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ON THE K -THEORY OF NILPOTENT ENDOMORPHISMS

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On the K -theory of nilpotent endomorphisms

Lars Hesselholt* and Ib Madsen**

For J. Peter May on his sixties birthday

Introduction

In this paper, we evaluate the relative K -theory of truncated polynomial algebras

$$\Lambda = A[x]/(x^n),$$

where A is a smooth algebra over a perfect field k of positive characteristic. This extends the calculation in [3], where the basic case $A = k$ was considered. Our motivation to consider the more general case is that it also leads to a calculation of the groups $\mathrm{Nil}_*(\Lambda)$. The results are stated in terms of the (big) de Rham-Witt complex of Deligne-Illusie. When A is a polynomial algebra, the structure of these groups is completely known.

Let A be a smooth k -algebra. The (p -typical) de Rham-Witt complex $W_s\Omega_A^*$ of Deligne-Illusie is a lift of the de Rham complex Ω_A^* to a differential graded algebra over $W_s(k)$ with zeroth term the p -typical Witt ring $W_s(A)$. In a similar way, the big de Rham-Witt complex $\mathbf{W}_m\Omega_A^*$ is a lift of Ω_A^* to a differential graded algebra over $\mathbf{W}_s(k)$ with zeroth term $\mathbf{W}_m(A)$. The Verschiebung $V_n: \mathbf{W}_m(A) \rightarrow \mathbf{W}_{mn}(A)$ extends to an additive map of complexes

$$V_n: \mathbf{W}_m\Omega_A^* \rightarrow \mathbf{W}_{mn}\Omega_A^*.$$

We note, however, that in positive degrees this map is usually not injective.

THEOREM A. *Let A be a smooth algebra over a perfect field of positive characteristic. Then there is a natural long exact sequence*

$$\dots \rightarrow \bigoplus_{m \geq 1} \mathbf{W}_m\Omega_A^{s-2m} \xrightarrow{V_n} \bigoplus_{m \geq 1} \mathbf{W}_{mn}\Omega_A^{s-2m} \rightarrow K_{s-1}(A[x]/(x^n), (x)) \rightarrow \dots$$

The decomposition of the middle and left hand terms in their p -typical parts is spelled out in the end of paragraph 1 below. When A is a polynomial algebra, the value of these components is given explicitly in [5, I.2.5].

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For any associative unital ring Λ , one considers the exact category $\mathbf{Nil}(\Lambda)$ with objects finitely generated projective (left) Λ -modules together with a nilpotent endomorphism. It contains the exact category $\mathbf{P}(\Lambda)$ of finitely generated projective Λ -modules as a retract, and this carries over to K -theory:

$$K_*(\mathbf{Nil}(\Lambda)) \simeq K_*(\Lambda) \oplus \mathbf{Nil}_*(\Lambda).$$

This defines the groups $\mathbf{Nil}_*(\Lambda)$. The fundamental theorem in algebraic K -theory states that

$$K_i(\Lambda[t]) \cong K_i(\Lambda) \oplus \mathbf{Nil}_{i-1}(\Lambda),$$

and hence $\mathbf{Nil}_*(\Lambda)$ is the obstruction to K -theory being homotopy invariant. By a theorem of Serre, a ring Λ is regular, if and only if every (left) Λ -module has a finite projective resolution. So the resolution theorem and the fact that K' -theory is homotopy invariant show that for a regular ring, $\mathbf{Nil}_*(\Lambda) = 0$. In general, one knows that the groups $\mathbf{Nil}_*(\Lambda)$ if non-zero are infinitely generated. It is also known that the groups $\mathbf{Nil}_*(\Lambda)$ are modules over the big Witt ring $\mathbf{W}(\Lambda)$, [8].

THEOREM B. *Let A be a smooth algebra over a perfect field of positive characteristic. Then there is a natural long exact sequence*

$$\cdots \rightarrow \bigoplus_{m \geq 1} \mathbf{W}_m \Omega_{(A[t], (t))}^{s-2m} \xrightarrow{V_n} \bigoplus_{m \geq 1} \mathbf{W}_{mn} \Omega_{(A[t], (t))}^{s-2m} \rightarrow \mathbf{Nil}_{s-2}(A[x]/(x^n)) \rightarrow \cdots$$

All rings are assumed commutative and unital without further notice.

1. Witt functors

1.1. A subset $S \subset \mathbb{N}$ is called a truncation set if it is stable under division, i.e. if $mn \in S$ then both $m \in S$ and $n \in S$. In particular, if S is non-empty then $1 \in S$. We denote by J the category of truncation sets and inclusions. It comes with a system of endo-functors $-/n: J \rightarrow J$, $n \geq 1$,

$$S \mapsto S/n = \{m \in \mathbb{N} : mn \in S\},$$

which is multiplicative in the sense that for all $m, n \geq 1$, $-/m \circ -/n = -/mn$. Note that the set S/n is non-empty if and only if $n \in S$. Every object S in J is a colimit of objects of the form

$$\langle n \rangle = \{d \in \mathbb{N} \mid d \text{ divides } n\}.$$

For every S in J , we have the big Witt ring $\mathbf{W}_S(A)$. As a set $\mathbf{W}_S(A) = A^S$, and the ring structure is characterized by the requirement that the ghost map

$$w: \mathbf{W}_S(A) \rightarrow A^S, \quad w_n(\bar{a}) = \sum_{d|n} d \bar{a}_d^{n/d},$$

be a natural ring homomorphism. There are natural maps

$$F_n: \mathbf{W}_S(A) \rightarrow \mathbf{W}_{S/n}(A), \quad V_n: \mathbf{W}_{S/n}(A) \rightarrow \mathbf{W}_S(A),$$

characterized by the formulas

$$F_n(\bar{w})_m = w_{mn}, \quad V_n(\bar{a})_m = \begin{cases} a_{m/n}, & \text{if } n|m, \\ 0, & \text{otherwise.} \end{cases}$$

The Teichmüller character is the multiplicative map

$$\underline{_S}: A \rightarrow \mathbf{W}_S(A)$$

with $\underline{a}_S \in A^S$ the function function that maps $1 \in S$ to a and the rest of S to zero.

Let DGA denote the category of differential graded algebras over \mathbb{Z} and let A be a ring. The de Rham-Witt complex $\mathbf{W}_S \Omega_A^*$ to be constructed below is the universal example of the following structure.

DEFINITION 1.1.1. A *Witt functor* over a A is a functor

$$M: J^{\text{op}} \rightarrow DGA,$$

which takes colimits to limits, together with, for every $n \geq 1$, additive natural transformations

$$F_n: M(S) \rightarrow M(S/n), \quad V_n: M(S/n) \rightarrow M(S),$$

such that F_n is a map of graded rings and V_n is a map of graded $M(S)$ -modules, when $M(S/n)$ is considered an $M(S)$ -module via F_n , and such that for all $m, n \geq 1$,

$$(1.1.2) \quad \begin{aligned} F_1 &= V_1 = \text{id}, & F_m F_n &= F_{mn}, & V_m V_n &= V_{mn}, \\ F_n V_n &= n, & F_m V_n &= V_n F_m & \text{if } (m, n) &= 1, \\ F_m dV_n &= kdF_{m/c}V_{n/c} + lF_{m/c}V_{n/c}d, \end{aligned}$$

where in the bottom line $c = (m, n)$ is the greatest common divisor and k and l is any pair of integers such that $km + ln = c$. In addition, it is required that as a functor to rings $M(S)^0 = \mathbf{W}_S(A)$ such that F_n and V_n extend the Frobenius and Verschiebung, respectively, and that for all $a \in A$,

$$(1.1.3) \quad F_n d\underline{a}_S = \underline{a}_{S/n}^{n-1} d\underline{a}_{S/n},$$

where $\underline{_S}: A \rightarrow \mathbf{W}_S(A)$ is the Teichmüller character. A map of Witt functors is a natural transformation which preserves all the relevant structure and which is the identity in degree zero.

The following relations are valid for every Witt functor

$$(1.1.4) \quad dF_n = nF_n d, \quad V_n d = n dV_n, \quad V_n(xdy) = V_n(x)dV_n y,$$

for all $x, y \in M(S/n)$. Indeed, $V_n(xdy) = V_n(xF_n dV_n y) = V_n(x)dV_n y$ and

$$\begin{aligned} V_n dx &= V_n(1)dV_n x = V_n(x)dV_n(1) + V_n(1)dV_n x \\ &= d(V_n(1)V_n x) = dV_n(F_n V_n x) = n dV_n x. \end{aligned}$$

The first relation is proved similarly. We note that since a Witt functor takes colimits to limits, it is determined by its values on the truncation sets $\langle n \rangle$, $n \geq 1$. This also implies that $M(\emptyset) = 0$ is the trivial ring concentrated in degree zero.

More generally, if T is a truncation set, we let $J_T = (J \downarrow T)$ be the category over T . The projection functor $J_T \rightarrow J$ is a full embedding which identifies J_T with the full subcategory of J which consists of all truncation sets $S \subset T$. We then define a *T-Witt functor* over A to be a functor

$$M: J_T^{\text{op}} \rightarrow DGA$$

which takes colimits to limits, together with additive natural transformations

$$F_n: M(S) \rightarrow M(S/n), \quad V_n: M(S/n) \rightarrow M(S)$$

subject to the same requirements as above.

For example, a $\{1\}$ -Witt functor over A is the same as a DGA whose degree zero term is equal to A , and the trivial ring 0 is the unique \emptyset -Witt functor. More importantly, for every prime p , we have the truncation set

$$P = \{1, p, p^2, \dots\}.$$

We call a P -Witt functor a p -typical Witt functor. For $T \subset U$ and $M: J_U^{\text{op}} \rightarrow DGA$ a U -Witt functor, we get a T -Witt functor i^*M by restriction.

PROPOSITION 1.1.5. *For every pair of truncation sets $U \subset T$, there is an adjunction $i_* \dashv i^*$,*

$$\{U\text{-Witt functors over } A\} \begin{matrix} \xrightarrow{i_*} \\ \xleftarrow{i^*} \end{matrix} \{T\text{-Witt functors over } A\},$$

where i^* is the forgetful functor.

PROOF. We use the Freyd adjoint functor theorem to prove the existence of a left adjoint, see [6]. The category \mathcal{W}_A^T of T -Witt functors over A obviously has all limits, and i^* preserves limits. We verify the solution set condition. Let $f: N \rightarrow i^*M$ be a map in \mathcal{W}_A^U . We shall define an S -Witt functor

$$\text{im}_S f: J_S^{\text{op}} \rightarrow DGA,$$

for all sub-truncation sets $S \subset T$. If $S \subset U$, we set $(\text{im}_S f)(S') = f(N(S'))$, for all $S' \subset S$. Suppose that $S - (S \cap U)$ is finite and assume inductively that $\text{im}_Q f$ has been defined, for all proper sub-truncation sets $Q \subset S$. We define $\text{im}_S f$ as follows: if $S' \subset S$ is a proper subset, we set $(\text{im}_S f)(S') = (\text{im}_{S'} f)(S')$, and $(\text{im}_S f)(S) \subset M(S)$ is defined to be the smallest DGA which contains both $V_n((\text{im}_{S/n} f)(S/n))$, for all $n > 1$, and the image of the Teichmüller character $\underline{-}_S: A \rightarrow \mathbf{W}_S(A)$. To prove that the functor $\text{im}_S f$ so defined is an S -Witt functor, we must show that $(\text{im}_S f)(S)^0 = \mathbf{W}_S(A)$ and that $F_n(\text{im}_S f(S)) \subset \text{im}_S f(S/n)$, for all $n \geq 1$. The first requirement follows from the fact that every $x \in \mathbf{W}_S(A)$ may be written (uniquely) as a sum

$$x = \sum_{n \in S} V_n \underline{a}_{n, S/n},$$

where $a_n \in A$ are the Witt coordinates of x . The second follows readily from (1.1.2), (1.1.3) and the fact that F_n is multiplicative. Finally, for a general $S \subset T$, we define $(\text{im}_S f)(S')$ as the limit of $(\text{im}_Q f)(Q)$ as Q ranges over the sub-truncation sets $Q \subset S'$ such that $Q - (Q \cap U)$ is finite.

By construction, there is a canonical map of T -Witt functors

$$g: \text{im}_T f \rightarrow M$$

such that $f: N \rightarrow i^*M$ factors through $i^*g: i^*\text{im}_T f \rightarrow i^*M$. Given N in \mathcal{W}_A^U , there is clearly only a set worth of isomorphism classes of T -Witt functors of the form $\text{im}_T f$ for some $f: N \rightarrow i^*M$. Hence the solution set condition is satisfied. \square

Taking $U = \emptyset$, we see that the category of T -Witt functors over A has an initial object, namely, i_*0 , where 0 is the unique \emptyset -Witt functor.

DEFINITION 1.1.6. The universal T -Witt functor over A is denoted

$$S \mapsto \mathbf{W}_S^T \Omega_A^*$$

and called the T -de Rham-Witt complex of A .

1.2. We will now study p -typical Witt functors more closely. Let us first restate the definition slightly different. A p -typical Witt functor is a functor

$$M: J_p^{\text{op}} \rightarrow DGA,$$

which takes colimits to limits, together with two additive natural transformations

$$F: M(S) \rightarrow M(S/p), \quad V: M(S/p) \rightarrow M(S),$$

such that F is a map of graded rings, V is a map of $M(S)$ -modules when $M(S/p)$ is considered an $M(S)$ -module via F , and such that

$$FV = p, \quad FdV = d.$$

Moreover, it is required that as a functor to rings $M(S)^0 = \mathbf{W}_S(A)$ such that F and V extend the Frobenius F_p and Verschiebung V_p , respectively, and that for all $a \in A$,

$$Fd\underline{a}_S = \underline{a}_{S/p} d\underline{a}_{S/p}.$$

We write $W_S \Omega_A^*$ instead of $\mathbf{W}_S^P \Omega_A^*$ and call it the p -typical de Rham-Witt complex of A . If A is an \mathbb{F}_p -algebra, this agrees with the de Rham-Witt complex of Deligne-Illusie, [5, I.1.3, I.2.17].

Let A be a $\mathbb{Z}_{(p)}$ -algebra and, for a truncation set S , let

$$I(S) = \{k \in S \mid (k, p) = 1\}.$$

Then the Witt ring decomposes

$$(1.2.1) \quad \mathbf{W}_S(A) = \prod_{k \in I(S)} \mathbf{W}_S(A) e_k,$$

with

$$e_k = \prod_{l \in I(S/k) \setminus \{1\}} \left(\frac{1}{k} V_k(1) - \frac{1}{kl} V_{kl}(1) \right).$$

Indeed, the ghost components for $V_k(1)$ are $w_n(V_k(1)) = k$, if $k|n$, and 0 otherwise, so $w(\frac{1}{k} V_k(1))$ is the indicator function $1_{S \cap k\mathbb{N}}$ and hence $w(e_k) = 1_{S \cap kP}$. Also,

$$F_k(e_{km}) = e_m, \quad V_k(e_m) = ke_m.$$

Moreover, the k th factor in (1.2.1) may be identified via the composite

$$\mathbf{W}_S(A) e_k \hookrightarrow \mathbf{W}_S(A) \xrightarrow{F_k} \mathbf{W}_{S/k}(A) \xrightarrow{R_{S/k \cap P}^{S/k}} \mathbf{W}_{S/k \cap P}(A)$$

which is an isomorphism. We define two new functors

$$(1.2.2) \quad \{p\text{-typical Witt functors over } A\} \xrightleftharpoons[i^!]{i_!} \{\text{Witt functors over } A\}$$

If L is a DGA over $\mathbb{Z}_{(p)}$ and k a natural number prime to p , we write $L(1/k)$ for the graded algebra L with the differential d replaced by $(1/k)d$. Let $N: J_P^{\text{op}} \rightarrow DGA$ be a p -typical Witt functor over A . Then $i_!N: J \rightarrow DGA$ is the functor

$$(1.2.3) \quad i_!N(S) = \prod_{k \in I(S)} N(S/k \cap P)(1/k)$$

and the natural transformations

$$\begin{aligned} F_n &: i_!N(S) \rightarrow i_!N(S/n), \\ V_n &: i_!N(S/n) \rightarrow i_!N(S), \end{aligned}$$

are defined as follows: write $n = p^s h$ with $(h, p) = 1$. Then F_n takes a factor $k = hl$ to the factor l by the map F^s and annihilates the remaining factors. Similarly, V_n takes the factor l to the factor $k = hl$ by the map hV^s . It follows from (1.2.1) that $i_!N(S)^0 = \mathbf{W}_S(A)$, and given this, one readily verifies that $i_!N$ is a Witt functor.

Conversely, for $M: J^{\text{op}} \rightarrow DGA$ a Witt functor, define $i^!M: J_P^{\text{op}} \rightarrow DGA$ by

$$(1.2.4) \quad i^!M(S) = M([S])e_1,$$

where $[S] = \{ks \in \mathbb{N} \mid s \in S, (k, p) = 1\}$ is the union of all truncation sets T with $T \cap P = S$. Then $i^!M$ is a p -typical Witt functor.

PROPOSITION 1.2.5. *Let A be a $\mathbb{Z}_{(p)}$ -algebra. Then there are adjunctions*

$$i_* \dashv i^* \dashv i_! \dashv i^!.$$

PROOF. The adjunction $i_* \dashv i^*$ follows from 1.1.5, and the composites $i^*i_!$ and $i^!i_!$ are both the identity. Indeed, by construction $i_!N$ has the property that $i_!N(T)e_k = N(T/k \cap P)$. We define $\eta: M(S) \rightarrow i_!i^*M(S)$ to be the map which on the k th factor is given by

$$M(S) \xrightarrow{F_k} M(S/k)(1/k) \xrightarrow{R_{S/k \cap P}^{S/k}} M(S/k \cap P)(1/k).$$

Then $i_! \dashv i^*$ is easily verified. Next, we define $\epsilon: i_!i^!M(S) \rightarrow M(S)$ as the composite

$$\begin{aligned} i_!i^!M(S) &= \prod_{k \in I(S)} M([S/k \cap P])e_1(1/k) \rightarrow \prod_{k \in I(S)} M([S/k \cap P])e_k \\ &\rightarrow \prod_{k \in I(S)} M(S)e_k = M(S), \end{aligned}$$

with the first map given by

$$\frac{1}{k}V_k: M([S])e_1(1/k) \rightarrow M([S])e_k$$

and with the second map induced from the inclusion $S \subset [S/k \cap P]$. Again the adjunction $i_! \dashv i^!$ is easily checked. \square

It follows that the functors i_* , i^* and $i_!$ all preserve colimits. In particular, they preserve initial objects and hence we have

COROLLARY 1.2.6. *There are canonical isomorphisms*

$$\begin{aligned} \mathbf{W}_S \Omega_A^* &= i_*(W_- \Omega_A^*)(S) = i_!(W_- \Omega_A^*)(S), \\ W_S \Omega_A^* &= i^*(\mathbf{W}_- \Omega_A^*)(S), \end{aligned}$$

valid for any $\mathbb{Z}_{(p)}$ -algebra A .

One may well expect that the last equality hold for every ring A . More generally, one would like that whenever $T \subset U$, the universal U -Witt functor restricts to the universal T -Witt functor. We leave this as an open question.

For the convinience of the reader, we spell out the statement of the corollary in the case of the truncated de Rham-Witt complexes which appear in theorems A and B of the introduction. First, for every natural number m , we have the truncation set $\underline{m} = \{l \in \mathbb{N} \mid l \leq m\}$, and we write

$$\mathbf{W}_m \Omega_A^* = \mathbf{W}_{\underline{m}} \Omega_A^*.$$

The set $I(\underline{m})$ is equal to the set of natural numbers d which are prime to p and less than or equal to m , and $\underline{m}/d \cap P = \{p^i \mid p^i d \leq m\}$. Finally, in the notation of [5],

$$\mathbf{W}_{\underline{m}/d \cap P} \Omega_A^* = W_s \Omega_A^*,$$

where $s = s(m, d) = \max\{i \mid p^i d \leq m\} + 1$. See also [3, pp. 95–96].

2. Truncated polynomial algebras

2.1. Let \mathbb{T} denote the circle group and let $C_r \subset \mathbb{T}$ denote the cyclic subgroup of order r . The topological Hochschild spectrum $T(A)$ is a \mathbb{T} -spectrum indexed on a complete universe \mathcal{U} . The reader is referred to [4] for the definition. Let $j: \mathcal{U}^{\mathbb{T}} \rightarrow \mathcal{U}$ be the inclusion of the trivial universe. We shall mostly be concerned with the underlying naive \mathbb{T} -spectrum $j^*T(A)$ but will not distinguish in notation. The obvious inclusion maps

$$F_r: T(A)^{C_{r^s}} \rightarrow T(A)^{C_s}$$

are accompanied by transfer maps going in the opposite direction,

$$V_r: T(A)^{C_s} \rightarrow T(A)^{C_{r^s}}.$$

We call these maps the r th Frobenius and Verschiebung, respectively.

In addition, $T(A)$ is *cyclotomic*. The cyclotomic structure gives a map

$$R_r: T(A)^{C_{r^s}} \rightarrow T(A)^{C_s},$$

called the r th restriction. It has the following equivariance property: Let $C_r \subset \mathbb{T}$ be a subgroup of order r and let $\rho_r: \mathbb{T} \rightarrow \mathbb{T}/C_r$, $\rho_r(z) = z^{1/r} C_r$, be the root isomorphism. If we view the naive \mathbb{T}/C_r -spectrum $T(A)^{C_r}$ as a naive \mathbb{T} -spectrum $\rho_r^*T(A)^{C_r}$ via ρ_r , then R_r is a map of \mathbb{T} -spectra

$$R_r: \rho_{r^s}^*T(A)^{C_{r^s}} \rightarrow \rho_s^*T(A)^{C_s}.$$

We define a functor

$$(2.1.1) \quad \mathbf{M}: J^{\text{op}} \rightarrow \text{DGA}$$

as follows. Its value on the elementary truncation set $\langle r \rangle$ is the graded abelian group

$$\mathbf{M}(\langle r \rangle) = \pi_* T(A)^{C_r},$$

with a differential to be specified shortly, and $\mathbf{M}(\langle rs \rangle) \rightarrow \mathbf{M}(\langle s \rangle)$ is the map of graded abelian group induced by R_r . To define the differential, let $\sigma, \eta \in \pi_1^S(\mathbb{T}_+)$ be the generators which under the obvious collaps maps restrict to $\text{id} \in \pi_1^S(\mathbb{T})$ and $\eta \in \pi_1^S(S^0)$, respectively, and consider the maps

$$\delta, \iota: \pi_i(\rho_r^*T(A)^{C_r}) \xrightarrow{\sigma, \eta} \pi_{i+1}(\mathbb{T}_+ \wedge \rho_r^*T(A)^{C_r}) \xrightarrow{\mu} \pi_{i+1}(\rho_r^*T(A)^{C_r}).$$

The left hand map is given by exterior multiplication by σ and η , respectively, and the right hand map is induced from the action by \mathbb{T} . One easily verifies that ι is equal to multiplication by η and that $\delta \circ \delta = \iota \circ \delta = \delta \circ \iota$. It follows that

$$d: \mathbf{M}(\langle n \rangle) \rightarrow \mathbf{M}(\langle n \rangle), \quad dx = \delta x + |x|\iota,$$

is a differential. Standard equivariant homotopy theory shows that δ is a derivation for the product on $\mathbf{M}(\langle n \rangle)$; hence so is d . We extend \mathbf{M} to general truncation sets by continuity,

$$\mathbf{M}(S) = \varprojlim \mathbf{M}(\langle n \rangle)$$

with the limit running over $n \in S$. The Frobenius and Verschiebung maps on $\mathbf{M}(\langle n \rangle)$ induce natural transformations

$$F_n: \mathbf{M}(S) \rightarrow \mathbf{M}(S/n), \quad V_n: \mathbf{M}(S/n) \rightarrow \mathbf{M}(S).$$

PROPOSITION 2.1.2. *The functor $\mathbf{M}: J^{\text{op}} \rightarrow \text{DGA}$ is a Witt functor. In particular, there is a preferred map*

$$\mathbf{W}_S \Omega_A^* \rightarrow \mathbf{M}(S).$$

PROOF. We proved in [4, addendum 3.3] that there is a canonical isomorphism

$$\mathbf{M}(S)^0 \xrightarrow{\sim} \mathbf{W}_S(A),$$

compatible with restriction, Frobenius and Verschiebung. The relation (1.1.3) was proved in [1, lemma 1.5.6], and the relations (1.1.2), except for the last one, are easy consequences of the fact that, for every G -ring spectrum, the functor which takes G/H to $\pi_* T^H$ is a Green functor. It remains to prove last relation in (1.1.2). The proof is similar to the proof given in [1, lemma 1.5.1] of the case $m = n$, where the relation reads $F_n dV_n = d$. We leave the general case to the reader. See also [2, 3.2.1]. \square

LEMMA 2.1.3. *If A is a $\mathbb{Z}_{(p)}$ -algebra then the Witt functor of proposition 2.1.2 is of the form $i_! \mathbf{N}$.*

PROOF. Let \mathbf{N} be the P -Witt functor defined by continuity from $\pi_* T(A)^{C_{p^s}}$. The lemma then follows from [3, proposition 4.2.5]. \square

2.2. Let k be a perfect field of characteristic $p > 0$ and let A be a k -algebra. It was proved in [3, 4.2.10] that there is a natural isomorphism

$$\pi_* \underset{\leftarrow R}{\text{holim}} T(k)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s} \cong \bigoplus_{m \geq 0} \mathbf{W}_{(m+1)n}(k)[2m].$$

Here and below, if M is a graded module, we write $M[i]$ for the i th suspension given by $M[i]_j = M_{j-i}$. The pairing

$$\underset{\leftarrow R}{\text{holim}} T(A)^{C_s} \wedge \underset{\leftarrow R}{\text{holim}} T(k)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s} \rightarrow \underset{\leftarrow R}{\text{holim}} T(A)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s}$$

and canonical map of proposition 2.1.2 induces a pairing

$$\mathbf{W}_S \Omega_A^* \otimes_{\mathbf{W}(k)} \pi_* \underset{\leftarrow R}{\text{holim}} T(k)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s} \rightarrow \pi_* \underset{\leftarrow R}{\text{holim}} T(A)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s},$$

where $\mathbf{W}\Omega_A^*$ is the limit of all $\mathbf{W}_{(n)}\Omega_A^*$. The $\mathbf{W}(k)$ -generator $\iota_{2m} \in \mathbf{W}_{(m+1)n}(k)[2m]$ defines a $\mathbf{W}(A)$ -linear map

$$(2.2.1) \quad \iota_{2m} : \mathbf{W}\Omega_A^*[2m] \rightarrow \pi_* \mathop{\mathrm{holim}}_{\leftarrow R} T(A)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s}.$$

We shall prove

THEOREM 2.2.2. *Let k be a perfect field of characteristic $p > 0$ and let A be a smooth k -algebra. Then the sum of the maps in (2.2.1) factors to a natural isomorphism*

$$\bigoplus_{m \geq 0} \mathbf{W}_{(m+1)n}\Omega_A^*[2m] \xrightarrow{\sim} \pi_* \mathop{\mathrm{holim}}_{\leftarrow R} T(A)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s}.$$

The proof which occupies the rest of the paragraph is based on [1] and [3]. It has two parts listed separately below. The first part is calculational and verifies the theorem when $A = k[x_1, \dots, x_d]$. The second part uses standard covering techniques for smooth algebras as in [1] and [5].

LEMMA 2.2.3. *The theorem holds for $A = k[x_1, \dots, x_d]$.*

PROOF. We prove the lemma for $A = \mathbb{F}_p[x]$ leaving the many variable case to the reader. It is only notationally more complicated, compare [1, §2.2]. The extension to a general perfect coefficient field of characteristic $p > 0$ is proved in a manner similar to op.cit. (2.4.5). It is convenient to break up the statement in its p -typical components. Then by [3, proposition 4.2.5] and corollary 1.2.6 above it suffices to prove that the maps (2.2.1) induce an isomorphism

$$\bigoplus_{m \geq 0} W_{s(m,d)}\Omega_A^*[2m] \xrightarrow{\sim} \pi_* \mathop{\mathrm{holim}}_{\leftarrow R} T(A)_{V_{\lfloor \frac{p^r d-1}{n} \rfloor}}^{C_{p^r}},$$

where $s(m, d) = \max\{i \mid p^i d \leq (m+1)n\}$.

Let us write V_r for $V_{\lfloor \frac{p^r d-1}{n} \rfloor}$. Then $V_r^{C_p} = V_{r-1}$ and there is a \mathbb{T} -equivalence

$$T(A)_{V_r} \simeq \bigvee_{s \geq 1} T(k)_{V_r} \wedge S^1(s)_+,$$

where $S^1(s)$ is the unit circle in the representation $\mathbb{C}^{\otimes s}$. Let $\rho_{p^r} : S^1 \rightarrow S^1/C_{p^r}$ be the p^r th root. Then we get a \mathbb{T} -equivalence

$$\begin{aligned} \rho_{p^r}^\# T(A)_{V_r}^{C_{p^r}} &\xrightarrow{\sim} \bigvee_{(l,p)=1} \bigvee_{u=r}^{\infty} \rho_{p^r}^\# T(\mathbb{F}_p)_{V_r}^{C_{p^r}} \wedge S^1(p^{u-r}l)_+ \vee \\ &\bigvee_{(l,p)=1} \bigvee_{u=0}^{r-1} \rho_{p^u}^\# (\rho_{p^{r-u}}^\# T(\mathbb{F}_p)_{V_r}^{C_{p^{r-u}}} \wedge_{C_{p^u}} S^1(l)_+). \end{aligned}$$

Moreover, there is a \mathbb{T} -equivalence

$$\rho_{C_{p^{r-u}}}^* T(\mathbb{F}_p)_{V_r}^{C_{p^{r-u}}} \wedge_{C_{p^u}} S^1(l)_+ \xrightarrow{\sim} |T(\mathbb{F}_p)_{V_r}^{C_{p^{r-u}}}| \wedge S^1(l)/C_{p^u}_+,$$

where the bars indicate trivial action. Indeed, the l th power map $\Delta_l : S^1(1) \rightarrow S^1(l)$ is a p -local homotopy equivalence, and we have the isomorphism

$$|T(\mathbb{F}_p)_{V_r}^{C_{p^{r-u}}}| \wedge S^1(1)_+ \xrightarrow{\sim} \rho_{p^{r-u}}^\# T(\mathbb{F}_p)_{V_r}^{C_{p^{r-u}}} \wedge S^1(1)_+$$

which maps (t, z) to (tz^{-1}, z) . The restriction map

$$R: T(A)_{V_r}^{C_{p^r}} \rightarrow T(A)_{V_{r-1}}^{C_{p^{r-1}}}$$

is the identity on the circle factors in the above decomposition, and equal to the restriction

$$R: T(\mathbb{F}_p)_{V_r}^{C_{p^{r-u}}} \rightarrow T(\mathbb{F}_p)_{V_{r-1}}^{C_{p^{r-1-u}}}, \quad 0 \leq u \leq r,$$

on the first factor in each sum. Finally, by [4, proposition 9.1],

$$\pi_{2m} \operatorname{holim}_{\leftarrow R} T(\mathbb{F}_p)_{V_r}^{C_{p^{r-u}}} \cong W_{s-u}(\mathbb{F}_p)$$

with $s = s(m, d)$. The spectrum $\operatorname{holim}_{\leftarrow} T(\mathbb{F}_p)_{V_r}^{C_{p^{r-u}}}$ is a module over $K(\mathbb{F}_p)_p^\wedge$ so is a wedge of the Eilenberg-MacLane spectra $H(W_{s(m,d)-u}(\mathbb{F}_p), 2m)$. Let us name generators as follows:

$$\begin{aligned} H_*(S^1(l)/C_{p^u}, \mathbb{Z}_{(p)}) &= \mathbb{Z}_{(p)} \{x^{l/p^u}, x^{l/p^u} d \log x\}, \\ H_*(S^1(p^{u-r}l), \mathbb{Z}_{(p)}) &= \mathbb{Z}_{(p)} \{x^{p^{u-r}l}, x^{p^{u-r}l} d \log x\}. \end{aligned}$$

Then we have

$$\begin{aligned} \pi_* \operatorname{holim}_{\leftarrow R} T(A)_{V_r}^{C_{p^r}} &\cong \bigoplus_{m \geq 0} (W_s(\mathbb{F}_p) \{x^k, x^k d \log x \mid v_p(k) \geq 0, k > 0\})[2m] \oplus \\ &\bigoplus_{m \geq 0} \left(\bigoplus_{u=1}^{s-1} W_s(\mathbb{F}_p) \{x^k, x^k d \log x \mid v_p(k) = -u\} \right)[2m], \end{aligned}$$

with $s = s(m, d)$. Since $W_t(\mathbb{F}_p) = \mathbb{Z}/p^t$ the right hand side is precisely equal to the direct sum of p -typical de Rham-Witt complexes,

$$\pi_* \operatorname{holim}_{\leftarrow R} T(A)_{V_r}^{C_{p^r}} \cong \bigoplus_{m \geq 0} W_{s(m,d)} \Omega_{\mathbb{F}_p[x]}^*[2m],$$

compare [1, §2.1] or [5]. It follows that the two sides of the statement in theorem 2.2.2 above are abstractly isomorphic. One argues as in [1, §2] that the stated map induces an isomorphism. \square

We next consider étale extensions, following [1] and [5]. If $A \rightarrow B$ is étale then so is $\mathbf{W}_S(A) \rightarrow \mathbf{W}_S(B)$, and for any pair of truncation sets $S \subset T$, there is a natural isomorphism

$$(2.2.4) \quad \mathbf{W}_T(B) \otimes_{\mathbf{W}_T(A)} \mathbf{W}_S \Omega_A^* \xrightarrow{\sim} \mathbf{W}_S \Omega_B^*,$$

see [5, p. 513, 549]. Let $\operatorname{Fil}^S \mathbf{W}_T \Omega_A^*$ denote the kernel of the restriction,

$$0 \rightarrow \operatorname{Fil}^S \mathbf{W}_T \Omega_A^* \rightarrow \mathbf{W}_T \Omega_A^* \rightarrow \mathbf{W}_S \Omega_A^* \rightarrow 0.$$

Then, more generally, (2.2.4) and the fact that étale maps are flat implies that the natural map

$$(2.2.5) \quad \mathbf{W}_T(B) \otimes_{\mathbf{W}_T(A)} \operatorname{Fil}^S \mathbf{W}_T \Omega_A^* \xrightarrow{\sim} \operatorname{Fil}^S \mathbf{W}_T \Omega_B^*$$

is an isomorphism.

Let us define

$$\mathbf{V}_{\langle s \rangle}^*(A) = \pi_* T(A)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s},$$

and extend to all truncation sets by continuity. An argument similar to [1, proposition 2.4.4] and [3, proposition 4.2.5] shows that for $A \rightarrow B$ étale,

$$(2.2.6) \quad \mathbf{W}_T(B) \otimes_{\mathbf{W}_T(A)} \mathbf{V}_S^*(A) \xrightarrow{\sim} \mathbf{V}_S^*(B).$$

We write

$$\mathbf{V}^*(A) = \varprojlim_{\langle s \rangle} \mathbf{V}_{\langle s \rangle}^*(A) = \varprojlim \mathbf{V}_T^*(A),$$

and recall that by [3, 4.2.7], the projection

$$(2.2.7) \quad \mathbf{V}^i(A) \rightarrow \mathbf{V}_T^i(A)$$

is an isomorphism for $i \leq [(p^{v_p(T)} - 1)/n]$. Here $v_p(T)$ denote the maximum of the p -adic valuations of elements of T . Suppose that A is a smooth k -algebra. Then the complex $\mathbf{W}\Omega_A^*$ is bounded, and hence (2.2.7) implies that the map

$$\iota_{2m}: \mathbf{W}\Omega_A^*[2m] \rightarrow \mathbf{V}^*(A)$$

factors over $\mathbf{W}_T\Omega_A^*[2m]$, for some finite T .

LEMMA 2.2.8. *Let $A = k[x_1, \dots, x_d]$ and let $A \rightarrow B$ be an étale map. Then the map (2.2.1) factors to a map*

$$\iota_{2m}: \mathbf{W}_{(m+1)n}\Omega_B^*[2m] \rightarrow \pi_* \underset{R}{\text{holim}} T(B)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^C.$$

PROOF. The statement is true for $B = A$ by lemma 2.2.3. We have isomorphisms

$$\begin{aligned} \text{Fil}^S \mathbf{W}\Omega_B^* &\xrightarrow{\sim} \varprojlim \text{Fil}^S \mathbf{W}_T\Omega_B^*, \\ \mathbf{V}^*(B) &\xrightarrow{\sim} \varprojlim \mathbf{W}_T(B) \otimes_{\mathbf{W}_T(A)} \mathbf{V}_T^*(A), \end{aligned}$$

and the lemma follows from 2.2.5 with $S = \underline{(m+1)n}$. \square

A k -algebra A is smooth if there exists relatively prime elements f_1, \dots, f_r such that the localizations $A_{f_i} = A[1/f_i]$ are étale extensions of a polynomial algebra $k[x_1, \dots, x_d]$. Consider $\mathbf{W}_T(A) \rightarrow \mathbf{W}_T(A_{f_i})$ as a cochain complex with the left hand term located in degree zero. Then the tensor complex

$$(2.2.9) \quad \bigotimes_{i=1}^r \mathbf{W}_T(A) (\mathbf{W}_T(A) \rightarrow \mathbf{W}_T(A_{f_i}))$$

is acyclic and flat over $\mathbf{W}_T(A)$, see [1, lemma 2.4.6]. Tensoring this complex with $\mathbf{W}\Omega_A^*$ over $\mathbf{W}_T(A)$ we thus get an exact sequence

$$0 \rightarrow \mathbf{W}_T\Omega_A^* \rightarrow \bigoplus_{i=1}^r \mathbf{W}_T\Omega_{A_{f_i}}^* \rightarrow \bigoplus_{i,j=1}^r \mathbf{W}_T\Omega_{A_{f_i f_j}}^* \rightarrow \dots$$

and similarly with $\mathbf{V}^*(A)$ in place of $\mathbf{W}\Omega_A^*$. (This uses that

$$\mathbf{W}_T(A_{f_i}) \otimes_{\mathbf{W}_T(A)} \mathbf{W}_T(A_{f_j}) = \mathbf{W}_T(A_{f_{ij}}),$$

which, in turn, is an immediate consequence of the fact that $\mathbf{W}_T(A_f) = \mathbf{W}_T(A)_{\underline{f}_T}$, see [5].) In particular, the horizontal maps in the diagram

$$\begin{array}{ccc} \mathbf{W}\Omega_A^* & \longrightarrow & \bigoplus_{i=1}^r \mathbf{W}\Omega_{A_{f_i}}^*[2m] \\ \downarrow \iota_{2m} & & \downarrow \iota_{2m} \\ \mathbf{V}^*(A) & \longrightarrow & \bigoplus_{i=1}^r \mathbf{V}^*(A_{f_i}) \end{array}$$

are injective. Indeed, taking limits is left exact. We conclude from lemma 2.2.8 that the left hand vertical map factors to

$$\iota_{2m}: \mathbf{W}_{(m+1)n}\Omega_A^*[2m] \rightarrow \pi_* \underset{\leftarrow R}{\operatorname{holim}} T(A)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s/n}.$$

Finally, the sum of the exact sequences

$$0 \rightarrow \mathbf{W}_{(m+1)n}\Omega_A^*[2m] \rightarrow \bigoplus_i \mathbf{W}_{(m+1)n}\Omega_{A_{f_i}}^*[2m] \rightarrow \bigoplus_{i,j} \mathbf{W}_{(m+1)n}\Omega_{A_{f_{ij}}}^*[2m]$$

for $m \geq 0$ maps to the exact sequence

$$0 \rightarrow \mathbf{V}^*(A) \rightarrow \bigoplus_i \mathbf{V}^*(A_{f_i}) \rightarrow \bigoplus_{i,j} \mathbf{V}^*(A_{f_{ij}}),$$

and the maps of the middle and right hand terms are isomorphisms. But then so is the left hand map. This finishes the proof of theorem 2.2.2.

We shall also need to know the following result.

THEOREM 2.2.10. *With the assumptions of 2.2.2 there is a natural isomorphism*

$$\iota_{2m}: \bigoplus_{m \geq 0} \mathbf{W}_{m+1}\Omega_A^*[2m] \xrightarrow{\sim} \pi_* \underset{\leftarrow R}{\operatorname{holim}} T(A)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s/n}.$$

PROOF. The proof, given [3, theorem 4.2.10], is entirely similar to the proof of theorem 2.2.2 above. \square

2.3. We can now prove theorems A and B of the introduction. The relative term $K(A[x]/(x^n), (x))$ is defined by the split cofibration sequence

$$K(A[x]/(x^n), (x)) \rightarrow K(A[x]/(x^n)) \rightarrow K(A), \quad x \mapsto 0,$$

and similarly for topological cyclic homology. A theorem of McCarthy, [7], implies that the cyclotomic trace induces an equivalence

$$K(A[x]/(x^n), (x)) \xrightarrow{\sim} \operatorname{TC}(A[x]/(x^n), (x)).$$

Indeed, it follows from results from [4] that both terms already are p -complete. On the other hand, from [3, proposition 4.2.3] we have the cofibration sequence

$$\Sigma \underset{\leftarrow R}{\operatorname{holim}} T(A)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s/n} \xrightarrow{V_n} \Sigma \underset{\leftarrow R}{\operatorname{holim}} T(A)_{V_{\lfloor \frac{s-1}{n} \rfloor}}^{C_s/n} \rightarrow \operatorname{TC}(A[x]/(x^n), (x)),$$

and theorems 2.2.2 and 2.2.10 above identifies the left hand and middle terms. Moreover, using the proof of [3, theorem 4.2.10], one identifies the map V_n with the map induced from the Verschiebung

$$V_n: \mathbf{W}_{m+1}\Omega_A^* \rightarrow \mathbf{W}_{(m+1)n}\Omega_A^*.$$

This completes the proof of theorem A.

To prove theorem B, recall that for any ring Λ , one defines $NK(\Lambda)$ by the split cofibration sequence

$$NK(\Lambda) \rightarrow K(\Lambda[t]) \rightarrow K(\Lambda), \quad t \mapsto 0.$$

The fundamental theorem in algebraic K -theory shows that $\text{Nil}_*(\Lambda) = NK(\Lambda)[-1]$. Now the nil groups vanish for regular rings and a smooth algebra over a field is regular. Therefore in the case at hand,

$$NK(A[x]/(x^n), (x)) \xrightarrow{\sim} NK(A[x]/(x^n)),$$

and hence we have a split cofibration sequence

$$NK(A[x]/(x^n)) \rightarrow K(A[t, x]/(x^n), (x)) \rightarrow K(A[x]/(x^n), (x)), \quad t \mapsto 0.$$

Thus theorem B follows from theorem A.

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