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Algebra of Principal Fibre Bundles, and Connections.

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The purpose of the present note is to advocate Ehresmann's groupoid PP^{-1}, derived from a principal G-bundle $P \to M$, and to enlarge it to a groupoid on $M + \{*\}$ (the "comprehensive groupoid of P"), in which many calculations with P and G become pure "multiplicative" algebra. This kind of principal-bundle algebra goes particularly well together with the combinatorial formulation which we elsewhere have advocated for the theory of connections and differential forms, in the context of synthetic differential geometry.

In so far as connection theory is concerned, this paper is a sequel to [14], and we presuppose some of the notions presented there. Part of the note may also be seen as a rewriting of [9]. A preliminary version of the first five Section appeared in [17]. The last section owes credit to discussions with Larry Breen and William Messing.

1 Principal bundles and groupoids

Consider a group object G in a left exact category $\underline{E}$, and let M be an object in $\underline{E}$. Recall that a principal G-bundle over M is an object P over M, $\pi : P \to M$ with a right G-action $P \times G \xrightarrow{\cdot} P$ so that

$$\begin{tikzcd} P \times G \arrow[r, shift left, "\cdot"] \arrow[r, shift right, "\mathrm{proj}"'] & P \arrow[r, "\pi"] & M \end{tikzcd}$$

is exact, i.e. it is a coequalizer diagram, and also a kernel pair diagram, and with π a universal effective descent epi.

In the category of smooth manifolds, say, any surjective submersion, in particular, a local product, is a universal effective descent epi. In the category of sets, and more generally, in any topos, any epi (= surjection) is a universal effective descent epi.

If we talk about $\underline{E}$ as if it were the category of sets – which we henceforth shall do – then a right G-action on P makes a surjection $\pi : P \to M$ into a principal G-bundle iff for all $a \in M$ and all $x, z \in P_a$ $(= \pi^{-1}(a))$, there exists a unique $g \in G$ with $x \cdot g = z$.

By the uniqueness of such g, it is natural to denote it $x^{-1}z$. We are going to present a result implying that this $x^{-1}z$ is not just a natural notation, but is a literal composite $x^{-1} \circ z$ in a groupoid Φ.

We compose from right to left in groupoids.

Theorem 1 *Given a principal G-bundle $P \to M$ as above, then there exists a transitive groupoid Φ with set of objects $M + \{*\}$ such that $G = \Phi(*, *)$ and $P_a = \Phi(*, a)$ for $a \in M$. The G-action is by composition in Φ; and for $x, z \in P_a$, $x^{-1}z = x^{-1} \circ z$,*

$$* \xrightarrow{\quad z \quad} a \xrightarrow{\quad x^{-1} \quad} *.$$

The groupoid described in the Theorem, one may call the *comprehensive* groupoid of the principal bundle $P \to M$. It will facilitate many calculations later on.

The full subgroupoid determined by M is a transitive subgroupoid with object set M, denoted, PP^{-1}. The notation comes about, because any arrow in it, say from $a \in M$ to $b \in M$, may be presented as a "fraction"

$$a \xrightarrow{\quad x^{-1} \quad} * \xrightarrow{\quad y \quad} b.$$

Similarly, since the elements of G may be presented as fractions $x^{-1}z$, G may be denoted $P^{-1}P$.

The method for constructing Φ is: one first gives an independent description of PP^{-1}, and of its G-invariant left action on P. This PP^{-1} and its notation is due to Ehresmann; we recall this construction in set theoretic terms; once this is done, the construction of the 'total' comprehensive groupoid Φ will become almost trivial.

So an arrow in PP^{-1} from a to b is given as an equivalence class of pairs (y, x) with $x \in P_a$, $y \in P_b$, under the equivalence relation

$$(y, x) \equiv (y \cdot g, x \cdot g)$$

for $g \in G$. The equivalence class ("fraction") of (y, x) is denoted yx^{-1}. If we have arrows

$$a \xrightarrow{yx^{-1}} b \xrightarrow{uz^{-1}} c$$

with $z \in P_b$, we find the unique $g \in G$ with $z \cdot g = y$, and then the composite is taken to be the fraction $(u \cdot g, x^{-1})$. If we denote the unique g with $z \cdot g = y$ by $z^{-1}y$, the definition of composition reads

$$(uz^{-1}) \circ (yx^{-1}) := (u \cdot z^{-1}y)x^{-1}.$$

Associativity of the composition thus defined comes from associativity of the G-action on P.

To describe the left PP^{-1}-action on P, let yx^{-1} be an arrow $a \to b$, as above, and let $v \in P_a$. Then $(yx^{-1}) \cdot v$ is defined by

$$(yx^{-1}) \cdot v := y \cdot (x^{-1}v), \tag{1}$$

where $x^{-1}v \in G$ is the unique g with $x \cdot g = v$. This action is equivariant for the right G-action; for, for any $h \in G$, the unique element of G which takes x to $v \cdot h$ is $g \cdot h$ (with x, v, g as before), so

$$(yx^{-1}) \cdot (v \cdot h) = y \cdot (g \cdot h) = (y \cdot g) \cdot h = (yx^{-1} \cdot v) \cdot h.$$

It is easy to see that, conversely, any map $\phi : P_a \to P_b$ which is equivariant for the right G-action, comes about as left multiplication by a unique arrow $a \to b$ in PP^{-1}. For, pick $x_\alpha \in P_a$ and let $y := \phi(x) \in P_b$. Then for arbitrary $v \in P_a$,

$$yx^{-1} \cdot v = y \cdot (x^{-1}v) = \phi(x) \cdot (x^{-1}v) = \phi(x \cdot x^{-1}v) = \phi(v),$$

the third equality sign by equivariance of ϕ. This proves existence, and uniqueness follows by taking $v = x$.

The construction of the groupoid Φ of the Theorem is now essentially trivial. The set of objects is $M + \{*\}$. The set of arrows is the disjoint union of PP^{-1}, $P^{-1}P$, and two copies of P which we denote P^+ and P^-,

$$\text{arrows}(\Phi) = PP^{-1} + P^+ + P^- + P^{-1}P,$$

with PP^{-1} as the full subgroupoid on the subset M with $P^{-1}P$ as the full subgroupoid (group, in fact) on the object $*$; and, for $a \in M$, $x \in P_a$, considered in the copy P^+ is considered to be an arrow $* \to a$, and considered in P^- is considered as its inverse $x^{-1} : a \to *$. Compositions in the full subgroupoids PP^{-1} and $P^{-1}P$ already have been defined, or are given, and the remaining compositions are essentially given by the actions. Thus, for $x, v \in P_a$, $y, w \in P_b$,

$$
\begin{aligned}
* \xrightarrow{\ v\ } a \xrightarrow{\ yx^{-1}\ } b \quad &:= \quad * \xrightarrow{\ (yx^{-1}) \cdot v\ } b, \\
a \xrightarrow{\ yx^{-1}\ } b \xrightarrow{\ w^{-1}\ } * \quad &:= \quad a \xrightarrow{\ (x \cdot y^{-1}w)^{-1}\ } *, \\
a \xrightarrow{\ x^{-1}\ } * \xrightarrow{\ y\ } b \quad &:= \quad a \xrightarrow{\ yx^{-1}\ } b \\
* \xrightarrow{\ v\ } a \xrightarrow{\ x^{-1}\ } * \quad &:= \quad * \xrightarrow{\ x^{-1}v\ } *.
\end{aligned}
$$

The verifications that these data provide a groupoid is trivial. The upshot is that all the defining formulas for PP^{-1}, Φ, etc. can safely be forgotten, since the algebra of the groupoid Φ takes care of it. In fact, what previously was just convenient notation, e.g. $z^{-1}y$ for the unique g with $z \cdot g = y$, is now literally the composition $z^{-1} \circ y$ in the groupoid Φ. Also, the defining equation (1) of $(yx^{-1}) \cdot v$ a $y \cdot (x^{-1}v)$ etc. now is an identity which follows from the associative law for the composition $\circ$ in Φ. Similarly for the above four defining equations. And finally, there is no need any more to distinguish between the composition $\circ$, and the dots $\cdot$ denoting the PP^{-1}- or $P^{-1}P$-actions on P; they may all be denoted $\cdot$, say, or omitted from notation altogether.

Remark 1. The notion of "principal bundle over M" may be axiomatized without apriori mention of any group G acting, through the notion of *pregroupoid* advocated in [9]; the primitive notion is a $\pi : P \to M$ equipped with a partially defined ternary operation $yx^{-1}z$ (defined whenever $\pi(x) = \pi(z)$). Out of this ternary operation, both the groupoid $PP^{-1} \rightrightarrows M$ and the group $P^{-1}P$ are constructed, with elements given by "fractions" yx^{-1} and $x^{-1}z$, respectively, – and acting on P, exactly as above. The theory we develop in the present article could be developed more symmetrically in these (less familiar) terms. For the purposes at hand, it makes no difference, but note that the *category* of pregroupois over M is different from the category of principal G-bundles over M. – The pregroupoids considered in [11] are more general, and geared rather to foliation theory, [13].

Remark 2. The notion of fibre bundle – rather than *principal* bundle – likewise admits an algebraic formulation, subordinate, though, to the notion of principal bundle. In fact, in the same way as principal bundles on M are seen as certain transitive groupoids Φ on $M + \{*\}$, we see fibre bundles on M as (left) actions by Φ on sets $E \to M + \{*\}$. We refer the reader to [10] for this viewpoint. – A principal bundle is itself a fibre bundle, and the same applies to its gauge group bundle, which we now describe.

2 Gauge group bundle

For any groupoid $\Psi \rightrightarrows M$ with object set M, there is a group bundle on M, namely

$$\coprod_{a \in M} \Psi(a, a) \to M,$$

sometimes ([18]) called the *gauge group bundle* of Ψ, gauge(Ψ). It carries a left conjugation action by Ψ: if $\psi : a \to b$ in Ψ, and $h \in \Psi(a, a)$, then $\psi \circ h \circ \psi^{-1} \in \Psi(b, b)$. In particular, the group bundle on M, defined in this way from the groupoid $PP^{-1} \rightrightarrows M$, is sometimes called the *adjoint bundle* of P, ad(P) or gauge(P).

In the case where $P^{-1}P = G$ is commutative, gauge(P) $\to M$ may canonically be identified with the constant group bundle $M \times G \to M$ (with trivial PP^{-1}-action): an element of gauge(P) over $a \in M$ is given by a fraction

$$yx^{-1} \in PP^{-1}(a, a)$$

with x and y both in P_a. But then also $x^{-1}y \in P^{-1}P$ makes sense, and the process $yx^{-1} \mapsto x^{-1}y$ is well defined if G is commutative: another presentation of the same fraction is $(yg)(xg)^{-1}$, but

$$(yg)(xg)^{-1} = g^{-1}(x^{-1}y)g = x^{-1}y,$$

the last equality by commutativity of G; so the identification of gauge(P) with $M \times G$ is

$$yx^{-1} \mapsto (a, x^{-1}y) \in M \times G,$$

for $x, y \in P_a$.

3 Connections versus connection forms

Consider a principal bundle $\pi : P \to M$, with group G, as above. We shall assume that M and P are equipped with reflexive symmetric relations $\sim$, called the *neighbour* relation. The set of pairs $(x, y) \in M \times M$ with $x \sim y$ is a subset $M_{(1)} \subseteq M \times M$, called the *first neighbourhood of the diagonal*, and similarly for $P_{(1)} \subseteq P \times P$. We assume that $\pi : P \to M$ preserves the relation $\sim$, and also that it is an "open submersion" in the sense that if $a \sim b$ in M, and $\pi(x) = a$, then there exists a $y \sim x$ in P with $\pi(y) = b$. In fact, we assume that for any "infinitesimal k-simplex" $a_0, \ldots, a_k$ in M (meaning a $k+1$-tuple of mutual neighbours), and for any $x_0 \in P$ above a_0, there exists an infinitesimal k-simplex $x_0, \ldots, x_k$ in P (with the given first vertex x_0) which by π maps to $a_0, \ldots, a_k$. Finally. the action of any $g \in G$ on P is assumed to preserve the relation $\sim$ on P.

This is motivated by Synthetic Differential Geometry (SDG), cf. [6], and more recently [14], where the notion of connection (infinitesimal parallel transport) and differential form is elaborated in these terms.

The groupoid viewpoint for connections is also in essence due to Ehresmann. In SDG, this connection notion becomes paraphrased (see [9], [12] or [14], Section 8): for a groupoid $\Phi \rightrightarrows M$, a connection in it is just a map $\nabla : M_{(1)} \to \Phi$ of reflexive symmetric graphs over M, so for $a \sim b \in M$, $\nabla(b, a)$ is an arrow $a \to b$ in Φ.

If $P \to M$ is a principal G-bundle, a connection in the groupoid $PP^{-1} \rightrightarrows M$ is sometimes called a *principal* connection in P. By the identification of arrows in $PP^{-1}(a, b)$ with right G-equivariant maps $P_a \to P_b$, described above, one may also describe principal connections in such more concrete (but less elementary) terms; explicitly, for each $a \sim b$, $\nabla(b, a)$ is a right G-equivariant map $P_a \to P_b$.

Let $\pi : P \to M$ be a principal fibre bundle. To any connection ∇ in the groupoid PP^{-1} (i.e. to any principal connection), one may associate a 1-form ω on P with values in the group $P^{-1}P$, as follows. For u and v neighbours in P, with $\pi(u) = a$, $\pi(v) = b$, put

$$\omega(u, v) := u^{-1}(\nabla(a, b) \cdot v). \tag{2}$$

Note that both u and $\nabla(a, b) \cdot v$ are in the π-fibre over a, so that the "fraction" $u^{-1}(\nabla(a, b) \cdot v)$ makes sense as an element of $P^{-1}P$.

The defining equation is equivalent to

$$u \cdot \underbrace{\omega(u,v)}_{\in P^{-1}P} = \underbrace{\nabla(\pi(u),\pi(v))}_{\in PP^{-1}} \cdot v. \tag{3}$$

If we agree that (for u, v in P a pair of neighbours in P) $\nabla(u,v)$ denotes $\nabla(\pi(u),\pi(v))$, this equation may be written more succinctly

$$u \cdot \omega(u,v) = \nabla(u,v) \cdot v. \tag{4}$$

It is possible to represent the relationship between ∇ and the associated ω by means of a simple figure:

$$\begin{tikzcd} \bullet & v \arrow[l, "\nabla(u,v)\cdot"'] \\ u \arrow[u, "\cdot\,\omega(u,v)"] & \end{tikzcd}$$

The figure reflects something geometric, namely that $\omega(u,v)$ acts inside the fibre (vertically), whereas ∇ defines a notion of horizontality.

We have the following two equations for ω. First, let $x \sim y$ in P, and assume that g has the property that also $xg \sim y$. Then

$$\omega(xg, y) = g^{-1}\omega(x,y). \tag{5}$$

Also, for $x \sim y$ and *any* $g \in G$

$$\omega(xg, yg) = g^{-1}\omega(x,y)g. \tag{6}$$

To prove (5), let us denote $\pi(x) = \pi(xg)$ by a and $\pi(y)$ by b. Then we have, using the defining equation (3) for ω twice,

$$xg\ \omega(xg, y) = \nabla(a,b)y = x\omega(x,y),$$

and now we may calculate in the comprehensive groupoid of P: first cancel the x on the left, then multiply the equation by g^{-1} on the left. To prove (6), we have, with a and b as above,

$$xg\ \omega(xg, yg) = \nabla(a,b)yg = x\omega(x,y)g,$$

by the defining equation (3) for $\omega(xg, yg)$, and by (3) for $\omega(x, y)$, multiplied on the right by g, respectively. From this, we get the result by first cancelling x and then multiplying the equation by g^{-1} on the left.

The following Proposition is now the rendering, in our context, of the relationship between a connection ∇ and its connection 1-form ω:

Proposition 1 *The process $\nabla \mapsto \omega$ just described, establishes a bijective corresondence between 1-forms ω on P, with values in the group $P^{-1}P$ and satisfying (5) and (6), and connections ∇ in the groupoid PP^{-1}.*

Proof. Given a 1-form ω satisfying (5) and (6), we construct a connection ∇ as follows. Let $a \sim b$ in M. To define the arrow $\nabla(a, b)$ in PP^{-1}, pick $u \sim v$ above $a \sim b$, and put

$$\nabla(a, b) = u(v\omega(v, u))^{-1}.$$

We first argue that this is independent of the choice of v, once u is chosen. Replacing v by $vg \sim u$, we are in the situation where (5) may be applied; we get

$$u(vg\,\omega(vg, u))^{-1} = u(vg\,g^{-1}\omega(v, u))^{-1} = u(v\omega(v, u))^{-1};$$

the left hand side is $\nabla(a, b)$ defined using u, vg, the right hand side is using u, v.

To prove independence of choice of u: any other choice is of form ug for some $g \in G$. For our new v, we now chose vg (the result will not depend on the choice, by the argument just given). Again we calculate. By (6), we have the first equality sign in

$$ug(vg\omega(vg, ug))^{-1} = ug(vgg^{-1}\omega(u, v)g)^{-1} = ug(v\omega(u, v)g)^{-1} = u(v\omega(u, v))^{-1},$$

and the two expressions here are $\nabla(a, b)$, defined using, respectively, ug, vg and u, v.

The calculation that the two processes are inverse of each other is trivial (using $\omega(u, v) = \omega(v, u)^{-1}$ and $\nabla(a, b) = \nabla(b, a)^{-1}$).

4 Gauge forms versus horizontal equivariant forms

We consider a principal fibre bundle $\pi : P \to M$ as in the previous section. The *horizontal* k-forms that we now consider, are k-forms on P with values in the group $G = P^{-1}P$. *Horizontality* means for a k-form θ that

$$\theta(u_0, u_1, \dots, u_k) = \theta(u_0, u_1 \cdot g_1, \dots, u_k \cdot g_k) \tag{7}$$

for any infinitesimal k-simplex $(u_0, u_1, \dots, u_k)$ in P, and any $g_1, \dots g_k \in P^{-1}P$ with the property that $(u_0, u_1 \cdot g_1, \dots, u_k \cdot g_k)$ is still an infinitesimal simplex (which is a strong "smallness" requirement on the g_i's).

Note that the connection form ω for a connection ∇ is *not* a horizontal 1-form, since $\omega(x, yg) = \omega(x, y)g$, not $= \omega(x, y)$.

We say that a k form θ, as above, is *equivariant* if for any infinitesimal k-simplex $(u_0, \dots, u_k)$, and *any* $g \in P^{-1}P$, we have

$$\theta(u_0 \cdot g, u_1 \cdot g, \dots, u_k \cdot g) = g^{-1}\theta(u_0, u_1, \dots, u_k)g. \tag{8}$$

Note that connection forms are equivariant in this sense, by (6).

Proposition 2 *Assume that the group $G = P^{-1}P$ is* commutative. *Then any horizontal equivariant k-form θ on P can be written $\pi^*(\Theta)$ for a unique G-valued k-form Θ on the base space M.*

Proof. It is evident that any form $\pi^*(\Theta)$ is horizontal and equivariant (which here is better called *invariant*, since the equivariance condition now reads $\theta(u_0 \cdot g, u_1 \cdot g, \dots, u_k \cdot g) = \theta(u_0, u_1, \dots, u_k)$). Conversely, given an equivariant (= invariant) k-form θ on P, and given an infinitesimal k-simplex $a_0, \dots, a_k$ in M, define

$$\Theta(a_0, \dots, a_k) := \theta(x_0, \dots, x_k)$$

where $x_0, \dots, x_k$ is any infinitesimal k-simplex above $a_0, \dots, a_k$. The proof that this value does not depend on the choice of the x_i's proceeds much like the proof of the well-definedness of a connection given a connection-form, in Proposition 1 above: First we prove, for fixed x_0 above a_0, that the value is independent of the choice of the remaining x_i's, and this is clear from the verticality assumption on θ. Next we prove that changing x_0 to $x_0 \cdot g$ (and picking $x_1 \cdot g, \dots, x_k \cdot g$ for the remaining vertices in the new k-simplex) does not change the value either, and this is clear from equivariance (= invariance).

Recall that a k-form with values in a group bundle $E \to M$ associates to an infinitesimal k-simplex $a_0, ..., a_1$ in M an element in the fibre of E_{a_0}. We are interested in the case where E is the gauge group bundle of a groupoid; such forms we call *gauge valued forms*, for brevity.

Proposition 3 *There is a natural bijective correspondence between horizontal equivariant k-forms on P with values in $G = P^{-1}P$, and k-forms on M with values in the gauge group bundle gauge(PP^{-1}).*

Proof/Construction. Given a horizontal equivariant k-form θ on P as above, we construct a gauge valued k-form $\check{\theta}$ on M by the formula

$$\check{\theta}(a_0, \dots, a_k) := (u_0 \cdot \theta(u_0, \dots, u_k))u_0^{-1}, \tag{9}$$

or equivalently

$$\check{\theta}(a_0, \dots, a_k) \cdot u_0 = u_0 \cdot \theta(u_0, \dots, u_k), \tag{10}$$

where $(u_0, \dots, u_k)$ is an arbitrary infinitesimal k-simplex mapping to the infinitesimal k-simplex $(a_0, \dots, a_k)$ by π (such exist, since π is a surjective submersion). Note that the enumerator and the denominator in the fraction defining the value of $\check{\theta}$ are both in the fibre over x_0, so that the value is an endo-map at a_0 in the groupoid PP^{-1}, thus does belong to the gauge group bundle. — We need to argue that this value does not depend on the choice of the infinitesimal simplex $(u_0, \dots u_k)$. We first argue that, once u_0 is chosen, the choice of the remaining u_i's in their respective fibres does not change the value. This follows from (7). To see that the value does not depend on the choice of u_0: choosing another one amounts to choosing some $u_0 \cdot g$, for some g. But then we just change $u_1, \dots, u_k$ by the same g; this will give the arrow in PP^{-1}

$$(u_0 \cdot g \cdot \theta(u_0 \cdot g, \dots, u_k \cdot g))(u_0 \cdot g)^{-1}.$$

Now we calculate using Theorem 1, i.e. we calculate in the comprehensive groupoid Φ; so we drop partentheses and multiplication dots; using the assumed equivariance (8), this expression then yields

$$u_0 g g^{-1} \theta(u_0, \dots, u_k) g g^{-1} u_0^{-1},$$

which clearly equals the expression in (9).

Conversely, given a gauge valued k-form α on M, we construct a $P^{-1}P$-valued k-form $\hat{\alpha}$ on P by putting

$$\hat{\alpha}(u_0, u_1, \dots, u_k) := u_0^{-1}(\alpha(a_0, a_1, \dots, a_k) \cdot u_0) \tag{11}$$

where a_i denotes $\pi(u_i)$. Since, for $i \geq 1$, this expression depends on u_i only through $\pi(u_i) = a_i$, it is clear that (7) holds, so the form $\hat{\alpha}$ is horizontal. Also,

$$\hat{\alpha}(u_0 \cdot g, \dots, u_k \cdot g) = (u_0 \cdot g)^{-1}(\alpha(a_0, \dots, a_k) \cdot (u_0 \cdot g));$$

by calculation in the comprehensive groupoid of P, this immediately calculates to the expression in (11).

Finally, a calculation with the the comprehensive groupoid again (cancelling u_0^{-1} with u_0) immediately gives that the two processes $\theta \mapsto \check{\theta}$ and $\alpha \mapsto \hat{\alpha}$ are inverse to each other.

We may summarize the bijection $\alpha \mapsto \hat{\alpha}$ from gauge(PP^{-1})-valued forms on M to horizontal equivariant $P^{-1}P$-valued forms on P by the formula

$$u_0 \cdot \hat{\alpha}(u_0, , , , , u_k) = (\pi^* \alpha)(u_0, ..., u_k) \cdot u_0 \tag{12}$$

(which is essentially just a rewriting of (10)). In the case that the group $G = P^{-1}P$ is commutative, we may cancel the "external" u_0's, and get

$$\hat{\alpha}(u_0, ..., u_k) = (\pi^* \alpha)(u_0, ..., u_k),$$

for all infinitesimal k-simplices $u_0, ..., u_k$. So under the identification (in the commutative case) of gauge valued forms with G-valued forms implied by Section 2, we have

$$\hat{\alpha} = \pi^* \alpha. \tag{13}$$

Recall that if ∇ and ∇_1 are two connections in a groupoid $\Phi \rightrightarrows M$, we may form a 1-form $\nabla_1 \nabla^{-1}$ with values in the gauge group bundle; it is given by

$$\nabla_1 \nabla^{-1}(a, b) = \nabla_1(a, b) \cdot \nabla(b, a).$$

For the case where the groupoid is PP^{-1}, we have the following Proposition, which we shall not use in the sequel, but include for possible future reference:

Proposition 4 *Let $P \to M$ be a principal bundle, and let ∇ and ∇_1 be two connections in the groupoid PP^{-1}. Then*

$$(\nabla_1 \nabla^{-1})\check{} = \omega_1 \cdot \omega^{-1}$$

where ω and ω_1 are the connection forms of ∇ and ∇_1, respectively.

Proof. Let $x \sim y$, over a and $b \in M$, respectively. Then

$$\begin{aligned}
(\nabla_1 \nabla^{-1})\check{}(x, y) &= x^{-1}(\nabla_1(a, b) \nabla(b, a) x) \\
&= x^{-1} \nabla_1(a, b) y \omega(y, x) \\
&= x^{-1} x \omega_1(x, y) \omega(y, x) \\
&= \omega_1(x, y) \omega(y, x) \\
&= (\omega_1 \omega^{-1})(x, y),
\end{aligned}$$

using the defining relation (12) for $(-)\check{}$, and the relation (3) for ∇ and ∇_1, respectively.

5 Curvature versus coboundary

Recall [14] that the *curvature* of a connection in a groupoid $\Phi \rightrightarrows M$ is the $\mathrm{gauge}(\Phi)$-valued 2-form $R = R_\nabla$ given by

$$R(a_0, a_1, a_2) = \nabla(a_0, a_1) \cdot \nabla(a_1, a_2) \cdot \nabla(a_2, a_0),$$

and recall [7], [14] that if ω is a 1-form with values in a group G, then $d\omega$ is the G-valued 2-form given by

$$d\omega(x_0, x_1, x_2) = \omega(x_0, x_1) \cdot \omega(x_1, x_2) \cdot \omega(x_2, x_0).$$

We apply this to the case where $\Phi = PP^{-1}$ and $G = P^{-1}P$, for a principal fibre bundle $\pi : P \to M$. Then the curvature R, which is a $\mathrm{gauge}(PP^{-1})$ -valued 2-form on M, gives, by Proposition 3, rise to a (horizontal and equivariant) $P^{-1}P$-valued 2-form $\hat{R}$ on P.

We then have the following:

Theorem 2 *Let $\pi : P \to M$ be a principal fibre bundle with group G, and let ∇ be a principal connection in it, i.e. a connection in the groupoid PP^{-1}. Let ω be its connection form (a G-valued 1-form on P), and let R be its curvature (a $\mathrm{gauge}(P)$-valued 2-form on M). Then*

$$\hat{R} = d\omega, \tag{14}$$

(as G-valued 2-forms on P, called the curvature form *of ∇), or equivalently*

$$R = (d\omega)\check{}. \tag{15}$$

(as $\mathrm{gauge}(P)$-valued 2-forms on M). In particular, the curvature form $d\omega$ is horizontal and equivariant.

For the case where G is commutative, we may identify $\mathrm{gauge}(P)$-valued forms with G-valued forms; in particular, the curvature R may be seen as a G-valued 2-form on M; and (14) then reads, by (13),

$$\pi^*(R) = d\omega. \tag{16}$$

Proof. Let x, y, z be an infinitesimal 2-simplex in P, and let $a = \pi(x)$, $b = \pi(y)$, and $c = \pi(z)$. We calculate the effect of the (left) action of the arrow $R(a, b, c)$ on x (note that $R(a, b, c)$ is an endo-arrow at a in the groupoid):

$$\begin{aligned} R(a,b,c) \cdot x &= \nabla(a,b) \cdot \nabla(b,c) \cdot \nabla(c,a) \cdot x \\ &= \nabla(a,b) \cdot \nabla(b,c) \cdot z \cdot \omega(z,x) \\ &= \nabla(a,b) \cdot y \cdot \omega(y,z) \cdot \omega(z,x) \\ &= x \cdot \omega(x,y) \cdot \omega(y,z) \cdot \omega(z,x) \\ &= x \cdot d\omega(x,y,z), \end{aligned}$$

using the defining equations for R and for dw for the two outer equality signs, and using (3) three times for the middle three ones. Then (15) follows by formula (10).

Remark. By [6] I.18, or in more detail, [7]), there is a bijective correspondence between G-valued k-forms θ on a manifold P (where G is a Lie group, say $P^{-1}P$), and differential k-forms $\overline{\theta}$, in the classical sense, with values in the Lie algebra $\underline{g}$ of G (i.e. multilinear alternating maps $TP \times_P ... \times_P TP \to \underline{g}$. Under this correspondence, the horizontal equivariant 2-form $d\omega$ considered in the Theorem corresponds to the classically considered "curvature 2-form" Ω on P, as in [20] II.4, [1] 5.3, or [4] V bis 4, (perhaps modulo a factor ± 2, depending on the conventions chosen). This is not completely obvious, since Ω differs from the exterior derivative $d\overline{\omega}$ of the classical connection form $\overline{\omega}$ by a "correction term" $1/2[\overline{\omega}, \overline{\omega}]$ involving the Lie Bracket of $\underline{g}$; or, alternatively, the curvature form comes about by modifying $d\overline{\omega}$ by a "horizontaliztion operator" (this "modification" also occurs in the treatment in [19]). The fact that this "correction term" (or the "modification") does not come up in our context can be explained by Theorem 5.4 in [7] (or see [6] Theorem 18.5); here it is proved that the formula $d\omega(x, y, z) = \omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x)$ already contains this correction term, when translated into "classical" Lie algebra valued forms.

The Theorem has the following Corollary, which is essentially what [19] call the infinitesimal version of Gauss-Bonnet Theorem (for the case where $G = SO(2)$):

Corollary 1 *Assume $P^{-1}P$ is commutative, and let the connection ∇ in PP^{-1} have connection form ω. Then the unique G-valued 2-form Ω on M with $\pi^*\Omega = d\omega$ is R (the curvature of ∇).*

Let us remark that [19] also gives a version of the Corollary for the non-commutative case, their Proposition 6.4.1; this, however, seems not correct. In this sense, our Theorem 2 is partly meant as a correction to Prop. 6.4.1, partly a "translation" of it into the pure multiplicative principal bundle calculus, which is our main concern.

6 Čech-de Rham theory of principal bundles

The characteristic classes of a smooth principal G-bundle $P \to M$ (G a Lie group) arise, by the Chern-Weil procedure, from the curvature of an arbitrary principal connection on P. The curvature itself is a gauge(PP^{-1})-valued 2-form R on the base manifold M.

Since the bundle P itself may be presented by a G-valued Čech 1-cocycle (=transition functions) on an open covering $\mathcal{U}$ on M, it is desirable to describe a connection, and hence its curvature, directly in terms of the G-valued Čech cocycle $\underline{g}$, together with the auxiliary data of a partitition of unity $\underline{f}$ relative to $\mathcal{U}$ (with values in R, the reals); cf. [2] §23, "Concluding Remarks".

The crux is the well-known possibility of forming *affine combinations* of connections. In the present context, this is a consequence of the possibility of forming affine combinations of a set of mutual 1-neighbours: if $x_0, ..., x_n \in M$ are mutual 1-neighbours, and $t_i \in R$ have $t_0 + t_1 + ... + t_n = 1$, then

$$t_0 \cdot x_0 + t_1 \cdot x_1 + ... + t_n \cdot x_n \in M$$

may be defined, using some chart $R^m \to M$ around the x_i's, but the resulting point in M does not depend on the chart chosen; see [15] Theorem 2.2.

Further, any (smooth) map preserves such affine combinations (and preserves the property of being 1-neighbours). In particular, in the case where $M = G$ is a Lie group, and $g \in G$ is an arbitrary element in it, we have for an arbitrary set $g_0, ..., g_n$ of mutual 1-neighbours

$$g \cdot (t_0 \cdot g_0 + ... + t_n \cdot g_n) = t_0 \cdot g \cdot g_0 + ... + t_n \cdot g \cdot g_n,$$

and similarly for multiplication by g on the right, so this looks deceptively like high school arithmetic with numbers.

Assume, then, that $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ is an open cover of a manifold M, and that $\underline{g} = \{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ ($\alpha, \beta \in I$) is a G-valued Čech 1-cocycle on $\mathcal{U}$. Then, as usual, a (right) principal G-bundle on M is constructed as the set of equivalence classes of pairs $(x_\alpha, g) = (x, \alpha, g)$ with $x \in U_\alpha$ and $g \in G$, under the equivalence relation

$$(x_\alpha, g) \equiv (x_\beta, g(x_\beta, x_\alpha) \cdot g),$$

where we write $g(x_\beta, x_\alpha)$ for $g_{\alpha\beta}(x)$.

Using a locally finite partition of unity $\underline{f} = \{f_\alpha : U_\alpha \to R\}$ $(\alpha \in I)$ relative to $\mathcal{U}$, we shall construct a principal connection ∇ on P. As an auxiliary data, we construct first a G-valued 1-form θ on $\coprod U_\alpha$, as follows. Note that if $y_\beta \sim x_\alpha$, then $\beta = \alpha$, by disjointness of the coproduct, so it suffices to define $\theta(y_\alpha, x_\alpha)$ for $x \sim y \in M$. We put

$$\theta(y_\alpha, x_\alpha) := \sum_\gamma f_\gamma(x) \cdot g(y_\alpha, y_\gamma) \cdot g(x_\gamma, x_\alpha).$$

We have first to argue that the combination makes sense. First, $\sum_\gamma f_\gamma(x) = 1$ (with only finitely many significant terms), since $\{f_\gamma\}$ is locally finite, so the combination is an affine one. Also, the $g(y_\alpha, y_\gamma) \cdot g(x_\gamma, x_\alpha)$, as γ ranges over I, are mutual neighbours. For fix α and x. Then, as a function of y,

$$(g(y_\alpha, y_\gamma) \cdot g(x_\gamma, x_\alpha))^{-1} \cdot (g(y_\alpha, y_\beta) \cdot g(x_\beta, x_\alpha))$$

is e if $y = x$, so is $\sim e$ if $y \sim x$, hence

$$g(y_\alpha, y_\gamma) \cdot g(x_\gamma, x_\alpha) \sim g(y_\alpha, y_\beta) \cdot g(x_\beta, x_\alpha)$$

for all γ ,$\beta \in I$. So the affine combination defining $\theta(y_\alpha, x_\alpha)$ makes sense.

We shall prove that

$$g(y_\beta, y_\alpha) \cdot \theta(y_\alpha, x_\alpha) = \theta(y_\beta, x_\beta) \cdot g(x_\beta, x_\alpha). \tag{17}$$

Starting on the left hand side, we calculate

$$g(y_\beta, y_\alpha) \cdot \theta(y_\alpha, x_\alpha) = g(y_\beta, y_\alpha) \cdot \sum_\gamma f_\gamma(x) \cdot g(y_\beta, y_\gamma) \cdot g(x_\gamma, x_\alpha)$$

$$= \sum_\gamma f_\gamma(x) \cdot g(y_\beta, y_\alpha) \cdot g(y_\alpha, y_\gamma) \cdot g(x_\gamma, x_\alpha) \text{ by high school arithmetic}$$

$$= \sum_\gamma f_\gamma(x) \cdot g(y_\beta, y_\gamma) \cdot g(x_\gamma, x_\alpha) \text{ by the cocycle condition}$$

$$= \sum_\gamma f_\gamma(x) \cdot g(y_\beta, y_\gamma) \cdot g(x_\gamma, x_\beta) \cdot g(x_\beta, x_\alpha) \text{ by the cocycle condition}$$

$$= (\sum_\gamma f_\gamma(x) \cdot g(y_\beta, y_\gamma) \cdot g(x_\gamma, x_\beta)) \cdot g(x_\beta, x_\alpha) \text{ by high school arithmetic}$$

$$= \theta(y_\beta, x_\beta) \cdot g(x_\beta, x_\alpha).$$

This is the right hand side in (17), which thus is proved.

Using the G-valued 1-form θ on $\coprod U_\alpha$, we define a principal connection ∇ on $P \to M$ as follows. Let $x \sim y$ in M, and let (x_α, g) represent an element in P_x. So $x \in U_\alpha$. By openness of U_α, $y \in U_\alpha$ as well, so y_α makes sense, and we can write down the following definition:

$$\nabla(y,x) \cdot (x_\alpha, g) := (y_\alpha, \theta(y_\alpha, x_\alpha) \cdot g). \tag{18}$$

We have to argue that this is well defined. Another representative for (x_α, g) is of the form $(x_\beta, g(x_\beta, x_\alpha) \cdot g)$; if we had used this representative, the defining equation would give

$$\nabla(y,x) \cdot (x_\beta, g(x_\beta, x_\alpha) \cdot g) = (y_\beta, \theta(y_\beta, x_\beta) \cdot g(x_\beta, x_\alpha) \cdot g)$$

$$= (y_\beta, g(y_\beta, y_\alpha) \cdot \theta(y_\alpha, x_\alpha) \cdot g),$$

(by (17)), but this is equivalent to the right hand side of (18), by the definition of $\equiv$.

It is clear from the formula (18) that the $\nabla(y,x)$ thus defined is right G-equivariant, thus gives a principal connection in P.

Let us calculate the curvature of ∇ in terms of θ, so let x, y, z be an infinitesimal 2-simplex in M. The curvature $R = R_\nabla$ takes this 2-simplex to an element of the gauge group bundle of P, sitting over x, namely the right G-equivariant map $P_x \to P_x$ given by $(x_\alpha, g) \mapsto \nabla(x,y) \cdot \nabla(y,z) \cdot \nabla(z,x) \cdot (x_\alpha, g)$; we calculate this using the defining equation of ∇ in terms of θ three times, as follows (cf. Section 5),

$$\nabla(x,y) \cdot \nabla(y,z) \cdot \nabla(z,x) \cdot (x_\alpha, g) = (x_\alpha, \theta(x_\alpha, y_\alpha) \cdot \theta(y_\alpha, z_\alpha) \cdot \theta(z_\alpha, x_\alpha) \cdot g) \tag{19}$$

which we recognize as $(x_\alpha, (d\theta)(x_\alpha, y_\alpha, z_\alpha) \cdot g)$, with the standard coboundary d of combinatorial differential forms.

Calculating instead the action of $\nabla(x,y) \cdot \nabla(y,z) \cdot \nabla(z,x)$ on another representative of (x_α, g), say $(x_\beta, g(x_\beta, x_\alpha) \cdot g)$, and utilizing that we know that the ∇'s are well defined, we conclude in particular, by a simple calculation, that

$$g(x_\beta, x_\alpha) \cdot (d\theta)(x_\alpha, y_\alpha, z_\alpha) = (d\theta)(x_\beta, y_\beta, z_\beta) \cdot g(x_\beta, x_\alpha). \tag{20}$$

We shall now be interested in the case where G is commutative, say $G = \mathbf{C}^*$, the nonzero complex numbers.

Then (20) implies that the G-valued 2-form $d\theta$ on $\coprod U_\alpha$ descends along $\delta : \coprod U_\alpha \to M$, i.e. that it is of the form $\delta^*(\Omega)$ for a G-valued 2-form Ω on M, and, in fact, under the identification of G-valued forms with gauge(P)-valued forms which is now available, the equation (19) just says that $d\theta = \delta^* R$

This gives us the possibility of describing the process which from $\underline{g}$, $\underline{f}$ leads to a connection (and hence to curvature and characteristic classes), in terms of the G-valued Čech-de Rham double complex $K^{\bullet\bullet}$, as expounded in [2], say. In the setting of SDG, this double complex arises from a bisimplicial set $K_{\bullet\bullet}$ which we now describe. An element of the set $K_{p,q}$, i.e. a (p,q)-simplex, is a $(p+1) \times (q+1)$ matrix where each row has the *row property*, meaning that its entries are mutual neighbours (so each row forms an infinitesimal q simplex, in the sense of [16]); and where each column has the *column property* Meaning that its entries are of the form

$$(x_{\alpha_0}, \ldots, x_{\alpha_p})$$

for some $x \in M$ (you may prefer to think of such a column as an $x \in M$ together with a $p+1$-tuple of indices α_i, such that $x \in U_{\alpha_i}$ for each $i = 0, \ldots, p$). We note that if a matrix has the column property for all columns, then the row property for just one row implies the row property for all rows.

The $K_{p,q}$ jointly have a bisimplicial structure, with the face operators δ_i and d_j being, respectively, "omit the i'th row", and "omit the jth column".

Now let $K^{p,q}$ be the set of maps from $K_{p,q}$ to G (so $K^{p,q}$ is the set of G-valued Čech-de Rham (p,q)-cochains). If we temporarily use additive notation in G (assumed commutative), we get the usual Čech coboundary operators

$$\delta := \sum_{i=0}^{p} (-1)^i \delta^i : K^{p,q} \to K^{p+1,q},$$

where δ^i is "composing with δ_i"; but we also get de Rham coboundary operators d by a similar formula

$$d := \sum_{j=0}^{q} (-1)^j d^j : K^{p,q} \to K^{p,q+1}.$$

Returning to multiplicative notation, we have

$$\theta \in K^{0,1} \text{ and } \underline{g} \in K^{1,0}$$

and the equation (17) may, by commutativity of G, be rewritten

$$\theta(y_\alpha, x_\alpha)^{-1} \cdot \theta(y_\beta, x_\beta) = g(x_\beta, x_\alpha)^{-1} \cdot g(y_\beta, y_\alpha),$$

which is to say

$$\delta(\theta) = d(\underline{g}) \in K^{1,1}.$$

The process which from the Čech cocycle $\underline{g}$ (and the partition of unity $\underline{f}$, as auxiliary data) leads to curvature R and hence characteristic classes, can be described in terms of "homological tic-tac-toe" ([2]) in the double complex $K^{\bullet\bullet}$, as follows:

$$\begin{tikzcd}
R \arrow[r, "\delta^*"] & \bullet & & \\
& \theta \arrow[u, "d"] \arrow[r, "\delta"] & \bullet & \\
& & \underline{g} \arrow[u, "d"] \arrow[r, "\delta"] & 0
\end{tikzcd}$$

where the (G-valued) deRham complex of M itself is the leftmost column, corresponding to $p = -1$. (The connection, of which R is the curvature, has been described during the construction, but does not fit into the diagram. The connection form could be displayed if we expanded the bisimplicial set into a tri-simplicial one, by considering also the simplicial kernel of $P \to M$.)

Remark. Unlike the case of the real-valued Čech-de Rham bicomplex, the zero row $K^{\bullet 0}$ in $K^{\bullet\bullet}$ is not exact, since partitions of unity with values in G are not in general available. However, the rows $K^{\bullet q}$ for $q > 0$ are exact, since the set of G-valued q-forms on M form a vector space, and so partitions of unity can be used.

References

[1] R. Bishop and R. Crittenden, Geometry of Manifolds, Academic Press 1964.

[2] R. Bott and L.W. Tu, Differential Forms in Algebraic Geometry, Springer 1982.

[3] L. Breen and W. Messing, Combinatorial Differential Forms, to appear in Advances in Math., arXiv:math.AG/0005087.

[4] Y. Choquet-Bruhat, C. DeWitt-Morette, M. Dillard-Bleick, Analysis, Manifolds and Physics, North-Holland 1982.

[5] C. Ehresmann, Les connexions infinitésimales dans un espace fibré differentiable, Colloque de Top. (Espaces Fibrées), Bruxelles 1950, 29-55, Masson et Cie 1951.

[6] A. Kock, Synthetic Differential Geometry, Cambridge Univ. Press 1981.

[7] A. Kock, Differential forms with values in groups, Bull. Austr. Math. Soc. 25 (1982), 357-386.

[8] A. Kock, A combinatorial theory of connections, Contemporary Mathematics 30 (1986), 132-144.

[9] A. Kock, Combinatorial notions relating to principal fibre bundles, J. Pure Appl. Alg. 39 (1986), 141-151.

[10] A. Kock, Fibre bundles in general categories, Journ. Pure Appl. Alg. 56 (1989), 233-245.

[11] A. Kock, Generalized fibre bundles, in Borceux (ed.), "Categorical Algebra and its Applications", Proceedings Louvain-la-Neuve 1987, Springer Lecture Notes 1348 (1988), 194-207.

[12] A. Kock, On the integration theorem for Lie groupoids, Czechoslovak Math. Journ. 39 (1989), 423-431.

[13] A. Kock, The maximal atlas of a foliation, (1996), available at ftp://ftp.imf.au.dk/pub/kock/atlas.ps

[14] A. Kock, Combinatorics of curvature, and the Bianchi identity, Theory and Appl. of Categories, Vol. 2 No. 7 (1996), 69-89.

[15] A. Kock Geometric construction of the Levi-Civita parallelism, Theory and Appl. of Categories, Vol. 4 No. 9 (1998), 195-207.

[16] A. Kock, Differential forms as infinitesimal cochains, Journ. Pure Appl. Alg. 154 (2000), 257-264.

[17] A. Kock, Algebra of principal fibre bundles, and connections. Preliminary version May 2000 in arXiv:math.CT/0005125.

[18] K. Mackenzie, Classification of principal bundles and Lie groupoids with prescribed gauge group bundle, Journ. Pure Appl. Alg. 58 (1989), 181-208.

[19] I. Moerdijk and G.E. Reyes, Smooth infinitesimal analysis, Springer Verlag 1991.

[20] K. Nomizu, Lie groups and differential geometry, The Math. Society of Japan, 1956.