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MULTIPLICITIES OF SECOND CELL TILTING MODULES

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Multiplicities of second cell tilting modules

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We consider three related representation theories: That of a quantum group at a complex root of unity, that of an almost simple algebraic group over an algebraically closed field of prime characteristic and that of the symmetric group.

Soergel has recently computed the characters of the quantum tilting modules. Applying Soergels result we determine some multiplicities of wallcrossed quantum tilting modules. This result is based on the combinatorics of right cells in the affine Weyl group, the group generated by reflections in the walls of the first alcove. One of these reflections, s_0, is not in the finite Weyl group; the right cell containing this reflection is central to this paper and we denote it by $\mathcal{C}(s_0)$. Identifying alcoves and Weyl group elements, we view $\mathcal{C}(s_0)$ as a set of alcoves. The multiplicities determined is the multiplicities of the indecomposable quantum tilting modules with highest weight in an alcove of $\mathcal{C}(s_0)$.

In type $A_{n\geq 2}$, D_n, E_6, E_7, E_8 and G_2 we obtain a generalization to the modular case. For any modular tilting module we give a formula describing the multiplicities of the modular indecomposable tilting modules with highest weight in an alcove of $\mathcal{C}(s_0)$. This formula is the main result of our paper.

As an application of the modular multiplicity formula, we determine the dimension of a set of simple representations of the symmetric group over a field of characteristic p. The dimension formula covers simple modules parametrized by partitions $(n_1, \ldots, n_n)$ with either $n_1 - n_{n-1} < p - n + 2$ or $n_2 - n_n < p - n + 2$. This generalizes a result of Mathieu [22] as well as a recent result by Jensen and Mathieu [12]. Further, it proves in part a conjecture by Mathieu [23].

Key Words: tilting modules, cells of affine groups, symmetric group

1. INTRODUCTION

The structure of the tilting modules is a highly interesting unsolved problem in the representation theory of reductive algebraic groups in prime characteristic. The notion of a tilting module was originally defined by

Ringel [25] in the setting of quasi hereditary algebras, and later adapted to reductive algebraic groups by Donkin [7]. In the latter setting, a tilting module is a module with a filtration of Weyl modules and a filtration of dual Weyl modules. The tilting modules form a family of modules with very interesting properties: It is closed under tensor products, and any summand of a tilting module is tilting. The indecomposable tilting modules can be parametrized by the dominant weights and are then denoted by $T(\lambda)$. The characters of these indecomposable tilting modules are in general unknown.

Let k denote an algebraically closed field of characteristic $p > 0$. We will mostly be concerned with an almost simple algebraic group G over k.

Identification of the indecomposable tilting modules of G poses serious problems. Their description would allow for the computation of the characters of the simple modules of G. It would also determine the dimensions of all simple representations of the symmetric group in prime characteristic. Even partial results on the characters of the tilting modules would shed light on the two problems above.

The problem of determining the tilting modules may be approached from several angles. We consider here formulae for the number of times an indecomposable tilting module occurs as a summand in a tilting module with known character. With Q denoting a known tilting module, we ask for the multiplicity $[Q : T(\lambda)]$ of all indecomposable tilting modules in Q. This is equivalent to finding the characters of the indecomposable tilting modules. If λ belongs to the first alcove (see Section 3 for the notation), the answer is well known, due to Georgiev and Mathieu [11] and Andersen and Paradowski [3].

$$[Q : T(\lambda)] \;=\; \sum_{x \in W,\ x.\lambda \in X^+} (-1)^{l(x)} [Q : V(x.\lambda)] \tag{1}$$

Little seems to be known if λ does not belong to the first alcove. In this paper we prove a formula for $[Q : T(z.\lambda)]$ with z in the right cell of s_0 and λ in the first alcove; the formula is valid in the simply laced types and in type G_2. To state the precise result we introduce the quantum group.

Consider U_q, the quantum group at a complex prime root of unity corresponding to G. The definition and the properties of the tilting U_q-modules resemble those of the modular tilting modules. Soergel [26] recently found a way to compute the characters of the indecomposable tilting U_q-modules, that relies on calculations in a Hecke algebra module, usually called the Hecke module. Central to the representation theory of Hecke algebras is the concept of right cells; the right cells provide right ideals and right modules of the Hecke algebra. From this point of view, right cells should provide insight into the structure of the quantum tilting modules. Along this line, Ostrik [24] used right cells to define tensor ideals of tilting modules. The first result we present use right cells to describe some of the summands in the wallcrossed quantum tilting modules. To be more precise, let $\mathcal{Z}$ denote the union of $\{e\}$ and the right cell containing s_0; as usual h denotes the Coxeter number of the root system of G.

THEOREM 1.1. *Let $p \geq h$. Consider a weight λ belonging to the first alcove and let s, t denote reflections in the walls of the first alcove.*

1.Assume $x \notin \mathcal{Z}$ and $z \in \mathcal{Z}$. Then

$$[\Theta_s T_q(x.\lambda) : T_q(z.\lambda)] \;=\; 0.$$

2.Assume $z \in \mathcal{Z}$ and $zs > z > zt$. There are non-negative numbers a_y such that

$$\Theta_s T_q(z.\lambda) \;=\; T_q(zs.\lambda) \oplus \delta T_q(zt.\lambda) \oplus \bigoplus_{y \notin \mathcal{Z}} a_y T_q(y.\lambda)$$

where δ equals 1 when $zts < zt$ and 0 when $zts > zt$.

Corresponding to a modular tilting module Q, we have a tilting U_q-module (denoted Q_q) with the same character, see Section 3. Since the quantum tilting modules are well understood, we can decompose Q_q in the quantum setup - even in situations where we do not know the modular decomposition of Q. In turn, the quantum decomposition yields information on the modular decomposition; as to the summands $T(z.\lambda)$ with $z \in \mathcal{Z}$, the quantum decomposition tells everything in the simply laced types as well as in type G_2. This is the main result of this paper, which we state in a theorem.

THEOREM 1.2. *Let $p \geq h$ and suppose that the root system of G is of type $A_{n\geq 2}, D_n, E_6, E_7, E_8$ or G_2. For a weight λ in the first alcove and $z \in \mathcal{Z}$ we have*

$$[Q : T(z.\lambda)] \;=\; [Q_q : T_q(z.\lambda)] \tag{2}$$

Note that the theorem covers the situation considered in equation (1). We can sharpen the theorem slightly; see Theorem 3.11.

Schur-Weyl duality is a link between the representation theories of the general linear group and the symmetric group. This duality determines the dimension of simple modules of the symmetric group in terms of tilting multiplicities in a $\mathrm{GL}_n(k)$-tilting module. As a result, the formula of

Theorem 1.2 provides information on the dimension of simple modules of the symmetric group. In section 4 we show that the formula determines the dimension of the simple modules parametrized by certain partitions:

THEOREM 1.3. *Let $p \geq n$ and consider a partition $(n_1, \dots n_n)$ with at least three lines such that $n_1 - n_{n-1} < p - n + 2$ or $n_2 - n_n < p - n + 2$.*

The dimension of the simple module of the symmetric group parametrized by $(n_1, \dots n_n)$ equals the dimension of the simple module of the Hecke algebra of the symmetric group at a primitive p'th root of unity parametrized by $(n_1, \dots n_n)$

This is a generalization of a result by Mathieu [22], determining the dimension of the simple modules parametrized by Young diagrams with $n_1 - n_n < p - n + 1$. Further, our result proves a special case of conjecture 15.4 in [23].

2. THE HECKE MODULE

The section serves several purposes. The Hecke algebra of an affine Weyl group is introduced; this allow us to define the Hecke module as well as right cells. The Hecke module calculations in 2.4 is vital for the various decompositions of tilting modules to follow. Whenever possible we will use the notation of [26], to which we also refer to for the missing proofs in 2.1 and 2.2.

2.1. The Hecke algebra

Let $S_0 = \{s_1, \dots, s_n\}$ denote the generators of the Weyl group W_0 of an irreducible rootsystem, and let $S = \{s_0, s_1, \dots, s_n\}$ denote the generators of the corresponding affine Weyl group W. On W we have the Bruhat order $<$ and a length function, l, mapping $w \in W$ to the length of a reduced expression. Let

$$\mathcal{L}(w) = \{s \in S \mid sw < w\}$$
$$\mathcal{R}(w) = \{s \in S \mid ws < w\}.$$

Let $\mathcal{H}$ denote the Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ associated to W. We choose $H_x = v^{l(x)} T_x$ as basis of $\mathcal{H}$. On $\mathcal{H}$ we have a ring homomorphism, $\bar{\ }$, taking H_x to $\overline{H}_x = H_{x^{-1}}^{-1}$ and v to v^{-1}. Since $\bar{\ }$ is an involution, we say that H is self-dual if $\overline{H} = H$.

THEOREM 2.1. *There exists a unique self-dual element*

$$\underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v]H_y.$$

We will (as is usual) denote $\underline{H}_s$ by C_s. We define polynomials $h_{y,x} \in \mathbb{Z}[v]$ by the formula

$$\underline{H}_x = \sum_{y \in W} h_{y,x} H_y.$$

Note that $h_{y,x}(0) \neq 0$ if and only if $x = y$. Further, if $y < x$ the leading coefficient of $h_{y,x}$ is 1 and $\deg h_{y,x} = l(x) - l(y)$.

Recall that W_0 is a parabolic subgroup of W, so each right coset of $W_0 \backslash W$ has a unique representative of minimal length. We denote the set of these representatives by W^0. Multiplication gives a bijection $W_0 \times W^0 \longleftrightarrow W$.

2.2. The Hecke module

Let $\mathcal{H}_0$ denote the Hecke algebra of S_0, W_0. It is a subalgebra of $\mathcal{H}$. We have a surjective $\mathbb{Z}[v, v^{-1}]$-algebra homomorphism, $\phi_{-v} : \mathcal{H}_0 \longrightarrow \mathbb{Z}[v, v^{-1}]$, mapping each generator $s_i \in S_0$ to $-v$. This gives $\mathbb{Z}[v, v^{-1}]$ a $\mathcal{H}_0$-module structure, and by induction we obtain a right $\mathcal{H}$-module

$$\mathcal{N} = \mathbb{Z}[v, v^{-1}] \otimes_{\mathcal{H}_0} \mathcal{H}.$$

So $\mathcal{N}$ has a basis consisting of $\{N_x = 1 \otimes H_x \mid x \in W^0\}$, and the action of $C_s \in \mathcal{H}$ is given by

LEMMA 2.2.

$$N_x C_s = \begin{cases} N_{xs} + vN_x & xs > x \text{ and } xs \in W^0 \\ N_{xs} + v^{-1}N_x & xs < x \text{ and } xs \in W^0 \\ 0 & xs \notin W^0 \end{cases}$$

The next step is to define an involution on $\mathcal{N}$ by $\overline{a \otimes H} = \overline{a} \otimes \overline{H}$. This involution is $\mathcal{H}$-skewlinear (ie $\overline{NH} = \overline{N}\ \overline{H}$ for $N \in \mathcal{N}, H \in \mathcal{H}$). We say that N is selfdual if $\overline{N} = N$.

THEOREM 2.3. *There is a unique selfdual element $\underline{N}_x$ in $\mathcal{N}$ such that*

$$\underline{N}_x \in N_x + \sum_{y<x} v\mathbb{Z}[v]N_y.$$

In this way we get a new basis of $\mathcal{N}$. Define polynomials $n_{y,x} \in \mathbb{Z}[v]$ by $\underline{N}_x = \sum_{y \in W^0} n_{y,x} N_y$. The next lemma links the polynomials $n_{y,x}$ and $h_{y,x}$.

LEMMA 2.4. *Let $x, y \in W^0$. Then*

$$n_{y,x} = \sum_{w \in W_0} (-v)^{l(w)} h_{wy,x}.$$

2.3. Right cells

Having defined the polynomials $h_{y,x}$, we can now define right cells. If $h_{y,x} = v^{l(x)-l(y)} + \cdots + a_1 v$ with $a_1 \neq 0$, we say that $h_{y,x}$ has a linear term. We write $x - y$ if either $h_{x,y}$ or $h_{y,x}$ has a linear term.

DEFINITION 2.5. *We define a pre-order: Write $x \leq_R y$ if there exist*

$$x = w_0 - w_1 - \cdots - w_r = y$$

such that $\mathcal{R}(w_i) \not\subseteq \mathcal{R}(w_{i+1})$ for each i. We write $x \sim_R y$ if $x \leq_R y \leq_R x$.

Note that $\sim_R$ is an equivalence relation. An equivalence class is called a **right cell**. As an example: $xs < x$ belongs to the same right cell when $\mathcal{R}(x) = \{s\}$, since $\mathcal{R}(xs)$ and $\mathcal{R}(x)$ intersect trivially and $h_{xs,x} = v$.

Let $\mathcal{C}$ denote the set of elements in W with a unique reduced expression, and let $\mathcal{C}(s_i)$ denote the subset with reduced expression beginning with s_i:

$$\mathcal{C}(s_i) \ = \ \{w \in \mathcal{C} \mid \mathcal{L}(w) = \{s_i\}\}$$

In the following proposition, 2. and 3. are direct consequences of the definition. As for 1. we refer to [18].

PROPOSITION 2.6 (Properties of $\mathcal{C}(s_i)$).

1.$\mathcal{C}(s_i)$ is a right cell.
2.$|\mathcal{L}(w)| = |\mathcal{R}(w)| = 1$ for each $w \in \mathcal{C}(s_i)$.
3.Suppose $w \in \mathcal{C}(s_i)$. If $ws < w$ then $ws \in \mathcal{C}(s_i)$ or $ws = e$.

We have $\mathcal{C}(s_0) \subset W^0$ and this right cell is central in the rest of this paper. The reduced expression of each element in $\mathcal{C}(s_0)$ may be obtained from table 1. The right order $\leq_R$ on the affine Weyl group induces a partial order on the set of right cells: $\mathcal{C}_1 \leq_R \mathcal{C}_2$ if $w_1 \leq_R w_2$ for all $w_1 \in \mathcal{C}_1$, $w_2 \in \mathcal{C}_2$. In this ordering $\mathcal{C}(s_0)$ is second among the right cells of W^0, since for any right cell $\mathcal{C}_1 \subset W^0$ different from $\{e\}$ we have

$$\mathcal{C}_1 \ \leq_R \ \mathcal{C}(s_0) \ \leq_R \ \{e\}.$$

TABLE 1.
The table list the reduced expression of elements in $\mathcal{C}(s_0)$. Used together with proposition 2.6 3., it yields the reduced expression of all elements of $\mathcal{C}(s_0)$. Note that s_0 corresponds to the highest *short* root in the rootsystem.

Type	Reduced expression	$\|\mathcal{C}(s_0)\|$
A_1	$(s_0s_1)^m$, $m \geq 1$	∞
A_n, $n \geq 2$	$(s_0s_n \dots s_1)^m$, $m \geq 1$	∞
	$(s_0s_1 \dots s_n)^m$, $m \geq 1$	
B_n	$s_0s_1s_0$	∞
	$(s_0s_1s_2 \dots s_{n-1}s_ns_{n-1} \dots s_2s_1)^m$, $m \geq 1$	
C_n, $n \geq 3$	$s_0s_2s_1$	$2n+1$
	$s_0s_2s_3 \dots s_{n-1}s_ns_{n-1} \dots s_2s_0$	
	$s_0s_2s_3 \dots s_{n-1}s_ns_{n-1} \dots s_2s_1$	
D_n	$s_0s_2s_1$	$n+1$
	$s_0s_2s_3 \dots s_{n-2}s_{n-1}$	
	$s_0s_2s_3 \dots s_{n-2}s_n$	
E_6	$s_0s_2s_4s_3s_1$	7
	$s_0s_2s_4s_5s_6$	
E_7	$s_0s_1s_3s_4s_2$	8
	$s_0s_1s_3s_4s_5s_6s_7$	
E_8	$s_0s_8s_7s_6s_5s_4s_2$	9
	$s_0s_8s_7s_6s_5s_4s_3s_1$	
F_4	$s_0s_4s_3s_2s_1$	8
	$s_0s_4s_3s_2s_3s_4s_0$	
G_2	$s_0s_1s_2s_1s_2s_1s_0$	8
	$s_0s_1s_2s_1s_0$	

2.4. Calculations in $\mathcal{N}$

For later use we derive some results about the coefficients of $\underline{N}_xC_s$, expressed in the basis $\{\underline{N}_y \mid y \in W^0\}$. For reasons, which become clear in section 3, we denote the coefficient of $\underline{N}_y$ by $[\underline{N}_xC_s : \underline{N}_y]$, such that

$$\underline{N}_xC_s \;=\; \sum_y [\underline{N}_xC_s : \underline{N}_y]\underline{N}_y.$$

The last assertion in the following proposition is the key to all the results in this paper.

Proposition 2.7. *Fix $x \in W^0$ and $s \in S$ such that $xs > x$.*

1.Let $y \in W^0$. $[\underline{N}_x C_s : \underline{N}_y] \neq 0$ implies $ys < y$.
2.Let $t \in S$, $xt < x$. Then

$$[\underline{N}_x C_s : \underline{N}_{xt}] = \begin{cases} 1 & \text{if } xts < xt \\ 0 & \text{if } xts > xt \end{cases}$$

3.$[\underline{N}_x C_s : \underline{N}_y] \neq 0$ implies $y \leq_R x$.
4.Suppose $z \in \mathcal{C}(s_0)$. Then $[\underline{N}_x C_s : \underline{N}_z] \neq 0$ if and only if $z = xt$ for some $t \in S$, $xts < xt$ and $x \in \mathcal{C}(s_0) \cup \{e\}$.

Proof. 1. and 2. are easy calculations.

We prove 3.. $[\underline{N}_x C_s : \underline{N}_y] \neq 0$ implies $ys < y$. Hence $\mathcal{R}(y) \ni s \notin \mathcal{R}(x)$ and so $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$. By lemma 2.2 we have

$$\underline{N}_x C_s \;=\; \sum_{w \neq y, ys} n_{w,x} N_w C_s + n_{y,x}(N_{ys} + v^{-1} N_y) + n_{ys,x}(N_y + v N_{ys}).$$

So $[\underline{N}_x C_s : \underline{N}_y] = (v^{-1} n_{y,x} + n_{ys,x})|_{v=0}$.

If $n_{ys,x}$ has a nonzero constant term, then $x = ys$. Now $x - y$ and $y \leq_R x$ follows.

If $n_{y,x}$ has a linear term, then, by lemma 2.4, so does $h_{y,x}$, hence $y - x$ and $y \leq_R x$.

We turn to part 4., where we leave the 'if' part to the reader. Assuming $[\underline{N}_x C_s : \underline{N}_z] \neq 0$ we have $z \leq_R x$ (hence $z - x$) and $zs < z$ (hence $\mathcal{R}(z) = \{s\}$). Then either $x = e$ or $\mathcal{R}(x) \not\subseteq \mathcal{R}(z)$; in the last case $x \leq_R z$ and therefore $x \in \mathcal{C}(s_0)$. We have $x \in \mathcal{C}(s_0) \cup \{e\}$.

It remains to prove $z = xt$ for some $t \in S$. If $z > x$ then $z = xs$. If $z < x$ then $n_{z,x}$ has a linear term. Pick a $t \in S$ such that $xt < x$. Then $zt > z$ since $\mathcal{R}(z) = \{s\}$. If we assume that $z \neq xt$ then the next lemma produces infinitely many y with $z < y < x$. This is a contradiction and we are done. ∎

Lemma 2.8. *Assume $zt > z$.*

Suppose $xt < x$, $x \neq zt$ and that $n_{z,x}$ has a linear term.

Then there is a $y < x$ with $yt < y$, $y \neq zt$ and such that $n_{z,y}$ has a linear term.

Proof. Recall from the construction of $\underline{N}_x$ that if

$$\underline{N}_{xt} C_t \;=\; N_x + \sum_{y<x} n'_y N_y$$

then

$$\underline{N}_x \;=\; \underline{N}_{xt}C_t - \sum_{y<x} n'_y(0)\underline{N}_y.$$

From this we get that

$$n_{z,x} \;=\; vn_{z,xt} + n_{zt,xt} - \sum_{y<x} n'_y(0)n_{z,y}. \tag{3}$$

Note that $x \neq zt$ implies that $n_{z,xt}$ has constant term 0. Now suppose $n_{zt,xt}$ has linear term av. We get

$$n'_{zt}(0) = (v^{-1}n_{zt,xt} + n_{z,xt})|_{v=0} = a.$$

That is, the linear term of $n_{zt,xt}$ in equation (3) cancels with that of $n'_{zt}(0)n_{z,zt}$. We have

$$\text{linear term}(n_{z,x}) \;=\; -\sum_{y<x,\ y\neq zt} n'_y(0)\,\text{linear term}(n_{z,y}).$$

When $n_{z,x}$ has a linear term, there is a $y<x$, $y \neq zt$ such that $n_{z,y}$ has a linear term. Note that $[\underline{N}_x C_s : \underline{N}_y] = n'_y(0)$ and hence that $n'_y(0) \neq 0$ implies $yt < y$. ∎

3. TILTING MODULES

Recall that G is an almost simple algebraic group over k. For a fixed torus, let X denote the weight lattice; now X contains the root system of G and, corresponding to a choice of simple roots $\{\alpha_1, \ldots \alpha_n\}$, we have the dominant weights, denoted X^+. Let ρ denote the half sum of the positive roots. Let s_α denote reflection in the hyperplane perpendicular to the root α, and abbreviate s_{α_i} to s_i. Then the finite Weyl group W_0 is generated by $\{s_1, \ldots s_n\}$. Let α_0 denote the highest short root and define s_0 by $s_0(\lambda) = s_{\alpha_0}(\lambda) + p\alpha_0$. The affine Weyl group is the group generated by $\{s_0, s_1, \ldots, s_n\}$. The affine Weyl group divides $E = X \otimes \mathbb{R}$ into alcoves, on which it acts through the dot-action $w.\lambda = w(\lambda+\rho)-\rho$. This action is simply transitive. Let

$$C_0 \;=\; \{\lambda \in E \mid 0 < \langle \lambda+\rho, \alpha^\vee\rangle < p \text{ for all positive roots } \alpha\}$$

denote the first (or standard) alcove. The first alcove contains a weight when $p \geq h$, h denoting the Coxeter number of the rootsystem of G.

For each dominant weight λ we have a Weyl-module $V(\lambda)$, an induced module (or dual Weyl module) $H^0(\lambda)$ and an indecomposable tilting module $T(\lambda)$, all with highest weight λ. A tilting module is, by definition, a module with a filtration by Weyl modules and a filtration by dual Weyl modules. Tilting modules have the following key properties ([7] is a convenient reference)

- $\dim T(\lambda)_\lambda = 1$.
- Any summand of a tilting module is tilting.
- A tilting module is fully determined by its character.
- The family of tilting modules is closed under tensor products.

Let $\mathcal{A} = \mathbb{Z}[v]_{(p,v-1)}$, and consider $U_\mathcal{A}$, the $\mathcal{A}$-form with divided powers of the quantum group corresponding to G. The fields k and $\mathbb{C}$ are $\mathcal{A}$-modules, with v acting by multiplication of $1 \in k$ and $q \in \mathbb{C}$, respectively; q denotes a primitive p'th root of unity. We consider two specializations of $U_\mathcal{A}$:

$$\begin{aligned} U_q &= U_\mathcal{A} \otimes_\mathcal{A} \mathbb{C} \\ U_k &= U_\mathcal{A} \otimes_\mathcal{A} k \end{aligned}$$

Each $U_\mathcal{A}$-module M gives rise to a U_q-module $M \otimes_\mathcal{A} \mathbb{C}$ and a U_k-module $M \otimes_\mathcal{A} k$. We identify the weight lattices of G and the three quantum groups in question. Using results of Lusztig [19, 20], the category of finite dimensional G-modules is identified with the category of finite dimensional U_k-modules. Thus $V(\lambda)$ and $T(\lambda)$ are regarded as U_k-modules. As for the quantum groups $U_\mathcal{A}$ and U_q we may to each dominant weight associate Weyl modules $V_\mathcal{A}(\lambda)$, $V_q(\lambda)$, induced modules (or dual Weyl modules) $H^0_\mathcal{A}(\lambda)$, $H^0_q(\lambda)$ together with indecomposable tilting modules $T_\mathcal{A}(\lambda), T_q(\lambda)$. The tilting modules are defined as for the algebraic group, and they have the same properties as their modular counterparts.

The Weyl module $V_\mathcal{A}(\lambda)$ is a free module over $\mathcal{A}$, and we have $V_\mathcal{A}(\lambda)_\mathcal{A} \otimes_\mathcal{A} k \simeq V(\lambda)$ and $V_\mathcal{A}(\lambda) \otimes_\mathcal{A} \mathbb{C} \simeq V_q(\lambda)$. Thus the character of the Weyl modules is the same in all three cases and given by Weyls character formula. It follows that the tilting modules of $U_\mathcal{A}$ are free over $\mathcal{A}$. Further $T_\mathcal{A}(\lambda) \otimes_\mathcal{A} \mathbb{C}$ and $T_\mathcal{A}(\lambda) \otimes_\mathcal{A} k$ are tilting. In fact $T_\mathcal{A}(\lambda) \otimes_\mathcal{A} k$ is indecomposable (see [1] for a proof); this does not necessarily hold for $T_\mathcal{A}(\lambda) \otimes_\mathcal{A} \mathbb{C}$. Now weight considerations give for some nonnegative integers a_μ

$$\begin{aligned} T_\mathcal{A}(\lambda) \otimes_\mathcal{A} k &\simeq T(\lambda) \\ T_\mathcal{A}(\lambda) \otimes_\mathcal{A} \mathbb{C} &\simeq T_q(\lambda) \oplus \bigoplus_{\mu < \lambda} a_\mu T_q(\mu). \end{aligned}$$

This shows that the tilting modules of G lift to tilting modules of $U_{\mathcal{A}}$. That is, for a tilting G-module Q there exists a tilting $U_{\mathcal{A}}$-module $Q_{\mathcal{A}}$ with the property $Q_{\mathcal{A}} \otimes_{\mathcal{A}} k \simeq Q$. We denote by Q_q the tilting U_q-module $Q_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$. In this way each tilting G-module Q gives rise to a tilting U_q-module Q_q with the same character.

We may now compare the characters of the modular tilting modules and tilting U_q-modules. This is particular interesting since the character of the indecomposable tilting U_q-modules is known. This is due to Soergel [26, 27]. In the notation of Section 2, the formula reads

THEOREM 3.1. *For a weight $\lambda \in C_0$ and $x, y \in W^0$ we have*

$$[T_q(x.\lambda) : V_q(y.\lambda)] \;=\; n_{y,x}(1).$$

REMARK 3.2. *There is a related formula for the multiplicities of the wall-crossed tilting modules. Keeping the notation of the theorem and assuming $xs > x$ we have*

$$[\Theta_s T_q(x.\lambda) : T_q(y.\lambda)] \;=\; [\underline{N}_x C_s : \underline{N}_y]. \tag{4}$$

The results of Soergel rely on an equivalence of categories between affine Lie algebra modules and quantum group modules established in [15], [16] together with results from [21] and [14].

3.1. Reductive groups

We will briefly consider reductive groups. So G denotes here a connected reductive group over k. Let T be a maximal torus and B a Borel subgroup containing T. By G' we denote the derived group (G, G). The intersection $B' = B \cap G'$ is a Borel subgroup of G' and $T' = T \cap G'$ is a maximal torus of G' contained in B'. It is convenient here to denote the induced modules by $\mathrm{Ind}_B^G(\lambda)$ instead of $H^0(\lambda)$.

The induced modules of G and G' are closely related. From [6] we have for any character, λ, of T

$$\mathrm{Ind}_B^G(\lambda)|_{G'} \;=\; \mathrm{Ind}_{B'}^{G'}(\lambda|_{B'}).$$

This remarkable result immediately implies that the tilting modules of G remains tilting under restriction to G'. Further a G-module is indecomposable if its restriction to G' is indecomposable. Thus

$$T_G(\lambda)|_{G'} \;=\; T_{G'}(\lambda|_{T'}).$$

Hence many questions on the tilting multiplicities of G may be answered by considering the restrictions to the connected semisimple group G'. In turn, the representation theory of a semisimple group is determined by that of its almost simple components.

3.2. Comparing modular and quantum tilting modules

We will use the translation functors, so assume throughout that p is at least the Coxeter number. Fix $\lambda \in C_0$ and write $T_q(x)$ for $T_q(x.\lambda)$, with similar notation for the modular tilting modules. Further, divide W^0 into three disjoint sets:

$$W^0 \;=\; \{e\} \cup \mathcal{C}(s_0) \cup C_{\mathrm{rest}}.$$

That is, C_{rest} is the union of the remaining right cells.

For completeness we begin with results about the “first” cell $\{e\}$.

Lemma 3.3.

1.Let Q be a modular tilting module. Then

$$[Q : T(e)] \;=\; [Q_q : T_q(e)].$$

2.For any $x \in W^0$ we have $[\Theta_s T_q(x) : T_q(e)] = 0$.

Proof. As noted in the introduction, there is a formula for $[Q : T(e)]$. There is a quantum analogue of this formula, see [3]. With Q denoting a quantum tilting module,

$$[Q : T_q(\lambda)] \;=\; \sum_{x\in W,\ x.\lambda\in X^+} (-1)^{l(x)}[Q : V_q(x.\lambda)]. \tag{5}$$

Now the first claim follows by a comparison with equation (1) from the introduction. The second claim follows from (5). ∎

The following result is merely a restatement of proposition 2.7, in the language of quantum tilting modules using remark 3.2.

Corollary 3.4. *Let $z \in \mathcal{C}(s_0)$. Then $[\Theta_s T_q(x) : T_q(z)] \neq 0$ iff $z = xt$ for some $t \in S$, $zs < z$ and $x \in \mathcal{C}(s_0) \cup \{e\}$.*

THEOREM 3.5.

1.Let $z \in \mathcal{C}(s_0)$ and $zs > z$. There is a unique $t \in S$ such that $zt < z$ and

$$\Theta_s T_q(z) \;=\; T_q(zs) \oplus \delta T_q(zt) \oplus \bigoplus_{y \notin \mathcal{C}(s_0)} a_y T_q(y),$$

Here a_y denotes nonnegative integers and δ equals 1 when $zts < zt$ and 0 when $zts > zt$.

2.Assume type $A_{n\geq 2}$, D_n, E_6, E_7, E_8. Let $z < zs \in \mathcal{C}(s_0)$. Then for some nonnegative integers a_y

$$\Theta_s T_q(z) \;=\; T_q(zs) \oplus \bigoplus_{y \in C_{rest}} a_y T_q(y).$$

3.All types. Let $e \neq x \notin \mathcal{C}(s_0)$. Then for some nonnegative integers a_y

$$\Theta_s T_q(x) \;=\; \bigoplus_{y \in C_{rest}} a_y T_q(y).$$

Proof. The first and the last assertion follows directly from corollary 3.4. To see the second it suffices to check that $\delta = 0$ for all $z \in \mathcal{C}(s_0)$ in type $A_{n\geq 2}$, D_n, E_6, E_7 and E_8. Use the description of $\mathcal{C}(s_0)$ given in table 1. ∎

Note that this settles Theorem 1.1 of the introduction.

THEOREM 3.6. *Assume type $A_{n\geq 2}$, D_n, E_6, E_7 or E_8. Let $z \in \mathcal{C}(s_0)$. Then for some nonnegative integers a_y*

$$T(z)_q \;=\; T_q(z) \oplus \bigoplus_{y \in C_{rest}} a_y T_q(y).$$

Proof. Note that $T(s_0)_q = T_q(s_0)$. Now pick $zs \in \mathcal{C}(s_0)$, $zs \neq s_0$ with $zs > z$. Then $z \in \mathcal{C}(s_0)$ by proposition 2.6. We use that the results holds for z by induction. Note the identity $(\Theta_s T(z))_q = \Theta_s(T(z)_q)$; both modules are quantum tilting modules with the same character. By induction and Theorem 3.5 2. and 3. we get

$$\begin{aligned}\Theta_s(T(z)_q) &= \Theta_s T_q(z) \oplus \bigoplus_{y \in C_{\mathrm{rest}}} b_y \Theta_s T_q(y)\\ &= T_q(zs) \oplus \bigoplus_{y \in C_{\mathrm{rest}}} b'_y T_q(y).\end{aligned}$$

We also have

$$(\Theta_s T(z))_q = T(zs)_q \oplus \bigoplus_{x \in W^0} c_x T(x)_q.$$

This proves the claim, since $T_q(zs)$ is a summand of $T(zs)_q$. ∎

Remark 3.7. *Consider type G_2. Let $z \in \mathcal{C}(s_0)$ and recall that $\mathcal{C}(s_0)$ consists of only 8 elements. We claim that $T(z)_q = T_q(z)$. In fact, if $zs < z$ (such s is necessarily unique) then*

$$\operatorname{ch} T(z) = \chi(z) + \chi(zs) = \operatorname{ch} T_q(z).$$

Here $\chi(z)$ denotes the character of the Weyl module. Pick a weight $\mu \in \overline{C}$ with $\mathrm{stab}_W\{\mu\} = \{s\}$; then the sumformula reveals that $V(z.\mu)$ is simple. The claim follows and we have actually proved that Theorem 3.6 holds in type G_2 too.

Remark 3.8. *Erdmann [10] has computed the characters of the modular tilting modules in type A_1. Outside the lowest p^2-alcove, the characters of the quantum and modular tilting modules disagree in general. This shows that Theorem 3.6 cannot hold in type A_1; the set C_{rest} is empty in type A_1.*

Theorem 3.9. *All types. Let $x \in C_{rest}$. Then for some nonnegative integers a_y*

$$T(x)_q = \bigoplus_{y \in C_{rest}} a_y T_q(y).$$

Proof. Suppose that $x > xs \in C_{\mathrm{rest}}$. By induction

$$T(xs)_q = \bigoplus_{y \in C_{\mathrm{rest}}} a_y T_q(y).$$

Then using theorem 3.5 3. we get

$$\begin{aligned} \Theta_s(T(xs)_q) &= \bigoplus_{y \in C_{\mathrm{rest}}} a_y \Theta_s T_q(y) \\ &= \bigoplus_{y \in C_{\mathrm{rest}}} a'_y T_q(y). \end{aligned}$$

On the other hand $T(x)_q$ is a summand of $(\Theta_s T(xs))_q$ and we conclude that

$$T(x)_q \;=\; \bigoplus_{y\in C_{\rm rest}} a''_y T_q(y).$$

It remains to consider x, where $x > xs$ implies $xs \notin C_{\rm rest}$. Then $xs \in \mathcal{C}(s_0)$. If $|\mathcal{R}(x)| = 1$, it is easy to see that $x \in \mathcal{C}(s_0)$. Thus, there is a $t \neq s$ such that $x > xs$ and $x > xt \in \mathcal{C}(s_0)$. Since $\mathcal{R}(x) \setminus \{s\} \subset \mathcal{R}(xs)$ we have $\mathcal{R}(xs) = \{t\}$ and similarly $\mathcal{R}(xt) = \{s\}$. We will have to consider two cases.

- $xtst < xts$ and $xsts < xst$.
- $xtst > xts$ or $xsts > xst$.

Suppose $xsts > xst$. Using Theorem 3.5 1. we get

$$\begin{aligned}\Theta_s(T(xs)_q) &= \Theta_s T_q(xs) \oplus \bigoplus_{y\in C_{\rm rest}} a_y \Theta_s T_q(y)\\ &= T_q(x) \oplus \bigoplus_{y\in C_{\rm rest}} a'_y T_q(y).\end{aligned}$$

Recall that $T(x)_q$ is a summand of $(\Theta_s T(xs))_q$. Hence we conclude that if $xsts > xst$ (or similarly $xtst > xts$) then $T(x)_q = \bigoplus_{y\in C_{\rm rest}} a''_y T_q(y)$.

Suppose that $xtst < xts$ and $xsts < xst$. Since $xts \in \mathcal{C}(s_0)$ we have $xts \neq xst$ by Proposition 2.6. Using Theorem 3.5 1. we get

$$\begin{aligned}\Theta_s(T(xs)_q) &= \Theta_s T_q(xs) \oplus \bigoplus_{y\in C_{\rm rest}} a_y \Theta_s T_q(y)\\ &= T_q(x) \oplus T_q(xst) \oplus \bigoplus_{y\in C_{\rm rest}} a'_y T_q(y).\end{aligned}$$

Similarly

$$\Theta_t(T(xt)_q) = T_q(x) \oplus T_q(xts) \oplus \bigoplus_{y\in C_{\rm rest}} a''_y T_q(y).$$

Recall that $xts \neq xst$ and that $T(x)_q$ is a summand of $(\Theta_s T(xs))_q$ as well as a summand of $(\Theta_t T(xt))_q$. The result follows. ∎

Theorem 3.10. *Assume type $A_{n\geq 2}$, D_n, E_6, E_7, E_8, G_2. Let $z \in \mathcal{C}(s_0)$ and let Q be a tilting G-module. Then*

$$[Q : T(z)] \;=\; [Q_q : T_q(z)].$$

Proof. Since Q is a direct sum of indecomposable tilting modules, it is enough to check the result for all $Q = T(x)$, $x \in W^0$. If $x = e$ both sides of the equality is zero. If $x \in \mathcal{C}(s_0)$ we may apply Theorem 3.6 and Remark 3.7. Finally $x \in C_{\text{rest}}$ is handled with Theorem 3.9. ▮

This proves Theorem 1.2 of the introduction. The last theorem in this section is a generalization of theorem 3.10 above to weights with nontrivial stabilizer.

THEOREM 3.11. *Assume type $A_{n\geq 2}$, D_n, E_6, E_7, E_8 or G_2.*
Consider a weight $\mu \in \overline{C}_0$ and assume that $z \in \mathcal{C}(s_0)$ is maximal among $\{zw \mid w.\mu = \mu\}$. Then, for any modular tilting module Q we have

$$[Q : T(z.\mu)] \;=\; [Q_q : T_q(z.\mu)].$$

Proof. The proof relies on the following results. With the notation from the theorem and with $\lambda \in C_0$ we have

$$\begin{aligned} T_\mu^\lambda T(z.\mu) &= T(z.\lambda) \\ T_\mu^\lambda T_q(z.\mu) &= T_q(z.\lambda). \end{aligned}$$

The first result is stated in [2] Proposition 5.2. For the quantum analogue, see [26] Remark 7.2 (2). To prove the claim it is enough to check the identity

$$[T_\mu^\lambda Q : T(z.\lambda)] \;=\; [(T_\mu^\lambda Q)_q : T_q(z.\lambda)].$$

But this identity follows directly from Theorem 3.10. ▮

4. SCHUR-WEYL DUALITY

In this section we apply the results of the first part of the paper to the special case of a rootsystem of type $A_{n\geq 2}$. Let $\{\alpha_{i,j} \mid 1 \leq i \neq j \leq n+1\}$ denote the set of roots. Let

$$\begin{aligned} \Sigma_1 &= \{(s_o s_1 \dots s_n)^m s_0 s_1 \dots s_k \mid m \geq 0, n \geq k \geq 0\} \cup \{e\} \\ \Sigma_2 &= \{(s_o s_n \dots s_1)^m s_0 s_n \dots s_k \mid m \geq 0, n \geq k \geq 0\} \cup \{e\} \end{aligned}$$

and observe that according to table 1 we have $\mathcal{C}(s_0) \cup \{e\} = \Sigma_1 \cup \Sigma_2$. Define

$$\begin{aligned} D_1 &= \{\lambda \in E \mid 0 < \langle \lambda + \rho, \alpha_{1,n}^\vee \rangle < p \text{ and } 0 < \langle \lambda, \alpha_i^\vee \rangle \text{ for } i = 1 \dots n\} \\ D_2 &= \{\lambda \in E \mid 0 < \langle \lambda + \rho, \alpha_{2,n+1}^\vee \rangle < p \text{ and } 0 < \langle \lambda, \alpha_i^\vee \rangle \text{ for } i = 1 \dots n\}. \end{aligned}$$

THEOREM 4.1. *Suppose that $\lambda \in D_1$ or $\lambda \in D_2$ and that Q is a tilting module. Then*

$$[Q : T(\lambda)] = [Q_q : T_q(\lambda)].$$

Proof. The Theorem follows from Theorem 3.10 and Theorem 3.11 together with the following claims:

$$\begin{aligned} x.C_0 \subset D_1 &\iff x \in \Sigma_1 \\ x.C_0 \subset D_2 &\iff x \in \Sigma_2 \end{aligned}$$

The claims are equivalent, so we sketch a proof of the first.

It is an essential ingredient in the proof to see that $x.C_0 \subset D_1$ and $xs < x$ implies $xs.C_0 \subset D_1$. Interpret $l(x)$ as the number of hyperplanes separating C_0 and $x.C_0$; then note that $x.C_0 \subset D_1$ and $xs.C_0 \not\subseteq D_1$ implies $l(xs) = l(x) + 1$.

A calculation shows that $(s_0 s_1 \cdots s_n)^m.C_0 \subset D_1$ thus establishing the 'if'-part. The 'only if'-part is proved by induction in the length of x. Choose $s \in S$ such that $xs < x$. Now $x.C_0 \subset D_1$ implies $xs.C_0 \subset D_1$ and hence $xs \in \Sigma_1$. This shows that $xt < x$ implies $xt \in \Sigma_1$. But Σ_1 has only one element of each length and we conclude that $\mathcal{R}(x) = \{s\}$. Then $x \sim_R xs$ and hence $x \in \Sigma_1 \cup \Sigma_2$. It is not difficult to exclude the case $x \in \Sigma_2$. ∎

4.1. The symmetric group

Consider the general linear group $\mathrm{GL}_n(k)$. Fix the subgroup of diagonal matrices as torus and let ϵ_i denote the character that takes a diagonal matrix to its (i,i)'th entry. The weight lattice of $\mathrm{GL}_n(k)$ is the free $\mathbb{Z}$-module with basis $\{\epsilon_1, \dots \epsilon_n\}$ and the rootsystem is of type A_{n-1}.

Let V denote the natural $\mathrm{GL}_n(k)$-module. It is a simple Weyl module and hence tilting. Then, by the properties of tilting modules, so is $V^{\otimes r}$. We thus obtain a decomposition

$$V^{\otimes r} = \bigoplus_{\lambda \in X^+} m_\lambda T_{\mathrm{GL}_n(k)}(\lambda).$$

Note that we denote the indecomposable tilting $\mathrm{GL}_n(k)$-modules by $T_{\mathrm{GL}_n(k)}(\lambda)$, thus reserving the notation $T(\lambda)$ for the indecomposable tilting modules of the almost simple group $\mathrm{SL}_n(k)$.

Let Σ_r denote the symmetric group on r letters. The $\mathrm{GL}_n(k)$-module $V^{\otimes r}$ carries also a structure of Σ_r-modules, given by permutation of the factors. The action of the general linear group and the action of the symmetric group commutes, and (see [5, Theorem 4.1]) we have a surjective ring homomorphism

$$k[\Sigma_r] \longrightarrow \mathrm{End}_{\mathrm{GL}_n(k)}(V^{\otimes r}). \tag{6}$$

By ring theory Σ_r has a simple module for each isomorphism class of indecomposable $\mathrm{GL}_n(k)$-modules appearing in the decomposition of $V^{\otimes r}$. The dimension of this simple Σ_r-module is given by the multiplicity of the indecomposable $\mathrm{GL}_n(k)$-module. We observed above that the indecomposable summands of $V^{\otimes r}$ are tilting; it remains to describe the simple Σ_r-module with dimension equal to the multiplicity in $V^{\otimes r}$ of a given indecomposable tilting module.

Recall that a weight $n_1\epsilon_1 + \cdots + n_n\epsilon_n$ of $\mathrm{GL}_n(k)$ is dominant when $n_1 \geq \cdots \geq n_n$ and polynomial if all $n_i \geq 0$. This establishes a bijection between polynomial dominant weights and partitions with at most n part; the weight $n_1\epsilon_1 + \cdots + n_n\epsilon_n$ corresponds to the partition $(n_1, n_2, \ldots n_n)$. With this bijection in mind, we make no distinction between elements of the two sets. The size of a weight $n_1\epsilon_1 + \cdots + n_n\epsilon_n$ is the sum $\sum_{1\leq i\leq n} n_i$. Note that the weights of $V^{\otimes r}$ are polynomial of size r; hence the tilting summands of $V^{\otimes r}$ are parametrized by dominant polynomial weights of size r.

We can now state the precise correspondence between simple Σ_r-modules and indecomposable tilting summands of $V^{\otimes r}$. Recall that the simple Σ_r-modules are parametrized by p-regular partitions; we denote the simple module corresponding to a partition λ by D^λ. In this notation we have

$$\dim D^\lambda \;=\; [V^{\otimes r} : T_{\mathrm{GL}_n(k)}(\lambda)]. \tag{7}$$

In fact $[V^{\otimes r} : T_{\mathrm{GL}_n(k)}(\lambda)] = 0$ when λ is p-singular. A convenient reference for the theory outlined above is [23, section 11].

Theorem 4.1 provides a formula for the multiplicities of a tilting module for the almost simple groups of type $A_{n\geq 2}$. Accordingly, we consider $\mathrm{SL}_n(k)$, the derived group of $\mathrm{GL}_n(k)$. We write $\overline{\lambda}$ for the restriction of a $\mathrm{GL}_n(k)$-weight λ; thus $\lambda = n_1\epsilon_1 + \cdots + n_n\epsilon_n$ gives $\overline{\lambda} = (n_1 - n_2)\omega_1 + \cdots + (n_{n-1} - n_n)\omega_{n-1}$, where ω_i denotes the i'th fundamental weight. By 3.1 we immediately have

$$T_{\mathrm{GL}_n(k)}(\lambda)|_{\mathrm{SL}_n(k)} \;=\; T(\overline{\lambda}). \tag{8}$$

Furthermore, at most one $\mathrm{GL}_n(k)$-weight of $V^{\otimes r}$ restricts to the $\mathrm{SL}_n(k)$-weight $\overline{\lambda}$. This shows

$$[V^{\otimes r} : T_{\mathrm{GL}_n(k)}(\lambda)] \;=\; [V^{\otimes r} : T(\overline{\lambda})]. \tag{9}$$

Theorem 4.2. *Let $p \geq n$ and let $\lambda = (n_1 \geq \cdots \geq n_n \geq 0)$ denote a partition with at least three parts. We can compute* $\dim D^\lambda$ *whenever*

- $n_1 - n_{n-1} < p - n + 2$ *or*
- $n_2 - n_n < p - n + 2$.

Explicitly we have

$$\dim D^\lambda \;=\; [V_q^{\otimes r} : T_q(\overline{\lambda})].$$

Proof. The partition λ satisfies $n_1 - n_{n-1} < p - n + 2$ if and only if the $\mathrm{SL}_n(k)$-weight $\overline{\lambda}$ belongs to D_1. Likewise λ satisfies $n_2 - n_n < p - n + 2$ if and only if $\overline{\lambda} \in D_2$. The theorem follows by (7), (9) and Theorem 4.1. ▮

4.2. Quantum Schur-Weyl duality

The formula of theorem 4.2 provides a closed formula for the dimension of D^λ. In practice the formula may be tedious to work with; we reformulate it in theorem 4.3 below. See also [23, section 15].

Let $\mathcal{H}_q$ denote the Hecke algebra of Σ_r, with q a primitive p'th root of unity. The representation theory of $\mathcal{H}_q$ resembles that of the symmetric group. In particular, the simple modules of $\mathcal{H}_q$ are indexed by p-regular partitions of r and denoted by D_q^λ. There is also a quantum analogue of (6): By [9, Theorem 6.3] we have a surjective homomorphism of rings

$$\mathcal{H}_q \longrightarrow \mathrm{End}_{U_q(\mathfrak{gl}_n)}(V^{\otimes r}). \tag{10}$$

As in the modular case, the multiplicities of $V^{\otimes r}$ determines the dimension of the simple modules of $\mathcal{H}_q$. By [8, Proposition 2.7] we may embed $U_q = U_q(\mathfrak{sl}_n)$ into $U_q(\mathfrak{gl}_n)$, and by imitating the arguments of (9) we get

$$[V^{\otimes r} : T_{U_q(\mathfrak{gl}_n)}(\lambda)] = [V^{\otimes r} : T_q(\overline{\lambda})]. \tag{11}$$

Theorem 4.3. *Let $p \geq n$ and let $\lambda = (n_1 \geq \cdots \geq n_n \geq 0)$ denote a partition with at least three parts. Assume $n_1 - n_{n-1} < p - n + 2$ or $n_2 - n_n < p - n + 2$. Then*

$$\dim D^\lambda = \dim D_q^\lambda. \tag{12}$$

Proof. We argue as follows, using the equations and theorems above.

$$\begin{aligned} \dim D^\lambda &= [V^{\otimes r} : T_{\mathrm{GL}_n(k)}(\lambda)] \\ &= [V^{\otimes r} : T(\overline{\lambda})] \\ &= [V_q^{\otimes r} : T_q(\overline{\lambda})] \\ &= [V_q^{\otimes r} : T_{U_q(\mathfrak{gl}_n)}(\lambda)] \\ &= \dim D_q^\lambda \end{aligned}$$

∎

This proves Theorem 1.3 of the introduction. The dimension of D_q^λ is known; see [17] and [4].

Remark 4.4. *Jensen [13] shows the multiplicity formula of Theorem 4.1 in type A_2; see Proposition 2.2.3 and the remark following it in loc.cit. The corresponding dimension result about simple Σ_r-modules parametrized by Young diagrams with three lines is the subject of [12].*

Note also that Theorem 4.3 proves a special case of conjecture 15.4 in [23].

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