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Abstract: Let $(T,\mathcal{B},\mu)$ be a measure space and let $f:\bar{\mathbf{R}}\times T\to\bar{\mathbf{R}}$ be a function. Then we say that f is an increasing μ-partition of unity if $f(x,t)$ is increasing in x, measurable in t and $\int_T f(x,t)\,\mu(dt)=x$ for all $x\in\bar{\mathbf{R}}$. Increasing partitions of unity have a variety of applications which will be explored in the paper. For instance, applications include the Fubini-Tonelli theorem for upper and lower integrals and Fubini-integrals, measurability or upper (lower) semicontinuity of integral transforms, and construction of functions with a prescribed integral transform and satisfying a given set of (in)equalities.

1. Introduction

Recall that $(X,\leq)$ is a *proset* if X is a non-empty set equipped with a relation $\leq$ satisfying $x\leq x\ \forall\,x\in X$ and $x\leq y$, $y\leq z\ \Rightarrow\ x\leq z$. Let $(M,\preceq)$ be a proset and let $\Sigma:M\to\bar{\mathbf{R}}$ be an increasing function where $\bar{\mathbf{R}}:=[-\infty,\infty]$ denotes *the extended real line* with its usual ordering. Then we let $m_\Sigma:=\inf_{\xi\in M}\Sigma(\xi)$ and $m^\Sigma:=\sup_{\xi\in M}\Sigma(\xi)$ denote the two extreme values of Σ. If S is a non-empty set and $\phi:S\to M$ is a given function, we let $\Sigma\phi(s):=\Sigma(\phi(s))$ denote the Σ-transform of ϕ for all $s\in S$. We say that $f:\bar{\mathbf{R}}\to M$ is an *increasing Σ-partition of unity* if f is increasing and $\Sigma f(x)=x$ for all $m_\Sigma\leq x\leq m^\Sigma$ or equivalently, if f is increasing and $\Sigma f(x)=m^\Sigma\wedge(x\vee m_\Sigma)$ for all $x\in\bar{\mathbf{R}}$. In Section 2, we shall apply the Hausdorff maximality principle to construct increasing partitions of unity satisfying a prescribed set of (in)equalities. Increasing partitions of unity have a variety of applications and in Section 3 and 4, we shall explore some of these applications.

Let $(T,\mathcal{B},\mu)$ be a measure space. Then we let $\bar{\mathbf{R}}^T$ denote the set of all functions $f:T\to\bar{\mathbf{R}}$, we let $\bar{M}(T,\mathcal{B})$ denote the set of all $f\in\bar{\mathbf{R}}^T$ which are $\mathcal{B}$-measurable, and we let $L^1(T,\mathcal{B},\mu)$ denote the set of all functions $f\in\bar{M}(T,\mathcal{B})$ which are μ-integrable. If $f,h:\in\bar{\mathbf{R}}^T$, we write $f\leq h$ if $f(t)\leq h(t)$ for all $t\in T$ and we write $f\leq_\mu h$ if $f(t)\leq h(t)$ for μ-a.a. $t\in T$. If $f\in\bar{\mathbf{R}}^T$, we let $\int^* f\,d\mu$ and $\int_* f\,d\mu$ denote *the upper and lower μ-integral* of f. If $f:\bar{\mathbf{R}}\times T\to\bar{\mathbf{R}}$ is a given function, we say that f is an *increasing μ-partition of unity* if $f(x,\cdot)$ is $\mathcal{B}$-measurable for all $x\in\bar{\mathbf{R}}$ and $f(\cdot,t)$ is increasing on $\bar{\mathbf{R}}$ for all $t\in T$ and we have $f(x,\cdot)\in L^1(T,\mathcal{B},\mu)$ and $\int_T f(x,t)\,\mu(dt)=x$ for all $x\in\mathbf{R}$. Note

$(\bar{M}(T,\mathcal{B}), \leq_\mu)$ is a proset and we say that $\Sigma : \bar{M}(T,\mathcal{B}) \to \bar{\mathbf{R}}$ is a μ-*integral* if Σ is increasing with respect to the preordering $\leq_\mu$ and satisfies

(1.1) $\Sigma(f) = \int_T f\,d\mu \;\; \forall\, f \in L^1(T,\mathcal{B},\mu)$ and if $f \in \bar{M}(T,\mathcal{B})$ and $|\Sigma(f)| < \infty$, then we have $f \in L^1(T,\mathcal{B},\mu)$

Let $(S,\mathcal{A},\nu)$ be a measure space. If ν and μ are sum-finite (see [3; p.171]), then the product measure $\nu \otimes \mu$ exists and we have (*the Fubini-Tonelli theorem*):

$$\int_* \phi d(\nu \otimes \mu) \leq \int_* \nu(ds) \int_* \phi(s,t)\mu(dt) \leq \int^* \nu(ds) \int^* \phi(s,t)\mu(dt) \leq \int^* \phi d(\nu \otimes \mu)$$

for all $\phi \in \bar{\mathbf{R}}^{S\times T}$. In Section 3, we shall how increasing partitions unity can be used to establish equality when $\phi(s,t)$ is measurable in t and increasing in s with respect some linear ordering on S . Moreover, we shall establish a Fubini-Tonelli inequality for the so-called Fubini integral:

Let S be a given set, let 2^S denote the set of all subsets of S and let $\rho : 2^S \to [0,\infty]$ be an increasing set function satisfying $\rho(\emptyset) = 0$. If $f : S \to [0,\infty]$ is a non-negative function, we let $\int^F f\,d\rho := \int_0^\infty \rho(s \mid f(s) > x)\,dx$ denote *the Fubini integral* of f ; see [5]. Let $\mathcal{A} \subseteq 2^S$ be an algebra on S and let $\nu : \mathcal{A} \to [0,\infty]$ be a finitely additive content. Then we set $\mathcal{A}^\circ := \{A \in \mathcal{A} \mid \nu(A) < \infty\}$ and if $C \subseteq S$, we define $\nu^*(C) := \inf_{A\in\mathcal{A}\ ,\ A\supseteq C} \nu(A)$ and $\nu_*(C) := \sup_{A\in\mathcal{A}\ ,\ A\subseteq C} \nu(A)$ and $\nu_\circ(C) := \sup_{A\in\mathcal{A}^\circ\ ,\ A\subseteq C} \nu(A)$. We let $L^1(\nu)$ denote the set of all ν-integrable functions in the sense of [1; Def.III.2.17 p.112] and we let $\int^* f\,d\nu$ and $\int_* f\,d\nu$ denote the upper and lower ν-integrals for all $f \in \bar{\mathbf{R}}^S$; see [4]. If $h : S \to [0,\infty]$ is a non-negative function, we have (see [5]):

(1.2) $\int^* h\,d\nu = \int^F h\,d\nu^*$ and $\int_* h\,d\nu = \int^F h\,d\nu_\circ$

If $x,y \in \bar{\mathbf{R}}$ are extended real numbers, we let $x \dot{+} y$ denote the usual extension of the addition with the convention $\infty \dot{+} (-\infty) := \infty$ and we let $x \underset{\cdot}{+} y$ denote the usual extension of the addition with the convention $\infty \underset{\cdot}{+} (-\infty) := -\infty$. We define $x \dot{-} y := x \dot{+} (-y)$ and $x \underset{\cdot}{-} y := x \underset{\cdot}{+} (-y)$. If $f : S \to \bar{\mathbf{R}}$ is an arbitrary function, we let $f_+(s) := f(s) \vee 0$ and $f_-(s) := f(s) \wedge 0$ denote *the positive and negative parts* of f for all $s \in S$. Then we have (see [5]):

(1.3) $\int^* f\,d\nu = \int^* f_+\,d\nu \dot{+} \int^* f_-\,d\nu$, $\int_* f\,d\nu = \int_* f_+\,d\nu \underset{\cdot}{+} \int_* f_-\,d\nu$

If $\mathcal{L} \subseteq 2^S$, we let $\bar{W}(S,\mathcal{L})$ denote the set of all *upper $\mathcal{L}$-functions*; that is, the set of all $f : S \to \bar{\mathbf{R}}$ such that for all $-\infty < x < y < \infty$, there exists $L_{xy} \in \mathcal{L} \cup \{\emptyset, S\}$ satisfying $\{f > y\} \subseteq L_{xy} \subseteq \{f > x\}$. If $\mathcal{L}$ is a σ-algebra on S , we have $\bar{W}(S,\mathcal{L}) = \bar{M}(S,\mathcal{L})$. If S is topological space and $\mathcal{L}$ is the set of all open (closed) subsets of S , then $\bar{W}(S,\mathcal{L})$ is the set of all lower (upper) semicontinuous functions $f : S \to \bar{\mathbf{R}}$. Let $(T,\mathcal{B},\mu)$ be a measure space and let $\phi : S \times T \to \bar{\mathbf{R}}$ be a given function such that $\phi(s,t)$ is an upper $\mathcal{L}$-function in s and $\mathcal{B}$-measurable in t . In Section3, we shall see that increasing μ-partitions can used to establish criteria for the integral transform $s \curvearrowright \int_T \phi(s,t)\,\mu(dt)$ to be an upper $\mathcal{L}$-function.

Let $(S, \leq)$ and $(M, \preceq)$ be prosets and let $\Sigma : M \to \bar{\mathbf{R}}$ be an increasing function. In Section 4, we shall see that increasing partitions unity can used to solve the following problem:

(IP) Let $\omega \in M$ be a given element and let $H : S \to \bar{\mathbf{R}}$ and $\phi : S \to M$ be increasing functions. Find necessary and/or sufficient conditions for the existence of a an increasing function $\psi : S \to M$ satisfying $\phi(s) \preceq \psi(s) \preceq \omega$ and $\Sigma\psi(s) = H(s) \ \ \forall \ s \in S$

Let me at this point recall the concepts concerning prosets, needed for our objective:

Let $(X, \leq)$ be a proset and let $x, y \in X$ be given. Then we write $x < y$ if $x \leq y$ and $y \not\leq x$, we write $x \approx y$ if $x \leq y$ and $y \leq x$, and we introduce the following *intervals*:

$$[*, x] = \{\, u \in X \mid u \leq x \,\} \ \ , \ \ [x, *] = \{\, u \in X \mid u \geq x \,\} \ \ , \ \ [x, y] = [x, *] \cap [*, y]$$

Let $A, B \subseteq X$ be a given sets. Then we write $A \leq B$ if $x \leq y$ for all $x \in A$ and all $y \in B$, and we introduce the following *intervals*:

$$[*, A] = \{\, u \in X \mid u \leq A \} \ \ , \ \ [A, *] = \{\, u \in X \mid u \geq A \} \ \ , \ \ [A, B] = [A, *] \cap [*, B]$$

We say that A is a *lower interval*, resp. an *upper interval*, if $[*, u] \subseteq A$, resp. $[u, *] \subseteq A$, for all $u \in A$. We let $\vee A$ denote the set of all *suprema* of A ; that is the set of all $x \in A$ satisfying $A \leq x$ and $x \leq y$ for all $y \in X$ satisfying $A \leq y$, and we define the set $\wedge A$ of all *infima* of A similarly. We say that A is *cofinal* in $(B, \leq)$ if $A \subseteq B \subseteq \cup_{u \in A} [*, u]$ and we say that $(A, \leq)$ is *countably cofinal* if $(A, \leq)$ admits a countable, cofinal subset. We say that $(X, \leq)$ is a *lattice* if $x \vee y \neq \emptyset$ and $x \wedge y \neq \emptyset$ for all $x, y \in X$, and we say that $(X, \leq)$ is a *σ-lattice* if $\vee A \neq \emptyset$ and $\wedge A \neq \emptyset$ for every non-empty countable set $A \subseteq X$. We say that A is *linear* if for all $x, y \in A$ we have either $x \leq y$ or $y \leq x$, and we say that A is a *maximal linearly ordered set* if A is linear and $A = B$ for every linear set $B \supseteq A$. By Hausdorff's maximality principle (see [6; p.248]), we have that every linear set $A \subseteq X$ is contained in some maximal linearly ordered set, and observe that we have

(1.4) If $A \subseteq X$ is a maximal linearly ordered set, then we have $\vee B \subseteq A$ and $\wedge B \subseteq A$ for all $B \subseteq A$

Let $x, x_1, x_2, \ldots \in X$ be given elements. Then we write $x_n \uparrow x$, if $x_1 \leq x_2 \leq \cdots$ and $x \in \vee\{x_n \mid n \geq 1\}$, and we write $x_n \downarrow x$, if $x_1 \geq x_2 \geq \cdots$ and $x \in \wedge\{x_n \mid n \geq 1\}$.

2. Smoothness and the Darboux property

Let $(M, \preceq)$ be a proset and let $\Sigma : M \to \bar{\mathbf{R}}$ be an increasing function. Then we let $m_\Sigma := \inf_{\xi \in M} \Sigma(\xi)$ and

$m^\Sigma := \sup_{\xi \in M} \Sigma(\xi)$ denote the two extreme values of Σ and we define

$$L^1(\Sigma) = \{\, \xi \in M \mid -\infty < \Sigma(\xi) < \infty \}$$

$$L_*(\Sigma) = \{\, \xi \in M \mid \Sigma(\xi) = -\infty \} \;\; , \;\; L^*(\Sigma) = \{\, \xi \in M \mid \Sigma(\xi) = \infty \}$$

$$\Sigma_\vee B = \inf_{\xi \in [B,*]} \Sigma(\xi) \;\; , \;\; \Sigma_\wedge B = \sup_{\xi \in [*,B]} \Sigma(\xi) \;\; \forall\, B \subseteq M$$

$$\sup \Sigma B = \sup_{\xi \in B} \Sigma(\xi) \;\; , \;\; \inf \Sigma B = \inf_{\xi \in B} \Sigma(\xi) \;\; \forall\, B \subseteq M$$

with the usual conventions $\inf \emptyset := \infty$ and $\sup \emptyset := -\infty$. Then we have

(2.1) $\quad \sup \Sigma B \leq \Sigma_\vee B$ and $\Sigma_\vee B = \Sigma(\xi) \;\; \forall\, \xi \in \vee B$

(2.2) $\quad \Sigma_\wedge B \leq \inf \Sigma B$ and $\Sigma_\wedge B = \Sigma(\xi) \;\; \forall\, \xi \in \wedge B$

We say that Σ is *smooth* if for every non-empty linear set $B \subseteq M$, we have

(2.3) $\quad -\infty < \sup \Sigma B < \infty \;\; \Rightarrow \;\; \exists\, \xi \in \vee B$ so that $\Sigma(\xi) = \sup \Sigma B$

(2.4) $\quad -\infty < \inf \Sigma B < \infty \;\; \Rightarrow \;\; \exists\, \xi \in \wedge B$ so that $\Sigma(\xi) = \inf \Sigma B$

We say that Σ has *the Darboux property* if for every pair $\xi, \eta \in M$, we have

(2.5) $\quad \xi \preceq \eta \;\; , \;\; \Sigma(\xi) < \Sigma(\eta) < \infty \;\; \Rightarrow \;\; \exists\, \kappa \in [\xi, \eta]$ so that $\Sigma(\xi) < \Sigma(\kappa) < \Sigma(\eta)$

(2.6) $\quad \xi \preceq \eta \;\; , \;\; -\infty < \Sigma(\xi) < \Sigma(\eta) \;\; \Rightarrow \;\; \exists\, \kappa \in [\xi, \eta]$ so that $\Sigma(\xi) < \Sigma(\kappa) < \Sigma(\eta)$

We say that Σ has *the strong Darboux property* if Σ has the Darboux property and satisfies the following condition:

(2.7) $\quad$ If $(\xi_n) \subseteq L_*(\Sigma)$ and $\xi_n \uparrow \xi$ for some $\xi \in L^1(\Sigma)$, then for every increasing sequence $(c_n) \subseteq \mathbf{R}$ satisfying $c_n \uparrow \Sigma(\xi)$ and $c_n < \Sigma(\xi)$ for all $n \geq 1$, there exists an increasing sequence $(\eta_n) \subseteq M$ such that $\xi_n \preceq \eta_n \preceq \xi$ and $-\infty < \Sigma(\eta_n) \leq c_n$ for all $n \geq 1$

We say that Σ is *order injective,* if $\xi \approx \eta$ for all $\xi, \eta \in L^1(\Sigma)$ satisfying $\xi \preceq \eta$ and $\Sigma(\xi) = \Sigma(\eta)$. If S is a non-empty set and $h : S \to \bar{\mathbf{R}}$ is a function, we let $D_h := \{s \in S \mid |h(s)| < \infty\}$ denote *the finite domain* of h and we let $D_h^\circ := \{s \in S \mid h(s) = -\infty\}$ and $D_h^* := \{s \in S \mid h(s) = \infty\}$ denote *the infinite domains* of h .

Lemma 2.1: *Let* $(M, \preceq)$ *be a* σ*-lattice, let* $\Sigma : M \to \bar{\mathbf{R}}$ *be an increasing function and let* $B \subseteq M$ *be a given set. Then we have*

(1) $\quad \forall\, \xi \in [B, *] \;\exists\, \psi \in [B, \xi]$ *so that* $\Sigma(\psi) = \Sigma_\vee B$

(2) $\quad \forall\, \xi \in [*, B] \;\exists\, \psi \in [\xi, B]$ *so that* $\Sigma(\psi) = \Sigma_\wedge B$

Proof: Let $\xi \in [B,*]$ be given. Since $[B,*]$ is non-empty, there exist $\psi_1, \psi_2, \ldots \in [B,*]$ such that $\Sigma(\psi_n) \to \Sigma_\vee B$ and since $(M, \preceq)$ is a σ-lattice, there exists an element $\psi \in \xi \wedge \wedge_{n\geq 1} \psi_n$. Since $B \preceq \xi$ and $B \preceq \psi_n$ for all $n \geq 1$, we have $\psi \in [B,\xi]$ and so we have $\Sigma_\vee B \leq \Sigma(\psi)$. Since $\psi \preceq \psi_n$, we have $\Sigma_\vee B \leq \Sigma(\psi) \leq \Sigma(\psi_n)$ for all $n \geq 1$ and since $\Sigma(\psi_n) \to \Sigma_\vee B$, we see that $\Sigma(\psi) = \Sigma_\vee B$ which proves (1) and (2) follows in the same manner. □

Lemma 2.2: *Let $(M, \preceq)$ be a σ-lattice and let $\Sigma : M \to \bar{\mathbf{R}}$ be an increasing smooth function. Let $B \subseteq M$ be a non-empty linear set and let us define $B^1 := B \cap L^1(\Sigma)$, $B_* := B \cap L_*(\Sigma)$ and $B^* := B \cap L^*(\Sigma)$. Then $B_* \preceq B^1 \preceq B^*$ and we have*

(1) $\Sigma_\vee B = \sup \Sigma B \quad \Leftrightarrow$ *either* $\sup \Sigma B > -\infty$ *or* $\Sigma_\vee B = -\infty$

(2) $\Sigma_\wedge B = \inf \Sigma B \quad \Leftrightarrow$ *either* $\inf \Sigma B < \infty$ *or* $\Sigma_\wedge B = \infty$

and if $B^1 \neq \emptyset$ *, then we have*

(3) $\vee B^1 = \vee(B^1 \cup B_*) \neq \emptyset$ *and* $\Sigma(\xi) = \sup \Sigma B^1 = \sup \Sigma(B^1 \cup B_*)$ $\forall \xi \in \vee B^1$

(4) $\wedge B^1 = \wedge(B^1 \cup B^*) \neq \emptyset$ *and* $\Sigma(\xi) = \inf \Sigma B^1 = \inf \Sigma(B^1 \cup B^*)$ $\forall \xi \in \wedge B^1$

Proof: (1+2): Since Σ is increasing and B is a linear set satisfying $\Sigma(\xi) = -\infty < \Sigma(\kappa) < \infty = \Sigma(\eta)$ for all $\xi \in B_*$, all $\kappa \in B^1$ and all $\eta \in B^*$, we have $B_* \preceq B^1 \preceq B^*$. By (2.1), we have $\sup \Sigma B \leq \Sigma_\vee B$. Hence, if $\sup \Sigma B = \infty$ or $\Sigma_\vee B = -\infty$, we have $\sup \Sigma B = \Sigma_\vee B$. Suppose that $-\infty < \sup \Sigma B < \infty$. By smoothness of Σ and linearity of B , there exists $\xi \in \vee B$ satisfying $\Sigma(\xi) = \sup \Sigma B$ and so by (2.1) we have $\sup \Sigma B = \Sigma_\vee B$. Hence, we see that the implication "$\Leftarrow$" in (1) holds and the converse implication is evident. Thus, (1) is proved and (2) follows in the same manner.

(3+4): Suppose that $B^1 \neq \emptyset$. Since $B_* \preceq B^1$, we have $\vee B^1 = \vee(B^1 \cup B_*)$ and $\sup \Sigma B^1 = \sup \Sigma(B^1 \cup B_*) > -\infty$. Suppose that $\sup \Sigma B^1 < \infty$. By smoothness of Σ , there exists $\eta \in \vee B^1$ such that $\Sigma(\eta) = \sup \Sigma B^1$. Hence, we see that (3) follows from (2.1). So suppose that $\sup \Sigma B^1 = \infty$. Then there exists a sequence $(\eta_n) \subseteq B^1$ such that $\Sigma(\eta_n) \to \infty$ and since M is a σ-lattice, there exists an element $\eta \in \vee_{n=1}^\infty \eta_n$. Let $\xi \in B^1$ be given. Since $\Sigma(\xi) < \infty$, there exists an integer $k \geq 1$ such that $\Sigma(\xi) < \Sigma(\eta_k)$. Since Σ is increasing and B is a linear set containing ξ and η_k , we have $\xi \preceq \eta_k \preceq \eta$ and since $(\eta_n) \subseteq B^1$, we have $\eta \in \vee B^1$. Hence, we see that (3) follows from (2.1) and (4) follows in the same manner. □

Theorem 2.3: *Let $(M, \preceq)$ be a lattice and let $\Sigma : M \to \bar{\mathbf{R}}$ be an increasing smooth function with the Darboux property. Let $B \subseteq M$ be a linear set such that $B^1 := B \cap L^1(\Sigma) \neq \emptyset$ and let us define $B_* := B \cap L_*(\Sigma)$ and $B^* := B \cap L^*(\Sigma)$. Then there exists a maximal linearly ordered set $L \subseteq M$ satisfying*

(1) $B \subseteq L \quad , \quad \Sigma_\vee B_* = \inf \Sigma L^1 = \Sigma_\vee L_* \quad , \quad \Sigma_\wedge B^* = \sup \Sigma L^1 = \Sigma_\wedge L^*$

where $L^1 := L \cap L^1(\Sigma)$, $L_* := L \cap L_*(\Sigma)$ *and* $L^* := L \cap L^*(\Sigma)$.

Proof: Let us define $M_0 := [B_*, B^1] \cap L^1(\Sigma)$ and $r := \Sigma_\vee B_*$. Then I claim that there exist a linear set $Q \subseteq M_0$ satisfying $\inf \Sigma(Q \cup B^1) = r$.

Since $B_* \preceq B^1 \neq \emptyset$, we have $r \leq \inf \Sigma(B^1)$ and $r < \infty$. Hence, if $\inf \Sigma(B^1) \leq r$, we see that $Q := \emptyset$ satisfies the claim. So suppose that $r < \inf \Sigma(B^1)$. Then we have $-\infty < \inf \Sigma(B^1) < \infty$ and so by smoothness of Σ , there exists $\pi \in \wedge B^1$ satisfying $\Sigma(\pi) = \inf \Sigma(B^1)$. In particular, we have $\pi \in L^1(\Sigma)$ and since $B_* \preceq B^1$ and $\pi \in \wedge B^1$, we have $B_* \preceq \pi \preceq B^1$. Hence, we see that $\pi \in M_0$ and that $(M_0, \preceq)$ is a non-empty proset. So by Hausdorff maximality principle there exists a maximal linearly ordered set $Q \subseteq M_0$ in the proset $(M_0, \preceq)$. Let us define $\alpha := \inf \Sigma(Q \cup B^1)$. Since Q and B^1 are linear and $Q \preceq B^1$, we see that $Q \cup B^1$ is linear and since $B^1 \neq \emptyset$ and $B_* \preceq Q \cup B^1$, we have $r \leq \alpha < \infty$. Suppose that $r < \alpha$. Then we have $-\infty < \alpha < \infty$ and so by linearity of $Q \cup B^1$ and smoothness of Σ , there exists $\eta \in \wedge(Q \cup B^1)$ such that $\Sigma(\eta) = \alpha$. In particular, we have $\eta \in L^1(\Sigma)$ and $B_* \preceq \eta \preceq Q \cup B^1$ and since $r < \alpha = \Sigma(\eta)$, there exists $\xi_0 \in [B_*, *]$ satisfying $\Sigma(\xi_0) < \alpha$. Since M is a lattice, there exists $\xi \in \xi_0 \wedge \eta$ and since $B_* \preceq \xi_0$ and $B_* \preceq \eta$, we have $\xi \in [B^*, \eta]$ and $r \leq \Sigma(\xi) \leq \Sigma(\xi_0) < \Sigma(\eta) < \infty$. Since Σ has the Darboux property there exists $\kappa \in [\xi, \eta]$ satisfying $\Sigma(\xi) < \Sigma(\kappa) < \Sigma(\eta)$. Since $B_* \preceq \xi \preceq \kappa \preceq \eta \preceq Q \cup B^1$, we see that $\kappa \in M_0$ and that $Q_0 := Q \cup \{\kappa\}$ is a linear subset of M_0 . Since $\Sigma(\kappa) < \Sigma(\eta) = \inf \Sigma(Q \cup B^1)$, we have $\kappa \notin Q$ and $Q \subsetneqq Q_0$. However, this contradicts the maximality of Q in M_0 and so we must have $\alpha \leq r$ and since $\alpha \geq r$, we see that Q satisfies the claim.

Hence, we see that there exists a linear set $Q \subseteq [B_*, B^1] \cap L^1(\Sigma)$ satisfying $\inf \Sigma(Q \cup B^1) = \Sigma_\vee B_*$. In the same manner, we see that there exists a linear set $R \subseteq [B^1, B^*] \cap L^1(\Sigma)$ satisfying $\sup \Sigma(R \cup B^1) = \Sigma_\wedge B^*$. Since B , Q and R are linear and

$$B_* \preceq Q \preceq B^1 \preceq R \preceq B^* \text{ and } B = B_* \cup B^1 \cup B^*$$

we see that $C := B \cup Q \cup R$ is a linear set containing B . So by Hausdorff's maximality principle there exists a maximal linearly ordered set L containing C . Let us define $L^1 := L \cap L^1(\Sigma)$, $L_* := L \cap L_*(\Sigma)$ and $L^* := L \cap L^*(\Sigma)$. Since L is linear, we have $L_* \preceq L^1 \preceq L^*$ and since $B \subseteq L$ and $Q \cup B^1 \subseteq L^1$, we have

$$\Sigma_\vee B_* \leq \Sigma_\vee L_* \leq \inf \Sigma L^1 \leq \inf \Sigma(Q \cup B^1) = \Sigma_\vee B_*$$

Hence, we see that $\Sigma_\vee B_* = \Sigma_\vee L_* = \inf \Sigma L^1$ and in the same manner, we see that $\Sigma_\wedge B^* = \Sigma_\wedge L^* = \sup \Sigma L^1$ which proves the theorem. □

Theorem 2.4: *Let* $(M, \preceq)$ *be a* σ*-lattice and let* $\Sigma : M \to \bar{\mathbf{R}}$ *be an increasing smooth function with the Darboux property. Let* $D \subseteq \bar{\mathbf{R}}$ *be a non-empty set and let* $h : D \to M$ *be a increasing function satisfying* $\Sigma h(x) = m^\Sigma \wedge (x \vee m_\Sigma)$ *for all* $x \in D$ *and* $h(D) \cap L^1(\Sigma) \neq \emptyset$. *Then there exists an increasing* Σ*-partition of unity* $f : \bar{\mathbf{R}} \to M$ *satisfying* $f(x) = h(x)$ *for all* $x \in D$.

Proof: Let us define $\lambda(x) := m^\Sigma \wedge (x \vee m_\Sigma)$ for all $x \in \bar{\mathbf{R}}$. Since h is increasing, we see that $h(D)$ is a linear, countably cofinal set and since M is a σ-lattice, we have that $\vee h(D)$ is non-empty. So by Lem.2.1 with $B := \emptyset$ there exists $\beta \in M$ such that $h(D) \preceq \beta$ and $\Sigma(\beta) = m^\Sigma$ and we may (and shall) take $\beta = h(\infty)$ if $\infty \in D$. Let $x \in \bar{\mathbf{R}}$ be given and let us define $D^x := D \cap [x, \infty]$ and $\Delta^x := h(D^x) \cup \{\beta\}$. Then Δ^x is countably cofinal and since h is increasing, we have $h(x) \in \wedge\Delta^x$ for all $x \in D$. Since M is a σ-lattice, there exists a function $h_0 : \bar{\mathbf{R}} \to M$ such that $h_0(x) \in \wedge\Delta^x$ for all $x \in \bar{\mathbf{R}}$ and $h_0(x) = h(x)$ for all $x \in D$. Since $x \curvearrowright \Delta^x$ is decreasing, we see that h_0 is increasing on $\bar{\mathbf{R}}$. Since $\Sigma h(y) < \infty$ for all $y \in D \cap [-\infty, \infty)$, we see that $\inf \Sigma\Delta^x = \infty$ implies $\Delta^x = \{\beta\}$ and so by (2.2) and Lem.2.2.(2), we have $\Sigma h_0(x) = \inf \Sigma\Delta^x$ for all $x \in \bar{\mathbf{R}}$. Since $\Sigma(\beta) = m^\Sigma = \lambda(x)$ for all $x \geq m^\Sigma$ and $\Sigma h(y) = \lambda(y)$ for all $y \in D$, we see that $\Sigma h_0(x) = \lambda(x)$ for all $x \in D \cup [m^\Sigma, \infty]$.

In the same manner, we see that there exists an increasing function $h_1 : \bar{\mathbf{R}} \to M$ satisfying $h_1(x) = h_0(x)$ for all $x \in D \cup [m^\Sigma, \infty]$ and $\Sigma h_1(x) = \lambda(x)$ for all $x \in D_1 := [-\infty, m_\Sigma] \cup D \cup [m^\Sigma, \infty]$. Hence, if $D_1 = \bar{\mathbf{R}}$, then h_1 is an increasing Σ-partition of unity satisfying $h_1(x) = h_0(x) = h(x)$ for all $x \in D$.

So suppose that $D_1 \neq \bar{\mathbf{R}}$. Then $m_\Sigma < m^\Sigma$ and $B := h_1(D_1)$ is a linear set containing $h(D)$. Since $\Sigma h_1(x) = \lambda(x) \neq \pm\infty$ for all $x \in D_1 \cap \mathbf{R}$, we see that the sets $B_* := B \cap L_*(\Sigma)$ and $B^* := B \cap L^*(\Sigma)$ contain at most one element and so we have $\Sigma_\vee B_* = m_\Sigma$ and $\Sigma_\wedge B^* = m^\Sigma$. Since $\emptyset \neq h(D) \cap L^1(\Sigma)$, we have $B^1 := B \cap L^1(\Sigma) \neq \emptyset$ and so by Thm.2.3 there exists a maximal linear set L satisfying

$$L \supseteq B \ , \ \inf \Sigma L^1 = \Sigma_\vee L_* = m_\Sigma \leq m^\Sigma = \Sigma_\wedge L^* = \sup \Sigma L^1$$

where $L^1 := L \cap L^1(\Sigma)$, $L_* := L \cap L_*(\Sigma)$ and $L^* := L \cap L^*(\Sigma)$.

Let $m_\Sigma < x < m^\Sigma$ be a given and let us define $A^x := \{\xi \in L \mid \Sigma(\xi) > x\}$ and $A_x := \{\xi \in L \mid \Sigma(\xi) \leq x\}$. Since $x < m^\Sigma = \sup \Sigma L^1$, we have $A^x \cap L^1(\Sigma) = A^x \cap L^1 \neq \emptyset$ and $A^x \cap L_*(\Sigma) = \emptyset$. So by Lem.2.2 there exists $f(x) \in \wedge A^x = \wedge(A^x \cap L^1)$ such that $x \leq \inf \Sigma A^x = \Sigma f(x) < \infty$. Since $m_\Sigma = \inf \Sigma L^1 < x$, we have $A_x \cap L^1(\Sigma) = A_x \cap L^1 \neq \emptyset$ and $A_x \cap L^*(\Sigma) = \emptyset$. So by Lem.2.2 there exists $g(x) \in \vee A_x = \vee(A_x \cap L^1)$ such that $-\infty < \Sigma g(x) = \sup \Sigma A_x \leq x$. Since $L = A^x \cup A_x$ is linear and $\Sigma(\xi) \leq x < \Sigma(\eta)$ for all $\xi \in A_x$ and all $\eta \in A^x$, we have $A_x \preceq A^x$ and so we have $g(x) \preceq f(x)$ and $-\infty < \Sigma g(x) \leq x \leq \Sigma f(x) < \infty$. Suppose that $\Sigma g(x) < \Sigma f(x)$. Since $g(x) \in L^1(\Sigma)$ and Σ has the Darboux property, there exists $\kappa \in [g(x), f(x)]$ such that $\sup \Sigma A_x = \Sigma g(x) < \Sigma(\kappa) < \Sigma f(x) = \inf \Sigma A^x$. Since $L = A_x \cup A^x$, we see that $\kappa \notin L$ and since L is linear and $A_x \preceq g(x) \preceq \kappa \preceq f(x) \preceq A^x$, we see that $L \cup \{\kappa\}$ is linear. However, this contradicts the maximality of L and so we must have $\Sigma g(x) \geq \Sigma f(x)$. Since $\Sigma g(x) \leq x \leq \Sigma f(x)$, we have $\Sigma g(x) = x = \Sigma f(x)$ for all $x \in (m_\Sigma, m^\Sigma)$ and by (1.4) and maximality of L , we have $f(x) \in L$ and $g(x) \in L$ for all $x \in (m_\Sigma, m^\Sigma)$.

Since $\mathbf{R} \setminus D_1 \subseteq (m_\Sigma, m^\Sigma)$, we may define $F(x) := f(x)$ if $x \in \bar{\mathbf{R}} \setminus D_1$ and $F(x) := h_1(x)$ if $x \in D_1$. Since $\Sigma h_1(x) = \lambda(x)$ for all $x \in D_1$ and

$\Sigma f(x) = x = \lambda(x)$ for all $x \in (m_\Sigma, m^\Sigma)$, we have $\Sigma F(x) = \lambda(x)$ for all $x \in \bar{\mathbf{R}}$ and since $h_1(D) \subseteq L$ and $f((m_\Sigma, m^\Sigma)) \subseteq L$, we see that $F(x) \in L$ for all $x \in L$. Let $x < y$ be given. Suppose that $\lambda(x) < \lambda(y)$. Then we have $\Sigma F(y) < \Sigma F(x)$ and since Σ is increasing and L is a linear set containing $F(x)$ and $F(y)$, we have $F(y) \preceq F(y)$. Suppose that $\lambda(x) = \lambda(y)$. Since $x < y$, we have either $x < y \leq m_\Sigma$ or $m^\Sigma \leq x < y$ and since h_1 is increasing, we have $F(x) = h_1(x) \preceq h_1(y) = F(y)$ in either case. Hence, we see that F is an increasing Σ-partition of unity satisfying $F(x) = h_1(x) = h(x)$ for all $x \in \bar{\mathbf{R}}$. □

Theorem 2.5: *Let $(M, \preceq)$ be a σ-lattice and let $\Sigma : M \to \bar{\mathbf{R}}$ be an increasing smooth functional with the Darboux property. Let $\omega \in M$ and $\kappa \in L^1(\Sigma)$ be given elements and let $A \subseteq [*, \omega]$ be a linear set such that $A_* \cup \{\kappa, \omega\}$ is linear where $A_* := A \cap L_*(\Sigma)$. Let $F \subseteq A_*$ be a given set and let us define $q := \Sigma_\vee F$ and $r := \Sigma_\vee A_*$. Then we have*

(1) $q \leq r \leq \Sigma(\kappa) < \infty$ *and* $q \leq r \leq \Sigma(\xi) \leq \Sigma(\omega)$ $\forall \xi \in A \setminus A_*$

(2) $q = r$ *if* F *is cofinal in* A_* *, and* $q = -\infty$ *if* F *is not cofinal in* A_*

and there exists an increasing Σ-partition of unity $f : \bar{\mathbf{R}} \to M$ satisfying

(3) $f(\Sigma(\omega)) = \omega$ *and* $\xi \preceq f(\Sigma(\xi))$ $\forall \xi \in A \setminus A_*$

(4) $\xi \preceq f(q)$ $\forall \xi \in F$ *and* $\xi \preceq f(r)$ $\forall \xi \in A_*$

Proof: (1): Since $F \subseteq A_*$, we have $q \leq r$ and since $A_* \cup \{\kappa, \omega\}$ is linear and $\Sigma(\xi) = -\infty < \Sigma(\kappa)$ for all $\xi \in A_*$, we have $A_* \preceq \kappa$. Hence, we have $q \leq r \leq \Sigma(\kappa) < \infty$ and by Lem.2.2, we have $A_* \preceq A \setminus A_*$. Hence, we have $r \leq \Sigma(\xi)$ for all $\xi \in A \setminus A_*$ and since $A \preceq \omega$, we have $q \leq r \leq \Sigma(\omega)$ which completes the proof of (1).

(2): If F is cofinal in A_* , we have $[F, *] = [A_*, *]$ and so we have $r = q$. Suppose that F is not cofinal in A_* . Then there exists $\eta \in A_*$ such that $\eta \not\preceq \xi$ for all $\xi \in F$ and since A_* is linear and contains F , we have $\xi \preceq \eta$ for all $\xi \in F$. Hence, we have $q \leq \Sigma(\eta)$ and since $\eta \in A_*$, we have $q = \Sigma(\eta) = -\infty$.

Suppose that $\Sigma(\omega) = -\infty$. By Thm.2.4 there exists an increasing Σ-partition of unity $f : \bar{\mathbf{R}} \to M$ such that $f(\Sigma(\omega)) = \omega$ and $f(\Sigma(\kappa)) = \kappa$ and since $A = A_*$ and $q = r = -\infty$, we see that f satisfies (3+4). So suppose that $\Sigma(\omega) > -\infty$. Set $A^1 := (A \cup \{\omega\}) \cap L^1(\Sigma)$ and let us define $C := A^1 \cup \{\omega\}$ if $A^1 \neq \emptyset$ and $C := \{\kappa, \omega\}$ if $A^1 = \emptyset$. Since $\{\kappa, \omega\}$ is linear and $\Sigma(\kappa) < \infty$, we see that $\kappa \preceq \omega$ if $\Sigma(\omega) = \infty$. Hence, we see that C is a linear set satisfying $C \cap L^1(\Sigma) \neq \emptyset$ and $A_* \preceq C \preceq \omega$. So by Lem.2.1 and Lem.2.2 there exists $\upsilon \in M$ satisfying $A_* \preceq \upsilon \preceq C$ and $\Sigma(\upsilon) = r$. Since $F \subseteq A_*$, we have $F \preceq \upsilon$ and so by Lem.2.1 there exists $\rho \in M$ such that $F \preceq \rho \preceq \upsilon$ and $\Sigma(\rho) = q$ and if $q = r$, we may (and shall) take $\rho = \upsilon$. Since C is linear and $\rho \preceq \upsilon \preceq C$, we see that $B := C \cup \{\rho, \upsilon\}$ is a linear set containing $C \cup \{\rho, \upsilon, \omega\}$ and so we have $B \cap L^1(\Sigma) \neq \emptyset$. Set $D := \Sigma(B)$, $b := \Sigma(\omega)$ and $B_x := \{\xi \in B \mid \Sigma(\xi) = x\}$ for all $x \in \bar{\mathbf{R}}$. Then we have $\emptyset \neq B_x \subseteq L^1(\Sigma)$ and $\sup \Sigma B_x$ for all $x \in D \cap \mathbf{R}$ and since $\Sigma(\omega) > -\infty$ and $\rho \preceq B \preceq \omega$, we have $\omega \in \vee B_b$ and $B_{-\infty} \subseteq \{\rho\}$.

So by Lem.2.2 there exists a function $h : D \to M$ such that $h(x) \in \vee B_x$ and $\Sigma h(x) = x$ for all $x \in D$ and $h(b) = \omega$. Since B is linear and Σ is increasing, we have $B_x \preceq B_y$ for all $x < y$ and so we see that h is increasing on D . Since $B^1 \neq \emptyset$, we have $h(D) \cap L^1(\Sigma) \neq \emptyset$ and so by Them.2.4 there exists an increasing Σ-partition of unity $f : \bar{\mathbf{R}} \to M$ such that $f(x) = h(x)$ for all $x \in D$. In particular, we have $f(b) = h(b) = \omega$. Let $\xi \in A \setminus A_*$ be given and set $x = \Sigma(\xi)$. If $x = b$, we have $\xi \preceq \omega = f(x)$. Suppose that $x < b$. Since $\xi \notin A_*$, we have $\xi \in A^1 \subseteq B$ and so we have $x \in D$ and $\xi \in B_x$. Since $f(x) = h(x) \in \vee B_x$, we have $\xi \preceq f(x)$. Thus, we see that f satisfies (3). Since $q, r \in D$ and $\rho \in B_q$ and $\upsilon \in B_r$, we have $\rho \preceq h(q) = f(q)$ and $\upsilon \preceq h(r) = f(r)$ and since $A_* \preceq \upsilon$ and $F \preceq \rho$, we see that f satisfies (4). □

Lemma 2.6: *Let $(M, \preceq)$ be a proset and let $\Sigma : M \to \bar{\mathbf{R}}$ be an increasing, order injective function. Then Σ is smooth if and only if*

(1) *If $(\xi_n) \subseteq L^1(\Sigma)$ is an increasing sequence satisfying $\sup_{n\geq 1} \Sigma(\xi_n) < \infty$, then there exists $\xi \in M$ such that $\xi_n \uparrow \xi$ and $\Sigma(\xi) = \sup_{n\geq 1} \Sigma(\xi_n)$*

(2) *If $(\xi_n) \subseteq L^1(\Sigma)$ is a decreasing sequence satisfying $\inf_{n\geq 1} \Sigma(\xi_n) > -\infty$, then there exists $\xi \in M$ such that $\xi_n \downarrow \xi$ and $\Sigma(\xi) = \inf_{n\geq 1} \Sigma(\xi_n)$*

Proof: The “only if” part is evident. So suppose that Σ satisfies (1+2) and let $B \subseteq M$ be a non-empty linear set satisfying $|\sup \Sigma B| < \infty$. Then there exists am increasing sequence $(\xi_n) \subseteq B$ such that $\Sigma(\xi_n) \uparrow \sup \Sigma B$ and $-\infty < \Sigma(\xi_n) \leq \sup \Sigma B < \infty$ for all $n \geq 1$. In particular, we see that $\xi_n \in L^1(\Sigma)$ and that $\sup_{n\geq 1} \Sigma(\xi_n) = \sup \Sigma B < \infty$. So by (1) there exists $\xi \in M$ such that $\xi_n \uparrow \xi$ and $\Sigma(\xi) = \sup \Sigma B$. Since $|\sup \Sigma B| < \infty$, we have $\xi \in L^1(\Sigma)$. Let $\eta \in B$ be given and let me show that $\eta \preceq \xi$. If $\eta \preceq \xi_n$ for some $n \geq 1$, this is evident. So suppose that $\eta \not\preceq \xi_n$ for all $n \geq 1$. Since B is linear and contains η and ξ_n , we have $\xi_n \preceq \eta$ for all $n \geq 1$ and since $\xi \in \vee_{n\geq 1} \xi_n$, we have $\xi \preceq \eta$. Hence, we have $\Sigma(\xi) \leq \Sigma(\eta) \leq \sup \Sigma B = \Sigma(\xi)$ and so we have $\Sigma(\xi) = \Sigma(\eta) = \sup \Sigma B \neq \pm\infty$. Hence, by order injectivity of Σ , we have $\eta \preceq \xi$ for all $\eta \in B$ and since $(\xi_n) \subseteq B$ and $\xi \in \vee_{n\geq 1} \xi_n$, we have $\xi \in \vee B$ and $\Sigma(\xi) = \sup \Sigma B$. Thus, we see that Σ satisfies (2.3) and in the same manner, we see that Σ satisfies (2.4). □

Theorem 2.7: *Let $(T, \mathcal{B}, \mu)$ be a measure space and let $\Sigma : \bar{M}(T, \mathcal{B}) \to \bar{\mathbf{R}}$ be a μ-integral. Then $(\bar{M}(T, \mathcal{B}), \leq_\mu)$ is a σ-lattice and Σ is an increasing, smooth, order injective function satisfying*

(1) $L^1(\Sigma) = L^1(T, \mathcal{B}, \mu)$, $\Sigma(f) = \int_T f \, d\mu \ \ \forall f \in \bar{L}(T, \mathcal{B}, \mu)$

(2) $\int_* f\,d\mu \le \Sigma(f) \le \int^* f\,d\mu \quad \forall\, f \in \bar{M}(T,\mathcal{B})$

(3) $\Sigma(f_+) \dot{+} \Sigma(f_-) \le \Sigma(f) \le \Sigma(f_+) \dot{+} \Sigma(f_-) \quad \forall\, f \in \bar{M}(T,\mathcal{B})$

(4) *If* $c \in \bar{\mathbf{R}}$ *and* $f : T \to \bar{\mathbf{R}}$ *and* $h \in \bar{M}(T,\mathcal{B},\mu)$ *are given functions satisfying* $\int^* f\,d\mu \le c \le \int_* h\,d\mu$ *and* $f(t) \le h(t)$ *for all* $t \in T$ *, then we have*

(a) $\exists\, g \in \bar{L}(T,\mathcal{B},\mu)$ *so that* $\int_T g\,d\mu = c$ *and* $f(t) \le g(t) \le h(t) \;\; \forall\, t \in T$

(5) Σ *has the Darboux property if and only if* μ *is finitely founded and if so then* Σ *has the increasing Darboux property*

Remark: Recall that μ is *finitely founded* if μ has no infinite atoms or equivalently, if $\mu_\circ(B) = \mu(B)$ for all $B \in \mathcal{B}$. Suppose that μ is finitely founded and let $f \in \bar{M}(T,\mathcal{B})$ be a given function. By (1.2) and (1.3), we see that f_+ and f_- belong to $\bar{L}(T,\mathcal{B},\mu)$ and that $f \in \bar{L}(T,\mathcal{B},\mu)$ if and only if either $\int^* f\,d\mu < \infty$ or $\int_* f\,d\mu > -\infty$. In particular, we see that the functionals $f \curvearrowright \int^* f\,d\mu$ and $f \curvearrowright \int_* f\,d\mu$ are μ-integrals whenever μ is finitely founded.

Proof: (1) and (2) are easy consequences of (1.1). In particular, we see that Σ is order injective. So by Lem.2.6 and the monotone convergence theorem we see that Σ is an increasing, smooth and order injective functional. Let $f \in \bar{M}(T,\mathcal{B},\mu)$ be given. If $\Sigma(f_-) = -\infty$ or $\Sigma(f) = \infty$, then the first inequality in (3) holds trivially. So suppose that $\Sigma(f_-) > -\infty$ and $\Sigma(f) < \infty$. Since $f_- \le f$ and Σ is increasing, we have $-\infty < \Sigma(f_-) \le \Sigma(f) < \infty$ and so by (1) we see that $f \in L^1(\Sigma) = L^1(T,\mathcal{B},\mu)$ and that the first inequality in (3) holds. The last inequality in (3) follows in the same manner.

(4): If $c = \infty$, we have $\int_* h\,d\mu = \infty = \int^* h\,d\mu = \infty$ and since $f(t) \le h(t)$ for all $t \in T$, we see that $g := h$ satisfies (4.a). So suppose that $c < \infty$. Then $\int^* f\,d\mu < \infty$ and so there exist functions $\phi_n \in L^1(T,\mathcal{B},\mu)$ and $\phi \in \bar{M}(T,\mathcal{B},\mu)$ such that $\int_T \phi_n\,d\mu \downarrow \int^* f\,d\mu$ and $\phi_n(t) \downarrow \phi(t) \ge f(t)$ for all $t \in T$. Then we have $\int^* \phi\,d\mu = \int^* f\,d\mu$ and since $h \in \bar{M}(T,\mathcal{B},\mu)$ and $f \le h$, we see that $\psi(t) := \phi(t) \wedge h(t)$ is $\mathcal{B}$-measurable and $f(t) \le \psi(t)$ for all $t \in T$. Hence, we have $\int^* f\,d\mu = \int^* \psi\,d\mu \le c \le \int_* h\,d\mu$ and I claim that $\psi \in \bar{L}(T,\mathcal{B},\mu)$. If $\int^* \psi\,d\mu = -\infty$, this is evident. If $\int^* \psi\,d\mu > -\infty$, we have $\inf_{n\ge 1} \int_T \phi_n\,d\mu > -\infty$ and $\int_* h\,d\mu > -\infty$. Hence, we have $h_- \in L^1(T,\mathcal{B},\mu)$ and by the monotone convergence theorem, we have $\phi \in L^1(T,\mathcal{B},\mu)$. Since $|\psi(t)| \le |\phi(t)| + |h_-(t)|$, we see that $\psi = \phi \wedge h \in L^1(T,\mathcal{B})$. Thus, we have $\psi \in \bar{L}(T,\mathcal{B},\mu)$, $\int_T \psi\,d\mu = \int^* f\,d\nu$ and $f(t) \le \psi(t) \le h(t)$ for all $t \in T$. In the same manner, we see that there exists $\xi \in \bar{L}(T,\mathcal{B},\mu)$ such that $\int_T \xi\,d\mu = \int_* h\,d\mu$ and $\psi(t) \le \xi(t) \le h(t)$ for all $t \in T$. If $c = \int^* f\,d\mu$, then $g := \psi$ satisfies (4.a), and if $c = \int_* h\,d\mu$, then $g := \xi$ satisfies. So suppose that $\int^* f\,d\mu < c < \int^* h\,d\mu$. Then we have $\int_T \psi\,d\mu < c < \int_T \xi\,d\mu$ and as above, we see that there exist $\psi_0, \xi_0 \in L^1(T,\mathcal{B},\mu)$ satisfying $\int_T \psi_0\,d\mu < c < \int_T \xi_0\,d\mu$ and $\psi(t) \le \psi_0(t) \le \xi_0(t) \le \xi(t)$ for all $t \in T$. Then it follows easily that $g(t) := \lambda\,\psi_0(t) + (1-\lambda)\,\xi_0(t)$ satisfies (4.a) if $0 < \lambda < 1$ is chosen such that $c = \lambda \int_T \psi_0\,d\mu + (1-\lambda) \int_T \xi_0\,d\mu$.

(5): Suppose that μ is not finitely founded and let $A \in \mathcal{B}$ be an infinite μ-atom. Then we have $\mu_\circ(A) = 0$ and $\mu(A) = \infty$. So by (1.1) we have $\Sigma(0) = 0 < \infty = \Sigma(1_A)$ and by (1.2), we see that $\int_* f\,d\mu \le 0$ for all $f \in \bar{M}(T, \mathcal{B})$ satisfying $f \le_\mu 1_A$. Hence, by (1) we see that Σ does not have the Darboux property. Suppose that μ is finitely founded. Let $f, h \in \bar{M}(T, \mathcal{B}, \mu)$ be given functions such that $f \le_\mu h$ and $\Sigma(f) < \Sigma(h) < \infty$. By (1.1), we have $h \in L^1(T, \mathcal{B}, \mu)$. Hence, we have $\int^* f\,d\mu < \infty$ and since μ is finitely founded, we have $f \in \bar{L}(T, \mathcal{B}, \mu)$. Hence, by (1) we have $\int^* f\,d\mu = \Sigma(f) < \Sigma(h) = \int_* h\,d\mu$ and so by (4) there exists $g \in L^1(T, \mathcal{B}, \mu)$ such that $f \le_\mu g \le_\mu h$ and $\Sigma(f) < \int_T g\,d\mu < \Sigma(h)$. Hence, by (1) we see that Σ satisfies (2.5) and in the same manner we see that Σ satisfies (2.6).

Let $\xi \in L^1(\Sigma)$ and $(\xi_n) \subseteq \bar{M}(T, \mathcal{B})$ be a given functions satisfying $\xi_n \uparrow \xi$ μ-a.e. and $\Sigma(\xi_n) = -\infty$ for all $n \ge 1$ and let $(c_n) \subseteq \mathbf{R}$ be an increasing sequence satisfying $c_n \uparrow c := \Sigma(\xi)$ and $c_n < \Sigma(\xi)$ for all $n \ge 1$. By (1.1), we have $\xi \in L^1(T, \mathcal{B}, \mu)$ and so redefining the functions on a μ-null set, we may assume that $|\xi(t)| < \infty$ and $\xi_n(t) \uparrow \xi(t)$ for all $t \in T$. Since μ is finitely founded and $\xi_n \le \xi$, we have $\xi_n \in \bar{L}(T, \mathcal{B}, \mu)$ for all $n \ge 1$. So by (1) we have $\int_T \xi\,d\mu = \Sigma(\xi)$ and $\int_T \xi_n\,d\mu = \Sigma(\xi_n) = -\infty$ for all $n \ge 1$. Let us define $a_n := c_{n+1} - c_n$ and $f_n(t) := \xi(t) - \xi_n(t)$ for all $n \ge 1$. Since $\xi(t)$ is finite and $\xi_n(t) \uparrow \xi(t)$, we have $f_n(t) \downarrow 0$ for all $t \in T$ and since $\xi \in L^1(T, \mathcal{B}, \mu)$ and $\int_T \xi_n\,d\mu = -\infty$, we have $f_n \in \bar{L}(T, \mathcal{B}, \mu)$ and $\int_T f_n\,d\mu = \infty$. Let me show that there exists functions $g_1, g_2, \ldots \in L^1(T, \mathcal{B}, \mu)$ satisfying

(i) $\quad \int_T g_n d\mu = a_n \ , \ 0 \le g_n(t) < \infty$ and $\sum\limits_{i=k}^{n} g_i(t) \le f_k(t) \ \ \forall\, t \in T \ \forall\, 1 \le k \le n$

I shall construct the g_n's recursively. By (4) with $(f, h, c) = (0, f_1, a_1)$, there exists $g_1 \in L^1(T, \mathcal{B}, \mu)$ such that $\int_T g_1\,d\mu = a_1$ and $0 \le g_1(t) \le f_1(t)$ and $g_1(t) < \infty$ for all $t \in T$. Then (i) holds for $n = 1$. Suppose that $g_1, \ldots, g_n \in L^1(T, \mathcal{B}, \mu)$ has been constructed such that $(g_k)_{1 \le k \le n}$ satisfies (i) and let us define $G_{n+1}(t) := 0$ and $G_k(t) := \sum_{k \le i \le n} g_i(t)$ for $k = 1, \ldots, n$. By (i), we have $0 \le G_k(t) \le f_k(t)$ for all $t \in T$ and all $1 \le k \le n+1$. Hence, we have $h_{n+1}(t) := \min_{1 \le k \le n+1} (f_k(t) - G_k(t)) \ge 0$ for all $t \in T$. Since $f_k(t) \ge f_{n+1}(t)$ and $G_k(t) \le G_1(t)$ for all $1 \le k \le n+1$, we have $h_{n+1}(t) \ge f_{n+1}(t) - G_1(t)$ for all $t \in T$ and since $G_1 \in L^1(T, \mathcal{B}, \mu)$ and $\int_T f_{n+1}\,d\mu = \infty$, we have $h_{n+1} \in \bar{L}(T, \mathcal{B}, \mu)$ and $\int_T h_{n+1}\,d\mu = \infty$. Hence, by (4) with $(f, h, c) = (0, h_{n+1}, a_{n+1})$, there exists $g_{n+1} \in L^1(T, \mathcal{B}, \mu)$ such that $\int_T g_{n+1}\,d\mu = a_{n+1}$ and $0 \le g_{n+1}(t) \le h_{n+1}(t)$ and $g_{n+1}(t) < \infty$ for all $t \in T$. Since $h_{n+1}(t) \le f_k(t) - G_k(t)$ for all $1 \le k \le n+1$, we see that $(g_k)_{1 \le k \le n+1}$ satisfies (i) which completes the recursive construction.

Let us define $g^n(t) := \sum_{i \ge n} g_i(t)$ for all $n \ge 1$ and all $t \in T$. Since $g_i \ge 0$ and $\sum_{i \ge n} a_i = c - c_n < \infty$, we see that $g^n \in L^1(T, \mathcal{B}, \mu)$ and $\int_T g^n\,d\mu = c - c_n$ and by (i), we have $0 \le g^n(t) \le f_n(t) = \xi(t) - \xi_n(t)$ for all $t \in T$ and all $n \ge 1$. Since $\xi \in L^1(T, \mathcal{B}, \mu)$ with $\int_T \xi\,d\mu = c$, we have $\eta_n := \xi - g^n \in L^1(T, \mathcal{B}, \mu)$ and $\int_T \eta_n\,d\mu = c_n$ for all $n \ge 1$ and since (g^n) is decreasing with $0 \le g^n(t) \le \xi(t) - \xi_n(t)$, we see that (η_n) is increasing with

$\xi_n(t) \le \eta_n(t) \le \xi(t)$ for all $t \in T$. Hence, by (1) we see that (η_n) satisfies the hypotheses in (2.7) and so we see that Σ has the strong Darboux property. $\square$

3. Integral functionals *Throughout this section, we let $(T, \mathcal{B}, \mu)$ denote a fixed finitely founded measure space with $\mu(T) > 0$ and we let $\Sigma : \bar{M}(T, \mathcal{B}) \to \bar{\mathbf{R}}$ denote a fixed μ-integral; see (1.1).*

Since $\mu(T) > 0$, we have $(m_\Sigma, m^\Sigma) = (-\infty, \infty)$ and by Thm.2.7, we see that $(\bar{M}(T, \mathcal{B}), \le_\mu)$ is a σ-lattice and that $\Sigma : \bar{M}(T, \mathcal{B}) \to \bar{\mathbf{R}}$ is an increasing, smooth, order injective functional with the strong Darboux property.

Let S be a non-empty set. Then we let $\bar{M}_S(T, \mathcal{B})$ denote the set of all functions $\phi : S \times T \to \bar{\mathbf{R}}$ satisfying $\phi(s, \cdot) \in \bar{M}(T, \mathcal{B})$ for all $s \in S$. If $(S, \le)$ is a proset and $\phi : S \times T \to \bar{\mathbf{R}}$ is a given function, we say that ϕ is *pointwise increasing* on S if $\phi(\cdot, t)$ is increasing on S for all $t \in T$, and we say that ϕ is μ*-a.e. increasing* on S, if $\phi(s, \cdot) \le_\mu \phi(u, \cdot)$ for all $s \le u$. By Thm.2.7, we see that $f : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ is an increasing Σ-partition if and only if f is μ-a.e. increasing on $\bar{\mathbf{R}}$ and we have $f(x, \cdot) \in \bar{L}(T, \mathcal{B}, \mu)$ and $\int_T f(x,t)\, \mu(dt) = x$ for all $x \in \bar{\mathbf{R}}$. In particular, we see that every increasing μ-partition of unity is an increasing Σ-partition. If $F : \bar{\mathbf{R}} \to \bar{\mathbf{R}}$ is an increasing function and $x \in \bar{\mathbf{R}}$, we set $F(x+) := \inf_{y>x} F(y)$ and $F(x-) := \sup_{y<x} F(y)$ with the conventions $F(\infty+) := F(\infty)$ and $F(-\infty-) := F(-\infty)$. If $f : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ is an increasing μ-partition of unity, we say that f is *right continuous*, resp. *left continuous*, if $f(x,t) = f(x+,t)$, resp. $f(x,t) = f(x-,t)$, for all $(x,t) \in \bar{\mathbf{R}} \times T$

If $(E, \le)$ is a proset, we say that μ is $(E, \le)$*-smooth* if $\mu^*(\cup_{u \in E} N_u) = 0$ for every increasing family $(N_u)_{u \in S}$ satisfying $N_u \in \mathcal{B}$ and $\mu(N_u) = 0$ for all $u \in E$. If $(E, \le)$ is countably cofinal, then every measure is $(E, \le)$-smooth. If $q : T \to [0, \infty)$ is a function such that $q^{-1}(0) \in \mathcal{B}$ and $\mu(B) = \sum_{t \in B} q(t)$ for all $B \in \mathcal{B}$, then μ is finitely founded and $(E, \le)$-smooth for every proset $(E, \le)$.

Lemma 3.1: *Let $f : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ be an increasing μ-partition of unity. Then the functions $(x,t) \curvearrowright f(x+,t)$ and $(x,t) \curvearrowright f(x-,t)$ are increasing μ-partitions unity satisfying*

(1) $f(x-,t) \le f(x,t) \le f(x+,t) \quad \forall\, (x,t) \in \bar{\mathbf{R}} \times T$

(2) $f(x-, \cdot) =_\mu f(x, \cdot) =_\mu f(x+, \cdot) \quad \forall\, x \in \mathbf{R}$

(3) *There exists a μ-null set $N \in \mathcal{B}$ and a set $B \in \mathcal{B}$ of σ-finite μ-measure such that $|f(x,t)| < \infty \ \ \forall\, (x,t) \in \mathbf{R} \times (T \setminus N)$ and $f(x,t) = 0 \ \ \forall\, (x,t) \in \mathbf{R} \times (T \setminus B)$*

Proof: (1) is evident and by the monotone convergence theorem, we see that $f(x+,t)$ and $f(x-,t)$ are increasing μ-partitions unity. Hence, we see that (2) follows from (1). Let Q denote the se of all rationals and let us define $N := \cup_{q \in Q} \{t \in T \mid |f(q,t)| = \infty\}$ and $B := \cup_{q \in Q} \{t \in T \mid f(q,t) \ne 0\}$. Then $N, B \in \mathcal{B}$ and since Q

is countable and $f(q,\cdot\,) \in L^1(T,\mathcal{B},\mu)$ for all $q \in Q$, we see that N is a μ-null set and that B is of σ-finite μ-measure. Since f is pointwise increasing, we see that the set N and B satisfies the claims in (3). $\square$

Theorem 3.2: *Let $S \subseteq \bar{\mathbf{R}}$ be a non-empty set and let $f,g : S \times T \to \bar{\mathbf{R}}$ be given functions such that g is pointwise increasing on S and f is μ-a.e. increasing on S and satisfies*

(1) $\quad f \in \bar{M}_S(T,\mathcal{B})$ *and* $g(s,t) \le f(s,t) \;\; \forall\,(s,t) \in S \times T$

Let $Q \subseteq S$ be a countable set and let $D \subseteq S$ be a set such that $f(\,\cdot\,,t)$ is increasing on D for all $t \in T$. Then there exists a function $h \in \bar{M}_S(T,\mathcal{B})$ such that h is pointwise increasing on S and satisfies

(2) $\quad f(s,\cdot\,) \le_\mu h(s,\cdot\,) \;\; \forall\, s \in S$ *and* $h(s,\cdot\,) \le_\mu f(u,\cdot\,) \;\; \forall\, s,u \in S$ *with* $s < u$

(3) $\quad g(s,t) \le h(s,t) \;\; \forall\,(s,t) \in S \times T$ *and* $h(s,t) = f(s,t) \;\; \forall\,(s,t) \in D \times T$

(4) $\quad \Sigma h(s) = \Sigma f(s) \;\; \forall\, s \in S$ *and* $h(s,\cdot\,) =_\mu f(s,\cdot\,) \;\; \forall\, s \in D_{\Sigma f} \cup Q$

Proof: Since f is μ-a.e. increasing, we have that $\Sigma f : S \to \bar{\mathbf{R}}$ is increasing and since $S \subseteq \bar{\mathbf{R}}$, we have that Δ is at most countably where Δ denotes the set of all discontinuity points of Σf. Let ρ denote the right Sorgenfrey topology on $\bar{\mathbf{R}}$. By [2; Exc.2.1.I p.103], there exists a countable set $C \subseteq S$ such that $Q \cup \Delta \subseteq C$ and C and $C \cap D$ are ρ-dense in S and D, respectively. Since C is countable and f is μ-a.e. increasing, there exists a μ-null set $N \in \mathcal{B}$ such that $f(\,\cdot\,,t)$ is increasing on C for all $t \in T \setminus N$.

Let $s \in S$ be given an let us define $D^s := D \cap [s,\infty]$, $C^s := C \cap [s,\infty]$ and

$$h(s,t) := \inf_{u \in D^s} f(u,t) \text{ if } t \in N \text{ and } h(s,t) := \inf_{u \in D^s \cup C^s} f(u,t) \text{ if } t \in T \setminus N$$

Then h is pointwise increasing on S and I claim that $h \in \bar{M}_S(T,\mathcal{B})$ and satisfies (2)–(4).

(2): Let $s \in S$ be given. Then there exists a countable set $L_s \subseteq D_s$ such that L_s is cofinal in $(D^s,\ge)$. Since f is pointwise increasing on D, we have $\inf_{u \in D^s} f(u,t) = \inf_{u \in L_s} f(u,t)$ for all $t \in T$. Since $f(u,\cdot\,)$ is $\mathcal{B}$-measurable and C and L_s are countable, we see that $h \in \bar{M}_S(T,\mathcal{B})$ and since f is μ-a.e. increasing, we have $f(s,\cdot\,) \le_\mu f(u,\cdot\,)$ for all $u \in S \cap [s,\infty]$. Hence, we have $f(s,\cdot\,) \le_\mu h(s,\cdot\,)$ for all $s \in S$. Let $s,u \in S$ be given such that $s < u$. Since C is ρ-dense in S, there exists $v \in C$ such that $s \le v < u$. Hence, we have $h(s,t) \le f(v,t)$ for all $t \in T \setminus N$ and since $f(v,\cdot\,) \le_\mu f(u,\cdot\,)$, we have $h(s,\cdot\,) \le_\mu f(u,\cdot\,)$. Thus, we see that h satisfies (2).

(3): Since g is pointwise increasing on S and $g \le f$, we have $g(s,t) \le g(u,t) \le f(u,t)$ for all $(s,t) \in S \times T$ and all $u \in S \cap [s,\infty]$. Hence, we see that

$g(s,t) \le h(s,t)$ for all $(s,t) \in S \times T$. Let $s \in D$ be given. Since $s \in D^s$, we have $h(s,t) \le f(s,t)$ and since f is pointwise increasing on D , we have $h(s,t) = f(s,t)$ for all $t \in N$ and $f(s,t) \le f(u,t)$ for all $(u,t) \in D^s \times T$. Let $t \in T \setminus N$ and $u \in C^s \setminus \{s\}$ be given. Since $s < u$ and $C \cap D$ is ρ-dense in D , there exists $v \in C \cap D$ such that $s \le v < u$ and since $s, v \in D$ and f is pointwise increasing on D , we have $f(s,t) \le f(v,t)$. Since $t \in T \setminus N$, we have that $f(\,\cdot\,,t)$ is increasing on C and since $v, u \in C$, we have $f(v,t) \le f(u,t)$. Hence, we see that $f(s,t) \le f(u,t)$ for all $u \in D^s \cup C^s$ and since $h(s,t) \le f(s,t)$, we have $f(s,t) = h(s,t)$ for all $(s,t) \in D \times (T \setminus N)$ which completes the proof of (3).

(4): By (2), we have $\Sigma f(s) \le \Sigma h(s)$ for all $s \in S$. Let $s \in C$ be given. Then we have $h(s,t) \le f(s,t)$ for all $t \in T \setminus N$ and so by (2) we have $h(s,\,\cdot\,) =_\mu f(s,\,\cdot\,)$ and $\Sigma f(s) = \Sigma h(s)$. Let $s \in S \setminus C$ be given. Since C is ρ-dense in S , there exists a decreasing sequence $(u_n) \subseteq C$ such that $u_n \downarrow s$. Since $u_n \in C^s$, we have $h(s,t) \le f(u_n,t)$ for all $t \in T \setminus N$ and so we have $\Sigma f(s) \le \Sigma h(s) \le \Sigma f(u_n)$ for all $n \ge 1$. Since $\Delta \subseteq C$ and $s \in S \setminus C$, we see that Σf is continuous at s and since $u_n \to s$, we see that $\Sigma f(s) = \Sigma h(s)$. Hence, we see that the first equality in (4) holds and so by (2) and order injectivity of Σ , we have $h(s,\,\cdot\,) =_\mu f(s,\,\cdot\,)$ for all $s \in D_{\Sigma f}$ and since $Q \subseteq C$, we see that h satisfies (4). □

Theorem 3.3: *Let* $S \subseteq \bar{\mathbf{R}}$ *be a non-empty set and let* $f, g : S \times T \to \bar{\mathbf{R}}$ *and* $\alpha, \beta \in \bar{L}(T, \mathcal{B}, \mu)$ *be given functions such that* g *is pointwise increasing on* S *and*

(1) $g(s,t) \le f(s,t) \le \beta(t) \quad \forall\, (s,t) \in S \times T$

(2) $f(s,\,\cdot\,) \in \bar{L}(T, \mathcal{B}, \mu)$ *and* $s = \int_T f(s,t)\,\mu(dt) = \int^* g(s,t)\,\mu(dt) \quad \forall\, s \in S$

Then f *is* μ*-a.e. increasing on* $S \setminus \{-\infty\}$ *and if* f *is* μ*-a.e. increasing on* S *and pointwise increasing on* D *for some set* $D \subseteq S$ *, then there exists an increasing* μ*-partition of unity* $h : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ *satisfying*

(3) $g(s,t) \le h(s,t) \le \beta(t) \;\; \forall\, (s,t) \in S \times T \;\; , \;\; h(s,t) = f(s,t) \;\; \forall\, (s,t) \in D \times T$

(4) $h(s,\,\cdot\,) =_\mu f(s,\,\cdot\,) \;\; \forall\, s \in S$ *and* $h(s,t) = f(s,t) \;\; \forall\, (s,t) \in D \times T$

Proof: Let $x, y \in S$ be given such that $-\infty < x < y$ and let define $\xi(t) := f(x,t) \wedge f(y,t)$ for all $t \in T$. Since g is pointwise increasing on S and $g \le f$, we have $g(x,t) \le \xi(t) \le f(x,t)$ for all $t \in T$ and so by (2) we see that ξ is $\mathcal{B}$-measurable with $\int^* \xi\, d\mu = \int_* \xi\, d\mu = x = \int_T f(x,t)\,\mu(dt)$. Since x is finite, we see that ξ and $f(x,\,\cdot\,)$ are μ-integrable and so we have $\xi =_\mu f(x,\,\cdot\,)$ or equivalently, $f(x,\,\cdot\,) \le_\mu f(y,\,\cdot\,)$. Hence we see that f is μ-a.e. increasing on $S \setminus \{-\infty\}$.

Suppose that f is μ-a.e. increasing on S and pointwise increasing on D . By (1) and Thm.3.2, there exists $f_0 \in \bar{M}_S(T, \mathcal{B})$ such that f_0 is pointwise increasing on S and satisfies $f_0(s,t) = f(s,t)$ for all $(s,t) \in D \times T$, $g(s,t) \le f_0(s,t) \le \beta(t)$ for all $(s,t) \in S \times T$ and $f_0(s,\,\cdot\,) =_\mu f(s,\,\cdot\,)$ for all $s \in S$. So by (2) and Thm.2.7, we have $f_0(s,\,\cdot\,) \in \bar{L}(T, \mathcal{B}, \mu)$ and $\Sigma f_0(s) = \int_T f_0(s,t)\,\mu(dt) = s$ for all $s \in S$.

Suppose that $S \cap \mathbf{R} \neq \emptyset$. By Thm.2.4, there exists an increasing Σ-partition of unity $f_1 : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ satisfying $f_1(s,t) = f_0(s,t)$ for all $(s,t) \in S \times T$. Then f_1 is pointwise increasing on S and so by Thm.3.2 with $g \equiv -\infty$, there exists an increasing μ-partition of unity $h : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ satisfying $h(s,t) = f_1(s,t)$ for all $(s,t) \in S \times T$. Since $f_1(s,t) = f_0(s,t)$ for all $(s,t) \in S \times T$, we see that h satisfies (3) and (4).

Suppose that $S = \{-\infty, \infty\}$. By (2) and Thm.2.7.(4) there exists $\xi \in L^1(T, \mathcal{B}, \mu)$ such that $\int_T \xi \, d\mu = 0$ and $f_0(-\infty, t) \leq \xi(t) \leq f_0(\infty, t)$ for all $t \in T$. Setting $\tilde{S} := \{-\infty, 0, \infty\}$, $\tilde{f}(\pm\infty, t) := f_1(\pm\infty, t)$, $\tilde{g}(\pm\infty, t) := g(\pm\infty, t)$ and $\tilde{f}(0,t) = \tilde{g}(t) := \xi(t)$, we see that $(\tilde{f}, \tilde{g}, \tilde{S})$ satisfies (1), (2) and $\tilde{S} \cap \mathbf{R} \neq \emptyset$. Hence, by the argument above we see that there exists an increasing μ-partition of unity $h : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ satisfying (3) and (4). The remaining two cases $S = \{\infty\}$ and $S = \{-\infty\}$ follow in the same manner. $\square$

Theorem 3.4: *Let $(S, \leq)$ be a linear proset and let $\phi \in \bar{M}_S(T, \mathcal{B})$ be a pointwise increasing function with Σ-transform $\Phi(s) := \Sigma\phi(s)$ for all $s \in S$. Let $\alpha, \beta \in \bar{L}(T, \mathcal{B}, \mu)$ be given functions satisfying $\alpha(t) \leq \phi(s,t) \leq \beta(t)$ for all $(s,t) \in S \times T$ and let us define*

$$a = \int_T \alpha \, d\mu \quad , \quad b = \int_T \beta \, d\mu \quad , \quad E_s = \{ u \in S \mid \Phi(u) = \Phi(s) \} \ \ \forall \, s \in S$$

$$\phi^*(s,t) = \sup_{u \in E_s} \phi(s,t) \quad , \quad \phi_*(s,t) = \inf_{u \in E_s} \phi(s,t) \ \ \forall \, (s,t) \in S \times T$$

$$\Phi^*(s) = \int^* \phi^*(s,t) \, \mu(dt) \quad , \quad \Phi_*(s) = \int_* \phi_*(s,t) \, \mu(dt) \ \ \forall \, s \in S$$

$$F_s = \{ u \in S \mid \Phi(u) < \Phi(s) \} \quad , \quad F^s = \{ u \in S \mid \Phi(u) > \Phi(s) \} \ \ \forall \, s \in S$$

Then ϕ^* and ϕ_* are pointwise increasing on S and there exists increasing μ-partitions of unity $h_0, h_1 : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ satisfying (see the remark below)

(1) $\quad a \vee \sup_{u \in F_u} \Phi^*(s) \leq \Phi_*(s) \leq \Phi(s) \leq \Phi^*(s) \leq b \wedge \inf_{u \in F^s} \Phi_*(u)$

(2) $\quad \alpha(t) \leq \phi_*(s,t) \leq \phi(s,t) \leq \phi^*(s,t) \leq \beta(t) \wedge \phi_*(u,t) \ \ \forall \, s \in S \ \forall \, u \in F^s$

(3) *If $s \in S \setminus D_\Phi^\circ$ and μ is $(E_s, \leq)$-smooth, then $\Phi(s) = \Phi^*(s)$*

(4) *If $s \in S \setminus D_\Phi^*$ and μ is $(E_s, \geq)$-smooth, then $\Phi(s) = \Phi_*(s)$*

(5) $\quad \alpha(t) \leq h_0(\Phi_*(s), t) \leq \phi(s,t) \leq h_1(\Phi^*(s), t) \leq \beta(t) \ \ \forall \, (s,t) \in S \times T$

(6) $\quad \alpha(t) = h_0(a,t) \leq h_1(a,t)$ *and* $h_0(b,t) \leq h_1(b,t) = \beta(t) \ \ \forall \, t \in T$

Proof: Let $x \in \bar{\mathbf{R}}$ be given and let us define $\gamma^*(x,t) := \sup_{s \in C_x} \phi(s,t)$ and $\gamma_*(x,t) := \inf_{s \in C^x} \phi(s,t)$ for all $t \in T$ where $C_x := \{s \in S \mid \Phi(s) \leq x\}$ and $C^x := \{s \in S \mid \Phi(s) \geq x\}$. Then γ^* and γ_* are pointwise increasing on $\mathbf{R}$. Let $s \in S$ be given and set $x := \Phi(s)$. Since $E_x \subseteq C_x \cap C^x$, we have $\gamma_*(x,t) \leq \phi_*(s,t) \leq \phi^*(s,t) \leq \gamma^*(x,t)$ for all $t \in T$. Let $s, u \in S$ be given elements satisfying $\Phi(s) < \Phi(u)$ and let $v \in E_s$ and $w \in E_u$ be given. Since S is linear and Φ is increasing with $\Phi(v) = \Phi(s) < \Phi(u) = \Phi(w)$, we have $v \le w$ and since ϕ is pointwise increasing, we have $\phi(v,t) \le \phi(w,t)$ for all $t \in T$. In particular, we see that (2) holds and that we have $\gamma_*(\Phi(s),t) = \phi_*(s,t) \le \phi^*(s,t) = \gamma^*(\Phi(s),t)$ for all $t \in T$ and so we have $\Phi_*(s) = \Gamma_*(\Phi(s))$ and $\Phi^*(s) = \Gamma^*(\Phi(s))$ for all $s \in S$ where $\Gamma^*(x) := \int^* \gamma^*(x,t)\,\mu(dt)$ and $\Gamma_*(x) := \int^* \gamma^*(x,t)\,\mu(dt)$ for all $x \in \bar{\mathbf{R}}$.

In particular, we see that ϕ_* and ϕ^* are pointwise increasing functions satisfying (2) and since $\alpha(t) \le \phi_*(s,t) \le \phi(s,t) \le \phi^*(s,t) \le \beta(t)$, we see that (1) follows from (2).

(3+4): Let $s \in S \setminus D_\Phi^\circ$ be a given element such that μ is $(E_s, \le)$-smooth. Then we have $-\infty < \Phi(s) \le \infty$ and by (1), we have $\Phi(s) = \Phi^*(s)$ if $\Phi(s) = \infty$. So suppose that $\Phi(s) \ne \pm\infty$ and let us define $N_u := \{t \in T \mid \phi(s,t) < \phi(u,t)\}$ for all $u \in S$. Then $N_u \in \mathcal{B}$ and since ϕ is pointwise increasing, we see that $u \curvearrowright N_u$ is increasing. Let $u \in E_s \cap [s,*]$ be given. Since $s \le u$, we have $\phi(s,t) \le \phi(u,t)$ for all $t \in T$ and since Σ is order injective and $\Sigma\phi(u) = \Phi(u) = \Phi(s) = \Sigma\phi(s) \ne \pm\infty$, we see $\mu(N_u) = 0$. Hence, by $(E_s, \le)$-smoothness of μ , we have $\mu^*(N^*) = 0$ where $N^* = \cup_{u \in E_s \cap [s,*]} N_u$ and since $N^* = \{t \in T \mid \phi(s,t) < \phi^*(s,t)\}$, we see that $\phi(s,\cdot\,) =_\mu \phi^*(s,\cdot\,)$ and $\Phi(s) = \Phi^*(s)$. Hence, we have proved (3), and (4) follows in the same manner.

Suppose that $a = b$. By (1), we have $\Phi_*(s) = \Phi(s) = \Phi^*(s) = a = b$ for all $s \in S$ and by Thm.3.3, there exists an increasing μ-partitions of unity $h_0, h_1 : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ such that $h_0(a,t) = \alpha(t)$ and $h_1(b,t) = \beta(t)$ for all $t \in T$. Hence, we see that (h_0, h_1) satisfies (5+6). So suppose that $a < b$ and let us define $\Lambda_x := \{y \in \bar{\mathbf{R}} \mid \Gamma^*(y) \le x\}$ and $g^*(x,t) := \sup_{y \in \Lambda_x} \gamma^*(y,t)$ for all $(x,t) \in \bar{\mathbf{R}} \times T$. Then $g^* : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ is pointwise increasing on $\bar{\mathbf{R}}$ and I claim that we have

(i) $\quad \int^* g^*(x,t)\,\mu(dt) = G^*(x) \ \ \forall\, x \in \bar{\mathbf{R}}$ where $G^*(x) = \sup_{y \in \Lambda_x} \Gamma^*(y)$

Proof of (i): Let $x \in \bar{\mathbf{R}}$ be given. If $\Lambda_x = \emptyset$, we have $G^*(x) = -\infty$ and $g^*(x,t) \equiv -\infty$ and so we see that (i) holds. Suppose that $\emptyset \ne \Lambda_x \subseteq \Lambda_{-\infty}$ and let $y \in \Lambda_x$ and $s \in C_y$ be given. Since $\Lambda_x \subseteq \Lambda_{-\infty}$, we have $\Gamma^*(y) = -\infty$ and so we have $G^*(x) = -\infty$. Since $\Phi(s) \le y$ and Γ^* is increasing with $\Phi(s) \le \Phi^*(s) = \Gamma^*(\Phi(s))$, we have $\Phi(s) = \Phi^*(s) = \Gamma^*(\Phi(s)) = \Gamma^*(y) = -\infty$. Hence, we have $C_y = C_{-\infty} = E_s$ and so we have $\gamma^*(y,t) = \phi^*(s,t)$ for all $t \in T$ and all $y \in \Lambda_x$. Hence, we have $g^*(x,t) = \phi^*(s,t)$ for all $t \in T$ and so we have $\int^* g(x,t)\,\mu(dt) = \Phi^*(s) = -\infty = G^*(x)$. Suppose that $\Lambda_x \not\subseteq \Lambda_{-\infty}$. Then there exists an increasing sequence $(y_n) \subseteq \Lambda_x \setminus \Lambda_{-\infty}$ such that (y_n) is cofinal in Λ_x . Since Γ^* and $\gamma^*(\,\cdot\,,t)$ are increasing, we have $\Gamma^*(y_n) \uparrow G^*(x)$ and $\gamma^*(y_n.t) \uparrow g^*(x,t)$ for all $t \in T$ and since $y_n \in \Lambda_x \setminus \Lambda_{-\infty}$, we have $-\infty < \Gamma^*(y_n) = \int^* \gamma^*(y_n,t)\,\mu(dt) \le x$ for all n . Since the upper integral satisfies the increasing monotone convergence theorem, we have $\Gamma^*(y_n) \uparrow \int^* g^*(x,t)\,\mu(dt)$ and since $\Gamma^*(y_n) \uparrow G^*(x)$, we have proved (i),

By (1), we have $S^\diamond := \{a,b\} \cup \Phi^*(S) \subseteq [a,b]$. Let $(x,t) \in S^\diamond$ be given and let us define $g(x,t) := g^*(x,t)$ if $x \in \Phi^*(S)$, $g(x,t) := \alpha(t)$ if $x = a \notin \Phi^*(S)$

and $g(x,t)=\beta(t)$ if $x=b\notin\Phi^*(S)$. Let $s\in S$ be given and set $y=\Phi(s)$ and $x=\Phi^*(s)$. Then we have $\phi(s,t)\le\phi^*(s,t)=\gamma^*(y,t)$ and $x=\Gamma^*(y)$ and so we have $G^*(x)=x$ and $\gamma^*(y,t)\le g^*(x,t)$ for all $t\in T$. Hence, we see that $\alpha(t)\le\phi(s,t)\le\phi^*(s,t)\le g^*(\Phi^*(s),t)$ for all $(s,t)\in S\times T$ and since g^* is pointwise increasing on $\bar{\mathbf{R}}$ with $g^*(x,t)\le\beta(t)$ for all $(x,t)\in\bar{\mathbf{R}}\times T$, we see that $g:S^\diamond\times T\to\bar{\mathbf{R}}$ is a pointwise function satisfying

$$\int^* g(x,t)\,\mu(dt)=x\quad,\quad\alpha(t)\le g(x,t)\le\beta(t)\quad,\quad\phi(s,t)\le g(\Phi^*(s),t)$$

for all $x\in S^\diamond$, all $s\in S$ and all $t\in T$. Hence, by Thm.2.7.(4) there exists $\xi_x\in\bar{L}(T,\mathcal{B},\mu)$ such that $\int_T\xi_x\,d\mu=x$ and $g(x,t)\le\xi_x(t)\le\beta(t)$ for all $(s,t)\in S^\diamond\times T$. By Thm.3.3, we see that $\xi_x(t)$ is μ-a.e. increasing on $S_0:=S^\diamond\setminus\{a\}$. Let $(z_n)\subseteq S_0$ be a decreasing sequence such that (z_n) is cofinal in $(S_0,\ge)$ and let us define $\eta(t):=\inf_{n\ge1}\xi_{z_n}(t)$ for all $t\in T$. Then we have $\xi_{z_n}\downarrow\eta$ μ-a.e. and $\eta\le_\mu\xi_x$ for all $x\in S_0$. Since $\xi_b\in\bar{L}(T,\mathcal{B},\mu)$ and $\xi_{z_n}\in L^1(T,\mathcal{B},\mu)$ if $z_n<b$, we have $\eta\in\bar{L}(T,\mathcal{B},\mu)$ and since g is pointwise increasing, we have $g(a,t)\le g(z_n,t)\le\xi_{z_n}(t)\le\beta(t)$ and so we see that $\alpha(t)\le g(a,t)\le\eta(t)\le\beta(t)$ for all $t\in T$.

Hence, by Thm.2.7.(4), there exists $f(a,\cdot)\in\bar{L}(T,\mathcal{B},\mu)$ such that $g(a,t)\le f(a,t)\le\eta(t)$ for all $t\in T$ and $\int_T f(a,t)\,\mu(dt)=\int^* g(a,t)\,\mu(dt)=a$. Let us define $f(b,t):=\beta(t)$ and $f(x,t):=\xi_x(t)$ for all $x\in S^\diamond\setminus\{a,b\}$ and all $t\in T$. Then we have $g(x,t)\le f(x,t)\le\beta(t)$ for all $(x,t)\in S^\diamond\times T$ and we have $f(x,\cdot)\in\bar{L}(T,\mathcal{B},\mu)$ and $\int_T f(x,t)\,\mu(dt)=x=\int^* g(x,t)\,\mu(dt)$ for all $x\in S^\diamond$. Since $\eta\le_\mu\xi_x\le\beta$ for all $x\in S_0$, we see that f is μ-a.e. increasing on $S^\diamond$ and so by Thm.3.3 with $D:=\{b\}$ there exists an increasing μ-partition of unity $h_1:\bar{\mathbf{R}}\times T\to\bar{\mathbf{R}}$ satisfying $g(x,t)\le h_1(x,t)\le\beta(t)$ for all $(x,t)\in S^\diamond\times T$ and $h_1(b,t)=\beta(t)$ for all $t\in T$. Since $a(t)\le g(a,t)$ and $\phi(s,t)\le g(\Phi^*(s),t)$, we have $\alpha(t)\le h_1(a,t)\le h_1(b,t)=\beta(t)$ and $\phi(s,t)\le h_1(\Phi^*(s),t)$ for all $(s,t)\in S\times T$.

Note that $\tilde{\phi}(s,t):=-\phi(s,t)$ is pointwise increasing on the linear proset $(S,\ge)$ satisfying $\tilde{\alpha}(t)\le\tilde{\phi}(s,t)\le\tilde{\beta}(t)$ where $\tilde{\alpha}(t):=-\beta(t)$ and $\tilde{\beta}(t):=-\alpha(t)$. Observe that $\tilde{\Sigma}(\xi):=-\Sigma(-\xi)$ is a μ-integral such that $\tilde{\Phi}(s):=\tilde{\Sigma}\tilde{\phi}(s)=-\Phi(s)$ for all $s\in S$. Applying the construction above on the pair $(\tilde{\phi},\tilde{\Sigma})$, we see that there exists an increasing μ-partition of unity $\tilde{h}_1:\bar{\mathbf{R}}\times T\to\bar{\mathbf{R}}$ satisfying $\tilde{\alpha}(t)\le\tilde{h}_1(\tilde{a},t)\le\tilde{h}_1(\tilde{b},t)=\tilde{\beta}(t)$ and $\tilde{\phi}(s,t)\le\tilde{h}_1(\tilde{\Phi}^*(s),t)$ for all $(s,t)\in S\times T$ where $\tilde{a}:=\int_T\tilde{\alpha}\,d\mu$ and $\tilde{b}:=\int_T\tilde{\beta}\,d\mu$. Let us define $h_0(x,t):=-\tilde{h}_1(-x,t)$ for all $(x,t)\in\bar{\mathbf{R}}\times T$. Then h_0 is an μ-partition of unity satisfying $h_0(a,t)=\alpha(t)\le h_0(b,t)\le\beta(t)$ and since $\tilde{\Phi}^*(s)=-\Phi_*(s)$, we have $h_0(\Phi_*(s),t)\le\phi(s,t)$ for all $(s,t)\in S\times T$. Thus, we see that the pair (h_0,h_1) satisfies (5+6). □

Theorem 3.5: *Let* $(S,\le)$ *be a linear proset and let* $\rho:2^S\to[0,\infty]$ *be an increasing set function satisfying* $\rho(\emptyset)=0$ *. Let* $\phi\in\bar{M}_S(T,\mathcal{B})$ *be a pointwise increasing function with* Σ*-transform* $\Phi(s):=\Sigma\phi(s)$ *for all* $s\in S$ *and let* $\Phi^*(s)$ *and* $\Phi_*(s)$ *be defined as in Thm.3.4. Then we have*

(1) $\quad\int^F\Phi_*\,d\rho\le\int_*\mu(dt)\int^F\phi(s,t)\,\rho(ds)\le\int^*\mu(dt)\int^F\phi(s,t)\,\rho(ds)\le\int^F\Phi^*\,d\rho$

Suppose that μ is sum-finite, let $\mathcal{A}$ be a σ-algebra on S and let ν be a sum-finite measure on $(S, \mathcal{A})$. If $\nu \otimes \mu$ denotes the product measure on the product space $(S \times T, \mathcal{A} \otimes \mathcal{B})$, then we have

(2) $\int_* \Phi_* \, d\nu \le \int_* \phi \, d(\nu \otimes \mu) \le \int_* \Phi \, d\nu \le \int^* \Phi \, d\nu \le \int^* \phi \, d(\nu \otimes \mu) \le \int^* \Phi^* \, d\nu$

Remarks: (a): If $F \subseteq S$, we say that F is ρ*-exhaustive* if $\rho(A) = \rho(A \cap F)$ for all $A \subseteq S$. If $f, g : S \to [0, \infty]$ are non-negative functions such that the set $\{f = g\}$ is ρ-exhaustive, then it follows easily that we have $\int^F f \, d\rho = \int^F g \, d\rho$. Hence, if $\{\Phi_* = \Phi^*\}$ is ρ-exhaustive, we have equality throughout in (1), and recall that (1), (3) and (4) in Thm.3.4 provide tools for verifying $\Phi_*(s) = \Phi(s)$ or $\Phi(s) = \Phi^*(s)$. Similarly, if $\Phi = \Phi^*$ ν-a.e., then the last two inequalities in (2) become equalities.

(b): Let $q : T \to [0, \infty)$ and $\phi : S \times T \to [0, \infty]$ be given functions such that ϕ is pointwise increasing on S . Then $\mu(B) = \sum_{t \in B} q(t)$ is a finitely founded measure on $(T, 2^T)$ and we have $\Sigma\phi(s) = \sum_{t \in T} q(t) \, \phi(s, t)$ for all $s \in S$. Hence, by Thm.3.4 and non-negativity of Φ we have $\Phi(s) = \Phi^*(s)$ for all $s \in S$ and $\Phi(s) = \Phi_*(s)$ for all $s \in \{\Phi < \infty\}$ and so by (1) we obtain the following remarkable inequality

$$\sum_{t \in T} q(t) \int^F \phi(s, t) \, \rho(ds) \le \int^F \sum_{t \in T} q(t) \, \phi(s, t) \, \rho(ds)$$

with equality if $\{\Phi < \infty\}$ is ρ-exhaustive.

(c): Let me give an example showing the we may have strict inequality in (2): Suppose that the continuum hypothesis holds. Then there exists a well-ordering $\preceq$ on the unit interval $I := [0, 1]$ such that $I_s := \{t \in I \mid t \preceq s\}$ is at most countable for all $s \in I$. Then $(I, \preceq)$ is a linear poset and we let λ denote the Lebesgue measure on the Borel σ-algebra on I . Let us define $\phi(s, t) := 1_{I_s}(t)$ for all $(s, t) \in I \times I$. Then $\phi(\,\cdot\,, t)$ is Borel measurable and increasing with respect to $\preceq$ and $\phi(s, \,\cdot\,)$ is Borel measurable and decreasing with respect to $\preceq$. Thus, we are in the setting of the theorem with $\nu = \mu := \lambda$ and observe that we have $\Phi(s) = \int_0^1 \phi(s, t) \, dt = 0$ and $\Phi^*(s) = 1$ for all $s \in I$. Hence, we have

$$\int_0^1 ds \int_0^1 \phi(s, t) \, dt = 0 < 1 = \int_0^1 dt \int_0^1 \phi(s, t) \, ds = \int^* \phi \, d(\lambda \otimes \lambda)$$

Proof: By Lem.3.1 and Thm.3.4 with $\alpha(t) := 0$ and $\beta(t) := \infty$, there exist increasing μ-partitions of unity $f, g : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ such that f is right continuous, g is left continuous, $g(\Phi_*(s), t) \le \phi(s, t) \le f(\Phi^*(s), t)$ for all $(s, t) \in S \times T$ and $g(0, t) \le 0 \le f(0, t)$ for all $t \in T$. In particular, we have $\int_T g(0, t) \, \mu(dt) = 0 = \int_T f(0, t) \, \mu(dt)$ and so by Lem.3.1 we see that there exists a μ-null set $N \in \mathcal{B}$ such that $g(0+, t) = g(0, t) = 0 = f(0, t) = f(0-, t)$ for all $t \in T \setminus N$ and $|f(x, t)| < \infty$ and $|g(x, t)| < \infty$ for all $(x, t) \in \mathbf{R} \times (T \setminus N)$.

Let $t \in T \setminus N$ be given. Then $f(\,\cdot\,, t)$ is a finite, increasing, right continuous function and we let λ_t denote the Lebesgue-Stieltjes measure induced by $f(\,\cdot\,, t)$. If $a < b$, we have $\lambda_t((a, b]) = f(b, t) - f(a, t)$ for all $t \in T$ and since f is an increasing μ-partition of unity and N is a μ-null set, we have $\int_{T \setminus N} \lambda_t((a, b]) \, \mu(dt) =$

$b - a = \lambda((a,b])$ where λ denotes the Lebesgue measure. Hence, by the standard proof we have

$$\text{(i)} \qquad \int_{T\setminus N} \mu(dt) \int_{\mathbf{R}} g(x)\,\lambda^t(dx) = \int_{\mathbf{R}} g(x)\,\mu(dx) = \int_{T\setminus N} \mu(dt) \int_{\mathbf{R}} g(x)\,\lambda_t(dx)$$

for every non-negative Borel function $g : \mathbf{R} \to [0,\infty]$.

Let us define $F(t) := \int^F \phi(s,t)\,\rho(ds)$ for all $t \in T$ and let me first show that $\int^* F\,d\mu \le \int^F \Phi^*\,d\rho$. If $\int^F \Phi^*\,d\rho = \infty$, this is evident. So suppose that $\int^F \Phi^*\,d\rho < \infty$. Let us define $R(x,t) := \rho(s \in S \mid \phi(s,t) > x)$ and $R_0(x,t) := \rho(s \in S \mid f(\Phi^*(s),t) > x)$ for all $(x,t) \in \mathbf{R} \times T$. Since $\phi(s,t) \le f(\Phi^*(s),t)$, we have $R(x,t) \le R_0(x,t)$. Let $t \in T \setminus N$ be given. Then we have $f(0,t) = 0$ and since $\int^F \Phi^*\,d\nu < \infty$, we have $\rho(s \in S \mid \Phi^*(s) = \infty) = 0$. Hence, we see that $R_0(x) = 0$ for all $x \ge f(\infty-,t)$ and since $R_0(\,\cdot\,,t)$ is decreasing we have (see [3; (3.29.7) p.205])

$$\begin{aligned} F(t) &= \int_0^\infty R(x,t)\,dx \le \int_0^\infty R_0(x,t)\,dx = \int_{f(0,t)}^{f(\infty-,t)} R_0(x,t)\,dx \\ &\le \int_0^\infty R_0(f(x-,t),t)\,\lambda_t(dx) \end{aligned}$$

Let $(s,t) \in S \times T$ and $x \in \mathbf{R}$ be given such that $f(x-,t) < f(\Phi^*(s),t)$. Since $f(y,t) \le f(x-,t)$ for all $y < x$, we must have $\Phi^*(s) \ge x$. Hence, we have $R_0(f(x-,t),t) \le R_1(x) := \rho(s \in S \mid \Phi^*(s) \ge x)$ and so we see that $F(t) \le \int_0^\infty R_1(x)\,\lambda_t(dx)$ for all $t \in T \setminus N$. So by (i) we have

$$\int^* F\,d\mu \le \int_{T\setminus N} \mu(dt) \int_0^\infty R_1(x)\,\lambda_t(dx) = \int_0^\infty R_1(x)\,dx = \int^F \Phi^*\,d\nu$$

which completes the proof of the last inequality in (1). The first inequality in (2) follows in the same manner using the increasing μ-partition of unity g and the mid-inequality is evident.

The last inequality in (2) holds trivially if $\int^* \Phi^* d\nu = \infty$. So suppose that $\int^* \Phi^* d\nu < \infty$ and let $a > \int^* \Phi^*\,d\nu$ be given. Then there exists $\xi \in L^1(S,\mathcal{A},\nu)$ such that $\int_S \xi\,d\nu < a$ and $\Phi^*(s) \le \xi(s)$ for all $s \in S$. Since $f(\,\cdot\,,t)$ is right continuous for all $t \in T$ and $f(x,\,\cdot\,)$ is $\mathcal{B}$-measurable for all $x \in \bar{\mathbf{R}}$, we see that f is measurable with respect to the product σ-algebra $\mathcal{B}(\bar{\mathbf{R}}) \otimes \mathcal{B}$ and since ξ is $\mathcal{A}$-measurable and $\Phi^* \le \xi$, we see that $f(\xi(s),t)$ is $(\mathcal{A} \otimes \mathcal{B})$-measurable and satisfies $0 \le \phi(s,t) \le f(\Phi^*(s),t) \le f(\xi(s),t)$. So by the Fubini-Tonelli theorem we have

$$\int^* \phi\,d(\nu \otimes \mu) \le \int_{S\times T} f(\xi(s),t)\,(\nu \otimes \mu)(ds,dt) = \int_S \nu(ds) \int_T f(\xi(s),t)\,\mu(dt)$$

Since f is an increasing μ-partition of unity, we have $\int_T f(\xi(s),t)\,\mu(dt) = \xi(s)$ for all $s \in S$ and so we see that $\int^* \phi\,d(\nu \otimes \mu) \le \int_S \xi\,d\nu < a$. Letting $a \downarrow \int^* \Phi^*\,d\nu$, we obtain the last inequality in (2). The first inequality in (2) follow in the same manner and the remaining inequalities in (2) are well-known and easy. $\square$

Theorem 3.6: *Let $(S, \leq)$ be a linear proset and let $\phi \in \bar{M}_S(T)$ be a given function with Σ-transform $\Phi(s) := \Sigma\phi(s)$. Suppose that ϕ is pointwise increasing on S and $\Phi_*(s) = \Phi^*(s)$ for all $s \in S$ where $\Phi^*(s)$ and $\Phi_*(s)$ are defined as in Thm.3.4. If $\mathcal{L} \subseteq 2^S$ is any given set such that $\phi(\,\cdot\,, t) \in \bar{W}(S, \mathcal{L})$ for μ-a.a. $t \in T$, then we have $\Phi \in \bar{W}(S, \mathcal{L})$.*

Proof: By Thm.3.4 with $\alpha(t) \equiv -\infty$ and $\beta(t) \equiv \infty$, there exist increasing μ-partitions of unity $f, g : \bar{\mathbf{R}} \times T \to \bar{\mathbf{R}}$ satisfying $g(\Phi(s), t) \leq \phi(s, t) \leq f(\Phi(s), t)$ for all $(s, t) \in S \times T$. Let $-\infty < x < y < \infty$ be given. Since $\int_T f(x, t)\,\mu(dt) = x < y = \int_T g(y, t)\,\mu(dt)$, we have $\mu(t \mid f(x, t) < g(y, t)) > 0$ and since $\phi(\,\cdot\,, t) \in \bar{W}(S, \mathcal{L})$ for μ-a.a. $t \in T$, there exists $t_0 \in T$ and $u, v \in \mathbf{R}$ such that $\phi(\,\cdot\,, t_0) \in \bar{W}(S, \mathcal{L})$ and $f(x, t_0) < u < v < g(y, t_0)$. Hence, there exists $L \in \mathcal{L} \cup \{\emptyset, S\}$ such that $\{s \mid \phi(s, t_0) > v\} \subseteq L \subseteq \{s \mid \phi(s, t_0) > u\}$. Let $s \in \{\Phi > y\}$ be given. Then we have $\phi(s, t_0) \geq g(\Phi(s), t_0) \geq g(y, t_0) > v$ and so we have $s \in L$. Let $s \in L$ be given. Then we have $f(\Phi(s), t_0) \geq \phi(s, t_0) > u > f(x, t_0)$ and since $f(\,\cdot\,, t_0)$ is increasing, we have $\Phi(s) > x$. Hence, we see that $\{\Phi > y\} \subseteq L \subseteq \{\Phi > x\}$ and so we have $\Phi \in \bar{W}(S, \mathcal{L})$. $\square$

4. Solutions to problem (IP)

Let $(M, \preceq)$ and $(S, \leq)$ be prosets, let $\omega \in M$ be a given element and let $\Sigma : M \to \bar{\mathbf{R}}$ and $H : S \to \bar{\mathbf{R}}$ be increasing functions. Then we let $I_\Sigma(H, \omega)$ denote the set of all increasing function $\phi : S \to M$ satisfying $\phi(s) \preceq \omega$ and $\Sigma\phi(s) \leq H(s) \leq \Sigma(\omega)$ for all $s \in S$. If $\phi \in I_\Sigma(H, \omega)$, we let $IP_\Sigma(\phi, H, \omega)$ denote the set of all increasing functions $\psi : S \to M$ satisfying $\phi(s) \preceq \psi(s) \preceq \omega$ and $\Sigma\psi(s) = H(s)$ for all $s \in S$. Note that $IP_\Sigma(\phi, H, \omega) \subseteq I_\Sigma(H, \omega)$ and that $IP_\Sigma(\phi, H, \omega)$ is exactly the set of all solution to problem (IP) of the introduction. We let $GI_\Sigma(H, \omega)$ denote the set of all $\phi \in I_\Sigma(H, \omega)$ for which there exists $\kappa \in L^1(\Sigma)$ such that $\phi(D^\circ_{\Sigma\phi}) \cup \{\kappa, \omega\}$ is a linear subset of $(M, \preceq)$ and if $\theta : S \to \bar{\mathbf{R}}$ is a function and $J \subseteq S$ is a given set, we define $\liminf_{s \uparrow J} \theta(s) := \sup_{u \in J} \inf_{s \in J \cap [u, *]} \theta(s)$ with the convention $\liminf_{s \uparrow \emptyset} \theta(s) := \infty$.

Theorem 4.1: *Let $(M, \preceq)$ be a σ-lattice and let $\Sigma : M \to \bar{\mathbf{R}}$ be an increasing smooth functional with the Darboux property. Let $\omega \in M$ be a given element, let $(S, \leq)$ be a linear proset and let $H : S \to \bar{\mathbf{R}}$ be an increasing function. Let $\phi \in I_\Sigma(H, \omega)$ be a given function and let us define $r := \Sigma_\vee\phi(D^\circ_{\Sigma\phi})$, $L := \{s \mid H(s) < r\}$ and $q := \Sigma_\vee\phi(L)$. Then we have*

(1) $L \cup D^\circ_H \subseteq D^\circ_{\Sigma\phi}$ *and* $q \leq r \leq \Sigma(\omega) \wedge \inf_{s \notin D^\circ_{\Sigma\phi}} \Sigma\phi(s) \leq \inf_{s \notin D^\circ_{\Sigma\phi}} H(s)$

(2) *If* $L \neq D^\circ_{\Sigma\phi}$ *, then we have* $q = -\infty$

(3) *If* $\phi \notin GI_\Sigma(H, \omega)$ *, then we have* $|\Sigma(\omega)| = |q| = |r| = |\Sigma\phi(s)| = \infty$ *for all* $s \in S$ *and* $\{H < \infty\} \subseteq D^\circ_{\Sigma\phi}$

and if $\phi \in GI_\Sigma(H,\omega)$ *, then* $r < \infty$ *and there exists an increasing* Σ*-partition of unity* $f : \bar{\mathbf{R}} \to M$ *satisfying*

(4) $f(\Sigma(\omega)) = \omega$ *and* $\phi(s) \preceq f(\Sigma\phi(s)) \ \ \forall\, s \in S \setminus D^\circ_{\Sigma\phi}$

(5) $\phi(s) \preceq f(r) \ \ \forall\, s \in D^\circ_{\Sigma\phi}$ *and* $\phi(s) \preceq f(q) \ \ \forall\, s \in L$

(6) $\phi(s) \preceq f(H(s)) \ \ \forall\, s \in \{ H \geq q\}$

Proof: (1): Since ϕ is increasing, we have that $\Sigma\phi$ is increasing and since $\Sigma\phi \leq H$, we have $D^\circ_H \subseteq D^\circ_{\Sigma\phi}$. Since S is linear we have that $A := \phi(S)$ is a linear subset of M satisfying $A \cap L_*(\Sigma) = \phi(D^\circ_{\Sigma\phi})$, $A \cap L^1(\Sigma) = \phi(D_{\Sigma\phi})$ and $A \cap L^*(\Sigma) = \phi(D^*_{\Sigma\phi})$. So by Lem.2.2 we have $\phi(u) \preceq \phi(s)$ for all $u \in D^\circ_{\Sigma\phi}$ and all $s \in S \setminus D^\circ_{\Sigma\phi}$. Hence, we have $r \leq \Sigma\phi(s)$ for all $s \in S \setminus D^\circ_{\Sigma\phi}$ and since $\Sigma\phi(s) \leq H(s) \leq \Sigma(\omega)$, we see that (1) holds.

(2): Suppose that $L \neq D^\circ_{\Sigma\phi}$. Since $L \subseteq D^\circ_{\Sigma\phi}$, there exists $u \in D^\circ_{\Sigma\phi} \setminus L$. Then we have $\Sigma\phi(u) = -\infty$ and $H(u) \geq r$. Since S is linear and H is increasing, we have $s \leq u$ for all $s \in L$ and since ϕ is increasing, we have $\phi(s) \preceq \phi(u)$ for all $s \in L$. Since $u \in D^\circ_{\Sigma\phi}$, we have $q \leq \Sigma\phi(u) = -\infty$.

(3): Suppose that $\phi \notin GI_\Sigma(H,\omega)$. Since $\phi(S) \preceq \omega$, we have $\omega \notin L^1(\Sigma)$; that is $|\Sigma(\omega)| = \infty$. Since $\phi(D_{\Sigma\phi}) \subseteq L^1(\Sigma)$ and $\phi(D^\circ_{\Sigma\phi}) \preceq \phi(D_{\Sigma\phi}) \preceq \omega$, we have $D_{\Sigma\phi} = \emptyset$; that is, $|\Sigma\phi(s)| = \infty$ for all $s \in S$. By Lem.2.1 there exists $v \in M$ such that $\phi(D^\circ_{\Sigma\phi}) \preceq v \preceq \omega$ and $\Sigma(v) = r$. Hence, we have $|r| = \infty$ and so by (2) we have $|q| = \infty$. Since $\Sigma\phi(s) \leq H(s)$ and $|\Sigma\phi(s)| = \infty$, we have $\{H < \infty\} \subseteq D^\circ_{\Sigma\phi}$.

(4)–(6): Suppose that $\phi \in GI_\Sigma(H,\omega)$. Then there exists $\kappa \in L^1(\Sigma)$ such that $\phi(D^\circ_{\Sigma\phi}) \cup \{\kappa,\omega\}$ is linear. Set $A := \phi(S)$. Then we have $A \cap L_*(\Sigma) = \phi(D^\circ_{\Sigma\phi})$ and so by Thm.2.5 with $F := \phi(L)$ we see that $r < \infty$ and that there exists an increasing Σ-partition $f : \bar{\mathbf{R}} \to M$ satisfying (4+5). Let $s \in S$ be a given element satisfying $H(s) \geq q$. By (4), we have $\phi(s) \preceq f(\Sigma\phi(s)) \preceq f(H(s))$ if $s \in S \setminus D^\circ\Sigma\phi$. By (5), we have $\phi(s) \preceq f(r) \preceq f(H(s))$ if $s \in D^\circ_{\Sigma\phi}$ and $r \leq H(s)$. So suppose that $s \in D^\circ_{\Sigma\phi}$ and $q \leq H(s) < r$. Then we have $s \in L$ and so by (5) we have $\phi(s) \preceq f(q) \preceq f(H(s))$ which completes the proof of (6). $\square$

Theorem 4.2: *Let* $(M,\preceq)$ *be a* σ*-lattice and let* $\Sigma : M \to \bar{\mathbf{R}}$ *be an increasing smooth functional with the Darboux property. Let* $\omega \in M$ *be a given element, let* $(S,\leq)$ *be a linear proset and let* $H : S \to \bar{\mathbf{R}}$ *be an increasing function. Let* $\phi, \sigma \in I_\Sigma(H,\omega)$ *be given functions satisfying* $\phi(s) \preceq \sigma(s)$ *for all* $s \in S$ *and let us define* $r := \Sigma_\vee\phi(D^\circ_{\Sigma\phi})$ *and*

$$S_H := \{ s \in S \mid -\infty < H(s) < \Sigma(\omega)\} \ \ , \ \ L := \{ s \in S \mid H(s) < r\}$$

and $q := \Sigma_\vee\phi(L)$ *. Let* $F : S \to \bar{\mathbf{R}}$ *and* $\theta : S \to M$ *be given function such that* θ *is increasing and* $\Sigma\theta(s) \dotplus F(s) \leq H(s)$ *for all* $s \in S$ *. Then we have*

(1) $\phi \in GI_\Sigma(H,\omega)$ *and* $\{ s \mid H(s) < q\} \subseteq D^\circ_H \ \Rightarrow \ IP_\Sigma(\phi,H,\omega) \neq \emptyset$

(2) *$IP_\Sigma(\sigma,H,\omega)\subseteq IP_\Sigma(\phi,H,\omega)$ and if $D^\circ_{\Sigma\sigma}\neq D^\circ_{\Sigma\phi}$ and either ϕ or σ belong to $GI_\Sigma(H,\omega)$, then we have $IP_\Sigma(\phi,H,\omega)\neq\emptyset$*

(3) $\liminf_{s\uparrow A}(H(s)\dot{-}F(s))\geq\Sigma_\vee\theta(A)>-\infty\quad\forall A\nsubseteq D^\circ_{\Sigma\theta}$

(4) *If $\phi\in GI_\Sigma(H,\omega)$ and $(M,\preceq)$ has the strong Darboux property, then the following two statements are equivalent:*

(a) $IP_\Sigma(\phi,H,\omega)\neq\emptyset$

(b) *Either $D^\circ_H=D^\circ_{\Sigma\phi}$ or $r\leq\sup_{s\in D^\circ_{\Sigma\phi}}H(s)$*

(5) $S_H\cap D^\circ_{\Sigma\phi}=\emptyset\ \Rightarrow\ IP_\Sigma(\phi,H,\omega)\neq\emptyset$

Proof: (1): Suppose that $\phi\in GI_\Sigma(H,\omega)$ and $\{H<q\}\subseteq D^\circ_H$. By Thm.4.1 there exists an increasing Σ-partition of unity $f:\bar{\mathbf{R}}\to M$ satisfying (4)–(6) in Thm.4.1. Let us define $\psi(s):=\phi(s)$ if $H(s)<q$ and $\psi(s):=f(H(s))$ if $H(s)\geq q$. By Thm.4.1, we have $\phi(s)\preceq\psi(s)\preceq\omega$ for all $s\in S$ and since S is linear and ϕ, H and f are increasing, we see that ψ is increasing. Since f is an increasing Σ-partition of unity, we have $\Sigma\psi(s)=H(s)$ for all $s\in\{H\geq q\}$. Since $\{H<q\}\subseteq D^\circ_H$ and $\phi\in I_\Sigma(H,\omega)$, we have $\Sigma\psi(s)=\Sigma\phi(s)\leq H(s)=-\infty$ for all $s\in\{H<q\}$. Hence, we have $\Sigma\psi(s)=H(s)$ for all $s\in S$ and $\psi\in IP_\Sigma(\phi,H,\omega)$.

(2): Since $\phi(s)\preceq\sigma(s)$ for all $s\in S$, we have $IP_\Sigma(\sigma,H,\omega)\subseteq P_\Sigma(\phi,H,\omega)$. So suppose that $D^\circ_{\Sigma\sigma}\neq D^\circ_{\Sigma\phi}$ and that either ϕ or σ belong to $GI_\Sigma(H,\omega)$. Let us define $\tau(s):=\phi(s)$ if $s\in D^\circ_{\Sigma\sigma}$ and $\tau(s):=\sigma(s)$ if $s\in S\setminus D^\circ_{\Sigma\sigma}$. Since $\phi(s)\preceq\sigma(s)$, we have $D^\circ_{\Sigma\sigma}\subseteq D^\circ_{\Sigma\phi}$ and since ϕ and σ are increasing with $\phi(s)\preceq\sigma(s)\preceq\omega$ and $\Sigma\sigma(s)\leq H(s)$ for all $s\in S$, we see that $\tau:S\to M$ is an increasing function satisfying $\phi(s)\preceq\tau(s)\preceq\omega$ and $\Sigma\tau(s)\leq H(s)$ for all $s\in S$. In particular, we have $\tau\in I_\Sigma(H,\omega)$ and since $D^\circ_{\Sigma\sigma}\subseteq D^\circ_{\Sigma\phi}$, we have $D^\circ_{\Sigma\tau}=D^\circ_{\Sigma\sigma}$. Since ϕ and τ coincide on $D^\circ_{\Sigma\sigma}$ and $\phi(s)\preceq\sigma(s)$, we have $\tau(D^\circ_{\Sigma\tau})=\phi(D^\circ_{\Sigma\sigma})\preceq\sigma(D^\circ_{\Sigma\sigma})$. Since either ϕ or σ belong to $GI_\Sigma(H,\omega)$, we see that $\tau\in GI_\Sigma(H,\omega)$. Since $D^\circ_{\Sigma\sigma}\subsetneqq D^\circ_{\Sigma\phi}$, there exists $u\in D^\circ_{\Sigma\phi}\setminus D^\circ_{\Sigma\sigma}$ and since S is linear and $\Sigma\phi$ is increasing, we have $\tau(D^\circ_{\Sigma\tau})=\phi(D^\circ_{\Sigma\sigma})\preceq\phi(u)$. Hence, we have $\Sigma_\vee\tau(D^\circ_{\Sigma\tau})\leq\Sigma\phi(u)=-\infty$ and so by (1) we have $IP_\Sigma(\tau,H,\omega)\neq\emptyset$. Since $\phi(s)\preceq\tau(s)$ for all $s\in S$, we see that $\emptyset\neq IP_\Sigma(\tau,H,\omega)\subseteq IP_\Sigma(\phi,H,\omega)$ which completes the proof of (2).

(3): Let $A\subseteq S$ be a given set satisfying $A\nsubseteq D^\circ_{\Sigma\theta}$ and let a denote the $\liminf$ in (3). Since $\Sigma\theta(s)\dot{+}F(s)\leq H(s)$, we have $\Sigma\theta(s)\leq H(s)\dot{-}F(s)$ for all $s\in S$ and since $\Sigma\theta$ is increasing, we have $\sup\Sigma\theta(A)\leq a$. Since $A\nsubseteq D^\circ_{\Sigma\theta}$., we have $\sup\Sigma\theta(A)>-\infty$ and so we see that (3) follows from Lem.2.2.

(4): Suppose that (4.a) holds and that $D^\circ_H\neq D^\circ_{\Sigma\phi}$. Then there exists an increasing function $\psi:S\to M$ satisfying $\phi(s)\preceq\psi(s)$ and $\Sigma\psi(s)=H(s)$ for all $s\in S$ and by Thm.4.1, we have $D^\circ_H\subsetneqq D^\circ_{\Sigma\phi}$. Hence, we have $D^\circ_{\Sigma\phi}\nsubseteq D^\circ_H=D^\circ_{\Sigma\psi}$ and so by (3) with $(\theta(s),F(s))=(\psi(s),0)$ and $A:=D^\circ_{\Sigma\phi}$, we see that $r\leq\sup_{s\in D^\circ_{\Sigma\phi}}H(s)$. Thus, we see that (4.a) implies (4.b)

Suppose that (4.b) holds and let me show that $IP_\Sigma(\phi,H,\omega)\neq\emptyset$. By (1), we see that this holds if $\{H<q\}\subseteq D_H^\circ$. So suppose that there exists $u\in S$ such that $-\infty<H(u)<q$. By Thm.4.1 we have $-\infty<q=r<\infty$ and $D_H^\circ\subseteq\{H<r\}=D_{\Sigma\phi}^\circ$ and since $-\infty<H(u)<q=r$, we have $D_H^\circ\neq D_{\Sigma\phi}^\circ$. Hence, by (4.b) we have $\sup_{s\in D_{\Sigma\phi}^\circ}H(s)=r$ and since $H(s)<r$ for all $s\in D_{\Sigma\phi}^\circ$, there exists $s_1,s_2,\ldots\in D_{\Sigma\phi}^\circ$ such that $-\infty<H(s_1)<H(s_2)<\cdots<r$ and $H(s_n)\uparrow r$. Since S is linear and H is increasing, we have $s_1\leq s_2\leq\cdots$. Let $s\in D_{\Sigma\phi}^\circ$ be given. Since $H(s)<r$ and $H(s_n)\uparrow r$, there exists an integer $n\geq 1$ such that $H(s)<H(s_n)$ and since S is linear and H is increasing, we have $s\leq s_n$. Hence, we see that (s_n) is cofinal in $D_{\Sigma\phi}^\circ$ and since ϕ is increasing, we have that $(\phi(s_n))$ is cofinal in $\phi(D_{\Sigma\phi}^\circ)$. Since M is a σ-lattice there exists an element $\kappa\in\vee\phi(D_{\Sigma\phi}^\circ)=\vee_{n=1}^\infty\phi(s_n)$. By (2.1), we have $\Sigma(\kappa)=r$ and since r is finite we have $\kappa\in L^1(\Sigma)$ and

$$\phi(s_n)\uparrow\kappa\ \ ,\ \ H(s_n)\uparrow r=\Sigma(\kappa)\ \ ,\ \ H(s_n)<r\ \text{ and }\ \Sigma\phi(s_n)=-\infty\ \ \forall\, n\geq 1$$

Since Σ has the strong Darboux property there exists an increasing sequence $(\eta_n)\subseteq M$ such that $\phi(s_{n+1})\preceq\eta_n\preceq\kappa$ and $-\infty<\Sigma(\eta_n)\leq H(s_n)$ for all $n\geq 1$. By Lem.2.2, we have $\phi(D_{\Sigma\phi}^\circ)\preceq\phi(S\setminus D_{\Sigma\phi}^\circ)$ and since $\kappa\in\vee\phi(D_{\Sigma\phi}^\circ)$, we have $\kappa\preceq\omega$ and $\kappa\preceq\phi(s)$ for all $s\in S\setminus D_{\Sigma\phi}^\circ$.

Let us define $\lambda(s):=\inf\{n\geq 0\mid s\leq s_{n+1}\}$ for all $s\in S$ with the usual convention $\inf\emptyset:=\infty$. Then $\lambda:S\to\{0,1,\ldots,\infty\}$ is an increasing function such that $\{\lambda=0\}=[*,s_1]$ and since (s_n) is cofinal in $D_{\Sigma\phi}^\circ$, we have $\{\lambda<\infty\}=D_{\Sigma\phi}^\circ$. In particular, we have $\phi(s)\preceq\phi(s_1)\preceq\eta_1$ for all $s\in\{\lambda=0\}$ and $\eta_n\preceq\kappa\preceq\phi(s)$ for all $s\in\{\lambda=\infty\}$ and since (η_n) is increasing, we see that

$$\psi(s):=\eta_{\lambda(s)}\ \text{ if }\ 1\leq\lambda(s)<\infty\ \text{ and }\ \psi(s):=\phi(s)\ \text{ if }\ \lambda(s)=0\ \text{ or }\ \lambda(s)=\infty$$

defines an increasing function from S into M satisfying $\psi(s)\preceq\omega$ for all $s\in S$. Let $s\in S$ be given such that $1\leq\lambda(s)<\infty$ and set $k:=\lambda(s)$. Then we have $s\leq s_{k+1}$ and $s\not\leq s_k$. Since ϕ is increasing, we have $\phi(s)\preceq\phi(s_{k+1})\preceq\eta_k=\psi(s)$ and since S is linear and H is increasing, we have $s_k\leq s$ and $\Sigma\psi(s)=\Sigma(\eta_k)\leq H(s_k)\leq H(s)$. Hence, we have $\phi(s)\preceq\psi(s)\preceq\omega$ and $\Sigma\psi(s)\leq H(s)$ for all $s\in S$. Since $H(s_1)<H(s_2)$, we have $\lambda(s_2)=1$ and $\psi(s_2)=\eta_1$ and since $\Sigma(\eta_1)>-\infty$ and $s_2\in D_{\Sigma\phi}^\circ$, we have $D_{\Sigma\psi}^\circ\neq D_{\Sigma\phi}^\circ$. Hence, by (2) we have $IP_\Sigma(\phi,H,\omega)\neq\emptyset$ which completes the proof of (4).

(5): Suppose that $S_H\cap D_{\Sigma\phi}^\circ=\emptyset$ and let us define $\psi(s):=\phi(s)$ if $H(s)<\Sigma(\omega)$ and $\psi(s):=\omega$ if $H(s)\geq\Sigma(\omega)$. Since S is linear and H and ϕ are increasing with $\phi(s)\preceq\omega$ and, we see that $\psi:S\to M$ is increasing and satisfies $\phi(s)\preceq\psi(s)\preceq\omega$ and $\Sigma\psi(s)\leq H(s)\leq\Sigma(\omega)$ for all $s\in S$. Suppose that $S_H=\emptyset$. Then we have $H(s)=-\infty=\Sigma\psi(s)$ if $H(s)<\Sigma(\omega)$ and $\Sigma\psi(s)=\Sigma(\omega)=H(s)$ if $H(s)\geq\Sigma(\omega)$ and so we see that $\psi\in IP_\Sigma(\phi,H,\omega)$. So suppose that $S_H\neq\emptyset$ and let $u\in S_H$ be given. Then we have $-\infty<H(u)<\Sigma(\omega)$ and so we have $D_H^\circ\subseteq D_{\Sigma\psi}^\circ=D_{\Sigma\phi}^\circ\cap\{H<\Sigma(\omega)\}$. Since $S_H\cap D_{\Sigma\phi}^\circ=\emptyset$, we see that $D_H^\circ = D_{\Sigma\psi}^\circ$ and $\sup_{s\in D_{\Sigma\psi}^\circ} H(s) = -\infty$ and since $u \in S_H$ and we have $-\infty < \Sigma\psi(u) \le H(u) < \Sigma(\omega)$. Hence, by Thm.4.1 we have $\psi \in GI_\Sigma(H,\omega)$ and so by (2) and (4), we have $IP_\Sigma(\phi, H, \omega) \ne \emptyset$. $\square$

Theorem 4.3: *Let $(M, \preceq)$ be a σ-lattice, let $\omega \in M$ be a given element and let $\Sigma : M \to \bar{\mathbf{R}}$ be an increasing smooth functional with the strong Darboux property. Let $(S, \le)$ be linear proset and let $H : S \to \bar{\mathbf{R}}$ be an increasing function. Let $\phi \in I_\Sigma(H,\omega)$ be a given function and let us define $J := \{s \in D_{\Sigma\phi}^\circ \mid H(s) < \infty\}$. Let $\xi \curvearrowright \xi^\diamond$ and $\xi \curvearrowright \xi_\diamond$ be increasing function from M into M satisfying*

(1) $\Sigma(\xi^\diamond) \underset{\cdot}{+} \Sigma(\xi_\diamond) \le \Sigma(\xi) \le \Sigma(\xi^\diamond) \dot{+} \Sigma(\xi_\diamond) \quad \forall \xi \in M$

(2) $\omega_\diamond \in L^1(\Sigma)$, $\xi_\diamond \preceq \xi \quad \forall \xi \in M$ *and* $\Sigma(\xi_\diamond) > -\infty \quad \forall \xi \in L^1(\Sigma)$

(3) *If $\xi, \eta, \tau \in M$ are given elements satisfying $\xi_\diamond \preceq \eta \preceq \omega_\diamond$ and $\tau \in \xi \vee \eta$, then we have $\tau_\diamond \preceq \eta$ and $\tau^\diamond \preceq \xi^\diamond$*

(4) $\liminf_{s\uparrow J} (H(s) \dot{-} \Sigma\phi^\diamond(s)) \ge \Sigma_\vee \phi_\diamond(J)$, $\liminf_{s\uparrow J} (H(s) \dot{-} \Sigma\phi^\diamond(s)) > -\infty$

Then we have $IP_\Sigma(\phi, H, \omega) \ne \emptyset$

Proof: Let us define $S_H := \{s \mid -\infty < H(s) < \Sigma(\omega)\}$. By Thm.4.2.(5), we have $IP_\Sigma(\phi, H, \omega) \ne \emptyset$ if $S_H \cap D_{\Sigma\phi}^\circ = \emptyset$. Suppose that $\inf \Sigma\phi(D_{\Sigma\phi}) = -\infty$. Then $D_{\Sigma\phi} \ne \emptyset$ and by Lem.2.2, we have $\phi(D_{\Sigma\phi}^\circ) \preceq \phi(D_{\Sigma\phi})$. Hence, we have $\Sigma_\vee \phi(D_{\Sigma\phi}^\circ) = -\infty$ and by Thm.4.1 we have $\phi \in GI_\Sigma(H,\omega)$. So by Thm.4.2.(4) we have $IP_\Sigma(\phi, H, \omega) \ne \emptyset$. So suppose that $S_H \cap D_{\Sigma\phi}^\circ \ne \emptyset$ and $a := \inf \Sigma\phi(D_{\Sigma\phi}) > -\infty$. Then we have $J \ne \emptyset$.

If $D_{\Sigma\phi} \ne \emptyset$, we have $-\infty < a < \infty$ and by Lem.2.2, there exists $v \in \vee\phi(D_{\Sigma\phi})$ such that $\Sigma(v) = a$ and $v \in L^1(\Sigma)$. If $D_{\Sigma\phi} = \emptyset$, we set $v := \omega$. By (2), we see that $v_\diamond \in L^1(\Sigma)$ and since $\phi(S) \preceq \omega$ and $\phi(D_{\Sigma\phi}^\circ) \preceq \phi(D_{\Sigma\phi})$, we have $\phi(D_{\Sigma\phi}^\circ) \preceq v \preceq \phi(D_{\Sigma\phi})$ and $v \preceq \omega$. Let us define

$$\kappa(s) := \phi_\diamond(s) \text{ if } s \in J \ , \ \kappa(s) := v_\diamond \text{ if } s \in S \setminus J$$

$$\theta(s) := H(s) \dot{-} \Sigma\phi^\diamond(s) \ , \ G(s) := \Sigma(v_\diamond) \wedge \inf\nolimits_{u\in J\cap[s,*]} \theta(u) \ \ \forall s \in S$$

Since ϕ and $\xi \curvearrowright \xi_\diamond$ are increasing, we see that $\phi_\diamond$ is increasing and since $\phi(J) \preceq v$, we have $\phi_\diamond(J) \preceq v_\diamond$. Since S is linear and J is a lower interval, we see that $\kappa : S \to M$ is an increasing function satisfying $\kappa(s) \preceq v_\diamond$ for all $s \in S$. Let $s \in S \setminus J$ be given. Since J is lower interval, we have $J \cap [s,*] = \emptyset$ and so we have $G(s) = \Sigma(v_\diamond) = \Sigma\kappa(s)$. Let $s \in J$ be given. By (1), we see that $\Sigma\phi_\diamond(u) \le \Sigma(v_\diamond) \wedge \theta(u)$ for all $u \in J$ and since $\Sigma\phi_\diamond$ is increasing, we have $\Sigma\kappa(s) = \Sigma\phi_\diamond(s) \le G(s)$. Hence, we see that $\kappa \in I_\Sigma(G, v_\diamond)$ and since $v_\diamond \in L^1(\Sigma)$, we have $\kappa \in GI_\Sigma(G, v_\diamond)$. By (2), we have $\kappa(s) = \phi_\diamond(s) \preceq \phi(s)$ for all $s \in J$ and since $J \subseteq D^\circ_{\Sigma\phi}$ and $v^\diamond \in L^1(\Sigma)$, we have $D^\circ_{\Sigma\kappa} = J$. Since $\kappa(J) \preceq v_\diamond$, we have $\Sigma_\vee\kappa(J) = \Sigma_\vee\phi_\diamond(J) \leq \Sigma(v_\diamond)$. Hence, by (4) we have

$$\sup_{s\in J} G(s) = \Sigma(v_\diamond) \wedge \liminf_{s\uparrow J}(H(s)\dot{-}\Sigma\phi^\diamond(s)) \geq \Sigma(v_\diamond) \wedge \Sigma_\vee\phi_\diamond(J) = \Sigma_\vee\kappa(J)$$

and so by Thm.4.2.(4) there exists an increasing function $\zeta : S \to M$ such that $\kappa(s) \preceq \zeta(s) \preceq v_\diamond$ and $\Sigma\zeta(s) = G(s)$ for all $s \in S$. Suppose that $\Sigma\phi^\diamond(s) = \infty$ for some $s \in J$. Since $\Sigma\phi^\diamond$ is increasing, we have $\Sigma\phi^\diamond(u) = \infty$ for all $u \geq s$ and since $H(u) < \infty$ for all $u \in J$, we have $\theta(u) = -\infty$ for all $u \in J \cap [s, *]$ which contradicts the last inequality in (4). Hence, we have $\Sigma\phi^\diamond(s) < \infty$ for all $s \in J$ and since $\Sigma(v_\diamond)$ is finite, there exists $v \in J$ such that $G(v) > -\infty$.

Since ϕ and ζ are increasing and M is a lattice, there exists an increasing function $\tau : S \to M$ satisfying $\tau(s) \in \phi(s) \vee \zeta(s)$ for all $s \in S$ and since $\phi(s) \preceq \omega$ and $\zeta(s) \preceq v_\diamond \preceq v \preceq \omega$, we have $\phi(s) \preceq \tau(s) \preceq \omega$ for all $s \in S$. Let $s \in S$ be given and let me show that $\Sigma\tau(s) \leq H(s)$. If $H(s) = \infty$, this is evident. Suppose $H(s) < \infty$ and $s \notin J$. Then we have $s \notin D^\circ_{\Sigma\phi}$ and $-\infty < \Sigma\phi(s) \leq H(s) < \infty$. Hence, we have $\zeta(s) \preceq v \preceq \phi(s)$ and so we see that $\tau(s) \approx \phi(s)$ and $\Sigma\tau(s) = \Sigma\phi(s) \leq H(s)$. Suppose that $s \in J$. Then we have $\phi_\diamond(s) = \kappa(s) \preceq \zeta(s) \preceq \omega_\diamond$ and so by (3) with $(\xi, \eta) = (\phi(s), \zeta(s))$ we have $\tau_\diamond(s) \preceq \zeta(s)$ and $\tau^\diamond(s) \preceq \phi^\diamond(s)$. Since $\Sigma\phi^\diamond(s) < \infty$ and $G(s) \leq \Sigma(v_\diamond) < \infty$, we have

$$G(s) \dot{+} \Sigma\phi^\diamond(s) = G(s) \underset{\cdot}{+} \Sigma\phi^\diamond(s) \leq G(s) \underset{\cdot}{+} (H(s) \dot{-} \Sigma\phi^\diamond(s)) \leq H(s)$$

and so by (1) we have

$$\Sigma\tau(s) \leq \Sigma\tau_\diamond(s) \dot{+} \Sigma\tau^\diamond(s) \leq \Sigma\zeta(s) \dot{+} \Sigma\phi^\diamond(s) \leq G(s) \dot{+} \Sigma\phi^\diamond(s) \leq H(s)$$

Hence, we have $\Sigma\tau(s) \leq H(s)$ and $\phi(s) \preceq \tau(s) \preceq \omega$ for all $s \in S$. Recall that $v \in J$ and $G(v) > -\infty$. Since $\zeta(v) \preceq \tau(v)$, we have $-\infty < G(v) = \Sigma\zeta(v) \leq \Sigma\tau(v) \leq H(v) < \infty$. Hence, we see that $v \in D_{\Sigma\tau} \cap D^\circ_{\Sigma\phi}$ and so by Thm.4.1.(3), we have $\tau \in GI_\Sigma(H, \omega)$. Hence, by Thm.4.2.(2) we have $IP_\Sigma(\phi, H, \omega) \neq \emptyset$. $\square$

Theorem 4.4: *Let $(T, \mathcal{B}, \mu)$ be finitely founded measure space with $\mu(T) > 0$ and let $\Sigma : \bar{M}(T, \mathcal{B}) \to \bar{\mathbf{R}}$ be a μ-integral. Let $(S, \leq)$ be a linear proset and let $\omega \in \bar{L}(T, \mathcal{B}, \mu)$ be a given function satisfying $\int_T \omega\, d\mu > -\infty$. Let $H : S \to \bar{\mathbf{R}}$ be an increasing function, let $\phi \in \mathrm{I}_\Sigma(H, \omega)$ be a given function and let us define $J := \{s \in D^\circ_{\Sigma\phi} \mid H(s) < \infty\}$ and $S_H := \{s \in S \mid -\infty < H(s) < \Sigma(\omega)\}$. Then the following three statements are equivalent:*

(1) $\mathrm{IP}_\Sigma(\phi, H, \omega) \neq \emptyset$

(2) *For every set $A \subseteq S$ satisfying $A \not\subseteq D^\circ_H$, we have*

(a) $\Sigma_\vee\phi(A) \leq \sup_{s\in A} H(s)$

(b) $\limsup_{s\uparrow A} (H(s) \dot{-} \Sigma\phi_+(s)) \geq \Sigma_\vee\phi_-(A)$, $\limsup_{s\uparrow A} (H(s) \dot{-} \Sigma\phi_+(s)) > -\infty$

(3) *Either* $D^\circ_{\Sigma\phi} \cap S_H = \emptyset$ *or*

(a) $\liminf\limits_{s \uparrow J} (H(s) \dot{-} \Sigma\phi_+(s)) \geq \Sigma_\vee \phi_-(J)$, $\liminf\limits_{s \uparrow J} (H(s) \dot{-} \Sigma\phi_+(s)) > -\infty$

Suppose that $IP_\Sigma(\phi, H, \omega) \neq \emptyset$ *and that* ϕ *is pointwise increasing on* S *and satisfies* $\phi(s,t) \leq \omega(t)$ *for all* $(s,t) \in S \times T$ *. Then there exists a function* $\psi \in \bar{M}_S(T, \mathcal{B})$ *such that* ψ *is pointwise increasing on* S *and*

(4) $\phi(s,t) \leq \psi(s,t) \leq \omega(t) \ \forall (s,t) \in S \times T$ *and* $\Sigma\psi(s) = H(s) \ \forall s \in S$

(5) $\psi(s,t) = \phi(s,t) \ \forall (s,t) \in D^\circ_H \times T$ *and* $\psi(s,t) = \omega(t) \ \forall (s,t) \in W \times T$

where $W := \{s \in S \mid H(s) \geq \Sigma(\omega)\}$.

Proof: (1) $\Rightarrow$ (2): Suppose that (1) holds and let $A \subseteq S$ be a given set satisfying $A \not\subseteq D^\circ_H$. Then there exists an μ-a.e. increasing function $\psi :\in \bar{M}_S(T, \mathcal{B})$ such that $\phi(s, \cdot) \leq_\mu \psi(s, \cdot) \leq_\mu \omega$ and $\Sigma\psi(s) = H(s)$ for all $s \in S$ and observe that we may take $\psi(s) = \omega$ for all $s \in \{H = \infty\}$. In particular, we have $D^\circ_H = D^\circ_{\Sigma\psi}$ and since $\Sigma\phi_+(s) \leq \Sigma\psi_+(s)$ and $\Sigma_\vee\phi(A) \leq \Sigma_\vee\psi(S)$, we see that (2.a) follows from Thm.4.2.(3) with $(\theta(s), F(s)) = (\psi(s), 0)$. Since $\int_T \omega \, d\mu > -\infty$, we have $\Sigma(\omega_-) > -\infty$ and since $\psi(s, \cdot) \in L^1(T, \mathcal{B}, \mu)$ for all $s \in D_H$, we have $D^\circ_H = D^\circ_{\Sigma\psi} = D^\circ_{\Sigma\psi_-}$. By Thm.2.7.(3), we have $\Sigma\psi_-(s) \dot{+} \Sigma\psi_+(s) \leq \Sigma\psi(s) = H(s)$ and so by Thm.4.2.(3) with $(\theta(s), F(s)) = (\psi_-(s), \Sigma\psi_+(s))$, we have

$$\liminf_{s \uparrow A} (H(s) \dot{-} \Sigma\psi_+(s)) \geq \Sigma_\vee \psi_-(A) > -\infty$$

and since $\Sigma\phi_+(s) \leq \Sigma\psi_+(s)$ and $\Sigma_\vee\phi(A) \leq \Sigma_\vee\psi(S)$, we see that (2.b) holds.

(2) $\Rightarrow$ (3): Suppose that (2) holds and that we have $S_H \cap D^\circ_{\Sigma\phi} \neq \emptyset$. Then we have $J \not\subseteq D^\circ_H$ and so we see that (3.a) follows from (2.b).

(3) $\Rightarrow$ (1): Suppose that (3) holds. If $S_H \cap D^\circ_{\Sigma\phi} = \emptyset$, then (1) follows from Thm.4.2.(5). So suppose that (3.a) holds. Since $\omega \in \bar{L}(T, \mathcal{B}, \mu)$ and $\int_T \omega \, d\mu > -\infty$, we have $\omega_- \in L^1(T, \mathcal{B}, \mu)$. But then it follows easily that the maps $\xi^\diamond(t) := \xi_+(t)$ and $\xi_\diamond(t) := \xi_-(t)$ satisfies the conditions (1)–(3) in Thm.4.3 and since (3.a) implies condition (4) in Thm.4.3, we see that $IP_\Sigma(\phi, H, \omega) \neq \emptyset$.

Thus, we see that (1)–(3) are equivalent. So suppose that $IP_\Sigma(\phi, H, \omega) \neq \emptyset$ and that ϕ is pointwise increasing and satisfies $\phi(s,t) \leq \omega(t)$ for all $(s,t) \in S \times T$. Suppose that $S_H = \emptyset$ and let us define $\psi(s,t) := \phi(s,t)$ if $(s,t) \in W^c \times T$ and $\psi(s,t) := \omega(t)$ if $(s,t) \in W \times T$. Then $\psi \in \bar{M}_S(T, \mathcal{B})$. Let $t \in T$ be given. Since S is linear and $\phi(\cdot, t)$ and H are increasing with $\phi(s,t) \leq \omega(t)$ and $H(s) \leq \Sigma(\omega)$ for all $s \in S$, we see that ψ is pointwise increasing on S and that we have $\phi(s,t) \leq \psi(s,t) \leq \omega(s)$ and $\Sigma\psi(s) \leq H(s) \leq \Sigma(\omega)$ for all $(s,t) \in S \times T$. Since $S_H = \emptyset$, we have $H(s) = -\infty = \Sigma\phi(s) = \Sigma\psi(s)$ for all $s \in W^c$ and $\Sigma\psi(s) = \Sigma(\omega) = H(s)$ for all $s \in W$ and since $\Sigma(\omega) > -\infty$, we see that ψ satisfies (4+5).

So suppose that $S_H \neq \emptyset$ and let $\xi \in IP_\Sigma(\phi, H, \omega)$ be given. Then we have $\phi(s,\cdot) \leq_\mu \xi(s,\cdot) \leq_\mu \omega$ and $\Sigma\xi(s) = H(s)$ for all $s \in S$ and since $\emptyset \neq S_H \subseteq D_H = D_{\Sigma\xi}$, we have $\xi \in GI_\Sigma(H,\omega)$ by Thm.4.1.(3). Hence, by Thm.4.1 and Thm.3.2 there exists a pointwise increasing μ-partition of unity $f : \mathbf{R} \times T \to \bar{\mathbf{R}}$ satisfying

$$\phi(s,\cdot) \leq_\mu \xi(s,\cdot) \leq_\mu f(H(s),\cdot) \ \ \forall s \in \{H \geq r\} \ \text{ and } \ f(\Sigma(\omega),t) = \omega(t) \ \ \forall t \in T$$

where $r := \Sigma_\vee \xi(D_H^\circ)$. Let us define $\psi(s,t) := \phi(s,t)$ if $(s,t) \in D_H^\circ \times T$ and $\psi(s,t) := \phi(s,t) \vee f(H(s),t)$ if $(s,t) \in (S \setminus D_H^\circ) \times T$. Then we have $\psi \in \bar{M}_S(T,\mathcal{B})$ and since D_H° is a lower interval and ϕ and f are pointwise increasing, we see that ψ is a pointwise increasing. Since $\phi(s,t) \leq \omega(t) = f(\Sigma(\omega),t)$ and $H(s) \leq \Sigma(\omega)$, we see that ψ satisfies (5) and that we have $\phi(s,t) \leq \psi(s,t) \leq \omega(t)$ for all $(s,t) \in S \times T$. In particular, we have $\Sigma\psi(s) = \Sigma\phi(s) = -\infty = H(s)$ if $s \in D_H^\circ$. Let $s \in S \setminus D_H^\circ$. Then we have $\psi(s,t) = \phi(s,t) \vee f(H(s),t)$ and by Thm.4.1.(1), we have $H(s) \geq r$. Hence, we have $\phi(s,\cdot) \leq_\mu f(H(s),\cdot)$ and so we see that $\psi(s,\cdot) = f(H(s),\cdot)$ and $\Sigma\psi(s) = \Sigma f(H(s)) = H(s)$. Thus, we see that ψ satisfies (4+5). $\square$

Example Let S and T be subsets of $\mathbf{R}$ with $\sup S = \sup T = \infty$. Let $\mathcal{B}$ denote the Borel σ-algebra on T and let μ be a finitely founded, Borel measure on T satisfying $\mu(T^s) = \infty$ for all $s \in S$ where $T^s := T \cap (s,\infty)$. Let $g : T \to [0,\infty)$ be a non-negative Borel function satisfying $G(s) := \int_{T_s} g\, d\mu < \infty$ for all $s \in S$ where $T_s := T \cap (-\infty, s]$. Let $\leq$ denote the usual ordering on S and let $\phi(s,t)$ denote the function given by

$$\phi(s,t) := g(t) \ \ \forall s \in S \ \forall t \in T_s \ \ , \ \ \phi(s,t) := -1 \ \ \forall s \in S \ \forall t \in T^s$$

Let Σ be any given μ-integral, let $\omega \in \bar{M}(T,\mathcal{B})$ be a given function satisfying $g(t) \leq \omega(t)$ for all $t \in T$ and let $H : S \to \bar{\mathbf{R}}$ an increasing function satisfying $H(s) \leq \int_T \omega\, d\mu$ for all $s \in S$. By Thm.2.7, we have $\Sigma\phi_+(s) = G(s)$ and $\Sigma\phi_-(s) = \Sigma\phi(s) = -\infty$ for all $s \in S$. Hence, we see that $\phi \in \mathrm{I}_\Sigma(H,\omega)$, $D_{\Sigma\phi}^\circ = S$ and $J = \{H < \infty\}$ where J and S_H are defined as in Thm.4.4. If $J \neq S$, there exists $u \in S$ such that $H(s) = \infty$ for all $s \in S \cap [u,\infty]$ and since $\Sigma\phi_+(s) = G(s) < \infty$, we have $\liminf_{s \uparrow J} (H(s) - \Sigma\phi_+(s)) = \infty$. Since $\phi_-(s,t) = -1_{T^s}(t)$ and $T^s \downarrow \emptyset$, we have $\Sigma_\vee \phi_-(S) = 0$. Hence, by Thm.4.4 we see that $\mathrm{IP}_\Sigma(\phi,H,\omega) \neq \emptyset$ if and only if H satisfies the following condition:

(A) Either $S_H = \emptyset$ or $\liminf_{s \uparrow S} (H(s) - G(s)) \geq 0$

and if so then there exists a function $\psi \in \bar{M}_S(T,\mathcal{B})$ such that ψ is pointwise increasing on S and satisfies (4+5) in Thm.4.4..

Let us take $T = [1,\infty)$, $\mu =$ the Lebesgue measure on T and $g(t) := \frac{1}{t}$ for all $t \in T$. Then we have $G(s) = \log_+ s$ and (A) takes the following form

(B) Either $S_H = \emptyset$ or $\liminf_{s \uparrow S} (H(s) - \log s) \geq 0$

Let us take $T = \mathbf{N}$, $\mu =$ the counting measure on $\mathbf{N}$ and $g(t) := \frac{1}{t}$ for all $t \in T$. Then we have $G(s) = \sum_{t=1}^{[s]} \frac{1}{t}$ where $[s]$ denotes the smallest integer $\geq s$. Hence if $\gamma = 0.5772156649\ldots$ denotes the Euler constant, then (A) takes the following form

(C) Either $S_H = \emptyset$ or $\liminf_{s \uparrow S} (H(s) - \log s) \geq \gamma$

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