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THE SCATTERING MATRIX FOR $\Gamma_0(N)$ WITH A PRIMITIVE, REAL CHARACTER

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THE SCATTERING MATRIX FOR $\Gamma_0(N)$ WITH A PRIMITIVE, REAL CHARACTER.

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Abstract. We study the scattering matrix for the congruence subgroups $\Gamma_0(N)$, with a character χ corresponding to a real, primitive character χ_N. The existence of such a character obviously puts restrictions on N. We obtain an explicit expression for the scattering matrix, which turns out to be "skew-diagonal".

Contents

1. Introduction

It is common knowledge among specialists in number theory that the Laplacian on certain hyperbolic surfaces contains information on deep number theoretic quantities. This insight is, among others, due to Maass and Selberg (see for instance [Hej83] or [Kub73]).

An important function in this context is the so-called *scattering matrix* $C(s)$ (for a connection to scattering theory see [LP76]), which is the object of study in this article. We will look at the Hecke subgroups $\Gamma_0(N)$, where

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in PSL(2,\mathbb{Z}) | N, c \in \mathbb{Z}\}.$$

On these groups we will define characters $\chi(\begin{pmatrix} a & b \\ Nc & d \end{pmatrix}) = \chi_N(d)$, where χ_N is a real, even, primitive character $\mod N$. Let $A(\Gamma_0(N),\chi)$ be the Laplacian on the hyperbolic plane $\mathcal{H}$ restricted to the subspace of functions f satisfying

$$f(g.z) = \chi(g) f(z),$$

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for all $g \in \, ,_0(N)$, then we will calculate the corresponding scattering matrix explicit. This is the main result of this article which will be stated more precisely below.

Now we will recall som notation and notions from number theory and Selberg theory and explain the structure of the article.

It is known that one can chose a fundamental domain for $,_0(N)$ as a hyperbolic polygon with a number of cusps. In each of these cusps κ we can chose a parabolic element P_κ of $,_0(N)$ that generates the stabilizer of κ i.e.

$$< P_\kappa >= \{g \in \, ,_0(N) | g.\kappa = \kappa\}.$$

The character χ is singular in κ if $\chi(P_\kappa) = 1$ and non-singular if $\chi(P_\kappa) \neq 1$. We will also say that χ leaves the cusp κ open if χ is singular there and say that χ closes the cusp κ if χ is non-singular in κ. In each open cusp κ we have a (family of) generalised eigenfunction(s) - the Eisenstein series - defined as

$$z \mapsto E(z; s; \chi) = \sum_{\gamma \in < P_\kappa > \backslash \Gamma_0(N)} y^s(g_\kappa.\gamma.z)\chi(\gamma),$$

where $y(w) = \Im(w) =$ the imaginary part of the complex number w, and where $g_\kappa \in PSL(2, \mathbb{R})$ satisfies:

$$g_\kappa.\kappa = \infty, \; g_\kappa P_\kappa g_\kappa^{-1} z = z + 1 \text{ for all } z \in \mathcal{H}.$$

Let $\mathcal{E}(z; s; \chi)$ be a vector consisting of the Eisenstein series for a complete set of inequivalent, open cusps. Then there exists a matrix $C(s)$ - the scattering matrix - such that

$$\mathcal{E}(z; s; \chi) = C(s)\mathcal{E}(z; 1 - s; \chi).$$

For all the groups $,_0(N)$ that we will consider the open cusps are a subset of $\{0, \infty, \frac{1}{d}\}$ where d runs over all divisors of N. Let us order the divisors corresponding to open cusps:

$$1 < d_1 < d_2 < \cdots < d_{h-2} < N,$$

and write

$$\mathcal{E}(z; s; \chi) = (E_0(z; s; \chi), E_{\frac{1}{d_1}}(z; s; \chi), \cdots, E_\infty(z; s; \chi)).$$

Then the result of this paper can be stated:

$$\begin{aligned} C(s) \;\; &= \;\; \left(\frac{N}{\pi}\right)^{1-2s} \frac{L(2-2s, \chi_N)}{L(2s, \chi_N)} \frac{(-s)!}{(s-1)!} \times \\ &\left(\begin{array}{ccccc} 0 & 0 & \cdots & 0 & N^{1-s}\frac{\tau(\chi_1)}{\tau(\chi_N)} \\ 0 & & & \frac{N_{d_1}^{1-s}}{d_1^s}\frac{\tau(\chi_{d_1})}{\tau(\chi_{N_{d_1}})} & 0 \\ \vdots & & \cdot & & \vdots \\ 0 & \frac{N_{d_{h-2}}^{1-s}}{d_{h-2}^s}\frac{\tau(\chi_{d_{h-2}})}{\tau(\chi_{N_{d_{h-2}}})} & & & 0 \\ N^{-s}\frac{\tau(\chi_N)}{\tau(\chi_1)} & 0 & \cdots & 0 & 0 \end{array} \right)., \end{aligned}$$

where $N_d = N/d$ and $\tau(\chi)$ is a Gauss sum.

For a more precise statement corresponding to each of the above 4 cases, see Theorem 6.2.

In this article we will calculate the scattering matrix for the following groups with their respective characters:

- $\Gamma_0(N_1)$, where $N_1 = \prod_{i=1}^n p_i \equiv 1 \mod 4$ and where the p_i are different, odd primes. The character is in this case $\chi_{N_1}(d) = \prod_i \chi_{p_i}(d)$, where

$$\chi_{p_i}(d) = \begin{cases} 0 & (d, p_i) > 1 \\ 1 & (d, p_i) = 1 \text{ and } \exists x \in \mathbb{Z} : x^2 \equiv d \mod p_i \\ -1 & \text{if not} \end{cases}$$

- $\Gamma_0(N_2)$, $N_2 = 4M_2$, $M_2 = \prod_{i=1}^n p_i \equiv 3 \mod 4$ and the p_i are again different, odd primes. In this case the character is $\chi_{N_2}(d) = \chi_4(d) \prod_i \chi_{p_i}(d)$, where χ_{p_i} is as above and

$$\chi_4(d) = \begin{cases} 0 & (d, 4) > 1 \\ 1 & d \equiv 1 \mod 4 \\ -1 & d \equiv 3 \mod 4 \end{cases}$$

- $\Gamma_0(N_3)$, $N_3 = 8M_3$, $M_3 = \prod_{i=1}^n p_i \equiv 1 \mod 4$, the p_i's are again different, odd primes. The character is $\chi_{N_3}(d) = \chi_8(d) \prod_i \chi_{p_i}(d)$, where

$$\chi_8(d) = \begin{cases} 0 & (d, 8) > 1 \\ 1 & d \equiv \pm 1 \mod 8 \\ -1 & d \equiv \pm 3 \mod 8 \end{cases}$$

- Finally, we will consider $\Gamma_0(N_4)$,$N_4 = 8M_4$, $M_4 = \prod_{i=1}^n p_i \equiv 3 \mod 4$, the p_i's are again different, odd primes. The character is $\chi_{N_4}(d) = \chi_4(d)\chi_8(d) \prod_i \chi_{p_i}(d)$.

These cases give all real, even[1], primitive characters (see [Dav67]).

For clarity of exposition, we will first go through the entire calculation for the first case above, i.e. $N = \prod_i p_i \equiv 1 \mod 4$, where the p_i's are different, odd primes. This is the subject of Sections 2 - 5 below. Then in Section 6 we will explain the few extra arguments needed in the remaining cases and state the corresponding results.

2. Cusps

We know from the general theory of Fuchsian groups (see [Shi71]) that the number of cusps h for $\Gamma_0(N)$ satisfies:

$$h = \sum_{d|N} \phi((N/d, d)),$$

where ϕ is Euler's ϕ function i.e.

$$\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^* = \#\{k \in \{0, 1, \cdots, n-1\} | (k, n) = 1\}.$$

Since N is a product of different primes $N = \prod_{i=1}^n p_i$, where the p_i's are primes and $p_i \neq p_j$ when $i \neq j$, we thus get:

$$h = \sum_{d|N} 1 = \#\{d \in 1, \cdot, N \Big| d|N\}.$$

An easy calculation gives that $\{0, \infty, 1/d$ where $d|N, 1 < d < N\}$ is a set of pairwise inequivalent cusps. Since the set contains the right number of elements it must obviously be a complete set.

[1]We need $\chi_N(-d) = \chi_N(d)$ for χ to be well-defined.

3. Eisenstein Series

In this section we want to prove that the Eisenstein series have the following form

Lemma 3.1. *The Eisenstein series for the cusp $\frac{1}{d}$ with character χ is given by the following expression*

$$E_{\frac{1}{d}}(z;s;\chi) = N_d^{-s}\chi_{N_d}(-1)\sum_{\gamma,\delta:(\gamma,\delta)=1}\frac{\chi_{N_d}(\gamma)\chi_d(\delta)y^s}{|d\gamma z+\delta|^{2s}},$$

where $N_d = N/d$.

Proof. Let d: $1 < d < N$ be a divisor of N. Then the parabolic generator of the subgroup that fixes $\frac{1}{d}$ is

$$\begin{aligned} P &= \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-wd & w \\ -d^2w & 1+dw \end{pmatrix}. \end{aligned}$$

Since this has to be an element of $\Gamma_0(N)$ we get $w = N_d$ and thus $\chi(P) = \chi(1) = 1$ i.e. the character is singular in the cusp $\frac{1}{d}$.

Let us furthermore write $\tilde{\Gamma}_\infty =< \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} >$ and calculate the cosets $\tilde{\Gamma}_\infty \setminus g^{-1}\Gamma_0(N)g$, where $g = \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix}$ sends $\frac{1}{d}$ to ∞:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ N\gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} &= \begin{pmatrix} \alpha+\beta d & \beta \\ N\gamma-\alpha d+d\delta-\beta d^2 & \delta-\beta d \end{pmatrix} \\ &= \begin{pmatrix} \alpha' & \beta' \\ d\gamma' & \delta' \end{pmatrix} \in \Gamma_0(d). \end{aligned}$$

Notice that $\gamma'-\delta' = N_d\gamma-\alpha$ and therefore $(\gamma'-\delta', N_d) = 1$. Let, on the other hand $\begin{pmatrix} \alpha & \beta \\ d\gamma & \delta \end{pmatrix} \in \Gamma_0(d)$ and $(\gamma-\delta, N_d) = 1$. Then we search $n \in \mathbb{Z}$ such that

$$\begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}\begin{pmatrix} \alpha+nd\gamma & \beta+n\delta \\ d\gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \in \Gamma_0(N).$$

Now

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}\begin{pmatrix} \alpha+nd\gamma & \beta+n\delta \\ d\gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \\ =& \begin{pmatrix} \alpha+nd\gamma-\beta d-nd\delta & \beta+n\delta \\ d\gamma+d\alpha+nd^2\gamma-\beta d^2-nd^2\delta-d\delta & \beta d+nd\delta+\delta \end{pmatrix}, \end{aligned}$$

so this is the case iff

$$N|d\gamma+d\alpha+nd^2\gamma-\beta d^2-nd^2\delta-d\delta,$$

i.e. iff

$$\gamma+\alpha-\beta d-\delta \equiv nd(\delta-\gamma) \mod N_d. \tag{3.1}$$

That equation surely has integer soultions since $(d(\delta-\gamma), N_d) = 1$. From (3.1) we also see that two different integer solutions n, n' satisfy

$$0 \equiv n-n' \mod N_d,$$

which is equivalent to the existence of an $m \in \mathbb{Z}$ such that

$$\begin{pmatrix} 1 & mw \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha + nd\gamma & \beta + n\delta \\ d\gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha + n'd\gamma & \beta + n'\delta \\ d\gamma & \delta \end{pmatrix}.$$

Finally, we need to calculate $\chi(g^{-1} \begin{pmatrix} \alpha + nd\gamma & \beta + n\delta \\ d\gamma & \delta \end{pmatrix} g)$, where n is as above. We get, by using (3.1) to eliminate n:

$$\begin{aligned}
\chi(g^{-1} \begin{pmatrix} \alpha + nd\gamma & \beta + n\delta \\ d\gamma & \delta \end{pmatrix} g) &= \chi_N(\beta d + nd\delta + \delta) \\
&= \chi_d(\delta)\chi_{N_d}(\beta d + nd\delta + \delta) \\
&= \chi_d(\delta)\chi_{N_d}(\gamma - \delta)\chi_{N_d}((\gamma - \delta)(\beta d + \delta) + nd\delta(\gamma - \delta)) \\
&= \chi_d(\delta)\chi_{N_d}(\gamma - \delta)\chi_{N_d}((\gamma - \delta)(\beta d + \delta) + \delta(\delta + \beta d - \gamma - \alpha)) \\
&= \chi_d(\delta)\chi_{N_d}(\gamma - \delta)\chi_{N_d}(\gamma\beta d - \alpha\delta) \\
&= \chi_d(\delta)\chi_{N_d}(\gamma - \delta)\chi_{N_d}(-1).
\end{aligned}$$

Thus, we get

$$E(g^{-1}z; s; \chi) = w^{-s}\chi_{N_d}(-1) \sum_{\gamma,\delta:(\gamma,\delta)=1} \frac{\chi_d(\delta)\chi_{N_d}(\gamma - \delta)y^s}{|d\gamma z + \delta|^{2s}}.$$

This is easily seen to imply the the theorem. □

For the cusp at 0 we have $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and get:

- $w = N$,
- $P = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix}$,
- $\tilde{,} = ,^0(N)$,

and therefore

$$E_0(g^{-1}z; s; \chi_N) = N^{-s} \sum_{\gamma,\delta:(\gamma,\delta)=1} \frac{\chi_N(\delta)y^s}{|\gamma z + \delta|^{2s}},$$

and finally

$$E_0(z; s; \chi_N) = N^{-s} \sum_{\gamma,\delta:(\gamma,\delta)=1} \frac{\chi_N(\gamma)y^s}{|\gamma z + \delta|^{2s}},$$

where we used that $\chi_N(-1) = 1$.

Finally, we can easily see that

$$E_\infty z; s; \chi_N) = \sum_{\gamma,\delta:(\gamma,\delta)=1} \frac{\chi_N(\delta)y^s}{|N\gamma z + \delta|^{2s}}.$$

4. Functional Equation for B_χ

Let us look at

$$B_d^{ch}(z; s) = \sum_{(\gamma,\delta)\neq(0,0)} \frac{\chi_{N_d}(\gamma)\chi_d(\delta)(dy)^s}{|\gamma dz + \delta|^{2s}}.$$

In the x variable this is a periodic function with period 1, so we calculate the corresponding Fourier series. For sufficiently large $\Re s$ the sums converge uniformly and the calculation below becomes justified.

$$\begin{aligned}
&\int_0^1 d^{-s} B_d^{ch}(z;s)\ dx \\
&= 2y^s \chi_{N_d}(0) \sum_{\delta=1}^{\infty} \frac{\chi_d(\delta)}{\delta^{2s}} + 2\sum_{\gamma=1}^{\infty} \chi_{N_d}(\gamma) \sum_{\delta=-\infty}^{\infty} \chi_d(\delta) \int_0^1 \frac{y^s\ dx}{[(\gamma d x+\delta)^2+\gamma^2 d^2 y^2]^s}.
\end{aligned}$$

We continue the calculation with the last term only:

$$\begin{aligned}
&2\sum_{\gamma=1}^{\infty} \chi_{N_d}(\gamma) \sum_{\delta=-\infty}^{\infty} \chi_d(\delta) \int_0^1 \frac{y^s\ dx}{[(\gamma d x+\delta)^2+\gamma^2 d^2 y^2]^s} \\
&= 2y^s \sum_{\gamma=1}^{\infty} \chi_{N_d}(\gamma) \sum_{\delta=1}^{\gamma d} \chi_d(\delta) \sum_{m=-\infty}^{\infty} \int_0^1 \frac{y^s\ dx}{[(\gamma d x+\delta+m\gamma d)^2+\gamma^2 d^2 y^2]^s} \\
&= 2y^s \sum_{\gamma=1}^{\infty} \chi_{N_d}(\gamma) \sum_{\delta=1}^{\gamma d} \chi_d(\delta) \int_{-\infty}^{\infty} \frac{y^s\ dx}{[(\gamma d x+\delta)^2+\gamma^2 d^2 y^2]^s} \\
&= 2y^{1-s} d^{-2s} \sum_{\gamma=1}^{\infty} \frac{\chi_{N_d}(\gamma)}{\gamma^{2s-1}} \int_{-\infty}^{\infty} \frac{dt}{(t^2+1)^s} \begin{cases} d & \text{if } \chi_d \equiv 1 \\ \phi(d) & \text{if } \chi_d = 1 \text{ on } (\mathbb{Z}/d\mathbb{Z})^* \\ 0 & \text{if not} \end{cases} \\
&= 2y^{1-s} d^{-2s} L(2s-1,\chi_{N_d}) \sqrt{\pi} \frac{(s-3/2)!}{(s-1)!} \begin{cases} d & \text{if } \chi_d \equiv 1 \\ \phi(d) & \text{if } \chi_d = 1 \text{ on } (\mathbb{Z}/d\mathbb{Z})^* \\ 0 & \text{if not} \end{cases} .
\end{aligned}$$

We also calculate the other Fourier coefficients:

$$\begin{aligned}
&\int_0^1 d^{-s} B_d^{ch}(z;s) e^{-2\pi i n x}\ dx \\
&= 2y^s \sum_{\gamma=1}^{\infty} \chi_{N_d}(\gamma) \sum_{\delta=1}^{d\gamma} \chi_d(\delta) \sum_{m=-\infty}^{\infty} \int_0^1 \frac{e^{-2\pi i n x}\ dx}{[(\gamma d(x+m)+\delta)^2+\gamma^2 d^2 y^2]^s} \\
&= 2y^s \sum_{\gamma=1}^{\infty} \frac{\chi_{N_d}(\gamma)}{\gamma^{2s}} \sum_{\delta=1}^{d\gamma} \chi_d(\delta) e^{2\pi i n \frac{\delta}{d\gamma}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i n y t}}{[t^2+1]^s} dt \\
&= 2y^{1-s} d^{-2s} 2\pi^s \frac{|ny|^{s-1/2}}{(s-1)!} K_{s-1/2}(2\pi|n|y) \sum_{\gamma=1}^{\infty} \frac{\chi_{N_d}(\gamma)}{\gamma^{2s}} \sum_{\delta=1}^{d\gamma} \chi_d(\delta) e^{2\pi i n \frac{\delta}{d\gamma}},
\end{aligned}$$

Where we have introduced the standard notation for the Bessel function K as in [Kub73]. Now

$$e^{2\pi i n/\gamma} \sum_{\delta=1}^{d\gamma} \chi_d(\delta) e^{2\pi i n \frac{\delta}{d\gamma}} = \sum_{\delta=1}^{d\gamma} \chi_d(\delta) e^{2\pi i n \frac{\delta}{d\gamma}},$$

so this is zero unless $\gamma | n$, and we get:

$$\sum_{\gamma=1}^{\infty} \frac{\chi_{N_d}(\gamma)}{\gamma^{2s}} \sum_{\delta=1}^{d\gamma} \chi_d(\delta) e^{2\pi i n \frac{\delta}{d\gamma}} = \sum_{\gamma k=n} \frac{\chi_{N_d}(\gamma)}{\gamma^{2s-1}} \sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i k \frac{\delta}{d}}.$$

We want to write

$$\sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i k \frac{\delta}{d}} = \chi_d(k)\tau(\chi_d),$$

where

$$\tau(\chi_d) = \sum_{j=1}^{d} e^{2\pi i \frac{j}{d}} \chi_d(j).$$

This is easily seen to be true, by a simple change of summation variable, when $(k,d) = 1$, so we only need to prove that if $(k,d) > 1$, then

$$\sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i k \frac{\delta}{d}} = 0.$$

This is not true for all characters - we will use the product structure of χ_d to prove it for the characters we are interested in:

Lemma 4.1. *Let $d = \prod_i p_i$, where the p_i are different, odd primes and let $\chi_d(\delta) = \prod_i \chi_{p_i}(\delta)$. Suppose $(k,d) > 1$, then*

$$\sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i k \frac{\delta}{d}} = 0.$$

Proof. Let $d = Pd'$, $k = Pk'$ where $(d', k') = 1$, then

$$\begin{aligned}
\sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i k \frac{\delta}{d}} &= \sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i k' \frac{\delta}{d'}} \\
&= \sum_{\delta=1}^{d'} e^{2\pi i k' \frac{\delta}{d'}} \sum_{j=0}^{P-1} \chi_d(\delta + jd') \\
&= \sum_{\delta=1}^{d'} e^{2\pi i k' \frac{\delta}{d'}} \sum_{j=0}^{P-1} \chi_{d'}(\delta)\chi_P(\delta + jd'), \qquad (4.1)
\end{aligned}$$

and $\sum_{j=0}^{P-1} \chi_P(\delta + jd') = \sum_{l \bmod P} \chi_P(l) = 0$, since P is a product of distinct, odd primes. □

So finally we get, for the characters we are interested in:

$$\int_0^1 B_d^{ch}(z;s) e^{-2\pi i n x}\ dx = 4\sqrt{y}\left(\frac{\pi}{d}\right)^s \frac{|n|^{s-1/2}}{(s-1)!} K_{s-1/2}(2\pi|n|y)\tau(\chi_d) \sum_{\gamma k = |n|} \frac{\chi_{N_d}(\gamma)}{\gamma^{2s-1}} \chi_d(k)$$

It can now easily be seen, at least formally, that

$$\frac{(s-1)!}{\tau(\chi_d)}\left(\frac{d}{\pi}\right)^s B_d^{ch}(z;s) = \frac{(-s)!}{\tau(\chi_{N_d})}\left(\frac{N_d}{\pi}\right)^{1-s} B_{N_d}^{ch}(z;1-s), \qquad (4.2)$$

since they have the same Fourier series. This formal argument can be made rigorous, since we will show in the next section that the $B_d^{ch}(z;s)$ can be written as a linear combination of Eisenstein series. Since the Eisenstein series can be analytically continued (as meromorphic functions, see [Kub73]) to the whole s-plane, this proves (4.2) for all but a discrete set of s.

Before we go on to calculate the scattering matrix, let us analyze the expression $\tau(\chi_d)$:

Lemma 4.2.

$$\tau(\chi_d)^2 = \chi_d(-1)|\tau(\chi_d)|^2 = \chi_d(-1)d.$$

Proof. Since χ_d is real we have:

$$\begin{aligned}\tau(\chi_d) &= \tau(\overline{\chi_d}) = \sum_{j=1}^{d} e^{2\pi i j/d}\overline{\chi_d(j)} \\ &= \overline{\sum_{j=1}^{d} e^{-2\pi i j/d}\chi_d(j)} \\ &= \chi_d(-1)\overline{\tau(\chi_d)}.\end{aligned}$$

This proves the first equality. The other will be proved using Lemma 4.3 below. Since $\frac{e_\alpha}{\sqrt{d}}$ form an orthonormal basis for $L^2(\mathbb{Z}/d\mathbb{Z})$ we have

$$\begin{aligned}\phi(d) &= \|\chi_d\|_2^2 = \sum_{\alpha=1}^{d} \frac{\langle e_\alpha, \chi_d\rangle}{d} \\ &= \frac{\phi(d)}{d}|\tau(\chi_d)|^2.\end{aligned}$$

This finishes the proof of the lemma. ☐

Lemma 4.3. *Let $e_\alpha(j) = e^{2\pi i \alpha j/d}$, where $\alpha \in \{1, 2, \cdots, d-1\}$. Then*

$$\langle e_\alpha, \chi_d\rangle = \chi_d(\alpha)\tau(\chi_d),$$

where $\langle , \rangle$ denotes the natural (unnormed) inner product on $\mathbb{Z}/d\mathbb{Z}$.

Proof. This is obvious for $\alpha \in (\mathbb{Z}/d\mathbb{Z})^*$. For $(\alpha, d) > 1$ it is proved by the same calculus as in (4.1). ☐

5. Scattering Matrix

We will use an idea by Huxley [Hux84] to calculate explicitly the scattering matrix. Let us define:

$$\begin{aligned}B_d^{ch}(dz; s) &\stackrel{def}{=} \sum_{(\gamma,\delta)\neq(0,0)} \frac{\chi_{N_d}(\gamma)\chi_d(\delta)(dy)^s}{|d\gamma z + \delta|^{2s}} \\ &= \sum_{n=1}^{\infty}\sum_{(\gamma,\delta)=1} \frac{\chi_{N_d}(n\gamma)\chi_d(n\delta)(dy)^s}{|dn\gamma z + n\delta|^{2s}} \\ &= L(2s, \chi_N)\sum_{(\gamma,\delta)=1} \frac{\chi_{N_d}(\gamma)\chi_d(\delta)(dy)^s}{|d\gamma z + \delta|^{2s}},\end{aligned}$$

where $L(2s, \chi_N) = \sum_n \frac{\chi_N(n)}{n^{2s}}$ is the Dirichlet L-series. Now, according to Section 3

$$\sum_{(\gamma,\delta)=1} \frac{\chi_{N_d}(\gamma)\chi_d(\delta)(dy)^s}{|d\gamma z + \delta|^{2s}} = N^s\chi_{N_d}(-1)E_{\frac{1}{d}}(z; s; \chi),$$

with appropriate interpretations in the cases $d = 1, N$. We get from (4.2)

$$\left(\frac{d}{\pi}\right)^s \frac{(s-1)!}{\tau(\chi_d)} B_d^{ch}(dz; s) = \left(\frac{N_d}{\pi}\right)^{1-s} \frac{(-s)!}{\tau(\chi_{N_d})} B_{N_d}^{ch}(N_d z; 1-s),$$

where

$$\tau(\chi_d) = \sum_{d' \mod d} e^{2\pi i d'/d}\chi_d(d').$$

Let us write

$$\mathcal{E}(z;s) = (E_0(z;s;\chi), E_{\frac{1}{d_1}}(z;s;\chi), \cdots, E_{\frac{1}{d_{h-2}}}(z;s;\chi), E_\infty(z;s;\chi)),$$

where the d_i's satisfy: $d_i|N$, $1 < d_1 < \cdots < d_{h-2} < N$. Let us likewise write:

$$\mathcal{B}^{ch}(z;s) = (B_1^{ch}(z;s), B_{d_1}^{ch}(d_1 z;s), \cdots, B_N^{ch}(Nz;s)),$$

then we have the relation

$$\mathcal{B}^{ch}(z;s) = N^s L(2s,\chi_N) D_1 \mathcal{E}(z;s),$$

where D_1 is the diagonal matrix

$$D_1 = \mathrm{diag}(\chi_N(-1), \chi_{N_{d_1}}(-1), \cdots, \chi_{N_{d_{h-2}}}(-1), 1).$$

Now the functional equation can be written

$$\mathcal{B}^{ch}(z;s) = \frac{(-s)!}{(s-1)!}\left(\frac{1}{\pi}\right)^{1-2s} D_2 P \mathcal{B}^{ch}(z;1-s),$$

where

$$D_2 = \mathrm{diag}(N^{1-s}\frac{\tau(\chi_1)}{\tau(\chi_N)}, \frac{N_{d_1}^{1-s}}{d_1^s}\frac{\tau(\chi_{d_1})}{\tau(\chi_{N_{d_1}})}, \cdots),$$

and

$$P = \begin{pmatrix} 0 & & \cdots & & 1 \\ & & & 1 & \\ \vdots & & \cdot & & \vdots \\ & 1 & & & \\ 1 & & \cdots & & 0 \end{pmatrix}.$$

Thus, we get the following expression for the scattering matrix: (since $D_1^{-1} = D_1$)

$$\begin{aligned} C(s) &= \left(\frac{1}{N^s L(2s,\chi_N)}\right)\left(\frac{(-s)!}{(s-1)!}\left(\frac{1}{\pi}\right)^{1-2s} D_2 P\right)\left(N^{1-s}L(2-2s,\chi_N)D_1\right) \\ &= \left(\frac{N}{\pi}\right)^{1-2s}\frac{L(2-2s,\chi_N)}{L(2s,\chi_N)}\frac{(-s)!}{(s-1)!}D_1 D_2 P D_1 \\ &= \left(\frac{N}{\pi}\right)^{1-2s}\frac{L(2-2s,\chi_N)}{L(2s,\chi_N)}\frac{(-s)!}{(s-1)!}P', \end{aligned}$$

where we used that $\chi_N(-1) = 1$ and where

$$P' = \begin{pmatrix} 0 & 0 & \cdots & 0 & N^{1-s}\frac{\tau(\chi_1)}{\tau(\chi_N)} \\ 0 & & & \frac{N_{d_1}^{1-s}}{d_1^s}\frac{\tau(\chi_{d_1})}{\tau(\chi_{N_{d_1}})} & 0 \\ \vdots & & \cdot & & \vdots \\ 0 & \frac{N_{d_{h-2}}^{1-s}}{d_{h-2}^s}\frac{\tau(\chi_{d_{h-2}})}{\tau(\chi_{N_{d_{h-2}}})} & & & 0 \\ N^{-s}\frac{\tau(\chi_N)}{\tau(\chi_1)} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

6. Generalisation

In the general case we will need a stronger version of Lemma 4.1. Let us look at a number d being of the form of N_1, N_2, N_3 or N_4 considered in the introduction. Let χ_d be the corresponding character. Then we will prove the following lemma:

Lemma 6.1. *Suppose* $(k,d) > 1$ *then*

$$\sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i k\delta/d} = 0.$$

Proof. We will need to consider two different cases. Let $P = (k,d)$ then one possibility is that P itself is of the form considered for d. This will be our first case below. If this is not the case we have 3 possibilities:

- $d = 4\prod_{i\in I} p_i$, $d/4 \equiv 3 \mod 4$ and $P = 2\prod_{i\in I'}$, where $I' \subset I$.
- $d = 8\prod_{i\in I} p_i$, $d/8 \equiv 1 \mod 4$ and $P = 2^a\prod_{i\in I'}$, where $I' \subset I$ and $a = 1$ or 2.
- $d = 8\prod_{i\in I} p_i$, $d/8 \equiv 3 \mod 4$ and $P = 2^a\prod_{i\in I'}$, where $I' \subset I$ and $a = 1$ or 2.

Each of these 4 possibilities will be treated below:
<u>1st case:</u> P itself is of the form considered for d.
Write $d = Pd', k = Pk'$ and calculate:

$$\begin{aligned}
\sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i k\delta/d} &= \sum \chi_{d'}(\delta)\chi_P(\delta) e^{2\pi i k'\delta/d'} \\
&= \sum_{\delta=1}^{d'}\sum_{j=0}^{P-1} \chi_{d'}(\delta + jd')\chi_P(\delta + jd') e^{2\pi i k'(\delta+jd')/d'} \\
&= \sum_{\delta=1}^{d'} \chi_{d'}(\delta) e^{2\pi i k'\delta/d'} \sum_{j=0}^{P-1} \chi_P(\delta + jd') \\
&= \sum_{\delta=1}^{d'} \chi_{d'}(\delta) e^{2\pi i k'\delta/d'} \sum_{l \mod P} \chi_P(l) \\
&= 0.
\end{aligned}$$

<u>2nd case:</u> $d = 4\prod_{i\in I} p_i$, $d/4 \equiv 3 \mod 4$ and $P = 2\prod_{i\in I'}$, where $I' \subset I$:
Here we write $k = 2k'$ and $d = 4d'$, where d' is odd, and get:

$$\begin{aligned}
\sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i 2k'\delta/d} &= \sum_{\delta=1}^{d'} e^{\pi i k'\delta/d'}\chi_{d'}(\delta) \sum_{j=0}^{3} \chi_4(\delta + jd') e^{\pi i k'(jd'+\delta)} e^{\pi i k'\delta} \\
&= \sum_{\delta=1}^{d'} e^{\pi i k'(\delta/d'+1)}\chi_{d'}(\delta) \sum_{l \mod 4} \chi_4(l) e^{\pi i k' l} \\
&= 0
\end{aligned}$$

since the last sum vanishes. Notice, that we used the fact that d' is odd to get the first equality.

<u>3rd case:</u> $d = 8\prod_{i\in I} p_i$, $d/8 \equiv 1 \mod 4$ and $P = 2^a \prod_{i\in I'}$, where $I' \subset I$ and $a = 1$ or 2. Here we write $d = 8d'$, $k = 2k'$ and get:

$$\begin{aligned}\sum_{\delta=1}^{d} \chi_d(\delta) e^{2\pi i 2k'\delta/d} &= \sum_{\delta=1}^{d'} \chi_{d'}(\delta) \sum_{j=0}^{7} \chi_8(\delta + jd') e^{2\pi i 2k'(\delta+jd')/d} \\ &= \sum_{\delta=1}^{d'} \chi_{d'}(\delta) e^{4\pi i k'\delta/d} \sum_{j=0}^{7} \chi_8(\delta + jd') e^{\pi i k' j/2}.\end{aligned}$$

Now, $d' \equiv 1 \mod 4$ and therefore $e^{\pi i l} = (e^{\pi i l})^{d'}$ for all integers l, so

$$\sum_{\delta=1}^{d'} \chi_{d'}(\delta) \sum_{j=0}^{7} \chi_8(\delta + jd') e^{2\pi i 2k'(\delta+jd')/d} = \sum_{\delta=1}^{d'} \chi_{d'}(\delta) e^{4\pi i k'\delta/d} e^{-\pi i k'\delta/2} \sum_{l \mod 8} \chi_8(l) e^{\pi i k' l/2},$$

and the sum $\mod 8$ is zero.
<u>4th case:</u>
This case follows by a calculus similar to the 3rd case. □

Now let us look at the different groups:

6.1. $,\ _0(N_2)$.
Here $N_2 = 4M_2$, $M_2 = \prod_{i=1}^{n} p_i \equiv 3 \mod 4$ and the p_i's are different, odd primes. It is easy to check that a complete set of inequivalent cusps is given by $\{0, \infty, \frac{1}{d}\}$ where d runs over all divisors $d|N_2$, $1 < d < N_2$. Let us now check which of these cusps are open:

$$P = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} = \begin{pmatrix} 1 - wd & w \\ -d^2 w & 1 + dw \end{pmatrix},$$

so because $P \in\ ,\ _0(N_2)$, we get:

$$w = \begin{cases} 4M_2/d & (d,2) = 1 \text{ or } 4|d \\ 2M_2/d & d \equiv 2 \mod 4 \end{cases}.$$

When $(d,2) = 1$ or $4|d$ it is obvious that $\chi(P) = 1$. For $d = 2\prod_{i=1}^{n_1}$ we get:

$$\chi(P) = \chi_{4M_2}(1 + 2M_2) = \chi_4(1 + 2M_2),$$

but $M_2 \equiv 3 \mod 4$ so $1 + 2M_2 \equiv 3 \mod 4$ and thus $\chi(P) = -1$.

Thus the only open cusps are $0, \infty, \frac{1}{d}$, where $d = 4\prod_{i\in I} p_I$ or $d = \prod_{i\in I} p_I$ where I runs over all subsets of $\{1, 2, \cdots, n\}$. For these cusps we can calculate the Eisenstein series exactly as in the proof of Lemma 3.1.

6.2. $,\ _0(N_3)$.
$N_3 = 8M_3$, $M_3 = \prod_{i=1}^{n} p_i \equiv 1 \mod 4$.

Here again the cusps are $\{0, \infty, \frac{1}{d}\}$ where d runs over all divisors $d|N_3$, $1 < d < N_3$. We get:

$$P_{\frac{1}{d}} = \begin{pmatrix} 1 - wd & w \\ -d^2 w & 1 + dw \end{pmatrix},$$

so

$$w = \begin{cases} 8M_3/d & (d,2) = 1 \text{ or } 8|d \\ 4M_3/d & d \equiv 2 \text{ or } 4 \mod 8 \end{cases}.$$

For $(d,2) = 1$ or $4|d$ it is again clear that $\chi(P_{\frac{1}{d}}) = 1$. For $d \equiv 2 \mod 4$ we get:

$$\chi(P_{\frac{1}{d}}) = \chi_{N_3}(1 + 4M_3) = \chi_8(1 + 4M_3) = -1,$$

since $M_3 \equiv 1 \mod 4$.

Once again the proof of Lemma 3.1 goes through for the open cusps.

6.3. $\Gamma_0(N_4)$.
In this case the cusps and their widths are as for $\Gamma_0(N_3)$. Write $N_4 = 8M_4$, $M_4 \equiv 3 \mod 4$. Since

$$\chi(P_{\frac{1}{d}}) = \chi_{N_4}(1+4M_4) = \chi_8(1+4M_4) = \chi_8(5) = -1,$$

Exactly the same cusps as for $\Gamma_0(N_3)$ are left open. Lemma 3.1 still holds, with the same proof, if the following change of notation is respected: If $d = 8\prod_i p_i$ then $\chi_d(\delta) = \chi_4(\delta)\chi_8(\delta)\prod_i \chi_{p_i}(\delta)$.

6.4. **Final Result.**
For the open cusps we get, in each of the above cases, that the calculations in Sections 4 and 5 go through with the only change that we have to appeal to Lemma 6.1 (instead of Lemma 4.1) in the proof of the functional equation for $B_d^{ch}(z;s)$. Thus we get:

Theorem 6.2. *Let N be any of the N_i considered in the introduction and let χ be the corresponding character on $\Gamma_0(N)$. Let $\{0, \infty, \frac{1}{d_1}, \cdots, \frac{1}{d_{h-2}}\}$ be a complete set of inequivalent, open cusps under χ, where $1 < d_1 < d_2 < \cdots < d_{h-2} < N$ Notice that if the cusp $\frac{1}{d}$ is open, then the same is true for $\frac{1}{N_d}$, where $N_d = N/d$. Let us write*

$$\mathcal{E}(z;s;\chi) = (E_0(z;s;\chi), E_{\frac{1}{d_1}}(z;s;\chi), \cdots, E_\infty(z;s;\chi)).$$

Then

$$\mathcal{E}(z;s;\chi) = C(s)\mathcal{E}(z;1-s;\chi),$$

where

$$\begin{aligned} C(s) &= \left(\frac{N}{\pi}\right)^{1-2s} \frac{L(2-2s,\chi_N)}{L(2s,\chi_N)} \frac{(-s)!}{(s-1)!} \times \\ & \begin{pmatrix} 0 & 0 & \cdots & 0 & N^{1-s}\frac{\tau(\chi_1)}{\tau(\chi_N)} \\ 0 & & & \frac{N_{d_1}^{1-s}}{d_1^s}\frac{\tau(\chi_{d_1})}{\tau(\chi_{N_{d_1}})} & 0 \\ \vdots & & . & & \vdots \\ 0 & \frac{N_{d_{h-2}}^{1-s}}{d_{h-2}^s}\frac{\tau(\chi_{d_{h-2}})}{\tau(\chi_{N_{d_{h-2}}})} & & & 0 \\ N^{-s}\frac{\tau(\chi_N)}{\tau(\chi_1)} & 0 & \cdots & 0 & 0 \end{pmatrix}. \end{aligned}$$

In the case $N = N_4$ the change of notation from Subsection 6.3 has to be respected.

References

[Dav67] H. Davenport, *Multiplicative number theory*, Markham, Chicago, 1967.

[Hej83] D. A. Hejhal, *The Selberg trace formula for $PSL(2,\mathbb{R})$*, Lecture Notes in Mathematics no. 1001, vol. II, Springer, 1983.

[Hux84] M. Huxley, *Scattering matrices for congruence subgroups*, Modular Forms (R. Rankin, ed.), Ellis Horwood Limited, 1984.

[Kub73] T. Kubota, *The elementary theory of Eisenstein series*, Halsted Press, 1973.

[LP76] P.D. Lax and R.S. Philips, *Scattering Theory for automorphic functions*, Annals of Mathematical Studies no. 87, Princeton University Press, 1976.

[Shi71] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten, Publishers and Princeton University Press, 1971.

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