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MASAKI KASHIWARA AND NIELS LAURITZEN

1. Introduction

Let k be a field. Consider the polynomial ring

$$R = k \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{bmatrix}$$

and $I \subset R$ the ideal generated by the three 2×2 minors

$$f_1 := \begin{vmatrix} X_{12} & X_{13} \\ X_{22} & X_{23} \end{vmatrix}, \quad f_2 := \begin{vmatrix} X_{13} & X_{11} \\ X_{23} & X_{21} \end{vmatrix} \quad \text{and} \quad f_3 := \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix}.$$

If k is a field of positive characteristic, the local cohomology modules $\mathrm{H}^j_I(R)$ vanish for $j > 2$ (see Chapitre III, Proposition (4.1) in [2]). However, if k is a field of characteristic zero, $\mathrm{H}^3_I(R)$ is non-vanishing (see Proposition 2.1 of this paper or Remark 3.13 in [4]).

Consider the Grassmann variety $X = \mathrm{Gr}(2, V)$ of 2-dimensional vector subspaces of a 5-dimensional vector space V over k. Let us take a two dimensional subspace W of V. Then the singularity R/I appears in the Schubert variety $Y \subset X$ of 2-dimensional subspaces E such that $\dim(E \cap W) \geq 1$. Therefore $\mathcal{H}^3_Y(\mathcal{O}_X)$ does not vanish in characteristic zero while it does vanish in positive characteristic.

In this paper we show how this difference in the vanishing of local cohomology translates into a non-vanishing first cohomology group for the $\mathcal{D}_X$-module $\mathcal{H}^2_Y(\mathcal{O}_X)$ in positive characteristic.

Previous work of Haastert ([3]) showed the Beilinson-Bernstein equivalence ([1]) to hold for projective spaces and the flag manifold of SL_3 in positive characteristic. However, as we show in this paper, $\mathcal{D}$-affinity breaks down for the flag manifold of SL_5 in all positive characteristics. The Beilinson-Bernstein equivalence, therefore, does not carry over to flag manifolds in positive characteristic.

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2. Local cohomology

Keep the notation from §1. A topological proof of the following proposition is given in §5.

Proposition 2.1. *$H^3_I(R)$ does not vanish in characteristic zero.*

Corollary 2.2. *$\mathcal{H}^3_Y(\mathcal{O}_X)$ does not vanish in characteristic zero.*

The local to global spectral sequence

$$\mathrm{H}^p(X, \mathcal{H}^q_Y(\mathcal{O}_X)) \Rightarrow \mathrm{H}^{p+q}_Y(X, \mathcal{O}_X)$$

and $\mathcal{D}$-affinity in characteristic zero ([1]) implies

$$\mathrm{H}^3_Y(X, \mathcal{O}_X) = \Gamma(X, \mathcal{H}^3_Y(\mathcal{O}_X)) \neq 0.$$

On the other hand, if k is a field of positive characteristic, $\mathcal{H}^q_Y(\mathcal{O}_X) = 0$ if $q \neq 2$, since Y is a codimension two Cohen Macaulay subvariety of the smooth variety X ([2], Chapitre III, Proposition (4.1)). This gives a totally different degeneration of the local to global spectral sequence. In the positive characteristic case we get

$$\mathrm{H}^p(X, \mathcal{H}^2_Y(\mathcal{O}_X)) \cong \mathrm{H}^{p+2}_Y(X, \mathcal{O}_X).$$

We will prove that $\mathrm{H}^3_Y(X, \mathcal{O}_X) \neq 0$ even if k is a field of positive characteristic. This will give the desired non-vanishing

$$\mathrm{H}^1(X, \mathcal{H}^2_Y(\mathcal{O}_X)) \neq 0$$

in positive characteristic.

3. Lifting to $\mathbb{Z}$

To deduce the non-vanishing of $\mathrm{H}^3_Y(X, \mathcal{O}_X)$ in positive characteristic, we need to compute the local cohomology over $\mathbb{Z}$ and proceed by base change. Flag manifolds and their Schubert varieties admit flat lifts to $\mathbb{Z}$-schemes. In this section $X_\mathbb{Z}$ and $Y_\mathbb{Z}$ will denote flat lifts of a flag manifold X and a Schubert variety $Y \subset X$ respectively.

The local Grothendieck-Cousin complex (cf. [5], §8) of the structure sheaf $\mathcal{O}_{X_\mathbb{Z}}$

$$\mathcal{H}^0_{X_0/X_1}(\mathcal{O}_{X_\mathbb{Z}}) \to \mathcal{H}^1_{X_1/X_2}(\mathcal{O}_{X_\mathbb{Z}}) \to \cdots, \tag{3.1}$$

where X_i denotes the union of Schubert schemes of codimension i, is a resolution of $\mathcal{O}_{X_\mathbb{Z}}$, since $\mathcal{O}_{X_\mathbb{Z}}$ is Cohen Macaulay, codim $X_i \geq i$ and $X_i \setminus X_{i+1} \to X$ are affine morphisms for all i (see [5], Theorem 10.9). The sheaves in this resolution decompose into direct sums

$$\mathcal{H}^i_{X_i/X_{i+1}}(\mathcal{O}_{X_\mathbb{Z}}) = \bigoplus_{\mathrm{codim}(C)=i} \mathcal{H}^i_C(\mathcal{O}_{X_\mathbb{Z}})$$

of local cohomology sheaves $\mathcal{H}^i_C(\mathcal{O}_{X_\mathbb{Z}})$ with support in Bruhat cells C of codimension i. The degeneration of the local to global spectral sequence gives

$$\mathrm{H}^p_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{H}^c_C(\mathcal{O}_{X_\mathbb{Z}})) = \mathrm{H}^{p+c}_{C\cap Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}),$$

since $\mathcal{H}^i_C(\mathcal{O}_{X_\mathbb{Z}}) = 0$ if $i \neq c = \mathrm{codim}(C)$.

Since the scheme $X_i \setminus X_{i+1}$ is affine it follows that $\mathrm{H}^p_C(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) = 0$ if $p \neq \mathrm{codim}(C)$ ([5], Theorem 10.9). This shows that the resolution (3.1) is acyclic for the functor $\Gamma_{Y_\mathbb{Z}}$ and

$$\Gamma_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{H}^c_C(\mathcal{O}_{X_\mathbb{Z}})) = \begin{cases} 0 & \text{if } C \not\subset Y_Z \\ \mathrm{H}^c_C(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) & \text{if } C \subset Y_\mathbb{Z}, \end{cases}$$

where $c = \mathrm{codim}(C)$. Applying $\Gamma_{Y_\mathbb{Z}}(X_\mathbb{Z}, -)$ to (3.1) we get the complex

$$M^\bullet_Y : \mathrm{H}^c_{C_Y}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \to \bigoplus_{\mathrm{codim}(C)=c+1,\, C\subset Y_\mathbb{Z}} \mathrm{H}^{c+1}_C(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \to \cdots,$$

where c is the codimension of $Y_\mathbb{Z}$, C_Y is the open Bruhat cell in $Y_\mathbb{Z}$ and $\mathrm{H}^c_{C_Y}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ sits in degree c. Notice that $\mathrm{H}^i(M^\bullet_Y) = \mathrm{H}^i_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ and that $M^\bullet_Y$ is a complex of free abelian groups. In fact the individual entries $\mathrm{H}^i_C(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ are direct sums of weight spaces, which are finitely generated free abelian groups (cf. [5], Theorem 13.4). By weight spaces we mean eigenspaces for a fixed $\mathbb{Z}$-split torus T. The differentials in $M^\bullet_Y$ being T-equivariant, the complex $M^\bullet_Y$ is a direct sum of complexes of finitely generated free abelian groups. Since $\mathrm{H}^i(M^\bullet_Y) = \mathrm{H}^i_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$, one obtains the following lemma.

Lemma 3.1. *Every local cohomology group $\mathrm{H}^i_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ is a direct sum of finitely generated abelian groups. In the codimension c of $Y_\mathbb{Z}$ in $X_\mathbb{Z}$, $\mathrm{H}^c_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ is a free abelian group.*

4. The counterexample

For a field k, let us set $X_k = X_\mathbb{Z} \otimes k$ and $Y_k = Y_\mathbb{Z} \otimes k$. Then one has $H^q_{Y_k}(X_k, \mathcal{O}_{X_k}) = \mathrm{H}^q_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}} \otimes k)$. Since $\mathcal{O}_{X_\mathbb{Z}}$ is flat over $\mathbb{Z}$, one has a spectral sequence

$$\mathrm{Tor}^\mathbb{Z}_{-p}(\mathrm{H}^q_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}), k) \Rightarrow \mathrm{H}^{p+q}_{Y_k}(X_k, \mathcal{O}_{X_k}).$$

This shows that the natural homomorphism

$$\mathrm{H}^i_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \otimes k \longrightarrow \mathrm{H}^i_{Y_k}(X_k, \mathcal{O}_{X_k})$$

is an injection, and it is an isomorphism if the field k is flat over $\mathbb{Z}$.

In our example (cf. §1), one has

$$\mathrm{H}^3_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \otimes \mathbb{C} \cong \mathrm{H}^3_{Y_\mathbb{C}}(X_\mathbb{C}, \mathcal{O}_{X_\mathbb{C}}) \neq 0.$$

By Lemma 3.1, the cohomology $\mathrm{H}^3_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}})$ must contain $\mathbb{Z}$ as a direct summand. Therefore the injection

$$\mathrm{H}^3_{Y_\mathbb{Z}}(X_\mathbb{Z}, \mathcal{O}_{X_\mathbb{Z}}) \otimes k \longrightarrow \mathrm{H}^3_{Y_k}(X_k, \mathcal{O}_{X_k})$$

shows that $\mathrm{H}^3_{Y_k}(X_k, \mathcal{O}_{X_k})$ is non-vanishing for any field k of positive characteristic. Since $\mathrm{H}^3_{Y_k}(X_k, \mathcal{O}_{X_k}) \cong \mathrm{H}^1(X_k, \mathcal{H}^2_{Y_k}(\mathcal{O}_{X_k}))$, one obtains the following result.

Proposition 4.1. $\mathrm{H}^1(X_k, \mathcal{H}^2_{Y_k}(\mathcal{O}_{X_k})) \neq 0$ *if k is of positive characteristic.*

5. Proof of non-vanishing of $H^3_I(R)$

In this section, we shall give a topological proof of Proposition 2.1. We may assume that the base field is the complex number field $\mathbb{C}$.

The local cohomologies $H^*_I(R)$ are the cohomology groups of the complex

$$\begin{aligned} R &\longrightarrow R[f_1^{-1}] \oplus R[f_2^{-1}] \oplus R[f_3^{-1}] \\ &\qquad \longrightarrow R[(f_1f_2)^{-1}] \oplus R[(f_2f_3)^{-1}] \oplus R[(f_1f_3)^{-1}] \longrightarrow R[(f_1f_2f_3)^{-1}]\,. \end{aligned}$$

Hence one has

$$H^3_I(R) = \frac{R[(f_1f_2f_3)^{-1}]}{R[(f_1f_2)^{-1}] + R[(f_2f_3)^{-1}] + R[(f_1f_3)^{-1}]}.$$

In order to prove the non-vanishing of $H^3_I(R)$, it is enough to show

$$\frac{1}{f_1f_2f_3} \notin R[(f_1f_2)^{-1}] + R[(f_2f_3)^{-1}] + R[(f_1f_3)^{-1}]\,. \tag{5.2}$$

Consider the 6-cycle

$$\begin{aligned} \gamma &= \Big\{ \begin{pmatrix} -t_2u + t_3\bar{v} & u & -t_1\bar{v} \\ -t_2v - t_3\bar{u} & v & t_1\bar{u} \end{pmatrix} \\ &\qquad\qquad ;\ |t_1| = |t_2| = |t_3| = 1,\ |u|^2 + |v|^2 = 1 \Big\} \\ &= \Big\{ k \begin{pmatrix} -t_2 & 1 & 0 \\ -t_3 & 0 & t_1 \end{pmatrix} \\ &\qquad\qquad ;\ |t_1| = |t_2| = |t_3| = 1,\ k = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \in SU(2) \Big\} \end{aligned}$$

in $X \setminus (f_1f_2f_3)^{-1}(0)$, where $X = \operatorname{Spec}(R) \cong \mathbb{C}^6$. Then on γ one has

$$f_1 = t_1,\ f_2 = t_1t_2 \text{ and } f_3 = t_3\,.$$

Set $\omega = \bigwedge dX_{ij}$. Then one has $\omega = t_1 dt_1 dt_2 dt_3 \theta$ on γ, where θ is a non-zero invariant form on $SU(2)$. Therefore one has

$$\int_\gamma \frac{\omega}{f_1f_2f_3} = \int_\gamma \frac{dt_1dt_2dt_3\theta}{t_1t_2t_3} \neq 0.$$

Hence, in order to show (5.2), it is enough to prove that

$$\int_\gamma \varphi\omega = 0 \tag{5.3}$$

for any $\varphi \in R[(f_1f_2)^{-1}] + R[(f_2f_3)^{-1}] + R[(f_1f_3)^{-1}]$.

For $\varphi \in R[(f_1f_2)^{-1}]$, the equation (5.3) holds because we can shrink the cycle γ by $|t_3| = \lambda$ from $\lambda = 1$ to $\lambda = 0$. For $\varphi \in R[(f_1f_3)^{-1}]$), the equation (5.3) holds because we can shrink the cycle γ by $|t_2| = \lambda$ from $\lambda = 1$ to $\lambda = 0$.

Let us show (5.3) for $\varphi \in R[(f_2f_3)^{-1}]$. Let us deform the cycle γ by

$$\gamma_\lambda = \left\{ k \begin{pmatrix} -(1-\lambda)t_2 & 1 & -\lambda t_1t_2t_3^{-1} \\ -t_3 & 0 & t_1 \end{pmatrix} \right. \left. ; |t_1| = |t_2| = |t_3| = 1,\ k \in SU(2) \right\}.$$

Note that the values of f_1, f_2 and f_3 do not change under this deformation. Hence γ_λ is a cycle in $X \setminus (f_1f_2f_3)^{-1}(0)$. One has

$$\begin{aligned} \gamma_1 &= \left\{ k \begin{pmatrix} 0 & 1 & -t_1t_2t_3^{-1} \\ -t_3 & 0 & t_1 \end{pmatrix} ; |t_1| = |t_2| = |t_3| = 1,\ k \in SU(2) \right\} \\ &= \left\{ k \begin{pmatrix} 0 & 1 & -t_2 \\ -t_3 & 0 & t_1 \end{pmatrix} ; |t_1| = |t_2| = |t_3| = 1,\ k \in SU(2) \right\}. \end{aligned}$$

In the last coordinates of γ_1, one has $f_2 = t_2t_3$ and $f_3 = t_3$. Hence, for $\varphi \in R[(f_2f_3)^{-1}]$,

$$\int_\gamma \varphi\omega = \int_{\gamma_1} \varphi\omega$$

vanishes because we can shrink the cycle γ_1 by $|t_1| = \lambda$ from $\lambda = 1$ to $\lambda = 0$.

Remark. Although we do not give a proof here, $H^3_I(R)$ is isomorphic to $H^6_J(R)$ as a D-module. Here J is the defining ideal of the origin.

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