinks in G. If G consists of three edges, e_i, and a vertex v with $r(e_i) = s(e_i) = v$, $i = 1, 2, 3$, and if $F(e_1) = F(e_2) = 0$ while $F(e_3) = 1$, we find that $C^*(G)$ is the Cuntz-algebra O_3 while $C_r^* \left(\mathcal{F}|_{\mathrm{Min}(F,G)} \right)$ is a copy of O_2.

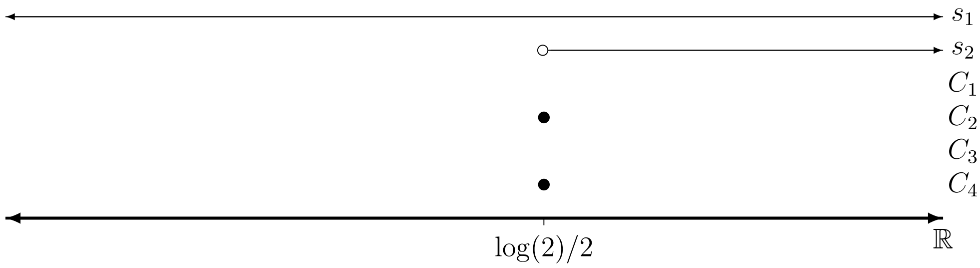
Which of the ground states are weak* limits, for $\beta \to \infty$, of β-KMS states, can be decided by combining Theorem 6.1 with Theorem 5.2. It follows, for instance, that they all are when $F = 1$, while none of them are in the last mentioned example.

7. An example

Consider the following graph G. The two sinks are s_1 and s_2 and there are four components labeled C_1 through C_4. In order to define various functions on the edge set we have labeled four edges a, b, c and d.

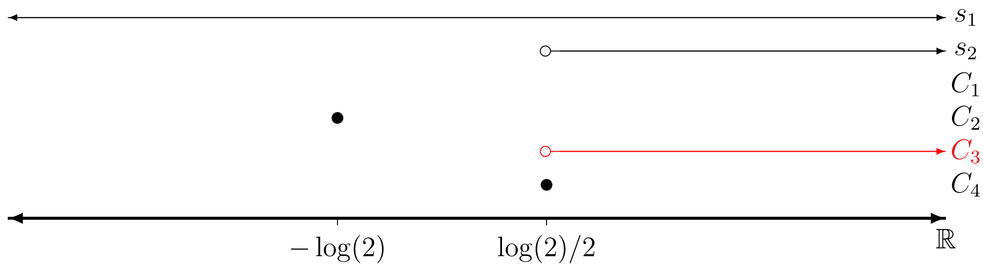


Consider first the gauge action where $F(e) = 1$ for all edges e. The two sinks are both KMS sinks in this case; with intervals $I_{s_1} = \mathbb{R}$ and $I_{s_2} = \left]\frac{\log 2}{2}, \infty\right[$. Of the components it is only C_2 and C_4 that are KMS components, both of positive type and with $\beta_{C_2} = \beta_{C_4} = \frac{\log 2}{2}$. There are three extremal β-KMS states when $\beta = \frac{\log 2}{2}$, coming from s_1, C_2 and C_4, one when $\beta < \frac{\log 2}{2}$, coming from s_1, and two when $\beta > \frac{\log 2}{2}$, coming from s_1 and s_2. This 'KMS spectrum' away from 0 can be described by the following figure.



KMS spectrum ($\beta \neq 0$) for the gauge action on $C^*(G)$.

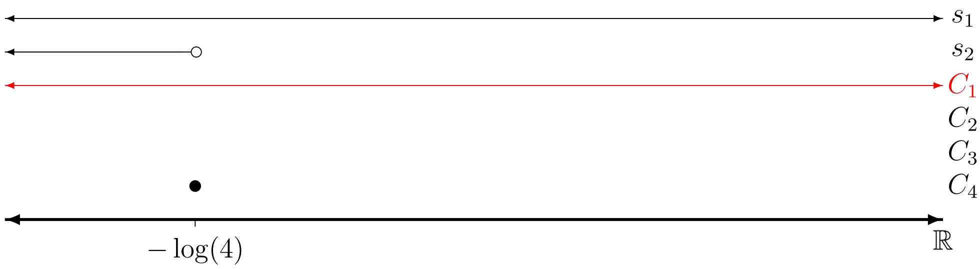
To define a different generalized gauge action, let E be the set of edges in G, and set $F_1(e) = 1$ when $e \in E\backslash\{a, b, c\}$ while $F_1(a) = F_1(b) = -2$ and $F_1(c) = 0$. If we describe the KMS-spectrum for the action α^{F_1} by a diagram as was done for the gauge action, the picture becomes the following. The red line[3] describes the contribution from the circular KMS component C_3 and hence each point on it represents a family of extremal KMS states parametrized by a circle.



KMS spectrum ($\beta \neq 0$) for the generalized gauge action α^{F_1} on $C^*(G)$.

Finally we consider F_2 defined such that $F_2(e) = 1$ when $e \in E\backslash\{a, d\}$, $F_2(a) = -1$ and $F_2(d) = -\frac{3}{2}$. For the generalized gauge action α^{F_2} we find the following KMS spectrum.

[3] For interpretation of the references to color in the text, the reader is referred to the web version of this article.



KMS spectrum ($\beta \neq 0$) for the generalized gauge action α^{F_2} on $C^*(G)$.

The structure of the ground states varies also for the three actions. For the gauge action there are two extremal ground states coming from the sinks, while for the actions α^{F_1} and α^{F_2} there are infinitely many. Concerning α^{F_1} the sinks still contribute two, but the infinite path c^∞ has minimal F_1-weight and contributes a family of extremal ground states naturally parametrized by a circle. The sink s_1 is the only sink which gives rise to an extremal ground state for the action α^{F_2}, but now the loop of period 2 beginning with the edge a is an element of $\mathrm{Min}(F_2, G)$ and gives rise to a family of extremal ground states naturally parametrized by a circle.

The 0-KMS states are of course the same for all three actions. They are the trace states on the algebra, and by using Theorem 5.3 we see that they can be identified with the trace states on $M_2(\mathbb{C}) \oplus M_3(C(\mathbb{T}))$, where the sink s_1 is responsible for the first summand and the component C_1 for the second.

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Equilibrium and ground states from Cayley graphs

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ABSTRACT

We present a general framework for the study of KMS states of generalized gauge actions on the C^*-algebra of a Cayley graph which is pointed by considering the neutral element of the group as a distinguished vertex. We use this framework to give a concrete description of a particular kind of KMS states on the C^*-algebras that we call abelian KMS states. If the group is nilpotent all KMS states are abelian and our analysis gives the full picture in this case. We then describe the ground states that are limits of abelian KMS states when the temperature goes to zero.

1. Introduction

The introduction of graph C^*-algebras more than 20 years ago by Pask, Kumjian, Raeburn and Renault, [10], initiated many exciting developments in C^*-algebra theory and gave opportunities for new connections to other fields of mathematics. One such is the connection to geometric group theory coming from Cayley graphs and the main purpose with this paper is to present a first investigation of the quantum statistical

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models which arise by considering generalized gauge actions on the C^*-algebra of a Cayley graph. From previous work involving infinite graphs it has become clear that the study of KMS states and weights for generalized gauge actions on the C^*-algebra of a infinite graph is closely related to the theory of denumerable Markov chains and random walks, and that results from these fields can provide solutions to some of the questions motivated by the physical interpretations. In many cases, however, these questions turn into problems that are notoriously very difficult, and only little help is offered by the results from other fields of mathematics. This is intriguing because the results in [15] show that infinite graphs offer a richness in the structure of KMS states and weights which can not be realized with finite graphs and which is only paralleled by the constructions made by Bratteli, Elliott and Kishimoto in the 80's, cf. [2]. Furthermore, several of the issues that arise from the operator algebra setting and its interpretation as a model in quantum statistical mechanics have not, or only very marginally been considered from the point of view of Markov chains or random walks and they call for new ideas.

In this paper we set up the general framework for the study of KMS states of generalized gauge actions on the graph C^*-algebra of a Cayley graph, or more precisely the restriction of these actions to the corner of the algebra obtained by considering the neutral element of the group as a distinguished vertex. These states are in one-to-one correspondence with the vectors that are normalized and harmonic for a matrix over the group which depends on the inverse temperature β. For fixed β one can in this way translate the problem of finding the β-KMS states to one which deals with a stochastic matrix and hence with a random walk on the group. This opens up the possibility of exploiting the substantial existing literature on harmonic functions for random walks on groups in the study of KMS states on associated graph C^*-algebras. However, the dependence on the inverse temperature β presents issues for which there are no analogues in the random walk setting, for example the question about the behavior of the equilibrium states as the temperature goes to zero which we study in detail in the present work.

For abelian and more generally nilpotent groups there are results which describe the harmonic vectors of all non-negative matrices over the group that are consistent with the Cayley graph and where the passage to a stochastic matrix is therefore not necessary. This allows us to give a complete description of the KMS and KMS_∞ states for the generalized gauge actions with a potential function defined from a strictly positive function on the set of generators when the graph in question is the Cayley graph of a nilpotent group. The structure depends almost entirely on the abelianization of the group and from the methods we employ it becomes clear that a similar structure is present for arbitrary groups. To make this clear, and because it shows that our results have bearing for all finitely generated groups, we introduce *abelian* KMS and KMS_∞ states in the general setting. They are the KMS and KMS_∞ states that arise from the harmonic vectors that factor through the abelianization of the group. We show that when the abelianization of the group is finite there is a unique abelian KMS state, and when the rank of the abelianization is $n \geq 1$ there is a critical inverse temperature $\beta_0 > 0$ such that there are no abelian β-KMS states when $\beta < \beta_0$, a unique abelian β_0-KMS state and for $\beta > \beta_0$ the simplex of abelian β-KMS states is affinely homeomorphic to the Bauer simplex of Borel probability measures on the $(n-1)$-sphere. Based on this we determine the abelian KMS_∞ states; the ground states in the quantum statistical model that are limits of abelian β-KMS states when β tends to infinity, [5]. The result is that they also form a Choquet simplex affinely homeomorphic to the Bauer simplex of Borel probability measures on the $(n-1)$-sphere. This may be expected given the description of the abelian β-KMS states, but one should bear in mind that it is a priori not even clear that the KMS_∞ states constitute a convex set and it is also not clear, in view of the massive collapse at the critical value β_0, that collapsing does not occur at infinity. The proof that the $(n-1)$-sphere 'survives to infinity' occupies almost half of the paper and involves a great deal of finite dimensional convex geometry.

In a final section we consider two examples; the Heisenberg group and the infinite dihedral group. In both examples we consider a canonical set of generators and the gauge action on the resulting graph C^*-algebras. Since the Heisenberg group is nilpotent we easily find all KMS and KMS_∞ states by using our general results. For the infinite dihedral group, which is not nilpotent, we need to do a bit more work and find that there is only one abelian KMS state and a richer collection of general KMS states. This illustrates that for groups that are not nilpotent the structure of KMS states and KMS_∞ states is more complicated, and the abelian states will only give a (small) part of the picture. For general finitely generated groups, the complete picture is presently way out of reach.

While this work is the first to study KMS states and ground states for actions on the C^*-algebra of Cayley graphs, KMS_∞ states have been investigated in other cases, for example in [3,11,12], following their introduction by Connes and Marcolli in [5]. In many cases the set of all ground states as they are usually defined, e.g. in [1], constitute a much larger set. This is also the case in our setting, where the set of ground states can be identified with the state space of a sub-quotient of the algebra, cf. [16].

2. Generalized gauge actions on a pointed Cayley graph

Given a group G and a finite set Y of generators of G there is a natural way of defining a directed graph $\Gamma = \Gamma(G,Y)$ whose vertexes are the elements of G and with an edge (or arrow) from $g \in G$ to $h \in G$ iff $g^{-1}h \in Y$. This is the *Cayley graph* and it provides the main tool for the geometric study of discrete finitely generated groups. Since the introduction of C^*-algebras from directed graphs, [10], the Cayley graphs have provided a way of associating to a finitely generated discrete group a C^*-algebra $C^*(\Gamma)$ very different from the full or reduced group C^*-algebra usually considered in relation to discrete groups. The algebra is the universal C^*-algebra generated by a set $V(g,s),\ g \in G,\ s \in Y$, of partial isometries such that

$$V(h,t)^*V(g,s) = \begin{cases} 0 & \text{when } h \neq g \text{ or } t \neq s, \\ \sum_{y \in Y} V(gy,y)V(gy,y)^* & \text{when } h = g \text{ and } t = s. \end{cases} \tag{2.1}$$

For $g \in G$ we let P_g denote the projection

$$P_g = \sum_{y \in Y} V(gy, y)V(gy, y)^*.$$

When G is infinite the graph Γ is also infinite and the C^*-algebra $C^*(\Gamma)$ is not unital. But the neutral element e_0 of G defines the canonical unital corner $P_{e_0}C^*(\Gamma)P_{e_0}$ stably isomorphic to $C^*(\Gamma)$, and in this paper we will focus attention to this corner of $C^*(\Gamma)$.

We emphasize that the only condition on Y is that it generates G as a semi-group, i.e. every element of G is a product of elements from Y. For various reasons it is convenient to exclude the case where Y only contains one element. G is a finite cyclic group when this happens, a case we do not exclude when Y contains at least two elements. Thus we make the following standing assumption:

Assumption 2.1. It is assumed that $Y \subseteq G$ is a finite set containing at least two elements and that it generates G as a semi-group.

Under this assumption $C^*(\Gamma)$ and the corner $P_{e_0}C^*(\Gamma)P_{e_0}$ are simple C^*-algebras by Corollary 6.8 of [10].

Let $F : Y \to \mathbb{R}$ be a function. The universal property of $C^*(\Gamma)$ guarantees the existence of a continuous one-parameter group of automorphisms $\gamma_t^F, t \in \mathbb{R}$, on $C^*(\Gamma)$ defined such that

$$\gamma_t^F(V(g, s)) = e^{iF(s)t}V(g, s).$$

Note that γ^F keeps the corner $P_{e_0}C^*(\Gamma)P_{e_0}$ globally invariant and therefore defines a continuous one-parameter group of automorphisms, still denoted by γ_t^F, on $P_{e_0}C^*(\Gamma)P_{e_0}$.

We aim now to identify $P_{e_0}C^*(\Gamma)P_{e_0}$ as a C^*-subalgebra of the Cuntz-algebra O_n where $n = \#Y$, cf. [6]. To simplify notation, set

$$O_Y = O_{\#Y}.$$

Let $R_g, g \in G$, be the right-regular representation of G on $l^2(G)$ and $V_s, s \in Y$, the canonical isometries generating O_Y. Let $1_g \in B(l^2(G))$ be the orthogonal projection onto the subspace of $l^2(G)$ spanned by the characteristic function of $g \in G$. The elements

$$V(g, s) = 1_{gs^{-1}}R_s \otimes V_s$$

in $B(l^2(G)) \otimes O_Y$ satisfy the relations (2.1) and hence they generate a copy of $C^*(\Gamma)$. For $t = (t_1, t_2, \cdots, t_n) \in Y^n$, set $\overline{t} = t_1t_2 \cdots t_n \in G$, and let $V_t \in O_Y$ be the isometry

$$V_t = V_{t_1}V_{t_2} \cdots V_{t_n}.$$

Set $Y^0 = \emptyset$ and $\overline{\emptyset} = e_0,\ V_\emptyset = 1 \in O_Y$. The elements

$$1_h R_{\overline{t}\ \overline{u}^{-1}} \otimes V_t V_u^*, \tag{2.2}$$

where $t \in Y^n,\ u \in Y^m,\ n,m \in \mathbb{N} \cup \{0\},\ h \in G$, span a $*$-subalgebra of $B(l^2(G)) \otimes O_Y$, and since

$$(1_{e_0} \otimes 1)(1_h R_{\overline{t}\ \overline{u}^{-1}} \otimes V_t V_u^*)(1_{e_0} \otimes 1) = \begin{cases} 1_{e_0} \otimes V_t V_u^* & \text{when } h = e_0 \text{ and } \overline{t} = \overline{u} \\ 0, & \text{otherwise,} \end{cases}$$

it follows that the elements $V_t V_u^*$ with $\overline{t} = \overline{u}$ span a $*$-subalgebra of O_Y whose closure, which we denote by $O_Y(G)$, is a copy of $P_{e_0} C^*(\Gamma) P_{e_0}$.

To formulate what the action γ^F looks like in this picture, set $F(\emptyset) = 0$ and

$$F(w) = \sum_{j=1}^{n} F(w_j)$$

when $w = (w_1, w_2, \cdots, w_n) \in Y^n$. It follows from the universal property of O_Y that there is a one-parameter group α^F of automorphisms on O_Y such that

$$\alpha_t^F\,(V_w V_u^*) = e^{it(F(w)-F(u))} V_w V_u^*$$

for all $w, u \in \bigcup_{n=0}^{\infty} Y^n$. Note that α^F leaves $O_Y(G)$ globally invariant and defines a one-parameter group of automorphisms on $O_Y(G)$.

We summarize the preceding considerations with the following

Proposition 2.2. *Let $O_Y(G)$ be the closed span in O_Y of the elements $V_w V_u^*$ with $\overline{w} = \overline{u}$. There is a $*$-isomorphism $\pi : P_{e_0} C^*(\Gamma) P_{e_0} \to O_Y(G)$ such that $\pi \circ \gamma_t^F = \alpha_t^F \circ \pi$ for all $t \in \mathbb{R}$.*

3. KMS measures and harmonic vectors

3.1. KMS states and KMS measures

Let $\beta \in \mathbb{R}$. A state ω on $O_Y(G)$ is a β-*KMS state* for α^F when there is a dense α^F-invariant $*$-subalgebra $\mathcal{A}$ of $O_Y(G)$ consisting of analytic elements for α^F such that

$$\omega(ab) = \omega\left(b \alpha_{i\beta}^F(a)\right) \tag{3.1}$$

for all $a, b \in \mathcal{A}$, cf. [1]. By Proposition 5.3.7 in [1] this condition is independent of $\mathcal{A}$, and since the elements $V_t V_u^*$ with $\overline{t} = \overline{u}$ span a $*$-subalgebra of $O_Y(G)$ consisting of analytic elements for α^F it follows that ω is a β-KMS state if and only if

$$\omega\left(V_{t_1} V_{u_1}^* V_{t_2} V_{u_2}^*\right) = e^{\beta(F(u_1)-F(t_1))} \omega\left(V_{t_2} V_{u_2}^* V_{t_1} V_{u_1}^*\right) \tag{3.2}$$

when $t_1, t_2, u_1, u_2 \in \bigcup_{n=0}^{\infty} Y^n$ and $\overline{t_i} = \overline{u_i},\ i = 1, 2$.

It is well-known that the elements $V_tV_t^*, t \in \bigcup_{n=0}^{\infty} Y^n$, generate a copy of $C\left(Y^{\mathbb{N}}\right)$ inside $O_Y(G)$ and that there is a conditional expectation $E : O_Y \to C\left(Y^{\mathbb{N}}\right)$ with the property that

$$E\left(V_tV_u^*\right) = \begin{cases} V_tV_t^* & \text{when } t = u \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Lemma 3.1. *Let $\beta \in \mathbb{R}$ and let ω be a β-KMS state for α^F on $O_Y(G)$. There is a unique Borel probability measure m on $Y^{\mathbb{N}}$ such that*

$$\omega(a) = \int_{Y^{\mathbb{N}}} E(a) \ \mathrm{d}m \tag{3.4}$$

for all $a \in O_Y(G)$.

Proof. The conclusion can be obtained by combining Proposition 5.6 in [4] with Theorem 2.2 in [15], but in the present setting there is a much shorter proof: Let $l \in \bigcup_{n=1}^{\infty} Y^n$ be an element such that $\overline{l} = e_0$. Since Y contains more than one element by assumption, there is an element $l' \in \bigcup_{n=1}^{\infty} Y^n$ with $\overline{l'} = e_0$ such that l and l' have different first entries. Then $V_lV_l^* + V_{l'}V_{l'}^* \leq 1$ and hence $\omega\left(V_lV_l^*\right) + \omega\left(V_{l'}V_{l'}^*\right) \leq 1$. The KMS condition (3.2) shows that

$$\omega\left(V_{l'}V_{l'}^*\right) = \omega\left(V_{l'}^*\alpha_{i\beta}^F\left(V_{l'}\right)\right) = e^{-\beta F(l')}\omega\left(V_{l'}^*V_{l'}\right) = e^{-\beta F(l')} > 0,$$

implying that $e^{-\beta F(l)} = \omega\left(V_lV_l^*\right) < 1$. Consider then general elements $t, u \in \bigcup_{n=0}^{\infty} Y^n$ with $\overline{t} = \overline{u}$ such that $t \neq u$. It follows from the KMS-condition (3.2) that

$$\omega(V_tV_u^*) = \omega(V_tV_t^*V_tV_u^*) = \omega(V_tV_u^*V_tV_t^*).$$

In particular, $\omega(V_tV_u^*) = 0$ if $V_u^*V_t = 0$. If $V_u^*V_t$ is not zero, there is an element $l \in \bigcup_{n=1}^{\infty} Y^n$ with $\overline{l} = e_0$ such that $u = tl$ or $t = ul$. Then

$$\omega(V_tV_u^*) = \omega\left(\alpha_{i\beta}^F\left(V_tV_u^*\right)\right) = e^{\pm\beta F(l)}\omega(V_tV_u^*),$$

where the sign depends on which of the two cases we are in. From the above we know that $e^{\beta F(l)} \neq 1$ and conclude therefore that $\omega(V_tV_u^*) = 0$ when $t \neq u$. This shows that $\omega = \omega \circ E$ and the statement in the lemma follows then from the Riesz representation theorem. $\square$

Definition 3.2. A Borel probability measure m on $Y^{\mathbb{N}}$ is a β-*KMS measure* for α^F when the state ω defined by (3.4) is a β-KMS state for α^F.

When $t = (t_1, t_2, \cdots, t_n) \in Y^n$ we denote in the following by $tY^{\mathbb{N}}$ the cylinder set

$$tY^{\mathbb{N}} = \left\{ (y_i)_{i=1}^{\infty} \in Y^{\mathbb{N}} : \ y_i = t_i, \ i = 1, 2, \cdots, n \right\},$$

and we set $\emptyset Y^{\mathbb{N}} = Y^{\mathbb{N}}$.

Lemma 3.3. *A Borel probability measure m on $Y^{\mathbb{N}}$ is a β-KMS measure if and only if*

$$e^{\beta F(t)} m\left(tY^{\mathbb{N}}\right) = e^{\beta F(u)} m\left(uY^{\mathbb{N}}\right) \tag{3.5}$$

whenever $t, u \in \bigcup_{n=0}^{\infty} Y^n$ satisfy that $\overline{t} = \overline{u}$.

Proof. Let ω be the state of $O_Y(G)$ defined by (3.4). Let $t_1, u_1, t_2, u_2 \in \bigcup_n Y^n$ such that $\overline{t_i} = \overline{u_i}, i = 1, 2$. Using the relations satisfied by the isometries V_s and (3.3) we find

$$\omega\left(V_{t_1} V_{u_1}^* V_{t_2} V_{u_2}^*\right) = \begin{cases} \omega\left(V_{u_2} V_{u_2}^*\right) & \text{if } t_2 = u_1 x \text{ and } u_2 = t_1 x \text{ for some } x \in \bigcup_{n=0}^{\infty} Y^n, \\ \omega\left(V_{t_1} V_{t_1}^*\right) & \text{if } u_1 = t_2 x \text{ and } t_1 = u_2 x \text{ for some } x \in \bigcup_{n=0}^{\infty} Y^n, \\ 0 & \text{in all other cases,} \end{cases}$$

while

$$\begin{aligned} &\omega\left(V_{t_2} V_{u_2}^* \alpha^F_{i\beta}\left(V_{t_1} V_{u_1}^*\right)\right) = e^{\beta(F(u_1) - F(t_1))} \omega\left(V_{t_2} V_{u_2}^* V_{t_1} V_{u_1}^*\right) \\ &= \begin{cases} e^{\beta(F(u_1) - F(t_1))} \omega\left(V_{u_1} V_{u_1}^*\right) & \text{if } t_1 = u_2 x \text{ and } u_1 = t_2 x \text{ for some } x \in \bigcup_{n=0}^{\infty} Y^n, \\ e^{\beta(F(u_1) - F(t_1))} \omega\left(V_{t_2} V_{t_2}^*\right) & \text{if } u_2 = t_1 x \text{ and } t_2 = u_1 x, \text{ for some } x \in \bigcup_{n=0}^{\infty} Y^n, \\ 0 & \text{in all other cases.} \end{cases} \end{aligned}$$

The two expressions agree for all choices of t_1, u_1, t_2 and u_2 if and only if (3.5) holds for all $t, u \in \bigcup_{n=0}^{\infty} Y^n$ with $\overline{t} = \overline{u}$. Since the elements $V_t V_u^*$ are analytic elements for α^F it follows that ω is a β-KMS state for α^F if and only if (3.5) holds for all $t, u \in \bigcup_{n=0}^{\infty} Y^n$ with $\overline{t} = \overline{u}$. □

Corollary 3.4. *The formula (3.4) gives an affine homeomorphism between the β-KMS measures on $Y^{\mathbb{N}}$ and the β-KMS states for α^F.*

In terms of the canonical generators from Proposition 2.2 the β-KMS state ω corresponding to a β-KMS measure m is given by the formula

$$\omega\left(V_t V_u^*\right) = \begin{cases} m\left(tY^{\mathbb{N}}\right) & \text{when } t = u \\ 0 & \text{otherwise.} \end{cases}$$

3.2. KMS measures and harmonic vectors

A vector (or function) $\psi : G \to [0, \infty)$ will be called *β-harmonic* when

$$\sum_{s \in Y} e^{-\beta F(s)} \psi_{gs} = \psi_g \tag{3.6}$$

for all $g \in G$, and *normalized* when $\psi_{e_0} = 1$.

Lemma 3.5. *Let ψ be a normalized β-harmonic vector. There is a unique β-KMS measure m on $Y^{\mathbb{N}}$ such that*

$$m\left(tY^{\mathbb{N}}\right) = e^{-\beta F(t)} \psi_{\overline{t}} \tag{3.7}$$

for all $t \in \bigcup_{n=0}^{\infty} Y^n$.

Proof. It is standard to construct from ψ a Borel probability measure on $Y^{\mathbb{N}}$ such that (3.7) holds. For example one can apply Theorem 1.12 in [17] with the stochastic matrix p over G defined such that

$$p(g, h) = \psi_g^{-1} e^{-\beta F(g^{-1}h)} \psi_h$$

when $g^{-1}h \in Y$ and $p(g, h) = 0$ otherwise, and with the initial distribution given by the Dirac measure supported on e_0. The resulting measure is clearly unique and it is a β-KMS measure by Lemma 3.3. □

As a converse to Lemma 3.5, note that a β-KMS measure m defines a vector $\psi : G \to [0, \infty)$ such that

$$\psi_g = e^{\beta F(t)} m\left(tY^{\mathbb{N}}\right) \tag{3.8}$$

for any choice of $t \in \bigcup_{n=0}^{\infty} Y^n$ with $\overline{t} = g$, and it is straightforward to check that ψ is β-harmonic. Therefore

Proposition 3.6. *The formula (3.7) establishes a bijection between the β-KMS measures on $Y^{\mathbb{N}}$ and the normalized β-harmonic vectors.*

In conclusion, there are affine homeomorphisms between

- the β-KMS states for α^F,
- the β-KMS measures on $Y^{\mathbb{N}}$, and
- the normalized β-harmonic vectors on G,

given by (3.4) and (3.7), respectively. As a consequence there are bijections between the extremal β-KMS states, the extremal β-KMS measures and the extremal normalized β-harmonic vectors.

4. The abelian KMS states

We say that a normalized β-harmonic vector ψ is *abelian* when

$$\psi_{hgk} = \psi_{hkg}$$

for all $h, g, k \in G$. The abelian elements constitute a closed convex subset in the set of normalized β-harmonic vectors and we denote this set by Δ. A KMS state for α^F is *abelian* when the associated normalized β-harmonic vector is abelian. Before we proceed with an investigation of abelian KMS states we point out that all KMS states are abelian when G is nilpotent. This follows from the Krein–Milman theorem and the following result of Margulis, cf. [14], first proved for abelian groups by Doob, Snell and Williamson, [7].

Theorem 4.1. *(Margulis) Assume that G is nilpotent. A normalized extremal β-harmonic vector $\psi : G \to [0, \infty)$ is multiplicative: $\psi_{gh} = \psi_g \psi_h$ for all $g, h \in G$.*

Lemma 4.2. *An element $\psi \in \Delta$ is extremal in Δ if and only if there is there is a homomorphism $c : G \to \mathbb{R}$ such that*

$$\psi_g = e^{c(g)} \quad \forall g \in G. \tag{4.1}$$

Proof. Assume first that ψ is extremal in Δ. Since ψ is abelian and β-harmonic,

$$\psi_g = \sum_{s \in Y} e^{-\beta F(s)} \psi_{sg} = \sum_{s \in Y} e^{-\beta F(s)} \psi_s \psi^s_g, \tag{4.2}$$

where $\psi^s : G \to [0, \infty)$ is defined by $\psi^s_g = \psi_s^{-1} \psi_{sg}$. Note that $\sum_{s \in Y} e^{-\beta F(s)} \psi_s = \psi_{e_0} = 1$ and that ψ^s is a normalized β-harmonic vector. Furthermore, ψ^s is abelian since ψ is. Since ψ is extremal by assumption it follows therefore from (4.2) that $\psi^s = \psi$ for all $s \in Y$. That is, $\psi_{sg} = \psi_s \psi_g$ for all $s \in Y$ and all $g \in G$. Since Y generates G as a semigroup it follows that $\psi_{hg} = \psi_h \psi_g$ for all $h, g \in G$. Set $c(g) = \log \psi_g$.

Conversely assume that there is a homomorphism $c : G \to \mathbb{R}$ such that (4.1) holds. Consider an element $\phi \in \Delta$ and a $t > 0$ such that $t\phi_g \leq \psi_g$ for all $g \in G$. We must show that $\phi = \psi$. To this end we use Choquet theory to write

$$\phi_g = \int_{\partial \Delta} \xi_g \ \mathrm{d}\nu(\xi)$$

where ν is a Borel probability measure on the set $\partial\Delta$ of extreme points in Δ, cf. e.g. Proposition 4.1.3 and Theorem 4.1.11 in [1]. When $\xi \in \partial\Delta\backslash\{\psi\}$ there is a $k \in G$ such that $\xi_k/\psi_k > 1$, and hence also an open neighborhood U of ξ in $\partial\Delta$ such that $\xi'_k/\psi_k > 1$ for all $\xi' \in U$. From the first part of the proof we know that the elements of $\partial\Delta$ are multiplicative and the same is true for ψ by assumption. Hence

$$\lim_{n\to\infty} \xi'_{nk} \left(\psi_{nk}\right)^{-1} = \lim_{n\to\infty} \left(\xi'_k \left(\psi_k\right)^{-1}\right)^n = \infty$$

for all $\xi' \in U$. But $t\phi \leq \psi$ by assumption, so we must have that

$$\int_U \xi'_{nk} \left(\psi_{nk}\right)^{-1} \, \mathrm{d}\nu(\xi') \leq \left(\psi_{nk}\right)^{-1} \int_{\partial\Delta} \xi'_{nk} \, \mathrm{d}\nu(\xi') = \left(\psi_{nk}\right)^{-1} \phi_{nk} \leq t^{-1}$$

for all $n \in \mathbb{N}$ and we conclude therefore that $\nu(U) = 0$. Since $\xi \in \partial\Delta\backslash\{\psi\}$ was arbitrary it follows first that $\nu\left(\partial\Delta\backslash\{\psi\}\right) = 0$, and then that $\phi = \psi$. $\square$

A Borel measure m on $Y^{\mathbb{N}}$ is *abelian* when $m\left(t_1t_2t_3Y^{\mathbb{N}}\right) = m\left(t_1t_3t_2Y^{\mathbb{N}}\right)$ for all $t_1, t_2, t_3 \in \bigcup_n Y^n$. Clearly, a β-KMS measure is abelian if and only if the corresponding β-harmonic vector is.

A Borel probability measure m on $Y^{\mathbb{N}}$ is *Bernoulli* when there is map, sometimes called a probability vector, $p : Y \to [0,1]$ with $\sum_{y\in Y} p(y) = 1$ such that m is the corresponding infinite product measure on $Y^{\mathbb{N}}$, i.e.

$$m\left(tY^{\mathbb{N}}\right) = \prod_{i=1}^{n} p(t_i)$$

when $t = (t_i)_{i=1}^n \in Y^n$.

Lemma 4.3. *An abelian β-KMS measure is extremal in the set of abelian β-KMS measures if and only if it is a Bernoulli measure.*

Proof. Consider an abelian β-KMS measure m and let ψ be the corresponding β-harmonic vector. If m is extremal in the set of abelian β-KMS measures ψ is extremal in Δ and by Lemma 4.2 there is a $c \in \mathrm{Hom}(G, \mathbb{R})$ such that $\psi_g = e^{c(g)}$ for all g. The condition $\sum_{s\in Y} e^{-\beta F(s)}\psi_s = \psi_{e_0} = 1$ implies that

$$\sum_{s\in Y} e^{c(s)-\beta F(s)} = 1.$$

Set $p(s) = e^{c(s)-\beta F(s)}$. For $t = (t_1, t_2, \cdots, t_n) \in Y^n$ we find that

$$m\left(tY^{\mathbb{N}}\right) = \prod_{i=1}^{n} e^{-\beta F(t_i)}\psi_{t_1t_2\cdots t_n} = \prod_{i=1}^{n} e^{-\beta F(t_i)+c(t_i)} = \prod_{i=1}^{n} p(t_i),$$

showing that m is the Bernoulli measure on $Y^{\mathbb{N}}$ defined from p. This proves one of the implications, and to prove the reverse assume that m is a β-KMS measure which happens to be Bernoulli. Let ψ be the β-harmonic vector corresponding to m. Then ψ is clearly abelian. When $t, u \in \bigcup_n Y^n$ we find from (3.7) that

$$\begin{aligned}\psi_{\overline{tu}} = \psi_{\overline{tu}} &= e^{\beta(F(t)+F(u))} m\left(tuY^{\mathbb{N}}\right)\\ &= e^{\beta F(t)} e^{\beta F(u)} m\left(tY^{\mathbb{N}}\right) m\left(uY^{\mathbb{N}}\right) = \psi_{\overline{t}}\psi_{\overline{u}}.\end{aligned}$$

Thus $g \to \psi_g$ is multiplicative and hence of the form (4.1) for some $c \in \operatorname{Hom}(G, \mathbb{R})$. It follows from Lemma 4.2 that ψ is extremal, and hence also that m is. □

Set

$$Q(\beta) = \left\{ c \in \operatorname{Hom}(G, \mathbb{R}) : \sum_{s \in Y} e^{c(s) - \beta F(s)} = 1 \right\}.$$

Equip $\mathbb{R}^G$ with the product topology and $Q(\beta) \subseteq \mathbb{R}^G$ with the relative topology. Then $Q(\beta)$ is a compact subset of $\mathbb{R}^G$. Given an element $c \in Q(\beta)$ we denote by b_c the Bernoulli measure on $Y^{\mathbb{N}}$ defined from the probability vector $p(y) = e^{c(y) - \beta F(y)}$.

Lemma 4.4. *The map $c \mapsto b_c$ is a homeomorphism from $Q(\beta)$ onto the set of extreme points of the abelian β-KMS measures equipped with the weak*-topology.*

Proof. b_c is a β-KMS measure by Lemma 3.5. It follows from Lemma 4.3 and its proof that $\{b_c : \ c \in Q(\beta)\}$ is the set of extreme points in the set of abelian β-KMS measures. Since the map $c \mapsto b_c$ is continuous it suffices to show that is it also injective; a fact which follows immediately from the observation that

$$e^{c(s)} = e^{\beta F(s)} b_c\left(sY^{\mathbb{N}}\right)$$

for all $s \in Y$. □

Theorem 4.5. *There is an affine homeomorphism $\nu \mapsto \omega_\nu$ from the Borel probability measures ν on $Q(\beta)$ onto the abelian β-KMS states for α^F such that*

$$\omega_\nu(a) = \int_{Q(\beta)} \int_{Y^{\mathbb{N}}} E(a) \ \mathrm{d}b_c \ \mathrm{d}\nu(c) \tag{4.3}$$

for $a \in O_Y(G)$.

Proof. Given a Borel probability measure ν on $Q(\beta)$ we can define a Borel probability measure m on $Y^{\mathbb{N}}$ such that

$$m(B) = \int_{Q(\beta)} b_c(B) \, \mathrm{d}\nu(c)$$

for every Borel subset $B \subseteq Y^{\mathbb{N}}$. It is clear that m is an abelian β-KMS measure since all $b_c, c \in Q(\beta)$ are, and it follows from Lemma 4.4 and Choquet theory that we obtain all abelian β-KMS measures m on $Y^{\mathbb{N}}$ this way. Set

$$\omega_\nu(a) = \int_{Y^{\mathbb{N}}} E(a) \, \mathrm{d}m,$$

and note that (4.3) holds. Hence the map under consideration is surjective onto the abelian β-KMS states for α^F. To see that it is also injective assume that ν and ν' are Borel probability measures on $Q(\beta)$ such that the corresponding states defined by (4.3) are the same. Then

$$\int_{Q(\beta)} \int_{Y^{\mathbb{N}}} f \, \mathrm{d}b_c \, \mathrm{d}\nu(c) = \int_{Q(\beta)} \int_{Y^{\mathbb{N}}} f \, \mathrm{d}b_c \, \mathrm{d}\nu'(c)$$

for all $f \in C\left(Y^{\mathbb{N}}\right)$. Taking f to be the characteristic function of $tY^{\mathbb{N}}$ we find that

$$\int_{Q(\beta)} \prod_{i=1}^{n} e^{c(t_i)} \, \mathrm{d}\nu'(c) = \int_{Q(\beta)} \prod_{i=1}^{n} e^{c(t_i)} \, \mathrm{d}\nu(c)$$

for all $t \in Y^n$ and all n. This shows that integration with respect to ν and ν' give the same functional on the algebra of functions on $Q(\beta)$ generated by the maps $Q(\beta) \ni c \mapsto e^{c(s)}, \ s \in Y$. This algebra is dense in $C(Q(\beta))$ by the Stone–Weierstrass theorem and it follows therefore that $\nu = \nu'$. $\square$

4.1. A closer look at $Q(\beta)$

First the case where the abelianization of G is finite:

Proposition 4.6. *When the abelianization $G/[G,G]$ of G is trivial or a finite group there is an abelian β-KMS measure if and only if*

$$\sum_{s \in Y} e^{-\beta F(s)} = 1. \tag{4.4}$$

When it exists, the abelian β-KMS measure is unique and it is the Bernoulli measure corresponding to the probability vector $p(s) = e^{-\beta F(s)}$.

Proof. This follows directly from Theorem 4.5 since $\mathrm{Hom}(G, \mathbb{R}) = \{0\}$ when $G/[G,G]$ is finite. $\square$

Corollary 4.7. *Assume that G is nilpotent and that the abelianization $G/[G,G]$ of G is trivial or a finite group. There is a β-KMS measure if and only if*

$$\sum_{s \in Y} e^{-\beta F(s)} = 1. \tag{4.5}$$

When it exists, the β-KMS measure is unique and it is the Bernoulli measure corresponding to the probability vector $p_s = e^{-\beta F(s)}$.

Note that the one-parameter group α^F is the restriction to $O_Y(G)$ of an action on O_Y. Any KMS state for the action on O_Y will restrict to a KMS state for α^F. It follows from work of Exel and Laca, [8], at least when F is strictly positive, that the action on O_Y has exactly one KMS state. The abelian KMS state in Proposition 4.6 is the restriction to $O_Y(G)$ of that state.

Consider now the case where the rank of $G/[G,G]$ is positive, say $n \geq 1$. Then $\mathrm{Hom}(G, \mathbb{R}) \simeq \mathbb{R}^n$ and we choose n linearly independent elements $c'_i \in \mathrm{Hom}(G, \mathbb{R})$, $i = 1, 2, \cdots, n$. For each $s \in Y$, set

$$c_s = (c'_1(s), c'_2(s), \cdots, c'_n(s)) \in \mathbb{R}^n.$$

Then

$$Q(\beta) \simeq \left\{ u \in \mathbb{R}^n : \sum_{s \in Y} \exp\left(u \cdot c_s - \beta F(s)\right) = 1 \right\} \tag{4.6}$$

when we let $\cdot$ denote the canonical inner product in $\mathbb{R}^n$.

Lemma 4.8. *There is a unique vector $u(\beta) \in \mathbb{R}^n$ such that*

$$\sum_{s \in Y} \exp\left(u \cdot c_s - \beta F(s)\right) \ > \ \sum_{s \in Y} \exp\left(u(\beta) \cdot c_s - \beta F(s)\right) \tag{4.7}$$

for all $u \in \mathbb{R}^n \backslash \{u(\beta)\}$. The vector $u(\beta)$ is determined by the condition that

$$\sum_{s \in Y} c_s \exp\left(u(\beta) \cdot c_s - \beta F(s)\right) = 0. \tag{4.8}$$

Proof. The function $\mathbb{R}^n \ni u \mapsto \sum_{s \in Y} e^{-\beta F(s)} e^{u \cdot c_s}$ is strictly convex since the exponential function is. It has therefore at most one local minimum, which is necessarily a global minimum. It follows that the global minimum, if it exists, occurs at the unique $u(\beta) \in \mathbb{R}^n$ where the gradient is 0, i.e. the vector $u(\beta)$ for which (4.8) holds. It suffices therefore to show that

$$\lim_{\|u\| \to \infty} \sum_{s \in Y} \exp\left(u \cdot c_s - \beta F(s)\right) = \infty. \tag{4.9}$$

To establish (4.9) it suffices to show that for each $v \in \mathbb{R}^n, \|v\| = 1$, there is an $s \in Y$ such that $v \cdot c_s > 0$. Assume for a contradiction that $\|v\| = 1$ and $v \cdot c_s \leq 0$ for all $s \in Y$. Define $c : G \to \mathbb{R}^n$ such that $c(g) = (c_1'(g), c_2'(g), \cdots, c_n'(g))$. Since $v \cdot c_s \leq 0$ it follows that $v \cdot c(g) \leq 0$ for all $g \in G$, and hence also $-v \cdot c(g) = v \cdot c\left(g^{-1}\right) \leq 0$ for all $g \in G$. This implies $v_1 c_1' + v_2 c_2' + \cdots + v_n c_n' = 0$, contradicting the choice of the c_i''s. □

In view of Lemma 4.8 and (4.6) we must distinguish between the following three cases:

4.1.1. $\sum_{s \in Y} e^{-\beta F(s)} e^{u(\beta) \cdot c_s} > 1$
Then $Q(\beta) = \emptyset$.

4.1.2. $\sum_{s \in Y} e^{-\beta F(s)} e^{u(\beta) \cdot c_s} = 1$
Then $Q(\beta) = \{u_0\}$, where $u_0 \in \mathrm{Hom}(G, \mathbb{R})$ is determined by the condition that $u_0(s) = u(\beta) \cdot c_s \ s \in Y$. The corresponding β-KMS measure is the Bernoulli measure on $Y^{\mathbb{N}}$ defined by the probability vector $p_s = e^{-\beta F(s)} e^{u(\beta) \cdot c_s}$.

4.1.3. $\sum_{s \in Y} e^{-\beta F(s)} e^{u(\beta) \cdot c_s} < 1$
Then $Q(\beta)$ is homeomorphic to the $(n-1)$-sphere

$$S^{n-1} = \{v \in \mathbb{R}^n : \ \|v\| = 1\}.$$

This follows from

Lemma 4.9. *Assume that $\sum_{s \in Y} e^{-\beta F(s)} e^{u(\beta) \cdot c_s} < 1$. For every $v \in S^{n-1}$ there is a unique positive number $t_\beta(v)$ such that*

$$u(\beta) + t_\beta(v) v \in Q(\beta)$$

and the map $S^{n-1} \ni v \mapsto u(\beta) + t_\beta(v) v$ is a homeomorphism from S^{n-1} onto $Q(\beta)$.

Proof. Let $v \in \mathbb{R}^n$ be a unit vector. The function $f_v : \mathbb{R} \to \mathbb{R}$ given by

$$f_v(t) = \sum_{s \in Y} \exp\left((u(\beta) + tv) \cdot c_s - \beta F(s)\right) = \sum_{s \in Y} e^{-\beta F(s)} e^{u(\beta) \cdot c_s} e^{tv \cdot c_s}$$

has a unique local minimum when $t = 0$ where $f_v(0) < 1$. It follows from (4.9) that $\lim_{t \to \pm\infty} f_v(t) = \infty$. There are therefore unique real numbers $t_-, t_+ \in \mathbb{R}$ with $t_- < 0 < t_+$ such that $f_v(t_-) = f_v(t_+) = 1$. Set $t_\beta(v) = t_+$ and note that $f_v'(t_\beta(v)) > 0$. It follows therefore from the implicit function theorem that $t_\beta(v)$ is a differentiable, in particular continuous function of v. As a consequence also the map

$$S^{n-1} \ni v \to u(\beta) + t_\beta(v) v \in Q(\beta)$$

is continuous. It is easy to see that it is injective. To prove surjectivity let $u \in Q(\beta)$. Then $u \neq u(\beta)$ and we set

$$v' := \frac{u - u(\beta)}{\|u - u(\beta)\|} \in S^{n-1}.$$

Observe that $t_\beta(v') = \|u - u(\beta)\|$ since this is a positive number and

$$\sum_{s \in Y} \exp\left((u(\beta) + \|u - u(\beta)\| v') \cdot c_s - \beta F(s)\right) = \sum_{s \in Y} e^{-\beta F(s)} e^{u \cdot c_s} = 1.$$

It follows that $u(\beta) + t_\beta(v')v' = u$. □

The abelian β-KMS measure on $Y^{\mathbb{N}}$ corresponding to the vector $v \in S^{n-1}$ is the Bernoulli measure given by the probability vector p, where

$$p(s) = \exp\left((u(\beta) + t_\beta(v)v) \cdot c_s - \beta F(s)\right).$$

Assumption 4.10. Assume now that $F(s) > 0$ for all $s \in Y$.

Lemma 4.11. *For each $u \in \mathbb{R}^n$ there exists a unique number $\beta(u) \in \mathbb{R}_+$ such that*

$$\sum_{s \in Y} e^{-\beta(u) F(s)} e^{u \cdot c_s} = 1\ .$$

The function $\mathbb{R}^n \ni u \to \beta(u) \in \mathbb{R}_+$ is continuous.

Proof. Let $u \in \mathbb{R}^n$ and set

$$g(t) = \sum_{s \in Y} e^{-tF(s)} e^{u \cdot c_s}.$$

As shown in the proof of Lemma 4.8 there must be some $s' \in Y$ such that $u \cdot c_{s'} \geq 0$. It follows that $g(0) > e^{u \cdot c_{s'}} \geq 1$. Since $F > 0$, the function g is strictly decreasing with limit 0 at infinity so there is a unique number $\beta(u) \in\ ]0, \infty[$ such that $g(\beta(u)) = 1$. Continuity of the function $u \mapsto \beta(u)$ follows from the implicit function theorem. □

We shall need the following observation regarding the function $\beta(u)$:

Lemma 4.12. *Let $\{u_m\} \subseteq \mathbb{R}^n$. Then $\beta(u_m) \to \infty$ if and only if $\|u_m\| \to \infty$.*

Proof. Set $L = \max\{\|c_s\| : s \in Y\}$. Then

$$1 = \sum_{s \in Y} e^{-\beta(u_m) F(s)} e^{u_m \cdot c_s} \leq \sum_{s \in Y} e^{-\beta(u_m) F(s)} e^{\|u_m\| L},$$

showing that $\beta(u_m) \to \infty \ \Rightarrow \ \|u_m\| \to \infty$. Assume then that $\|u_m\| \to \infty$ and for a contradiction also that $\beta(u_m) \nrightarrow \infty$. After passage to a subsequence we can assume that $\beta(u_m)$ is bounded by some K and that $u_m/\|u_m\| \to v$ for some $v \in S^{n-1}$. As shown in the proof of Lemma 4.8 there is a $s' \in Y$ such that $v \cdot c_{s'} > 0$. Hence we get that

$$1 \geq e^{-KF(s')} \exp\left(\|u_m\| \frac{u_m}{\|u_m\|} \cdot c_{s'}\right) \to \infty,$$

a contradiction. □

Theorem 4.13. *Assume that the abelianization $G/[G,G]$ has rank $n \geq 1$ and that $F(s) > 0$ for all $s \in Y$. It follows that there is a $\beta_0 > 0$ such that*

- *there are no abelian β-KMS states for α^F when $\beta < \beta_0$,*
- *there is a unique abelian β_0-KMS state for α^F, and*
- *for all $\beta > \beta_0$ the simplex of abelian β-KMS states for α^F is affinely homeomorphic to the simplex of Borel probability measures on the $(n-1)$ sphere S^{n-1}.*

Proof. First observe that the function $\beta \to u(\beta)$ defined by Lemma 4.8 is continuous. Indeed, assume that $\beta_n \to \beta$ in $\mathbb{R}$ and for a contradiction also that $u(\beta_n) \nrightarrow u(\beta)$. It follows from (4.9) that we can pass to a subsequence to arrange that $u(\beta_n) \to v \neq u(\beta)$. Then

$$\lim_{n\to\infty} \sum_{s\in Y} e^{-\beta_n F(s)} e^{u(\beta_n)\cdot c_s} = \sum_{s\in Y} e^{-\beta F(s)} e^{v\cdot c_s} > \sum_{s\in Y} e^{-\beta F(s)} e^{u(\beta)\cdot c_s},$$

where the last inequality follows from (4.7). It follows that for all large n,

$$\sum_{s\in Y} e^{-\beta_n F(s)} e^{u(\beta_n)\cdot c_s} > \sum_{s\in Y} e^{-\beta_n F(s)} e^{u(\beta)\cdot c_s},$$

in conflict with the definition of $u(\beta_n)$.

It follows from Lemma 4.11 that there is $\beta \geq 0$ such that $\sum_{s\in Y} e^{-\beta F(s)} e^{u(\beta)\cdot c_s} \leq 1$. Set

$$\beta_0 = \inf\left\{\beta \in \mathbb{R} : \sum_{s\in Y} e^{-\beta F(s)} e^{u(\beta)\cdot c_s} \leq 1\right\}.$$

By continuity we must have that $\sum_{s\in Y} e^{-\beta_0 F(s)} e^{u(\beta_0)\cdot c_s} = 1$. Note that $\beta_0 > 0$ since $\sum_{s\in Y} e^{u(\beta_0)\cdot c_s} > 1$. For $\beta < \beta_0$ we are in Case 4.1.1 and there are no β-KMS states since $Q(\beta) = \emptyset$. When $\beta = \beta_0$ we are in Case 4.1.2 and there is a unique β_0-KMS state because $Q(\beta)$ contains exactly one element. Finally, when $\beta > \beta_0$, it follows from (4.7) that

$$\sum_{s\in Y} e^{-\beta F(s)} e^{u(\beta)\cdot c_s} \leq \sum_{s\in Y} e^{-\beta F(s)} e^{u(\beta_0)\cdot c_s} < \sum_{s\in Y} e^{-\beta_0 F(s)} e^{u(\beta_0)\cdot c_s} = 1.$$

This means that we are in Case 4.1.3 when $\beta > \beta_0$. □

Corollary 4.14. *Assume that G is nilpotent, that the abelianization $G/[G,G]$ has rank $n \geq 1$ and that $F(s) > 0$ for all $s \in Y$. It follows that there is a $\beta_0 > 0$ such that*

- *there are no β-KMS states for α^F when $\beta < \beta_0$,*
- *there is a unique β_0-KMS state for α^F, and*
- *for all $\beta > \beta_0$ the simplex of β-KMS states for α^F is affinely homeomorphic to the simplex of Borel probability measures on the $(n-1)$ sphere S^{n-1}.*

5. The abelian KMS$_\infty$ states

Following [5] we say that a state ω on $O_Y(G)$ is a KMS$_\infty$ *state* when there is a sequence $\{\beta_n\} \subseteq \mathbb{R}$ and for each n a β_n-KMS state ω_n such that $\lim_{n\to\infty} \beta_n = \infty$ and $\lim_{n\to\infty} \omega_n = \omega$ in the weak*-topology. When the ω_n's can be chosen as abelian β_n-KMS states we say that ω is an *abelian* KMS$_\infty$ *state*. It follows from Proposition 4.6 that there are no abelian KMS$_\infty$ states when the abelianization of G is finite. We retain therefore here the assumption that the rank n of $G/[G,G]$ is at least 1. Furthermore, we assume also that F is strictly positive and we denote by β_0 the least inverse temperature β for which there are any abelian β-KMS states, cf. Theorem 4.13.

Let Δ_Y denote the simplex

$$\Delta_Y = \left\{ p \in [0,1]^Y : \sum_{s\in Y} p_s = 1 \right\}.$$

For $\beta > \beta_0$, set

$$N_\beta = \left\{ \left(e^{-\beta F(s)} e^{u\cdot c_s} \right)_{s\in Y} : u \in Q(\beta) \right\} \subseteq \Delta_Y.$$

Let N_∞ denote the limit set of N_β as $\beta \to \infty$; i.e.

$$N_\infty = \bigcap_{n\geq\beta_0} \overline{\bigcup_{\beta\geq n} N_\beta}.$$

Lemma 5.1. *Let $p \in N_\infty$. For all $\varepsilon > 0$ there exists a $\beta_\epsilon > 0$ such that for each $\beta \geq \beta_\epsilon$ there is a $x^\beta \in N_\beta$ with $\left|x_s^\beta - p_s\right| \leq \varepsilon \ \forall s \in Y$.*

Proof. By assumption there are sequences $\{u_n\}$ and $\{\beta_n\}$ with $u_n \in Q(\beta_n)$ such that $\beta_n \to \infty$ and $\lim_{n\to\infty} e^{-\beta_n F(s)} e^{u_n\cdot c_s} = p_s$ for all $s \in Y$. By choosing a subsequence, if necessary, we can assume that $\beta_n < \beta_{n+1}$ for all n. Set $\delta = \varepsilon/\#Y$ and choose $N \in \mathbb{N}$ such that

$$|e^{-\beta_n F(s)} e^{u_n \cdot c_s} - p_s| \leq \delta \qquad \forall s \in Y \tag{5.1}$$

when $n \geq N$. We claim that $\beta_\epsilon = \beta_N$ will do the job, so assume that $\beta \geq \beta_N$. There is an $n \geq N$ such that $\beta \in [\beta_n, \beta_{n+1}]$. Since the function $[0,1] \ni \lambda \to \beta((1-\lambda)u_n + \lambda u_{n+1})$ is continuous and $\beta(u_n) = \beta_n,\ \beta(u_{n+1}) = \beta_{n+1}$ by Lemma 4.11, it follows that there is a $\lambda \in [0,1]$ such that $\beta = \beta\left((1-\lambda)u_n + \lambda u_{n+1}\right)$. By convexity of the exponential function we have that

$$\begin{aligned}
&\sum_{s \in Y} e^{-[(1-\lambda)\beta_n + \lambda\beta_{n+1}]} e^{[(1-\lambda)u_n + \lambda u_{n+1}] \cdot c_s} \\
&\leq (1-\lambda) \sum_{s \in Y} e^{-\beta_n} e^{u_n \cdot c_s} + \lambda \sum_{s \in Y} e^{-\beta_{n+1}} e^{u_{n+1} \cdot c_s} = 1.
\end{aligned}$$

It follows that

$$\beta = \beta\left((1-\lambda)u_n + \lambda u_{n+1}\right) \leq (1-\lambda)\beta_n + \lambda\beta_{n+1},$$

and hence

$$\begin{aligned}
&-\beta F(s) + ((1-\lambda)u_n + \lambda u_{n+1}) \cdot c_s \\
&\geq (1-\lambda)\left(-\beta_n F(s) + u_n \cdot c_s\right) + \lambda\left(-\beta_{n+1} F(s) + u_{n+1} \cdot c_s\right) \\
&\geq \min\left\{-\beta_n F(s) + u_n \cdot c_s,\ -\beta_{n+1} F(s) + u_{n+1} \cdot c_s\right\}.
\end{aligned}$$

Therefore (5.1) implies that

$$e^{-\beta F(s)} e^{((1-\lambda)u_n + \lambda u_{n+1}) \cdot c_s} \ \geq\ p_s - \delta$$

for all $s \in Y$. On the other hand, if there was a $\tilde{s} \in Y$ such that

$$e^{-\beta F(\tilde{s})} e^{((1-\lambda)u_n + \lambda u_{n+1}) \cdot c_{\tilde{s}}} > p_{\tilde{s}} + \delta\,(\#Y),$$

it would follow that

$$1 = \sum_{s \in Y} e^{-\beta F(s)} e^{((1-\lambda)u_n + \lambda u_{n+1}) \cdot c_s} > p_{\tilde{s}} + \delta\,(\#Y) + \sum_{s \in Y \setminus \{\tilde{s}\}} (p_s - \delta) \geq \sum_{s \in Y} p_s = 1,$$

which is absurd. Hence, for all $s \in Y$,

$$p_s - \delta \leq e^{-\beta F(s)} e^{((1-\lambda)u_n + \lambda u_{n+1}) \cdot c_s} \leq p_s + \delta(\#Y).$$

Thus

$$x^\beta = \left(e^{-\beta F(s)} e^{((1-\lambda)u_n+\lambda u_{n+1})\cdot c_s}\right)_{s\in Y}$$

is an element of N_β such that $\left|p_s - x_s^\beta\right| \le \epsilon$ for all $s \in Y$. □

Proposition 5.2. *Assume that $G/[G,G]$ is not finite and assume that F is strictly positive. The abelian* KMS_∞ *states constitute a compact convex set affinely homeomorphic to the simplex of Borel probability measures on N_∞. The abelian* KMS_∞ *state ω on $O_Y(G)$ corresponding to a Borel probability measure ν on N_∞ is given by*

$$\omega(a) = \int\limits_{N_\infty}\int\limits_{Y^{\mathbb{N}}} E(a)\ \mathrm{d}n_p\ \mathrm{d}\nu(p) \tag{5.2}$$

for all $a \in O_Y(G)$, where n_p is the Bernoulli measure defined by $p \in N_\infty$.

Proof. Let $\beta > \beta_0$. Since the map $u \mapsto \left(e^{-\beta F(s)} e^{u\cdot c_s}\right)_{s\in Y}$ is a homeomorphism from $Q(\beta)$ onto N_β it follows from Theorem 4.5 that every abelian β-KMS state ω_β is given by a Borel probability measure ν on N_β such that

$$\omega_\beta(a) = \int\limits_{N_\beta}\int\limits_{Y^{\mathbb{N}}} E(a)\ \mathrm{d}n_p\ \mathrm{d}\nu(p)$$

for all $a \in O_Y(G)$. Let $\{\omega_n\}$ be a sequence of β_n-KMS states such that $\lim_{n\to\infty}\beta_n = \infty$ and $\lim_{n\to\infty}\omega_n = \omega$ in the weak* topology. Let ν_n be the Borel probability measure on N_{β_n} corresponding to ω_n. Extend ν_n to a Borel probability measure $\tilde{\nu_n}$ on Δ_Y such that

$$\tilde{\nu_n}(B) = \nu_n\left(B \cap N_{\beta_n}\right)$$

and let ν be a weak* condensation point of $\{\tilde{\nu_n}\}$ in the set of Borel probability measures on Δ_Y. Then ν is concentrated on N_∞ and (5.2) holds. This shows that the map from Borel probability measures on N_∞ to states on $O_Y(G)$ given by (5.2) hits every abelian KMS_∞ state. The proof that the map is injective is identical with the proof of injectivity in Theorem 4.5.

It remains to show that for an arbitrary Borel probability measure ν on N_∞ the state ω on $O_Y(G)$ defined by (5.2) is a KMS_∞ state. Since the set of KMS_∞ states is closed for the weak* topology it suffices to show this for a weak* dense subset of Borel probability measures on N_∞, e.g. for the set of convex combinations of Dirac measures. For this set the claim follows straightforwardly from Lemma 5.1. □

It remains to prove the following

Theorem 5.3. *The set $N_\infty \subseteq \Delta_Y$ is homeomorphic to the $(n-1)$-sphere S^{n-1}.*

The proof of Theorem 5.3 will occupy the next section, but we record here the following corollaries.

Corollary 5.4. *Assume that the abelianization $G/[G,G]$ has rank $n \geq 1$ and that $F(s) > 0$ for all $s \in Y$. The set of abelian* KMS_∞ *states for α^F on $O_Y(G)$ is a compact convex set affinely homeomorphic to the set of Borel probability measures on the $(n-1)$-sphere.*

Corollary 5.5. *Let G be a nilpotent group whose abelianization $G/[G,G]$ has rank $n \geq 1$ and assume that $F(s) > 0$ for all $s \in Y$. The set of* KMS_∞ *states for α^F on $O_Y(G)$ is a compact convex set affinely homeomorphic to the set of Borel probability measures on the $(n-1)$-sphere.*

6. Proof of Theorem 5.3

By definition $N_\beta \subseteq \Delta_Y$ is the image of the map

$$Q(\beta) \ni u \to \left(e^{-\beta F(s)} e^{u \cdot c_s}\right)_{s \in Y} \in \Delta_Y,$$

and N_∞ is the set of elements $t \in \Delta_Y$ for which there exist sequences $\{\beta_m\} \subseteq \mathbb{R}$ and $\{u_m\} \subseteq \mathbb{R}^n$ such that $u_m \in Q(\beta_m)$ for all m, $\beta_m \to \infty$ and

$$\left(e^{-\beta_m F(s)} e^{u_m \cdot c_s}\right)_{s \in Y} \to t \quad \text{for } m \to \infty \; .$$

6.1. Partitioning of S^{n-1} by polyhedral cones

For any non-empty subset $Z \subseteq Y$ we define $M(Z)$ to be the set

$$M(Z) = \Big\{ v \in \mathbb{R}^n : \; v \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)} \right) \leq 0 \quad \forall s \in Y \setminus Z, \; \forall z \in Z \; , \\ v \cdot \left(\frac{c_{z'}}{F(z')} - \frac{c_z}{F(z)} \right) = 0 \quad \forall z, z' \in Z \Big\}.$$

Notice that these sets are *convex polyhedral cones*. In the following 'polyhedral cone' will always mean a cone of this form. We refer to [9] and [13] for the facts we need on such cones and which we state in the following. A *face* of some $M(Z)$ is a subset T of $M(Z)$ obtained by changing some of the inequalities in the definition into equalities, i.e. a face of $M(Z)$ is a set of the form $M(Z')$ with $Z \subseteq Z'$. Equivalently a convex subset T is a face of $M(Z)$ if for any two distinct points $x, y \in M(Z)$ the implication $(x, y) \cap T \neq \emptyset \;\; \Rightarrow \;\; [x, y] \subseteq T$ holds, where (x, y) is the open line segment from x to y and $[x, y]$ is the closed line segment from x to y. Each face of $M(Z)$ is again a convex polyhedral cone, and a face of a face is a face. The intersection of two polyhedral cones $M(Z)$ and $M(Z')$ is again a polyhedral cone, equal to $M(Z \cup Z')$. In particular, for every polyhedral cone $M(Z)$ there is a unique subset $Z' \subseteq Y$, which we call *maximal* with the property that $M(Z) = M(Z')$ and $M(Z) = M(Z'') \Rightarrow Z'' \subseteq Z'$. For all $Z \subseteq Y$ we let $\mathcal{F}(M(Z))$ be the set of faces in $M(Z)$, which is a finite set. A *proper face* of $M(Z)$ is a face T of $M(Z)$ with $T \neq M(Z)$; we denote the set of these by $\mathcal{F}_0(M(Z))$. We define the dimension of a polyhedral cone $M(Z)$ to be $\dim(M(Z)-M(Z))$, i.e. the dimension of the smallest subspace of $\mathbb{R}^n$ containing $M(Z)$, and we call something a *k-face* if it is a face of dimension k. A *facet* of $M(Z)$ is then a face of $M(Z)$ of dimension $\dim(M(Z)) - 1$. It is well-known that any proper face of a polyhedral cone $M(Z)$ is contained in a facet of $M(Z)$.

A polyhedral cone $M(Z)$ is *strongly convex*, meaning that $M(Z) \cap (-M(Z)) = \{0\}$. To see this, take a $v \in M(Z) \cap (-M(Z))$ and a $z \in Z$. Then v satisfies:

$$v \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)} \right) = 0 \qquad \forall s \in Y$$

However if $v \cdot c_z \geq 0$ this would imply that $v \cdot c_s \geq 0$ for all $s \in Y$ which can not be true unless $v = 0$, cf. the proof of Lemma 4.8, and likewise $v \cdot c_z \leq 0$ would imply that $v \cdot c_s \leq 0$ for all $s \in Y$ which also implies $v = 0$.

When we use the expression $\mathrm{Int}(M(Z))$ we mean the topological interior of $M(Z)$ in the subspace $M(Z) - M(Z)$ with the relative topology. For each of our polyhedral cones $M(Z)$ of dimension at least 1 we define an element $c(M(Z)) \in \mathrm{Int}(M(Z)) \cap S^{n-1}$ which we call the *center* of $M(Z)$ as follows: Say $M(Z)$ has q 1-faces $T_1, \dots, T_q$. Since $M(Z)$ is strongly convex we can then write $M(Z) = \{r_1 v_1 + \cdots + r_q v_q \ : \ r_i \geq 0\}$ where each v_i is the unique $v_i \in T_i \cap S^{n-1}$, by (13) of Section 1.2 in [9]. We set

$$c(M(Z)) := \frac{\frac{1}{q}\sum_{i=1}^{q} v_i}{\|\frac{1}{q}\sum_{i=1}^{q} v_i\|}$$

and then $c(M(Z)) \in \mathrm{Int}(M(Z))$, cf. [9].

Lemma 6.1. *Let $M(Z)$ be a polyhedral cone with Z chosen maximal. The following holds:*

(1) *If $T \in \mathcal{F}(M(Z))$ and $T = M(Z')$ with Z' chosen maximal, then $Z \subseteq Z'$ and $M(Z) = T$ if and only if $Z = Z'$.*
(2) *For each $z \in Z$,*

$$\begin{aligned}
&\mathrm{Int}(M(Z)) \\
&= \left\{ v \in \mathbb{R}^n : \ v \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)} \right) < 0 \ \forall s \in Y \setminus Z, \ v \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)} \right) = 0 \ \forall s \in Z \right\} \\
&= M(Z) \setminus \left(\bigcup_{T \in \mathcal{F}_0(M(Z))} T \right).
\end{aligned}$$

Proof. (1): Since $M(Z') \subseteq M(Z)$ we have that $M(Z') = M(Z') \cap M(Z) = M(Z \cup Z')$, so $Z \cup Z' \subseteq Z'$ by maximality, and hence $Z \subseteq Z'$. That $Z = Z' \iff M(Z) = M(Z')$ follows directly from the way we defined M and the fact that Z and Z' both are maximal. (2): The second equality is (7) in section 1.2 of [9] and the first follows from the maximality of Z. $\square$

Set $\mathrm{bd}(M(Z)) = M(Z) \setminus \mathrm{Int}(M(Z))$.

Lemma 6.2. *Let $M(Z)$ be at least 2-dimensional. Fix $v \in \mathrm{Int}(M(Z)) \cap S^{n-1}$. For every $w \in (M(Z) \cap S^{n-1}) \setminus \{v\}$ there exists a unique pair (λ, u) where $\lambda \in \,]0,1]$ and $u \in \mathrm{bd}(M(Z)) \cap S^{n-1}$ such that*

$$w = \frac{(1-\lambda)v + \lambda u}{\|(1-\lambda)v + \lambda u\|}$$

Proof. We may assume that Z is maximal. Consider the subspace $W := \mathrm{span}\{v, w\}$. Note that $w \neq -v$ since $M(Z) \cap (-M(Z)) = \{0\}$. Hence $\dim(W) = 2$. Since $W \cap M(Z)$ is a closed convex cone in W and $-v \notin W \cap M(Z)$, the circle arch in W starting in v and continuing through w must reach the boundary of $M(Z) \cap W$ at some u with an angle to v less than π and

$$w = \frac{(1-\lambda)v + \lambda u}{\|(1-\lambda)v + \lambda u\|}, \tag{6.1}$$

for some $\lambda \in \,]0,1]$. Since u lies in the boundary of $M(Z) \cap W$ it can be approximated by elements from $M(Z)^c \cap W$ and hence $u \in \mathrm{bd}(M(Z))$. To establish the uniqueness part, assume (6.1) holds with (λ, u) replaced by the pair $(\lambda', u') \in \,]0,1] \times \mathrm{bd}(M(Z)) \cap S^{n-1}$. Fix a $z \in Z$. It follows from (1) in Lemma 6.1 that there is a subset $Z_1 \subseteq Y$ such that $u \in M(Z_1)$ and $Z \subsetneq Z_1$. We can assume that

$$\alpha = \frac{(1-\lambda')}{\|(1-\lambda')v + \lambda' u'\|} - \frac{(1-\lambda)}{\|(1-\lambda)v + \lambda u\|} \geq 0 \ .$$

Consider an element $s \in Z_1 \setminus Z$ and set $q := c_s/F(s) - c_z/F(z)$. Then

$$0 = q \cdot \frac{\lambda u}{\|(1-\lambda)v + \lambda u\|} = q \cdot \alpha v + q \cdot \frac{\lambda' u'}{\|(1-\lambda')v + \lambda' u'\|}.$$

But $q \cdot v < 0$ since $s \notin Z$ and $q \cdot \lambda' u' \leq 0$ since $u' \in M(Z)$, and hence $\alpha = 0$. This implies that

$$u \cdot \frac{\lambda}{\|(1-\lambda)v + \lambda u\|} = u' \cdot \frac{\lambda'}{\|(1-\lambda')v + \lambda' u'\|}.$$

Since $u, u' \in S^{n-1}$ and $\alpha = 0$ it follows first that $\|(1-\lambda)v + \lambda u\| = \|(1-\lambda')v + \lambda' u'\|$ and $\lambda = \lambda'$, and then also that $u = u'$. $\square$

We denote the u of Lemma 6.2 by $P(w)$ and the λ by λ_w. We suppress the v in the notation because it will always be $c(M(Z))$ in the following.

Lemma 6.3. *The maps $w \mapsto P(w)$ and $w \mapsto \lambda_w$ are continuous from $(M(Z) \cap S^{n-1}) \setminus \{v\}$ to S^{n-1} and $[0,1]$ respectively.*

Proof. Assume that $\{w_m\} \subseteq (M(Z) \cap S^{n-1}) \setminus \{v\}$ converges to a w in this set. If $\lambda_{w_m} \not\to \lambda_w$ or $P(w_m) \not\to P(w)$ then by compactness of $[0,1]$ and S^{n-1} we can take a subsequence $\{w_{m_i}\}$ of $\{w_m\}$ such that $\lambda_{w_{m_i}} \to \lambda$ and $P(w_{m_i}) \to u$, with either $\lambda \neq \lambda_w$ or $u \neq P(w)$. Since $\lim_i w_{m_i} = w$ we then have:

$$\frac{(1-\lambda_w)v + \lambda_w P(w)}{\|(1-\lambda_w)v + \lambda_w P(w)\|} = w = \lim_i \frac{(1-\lambda_{w_{m_i}})v + \lambda_{w_{m_i}} P(w_{m_i})}{\|(1-\lambda_{w_{m_i}})v + \lambda_{w_{m_i}} P(w_{m_i})\|} = \frac{(1-\lambda)v + \lambda u}{\|(1-\lambda)v + \lambda u\|}.$$

The uniqueness part of Lemma 6.2 implies that $\lambda = \lambda_w$ and $u = P(w)$, giving us the desired contradiction. $\square$

6.2. Constructing a homeomorphism $H : S^{n-1} \to N_\infty$

For each $l \in \{1, \dots, n\}$ we define the *l-skeleton* as the union of all sets $M(Z) \cap S^{n-1}$ with $M(Z)$ a l'-dimensional polyhedral cone for some $1 \leq l' \leq l$. The skeletons will be used to give a recursive definition of H, but we need some preparations for this.

Lemma 6.4. *The n-skeleton is all of S^{n-1}.*

Proof. Let $v \in S^{n-1}$ and choose $s \in Y$ such that $v \cdot \frac{c_y}{F(y)} \leq v \cdot \frac{c_s}{F(s)}$ for all $y \in Y$. Then $v \in M(\{s\})$. $\square$

Definition 6.5. We say a sequence $\{v_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}^n$ is *associated* with a $t \in N_\infty$ when $\beta(v_k) \to \infty$ and

$$\left(e^{-\beta(v_k)F(s) + v_k \cdot c_s}\right)_{s \in Y} \;\to\; t$$

for $k \to \infty$.

Lemma 6.6. *Assume $\{u_k\}_{k \in \mathbb{N}}$ is associated with $t \in N_\infty$ and that $u_k \in M(Z)$ for all n. Then*

(1) $t_s^{1/F(s)} \leq t_z^{1/F(z)}$ $\forall z \in Z$ $\forall s \in Y$, *and*
(2) $t_z \neq 0$ *and* $t_z^{1/F(z)} = t_y^{1/F(y)}$ *for all* $z, y \in Z$.

Proof. Since $u_k \in M(Z)$ it follows that $u_k \cdot \frac{c_s}{F(s)} \leq u_k \cdot \frac{c_z}{F(z)}$ and hence that

$$-\beta(u_k)F(s) + u_k \cdot c_s \leq \frac{F(s)}{F(z)}\left(-\beta(u_k)F(z) + u_k \cdot c_z\right)$$

for all k when $s \in Y$ and $z \in Z$. This shows that (1) holds. (2) follows from (1). □

Lemma 6.7. *Let $t, \tilde{t} \in \Delta_Y$. If there are $y, y' \in Y$ such that $t_y \neq 0$, $\tilde{t}_{y'} \neq 0$ and*

$$\frac{t_s^{1/F(s)}}{t_y^{1/F(y)}} = \frac{\tilde{t}_s^{1/F(s)}}{\tilde{t}_{y'}^{1/F(y')}} \qquad \forall s \in Y,$$

then $t = \tilde{t}$.

Proof. Note that $\tilde{t}_s = \tilde{t}_{y'}^{F(s)/F(y')}\left(t_s^{1/F(s)}/t_y^{1/F(y)}\right)^{F(s)}$ and hence

$$\begin{aligned}\sum_{s\in Y}\left(t_y^{1/F(y)}\right)^{F(s)}\left(\frac{t_s^{1/F(s)}}{t_y^{1/F(y)}}\right)^{F(s)} &= \sum_{s\in Y} t_s = 1 = \sum_{s\in Y}\tilde{t}_s \\ &= \sum_{s\in Y}\left(\tilde{t}_{y'}^{1/F(y')}\right)^{F(s)}\left(\frac{t_s^{1/F(s)}}{t_y^{1/F(y)}}\right)^{F(s)}. \end{aligned} \tag{6.2}$$

Since the function

$$]0,\infty[\ \ni x \mapsto \sum_{s\in Y} x^{F(s)}\left(\frac{t_s^{1/F(s)}}{t_y^{1/F(y)}}\right)^{F(s)}$$

is strictly increasing it follows from (6.2) that $t_y^{1/F(y)} = \tilde{t}_{y'}^{1/F(y')}$ which yields the conclusion. □

Lemma 6.8. *Assume $M(Z)$ is at least 1-dimensional and that $Z \subseteq Y$ is maximal. There is a unique element $t \in N_\infty$ satisfying the following two conditions:*

(1) *$t_s \neq 0$ if and only if $s \in Z$, and*
(2) *$t_{s_1}^{1/F(s_1)} = t_{s_2}^{1/F(s_2)}$ for all $s_1, s_2 \in Z$.*

Furthermore, $\lim_{r\to\infty} e^{-\beta(rv)F(s)+rv\cdot c_s} = t_s$ for all $s \in Y$ when $v \in \mathrm{Int}(M(Z))$.

Proof. Let $v \in \mathrm{Int}(M(Z))$. Then $v \neq 0$ and $\lim_{r\to\infty}\beta(rv) = \infty$ by Lemma 4.12. Since Δ_Y is compact there is a subsequence $\{r_i v\}_{i\in\mathbb{N}}$ and an element $t \in N_\infty$ such that $\{r_i v\}_{i\in\mathbb{N}}$ is associated with t. It follows as in the proof of Lemma 5.1 that $\lim_{r\to\infty} e^{-\beta(rv)F(s)+rv\cdot c_s} = t_s$ for all $s \in Y$. To prove (1) notice that $r_i v \in M(Z)$ and hence $t_z \neq 0$ for $z \in Z$ by Lemma 6.6. Assume then that $t_s \neq 0$ for a $s \in Y$, and assume for contradiction that $s \notin Z$. Fix a $y \in Z$. It follows from (2) of Lemma 6.1 that

$$\frac{v \cdot c_s}{F(s)} < \frac{v \cdot c_y}{F(y)}. \tag{6.3}$$

Since $t_s \neq 0$ and $\lim_{r\to\infty}(-\beta(rv)F(s) + rv \cdot c_s) = \log(t_s)$, it follows that $\lim_{r\to\infty}\beta(rv)/r = v\cdot c_s/F(s)$. It follows then from (6.3) that is a $L > 0$ such that $\beta(rv)/r < v\cdot c_y/F(y)$ for all $r > L$. But this means that

$$e^{-\beta(rv)F(y)+rv\cdot c_y} > 1,$$

when $r > L$, contradicting the definition of $\beta(rv)$, cf. Lemma 4.11. Hence (1) holds, and (2) follows from Lemma 6.6. For uniqueness, consider an element $t' \in \Delta_Y$ for which (1) and (2) hold. Let $z \in Z$ and note that

$$1 = \sum_{s\in Y} t'_s = \sum_{s\in Z} t'_s = \sum_{s\in Z} {t'}_z^{F(s)/F(z)} = \sum_{s\in Z} t_z^{F(s)/F(z)}.$$

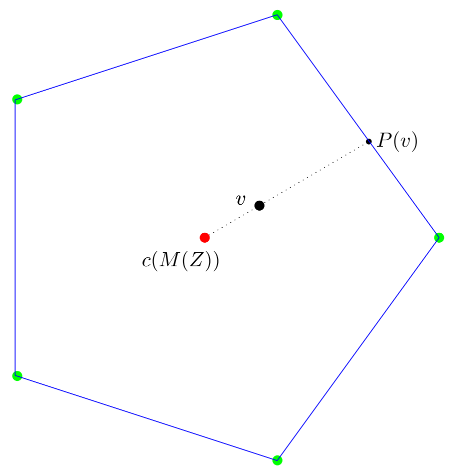
Since the function $]0,\infty[\ni x \to \sum_{s\in Z} x^{F(s)/F(y)}$ is strictly increasing it follows that $t_z = t'_z$, and hence that $t = t'$. □

6.2.1. H on the 1-skeleton

Let $M(Z)$ be a 1-dimensional polyhedral cone with Z chosen maximal. Then $M(Z)\cap S^{n-1}$ consists only of one point x. We define $H(x) = t$, where $t \in N_\infty$ is the unique element obtained from Lemma 6.8 using $M(Z)$.

6.2.2. H on the k-skeleton

H will be defined inductively and the basic idea is illustrated by the picture below. Everything inside the figure represents the intersection between S^2 and some 3-dimensional polyhedral cone $M(Z)$. The green dots (For interpretation of the references to color in this article, the reader is referred to the web version of this article.) are the 1-dimensional faces intersected with S^2, the blue lines (containing the green dots) are the 2-dimensional faces intersected with S^2. Assuming that we have defined H on the 2-skeleton, we have defined H on the blue lines and the green dots. As a step to define H on the 3-skeleton we first define H on $c(M(Z))$ which lies in the interior $\mathrm{Int}(M(Z))$, and hence is not in the 2-skeleton. Then for any $v \neq c(M(Z))$, we define H on v depending on $P(v)$ and how close it lies to the center $c(M(Z))$. We measure the distance to $c(M(Z))$ by using the unique decomposition obtained in Lemma 6.2.



For the procedure to work we have to impose conditions at each step. We say that H satisfies *the induction conditions on the l-skeleton* if H is defined on the l-skeleton and has the following properties.

(1) H is continuous on the l-skeleton.
(2) If $x \in M(Z) \cap S^{n-1}$ with $\dim(M(Z)) \leq l$, then there is a sequence $\{v_m\} \subseteq M(Z)$ associated with $H(x)$.
(3) For x in the l-skeleton, let $y \in Y$ satisfy that $H(x)_s^{1/F(s)} \leq H(x)_y^{1/F(y)}$ $\forall s \in Y$. Then $x \in \mathrm{Int}(M(Z))$ where

$$Z = \{s \in Y \ : \ H(x)_s^{1/F(s)} = H(x)_y^{1/F(y)}\},$$

and Z is maximal for $M(Z)$.

Lemma 6.9. *Assume H satisfy the induction condition on the l-skeleton. Let $x \in M(Z)$ with $\dim(M(Z)) \leq l$ and Z chosen maximal. Then $H(x)_z^{1/F(z)} \geq H(x)_s^{1/F(s)}$ for all $s \in Y$ and $z \in Z$. In particular $H(x)_z \neq 0$ for $z \in Z$.*

Proof. Combine induction condition (2) with Lemma 6.6. □

Notice that H satisfies the induction conditions on the 1-skeleton. For the induction step, assume that we have defined H on the $(k-1)$-skeleton for some $k > 1$ and that H satisfies the induction conditions on the $(k-1)$-skeleton.

Now we will define H on the k-skeleton. So let $M(Z)$ be a polyhedral cone of dimension k and let Z be chosen maximal. An element $v \in \mathrm{Int}(M(Z))$ does not lie in the $k-1$-skeleton by (2) of Lemma 6.1, while a $v \in \mathrm{bd}(M(Z)) \cap S^{n-1}$ does, so we want to define H on $\mathrm{Int}(M(Z)) \cap S^{n-1}$. Since $M(Z)$ is a strongly convex polyhedral cone we can consider $c(M(Z)) \in \mathrm{Int}(M(Z)) \cap S^{n-1}$, and set $H(c(M(Z))) = t$ where t is the unique element arising from Lemma 6.8 using $M(Z)$.

Lemma 6.10. *For any $v \in \big(S^{n-1} \cap M(Z)\big) \setminus \{c(M(Z))\}$ write*

$$v = \frac{(1-\lambda)c(M(Z)) + \lambda P(v)}{\|(1-\lambda)c(M(Z)) + \lambda P(v)\|},$$

where $\lambda \in \,]0,1]$ and $P(v) \in \mathrm{bd}(M(Z))$ are unique, cf. Lemma 6.2. There is a unique $t \in N_\infty$ such that

$$t_s^{1/F(s)} = t_z^{1/F(z)} \frac{H(P(v))_s^{1/F(s)}}{H(P(v))_z^{1/F(z)}} \exp\left(-\log(\lambda)c(M(Z)) \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right)\right) \tag{6.4}$$

for all $z \in Z$ and $s \in Y$. Furthermore, the following hold:

(1) *For any $\{v_m\}$ associated with $H(P(v))$ and any subsequence of $\{v_m - \log(\lambda)c(M(Z))\}$ associated to some $t' \in N_\infty$, we have that $t = t'$.*
(2) *$t_z^{1/F(z)} \geq t_s^{1/F(s)}$ for all $s \in Y$ and all $z \in Z$.*
(3) *There is a sequence $\{w_m\} \subseteq M(Z)$ associated with t.*
(4) *Assume that $v \in \mathrm{Int}(M(Z))$ and let $z \in Z$. Then $\{s \in Y \ : \ t_s^{1/F(s)} = t_z^{1/F(z)}\} = Z$.*

Proof. It follows from (2) of the induction conditions that there is a sequence $\{v_m\}$ which is associated with $H(P(v))$. By considering a sub-sequence we can assume that $\{v_m - \log(\lambda)c(M(Z))\}$ is associated with some $t \in N_\infty$. Choose Z' maximal such that $P(v) \in M(Z')$ with $\dim(M(Z')) \leq k-1$. Then $Z \subseteq Z'$ and Lemma 6.9 implies that $H(P(v))_z \neq 0$ for all $z \in Z$. Hence

$$\begin{aligned}\frac{H(P(v))_s^{1/F(s)}}{H(P(v))_z^{1/F(z)}} &= \lim_m \left(e^{-\beta(v_m)F(s)+v_m \cdot c_s}\right)^{1/F(s)} \left(e^{-\beta(v_m)F(z)+v_m \cdot c_z}\right)^{-1/F(z)} \\ &= \lim_m \exp\left(v_m \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right)\right)\end{aligned}$$

for all $z \in Z$, $s \in Y$. Set $w_m = v_m - \log(\lambda)c(M(Z))$, and note that

$$\begin{aligned}&\left(e^{-\beta(w_m)F(s)+w_m \cdot c_s}\right)^{1/F(s)} \left(e^{-\beta(w_m)F(z)+w_m \cdot c_z}\right)^{-1/F(z)} \\ &= \exp\left(-\log(\lambda)c(M(Z)) \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right)\right) \exp\left(v_m \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right)\right)\end{aligned}$$

for all $s \in Y$ and $z \in Z$. Considering the limit $m \to \infty$ in the last equation for a $s \in Y$ with $t_s \neq 0$ and a $z \in Z$, we see that $t_z \neq 0$ for $z \in Z$. Hence for all $s \in Y$, $z \in Z$ we get (6.4) by combining the two equations, proving the existence of t. The uniqueness follows from Lemma 6.7 since (6.4) implies that $t_z \neq 0$ for all $z \in Z$. To establish the additional properties, notice that (1) follows from the above. (3) follows from (1) and the definition of w_m since we can choose $v_m \in M(Z)$, and (2) follows from (3) and Lemma 6.6. The inclusion “$\supseteq$” in (4) follows from (2). For the opposite inclusion in (4), observe that $c(M(Z)) \cdot (c_s/F(s) - c_z/F(z)) < 0$ for $s \notin Z$ by (2) of Lemma 6.1. Furthermore, $\log(\lambda) < 0$ since $v \in \operatorname{Int}(M(Z))$ and it follows from Lemma 6.9 that $H(P(v))_s^{1/F(s)} \leq H(P(v))_z^{1/F(z)}$. Hence (6.4) shows that $t_s^{1/F(s)}/t_z^{1/F(z)} \neq 1$ for $s \notin Z$. $\square$

For $v \in \big(S^{n-1} \cap \operatorname{Int}(M(Z))\big) \setminus \{c(M(Z))\}$ we set $H(v) = t$, where $t \in N_\infty$ is the element determined by Lemma 6.10.

Lemma 6.11. *H is continuous on $M(Z) \cap S^{n-1}$.*

Proof. Let $\{x_m\}$ be a sequence such that $\lim_m x_m = x$ in $M(Z) \cap S^{n-1}$. To prove $\lim_{m\to\infty} H(x_m) = H(x)$, consider a subsequence $\{m_i\}$ such that $\lim_{i\to\infty} H(x_{m_i}) = u$ in N_∞. It suffices to show that $u = H(x)$. Assume first that $x \neq c(M(Z))$. By using Lemma 6.3 it follows that

$$x = \frac{(1-\lambda)c(M(Z)) + \lambda P(x)}{\|(1-\lambda)c(M(Z)) + \lambda P(x)\|},$$

where $\lambda = \lim_{i\to\infty} \lambda_{x_{m_i}}$, and from (6.4) by using the continuity of H on the $(k-1)$-skeleton, that

$$u_s^{1/F(s)} = u_z^{1/F(z)} \frac{H(P(x))_s^{1/F(s)}}{H(P(x))_z^{1/F(z)}} \exp\left(-\log(\lambda)c(M(Z)) \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right)\right)$$

for all $s \in Y$ and all $z \in Z$. Hence $u = H(x)$ by uniqueness in Lemma 6.10. Assume then that $x = c(M(Z))$. We may then assume that $x_{m_i} \neq c(M(Z))$ for all i, and (2) in Lemma 6.10 then implies that $u_z^{1/F(z)} \geq u_s^{1/F(s)}$ for all $s \in Y$ and all $z \in Z$. Hence $u_{s_1}^{1/F(s_1)} = u_{s_2}^{1/F(s_2)}$ when $s_1, s_2 \in Z$. Let $s \in Y \setminus Z$. To conclude from Lemma 6.8 that $u = H(c(M(Z)))$ we need only show that $u_s = 0$. Let $z \in Z$. Then

$$\lim_{i\to\infty} \frac{H(x_{m_i})_s^{1/F(s)}}{H(x_{m_i})_z^{1/F(z)}} = \frac{u_s^{1/F(s)}}{u_z^{1/F(z)}}$$

while

$$\frac{H\left((P(x_{m_i}))\right)_s^{1/F(s)}}{H\left((P(x_{m_i}))\right)_z^{1/F(z)}} \leq 1$$

for all i by Lemma 6.9. Furthermore, as observed in the proof of Lemma 6.10, $c(M(Z)) \cdot (c_s/F(s) - c_z/F(z)) < 0$ since $s \notin Z$. Since $\lim_{i\to\infty} \lambda_{x_{m_i}} = 0$ it follows that

$$\lim_{i\to\infty} \exp\left((-\log(\lambda_{x_{m_i}}))c(M(Z)) \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right)\right) = 0.$$

Therefore, by inserting x_{m_i} for v in (6.4) and taking the limit $i \to \infty$, it follows that $u_s = 0$ as desired. $\square$

We now have a continuous function $H : S^{n-1} \cap M(Z) \to N_\infty$ for every k-dimensional polyhedral cone $M(Z)$. If $M(Z)$ and $M(Z')$ are two distinct k-dimensional polyhedral cones such that $M(Z) \cap M(Z') \cap S^{n-1} \neq \emptyset$, the elements of $M(Z) \cap M(Z') \cap S^{n-1}$ will lie in the $(k-1)$-skeleton and hence the two H-functions, one arising from $M(Z)$ and the other from $M(Z')$, will agree on $M(Z) \cap M(Z') \cap S^{n-1}$. In this way we have a well-defined function H defined on the k-skeleton. H satisfies the induction condition (1) by Lemma 6.11, (2) by (3) in Lemma 6.10 and (3) by (4) and (2) in Lemma 6.10. Since the n-skeleton is all of S^{n-1} by Lemma 6.4, it follows that we have defined a continuous map $H : S^{n-1} \to N_\infty$.

6.2.3. H is a homeomorphism

It remains to show that H is injective and surjective.

Lemma 6.12. *$H : S^{n-1} \to N_\infty$ is injective.*

Proof. Note first that it follows from Lemma 6.8 that H is injective on the 1-skeleton. Assume then that H is injective on the $(k-1)$-skeleton, $k \leq n$. Let x_1, x_2 be elements in the k-skeleton such that $H(x_1) = H(x_2)$. Choose $y \in Y$ such that $H(x_1)_s^{1/F(s)} \leq H(x_1)_y^{1/F(y)}$ for all $s \in Y$. Set

$$Z_0 = \left\{ s \in Y : \ H(x_1)_s^{1/F(s)} = H(x_1)_y^{1/F(y)} \right\}.$$

It follows from the induction condition (3) that $x_1, x_2 \in \mathrm{Int}(M(Z_0))$ and that Z_0 is maximal for $M(Z_0)$. If $\dim(M(Z_0)) < k$ it follows from the induction hypothesis that $x_1 = x_2$ so we assume that $\dim(M(Z_0)) = k$. If $x_i \neq c(M(Z_0))$ it follows from (6.4) that there is an $s \notin Z_0$ such that $H(x_i)_s \neq 0$. Indeed, $P(x_i) \in M(Z')$ with $Z_0 \subsetneq Z'$ and by (3) from the induction conditions $H(P(x_i))_s \neq 0$ when $s \in Z'$. It follows from (6.4) that $H(x_i)_s \neq 0$ when $s \in Z' \backslash Z_0$. Therefore, if $H(x_1)_s = 0$ for all $s \notin Z_0$ it follows that $x_1 = x_2 = c(M(Z_0))$. We may therefore assume that $x_i \neq c(M(Z_0))$, $i = 1, 2$. Note that there is a face $M(Z_i') \subseteq M(Z_0)$ such that $Z_0 \subsetneq Z_i'$ and $P(x_i) \in M(Z_i')$. Then $Z_0 \subseteq Z_1' \cap Z_2'$ and in particular, $y \in Z_1' \cap Z_2'$. By symmetry we may assume that $\lambda_{x_1} \leq \lambda_{x_2}$. It follows then from (6.4) that

$$\frac{H(P(x_1))_s^{1/F(s)}}{H(P(x_1))_y^{1/F(y)}} \exp\left(\log(\lambda_{x_2}/\lambda_{x_1}) c(M(Z_0)) \cdot \left(\frac{c_s}{F(s)} - \frac{c_y}{F(y)} \right) \right) = \frac{H(P(x_2))_s^{1/F(s)}}{H(P(x_2))_y^{1/F(y)}} \tag{6.5}$$

for all $s \in Y$. Consider an element $s_2 \in Z_2' \backslash Z_0$. Since $y \in Z_1'$,

$$\frac{H(P(x_1))_{s_2}^{1/F(s_2)}}{H(P(x_1))_y^{1/F(y)}} \leq 1$$

and since $s_2 \in Z_2'$,

$$\frac{H(P(x_2))_{s_2}^{1/F(s_2)}}{H(P(x_2))_y^{1/F(y)}} = 1.$$

Furthermore, it follows from (2) in Lemma 6.1 that $c(M(Z_0)) \cdot \left(\frac{c_{s_2}}{F(s_2)} - \frac{c_y}{F(y)}\right) < 0$, and (6.5) therefore implies that $\lambda_{x_1} = \lambda_{x_2}$. It follows then first from (6.5) and Lemma 6.7 that $H(P(x_1)) = H(P(x_2))$, and then by the induction hypothesis that $P(x_1) = P(x_2)$. Hence $x_1 = x_2$. $\square$

To prove that H is surjective we use the following

Definition 6.13. Let $M(Z)$ be a polyhedral cone. We say that H is *surjective on* $M(Z)$ if for every $t \in N_\infty$ associated to a sequence $\{v_m\}_{m\in\mathbb{N}} \subseteq M(Z)$, there is some $x \in S^{n-1} \cap M(Z)$ with $H(x) = t$. We then say H is *surjective on the k-skeleton* when it is surjective on all polyhedral cones of dimension $\leq k$.

Lemma 6.14. *H is surjective on the 1-skeleton.*

Proof. Let $Z \subseteq Y$ satisfy that $M(Z)$ is 1-dimensional. Then $S^{n-1} \cap M(Z) = \{v\}$. Assume that some $t \in N_\infty$ has an associated sequence $\{v_m\}_{m\in\mathbb{N}} \subseteq M(Z)$. Then $v_m/\|v_m\| = v$ for all m and hence $v_m = \|v_m\| v$. It follows then from Lemma 6.8 that $t = H(v)$. $\square$

Lemma 6.15. *Let $k > 1$. If H is surjective on the $(k-1)$ skeleton, it is also surjective on the k-skeleton.*

Proof. Let $M(Z)$ be a k-dimensional polyhedral cone with Z maximal. Consider an element $t \in N_\infty$ with a sequence $\{v_m\} \subseteq M(Z)$ associated to t. If $v_m/\|v_m\| = c(M(Z))$ for infinitely many m it follows from Lemma 6.8 as in the proof of Lemma 6.14 that $t = H(c(M(Z)))$. By passing to a subsequence we may therefore assume that $v_m/\|v_m\| \neq c(M(Z))$ for all m. Since $M(Z)$ has a boundary consisting of a finite set of facets, we can by possibly going to a subsequence assume that there is some $Z' \supsetneq Z$ with $M(Z')$ a facet of $M(Z)$, and $P(v_m/\|v_m\|) \in M(Z')$ for all n. Set $\lambda_m = \lambda_{v_m/\|v_m\|}$ and $a_m = P(v_m/\|v_m\|)$. Then

$$\frac{v_m}{\|v_m\|} = \frac{(1-\lambda_m)c(M(Z)) + \lambda_m a_m}{\|(1-\lambda_m)c(M(Z)) + \lambda_m a_m\|}.$$

Setting

$$q_m := \frac{\|v_m\|}{\|(1-\lambda_m)c(M(Z)) + \lambda_m a_m\|}$$

we have that

$$v_m = q_m(1-\lambda_m)c(M(Z)) + q_m\lambda_m a_m \qquad \forall m \in \mathbb{N}.$$

We divide the proof into two cases.

Case 1: Assume $\{q_m(1-\lambda_m)\}_m$ *is bounded.* Passing to a subsequence we can then assume that there is a $q \geq 0$ with $\lim_m q_m(1-\lambda_m) = q$. Note that $\lambda_m \to 1$ since $q_m \to \infty$. We can therefore assume that $q_m\lambda_m a_m$ is associated with a $\tilde{t} \in N_\infty$. Since $q_m\lambda_m a_m \in M(Z')$ for all n, there is some $x \in M(Z') \cap S^{n-1}$ with $H(x) = \tilde{t}$ by our induction hypothesis. Set $\lambda := \exp(-q)$ and consider

$$u := \frac{(1-\lambda)c(M(Z)) + \lambda x}{\|(1-\lambda)c(M(Z)) + \lambda x\|} \in M(Z) \cap S^{n-1}.$$

Then $P(u) = x$ and by (1) of Lemma 6.10 $H(u)$ is associated with some subsequence of $w_m := q_m\lambda_m a_m + qc(M(Z)) \in M(Z)$. Fix $z \in Z$. For any $s \in Y$ we have that

$$\begin{aligned} &\frac{1}{F(s)}\left(-\beta(v_m)F(s) + v_m \cdot c_s\right) - \frac{1}{F(z)}\left(-\beta(v_m)F(z) + v_m \cdot c_z\right) = v_m \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right) \\ &= q_m\lambda_m a_m \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right) + q_m(1-\lambda_m)c(M(Z)) \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right) \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} &\frac{1}{F(s)}\left(-\beta(w_m)F(s) + w_m \cdot c_s\right) - \frac{1}{F(z)}\left(-\beta(w_m)F(z) + w_m \cdot c_z\right) = w_m \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right) \\ &= q_m\lambda_m a_m \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right) + qc(M(Z)) \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right) \end{aligned} \tag{6.7}$$

for all m. Note that $t_z \neq 0 \neq H(u)_z$ by Lemma 6.6. Since $q_m(1-\lambda_m) \to q$ it follows from (6.6) and (6.7) that

$$\frac{t_s^{1/F(s)}}{t_z^{1/F(z)}} = \frac{H(u)_s^{1/F(s)}}{H(u)_z^{1/F(z)}} \quad \forall s \in Y. \tag{6.8}$$

Then Lemma 6.7 implies that $H(u) = t$.

Case 2: Assume $\{q_m(1-\lambda_m)\}_m$ *is unbounded.* Passing to a subsequence we can assume that this sequence diverges to $+\infty$. Fix $z \in Z$ and let $s \notin Z$. Then

$$c(M(Z)) \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right) < 0 \quad \text{and} \quad a_m \cdot \left(\frac{c_s}{F(s)} - \frac{c_z}{F(z)}\right) \leq 0$$

and hence

$$\frac{1}{F(s)}\left(-\beta(v_m)F(s)+v_m\cdot c_s\right)-\frac{1}{F(z)}\left(-\beta(v_m)F(z)+v_m\cdot c_z\right)$$
$$=q_m\lambda_m a_m\cdot\left(\frac{c_s}{F(s)}-\frac{c_z}{F(z)}\right)+q_m(1-\lambda_m)c(M(Y))\cdot\left(\frac{c_s}{F(s)}-\frac{c_z}{F(z)}\right)\to-\infty. \tag{6.9}$$

Since $t_z \neq 0$ it follows from (6.9) that $t_s = 0$. It follows from Lemma 6.6 that $t_z^{1/F(z)} = t_s^{1/F(s)}$ when $s \in Z$ and we can therefore now conclude from Lemma 6.8 that $t = H(c(M(Z))$. □

Since the n-skeleton is all of S^{n-1} it follows from Lemma 6.14 and Lemma 6.15 that $H : S^{n-1} \to N_\infty$ is surjective, completing the proof of Theorem 5.3.

7. Two examples

7.1. The Heisenberg group

The Heisenberg group H_3 is the subgroup of $\mathrm{Sl}_3(\mathbb{Z})$ of matrices of the form

$$\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}. \tag{7.1}$$

It is wellknown that H_3 is nilpotent and finitely generated. The canonical set Y of 6 generators consists of the elements

$$\begin{pmatrix}1 & \pm1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & \pm1\\ 0 & 0 & 1\end{pmatrix} \text{ and } \begin{pmatrix}1 & 0 & \pm1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We consider the gauge action on $O_Y(\mathrm{H}_3)$, i.e. we set $F(y) = 1$ for all $y \in Y$. There is a homomorphism $\mathrm{H}_3 \to \mathbb{Z}^2$ sending the matrix (7.1) to (a, b) and the kernel of this homomorphism is the commutator group in H_3. It follows that $\mathrm{Hom}(\mathrm{H}_3, \mathbb{R})$ is spanned by c_1' and c_2' where

$$c_1'\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = a \text{ and } c_2'\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = b.$$

It is then straightforward to apply the results of the previous sections to deduce the following.

- There are no β-KMS states for the gauge action on $O_Y(\mathrm{H}_3)$ when $\beta < \log 6$.
- There is a unique $\log 6$-KMS state for the gauge action on $O_Y(\mathrm{H}_3)$.

- When $\beta > \log 6$ the simplex of β-KMS states for the gauge action on $O_Y(\mathrm{H}_3)$ is affinely homeomorphic to the simplex of Borel probability measures on the circle S^1.
- The set of KMS_∞ states for the gauge action on $O_Y(\mathrm{H}_3)$ is a convex set affinely homeomorphic to the simplex of Borel probability measures on the circle S^1.

7.2. The infinite dihedral group

The infinite dihedral group $\mathbb{D}_\infty$ is generated by two elements a, b where $b^2 = 1,\ bab = a^{-1}$. With $Y = \{a, b\}$ the Cayley graph $\Gamma(\mathbb{D}_\infty, Y)$ is the following graph.

$$
\begin{tikzcd}
\dots \arrow[r] & a^{-3} \arrow[r] \arrow[d, shift left] & a^{-2} \arrow[r] \arrow[d, shift left] & a^{-1} \arrow[r] \arrow[d, shift left] & e_0 \arrow[r] \arrow[d, shift left] & a \arrow[r] \arrow[d, shift left] & a^2 \arrow[r] \arrow[d, shift left] & a^3 \arrow[r] \arrow[d, shift left] & \dots \\
\dots & a^{-3}b \arrow[l] \arrow[u, shift left] & a^{-2}b \arrow[l] \arrow[u, shift left] & a^{-1}b \arrow[l] \arrow[u, shift left] & b \arrow[l] \arrow[u, shift left] & ab \arrow[l] \arrow[u, shift left] & a^2b \arrow[l] \arrow[u, shift left] & a^3b \arrow[l] \arrow[u, shift left] & \dots \arrow[l]
\end{tikzcd}
$$

We consider the gauge action on $O_Y(\mathbb{D}_\infty)$, i.e. we set $F(a) = F(b) = 1$. When $\psi : \mathbb{D}_\infty \to \mathbb{R}$ is a vector, set

$$\psi_{a^n} = x_n \text{ and } \psi_{a^n b} = y_n,\ \ n \in \mathbb{Z}. \tag{7.2}$$

Then ψ is a β-harmonic vector iff

a) $x_n \geq 0$, $y_n \geq 0$,
b) $e^\beta x_n = x_{n+1} + y_n$, and
c) $e^\beta y_n = x_n + y_{n-1}$

for all $n \in \mathbb{Z}$. Note that b) and c) are equivalent to b$'$) and c$'$), where

b$'$) $e^\beta x_n = x_{n+1} + x_{n-1}$

and

c$'$) $y_{n+1} = x_n$.

The positive solutions to b$'$) are exactly the β-harmonic functions on $\mathbb{Z}$ when we choose $Y = \{-1, 1\}$ and $F(-1) = F(1) = 1$ and can be found using the results from Section 4. In the notation used in that section, and with $c_1' \in \mathrm{Hom}(\mathbb{Z}, \mathbb{R})$ the identity map, we see that the $u(\beta)$ of Lemma 4.8 is the solution to the equation $e^{u(\beta)-\beta} - e^{-u(\beta)-\beta} = 0$, i.e. $u(\beta) = 0$. We are in Case 4.1.1 when $e^{u(\beta)-\beta} + e^{-u(\beta)-\beta} = 2e^{-\beta} > 1$, i.e. when $\beta < \log 2$, in Case 4.1.2 when $\beta = \log 2$ and in Case 4.1.3 when $\beta > \log 2$. When $\beta = \log 2$, $Q(\beta)$ contains only the zero homomorphism which means that $x_n = 1$ for all $n \in \mathbb{Z}$ and hence also $y_n = 1$ for all $n \in \mathbb{Z}$ according to c$'$). The corresponding $\log 2$-KMS measure on

$\{a,b\}^{\mathbb{N}}$ is the Bernoulli measure corresponding to the probability measure m_0 on $\{a,b\}$ such that $m_0(\{a\}) = m_0(\{b\}) = 1/2$. When $\beta > \log 2$,

$$Q(\beta) \simeq \left\{c \in \mathbb{R} : \ e^c + e^{-c} = e^{\beta}\right\} = \{-c(\beta), c(\beta)\},$$

where $c(\beta) > 0$ is the positive solution to $e^c + e^{-c} = e^{\beta}$. It follows therefore from Theorem 4.5 and c′) that the set of β-harmonic vectors on $\mathbb{D}_\infty$ are parametrized by the interval $[0,1]$ such that $0 \le t \le 1$ corresponds to the solution

$$x_n = te^{c(\beta)n} + (1-t)e^{-c(\beta)n}, \quad y_n = te^{c(\beta)(n-1)} + (1-t)e^{-c(\beta)(n-1)} \tag{7.3}$$

for all $n \in \mathbb{Z}$. It follows that for $\beta > \log 2$ the simplex of normalized β-KMS states is affinely homeomorphic to $[0,1]$. By Proposition 4.6 the only abelian KMS state is the $\log 2$-KMS measure.

It is not difficult to identify the set of KMS_∞ states as a convex set with two extreme points; limits of the two extreme β-KMS states as $\beta \to \infty$. The Borel probability measures on $Y^{\mathbb{N}} = \{a,b\}^{\mathbb{N}}$ corresponding to the extreme KMS_∞ states are the Dirac measures at $a^\infty \in \{a,b\}^{\mathbb{N}}$ and $ba^\infty \in \{a,b\}^{\mathbb{N}}$; the two geodesic paths emitted from e_0 in the graph above.

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SYMMETRIES OF THE KMS SIMPLEX

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Abstract. A continuous groupoid homomorphism c on a locally compact second countable Hausdorff étale groupoid $\mathcal{G}$ gives rise to a C^*-dynamical system in which every β-KMS state can be associated to a $e^{-\beta c}$-quasi-invariant measure μ on $\mathcal{G}^{(0)}$. Letting Δ_μ denote the set of KMS states associated to such a μ, we will prove that Δ_μ is a simplex for a large class of groupoids, and we will show that there is an abelian group that acts transitively and freely on the extremal points of Δ_μ. This group can be described using the support of μ, so our theory of symmetries can be used to obtain a description of all KMS states by describing the $e^{-\beta c}$-quasi-invariant measures. To illustrate this we will describe the KMS states for the Cuntz-Krieger algebras of all finite higher rank graphs without sources and a large class of continuous one-parameter groups.

1. Introduction

In recent years there has been a great deal of interest in describing KMS states for C^*-dynamical systems and many articles have been written about the subject. Often the C^*-dynamical systems investigated are given as a pair consisting of a groupoid C^*-algebra and a continuous one-parameter group arising from a continuous groupoid homomorphism. This is also the case for the articles about KMS states on C^*-algebras of higher rank graphs that have appeared the last several years, e.g. [aHKR1], [aHKR2], [aHLRS1] and [aHLRS2]. In [aHLRS2] the authors come to the conclusion that the simplex of KMS states for the C^*-dynamical systems they consider is "highly symmetric" in the sense that there is an abelian group that acts transitively and freely on the extremal points of the simplex. Inspired by this, the main purpose of this article is to investigate such symmetries using the groupoid picture of these C^*-algebras. We will do this by proving that the simplex of KMS states is symmetric for a large class of groupoid C^*-algebras and one-parameter groups given by continuous groupoid homomorphisms.

We will consider locally compact second countable Hausdorff étale groupoids $\mathcal{G}$ that admits a continuous homomorphism $\Phi : \mathcal{G} \to A$ where A is some countable discrete abelian group, and that satisfies that $\ker(\Phi) \cap \mathcal{G}_x^x = \{x\}$ for all $x \in \mathcal{G}^{(0)}$. Building on work of Renault, Neshveyev has described a bijection between the β-KMS states for one-parameter groups arising from a continuous groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$, and pairs consisting of a $e^{-\beta c}$-quasi-invariant probability measure μ on $\mathcal{G}^{(0)}$ and a specific kind of μ-measurable field. Our main theorem describes how each $e^{-\beta c}$-quasi-invariant probability measure μ gives rise to a simplex Δ_μ of KMS states associated to μ, and how there for each μ is a subgroup B of A with the dual $\hat{B}$ of B acting transitively and freely on the extremal points of Δ_μ. When there is only one $e^{-\beta c}$-quasi-invariant probability measure μ on $\mathcal{G}^{(0)}$ then Δ_μ is the set of KMS states, and then our theorem implies that $\hat{B}$ acts transitively and freely on the extremal points of the simplex of KMS states.

The subgroup B whose dual acts on the extreme points of Δ_μ has a very concrete description involving the support of the measure μ. This opens up the possibility of using these symmetries to describe the simplex of KMS states in cases where our methods so far have fallen short. When working with topological graphs of rank higher than 1 it seldom happens that the preferred method used for determining the KMS simplex is by using the description given by Neshveyev, because, even though there are often tools for describing the relevant measures, the μ-measurable fields are difficult to describe. However using our theory of symmetries it becomes redundant to describe the μ-measurable fields, and then one can determine all the extremal KMS states by determining the $e^{-\beta c}$-quasi-invariant probability measures. We believe that this makes our main theorem a useful tool for giving concrete descriptions of KMS states on the groupoids under consideration. To illustrate this point, we will use the theorem to describe the KMS states for all Cuntz-Krieger algebras of finite higher rank k-graphs without sources and all continuous one-parameter groups obtained by taking a $r \in \mathbb{R}^k$ and mapping $\mathbb{R}$ into $\mathbb{T}^k$ by $t \to (e^{itr_1}, \dots, e^{itr_k})$ and composing with the gauge-action.

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2. Notation and setting

2.1. **C^*-dynamical systems.** A C^*-dynamical system is a triple $(\mathcal{A}, \alpha, G)$ where $\mathcal{A}$ is a C^*-algebra, G is a locally compact group and α is a strongly continuous representation of G in $\operatorname{Aut}(\mathcal{A})$. To ease notation we will denote the systems where $G = \mathbb{R}$ as $(\mathcal{A}, \alpha)$, in which case we call $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ a continuous one-parameter group. For a C^*-dynamical system $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$ and a $\beta \in \mathbb{R}$, a *β-KMS state for α* or a *α-KMS$_\beta$ state* is a state ω on $\mathcal{A}$ satisfying:

$$\omega(xy) = \omega(y\alpha_{i\beta}(x))$$

for all elements x, y in a norm dense, α-invariant $*$-algebra of entire analytic elements of α, c.f. Definition 5.3.1 in [BR]. The definition is independent of choice of norm dense, α-invariant $*$-algebra of entire analytic elements. When there can be no confusion to which C^*-dynamical system and $\beta \in \mathbb{R}$ we work with, we will denote the set of KMS states by Δ. This is a simplex for unital C^*-algebras, and hence we can consider extremal KMS states, the set of which we will denote by $\partial\Delta$. In general when dealing with a compact and convex set C in a locally convex topological vector space, we will use ∂C to denote the extremal points of C.

2.2. **Groupoid C^*-algebras.** Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid with unit space $\mathcal{G}^{(0)}$ and range and source maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$. Since $\mathcal{G}$ is étale r and s are local homeomorphisms, and we call an open set $W \subseteq \mathcal{G}$ a bisection when $r(W)$ and $s(W)$ are open and the maps $r|_W : W \to r(W)$ and $s|_W : W \to s(W)$ are homeomorphisms. For $x \in \mathcal{G}^{(0)}$ we set $\mathcal{G}_x := s^{-1}(x)$ and $\mathcal{G}^x := r^{-1}(x)$. The isotropy group at x is then the set $\mathcal{G}_x \cap \mathcal{G}^x$ which we denote by

$\mathcal{G}_x^x$. Let $C_c(\mathcal{G})$ denote the space of compactly supported continuous functions on $\mathcal{G}$. We can make this space into a $*$-algebra by defining a product:

$$(f_1 * f_2)(g) = \sum_{h \in \mathcal{G}^{r(g)}} f_1(h) f_2(h^{-1}g) \qquad \forall g \in \mathcal{G}$$

and an involution by $f^*(g) = \overline{f(g^{-1})}$ for all $g \in \mathcal{G}$. When completing $C_c(\mathcal{G})$ in the full norm, see Definition 1.12 in chapter II of [Re], we obtain the full groupoid C^*-algebra $C^*(\mathcal{G})$. Since $\mathcal{G}$ is second countable it follows that $C^*(\mathcal{G})$ is separable. The full norm has the property that the map $C_c(\mathcal{G}) \to C_c(\mathcal{G}^{(0)})$ which restricts functions to $\mathcal{G}^{(0)}$ extends to a conditional expectation $P : C^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$.

Taking a continuous groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$, i.e. a continuous function $c : \mathcal{G} \to \mathbb{R}$ with $c(gh) = c(g) + c(h)$ when $s(g) = r(h)$, we can define an automorphism α_t^c of $C_c(\mathcal{G})$ for all $t \in \mathbb{R}$ by setting:

$$\alpha_t^c(f)(g) = e^{itc(g)} f(g) \qquad \forall g \in \mathcal{G}$$

The map α_t^c then extends to an automorphism of $C^*(\mathcal{G})$, and $\{\alpha_t^c\}_{t \in \mathbb{R}}$ becomes a continuous one-parameter group. For the C^*-dynamical system $(C^*(\mathcal{G}), \{\alpha_t^c\}_{t\in\mathbb{R}})$ the $*$-algebra $C_c(\mathcal{G})$ is norm-dense, α^c-invariant and consists of entire analytic elements for α^c, so it is sufficient to check the KMS condition on elements in $C_c(\mathcal{G})$.

2.3. **Neshveyevs Theorem.** In Theorem 1.3 in [N] Neshveyev provides a useful description of KMS states which we will outline in the following. For a continuous groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ on a locally compact second countable Hausdorff étale groupoid $\mathcal{G}$, we say that a finite Borel measure μ on $\mathcal{G}^{(0)}$ is $e^{-\beta c}$*-quasi-invariant* for some $\beta \in \mathbb{R} \setminus \{0\}$, if for every open bisection W of $\mathcal{G}$ we have:

$$\mu(s(W)) = \int_{r(W)} e^{\beta c\left(r_W^{-1}(x)\right)} \, d\mu(x)$$

where r_W^{-1} is the inverse of $r_W : W \to r(W)$. In the terminology used in [N] these measures are called quasi-invariant with Radon-Nikodym cocycle $e^{-\beta c}$. We will need the following observation about these measures: If μ is a $e^{-\beta c}$-quasi-invariant measure on $\mathcal{G}^{(0)}$ and $E \subseteq \mathcal{G}^{(0)}$ is an invariant Borel set, i.e. $r(s^{-1}(E)) = E = s(r^{-1}(E))$, then the Borel measure μ_E given by $\mu_E(\mathcal{B}) := \mu(E \cap \mathcal{B})$ is a $e^{-\beta c}$-quasi-invariant measure. For a proof of this we refer the reader to the proof of Lemma 2.2 in [Th].

Let μ be a $e^{-\beta c}$-quasi-invariant measure. We say that a collection $\{\varphi_x\}_{x \in \mathcal{G}^{(0)}}$ consisting for each $x \in \mathcal{G}^{(0)}$ of a state φ_x on $C^*(\mathcal{G}_x^x)$ is a μ-measurable field if for each $f \in C_c(\mathcal{G})$ the function:

$$\mathcal{G}^{(0)} \ni x \to \sum_{g \in \mathcal{G}_x^x} f(g) \varphi_x(u_g)$$

is μ-measurable, where u_g, $g \in \mathcal{G}_x^x$, denotes the canonical unitary generators of $C^*(\mathcal{G}_x^x)$. We do not distinguish between μ-measurable fields which agree for μ-a.e. $x \in \mathcal{G}^{(0)}$. For any $\beta \in \mathbb{R} \setminus \{0\}$ Neshveyevs Theorem establishes a bijection between the β-KMS states for α^c on $C^*(\mathcal{G})$ and the pairs $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ consisting of a $e^{-\beta c}$-quasi-invariant Borel probability measure μ on $\mathcal{G}^{(0)}$ and a μ-measurable field of states $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ satisfying:

$$\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}}) \quad \text{for } \mu\text{-a.e } x \in \mathcal{G}^{(0)} \text{ and all } g \in \mathcal{G}_x^x \text{ and } h \in \mathcal{G}_x \tag{2.1}$$

The KMS state ω corresponding to $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ satisfies:

$$\omega(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)\ d\mu(x) \qquad \forall f \in C_c(\mathcal{G})$$

2.4. **Duality of abelian groups.** For any locally compact abelian group A we let $\widehat{A}$ denote the dual of A, which is the set of continuous characters $\xi : A \to \mathbb{T}$. Setting $(\xi_1\xi_2)(a) = \xi_1(a)\xi_2(a)$ and $\xi^{-1}(a) = \overline{\xi(a)}$ for $a \in A$ defines a composition and inversion on $\hat{A}$ making it a group with the constant function 1 as the unit. Using the compact-open topology $\hat{A}$ becomes a locally compact abelian group. In this article we will consider abelian groups A that are discrete and countable, and then the compact-open topology on $\hat{A}$ is the topology of pointwise convergence, and $\hat{A}$ is compact with this topology. For any locally compact abelian group A we have the identification $\widehat{(\widehat{A})} \simeq A$, and for any closed subgroup H of A, defining the *annihilator* $H^\perp$ as:

$$H^\perp = \{\xi \in \widehat{A} \mid \xi(h) = 1 \text{ for all } h \in H\}$$

we also have that $(H^\perp)^\perp = H$, c.f. Lemma 2.1.3 in [Ru]. When there can be no confusion about which group A we work with, we denote its unit by e_0.

2.5. **Groupoids admitting an abelian valued homomorphism.** We will throughout this paper consider the following groupoids.

Definition 2.1. *We say that a groupoid $\mathcal{G}$ admits an abelian valued homomorphism $\Phi : \mathcal{G} \to A$, if $\mathcal{G}$ is a locally compact second countable Hausdroff étale groupoid with compact unitspace $\mathcal{G}^{(0)}$, A is some countable discrete abelian group and $\Phi : \mathcal{G} \to A$ is a continuous homomorphism satisfying that* $\ker(\Phi) \cap \mathcal{G}_x^x = \{x\}$ *for all $x \in \mathcal{G}^{(0)}$.*

Remark 2.2. For the rest of this paper all groupoids $\mathcal{G}$ will satisfy Definition 2.1 for some discrete countable abelian group A and continuous homomorphism $\Phi : \mathcal{G} \to A$.

It follows from Proposition 5.1 in chapter II in [Re] that for a groupoid $\mathcal{G}$ that satisfies Definition 2.1, there is an automorphism $\Psi_\xi \in \operatorname{Aut}(C^*(\mathcal{G}))$ for every $\xi \in \hat{A}$ satisfying:

$$\Psi_\xi(f)(g) = \xi(\Phi(g))f(g) \quad \forall g \in \mathcal{G} \tag{2.2}$$

whenever $f \in C_c(\mathcal{G})$. Letting $\Psi : \xi \to \Psi_\xi$ then $(C^*(\mathcal{G}), \Psi, \hat{A})$ is a C^*-dynamical system. When there can be no doubt about which group A we consider, we will often denote this action as the *gauge-action.*

Example 2.3. Let (Λ, d) be a compactly aligned topological k-graph for some $k \in \mathbb{N}$, see e.g. [Yeend]. Using (Λ, d) one can define a space of paths X_Λ and for each $m \in \mathbb{N}^k$ a map σ^m on $\{x \in X_\Lambda \mid d(x) \geq m\}$ and thereby obtain a groupoid:

$$G_\Lambda = \{(x, m, y) \in X_\Lambda \times \mathbb{Z}^k \times X_\Lambda \mid \exists p, q \in \mathbb{N}^k \text{ with } p \leq d(x), \\ q \leq d(y),\ p - q = m \text{ and } \sigma^p(x) = \sigma^q(y)\}$$

with composition $(x, m, y)(y, n, z) = (x, m + n, z)$, c.f. Definition 3.4 in [Yeend]. Using Proposition 3.6 and Theorem 3.16 in [Yeend] we can equip G_Λ with a topology such that the homomorphism $G_\Lambda \ni (x, m, y) \to m \in \mathbb{Z}^k$ becomes continuous and G_Λ satisfies Definition 2.1 when X_Λ is compact. So the groupoid for the Toeplitz algebra of a compactly aligned topological k-graph with compact unitspace satisfies Definition 2.1. Since the groupoid for the Cuntz-Krieger algebra for a compactly aligned topological k-graph is a reduction of G_Λ, it also satisfies Definiton 2.1 when it has a compact unitspace. This provides us with a lot of examples, see e.g. the ones listed in Example 7.1 in [Yeend] where the unitspace is compact. Most importantly for the content of this article, it implies that all groupoids of Cuntz-Krieger algebras of finite higher rank graphs without sources satisfy the criterion.

Example 2.4. Let X be a compact second countable Hausdorff space and A a countable abelian group, and denote by $\mathrm{End}(X)$ the semigroup of surjective local homeomorphisms from X to X. Let P be a subsemigroup of A cointaining the unit e_0 of A with $PP^{-1} = P^{-1}P = A$, and let θ be a right action of P on X in the sense that $\theta : P \to \mathrm{End}(X)$ satisfies $\theta_{e_0} = \mathrm{id}_X$ and $\theta_{nm} = \theta_m\theta_n (= \theta_n\theta_m)$ for all $n, m \in P$. Proposition 3.1 in [ER] then informs us that:

$$\mathcal{G} = \left\{ (x, g, y) \in X \times A \times X \mid \exists n, m \in P,\ g = nm^{-1},\ \theta_n(x) = \theta_m(y) \right\}$$

is a groupoid with composition $(x, a, y)(y, b, z) = (x, ab, z)$. In Proposition 3.2 in [ER] the authors define a topology that makes $\mathcal{G}$ a locally compact étale groupoid which is second countable and Hausdorff since X is. The topology furthermore makes the homomorphism $\mathcal{G} \ni (x, a, y) \to a \in A$ continuous, so since $\mathcal{G}^{(0)} \simeq X$ is compact $\mathcal{G}$ satisfies Definition 2.1.

3. The one-point Theorem

For this section we fix a groupoid $\mathcal{G}$ with an abelian valued homomorphism $\Phi : \mathcal{G} \to A$ as in Definition 2.1. The purpose of this section is to show how there is an interplay between the gauge-action and the KMS states.

Lemma 3.1. *Let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. Assume that ω is a α^c-KMS$_\beta$ state on $C^*(\mathcal{G})$ satisfying $\omega \circ \Psi_\xi = \omega$ for all $\xi \in \hat{A}$. Then $\omega = \omega \circ P$.*

Proof. Since $\Phi^{-1}(\{a\})$ is open for each $a \in A$ we can partition $\mathcal{G}$ into the open sets $\Phi^{-1}(\{a\})$, $a \in A \setminus \{e_0\}$, $\mathcal{G}^{(0)}$ and $\Phi^{-1}(\{e_0\}) \setminus \mathcal{G}^{(0)}$. Using a partition of unity and linearity and continuity of ω and $\omega \circ P$ it follows that it is enough to prove that $\omega(f) = \omega(P(f))$ for $f \in C_c(\mathcal{G})$ supported in any of the above three kinds of sets. Suppose first that $\mathrm{supp}(f) \subseteq \Phi^{-1}(\{a\})$ for a $a \neq e_0$. It follows that $P(f) = 0$. Let m denote the normalised Haar-measure on $\hat{A}$, the invariance of ω under Ψ implies that:

$$\omega(f) = \int_{\hat{A}} \omega(\Psi_\xi(f))\ dm(\xi) = \int_{\hat{A}} \omega(f)\xi(a)\ dm(\xi) = \omega(f) \int_{\hat{A}} \xi(a)\ dm(\xi) = 0$$

If $\mathrm{supp}(f) \subseteq \mathcal{G}^{(0)}$ then $P(f) = f$ and $\omega(f) = \omega(P(f))$. For the last case notice that if $g \in \Phi^{-1}(\{e_0\}) \setminus \mathcal{G}^{(0)}$ then $r(g) = s(g)$ would imply that $g \in \ker(\Phi) \cap \mathcal{G}_{r(g)}^{r(g)}$, contradicting that $g \notin \mathcal{G}^{(0)}$. Since $\mathcal{G}$ is étale it follows by linearity that we can assume $\mathrm{supp}(f)$ is contained in an open set U with $\overline{r(U)} \cap \overline{s(U)} = \emptyset$. Since $\mathcal{G}^{(0)}$ is compact we can pick $h \in C_c(\mathcal{G}^{(0)})$ with $h = 1$ on $\overline{r(U)}$ and $\mathrm{supp}(h) \subseteq \overline{s(U)}^C$. It follows using the definition of the product in $C_c(\mathcal{G})$ that $f = hf$ and $fh = 0$, so using that ω is a α^c-KMS$_\beta$ state and h is fixed by α^c we get:

$$\omega(f) = \omega(hf) = \omega(fh) = 0$$

which proves the Lemma. □

3.1. **The one-point theorem.** The purpose of the following theorem is to use the gauge-action to gain some control over the size of the set of extremal KMS states.

Theorem 3.2. *Let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism, $\beta \in \mathbb{R}$ and ω be an extremal β-KMS state for α^c on $C^*(\mathcal{G})$. Then for any extremal α^c-KMS$_\beta$ state ψ satisfying that $\psi \circ P = \omega \circ P$ there is a $\xi \in \hat{A}$ with $\psi = \omega \circ \Psi_\xi$.*

Proof. First we will argue that if some ψ is a β-KMS state for α^c then $\psi \circ \Psi_\xi$ is also a β-KMS state for α^c for all $\xi \in \hat{A}$. Equation (2.2) implies that $\Psi_\xi(C_c(\mathcal{G})) \subseteq C_c(\mathcal{G})$ and that $\alpha^c_t \circ \Psi_\xi = \Psi_\xi \circ \alpha^c_t$ for any $t \in \mathbb{R}$ and $\xi \in \hat{A}$. So for $f, g \in C_c(\mathcal{G})$ we get:

$$\begin{aligned}\psi \circ \Psi_\xi(fg) &= \psi(\Psi_\xi(f)\Psi_\xi(g)) = \psi(\Psi_\xi(g)\alpha^c_{i\beta}(\Psi_\xi(f)))\\ &= \psi(\Psi_\xi(g)\Psi_\xi(\alpha^c_{i\beta}(f))) = \psi \circ \Psi_\xi(g\alpha^c_{i\beta}(f))\end{aligned}$$

and since $\psi\circ\Psi_\xi$ is clearly a state it is a β-KMS state for α^c. That $\psi\circ\Psi_\xi$ is an extremal α^c-KMS$_\beta$ state for any extremal β-KMS state ψ and $\xi \in \hat{A}$ is straightforward to check using that Ψ_ξ has inverse $\Psi_{\xi^{-1}}$. Now assume for contradiction that there is an extremal β-KMS state for α^c, say ψ, with $\psi \circ P = \omega \circ P$ which is not on the form $\omega \circ \Psi_\xi$ for any $\xi \in \hat{A}$. It follows first that:

$$\{\psi \circ \Psi_\xi \mid \xi \in \hat{A}\}$$

is a set of extremal β-KMS states for α^c, and then that:

$$\{\psi \circ \Psi_\xi \mid \xi \in \hat{A}\} \cap \{\omega \circ \Psi_\eta \mid \eta \in \hat{A}\} = \emptyset$$

since if $\psi \circ \Psi_\xi = \omega \circ \Psi_\eta$ for some $\xi, \eta \in \hat{A}$, then $\psi = \omega \circ \Psi_{\eta\xi^{-1}}$, contradicting our choice of ψ. Denoting the β-KMS states for α^c by Δ, we can define two functions from $\hat{A}$ to Δ:

$$F_1(\xi) = \omega \circ \Psi_\xi \qquad F_2(\xi) = \psi \circ \Psi_\xi$$

Since Ψ is strongly continuous, F_1 and F_2 are continuous when Δ has the weak*-topology, so since $\hat{A}$ is compact $F_1(\hat{A}) \subseteq \partial\Delta$ and $F_2(\hat{A}) \subseteq \partial\Delta$ are two disjoint compact sets. Define two measures:

$$\nu_1 = m \circ F_1^{-1} \qquad , \qquad \nu_2 = m \circ F_2^{-1}$$

where m is the normalised Haar-measure on $\hat{A}$, then ν_1 and ν_2 become Borel probability measures on Δ supported on disjoint sets, and hence $\nu_1 \neq \nu_2$. Since Δ is metrizable Choquet theory informs us, c.f. Theorem 4.1.11 in [BR], that since $\nu_1(\partial\Delta) = 1 = \nu_2(\partial\Delta)$ both measures are maximal. So since Δ is a simplex they have two different barycenters $\omega_1 \neq \omega_2 \in \Delta$. For all $x \in C^*(\mathcal{G})_+$:

$$\omega_1(x) = \int_\Delta ev_x(\gamma)d\nu_1(\gamma) = \int_{\hat{A}} ev_x(\omega \circ \Psi_\xi)dm(\xi) = \int_{\hat{A}} \omega \circ \Psi_\xi(x)dm(\xi)$$

Notice that setting $\omega'(y) := \int_{\hat{A}} \omega \circ \Psi_\xi(y)dm(\xi)$ for $y \in C^*(\mathcal{G})$ defines a α^c-KMS$_\beta$ state that is invariant under Ψ, and hence $\omega'(y) = \omega'(P(y))$. However Ψ fixes $C(\mathcal{G}^{(0)})$ pointwise and hence $\omega'(y) = \omega'(P(y)) = \omega(P(y))$. So $\omega_1(x) = \omega \circ P(x)$, and likewise $\omega_2(x) = \psi \circ P(x)$, contradicting that $\omega \circ P = \psi \circ P$ but $\omega_1 \neq \omega_2$. $\square$

4. Extremal KMS states

In this section we again let $\mathcal{G}$ be a groupoid with an abelian valued homomorphism $\Phi : \mathcal{G} \to A$ as in Definition 2.1. To use Theorem 3.2 we need to obtain some extremal KMS state. The purpose of this section is to use Neshveyevs Theorem to obtain one extremal KMS state, and then use Theorem 3.2 to obtain the rest. To ease notation we will identify regular finite Borel measures on $\mathcal{G}^{(0)}$ with positive linear functionals on $C(\mathcal{G}^{(0)})$.

Lemma 4.1. *Fix a continuous groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ and a $\beta \in \mathbb{R}\backslash\{0\}$. Let $\tilde{\Delta}$ be the set of $e^{-\beta c}$-quasi-invariant probability measures on $\mathcal{G}^{(0)}$, and for any $\mu \in \tilde{\Delta}$ let Δ_μ be the set of α^c-KMS$_\beta$ states ω on $C^*(\mathcal{G})$ with $\omega|_{C(\mathcal{G}^{(0)})} = \mu$. Then:*

1. *$\tilde{\Delta}$ is a compact convex set.*
2. *Δ_μ is a compact convex set for any $\mu \in \tilde{\Delta}$.*
3. *A β-KMS state ω for α^c is extremal in the simplex of α^c-KMS$_\beta$ states Δ if and only if $\mu := \omega|_{C(\mathcal{G}^{(0)})} \in \partial\tilde{\Delta}$ and $\omega \in \partial\Delta_\mu$.*

Proof. That $\tilde{\Delta}$ is convex is straightforward to see. To see that it is closed, let $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \tilde{\Delta}$ be a sequence such that $\mu_n \to \mu$ in the weak* topology. Then $\omega_n(x) := \int_{\mathcal{G}^{(0)}} P(x)d\mu_n$ defines a sequence of β-KMS states that converges in the weak* topology, i.e. $\omega_n \to \omega$ for some β-KMS state ω. It then follows that $\omega|_{C(\mathcal{G}^{(0)})} = \mu$, so that $\mu \in \tilde{\Delta}$. We leave the verification of (2) to the reader.

For (3), assume that $\omega \in \partial\Delta$ and let $\mu = \omega|_{C(\mathcal{G}^{(0)})}$. By Theorem 1.3 in [N], ω is given by a pair $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ where $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field of states satisfying (2.1). Assume $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for some $\mu_1, \mu_2 \in \tilde{\Delta}$ and $\lambda \in]0, 1[$. Then $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ is also μ_1- and μ_2-measurable and satisfies (2.1) for μ_1 and μ_2, and then $(\mu_1, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ and $(\mu_2, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ represent two KMS states ω_1 and ω_2, satisfying:

$$\omega(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)\ d\mu(x) = \lambda\omega_1(f) + (1-\lambda)\omega_2(f)$$

for all $f \in C_c(\mathcal{G})$. Since $\omega \in \partial\Delta$ this implies $\omega = \omega_1 = \omega_2$, and hence $\mu = \mu_1 = \mu_2$, proving that $\mu \in \partial\tilde{\Delta}$, and since Δ_μ is contained in Δ, we get that $\omega \in \partial\Delta_\mu$. The other implication in (3) is straightforward. □

Using Lemma 4.1 we can now find an extremal KMS state for $(C^*(\mathcal{G}), \alpha^c)$.

Proposition 4.2. *Let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism, $\beta \in \mathbb{R} \setminus \{0\}$ and assume that μ is a $e^{-\beta c}$-quasi-invariant probability measure. Then for any $\xi \in \hat{A}$ there is a β-KMS state ω_ξ for α^c given by:*

$$\omega_\xi(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)\xi(\Phi(g))\ d\mu(x)$$

for all $f \in C_c(\mathcal{G})$. For the function $1 \in \hat{A}$ the state ω_1 is an extremal point in Δ_μ.

Proof. To prove the first claim it is enough to find a μ-measurable field of states $\{\psi_x\}_{x\in\mathcal{G}^{(0)}}$ satisfying (2.1) such that $\psi_x(u_g) = \xi(\Phi(g))$ for all $g \in \mathcal{G}_x^x$ and all $x \in \mathcal{G}^{(0)}$. To do this fix a $\xi \in \hat{A}$ and define a $*$-homomorphism $H_\xi : C^*(A) \to \mathbb{C}$ by specifying that $H_\xi(u_a) = \xi(a)$ for all unitary generators u_a and $a \in A$. In particular we have that H_ξ is a state on $C^*(A)$. The condition that $\ker(\Phi) \cap \mathcal{G}_x^x = \{x\}$ implies that

$\Phi : \mathcal{G}_x^x \to A$ is an injective group homomorphism for each $x \in \mathcal{G}^{(0)}$, which gives us an injective unital $*$-homomorphism $\iota_x : C^*(\mathcal{G}_x^x) \to C^*(A)$ satisfying $\iota_x(u_g) = u_{\Phi(g)}$ for all $g \in \mathcal{G}_x^x$. For each $x \in \mathcal{G}^{(0)}$ we define a state $\psi_x := H_\xi \circ \iota_x$ on $C^*(\mathcal{G}_x^x)$ and claim that $\{\psi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field of states. It suffices to prove that

$$\mathcal{G}^{(0)} \ni x \to \sum_{g\in\mathcal{G}_x^x} f(g)\psi_x(u_g)$$

is μ-measurable for $f \in C_c(\mathcal{G})$ with $\operatorname{supp}(f) \subseteq W \subseteq \overline{W} \subseteq U \subseteq \Phi^{-1}(\{a\})$ where W is open, $\overline{W}$ is compact, U is an open bisection and $a \in A$. The set $N := \{g \in \overline{W} \mid s(g) = r(g)\}$ is compact in $\mathcal{G}$, so $r(N)$ is closed in $\mathcal{G}^{(0)}$. However

$$\mathcal{G}^{(0)} \setminus r(N) \ni x \to \sum_{g\in\mathcal{G}_x^x} f(g)\psi_x(u_g) = 0$$

while for $x \in r(N)$ we have:

$$\sum_{g\in\mathcal{G}_x^x} f(g)\psi_x(u_g) = \sum_{g\in\mathcal{G}_x^x} f(g)\xi(\Phi(g)) = f(r_{\overline{W}}^{-1}(x))\xi(a)$$

So since $r(N) \ni x \to f(r_{\overline{W}}^{-1}(x))\xi(a)$ is continuous $\{\psi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field. For any $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x$ and $h \in \mathcal{G}_x$ we have $\psi_{r(h)}(u_{hgh^{-1}}) = \xi(\Phi(hgh^{-1})) = \psi_x(u_g)$, so $\{\psi_x\}_{x\in\mathcal{G}^{(0)}}$ satisfies (2.1). To prove ω_1 is extremal assume that $\omega_1 = \lambda\varphi' + (1-\lambda)\tilde{\varphi}$ with $\varphi', \tilde{\varphi} \in \Delta_\mu$. Now letting $\{\psi'_x\}_{x\in\mathcal{G}^{(0)}}$, $\{\tilde{\psi}_x\}_{x\in\mathcal{G}^{(0)}}$ be μ-measurable fields corresponding to respectively φ' and $\tilde{\varphi}$, then:

$$\omega_1(f) = \lambda\varphi'(f) + (1-\lambda)\tilde{\varphi}(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)(\lambda\psi'_x + (1-\lambda)\tilde{\psi}_x)(u_g)\, d\mu(x)$$

Using the uniqueness result of Neshveyev we get that $\lambda\psi'_x + (1-\lambda)\tilde{\psi}_x = \psi_x$ for μ-almost all x in $\mathcal{G}^{(0)}$. However $\psi_x = H_1 \circ \iota_x$ is multiplicative on an abelian C^*-algebra, giving that it is a pure-state by Corollary 2.3.21 in [BR]. So ψ'_x and $\tilde{\psi}_x$ has to be equal to ψ_x for μ-almost all x, and hence $\tilde{\varphi} = \omega_1 = \varphi'$ which proves the proposition. □

5. Symmetries of the KMS simplex

We now combine the results from the last two sections to obtain a description of the extremal points of the simplex of β-KMS states for $\beta \neq 0$. Throughout this section we again consider a groupoid $\mathcal{G}$ with an abelian valued homomorphism $\Phi : \mathcal{G} \to A$ as in Definition 2.1.

Theorem 5.1. *Let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R} \setminus \{0\}$. Then any extremal β-KMS state ω for α^c is on the form:*

$$\omega(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)\xi(\Phi(g))\, d\mu(x) \qquad \forall f \in C_c(\mathcal{G}) \tag{5.1}$$

where $\mu \in \partial\tilde{\Delta}$ and $\xi \in \hat{A}$. Conversely any state on this form is extremal.

Proof. Let ω be given by the pair $(\mu, \{\psi_x\}_{x\in\mathcal{G}^{(0)}})$ as in Theorem 1.3 in [N], then $\mu \in \partial\tilde{\Delta}$ by Lemma 4.1. Constructing ω_1 using μ as in Proposition 4.2 then ω_1 is extremal in Δ_μ, and Theorem 3.2 then implies that $\omega = \omega_1 \circ \Psi_\xi$ for some ξ. Since

$\omega_1 \circ \Psi_\xi$ is equal to ω_ξ from Proposition 4.2 this proves the formula. Conversely the state in (5.1) equals $\omega_1 \circ \Psi_\xi$ and hence it is extremal by Proposition 4.2. $\square$

We will say that an extremal KMS state ω is given by a pair $(\mu, \xi) \in \partial\tilde{\Delta} \times \hat{A}$, when ω can be written as in (5.1). The representation of the extremal KMS state is not necessarily unique: If a state is given by a pair (μ, ξ) and a pair (μ', ξ') then clearly $\mu = \mu'$, but we might not have $\xi = \xi'$. In the following Theorem we will address this issue.

Theorem 5.2. *Let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R} \setminus \{0\}$. Let $\mu \in \partial\tilde{\Delta}$ and let ω be the extremal β-KMS state for α^c given by the pair $(\mu, 1)$. Then:*

$$N := \{\xi \in \hat{A} \mid \omega \circ \Psi_\xi = \omega\}$$

is a closed subgroup in $\hat{A}$. Consider the subgroup:

$$B := N^\perp = \{a \in A \mid \xi(a) = 1 \text{ for all } \xi \in N\} \subseteq A$$

Then the following is true:

(1) *For any subgroup $C \subseteq A$ the set*

$$X(C) := \left\{x \in \mathcal{G}^{(0)} \mid \Phi(\mathcal{G}_x^x) = C\right\}$$

is a Borel set in $\mathcal{G}^{(0)}$, and:

$$\mu\left(X(C)\right) = \begin{cases} 1 & \text{if } C = B \\ 0 & \text{else} \end{cases}$$

(2) *Δ_μ is a simplex and $\hat{B} \simeq \hat{A}/N$ acts transitively and freely on $\partial\Delta_\mu$. This gives rise to a homeomorphism:*

$$\hat{B} \ni \xi \to \left[f \to \int_{X(B)} \sum_{g \in \mathcal{G}_x^x} f(g)\xi(\Phi(g))\, d\mu(x)\right] \in \partial\Delta_\mu \tag{5.2}$$

Proof. Checking that N is a closed subgroup is straightforward. To prove (1), we first claim that $X(a) := \left\{x \in \mathcal{G}^{(0)} \mid a \in \Phi(\mathcal{G}_x^x)\right\}$ is a Borel set in $\mathcal{G}^{(0)}$ for all $a \in A$. Since A is countable $\{g \in \Phi^{-1}(\{a\}) \mid r(g) = s(g)\}$ is closed in $\mathcal{G}$, so since $\mathcal{G}$ is second countable and étale this implies that $r(\{g \in \Phi^{-1}(\{a\}) \mid r(g) = s(g)\}) = X(a)$ is Borel. Clearly $X(a)$ is an invariant set, so if $\mu(X(a)) \in]0,1[$ then $X(a)^C$ would be an invariant Borel set with $\mu(X(a)^C) \in]0,1[$, which would imply that μ could be written as a convex combination of two elements in $\tilde{\Delta}$. However $\mu \in \partial\tilde{\Delta}$, so $\mu(X(a)) = 0$ or $\mu(X(a)) = 1$. If $\mu(X(a)) = 1$ and $\mu(X(b)) = 1$ then $\mu(X(a) \cap X(b)) = 1$, so since $X(a) \cap X(b) \subseteq X(ab)$ and $X(a) = X(a^{-1})$ we have that:

$$D := \{a \in A \mid \mu(X(a)) = 1\}$$

is a subgroup of A. For any subgroup C of A we can write:

$$X(C) = \left(\bigcap_{c \in C} X(c)\right) \setminus \left(\bigcup_{a \in A \setminus C} X(a)\right)$$

and hence $X(C)$ is Borel. From this equality it also follows that $\mu(X(D)) = 1$. Since $X(D) \cap X(C) = \emptyset$ for subgroups $C \neq D$, this implies $\mu(X(C)) = 0$ when $C \neq D$. By definition of N we have, using the notation of the proof of Proposition 4.2, that

$\xi \in N$ if and only if $H_1 \circ \iota_x = H_\xi \circ \iota_x$ for μ almost all $x \in \mathcal{G}^{(0)}$, so if and only if $D \subseteq \mathrm{Ker}(\xi)$. However $D \subseteq \mathrm{Ker}(\xi)$ if and only if $\xi \in D^\perp$, so combined we get that $D^\perp = N$, and hence $D = (D^\perp)^\perp = N^\perp = B$.

To prove (2) notice first that the map that sends $\phi B^\perp \in \hat{A}/B^\perp$ to $\phi|_B \in \widehat{B}$ is an isomorphism by Theorem 2.1.2 in [Ru], so since $N = B^\perp$ this proves $\hat{A}/N \simeq \hat{B}$. Since μ is extremal in $\tilde{\Delta}$ it follows by Theorem 5.1 that every $\psi \in \partial\Delta_\mu$ is on the form $\omega \circ \Psi_\xi$ for some $\xi \in \hat{A}$, so by definition of N we can define a transitive and free action of $\hat{A}/N$ on $\partial\Delta_\mu$ by:

$$\hat{A}/N \times \partial\Delta_\mu \ni (\xi N, \psi) \to \psi \circ \Psi_\xi \in \partial\Delta_\mu$$

So the map in (5.2) is a bijection, and since functions $f \in C_c(\mathcal{G})$ supported in some set $\Phi^{-1}(\{a\})$, $a \in A$, spans $C_c(\mathcal{G})$ it follows that the map is continuous, and hence a homeomorphism since $\hat{B}$ is compact. To see that Δ_μ is a simplex, let μ_1 and μ_2 be two different maximal regular Borel probability measures on Δ_μ, and assume for contradiction that they have the same barycenter. Then $\int_{\Delta_\mu} \gamma(x) d\mu_1(\gamma) = \int_{\Delta_\mu} \gamma(x) d\mu_2(\gamma)$ for all $x \in C^*(\mathcal{G})$. Since Δ is metrizable μ_1 and μ_2 are supported on $\partial\Delta_\mu$, so we consider them as measures on $\hat{B}$. It follows from Stone-Weierstrass that the span of $\{\mathrm{ev}_b \mid b \in B\}$ is a dense subalgebra of $C(\hat{B})$, so there exist a $b \in B$ with $\int_{\hat{B}} \xi(b) d\mu_1(\xi) \neq \int_{\hat{B}} \xi(b) d\mu_2(\xi)$. Since $\mathcal{G}$ is sigma-compact and $\Phi^{-1}(\{b\})$ is clopen, there is an increasing sequence of positive functions $f_n \in C_c(\mathcal{G})$ that converges pointwise to $1_{\Phi^{-1}(\{b\})}$, and hence the functions $x \to \sum_{g \in \mathcal{G}_x^x} f_n(g)$ increases pointwise to a function f' with $f' = 1$ on $X(B)$. Using monotone convergence there is a $f \in C_c(\mathcal{G})$ with $\mathrm{supp}(f) \subseteq \Phi^{-1}(\{b\})$ and $\omega(f) = \int_{X(B)} \sum_{g \in \mathcal{G}_x^x} f(g) \; d\mu(x) \neq 0$, and hence for $i = 1, 2$ we have:

$$\int_{\Delta_\mu} \gamma(f) d\mu_i(\gamma) = \int_{\hat{B}} \omega(\Psi_\xi(f)) d\mu_i(\xi) = \omega(f) \int_{\hat{B}} \xi(b) d\mu_i(\xi)$$

a contradiction. Hence Δ_μ is a simplex. □

Observation 5.3. This Lemma should be compared with Proposition 11.5 in [aHLRS2]. In [aHLRS2] the authors analyse the KMS states on the Cuntz-Krieger algebras of finite strongly connected higher-rank graphs, which are C^*-algebras of groupoids satisfying Definition 2.1, see section 6.3 below or section 12 in [aHLRS2]. Letting c be the continuous groupoid homomorphism giving rise to what the authors call the preferred dynamics, Lemma 12.1 in [aHLRS2] implies that there is exactly one $e^{-c \cdot 1}$-quasi-invariant measure and that the subgroup B described in Theorem 5.2 is the subgroup $\mathrm{Per}(\Lambda)$, see Proposition 5.2 in [aHLRS2] for the definition of $\mathrm{Per}(\Lambda)$. Then (2) in our Theorem 5.2 becomes Proposition 11.5 in [aHLRS2].

Theorem 1.3 in [N] is very useful for giving a concrete description of the KMS states when either the groupoids involved only have countably many points in the unitspace with non-trivial isotropy, or when it is possible to prove that all KMS states factors through the conditional expectation P. To illustrate how Theorem 5.2 can be used in more complex cases, we will now use it to analyse the KMS states for Cuntz-Krieger C^*-algebras of finite higher-rank graphs without sources, where neither of the two classical approaches suffices.

6. Background on higher-rank graphs

6.1. **The Cuntz-Krieger C^*-algebras of higher-rank graphs.** For $k \in \mathbb{N}$ we always denote the standard basis for $\mathbb{N}^k$ by $\{e_1, e_2, \ldots, e_k\}$, and for $n, m \in \mathbb{N}^k$ we write $n \leq m$ if $n_i \leq m_i$ for all $i = 1, 2, \ldots, k$, and $n \vee m$ for the vector in $\mathbb{N}^k$ with $(n \vee m)_i = \max\{n_i, m_i\}$ for all i. A higher-rank graph of rank $k \in \mathbb{N}$ is a pair (Λ, d) consisting of a countable small category Λ and a functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the factorisation property: for every $\lambda \in \Lambda$ and every decomposition $d(\lambda) = n + m$, $n, m \in \mathbb{N}^k$, there exists unique $\mu, \nu \in \Lambda$ with $d(\mu) = n$, $d(\nu) = m$ and $\lambda = \mu\nu$. For all $n \in \mathbb{N}^k$ we write $\Lambda^n := d^{-1}(\{n\})$ and we identify the objects of Λ with $\Lambda^0 \subseteq \Lambda$ and call these vertices. Elements of Λ are referred to as paths, and we use the range and source maps $r, s : \Lambda \to \Lambda^0$ to make sense of the start $s(\lambda)$ and the end $r(\lambda)$ of paths λ in Λ. For $0 \leq l \leq n \leq m$ and $\lambda \in \Lambda^m$ we denote by $\lambda(0, l) \in \Lambda^l$, $\lambda(l, n) \in \Lambda^{n-l}$ and $\lambda(n, m) \in \Lambda^{m-n}$ the unique paths with $\lambda = \lambda(0, l)\lambda(l, n)\lambda(n, m)$. We often abbreviate and write Λ for a higher-rank graph of rank k and simply call it a k-graph. For any $X, Y \subseteq \Lambda$ we write XY for the set:

$$XY := \{\mu\lambda \mid \mu \in X, \lambda \in Y \text{ and } s(\mu) = r(\lambda)\}$$

and we use variations on this theme to define sets throughout the next sections. We say that a k-graph Λ is *finite* if Λ^n is finite for all $n \in \mathbb{N}^k$ and *without sources* if $v\Lambda^n \neq \emptyset$ for all $n \in \mathbb{N}^k$ and $v \in \Lambda^0$. We can define a relation on Λ^0 by defining $v \leq w$ if $v\Lambda w \neq \emptyset$, i.e. if there is a path starting in w and ending in v. This gives an equivalence relation $\sim$ on Λ^0 by defining $v \sim w$ if $v \leq w$ and $w \leq v$. We call these equivalence classes *components*, and more specifically we call a component C *trivial* if $C\Lambda C = \{v\}$ for some $v \in \Lambda^0$ and *non-trivial* if this is not the case. The relation $\leq$ descends to a partial order on the set of components, i.e. $C \leq D$ if $C\Lambda D \neq \emptyset$. For sets $V \subseteq \Lambda^0$ we define the *closure* of V to be $\overline{V} = \{w \in \Lambda^0 \mid w\Lambda V \neq \emptyset\}$ and the *hereditary closure* to be $\widehat{V} = \{w \in \Lambda^0 \mid V\Lambda w \neq \emptyset\}$. For any set S that is closed, hereditary closed or a component we can define a new higher-rank graph (Λ_S, d) where $\Lambda_S = S\Lambda S$. A graph is called *strongly connected* if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$, and we notice that (Λ_C, d) is a strongly connected graph for all components C of Λ^0.

For a finite k-graph Λ we can define the $\Lambda^0 \times \Lambda^0$ vertex matrices $A_1, \ldots, A_k$ with entries $A_i(v, w) = |v\Lambda^{e_i}w|$. The factorisation property implies that these commute, and defining $A^n = \prod_{i=1}^k A_i^{n_i}$ for each $n \in \mathbb{N}^k$ one can prove that $A^n(v, w) = |v\Lambda^n w|$.

Definition 6.1. *Let Λ be a finite k-graph without sources. A Cuntz-Krieger Λ-family is a set of partial isometries $\{t_\lambda \mid \lambda \in \Lambda\}$ in a C^*-algebra satisfying:*

(CK1) $\{t_v \mid v \in \Lambda^0\}$ *is a set of mutually orthogonal projections.*
(CK2) $t_\lambda t_\gamma = t_{\lambda\gamma}$ *for all* $\lambda, \gamma \in \Lambda$ *with* $r(\gamma) = s(\lambda)$.
(CK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ *for all* $\lambda \in \Lambda$.
(CK4) $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ *for all* $v \in \Lambda^0$ *and* $n \in \mathbb{N}^k$.

We let $C^*(\Lambda)$ denote the C^*-algebra generated by a universal Cuntz-Krieger Λ-family.

To ease notation we define the projection $p_v := t_v$ for all $v \in \Lambda^0$, and we remind the reader that $C^*(\Lambda) = \overline{\operatorname{span}}\{t_\lambda t_\gamma^* \mid \lambda, \gamma \in \Lambda,\ s(\lambda) = s(\gamma)\}$ and that the universal property of $C^*(\Lambda)$ guarantees a strongly continuous action $\gamma : \mathbb{T}^k \to \operatorname{Aut}(C^*(\Lambda))$ by specifying that:

$$\gamma_z(t_\lambda) = z^{d(\lambda)} t_\lambda = \prod_{i=1}^{k} z_i^{d(\lambda)_i} t_\lambda \qquad \forall z \in \mathbb{T}^k, \quad \forall \lambda \in \Lambda \tag{6.1}$$

By setting:

$$\Lambda^{\min}(\lambda, \gamma) := \{(\delta, \nu) \in \Lambda \times \Lambda \mid \lambda\delta = \gamma\nu \text{ and } d(\lambda\delta) = d(\lambda) \vee d(\gamma)\}$$

for any $\lambda, \gamma \in \Lambda$, we furthermore have the equality:

$$t_\lambda^* t_\gamma = \sum_{(\delta,\nu)\in\Lambda^{\min}(\lambda,\gamma)} t_\delta t_\nu^* \tag{6.2}$$

6.2. **KMS states on Cuntz-Krieger algebras of higher-rank graphs.** Let Λ be a finite k-graph without sources. For any $r \in \mathbb{R}^k$ we can define a map $\mathbb{R} \ni t \to (e^{itr_1}, \dots, e^{itr_k}) \in \mathbb{T}^k$. Composing this map with the action γ from (6.1) yields a continuous one-parameter group $\{\alpha_t^r\}_{t\in\mathbb{R}}$ satisfying:

$$\alpha_t^r(t_\lambda t_\gamma^*) = \prod_{l=1}^{k} (e^{itr_l})^{d(\lambda)_l} \prod_{l=1}^{k} (e^{-itr_l})^{d(\gamma)_l} t_\lambda t_\gamma^* = e^{itr\cdot(d(\lambda)-d(\gamma))} t_\lambda t_\gamma^*$$

for all $\lambda, \gamma \in \Lambda$. We are interested in determining the β-KMS states for all $\beta \in \mathbb{R}$ and all C^*-dynamical systems $(C^*(\Lambda), \alpha^r)$ where $r \in \mathbb{R}^k$, and for this it suffices to check the KMS condition on pairs of elements on the form $t_\lambda t_\gamma^*$ with $\lambda, \gamma \in \Lambda$.

6.3. **The path groupoid for a finite higher-rank graph without sources.** For a finite k-graph Λ without sources we can realise $C^*(\Lambda)$ as a groupoid C^*-algebra. To do this, we first need to introduce the infinite path space Λ^∞ of Λ. The standard example of a k-graph Ω_k is constructed by considering morphisms:

$$\Omega_k := \{(n, m) \in \mathbb{N}^k \times \mathbb{N}^k \mid n \leq m\}$$

and objects $\Omega_k^0 := \mathbb{N}^k$ and then defining $s(n, m) = m$, $r(n, m) = n$, $d(n, m) = m - n$ and $(n, m)(m, q) = (n, q)$. An infinite path in the k-graph Λ is then a functor $x : \Omega_k \to \Lambda$ that intertwines the degree maps, and we denote the set of infinite paths in Λ by Λ^∞. Defining for each $\lambda \in \Lambda$ a set $Z(\lambda) = \{x \in \Lambda^\infty \mid x(0, d(\lambda)) = \lambda\}$ we get a basis $\{Z(\lambda)\}_{\lambda\in\Lambda}$ of compact and open sets, making Λ^∞ a second countable compact Hausdorff space. For each $p \in \mathbb{N}^k$ we can define a continuous map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by setting $\sigma^p(x)$ to be the infinite path $\sigma^p(x)(n, m) = x(n+p, m+p)$ for all $(n, m) \in \Omega_k$, and for any $p, q \in \mathbb{N}^k$ and $x \in \Lambda^\infty$ we then have that $\sigma^p(\sigma^q(x)) = \sigma^{p+q}(x) = \sigma^q(\sigma^p(x))$. Setting $r(x) = x((0, 0))$ for $x \in \Lambda^\infty$ we can compose $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ when $r(x) = s(\lambda)$ to get a new infinite path $\lambda x \in \Lambda^\infty$. Using Λ^∞ we can now obtain the path groupoid by defining:

$$\mathcal{G} = \left\{(x, m - n, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty \mid m, n \in \mathbb{N}^k \text{ and } \sigma^m(x) = \sigma^n(y)\right\}$$

one can check that this is in fact a groupoid when defining composition as:

$$(x, a, y)(y, b, z) = (x, a + b, z)$$

and inversion by $(x, a, y)^{-1} = (y, -a, x)$ and we then obtain range and source maps satisfying $r(x, a, y) = (x, 0, x)$ and $s(x, a, y) = (y, 0, y)$. The groupoid $\mathcal{G}$ becomes a locally compact second countable Hausdorff étale groupoid when we consider a basis $\{Z(\lambda, \gamma) \mid \lambda, \gamma \in \Lambda,\ s(\lambda) = s(\gamma)\}$ where:

$$Z(\lambda, \gamma) := \{(x, d(\lambda) - d(\gamma), z) \in \mathcal{G} \mid x \in Z(\lambda),\ z \in Z(\gamma),\ \sigma^{d(\lambda)}(x) = \sigma^{d(\gamma)}(z)\}$$

We can therefore consider the groupoid C^*-algebra $C^*(\mathcal{G})$, and it follows from Corollary 3.5 in [KP] that $C^*(\Lambda) \simeq C^*(\mathcal{G})$ under an isomorphism that maps $t_\lambda t_\gamma^*$ to $1_{Z(\lambda,\gamma)}$. Since $\mathcal{G}^{(0)} \simeq \Lambda^\infty$ we will identify the two spaces $C(\Lambda^\infty)$ and $C(\mathcal{G}^{(0)})$. The action introduced in (6.1) is then the same as the gauge-action introduced in equation (2.2), and the continuous one-parameter group $\{\alpha_t^r\}_{t\in\mathbb{R}}$ obtained using a vector $r \in \mathbb{R}^k$ is the same as the one obtained by considering the continuous groupoid homomorphism $c_r : \mathcal{G} \to \mathbb{R}$ given by $c_r(x,n,y) := r \cdot n$.

7. Harmonic vectors and KMS states

In this section we start our analysis of the KMS states by describing the gauge-invariant states.

Definition 7.1. *Let Λ be a finite k-graph without sources, $\beta \in \mathbb{R}$ and $r \in \mathbb{R}^k$. If $\psi \in [0,\infty[^{\Lambda^0}$ is a vector of unit 1-norm, i.e. $\sum_v |\psi_v| = 1$, and ψ satisfies that:*

$$A_i\psi = e^{\beta r_i}\psi \qquad \text{for all } i = 1,2,\dots,k$$

we call ψ a β-harmonic vector for α^r.

Lemma 7.2. *Let Λ be a finite k-graph without sources, $\beta \in \mathbb{R}$ and $r \in \mathbb{R}^k$. Let ω be a β-KMS state for α^r, then the vector:*

$$\{\omega(p_v)\}_{v\in\Lambda^0}$$

is a β-harmonic vector for α^r.

Proof. Set $\psi_w = \omega(p_w)$ for all $w \in \Lambda^0$, then clearly $\psi_w \in [0,\infty[^{\Lambda^0}$ is of unit 1-norm. Using $(CK4)$ we have for each $i \in \{1,2,\dots,k\}$ and $v \in \Lambda^0$ that:

$$\begin{aligned}\psi_v = \omega(p_v) &= \sum_{\lambda\in v\Lambda^{e_i}} \omega(t_\lambda t_\lambda^*) = \sum_{\lambda\in v\Lambda^{e_i}} \omega(t_\lambda^* \alpha_{i\beta}^r(t_\lambda)) = \sum_{\lambda\in v\Lambda^{e_i}} e^{-\beta r\cdot d(\lambda)}\omega(t_\lambda^* t_\lambda)\\ &= e^{-\beta r_i}\sum_{w\in\Lambda^0}\sum_{\lambda\in v\Lambda^{e_i}w}\omega(p_w) = e^{-\beta r_i}\sum_{w\in\Lambda^0} A_i(v,w)\psi_w = e^{-\beta r_i}(A_i\psi)_v\end{aligned}$$

proving the Lemma. □

Inspired by Proposition 8.1 in [aHLRS2] we will now associate a measure to a β-harmonic vector.

Proposition 7.3. *Let Λ be a finite k-graph without sources, $\beta \in \mathbb{R}$ and $r \in \mathbb{R}^k$. Let ψ be a β-harmonic vector for α^r, then there exists a unique Borel probability measure M_ψ on Λ^∞ satisfying:*

$$M_\psi(Z(\lambda)) = e^{-\beta r\cdot d(\lambda)}\psi_{s(\lambda)} \qquad \forall \lambda \in \Lambda$$

Proof. For all $m,n \in \mathbb{N}^k$ with $m \leq n$ we define $\pi_{m,n} : \Lambda^n \to \Lambda^m$ by $\pi_{m,n}(\lambda) = \lambda(0,m)$. Since Λ is without sources the maps $\pi_{m,n}$ are surjective. Giving Λ^n the discrete topology for each $n \in \mathbb{N}^k$, it follows that $(\Lambda^m, \pi_{m,n})$ is an inverse system of compact topological spaces and continuous surjective maps, and hence we get a topological space $\varprojlim(\Lambda^m, \pi_{m,n})$. It is straightforward to check that

$$\Lambda^\infty \ni x \to \{x(0,m)\}_{m\in\mathbb{N}^k} \in \varprojlim(\Lambda^m, \pi_{m,n})$$

is a homeomorphism. For each $m \in \mathbb{N}^k$ we now define a measure M_m on Λ^m by:

$$M_m(S) = e^{-\beta r\cdot m}\sum_{\lambda\in S}\psi_{s(\lambda)} \qquad \text{for } S \subseteq \Lambda^m$$

For $m \leq n$ and $\lambda \in \Lambda^m$ we have:

$$M_n(\pi_{m,n}^{-1}(\{\lambda\})) = e^{-\beta r \cdot n} \sum_{\nu \in \pi_{m,n}^{-1}(\{\lambda\})} \psi_{s(\nu)} = e^{-\beta r \cdot n} \sum_{\alpha \in s(\lambda)\Lambda^{n-m}} \psi_{s(\alpha)}$$
$$= e^{-\beta r \cdot n} \sum_{w \in \Lambda^0} A^{n-m}(s(\lambda), w)\psi_w = e^{-\beta r \cdot n}(A^{n-m}\psi)_{s(\lambda)} = e^{-\beta r \cdot m}\psi_{s(\lambda)}$$
$$= M_m(\{\lambda\})$$

Combining this calculation with Lemma 5.2 in [aHKR2] gives us a regular Borel measure M_ψ on Λ^∞ such that:

$$M_\psi(Z(\lambda)) = M_m(\{\lambda\}) = e^{-\beta r \cdot m}\psi_{s(\lambda)} = e^{-\beta r \cdot d(\lambda)}\psi_{s(\lambda)}$$

for $\lambda \in \Lambda^m$. Since ψ is of unit 1-norm M_ψ is a probability measure, and M_ψ is clearly unique. □

For each M_ψ we define a state ω_ψ on $C^*(\Lambda)$ by:

$$\omega_\psi(a) = \int_{\Lambda^\infty} P(a)\ dM_\psi$$

Theorem 7.4. *Assume Λ is a finite k-graph without sources, $\beta \in \mathbb{R}$ and $r \in \mathbb{R}^k$. The map $\psi \to \omega_\psi$ is an affine bijection from the β-harmonic vectors for α^r to the gauge-invariant β-KMS states for α^r.*

Proof. Let ψ be a β-harmonic vector for α^r and let M_ψ be the corresponding Borel probability measure on Λ^∞. Since the gauge-action fixes $C(\Lambda^\infty)$ it follows that ω_ψ is a gauge-invariant state. We will now argue that it is a β-KMS state for α^r, so let $\lambda, \gamma, \delta, \epsilon \in \Lambda$ with $s(\lambda) = s(\gamma)$ and $s(\delta) = s(\epsilon)$ with $d(\delta) \geq d(\gamma)$ and $d(\epsilon) \geq d(\lambda)$. Using equation (6.2) we have that:

$$\omega_\psi(t_\lambda t_\gamma^* t_\delta t_\epsilon^*) = \sum_{(\eta,\nu)\in\Lambda^{\min}(\gamma,\delta)} \omega_\psi(t_{\lambda\eta} t_{\epsilon\nu}^*) = \sum_{(\eta,\nu)\in\Lambda^{\min}(\gamma,\delta),\ \lambda\eta=\epsilon\nu} e^{-\beta r\cdot d(\lambda\eta)}\psi_{s(\eta)}$$

On the other hand:

$$\omega_\psi(t_\delta t_\epsilon^* \alpha_{i\beta}^r(t_\lambda t_\gamma^*)) = e^{-\beta r\cdot(d(\lambda)-d(\gamma))} \sum_{(\kappa,\tau)\in\Lambda^{\min}(\epsilon,\lambda)} \omega_\psi(t_{\delta\kappa} t_{\gamma\tau}^*)$$
$$= e^{-\beta r\cdot(d(\lambda)-d(\gamma))} \sum_{(\kappa,\tau)\in\Lambda^{\min}(\epsilon,\lambda),\ \delta\kappa=\gamma\tau} e^{-\beta r\cdot d(\gamma\tau)}\psi_{s(\tau)}$$
$$= \sum_{(\kappa,\tau)\in\Lambda^{\min}(\epsilon,\lambda),\ \delta\kappa=\gamma\tau} e^{-\beta r\cdot d(\lambda\tau)}\psi_{s(\tau)}$$

Now we claim that $(\eta, \nu) \to (\nu, \eta)$ is a bijection from $\{(\eta, \nu) \in \Lambda^{\min}(\gamma, \delta) \mid \lambda\eta = \epsilon\nu\}$ to $\{(\kappa, \tau) \in \Lambda^{\min}(\epsilon, \lambda) \mid \delta\kappa = \gamma\tau\}$. To see this, notice that by assumption $d(\gamma) \vee d(\delta) = d(\delta)$ and $d(\epsilon) \vee d(\lambda) = d(\epsilon)$, so $d(\nu) = 0$ and:

$$d(\lambda\eta) = d(\epsilon\nu) = d(\epsilon) + d(\nu) = d(\epsilon) = d(\epsilon) \vee d(\lambda)$$

So $(\nu, \eta) \in \Lambda^{\min}(\epsilon, \lambda)$ and by choice $\delta\nu = \gamma\eta$, proving that the map is well defined. It is straightforward to check that it has an inverse given by $(\kappa, \tau) \to (\tau, \kappa)$ and hence it is a bijection. This implies that:

$$\omega_\psi(t_\lambda t_\gamma^* t_\delta t_\epsilon^*) = \sum_{(\eta,\nu)\in\Lambda^{\min}(\gamma,\delta),\ \lambda\eta=\epsilon\nu} e^{-\beta r\cdot d(\lambda\eta)}\psi_{s(\eta)} = \sum_{(\kappa,\tau)\in\Lambda^{\min}(\epsilon,\lambda),\ \delta\kappa=\gamma\tau} e^{-\beta r\cdot d(\lambda\tau)}\psi_{s(\tau)}$$

This proves that ω_ψ satisfies the β-KMS condition for such pairs $t_\lambda t_\gamma^*$, $t_\delta t_\epsilon^*$. For such a pair not necessarily satisfying $d(\delta) \geq d(\gamma)$ and $d(\epsilon) \geq d(\lambda)$, taking a large n and using $(CK4)$ yield:

$$\omega_\psi(t_\lambda t_\gamma^* t_\delta t_\epsilon^*) = \sum_{v \in s(\delta)\Lambda^n} \omega_\psi(t_\lambda t_\gamma^* t_{\delta v} t_{\epsilon v}^*) = \sum_{v \in s(\delta)\Lambda^n} \omega_\psi(t_{\delta v} t_{\epsilon v}^* \alpha_{i\beta}^r(t_\lambda t_\gamma^*)) = \omega_\psi(t_\delta t_\epsilon^* \alpha_{i\beta}^r(t_\lambda t_\gamma^*))$$

proving that ω_ψ is a β-KMS state for α^r.

So $\psi \to \omega_\psi$ is well-defined. For injectivity, notice that:

$$\omega_\psi(p_v) = M_\psi(Z(v)) = \psi_v$$

by definition of M_ψ. To prove that it is surjective, take a gauge-invariant β-KMS state for α^r, say ω. It follows from Lemma 7.2 that setting $\psi_v = \omega(p_v)$ then ψ is a β-harmonic vector for α^r. It follows from Lemma 3.1 that $\omega = \omega \circ P$, so ω is given by a Borel probability measure M on Λ^∞. Since ω is a KMS state we have:

$$M(Z(\lambda)) = \omega(t_\lambda t_\lambda^*) = e^{-\beta r \cdot d(\lambda)} \omega(t_\lambda^* t_\lambda) = e^{-\beta r \cdot d(\lambda)} \psi_{s(\lambda)} = M_\psi(Z(\lambda))$$

proving that $\omega_\psi = \omega$, and hence surjectivity. □

7.1. **Decomposition of Harmonic vectors.** To describe the gauge-invariant KMS states Theorem 7.4 informs us that it is sufficient to analyse the set of harmonic vectors, which we will do in the following. It turns out, much like in the case for 1-graphs, that the set of harmonic vectors is a finite simplex, and that the extremal points in this simplex arise from certain components in the graph, see e.g. [aHLRS1] or [CT] for the 1-graph case (but be aware that [CT] uses a different convention for traversing paths). The technique used in [aHLRS1] required that the vertex set was ordered such that the vertex matrix was block upper diagonal, this is however difficult to do for graphs of rank $k > 1$, since one has to juggle numerous vertex matrices at once, c.f. Section 3 of [aHKR1]. To overcome this problem we define a new matrix A_F that incorporates all the vertex matrices as follows: Let F be a finite sequence $\{a_1, a_2, \ldots, a_m\}$ of elements in $\mathbb{N}^k \setminus \{0\}$ and set:

$$A_F := \sum_{n \in F} A^n = \sum_{j=1}^m A^{a_j}$$

Our reason for considering F as a sequence is that we allow for the same vector to occur multiple times in F. We call such a set F *well chosen* if for all $v, w \in \Lambda^0$, $A_F(v,w) > 0$ if and only if $v\Lambda^l w \neq \emptyset$ for some $l \in \mathbb{N}^k \setminus \{0\}$. Since $A^n(v,w) = |v\Lambda^n w|$ it follows that there always exist a well chosen set, and if F is well chosen and $S = \{b_1, \ldots b_q\}$ is a finite sequence in $\mathbb{N}^k \setminus \{0\}$, then $F \cup S = \{a_1, \ldots, a_m, b_1, \ldots, b_q\}$ is a well chosen set as well. Similarly to the strongly connected graph case it turns out that there is a connection between eigenvectors for A_F and eigenvectors for the vertex matrices A_i, $i = 1, \ldots, k$, c.f. Proposition 3.1 in [aHLRS2]. Given a k-graph Λ, a $\Lambda^0 \times \Lambda^0$ matrix B and $S, R \subseteq \Lambda^0$ we will write $B^{R,S}$ for the matrix B restricted to the rows R and columns S, and when we in the following write $A_F^{R,S}$ we specifically mean $(A_F)^{R,S}$. Set $B^S := B^{S,S}$. For vectors x we will denote the restriction to a set S by $x|_S$.

Definition 7.5. *Let F be a well chosen set for a finite k-graph Λ without sources. We say that a non-trivial component C is F-harmonic if either $\overline{C} \setminus C = \emptyset$ or:*

$$\rho(A_F^C) > \rho(A_F^{\overline{C} \setminus C})$$

We call a component C positive if $\rho(A_i^C) > 0$ for all $i \in \{1, 2, \ldots, k\}$.

Notice for the following Lemma that if C is a non-trivial component and F is well chosen, then A_F^C is a strictly positive integer matrix, and hence $\rho(A_F^C) > 0$.

Lemma 7.6. *Let F be a well chosen set for a finite k-graph Λ without sources, and let C be a F-harmonic component. Then there exists a unique vector $x_F^C \in [0,\infty[^{\Lambda^0}$ of unit 1-norm that satisfies:*

1. $A_F x_F^C = \rho(A_F^C) x_F^C$
2. $(x_F^C)_v = 0$ *for* $v \notin \overline{C}$.

This vector will furthermore satisfy that $(x_F^C)_v > 0$ for all $v \in \overline{C}$ and that $x_F^C|_C = cx$ where $c > 0$ and x is the unimodular Perron-Frobenious eigenvector for A_F^C.

Proof. A_F^C is strictly positive, so there exists a unique vector $x \in]0,\infty[^C$ with unit 1-norm such that $A_F^C x = \rho(A_F^C)x$. If $\overline{C} \setminus C \neq \emptyset$ it follows by choice of C that the matrix $\rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C}$ has spectral radius strictly less than 1. So:

$$\left(1^{\overline{C}\setminus C} - \rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C}\right)^{-1} = \sum_{n=0}^{\infty} \left(\rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C}\right)^n$$

We define a vector $x^C \in [0,\infty[^{\overline{C}}$ as follows; If $\overline{C} \setminus C = \emptyset$ we set $x^C = x$ and if $\overline{C} \setminus C \neq \emptyset$ we set:

$$x^C|_{\overline{C}\setminus C} = \left(1^{\overline{C}\setminus C} - \rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C}\right)^{-1} (\rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C, C})x \quad , \quad x^C|_C = x$$

By definition of F, $A_F^{\overline{C}\setminus C, C}$ is a matrix with strictly positive entries, so $x_v^C > 0$ for all $v \in \overline{C}$. If $v \in C$ and $w \in \overline{C} \setminus C$ then $v\Lambda w = \emptyset$. So $A_F^{C, \overline{C}\setminus C} = 0$, and this implies that $\left(A_F^{\overline{C}} x^C\right)|_C = A_F^C x = \rho(A_F^C) x^C|_C$ and:

$$\begin{aligned}
\left(A_F^{\overline{C}} x^C\right)|_{\overline{C}\setminus C} &= A_F^{\overline{C}\setminus C} \sum_{n=0}^{\infty} \left(\rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C}\right)^n (\rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C, C})x + A_F^{\overline{C}\setminus C, C} x \\
&= \rho(A_F^C) \sum_{n=0}^{\infty} \left(\rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C}\right)^{n+1} (\rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C, C})x + \rho(A_F^C) \left(\rho(A_F^C)^{-1} A_F^{\overline{C}\setminus C, C}\right) x \\
&= \rho(A_F^C) x^C|_{\overline{C}\setminus C}
\end{aligned}$$

Expand x^C to a vector $x_F^C \in [0,\infty[^{\Lambda^0}$ by setting $(x_F^C)_v = 0$ when $v \notin \overline{C}$ and $x_F^C|_{\overline{C}}$ to be the normalisation of x^C. Since $A_F^{\Lambda^0\setminus\overline{C}, \overline{C}} = 0$ it then follows that:

$$A_F x_F^C = \rho(A_F^C) x_F^C$$

which proves existence.

To prove uniqueness, assume $y \in [0,\infty[^{\Lambda^0}$ is of unit 1-norm and satisfies (1) and (2). Then by (2) and since $A_F^{C, \overline{C}\setminus C} = 0$ we have:

$$(A_F y)|_{\overline{C}} = A_F^{\overline{C}}(y|_{\overline{C}}) \qquad \left(A_F^{\overline{C}}(y|_{\overline{C}})\right)|_C = A_F^C(y|_C)$$

Combined this implies that $\rho(A_F^C)(y|_C) = (A_F y)|_C = A_F^C(y|_C)$, and hence $y|_C = \lambda x_F^C|_C$ for some $\lambda \in [0,\infty[$. If $\overline{C} \setminus C = \emptyset$ it follows that $y = x_F^C$ since they both have unit 1-norm, so assume $\overline{C} \setminus C \neq \emptyset$. Now $y - \lambda x_F^C$ is a vector supported on $\overline{C} \setminus C$ satisfying:

$$A_F(y - \lambda x_F^C) = \rho(A_F^C)(y - \lambda x_F^C)$$

but:

$$\rho(A_F^C)(y - \lambda x_F^C)|_{\overline{C}\setminus C} = \left(A_F(y - \lambda x_F^C)\right)|_{\overline{C}\setminus C} = A_F^{\overline{C}\setminus C}(y - \lambda x_F^C)|_{\overline{C}\setminus C}$$

since $\rho(A_F^{\overline{C}\setminus C}) < \rho(A_F^C)$ this implies that $y = \lambda x_F^C$, and since they both have unit 1-norm we must have that $y = x_F^C$. □

Lemma 7.7. *Let F be a well chosen set for a finite k-graph Λ without sources, and let C be a F-harmonic component. The vector x_F^C from Lemma 7.6 satisfies:*

$$A_i x_F^C = \rho(A_i^C) x_F^C$$

for $i = 1, \ldots, k$.

Proof. Since $A_i^{\Lambda^0\setminus\overline{C},\overline{C}} = 0$ and $x_F^C|_{\Lambda^0\setminus\overline{C}} = 0$ we have that:

$$\left(A_i x_F^C\right)|_{\Lambda^0\setminus\overline{C}} = 0$$

Now it follows that:

$$A_F(A_i x_F^C) = A_i(A_F x_F^C) = \rho(A_F^C)(A_i x_F^C)$$

Since $A_i x_F^C \in [0,\infty[^{\Lambda^0}$ Lemma 7.6 implies that $A_i x_F^C = \lambda_i x_F^C$ for some $\lambda_i \in [0,\infty[$ for each i, and hence:

$$\lambda_i x_F^C|_C = \left(A_i x_F^C\right)|_C = A_i^C(x_F^C|_C)$$

Since $x_F^C|_C$ is strictly positive we can conclude from Lemma 3.2 in [aHLRS2] that $\lambda_i = \rho(A_i^C)$. By definition of λ_i this proves the Lemma. □

Combining Lemma 7.7 and Lemma 7.6 it follows that a positive F-harmonic component C gives rise to a β-harmonic vector for α^r, x_F^C, when r is defined by:

$$r := \frac{1}{\beta}\left(\ln(\rho(A_1^C)), \ln(\rho(A_2^C)), \ldots, \ln(\rho(A_k^C))\right)$$

We will now prove that all β-harmonic vectors for α^r can be decomposed as convex combinations of such vectors. To do this we will need the following technical Lemma. Notice that it deals with graphs that might have sources, which will prove important in its utilization.

Lemma 7.8. *Let Λ be a finite k-graph for some $k \in \mathbb{N}$, and let $B \in M_{\Lambda^0}([0,\infty[)$ be a matrix satisfying that for all $v, w \in \Lambda^0$ then $B(v,w) > 0$ if and only if there exists a $n \in \mathbb{N}^k \setminus \{0\}$ with $v\Lambda^n w \neq \emptyset$. Then:*

$$\rho(B) = \max_{C \in \Lambda^0/\sim} \rho(B^C)$$

Proof. We will prove this by arranging the vertices of Λ^0 such that B appears in a block upper triangular form with the block matrices consisting of the matrices B^C with $C \in \Lambda^0/\sim$. This will prove the assertion in the Lemma, since the determinant of a block upper triangular matrix is the product of the determinants of the blocks.

To do this we define a directed 1-graph $|\Lambda| = (V, E, r, s)$ by setting $V = \Lambda^0$, $E = \Lambda^{e_1} \cup \cdots \cup \Lambda^{e_k}$ and letting s and r be the restriction of the source and range map on Λ. Let E^* denote the finite paths in $|\Lambda|$. Then using the factorisation property of d it follows that for all $v, w \in V$ we have $vE^*w \neq \emptyset$ if and only if

$v\Lambda w \neq \emptyset$. So defining a relation on V by $v \leq w$ if $vE^*w \neq \emptyset$ we get exactly the same relation on $V = \Lambda^0$ as defined in Section 6.1. Now we order the vertex set for the directed graph $|\Lambda|$ as it was done in Section 2.3 of [aHLRS1], giving us a numbering of the vertices $V = \{v_1, v_2, \dots, v_{|V|}\}$ satisfying that if $v_i \leq v_j$ then either $i \leq j$ or $v_i \sim v_j$, and that vertices in the same component are grouped together. For this order on Λ^0 it follows that B has the desired form. $\square$

The following proposition and its proof is inspired by Lemma 3.5 in [CT].

Proposition 7.9. *Let Λ be a finite k-graph without sources and let $\psi \in [0,\infty[^{\Lambda^0}$ be a β-harmonic vector for α^r for some $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. Let F be well chosen. Then there exists a unique collection $\mathcal{C}$ of F-harmonic components and positive numbers $t_C \in]0,1]$, $C \in \mathcal{C}$, such that:*

$$\psi = \sum_{C \in \mathcal{C}} t_C x_F^C$$

Furthermore each $C \in \mathcal{C}$ is positive with $C \nleq C'$ for $C' \in \mathcal{C} \setminus \{C\}$, and each $C \in \mathcal{C}$ satisfies:

$$r = \frac{1}{\beta}(\ln(\rho(A_1^C)), \dots, \ln(\rho(A_k^C)))$$

and that x_F^C is a β-harmonic vector for α^r.

Proof. We will first prove that such a decomposition exists. Since ψ is β-harmonic for α^r we know that $A_i \psi = e^{\beta r_i}\psi$ for all $i = 1, 2, \dots, k$. This implies that:

$$A_F \psi = \sum_{n \in F} A^n \psi = \sum_{n \in F} e^{\beta r \cdot n} \psi$$

Let $K := \sum_{n \in F} e^{\beta r \cdot n}$, then $K > 0$ and we let

$$B := K^{-1} A_F \in M_{\Lambda^0}([0,\infty[)$$

B is then a non-negative matrix with the property that $B_{v,w} > 0$ if and only if there is a non-trivial path from w to v, and:

$$B\psi = \psi$$

Set $W := \{v \in \Lambda^0 \mid \psi_v > 0\}$. We claim W is closed, i.e. $\overline{W} = W$. To see this, let $\lambda \in v\Lambda w$ for some $w \in W$ and $v \neq w$, then $B_{v,w} > 0$ and hence:

$$0 < B_{v,w}\psi_w \leq (B\psi)_v = \psi_v \Rightarrow v \in W$$

For any $v, w \in W$ we have $B^n_{v,w}\psi_w \leq (B^n\psi)_v \leq \psi_v$, so setting

$$L := \max\{\psi_v/\psi_w \mid v, w \in W\} > 0$$

we get that $B^n_{v,w} \leq L$ for all $n \in \mathbb{N}$ and $v, w \in W$. Using the properties of B and W we get that $(B^W)^n_{v,w} = B^n_{v,w}$ for all $v, w \in W$, so $\|(B^W)^n\|_F \leq |W| \cdot L$ for all $n \in \mathbb{N}$ where $\|\cdot\|_F$ denotes the Frobenius norm. By Gelfands Formula:

$$\rho(B^W) = \lim_{n\to\infty} \|(B^W)^n\|_F^{1/n} \leq 1$$

Since $B\psi = \psi$ we get that $\psi|_W = (B\psi)|_W = B^W(\psi|_W)$, and hence $\rho(B^W) = 1$. Using Lemma 7.8 on the graph Λ_W we get:

$$\rho(B^W) = \max_C \rho(B^C)$$

where max is taking over the components C in Λ_W. Let $\mathcal{C}'$ be the collection of components C in W with $\rho(B^C) = 1$, and let $\mathcal{C}$ be the minimal elements of $\mathcal{C}'$ with respect to the order $\leq$. We now claim that $\mathcal{C}$ consists of F-harmonic components C satisfying $\rho(A_F^C) = K$. For $C \in \mathcal{C}$ we have:

$$1 = \rho(B^C) = \rho((K^{-1}A_F)^C) = \rho(A_F^C)/K \Rightarrow \rho(A_F^C) = K$$

Since $K > 0$ this also implies that C is non-trivial. Since $\overline{C} \subseteq W$ we have as before that $\rho(B^{\overline{C}\setminus C}) \leq 1$ using Gelfand. If $\rho(B^{\overline{C}\setminus C}) = 1$ there must be some component $D \subseteq \overline{C} \setminus C$ with $\rho(B^D) = 1$, but since this implies that $D \leq C$, $D \neq C$ and $D \in \mathcal{C}'$ this cannot be the case. So $\rho(B^{\overline{C}\setminus C}) < 1$ and hence:

$$1 > \rho(B^{\overline{C}\setminus C}) = \rho((K^{-1}A_F)^{\overline{C}\setminus C}) = \rho(A_F^{\overline{C}\setminus C})/K \Rightarrow \rho(A_F^{\overline{C}\setminus C}) < K = \rho(A_F^C)$$

proving that C is in fact F-harmonic.

For any $D \in \mathcal{C}$ we have that $B^D(\psi|_D) \leq (B\psi)|_D = \psi|_D$, so:

$$A_F^D(\psi|_D) \leq K\psi|_D = \rho(A_F^D)\psi|_D$$

Since $\psi|_D$ is strictly positive the subinvariance theorem now imply that $A_F^D(\psi|_D) = \rho(A_F^D)\psi|_D$, and hence $\psi|_D$ is a positive eigenvector for A_F^D with eigenvalue $\rho(A_F^D)$. However this is also the case for $x_F^D|_D$, so there is a positive number $t_D > 0$ such that $\psi|_D = t_D x_F^D|_D$. Set:

$$\eta = \psi - \sum_{D\in\mathcal{C}} t_D x_F^D$$

Since $A_F x_F^D = \rho(A_F^D)x_F^D = Kx_F^D$ we see that $Bx_F^D = x_F^D$ and hence $B\eta = \eta$. The vector $\eta \in \mathbb{R}^{\Lambda^0}$ has $\eta_v = 0$ for $v \notin W$. By definition $D, D' \in \mathcal{C}$ has $D \cap \overline{D'} \neq \emptyset$ if and only if $D = D'$, so we also have that $\eta_v = 0$ for $v \in J := \bigcup_{D\in\mathcal{C}} D$. Set H to be the hereditary closure of J in Λ, $H = \widehat{J}$, and consider $D \in \mathcal{C}$. If $v \in (H \setminus J) \cap \overline{D}$, then there is $D' \in \mathcal{C}$ such that $D'\Lambda v \neq \emptyset$ and $v\Lambda D \neq \emptyset$, and hence by composing paths $D'\Lambda D \neq \emptyset$. So $D' \leq D$. If $D' = D$ then $v \in D \subseteq J$, so $D \neq D'$, and since $\mathcal{C}$ consists of minimal elements we reach a contradiction. So $(H \setminus J) \cap \overline{D} = \emptyset$ and hence $\eta|_{H\setminus J} = \psi|_{H\setminus J} \geq 0$. For any $w \in H \setminus J$ we have a $v \in J$ such that $v\Lambda w \neq \emptyset$, i.e. $B_{v,w} \neq 0$, and hence:

$$0 = \eta_v = (B\eta)_v = \sum_{u\in\Lambda^0} B_{v,u}\eta_u = \sum_{u\in H\setminus J} B_{v,u}\eta_u \geq B_{v,w}\eta_w \geq 0$$

So $\eta|_H = 0$. Since H contains all components C in W with $\rho(B^C) = 1$, we get that $\rho(B^{W\setminus H}) < 1$. However for $v \in W \setminus H$ we have:

$$(B^{W\setminus H}\eta|_{W\setminus H})_v = \sum_{w\in W\setminus H} B_{v,w}\eta_w = \sum_{w\in W} B_{v,w}\eta_w = \sum_{w\in\Lambda^0} B_{v,w}\eta_w = \eta_v$$

So $\eta|_{W\setminus H} = 0$, and hence $\eta = 0$, proving existence.

To prove uniqueness, assume that $\mathcal{D}$ is a collection of F-harmonic components and that there exists $s_D > 0$ for all $D \in \mathcal{D}$ such that:

$$\psi = \sum_{D\in\mathcal{D}} s_D x_F^D$$

So $W = \bigcup_{D\in\mathcal{D}} \overline{D} = \bigcup_{C\in\mathcal{C}} \overline{C}$ and hence for any $C \in \mathcal{C}$ there is a $D \in \mathcal{D}$ with $C \subseteq \overline{D}$. Assume $C \neq D$, then there is another $C' \in \mathcal{C}$ such that $D \subseteq \overline{C'}$, and hence $C \subseteq \overline{C'}$ with $C \neq C'$, contradicting the choice of $\mathcal{C}$. So $\mathcal{C} \subseteq \mathcal{D}$ and for $C \in \mathcal{C}$ the only $D \in \mathcal{D}$

with $\overline{D} \cap C \neq \emptyset$ is $D = C$, and hence we get $\psi|_C = s_C x_F^C|_C$. By choice of $\mathcal{C}$ we know $\overline{C'} \cap C = \emptyset$ for $C' \in \mathcal{C} \setminus \{C\}$ and hence we also have $\psi|_C = t_C x_F^C|_C$. This implies that $t_C = s_C$ for $C \in \mathcal{C}$, and hence:

$$0 = \psi - \psi = \sum_{D \in \mathcal{D}} s_D x_F^D - \sum_{C \in \mathcal{C}} t_C x_F^C = \sum_{D \in \mathcal{D} \setminus \mathcal{C}} s_D x_F^D$$

proving the uniqueness. Lemma 7.7 implies that each x_F^C satisfies:

$$A_i x_F^C = \rho(A_i^C) x_F^C$$

Fix a $i \in \{1, 2, \ldots, k\}$. By choice, $A_i \psi = e^{\beta r_i} \psi$, so multiplying with $A_i e^{-\beta r_i}$ gives:

$$\psi = \sum_{C \in \mathcal{C}} t_C \rho(A_i^C) e^{-\beta r_i} x_F^C$$

we now use the uniqueness result to get that:

$$\rho(A_i^C) e^{-\beta r_i} = 1 \Rightarrow \rho(A_i^C) = e^{\beta r_i}$$

for all $C \in \mathcal{C}$. Since i was arbitrary this proves the last statements. □

Corollary 7.10. *To be a positive F-harmonic component is independent of choice of well chosen F, and the vectors x_F^C are independent of choice of F.*

Proof. Assume that C is a positive F-harmonic component for some well chosen F, then there is a vector x_F^C such that $(x_F^C)_v = 0$ for $v \notin \overline{C}$ and that is 1-harmonic for α^r where $r = (\ln(\rho(A_1^C)), \ldots, \ln(\rho(A_k^C)))$. Let $\tilde{F}$ be another well chosen set, then by Proposition 7.9 there is a collection $\mathcal{D}$ of $\tilde{F}$-harmonic components with:

$$x_F^C = \sum_{D \in \mathcal{D}} t_D x_{\tilde{F}}^D$$

This implies that $\overline{C} = \bigcup_{D \in \mathcal{D}} \overline{D}$, so $C \in \mathcal{D}$ and hence C is a $\tilde{F}$-harmonic component. If there were a $D' \in \mathcal{D}$ with $D' \neq C$, then $D' \subseteq \overline{C} \setminus C$ which is impossible by choice of $\mathcal{D}$. So $\mathcal{D} = \{C\}$ and since x_F^C and $x_{\tilde{F}}^C$ have unit 1-norm $x_F^C = x_{\tilde{F}}^C$. □

Corollary 7.10 justifies that we drop the F and simply call it a *positive harmonic component*, and denote the vectors x^C. When we in the following write $r^1 \lneq r^2$ for vectors $r^1, r^2 \in \mathbb{R}^k$ we mean that $r_i^1 \leq r_i^2$ for all i, but that $r^1 \neq r^2$.

Lemma 7.11. *C is a positive harmonic component if and only if it is positive and*

$$\big(\rho(A_1^D), \rho(A_2^D), \ldots, \rho(A_k^D)\big) \lneq \big(\rho(A_1^C), \rho(A_2^C), \ldots, \rho(A_k^C)\big) \tag{7.1}$$

for all components $D \subseteq \overline{C} \setminus C$.

Proof. We will first argue that for all components D and well chosen F we have:

$$\rho(A_F^D) = \sum_{n \in F} \prod_{i=1}^{k} \rho(A_i^D)^{n_i} \tag{7.2}$$

If D is trivial this is true since both sides equal 0, so assume D is non-trivial. Then A_F^D is strictly positive and hence has a unimodular Perron-Frobenious eigenvector z, and since $A_i A_F = A_F A_i$ it follows that A_F^D and A_i^D commute, so $A_F^D A_i^D z =$

$\rho(A_F^D)A_i^D z$. Hence $A_i^D z = \lambda_i z$ with $\lambda_i \geq 0$ for all i. Lemma 3.2 in [aHLRS2] then implies that $\lambda_i = \rho(A_i^D)$, and hence:

$$\rho(A_F^D)z = A_F^D z = \Big(\sum_{n\in F}\prod_{i=1}^k A_i^{n_i}\Big)^D z = \sum_{n\in F}\Big(\prod_{i=1}^k A_i^{n_i}\Big)^D z$$

So if we can argue that $(\prod_i A_i^{n_i})^D = \prod_i (A_i^D)^{n_i}$ this proves (7.2). This equality follows from a straightforward induction argument on $l = n_1 + \cdots n_k$ and the fact that if $B_1, \dots, B_r$ is a set of matrices over Λ^0 satisfying that $B_i(v,w) > 0$ implies $v\Lambda w \neq \emptyset$, then $B_1 B_2 \cdots B_r$ has the same property.

Assume that C is a positive harmonic component, then for any well chosen F:

$$\rho(A_F^{\overline{C}\setminus C}) = \max_{D\subseteq \overline{C}\setminus C} \rho(A_F^D)$$

so $\rho(A_F^D) < \rho(A_F^C)$ for every component $D \subseteq \overline{C}\setminus C$ no matter the choice of well chosen F. Now fix a well chosen $F = \{a_1, \dots, a_m\}$. Assume for contradiction that there is a component $D \subseteq \overline{C}\setminus C$ and a j with $\rho(A_j^D) > \rho(A_j^C)$, and notice that this implies that D is non-trivial. Define for $s, l \in \mathbb{N}$, $F_s = \{a_1 + se_j, \dots, a_m + se_j\}$ and:

$$F_{l,s} := F \cup \bigcup_{i=1}^{l} F_s$$

Then $F_{l,s}$ is well chosen, and hence using (7.2) we get that:

$$\rho(A_F^D) + \rho(A_j^D)^s\rho(A_F^D)\cdot l = \rho(A_{F_{l,s}}^D) \ < \ \rho(A_{F_{l,s}}^C) = \rho(A_F^C) + \rho(A_j^C)^s\rho(A_F^C)\cdot l$$

for all $l, s \in \mathbb{N}$. This implies that:

$$\rho(A_F^D) \leq \frac{1+\rho(A_j^C)^s\cdot l}{1+\rho(A_j^D)^s\cdot l}\rho(A_F^C)$$

for all l, s and hence letting $l \to \infty$ we get that for all s:

$$\rho(A_F^D) \leq \frac{\rho(A_j^C)^s}{\rho(A_j^D)^s}\rho(A_F^C)$$

and hence letting $s \to \infty$ and using $\rho(A_j^D) > \rho(A_j^C)$ we get that $\rho(A_F^D) = 0$, in contradiction to the fact that A_F^D is a strictly positive integer matrix. If there were a $D \subseteq \overline{C}\setminus C$ with $\rho(A_i^D) = \rho(A_i^C)$ for each i then (7.2) would imply that $\rho(A_F^D) = \rho(A_F^C)$, also a contradiction.

Assume on the other hand that C is positive and satisfies (7.1), then (7.2) implies that $\rho(A_F^D) \leq \rho(A_F^C)$ for any $D \subseteq \overline{C}\setminus C$ and well chosen F. Fix a well chosen $F = \{a_1, \dots, a_m\}$ and define $F_i = \{a_1 + e_i, \dots, a_m + e_i\}$ for $i = 1, \dots, k$ and $\tilde{F} := F \cup F_1 \cup \cdots \cup F_k$. By definition of $\tilde{F}$, any component $D \subseteq \overline{C}\setminus C$ satisfies:

$$\rho(A_{\tilde{F}}^D) = \rho(A_F^D) + \sum_{i=1}^k \rho(A_i^D)\rho(A_F^D) < \rho(A_F^C) + \sum_{i=1}^k \rho(A_i^C)\rho(A_F^C) = \rho(A_{\tilde{F}}^C)$$

Since this is true for all $D \subseteq \overline{C}\setminus C$ we get that $\rho(A_{\tilde{F}}^{\overline{C}\setminus C}) < \rho(A_{\tilde{F}}^C)$ and hence C is a positive $\tilde{F}$-harmonic component. $\square$

Fix some $r \in \mathbb{R}^k$. For each $\beta \in \mathbb{R}$ we set $\mathcal{C}_r(\beta)$ to be the set of positive harmonic components C satisfying that $\beta r = (\ln(\rho(A_1^C)), \dots, \ln(\rho(A_k^C)))$.

Theorem 7.12. *Let Λ be a finite k-graph without sources and let $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. There is an affine bijective correspondence between the gauge-invariant β-KMS states ω for α^r and the functions $f : \mathcal{C}_r(\beta) \to [0,1]$ with $\sum_{C \in \mathcal{C}_r(\beta)} f(C) = 1$. A state ω corresponding to a function f is given by:*

$$\omega(t_\lambda t_\gamma^*) = \delta_{\lambda,\gamma} e^{-\beta r \cdot d(\lambda)} \psi_{s(\lambda)}$$

for all $\lambda, \gamma \in \Lambda$, where $\psi \in [0,\infty[^{\Lambda^0}$ is given by:

$$\psi = \sum_{C \in \mathcal{C}_r(\beta)} f(C) x^C$$

Proof. Let ω be a gauge-invariant β-KMS state for α^r and ψ be the corresponding unique β-harmonic vector for α^r given by Theorem 7.4, then for any $\lambda, \gamma \in \Lambda$:

$$\omega(t_\lambda t_\gamma^*) = \int_{\Lambda^\infty} P(t_\lambda t_\gamma^*)\, dM_\psi = \delta_{\lambda,\gamma} M_\psi(Z(\lambda)) = \delta_{\lambda,\gamma} e^{-\beta r \cdot d(\lambda)} \psi_{s(\lambda)}$$

That it is an affine bijection follows from Proposition 7.9 and the definition of $\mathcal{C}_r(\beta)$. □

8. The non gauge-invariant KMS states

We will now use Theorem 7.12 and the symmetries of the KMS-simplex to obtain a description of all the KMS states. The map $\psi \to M_\psi$ is an affine bijection from the β-harmonic vectors for α^r to the set of $e^{-\beta c_r}$-quasi-invariant measures, where $c_r(x,a,y) = a \cdot r$. So the extreme points of the simplex $\tilde{\Delta}$ of $e^{-\beta c_r}$-quasi-invariant probability measures are the measures $M_C := M_{x^C}$, where $C \in \mathcal{C}_r(\beta)$. To use Theorem 5.2 we first have to analyse the paths in Λ^∞. For a subset $S \subseteq \Lambda^0$ we say that a path $x \in \Lambda^\infty$ *eventually lies in* S if there exists a $n \in \mathbb{N}^k$ such that:

$$r(\sigma^m(x)) \in S \qquad \forall m \geq n$$

This concept proves important for describing the measures M_C, $C \in \mathcal{C}_r(\beta)$.

Lemma 8.1. *Let Λ be a finite k-graph without sources. For any component D the set:*

$$N_D := \{x \in \Lambda^\infty \mid x \text{ eventually lies in } D\}$$

is a Borel set in Λ^∞. For any $r \in \mathbb{R}^k$, $\beta \in \mathbb{R}$ and $D \in \mathcal{C}_r(\beta)$ we have $M_D(N_D) = 1$.

Proof. We first want to argue that:

$$N_{\overline{C}} := \{x \in \Lambda^\infty \mid x \text{ eventually lies in } \overline{C}\}$$

is a closed set for all components C. So let $y \in \Lambda^\infty \setminus N_{\overline{C}}$. Then there is a $m \in \mathbb{N}^k$ such that $r(\sigma^m(y)) \notin \overline{C}$, and we set $\lambda := y(0,m) \in \Lambda$. We claim that $Z(\lambda) \cap N_{\overline{C}} = \emptyset$. To see this, assume $z \in Z(\lambda) \cap N_{\overline{C}}$, then there exists a $N \geq m$ such that $r(\sigma^N(z)) \in \overline{C}$, however then $z(m,N)$ is a path with $r(z(m,N)) = s(\lambda) = r(\sigma^m(y))$ and $s(z(m,N)) = r((\sigma^N(z)) \in \overline{C}$, in contradiction to the fact that $r(\sigma^m(y)) \notin \overline{C}$. To prove that N_D is Borel, it suffices to prove that:

$$N_D = N_{\overline{D}} \setminus \left(\bigcup_{\text{components } C \subseteq \overline{D} \setminus D} N_{\overline{C}} \right)$$

If $N_D \cap N_{\overline{C}} \neq \emptyset$ for some $C \subseteq \overline{D}$, we must have $\overline{C} \cap D \neq \emptyset$, which implies that $D = C$. This proves " $\subseteq$ ". For a path z in the right hand side, numerate the finite collection of components $C_1, C_2, \dots, C_l \subseteq \overline{D} \setminus D$, and let $N_1, \dots, N_l \in \mathbb{N}^k$ be numbers such that $r(\sigma^{N_i}(z)) \notin \overline{C_i}$. It then follows that $r(\sigma^N(z)) \notin \bigcup_{i=1}^l \overline{C_i}$ for all $N \geq N_1 \vee \cdots \vee N_l$. There is a $N \geq N_1 \vee \cdots \vee N_l$ such that $r(\sigma^m(z)) \in \overline{D}$ for all $m \geq N$, so $r(\sigma^m(z)) \in D$ for all $m \geq N$, and hence $z \in N_D$.

Let $r \in \mathbb{R}^k$, $\beta \in \mathbb{R}$ and $D \in \mathcal{C}_r(\beta)$. We will first prove that $M_D(N_{\overline{D}}) = 1$. It is enough to prove that $M_D(Z(\lambda)) = 0$ for $\lambda \in \Lambda$ with $s(\lambda) \notin \overline{D}$. By definition:

$$M_D(Z(\lambda)) = e^{-\beta r \cdot d(\lambda)} x^D_{s(\lambda)}$$

however x^D is supported on $\overline{D}$, so $M_D(Z(\lambda)) = 0$, proving that $M_D(N_{\overline{D}}) = 1$. Now assume for contradiction that $M_D(N_{\overline{C}}) \neq 0$ for a $C \subseteq \overline{D} \setminus D$. If $M_D(N_{\overline{C}}) = 1$ then $M_D(Z(v)) = 0$ for $v \in D$, since then $Z(v) \cap N_{\overline{C}} = \emptyset$, however $M_D(Z(v)) = x^D_v > 0$, so $M_D(N_{\overline{C}}) \in]0,1[$. Notice that clearly $N_{\overline{C}}$ is invariant in the sense that $r(s^{-1}(N_{\overline{C}})) = s(r^{-1}(N_{\overline{C}})) = N_{\overline{C}}$, and hence as noted earlier we can decompose M_D as a non-trivial convex combination of two $e^{-\beta c_r}$-quasi-invariant measures, contradicting that M_D is extremal. So $M_D(N_{\overline{C}}) = 0$ which proves $M_D(N_D) = 1$. $\square$

Given a component $D \in \mathcal{C}_r(\beta)$ consider the graph Λ_D which is a strongly connected k-graph. Hence as in [aHLRS2] it has a Periodicity-group $\mathrm{Per}(\Lambda_D) \subseteq \mathbb{Z}^k$ associated with it. We denote this subgroup of $\mathbb{Z}^k$ as $\mathrm{Per}(D) := \mathrm{Per}(\Lambda_D)$, and remind the reader that:

$$\mathrm{Per}(D) = \{m - n \mid m, n \in \mathbb{N}^k,\ \sigma^m(x) = \sigma^n(x) \text{ for all } x \in \Lambda_D^\infty\}$$

c.f. Proposition 5.2 in [aHLRS2]. We let $\Phi : \mathcal{G} \to \mathbb{Z}^k$ denote the map $(x, a, y) \to a$. We can now obtain the entire description of KMS states.

Theorem 8.2. *Let Λ be a finite k-graph without sources and $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. There is a bijection from the pairs (C, ξ) consisting of a $C \in \mathcal{C}_r(\beta)$ and a $\xi \in \widehat{Per(C)}$ to the set of extremal β-KMS states for α^r given by:*

$$(C, \xi) \to \omega_{C,\xi}$$

where:

$$\omega_{C,\xi}(f) = \int_{X(Per(C))} \sum_{g \in \mathcal{G}_x^x} f(g) \xi(\Phi(g))\ dM_C(x) \qquad \forall f \in C_c(\mathcal{G})$$

Proof. To use Theorem 5.2 we assume for now that $\beta \neq 0$. We will prove that for a $D \in \mathcal{C}_r(\beta)$, the unique subgroup of $\mathbb{Z}^k$ described in (1) in Theorem 5.2 for the measure M_D is $\mathrm{Per}(D)$. Assume that $y \in N_D$, then there is a $m \in \mathbb{N}^k$ such that $r(\sigma^l(y)) \in D$ for all $l \geq m$, and hence $\sigma^m(y)$ can be considered as an infinite path in the graph Λ_D. It follows that for $n_1 - n_2 \in \mathrm{Per}(D)$ we have:

$$\sigma^{n_1+m}(y) = \sigma^{n_1}(\sigma^m(y)) = \sigma^{n_2}(\sigma^m(y)) = \sigma^{n_2+m}(y)$$

hence $(y, n_1 - n_2, y) \in \mathcal{G}_y^y$. So:

$$1 = M_D(N_D) = M_D(\{x \in \Lambda^\infty \mid \{x\} \times \mathrm{Per}(D) \times \{x\} \subseteq \mathcal{G}_x^x\})$$

and hence the subgroup B from Theorem 5.2 satisfies $\mathrm{Per}(D) \subseteq B$. Assume for a contradiction that $\mathrm{Per}(D) \subsetneq B$, then for $l \in B \setminus \mathrm{Per}(D)$ we have:

$$M_D(\{x \in N_D \mid (x, l, x) \in \mathcal{G}_x^x\}) = 1$$

Since $M_D(Z(v)) = x^D_v > 0$ for a $v \in D$ and since:

$$\{x \in vN_D \mid (x, l, x) \in \mathcal{G}^x_x\} \subseteq \bigcup_{n,m \in \mathbb{N}^k,\ n-m=l} \{x \in vN_D \mid \sigma^n(x) = \sigma^m(x)\}$$

there must be a $n_1, n_2 \in \mathbb{N}^k$ with $M_D(\{x \in vN_D \mid \sigma^{n_1}(x) = \sigma^{n_2}(x)\}) > 0$ and $n_1 - n_2 = l$. Now consider the measure M defined on the strongly connected graph Λ_D as in [aHLRS2], since $l \notin \mathrm{Per}(D)$ we have by Proposition 8.2 in [aHLRS2] that:

$$M(\{x \in v\Lambda^\infty_D \mid \sigma^{n_1}(x) = \sigma^{n_2}(x)\}) = 0$$

Since this set is compact we can choose an arbitrary $\varepsilon > 0$ and find a finite number of paths $\delta_i \in \Lambda_D$, $i = 1, \ldots, n$ such that letting $Z_D(\delta_i) = \{x \in \Lambda^\infty_D \mid x(0, d(\delta_i)) = \delta_i\}$ for each i we have $Z_D(\delta_i) \cap Z_D(\delta_j) = \emptyset$ for $i \neq j$ and:

$$\{x \in v\Lambda^\infty_D \mid \sigma^{n_1}(x) = \sigma^{n_2}(x)\} \subseteq \bigcup_{i=1}^n Z_D(\delta_i) \quad , \quad \sum_{i=1}^n M(Z_D(\delta_i)) \leq \varepsilon$$

The paths $\delta_i \in \Lambda_D$ can be considered as paths in Λ, and hence denoting by $Z(\delta_i) = \{x \in \Lambda^\infty \mid x(0, d(\delta_i)) = \delta_i\}$ it is straightforward to check that:

$$\{x \in vN_D \mid \sigma^{n_1}(x) = \sigma^{n_2}(x)\} \subseteq \bigcup_{i=1}^n Z(\delta_i)$$

By definition of M_D and x^D there is a $c \in]0, 1]$ such that $x^D|_D = cx$ where x is the unimodular Perron-Frobenious eigenvector for A^D_F. Since $A_i x^D = \rho(A^D_i)x^D$ it follows that $A^D_i(x^D|_D) = \rho(A^D_i)(x^D|_D)$, so since $A^D_1, \ldots, A^D_k$ are the vertex matrices for Λ_D, it follows that x is the unimodular Perron-Frobenius eigenvector of Λ_D, c.f. Definition 4.4 in [aHLRS2]. So by definition of M in Section 8 of [aHLRS2] we get:

$$\begin{aligned} M_D\left(\bigcup_{i=1}^n Z(\delta_i)\right) &\leq \sum_{i=1}^n M_D(Z(\delta_i)) = \sum_{i=1}^n e^{-\beta r \cdot d(\delta_i)} x^D_{s(\delta_i)} = c \sum_{i=1}^n e^{-\beta r \cdot d(\delta_i)} x_{s(\delta_i)} \\ &= c \sum_{i=1}^n M(Z_D(\delta_i)) \leq c\varepsilon \leq \varepsilon \end{aligned}$$

since ε was arbitrary, we reach our contradiction, and hence $B = \mathrm{Per}(D)$. In the case where $\beta = 0$ we notice that the β-KMS states for α^r are the same as the 1-KMS states for α^0, with $0 \in \mathbb{R}^k$, since α^0 is the trivial one-parameter group. However $\mathcal{C}_r(0) = \mathcal{C}_0(1)$, so we also have a bijection in this case. □

Remark 8.3. In our setting the C^*-algebra $C^*(\Lambda)$ is simple if and only if Λ is cofinal and has no local periodicity, c.f. Theorem 3.1 in [RS]. Since Λ has no sources it has to contain some positive component C, and since it is cofinal C has to be the only positive component and it has to satisfy $\hat{C} = C$. Since Λ has no local periodicity it follows that $\mathrm{Per}(C)$ is trivial. To see that C is a harmonic component, assume D is another component and $i \in \{1, 2, \ldots, k\}$. Since C is the only positive component, there exists a $n \in \mathbb{N}^k$ such that $\Lambda^n = \Lambda^n C$. Setting $N := |D\Lambda^n|$ and letting $l \in \mathbb{N}$ be arbitrary, it follows for each $v, w \in D$ and fixed $\gamma \in w\Lambda^n$ that the map:

$$v\Lambda^{le_i}w \ni \lambda \xrightarrow{\varphi} (\lambda\gamma)(n, n + le_i) \in C\Lambda^{le_i}C$$

has at most N points in $\varphi^{-1}(\{\nu\})$ for each $\nu \in C\Lambda^{le_i}C$, so $|v\Lambda^{le_i}w| \leq N|\varphi\left(v\Lambda^{le_i}w\right)| \leq N|C|^2 \cdot \|(A_i^C)^l\|_{\max}$. It follows that:

$$\|(A_i^D)^l\|_F \leq |\Lambda^0|\|(A_i^D)^l\|_{\max} \leq |\Lambda^0|N|C|^2 \cdot \|(A_i^C)^l\|_F$$

By Gelfand's formula $\rho(A_i^D) \leq \rho(A_i^C)$, so since D is not positive we conclude that C is harmonic, and Theorem 8.2 then implies that there is exactly one β-KMS state for α^r if $r = \frac{1}{\beta}(\ln(A_1^C), \dots, \ln(A_k^C))$ and no β-KMS states for α^r any other choices of β.

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KMS states on the Toeplitz algebras of higher-rank graphs

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Abstract

The Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ for a finite k-graph Λ is equipped with a continuous one-parameter group α^r for each $r \in \mathbb{R}^k$, obtained by composing the map $\mathbb{R} \ni t \to (e^{itr_1}, \dots, e^{itr_k}) \in \mathbb{T}^k$ with the gauge action on $\mathcal{T}C^*(\Lambda)$. In this paper we give a complete description of the β-KMS states for the C^*-dynamical system $(\mathcal{T}C^*(\Lambda), \alpha^r)$ for all finite k-graphs Λ and all values of $\beta \in \mathbb{R}$ and $r \in \mathbb{R}^k$.

1. Introduction

An easy way to construct a C^*-dynamical system is by considering the C^*-algebra of a directed graph and a continuous one-parameter group defined using the gauge action on the graph C^*-algebra. Describing the KMS states for such a system is a difficult but rewarding task - the KMS states usually remembers thought-provoking information about the C^*-dynamical system, and the insight obtained can frequently be generalised to a much broader class of C^*-dynamical systems. Higher-rank graphs is a natural generalisation of directed graphs, and since one would expect that they give rise to equally intriguing C^*-dynamical systems, the question of describing their KMS states has been studied intensely in recent years.

For the Toeplitz algebra of a finite strongly connected k-graph Λ without sources and sinks the simplex of KMS states was described in [5] for a specific dynamics defined using the vertex matrices of Λ (this dynamics is called "preferred" in [5]). Since then there has been contributions from a handful of papers where the objective has been to describe the KMS states for more general graphs and more general continuous one-parameter groups. The most recent contribution is [2], where the authors describe an algorithm for determining the β-KMS simplex on the Toeplitz algebra of a finite k-graph Λ and

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a continuous one-parameter group defined using a vector $r \in \mathbb{R}^k$ subject to the conditions:

1. Λ has no sinks and no sources.
2. $\beta > 0$, $r \in (0, \infty)^k$ and r has rationally independent coordinates.
3. There are no trivial strongly connected components and no isolated subgraphs in Λ
4. For all components C in Λ the graph restricted to C, Λ_C, is coordinate-wise irreducible and each vertex matrix for Λ restricted to a component C has spectral radius greater than 1.
5. If two components C and D are connected by an edge of any color in the skeleton of Λ, then they are connected by edges of all k colors.

The aim of this paper is to describe the simplex of β-KMS states on the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ for the continuous one-parameter group α^r for all values of $\beta \in \mathbb{R}$, $r \in \mathbb{R}^k$ and all finite k-graphs Λ. Our results reveal that some of the above restrictions imposed in [2] greatly reduce the size and complexity of the simplex of β-KMS states, for example they imply that the simplex is finite-dimensional, while our more general approach gives rise to simplexes with uncountably many extreme points. Furthermore our description does not involve any repeated algorithm, and we believe that this makes it much easier to use in concrete calculations.

To describe the KMS states on $\mathcal{T}C^*(\Lambda)$ for a finite k-graph Λ we proceed as follows: In section 2 we present the theory on higher-rank graphs, groupoids and C^*-dynamical systems that we will need in the paper. Section 3 is devoted to a linear algebraic result concerning vectors that are almost invariant under a family of commuting matrices. In section 4 we use the general result from section 3 to describe a bijection between certain vectors over Λ^0 and gauge-invariant KMS states on the Toeplitz algebra. We then proceed in section 5 to describe a decomposition of the gauge-invariant KMS states, which in section 6 allows us to use the theory developed in [1] to describe all KMS states. To illustrate our results we use section 7 to present a few examples and compare our results with the literature.

The techniques and approach in this paper are similar to the ones used in [1] to describe the KMS states on the Cuntz-Kriger algebras of finite higher-rank graphs without sources, and especially the analysis in section 6 that describes the non gauge-invariant KMS states is heavily inspired by ideas in [1]. The description of the gauge-invariant KMS states uses many ideas and techniques already described in the literature on the subject (e.g. in [3],

[4] and [5]). We do however find the new insight obtained regarding gauge-invariant KMS states both interesting and non-trivial, and we consider this the main contribution of this paper.

2. Background

Higher-rank graphs and their Toeplitz algebra

Throughout $\mathbb{N}$ denotes the natural numbers including zero. For $k \in \mathbb{N}$ with $k \geq 1$ we write $\{e_1, \dots, e_k\}$ for the standard generators for $\mathbb{N}^k$ and for $n, m \in \mathbb{N}^k$ we write $n \vee m$ for the pointwise maximum of n and m. A higher-rank graph (Λ, d) of rank $k \in \mathbb{N}$ with $k \geq 1$ is a pair consisting of a countable small category Λ and a functor $d : \Lambda \to \mathbb{N}^k$ that has the factorisation property, i.e. if $d(\lambda) = n + m$ for some $\lambda \in \Lambda$ and $n, m \in \mathbb{N}^k$, then there exists unique $\mu, \eta \in \Lambda$ with $d(\mu) = n$, $d(\eta) = m$ and $\lambda = \mu\eta$. We define $\Lambda^n := d^{-1}(\{n\})$ for each $n \in \mathbb{N}^k$. The factorisation property guarantees that we can identify the objects of the category Λ with Λ^0, and we call them vertexes. Likewise we think of elements λ of Λ as paths in a graph with degree $d(\lambda)$, and we use the range and the source maps $r, s : \Lambda \to \Lambda^0$ to make sense of the start $s(\lambda)$ and the end $r(\lambda)$ of our path. Some times we will write Λ instead of (Λ, d) and simply call it a k-graph, in which case it is implicit that $k \geq 1$. Whenever $X, Y \subseteq \Lambda$ we let XY denote the set of composed paths, and we use the usual conventions for defining sets of paths, e.g. $v\Lambda w := \{w\}\Lambda\{w\}$ for $v, w \in \Lambda^0$. For $I \subseteq \{1, \dots, k\}$ we set:

$$\Lambda_I := \{\lambda \in \Lambda \ : \ d(\lambda)_j = 0 \text{ for all } j \in \{1, \dots, k\} \setminus I\}$$

When $I \neq \emptyset$ Λ_I can then be considered as a $|I|$-graph by defining a $d' : \Lambda_I \to \mathbb{N}^{|I|}$ in the obvious way, but to make the notation more fluid we will let the degree functor be the restriction of d to Λ_I, i.e. we identify $\mathbb{N}^{|I|}$ with

$$\mathbb{N}^I := \{n \in \mathbb{N}^k \ : \ n_j = 0 \text{ for } j \notin I\}$$

Keeping in line with this notation, we will identify $\mathbb{N}^k$ with $\mathbb{N}^I \oplus \mathbb{N}^J$ whenever we have a partition $I \sqcup J = \{1, \dots, k\}$, and write $d(x) = (d(x)_I, d(x)_J)$. For each subset $I \subseteq \{1, \dots, k\}$ we can define a relation $\leq_I$ on Λ^0 by letting $v \leq_I w$ if $v\Lambda_I w := \{w\}\Lambda_I\{w\} \neq \emptyset$, and we can then define an equivalence relation $\sim_I$ on Λ^0 by defining $v \sim_I w$ when $v \leq_I w$ and $w \leq_I v$. We write $\sim$ instead of $\sim_{\{1,\dots,k\}}$, and when there can be no confusion about which relation $\sim_I$ we refer to we call the equivalence classes *components*. When a graph only has one component in $\sim$ we call it *strongly connected*. Our k-graph Λ is *finite* when Λ^n is a finite set for each $n \in \mathbb{N}^k$, and without sources when for each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ there is a $\lambda \in \Lambda^n$ with $r(\lambda) = v$, i.e $v\Lambda^n \neq \emptyset$. If Λ is a finite k-graph, then Λ_I is a finite $|I|$-graph for each $I \subseteq \{1, \dots, k\}$ with $I \neq \emptyset$. For $I \subseteq \{1, \dots, k\}$ and $V \subseteq \Lambda^0$ we define the *I-closure* $\overline{V}^I$ of V and the *hereditary I-closure* $\widehat{V}^I$ as:

$$\overline{V}^I := \{w \in \Lambda^0 \ : \exists v \in V,\ w \leq_I v\} \quad , \quad \widehat{V}^I := \{w \in \Lambda^0 \ : \exists v \in V,\ v \leq_I w\}$$

We write $\overline{V} := \overline{V}^{\{1,\dots,k\}}$ and $\widehat{V} := \widehat{V}^{\{1,\dots,k\}}$ and call it the closure and the hereditary closure of V. Letting $M_S(\mathbb{F})$ be the set of matrices over the finite set S with entries in $\mathbb{F}$, the *vertex matrices* $A_1, \dots, A_k \in M_{\Lambda^0}(\mathbb{N})$ for a finite k-graph Λ are the matrices with entries $A_i(v,w) = |v\Lambda^{e_i}w|$. They commute pairwise, and setting $A^n := \prod_{i=1}^k A_i^{n_i}$ for $n \in \mathbb{N}^k$ it follows that $A^n(v,w) = |v\Lambda^n w|$.

For a finite k-graph Λ, a *Toeplitz-Cuntz-Krieger Λ-family* consists of partial isometries $\{S_\lambda : \ \lambda \in \Lambda\}$ subject to the conditions:

1. $\{p_v := S_v \ : v \in \Lambda^0\}$ are mutually orthogonal projections.
2. When $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$ we have $S_{\lambda\mu} = S_\lambda S_\mu$.
3. $S_\lambda^* S_\lambda = p_{s(\lambda)}$ for every $\lambda \in \Lambda$.
4. $p_v \geq \sum_{\lambda \in v\Lambda^n} S_\lambda S_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.
5. $S_\mu^* S_\lambda = \sum_{(\kappa,\eta) \in \Lambda^{\min}(\mu,\lambda)} S_\kappa S_\eta^*$ for all $\mu, \lambda \in \Lambda$.

where $\Lambda^{\min}(\mu,\lambda) := \{(\kappa,\eta) \in \Lambda \times \Lambda \ : \mu\kappa = \lambda\eta,\ d(\mu\kappa) = d(\mu) \vee d(\lambda)\}$. The Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ of Λ is then the C^*-algebra generated by a universal Toeplitz-Cuntz-Krieger Λ-family. It follows from the definition of $\mathcal{T}C^*(\Lambda)$ that $\mathcal{T}C^*(\Lambda) = \overline{\operatorname{span}}\{S_\lambda S_\mu^* \ : \ \lambda, \mu \in \Lambda\}$ and that we have a strongly continuous action $\gamma : \mathbb{T}^k \to \operatorname{Aut}(\mathcal{T}C^*(\Lambda))$ with:

$$\gamma_z(S_\lambda) = z^{d(\lambda)} S_\lambda \qquad \text{for all } z \in \mathbb{T}^k \text{ and } \lambda \in \Lambda$$

where $z^{d(\lambda)} := \prod_{i=1}^k z_i^{d(\lambda)_i}$.

C^*-dynamical systems and KMS states

In this paper a C^*-dynamical system is a pair $(\mathcal{A}, \alpha)$ consisting of a C^*-algebra $\mathcal{A}$ and a continuous one-parameter group α, i.e. a strongly continuous representation of $\mathbb{R}$ in $\operatorname{Aut}(\mathcal{A})$. An element $a \in \mathcal{A}$ is analytic for α when there is an analytic extension of the map $\mathbb{R} \ni t \to \alpha_t(a) \in \mathcal{A}$ to the entire complex plane $\mathbb{C}$, and we then denote the value of this map at $z \in \mathbb{C}$ as $\alpha_z(a)$. A β-KMS state for the C^*-dynamical system $(\mathcal{A}, \alpha)$ is a state ω on $\mathcal{A}$ satisfying:

$$\omega(xy) = \omega(y\alpha_{i\beta}(x))$$

for all elements x, y in a norm dense, α-invariant $*$-algebra of $\mathcal{A}$ consisting of analytic elements for α.

For any $r \in \mathbb{R}^k$ we can compose the map $\mathbb{R} \ni t \to e^{itr} := (e^{itr_j})_{j=1}^k \in \mathbb{T}^k$ with the gauge action γ on $\mathcal{T}C^*(\Lambda)$ to obtain a continuous one-parameter group α^r. For all $\lambda, \mu \in \Lambda$ the map:

$$\mathbb{R} \ni t \to \alpha_t^r(S_\lambda S_\mu^*) = e^{itr\cdot(d(\lambda)-d(\mu))} S_\lambda S_\mu^*$$

has an analytic extension to $\mathbb{C}$, and hence $S_\lambda S_\mu^*$ is an analytic element for $(\mathcal{T}C^*(\Lambda), \alpha^r)$.

Realising $\mathcal{T}C^*(\Lambda)$ as a groupoid C^*-algebra

We follow [3] when introducing the groupoid of the Toeplitz algebra. We write $n \leq m$ for elements $n, m \in (\mathbb{R} \cup \{\infty\})^k$ when $n_i \leq m_i$ for all $i \in \{1, \dots, k\}$ and $n \lneq m$ when $n \leq m$ and $n \neq m$, and we use the same notation for the relation restricted to the subsets $(\mathbb{N} \cup \{\infty\})^k$ and $\mathbb{N}^k$. $\Omega_k = \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k \ : p \leq q\}$ is the standard example of a k-graph Λ without sources, and for $n \in (\mathbb{N} \cup \{\infty\})^k$ we set $\Omega_{k,n}$ equal to the subgraph $\{(p, q) \in \Omega_k \ : q \leq n\}$. For a finite k-graph Λ and each $n \in (\mathbb{N} \cup \{\infty\})^k$ we let Λ^n denote the set of degree preserving functors $x : \Omega_{k,n} \to \Lambda$ and set $d(x) := n$ and $r(x) := x(0, 0)$. When n has finite entries this set can be identified with $d^{-1}(\{n\})$, so the notation does not collide with the one already introduced. Let:

$$\Lambda^* := \bigcup_{n \in (\mathbb{N} \cup \{\infty\})^k} \Lambda^n$$

and define for each $\lambda \in \Lambda$ the cylinder set $Z(\lambda) := \{x \in \Lambda^* \ : x(0, d(\lambda)) = \lambda\}$. For each finite set $F \subseteq s(\lambda)\Lambda$ set

$$Z(\lambda \setminus F) := Z(\lambda) \setminus \left(\bigcup_{\mu \in F} Z(\lambda\mu) \right)$$

The sets $Z(\lambda \setminus F)$ then form a basis of compact open sets for a second countable locally compact Hausdorff topology on Λ^*, and since $\Lambda^* = \bigcup_{v \in \Lambda^0} Z(v)$ it follows that Λ^* is compact. Whenever we have a partition $I \sqcup J = \{1, \dots, k\}$ we write $\infty_I := (\infty)_{i\in I} \in (\mathbb{N} \cup \{\infty\})^I$ for the element with $(\infty_I)_i = \infty$ for all $i \in \mathbb{I}$ and for $m \in \mathbb{N}^J$ we set:

$$\Lambda^{\infty_I, m} := \{x \in \Lambda^* \ : \ d(x) = (\infty_I, m)\}$$

which is a Borel set by Proposition 3.2 in [3], and we set $\partial^I \Lambda := \bigcup_{m\in\mathbb{N}^J} \Lambda^{\infty_I, m}$. When $I = \emptyset$ then $\Lambda^{\infty_I, m} = \Lambda^m$ and $\partial^I\Lambda = \Lambda$. When $I = \{1, \dots, k\}$ then $\Lambda^\infty := \Lambda^{\infty_I, 0}$ is the *infinite path space* of Λ. It follows that we have a Borel partition of Λ^*, i.e.:

$$\Lambda^* = \bigsqcup_{I\subseteq\{1,\dots,k\}} \partial^I\Lambda$$

For each $n \in \mathbb{N}^k$ the formula $\sigma^n(x)(p, q) = x(p + n, q + n)$ defines a map σ^n on $\{x \in \Lambda^* \ : d(x) \geq n\}$ which we call the *shift map.* We can then define a groupoid $\mathcal{G}_\Lambda$ as:

$$\mathcal{G}_\Lambda := \{(x, p - q, y) \in \Lambda^* \times \mathbb{Z}^k \times \Lambda^* \ : \ p \leq d(x),\ q \leq d(y),\ \sigma^p(x) = \sigma^q(y)\}$$

with the usual composition and inverse. We equip $\mathcal{G}_\Lambda$ with a topology such that it becomes a locally compact second countable Hausdorff étale groupoid, satisfying that the full groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$ is isomorphic to $\mathcal{T}C^*(\Lambda)$, that the unitspace $\mathcal{G}_\Lambda^{(0)}$ is isomorphic to Λ^* with the topology generated by the sets $Z(\lambda \setminus F)$, and that $C(\Lambda^*) \simeq \overline{\operatorname{span}}\{S_\lambda S_\lambda^* \ : \lambda \in \Lambda\}$ under an isomorphism that maps $1_{Z(\lambda)} \to S_\lambda S_\lambda^*$ for each $\lambda \in \Lambda$. Furthermore the continuous one-parameter group α^r corresponds to the one arising from the groupoid homomorphism $c_r(x, n, y) = r \cdot n$ in the groupoid picture of $\mathcal{T}C^*(\Lambda)$, and the topology on $\mathcal{G}_\Lambda$ makes the map $\Phi : \mathcal{G}_\Lambda \to \mathbb{Z}^k$ given by $\Phi(x, n, y) := n$ continuous. Since the definition of the topology on $\mathcal{G}_\Lambda$ is not crucial for our exposition, we refer the reader to Appendix B in [3] for the details.

When considering the groupoid picture of $\mathcal{T}C^*(\Lambda)$ every KMS state ω on $\mathcal{T}C^*(\Lambda)$ gives rise to a Borel probability measure m on Λ^* by using the Riesz Representatiom Theorem on ω restricted to $C(\Lambda^*)$. We say that this measure is *the measure associated to* ω, and by Theorem 1.3 in [7] such measure are exactly the probability measures that are quasi-invariant with Radon-Nikodym cocycle $e^{-\beta c_r}$. Since any such measure restricted to an invariant Borel subset of Λ^*, i.e. a Borel set B with $s(r^{-1}(B)) = B$, is again a quasi-invariant measure with Radon-Nikodym cocycle $e^{-\beta c_r}$, it follows that the extremal quasi-invariant probability measure with Radon-Nikodym cocycle $e^{-\beta c_r}$ maps invariant sets of Λ^* into $\{0, 1\}$.

3. Decomposition of almost invariant vectors

To decompose our KMS states it is necessary to decompose certain vectors over the set of vertexes, and since our solution to this problem is purely linear algebraic and works for very general sets and vectors, we have devoted this section to present it in its full generality. Regarding notation $\mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$ and we write $\prod_{i=1}^{l} H_i x$ for matrices H_i and an expression x to mean $H_1 \cdots H_l x$.

Definition 3.1. *Let S be a finite set and $B_1, \ldots, B_k \in M_S(\mathbb{R}_+)$ be pairwise commuting, i.e. $B_iB_j = B_jB_i$ for all i, j. We say a vector $\psi \in [0, \infty[^S$ is almost invariant for the family $\{B_i\}_{i=1}^k$ if:*

$$\prod_{i \in I}(1_S - B_i)\psi \geq 0 \qquad \textit{for each subset } I \subseteq \{1, \ldots, k\} \tag{3.1}$$

Proposition 3.2. *Let S be a finite set, $B_1, \ldots, B_k \in M_S(\mathbb{R}_+)$ be pairwise commuting and ψ be an almost invariant vector for the family $\{B_i\}_{i=1}^k$. For each subset $I \subseteq \{1, 2, \ldots, k\}$ there exists a vector h^I that is almost invariant for the family $\{B_i\}_{i=1}^k$ such that:*

1. *$B_i h^I = h^I$ for all $i \in I$.*
2. *$\lim_{n\to\infty} B_j^n h^I = 0$ for $j \in \{1, \ldots, k\} \setminus I$.*
3. *$\psi = \sum_{I \subseteq \{1,\ldots,k\}} h^I$*

Furthermore this decomposition is unique in the sense that there is only one family of almost invariant vectors satisfying 1-3.

Proof. Fix a $i \in \{1, \ldots, k\}$ and let ψ be some almost invariant vector. Using (3.1) with $I = \{i\}$ we get $B_i\psi \leq \psi$. It follows from the Riesz decomposition of vectors (see e.g. Theorem 5.6 [8]) that we can write $\psi = \psi_1 + \psi_2$ where:

$$\psi_1 := \lim_{n\to\infty} B_i^n \psi \quad , \quad \psi_2 := \sum_{n=0}^{\infty} B_i^n(\psi - B_i\psi)$$

Clearly $B_i\psi_1 = \psi_1$ and $\lim_{n\to\infty} B_i^n \psi_2 = 0$. To see that ψ_1 and ψ_2 are almost invariant let $J \subseteq \{1, 2, \ldots, k\}$ be arbitrary. Since:

$$\prod_{j \in J}(1_S - B_j)\,[B_i^n\psi] = B_i^n \prod_{j \in J}(1_S - B_j)\psi \geq 0$$

for each $n \in \mathbb{N}$ it follows that ψ_1 is almost invariant. If $i \notin J$, then:

$$\prod_{j\in J}(1_S - B_j)\sum_{n=0}^{\infty} B_i^n(\psi - B_i\psi) = \sum_{n=0}^{\infty} B_i^n \prod_{j\in J\cup\{i\}}(1_S - B_j)\psi$$

Every vector in the sum is non-negative by assumption, so this is a non-negative vector. If $i \in J$ then:

$$\begin{aligned}\prod_{j\in J}(1_S - B_j)\sum_{n=0}^{\infty} B_i^n(\psi - B_i\psi) &= (1_S - B_i)\sum_{n=0}^{\infty} B_i^n \prod_{j\in J}(1_S - B_j)\psi \\ &= \prod_{j\in J}(1_S - B_j)\psi \geq 0\end{aligned}$$

Hence ψ_1 and ψ_2 are almost invariant. If $B_j\psi = \psi$ for some j then clearly $B_j\psi_l = \psi_l$ for $l = 1, 2$. If $\lim_{n\to\infty} B_j^n\psi = 0$ the same would be true for ψ_l for $l = 1, 2$ since $0 \leq B_j^n\psi_l \leq B_j^n\psi$ for each n. Hence repeated decomposition gives us the existence of the family h^I, $I \subseteq \{1, \dots, k\}$.

To prove that the decomposition is unique, assume that the functions $\tilde{h}^I$, $I \subseteq \{1, \dots, k\}$ are almost invariant for the family $\{B_i\}_{i=1}^k$ and satisfies 1-3. It then follows that the expression:

$$\lim_{n_1\to\infty} B_1^{n_1} \lim_{n_2\to\infty} B_2^{n_2} \cdots \lim_{n_k\to\infty} B_k^{n_k}\psi$$

is equal to both $h^{\{1,\dots,k\}}$ and $\tilde{h}^{\{1,\dots,k\}}$. Assume now that $h^I = \tilde{h}^I$ for all subsets $I \subseteq \{1, \dots, k\}$ with $|I| \geq n$ for some $1 \leq n \leq k$, and take a set $J \subseteq \{1, \dots, k\}$ with $|J| = n - 1$. We write $J = \{j_1, \dots, j_{n-1}\}$. Taking the limits:

$$\lim_{m_1\to\infty} B_{j_1}^{m_1} \lim_{m_2\to\infty} B_{j_2}^{m_2} \cdots \lim_{m_{n-1}\to\infty} B_{j_{n-1}}^{m_{n-1}}\psi$$

we get from 1 and 2 that:

$$\sum_{I\supseteq J} \tilde{h}^I = \sum_{I\supseteq J} h^I$$

By assumption $\tilde{h}^I = h^I$ for all $I \neq J$ in this sum, so we must have that $\tilde{h}^J = h^J$. It now follows from induction that the family h^I is unique. □

4. A description of the gauge-invariant KMS states

The first step in our analysis of the KMS states on the Toeplitz algebra of a finite k-graph Λ is to describe the ones that are gauge-invariant. In this section we will reduce the problem of finding gauge-invariant KMS states to the much simpler problem of finding certain invariant vectors over Λ^0. We remind the reader that for a finite set S the 1-norm for a vector $\psi \in \mathbb{R}^S$ is given by $\|\psi\|_1 = \sum_{s\in S}|\psi_s|$.

Lemma 4.1. *Let Λ be a finite k-graph and let $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. Let ω be a β-KMS state for α^r and set $\psi_v := \omega(p_v)$ for each $v \in \Lambda^0$. Then $\psi \in [0,\infty[^{\Lambda^0}$ is an almost invariant vector for the family $\{e^{-\beta r_i}A_i\}_{i=1}^k$ of unit 1-norm.*

Proof. When $\beta \geq 0$, $r \in]0,\infty[^k$ and Λ has no sources this statement is part (a) of Proposition 4.1 in [4]. When interpreting empty sums a 0 the proof given there works for general $\beta \in \mathbb{R}$, $r \in \mathbb{R}^k$ and finite k-graphs, so we will not give it here. □

Lemma 4.1 gives us an affine map from the set of gauge-invariant β-KMS states for α^r on $\mathcal{T}C^*(\Lambda)$ to the set of non-negative almost invariant vectors for the family $\{e^{-\beta r_i}A_i\}_{i=1}^k$ of unit 1-norm. Proposition 4.3 below implies that it is a bijection. To prove this we need the following description of $\Lambda^{\infty_I,m}$ when we have a partition $I \sqcup J = \{1,\dots,k\}$ with $I \neq \emptyset$. Set:

$$\Lambda^0(I) := \{v \in \Lambda^0 \ : \ v\Lambda^{(n,0)} \neq \emptyset \text{ for all } n \in \mathbb{N}^I\}$$

i.e. $\Lambda^0(I)$ are the vertexes that are not sources in Λ_I. For $(n,m) \in \mathbb{N}^I \oplus \mathbb{N}^J$ we set:

$$\mathcal{U}_I^{(n,m)} = \{\lambda \in \Lambda^{(n,m)} \ : \ s(\lambda(0,l)) \in \Lambda^0(I) \text{ for each } 0 \leq l \leq (n,m)\}$$

Giving $\mathcal{U}_I^{(n,m)}$ the discrete topology we can for each $n,l \in \mathbb{N}^I$ with $n \leq l$ define a continuous map $\pi_{l,n} : \mathcal{U}_I^{(l,m)} \to \mathcal{U}_I^{(n,m)}$ by $\pi_{l,n}(\lambda) = \lambda(0,(n,m))$.

Lemma 4.2. *Assume $I \neq \emptyset$. Then $\Lambda^0(I)$ is closed in Λ, $\pi_{l,n}$ is surjective and $\lim_{\leftarrow n\in\mathbb{N}^I} \mathcal{U}_I^{(n,m)}$ is homeomorphic to $\Lambda^{\infty_I,m}$.*

Proof. To see that $\Lambda^0(I)$ is closed, let $\lambda \in v\Lambda w$ with $w \in \Lambda^0(I)$ and let $n \in \mathbb{N}^I$. For $\mu \in w\Lambda^{(n,0)}$ then $\lambda\mu \in v\Lambda^{(n,0)+d(\lambda)}$, and hence by the unique factorisation property there are paths $\lambda' \in v\Lambda^{(n,0)}$ and $\mu' \in \Lambda^{d(\lambda)}$ such that

$\lambda\mu = \lambda'\mu'$, so $v\Lambda^{(n,0)} \neq \emptyset$. It follows that $\Lambda^0(I)$ is closed. When $\mu \in \mathcal{U}_I^{(n,m)}$ then $s(\mu) \in \Lambda^0(I)$, so for each $s \in \mathbb{N}$ we can choose $\lambda_s \in s(\mu)\Lambda$ with $d(\lambda_s) = l - n + \sum_{i\in I} se_i$. Since Λ^{l-n} is finite, there is a $\lambda \in s(\mu)\Lambda^{l-n}$ with $\lambda_s(0, l-n) = \lambda$ for infinitely many s, and it follows that $\mu\lambda \in \mathcal{U}_I^{(l,m)}$, proving that $\pi_{l,n}$ is surjective.

Denote by $\tilde{\pi}_{l,n}$ the map $\Lambda^{(l,m)} \to \Lambda^{(n,m)}$ given by $\tilde{\pi}_{l,n}(\lambda) = \lambda(0,(n,m))$, so π is a restriction of $\tilde{\pi}$. The map:

$$\lim_{\leftarrow n\in\mathbb{N}^I} \Lambda^{(n,m)} \ni \{\lambda_n\}_{n\in\mathbb{N}^I} \to \{\lambda_n\}_{n\in\mathbb{N}^I} \in \lim_{\leftarrow n\in\mathbb{N}^I} \mathcal{U}_I^{(n,m)} \tag{4.1}$$

is well defined, because for each $n, n' \in \mathbb{N}^I$ the element $\lambda_n \in \Lambda^{(n,m)}$ satisfies that $\lambda_{n+n'} \in \Lambda^{(n+n',m)}$ can be decomposed $\lambda_{n+n'} = \lambda_n\mu$ with $\mu \in s(\lambda_n)\Lambda^{(n',0)}$, so since n' was arbitrary $s(\lambda_n) \in \Lambda^0(I)$. Standard arguments imply that (4.1) is a continuous bijection, and so since $\lim_{\leftarrow n\in\mathbb{N}^I} \Lambda^{(n,m)}$ is compact it is also a homeomorphism. Since Proposition 3.2 in [3] implies that $\lim_{\leftarrow n\in\mathbb{N}^I} \Lambda^{(n,m)}$ is homeomorphic to $\Lambda^{\infty_I,m}$ this proves the Lemma. □

The construction of the KMS state in the proof of Proposition 4.3 has a predecessor in Theorem 5.1 in [3].

Proposition 4.3. *Let Λ be a finite k-graph, $r \in \mathbb{R}^k$, $\beta \in \mathbb{R}$ and let $\psi \in [0,\infty[^{\Lambda^0}$ be an almost invariant vector for the family $\{e^{-\beta r_i}A_i\}_{i=1}^k$ of unit 1-norm. Then there exists a unique gauge-invariant β-KMS state ω_ψ for α^r on $\mathcal{T}C^*(\Lambda)$ such that $\omega_\psi(p_v) = \psi_v$ for each $v \in \Lambda^0$.*

Proof. Assume ω and ω' are gauge-invariant β-KMS states for α^r with $\omega(p_v) = \omega'(p_v)$ for all $v \in \Lambda^0$. Lemma 3.1 in [1] implies that both ω and ω' is then determined by their values on the elements $S_\lambda S_\lambda^*$, $\lambda \in \Lambda$. Since:

$$\omega(S_\lambda S_\lambda^*) = e^{-\beta r\cdot d(\lambda)}\omega(p_{s(\lambda)}) = e^{-\beta r\cdot d(\lambda)}\omega'(p_{s(\lambda)}) = \omega'(S_\lambda S_\lambda^*)$$

we must have $\omega = \omega'$, which proves that if the state ω_ψ exists it is unique. Proposition 3.2 implies that it is enough to prove that ω_ψ exists when there is a partition $I \sqcup J = \{1, 2, \dots, k\}$ with $e^{-\beta r_i}A_i\psi = \psi$ for $i \in I$ and $\lim_{n\to\infty}(e^{-\beta r_j}A_j)^n\psi = 0$ for $j \in J$, so we assume this is the case. We now define a vector ϕ by:

$$\phi := \prod_{j\in J}(1_{\Lambda^0} - e^{-\beta r_j}A_j)\psi$$

When $J = \emptyset$ we interpret this as $\phi := \psi$. Notice $\phi \in [0,\infty[^{\Lambda^0}$ since ψ is almost invariant, it is however not clear yet that $\phi \neq 0$. We will now define measures ν^m on $\Lambda^{\infty_I,m}$ for each $m \in \mathbb{N}^J$ using ϕ. When $I = \emptyset$, we define ν^m on $\Lambda^{\infty_I,m} = \Lambda^m$ by $\nu^m(\{\lambda\}) = e^{-\beta r\cdot m}\phi_{s(\lambda)}$. When $I \neq \emptyset$ give the finite set $\mathcal{U}_I^{(n,m)}$ the discrete topology for each $(n,m) \in \mathbb{N}^I \oplus \mathbb{N}^J$, and define a measure $\nu^{n,m}$ on $\mathcal{U}_I^{(n,m)}$ by:

$$\nu^{n,m}(\{\lambda\}) = e^{-\beta r\cdot(n,m)}\phi_{s(\lambda)} \qquad \text{for } \lambda \in \mathcal{U}_I^{(n,m)} \tag{4.2}$$

Since the vertex matrices commute it follows from the definition of ϕ that $e^{-\beta r_i}A_i\phi = \phi$ for $i \in I$. For $v \in \Lambda^0 \backslash \Lambda^0(I)$ there is a $n \in \mathbb{N}^I$ with $A^{(n,0)}(v,u) = 0$ for all u, and hence $\phi_v = e^{-\beta r\cdot(n,0)}(A^{(n,0)}\phi)_v = 0$. Since $\Lambda^0(I)$ is closed by Lemma 4.2 we get for any $\lambda \in \mathcal{U}_I^{(n,m)}$:

$$\begin{aligned}
\nu^{l,m}(\pi_{l,n}^{-1}(\{\lambda\})) &= \sum_{\mu \in \pi_{l,n}^{-1}(\{\lambda\})} e^{-\beta r\cdot(l,m)}\phi_{s(\mu)} = \sum_{\eta \in s(\lambda)\mathcal{U}_I^{(l-n,0)}} e^{-\beta r\cdot(l,m)}\phi_{s(\eta)} \\
&= \sum_{w\in\Lambda^0} \sum_{\eta \in s(\lambda)\mathcal{U}_I^{(l-n,0)}w} e^{-\beta r\cdot(l,m)}\phi_w = \sum_{w\in\Lambda^0} \sum_{\eta \in s(\lambda)\Lambda^{(l-n,0)}w} e^{-\beta r\cdot(l,m)}\phi_w \\
&= e^{-\beta r\cdot(l,m)}(A^{(l-n,0)}\phi)_{s(\lambda)} = e^{-\beta r\cdot(n,m)}\phi_{s(\lambda)} = \nu^{n,m}(\{\lambda\})
\end{aligned}$$

A standard argument (using e.g. Lemma 5.2 in [3]) gives the existence of a measure ν^m on $\lim_{\leftarrow n\in\mathbb{N}^I}\mathcal{U}_I^{(n,m)}$. We consider ν^m as a Borel measure on Λ^* with $\nu^m(\Lambda^{\infty_I,m}) = \nu^m(\Lambda^*)$ and by construction it satisfies:

$$\nu^m(Z(\lambda) \cap \Lambda^{\infty_I,m}) = \nu^{n,m}(\{\lambda\}) = e^{-\beta r\cdot d(\lambda)}\phi_{s(\lambda)} \tag{4.3}$$

for each $\lambda \in \mathcal{U}_I^{(n,m)}$. If $\lambda \in \Lambda^{(n,m)}\backslash\mathcal{U}_I^{(n,m)}$ then (4.3) still holds true since both sides are 0. The measure ν^m constructed when $I = \emptyset$ also satisfies (4.3). We will now construct a measure ν on $\partial^I\Lambda$ by summing all of the measures ν^m, $m \in \mathbb{N}^J$. When $J = \emptyset$ we have only constructed a measure ν^0 on $\Lambda^{\infty_I,0}$, so we set $\nu = \nu^0$ and notice that by (4.3) $\nu(Z(v)) = \psi_v$ for each $v \in \Lambda^0$. When $J \neq \emptyset$ we can use (4.3) for any $\mu \in \Lambda$ with $l := d(\mu)_J \leq m$ to see that:

$$\begin{aligned}
\nu^m(Z(\mu)) = \nu^m(Z(\mu) \cap \Lambda^{\infty_I,m}) &= \sum_{\lambda \in s(\mu)\Lambda^{(0,m-l)}} \nu^m(Z(\mu\lambda) \cap \Lambda^{\infty_I,m}) \qquad (4.4) \\
&= \sum_{\lambda \in s(\mu)\Lambda^{(0,m-l)}} e^{-\beta r\cdot(d(\mu)_I,m)}\phi_{s(\lambda)} = e^{-\beta r\cdot(d(\mu)_I,m)}(A^{(0,m-l)}\phi)_{s(\mu)}
\end{aligned}$$

In particular, we have that $\nu^m(Z(v)) = e^{-\beta r\cdot(0,m)}(A^{(0,m)}\phi)_v$ for all $v \in \Lambda^0$. Taking a $M \in \mathbb{N}^J$ we see that:

$$\sum_{0\leq m\leq M} e^{-\beta r\cdot(0,m)} A^{(0,m)}\phi = \sum_{0\leq m\leq M}\prod_{j\in J}\left(e^{-\beta r_j}A_j\right)^{m_j}\phi = \prod_{j\in J}\sum_{m_j=0}^{M_j}\left(e^{-\beta r_j}A_j\right)^{m_j}\phi$$
$$= \left[\prod_{j\in J}\sum_{m_j=0}^{M_j}\left(e^{-\beta r_j}A_j\right)^{m_j}(1_{\Lambda^0} - e^{-\beta r_j}A_j)\right]\psi = \left[\prod_{j\in J}(1_{\Lambda^0} - (e^{-\beta r_j}A_j)^{M_j+1})\right]\psi$$
$$= \left[\sum_{L\subseteq J}(-1)^{|L|}\prod_{j\in L}(e^{-\beta r_j}A_j)^{M_j+1}\right]\psi$$

By choice of J we have that $\prod_{j\in L}(e^{-\beta r_j}A_j)^{M_j+1}\psi \to 0$ for $M_j \to \infty$ for any $j \in L$, so when we consider the limit all terms in the sum except for the one where $L = \emptyset$ vanishes, so:

$$\sum_{m\in\mathbb{N}^J} e^{-\beta r\cdot(0,m)}A^{(0,m)}\phi = \psi$$

This implies $\phi \neq 0$ and it implies that we can define a Borel probability measure ν on Λ^* by $\nu = \sum_{m\in\mathbb{N}^J}\nu^m$ that as in the case where $J = \emptyset$ satisfies $\nu(Z(v)) = \psi_v$ for each $v \in \Lambda^0$. Since ν is a Borel probability measure on the second countable locally compact Hausdorff space Λ^* it is also a regular measure. We define a state ω_ψ by:

$$\omega_\psi(a) = \int_{\Lambda^*} P(a)\ d\nu \qquad \forall a \in \mathcal{T}C^*(\Lambda)$$

where $P : \mathcal{T}C^*(\Lambda) \to C(\Lambda^*)$ is the canonical conditional expectation. Since $P(S_\lambda S_\mu^*) = 0$ when $\mu \neq \lambda$ it follows that ω_ψ is gauge-invariant. For any path $\lambda \in \Lambda^{(n,l)}$ for some $n \in \mathbb{N}^I$ and $l \in \mathbb{N}^J$ we have by (4.4) when $J \neq \emptyset$:

$$\nu(Z(\lambda)) = \sum_{m\in\mathbb{N}^J}\nu^m(Z(\lambda)) = \sum_{m\geq l}\nu^m(Z(\lambda)) = \sum_{m\geq l} e^{-\beta r\cdot(n,m)}(A^{(0,m-l)}\phi)_{s(\lambda)}$$
$$= \sum_{m\in\mathbb{N}^J} e^{-\beta r\cdot(n,m+l)}(A^{(0,m)}\phi)_{s(\lambda)} = e^{-\beta r\cdot d(\lambda)}\sum_{m\in\mathbb{N}^J} e^{-\beta r\cdot(0,m)}(A^{(0,m)}\phi)_{s(\lambda)}$$
$$= e^{-\beta r\cdot d(\lambda)}\nu(Z(s(\lambda)))$$

When $J = \emptyset$ we also have $\nu(Z(\lambda)) = e^{-\beta r \cdot d(\lambda)}\nu(Z(s(\lambda)))$, so in both cases this implies that:

$$\omega_\psi(S_\lambda S_\mu^*) = \delta_{\lambda,\mu}\nu(Z(\lambda)) = \delta_{\lambda,\mu}e^{-\beta r\cdot d(\lambda)}\nu(Z(s(\lambda))) = \delta_{\lambda,\mu}e^{-\beta r\cdot d(\lambda)}\psi_{s(\lambda)}$$

It now follows, for example as in the proof of part (b) of Proposition 3.1 in [4], that ω_ψ is a β-KMS state for α^r. $\square$

The proof of Proposition 4.3 yields the following corollary.

Corollary 4.4. *In the setting of Proposition 4.3 assume that there exist sets I, J such that $I \sqcup J = \{1, 2, \ldots, k\}$ and $e^{-\beta r_i}A_i\psi = \psi$ for $i \in I$ and $\lim_{n\to\infty}(e^{-\beta r_j}A_j)^n\psi = 0$ for $j \in J$. Then the measure m_ψ on Λ^* associated to ω_ψ satisfies $m_\psi(\partial^I\Lambda) = 1$.*

5. Decomposition of gauge-invariant KMS states

In this section we will analyse the gauge-invariant KMS states by analysing the almost invariant vectors. The first step in this analysis is to construct almost invariant vectors using components in different equivalences $\sim_I$ in the k-graph. The next step is to prove that all invariant vectors can be realised as convex combinations of the invariant vectors constructed.

First let us introduce some notation. For a set $S \subseteq \Lambda^0$ and $B \in M_{\Lambda^0}(\mathbb{R})$ we let $B^S \in M_S(\mathbb{R})$ denote the restriction of B to $S \times S$ and for any matrix B we write $\rho(B)$ for its spectral radius. Whenever we have a k-graph Λ with vertex matrices $A_1, \ldots, A_k$ and some $S \subseteq \Lambda^0$ we set:

$$\rho(A^S) := (\rho(A_1^S), \rho(A_2^S), \ldots, \rho(A_k^S)) \in \mathbb{R}^k$$

Definition 5.1. *Let Λ be a finite k-graph, $r \in \mathbb{R}^k$, $\beta \in \mathbb{R}$ and let $I \subseteq \{1, \ldots, k\}$. A component C in Λ_I (i.e. an equivalence class for $\sim_I$) is called a (I, β, r)-subharmonic component, if it satisfies:*

1. *All equivalence classes D in $\sim_I$ with $D \neq C$ and $D \subseteq \overline{C}^I$ satisfies:*

$$\rho(A^D)_I \lneq \rho(A^C)_I$$

2. *$\rho(A_i^C) = e^{\beta r_i}$ for $i \in I$.*
3. *$\rho(A_j^{\overline{C}}) < e^{\beta r_j}$ for $j \in J := \{1, \ldots, k\} \setminus I$.*

When $I = \emptyset$ then $\Lambda_I = \Lambda^0$ and the different equivalence classes are just the sets $\{v\}$, $v \in \Lambda^0$, so condition 3 is the only one that is not trivially fulfilled. We will need some results from [1] regarding the construction of vectors over Λ^0 which we will summarise in the following Lemma 5.2.

Lemma 5.2. *Let Λ be a finite k-graph and let $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. For each $(\{1,\dots,k\},\beta,r)$-subharmonic component C there exists a unique vector $z^C \in [0,\infty[^{\Lambda^0}$ of unit 1-norm satisfying 1. and 2.:*

1. *$z^C_v = 0$ for $v \notin \overline{C}$.*
2. *$A_i z^C = e^{\beta r_i} z^C$ for all $i \in \{1,\dots,k\}$.*

Furthermore $z^C_v > 0$ for $v \in \overline{C}$. For any $x \in [0,\infty[^{\Lambda^0}$ of unit 1-norm with $A_i x = e^{\beta r_i} x$ for all $i \in \{1,\dots,k\}$ there is a unique collection of $(\{1,\dots,k\},\beta,r)$-subharmonic components $\mathcal{C}$ in Λ and numbers $t_C > 0$, $C \in \mathcal{C}$, such that:

$$x = \sum_{C \in \mathcal{C}} t_C z^C$$

Proof. Since the construction of the vectors in [1] is for graphs with no sources, we will start by proving the Lemma when Λ is without sources. Let C be a $(\{1,\dots,k\},\beta,r)$-subharmonic component in Λ, then C satisfies the criterion in Lemma 7.11 in [1]. Choosing a finite set $F \subseteq \mathbb{N}^k \setminus \{0\}$ with the property that for all $v,w \in \Lambda^0$ then $\sum_{n \in F} A^n(v,w) > 0$ if and only if $v\Lambda^l w \neq \emptyset$ for some $l \in \mathbb{N}^k \setminus \{0\}$ (such a set is called *well chosen* in [1]), Corollary 7.10 implies that C in the terminology of [1] is *F-harmonic*. By Lemma 7.6 in [1] a F-harmonic component gives rise to a unique vector $\chi^C \in [0,\infty[^{\Lambda^0}$ of unit 1-norm, and by Lemma 7.6 and Lemma 7.7 χ^C satisfies 1 and 2 and $\chi^C_v > 0$ for $v \in \overline{C}$, proving existence of z^C. If $z' \in [0,\infty[^{\Lambda^0}$ is a vector of unit 1-norm satisfying 1 and 2, then by Proposition 7.9 in [1] there is a unique collection of F-harmonic components $\mathcal{C}$ such that z' is a convex combination of the vectors χ^D, $D \in \mathcal{C}$, and furthermore $\rho(A^D) = e^{\beta r}$ for each $D \in \mathcal{C}$. Combining 1 and the fact that χ^D is positive on $\overline{D}$, we get that each $D \in \mathcal{C}$ satisfies $D \subseteq \overline{C}$, but then condition 1 and 2 in Definition 5.1 combined with $\rho(A^D) = e^{\beta r}$ imply that $\mathcal{C} = \{C\}$, so $z' = \chi^C$, proving uniqueness. For the unique decomposition of x, notice that by Lemma 7.11 in [1] a component C is $(\{1,\dots,k\},\beta,r)$-subharmonic if and only if $\rho(A^C) = e^{\beta r}$ and C is F-harmonic. The statement therefore follows from Proposition 7.9 in [1] and the construction of the vectors z^C.

Assume now that Λ is a general finite k-graph, and let C be a $(\{1,\dots,k\},\beta,r)$-subharmonic component in Λ. Then $A_i^C \neq 0$ for all i, so taking a $v \in C$ and a $i \in \{1,\dots,k\}$ there is a $\mu \in v\Lambda C$ with $d(\mu)_i > 0$. The factorisation property then implies that $v\Lambda^{e_i}C \neq \emptyset$. Since this is true for all $v \in C$, it follows that $C \subseteq \widetilde{\Lambda^0} := \Lambda^0(\{1,\dots,k\})$, and hence by Lemma 4.2 it follows that $\overline{C} \subseteq \widetilde{\Lambda^0}$. Since $\widetilde{\Lambda^0}$ is closed we can consider the finite k-graph $\widetilde{\Lambda} := \Lambda\widetilde{\Lambda^0}$, which has vertex matrices $A_1^{\widetilde{\Lambda^0}},\dots,A_k^{\widetilde{\Lambda^0}}$. To see that $\widetilde{\Lambda}$ has no sources take $v \in \widetilde{\Lambda^0}$, $m \in \mathbb{N}^k$ and $\lambda_l \in v\Lambda^{le_1+\cdot+le_k+m}$ for each $l \in \mathbb{N}$. Since $\lambda_l(0,m) \in v\Lambda^m$ for each $l \in \mathbb{N}$, there is a $\lambda \in v\Lambda^m$ with $\lambda_l(0,m) = \lambda$ for infinitely many l, which implies that $s(\lambda) \in \widetilde{\Lambda^0}$ and hence $v\widetilde{\Lambda}^m \neq \emptyset$. Since components in $\widetilde{\Lambda}$ are exactly components in Λ contained in $\widetilde{\Lambda^0}$, C is a $(\{1,\dots,k\},\beta,r)$-subharmonic component in $\tilde{\Lambda}$, so there exists a unique vector $\tilde{z}^C \in [0,\infty[^{\widetilde{\Lambda^0}}$ of unit 1-norm with $\tilde{z}^C_v = 0$ when $v \in \widetilde{\Lambda^0} \setminus \overline{C}$ and $A_i^{\widetilde{\Lambda^0}}\tilde{z}^C = e^{\beta r_i}\tilde{z}^C$ for all i. Furthermore $\tilde{z}^C_v > 0$ for $v \in \overline{C}$. It is now straightforward to check that defining $z^C \in [0,\infty[^{\Lambda^0}$ by $z^C|_{\widetilde{\Lambda^0}} = \tilde{z}^C$ and $z^C_v = 0$ for $v \notin \widetilde{\Lambda^0}$ gives the desired vector.

Assume $z' \in [0,\infty[^{\Lambda^0}$ satisfies 1 and 2 and is of unit 1-norm, then $z'|_{\widetilde{\Lambda^0}} \in [0,\infty[^{\widetilde{\Lambda^0}}$ also has unit 1-norm. By 1 $(z'|_{\widetilde{\Lambda^0}})_v = 0$ for $v \in \widetilde{\Lambda^0} \setminus \overline{C}$ and:

$$A_i^{\widetilde{\Lambda^0}} z'|_{\widetilde{\Lambda^0}} = (A_i z')|_{\widetilde{\Lambda^0}} = e^{\beta r_i} z'|_{\widetilde{\Lambda^0}} \qquad \text{for all } i$$

It follows that $z'|_{\widetilde{\Lambda^0}} = \tilde{z}^C$, which proves uniqueness.

For the last statement let $x \in [0,\infty[^{\Lambda^0}$ of unit 1-norm satisfy $A_i x = e^{\beta r_i}x$ for all i. If $v\Lambda^n = \emptyset$ for some $n \in \mathbb{N}$ then $x_v = e^{-\beta r\cdot n}(A^n x)_v = 0$, so $x_v = 0$ for $v \notin \widetilde{\Lambda^0}$ and hence $A_i^{\widetilde{\Lambda^0}} x|_{\widetilde{\Lambda^0}} = e^{\beta r_i} x|_{\widetilde{\Lambda^0}}$ for all i. Using the Lemma on $x|_{\widetilde{\Lambda^0}}$ we get a unique collection $\mathcal{C}$ of $(\{1,\dots,k\},\beta,r)$-subharmonic component in $\tilde{\Lambda}$ with corresponding unique vectors $\tilde{z}^C$, $C \in \mathcal{C}$, and numbers $t_C > 0$, $C \in \mathcal{C}$, such that:

$$x|_{\widetilde{\Lambda^0}} = \sum_{C \in \mathcal{C}} t_C \tilde{z}^C$$

Since C is an $(\{1,\dots,k\},\beta,r)$-subharmonic component in $\tilde{\Lambda}$ if and only if it is a $(\{1,\dots,k\},\beta,r)$-subharmonic component in Λ, it follows from the definition of z^C that we have a unique decomposition:

$$x = \sum_{C \in \mathcal{C}} t_C z^C$$

which proves the Lemma. $\square$

Lemma 5.3. *Let Λ be a finite k-graph, $r \in \mathbb{R}^k$, $\beta \in \mathbb{R}$, $I \subseteq \{1, \dots, k\}$ and C be a (I, β, r)-subharmonic component. There exists a unique vector $x^C \in [0, \infty[^{\Lambda^0}$ of unit 1-norm satisfying 1. and 2.:*

1. $x^C_v = 0$ *for* $v \notin \overline{C}^I$.
2. $A_i x^C = e^{\beta r_i} x^C$ *for all* $i \in I$.

Furthermore $x^C_v > 0$ *for* $v \in \overline{C}^I$.

Proof. If $I = \emptyset$ then $C = \{v\}$ for some $v \in \Lambda^0$, and x^C is the vector with $x^C_w = 0$ for $w \neq v$ and $x^C_v = 1$. If $I \neq \emptyset$ consider the finite graph Λ_I with vertex matrices $(A_i)_{i \in I}$. Setting $r_I = (r_i)_{i \in I} \in \mathbb{R}^I$, it follows from Definition 5.1 that C is a $(\{i\}_{i \in I}, \beta, r_I)$-subharmonic component in the I-graph Λ_I, and hence we get the unique vector from Lemma 5.2. $\square$

Proposition 5.4. *Let Λ be a finite k-graph, $r \in \mathbb{R}^k$, $\beta \in \mathbb{R}$, $I \sqcup J = \{1, \dots, k\}$ be a partition and C be a (I, β, r)-subharmonic component. Denote by $x^C \in [0, \infty[^{\Lambda^0}$ the unique vector given in Lemma 5.3 using C. Set:*

$$\tilde{x}^C|_{\overline{C}} := \prod_{j \in J} (1_{\overline{C}} - e^{-\beta r_j} A_j^{\overline{C}})^{-1} x^C|_{\overline{C}} \tag{5.1}$$

and $\tilde{x}^C|_{\Lambda^0 \setminus \overline{C}} = 0$. Then $\tilde{x}^C$ is almost invariant for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$.

Proof. If $J = \emptyset$ then $\tilde{x}^C = x^C$ which is clearly invariant for $\{e^{-\beta r_i} A_i\}_{i=1}^k$, so assume $J \neq \emptyset$. Notice first that condition 3 in Definition 5.1 implies that $(1_{\overline{C}} - e^{-\beta r_j} A_j^{\overline{C}})^{-1}$ exists for each $j \in J$, so (5.1) makes sense. To express $\tilde{x}^C$ differently, assume that $J_0 \subseteq J$ is an arbitrary non-empty subset, then for any $N \in \mathbb{N}^{J_0}$ we have that:

$$\sum_{0 \leq n \leq N} \prod_{j \in J_0} e^{-\beta r_j n_j} (A_j^{\overline{C}})^{n_j} = \prod_{j \in J_0} \left(\sum_{n_j = 0}^{N_j} e^{-\beta r_j n_j} (A_j^{\overline{C}})^{n_j} \right)$$

Hence as $N \to \infty$ in $\mathbb{N}^{J_0}$ we get that:

$$\prod_{j \in J_0} (1_{\overline{C}} - e^{-\beta r_j} A_j^{\overline{C}})^{-1} = \sum_{n \in \mathbb{N}^{J_0}} \prod_{j \in J_0} e^{-\beta r_j n_j} (A_j^{\overline{C}})^{n_j} \tag{5.2}$$

Let $L \subseteq \{1, \dots, k\}$, to prove that $\tilde{x}^C$ is almost invariant we then want to verify (3.1) for the set L and the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$. Since $x^C \in [0, \infty[^{\Lambda^0}$

it follows from (5.2) with $J = J_0$ that $\tilde{x}^C \in [0,\infty[^{\Lambda^0}$, proving (3.1) when $L = \emptyset$. Assume then that $L \neq \emptyset$. If $y \in [0,\infty[^{\Lambda^0}$ is a vector with $y|_{\Lambda^0 \setminus \overline{C}} = 0$ and $B \in M_{\Lambda^0}(\mathbb{R}_+)$ has the property that $B(v,w) > 0$ implies $v \leq w$, then it follows that:

$$(By)|_{\Lambda^0 \setminus \overline{C}} = 0 \quad , \quad (By)|_{\overline{C}} = B^{\overline{C}}(y|_{\overline{C}}) \tag{5.3}$$

This implies that $(A^n \tilde{x}^C)|_{\Lambda^0 \setminus \overline{C}} = 0$ for all $n \in \mathbb{N}^k$, and hence for $v \in \Lambda^0 \setminus \overline{C}$ we get:

$$\left[\prod_{l \in L}(1_{\Lambda^0} - e^{-\beta r_l} A_l)\tilde{x}^C\right]_v = \left[\sum_{S \subseteq L}(-1)^{|S|}\prod_{l \in S} e^{-\beta r_l} A_l \tilde{x}^C\right]_v = \tilde{x}^C_v \geq 0$$

Using the second equality in (5.3) we obtain:

$$\begin{aligned}\left[\prod_{l \in L}(1_{\Lambda^0} - e^{-\beta r_l} A_l)\tilde{x}^C\right]_{\overline{C}} &= \left[\sum_{S \subseteq L}(-1)^{|S|}\prod_{l \in S} e^{-\beta r_l} A_l \tilde{x}^C\right]_{\overline{C}} \\ &= \sum_{S \subseteq L}(-1)^{|S|}\prod_{l \in S} e^{-\beta r_l} A_l^{\overline{C}}(\tilde{x}^C|_{\overline{C}}) \\ &= \prod_{l \in L}(1_{\overline{C}} - e^{-\beta r_l} A_l^{\overline{C}})(\tilde{x}^C|_{\overline{C}}) \end{aligned} \tag{5.4}$$

It now follows from (5.1) that if there is a $i \in L \cap I$, then since $A_i^{\overline{C}} x^C|_{\overline{C}} = (A_i x^C)|_{\overline{C}} = e^{\beta r_i} x^C|_{\overline{C}}$ we get that $(1_{\overline{C}} - e^{-\beta r_i} A_i^{\overline{C}})\tilde{x}^C|_{\overline{C}} = 0$, and hence the expression in (5.4) is zero. If $L \cap I = \emptyset$ then $L \subseteq J$, and:

$$\prod_{l \in L}(1_{\overline{C}} - e^{-\beta r_l} A_l^{\overline{C}})(\tilde{x}^C|_{\overline{C}}) = \prod_{j \in J \setminus L}(1_{\overline{C}} - e^{-\beta r_j} A_j^{\overline{C}})^{-1} x^C|_{\overline{C}}$$

It follows from (5.2) with $J_0 = J \setminus L$ that this is a non-negative vector, and combined with (5.4) this implies that $\tilde{x}^C$ is almost invariant. $\square$

Definition 5.5. *When C is a (I,β,r)-subharmonic component we set $y^C := \tilde{x}^C / \|\tilde{x}^C\|_1$.*

The notation in Definition 5.5 is not well defined since a set $C \subseteq \Lambda^0$ can both be a (I,β,r)-subharmonic component and a (I',β',r')-subharmonic component with $(I,\beta,r) \neq (I',\beta',r')$. If however C is (I,β,r)-subharmonic and $i \in I$ then $A_i(v,w) = 0$ for $v \in \overline{C}$ and $w \in \overline{C}^I$, so we get that $\rho(A_i^{\overline{C}}) \geq$

$\rho(A_i^{\overline{C}^I})$, and since $A_i(v,w) = 0$ for $v \in C$ and $w \in \overline{C}^I$ we furthermore get that $\rho(A_i^{\overline{C}^I}) \geq \rho(A_i^C)$, so by Definition 5.1 C can not be (I', β, r)-subharmonic for an $I' \neq I$. Since we will formulate our results for some fixed values of r and β, we therefore abuse notation and simply write y^C.

Proposition 5.4 implies that a (I, β, r)-subharmonic component gives rise to a gauge-invariant β-KMS state ω for α^r. To prove that all gauge-invariant states are given by convex combinations of states arising from such components, it becomes essential that we can rediscover the vector x^C from ω. To do this we need the following technical result.

Lemma 5.6. *Let Λ be a finite k-graph and let ω be a β-KMS state for α^r for some $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. Let m be the measure on Λ^* associated to ω and $I \sqcup J \subseteq \{1, \dots, k\}$ be some partition. For each $\lambda \in \Lambda$ the set:*

$$\lambda\Lambda^{\infty_I,0} = \{x \in \Lambda^* \ : \ x = \lambda x' \text{ for some } x' \in \Lambda^{\infty_I,0}\}$$

is Borel and:

$$m(\lambda\Lambda^{\infty_I,0}) = e^{-\beta r \cdot d(\lambda)} m(s(\lambda)\Lambda^{\infty_I,0})$$

Proof. To see that $\lambda\Lambda^{\infty_I,0}$ is Borel set $p := 1^I \in \mathbb{N}^I$ if $I \neq \emptyset$ and set $p = 0$ if $I = \emptyset$, then:

$$\lambda\Lambda^{\infty_I,0} = \bigcap_{n \in \mathbb{N}} \bigcup_{\mu \in s(\lambda)\Lambda^{n \cdot p}} \left[Z(\lambda\mu) \setminus \left(\bigcup_{j \in J} \bigcup_{e \in s(\lambda)\Lambda^{e_j}} Z(\lambda e) \right) \right]$$

Since we take the union over decreasing sets we get that:

$$\begin{aligned} m(\lambda\Lambda^{\infty_I,0}) &= \lim_{n \to \infty} \sum_{\mu \in s(\lambda)\Lambda^{n \cdot p}} m \left(Z(\lambda\mu) \setminus \left(\bigcup_{j \in J} \bigcup_{e \in s(\lambda)\Lambda^{e_j}} Z(\lambda e) \right) \right) \\ &= \lim_{n \to \infty} \sum_{\mu \in s(\lambda)\Lambda^{n \cdot p}} m \left(Z(\lambda\mu) \setminus \left(\bigcup_{j \in J} \bigcup_{e \in s(\mu)\Lambda^{e_j}} Z(\lambda\mu e) \right) \right) \end{aligned} \tag{5.5}$$

Set $e_L = \sum_{l \in L} e_l$ for any $L \subseteq J$ and $\psi_v = \omega(p_v)$ for $v \in \Lambda^0$. We claim that for any path $\eta \in \Lambda$:

$$m \left(Z(\eta) \setminus \left(\bigcup_{j \in J} \bigcup_{e \in s(\eta)\Lambda^{e_j}} Z(\eta e) \right) \right) = \sum_{L \subseteq J} (-1)^{|L|} e^{-\beta r \cdot d(\eta)} e^{-\beta r \cdot e_L} (A^{e_L}\psi)_{s(\eta)} \tag{5.6}$$

The Lemma follows from (5.6), because using it twice on (5.5) yields:

$$\begin{aligned}m(\lambda\Lambda^{\infty_I,0}) &= \lim_{n\to\infty}\sum_{\mu\in s(\lambda)\Lambda^{n\cdot p}}\sum_{L\subseteq J}(-1)^{|L|}e^{-\beta r\cdot d(\lambda\mu)}e^{-\beta r\cdot e_L}(A^{e_L}\psi)_{s(\mu)}\\ &= e^{-\beta r\cdot d(\lambda)}\lim_{n\to\infty}\sum_{\mu\in s(\lambda)\Lambda^{n\cdot p}}\sum_{L\subseteq J}(-1)^{|L|}e^{-\beta r\cdot d(\mu)}e^{-\beta r\cdot e_L}(A^{e_L}\psi)_{s(\mu)}\\ &= e^{-\beta r\cdot d(\lambda)}m(s(\lambda)\Lambda^{\infty_I,0})\end{aligned}$$

To prove (5.6) set $\mathcal{M}(e_j) := \bigcup_{e\in s(\eta)\Lambda^{e_j}} Z(\eta e)$. We use that $Z(\eta e)\subseteq Z(\eta)$ for each $e\in s(\eta)\Lambda^{e_j}$ and $j\in J$ to get the equality:

$$1_{Z(\eta)\setminus\left(\bigcup_{j\in J}\mathcal{M}(e_j)\right)} = \prod_{j\in J}(1_{Z(\eta)} - 1_{\mathcal{M}(e_j)}) = \sum_{L\subseteq J}(-1)^{|L|}\prod_{j\in L}1_{\mathcal{M}(e_j)} \tag{5.7}$$

Since $\prod_{j\in L}1_{\mathcal{M}(e_j)} = 1_{\bigcap_{j\in L}\mathcal{M}(e_j)}$ we get (5.6) by combining (5.7) with:

$$\begin{aligned}m\left(\bigcap_{j\in L}\mathcal{M}(e_j)\right) &= m\left(\bigcup_{e\in s(\eta)\Lambda^{e_L}} Z(\eta e)\right) = \sum_{e\in s(\eta)\Lambda^{e_L}} m(Z(\eta e))\\ &= e^{-\beta r\cdot(d(\eta)+e_L)}(A^{e_L}\psi)_{s(\eta)}\end{aligned}$$

□

Lemma 5.7. *Let Λ be a finite k-graph, $r\in\mathbb{R}^k$ and $\beta\in\mathbb{R}$. Let $I\subseteq\{1,\dots,k\}$ and C be a (I,β,r)-subharmonic component. Let ω be the KMS state associated to the vector y^C, and let m_C be the measure associated to ω, then:*

$$x_v^C = \|\tilde{x}^C\|_1 m_C(v\Lambda^{\infty_I,0}) \qquad \forall v\in\Lambda^0$$

Proof. Since $m_C(\partial^I\Lambda)=1$ by Corollary 4.4, the formula (5.6) implies:

$$m_C(v\Lambda^{\infty_I,0}) = m_C\left(Z(v)\setminus\left(\bigcup_{j\in J}\bigcup_{e\in v\Lambda^{e_j}}Z(e)\right)\right) = \sum_{L\subseteq J}(-1)^{|L|}e^{-\beta r\cdot e_L}(A^{e_L}y^C)_v$$

for each $v\in\Lambda^0$. When $v\notin\overline{C}$ then $(A^{e_L}y^C)_v = 0$ for each L, and hence $m_C(v\Lambda^{\infty_I,0}) = 0$. When $v\in\overline{C}$ then $(A^{e_L}y^C)_v = ((A^{\overline{C}})^{e_L}y^C|_{\overline{C}})_v$ and hence:

$$\begin{aligned}m_C(v\Lambda^{\infty_I,0}) &= \sum_{L\subseteq J}(-1)^{|L|}e^{-\beta r\cdot e_L}((A^{\overline{C}})^{e_L}y^C|_{\overline{C}})_v\\ &= \|\tilde{x}^C\|_1^{-1}\left[\prod_{j\in J}(1_{\overline{C}} - e^{-\beta r_j}A_j^{\overline{C}})\tilde{x}^C|_{\overline{C}}\right]_v = \|\tilde{x}^C\|_1^{-1}x_v^C\end{aligned}$$

Since $x_v^C = 0$ for $v\notin\overline{C}$ this proves the Lemma. □

By Proposition 3.2 we already have a decomposition of a general almost invariant vector, so we can focus on decomposing the vectors appearing in Proposition 3.2.

Proposition 5.8. *Let Λ be a finite k-graph, $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. Let $\psi \in [0,\infty[^{\Lambda^0}$ be an almost invariant vector for the family $\{e^{-\beta r_i} A_i\}_{i=1}^k$ and assume that there is a partition $I \sqcup J = \{1,\dots,k\}$ such that $e^{-\beta r_i} A_i \psi = \psi$ for $i \in I$ and $\lim_{n\to\infty}(e^{-\beta r_j} A_j)^n \psi = 0$ for $j \in J$. There exists a unique collection of (I,β,r)-subharmonic components $\mathcal{C}$ and numbers $t_C > 0$, $C \in \mathcal{C}$, such that:*

$$\psi = \sum_{C\in\mathcal{C}} t_C y^C$$

Proof. Let ω_ψ be the gauge-invariant β-KMS state for α^r given by ψ, and let m_ψ be the associated measure on Λ^*. Define a vector $\psi' \in [0,\infty[^{\Lambda^0}$ by:

$$\psi'_v = m_\psi(v\Lambda^{\infty_I,0}) \qquad \text{for } v \in \Lambda^0$$

If $I = \emptyset$ then we can uniquely write ψ' as in (5.8) below where each z^C is the indicator function for the v with $C = \{v\}$. When $I \neq \emptyset$ it follows from Lemma 5.6 that for each $i \in I$ and $v \in \Lambda^0$:

$$\psi'_v = m_\psi\left(\bigcup_{\mu\in v\Lambda^{e_i}} \mu\Lambda^{\infty_I,0}\right) = e^{-\beta r_i}\sum_{w\in\Lambda^0} A_i(v,w) m_\psi(w\Lambda^{\infty_I,0}) = e^{-\beta r_i}(A_i\psi')_v$$

So $A_i\psi' = e^{\beta r_i}\psi'$ for each $i \in I$, and if $\psi' \neq 0$ considering the graph Λ_I and the action given by $r_I = (r_i)_{i\in I}$, Lemma 5.2 gives us a unique collection $\mathcal{C}$ of $(\{i\}_{i\in I},\beta,r_I)$-subharmonic components in Λ_I and numbers $t'_C > 0$ for $C \in \mathcal{C}$ such that:

$$\psi' = \sum_{C\in\mathcal{C}} t'_C z^C \tag{5.8}$$

If $\psi' = 0$, we set $\mathcal{C} = \emptyset$ and (5.8) holds true.

We now want to prove that $e^{\beta r_j} > \rho(A_j^{\overline{C}})$ for all $j \in J$ and $C \in \mathcal{C}$, since that would imply that each $C \in \mathcal{C}$ was (I,β,r)-subharmonic and that each vector z^C from (5.8) was equal to x^C from Lemma 5.3. When $J = \emptyset$ this is trivial, so assume this is not the case. Since $m_\psi(\partial^I\Lambda) = 1$, we get for any $v \in \Lambda^0$ that:

$$\psi_v = m_\psi(Z(v)) = m_\psi\left(\bigcup_{n\in\mathbb{N}^J} v\Lambda^{\infty_I,n}\right) = \sum_{n\in\mathbb{N}^J} m_\psi\left(v\Lambda^{\infty_I,n}\right)$$

Looking at just one of the terms in the sum and using Lemma 5.6 we get:

$$m_\psi\left(v\Lambda^{\infty_I,n}\right)=\sum_{w\in\Lambda^0} m_\psi\left(\bigcup_{\mu\in v\Lambda^{(0,n)}w}\mu\Lambda^{\infty_I,0}\right)=\sum_{w\in\Lambda^0}\sum_{\mu\in v\Lambda^{(0,n)}w} m_\psi\left(\mu\Lambda^{\infty_I,0}\right)$$
$$=\sum_{w\in\Lambda^0} e^{-\beta r\cdot(0,n)}A^{(0,n)}(v,w)\psi'_w=e^{-\beta r\cdot(0,n)}\left(A^{(0,n)}\psi'\right)_v$$

Let $v\in\overline{C}$ and $w\in\overline{C}^I$ for a $C\in\mathcal{C}$, then $\psi'_w>0$, and by the above calculations:

$$\psi_v=\sum_{n\in\mathbb{N}^J}\sum_{u\in\Lambda^0} e^{-\beta r\cdot(0,n)}A^{(0,n)}(v,u)\psi'_u$$

This implies that $\sum_{n\in\mathbb{N}^J} e^{-\beta r\cdot(0,n)}A^{(0,n)}(v,w)<\infty$ for such w and v. Now let $u\in\overline{C}$, then there exists a $m\in\mathbb{N}^J$ and $w\in\overline{C}^I$ such that $A^{(0,m)}(u,w)>0$. For each $n\in\mathbb{N}^J$ and $v\in\overline{C}$ we have:

$$e^{-\beta r\cdot(0,n)}A^{(0,n)}(v,u)A^{(0,m)}(u,w)\le e^{\beta r\cdot(0,m)}\left(e^{-\beta r\cdot(0,n+m)}A^{(0,n+m)}(v,w)\right)$$

So $\sum_{n\in\mathbb{N}^J} e^{-\beta r\cdot(0,n)}A^{(0,n)}(v,u)<\infty$ for all $v,u\in\overline{C}$, and hence the sum $\sum_{l=0}^{\infty}(e^{-\beta r_j}A_j^{\overline{C}})^l$ converges for each $j\in J$, proving $\rho(A_j^{\overline{C}})<e^{\beta r_j}$.

We now know that each $C\in\mathcal{C}$ is a (I,β,r)-subharmonic component, so the vectors y^C exist for each such C. For any $v\in\Lambda^0$:

$$\psi_v=\sum_{n\in\mathbb{N}^J} e^{-\beta r\cdot(0,n)}\left(A^{(0,n)}\psi'\right)_v=\sum_{C\in\mathcal{C}} t'_C\sum_{n\in\mathbb{N}^J} e^{-\beta r\cdot(0,n)}\left(A^{(0,n)}x^C\right)_v$$
$$=\sum_{C\in\mathcal{C},v\in\overline{C}} t'_C\left[\left(\sum_{n\in\mathbb{N}^J}\prod_{j\in J}\left(e^{-\beta r_j}A_j^{\overline{C}}\right)^{n_j}\right)x^C|_{\overline{C}}\right]_v=\sum_{C\in\mathcal{C}} t'_C\tilde{x}^C_v=\sum_{C\in\mathcal{C}} t_C y^C_v$$

where $t_C=t'_C\|\tilde{x}^C\|_1>0$. We have now proved that the decomposition exists.

To prove the uniqueness statement assume that $\mathcal{D}$ is a collection of (I,β,r)-subharmonic components and that there exists $s_D>0$ for each $D\in\mathcal{D}$ such that:

$$\psi=\sum_{D\in\mathcal{D}} s_D y^D$$

Let m_D be the measure on Λ^* associated to y^D for each $D\in\mathcal{D}$, since $m_\psi=\sum_{D\in\mathcal{D}} s_D m_D$ it follows by Lemma 5.7 that considering these measures on

$\Lambda^{\infty_I,0}$ give:

$$\psi' = \sum_{D\in\mathcal{D}} s_D \|\tilde{x}^D\|_1^{-1} x^D$$

When $I \neq \emptyset$ then $\mathcal{D}$ can be considered a collection of $(\{i\}_{i\in I}, \beta, r_I)$-subharmonic components in Λ_I and x^D are then by construction the unique vectors from Lemma 5.2. For all I uniqueness of the decomposition in (5.8) of ψ' then gives $\mathcal{D} = \mathcal{C}$ and $s_C\|\tilde{x}^C\|_1^{-1} = t'_C$, and hence $t_C = s_C$, for each $C \in \mathcal{C}$. □

Combining Proposition 5.8, Proposition 4.3 and Proposition 3.2 we get the following

Theorem 5.9. *Let Λ be a finite k-graph, $r \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. For $I \subseteq \{1,\dots,k\}$ let $\mathcal{C}_r^I(\beta)$ be the (I,β,r)-subharmonic components and set:*

$$\mathcal{C}_r(\beta) := \bigsqcup_{I\subseteq\{1,\dots,k\}} \mathcal{C}_r^I(\beta)$$

There is an affine bijective correspondence between functions $f : \mathcal{C}_r(\beta) \to [0,1]$ with $\sum_{C\in\mathcal{C}_r(\beta)} f(C) = 1$ and the gauge-invariant β-KMS states for α^r. A KMS state ω corresponding to a function f is given by:

$$\omega(S_\lambda S_\mu^*) = \delta_{\lambda,\mu} e^{-\beta r\cdot d(\lambda)} \psi_{s(\lambda)}$$

where:

$$\psi = \sum_{C\in\mathcal{C}_r(\beta)} f(C) y^C$$

Remark 5.10. Notice that the face of the simplex of gauge-invariant KMS states given by components in $\mathcal{C}_r^I(\beta)$ corresponds to the face in the simplex of almost invariant vectors of unit 1-norm satisfying $A_i x = e^{\beta r_i} x$ for $i \in I$ and $(e^{-\beta r_j} A_j)^l x \to 0$ for $l \to \infty$ for $j \notin I$, which again corresponds to the face in the simplex of quasi-invariant Borel probability measures m with Radon Nikodym derivative $e^{-\beta c_r}$ satisfying $m(\partial^I \Lambda) = 1$.

Remark 5.11. Theorem 5.9 is already an improvement of the results obtained in [2]. To see this, notice that if r and β satisfies condition 2 mentioned in the introduction and ω is a β-KMS state for α^r, then for any $\lambda, \mu \in \Lambda$ and $t \in \mathbb{R}$ we get:

$$\omega(S_\lambda S_\mu^*) = \omega(\alpha_t^r(S_\lambda S_\mu^*)) = e^{itr\cdot(d(\lambda)-d(\mu))}\omega(S_\lambda S_\mu^*)$$

If $\omega(S_\lambda S_\mu^*) \neq 0$ then this is only possible if $r \cdot (d(\lambda) - d(\mu)) = 0$, implying that $d(\lambda) = d(\mu)$. So $S_\lambda S_\lambda^*$ and $S_\mu S_\mu^*$ are mutually orthogonal when $\lambda \neq \mu$, and since $\omega(S_\lambda S_\mu^*) = e^{-\beta r \cdot d(\lambda)} \omega(S_\mu^* S_\lambda)$, we get that $\omega(S_\lambda S_\mu^*) = 0$ when $\lambda \neq \mu$. This implies that ω is gauge-invariant, so Theorem 5.9 gives a complete description of the β-KMS states for finite k-graphs satisfying condition 2 from the introduction.

6. Including the non gauge-invariant KMS states

We are now interested in determining the KMS states that are not gauge-invariant. To do this we will use the ideas developed in [1]. Theorem 5.9 gives us a complete description of the gauge-invariant KMS states, but by Lemma 3.1 in [1] this is exactly the KMS states ω satisfying $\omega \circ P = \omega$. So in the terminology of [7] we can consider Theorem 5.9 as a description of the quasi-invariant Borel probability measures with Radon-Nikodym cocycle $e^{-\beta c_r}$, where c_r is the 1-cocycle $c_r(x, n, y) := r \cdot n$. Hence we can use Theorem 5.2 in [1] to obtain a description of all KMS states. We follow the outline and ideas in [1] to do this, and start by analysing the relationship between the paths in Λ^* and the measures associated to extremal KMS states.

Definition 6.1. *We say that a path $x \in \Lambda^*$ eventually lies in S for some set $S \subseteq \Lambda^0$, if there exists n $n \in \mathbb{N}^k$ with $n \leq d(x)$ such that $r(\sigma^m(x)) \in S$ for all $m \in \mathbb{N}^k$ with $n \leq m \leq d(x)$.*

Lemma 6.2. *Let Λ be a finite k-graph, $r \in \mathbb{R}^k$, $\beta \in \mathbb{R}$ and let $I \subseteq \{1, 2, \cdots, k\}$. If D is an equivalence class in the relation $\sim_I$, then the set:*

$$N_D^I = \{x \in \partial^I \Lambda \ : \ x \text{ eventually lies in } D\}$$

is a Borel set. If D is a (I, β, r)-subharmonic component, then the measure m_D associated to the corresponding β-KMS state for α^r satisfies $m_D(N_D^I) = 1$.

Proof. If $I = \emptyset$ then $\partial^I \Lambda = \Lambda$, $D = \{v\}$ for some vertex $v \in \Lambda^0$ and the set N_D^I is the countable set of paths $\lambda \in \Lambda$ with $s(\lambda) = v$, hence in particular it is a Borel set. Assume that $I \neq \emptyset$, and set:

$$N_{\overline{D}^I}^I := \{x \in \partial^I \Lambda \ : x \text{ eventually lies in } \overline{D}^I\} \tag{6.1}$$

Take $x \in \partial^I\Lambda$ with $x \notin N^I_{\overline{D}^I}$. In particular there is a $m \in \mathbb{N}^k$ with $(0, d(x)_J) \leq m \leq d(x)$ such that $r(\sigma^m(x)) \notin \overline{D}^I$. Setting $\lambda = x(0,m) \in \Lambda^m$ and letting $E := \{e \in s(\lambda)\Lambda \ : \ d(e) = e_j \text{ for some } j \notin I\}$, we see that:

$$\left[Z(\lambda) \setminus \left(\bigcup_{e\in E} Z(\lambda e)\right)\right] \cap N^I_{\overline{D}^I} = \emptyset$$

however x is contained in the set we intersect with $N^I_{\overline{D}^I}$, so $\left[\Lambda^* \setminus N^I_{\overline{D}^I}\right] \cap \partial^I\Lambda$ is an open set in $\partial^I\Lambda$, which implies that $N^I_{\overline{D}^I}$ is a Borel set. Letting $\mathcal{M}$ be the set of equivalence classes in $\sim_I$ contained in $\overline{D}^I \setminus D$, then:

$$N^I_D = N^I_{\overline{D}^I} \setminus \left(\bigcup_{C\in\mathcal{M}} N^I_{\overline{C}^I}\right)$$

and hence N^I_D is Borel. The sets N^I_D, where D is an equivalence set in $\sim_I$, is a disjoint Borel partition of $\partial^I\Lambda$, also when $I = \emptyset$. It follows from Theorem 5.9 that m_D is extremal in the set of quasi-invariant Borel probability measures with Radon-Nikodym cocycle $e^{-\beta c_r}$, and hence m_D maps invariant Borel sets to $\{0,1\}$. Since $m_D(\partial^I\Lambda) = 1$ there must therefore exist exactly one equivalence class C in $\sim_I$ such that $m_D(N^I_C) = 1$. We know from Lemma 5.7 that $m_D(v\Lambda^{\infty_I,0}) > 0$ if and only if $v \in \overline{D}^I$. However this must imply that for each $v \in \overline{D}^I$ we have $m_D(v\Lambda^{\infty_I,0} \cap N^I_C) > 0$, which implies that $v \in \overline{C}^I$, so that in particular $D \subseteq \overline{C}^I$. Considering a $v \in D$, it follows since $m_D(v\Lambda^{\infty_I,0} \cap N^I_C) > 0$ that there is an $\alpha \in v\Lambda C$ such that $m_D(\alpha\Lambda^{\infty_I,0}) > 0$, and hence using Lemma 5.6 we get that $m_D(s(\alpha)\Lambda^{\infty_I,0}) > 0$. This implies that $s(\alpha) \in \overline{D}^I$, so we also get $C \supseteq \overline{D}^I$, and hence $C = D$. □

To describe the non gauge-invariant KMS states, fix a $\beta \in \mathbb{R}$ and $r \in \mathbb{R}^k$, and let C be a (I,β,r)-subharmonic component for some $I \subseteq \{1,\dots,k\}$. When $I \neq \emptyset$ we define the Periodicity group $\mathrm{Per}_I(C)$ as:

$$\{(m,0)-(n,0) \ : \ m,n \in \mathbb{N}^I \ , \ \sigma^{(m,0)}(x) = \sigma^{(n,0)}(x) \text{ for all } x \in C\Lambda^{\infty_I,0} \cap N^I_C\}$$

Since C is a component in the $|I|$-graph Λ_I, $C\Lambda_I C$ is a strongly connected $|I|$-graph without sources and sinks, and hence it has an infinite path space $(C\Lambda_I C)^\infty$ consisting of functors from $\Omega_{|I|}$ to $C\Lambda_I C$. By identifying $\Omega_{|I|}$ with

$\Omega_{k,(\infty_I,0)}$ we get a homeomorphism from $(C\Lambda_I C)^\infty$ to $C\Lambda^{\infty_I,0}\cap N_C^I$ that sends the shift map of degree $n\in\mathbb{N}^I$ on $(C\Lambda_I C)^\infty$ to the shift $\sigma^{(n,0)}$ on $C\Lambda^{\infty_I,0}\cap N_C^I$, and that sends a cylinder set in $(C\Lambda_I C)^\infty$ given by $\lambda\in C\Lambda_I C$ to the relatively open set $Z(\lambda)\cap\big(C\Lambda^{\infty_I,0}\cap N_C^I\big)$. It follows from this that our periodicity group $\mathrm{Per}_I(C)$ is isomorphic to the periodicity group $\mathrm{Per}(C\Lambda_I C)$ for the I-graph $C\Lambda_I C$ introduced in Section 5 in [5], and so by Proposition 5.2 in [5] it is in fact a group. When $I=\emptyset$, we let $\mathrm{Per}_I(C)=\{0\}$ with $0\in\mathbb{Z}^k$. Using the continuous map $\Phi:\mathcal{G}_\Lambda\to\mathbb{Z}^k$ defined in section 2 we can now describe the non gauge-invariant KMS states.

Theorem 6.3. *Let Λ be a finite k-graph and fix $r\in\mathbb{R}^k$ and $\beta\in\mathbb{R}\setminus\{0\}$. There is a bijection between pairs (C,ξ), where $C\in\mathcal{C}_r^I(\beta)$ for some $I\subseteq\{1,\dots,k\}$ and ξ lies in the dual $\widehat{Per_I(C)}$ of $Per_I(C)$, to the set of extremal β-KMS states for α^r:*

$$(C,\xi)\to\omega_{C,\xi}$$

where:

$$\omega_{C,\xi}(f)=\int_{\Lambda^*}\sum_{g\in\mathcal{G}_x^x}f(g)\xi(\Phi(g))\ dm_C(x)$$

Remark 6.4. The observation made after Definition 5.5 is also true here; The notations $\omega_{C,\xi}$ and m_C are only well defined because we have fixed β and r.

Remark 6.5. Theorem 6.3 also gives a complete description of the 0-KMS states for α^r. The 0-KMS states are the tracial states on $\mathcal{T}C^*(\Lambda)$, but choosing $0\in\mathbb{R}^k$ this is the same as the 1-KMS states for α^0.

Proof of Theorem 6.3. Let C be (I,β,r)-subharmonic. Let A denote the subgroup of $\mathbb{Z}^k$ ensured by Theorem 5.2 in [1] that satisfies:

$$m_C(\{x\in\Lambda^*\ :\ \Phi(\mathcal{G}_x^x)=A\})=1$$

and denote this Borel set by $X(A)$. Using Lemma 4.1 and Theorem 5.2 in [1] it is enough to prove that $A=\mathrm{Per}_I(C)$ to prove the Theorem. Since $m_C(X(A)\cap N_C^I)=1$, we can pick a $x\in X(A)\cap N_C^I$, then setting $J=\{1,\dots,k\}\setminus I$ there is a $m\in\mathbb{N}^k$ such that $(0,d(x)_J)\le m\le d(x)$ and $r(\sigma^l(x))\in C$ for all $m\le l\le d(x)$, i.e. $\sigma^m(x)\in C\Lambda^{\infty_I,0}\cap N_C^I$. For $l\in\mathrm{Per}_I(C)$ we can write $l=(s,0)-(p,0)$ for some $s,p\in\mathbb{N}^I$ such that $\sigma^{(s,0)}(\sigma^m(x))=\sigma^{(p,0)}(\sigma^m(x))$, so $l\in\Phi(\mathcal{G}_x^x)$. Since $x\in X(A)$ this implies that $\mathrm{Per}_I(C)\subseteq A$.

For the other inclusion, let $x\in X(A)\cap N_C^I$ and assume that there is a $a\in A$ with $a_j>0$ for a $j\in J$. Since $a\in A$ there exists n,m such that $a=$

$n-m$, $n_j > m_j$ and $\sigma^n(x) = \sigma^m(x)$. This implies that $d(\sigma^n(x)) = d(\sigma^m(x))$, so $d(x)_j = \infty$. Since $j \in J$ and $x \in N_C^I \subseteq \partial^I\Lambda$ this is a contradiction. So $a_j = 0$ for $j \in J$, which in particular proves the Theorem when $I = \emptyset$. So assume $I \neq \emptyset$ and that there exists $a \in A \setminus \mathrm{Per}_I(C)$. Now fix $v \in C$, then:

$$v\Lambda^{\infty_I,0} \cap X(A) \subseteq \bigcup_{n,m\in\mathbb{N}^k, n-m=a} \{x \in v\Lambda^{\infty_I,0} \ : \ \sigma^n(x) = \sigma^m(x)\}$$

so since $m_C(v\Lambda^{\infty_I,0} \cap X(A)) > 0$ we can find a $n_1, n_2 \in \mathbb{N}^I$ with $(n_1-n_2, 0) = a$ and

$$m_C(\{x \in v\Lambda^{\infty_I,0} \ : \ \sigma^{(n_1,0)}(x) = \sigma^{(n_2,0)}(x)\}) > 0 \tag{6.2}$$

For the vector y^C corresponding to m_C we have $A_i y^C = e^{\beta r_i} y^C = \rho(A_i^C) y^C$ for $i \in I$, which implies that

$$A_i^C y^C|_C = e^{\beta r_i} y^C|_C = \rho(A_i^C) y^C|_C$$

Since $(A_i^C)_{i\in I}$ are the vertex matrices for $C\Lambda_I C$, it follows from (*b*) in Corollary 4.2, Proposition 8.1 and Proposition 8.2 in [5] that there is a Borel probability measure M on $(C\Lambda_I C)^\infty \simeq C\Lambda^{\infty_I,0} \cap N_C^I$, with

$$M(\{x \in C\Lambda^{\infty_I,0} \cap N_C^I \ : \ \sigma^{(n_1,0)}(x) = \sigma^{(n_2,0)}(x)\}) = 0$$

and that for each $\lambda \in C\Lambda_I C$ satisfies:

$$M(\ C\Lambda^{\infty_I,0} \cap N_C^I \cap Z(\lambda)\} = e^{-\beta r\cdot d(\lambda)} y^C_{s(\lambda)} \|y^C|_C\|_1^{-1}$$

Let $\varepsilon > 0$. By compactness there are paths $\delta_1, \dots, \delta_q \in C\Lambda_I C$ of the same degree such that:

$$\{x \in C\Lambda^{\infty_I,0} \cap N_C^I \ : \ \sigma^{(n_1,0)}(x) = \sigma^{(n_2,0)}(x)\} \subseteq \bigsqcup_{i=1}^q Z(\delta_i) \cap C\Lambda^{\infty_I,0} \cap N_C^I$$

and such that $\sum_{i=1}^q M(Z(\delta_i) \cap C\Lambda^{\infty_I,0} \cap N_C^I) < \varepsilon$. Since ε was arbitrary the calculation:

$$\begin{aligned}\sum_{i=1}^n m_C(Z(\delta_i)) &= \sum_{i=1}^n e^{-\beta r\cdot d(\delta_i)} m_C(Z(s(\delta_i))) = \sum_{i=1}^n e^{-\beta r\cdot d(\delta_i)} y^C_{s(\delta_i)} \\ &= \|y^C|_C\|_1 \sum_{i=1}^n M(Z(\delta_i) \cap C\Lambda^{\infty_I,0} \cap N_C^I) < \varepsilon\end{aligned}$$

implies that:

$$m_C(\{x \in C\Lambda^{\infty_I,0} \cap N_C^I \ : \ \sigma^{(n_1,0)}(x) = \sigma^{(n_2,0)}(x)\}) = 0$$

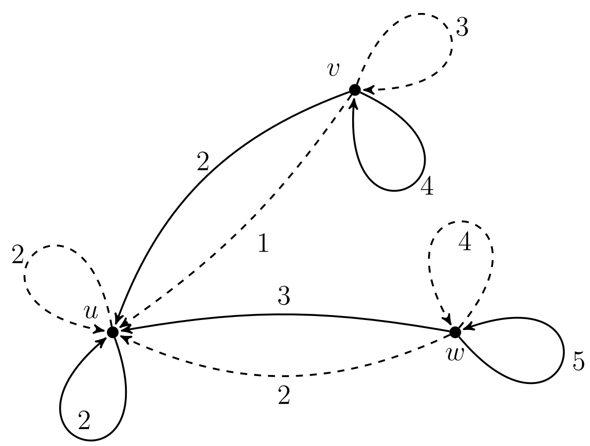
Combining this with (6.2) gives us a contradiction since any $x \in v\Lambda^{\infty_I,0}$ with $x \notin N_C^I$ satisfies $x \in Z(\alpha)$ for some α with $m_C(Z(\alpha)) = 0$. In conclusion $A = \mathrm{Per}_I(C)$. □

7. Examples and comparison with the literature

7.1. Examples

To illustrate how to describe the KMS states for a given graph we will use our machinery on a few examples. The first graph we consider is from Example 9.1 in [2] where the KMS states for the action given by $r = (\ln(5), \ln(4))$ were calculated. We have included this example to illustrate that our results give the same KMS states as the ones in [2], but also to show the strength of our approach when it comes to concrete calculations.

Example 7.1. Consider a 2-graph given by the graph below, where normal edges have degree e_1 and dashed edges have degree e_2, and the number at each edge denote the number of edges:



No matter which $I \subseteq \{1,2\}$ we choose there are three components $\{u\}$, $\{v\}$ and $\{w\}$ for $\sim_I$, so we will analyse for which β and r each is (I,β,r)-subharmonic. For this, notice that $\overline{\{v\}} = \{u,v\}$, $\overline{\{u\}} = \{u\}$ and $\overline{\{w\}} = \{u,w\}$. Considering the graph it follows that:

$$\begin{aligned}
\rho(A_1^{\overline{\{v\}}}) &= \rho(A_1^{\{v\}}) = 4 \quad , \quad \rho(A_2^{\overline{\{v\}}}) = \rho(A_2^{\{v\}}) = 3 \\
\rho(A_1^{\overline{\{u\}}}) &= \rho(A_1^{\{u\}}) = 2 \quad , \quad \rho(A_2^{\overline{\{u\}}}) = \rho(A_2^{\{u\}}) = 2 \\
\rho(A_1^{\overline{\{w\}}}) &= \rho(A_1^{\{w\}}) = 5 \quad , \quad \rho(A_2^{\overline{\{w\}}}) = \rho(A_2^{\{w\}}) = 4
\end{aligned}$$

Hence by Definition 5.1 the different components give KMS states for βr in the sets as indicated in the table below.

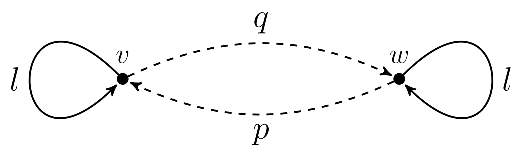
I\C	$\{v\}$	$\{u\}$	$\{w\}$
$\emptyset$	$]\ln(4),\infty[\times]\ln(3),\infty[$	$]\ln(2),\infty[\times]\ln(2),\infty[$	$]\ln(5),\infty[\times]\ln(4),\infty[$
$\{1\}$	$\{\ln(4)\}\times]\ln(3),\infty[$	$\{\ln(2)\}\times]\ln(2),\infty[$	$\{\ln(5)\}\times]\ln(4),\infty[$
$\{2\}$	$]\ln(4),\infty[\times\{\ln(3)\}$	$]\ln(2),\infty[\times\{\ln(2)\}$	$]\ln(5),\infty[\times\{\ln(4)\}$
$\{1,2\}$	$\{\ln(4)\}\times\{\ln(3)\}$	$\{\ln(2)\}\times\{\ln(2)\}$	$\{\ln(5)\}\times\{\ln(4)\}$

As in [2] we now consider the action given by $r = (\ln(5), \ln(4))$ which has rationally independent coordinates. Theorem 5.9 then implies that we get a complete description of the β-KMS states for α^r by describing the (I,β,r)-subharmonic components for different $I \subseteq \{1,2\}$. So we go through each entry of the table and consider for which value of β that βr lies in the set at that entry. This gives the following result (notice $\ln(2)/\ln(4) = 1/2$):

I\C	$\{v\}$	$\{u\}$	$\{w\}$
$\emptyset$	$]\ln(4)/\ln(5),\infty[$	$]1/2,\infty[$	$]1,\infty[$
$\{1\}$	$\{\ln(4)/\ln(5)\}$	$\emptyset$	$\emptyset$
$\{2\}$	$\emptyset$	$\{1/2\}$	$\emptyset$
$\{1,2\}$	$\emptyset$	$\emptyset$	$\{1\}$

This is exactly the same as obtained in Example 9.1 in [2].

Example 7.2. Using Theorem 5.9 we will give an example of a strongly connected graph without sources and sinks and a one-parameter group α^r with two different gauge-invariant β-KMS states for α^r for the critical temperature β. To do this consider the following skeleton:



The full edges have degree e_1 and the dashed edges have degree e_2, and the numbers $l, p, q \geq 1$ denote the number of edges . Since $A_1 = l1_{\{v,w\}}$ then A_1 and A_2 must commute, and hence there exists a 2-graph with this skeleton, c.f. Section 6 in [6]. In the equivalence relation $\sim_{\{1\}}$ both $\{v\}$ and $\{w\}$ are components, and choosing $r = (\ln(l), \ln(2\sqrt{p \cdot q}))$ we see that $e^{1r_1} = \rho(A_1^{\{v\}}) = \rho(A_1^{\{w\}})$. Since $\overline{\{v\}} = \overline{\{w\}} = \{v, w\}$, and since $\rho(A_2) = \sqrt{p \cdot q}$, both $\{v\}$ and $\{w\}$ are $(\{1\}, 1, r)$-subharmonic components, and since there are no $(\emptyset, 1, r)$- and $(\{1, 2\}, 1, r)$-subharmonic components, Theorem 5.9 implies that they give rise to only gauge-invariant 1 -KMS states for α^r. Ordering the set of vertexes by $\{v, w\}$ then the vectors given in Proposition 5.4 are:

$$\tilde{x}^{\{v\}} = \frac{2}{3} \begin{pmatrix} 2 \\ \sqrt{q}/\sqrt{p} \end{pmatrix} \quad , \quad \tilde{x}^{\{w\}} = \frac{2}{3} \begin{pmatrix} \sqrt{p}/\sqrt{q} \\ 2 \end{pmatrix}$$

Both vectors are, as predicted, invariant for the family $\{l^{-1}A_1, (2\sqrt{p \cdot q})^{-1}A_2\}$. Hence their normalizations $y^{\{v\}}$ and $y^{\{w\}}$ give rise to two different gauge-invariant 1-KMS states for α^r. When $l > 1$ then $\mathrm{Per}_{\{1\}}(\{v\}) = \mathrm{Per}_{\{1\}}(\{w\}) = \{0\}$, and the 1-KMS states for α^r are given by convex combinations of the two states $\omega_{y^{\{v\}}}$ and $\omega_{y^{\{w\}}}$, with:

$$\omega_{y^{\{v\}}}(S_\lambda S_\mu^*) = \delta_{\lambda,\mu} e^{-\beta r \cdot d(\lambda)} y^{\{v\}} \quad , \quad \omega_{y^{\{w\}}}(S_\lambda S_\mu^*) = \delta_{\lambda,\mu} e^{-\beta r \cdot d(\lambda)} y^{\{w\}}$$

If $l = 1$ then $\mathrm{Per}_{\{1\}}(\{v\}) = \mathrm{Per}_{\{1\}}(\{w\}) = \mathbb{Z} \times \{0\}$, so letting m_v and m_w be the measures corresponding to respectively $\{v\}$ and $\{w\}$, then $\Phi(\mathcal{G}_x^x) = \mathbb{Z} \times \{0\}$ for almost all $x \in \Lambda^*$ and the extremal 1-KMS states for α^r are:

$$\omega_{\{u\},\lambda}(f) = \int_{\Lambda^*} \sum_{(x,(n,m),x) \in \mathcal{G}_x^x} f(x, (n, m), x) \lambda^n \, dm_u(x)$$

for all $\lambda \in \mathbb{T}$ and $u = v, w$.

7.2. Comparison with the literature

We will now compare our results to the ones in [2]. To do this we will need the following Lemma:

Lemma 7.3. *Let Λ be a finite k-graph without sources that has the property, that when $v, w \in \Lambda^0$ satisfies $v\Lambda^m w \neq \emptyset$ for some $m \in \mathbb{N}^k \setminus \{0\}$, then they also satisfy $v\Lambda^m w \neq \emptyset$ for some $m \in \mathbb{N}^{\{i\}} \setminus \{0\}$ for each $i \in \{1, \dots, k\}$. Assume C is a component in $\sim$ with $\rho(A_j^C) > 0$ for each $j \in \{1, \dots, k\}$, then:*

1. *If $\rho(A^D) \lneq \rho(A^C)$ for each $D \subseteq \overline{C} \setminus C$, then for all $i \in \{1, \dots, k\}$:*

$$\rho(A_i^D) < \rho(A_i^C) \quad \text{for all components } D \subseteq \overline{C} \setminus C \tag{7.1}$$

2. *If C satisfies (7.1) for some i then it satisfies it for all $i \in \{1, \dots, k\}$.*

Proof. Fix an i. The condition on the graph implies that we can find a finite set of numbers $F \subseteq \mathbb{N}^{\{i\}} \setminus \{0\}$ such that $A_F(v, w) := \sum_{n \in F} A^n(v, w) > 0$ if and only if $v\Lambda^m w \neq \emptyset$ for some $m \in \mathbb{N}^k$. Such a set is called well-chosen in the terminology of [1], and it follows from Lemma 7.11 and Definition 7.5 in that article that $\rho(A_F^{\overline{C} \setminus C}) < \rho(A_F^C)$. Using Lemma 7.8 in [1] on the graph $\Lambda\overline{C} \setminus C$ we get $\rho(A_F^D) < \rho(A_F^C)$ for each $D \subseteq \overline{C} \setminus C$. Since $\rho(A_F^D) = \sum_{n \in F} \rho(A_i^D)^{n_i}$ by equation (7.2) in [1] (7.1) follows.

To prove 2 assume C satisfies (7.1) for a i and choose F as above for this i. Equation (7.2) in [1] gives $\rho(A_F^D) < \rho(A_F^C)$ for all components $D \subseteq \overline{C} \setminus C$, and hence combining Lemma 7.8, Definition 7.5 and Lemma 7.11 in [1] imply that C satisfies the criterion in 1. ☐

To follow the set-up in [2] we consider a finite k-graph Λ and a $r \in \mathbb{R}^k$ satisfying condition $1-5$ from the introduction, and we assume that $K \sqcup L = \{1, \dots, k\}$ is a partition with $r_i = \ln(\rho(A_i))$ for $i \in K \neq \emptyset$ and $r_l > \ln(\rho(A_l))$ for $l \in L$. We let $\mathcal{C}_{crit}$ be the components C in $\sim$ with $\ln(\rho(A_j^C)) = r_j$ for some j, and $\mathcal{C}_{mincrit}$ be the minimal elements in $\mathcal{C}_{crit}$ for the order $\leq$. Notice that condition 4 and 5 imply that the condition imposed on Λ in Lemma 7.3 is satisfied, and that the relations $\leq_I$, $I \neq \emptyset$ are all equal.

Let $C \in \mathcal{C}_{mincrit}$ and set $I = \{i : \ln(\rho(A_i^C)) = r_i\}$ then $I \neq \emptyset$. For $i \in I$ C satisfies 2 in Lemma 7.3, so $\rho(A^D)_I \lneq \rho(A^C)_I$ for all $D \subseteq \overline{C} \setminus C$, and by (7.1) then $\rho(A_j^{\overline{C}}) = \rho(A_j^C) < e^{r_j}$ for $j \notin I$, so C is $(I, 1, r)$-subharmonic. Assume on the other hand that C is a $(I, 1, r)$-subharmonic component for a $I \neq \emptyset$, then for $i \in I$ we have $\ln(\rho(A_i^C)) = r_i$, so $C \in \mathcal{C}_{crit}$. If $C \notin \mathcal{C}_{mincrit}$ then there is a $(J,1,r)$-subharmonic component D with $D \subseteq \overline{C} \setminus C$ and $J \subseteq \{1,\dots,k\}$ not empty. If $l \notin I$ then:

$$\rho(A_l^D) \leq \rho(A_l^{\overline{C}}) < e^{r_l}$$

so then $l \notin J$, giving $J \subseteq I$. Since Λ_I as an I-graph satisfies the criterion of Lemma 7.3, C satisfies the criterion in 1 for the graph Λ_I and $D \subseteq \overline{C}^I \setminus C$ we get that $\rho(A_i^D) < \rho(A_i^C)$ for all $i \in I$. For $i \in J$ this implies $e^{r_i} = \rho(A_i^D) < \rho(A_i^C) = e^{r_i}$, a contradiction. So $\mathcal{C}_{mincrit}$ is the set of $(I,1,r)$-subharmonic components with $I \neq \emptyset$.

If $\{v\}$ is a $(\emptyset,1,r)$-subharmonic component, then $\rho(A_j^{\overline{\{v\}}}) < e^{r_j}$ for each $j \in \{1,\dots,k\}$, so $\overline{\{v\}}$ contains no components from $\mathcal{C}_{crit}$, and hence $v \notin \widehat{\mathcal{C}_{crit}}$. If on the other hand $v \notin \widehat{\mathcal{C}_{crit}}$ then $\overline{\{v\}}$ contains no critical components, so $\rho(A_j^{\overline{\{v\}}}) < e^{r_j}$ for each $j \in \{1,\dots,k\}$, implying that $\{v\}$ is $(\emptyset,1,r)$-subharmonic.

Comparing Theorem 5.9 with Section 7 in [2] we now see that the vertexes and components giving rise to extremal 1-KMS states for α^r are the same in the two expositions. To see that the states also agree it suffices to argue that the corresponding vectors over Λ^0 agree. For $v \notin \widehat{\mathcal{C}_{crit}}$ this follows from comparing the vector defined in Proposition 5.4 when considerind $\{v\}$ a $(\emptyset,1,r)$-subharmonic component with the one constructed in Theorem 6.1 in [4]. For $C \in \mathcal{C}_{mincrit}$ define $H \subseteq \Lambda^0$ as in Proposition 3.4 in [2], then the vector z of unit 1-norm constructed in [2] corresponding to C satisfies $z_v = 0$ for $v \notin \overline{C}$ and for all $i \in \{1,\dots,k\}$ that:

$$A_i^{\Lambda^0\setminus H} z|_{\Lambda^0\setminus H} = \rho(A_i^C) z|_{\Lambda^0\setminus H}$$

For $I := \{i : r_i = \ln(\rho(A_i^C))\}$ we get $A_i z = e^{r_i} z$ for $i \in I$, so z is the unique vector x^C from Lemma 5.3. Since $\tilde{x}^C$ is supported on $\overline{C} = \overline{C}^I$ and satisfies $A_i \tilde{x}^C = e^{r_i} \tilde{x}^C$, it has to be a scalar of x^C, so $y^C = z$. This proves that the states obtained in [2] are the same as the ones obtained in Theorem 5.9.

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DIAGONALITY OF ACTIONS AND KMS WEIGHTS

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ABSTRACT. We show that in many cases a one-parameter group of automorphisms on a C^*-algebra of an étale groupoid is given by a real-valued homomorphism on the groupoid if and only if the KMS weights of the one-parameter group is given by measures on the unit space. The results are applied to graph C^*-algebras.

KEYWORDS: *KMS weights, one-parameter groups, diagonality, graph C^*-algebras.*

MSC (2010): 46L55, 46L60, 46L30.

1. INTRODUCTION

Recent years have seen an increasing interest in the investigation of KMS states for one-parameter actions on C^*-algebras. While the original motivation for the introduction of KMS states came from the interpretation of these states as equilibrium states in models from quantum statistical mechanics, the renewed interest stems also from more purely mathematical considerations, where the KMS states have been related to objects and structures from other fields, such as number theory or dynamical systems. In the present paper we investigate relations between properties of the KMS states and properties of the one-parameter action giving rise to them. As we shall now explain, we show that the existence of a "diagonal" KMS state or weight implies that the action itself must be "diagonal".

For most if not all the one-parameter actions on C^*-algebras for which we have been able to determine the structure of the KMS states or KMS weights, the underlying C^*-algebra can be presented as the C^*-algebra of a locally compact groupoid, as introduced by Renault in [9], and the action described as arising from a continuous real-valued homomorphism on the groupoid by a canonical procedure also introduced in [9]. For this reason the results of Neshveyev, [7], which extend results of Renault and give a general and abstract description of the KMS states for such actions on a groupoid C^*-algebra are of utmost importance. In the following we call these actions *diagonal*.

When the groupoid and the associated C^*-algebra is fixed, it is certainly not all one-parameter actions that are diagonal. It follows from Neshveyev's theorem, Theorem 1.3 in [7], that a diagonal action has the property that if a KMS state exists, there will also be one which factorizes through the canonical conditional expectation onto the abelian C^*-subalgebra generated by the continuous compactly supported functions on the unit space. In the following we call these states *diagonal*. The present work sprang from the realization that in many cases the property that there is a diagonal KMS state characterizes the diagonal actions. That is, for many groupoid C^*-algebras a one-parameter action is diagonal if and only if the action admits a diagonal KMS state. The simplest example of this is perhaps the following.

Consider the C^*-algebra M_n of complex n by n matrices. A continuous one-parameter group α of automorphisms on M_n is inner in the sense that there is a self-adjoint matrix $A \in M_n$ such that

$$\alpha_t(B) = \mathrm{e}^{\mathrm{i}tA} B \mathrm{e}^{-\mathrm{i}tA}$$

for all $t \in \mathbb{R}$ and all $B \in M_n$. For each $\beta \in \mathbb{R}$ there is a unique β-KMS state ω_β for α given by

$$\omega_\beta(B) = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta A} B)}{\mathrm{Tr}(\mathrm{e}^{-\beta A})}.$$

It can be shown that for $\beta \neq 0$ the state ω_β factorizes through the canonical (and unique) conditional expectation from M_n onto the C^*-subalgebra of diagonal matrices if and only if A is diagonal. It is this fact we will generalize. For this note that M_n is the groupoid C^*-algebra of the groupoid $\mathcal{G} = \{1, 2, 3, \dots, n\} \times \{1, 2, 3, \dots, n\}$ with operations

$$(a, b)(b, c) = (a, c) \quad \text{and} \quad (a, b)^{-1} = (b, a).$$

When M_n is identified with the C^*-algebra $C^*(\mathcal{G})$ of $\mathcal{G}$, the diagonal matrices in M_n constitute the C^*-algebra $C(\mathcal{G}^{(0)})$ of (continuous) functions on $\mathcal{G}$ whose support is contained in the unit space

$$\mathcal{G}^{(0)} = \{(k, k) : k \in \{1, 2, \dots, n\}\}$$

of $\mathcal{G}$. In this picture the conditional expectation onto the diagonal matrices is the map

$$P : C^*(\mathcal{G}) \to C(\mathcal{G}^{(0)})$$

which restricts functions to $\mathcal{G}^{(0)}$. Furthermore, the matrix A will be diagonal if and only if there is a groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ such that

$$\alpha_t(f)(a, b) = \mathrm{e}^{\mathrm{i}c(a,b)t} f(a, b) \tag{1.1}$$

for all $t \in \mathbb{R}$, $(a, b) \in \mathcal{G}$ and all $f \in C^*(\mathcal{G})$. Because the whole setup is so transparent in this case, we can easily conclude that there is an equivalence between the following conditions:

(1) α is diagonal in the sense that there is a groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ such that (1.1) holds.

(2) There is a $\beta \neq 0$ and a β-KMS state ω_β for α which is diagonal in the sense that it factorizes through the conditional expectation $C^*(\mathcal{G}) \to C(\mathcal{G}^{(0)})$.

(3) $\alpha_t(f) = f$ for all $t \in \mathbb{R}$ and all $f \in C(\mathcal{G}^{(0)})$.

(4) All β-KMS states of α, for $\beta \neq 0$, are diagonal.

Our main result is that these equivalences hold much more generally as we shall now explain.

2. NOTATION AND MAIN RESULT

Let $\mathcal{G}$ be a second countable locally compact Hausdorff étale groupoid with unit space $\mathcal{G}^{(0)}$. Let $r : \mathcal{G} \to \mathcal{G}^{(0)}$ and $s : \mathcal{G} \to \mathcal{G}^{(0)}$ be the range and source maps, respectively. For $x \in \mathcal{G}^{(0)}$ put $\mathcal{G}^x = r^{-1}(x)$, $\mathcal{G}_x = s^{-1}(x)$ and $\mathcal{G}_x^x = s^{-1}(x) \cap r^{-1}(x)$. Note that $\mathcal{G}_x^x$ is a group, the *isotropy group* at x. The space $C_c(\mathcal{G})$ of continuous compactly supported functions is a $*$-algebra when the product is defined by

$$(f_1 * f_2)(g) = \sum_{h \in \mathcal{G}^{r(g)}} f_1(h) f_2(h^{-1}g)$$

and the involution by $f^*(g) = \overline{f(g^{-1})}$. To define the *reduced groupoid C^*-algebra* $C_r^*(\mathcal{G})$, let $x \in \mathcal{G}^{(0)}$. There is a representation π_x of $C_c(\mathcal{G})$ on the Hilbert space $l^2(\mathcal{G}_x)$ of square-summable functions on $\mathcal{G}_x$ given by

$$\pi_x(f)\psi(g) = \sum_{h \in \mathcal{G}^{r(g)}} f(h)\psi(h^{-1}g).$$

$C_r^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ with respect to the norm

$$\|f\|_r = \sup_{x \in \mathcal{G}^{(0)}} \|\pi_x(f)\|.$$

Note that $C_r^*(\mathcal{G})$ is separable since we assume that the topology of $\mathcal{G}$ is second countable.

We shall here be concerned not only with KMS states, but more generally with KMS weights. Let A be a C^*-algebra and A_+ the convex cone of positive elements in A. A *weight* on A is a map $\psi : A_+ \to [0, \infty]$ with the properties that $\psi(a+b) = \psi(a) + \psi(b)$ and $\psi(\lambda a) = \lambda\psi(a)$ for all $a, b \in A_+$ and all $\lambda \in \mathbb{R}$, $\lambda > 0$. By definition ψ is *densely defined* when $\{a \in A_+ : \psi(a) < \infty\}$ is dense in A_+ and *lower semi-continuous* when $\{a \in A_+ : \psi(a) \leqslant \alpha\}$ is closed for all $\alpha \geqslant 0$. We will use [5], [6] as our source for information on weights, and we say that a weight is *proper* when it is non-zero, densely defined and lower semi-continuous. Let ψ be a proper weight on A. Set $\mathcal{N}_\psi = \{a \in A : \psi(a^*a) < \infty\}$ and note that

$$\mathcal{M}_\psi = \operatorname{Span}\{a^*b : a, b \in \mathcal{N}_\psi\}$$

is a dense $*$-subalgebra of A, and that there is a unique well-defined linear map $\mathcal{M}_\psi \to \mathbb{C}$ which extends $\psi : \mathcal{M}_\psi \cap A_+ \to [0, \infty)$. We denote also this densely defined linear map by ψ.

Let $\alpha : \mathbb{R} \to \operatorname{Aut} A$ be a continuous one-parameter group of automorphisms on A. Let $\beta \in \mathbb{R}$. Following [2] we say that a proper weight ψ on A is a *β-KMS weight* for α when

(i) $\psi \circ \alpha_t = \psi$ for all $t \in \mathbb{R}$, and

(ii) for every pair $a, b \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$ there is a continuous and bounded function F defined on the closed strip D_β in $\mathbb{C}$ consisting of the numbers $z \in \mathbb{C}$ whose imaginary part lies between 0 and β, and is holomorphic in the interior of the strip and satisfies that

$$F(t) = \psi(a\alpha_t(b)),\ F(t + \mathrm{i}\beta) = \psi(\alpha_t(b)a)$$

for all $t \in \mathbb{R}$.

Compared to [2] we have changed the orientation in order to have the same sign convention as in [1], for example. It will be important for us that there is an alternative characterization of when a proper weight is a β-KMS weight. Specifically, by Proposition 1.11 in [6] a proper weight ψ is a β-KMS weight for α if and only if it is α-invariant (as in (i) above) and

$$\psi(a^*a) = \psi(\alpha_{\mathrm{i}\beta/2}(a)\alpha_{\mathrm{i}\beta/2}(a)^*) \tag{2.1}$$

for all a in the domain $D(\alpha_{\mathrm{i}\beta/2})$ of $\alpha_{\mathrm{i}\beta/2}$; the closure of the restriction of $\alpha_{\mathrm{i}\beta/2}$ to the analytic elements for α, cf. [5]. A β-KMS weight ψ with the property that

$$\sup\{\psi(a) : 0 \leqslant a \leqslant 1\} = 1$$

will be called a *β-KMS state*.

Returning to the case $A = C_r^*(\mathcal{G})$, note that the map $C_c(\mathcal{G}) \to C_c(\mathcal{G}^{(0)})$ which restricts functions to $\mathcal{G}^{(0)}$ extends to a conditional expectation $P : C_r^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$. Via P a regular Borel measure m on $\mathcal{G}^{(0)}$ gives rise to a weight $\varphi_m : C_r^*(\mathcal{G})_+ \to [0, \infty]$ defined by the formula

$$\varphi_m(a) = \int_{\mathcal{G}^{(0)}} P(a)\, \mathrm{d}m. \tag{2.2}$$

It follows from Fatou's lemma that φ_m is lower semi-continuous. Since $\varphi_m(faf) < \infty$ for every non-negative function f in $C_c(\mathcal{G}^{(0)})$, it follows that φ_m is also densely defined, i.e. φ_m is a proper weight on $C_r^*(\mathcal{G})$ if and only if m is not the zero measure. In the following we say that a proper weight ψ on $C_r^*(\mathcal{G})$ is *diagonal* when $\psi = \varphi_m$ for some regular Borel measure m on $\mathcal{G}^{(0)}$. By the Riesz representation theorem this occurs if and only if $\psi \circ P = \psi$.

Given a continuous homomorphism $c : \mathcal{G} \to \mathbb{R}$ there is a continuous one-parameter group σ^c on $C_r^*(\mathcal{G})$ such that

$$\sigma_t^c(g)(\xi) = \mathrm{e}^{itc(\xi)}g(\xi) \tag{2.3}$$

for all $t \in \mathbb{R}$, all $g \in C_c(\mathcal{G})$ and all $\xi \in \mathcal{G}$, cf. [9]. A one-parameter action of this kind will be called *diagonal* in the following. We can then formulate our main result as follows.

THEOREM 2.1. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid such that for at least one element $x \in \mathcal{G}^{(0)}$ the isotropy group $\mathcal{G}_x^x$ is trivial, i.e. $\mathcal{G}_x^x = \{x\}$, and that $\mathcal{G}$ is minimal in the sense that $s(r^{-1}(y))$ is dense in $\mathcal{G}^{(0)}$ for all $y \in \mathcal{G}^{(0)}$. Furthermore, assume that $\mathcal{G}^{(0)}$ is totally disconnected.*

*Let $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group of automorphisms on $C_r^*(\mathcal{G})$ and assume that for some $\beta_0 \neq 0$ there is a β_0-*KMS *weight for α.*

The following are equivalent:

(i) *There is a $\beta_1 \neq 0$ and a diagonal β_1-*KMS *weight for α.*

(ii) *Whenever $\beta \neq 0$ and there is a β-*KMS *weight for α, there is also a diagonal β-*KMS *weight for α.*

(iii) *$\alpha_t(f) = f$ for all $t \in \mathbb{R}$ and all $f \in C_0(\mathcal{G}^{(0)})$.*

(iv) *$\alpha_t(C_0(\mathcal{G}^{(0)})) \subseteq C_0(\mathcal{G}^{(0)})$ for all $t \in \mathbb{R}$.*

(v) *α is diagonal.*

Some of the (non-trivial) implications hold with fewer assumptions. Specifically, (i)⇒(iii) holds without the assumption that the unit space is totally disconnected by Proposition 4.1, and the implication (iii)⇒(v) holds assuming only that the points with trivial isotropy are dense in $\mathcal{G}^{(0)}$ (i.e. if $\mathcal{G}$ is topologically principal) by Proposition 4.3. The implication (v)⇒(ii) holds whenever $\mathcal{G}^{(0)}$ is totally disconnected, without any further assumptions, as it follows from Corollary 3.4. It may be that this implication is true in general and if so the theorem with (iv) removed is true also when $\mathcal{G}^{(0)}$ is not totally disconnected. However, the first two assumptions on $\mathcal{G}$ which are equivalent to topological principality and minimality of $\mathcal{G}$ are certainly necessary for the implication (iii)⇒(i) to hold, cf. Example 4.9. Finally, the gauge action on the C^*-algebra of a strongly connected (row-finite) graph with infinite Gurevich entropy does not admit any KMS weights at all, cf. [14], showing that it is necessary to assume the existence of some KMS-weight for the implication (v)⇒(i) to hold.

3. NESHVEYEV'S THEOREM FOR KMS WEIGHTS

LEMMA 3.1. *Let A be a C^*-algebra, α a continuous one-parameter group of automorphisms on A and ψ a* KMS *weight for α. Let $p \in A$ be a projection in the fixed point algebra of α. Then $\psi(p) < \infty$.*

Proof. Assume that $a \geqslant 0$, $\psi(a) < \infty$ and that $a^{1/2}$ is analytic for α. Then Proposition 1.11 in [6] applies to conclude that

$$\psi(pap) = \psi(\alpha_{i\beta/2}(a^{1/2})p\alpha_{i\beta/2}(a^{1/2})^*) \leqslant \psi(\alpha_{i\beta/2}(a^{1/2})\alpha_{i\beta/2}(a^{1/2})^*) = \psi(a). \tag{3.1}$$

Let $\{b_k\}$ be a sequence of positive elements in A such that $\lim_{k\to\infty} b_k = p$ and $\psi(b_k) < \infty$ for all k. For each $n \in \mathbb{N}$, set

$$c_{k,n} = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t(b_k) \mathrm{e}^{-nt^2} \, \mathrm{d}t.$$

Then $c_{k,n}$ is analytic for α and $\psi(c_{k,n}^2) \leqslant \|c_{k,n}\|\psi(c_{k,n}) \leqslant \|c_{k,n}\|\psi(b_k) < \infty$ for all k, n. It follows therefore from (3.1) that $\psi(pc_{k,n}^2 p) \leqslant \psi(c_{k,n}^2) < \infty$ for all k, n. Note that

$$\lim_{k\to\infty} \lim_{n\to\infty} c_{k,n}^2 = \lim_{k\to\infty} b_k^2 = p^2 = p.$$

It follows that there are k, n such that $\|p - pc_{k,n}^2 p\| \leqslant 1/2$, and then spectral theory tells us that $pc_{k,n}^2 p \geqslant (1/2)p$. Hence $\psi(p) \leqslant 2\psi(pc_{k,n}^2 p) < \infty$. ∎

Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and $c : \mathcal{G} \to \mathbb{R}$ a continuous homomorphism. Let μ be a regular Borel measure on $\mathcal{G}^{(0)}$ and $\beta \in \mathbb{R}$ a real number. We say that μ is *$(\mathcal{G}, c)$-conformal with exponent β*, as in [14], or that μ is *quasi-invariant with Radon–Nikodym cocycle* $\mathrm{e}^{-\beta c}$, as in [7], when

$$\mu(s(W)) = \int_{r(W)} \mathrm{e}^{\beta c(r_W^{-1}(x))} \, \mathrm{d}\mu(x) \tag{3.2}$$

for every open bi-section $W \subseteq \mathcal{G}$, where r_W^{-1} denotes the inverse $r : W \to r(W)$. For each $x \in \mathcal{G}^{(0)}$ we can consider the full group C^*-algebra $C^*(\mathcal{G}_x^x)$ of the discrete group $\mathcal{G}_x^x$, the isotropy group at x. As in [7] we denote for $g \in \mathcal{G}_x^x$ by u_g the characteristic function of the element g when we consider $C^*(\mathcal{G}_x^x)$ as a completion of $C_c(\mathcal{G}_x^x)$. Thus $u_g, g \in \mathcal{G}_x^x$, are the canonical unitary generators of $C^*(\mathcal{G}_x^x)$. Following [7] we say that a collection $\varphi_x, x \in \mathcal{G}^{(0)}$, of states on $C^*(\mathcal{G}_x^x)$ is a *μ-measurable field* when the function

$$\mathcal{G}^{(0)} \ni x \mapsto \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)$$

is μ-measurable for all $f \in C_c(\mathcal{G})$. We identify two μ-measurable fields $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ and $\{\varphi'_x\}_{x\in\mathcal{G}^{(0)}}$ when $\varphi_x = \varphi'_x$ for μ-almost every x.

The following theorem is a version for weights of Theorem 1.3 in [7]. Note that it deals with the full groupoid C^*-algebra $C^*(\mathcal{G})$ which is an extension of the reduced groupoid $C_r^*(\mathcal{G})$. We refer to [9] for the definition of the full groupoid C^*-algebra. To understand the following theorem and its proof it suffices to know that $C^*(\mathcal{G})$, like $C_r^*(\mathcal{G})$, is a completion of $C_c(\mathcal{G})$ and that a continuous homomorphism $c : \mathcal{G} \to \mathbb{R}$ also defines a continuous one-parameter group σ^c on $C^*(\mathcal{G})$ via the formula (2.3).

THEOREM 3.2 (Neshveyev's theorem for weights). *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous homomorphism. Assume that the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is totally disconnected.*

There is a bijective correspondence between the β-KMS weights for σ^c on $C^*(\mathcal{G})$ and the pairs $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$, where μ is a regular Borel measure on $\mathcal{G}^{(0)}$ and $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field of states φ_x on $C^*(\mathcal{G}_x^x)$ such that:

(i) *μ is quasi-invariant with Radon–Nikodym cocycle $\mathrm{e}^{-\beta c}$,*

(ii) *$\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}})$ for μ-almost every $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x$, $h \in \mathcal{G}_x$, and*

(iii) *$\varphi_x(u_g) = 0$ for μ-almost every $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x \backslash c^{-1}(0)$.*

The β-KMS weight ϕ corresponding to the pair $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ has the properties that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\phi$ and

$$\phi(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)\, \mathrm{d}\mu(x) \tag{3.3}$$

when $f \in C_c(\mathcal{G})$.

Proof. Let ϕ be a β-KMS weight for σ^c. Since $\mathcal{G}^{(0)}$ is totally disconnected by assumption there is a sequence $p_1 \leqslant p_2 \leqslant p_3 \leqslant \cdots$ of projections in $C_c(\mathcal{G}^{(0)})$ with the property that $\{p_n\}$ is an approximate unit for $C^*(\mathcal{G})$. It follows from Lemma 3.1 that $\phi(p_n) < \infty$ for all n. Since $\phi \neq 0$ we can assume, without loss of generality, that $\phi(p_n) > 0$ for all n. Since $\phi(f) < \infty$ for every non-negative function in $C_c(\mathcal{G}^{(0)})$ it follows that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\phi$ and from the Riesz representation theorem that there is a unique regular Borel measure μ on $\mathcal{G}^{(0)}$ such that

$$\phi(f) = \int_{\mathcal{G}^{(0)}} f(x)\, \mathrm{d}\mu(x)$$

for all $f \in C_c(\mathcal{G}^{(0)})$. Let U_n be the compact and open support of p_n, and set

$$\mathcal{G}^n = \mathcal{G}|_{U_n} = \{\xi \in \mathcal{G} : \ r(\xi), s(\xi) \in U_n\}$$

and $c_n = c|_{\mathcal{G}^n}$. Note that $\phi(p_n)^{-1}\phi$ restricts to a β-KMS state on $p_nC^*(\mathcal{G})p_n = C^*(\mathcal{G}^n)$. It follows from Neshveyev's theorem [7] that there is a probability measure μ_n on U_n, and a μ_n-measurable field $\{\varphi_x^n\}_{x\in U_n}$ of states such that:

(an) μ_n is quasi-invariant on $\mathcal{G}^n$ with cocycle $\mathrm{e}^{-\beta c_n}$,

(bn) $\varphi_x^n(u_g) = \varphi_{r(h)}^n(u_{hgh^{-1}})$ for μ_n-almost every $x \in U_n$ and all $g \in \mathcal{G}_x^x$, $h \in (\mathcal{G}^n)_x$,

(cn) $\varphi_x^n(u_g) = 0$ for μ_n-almost every $x \in U_n$ and all $g \in \mathcal{G}_x^x \backslash c_n^{-1}(0)$,

and

$$\phi(p_n)^{-1}\phi(f) = \int_{U_n} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x^n(u_g)\, \mathrm{d}\mu_n(x)$$

when $f \in C_c(\mathcal{G}^n)$. For every $f \in C_c(U_n)$ we get that:

$$\phi(p_n)^{-1} \int_{U_n} f(x) \,\mathrm{d}\mu(x) = \phi(p_n)^{-1} \int_{\mathcal{G}^{(0)}} f(x) \,\mathrm{d}\mu(x) = \phi(p_n)^{-1}\phi(f)$$
$$= \int_{U_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^n(u_g) \,\mathrm{d}\mu_n(x) = \int_{U_n} f(x) \,\mathrm{d}\mu_n(x)$$

so $\mu|_{U_n} = \phi(p_n)\mu_n$. Notice that since $\phi(p_n) > 0$, being a μ null set in U_n is the same as being a μ_n null set. For a Borel set $V \subseteq U_n \subseteq U_{n+1}$ we have that:

$$\phi(p_{n+1})\mu_{n+1}(V) = \mu(V) = \phi(p_n)\mu_n(V).$$

So $\mu_n = \phi(p_{n+1})/\phi(p_n)\mu_{n+1}|_{U_n}$. For every $f \in C_c(\mathcal{G}^n)$ we get that:

$$\int_{U_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^n(u_g) \,\mathrm{d}\mu_n(x) = \phi(p_n)^{-1}\phi(f) = \frac{\phi(p_{n+1})}{\phi(p_n)}\phi(p_{n+1})^{-1}\phi(f)$$
$$= \frac{\phi(p_{n+1})}{\phi(p_n)} \int_{U_{n+1}} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^{n+1}(u_g) \,\mathrm{d}\mu_{n+1}(x)$$
$$= \int_{U_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^{n+1}(u_g) \,\mathrm{d}\mu_n(x).$$

Since μ_n by choice satisfy (an), and since it is easily seen that $\{\varphi_x^{n+1}\}_{x \in U_n}$ satisfy (bn) and (cn), the uniqueness statement in Neshveyev's theorem gives that $\varphi_x^n = \varphi_x^{n+1}$ for a.e. $x \in U_n$. Hence for a.e. $x \in \mathcal{G}^{(0)}$ we can define a state on $C^*(\mathcal{G}_x^x)$ by:

$$\varphi_x(d) = \lim_{n\to\infty} \varphi_x^n(d).$$

For every $f \in C_c(\mathcal{G})$ there is a $N \in \mathbb{N}$ such that $f \in C_c(\mathcal{G}^N)$, and hence:

$$\phi(f) = \phi(P_N) \int_{U_N} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^N(u_g) \,\mathrm{d}\mu_N(x) = \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g) \,\mathrm{d}\mu(x).$$

The properties (i)–(iii) follow from (an)–(cn), and measurability of $x \mapsto \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)$ follows from measurability of $x \mapsto \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^n(u_g)$.

For the converse, assume we are given a pair $(\mu, \{\varphi_x\}_{x \in \mathcal{G}^{(0)}})$ for which (i), (ii) and (iii) hold. As shown by Neshveyev in the proof of Theorem 1.1 in [7] every x gives rise to a state ψ_x on $C^*(\mathcal{G})$ such that

$$\psi_x(f) = \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)$$

when $f \in C_c(\mathcal{G})$. Note that $x \to \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)$ is μ-measurable by assumption, and then $x \to \psi_x(a)$ is also for each $a \in C^*(\mathcal{G})$. For $a \geqslant 0$ we can therefore define

$$\phi(a) = \int_{\mathcal{G}^{(0)}} \psi_x(a) \,\mathrm{d}\mu(x).$$

ϕ is a lower semi-continuous weight by Fatous lemma and by regularity

$$\phi(p_n a p_n) = \int_{U_n} \psi_x(a) \, \mathrm{d}\mu(x) \leqslant \|a\| \mu(U_n) < \infty$$

for all n, so it is also densely defined. Note that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\phi$ and that (3.3) holds by construction. Since the pair $(\phi(p_n)^{-1}\mu, \{\varphi_x\}_{x\in U_n})$ represents $\phi(p_n)^{-1}\phi$ in the sense of Theorem 1.1 in [7] it follows from Theorem 1.3 in [7] that ϕ is a bounded β-KMS weight on $p_n C^*(\mathcal{G}) p_n$. Since

$$\psi_x(p_n a p_n) = \begin{cases} \psi_x(a) & x \in U_n, \\ 0 & x \notin U_n, \end{cases}$$

we find that $\lim_{n\to\infty} \phi(p_n a p_n) = \lim_{n\to\infty} \int_{U_n} \psi_x(a) \, \mathrm{d}\mu(x) = \phi(a)$ for all $a \geqslant 0$ in $C^*(\mathcal{G})$. Now note that for every a in the domain of $\sigma^c_{-\mathrm{i}\beta/2}$,

$$\phi(p_n a^* a p_n) = \phi(\sigma^c_{-\mathrm{i}\beta/2}(a p_n) \sigma^c_{-\mathrm{i}\beta/2}(a p_n)^*) = \phi(\sigma^c_{-\mathrm{i}\beta/2}(a) p_n \sigma^c_{-\mathrm{i}\beta/2}(a)^*)$$

since ϕ is a bounded β-KMS weight on $p_n C^*(\mathcal{G}) p_n$. Since

$$\lim_{n\to\infty} \phi(\sigma^c_{-\mathrm{i}\beta/2}(a) p_n \sigma^c_{-\mathrm{i}\beta/2}(a)^*) = \phi(\sigma^c_{-\mathrm{i}\beta/2}(a) \sigma^c_{-\mathrm{i}\beta/2}(a)^*)$$

by the lower semi-continuity of ϕ, we conclude that

$$\phi(a^* a) = \phi(\sigma^c_{-\mathrm{i}\beta/2}(a) \sigma^c_{-\mathrm{i}\beta/2}(a)^*),$$

showing that ϕ is indeed a β-KMS weight for σ^c.

If $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ and $(\mu', \{\varphi'_x\}_{x\in\mathcal{G}^{(0)}})$ represent the same β-KMS weight it follows from the uniqueness part of the Riesz representation theorem that $\mu = \mu'$. By using (3.3) we find that

$$\int_{\mathcal{G}^{(0)}} k(x) \sum_{g\in\mathcal{G}^x_x} f(g)\varphi_x(u_g) \, \mathrm{d}\mu(x) = \int_{\mathcal{G}^{(0)}} k(x) \sum_{g\in\mathcal{G}^x_x} f(g)\varphi'_x(u_g) \, \mathrm{d}\mu(x) \tag{3.4}$$

when $f \in C_c(\mathcal{G})$ and $k \in C_c(\mathcal{G}^{(0)})$. It follows from this that

$$\sum_{g\in\mathcal{G}^x_x} f(g)\varphi_x(u_g) = \sum_{g\in\mathcal{G}^x_x} f(g)\varphi'_x(u_g)$$

for μ-almost all $x \in \mathcal{G}^{(0)}$ and all $f \in C_c(\mathcal{G})$. Thanks to the separability of $C^*(\mathcal{G})$ we conclude that $\varphi_x = \varphi'_x$ for μ-almost all x. ∎

COROLLARY 3.3. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous homomorphism. Assume that the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is totally disconnected and that the isotropy groups $\mathcal{G}^x_x, x \in \mathcal{G}^{(0)}$, are all amenable.*

There is a bijective correspondence between the β-KMS weights for σ^c on $C^*_r(\mathcal{G})$ and the pairs $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$, where μ is a regular Borel measure on $\mathcal{G}^{(0)}$ and $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field of states φ_x on $C^*_r(\mathcal{G}^x_x)$ such that:

(i) *μ is quasi-invariant with cocycle* $\mathrm{e}^{-\beta c}$,

(ii) $\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}})$ *for μ-almost every $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x$, $h \in \mathcal{G}_x$, and*

(iii) $\varphi_x(u_g) = 0$ *for μ-almost every $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x \backslash c^{-1}(0)$.*

*The β-*KMS *weight ϕ corresponding to the pair $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ has the properties that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\phi$ and*

$$\phi(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)\, \mathrm{d}\mu(x) \tag{3.5}$$

when $f \in C_c(\mathcal{G})$.

Proof. It suffices to show that the assumption on the isotropy groups implies that every β-KMS weight ϕ on $C^*(\mathcal{G})$ factorises through $C_r^*(\mathcal{G})$. To this end note that it follows from Lemma 2.1 in [13] that for each $n \in \mathbb{N}$ there is a bounded β-KMS weight $\widetilde{\phi}_n$ on $p_n C_r^*(\mathcal{G}) p_n$ such that $\widetilde{\phi}_n(p_n\pi(a)p_n) = \phi(p_n a p_n)$ for all $a \in C^*(\mathcal{G})$ where $\pi : C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$ is the canonical surjection. Then $\widetilde{\phi}_n(p_n b p_n) \leqslant \widetilde{\phi}_{n+1}(p_{n+1} b p_{n+1})$ for all $b \geqslant 0$ in $C_r^*(\mathcal{G})$ and we can define a lower semi-continuous weight $\widetilde{\phi}$ on $C_r^*(\mathcal{G})$ such that $\widetilde{\phi}(b) = \lim_{n\to\infty} \widetilde{\phi}_n(p_n b p_n)$. It follows that $\widetilde{\phi} \circ \pi = \phi$. ∎

It is an interesting problem if Corollary 3.3 remains true without the amenability assumption on the isotropy groups. For the proof of our main result the following suffices.

COROLLARY 3.4. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous homomorphism. Assume that the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is totally disconnected. If there is a β-*KMS *weight for σ^c on $C_r^*(\mathcal{G})$ there is also one which is diagonal.*

Proof. Let ϕ be a β-KMS weight for σ^c on $C_r^*(\mathcal{G})$ and let $\pi : C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$ be the canonical surjection. Then $\phi \circ \pi$ is a β-KMS weight for σ^c on $C^*(\mathcal{G})$ and we can consider the corresponding regular Borel measure μ. Since μ is quasi-invariant with cocycle $\mathrm{e}^{-\beta c}$ it follows from Proposition 2.1 in [14] that μ defines a diagonal β-KMS weight by the formula (2.2). ∎

4. CONDITIONS ON A KMS WEIGHT THAT IMPLY DIAGONALITY OF THE ACTION

4.1. WHEN KMS WEIGHTS FACTOR THROUGH THE CONDITIONAL EXPECTATION ONTO AN ABELIAN SUBALGEBRA. A weight ω is *faithful* when $a \geqslant 0$, $\omega(a) = 0 \Rightarrow a = 0$.

PROPOSITION 4.1. *Let A be a C^*-algebra and γ a continuous one-parameter group of automorphisms on A. Let $D \subseteq A$ be an abelian C^*-subalgebra and $P : A \to D$ a conditional expectation.*

Assume that ω is a faithful β-KMS weight for γ, $\beta \neq 0$, such that $\omega \circ P = \omega$. It follows that $\gamma_t(d) = d$ for all $t \in \mathbb{R}$ and all $d \in D$.

Proof. Let $f \in D$, $f \geqslant 0$. Since ω is densely defined there is a sequence $\{a_n\}$ of positive elements in A such that $\lim_{n\to\infty} a_n = f$ and $\omega(a_n) < \infty$ for all n. Then $\lim_{n\to\infty} P(a_n) = f$ and $\omega(P(a_n)) = \omega(a_n) < \infty$. It suffices therefore to consider $f \in D$, $f \geqslant 0$ such that $\omega(f) < \infty$ and show that $\gamma_t(f) = f$ for all $t \in \mathbb{R}$.

We find that

$$\omega(af) = \omega(P(a)f) = \omega(fP(a)) = \omega(fa) \tag{4.1}$$

for all $a \in \mathcal{M}_\omega$. Since $f \in \mathcal{M}_\omega$ and this is a subalgebra, the desired conclusion follows from Result 6.29 in [5]. ∎

COROLLARY 4.2. *Let A be a simple C^*-algebra and γ a continuous one-parameter group of automorphisms on A. Let $D \subseteq A$ be an abelian C^*-subalgebra and $P : A \to D$ a conditional expectation.*

Assume that ω is a β-KMS weight for γ, $\beta \neq 0$, such that $\omega \circ P = \omega$. It follows that $\gamma_t(d) = d$ for all $t \in \mathbb{R}$ and all $d \in D$.

Proof. It suffices to show that ω is faithful. For $a \in A$ and $k \in \mathbb{N}$, set:

$$Q_k(a) = \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-kt^2} \gamma_t(a) \mathrm{d}t.$$

Note that $Q_k(a)$ is analytic for γ and that $\lim_{k\to\infty} Q_k(a) = a$. Standard approximation arguments establish the following orservation: Assume that $a \in \mathcal{M}_\omega$. It follows that

$$\sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-k(t+\mathrm{i}s)^2} \gamma_t(a) \mathrm{d}t \in \mathcal{M}_\omega$$

for all $s \in \mathbb{R}$.

This will be used to show that ω is faithful in the following way: Assume that $b = b^* \in A$ and that $\omega(b^2) = 0$. For a $c \in \mathcal{M}_\omega$ it follows from the observation that $Q_k(c), \gamma_{\mathrm{i}\beta}(Q_k(c)^*) \in \mathcal{M}_\omega$, hence by an application of the Cauchy–Schwarz inequality

$$|\omega(Q_k(c)^* b^2 Q_k(c))|^2 = |\omega(b^2 Q_k(c) \gamma_{\mathrm{i}\beta}(Q_k(c)^*))|^2 \leqslant 0.$$

Lower semi-continuity now implies that $\omega(c^* b^2 c) = 0$ and by using Cauchy–Schwarz again we deduce that

$$\omega(\operatorname{Span} \mathcal{M}_\omega b^2 \mathcal{M}_\omega) = \{0\}. \tag{4.2}$$

Since $\mathcal{M}_\omega$ is dense in A the closure of $\operatorname{Span} \mathcal{M}_\omega b^2 \mathcal{M}_\omega$ is a (closed two sided) ideal in A. If $b \neq 0$ this ideal must be all of A because we assume that A is simple. But then we reach a contradiction the following way: let $a \geqslant 0$. Choose a sequence $\{x_n\} \subseteq \operatorname{Span} \mathcal{M}_\omega b^2 \mathcal{M}_\omega$ such that $\lim_{n\to\infty} x_n = \sqrt{a}$. Since

$x_n x_n^* \in \operatorname{Span} \mathcal{M}_\omega b^2 \mathcal{M}_\omega$ and $\lim_{n\to\infty} x_n x_n^* = a$, it follows from (4.2) and the lower semi-continuity of ω that $\omega(a) = 0$. This is a contradiction because $\omega \neq 0$. Hence $b = 0$. ∎

4.2. ONE-PARAMETER GROUPS TRIVIAL ON THE DIAGONAL. The following result has a predecessor in the von Neumann algebra setting in Theorem 2 of [3].

PROPOSITION 4.3. *Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid and $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ a continuous one-parameter group of automorphisms on $C_r^*(\mathcal{G})$ such that*

$$\alpha_t(f) = f$$

for all $f \in C_0(\mathcal{G}^{(0)})$ and all $t \in \mathbb{R}$. Assume that the elements of $\mathcal{G}^{(0)}$ with trivial isotropy group in $\mathcal{G}$ are dense in $\mathcal{G}^{(0)}$. There is a continuous homomorphism $c : \mathcal{G} \to \mathbb{R}$ such that

$$\alpha_t(g)(\xi) = \mathrm{e}^{\mathrm{i}tc(\xi)} g(\xi)$$

for all $t \in \mathbb{R}$, all $g \in C_c(\mathcal{G})$ and all $\xi \in \mathcal{G}$.

Proof. We shall use the continuous linear embedding $j : C_r^*(\mathcal{G}) \to C_0(\mathcal{G})$ introduced by Renault in Proposition 4.2 in [9].

OBSERVATION 4.4. Let $f \in C_c(\mathcal{G})$ be supported in an open subset $U \subseteq \mathcal{G}$ such that $r : U \to \mathcal{G}^{(0)}$ is injective. Assume that $f(\xi) = 0$ for some $\xi \in U$. It follows that $j(\alpha_t(f))(\xi) = 0$ for all $t \in \mathbb{R}$.

To prove this, let $\varepsilon > 0$. There is an open bisection W of ξ such that $W \subseteq U$ and $|f(\mu)| \leqslant \varepsilon$ for all $\mu \in W$. Let $\varphi \in C_c(\mathcal{G}^{(0)})$ be such that $0 \leqslant \varphi \leqslant 1$, $\operatorname{supp} \varphi \subseteq r(W)$ and $\varphi(r(\xi)) = 1$. By use of Proposition 4.2 in [9] we find that

$$j(\alpha_t(f))(\xi) = \varphi(r(\xi)) j(\alpha_t(f))(\xi) = j(\varphi\alpha_t(f))(\xi) = j(\alpha_t(\varphi f))(\xi). \tag{4.3}$$

Note that $\operatorname{supp}(\varphi f) \subseteq W$ and that $\|\varphi f\|_\infty \leqslant \varepsilon$. It follows that

$$\|j(\alpha_t(\varphi f))\|_\infty \leqslant \|\alpha_t(\varphi f)\| = \|\varphi f\| = \|\varphi f\|_\infty \leqslant \varepsilon,$$

where the last identity follows from Lemma 2.4 in [12]. In particular, $|j(\alpha_t(\varphi f))(\xi)| \leqslant \varepsilon$, and then (4.3) shows that $|j(\alpha_t(f))(\xi)| \leqslant \varepsilon$. This proves Observation 4.4.

In the same way we obtain the following.

OBSERVATION 4.5. Let $f \in C_c(\mathcal{G})$ be supported in an open subset $U \subseteq \mathcal{G}$ such that $s : U \to \mathcal{G}^{(0)}$ is injective. Assume that $f(\xi) = 0$ for some $\xi \in U$. It follows that $j(\alpha_t(f))(\xi) = 0$ for all $t \in \mathbb{R}$.

OBSERVATION 4.6. Let $\xi \in \mathcal{G}$, and let $h, h' \in C_c(\mathcal{G})$ be supported in (not necessarily the same) open bisections in $\mathcal{G}$. Assume that $h(\xi) = h'(\xi) = 1$. Then

$$j(\alpha_t(h))(\xi) = j(\alpha_t(h'))(\xi) \tag{4.4}$$

for all $t \in \mathbb{R}$.

To show this, let $h \cdot h'$ be the point wise product of h and h'. It follows from Observation 4.4 that

$$j(\alpha_t(h \cdot h' - h'))(\xi) = j(\alpha_t(h \cdot h' - h))(\xi) = 0,$$

which yields (4.4): $j(\alpha_t(h))(\xi) = j(\alpha_t(h \cdot h'))(\xi) = j(\alpha_t(h'))(\xi)$.

It follows from Observation 4.6 that we can define a map $G_t : \mathcal{G} \to \mathbb{C}$ such that

$$G_t(\xi) = j(\alpha_t(h))(\xi),$$

where h is any element of $C_c(\mathcal{G})$ which is supported in an open bisection and takes the value 1 at ξ. Note that G_t is continuous by construction.

OBSERVATION 4.7. For every $f \in C_c(\mathcal{G})$ and every $\xi \in \mathcal{G}$,

$$j(\alpha_t(f))(\xi) = G_t(\xi)f(\xi). \tag{4.5}$$

To show this, we may assume that there are open bisections $U \subseteq V$ such that $\operatorname{supp} f \subseteq U$ and $\overline{U} \subseteq V$. Assume first that $\xi \notin \overline{U}$. We must show that $j(\alpha_t(f))(\xi) = 0$ in this case. By continuity and the assumption on $\mathcal{G}$ we may assume that $s(\xi)$ has trivial isotropy. If $\mu \in U$ and $r(\mu) = r(\xi)$, $s(\mu) = s(\xi)$, we see that

$$r(\mu^{-1}\xi) = s(\mu) = s(\xi) \quad \text{and} \quad s(\mu^{-1}\xi) = s(\xi)$$

which is impossible since $\xi \neq \mu$. It follows that we can write f as a finite sum

$$f = \sum_i f_i$$

such that each $f_i \in C_c(U)$ is supported in an open set $W_i \subseteq U$ such that either $s(\xi) \notin s(\overline{W_i})$ or $r(\xi) \notin r(\overline{W_i})$. It follows that $j(\alpha_t(f_i))(\xi) = 0$; in the first case thanks to Observation 4.5, in the second thanks to Observation 4.4. Hence

$$j(\alpha_t(f))(\xi) = \sum_i j(\alpha_t(f_i))(\xi) = 0,$$

as desired. Assume then that $\xi \in \overline{U} \subseteq V$. Choose $\varepsilon > 0$ such that $f(\xi) + \varepsilon \neq 0$ and a function $\varphi \in C_c(V)$ such that $\varphi(\xi) = 1$. Then

$$j(\alpha_t(f + \varepsilon\varphi))(\xi) = j\Big(\alpha_t\Big(\frac{f + \varepsilon\varphi}{f(\xi) + \varepsilon}\Big)\Big)(\xi)(f(\xi) + \varepsilon) = G_t(\xi)(f(\xi) + \varepsilon).$$

Letting $\varepsilon \to 0$ we obtain (4.5).

Note that it follows from Observation 4.7 that $\alpha_t(C_c(\mathcal{G})) \subseteq C_c(\mathcal{G})$, and

$$\alpha_t(f)(\xi) = G_t(\xi)f(\xi)$$

for all $f \in C_c(\mathcal{G})$ and all $\xi \in \mathcal{G}$. Since $\|f\| = \|\alpha_t(f)\|$ this implies that $|G_t(\xi)| = 1$. Furthermore, if $h \in C_c(\mathcal{G})$ is supported in a bisection and $h(\xi) = 1$, we find that

$$\begin{aligned} G_{t+s}(\xi) &= \alpha_t(\alpha_s(h))(\xi) = \alpha_s(h)(\xi)\alpha_t\Big(\frac{\alpha_s(h)}{\alpha_s(h)(\xi)}\Big)(\xi) \\ &= \alpha_s(h)(\xi)G_t(\xi) = G_s(\xi)G_t(\xi). \end{aligned}$$

Since $t \mapsto G_t(\xi)$ is continuous, this implies that there is a unique real-valued function $c : \mathcal{G} \to \mathbb{R}$ such that

$$G_t(\xi) = \mathrm{e}^{itc(\xi)}. \tag{4.6}$$

To show that c is a homomorphism, let $\gamma_1, \gamma_2 \in \mathcal{G}$ such that $s(\gamma_1) = r(\gamma_2)$. Set $\gamma = \gamma_1\gamma_2$. Let U be an open bisection containing γ and U_i an open bisection containing γ_i, $i = 1,2$, such that $\mu_1\mu_2 \in U$ when $(\mu_1, \mu_2) \in \mathcal{G}^{(2)} \cap (U_1 \times U_2)$. Choose $h_i \in C_{\mathrm{c}}(U_i)$ such that $h_i(\gamma_i) = 1$. Then $h_1h_2(\gamma) = 1$ and

$$\begin{aligned} G_t(\gamma) &= j(\alpha_t(h_1h_2))(\gamma) = \alpha_t(h_1)\alpha_t(h_2)(\gamma) \\ &= \alpha_t(h_1)(\gamma_1)\alpha_t(h_2)(\gamma_2) = G_t(\gamma_1)G_t(\gamma_2). \end{aligned} \tag{4.7}$$

Hence G_t is a homomorphism as asserted. Combining (4.6) and (4.7) and taking derivatives with respect to t, it follows that c is a homomorphism, i.e.

$$c(\gamma_1\gamma_2) = c(\gamma_1) + c(\gamma_2)$$

when $s(\gamma_1) = r(\gamma_2)$.

Finally, to show that c is continuous, let $\xi \in \mathcal{G}$ and $\varepsilon > 0$ be given. Choose open bisections $U \subseteq V$ such that $\xi \in U \subseteq \overline{U} \subseteq V$ and $h \in C_{\mathrm{c}}(V)$ a function such that $h = 1$ on $\overline{U}$. Then

$$G_t(\gamma) = \alpha_t(h)(\gamma)$$

for all $t \in \mathbb{R}$ and all $\gamma \in U$. Let $K \subseteq \mathbb{R}$ be a compact set. There are finitely many points $t_i \in K, i = 1,2,\ldots,N$, such that for every $t \in K$ there is an i such that

$$\|\alpha_t(h) - \alpha_{t_i}(h)\|_\infty = \|\alpha_t(h) - \alpha_{t_i}(h)\| \leqslant \varepsilon.$$

By continuity of $\alpha_{t_i}(h)$ there is an open neighborhood $W \subseteq U$ of ξ such that

$$|\alpha_{t_i}(h)(\gamma) - \alpha_{t_i}(h)(\xi)| \leqslant \varepsilon$$

for all $\gamma \in W$ and $i = 1,2,\ldots,N$. It follows that $|G_t(\gamma) - G_t(\xi)| \leqslant 3\varepsilon$ for all $t \in K$ and all $\gamma \in W$. By Pontryagin duality this implies that c is continuous. ∎

THEOREM 4.8. *Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid such that for at least one element $x \in \mathcal{G}^{(0)}$ the isotropy $\mathcal{G}_x^x$ is trivial, i.e. $\mathcal{G}_x^x = \{x\}$, and that $\mathcal{G}$ is minimal in the sense that $s(r^{-1}(y))$ is dense in $\mathcal{G}^{(0)}$ for all $y \in \mathcal{G}^{(0)}$. Let $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group of automorphisms on $C^*_{\mathrm{r}}(\mathcal{G})$ and assume that for some $\beta \in \mathbb{R}\backslash\{0\}$ there is a diagonal β-KMS weight for α. Then α is diagonal, i.e. there is a continuous homomorphism $c : \mathcal{G} \to \mathbb{R}$ such that*

$$\alpha_t(g)(\xi) = \mathrm{e}^{itc(\xi)}g(\xi) \tag{4.8}$$

for all $t \in \mathbb{R}$, all $g \in C_{\mathrm{c}}(\mathcal{G})$ and all $\xi \in \mathcal{G}$.

Proof. Combine Corollary 4.2 and Proposition 4.3, using that in the presence of a single unit with trivial isotropy group the minimality of $\mathcal{G}$ is equivalent to the simplicity of $C^*_{\mathrm{r}}(\mathcal{G})$ by Corollary 2.18 in [12]. ∎

We can now put the pieces together for a *Proof of Theorem* 2.1. (i)$\Rightarrow$(iii) follows from Proposition 4.1. That (iii) is equivalent to (iv) follows from a standard argument using that $\mathcal{G}^{(0)}$ is totally disconnected. The implication (iii)$\Rightarrow$(v) follows from Proposition 4.3 and (v)$\Rightarrow$(ii) from Corollary 3.4. This gives the equivalence of all five conditions since (ii)$\Rightarrow$(i) is trivial.

EXAMPLE 4.9. Let $\mathcal{G} = \mathbb{F}_2$ be the free group on two generators. Then $C^*_r(\mathbb{F}_2)$ is a simple C^*-algebra and $C_0(\mathcal{G}^{(0)}) = \mathbb{C}1$. Let $A = A^* \in C^*_r(\mathbb{F}_2)$ and set

$$\alpha_t(a) = e^{itA} a e^{-itA}.$$

Note that α_t acts trivially on $C_0(\mathcal{G}^{(0)}) = \mathbb{C}1$. Let $U_x, x \in \mathbb{F}_2$, be the canonical unitaries generating $C^*_r(\mathbb{F}_2)$. Assume that there is a homomorphism $c : \mathbb{F}_2 \to \mathbb{R}$ such that $\alpha_t(U_x) = e^{itc(x)} U_x$ for all t, x. By differentiation this leads to the conclusion that $AU_x - U_x A = c(x) U_x$ and hence that $U^*_x A U_x = A + c(x)1$. The last equation implies that the spectrum $\sigma(A)$ of A satisfies $\sigma(A) = \sigma(A) + c(x)$, i.e. $c(x) = 0$. But then $\alpha_t(U_x) = U_x$ for all t, x, i.e. $\alpha_t = \mathrm{id}$ for all $t \in \mathbb{R}$. This implies by differentiation that $AX = XA$ for all $X \in C^*_r(\mathbb{F}_2)$, i.e. A is in the center of $C^*_r(\mathbb{F}_2)$. So by choosing $A \notin \mathbb{R}1$, we have an example showing that Proposition 4.3 does not always hold when there are no units with trivial isotropy in $\mathcal{G}$. In relation to Theorem 2.1 note that there are β-KMS weights for α for all $\beta \in \mathbb{R}$. Indeed, when ω is the tracial state on $C^*_r(\mathbb{F}_2)$, the functional

$$C^*_r(\mathbb{F}_2) \ni a \mapsto \omega(e^{-\beta A} a)$$

is a bounded β-KMS weight. Since condition (iii) in Theorem 2.1 holds while (v) does not, it follows that it is necessary, in Theorem 2.1, to assume the existence of a unit with trivial isotropy group.

Similarly, by considering a disjoint union $\mathbb{F}_2 \sqcup \mathcal{H}$, where $\mathcal{H}$ is an appropriate groupoid, it is easy to obtain examples showing that the implication (iv)$\Rightarrow$(i) in Theorem 2.1 fails in general if $\mathcal{G}$ is not minimal.

5. APPLICATIONS TO GRAPH C^*-ALGEBRAS

In this section we apply the results obtained above to the study of KMS weights on graph C^*-algebras. For this we first show how a graph C^*-algebra can be realized as the groupoid C^*-algebra of a locally defined local homeomorphism as it was introduced by Renault in [10]. Recall that graph C^*-algebras were originally introduced for row-finite graphs in [4] as the C^*-algebra of the left-shift on the space of infinite paths in the graph. We show that in general, when the graph may have infinite emitters, its C^*-algebra is still the groupoid C^*-algebra of a local homeomorphism which is generally only defined on a dense open subset of a locally compact Hausdorff space.

5.1. THE RENAULT GROUPOID OF A LOCAL HOMEOMORPHISM. Let X be a locally compact second countable Hausdorff space. Let $U \subseteq X$ be an open subset and $\varphi : U \to X$ a local homeomorphism, i.e. for every $u \in U$ there is an open subset $V \subseteq U$ such that $u \in V$, $\varphi(V)$ is open and $\varphi : V \to \varphi(V)$ is a homeomorphism. Set $\varphi^0 = \mathrm{id}_X$ (with domain $D(\varphi^0) = X$) and for $n \geqslant 1$, set

$$D(\varphi^n) = U \cap \varphi^{-1}(U) \cap \varphi^{-2}(U) \cap \cdots \cap \varphi^{-n+1}(U)$$

and let φ^n be the map

$$\varphi^n = \varphi \circ \varphi \circ \cdots \circ \varphi : D(\varphi^n) \to X.$$

Set

$$\mathcal{G}_\varphi = \{(x, n-m, y) \in X \times \mathbb{Z} \times X : x \in D(\varphi^n),\ y \in D(\varphi^m),\ \varphi^n(x) = \varphi^m(y)\}$$

which is a groupoid with product $(x,k,y)(y,l,z) = (x,k+l,z)$ and inversion $(x,k,y)^{-1} = (y,-k,x)$. Sets of the form

$$\{(x, n-m, y) :\ \varphi^n(x) = \varphi^m(y),\ x \in W,\ y \in V\}$$

for some open subsets $W \subseteq D(\varphi^n)$, $V \subseteq D(\varphi^m)$, constitute a basis for a topology in $\mathcal{G}_\varphi$ which turns it into a locally compact second countable Hausdorff étale groupoid.

Let $F : X \to \mathbb{R}$ be a function which is continuous on U. We can then define $c_F : \mathcal{G}_\varphi \to \mathbb{R}$ such that

$$c_F(x, n-m, y) = \sum_{i=0}^{n} F(\varphi^i(x)) - \sum_{i=0}^{m} F(\varphi^i(y)).$$

Note that c_F is a continuous homomorphism, and if $F' : X \to \mathbb{R}$ is a function which agrees with F on U, then $c_{F'} = c_F$.

PROPOSITION 5.1. *Let $c : \mathcal{G}_\varphi \to \mathbb{R}$ be a continuous homomorphism. There is a map $F : X \to \mathbb{R}$ which is continuous on U such that $c = c_F$.*

Proof. Define $F : X \to \mathbb{R}$ such that

$$F(x) = \begin{cases} c(x, 1, \varphi(x)) & x \in U, \\ 0 & x \notin U. \end{cases}$$

It is straightforward to verify that F is continuous on U and that $c = c_F$. ∎

It follows that the continuous homomorphisms $\mathcal{G}_\varphi \to \mathbb{R}$ are in bijective correspondence with the continuous maps $U \to \mathbb{R}$.

A point $x \in X$ is *aperiodic* under φ when

$$x \in D(\varphi^n) \cap D(\varphi^m),\ \varphi^n(x) = \varphi^m(x)\ \Rightarrow\ n = m.$$

Under the identification of X with the unit space of $\mathcal{G}_\varphi$ the aperiodic points are the elements with trivial isotropy group. We can therefore combine Proposition 5.1 with Proposition 4.3 to obtain the following.

PROPOSITION 5.2. *Let X be a locally compact second countable Hausdorff space, $U \subseteq X$ an open subset and $\varphi : U \to X$ a local homeomorphism. Assume that $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ is a continuous one-parameter group of automorphisms on $C_r^*(\mathcal{G}_\varphi)$ such that*

$$\alpha_t(f) = f$$

for all $f \in C_0(X) \subseteq C_r^*(\mathcal{G}_\varphi)$ and all $t \in \mathbb{R}$. Assume also that the aperiodic points of φ are dense in X.

There is a continuous map $F : U \to \mathbb{R}$ such that

$$\alpha_t(g)(\xi) = \mathrm{e}^{\mathrm{i}tc_F(\xi)}g(\xi)$$

for all $t \in \mathbb{R}$, all $g \in C_c(\mathcal{G}_\varphi)$ and all $\xi \in \mathcal{G}_\varphi$.

For $n, m \in \mathbb{N} \cup \{0\}$, set

$$\varphi^{-m}(\varphi^n(x)) = \begin{cases} \emptyset & \text{when } x \notin D(\varphi^n), \\ \{y \in D(\varphi^m) : \varphi^m(y) = \varphi^n(x)\} & \text{when } x \in D(\varphi^n). \end{cases}$$

We say that φ is *minimal* when

$$\bigcup_{n,m\in\mathbb{N}\cup\{0\}} \varphi^{-m}(\varphi^n(x)) \tag{5.1}$$

is dense in X for all $x \in X$. Note that (5.1) is the orbit of x under the action of $\mathcal{G}_\varphi$ on its unit space. Thus φ is minimal if and only if $\mathcal{G}_\varphi$ is.

PROPOSITION 5.3. *Let X be a locally compact second countable Hausdorff space, $U \subseteq X$ an open subset and $\varphi : U \to X$ a local homeomorphism. Assume that φ is minimal and that there is at least one aperiodic point for φ. Let $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group of automorphisms on $C_r^*(\mathcal{G}_\varphi)$.*

*If, for some $\beta \neq 0$, there is a diagonal β-*KMS *weight for α, then there is a continuous map $F : U \to \mathbb{R}$ such that*

$$\alpha_t(g)(\xi) = \mathrm{e}^{\mathrm{i}tc_F(\xi)}g(\xi) \tag{5.2}$$

for all $t \in \mathbb{R}$, all $g \in C_c(\mathcal{G}_\varphi)$ and all $\xi \in \mathcal{G}_\varphi$.

Proof. In view of Corollary 4.2 and Proposition 5.2 it suffices to observe that $C_r^*(\mathcal{G}_\varphi)$ is simple under the present assumptions, cf. Proposition 2.5 in [10]. ∎

5.2. A LOCAL HOMEOMORPHISM FROM AN INFINITE GRAPH. Let G be a directed graph with vertexes V and edges E. We assume that G is countable in the sense that V and E are both countable sets. We let r and s denote the maps $r : E \to V$ and $s : E \to V$ which associate to an edge $e \in E$ its target vertex $r(e)$ and source vertex $s(e)$, respectively. A vertex v is an *infinite emitter* when $s^{-1}(v)$ contains infinitely many edges and a *sink* when $s^{-1}(v)$ is empty. The union of sinks and infinite emitters constitute a set which will be denoted by V_∞. The graph C^*-algebra $C^*(G)$ is by definition the universal C^*-algebra generated by a collection $S_e, e \in E$, of partial isometries and a collection $P_v, v \in V$, of mutually orthogonal projections subject to the conditions that:

(1) $S_e^* S_e = P_{r(e)}$, $\forall e \in E$,

(2) $\sum_{e \in F} S_e S_e^* \leqslant P_v$ for every finite subset $F \subseteq s^{-1}(v)$ and all $v \in V$, and

(3) $P_v = \sum_{e \in s^{-1}(v)} S_e S_e^*$, $\forall v \in V \backslash V_\infty$.

Let $P_f(G)$ and $P(G)$ denote the set of finite and infinite paths in G, respectively. The range and source maps, r and s, extend in the natural way to $P_f(G)$; the source map also to $P(G)$. Set $\Omega_G = P(G) \cup Q(G)$, where

$$Q(G) = \{p \in P_f(G) : r(p) \in V_\infty\}$$

is the set of finite paths that terminate at a vertex in V_∞. In particular, $V_\infty \subseteq Q(G)$ because vertexes are considered to be finite paths of length 0. For any $p \in P_f(G)$, let $|p|$ denote the length of p. When $|p| \geqslant 1$, set

$$\begin{aligned} &Z(p) = \{q \in \Omega_G : |q| \geqslant |p|,\ q_i = p_i,\ i = 1, 2, \ldots, |p|\}, \quad \text{and} \\ &Z(v) = \{q \in \Omega_G : s(q) = v\}, \end{aligned}$$

when $v \in V$. When $\nu \in P_f(G)$ and F is a finite subset of $P_f(G)$, set

$$Z_F(\nu) = Z(\nu) \backslash \Big(\bigcup_{\mu \in F} Z(\mu) \Big). \tag{5.3}$$

The sets $Z_F(\nu)$ form a basis of compact and open subsets for a locally compact Hausdorff topology on Ω_G. When $\mu \in P_f(G)$ and $x \in \Omega_G$, we can define the concatenation $\mu x \in \Omega_G$ in the obvious way when $r(\mu) = s(x)$. The groupoid $\mathcal{G}_G$ consists of the elements in $\Omega_G \times \mathbb{Z} \times \Omega_G$ of the form

$$(\mu x, |\mu| - |\mu'|, \mu' x),$$

for some $x \in \Omega_G$ and some $\mu, \mu' \in P_f(G)$. The product in $\mathcal{G}_G$ is defined by

$$(\mu x, |\mu| - |\mu'|, \mu' x)(\nu y, |\nu| - |\nu'|, \nu' y) = (\mu x,\ |\mu| + |\nu| - |\mu'| - |\nu'|, \nu' y),$$

when $\mu' x = \nu y$, and the involution by $(\mu x, |\mu| - |\mu'|, \mu' x)^{-1} = (\mu' x, |\mu'| - |\mu|, \mu x)$. To describe the topology on $\mathcal{G}_G$, let $Z_F(\mu)$ and $Z_{F'}(\mu')$ be two sets of the form (5.3) with $r(\mu) = r(\mu')$. The topology we shall consider has as a basis the sets of the form

$$\{(\mu x, |\mu| - |\mu'|, \mu' x) : \mu x \in Z_F(\mu),\ \mu' x \in Z_{F'}(\mu')\}. \tag{5.4}$$

With this topology $\mathcal{G}_G$ becomes an étale locally compact second countable Hausdorff groupoid and we can consider the reduced C^*-algebra $C^*_r(\mathcal{G}_G)$ as in [9]. As shown by Paterson in [8] there is an isomorphism $C^*(G) \to C^*_r(\mathcal{G}_G)$ which sends S_e to 1_e, where 1_e is the characteristic function of the compact and open set

$$\{(ex, 1, r(e)x) : x \in \Omega_G\} \ \subseteq\ \mathcal{G}_G,$$

and P_v to 1_v, where 1_v is the characteristic function of the compact and open set

$$\{(vx, 0, vx)\ : x \in \Omega_G\} \ \subseteq\ \mathcal{G}_G.$$

In the following we use the identification $C^*(G) = C^*_r(\mathcal{G}_G)$ and identify Ω_G with the unit space of $\mathcal{G}_G$ via the embedding $\Omega_G \ni x \mapsto (x, 0, x)$.

Note that $\Omega_G \backslash V_\infty$ is an open subset of Ω_G and that we can define a local homeomorphism

$$\sigma : \Omega_G \backslash V_\infty \to \Omega_G$$

such that σ is the usual left shift on $P(G)$, defined such that $\sigma(x)_i = x_{i+1}$, while $\sigma(e_1 e_2 \cdots e_n)$ is defined as follows when $e_1 e_2 \cdots e_n \in Q(G)$:

$$\sigma(e_1 e_2 \cdots e_n) = \begin{cases} e_2 e_3 \cdots e_n & \text{when } n \geqslant 2, \\ r(e_1) & \text{when } n = 1. \end{cases}$$

It is straightforward to check that there is an identification

$$\mathcal{G}_G = \mathcal{G}_\sigma,$$

as topological groupoids. In particular, it follows that any continuous function $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ defines a continuous homomorphism $c_F : \mathcal{G}_G \to \mathbb{R}$ such that

$$c_F(\mu x, |\mu| - |\mu'|, \mu' x) = \sum_{n=0}^{|\mu|} F(\sigma^n(\mu x)) - \sum_{n=0}^{|\mu'|} F(\sigma^n(\mu' x)).$$

To simplify notation the one-parameter group σ^{c_F} defined from c_F will be denoted by σ^F. It follows from Proposition 5.1 that every continuous homomorphism $\mathcal{G}_G \to \mathbb{R}$ arises from a continuous function $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ as above. We can therefore formulate Corollary 3.3 in the following way for graph C^*-algebras.

THEOREM 5.4. *Let $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ be a continuous function. There is a bijective correspondence between the β-KMS weights for σ^F on $C^*(G)$ and the pairs $(\mu, \{\varphi_x\}_{x \in \Omega_G})$, where μ is a regular Borel measure on Ω_G and $\{\varphi_x\}_{x \in \Omega_G}$ is a μ-measurable field of states φ_x on $C^*_r((\mathcal{G}_G)^x_x)$ such that:*

(i) *μ is $e^{\beta F}$-conformal for σ,*

(ii) *$\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}})$ for μ-almost every $x \in \Omega_G$ and all $g \in (\mathcal{G}_G)^x_x$, $h \in (\mathcal{G}_G)_x$, and*

(iii) *$\varphi_x(u_g) = 0$ for μ-almost every $x \in \Omega_G$ and all $g \in (\mathcal{G}_G)^x_x \backslash c_F^{-1}(0)$.*

The β-KMS weight ϕ corresponding to the pair $(\mu, \{\varphi_x\}_{x \in \Omega_G})$ has the properties that $C_c(\mathcal{G}_G) \subseteq \mathcal{M}_\phi$ and

$$\phi(f) = \int_{\Omega_G} \sum_{g \in (\mathcal{G}_G)^x_x} f(g) \varphi_x(u_g) \, \mathrm{d}\mu(x) \tag{5.5}$$

when $f \in C_c(\mathcal{G}_G)$.

Similarly, for graph C^*-algebras our main result takes the following form.

THEOREM 5.5. *Let G be a countable directed graph such that $C^*(G)$ is simple. Let $\alpha = (\alpha_t)_{t \in \mathbb{R}}$ be a continuous one-parameter group of automorphisms on $C^*(G)$ and assume that for some $\beta_0 \neq 0$ there is a β_0-KMS weight for α.*

The following are equivalent:

(i) *There is a* $\beta_1 \neq 0$ *and a diagonal* β_1-KMS *weight for* α.

(ii) *Whenever* $\beta \neq 0$ *and there is a* β-KMS *weight for* α, *there is also a diagonal* β-KMS *weight for* α.

(iii) $\alpha_t(f) = f$ *for all* $t \in \mathbb{R}$ *and all* $f \in C_0(\Omega_G)$.

(iv) *There is a continuous function* $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ *such that* $\alpha = \sigma^F$.

It follows from Theorem 5.4 (and Proposition 5.1) that all KMS weights for a diagonal action on the C^*-algebra of a graph without loops are diagonal. This is not true in general; not even for finite strongly connected graphs as shown in [15]. However, we can now show that it holds for strongly connected graphs when the function F has bounded local variation in the a sense we now make precise.

Let v be a vertex in G and set

$$\mathrm{Var}_{n,v}(F) = \sup_{x,y} \Big| \sum_{j=0}^{n-1} F(\sigma^j(x)) - \sum_{j=0}^{n-1} F(\sigma^j(y)) \Big|$$

where we take the supremum over all pairs $x, y \in P(G)$ with the property that $x_i = y_i$, $i = 1, 2, \dots, n$, and $s(x_1) = s(y_1) = v$. The following condition (5.6) should be compared with *Bowen's condition* used by Walters, cf. [16].

PROPOSITION 5.6. *Let G be a countable directed graph such that $C^*(G)$ is simple and let $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ be a continuous function such that for some vertex v,*

$$\sup_n \mathrm{Var}_{n,v}(F) < \infty. \tag{5.6}$$

Then every KMS *weight for* σ^F *is diagonal.*

Proof. The assumption that $C^*(G)$ is simple means that G is cofinal in the sense used (e.g.) in [14] and that every minimal loop in G has an exit, cf. [11]. It is easily seen that the set of vertexes v for which (5.6) holds is both hereditary and saturated. Under the present assumptions it will therefore hold for all v. Consider a β-KMS weight ϕ and the pair $(\mu, \{\varphi_x\}_{x \in \Omega_G})$ associated to it by Theorem 5.4. It suffices to show that the elements $x \in \Omega_G$ for which the isotropy group $(\mathcal{G}_G)_x^x$ is non-trivial is a null set with respect to μ. The isotropy group of a point $x \in \Omega_G$ is non-trivial if and only if x is an infinite pre-periodic path in G, and there are at most countably many such points. It suffices therefore to show that $\mu(\{x\}) = 0$ for any infinite pre-periodic path x. There is an $m \in \mathbb{N}$ such that $x_0 = \sigma^m(x)$ is periodic. It follows from (3.2) that

$$\mu(\{x\}) = e^{-\beta \sum_{j=0}^{m-1} F(\sigma^j(x))} \mu(\{x_0\}),$$

so it suffices to show that $\mu(\{x_0\}) = 0$. Since x_0 is periodic there is a finite loop δ in G such that $x_0 = \delta^\infty$, and since G is cofinal and every loop in G has an exit there is also a finite loop δ' in G such that $\delta' \nsubseteq x_0$ and $s(\delta') = s(\delta)$. By prolonging δ and δ' if necessary we may assume that the length of δ and δ' are the same, say p. For each $k \in \mathbb{N}$ set

$$y_k = \delta^k \delta' x_0.$$

Since x_0 is p-periodic it follows from (3.2) that

$$\mu(\{x_0\}) = \mathrm{e}^{-\beta \sum_{j=0}^{kp-1} F(\sigma^j(x_0))}\mu(\{x_0\}),$$

for all $k \in \mathbb{N}$, and the desired conclusion follows if $-\beta \sum_{j=0}^{kp-1} F(\sigma^j(x_0))$ is not zero for some k. Consider therefore now the case where $-\beta \sum_{j=0}^{kp-1} F(\sigma^j(x_0)) = 0$ for all $k \in \mathbb{N}$. Since (5.6) holds we find then that

$$\left|\beta \sum_{j=0}^{kp-1} F(\sigma^j(y_k))\right| = \left|\beta \sum_{j=0}^{kp-1} F(\sigma^j(y_k)) - \beta \sum_{j=0}^{kp-1} F(\sigma^j(x_0))\right| \leqslant |\beta|K \tag{5.7}$$

for all k, where $K = \sup_n \mathrm{Var}_{n,v}(F)$ and $v = s(\delta)$ is the source of δ. Now apply (3.2) again to find that

$$\mu(\{y_k\}) = \mathrm{e}^{-\beta \sum_{j=0}^{(k+1)p-1} F(\sigma^j(y_k))}\mu(\{x_0\}).$$

Inserting (5.7) this leads to the conclusion that

$$\mu(\{y_k\}) \geqslant \mathrm{e}^{-|\beta|K}\mathrm{e}^{-\beta \sum_{j=0}^{p-1} F(\sigma^j(z))}\mu(\{x_0\}),$$

where $z = \delta' x_0 = \sigma^{kp}(y_k)$. Since

$$\sum_{k=1}^{\infty} \mu(\{y_k\}) \leqslant \mu(Z(v)) < \infty,$$

we conclude that $\mu(\{x_0\}) = 0$, as desired. ∎

It follows from Proposition 5.6 that a generalized gauge action on a graph C^*-algebra, considered for example in [14], where F only depends on the first edge only has gauge-invariant KMS weights, at least as long as the algebra is simple.

REMARK 5.7. It should be emphasized that the conclusion in Proposition 5.6 does not hold without some condition on F. To see this observe that the example presented in [15] shows that already for the canonical finite graph G for which $C^*(G)$ is a copy of the Cuntz algebra O_2, namely the graph consisting of one vertex and two arrows, there are continuous non-negative functions $F : \Omega_G \to \mathbb{R}$ such that σ^F admits non-diagonal KMS states. In the example from [15] there is at least a single extremal KMS state which is diagonal, namely the extremal KMS-state corresponding to the lowest inverse temperature β_0. Here we want to indicate how to modify the example in [15] to get an example where no *extremal* KMS state is diagonal. The basis for this is a sequence $\{b_n\}_{n=1}^{\infty}$ of positive numbers with the following properties:

(a) $b_n \geqslant b_{n+1}\ \forall n$,

(b) $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1$,

(c) $\sum_{n=1}^{\infty} b_n < 1$, and

(d) $\sum_{n=1}^{\infty} b_n^s = \infty$ for all $s < 1$.

We leave the reader to verify the existence of such a sequence. Set $a_1 = -\log b_1$ and $a_k = \log b_{k-1} - \log b_k$, $k \geqslant 2$, and identify the infinite path space Ω_G with $\{0,1\}^{\mathbb{N}}$ by labelling the two arrows in G by 0 and 1. Define then $T : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ such that $T((x_i)_{i=1}^{\infty}) = a_k$ where $k = \min\{i : x_i = 0\}$ when $(x_i)_{i=1}^{\infty} \neq 1^{\infty}$, and set $T(1^{\infty}) = 0$. (As in [15] 1^{∞} is the infinite string of 1's.) This is a continuous non-negative function. By using Theorem 2.2 in [15] and arguing exactly as in Section 3 of [15], but with the sequence $\{n^{-1}\}$ replaced by $\{a_n\}$, it follows that there are β-KMS states for σ^T if and only if $\beta \geqslant 1$, and for each $\beta \geqslant 1$ the extremal KMS states are parametrised by the circle, and none are diagonal. As guaranteed by Theorem 5.5 there are for each $\beta \geqslant 1$ also one which is diagonal. As explained in [15] it arises by integrating the extremal ones with respect to Lebesgue measure on the circle.

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