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1. Introduction

It is wellknown that the equivariant KK-theory of Kasparov, [K], has excision with respect to equivariant extensions for which the quotient map admits a completely positive and equivariant section. In other words, when G is a second countable locally compact group, an extension of G-algebras,

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \longrightarrow 0, \tag{1.1}$$

with the property that there is a completely positive and equivariant linear map $s : A \to E$ such that $p \circ s = \mathrm{id}_A$, will give rise to the two six-term exact sequences

$$\begin{tikzcd} KK_G(D,B) \arrow[r] & KK_G(D,E) \arrow[r] & KK_G(D,A) \arrow[d] \\ KK^1_G(D,A) \arrow[u] & KK^1_G(D,E) \arrow[l] & KK^1_G(D,B) \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} KK_G(A,D) \arrow[r] & KK_G(E,D) \arrow[r] & KK_G(B,D) \arrow[d] \\ KK^1_G(B,D) \arrow[u] & KK^1_G(E,D) \arrow[l] & KK^1_G(A,D) \arrow[l] \end{tikzcd}$$

for any G-algebra D. This important property of equivariant KK-theory was established in [BS]. Already in the non-equivariant case it is in general necessary to have the completely positive section, cf. [S], but it was a question if it was necessary that the completely positive section be equivariant. This question has been addressed in [Ma], [C], and partial results - all indicating that equivariance may not be necessary - have been obtained in [CS], [BS] and [Ma]. The purpose with this note is to prove that equivariance of s is in fact not necessary. This will be done by proving the following theorem.

Theorem 1.1. *Let G be a second countable locally compact group. Let A and B be separable G-algebras and*

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \longrightarrow 0$$

a G-extension (i.e. a short exact sequence where all $*$-homomorphisms are equivariant). Assume that there is a completely positive linear map $s : A \to E$ such that $p \circ s = \mathrm{id}_A$. ($s$ need not be equivariant !) It follows that there is a completely positive

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equivariant linear contraction $r : A\otimes\mathbb{K}_G \to E\otimes\mathbb{K}_G$ *such that* $(p\otimes\mathrm{id}_{\mathbb{K}_G})\circ r = \mathrm{id}_{A\otimes\mathbb{K}_G}$. *In other words, the* G*-extension*

$$0 \longrightarrow B\otimes\mathbb{K}_G \longrightarrow E\otimes\mathbb{K}_G \xrightarrow{p\otimes\mathrm{id}_{\mathbb{K}_G}} A\otimes\mathbb{K}_G \longrightarrow 0 \tag{1.2}$$

is equivariantly semi-split.

$\mathbb{K}_G$ is here the G-algebra of compact operators on the countable direct sum of copies of $L^2(G)$, equipped with the G-action implemented by the direct sum of the left regular representation. Since $\mathbb{K}_G$ is equivalent to $\mathbb{C}$ in KK_G the general excision property of equivariant KK-theory follows from Theorem 1.1 and [BS].

Theorem 1.1 has other applications, both in K-theory and beyond. Since the notion of exactness for groups is currently under extensive study we mention the following corollary.

Corollary 1.2. *For any* G*-extension (1.1) which is semi-split in the sense that there is a completely positive linear map* $s : A \to E$ *such that* $p\circ s = \mathrm{id}_A$*, the sequence*

$$0 \longrightarrow B\rtimes_r G \longrightarrow E\rtimes_r G \longrightarrow A\rtimes_r G \longrightarrow 0,$$

is also exact and semi-split.

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2. Proofs

In the following G is a fixed second countable locally compact group.

Lemma 2.1. *There is a non-negative function* $\varphi \in C_c(G)$ *and a sequence* $\{\psi_n\}_{n=1}^\infty$ *of non-negative functions in* $C_c(G\times G)$ *such that,*

a) $\int_G \varphi(h)\int_G \psi_n(g^{-1}h,g)^2\ dg\ dh = 1$ *for all* $n\in\mathbb{N}$*, and*

b) *for any compact subset* $K\subseteq G$ *there is an* $N\in\mathbb{N}$ *such that*

$$\int_G \varphi(h)\int_G |\psi_n(k^{-1}h,gk)-\psi_n(h,g)|^2\ dg\ dh = 0$$

for all $k\in K$ *and all* $n\geq N$.

Proof. Let $J\in C_c(G)$ be a non-negative function such that $J(e)\neq 0$, where $e\in G$ is the neutral element. Define $\lambda : G\times G\to\mathbb{R}$ by

$$\lambda(h,s) = \int_G J(g^{-1}s^{-1})J(h^{-1}g)\ dg.$$

Then λ is continuous and non-negative. Furthermore,

$$\lambda(k^{-1}h,sk) = \lambda(h,s) \tag{2.1}$$

for all $h,s\in G$. Set $\varphi(h) = \lambda(h,e)$. φ is non-negative, has compact support and $\int_G\varphi(h)^2\ dh > 0$. Let $U_1\subseteq U_2\subseteq U_3\subseteq\cdots$ be a sequence of open sets in G with compact closures such that $\bigcup_n U_n = G$ and let $\lambda_n : G\to[0,1], n\in\mathbb{N}$, be a monotonely increasing sequence in $C_c(G)$ such that $\lambda_n(g) = 1, g\in U_n$. Set $l_n(h,s) = \lambda_n(s)\lambda(h,s)$. Each l_n is non-negative and has compact support. Set $\kappa_n = \sqrt{l_n}$. Then

$$\lim_{n\to\infty}\int_G\varphi(h)\int_G\kappa_n(g^{-1}h,g)^2\ dg\ dh > 0. \tag{2.2}$$

Indeed, by Lebesgues theorem on monotone convergence,

$$\lim_{n\to\infty} \int_G \varphi(h) \int_G \kappa_n(g^{-1}h, g)^2 \ dg \ dh = \int_G \varphi(h) \int_G \lambda(g^{-1}h, g) \ dg \ dh.$$

By (2.1) the latter equals $\int_G \varphi(h) \int_G \lambda(h, e) \ dg \ dh = m(G) \int_G \varphi(h)^2 \ dh$, where m is the Haar-measure. The last expression is strictly positive (and $= +\infty$ when G is not compact). It follows that if we ignore finitely many n's we can set $\psi_n(h, s) = \delta_n^{-1/2} \kappa_n(h, s)$ where $\delta_n = \int_G \varphi(h) \int_G \kappa_n(g^{-1}h, g)^2 \ dg \ dh$. Then a) holds by construction. To see that b) holds, consider a compact subset $K \subseteq G$ and set $L = K \cup \operatorname{supp} \varphi$. The support of $g \mapsto \lambda(h, g)$ is contained in $M = \{g_1^{-1} g_2^{-1} g_3^{-1} : g_1, g_2 \in \operatorname{supp} J, \ g_3 \in L\}$ when $h \in L$. Choose N so large that $M \cup ML \subseteq U_N$. It follows from (2.1) that

$$\int_G |l_n(k^{-1}h, gk) - l_n(h, g)| \ dg = \int_G |\lambda_n(gk) - \lambda_n(g)| \lambda(h, g) \ dg,$$

so we see that

$$\int_G |l_n(k^{-1}h, gk) - l_n(h, g)| \ dg = 0, \ n \geq N, \ k, h \in L. \tag{2.3}$$

Since $|a - b|^2 \leq |a^2 - b^2|, \ a, b \geq 0$, we have the estimate

$$\int_G |\kappa_n(k^{-1}h, gk) - \kappa_n(h, g)|^2 \ dg \leq \int_G |l_n(k^{-1}h, gk) - l_n(h, g)| \ dg$$

for all n. In combination with (2.3) this implies that $\int_G \varphi(h) \int_G |\psi_n(k^{-1}h, gk) - \psi_n(h, g)|^2 \ dg \ dh = 0$ for all $k \in K$ and all $n \geq N$. □

By a G-algebra A we mean a separable C^*-algebra equipped with a point-wise normcontinuous action of G by automorphisms. Such an action extends to an action of G by automorphisms of the multiplier algebra $M(A)$, although the extended action can fail to be point-wise normcontinuous. In the following we shall also consider Hilbert modules over a G-algebra equipped with a comparable G-action. Since the definitions and terminology will be familiar to most readers (if not all), we refer to [Ma], [T] or [MP] for this. We denote the action of a group element $g \in G$ on an element x of a C^*-algebra or a Hilbert module over a C^*-algebra by $g \cdot x$. A sequence $\{\varphi_n\}$ of maps $\varphi_n : \mathcal{E} \to \mathcal{F}$ between Hilbert B, G-modules will be called *eventually equivariant* when the following holds: For any compact subset $K \subseteq G$ there is an $N \in \mathbb{N}$ such that $g \cdot \varphi_n(x) = \varphi_n(g \cdot x)$ for all $x \in \mathcal{E}, g \in K$, when $n \geq N$.

Lemma 2.2. *Let A and B be G-algebras. Let $s : A \to M(B)$ be a completely positive contraction such that $s(g \cdot a) - g \cdot s(a) \in B$ for all $a \in A$ and all $g \in G$. There is then a countably generated Hilbert B, G-module $\mathcal{E}$, an equivariant $*$-homomorphism $\pi : A \to \mathbb{L}_B(\mathcal{E})$ and an eventually equivariant sequence $\{W_n\}$ of adjoinable isometries $W_n : B \to \mathcal{E}$ such that $s(a) - W_n^* \pi(a) W_n \in B$ for all $a \in A$ and all $n \in \mathbb{N}$.*

Proof. After adding a unit to A and extending s we may assume that A and s are unital. Let φ and $\{\psi_n\}$ be the functions from Lemma 2.1. Let α and β denote the given representations of G as automorphisms on A and B, respectively. We give $C_c(G, A) \otimes C_c(G, B)$ a right B-module structure such that $(h \otimes f)b = h \otimes fb$, where $fb(g) = f(g)\beta_g(b)$ and make it into a semi-inner-product B-module in the sense of

[L] such that

$$< j \otimes \psi, j_1 \otimes \psi_1 > = \int_G \varphi(h) \int_G g^{-1} \cdot [\psi(g)^* s(g \cdot (j(g^{-1}h)^* j_1(g^{-1}h)))\psi_1(g)] \ dg \ dh \ .$$

Let $\mathcal{E}$ denote the resulting Hilbert B-module, cf. pp 3-4 in [L]. $\mathcal{E}$ is countably generated because G is second countable and A and B separable. Define a representation S of G on $\mathcal{E}$ such that

$$S_k(j \otimes f) = \alpha_k(j(k^{-1} \cdot)) \otimes f(\cdot k).$$

Then $\mathcal{E}$ is a Hilbert B, G-module. Define $\pi : A \to \mathbb{L}_B(\mathcal{E})$ such that $\pi(a)(j \otimes f) = aj \otimes f$, where $(aj)(g) = aj(g)$. Then $\pi(a)S_{k^{-1}}(j \otimes f) = a\alpha_{k^{-1}}(j(k\cdot)) \otimes f(\cdot k^{-1})$ and hence $(S_k \pi(a) S_{k^{-1}})(j \otimes f) = \alpha_k(a) j(\cdot) \otimes f(\cdot) = \pi(k \cdot a)(j \otimes f)$, proving that π is an equivariant $*$-homomorphism. Since ψ_n can be uniformly approximated by elements from $C_c(G) \otimes C_c(G)$ and $1 \in A$, the function $(h, s) \mapsto \psi_n(h, s)\beta_s(b)$ is an element of $\mathcal{E}$ which we define to be $W_n b$. To see that $W_n : B \to \mathcal{E}$ is adjoinable, define first $T : C_c(G, A) \otimes C_c(G, B) \to B$ such that

$$T(j \otimes f) = \int_G \varphi(h) \int_G \psi_n(g^{-1}h, g) g^{-1} \cdot [s(g \cdot j(g^{-1}h)) f(g)] \ dg \ dh.$$

It is then straightforward to check that

$$< W_n b, j \otimes f > = < b, T(j \otimes f) > . \tag{2.4}$$

Let $\sum_i j_i \otimes f_i$ be a finite sum of simple tensors in $C_c(G, A) \otimes C_c(G, B)$. By using that s is a completely positive contraction we find that

$$\begin{aligned}
&\|T(\sum_i j_i \otimes f_i)\| \\
&= \| \int_G \varphi(h) \int_G \psi_n(g^{-1}h, g) \sum_i g^{-1} \cdot [s(g \cdot (j_i(g^{-1}h))) f_i(g)] \ dg \ dh\| \\
&\leq \| \int_G \varphi(h) \int_G g^{-1} \cdot [\sum_i f_i(g)^* s(g \cdot (j_i(g^{-1}h)^*)) \sum_l s(g \cdot (j_l(g^{-1}h))) f_l(g)] \ dg \ dh\|^{1/2} \times \\
&\qquad (\int_G \varphi(h) \int_G \psi_n(g^{-1}h, g)^2 \ dg \ dh)^{1/2} \\
&\leq \| \int_G \varphi(h) \int_G g^{-1} \cdot [\sum_{i,l} f_i(g)^* s(g \cdot (j_i(g^{-1}h)^* j_l(g^{-1}h))) f_l(g)] \ dg \ dh\|^{1/2} \\
&= \| \sum_i j_i \otimes f_i\|.
\end{aligned}$$

It follows that T extends to a linear contraction $T : \mathcal{E} \to B$ which then, by (2.4), is W_n^*. Since

$$W_n^* W_n b = b \int_G \varphi(h) \int_G \psi_n(g^{-1}h, g)^2 \ dg \ dh = b,$$

we see that W_n is an isometry thanks to a) of Lemma 2.1. Since $g \mapsto g^{-1} \cdot s(g \cdot a)$ is normcontinuous by Theorem 2.1 of [T], the identity

$$W_n^* \pi(a) W_n b - s(a)b = \int_G \varphi(h) \int_G \psi_n(g^{-1}h, g)^2 [g^{-1} \cdot (s(g \cdot a)) - s(a)] b \; dg \; dh$$

shows that $W_n^* \pi(a) W_n - s(a) \in B$ for all a, n. Since $(S_k W_n b)(h, s) = \psi_n(k^{-1}h, sk)\beta_{sk}(b)$ and $W_n \beta_k(b)(h, s) = \psi_n(h, s)\beta_{sk}(b)$, we find that

$$\|S_k W_n b - W_n \beta_k b\|^2 = \| \int_G \varphi(h) \int_G \beta_k(b^* b) |\psi_n(k^{-1}h, gk) - \psi_n(h, g)|^2 \; dg \; dh \|$$

$$\leq \|b\|^2 \int_G \varphi(h) \int_G |\psi_n(k^{-1}h, gk) - \psi_n(h, g)|^2 \; dg \; dh.$$

It follows that $\{W_n\}$ is eventually equivariant thanks to property b) of Lemma 2.1. □

In the following we shall manipulate the Hilbert B, G-module from Lemma 2.2 further, and to do this we need to assume that B is 'stable'. For G-algebras there are at least two different notions of stability which are relevant. We will say that B is *weakly stable* when $B \otimes \mathcal{K}$ is equivariantly isomorphic to B, where $\mathcal{K}$ denotes the G-algebra of compact operators on a separable infinite dimensional Hilbert space with the trivial G-action. Thus any G-algebra can be weakly stabilized; $B \otimes \mathcal{K}$ is weakly stable. A stronger notion of stability is the following. Let $\mathbb{K}_G$ be the G-algebra of compact operators on the countable direct sum of copies of $L^2(G)$, equipped with the G-action implemented by the direct sum of the left regular representation. We say that B is *stable* when $B \otimes \mathbb{K}_G$ is equivariantly isomorphic to B. An arbitrary G-algebra can also be stabilized; $B \otimes \mathbb{K}_G$ is stable, cf. [MP].

In the following we shall consider the Hilbert B, G-module $L^2(G, B)^\infty$, cf. [MP]. Note that $L^2(G, B)^\infty \simeq L^2(G, B) \simeq B \otimes L^2(G)$ when B is weakly stable.

Lemma 2.3. *Let A and B be G-algebras, B weakly stable. Let $s : A \to M(B)$ be a completely positive contraction such that $s(g \cdot a) - g \cdot s(a) \in B$ for all $a \in A$ and all $g \in G$. It follows that there is an equivariant $*$-homomorphism $\pi : A \otimes \mathcal{K}_G \to M(B \otimes \mathcal{K}_G)$ and an eventually equivariant sequence $\{W_n\}$ of isometries in $M(B \otimes \mathcal{K}_G)$ such that $W_n^* \pi(x) W_n - s \otimes \mathrm{id}_{\mathcal{K}_G}(x) \in B \otimes \mathcal{K}_G$ for all n and all $x \in A \otimes \mathcal{K}_G$.*

Proof. It follows from Lemma 2.2 that there is a countably generated Hilbert B, G-module $\mathcal{E}$, a $*$-homomorphism $\pi' : A \to \mathbb{L}_B(\mathcal{E})$ and an eventually equivariant sequence of isometries $\tilde{W_n} : B \to \mathcal{E}$ such that $\tilde{W_n}^* \pi'(a) \tilde{W_n} - s(a) \in B$ for all $a \in A$. Set $V_n = \tilde{W_n} \otimes \mathrm{id}_{L^2(G)} : B \otimes L^2(G) \to \mathcal{E} \otimes L^2(G)$ and $\pi'' = \pi' \otimes \mathrm{id}_{\mathcal{K}_G} : A \otimes \mathcal{K}_G \to \mathbb{L}_B(\mathcal{E} \otimes L^2(G))$. We have the following isomorphisms of Hilbert B, G-modules:

$$\begin{aligned} (\mathcal{E} \otimes L^2(G))^\infty \oplus L^2(G, B)^\infty &\simeq (L^2(G, \mathcal{E} \oplus B))^\infty \qquad \text{(by definition)} \\ &\simeq L^2(G, B)^\infty \qquad \text{(by Theorem 2.4 of [MP])} \\ &\simeq B \otimes L^2(G) \qquad \text{(since } B \text{ is weakly stable)} . \end{aligned}$$

It follows that there is an adjoinable equivariant isometry $V : \mathcal{E} \otimes L^2(G) \to B \otimes L^2(G)$. Let $\pi = V \pi''(\cdot) V^*$ and set $W_n = V V_n$. Via the identification $M(B \otimes \mathcal{K}_G) = \mathbb{L}_B(B \otimes L^2(G))$ we get the desired things. □

Proposition 2.4. *Let A and B be G-algebras, B weakly stable. Let $s : A \to M(B)$ be a completely positive contraction such that $s(g \cdot a) - g \cdot s(a) \in B$ for all $a \in A$ and all $g \in G$. It follows that there is an equivariant $*$-homomorphism $\pi : A \otimes \mathcal{K}_G \to M(B \otimes \mathcal{K}_G)$ and a sequence of isometries $\{S_n\} \subseteq M(B \otimes \mathcal{K}_G)$ such that $S_n^* \pi(x) S_n - s \otimes \mathrm{id}_{\mathcal{K}_G}(x) \in B \otimes \mathcal{K}_G$ for all $x \in A \otimes \mathcal{K}_G$ and all n, $g \cdot S_n - S_n \in B \otimes \mathcal{K}_G$ for all g, n, and $\lim_{n \to \infty} g \cdot S_n - S_n = 0$, uniformly on compact subsets of G.*

Proof. Set $\mathcal{A} = A \otimes \mathcal{K}_G$, $\mathcal{B} = B \otimes \mathcal{K}_G$. By Lemma 2.3 there is an equivariant $*$-homomorphism $\tilde{\pi} : \mathcal{A} \to M(\mathcal{B})$ and an eventually equivariant sequence $\{\tilde{W}_n\}$ of isometries in $M(\mathcal{B})$ such that $\tilde{W}_n{}^* \tilde{\pi}(x) \tilde{W}_n - s \otimes \mathrm{id}_{\mathcal{K}_G}(x) \in \mathcal{B}$, $x \in A \otimes \mathcal{K}_G$. Since $\mathcal{B}$ is weakly stable there is a sequence $\{V_n\}$ of G-invariant isometries in $M(\mathcal{B})$ such that $\sum_{n=1}^{\infty} V_n V_n^* = 1$, with convergence in the strict topology. Set $\pi(x) = \sum_{n=1}^{\infty} V_n \tilde{\pi}(x) V_n^*$ and $W_n = V_n \tilde{W}_n$. The W_n's are isometries, $W_n^* \pi(x) W_n = \tilde{W}_n{}^* \tilde{\pi}(x) \tilde{W}_n$ for all x, n, and $\{W_n\}$ is eventually equivariant. In addition,

$$W_i^* \pi(\mathcal{A}) W_j = \{0\}, \text{ and } W_i^* W_j = 0, \quad i \neq j \tag{2.5}$$

and

$$\lim_{n \to \infty} W_n^* b = 0, \ b \in \mathcal{B}. \tag{2.6}$$

Fix a compact subset X with dense span in $\mathcal{A}$, an $\epsilon > 0$ and a compact subset $K \subseteq G$. To complete the proof it suffices to construct an isometry $S \in M(\mathcal{B})$ such that $g \cdot S - S \in \mathcal{B}$ for all $g \in G$, $S^* \pi(x) S - s \otimes \mathrm{id}_{\mathcal{K}_G}(x) \in B, x \in X$, and $\|k \cdot S - S\| \leq \epsilon$ for all $k \in K$. Let $K = K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be a sequence of compact sets such that $G = \bigcup_n K_n$. Let b be a strictly positive element in $\mathcal{B}$. Choose a sub-sequence $\{T_k\}$ of $\{W_k\}$ such that

$$g \cdot T_i = T_i, \ g \in K_i, \tag{2.7}$$

and, using (2.6),

$$\|T_i^* b\| \leq 2^{-i}, \tag{2.8}$$

for all i. Let $\{e_i\}_{i=1}^{\infty}$ be an approximate unit for $\mathcal{B}$ which is asymptotically G-invariant and asymptotically commutes with $s \otimes \mathrm{id}_{\mathcal{K}_G}(\mathcal{A})$, cf. Lemma 1.4 of [K]. Let $n_1 < n_2 < n_3 < \cdots$ be a sequence in $\mathbb{N}$ and set

$$f_1 = e_{n_1}^{1/2}, \ f_k = (e_{n_k} - e_{n_{k-1}})^{1/2}, \ k \geq 2.$$

We shall assume that $\{n_i\}$ increases so fast that

$$\|g \cdot f_i - f_i\| \leq 2^{-i} \epsilon \tag{2.9}$$

for all $g \in K_i$ and all $i \in \mathbb{N}$,

$$\|f_i b\| \leq 2^{-i}, \ i \geq 2, \tag{2.10}$$

and

$$\|f_k s \otimes \mathrm{id}_{\mathcal{K}_G}(x) - s \otimes \mathrm{id}_{\mathcal{K}_G}(x) f_k\| \leq 2^{-k} \tag{2.11}$$

for all $x \in X$ and $k \in \mathbb{N}$. Furthermore, since $T_i^* \pi(x) T_i - s \otimes \mathrm{id}_{\mathcal{K}_G}(x) \in \mathcal{B}$ for all $x \in X$, we can arrange that

$$\|f_i (T_i^* \pi(x) T_i - s \otimes \mathrm{id}_{\mathcal{K}_G}(x)) f_i\| \leq 2^{-i}, \ x \in X, i \geq 2. \tag{2.12}$$

It follows then from (2.10), (2.8) and (2.5) that $\sum_{k=1}^{\infty} T_k f_k$ converges in the strict topology to an isometry S in $M(\mathcal{B})$. By (2.9) and (2.7),

$$\|g \cdot (T_k f_k) - T_k f_k\| \leq 2^{-k} \epsilon, \ k \geq n, \ g \in K_n,$$

from which we conclude that $g \cdot S - S \in \mathcal{B}$ for all $g \in G$ and that $\|k \cdot S - S\| \leq \epsilon, \ k \in K$. It follows from (2.5) that $S^* \pi(x) S - s \otimes \mathrm{id}_{\mathcal{K}_G}(x) = \sum_{i=1}^{\infty} f_i T_i^* \pi(x) T_i f_i - s \otimes \mathrm{id}_{\mathcal{K}_G}(x)$ for all $x \in X$. Furthermore, (2.11) ensures that $s \otimes \mathrm{id}_{\mathcal{K}_G}(x) - \sum_{i=1}^{\infty} f_i s \otimes \mathrm{id}_{\mathcal{K}_G}(x) f_i \in \mathcal{B}$. Hence

$$S^* \pi(x) S - s \otimes \mathrm{id}_{\mathcal{K}_G}(x) = \sum_{i=1}^{\infty} f_i (T_i^* \pi(x) T_i - s \otimes \mathrm{id}_{\mathcal{K}_G}(x)) f_i$$

modulo $\mathcal{B}$ for all $x \in X$. The last sum is in $\mathcal{B}$ by (2.12).

□

Proof. (Of Theorem 1.1.) By tensoring the entire extension with $\mathbb{K}_G$ we may assume that A, E and B are all stable. Furthermore, we may assume that s is a contraction, cf. Remark 2.5.1 of [CS]. By combining Proposition 2.4 here with Lemma 3.2 of [T] we conclude that the extension (1.2) is invertible in the sense of [T]. The result follows now from Theorem 8.1 of [T]. □

Proof. (Of Corollary 1.2.) It follows from Theorem 1.1 that

$$0 \longrightarrow (B \otimes \mathbb{K}_G) \rtimes_r G \longrightarrow (E \otimes \mathbb{K}_G) \rtimes_r G \longrightarrow (A \otimes \mathbb{K}_G) \rtimes_r G \longrightarrow 0,$$

is exact and semi-split. But the action of G on $B \otimes \mathbb{K}_G$ is exterior equivalent to the action which is trivial on the tensor factor $\mathbb{K}_G$, and consequently $(B \otimes \mathbb{K}_G) \rtimes_r G \simeq (B \rtimes_r G) \otimes \mathcal{K}$. The same is of course true with E or A in place of B, and the isomorphisms are natural. Hence

$$0 \longrightarrow (B \rtimes_r G) \otimes \mathcal{K} \longrightarrow (E \rtimes_r G) \otimes \mathcal{K} \longrightarrow (A \rtimes_r G) \otimes \mathcal{K} \longrightarrow 0,$$

is also exact and semi-split, and the corollary follows straightforwardly from this. □

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