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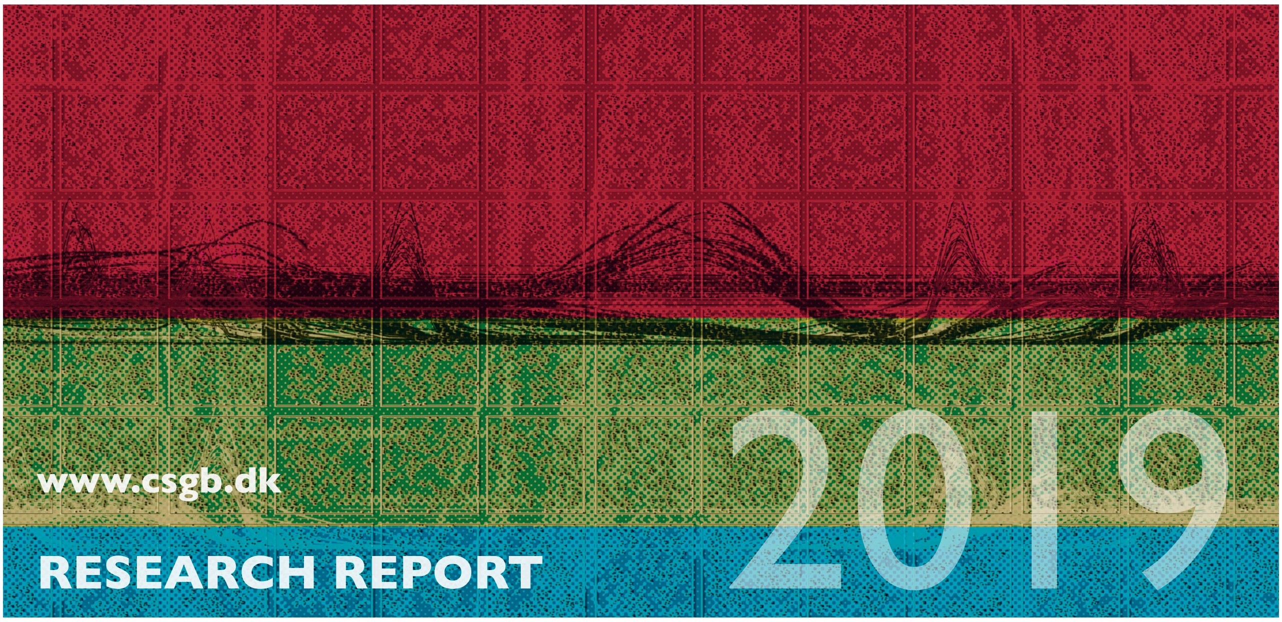

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Abstract

Crofton's intersection formula states that the $(n-j)$'th intrinsic volume of a compact convex set in $\mathbb{R}^n$ can be obtained as an invariant integral of the $(k-j)$'th intrinsic volume of sections with k-planes. This paper discusses the question if the $(k-j)$'th intrinsic volume can be replaced by other functionals, that is, if the measurement function in Crofton's formula is unique.

The answer is negative: we show that the sums of the $(k-j)$'th intrinsic volume and certain translation invariant continuous valuations of homogeneity degree k yield counterexamples. If the measurement function is local, these turn out to be the only examples when $k=1$ or when $k=2$ and we restrict considerations to even measurement functions. Additional examples of local functionals can be constructed when $k \geq 2$.

Keywords: Crofton's formula, Klain functional, Local functions, Spherical lifting, Uniqueness, Valuation

This paper is dedicated to the memory of Wolfgang Weil, a kind colleague and teacher.

1 Introduction and main results

1.1 Uniqueness of local measurement functions in Crofton's formula

The classical Crofton formula [18] for compact convex sets K states that the invariantly integrated j-th intrinsic volume V_j of the intersection of K with a k-dimensional flat E is essentially an intrinsic volume of K:

$$\int_{A(n,k)} V_j(K \cap E) d\mu_k(E) = \alpha_{n,j,k} V_{n+j-k}(K). \tag{1.1}$$

Here μ_k is an (appropriately normalized) invariant measure on the space $A(n,k)$ of all k-flats (k-dimensional affine subspaces of $\mathbb{R}^n$), $\alpha_{n,j,k} > 0$ is a known constant and $0 \leq j \leq k \leq n-1$.

We will make use of the following notation. For a linear topological space X of finite dimension, we will write $\mathcal{B}(X)$ for the Borel σ-algebra on X and denote the family of all compact convex subsets of X by $\mathcal{K}(X)$. We will write $\mathcal{K}_k(X)$ for the subfamily of all such sets of dimension at most k, $0 \leq k \leq \dim X$. Clearly, $\mathcal{K}_{\dim X}(X) = \mathcal{K}(X)$. In contrast to the standard literature (e.g. [18]) we include the empty set in these classes.

To simplify notation, we introduce *the Crofton operator* $\mathrm{C_k} : (\mathcal{K}_k(\mathbb{R}^n))^{\mathbb{R}} \to (\mathcal{K}(\mathbb{R}^n))^{\mathbb{R}}$ by

$$(\mathrm{C_k}\,\varphi)(K) = \int_{A(n,k)} \varphi(K \cap E) d\mu_k(E), \qquad K \in \mathcal{K}(\mathbb{R}^n) \tag{1.2}$$

for a *measurement function* $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$. Here and in the rest of the paper we assume that $E \mapsto \varphi(K \cap E)$ is integrable for all $K \in \mathcal{K}(\mathbb{R}^n)$. Due to (1.1) there exists a measurement function φ solving $\mathrm{C_k}(\varphi) = V_j$ for any $j \in \{n-k, \dots, n\}$. The purpose of the present paper is to discuss uniqueness of such a solution, possibly under additional restrictions on φ. As $\mathrm{C_k}$ is linear, the equality $\mathrm{C_k}(\varphi) = V_j$ has at most one solution if and only if its kernel $\ker \mathrm{C_k}$ is trivial. We will therefore describe properties of the kernel of the Crofton operator.

Unless otherwise stated, we will assume that $n \in \{2, 3, \dots\}$ and $k \in \{1, \dots, n-1\}$, thereby excluding the trivial case $k = 0$. However, the case $k = 0$ will be discussed when measurement functions on smaller domains are considered; see Section 1.2.

For general $k \in \{1, \dots, n-1\}$ the kernel of $\mathrm{C_k}$ is not trivial and we will give non-vanishing examples of measurement functions in $\ker \mathrm{C_k}$ later. We therefore impose additional assumption on φ, which are typically geometrically motivated. A set of rather strong assumptions would be the defining properties of the intrinsic volumes: continuity, motion invariance and additivity. However, due to Hadwiger's characterization theorem of the intrinsic volumes, applied in k-flats, such a measurement function must be a linear combination of $V_0, \dots, V_k$, and thus $\varphi \in \ker \mathrm{C_k}$ if and only if $\varphi \equiv 0$ by (1.1).

Can the assumptions imposed on φ be relaxed? The first result shows that there are non-trivial elements in $\ker \mathrm{C_k}$, if the motion invariance is weakened and replaced by translation invariance.

To state the result let $\operatorname{lin} K = \operatorname{aff} K - x$, where $\operatorname{aff} K$ is the affine hull of K, and x is an arbitrary element of $K \in \mathcal{K}(\mathbb{R}^n) \setminus \{\emptyset\}$. We will write ν_k for the invariant probability measure on the Grassmannian $G(n,k)$ of k-dimensional linear subspaces of $\mathbb{R}^n$.

Proposition 1. *Let $f : G(n,k) \to \mathbb{R}$ be a ν_k-integrable function and define*

$$\varphi_f(K) = \begin{cases} V_k(K) f(\operatorname{lin} K), & \text{if } \dim K = k, \\ 0, & \text{otherwise,} \end{cases}$$

for $K \in \mathcal{K}_k(\mathbb{R}^n)$. Then

(i) *φ_f is translation invariant and additive.*

(ii) *If f is continuous, then φ_f is continuous.*

(iii) If $f \not\equiv 0$ then $\varphi_f \not\equiv 0$.

(iv) We have

$$\int_{G(n,k)} f(L)\nu_k(dL) = 0 \tag{1.3}$$

if and only if $\mathrm{C_k}(\varphi_f) \equiv 0$ on $\mathcal{K}(\mathbb{R}^n)$.

The aim of the following considerations is to introduce a natural geometric property and impose it to the measurement function φ. The Crofton formula and a number of other integral geometric relations are widely used in geometric sampling. Many of the stereological estimators obtained this way share a local property. Roughly speaking, this means that they can be seen as sums or integrals of contributions which only depend on an infinitesimal neighbourhood of the location considered. This is not only true for volume and surface area estimators under IUR sampling [2] but also under vertical and local designs. For instance [9], the nucleator and the surfactor are of this type. Wolfgang Weil [22] gave a formal definition of the local property, which we will recall below. He named functionals which have the local property and are in addition continuous and translation invariant, *local functionals* on $\mathcal{K}(\mathbb{R}^n)$. Among other things, he showed that any local functional φ is a *standard functional* in the sense of translation invariant valuation theory, that is, φ is continuous, translation invariant and additive on $\mathcal{K}(\mathbb{R}^n)$. It is an open problem if any standard functional is local, however the two notions are indeed equivalent for $n \in \{1, 2\}$; see for instance Proposition 7 combined with equation (1.8). As our main focus are Crofton formulae in $\mathbb{R}^3$, where only planes of dimension $k \in \{1, 2\}$ are of practical interest, we thus could have developed our theory using standard functionals. An exact definition of local functionals on $\mathcal{K}(\mathbb{R}^n)$ is given in Definition 5 in Section 2.3, below. We already mention here that φ has the local property if it satisfies the following condition. For each $K \in \mathcal{K}(\mathbb{R}^n)$ there exists a finite signed Borel measure $\Phi(K, \cdot)$, such that $\varphi(K) = \Phi(K, \mathbb{R}^n)$, and $\Phi(K, \cdot)$ is local, meaning that the intersection of K with an open neighbourhood of a Borel set A already determines $\Phi(K, A)$. The transition kernel Φ is called a local extension of φ. For our considerations we need to extend this definition to include functionals only acting on compact convex sets of dimension at most k. A natural way of doing this is to consider the restrictions of φ to compact convex subsets of linear subspaces $L \in G(n, k)$ and to require them to be local in the sense of Definition 5, when identifying L with $\mathbb{R}^k$. However, this would only give us translation invariance of the functional in each $L \in G(n, k)$ and not necessarily in all of $\mathbb{R}^n$. We therefore say that a functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is local if it is translation invariant and each restriction of φ to subsets of a k-dimensional linear subspace L are local in L in the sense of Wolfgang Weil; see Definitions 5 and 6 for details. Note that a local functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is continuous on $\mathcal{K}(L)$ for all $L \in G(n, k)$.

Our first main result is an extension of [22, Theorem 2.1] to local functionals on $\mathcal{K}_k(\mathbb{R}^n)$, $k \in \{1, \dots, n\}$ and it gives a decomposition of a local functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ into homogeneous, local functionals $\varphi^{(j)}$, $j = 0, \dots, k$.

Before stating this result we need to fix some notation. Let κ_k be the k-dimensional volume of the unit ball in $\mathbb{R}^k$. For each $L \in G(n, k)$, define the Euclidean unit ball in L to be $B_L = B^n \cap L$, where B^n is the unit ball in $\mathbb{R}^n$ and let $S^L_{k-1}(K, \cdot)$ be the $(k-1)$'th surface measure of a compact convex set $K \in \mathcal{K}(L)$ with L as ambient space. Let $\tilde{\wp}^{\text{k-1}}_{\text{j-1}}(L)$ be the family of spherical polytopes of dimension at most $j-1$ in L, where the dimension of a spherical polytope is defined to be one smaller than the dimension j of its positive hull in L. For $k = n$, we simplify notation and write $\tilde{\wp}^{\text{n-1}}_{\text{j-1}} = \tilde{\wp}^{\text{n-1}}_{\text{j-1}}(\mathbb{R}^n)$. For a polytope P, let $\mathcal{F}_j(P)$ be the collection of j-faces of P, $j = 0, \dots, n$ and for $F \in \mathcal{F}_j(P)$ we denote by λ_F the restiction to F of the Lebesgue measure in the affine hull of F. We further let $N_L(P, F)$ be the normal cone of $P \subset L$ at F in the subspace L and let $n_L(P, F)$ denote the intersection of the unit sphere S^{n-1} with $N_L(P, F)$. For $k = n$, we write, $n(P, F) = n_{\mathbb{R}^n}(P, F)$. A function $\psi : S^{n-1} \cap L \to \mathbb{R}$ is called *centered* if

$$\int_{S^{n-1} \cap L} u\psi(u)\mathcal{H}^{k-1}(du) = 0,$$

where $\mathcal{H}^{k-1}$ is the $(k-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

Theorem 2. *Let $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ be a local functional with local extension $\Phi_L : \mathcal{K}(L) \times \mathcal{B}(L) \to \mathbb{R}$ for each $L \in G(n, k)$. Then φ has a unique representation*

$$\varphi(K) = \sum_{j=0}^{k} \varphi^{(j)}(K) \tag{1.4}$$

with j-homogeneous local functionals $\varphi^{(j)}$ on $\mathcal{K}_k(\mathbb{R}^n)$.

In addition, for each $L \in G(n, k)$ there is a decomposition

$$\Phi_L(K, \cdot) = \sum_{j=0}^{k} \Phi_L^{(j)}(K, \cdot), \tag{1.5}$$

$K \in \mathcal{K}(L)$, such that $\Phi_L^{(j)}$ is a local extension of $\varphi^{(j)}$ restricted to L, for $j = 0, \dots, k$.

For a polytope $P \in \mathcal{K}(L)$, each $\Phi_L^{(j)}$ has the form

$$\Phi_L^{(j)}(P, \cdot) = \sum_{F \in \mathcal{F}_j(P)} g_L^{(j)}(n_L(P, F))\lambda_F, \tag{1.6}$$

where $g_L^{(j)} : \tilde{\wp}^{k-1}_{k-j-1}(L) \to \mathbb{R}$, $j = 0, \dots, k$, are (uniquely determined by $\Phi_L^{(j)}$) simple additive and continuous functions, the so-called associated functions of Φ_L.

Moreover we have

$$\varphi^{(k)}(K) = c_L^{(k)} V_k(K) \tag{1.7}$$

for all $K \in \mathcal{K}(L)$, where $c_L^{(k)} = \varphi^{(k)}(\kappa_k^{-1/k} B_L)$ and

$$\varphi^{(0)}(K) = c^{(0)} V_0(K) \tag{1.8}$$

for $K \in \mathcal{K}_k(\mathbb{R}^n)$, where $c^{(0)} = \varphi^{(0)}(\{0\})$. If further $x \in L^\perp$, then

$$\varphi^{(k-1)}(K) = \int_{S^{n-1} \cap L} \theta(L, v) S_{k-1}^L(K - x, dv) \tag{1.9}$$

for all $K \in \mathcal{K}(L + x)$, where $\theta(L, v) = g_L^{(k-1)}(\{v\})$, is continuous in v and centered for fixed L.

Note that (1.4)-(1.7) is an extension of [22, Theorem 2.1] to functionals acting on the subfamilies $\mathcal{K}_k(\mathbb{R}^n)$ of $\mathcal{K}(\mathbb{R}^n)$, $k < n$. However, continuity of $g_L^{(j)}$ appears to be a new result.

By linearity of the Crofton operator $\mathrm{C_k}$, a local functional is in the kernel of $\mathrm{C_k}$ if and only if each homogeneous functional in its decomposition is in the kernel of $\mathrm{C_k}$. By (1.7), (1.8) and (1.9) we have explicit descriptions of local functionals acting on compact convex sets of dimension at most 2. Using these descriptions our main result can be proven. It shows that the only local functionals in $\ker \mathrm{C}_1$ and the only even local functionals in $\ker \mathrm{C}_2$ are the examples given in Proposition 1. This result cannot be improved as there are other examples in all remaining cases.

Theorem 3. *Let $n \in \mathbb{N}$ and $k \leq n-1$ be given.*

1. *For $k = 1$ the local functionals in $\ker \mathrm{C}_1$ are precisely the functionals $\varphi = \varphi_f$ with some $f : G(n,k) \to \mathbb{R}$ satisfying* (1.3)*.*
2. *For $k = 2$ the* **even** *local functionals in $\ker \mathrm{C}_2$ are precisely the functionals $\varphi = \varphi_f$ with some $f : G(n,k) \to \mathbb{R}$ satisfying* (1.3)*.*
3. *For $k \geq 2$ there is a local functional φ of homogeneity degree $k-1$ in $\ker \mathrm{C_k}$, which is not trivial, as there exists $K \in \mathcal{K}(\mathbb{R}^n)$ such that $\{E \in A(n,k) : \varphi(K \cap E) \neq 0\}$ is not a set of μ_k-measure zero.*

Thus we have a complete description of the kernel of the Crofton operator, when considering local and even functionals acting on intersections of compact convex sets in $\mathbb{R}^n$ with either 1- or 2-dimensional flats. The proof of the theorem makes use of the decomposition of φ into homogeneous parts (Theorem 2), which reduces the problem to considering homogeneous local functionals in the kernel of the Crofton operator. The explicit expression for the 0-homogeneous functional $\varphi^{(0)}$ in (1.7) shows that $\varphi^{(0)} \in \ker \mathrm{C_k}$ if and only if $\varphi^{(0)} = 0$. Together with (1.8) this gives the claim for $k = 1$. For $k = 2$ and φ being even rewriting the 1-homogeneous functional gives a connection to the kernel of the Radon transform on Grassmannians, which has well understood properties (see, for instance [5]). Using injectivity properties of this transform, the 1-homogeneous functional must vanish, yielding Theorem 3.2.

The relation to the Radon transform also leads to the existence of a non-trivial $(k-1)$-homogeneous local functional in the kernel of the Crofton operator, when $k > 2$, hence yielding Theorem 3.3 for $k > 2$. For $k = 2$ we start by explicitly constructing a 1-homogeneous local functional in the kernel of the Crofton operator when $n = 3$. Examples in higher dimensions are then constructed by averaging a three-dimensional counterexample over all three-dimensional subspaces containing a given $K \in \mathcal{K}_2(\mathbb{R}^n)$; for details see Section 2.3. This yields Theorem 3.3.

1.2 Variations: Measurement functions on smaller domains

The problem becomes more involved when considering functionals acting on specific subsets of $\mathcal{K}(\mathbb{R}^n)$. To discuss this more general setting, we fix $\mathcal{M} \subset \mathcal{K}(\mathbb{R}^n)$ and consider the collection

$$\mathcal{M}_k = \{K \cap E \,:\, K \in \mathcal{M}, E \in A(n,k)\}$$

of section profiles of sets in $\mathcal{M}$ with k-flats. The uniqueness of the measurement function $\varphi : \mathcal{M}_k \to \mathbb{R}$ in a suitable Crofton formula is again equivalent to a trivial kernel $\ker \mathrm{C_k}$, where the Crofton operator $\mathrm{C_k}$ is now a function from $\mathcal{M}_k^{\mathbb{R}}$ to $\mathcal{M}^{\mathbb{R}}$ defined by (1.2), but with $K \in \mathcal{M}$.

To appreciate the difficulty of the problem consider $k = 0$ and let $f(x) = \varphi(\{x\})$. The measurement function $\varphi : \mathcal{M}_k \to \mathbb{R}$ is an element of $\ker \mathrm{C_0}$ if and only if $\varphi(\emptyset) = 0$ and

$$\int_K f(x)dx = 0 \tag{1.10}$$

for all $K \in \mathcal{M}$. When $\mathcal{M}$ is 'large', for instance when it contains all axis-parallel cubes, we obviously have $f = 0$ almost everywhere by Dynkin's lemma (see, [3, Theorem 1.6.1]). However if

$$\mathcal{M} = \{gK_0 \,:\, g \text{ is a rigid motion in } \mathbb{R}^n\}$$

consists of all rigid motions of a fixed non-empty compact convex set K_0, the existence of non-vanishing functions f is not trivial at all. When K_0 is a Euclidean ball, a non-vanishing solution f of (1.10) can be given in terms of a Bessel function, and all solutions (within the Schwartz class of distributions) can be characterized. Whether there are other sets K_0 for which (1.10) has a non-vanishing solution f, is the *Pompeiu problem*; see, for instance [14]. This long-standing problem is still open in arbitrary dimension, but the case $n = 3$ has been settled by Ramm [15] even without convexity assumptions. His result implies that (1.10) holds for some K_0 with C^1-smooth boundary if and only if $f = 0$ almost everywhere.

We cannot solve this uniqueness problem in full generality, but state a result in the special case where $\mathcal{M}$ consist of all n-dimensional balls. We restrict attention to motion invariant measurement functions.

Theorem 4. *Let $n \in \mathbb{N}$, $k \in \{0, \dots, n-1\}$ and $\mathcal{M}$ be the set of all n-dimensional balls and assume that $\varphi : \mathcal{M}_k \to \mathbb{R}$ is motion invariant. Then*

$$\int_{A(n,k)} \varphi(K \cap E) d\mu_k(E) = 0 \tag{1.11}$$

for all $K \in \mathcal{M}$ if and only if

$$\varphi(K \cap E) = 0$$

for μ_k-almost all $E \in A(n,k)$.

Remark that all compact convex sets of dimension 1 are balls and hence the above theorem states that the kernel of the Crofton operator is trivial when considering motion invariant functionals defined on all 1-dimensional compact, convex sets in $\mathbb{R}^n$.

The proof of this theorem makes use of the fact that a translation invariant functional of a lower dimensional ball does not depend on the center of the ball. Furthermore, due to rotation invariance each intersection $K \cap E$ can be replaced by a k-dimensional ball of equal radius within a fixed flat in $G(n,k)$. Hence φ only depends on the radius of the ball $E \cap K$. The proof of Theorem 4 will be given in Section 2.4. It exploits that the left side of (1.11) can be written as a Riemann-Liouville integral whose injectivity properties are known.

1.3 Table of contents

The paper is structured as follows. In Section 2.1 some preliminary definitions and notations are introduced. In Section 2.2 the proof of Proposition 1 is given by constructing non-zero functionals in the kernel of the Crofton operator. Section 2.3 is devoted to considering local functionals. The definition of these is given and the first main result, Theorem 2 is proven. The section ends with a proof of Theorem 3, using the results of Theorem 2. Finally, in Section 2.4, we consider functionals on subsets of $\mathcal{K}_k(\mathbb{R}^n)$ and give a proof of Theorem 4.

2 Proofs

2.1 Notation and preliminaries

Before giving the proofs of the above stated results we will introduce some further notation. Let $A \subset \mathbb{R}^n$, we will denote its boundary by $\operatorname{bd} A$, its interior by $\operatorname{int} A$ and its relative interior by $\operatorname{relint} A$. The orthogonal complement of A is given by $A^\perp$ and the convex hull by $\operatorname{conv}(A)$. If A is convex, its dimension is defined to be the dimension of its affine hull. The dual cone of A is given by

$$A^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 0 \,\forall y \in A\}.$$

Note that the dual of the convex cone $C = \{\alpha a : a \in A \text{ and } \alpha \geq 0\}$ of A satisfies $C^\circ = A^\circ$.

We have already introduced the notation $\tilde{\wp}^{\text{n-1}}_{\text{j-1}}$, the set of spherical polytopes in $\mathbb{R}^n$ of dimension at most $j-1$. We will make use of the subset $\wp^{\text{n-1}}_{\text{j-1}}$, consisting of all spherical polytopes of exact dimension $j-1$.

Throughout this paper we consider $G(n,k)$ and $A(n,k)$ endowed with the Fell topology, which induces Borel σ-algebras as in [17, p. 582]. Let $j, l \in \mathbb{N}$ with $j, l < n$ and consider $E \in A(n,l)$, $(E \in G(n,k))$. We define the space of all affine (linear) subspaces of dimension j incident to E by $A(E,j)$, $(G(E,j))$. As in [17, Section 13.2] we denote the appropriately normalized, invariant measures of these spaces by μ_j^E and ν_j^E, respectively. Further, for $L \in G(n,k)$, we let the set of all convex polytopes in L be denoted by $\mathcal{P}(L)$.

2.2 Construction of non-zero kernel functionals

Proof of Proposition 1: Fix $f : G(n,k) \to \mathbb{R}$ and let φ_f be defined as in Proposition 1. We show (i). By translation invariance of the intrinsic volume and the function $K \mapsto \operatorname{lin} K$, φ_f becomes translation invariant. To show additivity, we need to prove that

$$\varphi_f(K) + \varphi_f(M) - \varphi_f(K \cap M) = \varphi_f(K \cup M). \tag{2.1}$$

holds for all $K, M \in \mathcal{K}_k(\mathbb{R}^n)$ with $K \cup M \in \mathcal{K}_k(\mathbb{R}^n)$. Equation (2.1) is trivially true when $\dim(K \cup M) < k$, so we may assume $\dim(K \cup M) = k$. If $\dim K = \dim M = k$, then $\operatorname{aff} M = \operatorname{aff} K = \operatorname{aff}(K \cup M)$ and (2.1) follows by additivity of the intrinsic volume. This leaves us with the case where $\dim(K \cup M) = k$ and one of the sets has dimension strictly less than k. Without loss of generality we may assume that $j = \dim M < k$. As $\dim(K \cup M) = k$ then there exists $z \in K \setminus \operatorname{aff} M$. Since $K \cup M$ is convex we have

$$\{\alpha M + (1-\alpha)z : \alpha \in [0,1)\} \subset K \cup M \setminus \operatorname{aff} M.$$

As K is closed, we obtain $M \subset K$ and hence in this case (2.1) trivially follows. Thus φ_f is additive.

We now show (ii). Assume that f is continuous and let (K_m) be a sequence with $K_m \in \mathcal{K}_k(\mathbb{R}^n)$ converging to $K \in \mathcal{K}_k(\mathbb{R}^n)$. If $\dim K = k$ then $\operatorname{lin}(K_m)$ converges to $\operatorname{lin}(K)$ and so by continuity of the intrinsic volumes and f we get

$$\varphi_f(K_m) \to \varphi_f(K) \quad \text{for } m \to \infty.$$

If $\dim K < k$ then

$$|\varphi_f(K_m) - \varphi_f(K)| = |\varphi_f(K_m)| \leq \|f\|_\infty V_k(K_m) \to 0$$

as $m \to \infty$, where we used the facts that the continuous function f has a finite maximum norm on the compact set $G(n,k)$, and that V_k is continuous. Hence φ is continuous yielding (ii). As (iii) is obvious it remains to show (iv). For fixed $K \in \mathcal{K}(\mathbb{R}^n)$, $L \in G(n,k)$ and $x \in L^\perp$ remark that if $\dim(K \cap (L+x)) = k$ then $\operatorname{lin}(K \cap (L+x)) = L$ and otherwise $V_k(K \cap (L+x)) = 0$. Hence using Fubini's Theorem

$$\int_{A(n,k)} |\varphi_f(K \cap E)| \mu_k(dE) = V_n(K) \int_{G(n,k)} |f(L)| \nu_k(dL).$$

Thus, the integrability of f implies the integrability of φ_f. The same arguments also show $[\mathrm{C_k}(\varphi_f)](K) = V_n(K) \int_{G(n,k)} f(L) \nu_k(dL)$, which clearly implies (iv). This finishes the proof of Proposition 1. $\square$

It should be remarked that the vector space of integrable real functions f on $G(n,k)$ satisfying (1.3) is infinite dimensional. Hence Proposition 1 yields a large number of nontrivial functionals in $\ker \mathrm{C_k}$.

2.3 Local functionals

For the reader's convenience, we recall the definition of local functionals φ due to Wolfgang Weil in [22]. In contrast to [22] the empty set is an element of the domain of φ here.

Definition 5. *A functional $\varphi : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ is called* ***local****, if it has a* ***local extension*** *$\Phi : \mathcal{K}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$, which is a measurable function on $\mathcal{K}(\mathbb{R}^n)$ in the first variable and a finite signed Borel measure on $\mathbb{R}^n$ in the second variable and such that Φ has the following properties:*

(i) $\varphi(K) = \Phi(K, \mathbb{R}^n)$ *for all* $K \in \mathcal{K}(\mathbb{R}^n)$,

(ii) Φ *is translation covariant, that is,* $\Phi(K+x, A+x) = \Phi(K, A)$ *for* $K \in \mathcal{K}(\mathbb{R}^n)$, $A \in \mathcal{B}(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$,

(iii) Φ *is locally determined, that is,* $\Phi(K,A) = \Phi(M,A)$ *for* $K, M \in \mathcal{K}(\mathbb{R}^n)$, $A \in \mathcal{B}(\mathbb{R}^n)$, *if there is an open set* $U \subset \mathbb{R}^n$ *with* $K \cap U = M \cap U$ *and* $A \subset U$,

(iv) $K \mapsto \Phi(K, \cdot)$ *is weakly continuous on* $\mathcal{K}(\mathbb{R}^n)$ *(w.r.t. the Hausdorff metric).*

If φ is local with local extension Φ, then $\Phi(\emptyset, \cdot) = 0$ due to (ii) and (iii), and hence $\varphi(\emptyset) = 0$ by (i).

To prove Theorem 2 we first need to extend the above definition to include functionals acting on compact convex subsets of $\mathbb{R}^n$ of dimension at most k, $k \in \{1, \dots, n\}$.

Definition 6. *A functional* $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ *is called* ***local*** *if and only if* φ *is translation invariant and for all* $L \in G(n,k)$, φ *restricted to* $\mathcal{K}(L)$ *is local, i.e.* $\varphi_L : \mathcal{K}(L) \to \mathbb{R}$, $K \mapsto \varphi(K)$ *is local in the sense of Definition 5 where* L *is identified with* $\mathbb{R}^k$. *The local extension of* φ_L *is denoted by* $\Phi_L : \mathcal{K}(L) \times \mathcal{B}(L) \to \mathbb{R}$.

More explicitly, choosing an orthonormal basis $u_1, \dots, u_k$ of $L \in G(n,k)$, we can identify L with $\mathbb{R}^k$ using the isometry $\hat{}: \mathbb{R}^k \to L$, $a \mapsto \sum_{i=1}^k a_i u_i$. Then φ_L is local on L if and only if $\hat{\varphi}_L : \mathcal{K}(\mathbb{R}^k) \to \mathbb{R}$, $K \mapsto \varphi(\hat{K})$ is local on $\mathbb{R}^k$.

Now for $k = n$ Theorem 2 was proven by Wolfgang Weil in [22] except for the continuity of the associated functions and equations (1.8) and (1.9), which both are consequences of this continuity property. We will therefore start out by proving the mentioned continuity in the case $k = n$. This will afterwards be used to prove Theorem 2 for general $k \in \{1, \dots, n\}$.

Proof of continuity of associated functions: Let $\varphi : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ be local with extension $\Phi : \mathcal{K}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$. The associated functions of Φ, $f^{(j)} : \tilde{\wp}^{\text{n-1}}_{\text{n-j-1}} \to \mathbb{R}$ are shown in [22] to vanish on $\tilde{\wp}^{\text{n-1}}_{\text{n-j-1}} / \wp^{\text{n-1}}_{\text{n-j-1}}$ for all $j \in \{0, \dots, n\}$. Fix $j \in \{0, \dots, n\}$ and consider the mapping $P : \tilde{\wp}^{\text{n-1}}_{\text{n-j-1}} \to \mathcal{P}(\mathbb{R}^n)$, $p \mapsto P(p) = p^\circ \cap Q$, where $Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$. This mapping is continuous which can be seen by decomposing it into the following three maps

$$p \mapsto \operatorname{conv}(p \cup \{0\}) \mapsto [\operatorname{conv}(p \cup \{0\})]^\circ = p^\circ \mapsto p^\circ \cap Q.$$

Continuity of the first and the last map is due to the continuity of the convex hull operator and the intersection operation of compact convex sets which cannot be separated by a hyperplane (see for instance [18, Theorem 1.8.10]), respectively. We note that the dual cone map on convex sets in the unit ball B^n is continuous in the Hausdorff metric by [19, Theorem 1]. The relationship between the Hausdorff metric and the Fell topology combined with the characterization of the Fell topology (see for instance [17, Theorem 12.2.2, 12.3.3]), yields continuity of the second map and therefore also continuity of P.

If $p \in \wp^{n-1}_{n-i-1}$ with $i \geq j$, then there is an i-face $F \in \mathcal{F}_i(P(p))$ with $0 \in \operatorname{relint} F$ and $n(P(p), F) = p$. Furthermore $F \cap \operatorname{int} Q \subset \operatorname{relint} F$ and all other i-faces of $P(p)$ do not hit $\operatorname{int} Q$. Now let (p_m) be a sequence in $\tilde{\wp}^{\text{n-1}}_{\text{n-j-1}}$ converging to $p \in \tilde{\wp}^{\text{n-1}}_{\text{n-j-1}}$ and put $A = \epsilon B^n$ with $\epsilon < \frac{1}{2}$. The fact that $A \subset \operatorname{int} Q$ and the above observations imply

$$\begin{aligned} \mathbf{1}_{\{n-j-1\}}(\dim p_m) f^{(j)}(p_m) \epsilon^j \kappa_j &= \Phi^{(j)}(P(p_m), A) \\ \to \Phi^{(j)}(P(p), A) &= \mathbf{1}_{\{n-j-1\}}(\dim p) f^{(j)}(p) \epsilon^j \kappa_j \end{aligned} \tag{2.2}$$

as $m \to \infty$, where the weak continuity of $K \mapsto \Phi(K, \cdot)$ combined with the Portmanteau theorem was used, as $\Phi^{(j)}(P(p), \operatorname{bd} A) = 0$. As $f^{(j)}$ vanishes on $\tilde{\wp}^{\text{n-1}}_{\text{n-j-1}} \setminus \wp^{\text{n-1}}_{\text{n-j-1}}$, we have $\mathbf{1}_{\{n-j-1\}}(\dim q) f^{(j)}(q) = f^{(j)}(q)$ for all $q \in \tilde{\wp}^{\text{n-1}}_{\text{n-j-1}}$, so (2.2) implies the continuity of $f^{(j)}$. $\square$

Proof of Theorem 2: Let $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ be local in the sense of Definition 6. Fix $L \in G(n,k)$ and $x \in L^\perp$. Let E denote the affine subspace $L + x$ and let $u_1, \dots, u_k$ denote an orthonormal basis of L. It follows that $\hat{\varphi}_L : \mathcal{K}(\mathbb{R}^k) \to \mathbb{R}$ is local by Definition 6. Due to [22, Theorem 2.1], applied to $\mathbb{R}^k$, there are j-homogeneous local functionals $\hat{\varphi}_L^{(j)}$ on $\mathcal{K}(\mathbb{R}^k)$ such that $\hat{\varphi}_L = \sum_{j=0}^k \hat{\varphi}_L^{(j)}$. Furthermore, $\varphi_L^{(k)} = c_L^{(k)} V_k$ with some constant $c_L^{(k)} \in \mathbb{R}$, possibly depending on L. Let $\tilde{} : L \to \mathbb{R}^k$ be the inverse of $\hat{} : \mathbb{R}^k \to L$ and put $\varphi_L^{(j)}(K) = \hat{\varphi}_L(\widetilde{K - x})$ for $K \in \mathcal{K}(E)$ and $j = 0, \dots, k$. Then

$$\varphi(K) = \varphi_L(K - x) = \hat{\varphi}_L(\widetilde{K - x}) = \sum_{j=0}^{k-1} \varphi_L^{(j)}(K), \tag{2.3}$$

where $\varphi_L^{(j)}$ is local on L and j-homogeneous, and $\varphi_L^{(k)} = c_L^{(k)} V_k$.

For each $j \in \{0, \dots, k\}$ we make the above definition independent of L by defining the functionals $\varphi^{(j)}(K) = \varphi_L^{(j)}(K)$ for all $K \in \mathcal{K}_k(\mathbb{R}^n)$ contained in a translation of $L \in G(n,k)$. Note that this definition is independent of the choice of L. If $L, L' \in G(n,k)$ are such that K is contained in translates of L and L' then (2.3) and the homogeneity of $\varphi_L^{(j)}$ give

$$0 = \sum_{j=0}^{k} \alpha^j \big(\varphi_L^{(j)}(K) - \varphi_{L'}^{(j)}(K)\big),$$

for all $\alpha > 0$, implying $\varphi_L^{(j)}(K) = \varphi_{L'}^{(j)}(K)$. Hence (2.3) becomes (1.4). Representation (1.4) is unique, due to a standard homogeneity argument.

For the proof of the second part of the theorem, we let $L \in G(n,k)$. By [22, Theorem 2.1] there is a local extension, $\hat{\Phi}_L : \mathcal{K}(\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^k) \to \mathbb{R}$ of $\hat{\varphi}_L$ with a unique representation

$$\hat{\Phi}_L(K, \cdot) = \sum_{j=0}^{k} \hat{\Phi}_L^{(j)}(K, \cdot)$$

for $K \in \mathcal{K}(\mathbb{R}^k)$, such that $\hat{\Phi}_L^{(j)}$ is a local extension of $\hat{\varphi}_L^{(j)}$. Using the identification of L with $\mathbb{R}^k$ yields (1.5). Furthermore [22, Theorem 2.1] also gives that for a polytope $P \subset L$ and $A \in \mathcal{B}(L)$

$$\hat{\Phi}_L^{(j)}(\tilde{P}, \tilde{A}) = \sum_{\tilde{F} \in \mathcal{F}_j(\tilde{P})} f_L^{(j)}\big(n(\tilde{P}, \tilde{F})\big) \lambda_{\tilde{F}}(\tilde{A}).$$

We remark that $\tilde{F} \in \mathcal{F}_j(\tilde{P})$ if and only if $F \in \mathcal{F}_j(P)$ and by defining $g_L^{(j)} : \tilde{\wp}^{k-1}_{k-j-1}(L) \to \mathbb{R}$, $g_L^{(j)}(p) = f_L^{(j)}(\tilde{p})$, equation (1.6) follows from $n(\tilde{P}, \tilde{F}) = \widetilde{n_L(P, F)}$ and

$$\Phi_L^{(j)}(P, A) = \hat{\Phi}_L^{(j)}(\tilde{P}, \tilde{A}) = \sum_{F \in \mathcal{F}_j(P)} g_L^{(j)}(n_L(P, F)) \lambda_F(A).$$

We have previously proved that $f^{(j)}$ is continuous for all j and so $g_L^{(j)}$ is continuous. Also by direct calculation it follows that $g_L^{(j)}$ inherits simple additivity from $f^{(j)}$ for all j and for any fixed local extension uniqueness follows by uniqueness of $f^{(j)}$ given in [22, Theorem 2.1]. This proves the asserted properties of the decomposition (1.6). We remark that

$$\varphi_L^{(j)}(P) = \hat{\Phi}_L^{(j)}(\tilde{P}, \mathbb{R}^k) = \sum_{F \in \mathcal{F}_j(P)} g_L^{(j)}(n_L(P,F))V_j(F). \tag{2.4}$$

For $j = 0$, $V_0(\emptyset) = 0 = \varphi^{(0)}(\emptyset)$ and hence, for fixed $L \in G(n,k)$, we may consider a non-empty convex polytope $P \subset L$. Using (2.4) combined with the simple additivity of $g_L^{(0)}$ yields

$$\varphi_L^{(0)}(P) = \sum_{x \in \mathrm{vert}(P)} g_L^{(0)}(n_L(P,x)) = g_L^{(0)}(S^{n-1} \cap L),$$

where $\mathrm{vert}(P)$ denotes the set of vertices of the polytope P. Applying this twice, first with $P = \{0\}$, and then with an arbitrary $P \in \mathcal{P}(L)$, shows $\varphi_L^{(0)}(P) = \varphi^{(0)}(\{0\})V_0(P)$ for all $P \in \mathcal{P}(L)$. The functional $\varphi_L^{(0)}$ is local, so Definition 5 (i) and (iv) imply that it is continuous on $\mathcal{K}(L)$. A standard approximation argument in L now shows $\varphi_L^{(0)} = \varphi^{(0)}(\{0\})V_0$ on $\mathcal{K}(L)$, which yields (1.8).

Concerning the case $j = k$, we have already remarked after (2.3), that $\varphi^{(k)}(K) = \varphi_L^{(k)}(K) = c_L^{(k)}V_k(K)$ for all $K \in \mathcal{K}(L)$ holds. $\square$

For the proof of the case $j = k-1$ we will show the following more general result which essentially follows from Wolfgang Weil's paper [22] and McMullen's characterization of standard functionals of homogeneity degree $n-1$; see [13] and e.g. [1, Theorem 3.1(iii)]. Recall that a *standard* functional $\psi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is a continuous translation invariant valuation.

Proposition 7. *Let $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ be given. The following statements are equivalent:*

1. *φ is a local functional of homogeneity degree $k-1$.*
2. *φ is translation invariant and homogeneous of degree $k-1$. For each $L \in G(n,k)$ the restriction of φ to L is a standard functional.*
3. *There is a function*

$$\theta : \{(L,v) :\ L \in G(n,k),\ v \in S^{n-1} \cap L\} \to \mathbb{R}$$

such that $\theta(L,\cdot)$ is continuous and centered on $S^{n-1} \cap L$ for all $L \in G(n,k)$, and

$$\varphi(K) = \int_{S^{n-1} \cap L} \theta(L,v) S_{k-1}^L(K-x, dv) \tag{2.5}$$

for all $K \in \mathcal{K}(L+x)$, where $x \in L^\perp$ and $S_{k-1}^L(K-x,\cdot)$ is the $(k-1)$'th surface measure of $K-x$ with L as ambient space.

The function θ in 3. is uniquely determined by φ.

Note that uniqueness of the function θ in 3. implies that φ is even if and only if $\theta(L,\cdot)$ is even for all $L \in G(n,k)$.

Another consequence of Proposition 7 is the fact that every $(k-1)$-homogeneous local functional on $\mathcal{K}_k(\mathbb{R}^n)$ is a valuation. This can be seen as in the proof of Proposition 1 taking into account the additivity in the first variable of the surface area measure (see, for instance [17, Theorem 14.2.2]).

Proof of Proposition 7: Start by assuming that $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is local and $(k-1)$-homogeneous. Identifying L with $\mathbb{R}^k$ it follows that $\hat{\varphi}_L : \mathcal{K}(\mathbb{R}^k) \to \mathbb{R}$ is local, hence it satisfies Definition 5 and so by [22] it is a standard functional yielding the second statement.

Assuming 2. it follows by translation invariance of φ that it is enough to consider compact convex subsets of $L \in G(n,k)$. For each $L \in G(n,k)$ the restriction of φ to $\mathcal{K}(L)$ has a representation (2.5) with a function $\theta(L,\cdot)$ that is centered and continuous on $S^{n-1} \cap L$ by McMullen's characterization of standard functionals applied in L; see [1, Theorem 3.1(iii)]. This proves the implication 2. $\Rightarrow$ 3.

Assume now that 3. holds. Due to [1, Theorem 3.1(*iii*)] the restriction of φ to $\mathcal{K}(L)$ is a standard functional on $\mathcal{K}(L)$ of homogeneity degree $k-1$. Relation (2.5) implies $\varphi(K+x) = \varphi(K)$ for all $x \in (\operatorname{lin} K)^\perp$ and therefore φ is translation invariant and homogeneous of degree $k-1$. By [22, Theorem 3.1] it follows that each restriction of φ to $L \in G(n,k)$ is local and so φ is local of homogeneity degree $k-1$. $\square$

We remark the following consequences of Theorem 2 for the solution of our uniqueness problem.

Lemma 8. *For $k \in \{1,\dots,n-1\}$, a local functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ with decomposition* (1.4) *satisfies*

1. $\varphi \in \ker \mathrm{C_k}$ *if and only if* $\varphi^{(0)},\dots,\varphi^{(k)} \in \ker \mathrm{C_k}$.
2. $\varphi^{(0)} \in \ker \mathrm{C_k}$ *if and only if* $\varphi^{(0)} = 0$.
3. $\varphi^{(k)} \in \ker \mathrm{C_k}$ *if and only if* $\varphi^{(k)} = \varphi_f$ *with* $f : G(n,k) \to \mathbb{R}$ *satisfying* (1.3).

Proof. Let $k \in \{1,\dots,n-1\}$ and φ be given as in the lemma. By decomposition (1.4) we have

$$\begin{aligned}
&\int_{A(n,k)} \varphi(\alpha K \cap E)\mu_k(dE) \\
&\qquad = \sum_{j=0}^{k} \alpha^{n-k+j} \int_{G(n,k)} \int_{L^\perp} \varphi^{(j)}(K \cap (L+t))\lambda_{L^\perp}(dt)\nu_k(dL) \\
&\qquad = \sum_{j=0}^{k} \alpha^{n-k+j} \int_{A(n,k)} \varphi^{(j)}(K \cap E)\mu_k(dE)
\end{aligned}$$

for all $\alpha > 0$, which yields 1. by comparing coefficients. Due to (1.8), combined with Crofton's formula (1.1), we have $\varphi^{(0)} \in \ker \mathrm{C_k}$ if and only if

$$0 = \int_{A(n,k)} \varphi^{(0)}(\{0\})V_0(K \cap E)\mu_k(dE) = \varphi^{(0)}(\{0\})\alpha_{n,0,k}V_{n-k}(K)$$

for all $K \in \mathcal{K}(\mathbb{R}^n)$, where the constant $\alpha_{n,0,k}$ is positive. Thus 2. holds.

In view of (1.7) we have $\varphi^{(k)} = \varphi_f$ with $f(L) = c_L^{(k)}$ for all $L \in G(n,k)$, and the last claim follows from Proposition 1(iv). $\square$

This lemma implies Theorem 3(i). For (ii) we need to treat the $(k-1)$-homogeneous part. Let therefore φ be a local functional of homogeneity degree $k-1$, and assume that φ is even. In particular, due to Theorem 2 and the remark given below the Theorem, φ is a translation invariant even valuation. Its *Klain function* $\mathbf{K}_\varphi : G(n,k-1) \to \mathbb{R}$ is defined by

$$\mathbf{K}_\varphi(M) = \frac{1}{\kappa_{k-1}} \varphi\big(B^n \cap M\big),$$

for $M \in G(n,k-1)$. Strictly speaking, this is a slight extension of Klain's [10] original definition (he calls $\mathbf{K}_\varphi$ the *generating function* of φ), which was only formulated for *continuous* translation invariant even valuations. In the present context φ need not be continuous, but (2.5) and the fact that $\theta(L,\cdot)$ in this formula must be even, imply

$$\mathbf{K}_\varphi(M) = 2\theta\big(\operatorname{span}(M \cup \{u\}), u\big) \tag{2.6}$$

for all unit vectors $u \in M^\perp$, and thus, using (2.5) again,

$$\varphi(K) = \mathbf{K}_\varphi(M) V_{k-1}(K) \tag{2.7}$$

for all $K \in \mathcal{K}(M)$. Although $\mathbf{K}_\varphi$ need not be continuous on $G(n,k-1)$, it determines φ like in the classical case. In fact, (2.5) and (2.6) yield the explicit inversion formula

$$\varphi(K) = \frac{1}{2} \int_{S^{n-1} \cap L} \mathbf{K}_\varphi(L \cap v^\perp) S_{k-1}^L(K, dv)$$

for all $K \in \mathcal{K}(L)$ and $L \in G(n,k)$.

For $i, j \in \{1, \dots, n-1\}$ *the Radon transform on Grassmannians* $\mathrm{R}_{i,j} : \mathrm{L}^1(G(n,i)) \to \mathrm{L}^1(G(n,j))$ is defined by

$$(\mathrm{R}_{i,j}\, f)(L) = \int_{G(L,i)} f(M) \nu_i^L(dM)$$

for $L \in G(n,j)$ and an integrable function $f \in L^1(G(n,i))$. We remark that $\mathrm{R}_{i,j}$ is well defined since for $f \in \mathrm{L}^1(G(n,i))$

$$\int_{G(n,j)} |(\mathrm{R}_{i,j}\, f)(L)| \nu_j(dL) \leq \int_{G(n,i)} |f(M)| \nu_i(dM).$$

This also implies that the Radon transform is Lipschitz continuous.

Proposition 9. *Let $k \in \{1, \dots, n-1\}$. Assume that the even $(k-1)$-homogeneous local functional on $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ has Klain function $\mathbf{K}_\varphi$. Then*

$$\int_{A(n,k)} \varphi(P \cap E) \mu_k(dE) = 0 \tag{2.8}$$

for all convex polytopes $P \in \mathcal{K}_{n-1}(\mathbb{R}^n)$ if and only if $R_{k-1,n-1}\big(\mathbf{K}_\varphi\big) = 0$.

Proof. Let $P \subset v^\perp$ be a convex polytope of dimension $n-1$ with $v \in S^{n-1}$. For any fixed $L \in G(n,k)$ (2.7) implies

$$\int_{L^\perp} \varphi(P \cap (L+x))\lambda_{L^\perp}(dx) = \mathbf{K}_\varphi(L \cap v^\perp) \int_{L^\perp} V_{k-1}(P \cap (L+x))\lambda_{L^\perp}(dx).$$

The translative integral on the right is proportional to the mixed volume

$$V\big(P[n-1], B_L[1]\big) = \frac{2}{n}\|v|L\| V_{n-1}(P);$$

see, e.g. [20, p. 177] and [18, Section 5.1]. Hence, (2.8) holds for all convex polytopes of dimension $n-1$ if and only if

$$\int_{G(n,n-k)} \mathbf{K}_\varphi(L^\perp \cap v^\perp)\|v|L^\perp\|\nu_{n-k}(dL) = 0$$

for all $v \in S^{n-1}$. Here we also replaced the integration with respect to ν_k by an integration with respect to ν_{n-k} by taking orthogonal complements. Using a Blaschke-Petkantschin formula (see, for instance [17, Theorem 7.2.4]) this can be shown to be equivalent to

$$\int_{G(\mathrm{span}\{v\},n-k+1)} \mathbf{K}_\varphi(L^\perp)\, h(L,v)\, \nu_{n-k+1}^{\mathrm{span}\{v\}}(dL) = 0,$$

where

$$h(L,v) = \int_{G(L,n-k)} \|v|M^\perp\| [M, \mathrm{span}\{v\}]^{k-1} \nu_{n-k}^L(dM)$$

depends on the subspace determinant $[M, \mathrm{span}\{v\}]$, which is here the sine of the angle between v and M. The function $h(L,v)$ is clearly positive and independent of $L \in G(\mathrm{span}\{v\}, n-k+1)$, as any plane in this space can be rotated to any other plane by a rotation fixing v.

Putting things together, we see that (2.8) holds for all convex polytopes of dimension $n-1$ if and only if

$$\int_{G(v^\perp,k-1)} \mathbf{K}_\varphi(L)\nu_{k-1}^{v^\perp}(dL) = 0$$

for all unit vectors v, where we again took orthogonal complements. This is the assertion. □

By [5], the Radon transform, $\mathrm{R}_{i,j}$, $i < j$, on the set of all square integrable functions $\mathrm{L}^2(G(n,i))$ is injective if and only if $i + j \le n$. Hence $\mathrm{R}_{k-1,n-1}$ is injective when acting on $\mathrm{L}^2(G(n,i))$ if and only if $k \in \{1,2\}$. We note that the kernel of $\mathrm{R}_{i,j}$ when $\mathrm{R}_{i,j}$ acts on $\mathrm{L}^2(G(n,i))$ is trivial if and only if its kernel is trivial when it acts on $\mathrm{L}^1(G(n,i))$. This can be proven using similar arguments as given in the proof of the theorem below.

Theorem 10. *There is no non-trivial even local functional on $\mathcal{K}_2(\mathbb{R}^n)$ of homogeneity degree 1 in the kernel of the Crofton operator with 2-flats.*

Let $2 < k < n$. There are non-trivial even local functionals on $\mathcal{K}_k(\mathbb{R}^n)$ of homogeneity degree $k-1$ in the kernel of the Crofton operator with k-flats.

Proof. For $k = 2$ the kernel of $\mathrm{R}_{k-1,n-1}$ is trivial and for any $\varphi \in \ker \mathrm{C_k}$, Proposition 9 gives that its Klain function $\mathbf{K}_\varphi$ is in the kernel of $\mathrm{R}_{k-1,n-1}$, which implies that $\mathbf{K}_\varphi \equiv 0$ and hence φ is trivial. This proves that there are no non-trivial even local functionals on $\mathcal{K}_2(\mathbb{R}^n)$ of homogeneity degree 1 in the kernel of the Crofton operator with 2-flats.

On the other hand if $2 < k < n$ then there exists a non-trivial function $f \in \ker \mathrm{R}_{k-1,n-1}$. Using convolution on the compact Lie group $\mathrm{SO}(n)$ of all proper rotations and the subgroup $\mathrm{SO}(n)/\mathrm{SO}(k \times (n-k))$, which can be identified with $G(n,k)$ (see [4]) we can approximate f with continuous functions in the kernel of $\mathrm{R}_{k-1,n-1}$. This implies that there exist non-trivial continuous functions in $\ker \mathrm{R}_{k-1,n-1}$. Letting $\mathbf{K}_\varphi$ be one such function we can construct φ by (2.7) yielding a continuous even local $(k-1)$-homogeneous function. By Proposition 9 we have $(\mathrm{C_k}\,\varphi)(P) = 0$ for all polytopes $P \in \mathcal{K}_{n-1}(\mathbb{R}^n)$. Using approximation of compact convex sets by polytopes from the outside implies that $\varphi \in \ker \mathrm{C_k}$ and so the last statement of the theorem follows. □

Proof of Theorem 3.1 and 3.2: Note that by Lemma 8 a local functional φ is in $\ker \mathrm{C_k}$ if and only if $\varphi^{(0)} = 0$, $\varphi^{(k)} = \varphi_f$ for some f satisfying (1.3). If $k = 1$ then $\varphi = \varphi_f$ by Theorem 2 proving Theorem 3.1. Assuming that $k = 2$ and that φ is in addition even, $\varphi^{(1)} = 0$ by Theorem 10 and hence $\varphi = \varphi_f$. □

For the poof of Theorem 3.3 we note that Theorem 10 states that if $k > 2$ then there exists non-trivial $(k-1)$-homogeneous even local functionals $\varphi \in \ker \mathrm{C_k}$ and hence we only need to construct examples of non-trivial 1-homogeneous local functionals in $\ker \mathrm{C_2}$. To explicitly make such a construction, we need to consider local functionals which are not necessarily even. For this we will make use of the translational Crofton formula for the surface area measures; see [6, Theorem 3.1]: For $K \in \mathcal{K}(\mathbb{R}^n)$ and $L \in G(n,k)$, we have

$$\int_{L^\perp} S^L_{k-1}((K-x)\cap L, \cdot)\lambda_{L^\perp}(dx) = \pi_{L,1} S_{n-1}(K,\cdot). \tag{2.9}$$

This relation makes use of the operator $\pi_{L,m}$ with $m = 1$, which is described in the following. For $m \in \mathbb{Z}$, $m > -k$, the *m-weighted spherical projection* $\pi_{L,m}$ maps any finite signed Borel measure μ on S^{n-1} into the space of finite signed Borel measures on $S^{n-1} \cap L$. The measure $\pi_{L,m}\mu$ is defined as the image of the measure μ_m, given by

$$\mu_m(A) = \int_A \|u|L\|^m \, \mu(du),$$

(where $A \subset S^{n-1}$ is a Borel set) under the *spherical projection*

$$p_L : S^{n-1} \setminus L^\perp \to S^{n-1} \cap L, v \mapsto \frac{v|L}{\|v|L\|}.$$

For $m \leq 0$ one must restrict considerations to a subclass of measures to assure that $\pi_{L,m}\mu$ is a well-defined finite signed measure.

If f is a continuous function on the sphere, its m-weighted spherical projection on L is the density of the m-weighted spherical projection of the measure $\int_{(\cdot)} f(v)\,dv$.

More explicitly, this is the function given by

$$[\pi_{L,m} f](u) = \int_{p_L^{-1}(\{u\})} f(v) \langle u, v \rangle^{k+m-1} \, dv, \tag{2.10}$$

for $u \in S^{n-1} \cap L$. For more details on $\pi_{L,m}$ and the *m-weighted spherical lifting* $\pi^*_{L,m}$, see [7]. For later use, we remark that

$$[\pi^*_{L,m} f(\cdot)](u) = \|u|L\|^m f(p_L(u)) \tag{2.11}$$

for all integrable functions $f : S^{n-1} \cap L \to \mathbb{R}$, $u \in S^{n-1} \setminus L^\perp$ and $L \in G(n,k)$. By [7, (5.4) and (5.5)], $\pi_{L,m}$ and $\pi^*_{L,m}$ can be considered as transpose operators, as

$$\int_{S^{n-1} \cap L} f \, d(\pi_{L,m} \mu) = \int_{S^{n-1}} (\pi^*_{L,m} f) \, d\mu \tag{2.12}$$

for all integrable functions f on $S^{n-1} \cap L$ and all finite signed measures μ, and

$$\int_{S^{n-1} \cap L} (\pi_{L,m} f) \, d\mu = \int_{S^{n-1}} f \, d(\pi^*_{L,m} \mu) \tag{2.13}$$

for all integrable functions f on S^{n-1} and all finite signed measures μ on $S^{n-1} \cap L$.

The following proposition is a counterpart to Proposition 9 for $(k-1)$-homogeneous local functionals, but without the evenness assumption. We recall that, according to (1.9) for any $(k-1)$-homogeneous local functional φ there is an associated function θ on the compact domain $D = \{(L,v) \in G(n,k) \times S^{n-1} : v \in L\}$ such that

$$\varphi(K) = \int_{S^{n-1} \cap L} \theta(L,v) S^L_{k-1}(K - x, dv) \tag{2.14}$$

for all compact convex sets K contained in $x + L$, where $L \in G(n,k)$ and $x \in L^\perp$. To avoid technicalities, we consider only the case where the associated function θ is continuous on D.

Theorem 11. *Let φ be a $(k-1)$-homogeneous local functional on $\mathcal{K}_k(\mathbb{R}^n)$ such that its associated function $\theta(L,v)$ given by* (2.14) *is continuous.*

Then $\varphi \in \ker \mathrm{C_k}$ if and only if

$$\int_{G(n,k)} \left[\pi^*_{L,1} \theta\big(L, \cdot\big)\right] \nu_k(dL) = 0 \tag{2.15}$$

on S^{n-1}.

Proof. Let φ be a $(k-1)$-homogeneous local functional on $\mathcal{K}_k(\mathbb{R}^n)$ and let $\theta(L,v)$ given by (2.14). Due to (2.14) and (2.9), we have

$$\begin{aligned}
(\mathrm{C_k}\, \varphi)(K) &= \int_{G(n,k)} \int_{S^{n-1} \cap L} \theta\big(L, v\big) \left[\pi_{L,1} S_{n-1}(K, \cdot)\right](dv) \, \nu_k(dL) \\
&= \int_{G(n,k)} \int_{S^{n-1}} \left[\pi^*_{L,1} \theta\big(L, \cdot\big)\right](v) \, S_{n-1}(K, dv) \, \nu_k(dL)
\end{aligned}$$

for all $K \in \mathcal{K}(\mathbb{R}^n)$, where we used (2.12) for the last equality. In view of (2.11) and the assumption that θ is continuous, $(L, v) \mapsto [\pi^*_{L,1}\theta(L, \cdot)](v)$ is continuous and hence bounded, so an application of Fubini's theorem implies

$$(\mathrm{C_k}\,\varphi)(K) = \int_{S^{n-1}} h(v)\, S_{n-1}(K, dv), \tag{2.16}$$

where

$$h(v) = \int_{G(n,k)} \left[\pi^*_{L,1}\theta(L, \cdot)\right](v)\, \nu_k(dL)$$

defines a continuous function on S^{n-1}. For instance by decomposing the Hausdorff measure on the sphere like in [7, (3.3)], one can show that the function h is centered.

Concluding, we see that $\varphi \in \ker \mathrm{C_k}$ if and only if the right hand side of (2.16) is zero for all $K \in \mathcal{K}(\mathbb{R}^n)$. [13, Theorem 3] now shows that this is equivalent to (2.15). □

We use the notation in [7] to give an example of a function θ satisfying (2.15) by writing it as linear combination of spherical projections of spherical harmonics.

Let ω be a continuous function on S^{n-1}. If we put $\theta(L, \cdot) = \pi_{L,m}\omega$, then (2.15) is equivalent to $\pi^{(k)}_{1,m}\omega = 0$ on S^{n-1}, where

$$\pi^{(k)}_{1,m}\omega = \int_{G(n,k)} [\pi^*_{L,1}\pi_{L,m}\omega]\, \nu_k(dL). \tag{2.17}$$

The operators $\pi^{(k)}_{1,m}$ act as multiples of the identity on any space of spherical harmonics of order $r \in \mathbb{N}_0$: if ω_r is a spherical harmonic of order r, then

$$\pi^{(k)}_{1,m}\omega_r = a_{n,k,1,m,r}\, \omega_r, \tag{2.18}$$

with the multiplier $a_{n,k,1,m,r}$ given as a rather complicated finite sum in [7, Theorem 9.1]. This suggests to finding m, r such that the multipliers vanishes. However, the choice of m is nontrivial. If we for instance choose $m = 1 - k$, which can be shown to correspond to

$$\varphi(K) = \binom{n-1}{k-1} \int_{S^{n-1}} \omega(u) S_{k-1}(K, du), \tag{2.19}$$

$K \in \mathcal{K}(\mathbb{R}^n)$, by [7, Theorem 6.2], then all multipliers in (2.18) are non-vanishing by [7, p. 41]. Hence there are only trivial maps φ of type (2.19) in $\ker \mathrm{C_k}$. For many other choices of m explicit formulae for $a_{n,k,1,m,r}$ are not available.

We therefore construct counterexamples as follows: Fix $r \in \mathbb{N}_0 \setminus \{1\}$ and choose different integers m and m'. As two numbers are always linearly dependent, there are $\alpha, \beta \in \mathbb{R}$ with $(\alpha, \beta) \neq 0$ such that

$$\alpha a_{n,k,1,m,r} + \beta a_{n,k,1,m',r} = 0. \tag{2.20}$$

Let ω_r be a spherical harmonic of order r and set

$$\theta(L, \cdot) = [\alpha \pi_{L,m} + \beta \pi_{L,m'}]\omega_r - \langle x_L, \cdot \rangle, \tag{2.21}$$

where $x_L \in L$ is chosen such that $\theta(L, \cdot)$ is centered. This defines a continuous function θ. Due to (2.18) and (2.20), equation (2.15) holds on S^{n-1}, and Theorem 11 thus implies that the $(k-1)$-homogeneous local functional φ with this associated function θ is an element of $\ker \mathrm{C}_k$.

Proposition 12. *There exist non-trivial local functionals φ on $\mathcal{K}_2(\mathbb{R}^n)$ of homogeneity degree* 1 *in* $\ker \mathrm{C}_2$.

Before giving the proof we will introduce some notation. Let $j, l \in \mathbb{N}$ with $j < l \leq n$ and $L \in G(n,l)$. The spherical projection on the sphere in $L' \in G(L,j)$ in the ambient space L is

$$p^L_{L'} : (S^{n-1} \setminus (L')^\perp) \cap L \to S^{n-1} \cap L', v \mapsto \frac{v|L'}{\|v|L'\|}.$$

If $L = \mathbb{R}^n$ then $p^L_{L'}$ coincides with the already defined spherical projection.

Proof. Let $n \in \mathbb{N}$, $n \geq 3$ and fix $u_0 \in S^{n-1}$. For each $L \in G(n,3)$ define $\psi^L : \mathcal{K}_2(L) \to \mathbb{R}$ by for $L' \in G(L,2)$

$$\psi^L(K') = \int_{S^{n-1}\cap L'} \theta^L(L',v) S^{L'}_1(K',dv)$$

for $K' \in \mathcal{K}_2(L)$ with $K \subset L'$ and $L \not\subseteq u_0^\perp$. When $L \subseteq u_0^\perp$ put $\psi^L(K') = 0$. Here $\theta^L : \{(L',v) \in G(L,2) \times S^{n-1} : v \in L'\} \to \mathbb{R}$ is given as in (2.21) when identifying L with $\mathbb{R}^3$ and choosing ω_r to be a certain spherical harmonic of order $r = 5$. More explicitly, we set

$$\theta^L(L',\cdot) = (\alpha\pi^L_{L',1} + \beta\pi^L_{L',2}) P^3_5(\langle p_L(u_0),\cdot\rangle) - \langle x_{L'},\cdot\rangle. \tag{2.22}$$

with

$$(\pi^L_{L',m} f)(v) = \int_{(p^L_{L'})^{-1}(\{v\})} f(u)\langle v,u\rangle^{k+m-1} du$$

and $P^3_5(t) = \frac{1}{8}(63t^5 - 70t^3 + 15t)$ being the fifth order Legendre polynomial of dimension 3; see for instance [8, p. 85]. Note that $P^3_5(\langle p_L(u_0),\cdot\rangle)$ is a spherical harmonic of order 5 on $S^{n-1} \cap L'$ for any $u_0 \in S^{n-1} \cap L$. As P^3_5 is odd also $\theta^L(L',\cdot)$ is an odd function. Furthermore, we define $\psi^{L+x}(K) = \psi^L(K - x)$ for all $K \in \mathcal{K}_2(L)$ and $x \in \mathbb{R}^n$. Using this we define $\varphi : \mathcal{K}_2(\mathbb{R}^n) \to \mathbb{R}$ as

$$\varphi(K') = \int_{A(\operatorname{aff} K',3)} \psi^F(K') \mu_2^{\operatorname{aff} K'}(dF) \tag{2.23}$$

for $K' \in \mathcal{K}_2(\mathbb{R}^n)$. Note that by definition of ψ^F, the map φ is translation invariant. As ψ^F is 1-homogeneous and local for each $F \in A(n,3)$ the map φ is 1-homogeneous and local. Also, if θ^L satisfies (2.15) for all $L \in G(n,3)$ then by Theorem 11, $\psi^F \in \ker \mathrm{C}_2$ for all $F \in A(n,3)$ and using [17, Theorem 7.1.2] we get

$$\int_{A(n,2)} \varphi(K \cap E)\mu_2(dE) = \int_{A(n,3)} \int_{A(F,2)} \psi^F(K \cap E)\mu_2^F(dE)\mu_3(dF) = 0$$

for $K \in \mathcal{K}(\mathbb{R}^n)$. Hence we need to construct functionals θ^L satisfying (2.15) for all $L \in G(n,3)$ such that φ is non-trivial. In view of Lemma 13, below, we need to find $L'_0 \in G(n,2)$, $L_0 \in G(n,3)$ with $u_0 \in L'_0 \subset L_0$ and a set $K' \in \mathcal{K}(L'_0)$ such that $\psi^{L_0}(K') > 0$ holds. From now on we fix $L'_0 \in G(n,2)$ and $L_0 \in G(n,3)$ with

$u_0 \in L_0' \subset L_0$ (implying $p_{L_0}(u_0) = u_0$) and show the existence of $K' \subset L_0'$ with the required properties. We identify these spaces with $\mathbb{R}^3$ and a two-dimensional subspace L of $\mathbb{R}^3$, respectively. With this identification in mind, the map $K' \mapsto \psi(K')$ given by the right hand side of (2.14) corresponds to ψ^{L_0}. Let $\theta^{\mathbb{R}^3} = \theta$ be given as in (2.21) with $m = 1$, $m' = 2$ and $r = 5$. Explicit calculation gives

$$a_{3,2,1,2,5} = \frac{97}{1536}\pi \qquad \text{and} \qquad a_{3,2,1,1,5} = \frac{344}{1575}.$$

We therefore put

$$\alpha = -176128 \qquad \text{and} \qquad \beta = 50925\pi. \tag{2.24}$$

This implies that the function θ satisfies (2.15) when $n = 3$ and $k = 2$. Let u_1 be a unit vector in L orthogonal to u_0. For abbreviation, we put $f_m(\cdot) = \pi_{L,m} P_5^3(\langle p_L(u_0), \cdot\rangle)$. Note that $\theta(L, \cdot)$ is the sum of the function $f = \alpha f_1 + \beta f_2$ and the linear function $\langle x_L, \cdot\rangle$. As x_L is a multiple of u_0, it is enough to show that $s \mapsto f(su_0 + \sqrt{1-s^2}u_1)$ is not linear. Now, for $m \in \mathbb{N}$ and $s \in (0,1)$ we have

$$\begin{aligned}
g_m(s) &= \frac{\partial^4}{\partial s^4} f_m(su_0 + \sqrt{1-s^2}u_1) \\
&= \frac{\partial^4}{\partial s^4} \int_{p_L^{-1}(\{su_0+\sqrt{1-s^2}u_1\})} P_5^3(\langle u_0, v\rangle)\langle su_0 + \sqrt{1-s^2}u_1, v\rangle^{m+1} dv \\
&= 2\int_0^1 \frac{\partial^4}{\partial s^4} P_5^3\big(st\big) t^{m+1}(1-t^2)^{-1/2} dt \\
&= 1890 s \int_0^1 t^{m+6}(1-t^2)^{-1/2} dt.
\end{aligned}$$

Now

$$\int_0^1 t^\gamma (1-t^2)^{-1/2} dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma\big(\frac{\gamma+1}{2}\big)}{\Gamma\big(\frac{\gamma+2}{2}\big)}, \qquad \gamma > -1,$$

which yields

$$g_m(s) = s\frac{1890\sqrt{\pi}}{2}\frac{\Gamma(\frac{m+7}{2})}{\Gamma(\frac{m+8}{2})},$$

so $g_2(s) = \frac{33075\pi}{128} s$ and $g_1(s) = 864s$. Due to (2.24), we obtain

$$\frac{\partial^4}{\partial s^4} f(su_0 + \sqrt{1-s^2}u_1) = \alpha g_1(s) + \beta g_2(s) = -1512000\pi s \neq 0,$$

for $s \neq 0$. So $\theta(L, \cdot)$ does not vanish. As $\theta(L, \cdot)$ is centered and non-vanishing, the associated local functional ψ given by (2.14) cannot be vanishing by Proposition 7. Therefore, there must be a set $K' \in \mathcal{K}(L)$ with $\psi(K') \neq 0$. Possibly changing the signs of α and β, we can assure $\psi(K') > 0$ and the proposition is shown. $\square$

In the above proof Lemma 13 was used. In the proof of this lemma we will work with Lipschitz functions and therefore introduce metrics that induce the natural topologies. Identifying rotations with their matrix representations with respect to the standard basis, we can for instance use the Frobenius norm on $\mathrm{SO}(n)$. On $G(n,k)$ we can work with the metric d_k given by

$$d_k(L,\tilde{L}) = d(B^n \cap L, B^n \cap \tilde{L}), \quad L,\tilde{L} \in G(n,k),$$

where d is the Hausdorff metric on $\mathcal{K}(\mathbb{R}^n)$. It is now clear that for $L_0 \in G(n,k)$, $\vartheta \mapsto \vartheta L_0'$ is non-expansive, that is, this mapping is Lipschitz with constant at most 1.

Lemma 13. *Let $n \geq 3$, $u_0 \in S^{n-1}$, $L_0' \in G(n,2)$ and $L_0 \in G(n,3)$ with $u_0 \in L_0' \subset L_0$ be given. Furthermore, let φ be defined as in (2.23).*

If there is a compact convex set $K' \in \mathcal{K}_2(L_0')$ with $\psi^{L_0}(K') > 0$, then there is a compact convex set K_0 in $\mathbb{R}^n$ such that

$$G = \{E \in G(n,2) : \varphi(K_0 \cap E) > 0\} \tag{2.25}$$

contains an open neighbourhood of L_0' and has therefore positive μ_2-measure.

Proof. Let L_0, L_0', K' and u_0 as assumed in the Lemma. The assumption on K' implies

$$\psi^L(K') = \psi^{L_0}(K') > 0 \tag{2.26}$$

for all $L \in G(L_0',3)$. In fact, if ϑ is a rotation fixing L_0' pointwise and satisfying $\vartheta L = L_0$, then

$$[\pi^{L_0}_{L_0',m} P_5^3(\langle u_0,\cdot\rangle)](v) = [\pi^{\vartheta L}_{\vartheta L_0',m} P_5^3(\langle \vartheta u_0,\cdot\rangle)](\vartheta v) = [\pi^{L}_{L_0',m} P_5^3(\langle u_0,\cdot\rangle)](v),$$

$v \in S^{n-1} \cap L_0'$, implying $\theta^L(L_0',\cdot) = \theta^{L_0}(L_0',\cdot)$ and hence $\psi^L(K') = \psi^{L_0}(K')$.

As $\theta^{L_0}(L_0',\cdot)$ is an odd function, $K' \in \mathcal{K}_2(L_0')$ must be two-dimensional by (2.26) and without loss of generality, we may assume that 0 is a relative interior point of K'.

Let $R_0 > 0$ be such that K' is contained in the interior of $R_0 B^n$, and put $K_0 = (K' + (L_0')^\perp) \cap R B^n$. Then L_0' hits the interior of K_0 and $K_0 \cap L_0' = K'$. More generally, for all $z \in (L_0')^\perp$ in a bounded neighbourhood U of 0 the plane L_0' hits the interior of $K_0 - z$ and $(K_0 - z) \cap L_0' = K'$. All the sets $K_0 - z$ with $z \in U$ are contained in a ball of radius $R > 0$, say. In addition, we may assume that all these sets contain the ball rB^n for some $r > 0$. We will later require that all translations $K_0 - x$ with $x \in \mathbb{R}^n$ and $\|x\| \leq \sup_{z \in U} \|z\|$ are contained in a given neighbourhood W of K_0. This can be achieved by shrinking U even further, if necessary.

For all $L \in G(L_0',3)$ we have $u_0|L = u_0$, so $\|u_0|\vartheta L_0\| \geq \frac{1}{2}$ in a neighbourhood of the identity id in $\mathrm{SO}(n)$. It is therefore easy to show that $\vartheta \mapsto p_{\vartheta L}(u_0)$ is Lipschitz in this neighbourhood with a Lipschitz constant that is independent of L. As P_5^3 is Lipschitz on $[-1,1]$, the function

$$\vartheta \mapsto P_5^3\big(\langle \vartheta^{-1} p_{\vartheta L}(u_0), v\rangle\big)$$

is Lipschitz in a neighbourhood of id in $\mathrm{SO}(n)$ with a Lipschitz constant that can be chosen independent of $v \in S^{n-1}$ and $L \in G(L_0',3)$. It follows that

$$\begin{aligned}\vartheta \mapsto & \pi^{\vartheta L}_{\vartheta L_0',m} P_5^3\big(\langle p_{\vartheta L}(u_0),\cdot\rangle\big)(\vartheta w)\\ &= \int_{(p^L_{L_0'})^{-1}(\{w\})} P_5^3\big(\langle \vartheta^{-1} p_{\vartheta L}(u_0), v\rangle\big)\langle v,w\rangle^{m+1} dv\end{aligned}$$

and hence $\vartheta \mapsto \theta^{\vartheta L}(\vartheta L_0', \vartheta w)$ is Lipschitz in a neighbourhood of id with a Lipschitz constant that can be chosen independent of $w \in S^{n-1} \cap L_0'$ and $L \in G(L_0', 3)$. Now let K be a convex body in RB^n containing the ball rB^n and observe that

$$\psi^{\vartheta L}(K \cap \vartheta L_0') = \int_{S^{n-1} \cap L_0'} \theta^{\vartheta L}(\vartheta L_0', \vartheta w) S_1^{L_0'}\big((\vartheta^{-1} K) \cap L_0', dw\big).$$

Hence,

$$\begin{aligned} &|\psi^{\vartheta L}(K \cap \vartheta L_0') - \psi^L(K_0 \cap L_0')| \\ &\quad \le \int_{S^{n-1} \cap L_0'} \sup_{w \in S^{n-1} \cap L_0'} |\theta^{\vartheta L}(\vartheta L_0', \vartheta w) - \theta^L(L_0', w)| S_1^{L_0'}\big((\vartheta^{-1} K) \cap L_0', dw\big) \\ &\qquad + \left| \int_{S^{n-1} \cap L_0'} \theta^L(L_0', w)[S_1^{L_0'}\big((\vartheta^{-1} K) \cap L_0', \cdot\big) - S_1^{L_0'}\big(K_0 \cap L_0', \cdot\big)](dw) \right| \end{aligned} \tag{2.27}$$

Due to the monotonicity and motion invariance of the first intrinsic volume V_1, the total mass of $S_1^{L_0'}\big((\vartheta^{-1} K) \cap L_0', \cdot\big)$ is

$$2V_1((\vartheta^{-1} K) \cap L_0') \le 2V_1(\vartheta^{-1} K) \le 2V_1(RB^n),$$

so the first expression in (2.27) is Lipschitz in ϑ in a neighbourhood of id with a Lipschitz constant that does not depend on K or L. The expression of θ in (2.22) and continuity of P_5^3 implies that the integrand in the second term of (2.27) is bounded by a constant M, which does not depend on L, and thus this second term is bounded by

$$M \left| V_1\big(K \cap \vartheta L_0'\big) - V_1\big(K_0 \cap L_0'\big) \right|.$$

Now $(\vartheta, K) \mapsto V_1(K \cap \vartheta L_0')$ is a Lipschitz function when we restrict considerations to compact convex sets contained in RB^n and containing rB^n, with a Lipschitz constant that depends on r and R only. In fact, that $L \mapsto K \cap L$ is Lipschitz with a constant that depends on r and R only, follows for instance from [12, Lemma 2.2] and the explicit form of the Lipschitz constant in [11]. Due to its interpretation as multiple of the mean width (see for instance [18, pages 50, 297 and 231]) the first intrinsic volume is Lipschitz with constant $n\kappa_n/\kappa_{n-1}$.

Concluding, as (2.26) implies that $2\epsilon = \psi^L(K_0 \cap L_0')$ is positive, the bound (2.27) yields

$$\psi^{\vartheta L}(K \cap \vartheta L_0') > \epsilon > 0 \tag{2.28}$$

for all ϑ in a sufficiently small open neighbourhood V of id, all $L \in G(L_0', 3)$, and all K in a sufficiently small neighbourhood W of K_0. Hence, if $z \in U$ and $\vartheta \in V$, we get

$$\varphi(K_0 \cap \vartheta(L_0' + z)) = \int_{G(L_0', 3)} \psi^{\vartheta L}((K_0 - \vartheta z) \cap \vartheta L_0') \nu_2^{L_0'}(dL) > \epsilon,$$

where we used $K = K_0 - \vartheta z$ in (2.28). It follows that G in (2.25) contains an open neighbourhood of L_0' and the assertion is shown. $\square$

2.4 Motion invariant functionals

The proof of Theorem 4 makes use of the Riemann-Liouville integral

$$(I^\alpha g)(x) = \Gamma(\alpha)^{-1} \int_0^x g(t)(x-t)^{\alpha-1}\lambda(dt)$$

of locally integrable functions $g : [0,\infty) \to \mathbb{R}$, where $\alpha > 0$ is a parameter. For arbitrary $\alpha, \beta > 0$ we have

$$I^{\alpha+\beta} g = I^\alpha I^\beta g \tag{2.29}$$

and

$$\frac{d}{dx}(I^{\alpha+1} g) = I^\alpha g; \tag{2.30}$$

see [16] for details.

Lemma 14. *Let $\alpha > 0$ and a locally integrable function $g : [0,\infty) \to \mathbb{R}$ be given. If $I^\alpha g(x) \equiv 0$ on $[0,\infty)$ then $g \equiv 0$ almost everywhere on $[0,\infty)$.*

Proof. Let $m > \alpha$ be an integer and set $\beta = m - \alpha > 0$. The assumption $I^\alpha g \equiv 0$ and (2.29) yield $I^m g \equiv 0$ on $[0,\infty)$, and applying (2.30) $(m-1)$ times gives

$$0 = I^1 g(x) = \int_0^x g(t)dt$$

for all $x \geq 0$. As the Radon-Nikodym derivative of a measure is uniquely determined almost everywhere, the claim follows. □

This can now be used to prove Theorem 4.

Proof of Theorem 4: Let $L \in G(n,k)$ and let $\varphi : \mathcal{M}_k \to \mathbb{R}$ be a motion invariant map. Let $h : (0,\infty) \to \mathbb{R}$ be given by $h(r) = \varphi(r^{1/2}(B^n \cap L))$. By the motion invariance of φ, h is independent of $L \in G(n,k)$. Since $rB^n \cap (L+x)$ is either empty, a point or a k-dimensional ball of radius $\sqrt{r^2 - \|x\|^2}$ for $x \in L^\perp$ we get that

$$\begin{aligned}\int_{A(n,k)} \varphi(rB^n \cap E)\mu_k(dE) &= \int_{G(n,k)} \int_{L^\perp \cap rB^n} h(r^2 - \|x\|^2)\lambda_{L^\perp}(dx)\nu_k(dL) \\ &= \int_0^r h(r^2 - s^2)s^{n-k-1}ds(n-k)\kappa_{n-k}\end{aligned}$$

for all $rB^n \in \mathcal{M}$ with $r > 0$. The last equality follows by identifying $L^\perp$ with $\mathbb{R}^{n-k}$ and introducing spherical coordinates (see, for instance [9]). As ν_k is a probability measure and $(n-k)\kappa_{n-k} \neq 0$, a substitution shows that the left hand side of the last displayed formula is zero if and only if

$$0 = \int_0^{r^2} h(t)(r^2 - t)^{\alpha-1}(dt) = \Gamma(\alpha)(I^\alpha h)(r^2),$$

where $\alpha = \frac{n-k}{2} > 0$. Concluding, (1.11) is equivalent to $(I^\alpha h)(t) = 0$ for all $t > 0$, and Lemma 14 proves the claim. □

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