THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

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Research Report

No. 07 | June 2007

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This Thiele Research Report is also Research Report number 494 in the Stochastics Series at Department of Mathematical Sciences, University of Aarhus, Denmark.

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June 14, 2007

Abstract

Upsilon transformations satisfying certain regularity conditions are shown to generate semigroups of such transformations. This is based on a general commutativity property of the Upsilon transformations, and uses log infinite divisibility. The existence of stochastic integral representations of Upsilon transformations and of the generated semigroups is also discussed.

1 Introduction

Υ (or Upsilon) transformations are a special type of mappings that send Lévy measures to Lévy measures. Closely associated to these are mappings of the set $\mathrm{ID}(\mathbb{R}^d)$ of all infinitely divisible laws on $\mathbb{R}^d$ into itself. Furthermore, in many cases, the transformations have a stochastic interpretation in terms of stochastic integrals with respect to Lévy processes.

Various special cases of Υ transformations have been studied in [BNTh02a], [BNTh02b], [BNTh02c], [BNTh04], [BNTh05], [BNTh06a], [BNTh06b], [BNMaSa06], [BNPA06], [BNLi06] and general formulations are given in [BNRoTh07]. The early works cited above arose out of a study of the connection between free and classical probability.

In wide generality the Υ transformations are smooth one-to-one mappings with absolutely continuous Lévy measures as images. Moreover, the image measures often have important monotonicity properties.

In the present paper, restricting attention to the case of one-dimensional measures and distributions, we introduce and discuss semigroups $\{\Upsilon_t\}_{t\geq 0}$ of Υ transformations that, in considerable generality, are representable in terms of stochastic integrals with respect to Lévy processes and which interpolate smoothly between some previously studied iterates of specific Υ transformations.

The dynamic Upsilon transformations are defined and discussed in Section 3, while Section 2 presents a number of relevant background results and examples on Upsilon transformations.

For comprehensive information on the theory of Upsilon transformations and on the relation to stochastic integrals of deterministic functions with respect to Lévy processes, see [BNRoTh07], [Sa06] and [Sa07], and references given there.

2 Background on Upsilon transformations

We denote the sets of σ-finite Borel measures on $\mathbb{R}$ and on $\mathbb{R}_+$ $(= (0, \infty))$ by $\mathfrak{M}$ and $\mathfrak{M}^+$, respectively, while $\mathfrak{M}_L$ and $\mathfrak{M}_L^+$ will stand for the sets of Lévy measures in $\mathfrak{M}$, respectively the set of Lévy measures corresponding to positive infinitely divisible laws, i.e.

$$\mathfrak{M}_L^+ = \left\{ \rho \in \mathfrak{M}^+ : \int_0^\infty (x \wedge 1) \rho (\mathrm{d}x) < \infty \right\}.$$

For any γ in $\mathfrak{M}^+$, let $\Upsilon_\gamma : \rho \to \Upsilon_\gamma (\rho)$ be the mapping given on $\mathfrak{M}$ such that, for any Borel set A of $\mathbb{R}$,

$$\Upsilon_\gamma (\rho) (A) = \int_0^\infty \rho \left(\xi^{-1} A \right) \gamma (\mathrm{d}\xi) \tag{1}$$

or, in infinitesimal form,

$$\Upsilon_\gamma (\rho) (\mathrm{d}x) = \int_0^\infty \rho \left(\xi^{-1} \mathrm{d}x \right) \gamma (\mathrm{d}\xi) . \tag{2}$$

We shall also use the notation ρ_γ for $\Upsilon_\gamma (\rho)$. The measure γ is referred to as the *dilation measure* of the Upsilon transformation.

In case γ is absolutely continuous with density g we sometimes write Υ_g and ρ_g for Υ_γ and ρ_γ, respectively. If, moreover, ρ has a density r with respect to Lebesgue measure then the same is true of ρ_g, and, letting r_g denote the density of ρ_g, we may reexpress (2) as

$$r_g (x) = \int_0^\infty r \left(\xi^{-1} x \right) \xi^{-1} g (\xi) \mathrm{d}\xi.$$

The image $\Upsilon_\gamma (\rho)$ is a, not necessarily finite, measure on $\mathbb{R}$. We define the *Lévy domain* of Υ_γ to be the set

$$\mathrm{dom}_L \Upsilon_\gamma = \{ \rho \in \mathfrak{M}_L : \Upsilon_\gamma (\rho) \in \mathfrak{M}_L \}$$

and shall also refer to the *positive Lévy domain* as given by

$$\mathrm{dom}_L^+ \Upsilon_\gamma = \left\{ \rho \in \mathfrak{M}_L^+ : \Upsilon_\gamma (\rho) \in \mathfrak{M}_L^+ \right\}$$

Thus $\mathrm{dom}_L \Upsilon_\gamma \subset \mathfrak{M}_L$ and $\mathrm{dom}_L^+ \Upsilon_\gamma \subset \mathfrak{M}_L^+$.

Theorem 1. *We have*

$$\mathrm{dom}_L \Upsilon_\gamma = \mathfrak{M}_L \tag{3}$$

if and only if

$$\int_0^\infty (\xi^2+1)\gamma(\mathrm{d}\xi) < \infty. \tag{4}$$

Furthermore,

$$\mathrm{dom}_L^+ \Upsilon_\gamma {=} \mathfrak{M}_L^+ \tag{5}$$

if and only if

$$\int_0^\infty (\xi+1)\gamma(\mathrm{d}\xi) < \infty. \tag{6}$$

Thus, in both cases, the measure γ is necessarily finite.

A proof of the equivalence of (5) and (6) was given in [Sa05], and a statement similar to the equivalence of (3) and (4) was also proved. For a detailed verification of Theorem 1, see [BNRoTh07].

Example 2. *Suppose γ has density $g(\xi) = e^{-\xi}$. The corresponding Upsilon transformation (first considered in [BNTh04], cf. also [BNTh06b]), is denoted by Υ. It has Lévy domain*

$$\mathrm{dom}_L \Upsilon = \mathfrak{M}_L,$$

and the range

$$\mathfrak{B} = \mathrm{ran}_L \Upsilon = \Upsilon(\mathfrak{M}_L)$$

equals the family of Lévy measures of the Goldie-Steutel-Bondesson class of infinitely divisible distributions, cf. [BNMaSa06].

Example 3. *For any λ in $(-1,\infty)$, let*

$$g(x) = x^{\lambda-1}\mathrm{e}^{-x} \qquad (x \in (0,\infty)).$$

The corresponding Upsilon mappings Υ_λ were introduced and studied in [Sa05] and [BNPA06]; see also [Sa06]. Here

$$\mathrm{dom}_L \Upsilon_\lambda = \begin{cases} \mathfrak{M}_L & \text{if } \lambda > 0 \\ \mathfrak{M}_{\log} & \text{if } \lambda = 0 \\ \mathfrak{M}_\lambda & \text{if } \lambda \in (-1,0) \end{cases}$$

where the classes $\mathfrak{M}_{\log}(\mathbb{R})$ and $\mathfrak{M}_\lambda(\mathbb{R})$, $\lambda \in (-1,0)$, are defined by:

$$\mathfrak{M}_{\log} = \left\{\rho \in \mathfrak{M}(\mathbb{R}) \;\middle|\; \int_1^\infty \log y \,\rho(\mathrm{d}y) < \infty\right\}$$

and

$$\mathfrak{M}_\lambda = \left\{\rho \in \mathfrak{M}(\mathbb{R}) \;\middle|\; \int_1^\infty y^{-\lambda}\,\rho(\mathrm{d}y) < \infty\right\},$$

respectively.

For an extension to Upsilon mappings of Lévy measures on the cone of positive definite matrices, see [BNPA06].

Example 4. *Let Φ be the Upsilon transformation determined by $g(\xi) = \xi^{-1}\mathbf{1}_{[0,1]}(\xi)$. Its domain of definition is*

$$\mathrm{dom}_L \Phi = \mathfrak{M}_{\log}.$$

The range

$$\mathfrak{L} = \mathrm{ran}_L \Phi = \Phi(\mathfrak{M}_{\log})$$

is the family of Lévy measures of the class of selfdecomposable laws (cf. [BNMaSa06]).

Henceforth in all cases we assume that γ has finite upper tail measure, i.e. letting

$$\varepsilon_\gamma(\xi) = \gamma([\xi, \infty)),$$

we require that $\varepsilon_\gamma(\xi) < \infty$ for all $\xi > 0$. Then

$$\gamma(\mathrm{d}\xi) = -\mathrm{d}\varepsilon_\gamma(\xi). \tag{7}$$

The inverse function of ε_γ, denoted ε_γ^*, is defined by

$$\varepsilon_\gamma^*(x) = \inf\{\xi > 0 : \varepsilon_\gamma(\xi) \le x\}. \tag{8}$$

Note that both functions $\xi \to \varepsilon_\gamma(\xi)$ and $x \to \varepsilon_\gamma^*(x)$ are decreasing and càglàd. In case γ is absolute continuous with density g, we also write ε_g for ε_γ and ε_g^* for ε_γ^*.

Given a $\gamma \in \mathfrak{M}^+$ and a Lévy measure ρ, consider the stochastic integral

$$Y = \int_0^{\varepsilon_\gamma(0)} \varepsilon_\gamma^*(s)\,\mathrm{d}Z_s \tag{9}$$

where $Z = \{Z_t\}$ is the Lévy process for which the cumulant function of Z_1 is given by

$$\mathrm{C}_\rho(z) = \mathrm{i}\eta z + \int_{\mathbb{R}} \left(e^{\mathrm{i}zt} - 1 - \mathrm{i}zt\left(1+t^2\right)^{-1}\right)\rho(\mathrm{d}t) \tag{10}$$

for some $\eta \in \mathbb{R}$.

As in [BNRoTh07], we say that (9) is a *stochastic integral representation* (SIR) of Υ_γ at $\rho \in \mathrm{dom}_L \Upsilon_\gamma$ provided the integral (9) exists as the limit in probability of the Riemann sums and the random variable Y (which is then necessarily infinitely divisible) has Lévy measure $\rho_\gamma = \Upsilon_\gamma(\rho)$ and cumulant function

$$\mathrm{C}_{\rho_\gamma}(z) = \mathrm{i}\tilde{\eta} z + \int_{\mathbb{R}} \left(e^{\mathrm{i}zt} - 1 - \mathrm{i}zt\left(1+t^2\right)^{-1}\right)\rho_\gamma(\mathrm{d}t) \tag{11}$$

where

$$\tilde{\eta} = \int_0^\infty x\left(\eta + \int_{\mathbb{R}} y\left(\left(1+(xy)^2\right)^{-1} - \left(1+y^2\right)^{-1}\right)\rho(\mathrm{d}y)\right)\gamma(\mathrm{d}x).$$

Note that, in this case, the cumulant function $\mathrm{C}\{z \ddagger Y\}$ of Y is expressible in terms of C_ρ and ε_γ^* as

$$\mathrm{C}\{z \ddagger Y\} = \int_0^\infty \mathrm{C}_\rho\left(\varepsilon_\gamma^*(s)\,z\right)\mathrm{d}s. \tag{12}$$

The representation property holds, in particular, for all ρ in $\mathrm{dom}_L \Upsilon_\gamma$ provided ε_γ^* is continuous and γ is a probability measure with finite second moment. This follows by direct extension of the proof given by [BNTh04] in the special case $\gamma(\mathrm{d}x) = e^{-x}\mathrm{d}x$. Extensions and ramifications are also discussed in [BNTh06b], [Sa05], [Sa06] and [Sa07].

The following proposition gives a set of simple sufficient conditions for existence of the stochastic integral (9) in tems of the dilation measure γ. The proof, which relies on a result from the general theory developed in [Sa06], is given in the Appendix.

Proposition 5. *The following conditions are sufficient for the existence of the stochastic integral in* (9)*.*

$$\int_0^\infty (\xi + \xi^2)\gamma(\mathrm{d}\xi) < \infty, \tag{13}$$

$$\int_0^a \xi^2\gamma(\mathrm{d}\xi) = O(a^2) \quad \text{as } a \downarrow 0 \tag{14}$$

$$\int_0^a \xi\gamma(\mathrm{d}\xi) = O(a) \quad \text{as } a \downarrow 0 \tag{15}$$

and

$$\int_{|x|>1} \gamma([1/|x|, \infty))\rho(\mathrm{d}x) < \infty. \tag{16}$$

Remark 6. *The measures γ in Examples 2, 3 and 4 satisfy the conditions in Proposition 5. In particular, in Example 4, since $\gamma([1/|x|, \infty)) = \int_{1/|x|}^1 \xi^{-1}\mathbf{1}_{[0,1]}(\xi)\mathrm{d}\xi$,* (16) *turns out to be $\int_{|x|>1} \log|x|\rho(dx) < \infty$, meaning that $\rho \in \mathfrak{M}_{\log}$. And similarly for Example 3.*

Next we recall some results from [BNRoTh07]. Direct calculation shows that the Upsilon transformations commute, that is

$$\Upsilon_\eta \circ \Upsilon_\gamma = \Upsilon_\gamma \circ \Upsilon_\eta.$$

Furthermore, introducing the product convolution $\circledast$ of measures $\eta, \gamma \in \mathfrak{M}^+$, defined by

$$(\eta \circledast \gamma)(\mathrm{d}x) := \int_0^\infty \eta(\xi^{-1}\mathrm{d}x)\,\gamma(\mathrm{d}\xi), \tag{17}$$

one finds that $\eta \circledast \gamma = \Upsilon_\gamma(\eta)$ and

$$\Upsilon_\eta \circ \Upsilon_\gamma = \Upsilon_\gamma \circ \Upsilon_\eta = \Upsilon_{\eta\circledast\gamma}. \tag{18}$$

Note that there is no guarantee that the measure $\eta \circledast \gamma$ is finite. However, if γ and η are probability measures then so is $\eta \circledast \gamma$.

Using the notation (7) we may reexpress formula (17) as

$$\varepsilon_{\gamma\circledast\eta}(x) = -\int_0^\infty \varepsilon_\gamma\left(\xi^{-1}x\right)\mathrm{d}\varepsilon_\eta(\xi). \tag{19}$$

This in turn can be rewritten

$$\varepsilon_{\gamma\circledast\eta} = \varepsilon_\gamma \circledast \varepsilon_\eta$$

implying

$$\varepsilon^*_{\gamma\circledast\eta} = (\varepsilon_\gamma \circledast \varepsilon_\eta)^* .$$

If γ and η have densities g and h, then $(\eta \circledast \gamma)(\mathrm{d}x) = (g \circledast h)(x)\mathrm{d}x$, where

$$(g \circledast h)(x) = \int_0^\infty h(\xi^{-1}x)\xi^{-1}g(\xi)\,\mathrm{d}\xi. \tag{20}$$

Note also that the operation $\circledast$ is converted into ordinary convolution by exponential transformation.

3 Dynamic Upsilon transformations

Henceforth the dilation measures considered are assumed to be probability measures.

Let γ be such a measure, let $\acute{\gamma}$ be the probability measure obtained from γ by the transformation $\xi \to e^x$, i.e. $\acute{\gamma}\,(\mathrm{d}x) = \gamma\,(x^{-1}\mathrm{d}x)$, and suppose that $\acute{\gamma}$ is infinitely divisible. We then say that γ is *log infinitely divisible.* Let V be a Lévy process such that V_1 has law $\acute{\gamma}$, let $\acute{\gamma}_t$ and γ_t denote the law of V_t and $U_t = e^{V_t}$, respectively, and write ρ_t and Υ_t as a shorthand for ρ_{γ_t} and Υ_{γ_t}; then

$$\rho_t\,(\mathrm{d}x) = \int_0^\infty \rho\left(\xi^{-1}\mathrm{d}x\right)\gamma_t\,(\mathrm{d}\xi)\,. \tag{21}$$

Given such a γ, we shall, for brevity, write ε_t for ε_{γ_t} and ε_t^* for $\varepsilon^*_{\gamma_t}$.

Theorem 7. *We have*

$$\Upsilon_{t+s} = \Upsilon_t \circ \Upsilon_s, \quad t, s \geq 0 \tag{22}$$

i.e. the family $\{\Upsilon_t\}$ *constitutes a semigroup. Furthermore,*

$$\varepsilon_{t+s}(x) = \varepsilon_t \circledast \varepsilon_s. \tag{23}$$

Proof. By (18), $\Upsilon_t \circ \Upsilon_s = \Upsilon_{\gamma_t \circledast \gamma_s}$. Thus, to verify (22) it is enough to show that

$$\gamma_t \circledast \gamma_s = \gamma_{t+s}. \tag{24}$$

Now, for any Borel set A in $\mathbb{R}_+$,

$$\begin{aligned}
&(\gamma_t \circledast \gamma_s)(A) \\
&\quad= \int_0^\infty \gamma_t(u^{-1}A)\gamma_s(\mathrm{d}u) \\
&\quad= \int_0^\infty P(e^{V_t} \in u^{-1}A)P(e^{V_s} \in \mathrm{d}u) \\
&\quad= \int_{-\infty}^\infty P(V_t \in \log A - \log u)P(V_s \in \mathrm{d}(\log u)) \\
&\quad= P(V_{t+s} - V_t + V_t \in \log A) \\
&\quad= P(V_{t+s} \in \log A) \\
&\quad= P(e^{V_{t+s}} \in A) \\
&\quad= \gamma_{t+s}(A)
\end{aligned}$$

proving (24). Formula (23) then follows from (24) and (19). $\square$

Theorem 8. *Suppose the log infinitely divisible probability measure γ has finite second moment. Then* $\mathrm{dom}_L \Upsilon_{\gamma_t} = \mathfrak{M}_L$ *for all* $t \geq 0$ *and* $\Upsilon_t(\mathfrak{M}_L)$ *is a decreasing process of subsets of* $\mathfrak{M}_L$.

Proof. The first assertion follows from Theorem 1 and the fact that existence of g-moments for g submultiplicative (here $g(x) = e^x$) is a time independent property of Lévy processes (cf. [Sa99], Theorem 25.3). The second assertion is a consequence of Theorem 7, on noting that

$$\Upsilon_t(\mathfrak{M}_L) = \Upsilon_s \circ \Upsilon_{t-s}(\mathfrak{M}_L) = \Upsilon_s(\Upsilon_{t-s}(\mathfrak{M}_L)) \subset \Upsilon_s(\mathfrak{M}_L).$$

□

Let $\{\Upsilon_t\}$ denote the semigroup of Upsilon transformations generated by a log infinitely divisible probability measure γ. We proceed to derive a set of sufficient conditions ensuring that Υ_t has the stochastic integral representation, for all $t > 0$. When this is the case we speak of $\{\Upsilon_t\}$ as a *dynamic Upsilon transformation.* In the present setting, where $\varepsilon_t(0) = 1$ for all t, (9) becomes

$$Y_t = \int_0^1 \varepsilon_t^*(s)\,\mathrm{d}Z_s. \tag{25}$$

Theorem 9. *Suppose that γ has no point mass and that*

$$\int_0^\infty \left(\xi^{-2} + \xi^2\right)\gamma(\mathrm{d}\xi) < \infty. \tag{26}$$

Then $\{\Upsilon_t\}$ *is a dynamic Upsilon transformation.*

Proof. Since γ is continuous the same holds for $\acute{\gamma}_1$ and hence, by [Sa99], point (b)(1) page 146, the same is true of $\acute{\gamma}_t$ for all $t > 0$. This in turn implies that ε_t^* is continuous for all $t > 0$. Thus, by the remark following (12), it is enough to show that $\acute{\gamma}_t$ has second moment, i.e.

$$\int_{\mathbb{R}} e^{2x}\acute{\gamma}_t(\mathrm{d}x) < \infty.$$

On account of (26), this is the case for $t = 1$. In fact, (26) is equivalent to the stronger statement that

$$\int_{\mathbb{R}} e^{2|x|}\acute{\gamma}_1(\mathrm{d}x) < \infty. \tag{27}$$

But (27) is a time independent distributional property, see [Sa99], point (b)(6) page 146, and consequently $\acute{\gamma}_t$ does have second moment. □

Example 10. *Let γ be the log normal law. Then condition* (26) *is satisfied and consequently γ generates a dynamic Upsilon transformation* $\{\Upsilon_t\}$ *with stochastic integral representation* (25).

Example 11. *Let γ be the gamma law with probability density* $g(\xi) = \Gamma(\lambda)^{-1}\xi^{\lambda-1}e^{-\xi}$. *The log gamma distribution is infinitely divisible, with Lévy density*

$$\nu(\mathrm{d}x) = |x|^{-1} e^{\lambda x}(1 - e^x)^{-1} 1_{(-\infty,0)}(x)\,\mathrm{d}x$$

(see, for instance, [Sa99], Example 18.19). For $\lambda > 2$ condition (26) *is satisfied and hence $\{\Upsilon_t\}$ is a dynamic Upsilon transformation.*

It seems quite possible that the same is true for any $\lambda > 0$. An indication of this is the following result, proved in [Ma07]: For $\lambda = 1$ (this is the case considered also in Example 2) and $n = 1, 2, ...$ the transformation Υ_n does have the stochastic integral representation.

4 Appendix

Here we give the proof of Proposition 5.

The following proposition is a direct consequence of the general results in [Sa06].

Proposition 12. *The existence of the stochastic integral (9) is assured under the following conditions:*

$$\int_0^\infty \varepsilon_\gamma^*(s)^2 \mathrm{d}s < \infty, \tag{28}$$

$$\int_0^\infty \mathrm{d}s \int_{\mathbb{R}} (|\varepsilon_\gamma^*(s)x|^2 \wedge 1)\rho(\mathrm{d}x) < \infty \tag{29}$$

and

$$\int_0^\infty \left(1 + \left| \int_{\mathbb{R}} x \left(\frac{1}{1 + |\varepsilon_\gamma^*(s)x|^2} - \frac{1}{1+|x|^2} \right) \rho(\mathrm{d}x) \right| \right) \varepsilon_\gamma^*(s) \mathrm{d}s < \infty. \tag{30}$$

The conditions stated in Proposition 5 imply (28)-(30), as we now show.

Condition (13) entails (28), since

$$\int_0^\infty \varepsilon_\gamma^*(s)^2 \mathrm{d}s = \int_0^\infty \xi^2 \gamma(\mathrm{d}\xi) < \infty.$$

As to (29), we have

$$\begin{aligned}
&\int_0^\infty \mathrm{d}s \int_{\mathbb{R}} (|\varepsilon_\gamma^*(s)x|^2 \wedge 1)\rho(\mathrm{d}x) \\
&\quad = - \int_0^\infty d\varepsilon_\gamma(\xi) \int_{\mathbb{R}} (|\xi x|^2 \wedge 1) \rho(\mathrm{d}x) \\
&\quad = \int_0^\infty \gamma(\mathrm{d}\xi) \left(\int_{|x| \le 1/\xi} |\xi x|^2 \rho(\mathrm{d}x) + \int_{|x|>1/\xi} \rho(\mathrm{d}x) \right) \\
&\quad =: I_1 + I_2,
\end{aligned}$$

say. Here

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}} |x|^2 \rho(\mathrm{d}x) \int_0^{1/|x|} \xi^2 \gamma(\mathrm{d}\xi) \\
&= \left(\int_{|x|\le 1} + \int_{|x|>1} \right) |x|^2 \rho(\mathrm{d}x) \int_0^{1/|x|} \xi^2 \gamma(\mathrm{d}\xi) \\
&=: I_{11} + I_{12},
\end{aligned}$$

say, and

$$I_{11} \le \int_{|x|\le 1} |x|^2 \rho(\mathrm{d}x) \int_0^\infty \xi^2 \gamma(\mathrm{d}\xi) < \infty \quad \text{(by (13))},$$
$$I_{12} = \int_{|x|>1} |x|^2 \rho(\mathrm{d}x) \int_0^{1/|x|} \xi^2 \gamma(\mathrm{d}\xi) < \infty \quad \text{(by (14))}.$$

Also,

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} \rho(\mathrm{d}x) \int_{1/|x|}^\infty \gamma(\mathrm{d}\xi) = \left(\int_{|x|\le 1} + \int_{|x|>1} \right) \rho(\mathrm{d}x) \int_{1/|x|}^\infty \gamma(\mathrm{d}\xi) \\ &=: I_{21} + I_{22}, \end{aligned}$$

say, and

$$I_{21} \le \int_{|x|\le 1} |x|^2 \rho(\mathrm{d}x) \int_0^\infty \xi^2 \gamma(\mathrm{d}\xi) < \infty \quad \text{(by (13))},$$
$$I_{22} \le \int_{|x|>1} \rho(\mathrm{d}x) \int_{1/|x|}^\infty \gamma(\mathrm{d}\xi) = \int_{|x|>1} \gamma([1/|x|, \infty)) \rho(\mathrm{d}x) < \infty, \quad \text{(by (16))}.$$

This shows (29). For (30), we have

$$\begin{aligned} &\int_0^\infty \left(1 + \left| \int_{\mathbb{R}} x \left(\frac{1}{1+|\varepsilon_\gamma^*(s) x|^2} - \frac{1}{1+|x|^2} \right) \rho(\mathrm{d}x) \right| \right) \varepsilon_\gamma^*(s) \mathrm{d}s \\ &\quad \le -\int_0^\infty \xi \mathrm{d}\varepsilon_\gamma(\xi) - \int_0^\infty \left| \xi \int_{\mathbb{R}} x \left(\frac{1}{1+|\xi x|^2} - \frac{1}{1+|x|^2} \right) \rho(\mathrm{d}x) \right| \mathrm{d}\varepsilon_\gamma(\xi) \\ &\quad =: I_3 + I_4, \end{aligned}$$

say, where

$$\begin{aligned} I_3 &= \int_0^\infty \xi \gamma(\mathrm{d}\xi) < \infty, \quad \text{(by (13))} \\ I_4 &\le \int_0^\infty \xi \gamma(\mathrm{d}\xi) \left| \int_{\mathbb{R}} \left(\frac{x|x|^2|\xi^2-1|}{(1+|\xi x|^2)(1+|x|^2)} \right) \rho(\mathrm{d}x) \right| \\ &\le \int_0^\infty \xi|\xi^2-1| \gamma(\mathrm{d}\xi) \int_{\mathbb{R}} \frac{|x|^3}{(1+|\xi x|^2)(1+|x|^2)} \rho(\mathrm{d}x) \\ &= \int_0^\infty \xi|\xi^2-1| \gamma(\mathrm{d}\xi) \left(\int_{|x|\le 1} + \int_{|x|>1} \right) \frac{|x|^3}{(1+|\xi x|^2)(1+|x|^2)} \rho(\mathrm{d}x) \\ &=: I_{41} + I_{42}, \end{aligned}$$

say. Here

$$\begin{aligned} I_{41} &= \int_0^\infty \xi|\xi^2-1| \gamma(\mathrm{d}\xi) \int_{|x|\le 1} \frac{|x|^3}{1+|x|^2} \rho(\mathrm{d}x) \\ &= \left(\int_0^1 + \int_1^\infty \right) \xi|\xi^2-1| \gamma(\mathrm{d}\xi) \int_{|x|\le 1} \frac{|x|^3}{1+|x|^2} \rho(\mathrm{d}x) \\ &=: I_{411} + I_{412}, \end{aligned}$$

say. We have

$$I_{411} \le \int_0^1 \xi\gamma(\mathrm{d}\xi) \int_{|x|\le 1} |x|^2 \rho(\mathrm{d}x) < \infty, \quad \text{(by (13))}$$

and

$$\begin{aligned} I_{412} &\le \int_1^\infty (\xi^2+1)\gamma(\mathrm{d}\xi) \int_{|x|\le 1} \frac{|\xi x||x|^2}{1+|\xi x|^2}\rho(\mathrm{d}x) \\ &\le \int_1^\infty (\xi^2+1)\gamma(\mathrm{d}\xi) \int_{|x|\le 1} |x|^2\rho(\mathrm{d}x) < \infty, \quad \text{(by (13))}. \end{aligned}$$

Also,

$$\begin{aligned} I_{42} &= \int_{|x|>1} \frac{|x|^3}{1+|x|^2}\rho(\mathrm{d}x) \int_0^\infty \frac{\xi|\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi) \\ &= \int_{|x|>1} \frac{|x|^3}{1+|x|^2}\rho(\mathrm{d}x) \left(\int_0^1 + \int_1^\infty\right) \frac{\xi|\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi) \\ &=: I_{421} + I_{422}, \end{aligned}$$

say. Furthermore,

$$\begin{aligned} I_{421} &= \int_{|x|>1} \frac{|x|^3}{1+|x|^2}\rho(\mathrm{d}x) \left(\int_0^{1/|x|} + \int_{1/|x|}^1\right) \frac{\xi|\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi) \\ &=: I_{4211} + I_{4212}, \end{aligned}$$

say. We have

$$I_{4211} \le \int_{|x|>1} |x|\rho(\mathrm{d}x) \int_0^{1/|x|} \xi\gamma(\mathrm{d}\xi) < \infty, \quad \text{(by (15))},$$

and

$$\begin{aligned} I_{4212} &= \int_{|x|>1} \frac{|x|^2}{1+|x|^2}\rho(\mathrm{d}x) \int_{1/|x|}^1 \frac{|\xi x||\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi) \\ &\le \int_{|x|>1} \gamma([1/|x|,1])\rho(\mathrm{d}x) < \infty, \quad \text{(by (16))}. \end{aligned}$$

Also

$$\begin{aligned} I_{422} &= \int_{|x|>1} \frac{|x|^2}{1+|x|^2}\rho(\mathrm{d}x) \int_1^\infty \frac{|\xi x||\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi) \\ &\le \int_{|x|>1} \frac{|x|^2}{1+|x|^2}\rho(\mathrm{d}x) \int_1^\infty (\xi^2+1)\gamma(\mathrm{d}\xi) < \infty \quad \text{(by (13))}. \end{aligned}$$

Thus we have (30). This completes the verification.

Acknowledgement

We are very grateful to Ken-iti Sato for helpful discussions and comments.

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