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Abstract

In the present paper we study the statistical properties of local stereological estimators of particle volume. It is shown that the variance of the estimators can be decomposed into the variance due to the local stereological estimation procedure and the variance due to the variability in the particle population. It turns out that these two variance components can be estimated separately, from sectional data. We present further results on the variances that can be used to determine the variance by numerical integration for particular choices of particle shapes.

Keywords: local stereology, marked point process, model-based setting, star-shaped set, variance

1 Introduction

One of the important unsolved problems in stereology concerns the stereological estimation of particle size distributions without specific assumptions about particle shape. It has been known for some time how to estimate stereologically the mean particle volume for particles of varying shape, cf. Jensen (1998). The resulting distribution of estimated particle volumes has been used as an estimate of the distribution of the true particle volumes. It is clearly important to be able to judge when such a procedure is justified.

The particular case of estimating the volume-weighted mean particle volume has recently been treated in Cabo and Baddeley (2003). It is here shown that an estimator based on planar observation is one order of magnitude more efficient than the traditional one based on observation along lines. In the present paper, we concentrate on the ordinary (unweighted) particle volume distribution. It is shown that an estimator of mean particle volume based on planar observations is superior to one based on line observations, especially for elongated particles. The variance of the estimator can be decomposed into the variance due to the stereological estimation procedure and the variance due to the variability in the particle volumes. We will show how to estimate these variance components separately, from sectional data. If the variance due to the stereological estimation procedure is small compared to the variance due to the variability in the particle volumes, the distribution of estimated particle volumes can be regarded as an estimate of the distribution of the true particle volumes.

In Section 2, we define the particle model by means of marked point processes. In Section 3, local stereological estimators of Hausdorff measure are presented. Explicit results concerning the variance of local volume estimators are discussed in Section 4. Particular particle shapes are treated in Section 5 while Section 6 deals with the estimation of the variance components.

2 Marked point processes

In this section we describe the particle model which is defined by means of marked point processes. For more details, we refer to Stoyan et al. (1995). Let $\Psi_m = \{[X_i; \Xi_i]\}$ be a marked point process such that X_i is a point in $\mathbb{R}^n$ and Ξ_i belongs to the space $\mathcal{M}_d$ of d-dimensional differentiable manifolds in $\mathbb{R}^n$ with finite d-dimensional Hausdorff measure. We assume that $O \in \Xi_i$. The point X_i then serves as a reference point of $X_i + \Xi_i$, the i-th particle.

The marked point process Ψ_m will be assumed to be stationary, i.e. $\Psi_m + x = \{[X_i + x; \Xi_i]\}$ has the same distribution as Ψ_m for every $x \in \mathbb{R}^n$. Stationarity of Ψ_m implies stationarity of the unmarked point process $\Psi = \{X_i\}$. Denote by λ its intensity and assume that $0 < \lambda < \infty$. Let $SO(n, L_r)$ be the subgroup of $SO(n)$ consisting of rotations keeping an r-dimensional linear subspace L_r fixed. Then, Ψ_m is said to be invariant under rotations in $SO(n, L_r)$ if $B\Psi_m = \{[BX_i; B\Xi_i]\}$ and Ψ_m have the same distribution for all rotations $B \in SO(n, L_r)$.

The intensity measure of the marked point process is defined as

$$\Lambda_m(A \times U) = \mathrm{E} \sum_i \mathbf{1}_A(X_i)\mathbf{1}_U(\Xi_i), \quad A \in \mathcal{B}(\mathbb{R}^n),\ U \in \mathcal{B}(\mathcal{M}_d),$$

where $\mathbf{1}_A(\cdot)$ stands for the indicator function of the set A and $\mathcal{B}$ denotes the Borel σ-algebra. The stationarity of Ψ_m implies the following decomposition

$$\Lambda_m(A \times U) = \lambda V(A) P_m(U), \quad A \in \mathcal{B}(\mathbb{R}^n),\ U \in \mathcal{B}(\mathcal{M}_d),$$

where $V = \lambda_n$ is the volume and P_m is the mark distribution. By Ξ_0 we denote a random manifold with distribution P_m. If Ψ_m is invariant under rotations in $SO(n, L_r)$, then $B\Xi_0$ has the same distribution as Ξ_0 for all $B \in SO(n, L_r)$.

3 Local stereological estimators

Local stereological estimators are based on information from section planes in $\mathbb{R}^n$ through a reference point of the particle. In this section, we present the actual form of the estimators for a generic particle $K \in \mathcal{M}_d$ with reference point at the origin O. Then a section plane of dimension p is a p-dimensional linear subspace (for brevity called p-subspace) of $\mathbb{R}^n$, $p = 0, 1, \ldots, n$. For comprehensive exposition of local stereology, see Jensen (1998).

There are various forms of the local estimators, depending on the restriction put on the p-subspace. Denote by $\mathcal{L}^n_{p,L_r}$ the set of p-subspaces containing a fixed r-subspace L_r, $0 \le r < p \le n$. Let λ^d_n be the d-dimensional Hausdorff measure in $\mathbb{R}^n$ and let us use the short notation $\mathrm{d}x^d$ instead of $\lambda^d_n(\mathrm{d}x)$. Note that the ordinary Lebesgue measure is $\lambda^n_n = \lambda_n$. For $K \in \mathcal{M}_d$, the local stereological estimator of $\lambda^d_n(K)$, based on a p-subspace $L_p \in \mathcal{L}^n_{p,L_r}$, $d - n + p \ge 0$, has the form, cf. Jensen (1998, (5.24)),

$$m^{(n,d)}_{p,L_r}(K, L_p) = \frac{\sigma_{n-r}}{\sigma_{p-r}} \int_{K \cap L_p} \|\pi_{L_r^\perp} x\|^{n-p} G(\mathrm{Tan}[K,x], L_p)^{-1} \, \mathrm{d}x^{d-n+p}, \tag{1}$$

where $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, $\pi_{L_r^\perp}$ is the orthogonal projection onto $L_r^\perp$ and $G(\mathrm{Tan}[K,x], L_p)$ is the so-called G-factor defined in Jensen (1998, Definition 2.9). Note that in the case $d = n$, $G(\mathrm{Tan}[K,x], L_p) = 1$ for arbitrary L_p.

In a design-based setting, (1) is an unbiased estimator of $\lambda^d_n(K)$. Thus, let μ^n_{p,L_r} be the unique measure on $\mathcal{L}^n_{p,L_r}$, invariant under rotations from $SO(n, L_r)$, satisfying

$$\mu^n_{p,L_r}(\mathcal{L}^n_{p,L_r}) = c(n-r, p-r),$$

where

$$c(n,p) = \frac{\sigma_n \sigma_{n-1} \cdots \sigma_{n-p+1}}{\sigma_p \sigma_{p-1} \cdots \sigma_1}.$$

In what follows, we will write $\mathrm{d}L^n_{p,L_r}$ as short for $\mu^n_{p,L_r}(\mathrm{d}L_p)$. By an isotropic p-subspace in $\mathbb{R}^n$, containing the fixed r-subspace L_r, we mean a random p-subspace with constant density with respect to μ^n_{p,L_r}. Then, if K satisfies the regularity conditions stated in Jensen (1998, Proposition 5.4) and $\tilde{L}_p$ is an isotropic p-subspace containing L_r, the local estimator $m^{(n,d)}_{p,L_r}(K, \tilde{L}_p)$ is an unbiased estimator of $\lambda^d_n(K)$, i.e.

$$\lambda^d_n(K) = \int_{\mathcal{L}^n_{p,L_r}} m^{(n,d)}_{p,L_r}(K, L_p) \frac{\mathrm{d}L^n_{p,L_r}}{c(n-r, p-r)}. \tag{2}$$

Example 1. For $K \in \mathcal{M}_3$ in $\mathbb{R}^3$ there are three local stereological estimators of the volume $V(K)$,

$$m^{(3,3)}_{2,O}(K, L_2) = 2 \int_{K \cap L_2} \|x\| \, \mathrm{d}x^2, \tag{3}$$

$$m^{(3,3)}_{1,O}(K, L_1) = 2\pi \int_{K \cap L_1} \|x\|^2 \, \mathrm{d}x^1, \tag{4}$$

$$m^{(3,3)}_{2,L_1}(K, L_2) = \pi \int_{K \cap L_2} \|\pi_{L_1^\perp} x\| \, \mathrm{d}x^2. \tag{5}$$

The estimators (3) and (4) are related by a so-called Rao-Blackwell procedure. We have

$$m^{(3,3)}_{2,O}(K, L_2) = \mathrm{E}\left[m^{(3,3)}_{1,O}(K, \tilde{L}_1) \mid L_2 \right], \tag{6}$$

if $\tilde{L}_1$ is an isotropic line in L_2. □

For later reference, we also present the local estimator of $\lambda_n^d(K)^2$,

$$\tilde{m}_{p,L_r}^{(n,d)}(K, L_p) = \frac{\sigma_{n-r}\sigma_{n-r-1}}{\sigma_{p-r}\sigma_{p-r-1}} \int_{K\cap L_p} \int_{K\cap L_p} \nabla_{r+2}(e_1, \ldots, e_r, x_1, x_2)^{n-p} \times \prod_{i=1}^{2} G(\mathrm{Tan}[K, x_i], L_p)^{-1} \prod_{i=1}^{2} \mathrm{d}x_i^{d-n+p}, \quad (7)$$

where $1 \leq r+1 < p \leq n$, $d-n+p \geq 0$ and $\nabla_q(y_1, \ldots, y_q)$ denotes q-dimensional Hausdorff measure of the parallelepiped spanned by the vectors $y_1, \ldots, y_q$. Assuming that K satisfies the regularity conditions from Jensen (1998, Theorem 5.6), $\tilde{m}_{p,L_r}^{(n,d)}(K, \tilde{L}_p)$ is an unbiased estimator of $\lambda_n^d(K)^2$ if $\tilde{L}_p$ is an isotropic p-subspace, containing L_r.

Example 2. For $d = n = 3$, $p = 2$ and $r = 0$, the estimator of $V(K)^2$ has the form

$$\tilde{m}_{2,O}^{(3,3)}(K, L_2) = 2\pi \int_{K\cap L_2} \int_{K\cap L_2} \nabla_2(x, y)\, \mathrm{d}y^2\, \mathrm{d}x^2$$

and $\nabla_2(x, y)$ is twice the area of the triangle with vertices O, x and y. □

4 The variance of local estimators

We will now return to the model-based case, described in Section 2. We let Ξ_0 be a generic random particle, distributed according to P_m and denote by E_m the expectation with respect to this distribution. Let $L_{p(0)}$ be a fixed p-subspace in $\mathbb{R}^n$, containing an r-subspace L_r, $0 \leq r < p \leq n$, $d-n+p \geq 0$.

Below, we give explicit results for the second moment of $m_{p,L_r}^{(n,d)}\left(\Xi_0, L_{p(0)}\right)$. For this purpose, the following proposition is very useful.

Proposition 1. *Let P_m be invariant under rotations in $SO(n, L_r)$. For fixed $L_{p(0)} \in \mathcal{L}_{p,L_r}^n$, the estimator $m_{p,L_r}^{(n,d)}\left(\Xi_0, L_{p(0)}\right)$ has the same distribution as $m_{p,L_r}^{(n,d)}(\Xi_0, \tilde{L}_p)$, where $\tilde{L}_p$ is an isotropic p-subspace containing L_r and Ξ_0 and $\tilde{L}_p$ are independent.*

Proof. The result follows from the fact that for any non-negative measurable function h,

$$\mathrm{E}_m\, h\left(m_{p,L_r}^{(n,d)}\left(\Xi_0, L_{p(0)}\right)\right) = \mathrm{E}_m \int_{\mathcal{L}_{p,L_r}^n} h\left(m_{p,L_r}^{(n,d)}\left(\Xi_0, L_p\right)\right) \frac{\mathrm{d}L_{p,L_r}^n}{c(n-r, p-r)}.$$

The left-hand side can be rewritten using the invariance of P_m under rotations in $SO(n, L_r)$ and Jensen (1998, Lemma 8.4),

$$\begin{aligned}\mathrm{E}_m\, h\left(m_{p,L_r}^{(n,d)}\left(\Xi_0, L_{p(0)}\right)\right) &= \mathrm{E}_m\, h\left(m_{p,L_r}^{(n,d)}\left(B^T\Xi_0, L_{p(0)}\right)\right) \\ &= \mathrm{E}_m\, h\left(m_{p,L_r}^{(n,d)}(\Xi_0, BL_{p(0)})\right),\end{aligned}$$

where $B \in SO(n, L_r)$. From invariant measure theory there exists an invariant probability measure $\alpha_{(r)}^n$ on $SO(n, L_r)$. Note that $\mathcal{L}_{p,L_r}^n = \{BL_{p(0)} : B \in SO(n, L_r)\}$ and

$$\int_{SO(n,L_r)} g(BL_{p(0)})\, \alpha_{(r)}^n(\mathrm{d}B) = \int_{\mathcal{L}_{p,L_r}^n} g(L_p) \frac{\mathrm{d}L_{p,L_r}^n}{c(n-r, p-r)}$$

for any non-negative measurable function g on $\mathcal{L}^n_{p,L_r}$. Thus,

$$\begin{aligned}\mathrm{E}_m\, h\left(m^{(n,d)}_{p,L_r}\left(\Xi_0, L_{p(0)}\right)\right) &= \int_{SO(n,L_r)} \mathrm{E}_m\, h\left(m^{(n,d)}_{p,L_r}(\Xi_0, BL_{p(0)})\right)\, \alpha^n_{(r)}(\mathrm{d}B)\\ &= \mathrm{E}_m \int_{SO(n,L_r)} h\left(m^{(n,d)}_{p,L_r}(\Xi_0, BL_{p(0)})\right)\, \alpha^n_{(r)}(\mathrm{d}B)\\ &= \mathrm{E}_m \int_{\mathcal{L}^n_{p,L_r}} h\left(m^{(n,d)}_{p,L_r}(\Xi_0, L_p)\right)\, \frac{\mathrm{d}L^n_{p,L_r}}{c(n-r,p-r)}. \qquad \square\end{aligned}$$

When convenient we use the short notation $m(\Xi_0)$ for $m^{(n,d)}_{p,L_r}\left(\Xi_0, L_{p(0)}\right)$ and $m(\Xi_0, \tilde{L}_p)$ for $m^{(n,d)}_{p,L_r}(\Xi_0, \tilde{L}_p)$. Using Proposition 1 and (2), we get

$$\mathrm{E}_m\, m(\Xi_0) = \mathrm{E}\, m(\Xi_0, \tilde{L}_p) = \mathrm{E}_m\, \lambda^d_n(\Xi_0). \tag{8}$$

Moreover, the relation (2) for a fixed Ξ_0 can be written as

$$\mathrm{E}\left[m(\Xi_0, \tilde{L}_p) \mid \Xi_0\right] = \lambda^d_n(\Xi_0),$$

almost surely, and for the variance of $m(\Xi_0)$, we get

$$\begin{aligned}\mathrm{var}_m\, m(\Xi_0) &= \mathrm{var}\, m(\Xi_0, \tilde{L}_p)\\ &= \mathrm{E}_m\, \mathrm{var}\left[m(\Xi_0, \tilde{L}_p) \mid \Xi_0\right] + \mathrm{var}_m\, \mathrm{E}\left[m(\Xi_0, \tilde{L}_p) \mid \Xi_0\right]\\ &= \mathrm{E}_m\, \mathrm{var}\left[m(\Xi_0, \tilde{L}_p) \mid \Xi_0\right] + \mathrm{var}_m\, \lambda^d_n(\Xi_0)\\ &\geq \mathrm{var}_m\, \lambda^d_n(\Xi_0).\end{aligned}$$

Hence,

$$\mathrm{var}_m\, \lambda^d_n(\Xi_0) \leq \mathrm{var}_m\, m(\Xi_0). \tag{9}$$

Generally, two random variables with the same expectation and variance doesn't have to be equal almost surely. But in our situation we can show that the equality of variances suffices.

Proposition 2. *Let Ξ_0 be a typical manifold with distribution P_m which is invariant under rotations in $SO(n, L_r)$. Then $\mathrm{var}_m\, \lambda^d_n(\Xi_0) = \mathrm{var}_m\, m(\Xi_0)$ if and only if $m(\Xi_0) = \lambda^d_n(\Xi_0)$ almost surely.*

Proof. The equality in (9) happens if $\mathrm{E}_m\, \mathrm{var}\left[m(\Xi_0, \tilde{L}_p) \mid \Xi_0\right] = 0$ which can be rewritten (using the independence of Ξ_0 and $\tilde{L}_p$) as

$$\int_{\mathcal{M}_d} \int_{\mathcal{L}^n_{p,L_r}} \left(m(K_0, L_p) - \lambda^d_n(K_0)\right)^2 \frac{\mathrm{d}L^n_{p,L_r}}{c(n-r,p-r)}\, P_m(\mathrm{d}K_0) = 0.$$

This in turn implies that $m(K_0, L_p) = \lambda^d_n(K_0)$ for $(P_m \times \mu^n_{p(r)})$-a.a. (K_0, L_p). It can be deduced that for $\mu^n_{p(r)}$-a.a. $L_{p(0)}$ we have $m(K_0) = \lambda^d_n(K_0)$ for P_m-a.a. $K_0 \in \mathcal{M}_d$. We would like to show this for all p-subspaces $L_{p(0)}$.

Let us suppose that there exists a subspace $L_{p(0)}$ and a set $A(L_{p(0)}) \in \mathcal{B}(\mathcal{M}_d)$ with $P_m(A(L_{p(0)})) > 0$ such that $m(K_0, L_{p(0)}) \neq \lambda^d_n(K_0)$ for $K_0 \in A(L_{p(0)})$. For any

$L_p \in \mathcal{L}^n_{p,L_r}$, we can find $B \in SO(n, L_r)$ such that $L_p = BL_{p(0)}$. Since $m(K_0, L_{p(0)}) = m(BK_0, BL_{p(0)})$, see Jensen (1998, Lemma 8.4), we have $m(BK_0, L_p) \neq \lambda^d_n(K_0)$ for $K_0 \in A(L_{p(0)})$. Therefore, $m(K_0, L_p) \neq \lambda^d_n(K_0)$ for $K_0 \in A(L_p) = B^{-1}A(L_{p(0)})$. Thus, $m(K_0, L_p) \neq \lambda^d_n(K_0)$ on the set $A = \{(K_0, L_p) : K_0 \in A(L_p), L_p \in \mathcal{L}^n_{p,L_r}\}$. But from the invariance of P_m under rotations in $SO(n, L_r)$ we obtain $P_m(A(L_p)) = P_m(A(L_{p(0)})) > 0$ which means that $(P_m \times \mu^n_{p(r)})(A) > 0$ and this leads us to a contradiction. $\square$

If $\mathrm{var}_m\, m(\Xi_0) = \mathrm{var}_m\, \lambda^d_n(\Xi_0)$, the Hausdorff measure of the manifold is determined from the local section without error. Such local stereological estimators are exact, i.e. the variance of the estimator is created only by the randomness of particles. The simplest example of a particle with exact local volume estimator is a ball.

Proposition 3. *Let Ξ_0 be an n-dimensional ball in $\mathbb{R}^n$ centred at O with probability one. Then*

$$\mathrm{var}_m\, m^{(n,n)}_{p,L_r}\left(\Xi_0, L_{p(0)}\right) = \mathrm{var}_m\, V(\Xi_0). \tag{10}$$

Proof. If $K = b_n(O, R)$ is an n-dimensional ball of radius R with centre at the origin, then for all $L_p \in \mathcal{L}^n_{p,L_r}$,

$$\int_{K\cap L_p} \|\pi_{L_r^\perp} x\|^{n-p}\, \mathrm{d}x^p = \underset{\{x_1^2+\cdots+x_p^2\leq R\}}{\int\cdots\int} (x^2_{r+1} + \cdots + x^2_p)^{\frac{n-p}{2}}\, \mathrm{d}x_p \cdots \mathrm{d}x_1.$$

Since the right-hand side is $\omega_n R^n \frac{\sigma_{p-r}}{\sigma_{n-r}}$, where

$$\omega_n = \lambda_n(b_n(O, 1)) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}$$

is the volume of the unit ball in $\mathbb{R}^n$, it follows that $m(K, L_p) = \lambda_n(K)$. Applying Proposition 1, (10) follows immediately.

$\square$

In Jensen and Petersen (1999) the class of particles having an exact volume estimator (called quasi-spherical bodies) is studied.

By the similar reasoning as in the previous proof we can show that a sphere has exact surface area estimator.

Proposition 4. *Let Ξ_0 be an $(n-1)$-dimensional sphere in $\mathbb{R}^n$ with centre O almost surely. Then*

$$\mathrm{var}_m\, m^{(n,n-1)}_{p,L_r}(\Xi_0, L_{p(0)}) = \mathrm{var}_m\, \lambda^{n-1}_n(\Xi_0).$$

Proof. It can be shown that $G(\mathrm{Tan}[K, x], L_p) = 1$ if K is an $(n-1)$-dimensional sphere in $\mathbb{R}^n$ and $x \in K \cap L_p$. Now the proof proceeds along the same lines as the proof of Proposition 3, the integration over a p-dimensional ball is replaced by the integration over a $(p-1)$-dimensional sphere. $\square$

Remark 1. Lower dimensional spheres are not necessarily quasi-spherical. For example, let us consider $n = 3$, $d = 1$, $p = 2$ and $r = 0$. Then the local estimator of $\lambda^1_3(\Xi_0)$ has the form

$$m^{(3,1)}_{2,O}(\Xi_0, L_{2(0)}) = \frac{4R}{\sin\alpha},$$

where R is the radius of Ξ_0 and α is the angle between $L_{2(0)}$ and the plane containing the circle Ξ_0. $\square$

The local estimator (1) can be simplified if K is star-shaped at O, i.e. $K \cap L_1$ is a line-segment for all $L_1 \in \mathcal{L}_1^n$. Let $\rho_K(\omega)$ denote the radial function of K,

$$\rho_K(\omega) = \max\{c : c\omega \in K\}, \quad \omega \in \mathbb{S}^{n-1},$$

cf. Gardner (1995, p. 18). Let for $\omega \in \mathbb{S}^{n-1}$,

$$\rho_{n,K}(\omega) = \begin{cases} \rho_K(\omega)^n + \rho_K(-\omega)^n & \text{for } O \in K, \\ \big||\rho_K(\omega)|^n - |\rho_K(-\omega)|^n\big| & \text{for } O \notin K, \end{cases}$$

be the n-chord function of the set K at O, cf. Gardner (1995, Definition 6.1.1). Furthermore, let

$$\tilde{\rho}_{n,K}(L_p) = \frac{1}{2p} \int_{\mathbb{S}^{n-1} \cap L_p} \rho_{n,K}(\omega) \, \mathrm{d}\omega^{p-1}$$

be the section function, cf. Gardner (1995, Chapter 7).

Proposition 5. *Let K be a star-shaped set at O. Then*

$$m_{p,L_r}^{(n,n)}(K, L_p) = \frac{\sigma_{n-r}}{\sigma_{p-r}} \frac{1}{2n} \int_{\mathbb{S}^{n-1} \cap L_p} \rho_{n,K}(\omega) \|\pi_{L_r^\perp} \omega\|^{n-p} \, \mathrm{d}\omega^{p-1}. \tag{11}$$

Proof. Using the polar decomposition of Lebesgue measure we obtain

$$\begin{aligned} \int_{K \cap L_p} \|\pi_{L_r^\perp} x\|^{n-p} \, \mathrm{d}x^p &= \frac{1}{2} \int_{\mathbb{S}^{n-1} \cap L_p} \int_{K \cap \mathrm{span}\{\omega\}} \|x\|^{n-1} \|\pi_{L_r^\perp} \omega\|^{n-p} \, \mathrm{d}x^1 \, \mathrm{d}\omega^{p-1} \\ &= \frac{1}{2n} \int_{\mathbb{S}^{n-1} \cap L_p} \rho_{n,K}(\omega) \|\pi_{L_r^\perp} \omega\|^{n-p} \, \mathrm{d}\omega^{p-1}. \end{aligned}$$

□

In particular, for $r = 0$, the local stereological estimator is proportional to the section function

$$m_{p,O}^{(n,n)}(K, L_p) = \frac{\omega_n}{\omega_p} \tilde{\rho}_{n,K}(L_p).$$

Our aim is now to derive some explicit results for the second moment of $m(\Xi_0)$. This will give an easy way of finding $\mathrm{var}_m \, m(\Xi_0)$ (without simulation) for particular choices of shapes of Ξ_0 and will give insight into what kind of shapes of Ξ_0 result in an estimator with large variance. In what follows we always assume that Ξ_0 is invariant under rotations in $SO(n, L_r)$.

Proposition 6. *Let Ξ_0 be a symmetric and star-shaped set at O. Then for the local estimator with $d = n$, $p = 1$ and $r = 0$ we have*

$$\mathrm{E}_m \, m_{1,O}^{(n,n)}(\Xi_0, L_{1(0)})^2 = \omega_n^2 \, \mathrm{E}_m \int_{\mathbb{S}^{n-1}} \rho_{\Xi_0}(\omega)^{2n} \frac{\mathrm{d}\omega^{n-1}}{\sigma_n},$$

Proof. Since Ξ_0 is star-shaped at O, we see from (11) that the local estimator is proportional to the n-chord function,

$$m_{1,O}^{(n,n)}(\Xi_0, \mathrm{span}\{\omega\}) = \frac{\omega_n}{2} \rho_{n,\Xi_0}(\omega), \quad \omega \in \mathbb{S}^{n-1}.$$

Using the symmetry of Ξ_0 ($\rho_{\Xi_0}(\omega) = \rho_{\Xi_0}(-\omega)$) and Proposition 1 we obtain the stated result. □

Sometimes an alternative expression of $\mathrm{E}_m\, m_{1,O}^{(n,n)}(\Xi_0, L_{1(0)})^2$ can be useful.

Proposition 7. *Let Ξ_0 be symmetric and star-shaped set at O. Then for the local estimator with $d = n$, $p = 1$ and $r = 0$ we have*

$$\mathrm{E}_m\, m_{1,O}^{(n,n)}(\Xi_0, L_{1(0)})^2 = 2\omega_n\, \mathrm{E}_m \int_{\Xi_0} \|x\|^n\, \mathrm{d}x^n.$$

Proof. The formula in Proposition 6 can be rewritten as

$$\begin{aligned}
\mathrm{E}_m\, m(\Xi_0)^2 &= \omega_n^2\, \mathrm{E}_m \int_{\mathbb{S}^{n-1}} \int_{\Xi_0 \cap \mathrm{span}\{\omega\}} n\|x\|^{2n-1}\, \mathrm{d}x^1\, \frac{\mathrm{d}\omega^{n-1}}{\sigma_n} \\
&= 2\omega_n\, \mathrm{E}_m \int_{\mathcal{L}_{1,O}^n} \int_{\Xi_0 \cap L_1} \|x\|^{2n-1}\, \mathrm{d}x^1\, \mathrm{d}L_{1,O}^n \\
&= 2\omega_n\, \mathrm{E}_m \int_{\Xi_0} \|x\|^n\, \mathrm{d}x^n,
\end{aligned}$$

where in the last step we have used Jensen (1998, Proposition 4.1) with $g(x) = \|x\|^n$. $\square$

Remark 2. The assumptions of the previous two propositions are not restrictive as they may appear. If Ξ_0 is not a symmetric and star-shaped set, we can define an equivalent symmetric star-shaped set $\mathrm{star}(\Xi_0)$, cf. Jensen (2000), by

$$\rho_{\mathrm{star}(\Xi_0)}(\omega) = \omega_n^{-1/n} m_{1,O}^{(n,n)}(\Xi_0, \mathrm{span}\{\omega\})^{1/n}, \quad \omega \in \mathbb{S}^{n-1}.$$

Obviously, $m(\Xi_0) = m(\mathrm{star}(\Xi_0))$, thus Proposition 6 and Proposition 7 can be used for any Ξ_0 if Ξ_0 is replaced by $\mathrm{star}(\Xi_0)$ in the right-hand side of Propositions 6 and 7. $\square$

Now we turn to the case $p \geq r + 2$.

Proposition 8. *For $p \geq r + 2$ the second moment of the local estimator is*

$$\begin{aligned}
&\mathrm{E}_m\, m_{p,L_r}^{(n,d)}\left(\Xi_0, L_{p(0)}\right)^2 \\
&= \frac{\sigma_{n-r}\sigma_{p-r-1}}{\sigma_{n-r-1}\sigma_{p-r}}\, \mathrm{E}_m \int\limits_{\Xi_0} \int\limits_{\Xi_0} \left(1 - \left\langle \frac{\pi_{L_r^\perp} x}{\|\pi_{L_r^\perp} x\|}, \frac{\pi_{L_r^\perp} y}{\|\pi_{L_r^\perp} y\|} \right\rangle^2 \right)^{-\frac{n-p}{2}} \mathrm{d}x^d\, \mathrm{d}y^d.
\end{aligned}$$

Proof. Under the regularity conditions of Jensen (1998, Theorem 5.6), we know that

$$\begin{aligned}
\mathrm{E}_m\, m(\Xi_0)^2 = \frac{\sigma_{n-r}^2}{\sigma_{p-r}^2}\, \mathrm{E}_m &\int\limits_{\mathcal{L}_{p,L_r}^n} \int\limits_{\Xi_0 \cap L_p} \int\limits_{\Xi_0 \cap L_p} \|\pi_{L_r^\perp} x\|^{n-p} G(\mathrm{Tan}[\Xi_0, x], L_p)^{-1} \\
&\times \|\pi_{L_r^\perp} y\|^{n-p} G(\mathrm{Tan}[\Xi_0, y], L_p)^{-1}\, \mathrm{d}x^{d-n+p}\, \mathrm{d}y^{d-n+p}\, \frac{\mathrm{d}L_{p,L_r}^n}{c(n-r, p-r)}.
\end{aligned}$$

Using the generalized Blaschke-Petkantschin formula (Jensen (1998, Theorem 5.6)) with

$$g(x, y) = \frac{\|\pi_{L_r^\perp} x\|^{n-p} \|\pi_{L_r^\perp} y\|^{n-p}}{\nabla_2(\pi_{L_r^\perp} x, \pi_{L_r^\perp} y)^{n-p}}$$

Table 1: Variances of the local area estimator based on a line through O, chosen as the centre of gravity.

Ξ_0	$\mathrm{var}_m\, m_{1,O}^{(2,2)}(\Xi_0, L_{1(0)})$
rectangle with sides of lengths a, b	$\frac{\pi}{6}\,\mathrm{E}\,ab(a^2+b^2) - (\mathrm{E}\,ab)^2$
ellipse with semiaxes of lengths a, b	$\frac{\pi^2}{2}\,\mathrm{E}\,ab(a^2+b^2) - \pi^2(\mathrm{E}\,ab)^2$
equilateral triangle with the side of length a	$\frac{\pi\sqrt{3}}{24}\,\mathrm{E}\,a^4 - \frac{3}{16}(\mathrm{E}\,a^2)^2$

we get $(d-n+p \geq 0)$

$$\mathrm{E}_m\, m_{p,L_r}^{(n,d)}\left(\Xi_0, L_{p(0)}\right)^2 = \frac{\sigma_{n-r}\sigma_{p-r-1}}{\sigma_{n-r-1}\sigma_{p-r}}\ \mathrm{E}_m \int\limits_{\Xi_0}\int\limits_{\Xi_0} \frac{\|\pi_{L_r^\perp} x\|^{n-p}\|\pi_{L_r^\perp} y\|^{n-p}}{\nabla_2(\pi_{L_r^\perp} x, \pi_{L_r^\perp} y)^{n-p}}\,\mathrm{d}x^d\,\mathrm{d}y^d.$$

Notice that due to the regularity conditions $\nabla_2(\pi_{L_r^\perp} x, \pi_{L_r^\perp} y) = 0$ on a set of $(\lambda_n^d \times \lambda_n^d)$-measure zero and the integral on the right-hand side is well-defined. The result now follows immediately from $\nabla_2(x,y) = (\|x\|^2\|y\|^2 - \langle x, y\rangle^2)^{1/2}$. $\square$

Note that the second moment of $m_{p,L_r}^{(n,d)}\left(\Xi_0, L_{p(0)}\right)$ does not depend on the G-factor. For n-dimensional particles the formula for the variance given in Proposition 7 was expressed through integrals over $\mathbb{S}^{n-1} \cap L_r^\perp$ in Jensen and Petersen (1999). Finally, we consider the special case of star-shaped particles and $r=0$.

Proposition 9. *Let Ξ_0 be a star-shaped set at O with $O \in \Xi_0$ almost surely. Suppose that $p \geq 2$. Then*

$$\begin{aligned}&\mathrm{E}_m\, m_{p,O}^{(n,n)}(\Xi_0, L_{p(0)})^2 \\ &\quad = \frac{\sigma_n\sigma_{p-1}}{4n^2\sigma_{n-1}\sigma_p}\,\mathrm{E}_m \int\limits_{\mathbb{S}^{n-1}}\int\limits_{\mathbb{S}^{n-1}} \rho_{n,\Xi_0}(\omega_1)\rho_{n,\Xi_0}(\omega_2)\left(1 - \langle\omega_1,\omega_2\rangle^2\right)^{-\frac{n-p}{2}}\,\mathrm{d}x^n\,\mathrm{d}y^n.\end{aligned}$$

Proof. The result follows immediately from Proposition 8 and the following formulation of polar decomposition of Lebesgue measure

$$\int_K g\left(\frac{x}{\|x\|}\right)\mathrm{d}x^n = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\mathrm{span}\{\omega\}} \mathbf{1}_K(x)g(\omega)\|x\|^{n-1}\,\mathrm{d}x^1\,\mathrm{d}\omega^{n-1}.$$ $\square$

5 Examples

In this section we use the results from Section 4 to find explicit expressions of the variance of local stereological estimators for specific particle shapes.

5.1 The planar case

For $n=d=2$, $p=1$ and $r=0$, the variance can easily be determined, using Proposition 6 or Proposition 7 for various shapes of Ξ_0. In Table 1 we give the formulas for three particular choices of Ξ_0.

5.2 Triaxial ellipsoids

We suppose that Ξ_0 is an ellipsoid centred at O and with semiaxes of lengths a, b and c. In $\mathbb{R}^3$ there are three local stereological volume estimators, namely (3), (4) and (5).

For $p = 1$ the second moment of $m_{1,O}^{(3,3)}(\Xi_0, L_{1(0)})$ can be written as (using Proposition 6 and spherical coordinates)

$$\frac{8\pi}{9}\,\mathrm{E}\int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos^2\theta\cos^2\varphi}{a^2} + \frac{\cos^2\theta\sin^2\varphi}{b^2} + \frac{\sin^2\theta}{c^2}\right)^{-3} \cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{12}$$

or in the form

$$\frac{8\pi}{9}\,\mathrm{E}\,abc\int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(a^2\cos^2\theta\cos^2\varphi + b^2\cos^2\theta\sin^2\varphi + c^2\sin^2\theta\right)^{3/2} \cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi, \tag{13}$$

where we used Proposition 7 and the transformation

$$x = (ar\cos\theta\cos\varphi, br\cos\theta\sin\varphi, cr\sin\theta)^T,\ r \in (0,1), \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \varphi \in (0, 2\pi).$$

If we suppose that there exist constants a_0, b_0, c_0 and a non-negative random variable ϱ such that $a = \varrho a_0$, $b = \varrho b_0$ and $c = \varrho c_0$ (i.e. the typical particle shape is fixed, only size and direction are random), then the variance of the local estimator becomes

$$\mathrm{var}_m\, m(\Xi_0)^2 = V_0^2\left(\kappa\,\mathrm{E}\,\varrho^6 - \left(\mathrm{E}\,\varrho^3\right)^2\right), \tag{14}$$

where $V_0 = \frac{4\pi}{3}a_0b_0c_0$ is the volume of an ellipsoid with semiaxes a_0, b_0, c_0 and the constant κ can be determined from either (12) or (13) by means of numerical integration.

For $p = 2$ and $r = 0$ we can proceed in similar way. From Proposition 9 we have

$$\mathrm{E}_m\, m_{2,O}^{(3,3)}(\Xi_0, L_{2(0)})^2 = \frac{2}{9\pi}\,\mathrm{E}_m \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \left(1 - \langle\omega_1, \omega_2\rangle^2\right)^{-\frac{1}{2}} \\ \times \rho_{\Xi_0}(\omega_1)^3 \rho_{\Xi_0}(\omega_2)^3\,\mathrm{d}\omega_1^2\,\mathrm{d}\omega_2^2.$$

Note that there is a mistake in Jensen (2000), the constant $\frac{8}{\pi^3}$ should be replaced by $\frac{1}{8\pi^3}$. For fixed Ξ_0 the double integral can again be computed numerically. The formula (14) still holds, the values of κ for several choices of ratios a_0/b_0 and b_0/c_0 are summarized in Table 2. We have also computed κ for an intermediate estimator, usually called the nucleator, cf. Gundersen (1988),

$$\bar{m}_{1,O}^{(3,3)}(\Xi_0) = \frac{1}{2}\left[m_{1,O}^{(3,3)}(\Xi_0, \mathrm{span}\{\omega_1\}) + m_{1,O}^{(3,3)}(\Xi_0, \mathrm{span}\{\omega_2\})\right],$$

where $\omega_1 \in \mathbb{S}^2 \cap L_2$ is an isotropic direction in an isotropic plane L_2 and $\omega_2 \in \mathbb{S}^2 \cap L_2$ is orthogonal to ω_1. Our approach based on numerical integration enables more precise results than those obtained by simulation in Jensen (2000).

Higher values of κ mean higher variance caused by the local stereological estimation. For ball ($\kappa = 1$) we have an exact estimator with

$$\mathrm{var}_m\, m(\Xi_0) = V_0^2\,\mathrm{var}\,\varrho^3 = \mathrm{var}_m\,\lambda_3(\Xi_0).$$

Table 2: The values of κ from (14) for three types of volume estimators and various shapes of ellipsoids.

a_0/b_0	b_0/c_0	$m_{1,O}^{(3,3)}(\Xi_0)$	$\bar{m}_{1,O}^{(3,3)}(\Xi_0)$	$m_{2,O}^{(3,3)}(\Xi_0)$
1	1	1	1	1
1	2	1.34440	1.10854	1.07945
1	4	2.43264	1.58048	1.26608
2	1	1.51757	1.16635	1.11835
2	2	2.60921	1.63012	1.31307
2	4	5.03481	2.79692	1.59320
4	1	4.42495	2.45371	1.59821
4	2	8.51728	4.44465	2.03585
4	4	16.87915	8.60156	2.56130

Note that the error is larger for prolate spheroids ($b = c$) than for the corresponding oblate spheroids ($a = b$). In view of (6), it is not surprising that smaller values of error are obtained for the local estimator based on plane sections.

In the remainder of this subsection we consider the last local volume estimator (5). Obviously, it depends on the choice of the fixed line L_1 (usually called vertical axis) relative to the ellipsoid. We assume that the vertical axis has the same direction as one of the semiaxes of the ellipsoid (say the one of length c). Then the profile $\Xi_0 \cap L_{2(0)}$ is a planar ellipse with semiaxes of length A and c. Hence, the local estimator has the following form

$$m_{2,L_1}^{(3,3)}(\Xi_0, L_{2(0)}) = \pi \int_0^{2\pi} \int_0^1 A^2 c r^2 |\cos\varphi| \,\mathrm{d}r\,\mathrm{d}\varphi = \frac{4\pi}{3} A^2 c.$$

Let α be the angle between $L_{2(0)}$ and the semiaxis of length a. Then A can be expressed as the function of a, b and α and the second moment of $m(\Xi_0)$ is

$$\begin{aligned} \mathrm{E}\, m_{2,L_1}^{(3,3)}(\Xi_0, L_{2(0)})^2 &= \frac{16\pi^2}{9} c^2\, \mathrm{E} \int_0^{\pi} \frac{1}{\left(\frac{\cos^2\alpha}{a^2} + \frac{\sin^2\alpha}{b^2}\right)^2} \,\mathrm{d}\alpha \\ &= \frac{8\pi^2}{9}\, \mathrm{E}\, abc^2(a^2+b^2). \end{aligned}$$

For fixed shape of Ξ_0 the constant κ in (14) does not depend on c_0,

$$\kappa = \frac{a_0^2 + b_0^2}{2a_0 b_0}.$$

For the values mentioned in Table 2 we get $\kappa = 1$ if $a_0/b_0 = 1$, $\kappa = 1.25$ if $a_0/b_0 = 2$ and $\kappa = 2.125$ if $a_0/b_0 = 4$.

5.3 Other spatial particles

A table similar to Table 2 can be determined for other choices of particle shape.

As an example, let Ξ_0 be obtained by scaling the prototype cuboid with edges of lenghts a_0, b_0, c_0. It means that Ξ_0 is an isotropically oriented cuboid with edges of

Table 3: The values of κ for various shapes of cuboids.

a_0/b_0	b_0/c_0	$m_{1,O}^{(3,3)}(\Xi_0)$	$m_{2,O}^{(3,3)}(\Xi_0)$
1	1	1.15338	1.01883
1	2	1.52818	1.10735
1	4	2.72926	1.31183
2	1	1.70178	1.15148
2	2	2.87074	1.36691
2	4	5.49895	1.67056
4	1	4.75547	1.68498
4	2	9.07151	2.16437
4	4	17.93156	2.72906

lengths ϱa_0, ϱb_0, ϱc_0, where ϱ is a non-negative random variable. Then (14) holds with $V_0 = a_0 b_0 c_0$ and κ can be calculated numerically (see Table 3). The obtained values are slightly larger than for ellipsoids.

As the next example consider a regular tetrahedron of random size. For the estimator based on line section $\kappa = 1.20049$ and for the estimator based on plane section $\kappa = 1.01775$.

5.4 Higher dimensions

If $n > 3$ we do not always have to use numerical integration in order to derive the explicit formula for the variance. For instance, if Ξ_0 is an ellipsoid in $\mathbb{R}^4$ with semiaxes a_1, a_2, a_3, a_4, then

$$\begin{aligned}&\operatorname{var}_m m_{1,O}^{(4,4)}(\Xi_0, L_{1(0)})\\&\quad= \frac{1}{8}\pi^4 \operatorname{E} a_1a_2a_3a_4 \left(\frac{1}{6}\sum_{i=1}^{4} a_i^4 + \frac{1}{12}\left(\sum_{i=1}^{4} a_i^2\right)^2\right) - \frac{1}{4}\pi^4 \left(\operatorname{E} a_1a_2a_3a_4\right)^2 .\end{aligned}$$

This result can be derived from Proposition 7, using elliptical coordinates and lengthy but straightforward calculations.

6 Estimation of variances

It is interesting to find estimators of the variances $\sigma_m^2 = \operatorname{var}_m m_{p,L_r}^{(n,d)}\left(\Xi_0, L_{p(0)}\right)$ and $\sigma_\lambda^2 = \operatorname{var}_m \lambda_n^d(\Xi_0)$, separately, from sectional data. First we introduce ratio-unbiased estimators of $\mu = \operatorname{E}_m \lambda_n^d(\Xi_0)$, $\alpha_m^2 = \operatorname{E}_m m_{p,L_r}^{(n,d)}\left(\Xi_0, L_{p(0)}\right)^2$ and $\alpha_\lambda^2 = \operatorname{E}_m \lambda_n^d(\Xi_0)^2$, respectively.

Let W be a fixed bounded Borel set in $\mathbb{R}^n$ with positive volume. We consider a sample of particles $X_i + \Xi_i$ with X_i in the sampling window W. Then the local estimator is determined from the central section $(X_i + \Xi_i) \cap (X_i + L_{p(0)})$ for each sampled particle. In order to be in accordance with (1) we can think that the centred particle Ξ_i is sectioned by a fixed p-subspace $L_{p(0)} \in \mathcal{L}_{p,L_r}^n$, i.e.

$$m_{p,L_r}^{(n,d)}(X_i + \Xi_i, X_i + L_{p(0)}) = m_{p,L_r}^{(n,d)}(\Xi_i, L_{p(0)}).$$

In what follows we assume that the mark distribution P_m of a stationary marked point process Ψ_m is invariant under rotations in $SO(n, L_r)$. It can be shown that (see Jensen (1998, Proposition 8.5))

$$\widehat{\mu} = \sum_i \mathbf{1}_W(X_i) m^{(n,d)}_{p,L_r}(X_i + \Xi_i, X_i + L_{p(0)})/\Psi(W)$$

is a ratio-unbiased estimator of mean particle Hausdorff measure

$$\mathrm{E}_m\, \lambda^d_n(\Xi_0) = \int_{\mathcal{M}_d} \lambda^d_n(K)\, P_m(\mathrm{d}K).$$

The proof is based on Campbell's formula for marked point processes (see Stoyan et al. (1995, Section 4.2)) and (8).

Using (7), we can estimate the second moment in the mark distribution, i.e. $\mathrm{E}_m\, \lambda^d_n(\Xi_0)^2$. Let Ξ_0 satisfy the regularity conditions from Jensen (1998, Theorem 5.6), and let $L_{p(0)}$ be a fixed p-subspace in $\mathbb{R}^n$, containing L_r, $1 \leq r+1 < p \leq n$, $d - n + p \geq 0$. Jensen (1998, Proposition 8.7) states that if W is a bounded Borel set with positive volume, then

$$\widehat{\alpha^2_\lambda} = \sum_i \mathbf{1}_W(X_i) \tilde{m}^{(n,d)}_{p,L_r}(X_i + \Xi_i, X_i + L_{p(0)})/\Psi(W)$$

is a ratio-unbiased estimator of $\mathrm{E}_m\, \lambda^d_n(\Xi_0)^2$.

We can also estimate $\mathrm{E}_m\, m^{(n,d)}_{p,L_r}\left(\Xi_0, L_{p(0)}\right)^2$ by

$$\widehat{\alpha^2_m} = \sum_i \mathbf{1}_W(X_i) m^{(n,d)}_{p,L_r}(X_i + \Xi_i, X_i + L_{p(0)})^2/\Psi(W).$$

This estimator is again ratio-unbiased, as can be easily seen from Campbell's formula for marked point processes.

The problem is how to estimate

$$\mu^2 = \left(\mathrm{E}_m\, m^{(n,d)}_{p,L_r}\left(\Xi_0, L_{p(0)}\right)\right)^2 = \left(\mathrm{E}_m\, \lambda^d_n(\Xi_0)\right)^2.$$

As long as we restrict ourselves to the case of independently marked point process (i.e. the Ξ_i are independent and identically distributed and independent of Ψ),

$$\frac{1}{\Psi(W)\,(\Psi(W)-1)} \sum_{i \neq j} \mathbf{1}_W(X_i)\mathbf{1}_W(X_j) m(X_i + \Xi_i) m(X_j + \Xi_j)$$

is an unbiased estimator of $\left(\mathrm{E}_m\, \lambda^d_n(\Xi_0)\right)^2$. Accordingly,

$$\widehat{\sigma^2_m} = \frac{1}{\Psi(W)-1} \sum_i \left(m(X_i + \Xi_i) - \widehat{\mu}\right)^2 \tag{15}$$

and

$$\widehat{\sigma^2_\lambda} = \widehat{\alpha^2_\lambda} - (\widehat{\mu})^2 + \frac{1}{\Psi(W)-1}\left(\widehat{\alpha^2_m} - (\widehat{\mu})^2\right) \tag{16}$$

are unbiased estimators of $\sigma^2_m = \mathrm{var}_m\, m(\Xi_0)$ and $\sigma^2_\lambda\, \mathrm{var}_m\, \lambda^d_n(\Xi_0)$, respectively.

In the general case we propose the following estimator of σ_m^2,

$$\widehat{\sigma_m^2} = \frac{\displaystyle\sum_{i,j} \frac{\mathbf{1}_W(X_i)\mathbf{1}_W(X_j)\mathbf{1}_{[h_0,\infty)}(\|X_i - X_j\|)\,(m(X_i + \Xi_i) - m(X_j + \Xi_j))^2}{V\,((W - X_i) \cap (W - X_j))}}{\displaystyle 2\sum_{i,j} \frac{\mathbf{1}_W(X_i)\mathbf{1}_W(X_j)\mathbf{1}_{[h_0,\infty)}(\|X_i - X_j\|)}{V\,((W - X_i) \cap (W - X_j))}}, \tag{17}$$

with an appropriate choice of $h_0 \geq 0$. Using Campbell's formula it is easy to show that $\widehat{\sigma_m^2}$ is ratio-unbiased if $m(X_i + \Xi_i)$ and $m(X_j + \Xi_j)$ are independent whenever $\|X_i - X_j\| \geq h_0$. The estimate of σ_λ^2 has then the form

$$\widehat{\sigma_\lambda^2} = \widehat{\sigma_m^2} - \widehat{\alpha_m^2} + \widehat{\alpha_V^2}. \tag{18}$$

In applications, the distribution of estimated particle volumes (or other size parameters) has been used as an estimate of the true particle volume distribution. This procedure is justified if the variance due to the stereological estimation procedure is small compared to the variance due to the variability in the particle population. We can estimate both variances from central sections using (15) and (16) or (17) and (18). If the estimates are closed we can expect that the distribution of estimated sizes will be close to the true size distribution. The practical implications of this observation will be investigated elsewhere.

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