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Abstract

A number of authors have reported empirically observed scaling laws of the absolute values of log returns of stocks and exchange rates, with a scaling coefficient in the order of 0.58–0.59. It is suggested here that this phenomenon is largely due to the semi-heavy tailedness of the distributions concerned rather than to real scaling.

Keywords: *NIG* Lévy processes; *NIG* shape triangle; normal inverse Gaussian distribution; scaling power laws; high-frequency data in finance.

1 Introduction

Scaling laws, empirical or theoretical, are of widespread interest in mathematics, physics, geology, and other fields, and in particular in turbulence (cf. for instance Frisch 1995). Fairly recently, with the access to large and detailed data sets from the financial markets, and also with the recognition of striking similarities between key observational features of such data and data from studies of turbulence, a search for scaling laws in finance has begun, see Müller et al. (1990, more precisely, p. 58), Ghashgaie et al. (1996), Guillaume et al. (1997), and Piccinato et al. (1998) (cf. also Barndorff-Nielsen 1998a).

The papers by Müller et al. (1990), Guillaume et al. (1997), and Piccinato et al. (1998) report empirical evidence for scaling of absolute log returns with a scaling index H close to 0.58, both for foreign exchange rates and for short term interest rate futures contracts. A summary of their findings is given in Section 2.

The purpose of the present note is to point out that these latter findings may, at least to a significant extent, be explained not as a real scaling phenomenon but, somewhat surprisingly, as being due to (semi-)heavy-tailedness and, to a lesser extent, skewness of the relevant empirical distributions determined by the rates.

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Distributions of returns of financial assets can generally be well fitted by the normal inverse Gaussian (*NIG*) law (Barndorff-Nielsen 1997, 1998b; Barndorff-Nielsen and Jiang 1998; Prause 1997; Rydberg 1997), and we base our discussion on this law which, furthermore, allows explicit analytic calculations.

In Section 3 relevant properties of the *NIG* law are reviewed and in Section 4 we consider the behaviour of the absolute first moment of the *NIG* Lévy process. The second order cumulant of the symmetric *NIG* Lévy process behaves according to an exact scaling with index $H = \frac{1}{2}$ but for the absolute moment this is only true asymptotically for t, the time indicator, tending to ∞. And for parameter values in the range that is typical of many financial regimes the absolute moments are in fact close to 0.58, as will be demonstrated in Subsection 4.1.

Subsection 4.2 provides illustrations of the effect of skewness of the distributions. The conclusion is that skewness will make adherence to the apparent scaling behaviour more manifest.

As a model for the development of a financial asset the *NIG* Lévy process is generally more realistic than the Brownian motion; but, it does not capture the prevalent quasi long range dependence of financial time series. However, this appears of secondary importance for the point we wish to make.

2 Empirical Scaling Power Laws in Finance

One important point in the modelling of high-frequency data in Finance is to analyze volatility on different time scales. Volatility of high-frequency data is sometimes (e.g in Mandelbrot 1963, Müller et al. 1990 and Guillaume et al. 1997) defined not as a standard deviation but as the average of absolute logarithmic price changes

$$v(t_i, \Delta t) = \frac{1}{N(\Delta t)} \sum_{i=1}^{N(\Delta t)} |x(t_i) - x(t_i - \Delta t)| \tag{1}$$

where $x(t_i) = \log(S_{t_i}/S_0)$ and $t_i = i\Delta t, \quad i = 1, \ldots, N(\Delta t)$. We choose $N(\Delta t)$ such that the length of the observation period $\Delta t \cdot N(\Delta t)$ is constant. Considering the absolute logarithmic price changes of FX rates one often empirically finds a *scaling power law* which relates the volatility over a given time interval Δt to the size of this interval by the formula

$$v(t_i, \Delta t) = \left(\frac{\Delta t}{\Delta T}\right)^H \tag{2}$$

for an arbitrarily chosen ΔT. Equation (2) yields

$$H = \frac{\log(v(t_i, \Delta t))}{\log(\Delta t/\Delta T)}. \tag{3}$$

Hence we can estimate the exponent H by the linear regression coefficient of $\log(v(t_i, \Delta t))$ against $\log(n)$ where $n = \Delta t/\Delta T$. The scaling power law is observed over several orders of magnitude in time. For major free floating currencies the empirical exponent H is usually about 0.58 (Müller et al. 1990; Guillaume et al. 1997). See Schnidrig and Würtz (1995) for USD/DEM exchange rates. Therefore it deviates markedly from the exponent 1/2 which is obtained in the case of the Brownian motion.

3 Normal Inverse Gaussian Laws

We summarize some properties of the normal inverse Gaussian laws. For further information see Barndorff-Nielsen (1997, 1998a).

3.1 Theoretical Properties

The *normal inverse Gaussian distribution* with parameters α, β, μ and δ is denoted $NIG(\alpha,\beta,\mu,\delta)$ and is the distribution on $\mathbb{R}$ having density function

$$g(x;\alpha,\beta,\mu,\delta) = a(\alpha,\beta,\mu,\delta)\, q\Big(\frac{x-\mu}{\delta}\Big)^{-1} K_1\{\delta\alpha\, q\Big(\frac{x-\mu}{\delta}\Big)\}\ \exp\{\beta x\} \tag{4}$$

where $q(x) = \sqrt{1+x^2}$ and the norming constant has the simple form

$$a(\alpha,\beta,\mu,\delta) = \pi^{-1}\alpha\exp\{\delta\sqrt{\alpha^2-\beta^2} - \beta\mu\}; \tag{5}$$

furthermore K_1 is the modified Bessel function of the third kind and index 1. The domain of variation of the parameters is given by $\mu \in \mathbb{R}$, $\delta \in \mathbb{R}_+$, and $0 \leq |\beta| < \alpha$. Obviously δ is a scaling and μ a location parameter, whereas β is an asymmetry parameter and $\alpha \pm \beta$ determines the heaviness of the tails, cf. formula (7) below.

It follows immediately from (4) and (5) that the cumulant generating function of the normal inverse Gaussian distribution is

$$\mathrm{K}(u;\alpha,\beta,\mu,\delta) = \delta\{\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+u)^2}\} + \mu u. \tag{6}$$

Thus, in particular, if $x_1,\ldots,x_m$ are independent normal inverse Gaussian random variables with common parameters α and β but having individual location-scale parameters μ_i and δ_i $(i = 1,\ldots,m)$ then $x_+ = x_1 + \cdots + x_m$ is again distributed according to a normal inverse Gaussian law, with parameters $(\alpha,\beta,\mu_+,\delta_+)$, $\mu_+ = \sum_{i=1}^m \mu_i$ and $\delta_+ = \sum_{i=1}^m \delta_i$.

The *NIG* distribution (4) has semiheavy tails; specifically,

$$g(x;\alpha,\beta,\mu,\delta) \sim \mathit{const.}\, |x|^{-3/2}\ \exp\{-\alpha\,|x| + \beta x\} \quad \text{as} \quad x \to \pm\infty\ , \tag{7}$$

as follows from the well known asymptotic relation for the Bessel functions $K_\nu(x)$

$$K_\nu(x) \sim \sqrt{\pi/2}\ x^{-1/2}e^{-x} \quad \text{as} \quad x \to \infty\ . \tag{8}$$

A further important characterization of the normal inverse Gaussian law $NIG(\alpha,\beta,\mu,\delta)$ is the following. Let $b(t) = \big(b_1(t), b_2(t)\big)$ be a bivariate Brownian motion starting at $(\mu, 0)$ and having drift vector (β,γ) where $\beta \in \mathbb{R}$ and $\gamma \geq 0$. Furthermore, let u denote the time when b_1 first reaches level $\delta > 0$ and let $x = b_2(u)$. Then $x \sim NIG(\alpha,\beta,\mu,\delta)$ with $\alpha = \sqrt{\beta^2+\gamma^2}$.

It is often of interest to consider alternative parametrisations of the normal inverse Gaussian laws. In particular, letting $\overline{\alpha} = \delta\alpha$ and $\overline{\beta} = \delta\beta$, we have that $\overline{\alpha}$ and $\overline{\beta}$ are invariant under location-scale changes, and when $\overline{\alpha}, \overline{\beta}, \mu, \delta$ constitute the parametrisation of interest we shall write $NIG[\overline{\alpha}, \overline{\beta}, \mu, \delta]$ instead of $NIG(\alpha,\beta,\mu,\delta)$. In terms of this alternative parametrisation the first four cumulants of $NIG[\overline{\alpha}, \overline{\beta}, \mu, \delta]$ are

$$\kappa_1 = \mu + \delta\varrho/(1-\varrho^2)^{1/2} \tag{9}$$

$$\kappa_2 = \delta^2/\{\overline{\alpha}\,(1-\varrho^2)^{3/2}\} \tag{10}$$

$$\kappa_3 = 3\delta^3\varrho/\{\overline{\alpha}^{\,2}(1-\varrho^2)^{5/2}\} \tag{11}$$

$$\kappa_4 = 3\delta^4(1+4\varrho^2)/\{\overline{\alpha}^{\,3}(1-\varrho^2)^{7/2}\} \tag{12}$$

where $\varrho = \beta/\alpha$, which is invariant since $\beta/\alpha = \overline{\beta}/\overline{\alpha}$. Recall that κ_1 and κ_2 denote mean and variance. It follows that the standardised third and fourth cumulants, i.e. skewness and kurtosis, have the form

$$\gamma_1 = \kappa_3/\kappa_2^{3/2} = 3\overline{\alpha}^{\,-1/2}\varrho/(1-\varrho^2)^{1/4} \tag{13}$$

$$\gamma_2 = \kappa_4/\kappa_2^2 = 3\overline{\alpha}^{\,-1}(1+4\varrho^2)/(1-\varrho^2)^{5/4}. \tag{14}$$

For some purposes it is useful, instead of the classical skewness and kurtosis quantities (11) and (12), to work with the alternative steepness and asymmetry parameters ξ and χ defined by

$$\xi = \left(1 + \delta\sqrt{\alpha^2 - \beta^2}\,\right)^{-1/2} \tag{15}$$

and

$$\chi = \frac{\beta}{\alpha}\,\xi. \tag{16}$$

Like γ_1 and γ_2, these parameters are invariant under location-scale changes and the domain of variation for (χ, ξ) is the *normal inverse Gaussian shape triangle*

$$\{(\chi, \xi) : -1 < \chi < 1, \quad 0 < \xi < 1\}.$$

The distributions with $\chi = 0$ are symmetric, and the normal and Cauchy laws occur as limiting cases for (χ, ξ) near to $(0, 0)$ and $(0, 1)$, respectively.

3.2 Applicability to Finance

As illustration of the applicability of the *NIG* laws in finance we consider some data sets, on foreign exchange rates and on stocks.

The analyzed intraday (FX) rate is part of the HFDF96 data set provided by Olsen & Associates (1997) and covers the year 1996. To cope with the bid- and ask-spreads in the high-frequency data set we define the log-price at time t as usual as

$$x_t = \frac{\log P_{\mathrm{bid},t} + \log P_{\mathrm{ask},t}}{2}.$$

Finally, to take periods without trading into account, we have removed the zero-returns in these periods. Here it would be interesting to apply more sophisticated methods, e.g. the transformation to a business time scale, but this would require additional information which is not available. We estimated the *NIG* distribution from 3 hour returns of the USD/DEM exchange rate using the algorithm described in Prause (1997) and obtain the following *NIG* distribution

$$\begin{aligned} &\alpha = 415.9049, \quad \beta = 1.512, \quad \delta = 0.0011, \quad \mu = 0.000026, \\ &\overline{\alpha} = 0.4574, \quad \overline{\beta} = 0.00166, \quad \chi = 0.00301, \quad \xi = 0.8282. \end{aligned} \tag{17}$$

Figure 1 shows the empirical and the estimated *NIG* and normal distributions and the corresponding Q-Q plot as well as a Q-Q plot for Deutsche Bank stock returns are given as Figure 2. Obviously, the estimated *NIG* distributions fit the return distributions very well.

Figures 3 and 4 exhibit the maximum likelihood estimates of the shape parameters χ and ξ for the USD/DEM exchange rates and for those stocks incorporated in the German stock market index DAX, respectively. For later, note in particular that the ξ values lie in the neighbourhood of 0.7–0.9.

Note also that ξ decreases with increasing time lag, reflecting the stylized feature of aggregational Gaussianity (the normal law corresponds to $(\chi, \xi) = (0, 0)$).

4 NIG Lévy Processes

Let $z(t)$ be the *NIG Lévy process*, i.e. the process with stationary independent increments for which $z(1)$ is distributed as $NIG(\alpha, \beta, \mu, \delta)$. See Protter (1990) for an introduction to Lévy processes. From the form (6) of the cumulant function of $NIG(\alpha, \beta, \mu, \delta)$ it is immediate that for any $t > 0$, $z(t)$ has distribution $NIG(\alpha, \beta, t\mu, t\delta)$.

Note that for increasing time t the corresponding value of ξ converges to zero. Hence, in accordance with the empirical results shown in Figure 3, the kurtosis of the distribution decreases.

4.1 Symmetric NIG Lévy Processes

Initially, let us consider the case $\mu = \beta = 0$, in which instance the distribution $NIG(\alpha, 0, 0, \delta)$ of $z(1)$ is symmetric around 0. To calculate its absolute first moment we recall that the Bessel functions $K_\nu(x)$ satisfy the equation

$$K'_\nu(x) = -\frac{1}{2}\{K_{\nu-1}(x) + K_{\nu+1}(x)\}.$$

Hence, in particular, we have

$$K'_0(x) = -K_1(x). \tag{18}$$

By the substitution $u = (1 + x^2/\delta^2)^{1/2}$ and using (18) we find

$$\begin{aligned}
\mathrm{E}\{|z(1)|\} &= 2\pi^{-1}\alpha e^{\bar{\alpha}} \int_0^\infty x(1 + x^2/\delta^2)^{-1/2} K_1\big(\bar{\alpha}(1 + x^2/\delta^2)^{1/2}\big)\mathrm{d}x \\
&= 2\pi^{-1}\delta\alpha e^{\bar{\alpha}} \int_1^\infty K_1(\bar{\alpha}u)\mathrm{d}u \\
&= 2\pi^{-1}\bar{\alpha}e^{\bar{\alpha}} \int_{\bar{\alpha}}^\infty K_1(u)\mathrm{d}u \\
&= 2\pi^{-1}\bar{\alpha}e^{\bar{\alpha}} K_0(\bar{\alpha}).
\end{aligned}$$

Hence, letting

$$\psi(x) = xe^x K_0(x) \tag{19}$$

we have

$$\mathrm{E}\{|z(t)|\} = 2\pi^{-1}\psi(t\bar{\alpha}). \tag{20}$$

It is well-known that

$$K_0(x) \sim \begin{array}{ll} -\log x & \text{for} \quad x \downarrow 0 \\ \sqrt{2/\pi}\, x^{-1/2} e^{-x} & \text{for} \quad x \to \infty. \end{array} \tag{21}$$

It follows that $\phi(x) = \log \psi(e^x)$ has linear asymptotes both for $x \to 0$ and for $x \to \infty$, the slopes of the asymptotes being, respectively, 1 and 1/2. The formula for the slope is

$$\phi'(x) = 1 + e^x - e^x \frac{K_1(e^x)}{K_0(e^x)} \tag{22}$$

and a number of values of $\phi'(x)$ is given in Table 1.

ξ	$\overline{\alpha}$	$\log \overline{\alpha}$	$\phi'(\log \overline{\alpha})$
.99	.0203	−3.9	.771
.95	.108	−2.23	.690
.90	.235	−1.45	.646
.85	.384	−.957	.618
.80	.563	−.575	.598
.75	.778	−.251	.582
.70	1.05	.049	.568
.65	1.37	.315	.557
.60	1.78	.577	.548
.55	2.31	.837	.539
.50	3.00	1.10	.532
.45	3.94	1.37	.526
.40	5.25	1.66	.520
.35	7.16	1.97	.515
.30	10.1	2.31	.511
.25	15.0	2.71	.508

Table 1: Slope of $\log \mathrm{E}\{|z(t)|\}$.

The remarkable point is that for ξ values in the range 0.7–0.9 (cf. Figures 3 and 4) the slope of $\log \mathrm{E}\{|z(t)|\}$ is close to the empirically observed of 0.58 (cf. Section 2).

4.2 Skewed NIG Lévy Processes

In this section we describe the approximate scaling power law of Lévy processes based on *NIG* estimates for the USD/DEM foreign exchange (FX) rate described in Section 3.2.

We define the Lévy process $z(t)$ by taking the *NIG* estimate (17) as the distribution of $z(1)$ where $t = 1$ corresponds to 3 hours. The distribution of $z(t)$ corresponding to time points from 10 minutes to two months is given by $NIG(\alpha, \beta, t\mu, t\delta)$. In contrast to the symmetric case, considered in the previous subsection, an explicit expression for the mean of $|z(t)|$ is not available and we have therefore to determine $E\{|z(t)|\}$ by numerical integration. The result is shown in Figure 5 together with the fitted regression line which has slope 0.5705, close to the empirically observed slopes reported in Section 2. Compare Figure 5 with the graphs in Müller et al. (1990, p. 1196) and Guillaume et al. (1997, p. 115). From a statistical point of view, the empirical scaling power law and the approximate scaling power laws of *NIG* Lévy processes are difficult to distinguish.

Log-returns of financial assets are usually only slightly skewed. A question of interest is, what the scaling behaviour of *NIG* Lévy processes looks like if the *NIG* distribution is more skewed. To analyze approximate scaling power laws of more skewed *NIG* processes, we consider several values of the skewness parameter χ, different from the value 0.00301 used in Figure 5, but keeping the same values of μ, δ, ξ where $\xi = 0.828$.

The results are shown in Figure 6. The adherence to a scaling law is enhanced by the skewness. Note also, that the skewed *NIG* Lévy motions have higher scaling coefficients. It does not matter if the skewness is negative or positive.

5 Conclusion

Although the *NIG* Lévy processes considered here do not capture all "stylized features" of, for instance, foreign exchange markets, they do represent the observed return distributions well and provide an explanation of the empirical scaling power law. Recently, models of this type have been examined with a view towards option pricing and risk measurement (see e.g. Eberlein and Prause 1998). Developments of the models, that incorporate further stylized features—in particular the volatility clustering and the quasi long range dependence—are considered in Barndorff-Nielsen (1998b), Barndorff-Nielsen and Shephard (1999) (see also Nicolato and Prause 1999).

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References

Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics 24*, 1–13.

Barndorff-Nielsen, O. E. (1998a). Probability and statistics: self-decomposability, finance and turbulence. In L. Acccardi and C. C. Heyde (Eds.), *Probability Towards 2000*, New York. Springer. Proceedings of a Symposium held 2–5 October 1995 at Columbia University.

Barndorff-Nielsen, O. E. (1998b). Processes of normal inverse Gaussian type. *Finance & Stochastics 2*, 41–68.

Barndorff-Nielsen, O. E. and W. Jiang (1998). An initial analysis of some German stock price series. Working Paper 15, Centre for Analytical Finance.

Barndorff-Nielsen, O. E. and N. Shephard (1999). Non-Gaussian OU based models and some of their uses in financial economics. Working Paper 37, Centre for Analytical Finance, University of Århus.

Eberlein, E. and K. Prause (1998). The generalized hyperbolic model: financial derivatives and risk measures. FDM Preprint 56, University of Freiburg.

Frisch, U. (1995). *Turbulence*. Cambridge: Cambridge University Press.

Ghashgaie, S., W. Breymann, J. Peinke, P. Talkner, and Y. Dodge (1996). Turbulent cascades in foreign exchange markets. *Nature 381*, 767–770.

Guillaume, D. M., M. M. Dacorogna, R. D. Davé, U. A. Müller, R. B. Olsen, and O. V. Pictet (1997). From the bird's eye to the microscope: a survey of new stylized facts of the intra-daily foreign exchange markets. *Finance and Stochastics 1*, 95–129.

Mandelbrot, B. B. (1963). The variation of certain speculative prices. *Journal of Business 36*, 394–419.

Müller, U. A., M. M. Dacorogna, R. B. Olsen, O. V. Pictet, M. Schwarz, and C. Morgenegg (1990). Statistical study of foreign exchange rates, empirical evidence of a price change scaling law, and intraday analysis. *Journal of Banking and Finance 14*, 1189–1208.

Nicolato, E. and K. Prause (1999). Derivative pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. Working Paper, University of Århus.

Olsen & Associates (1997). HFDF96. High frequency data in preparation for the Second International Conference on High Frequency Data in Finance (HFDF II), Zürich, April 1–3, 1998.

Piccinato, B., G. Ballocchi, and M. Dacorogna (1998). A closer look at the Eurofutures market: Intraday statistical analysis. Preprint. The O&A Research Group, Zürich.

Prause, K. (1997). Modelling financial data using generalized hyperbolic distributions. FDM Preprint 48, University of Freiburg.

Protter, P. (1990). *Stochastic Integration and Differential Equations. A new Approach*. Heidelberg: Springer.

Rydberg, T. H. (1997). The normal inverse Gaussian Lévy process: Simulation and approximation. *Communications in Statistics: Stochastic Models 13*, 887–910.

Schnidrig, R. and D. Würtz (1995). Investigation of the volatility and autocorrelation function of the USDDEM exchange rate on operational time scales. In *Proceedings of the "1st International Conference on High Frequency Data"*, Zürich. March 29–31, 1995.

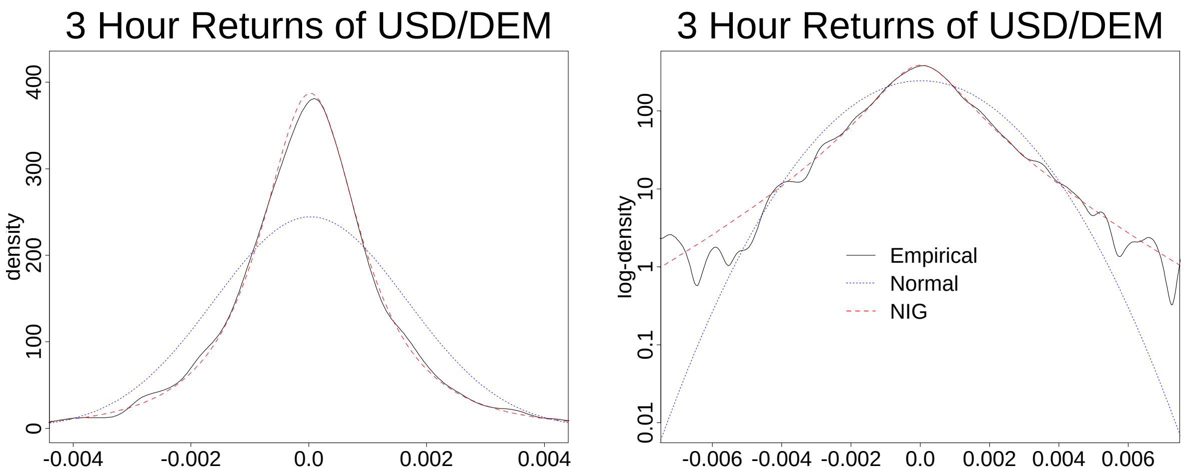

Figure 1: Density and log-density of the *NIG* distributions estimated for 3 hour returns of USD/DEM exchange rates in 1996.

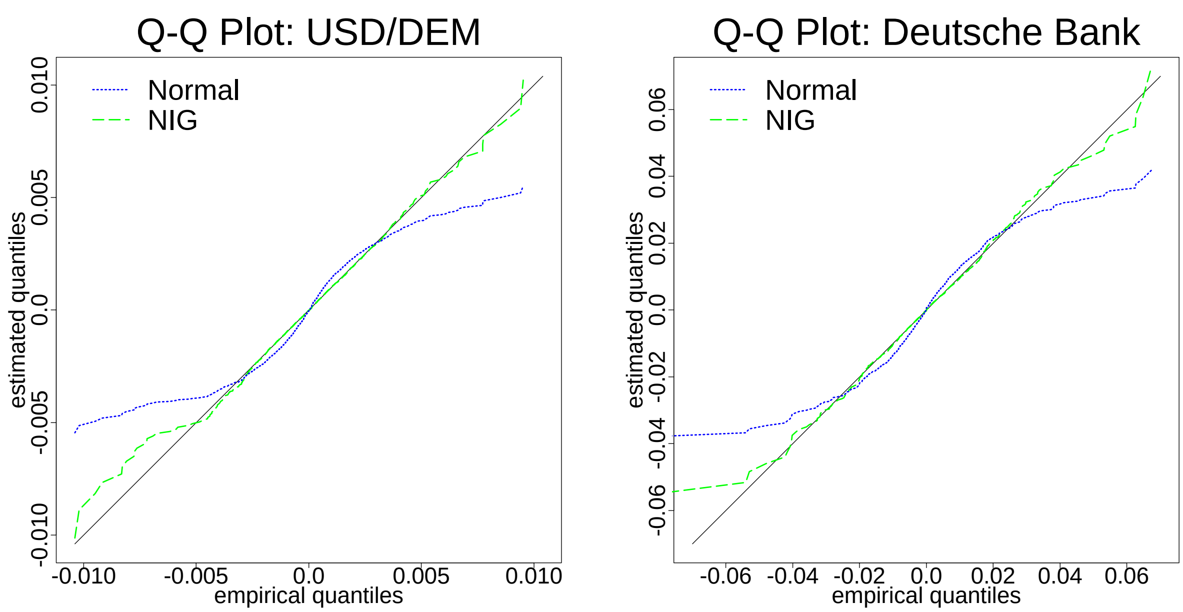

Figure 2: Q-Q Plots of 3 hour USD/DEM exchange rate returns and of daily returns of Deutsche Bank stocks (1988–94).

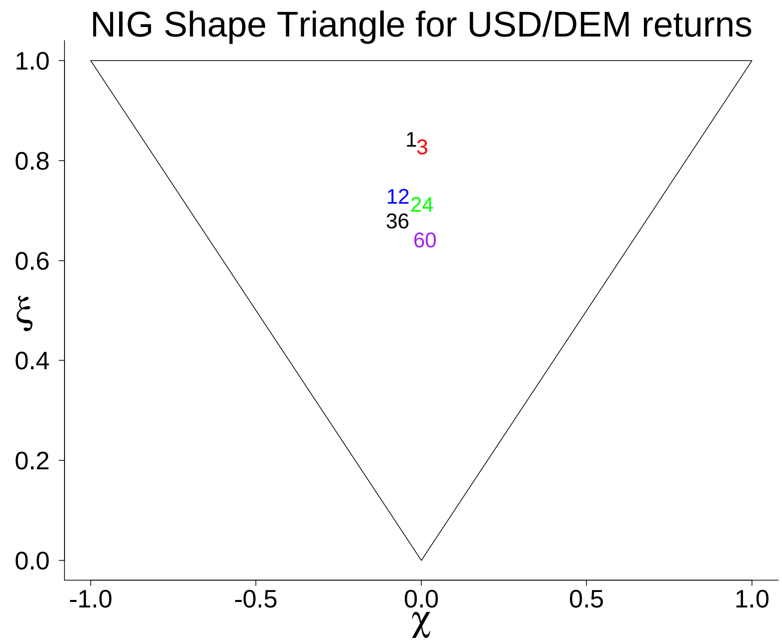

Figure 3: Shape triangle of the *NIG* distributions and positions of the maximum likelihood estimates of (χ, ξ) for USD/DEM returns with different time lags (1, 3, 12, 24, 36, 60 measured in hours).

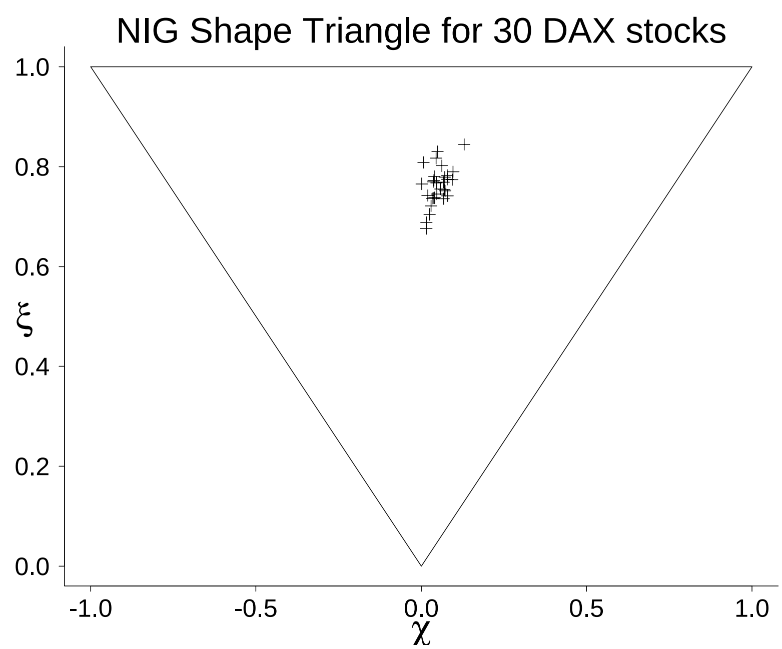

Figure 4: Shape triangle of the *NIG* distributions and positions of the maximum likelihood estimates of (χ, ξ) for daily returns of the 30 German stocks in the DAX (1988–94).

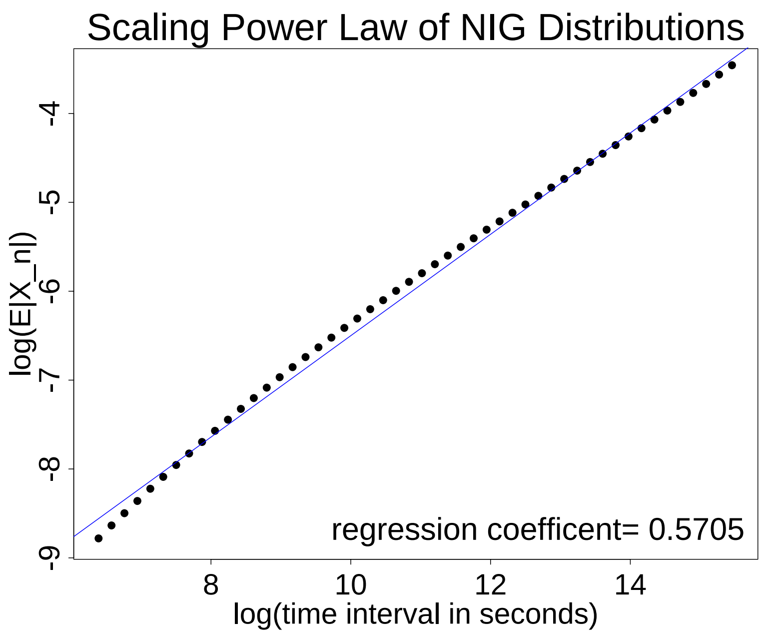

Figure 5: Approximate scaling power law of the *NIG* Lévy process with parameters estimated from the 3 hour returns of USD/DEM exchange rates.

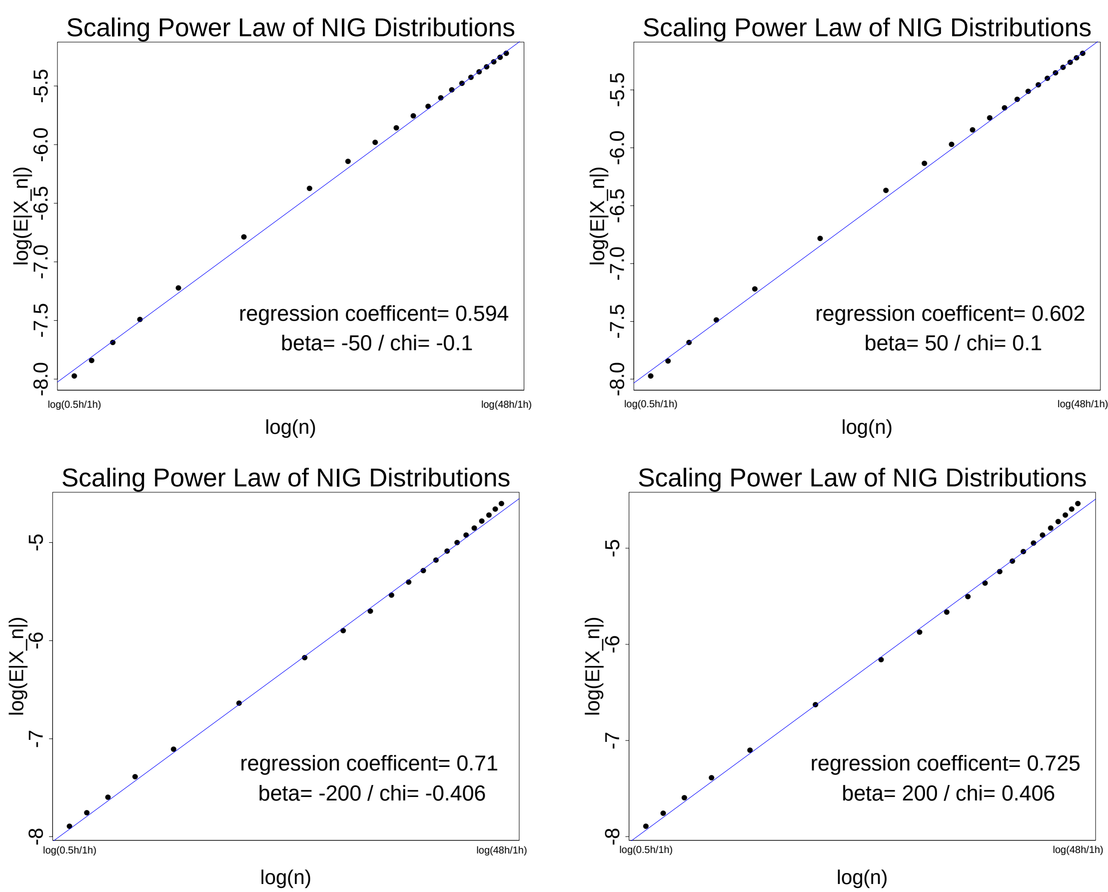

Figure 6: Scaling Power Law of *NIG* distributions with different skewness parameters.