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Homology of orthogonal groups; Topological Hochschild homology of $\mathbb{Z}/p^2$.

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Contents

Introduction

This Ph.D.-thesis contains two papers. The only thing these papers have in common is that they are concerned with some kind of homology, and that the results are related to algebraic K-theory. To explain what homology is, let me quote from MacLane [19]:

> Homology provides an algebraic "picture" of topological spaces, assigning to each space X a family of abelian groups $H_0(X), \dots, H_n(X), \dots,$ to each continous map $f : X \to Y$ a family of group homomorphisms $f_n : H_n(X) \to H_n(Y)$. Properties of the space or the map can often be effectively found from properties of the groups H_n or the homomorphisms f_n. A similar process associates homology groups to other Mathematical objects; for example, to a group Π or to an associative algebra Λ.

In this generality, K-theory and topological Hochschild homology are particular kinds of homology for associative algebras, and the two homology theories are related by maps $K_n(R) \to \mathrm{THH}_n(R)$ from the K-(homology) groups to the topological Hochschild homology groups. One direction in the subject of algebraic K-theory is to compute the K-groups for as many rings as possible. Paper 2 can be viewed as a step towards a computation of the groups $K_n(\mathbb{Z}/p^2)$. Another direction in algebraic K-theory is to relate the K-groups to problems in geometry. This is one of the motivations for paper 1.

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1. Homology of $O(n)$ and $O^1(1,n)$ made discrete: an application of Edgewise Subdivision. This paper is concerned with homology of orthogonal groups. It is a joint paper with M. Bökstedt and J. Dupont. The main result is the following theorem conjectured by C.-H. Sah:

Theorem . *The inclusion $O(n) \subseteq O^1(1,n)$ induces an isomorphism $H_k(O(n)) \to H_k(O^1(1,n))$ for $k \leq n-1$.*

Here $O(n)$ denotes the group of orthogonal $n \times n$-matrices, $O^1(1,n)$ denotes the group of isometries of hyperbolic n-space and H_k denotes group homology. In their paper on scissors congruences, Dupont, Parry and Sah found that this theorem implies that the scissors congruence group in spherical 3-space is a rational vectorspace [6]. The theorem can also be related to a conjecture of Friedlander and Milnor on homology of topological groups [21]. Let me explain what the scissors congruence group is:

Let X be either the Euclidean space $\mathbb{R}^n$, the sphere S^n or the hyperbolic space $\mathcal{H}^n$. A *k-simplex* σ in X is just an ordered set of points $\sigma = \{a_0, a_1, \dots a_k\}$, $a_i \in X$, where in the spherical case we assume the distance between any two of these points to be less than π. σ will be said to be generic if each of its j-faces with $j \leq \min(n,k)$ spans a geodesic subspace of dimension j. The underlying *geometric simplex* $|\sigma|$ is the geodesic convex hull (in the spherical case of diameter $< \pi$). A *polytope* $P \subseteq X$ is a finite union $P = \bigcup_i |\sigma_i|$ of generic geometric n-simplices such that any two intersect in a common face of dimension less than n.

Now define the scissors congruence group $\mathcal{P}(X)$ to be the free abelian group generated by all polytopes P modulo the relations

1. $[P] = [P'] + [P'']$ if $P = P' \cup P''$ and $P' \cap P''$ has no interior points,
2. $[P] = [gP]$, for g an isometry of X.

As the 3rd among 23 problems Hilbert asked for two tetrahedra in $\mathbb{R}^3$ with the same volume, that are not scissors congruent [12]. Such tetrahedra were found a few months later by Dehn [4]; he showed that the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ and $(1,1,1)$ are not scissors congruent. The way to prove this is to define an invariant, called the Dehn invariant, on the scissors congruence group. In the modern formulation of Jessen [14], the Dehn invariant takes values in the tensor product of the abelian groups $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$. For a tetrahedron the Dehn invariant is the sum over all the edges of the elements $l \otimes (\theta/\pi)$, where l denotes the length of the edge and θ denotes the interior dihedral angle along this edge, that is the angle obtained by intersecting the tetrahedron with a plane perpendicular to the edge, measured in radians. Later Sydler proved that the scissors congruence class of a polyhedron is completely determined by its volume and the value of the Dehn invariant [26]. The analogous result in dimension 4 was proven by Jessen [15], but in higher dimensional Euclidean spaces it is not known weather the volume and the Dehn invariant determines the scissors congruence class. There are similar Dehn invariants for polyhedra in hyperbolic and spherical 3-space, and the third problem of Hilbert can be generalized to these geometries: *Is the scissors congruence class for a polyhedron in spherical (respective hyperbolic)* 3*-space determined by the volume and the Dehn invariant?* This question can also be posed in higher dimensions, but then there are several reasonable choices of Dehn invariants. For hyperbolic 3-space the generalized 3rd problem of Hilbert is partially solved by the exact sequence

$$0 \to H_3(\mathrm{SL}(2,\mathbb{C}))^- \to \mathcal{P}(\mathcal{H}^3) \xrightarrow{D} \mathbb{R} \otimes \mathbb{R}/\mathbb{Z} \to H_2(\mathrm{SL}(2,\mathbb{C}))^- \to 0,$$

given by Dupont, Parry and Sah in [6]. Here D denotes the Dehn invariant, and the superscript $-$ denotes the (-1)-eigenspace with respect to complex conjugation. For spherical 3-space the exact sequence

$$0 \to H_3(\mathrm{SU}(2)) \to \mathcal{P}(S^3) \xrightarrow{D} \mathbb{R} \otimes \mathbb{R}/\mathbb{Z} \to H_2(\mathrm{SU}(2)) \to 0,$$

also given in [6], solves the generalized 3rd Hilbert problem partially. As a corollary to the theorem about homology of orthogonal groups stated above, we have identified $H_3(\mathrm{SU}(2))$ with a subgroup of the algebraic K-group $K_3(\mathbb{C})$. The rationality of $\mathcal{P}(S^3)$ follows from results of Suslin about $K_3(\mathbb{C})$ (see [24, p. 304]). For a more complete survey of scissors congruences in 3-space, see [8].

In order to give a summary of the proof of the main theorem of the paper, let $C'_*(\mathcal{H}^3)$ denote the chain complex where a generator in degree k is a k-simplex σ in $\mathcal{H}^3$, and let $C_*(\mathcal{H}^3)$ denote the subcomplex of $C'_*(\mathcal{H}^3)$ spanned by the set of generic simplices. The theorem follows from a stabilization result of C.-H. Sah [23] and the following lemma:

Lemma . *The chain complex $\mathbb{Z} \otimes_{\mathbb{Z}[O^1(1,n)]} C_*(\mathcal{H}^n)$ is $(n-1)$-acyclic with augmentation $\mathbb{Z}$.*

There is an elegant geometric proof of the analogous lemma for Euclidean space given in [7]. In the hyperbolic case it has to be modified because the circumscribed sphere for a generic n-simplex is used, and such a circumscribed sphere does not in general exist for generic hyperbolic n-simplices. The solution is to divide a generic hyperbolic n-simplex into smaller generic hyperbolic n-simplices each possessing a circumscribed sphere. The way we do this is by edgewise subdivision, cutting a simplex into smaller simplices by hyperplanes that intersect the edges of the original simplex at the middle. The main body of the paper consists of the construction of edgewise subdivision as a chain map $\mathrm{Sd} : C_*(\mathcal{H}^n) \to C_*(\mathcal{H}^n)$, inducing a chain map $\mathrm{Sd} : \mathbb{Z} \otimes_{\mathbb{Z}[O^1(1,n)]} C_*(\mathcal{H}^n) \to \mathbb{Z} \otimes_{\mathbb{Z}[O^1(1,n)]} C_*(\mathcal{H}^n)$ with the property that $\mathrm{Sd}x - x$ is a boundary for $x \in \mathbb{Z} \otimes_{\mathbb{Z}[O^1(1,n)]} C_m(\mathcal{H}^n)$, $m \leq n-1$. The traditional barycentric subdivision is not suited for this purpose because it changes the shape of the simplices too much.

This paper is accepted for publication by the Journal of Pure and Applied Algebra. This version of the paper differs from the version submitted to the journal at the proof of theorem 4.1.6, that unfortunately was incomplete.

2. Topological Hochschild homology of the integers modulo the square of an odd prime. In this paper I compute topological Hochschild homology, THH, of $\mathbb{Z}/p^2$ for p an odd prime: For $i \geq 0$ there is an isomorphism

$$\pi_i \operatorname{THH}(\mathbb{Z}/p^2) \cong \bigoplus_{k \geq 0} \pi_{i-2k} \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p^2).$$

To work out this formula one needs to know that $\pi_0 \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p^2) \cong \mathbb{Z}/p^2$, and that for $k \geq 1$

$$\pi_{2k} \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p^2) \cong \pi_{2k-1} \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p^2) \cong \mathbb{Z}/(k, p^2),$$

where (k, p^2) denotes the greatest common divisor of p^2 and k.

Topological Hochschild homology is a particular kind of homology for rings up to homotopy. (Homology in the general sense given in the introduction.) A ring up to homotopy is roughly a topological space with operations like addition and multiplication for an associative algebra, for which certain of the identities, that an associative algebra satisfies, only are satisfied up to homotopy. It is quite complicated to give a good notion of a ring up to homotopy, and several notions of rings up to homotopy are available. Some of these notions are new and not yet published. (For recent work on rings up to homotopy, see the work of Hovey, Shipley and Smith [13], the work of Elmendorf, Kriz, Mandell and Kay [9], or the work of Lydakis [18] and Schwede [25].) The most basic ring up to homotopy is the infinite loop space $QS^0 = \varinjlim_n \Omega^n S^n$; for rings up to homotopy it plays the same role as the ring of integers does for associative algebras. An associative algebra is another example of a ring up to homotopy.

Topological Hochschild homology is constructed by imitating the classical definition of homology of associative algebras, due to Hochschild, in the framework of rings up to homotopy. Goodwillie was the first to suggest to do this, and it was Bökstedt who wrote down the first precise definition of topological Hochschild homology [1]. To do this he needed a good notion of rings up to homotopy, and the concept of a functor with smash product turned out to be well suited for this purpose. In fact Goodwillie conjectured what these topological Hochschild homology groups should be for the integers and $\mathbb{Z}/p$ for any prime p, and he conjectured that topological Hochschild homology is equivalent to stable K-theory for any associative algebra. Bökstedt computed $\operatorname{THH}(\mathbb{Z})$ and $\operatorname{THH}(\mathbb{Z}/p)$ in [2], and showed that they agreed with Goodwillie's original conjecture. Dundas and McCarthy have in [5] shown that topological Hochschild homology and stable K-theory are equivalent for any associative algebra. By use of an action of the cyclic groups on topological Hochschild homology, another homology theory, topological cyclic homology, TC, has been constructed by Bökstedt, Hsiang and Madsen in [3]. They show that for an associative algebra R, there are maps

$$K_i(R) \xrightarrow{\operatorname{trc}} \operatorname{TC}_i(R) \to \operatorname{THH}_i(R),$$

connecting topological Hochschild homology and topological cyclic homology to algebraic K-theory. These maps are factorizations of a map $K_i(R) \to \mathrm{HH}_i(R)$ from algebraic K-theory to Hochschild homology (the classical homology for associative algebras), often called the Dennis trace map. They are in general non-trivial, and Hesselholt and Madsen have proven that in situations including the case $R = \mathbb{Z}/p^2$, the map trc is almost an isomorphism [11]. In this spirit the computation of $\mathrm{THH}(\mathbb{Z}/p^2)$ can be viewed as partial information about algebraic K-theory for $\mathbb{Z}/p^2$.

Pirashvili and Waldhausen have shown in [22] that topological Hochschild homology for an associative algebra is isomorphic to another kind of homology for associative algebras, constructed much earlier by MacLane [20]. This MacLane homology was constructed purely algebraically, but initially nobody was able to compute it. Pirashvili and Jibladze gave an alternative description of MacLane homology in [16], and Franjou, Lannes and Schwartz have now computed MacLane homology for finite fields in [10].

Until now I have mentioned that topological Hochschild homology is known for the integers and for finite fields. It is not hard to see that for an associative algebra containing the rational numbers topological Hochschild homology and Hochschild homology agree. Also if R is an associative algebra and G is a group, then topological Hochschild homology for the group ring $R[G]$ can be expressed in terms of topological Hochschild homology for R and the homology of the group G. For perfect fields of characteristic $p > 0$ (that is fields for which the map $x \mapsto x^p$ is an isomorphism), topological Hochschild homology is described in [11], and for integers in number rings it has been computed by Lindenstrauss and Madsen [17]. It is a fundamental fact that topological Hochschild homology of the ring up to homotopy QS^0 is simply the homotopy groups of QS^0, but these are not known!

The main observation of this paper is that the filtration $0 \subseteq p\mathbb{Z}/p^2 \subseteq \mathbb{Z}/p^2$ of $\mathbb{Z}/p^2$ gives rise to a filtration of $\mathrm{THH}(\mathbb{Z}/p^2)$. Comparing topological Hochschild homology to Hochschild homology, the computation can be made by use of the multiplicative structure of a spectral sequence first considered by Pirashvili and Waldhausen [22]. To describe this multiplicative structure, I have changed Bökstedt's construction of topological Hochschild homology slightly in order to be able to describe the product on topological Hochschild homology in a direct way.

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1 Homology of $O(n)$ and $O^1(1, n)$ made discrete: an application of Edgewise Subdivision.

2 Topological Hochschild homology of the integers modulo the square of an odd prime.

HOMOLOGY OF $O(n)$ AND $O^1(1,n)$ MADE DISCRETE: AN APPLICATION OF EDGEWISE SUBDIVISION

MARCEL BÖKSTEDT, MORTEN BRUN, AND JOHAN DUPONT

ABSTRACT. The group of isometries of hyperbolic n-space contains the orthogonal group $O(n)$ as a subgroup. We prove that this inclusion induces a stable isomorphism of discrete group homology. The unstable version of this result implies in particular that the scissors congruence group $\mathcal{P}(S^3)$ in spherical 3-space is a rational vectorspace.

1. Introduction

Let $O(n)$ denote the group of orthogonal $n \times n$-matrices and let $O^1(1,n)$ be the group of isometries of hyperbolic n-space $\mathcal{H}^n$. The main result of this paper is the following theorem conjectured by C.-H. Sah (cf. [11, appendix A], [4, §4], where it is shown in a few low dimensional cases):

Theorem (4.1.6). *The inclusion $O(n) \subseteq O^1(1,n)$ induces an isomorphism $H_k(O(n)) \to H_k(O^1(1,n))$ for $k \leq n-1$.*

Here $H_k(G)$ for G any Lie group denotes the Eilenberg-MacLane homology of the underlying uncountably infinite discrete group G^δ with integer coefficients. This result has a number of corollaries. Thus for $n = 3$ it answers affirmatively Problem 4.14 in [12]:

Corollary 1.0.1. *The natural map $H_k(\mathrm{SU}(2)) \to H_k(\mathrm{Sl}(2,\mathbb{C}))^+$ is an isomorphism for $k \leq 3$.*

Here $+$ indicates the $+1$-eigenspace for complex conjugation. As explained in [4] this result in turn has implications for the Scissors Congruence Problem for polyhedra in spherical 3-space (Extended 3rd Problem of Hilbert). In particular we conclude:

Corollary 1.0.2. *The scissors congruence group $\mathcal{P}(S^3)$ in spherical 3-space is a $\mathbb{Q}$-vector space.*

For the analogous result in hyperbolic 3-space see Sah [12, thm. 4.16].

These results are related to a well-known conjecture of Friedlander and Milnor (see [10]). In general for a Lie group G the natural homomorphism $G^\delta \to G$ gives rise to a continuous mapping $BG^\delta \to BG$ between classifying spaces.

Isomorphism Conjecture (Friedlander-Milnor) . *The canonical mapping $BG^\delta \to BG$ induces isomorphism of homology with mod p coefficients.*

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Now the homology of BG^δ is just the Eilenberg MacLane homology of G^δ. Since the subgroup of $O^1(1,n)$ fixing one point in $\mathcal{H}^n$ is isomorphic to $O(n)$, there is an identification $O^1(1,n)/O(n) = \mathcal{H}^n$. Therefore there is a fibration sequence

$$O(n) \to O^1(1,n) \to \mathcal{H}^n.$$

Since $\mathcal{H}^n$ is contractible the inclusion $O(n) \hookrightarrow O^1(1,n)$ is a homotopy equivalence. Hence also the map $BO(n) \to BO^1(1,n)$ of classifying spaces is a homotopy equivalence. In the commutative diagram

$$\begin{tikzcd}
H_*(BO(n)^\delta, \mathbb{Z}/p) \arrow[r] \arrow[d] & H_*(BO^1(1,n)^\delta, \mathbb{Z}/p) \arrow[d] \\
H_*(BO(n), \mathbb{Z}/p) \arrow[r, "\cong"] & H_*(BO^1(1,n), \mathbb{Z}/p)
\end{tikzcd}$$

the upper horizontal map is an isomorphism for $k \leq n-1$ by our theorem. Therefore it follows that in this range the isomorphism conjecture for $O(n)$ is equivalent to the conjecture for $O^1(1,n)$.

A main ingredient in the proof of the theorem above is the following stability result of Sah [11, theorem 3.8]:

Theorem (Sah). *Let $G(p,q) = U(p,q,\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Fix $p \geq 0$ and consider the inclusion of $G(p,q)$ into $G(p,q+1)$. The induced map from $H_i(G(p,q))$ to $H_i(G(p,q+1))$ is then surjective for $i \leq q$ and bijective for $i < q$.*

Notice that if we take $\mathbb{F} = \mathbb{R}$ then $G(0,q) = O(q)$ and $G(1,q) = O(1,q)$. Since $O(1,q) = O^1(1,q) \times \{\pm I_{q+1}\}$ it follows that Sah's theorem is also true for $G(1,q) = O^1(1,q)$. Furthermore in the proof of theorem 4.1.6 we shall use the same basic strategy as used in [5] for the result analogous to theorem 4.1.6 with $O^1(1,n)$ replaced by the Euclidean isometry group. However the geometry of $\mathcal{H}^n$ makes some changes necessary. The problem is that the existence of a circumscribed sphere for $(n+1)$ points in $\mathbb{R}^n$ not lying on a common hyperplane is used. Here a circumscribed sphere for $(n+1)$ points in a metric space X is the set $\{x \mid d(x,a) = r\}$ for a a point with equal distance r to all of the $(n+1)$ given points. Such a *circumcenter* a does not in general exist, and in particular it does not always exist for $(n+1)$ points in hyperbolic n-space $\mathcal{H}^n$ not lying on a common geodesic hyperplane. Since we will be working in a chain complex this problem can be solved by replacing an n-simplex not possessing a circumcenter with a chain-equivalent sum of simplices for which there does exist circumcenters. This is what edgewise subdivision is used for.

The edgewise subdivision, Sd, is a chain map from the singular chain complex (of any topological space) into itself. It is a subdivision in the sense that it is chain homotopic to the identity, and it divides a simplex into smaller simplices. The property that distinguishes it from the well-known barycentric subdivision is that it divides a simplex into simplices which are more round, or more precisely, in the Euclidean model we have the following:

Lemma (2.4.2). *All the simplices in $\mathrm{Sd}_{\Delta^n}(\Delta^n)$ are isometric to $\frac{1}{2}\Delta^n$ by a permutation of the basis vectors i_τ for $\tau \in \Sigma_n$ followed by a translation by a vector in $\frac{1}{2}\{0,1\}^n$.*

Here Δ^n denotes both the space

$$\Delta^n = \{(x_1, \dots, x_n) \mid 0 \leq x_1 \leq \cdots \leq x_n \leq 1\} \subseteq \mathbb{R}^n$$

and the identity map $\Delta^n = \mathrm{id}_{\Delta^n}$. Similarly $\frac{1}{2}\Delta^n$ is the map multiplying by $\frac{1}{2}$. The map i_τ is given by $i_\tau(x_1,\dots,x_n) = (x_{\tau^{-1}(1)},\dots,x_{\tau^{-1}(n)})$ for some permutation τ of the numbers $\{0,\dots,n\}$.

Besides from this nice property, edgewise subdivision has the advantage that it is simple to define explicitly, and it is very simple to see that the simplices in the i-fold iterated subdivisions Sd^i becomes small (cf. §2.1) for i big. Hsiang, Bökstedt and Madsen have defined a variation of this subdivision as an operation on simplicial sets [1]. The subdivision defined in this paper has been developed from this.

In §2 we define the edgewise subdivision, Sd. We prove that Sd is a subdivision in the technical meaning described below. We also prove lemma 2.4.2. This section is quite technical but self-contained.

In §3 we discuss the geometrical consequences of lemma 2.4.2 for geodesic hyperbolic simplices. To be able to do this we introduce a parametrization of hyperbolic simplices in §3.1. In §3.2 we discuss criteria for a geodesic simplex to possess a circumscribed sphere and we show that the simplices in the iterated edgewise subdivision eventually will possess one.

In §4 we prove theorem 4.1.6, and in §5 we make a few applications in particular to scissors congruences.

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2. Edgewise Subdivision

2.1. **Notation.** A singular simplex $\sigma \in S_n(X)$ is a map from Δ^n to the space X, where $\Delta^n = \{x = (x_1,\dots,x_n) \mid 0 \le x_1 \le \cdots \le x_n \le 1\} \subseteq \mathbb{R}^n$. The boundary maps $\partial_i : \Delta^{n-1} \to \Delta^n$, $i = 0,\dots,n$ are given by

$$\begin{aligned}
\partial_0(x_1,\dots,x_{n-1}) &= (0, x_1,\dots,x_{n-1}) \\
\partial_k(x_1,\dots,x_{n-1}) &= (x_1,\dots,x_k,x_k,\dots,x_{n-1}) \qquad 0<k<n \\
\partial_n(x_1,\dots,x_{n-1}) &= (x_1,\dots,x_{n-1},1)
\end{aligned}$$

We will denote the singular functor from spaces to chain complexes by S_* or sometimes just by S. As usual the boundary of σ is $d\sigma = \sum_{i=0}^n (-1)^i \sigma\partial_i$.

Given an open covering $X = \cup_{\alpha\in A} V_\alpha$, a singular simplex $\sigma \in S_n(X)$ is small (with respect to V_α, $\alpha \in A$) if the image of σ is contained in V_α for some in $\alpha \in A$. A subdivision is a set of natural transformations $\mathrm{Sd} : S_n \to S_n$, $H : S_n \to S_{n+1}$ such that for any space X with open covering $X = \cup_{\alpha\in A} V_\alpha$

1. Sd_X is a chain map
2. H_X is a chain homotopy from Sd_X to $\mathrm{id}_{S_*(X)}$ which preserves the subcomplex of small simplices.
3. For any $\sigma \in S_n(X)$ there exists $i \in \mathbb{N}$ such that the i-fold iteration $(\mathrm{Sd}_X)^i(\sigma)$ is a sum of small simplices.

The unit cube in $\mathbb{R}^n$ is

$$\left(\Delta^1\right)^n = \{(x_1,\dots,x_n) \mid 0 \le x_i \le 1, i = 1,\dots,n\}.$$

Any permutation $\tau \in \Sigma_n$ of the axes of $\mathbb{R}^n$ induces an affine map $i_\tau : (\Delta^1)^n \to (\Delta^1)^n$ by $i_\tau(e_k) = e_{\tau(k)}$ where $e_k = (0,\dots,0,\underset{k}{1},0,\dots,0)$. Given $\nu \in \{0,1\}^n \subseteq (\Delta^1)^n$ we define $\alpha_\nu : (\Delta^1)^n \to (\Delta^1)^n$ by $\alpha_\nu(x) = \frac{1}{2}(x+\nu)$. Let us denote the restrictions of α_ν

and i_τ to Δ^n by the same names. Finally let $V \subseteq \{0,1\}^n \times \Sigma_n$ denote the elements (ν, τ) such that $\alpha_\nu i_\tau(\Delta^n) \subseteq \Delta^n$.

We will adopt the notation that the symbol X denotes both the space X and the identity map on X.

2.2. Construction of edgewise subdivision.

Definition 2.2.1. The natural map $\mathrm{Sd}_X : S_*(X) \to S_*(X)$ is defined by

$$\mathrm{Sd}_X(\sigma : \Delta^n \to X) = \sum_{(\nu,\tau)\in V} \mathrm{sign}(\tau)\sigma\alpha_\nu i_\tau$$

We often just write Sd instead of Sd_X. The definition has a close relation to the Eilenberg-Zilber map $\mathrm{EZ} : S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ (cf. MacLane [9, chapter VIII, §8]). Extending this to n factors we obtain in particular

$$\mathrm{EZ} : \left(S_*(\Delta^1)\right)^{\otimes n} \to S_*\left(\left(\Delta^1\right)^n\right),$$

and an easy calculation shows that

$$\mathrm{EZ}((\Delta^1)^{\otimes n}) = \sum_{\omega\in\Sigma_n} \mathrm{sign}(\omega) i_\omega.$$

We need to show

Proposition 2.2.2. Sd *is a chain map.*

The main step of the proof of proposition 2.2.2 is given by

Lemma 2.2.3. $\mathrm{Sd}_{(\Delta^1)^n}\,\mathrm{EZ}((\Delta^1)^{\otimes n}) = \mathrm{EZ}\,\mathrm{Sd}^{\otimes n}_{(\Delta^1)}((\Delta^1)^{\otimes n})$.

Proof. Let us note the relation $i_\omega \alpha_\nu = \alpha_{\omega\cdot\nu} i_\omega$, where $\omega \cdot \nu = i_\omega(\nu)$. Using this we get

$$\mathrm{Sd}\,\mathrm{EZ}((\Delta^1)^{\otimes n}) = \sum_{\substack{\omega\in\Sigma_n \\ (\nu,\tau)\in V_n}} \mathrm{sign}(\omega\tau)\alpha_{\omega\cdot\nu} i_{\omega\tau}$$

$$\mathrm{EZ}\,\mathrm{Sd}^{\otimes n}((\Delta^1)^{\otimes n}) = \sum_{\substack{\epsilon\in\{0,1\}^n \\ \sigma\in\Sigma_n}} \mathrm{sign}(\sigma)\alpha_\epsilon i_\sigma.$$

Let us show that the map $V \times \Sigma_n \to \{0,1\}^n \times \Sigma_n$ given by

$$((\nu,\tau),\omega) \mapsto (\omega\cdot\nu, \omega\tau)$$

is a bijection. Given $(\epsilon,\sigma) \in \{0,1\}^n \times \Sigma_n$, let $\tau \in \Sigma_n$ be the element such that τ^{-1} is a shuffle map (i.e. there exists $i \le n$ such that $\tau^{-1}(1) < \cdots < \tau^{-1}(i)$ and $\tau^{-1}(i+1) < \cdots < \tau^{-1}(n)$), and such that $i_\tau(\sigma^{-1}\cdot\epsilon) \in \Delta^n$. Here τ is uniquely determined, and i is the number of zeros in $\sigma^{-1}\cdot\epsilon$. It is easy to see that $i_\tau \alpha_{\sigma^{-1}\cdot\epsilon}(\Delta^n) \subseteq \Delta^n$. Hence by the above relation $\alpha_{\tau\sigma^{-1}\cdot\epsilon} i_\tau(\Delta^n) \subseteq \Delta^n$. The map $\{0,1\}^n \times \Sigma_n \to V \times \Sigma_n$ given by

$$(\epsilon,\sigma) \mapsto ((\tau\sigma^{-1}\cdot\epsilon, \tau), \sigma\tau^{-1}).$$

is an inverse to the above map. By one of the compositions $((\nu,\tau'),\omega)$ is sent to $((\tau\tau'^{-1}\omega^{-1}\omega\cdot\nu,\tau),\omega\tau'\tau^{-1})$. To see that this is actually the identity we need to observe that if $i_\tau\alpha_{\sigma^{-1}\cdot\epsilon}(\Delta^n) \subseteq \Delta^n$ and $i_{\tau'}\alpha_{\sigma^{-1}\cdot\epsilon}(\Delta^n) \subseteq \Delta^n$ then $\tau = \tau'$. For this use that if $\tau \neq \tau'$ then $i_{\tau^{-1}}(\Delta^n) \cap i_{\tau'^{-1}}(\Delta^n)$ is contained in an affine hyperplane. The other composition is clearly the identity. ∎

Corollary 2.2.4. $d\,\mathrm{Sd}_{(\Delta^1)^n}\,\mathrm{EZ}((\Delta^1)^{\otimes n}) = \mathrm{Sd}_{(\Delta^1)^n}\,d\,\mathrm{EZ}((\Delta^1)^{\otimes n})$

Proof. As an immediate consequence of the definitions we get

$$\mathrm{Sd}_{\Delta^0\times X}\,\mathrm{EZ}(\Delta^0\otimes\sigma) = \mathrm{EZ}(\mathrm{Sd}_{\Delta^0}\otimes\mathrm{Sd}_X)(\Delta^0\otimes\sigma)$$

for any space X and any singular simplex $\sigma : \Delta^n \to X$. It now follows from lemma 2.2.3 and the associativity of EZ that

$$\mathrm{Sd}_{\Delta^0\times(\Delta^1)^{n-1}}\,\mathrm{EZ}(\Delta^0\otimes\left(\Delta^1\right)^{\otimes(n-1)}) = \mathrm{EZ}(\mathrm{Sd}_{\Delta^0}\otimes\mathrm{Sd}_{(\Delta^1)^{\otimes(n-1)}})(\Delta^0\otimes\left(\Delta^1\right)^{\otimes(n-1)})$$

The corollary follows from this formula and lemma 2.2.3 by a calculation. □

Proof of proposition 2.2.2. Let $\pi : (\Delta^1)^n \to \Delta^n$ be a retraction of the continuous map $i_{\mathrm{id}} : \Delta^n \to (\Delta^1)^n$ so that π is affine on the simplices $i_\tau(\Delta^n)$ and maps $i_\tau(\Delta^n)$ to a degenerate simplex for $\tau \neq \mathrm{id}$. If $W \subseteq \{0,1\}^n$ denotes the vertices of Δ^n then such a map is determined by the retraction $\{0,1\}^n \to W$, sending an element of $\{0,1\}^n\backslash W$ with the first 1 occuring in the $(i+1)$'th entry to the element with zero in the first i entries and 1 in the rest of the entries. Notice that $S(\pi)\,\mathrm{EZ}((\Delta^1)^{\otimes n}) = \Delta^n + x$ where x is a sum of degenerate simplices. From corollary 2.2.4 it follows that

$$d\,\mathrm{Sd}_{\Delta^n}(\Delta^n + x) = \mathrm{Sd}_{\Delta^n}(d(\Delta^n + x))$$

or

$$d\,\mathrm{Sd}_{\Delta^n}(\Delta^n) - \mathrm{Sd}_{\Delta^n}(d\Delta^n) = \mathrm{Sd}_{\Delta^n}(dx) - d\,\mathrm{Sd}_{\Delta^n}(x) \tag{2.2.5}$$

Now $\mathrm{Sd}_{\Delta^n}(\Delta^n)$ consists of non-degenerate simplices. Therefore the left hand side of (2.2.5) consists of non-degenerate simplices. Since d preserves the subcomplex of degenerate simplices the right hand side of 2.2.5 is a sum of degenerate simplices. From this we conclude that

$$d\,\mathrm{Sd}_{\Delta^n}(\Delta^n) - \mathrm{Sd}_{\Delta^n}(d\Delta^n) = 0$$

By naturality this concludes the proof. □

2.3. **The homotopy.** A chain homotopy H_X from Sd_X to $\mathrm{id}_{S_*(X)}$ consists of maps $H_X : S_n(X) \to S_{n+1}(X)$, $n \geq 0$ satisfying $dH_X + H_X d = \mathrm{Sd}_X - \mathrm{id}_{S_*(X)}$. We will construct such maps naturally in X. For this we will use the map $\phi_{n+1} : \Delta^{n+1}\setminus\{(1,\ldots,1)\} \to I\times\Delta^n$ given by

$$\phi_{n+1}(x_1,\ldots,x_{n+1}) = \begin{cases}(2x_1, \frac{(x_2-x_1,\ldots,x_{n+1}-x_1)}{1-x_1}) & x_1 \leq \frac{1}{2}\\ (1, \frac{(x_2-x_1,\ldots,x_{n+1}-x_1)}{1-x_1}) & 1 > x_1 \geq \frac{1}{2}\end{cases}$$

Here I denotes the unit interval. Let $\alpha_n = \alpha_{(1,\ldots,1)} : \Delta^n \to \Delta^n$. Note the relations

$$\begin{aligned}\phi_{n+1}\partial_0 &= (0,\Delta^n)\\ \phi_{n+1}\partial_k &= (I,\partial_{k-1})\phi_n \qquad k>0\\ \phi_{n+1}\alpha_{n+1}\partial_0 &= (1,\Delta^n)\\ \alpha_{n+1}\partial_k &= \partial_k\alpha_n \qquad k>0\end{aligned}$$

where 0 and 1 are the constant maps from Δ^n to I.

Definition 2.3.1. Let $\widetilde{H}_X : S_n(X) \to S_{n+1}(I\times X)$ be the natural map given by

$$\widetilde{H}_X(\sigma : \Delta^n \to X) = \sum_{(\nu,\tau)\in(\{0,1\}^{n+1}\times\Sigma_{n+1})\setminus\{((1,\ldots,1),\mathrm{id})\}} \mathrm{sign}(\tau)(I,\sigma)\phi_{n+1}\alpha_\nu i_\tau$$

This definition may be rewritten

$$\widetilde{H}_X(\sigma) = S((I,\sigma)\phi_{n+1})(\operatorname{Sd}\Delta^{n+1} - \alpha_{n+1}).$$

Note that we have subtracted the term $S((I,\sigma)\phi_{n+1})\alpha_{n+1}$ which is not defined. Let $\mathrm{pr} : I \times X \to X$ be the projection on X.

Definition 2.3.2. $H_X = S(\mathrm{pr})\widetilde{H}_X : S_n(X) \to S_{n+1}(X)$.

Proposition 2.3.3. $dH_X + H_X d = \operatorname{Sd}_X - \mathrm{id}_{S(X)}$.

This proposition follows from

Lemma 2.3.4. $d\widetilde{H}_X\sigma + \widetilde{H}_X d\sigma = S(0,\sigma)\operatorname{Sd}\Delta^n - S(1,\sigma)\Delta^n$.

Proof. By naturality it is enough to prove the lemma for $\sigma = \Delta^n$. Now

$$d\widetilde{H}_{\Delta^n}(\Delta^n) = dS(\phi_{n+1})\left(\operatorname{Sd}\Delta^{n+1} - \alpha_{n+1}\right) = S(\phi_{n+1})\left(\operatorname{Sd}(d\Delta^{n+1}) - d\alpha_{n+1}\right)$$

and

$$\begin{aligned}\widetilde{H}_{\Delta^n}(d\Delta^n) &= \sum_{i=0}^{n}(-1)^i S((I,\partial_i)\phi_n)\left(\operatorname{Sd}\Delta^n - \alpha_n\right)\\ &= -\sum_{i=0}^{n}(-1)^{i+1} S(\phi_{n+1}\partial_{i+1})\left(\operatorname{Sd}\Delta^n - \alpha_n\right)\\ &= S(\phi_{n+1})\left(-\operatorname{Sd}(d\Delta^{n+1}) + \partial_0\operatorname{Sd}\Delta^n + d\alpha_{n+1} - \alpha_{n+1}\partial_0\right).\end{aligned}$$

Hence

$$\begin{aligned}d\widetilde{H}_{\Delta^n} + \widetilde{H}_{\Delta^n}d\Delta^n &= S(\phi_{n+1})\left(\partial_0\operatorname{Sd}\Delta^n - \alpha_{n+1}\partial_0\right)\\ &= (0,\Delta^n)\operatorname{Sd}\Delta^n - (1,\Delta^n)\Delta^n.\end{aligned}$$

□

2.4. Sd **is a subdivision.** We have now found natural transformations $\operatorname{Sd} : S_n \to S_n$ and $H : S_n \to S_{n+1}$ such that for any space X with an open covering $X = \cup_{\alpha\in A}V_\alpha$

1. Sd_X is a chain map.
2. H_X is a chain homotopy from Sd_X to $\mathrm{id}_{S_*(X)}$ which preserves the subcomplex of small simplices.

Theorem 2.4.1. Sd *is a subdivision.*

Proof. We only need to show that for any $\sigma \in S_n(X)$ there exists $i \in \mathbb{N}$ such that $(\operatorname{Sd}_X)^i(\sigma)$ is a sum of small simplices. This follows from our next lemma. □

Lemma 2.4.2. *All the simplices in* $\operatorname{Sd}_{\Delta^n}(\Delta^n)$ *are isometric to* $\frac{1}{2}\Delta^n$ *by a permutation of the basis vectors* i_τ *for* $\tau \in \Sigma_n$ *followed by a translation by a vector in* $\frac{1}{2}\{0,1\}^n$.

Proof. The simplices in $\operatorname{Sd}_{\Delta^n}(\Delta^n)$ are of the form $\alpha_\nu i_{\tau^{-1}}$ for $\nu \in \{0,1\}^n$ and $\tau \in \Sigma_n$. In lemma 2.2.3 we found that $i_\tau\alpha_\nu i_{\tau^{-1}} = \alpha_{\tau\cdot\nu}$. This map differs from $\frac{1}{2}\Delta^n$ by a translation. □

3. Geometrical properties of $\mathrm{Sd}_{\mathcal{H}^n}$

Let $\mathcal{H}^n$ denote hyperbolic n-space. The aim of this section is to show that any non-degenerate geodesic hyperbolic simplex after applying Sd sufficiently many times will be divided into simplices possessing a circumscribed sphere. We start by parametrizing geodesic simplices.

3.1. **A good parametrization of a simplex.** Given points $a_0, \dots, a_n \in \mathcal{H}^n$ we will construct a parametrization of the geodesic simplex $a = (a_0, \dots, a_n)$.

Lemma 3.1.1. *There is a map f_a from Δ^n to $\mathcal{H}^n$ such that*

1. *The image of f_a is the geodesic span of $a_0, \dots, a_n$.*
2. *For any intersection K of Δ^n with an affine subset of $\mathbb{R}^n$ the image $f_a(K)$ is a geodesically convex subset of $\mathcal{H}^n$. In particular, the image of an affine subdivision of Δ^n maps to a subdivision of a by hyperbolic simplices.*
3. *For any isometry $\gamma : \mathcal{H}^n \to \mathcal{H}^n$, f satisfies that $f_{\gamma a} = \gamma f_a$.*

Proof. We consider the hyperbolic model for $\mathcal{H}^n$. In $\mathbb{R}^{n+1}$ consider the inner product

$$F(u,v) = -u_0v_0 + u_1v_1 + \cdots + u_nv_n$$

and let $\mathcal{H}^n$ be the set

$$\mathcal{H}^n = \{u \in \mathbb{R}^{n+1} \mid u_0 > 0,\ F(u,u) = -1\}.$$

The geodesic curves in this model are all curves of the form $\mathcal{H}^n \cap E$ where $E \subseteq \mathbb{R}^{n+1}$ is a 2-plane through 0 such that $F|_E$ is non-degenerate of type $(1,1)$ (cf. Iversen [7, section II.4]).

Let $\underline{\Delta}^n$ denote the convex hull of the canonical basis $\overline{e}_0, \dots, \overline{e}_n$ for $\mathbb{R}^{n+1}$. Let $h : \Delta^n \to \underline{\Delta}^n$ be the affine map given by $h(x_1, \dots, x_n) = (t_0, \dots, t_n)$ where $t_0 = x_1$, $t_i = (x_{i+1} - x_i)$ for $1 \le i \le n-1$ and $t_n = 1 - x_n$.

Definition 3.1.2. The *good parametrization* f_a of a simplex $a = (a_0, \dots, a_n)$ in $\mathcal{H}^n$ is the map $f_a : \Delta^n \to \mathcal{H}^n$ given by

$$f_a(x) = \frac{a(t)}{\sqrt{-F(a(t), a(t))}}$$

where $t = (t_0, \dots, t_n) = h(x)$ and $a(t) = \sum_{i=0}^n t_i a_i$.

Notice that $F(a(t), a(t)) = \sum_{i,j} t_i t_j F(a_i, a_j) < 0$ since $F(a_i, a_j) \le 0$ for $i \ne j$ and $F(a_i, a_i) = -1$.

Since $f_a(\overline{e}_i) = a_i$ and f_a maps affine lines either to images of geodesic curves or to a point f satisfies (1) and (2) above. Since an isometry $\gamma : \mathcal{H}^n \to \mathcal{H}^n$ is induced by an element $A \in O^1(1,n) \subseteq \mathrm{Gl}(\mathbb{R}^{n+1})$ (cf. Iversen [7, section II.4]), we have

$$(\gamma a)(t) = \sum_{i=0}^n t_i(\gamma a_i) = \sum_{i=0}^n t_i(Aa_i) = A(a(t))$$

and hence

$$\gamma f_a(z) = \gamma\left(\frac{a(t)}{\sqrt{-F(a(t), a(t))}}\right) = \frac{(\gamma a)(t)}{\sqrt{-F((\gamma a)(t), (\gamma a)(t))}} = f_{\gamma a}(z)$$

which proves (3). $\square$

Note that if $a_0, \dots, a_n$ are in general position (i.e. not lying on a common geodesic hyperplane) then f_a maps affine lines to geodesic curves.

3.2. **Hyperbolic and Euclidean circumscribed spheres.** In this section we will use the disc model for hyperbolic n-space $\mathcal{H}^n$. In this model $\mathcal{H}^n = D^n = \{x \mid |x| < 1\} \subseteq \mathbb{R}^n$. For a detailed description see Iversen [7, section II.6]. We start by making a simple observation:

Proposition 3.2.1. *In the disc model for $\mathcal{H}^n$ a hyperbolic sphere is a Euclidean sphere contained in D^n and conversely.*

Proof. By definition a hyperbolic sphere of radius $r > 0$ is the set of points x of distance r from a given center point $c \in \mathcal{H}^n$. By rotational symmetry of the metric it suffices to consider the case $n = 2$. Since isometries of the disc model of $\mathcal{H}^2$ are induced by Möbius transforms which preserves circles we can assume $c = 0$. In this case the statement is obvious by rotational symmetry. Conversely let S be an Euclidean sphere contained in the unit disc. Let p denote the Euclidean center for S, and let L denote the line connecting p and the center of the disc model. L intersects the sphere S at two points, a and b. Let S' denote the hyperbolic sphere with center on L, passing through a and b. By the first part of the proposition S' is an Euclidean sphere, and by symmetry the Euclidean center of S' must lie on L. Since there is only one Euclidean sphere with center on L containing a and b, we conclude that $S' = S$. Thus S is a hyperbolic sphere. □

Corollary 3.2.2. *For $\mathcal{H}^n = D^n$, $n+1$ points in $\mathcal{H}^n$ have a hyperbolic circumscribed sphere if and only if they have an Euclidean circumscribed sphere fully contained in the model.*

Hence we are reduced to finding circumscribed spheres in Euclidean space. Given a non-degenerate affine n-simplex with vertices $v_0, \dots, v_n \in \mathbb{R}^n$ we want to find the circumcenter c and the radius r of the circumscribed sphere. For convenience put $v_0 = 0$. If $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denotes the usual norm and inner product in $\mathbb{R}^n$, then r and c are determined by

$$\begin{aligned} r^2 &= |c - v_0|^2 = |c|^2 = \langle c, c \rangle \\ &= |c - v_i|^2 = \langle c - v_i, c - v_i \rangle = \langle c, c \rangle + \langle v_i, v_i - 2c \rangle, \qquad i = 1, \dots, n \end{aligned}$$

Subtracting, we obtain

$$\langle v_i, c \rangle = \frac{1}{2} \langle v_i, v_i \rangle = \frac{1}{2} |v_i|^2$$

or

$$\left\langle \frac{v_i}{|v_i|}, c \right\rangle = \frac{1}{2} |v_i|.$$

Put $u_i = \frac{v_i}{|v_i|}$ for $i = 1, \dots, n$ and consider the non-singular matrix of row vectors

$$U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Then

$$Uc = \frac{1}{2} \begin{pmatrix} |v_1| \\ \vdots \\ |v_n| \end{pmatrix}.$$

or

$$c = \frac{1}{2}U^{-1}\begin{pmatrix}|v_1|\\ \vdots \\ |v_n|\end{pmatrix}.$$

Hence if we put $A = UU^t = [\langle u_i, u_j\rangle]$ then

$$r^2 = |c|^2 = \frac{1}{4}(|v_1|,\dots,|v_n|)A^{-1}\begin{pmatrix}|v_1|\\ \vdots \\ |v_n|\end{pmatrix}$$

Let $\|\cdot\|$ denote the operator norm on the set of real $n\times n$-matrices. It follows that if $m = \sqrt{n}\max\{|v_1|,\dots,|v_n|\}$ then $r \le \frac{1}{2}m\sqrt{\|A^{-1}\|}$. The sphere will be contained in the unit ball of $\mathbb{R}^n$ if $r < \frac{1}{2}$ hence if $m\sqrt{\|A^{-1}\|} < 1$.

We now consider $v = (v_0,\dots v_n)$ as a hyperbolic simplex in the disc model D^n for $\mathcal{H}^n$ (again $v_0 = 0$). It follows that v has a circumscribed sphere if $m\sqrt{\|A^{-1}\|} < 1$. Let d denote the (hyperbolic) diameter of v. Since $d(0,x) = \log\left(\frac{1+|x|}{1-|x|}\right)$ for $x \in D^n$ we see that $d \ge \log\left(\frac{1+m/\sqrt{n}}{1-m/\sqrt{n}}\right)$. Therefore v has a circumscribed sphere if

$$\frac{e^d-1}{e^d+1}\sqrt{n\|A^{-1}\|} < 1.$$

Notice that $\frac{e^d-1}{e^d+1} \to 0$ for $d \to 0$.

Proposition 3.2.3. *Let $\omega = (\omega_0,\dots,\omega_n)$ denote an affine simplex and $f : \omega \to \mathcal{H}^n$ a differentiable parametrization of a non-degenerate hyperbolic simplex which maps lines to geodesics. Let d denote the hyperbolic diameter of $f(\omega)$ and put*

$$A = \left[\left\langle \frac{T_{\omega_0}f(\omega_i-\omega_0)}{|T_{\omega_0}f(\omega_i-\omega_0)|}, \frac{T_{\omega_0}f(\omega_j-\omega_0)}{|T_{\omega_0}f(\omega_j-\omega_0)|}\right\rangle\right],$$

where T_{ω_0} denotes the differential at ω_0 and the bracket $\langle\cdot,\cdot\rangle$ and the norm $|\cdot|$ denotes the hyperbolic inner product and norm on $T_{\omega_0}\mathcal{H}^n$. Then A is invertible and the simplex has a circumscribed sphere if $\frac{e^d-1}{e^d+1}\sqrt{n\|A^{-1}\|} < 1$.

Notation: We say that A is the matrix associated to f.

Proof. Note that in the disc model the Euclidean and the hyperbolic inner product on $T_{\omega_0}\mathcal{H}^n$ are conformally equivalent (i.e. they differ by multiplication of a smooth function). Therefore when replacing the hyperbolic norm and inner product by Euclidean ones, we get the same associated matrix. In the following we will therefore let $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote Euclidean norm and inner product. There exists an isometry $\gamma : D^n \to D^n$ such that $g = \gamma f$ has $g(\omega_0) = 0$. Let $v = (v_0,\dots,v_n) = (g(\omega_0),\dots,g(\omega_n))$. Since geodesic rays starting at 0 agree with Euclidean lines there exists $l_1,\dots,l_n > 0$ such that $v_i = g(\omega_i) = l_i T_{\omega_0}g(\omega_i-\omega_0)$ for $i = 1,\dots,n$. We put

$$u_i = \frac{v_i}{|v_i|} = \frac{T_{\omega_0}g(\omega_i-\omega_0)}{|T_{\omega_0}g(\omega_i-\omega_0)|}$$

Since γ is an isometry f possesses a circumscribed sphere if and only if g does, and since

$$\left\langle \frac{T_{\omega_0}f(\omega_i-\omega_0)}{|T_{\omega_0}f(\omega_i-\omega_0)|}, \frac{T_{\omega_0}f(\omega_j-\omega_0)}{|T_{\omega_0}f(\omega_j-\omega_0)|}\right\rangle = \left\langle \frac{T_{\omega_0}g(\omega_i-\omega_0)}{|T_{\omega_0}g(\omega_i-\omega_0)|}, \frac{T_{\omega_0}g(\omega_j-\omega_0)}{|T_{\omega_0}g(\omega_j-\omega_0)|}\right\rangle$$

for $i, j = 1, \dots, n$ (by conformal equivalence of the inner products), the result follows from the above discussion. $\square$

Corollary 3.2.4. *Let $f_{(j,m)} : \widetilde{\Delta}_{(j,m)} \to \mathcal{H}^n$, $m \in \mathbb{N}$, $j \in J_m$ denote a family of parametrized hyperbolic simplices as in proposition 3.2.3. Let $A_{(j,m)}$ denote the matrix associated to $f_{(j,m)}$. Suppose that*

1. *$\{\|A_{(j,m)}^{-1}\| \mid m \in \mathbb{N},\ j \in J_m\}$ is bounded*
2. *For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that the diameter of $f_{(j,m)}$ is less than ϵ for $m > N$ and $j \in J_m$.*

Then there exists $M \in \mathbb{N}$ such that $f_{(j,m)}$ possesses a circumscribed sphere for $m > M$.

3.3. **Circumscribed spheres for edgewise subdivision.**

Theorem 3.3.1. *Let $a_0, \dots, a_n \in \mathcal{H}^n$ in general position. There exists $M \in \mathbb{N}$ such that if a denote the geodesic simplex determined by these points, then for $m \geq M$ every simplex in $(\mathrm{Sd}_{\mathcal{H}^n})^m f_a$ possesses a circumscribed sphere.*

Proof. By naturality $(\mathrm{Sd}_{\mathcal{H}^n})^m f_a = S(f_a)(\mathrm{Sd}_{\Delta^n})^m(\Delta^n)$. There is a finite set J_m and affine simplices $g_{(j,m)} : \Delta^n \to \Delta^n$ such that $(\mathrm{Sd}_{\Delta^n})^m(\Delta^n) = \sum_{j \in J_m} g_{(j,m)}$. Let $\widetilde{\Delta}_{(j,m)} = g_{(j,m)}(\Delta^n)$, and let $f_{(j,m)} = f_a|_{\widetilde{\Delta}_{(j,m)}}$. In order to apply the above corollary it suffices to show that

1. $B = \{\|A_{(j,m)}^{-1}\| \mid m \in \mathbb{N}, j \in J_m\}$ is bounded
2. For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that the diameter of $f_{(j,m)}$ is less than ϵ for $m > N$, $j \in J_m$.

We start by (1). Lemma 2.4.2 tells that up to a translation (by a point i Δ^n) and a dilation (by a power of 2) there are at most $n!$ affine simplices $g_{(j,m)}$. In order to see that B is bounded we can without loss of generality assume that all simplices $g_{(j,m)}$ are similar up to translation and dilation. More precisely we can assume that there exists $g : \Delta^n \to \Delta^n$ such that for all (j, m) there exists $x \in \Delta^n$ such that $g_{(j,m)}(z) = x + 2^{-m} g(z)$ for all $z \in \Delta^n$. For $y \in \Delta^n$ we put $v_i(y) = T_y f_a(g(e_i) - g(e_0))$. Let A_y denote the matrix $\left[\left\langle \frac{v_i(y)}{|v_i(y)|}, \frac{v_j(y)}{|v_j(y)|} \right\rangle\right]$. We notice that

$$T_x f_{(j,m)}(g_{(j,m)}(e_i) - g_{(j,m)}(e_0)) = 2^{-m} v_i(x).$$

Therefore A_x is the matrix associated to $f_{(j,m)}$. Since Δ^n is compact and the real function $y \mapsto \|A_y^{-1}\|$ is continuous, the image $\{\|A_y^{-1}\| \mid y \in \Delta^n\}$ is bounded. Therefore also the subset $\{\|A_{(j,m)}^{-1}\| \mid m \in \mathbb{N}, j \in J_m\}$ is bounded. Thus (1) is proved.

Since Δ^n is compact and f_a is continous, (2) follows from the fact that the diameter of $g_{(j,m)}$ tends to 0 for $m \to \infty$. $\square$

4. The Hyperbolic isometry group

4.1. **The complex of generic simplices.** Let $C'_* = C'_*(\mathcal{H}^n)$ denote the chain complex where a generator or simplex in degree k is a $(k+1)$-tuple $\sigma = (v_0, \dots, v_k)$ of points in $\mathcal{H}^n$. The acyclicity (with augmentation $\mathbb{Z}$) of this chain complex is well-known. Let C_* denote the subcomplex of C'_* spanned by the set of all generic simplices. Here a k-simplex $(v_0, \dots, v_k)$ is called *generic* when each of its j-faces with $j \leq \min(n, k)$ spans a hyperbolic subspace of dimension j. It is easy to see that also C_* is acyclic with augmentation $\mathbb{Z}$. The main step in the proof of theorem 4.1.6 is to prove that the chain complex $\mathbb{Z}\otimes_{O^1(1,n)}C_*$ is $(n-1)$-acyclic (lemma 4.1.5).

Using the good parametrization, definition 3.1.2, we can define edgewise subdivision Sd on C'_* as the composition

$$C'_* \xrightarrow[\text{}]{\substack{\text{good}\\ \text{parametrization}}} S_* \xrightarrow{\mathrm{Sd}} S_* \xrightarrow{\text{vertices}} C'_*,$$

where $S_* = S_*(\mathcal{H}^n)$. We shall see that Sd preserves the complex C_* and induces the identity on the homology of $\mathbb{Z}\otimes_{O^1(1,n)} C_*$ (lemma 4.1.2).

Let $a_0,\dots,a_m \in \mathcal{H}^n$. The parametrization $f_a : \Delta^m \to \mathcal{H}^n$ can be viewed as the following composition

$$f_a : \Delta^m \xrightarrow{h} \underline{\Delta}^m \xrightarrow{a} q^{-1}(\mathbb{R}_-)^+ \to \mathcal{H}^n = q^{-1}(-1)^+.$$

Here $q(x) = F(x,x)$ for $x \in \mathbb{R}^{n+1}$ where F is the inner product defined in §3.1. Also h and $\underline{\Delta}^m$ are defined there. Next $q^{-1}(\mathbb{R}_-)^+$ denotes the connected component of $q^{-1}(\mathbb{R}_-)$ containing $(1,0,\dots,0)$ and the map a is given by $a(t_0,\dots,t_m) = \sum_{i=0}^m t_i a_i$. The last map is given by $y \mapsto y/\sqrt{-q(y)}$. Since $q^{-1}(\mathbb{R}_-)^+ \subseteq \mathbb{R}^{n+1}$ is an open subset, there exists $\epsilon > 0$ such that the map $\widetilde{a}_\epsilon : I \times \underline{\Delta}^m \to \mathbb{R}^{n+1}$ given by $\widetilde{a}_\epsilon(s,x) = a(x) + \epsilon s(0,\dots,0,1)$ maps into $q^{-1}(\mathbb{R}_-)^+$. We can now define $g_a^\epsilon : I \times \Delta^m \to \mathcal{H}^n$ as the composition

$$I \times \Delta^m \xrightarrow{I\times h} I \times \underline{\Delta}^m \xrightarrow{\widetilde{a}_\epsilon} q^{-1}(\mathbb{R}_-)^+ \longrightarrow \mathcal{H}^n.$$

where the last map is $y \mapsto y/\sqrt{-q(y)}$ as above. With this definition $g_a^\epsilon(0,x) = f_a(x)$, and for a given a, g_a^ϵ is defined for some $\epsilon > 0$.

Lemma 4.1.1. *Let $a_0,\dots,a_m \in \mathcal{H}^{n-1} \subseteq \mathcal{H}^n$. If $a = (a_0,\dots,a_m) \in C_m$ and if ϵ is such that g_a^ϵ is defined then*

1. *For any isometry $\gamma : \mathcal{H}^n \to \mathcal{H}^n$, keeping $\mathcal{H}^{n-1}$ invariant, $g_{\gamma a}^\epsilon = \gamma g_a^\epsilon$.*
2. *For any convex subset K of $I \times \Delta^m \subseteq \mathbb{R}^{m+1}$, the image $g_a^\epsilon(K)$ is a geodesically convex subspace of $\mathcal{H}^{n+1}$ of the same geodesical dimension as K.*

Proof. (1) Since $(0,\dots,0,1)$ is fixed by γ, this is exactly the same calculation as in lemma 3.1.1.

(2) As in lemma 3.1.1 it is clear that g_a^ϵ sends lines to geodesic curves. Since no line is sent to a point, it must preserve the dimension. □

Observe that $g_{d_i a} = g_a(I \times \partial_i) : I \times \Delta^{m-1} \to \mathcal{H}^n$ for $i = 0,\dots,m$.

Lemma 4.1.2. $\mathrm{Sd}\, x - x$ *is a boundary for any cycle $x \in \mathbb{Z}\otimes_{O^1(1,n)} C_m$, $m \le n-1$.*

Proof. The idea behind this somewhat technical proof is to observe that the subdivisions of the prism $I \times \Delta^m$ introduced by $\widetilde{H}$ from definition 2.3.1 and EZ agree at one end, and at the other end $\mathrm{Sd}\, x$ respectively x are represented. Since x is represented by a finite formal sum of generators $a = (a_0,\dots,a_m) \in C_m$, there exists $\epsilon > 0$ such that $g_a = g_a^\epsilon$ is defined and parametrizes the prism for each such a appearing in x. Let $\mathrm{EZ} : S(I) \otimes S(\Delta^m) \to S(I \times \Delta^m)$ be the Eilenberg-Zilber map. For any a we can form the chain

$$Ga = S(g_a)(\mathrm{EZ}(I \otimes \Delta^m) + \widetilde{H}\Delta^m).$$

We find that

$$\begin{aligned}
Gda &= \sum_{i=0}^{m}(-1)^i G d_i a \\
&= \sum_{i=0}^{m}(-1)^i S(g_{d_i a})(\mathrm{EZ}(I \otimes \Delta^{m-1}) + \widetilde{H}(\Delta^{m-1})) \\
&= \sum_{i=0}^{m}(-1)^i S(g_a(I \times \partial_i))(\mathrm{EZ}(I \otimes \Delta^{m-1}) + \widetilde{H}(\Delta^{m-1})) \\
&= S(g_a)(\mathrm{EZ}(I \otimes d\Delta^m) + \widetilde{H} d\Delta^m)
\end{aligned}$$

and

$$\begin{aligned}
dGa =& dS(g_a)(\mathrm{EZ}(I \otimes \Delta^m) + \widetilde{H}\Delta^m) \\
&= S(g_a)(\mathrm{EZ}(d(I \otimes \Delta^m)) + d\widetilde{H}\Delta^m) \\
&= S(g_a)(\mathrm{EZ}(1 \otimes \Delta^m - 0 \otimes \Delta^m - I \otimes d\Delta^m) + d\widetilde{H}\Delta^m) \\
&= S(g_a)(1 \times \Delta^m - 0 \times \Delta^m - \mathrm{EZ}(I \otimes d\Delta^m) + d\widetilde{H}\Delta^m)
\end{aligned}$$

Hence

$$\begin{aligned}
dGa + Gda &= S(g_a)(1 \times \Delta^m - 0 \times \Delta^m + d\widetilde{H}\Delta^m + \widetilde{H} d\Delta^m) \\
&= S(g_a)(1 \times \Delta^m - 0 \times \Delta^m + 0 \times \mathrm{Sd}\,\Delta^m - 1 \times \Delta^m) \\
&= S(g_a)(0 \times \mathrm{Sd}\,\Delta^m - 0 \times \Delta^m) \\
&= \mathrm{Sd}\, a - a
\end{aligned} \tag{4.1.3}$$

By composing with the vertex map from S_* to C'_* we can read (4.1.3) as an equation in C'_m. Since $\mathrm{EZ}(I \otimes \Delta^m) + \widetilde{H}\Delta^m$ is the formal sum of convex affine simplices in $I \times \Delta^m$, the previous lemma implies that each simplex occuring in Ga is generic, so that $Ga \in C_{m+1}$. Since $x \in \mathbb{Z} \otimes_{O^1(1,n)} C_m$ we can represent x by generators $a \subset \mathcal{H}^{n-1} \subseteq \mathcal{H}^n$ such that the generators of dx cancel by using isometries fixing $\mathcal{H}^{n-1}$ (this requires $m \leq n-1$). The previous lemma now implies that $Gdx = 0$. The proof is complete. ☐

In order to prove theorem 4.1.6 we consider a spectral sequence associated to the action of $O^1(1,n)$ on $C_*(n)$. Put $G = O^1(1,n)$, and let F_* be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$. Let us consider the two spectral sequences associated to the double complex $F_j \otimes_{\mathbb{Z}[G]} C_i$. Using that C_* is acyclic the E^2-term of one of these is

$$``E^2_{p,q} = \begin{cases} 0 & q \neq 0 \\ H_p(G) & q = 0 \end{cases}$$

The E^1-term of the other spectral sequence is given by the chain complexes

$$`E^1_{*,q} = H_q(G, C_*).$$

Observe that since $C_0 = \mathbb{Z}[G/O(n)]$ we get $H_q(G, C_0) \cong H_q(O(n))$ by Shapiro's lemma.

Lemma 4.1.4. *The complex* $`E^1_{*,q}$ *is* $(n-q-1)$*-acyclic with augmentation* $`E^1_{0,q} = H_q(O(n))$.

The proof of this lemma begins by the special case when $q = 0$:

Lemma 4.1.5. *$'E^1_{*,0} = \mathbb{Z} \otimes_{\mathbb{Z}[G]} C_*$ is $(n-1)$-acyclic with augmentation $\mathbb{Z}$.*

Proof. Any cycle of $\mathbb{Z} \otimes_{\mathbb{Z}[G]} C_*$ is made up of a finite number of generic simplices. For $j \leq n-1$ lemma 4.1.2 and theorem 3.3.1 shows that we can assume without loss of generality that each such j-simplex has a unique circumscribed $(j-1)$-sphere with a positive radius r and circumcenter p in the hyperbolic j-subspace spanned by its vertices in $\mathcal{H}^n$. By moving p in the direction perpendicular to this j-subspace the radius is increased continously. Since G is transitive on $\mathcal{H}^n$, we may move the j-simplices appearing in our j-cycle c until all of them are circumscribed by a common j-sphere of radius strictly larger than the finite number of radii associated to the j-simplices appearing in c. If we let p denote the center of this circumscribed sphere, then we form the $(j+1)$-chain $p * c$. Since p is not on the hyperbolic j-subspace spanned by the vertices of any of the j-simplices appearing in c, $p * c$ is a generic $(j+1)$-chain. We can now compute the boundary $\partial(p * c) = c - p * \partial c$. Since c is a j-cycle, the $(j-1)$-simplices appearing in ∂c must cancel in pairs by using the action of G and since p has the same distance to all vertices the same must be true for $p * \partial c$. In other words, c is the boundary of $p * c$. □

Proof of lemma 4.1.4. Let $q \leq n$. Then the isotropy group for a generic q-simplex in $\mathcal{H}^n$ is $O(n-q)$, and hence there is a canonical isomorphism of $\mathbb{Z}[G]$-modules

$$C_q \cong \bigoplus_a \mathbb{Z}[G/O(n-q)]a$$

where a runs through a set of representatives for the orbits in the set of all these simplices. It then follows from Shapiro's lemma that

$$H_i(G, C_q) \cong \bigoplus_a H_i(G, \mathbb{Z}[G/O(n-q)]) \cong H_0(G, C_q) \otimes H_i(O(n-q)).$$

By using the stability theorem of Sah [11, theorem 3.8], mentioned in the introduction, we find that for $q < n$ the complex $'E^1_{*,q}$ has constant coefficients $H_q(O(n))$ in dimensions less than $n - q$. We can therefore apply lemma 4.1.5 to finish the proof of lemma 4.1.4 □

We are now ready to prove our main result:

Theorem 4.1.6. *The inclusion $O(n) \subseteq O^1(1,n)$ induces an isomorphism $H_k(O(n)) \to H_k(O^1(1,n))$ for $k \leq n-1$.*

Proof. From lemma 4.1.4 follows that the map

$$H_q(O(n)) \to H_q(O^1(1,n))$$

is an isomorphism for $q \leq n-2$ and that it is onto for $q = n-1$. To complete the proof we can apply the stability theorem of Sah in the commutative diagram

$$\begin{array}{ccc} H_{n-1}(O(n)) & \longrightarrow & H_{n-1}(O^1(1,n)) \\ \downarrow \cong & & \downarrow \cong \\ H_{n-1}(O(n+1)) & \xrightarrow{\cong} & H_{n-1}(O^1(1,n+1)). \end{array}$$

□

Remark. In low dimensions ($k = 1, 2$) theorem 4.1.6 is contained in [4, theorem 4.4], where it was furthermore shown that the map $H_3(O(3)) \to H_3(O^1(1,3))$ is surjective. Combining this with the injectivity of $H_3(O(3)) \to H_3(O(4))$ (see [4, theorem 4.4 (a)]) and the above theorem 4.1.6 (for n = 4, k = 3) we conclude from the diagram

$$\begin{tikzcd} H_3(O(3)) \arrow[r] \arrow[d] & H_3(O^1(1,3)) \arrow[d] \\ H_3(O(4)) \arrow[r, "\cong"] & H_3(O^1(1,4)) \end{tikzcd}$$

that the following holds:

Corollary 4.1.7. *The inclusion $O(3) \subseteq O^1(1,3)$ induces an isomorphism $H_k(O(3)) \to H_k(O^1(1,3))$ for $k \leq 3$.*

Remark. This is in contrast with the case $n = 2$ where the map $H_2(O(2)) \to H_2(O^1(1,2))$ is *not* injective. In fact the kernel turns out to be the image of the spherical Dehn-invariant in $\Lambda^2_{\mathbb{Z}}(\mathbb{R}/\mathbb{Z}) = H_2(O(2))$. (cf. [4, theorem 5.10] or section 5.2 below).

5. Applications

5.1. **More homology isomorphisms.** For the "stable" groups $O(\infty) = \lim_n O(n)$, $O^1(1,\infty) = \lim_n O^1(1,n)$ and the subgroups $\mathrm{SO}(\infty) \subseteq O(\infty)$ and $\mathrm{SO}^1(1,\infty) \subseteq O^1(1,\infty)$ of elements with determinants $+1$, we now obtain from theorem 4.1.6:

Corollary 5.1.1. *The inclusions $O(\infty) \subseteq O(1,\infty)$ and $\mathrm{SO}(\infty) \subseteq \mathrm{SO}^1(1,\infty)$ induce isomorphisms*

$$H_*(O(\infty)) \cong H_*(O(1,\infty)), \qquad H_*(\mathrm{SO}(\infty)) \cong H_*(\mathrm{SO}^1(1,\infty))$$

Proof. The first isomorphism is obvious from the theorem. The second follows from this and the comparison theorem for the spectral sequence associated to the extensions

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SO}(\infty) \arrow[r] \arrow[d] & O(\infty) \arrow[r, "\det"] \arrow[d] & \{\pm 1\} \arrow[r] \arrow[d, "="'] & 1 \\ 1 \arrow[r] & \mathrm{SO}^1(1,\infty) \arrow[r] & O^1(1,\infty) \arrow[r, "\det"] & \{\pm 1\} \arrow[r] & 1. \end{tikzcd}$$

For this we need that the action by the quotient group is trivial on both $H_*(\mathrm{SO}(\infty))$ and $H_*(\mathrm{SO}^1(1,\infty))$. In fact in the first case the action on the image of $H_*(\mathrm{SO}(n))$ is induced by inner conjugation by the reflection multiplying by (-1) on the $(n+1)$st coordinate. The other case is similar. □

In the unstable case the action by the quotient group $\{1,\tau\} = O^1(1,n)/\mathrm{SO}^1(1,n)$ on $H_*(\mathrm{SO}^1(1,n))$ may very well be non-trivial, and we let $H_*(\mathrm{SO}^1(1,n))^+$ denote the invariant part. For n *odd* $O(n) = \mathrm{SO}(n) \times \{\pm I_n\}$, and since the action by τ on $\mathrm{SO}(1,n)$ is induced by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix}$$

it follows that the inclusion $i : \mathrm{SO}(n) \subseteq \mathrm{SO}^1(1,n)$ maps $H_*(\mathrm{SO}(n))$ into $H_*(\mathrm{SO}^1(1,n))^+$. We now have:

Corollary 5.1.2. *For n odd and $k \leq n-1$ or $k = n = 3$ the inclusion $i : \mathrm{SO}(n) \subseteq \mathrm{SO}^1(1,n)$ gives a split exact sequence*

$$0 \longrightarrow H_k(\mathrm{SO}(n)) \underset{\longleftarrow}{\xrightarrow{i_*}} H_k(\mathrm{SO}^1(1,n))^+ \longrightarrow A_k \longrightarrow 0$$

where A_k has exponent 2.

Proof. The diagram of inclusion maps

$$\begin{array}{ccc} \mathrm{SO}(n) & \xrightarrow{i} & \mathrm{SO}^1(1,n) \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ O(n) & \xrightarrow{i'} & O^1(1,n) \end{array}$$

gives rise to the following commutative diagram

$$\begin{array}{ccc} H_k(\mathrm{SO}(n)) & \xrightarrow{i_*} & H_k(\mathrm{SO}^1(1,n))^+ \\ {\scriptstyle \mathrm{Tr}}\uparrow\ \downarrow{\scriptstyle j'_*} & & {\scriptstyle \mathrm{Tr}}\uparrow\ \downarrow{\scriptstyle j_*} \\ H_k(O(n)) & \underset{\cong}{\xrightarrow{i'_*}} & H_k(O^1(1,n)) \end{array}$$

where Tr denotes the transfer map (cf. Eckmann [6]) and i'_* is an isomorphism by theorem 4.1.6. Since $O(n) = \mathrm{SO}(n) \times \{\pm I_n\}$ it follows that j'_* on the left maps $H_k(\mathrm{SO}(n))$ injectively onto a direct summand in $H_k(O(n))$ and hence, by the diagram, i_* in the top row maps $H_k(\mathrm{SO}(n))$ injectively onto a direct summand of $H_k(\mathrm{SO}^1(1,n))^+$. It remains to show that the cokernel of i_* has exponent 2. This however follows from the identity $\mathrm{Tr} \circ j_*(x) = x + \tau_* x = 2x$, for $x \in H_k(\mathrm{SO}^1(1,n))^+$, since by the diagram $\mathrm{im}\,\mathrm{Tr} \subseteq \mathrm{im}\, i_*$. □

We next want to lift this result to the double covering groups $\mathrm{Spin}(n) \subseteq \mathrm{Spin}^1(1,n)$. Notice that the isometry $(x_0, x_1, \ldots, x_n) \mapsto (x_0, -x_1, \ldots, -x_n)$ induces an involutive automorphism $\widetilde{\tau}$ of $\mathrm{Spin}^1(1,n)$ which restricts to the identity on $\mathrm{Spin}(n)$. (See e.g. Lawson-Michelsohn [8, chapter 1] for the definitions of Spin-groups.) We shall now prove

Corollary 5.1.3. *For n odd and $k \leq n-1$ or $k = n = 3$ the inclusion $\widetilde{i} : \mathrm{Spin}(n) \subseteq \mathrm{Spin}^1(1,n)$ gives a split exact sequence*

$$0 \longrightarrow H_k(\mathrm{Spin}(n)) \underset{\longleftarrow}{\xrightarrow{\widetilde{i}_*}} H_k(\mathrm{Spin}^1(1,n))^+ \longrightarrow B_k \longrightarrow 0$$

where the superscript $+$ indicates the set of elements invariant under $\widetilde{\tau}_*$ and B_k is a group of exponent 2.

Proof. Let $C_2 = \{1, \widetilde{\tau}\}$ and let $\mathrm{Pin}^*(n) = \mathrm{Spin}(n) \times C_2$ whereas $\mathrm{Pin}^*(1,n)$ denotes the semidirect product of $\mathrm{Spin}^1(1,n)$ and C_2 using the above action of $\widetilde{\tau}$. (These groups agree with the usual Pin-groups for $n \equiv 3 \mod 4$, but are the "other" Pin-groups for $n \equiv 1 \mod 4$.) We then have the following exact sequences of groups

$$\begin{tikzcd}
1 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, "="] & \mathrm{Pin}^*(n) \arrow[r] \arrow[d, "\widetilde{i}'"] & O(n) \arrow[r] \arrow[d, "i'"] & 1 \\
1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{Pin}^*(1,n) \arrow[r] & O^1(1,n) \arrow[r] & 1.
\end{tikzcd}$$

By the comparison theorem applied to the associated spectral sequences we deduce from theorem 4.1.6 that $\widetilde{i}'$ induces isomorphisms

$$H_k(\mathrm{Pin}^*(n)) \to H_k(\mathrm{Pin}^*(1,n)), \qquad \text{for } k \leq n-1 \text{ or } k = n = 3.$$

We can now finish the proof by arguments similar to the proof of corollary 5.1.2. □

Let us exemplify this result for $n = 3$ and 5. Thus there are well-known isomorphisms (cf. [4, page 226])

$$\mathrm{Spin}^1(1,3) \cong \mathrm{Sl}(2,\mathbb{C}); \qquad \mathrm{Spin}^1(1,5) \cong \mathrm{Sl}(2,\mathbb{H})$$

and the involution $\widetilde{\tau}$ is in these terms given by

$$\widetilde{\tau}(g) = (g^*)^{-1}, \qquad g \in \mathrm{Sl}(2,\mathbb{F}), \quad \mathbb{F} = \mathbb{C}, \mathbb{H},$$

where g^* denotes the conjugate transposed of the matrix g. The fixed points of this involution is of course $\mathrm{SU}(2)$ respectively $\mathrm{Sp}(2)$. Hence we conclude using the results of Sah [13, lemma 1.22 and prop. 1.12], [12, prop. 4.2 and thm. 4.1 a)] and [4, thm. 4.11]:

Corollary 5.1.4. *a) The natural inclusion* $\mathrm{SU}(2) \subset \mathrm{Sl}(2,\mathbb{C})$ *induces an isomorphism*

$$H_k(\mathrm{SU}(2)) \cong H_k(\mathrm{Sl}(2,\mathbb{C}))^+, \qquad k \leq 3.$$

b) The natural inclusion $\mathrm{Sp}(2) \subset \mathrm{Sl}(2,\mathbb{H})$ *gives a direct decomposition*

$$H_k(\mathrm{Sl}(2,\mathbb{H}))^+ \cong H_k(\mathrm{Sp}(2)) \oplus B_k \qquad k \leq 4$$

where $B_k = 0$ *for* $k \leq 3$ *and is a group of exponent* 2 *for* $k = 4$.

Remark. In the complex case the involution $\widetilde{\tau}$ is conjugate (using the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$) to the involution $g \mapsto \overline{g}$ given by complex conjugation. It follows that these induce the same involution on homology and hence $H_k(\mathrm{Sl}(2,\mathbb{C}))^+$ is also the invariant part by complex conjugation, as stated in corollary 1.0.1.

5.2. **Scissors congruence.** Finally we briefly indicate the implications for the Scissors Congruence Problem for polyhedra in spherical 3-space. We refer to [4, section 5] for a more detailed account. Following the notation there we let $\mathcal{P}(S^3)$ denote the scissors congruence group for the 3-sphere and we recall the exact sequence

$$0 \to H_3(\mathrm{SU}(2)) \xrightarrow{\sigma} \mathcal{P}(S^3)/\mathbb{Z} \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\mathbb{Z}) \to H_2(\mathrm{SU}(2)) \to 0. \tag{5.2.1}$$

Here $\mathbb{Z} \subseteq \mathcal{P}(S^3)$ is generated by $[S^3]$ and D is the Dehn-invariant whereas σ is defined using the natural action of $\mathrm{SU}(2)$ on S^3. By corollary 5.1.4 and by [4, thm. 5.10] and [12, thm. 4.1] we therefore conclude

Corollary 5.2.2. *There is an exact sequence*

$$0 \to K_3^{\mathrm{ind}}(\mathbb{C})^+ \xrightarrow{\sigma} \mathcal{P}(S^3)/\mathbb{Z} \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\mathbb{Z}) \to K_2(\mathbb{C})^+ \to 0$$

In this sequence $K_3^{\text{ind}}(\mathbb{C})$ is a direct summand in $K_3(\mathbb{C})$, in fact, $K_3(\mathbb{C}) = K_3^{\text{ind}}(\mathbb{C}) \oplus K_3^M(\mathbb{C})$ where $K_3^M(\mathbb{C})$ denotes the decomposable part (Milnor's K-theory). As usual the superscript $+$ indicates the fixed points for complex conjugation. From this sequence we now easily prove the following result mentioned in the introduction:

Corollary 5.2.3. *The scissors congruence group $\mathcal{P}(S^3)$ is a rational vector space.*

Proof. It is well-known that $K_2(\mathbb{C})$ is a $\mathbb{Q}$-vector space and also by results of Suslin (cf. Sah [12]), $K_3(\mathbb{C})$ is the direct sum of a $\mathbb{Q}$-vector space and a copy of $\mathbb{Q}/\mathbb{Z}$. By the exact sequence in corollary 5.2.2 the same statement is true for $\mathcal{P}(S^3)/\mathbb{Z}$. Furthermore it is easy to check that the element in $K_3(\mathbb{C})$ corresponding to $p/q \in \mathbb{Q}/\mathbb{Z}$ via the map σ is represented in $\mathcal{P}(S^3)$ by the double suspension of an arc in S^1 of length $(p/q)\cdot 2\pi$. Using the volume function $\mathrm{Vol}_{S^3} : \mathcal{P}(S^3) \to \mathbb{R}$ it now follows that these rational "lunes" generate a copy of $\mathbb{Q}$ inside $\mathcal{P}(S^3)$ which ends the proof. $\square$

In the case of the scissors congruence group $\mathcal{P}(\mathcal{H}^3)$ for the hyperbolic 3-space there is the following exact sequence analogous to corollary 5.2.2 (cf. [4, section 5] and [12, thm. 4.1]:

$$0 \to K_3^{\text{ind}}(\mathbb{C})^- \xrightarrow{\sigma} \mathcal{P}(\mathcal{H}^3) \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\mathbb{Z}) \to K_2(\mathbb{C})^- \to 0 \tag{5.2.4}$$

where again D denotes the hyperbolic Dehn-invariant and the superscript indicates the negative eigenspace for complex conjugation. In either geometry (hyperbolic or spherical) the Extended Hilbert's 3rd Problem is to determine when 2 polyhedra are scissors congruent, and thus the two exact sequences 5.2.2 and (5.2.4) reduce this problem to a matter of detecting elements in $K_3^{\text{ind}}(\mathbb{C})$. That is, suppose P_1, P_2 are two such polyhedra; then of course the first condition for scissors congruence is $D(P_1) = D(P_2)$ and granted this we obtain a unique "difference element"

$$d(P_1, P_2) \in K_3^{\text{ind}}(\mathbb{C})^{\pm}$$

such that

$$\sigma(d(P_1,P_2)) = [P_1] - [P_2] \qquad \text{in } \mathcal{P}(S^3)/\mathbb{Z} \text{ resp. } \mathcal{P}(\mathcal{H}^3).$$

To determine elements in $K_3(\mathbb{C})$ there is the 2nd Cheeger-Chern-Simons invariant $\widehat{C}$ with values in $\mathbb{C}/\mathbb{Z}$ (cf. [3]). In terms of the above exact sequences this is given by

$$\widehat{C}(z) = \frac{1}{4\pi^2}\left(\mathrm{Vol}_{S^3} \circ \sigma(z_+) + i\,\mathrm{Vol}_{\mathcal{H}^3} \circ \sigma(z_-)\right)$$

where $z = z_+ + z_-$ in $K_3^{\text{ind}}(\mathbb{C})$ is the decomposition with respect to complex conjugation. Furthermore if $\alpha \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ is a field automorphism we get another invariant by

$$\widehat{C}_\alpha(z) = \widehat{C}(\alpha_*(z)) \qquad z \in K_3^{\text{ind}}(\mathbb{C}).$$

Conjecturally all these invariants suffice to determine elements of $K_3^{\text{ind}}(\mathbb{C})$. The analogous situation is completely clarified if $\mathbb{C}$ is replaced by $\overline{\mathbb{Q}}$ the field of algebraic numbers. Notice that the natural map $K_3(\overline{\mathbb{Q}}) \to K_3(\mathbb{C})$ is injective (by Hilbert's Nullstellensatz) so in particular $K_3(\overline{\mathbb{Q}})/(\mathbb{Q}/\mathbb{Z})$ is torsion free. It now follows from Borel's theorem [2] that elements in $K_3^{\text{ind}}(\overline{\mathbb{Q}})/(\mathbb{Q}/\mathbb{Z})$ are determined by the Borel *regulators*, i.e. by the real valued invariants r_α on $K_3^{\mathrm{ind}}(\overline{\mathbb{Q}})$ defined for each $\alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by

$$r_\alpha(z) = \mathrm{im}(\widehat{C}(\alpha_*(z)) = (1/4\pi^2)\,\mathrm{Vol}_{\mathcal{H}^3} \circ\sigma(\alpha_*(z)_-), \qquad z \in K_3^{\mathrm{ind}}(\overline{\mathbb{Q}}).$$

Hence we conclude:

Corollary 5.2.5. *Let P_1 and P_2 be two spherical or two hyperbolic polyhedra with all vertices defined over $\overline{\mathbb{Q}}$. Then P_1 and P_2 are scissors congruent iff*
(i) $D(P_1) = D(P_2)$
(ii) $\mathrm{Vol}(P_1) = \mathrm{Vol}(P_2)$
(iii) $r_\alpha(d(P_1, P_2)) = 0 \qquad \forall\alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\ \alpha \neq \mathrm{id}.$

Remark. We have excluded $\alpha = \mathrm{id}$ in (iii) since it is listed separately in (ii). It is however curious to note that in the spherical case r_α are all defined using the *hyperbolic* volume function. The spherical volume function is only used to distinguish rational multiples of $[S^3]$.

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TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE INTEGERS MODULO THE SQUARE OF AN ODD PRIME

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ABSTRACT. This paper gives a computation of topological Hochschild homology for the integers modulo the square of an odd prime. The most important idea in the computation is to consider $\mathbb{Z}/p^2$ as a filtered ring, and to construct a spectral sequence for topological Hochschild homology of a filtered ring. The product of topological Hochschild homology is used extensively, and a direct description of this product is given.

1. INTRODUCTION

Topological Hochschild homology (THH) was introduced by M. Bökstedt in [1]. B.I. Dundas and R. McCarty have proven that THH and stable K-theory in the sense of F. Waldhausen are isomorphic. In this paper we shall give a calculation of $\mathrm{THH}(\mathbb{Z}/p^2)$.

The main result of this paper is the following.

Theorem (7.1.1). *Let p be an odd prime. For $i \geq 0$ there is an isomorphism*

$$\pi_i \,\mathrm{THH}(\mathbb{Z}/p^2) \cong \bigoplus_{k\geq 0} \pi_{i-2k} \,\mathrm{THH}(\mathbb{Z}, \mathbb{Z}/p^2).$$

Bökstedt has computed $\mathrm{THH}(\mathbb{Z})$, and from his result follows that $\pi_0 \,\mathrm{THH}(\mathbb{Z}, \mathbb{Z}/p^2) \cong \mathbb{Z}/p^2$, and that for $k \geq 1$

$$\pi_{2k} \,\mathrm{THH}(\mathbb{Z}, \mathbb{Z}/p^2) \cong \pi_{2k-1} \,\mathrm{THH}(\mathbb{Z}, \mathbb{Z}/p^2) \cong \mathbb{Z}/(k, p^2)$$

where (k, p^2) denotes the greatest common divisor of p^2 and k.

This result can be regarded as a step towards a computation of algebraic K-theory for $\mathbb{Z}/p^2$. Bökstedt, Hsiang and Madsen defined topological cyclic homology (TC) in [3] as a refinement of THH by use of the action of the circle group on THH, and on its deloopings. For a ring R, $\mathrm{TC}(R)$ is a (-1)-connected spectrum. Hesselholt and Madsen have proven the following theorem.

Theorem ([6]). *Suppose that R is a finite algebra over the Witt vectors of a perfect field of characteristic $p > 0$. Then there is a weak equivalence of p-adic completions*

$$K(R)^\wedge_p \simeq \mathrm{TC}(R)^\wedge_p[0, \infty),$$

where $\mathrm{TC}(R)^\wedge_p[0, \infty)$ denotes the connective cover of the p-adic completion of $\mathrm{TC}(R)$.

This theorem in particular applies in the situation where $R = \mathbb{Z}/p^2$.

One idea in the proof of theorem 7.1.1 is to consider $\mathbb{Z}/p^2$ as a filtered ring $0 \subseteq p\mathbb{Z}/p^2 \subseteq \mathbb{Z}/p^2$ and to construct a filtration of $\mathrm{THH}(\mathbb{Z}/p^2)$ induced from this filtration.

Another ingredient is the multiplicative structure of THH of a commutative ring. In fact we shall construct a model for THH with the property that THH of a commutative ring is a simplicial ring. With this model of THH at hand we can construct the following spectral sequences of graded rings:

Theorem (6.2.5). *Let R be a commutative ring with no elements of finite order and let A be an algebra over R. There is a spectral sequence of graded algebras over R with E^2-terms*

$$E^2_{s,t} = \pi_s \operatorname{HH}(R, \pi_t \operatorname{THH}(\mathbb{Z}, A))$$

converging towards $\pi_{s+t} \operatorname{THH}(R, A)$.

Theorem (6.2.7). *Let R be a commutative ring and let S be a commutative algebra over R. There is a spectral sequence of graded rings with E^2-terms*

$$E^2_{s,t} = \pi_s \operatorname{THH}(S, \operatorname{Tor}^R_t(S, S))$$

converging towards $\pi_{s+t} \operatorname{THH}(R, S)$.

Additively theorem 6.2.7 was proven by Pirashvili and Waldhausen in [9]. The multiplicative structure of these spectral sequences may be usefull for other computations. In fact the multiplicative structure of the spectral sequence in theorem 6.2.7 goes into the computation of THH for the ring of integers in a finite extension of the field of rational numbers by Lindenstrauss and Madsen [7].

Theorem ([7]). *Let K be a finite extension of the field of rational numbers, and A its ring of integers. The only non-zero homotopy groups of $\operatorname{THH}(A)$ are*

$$\pi_0 \operatorname{THH}(A) = A, \qquad \pi_{2n-1} \operatorname{THH}(A) = A/n\mathcal{D}_A,$$

where $\mathcal{D}_A = \mathcal{D}_{A/\mathbb{Z}}$ denotes the different ideal of K over $\mathbb{Q}$.

One way to describe $\mathcal{D}_A$ is by the isomorphism

$$A/\mathcal{D}_A \cong \pi_1 \operatorname{THH}(A) \cong \operatorname{HH}_1(A),$$

where $\operatorname{HH}_1(A)$ denotes the first Hochschild homology group of A.

Acknowledgements . This paper is deeply influenced by ideas of other people. First of all it was L. Hesselholt who suggested to consider $\mathbb{Z}/p^2$ as a filtered ring. The structure of THH as an abelian group was discovered by B. I. Dundas and R. McCarthy in [5] and this presentation is very much inspired by that paper. Not least M. Bökstedt has influenced my work with many helpful ideas. Also I. Madsen, C. Schlichtkrull, T. Pirashvili and S. Schwede have helped me with this work.

2. Preliminaries

2.1. **Simplicial objects.** Let Δ be the category of standard ordered finite sets $[n] = \{0 < 1 < \cdots < n\}$ and monotone maps. A simplicial object in a category $\mathcal{C}$ is a functor from Δ^o to $\mathcal{C}$. The category of simplicial objects in $\mathcal{C}$ will be denoted $s\mathcal{C}$. We will denote the category of pointed simplicial sets by s.Set$_*$. $\Delta(n)$ denotes the pointed simplicial set $\{q \mapsto \Delta([q], [n])\}$.

2.2. **Simplicial categories.** A simplicial category is an object of the functor category from Δ^o to some category of categories. A simplicial functor is a natural transformation between two simplicial categories. If a pointed simplicial category $\mathcal{C}$ has a representation of the functor $\mathrm{s.Set}_*(X, \mathcal{C}(c,-))$, say $\mathcal{C}(X \otimes c, -) \cong \mathrm{s.Set}_*(X, \mathcal{C}(c,-))$ where $c \in \mathcal{C}$ and $X \in \mathrm{s.Set}_*$, we will say that $\mathcal{C}$ has products with pointed simplicial sets.

Suppose that the pointed simplicial categories $\mathcal{C}$ and $\mathcal{D}$ have products with pointed simplicial sets. Then any pointed simplicial functor $F : \mathcal{C} \to \mathcal{D}$ has associated to it a natural transformation $\lambda_{X,c} : X \otimes F(c) \to F(X \otimes c)$ induced by the identity via

$$\begin{array}{ccc} \mathrm{id} \in \mathcal{C}(X \otimes c, X \otimes c) & \xrightarrow{\cong} & \mathrm{s.Set}_*(X, \mathcal{C}(c, X \otimes c)) \\ & & \downarrow \\ \lambda_{X,c} \in \mathcal{D}(X \otimes F(c), F(X \otimes c)) & \xleftarrow{\cong} & \mathrm{s.Set}_*(X, \mathcal{D}(F(c), F(X \otimes c)). \end{array}$$

In this paper we shall only consider the case where $\mathcal{C} = \mathrm{s.Set}_*$ and $\mathcal{D}$ is either $\underline{\mathrm{s.Ab}}$ or $\underline{\mathrm{s.Set}}_*$.

2.3. **Simplicial sets.** The category of pointed simplicial sets is the category $\underline{\mathrm{s.Set}}_*$ with $\underline{\mathrm{s.Set}}_*([n]) = \mathrm{s.Set}_*$ and $\underline{\mathrm{s.Set}}_*(X,Y)([n]) = \mathrm{s.Set}_*(X \wedge \Delta(n)_+, Y)$. Clearly $\underline{\mathrm{s.Set}}_*$ has products with simplicial sets. Let $F(X,Y) = \underline{\mathrm{s.Set}}_*(X, \mathrm{Sin}\,|Y|)$, where $\mathrm{Sin}\,|Y|$ denotes the singular complex on the geometric realization of Y. There are adjunction maps

$$F(X, F(Y,Z)) \rightleftarrows F(X \wedge Y, Z).$$

These maps are not isomorphisms but they are homotopy inverse to each other.

We shall say that a pointed simplicial set X is n-connected if $\pi_q(X) = 0$ for $q \leq n$, and if $n \geq 0$ that a map $f : X \to Y$ is n-connected if the homotopy fibre is $(n-1)$-connected and $\pi_0(X) \to \pi_0(Y)$ is surjective. As models for spheres we choose $S^1 = \Delta(1)/\partial\Delta(1)$ and

$$S^U = \underline{\mathrm{s.Set}}_*(U_+, S^1)/\{\alpha \mid * \in \alpha(U)\}$$

for a finite set U. We shall say that a map $X \to Y$ of pointed simplicial sets is a weak equivalence if it induces isomorphism on all homotopy groups. We shall say that two pointed simplicial sets X and Y are weakly equivalent, $X \simeq Y$, if there is a sequence of weak equivalences connecting X and Y. (See [5] for further details.)

2.4. **Simplicial abelian groups.** Let s.Ab denote the category of simplicial objects in the category of abelian groups. For a pointed simplicial set X let $\mathbb{Z}(X)$ denote the simplicial abelian group with n-simplices $\mathbb{Z}[X_n]/\mathbb{Z}[*]$. The simplicial category $\underline{\mathrm{s.Ab}}$ is given by $\underline{\mathrm{s.Ab}}([n]) = \mathrm{s.Ab}$ and $\underline{\mathrm{s.Ab}}(G,H)([n]) = \mathrm{s.Ab}(G \otimes \mathbb{Z}(\Delta(n)), H)$. Note that there is a forgetful functor from $\underline{\mathrm{s.Ab}}$ to $\underline{\mathrm{s.Set}}_*$, and that $\underline{\mathrm{s.Ab}}$ has products with pointed simplicial sets given by $X \otimes G = \mathbb{Z}(X) \otimes G$.

2.5. **Homotopy colimits.** For a small category $\mathcal{C}$ and a functor Z from $\mathcal{C}$ to $\mathrm{s.Set}_*$ the homotopy colimit of Z, $\operatorname{hocolim}_{\overrightarrow{\mathcal{C}}} Z$, is the diagonal of the bisimplicial set associated to the simplicial simplicial set with r-simplices

$$\bigvee_{(c_0 \leftarrow \cdots \leftarrow c_r)} Z(c_r).$$

Here $(c_0 \leftarrow \cdots \leftarrow c_r)$ denotes r arrows in $\mathcal{C}$. For $0 \leq i < r$ the face maps are induced by composition of maps,

$$d_i(c_0 \leftarrow \cdots \leftarrow c_r; z) = (c_0 \leftarrow \cdots \leftarrow c_{i-1} \leftarrow c_{i+1} \leftarrow \cdots \leftarrow c_r; z)$$

and d_r forgets the last map

$$d_r(c_0 \leftarrow \cdots \leftarrow c_r; z) = c_0 \leftarrow \cdots \leftarrow c_{r-1}; (c_r \to c_{r-1})_* z).$$

The degeneracy maps simply put in the identity map on an appropriate place,

$$s_i(c_0 \leftarrow \cdots \leftarrow c_r; z) = (c_0 \leftarrow \cdots \leftarrow c_i \leftarrow c_i \leftarrow c_{i+1} \leftarrow \cdots \leftarrow c_r; z).$$

Let cat denote some category of small categories. A monoid in cat is a small category $\mathcal{C}$ together with an associative product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with a unit $+$. A filtered monoid in cat is a monoid in cat together with subcategories

$$\mathcal{C} = F_0\mathcal{C} \supseteq F_1\mathcal{C} \supseteq \cdots \supseteq F_i\mathcal{C} \supseteq \dots$$

satisfying that $\mu(F_i\mathcal{C}, F_j\mathcal{C}) \subseteq F_{i+j}\mathcal{C}$. The following lemma is a reformulation of theorem 1.5 of [1].

Lemma 2.5.1. *Let $Z : \mathcal{C} \to \mathrm{s.Set}_*$ be a functor defined on a filtered monoid in* cat. *If the unit for the multiplication in $\mathcal{C}$ is an initial object, and if $\lambda : \mathbb{N} \to \mathbb{N}$ is a function such that every morphism $c_1 \to c_2$ in $F_i\mathcal{C}$ induces a $\lambda(i)$-connected map $Z(c_1) \to Z(c_2)$, then the map*

$$Z(c) \to \underset{\overrightarrow{\mathcal{C}}}{\mathrm{hocolim}}\, Z$$

induced by inclusion in the 0-skeleton is $(\lambda(i) - 1)$-connected for every $c \in F_i\mathcal{C}$.

Proof. For $c \in F_i\mathcal{C}$ there is a natural map

$$\mathrm{id} = \mu(+, \mathrm{id}) \to \mu(c, \mathrm{id})$$

of functors from $\mathcal{C}$ to $\mathcal{C}$. Restricting to $F_i\mathcal{C}$ we obtain a map of functors from $F_i\mathcal{C}$ to $F_i\mathcal{C}$. By standard arguments (see [11], section 1) these maps can be used to prove that the inclusion $F_i\mathcal{C} \subseteq \mathcal{C}$ induces a homotopy equivalence

$$\underset{\overrightarrow{F_i\mathcal{C}}}{\mathrm{hocolim}}\, Z \to \underset{\overrightarrow{\mathcal{C}}}{\mathrm{hocolim}}\, Z.$$

Therefore it suffices to see that the map

$$Z(c) \approx \underset{\overrightarrow{\{c\}\times\{c\}}}{\mathrm{hocolim}}\, Z \circ \mathrm{pr}_1 \to \underset{\overrightarrow{F_i\mathcal{C}\times\{c\}}}{\mathrm{hocolim}}\, Z \circ \mathrm{pr}_1$$

is $(\lambda(i) - 1)$-connected. Here $\mathrm{pr}_1 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ denotes projection on the first factor. Since $\mathcal{C}$ has an initial object, we have that

$$\underset{\overrightarrow{F_i\mathcal{C}}}{\mathrm{hocolim}} * \simeq \underset{\overrightarrow{\mathcal{C}}}{\mathrm{hocolim}} * \simeq *.$$

Therefore the vertical maps in the commutative diagram

$$\begin{array}{ccc}
\underset{\{c\}\times\{c\}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mathrm{pr}_1 & \longrightarrow & \underset{F_i\mathcal{C}\times\{c\}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mathrm{pr}_1 \\
\downarrow & & \downarrow \\
\underset{\{c\}\times F_i\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mathrm{pr}_1 & \longrightarrow & \underset{F_i\mathcal{C}\times F_i\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mathrm{pr}_1
\end{array}$$

are homotopy equivalences. Since $\mathrm{pr}_1 = \mu(\mathrm{id}, +)$, there is a natural map $\mathrm{pr}_1 \to \mu$, and similarly there is a natural map $\mathrm{pr}_2 \to \mu$. By the glueing lemma a wedge of $\lambda(i)$-connected maps is $\lambda(i)$-connedted, and therefore the vertical maps in the commutative diagram

$$\begin{array}{ccc}
\underset{\{c\}\times F_i\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mathrm{pr}_1 & \longrightarrow & \underset{F_i\mathcal{C}\times F_i\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mathrm{pr}_1 \\
\downarrow & & \downarrow \\
\underset{\{c\}\times F_i\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mu & \longrightarrow & \underset{F_i\mathcal{C}\times F_i\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mu \\
\uparrow & & \uparrow \\
\underset{\{c\}\times F_i\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mathrm{pr}_2 & \longrightarrow & \underset{F_i\mathcal{C}\times F_i\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, Z\circ \mathrm{pr}_2
\end{array}$$

are degreewise $\lambda(i)$-connedted, and hence they are $\lambda(i)$-connedted. Since the lower horizontal map in this diagram is a homotopy equivalence, the lemma follows. $\square$

Example 2.5.2. (1) The category $\mathcal{J}$ of finite subsets of $\{1, 2, 3, \dots\}$ with morphisms injective maps satisfies the conditions of lemma 2.5.1: Let $F_i\mathcal{J}$ denote the full subcategory of sets with cardinality $\geq i$, and let $\mu : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ be the functor with $\mu(m, n) = m \cup (\max(m) + n)$, with the convention that the maximum of the empty set is 0.

(2) Let $\mathcal{J}(k) \subseteq \mathcal{J}^k$ denote the full subcategory of tuples of mutually disjoint subsets of $\{1, 2, 3, \dots\}$. $\mathcal{J}(k)$ satisfies the conditions of lemma 2.5.1: Let $F_i\mathcal{J}(k) = \mathcal{J}(k) \cap (F_i\mathcal{J})^k$, and let $\mu : \mathcal{J}(k) \times \mathcal{J}(k) \to \mathcal{J}(k)$ be the functor with

$$\mu((m_1, \dots, m_k), (n_1, \dots, n_k)) = (m_1 \cup (M + n_1), m_2 \cup (M + n_2), \dots, m_k \cup (M + n_k)),$$

where $M = \max(m_1 \cup \dots \cup m_k)$.

For a functor $G : \mathcal{C} \to \mathrm{s.Ab}$ there is a similar homotopy colimit, $\underset{\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, G$, taking values in abelian groups, defined as the diagonal of the simplicial simplicial set with r-simplices

$$\bigoplus_{(c_0 \leftarrow \cdots \leftarrow c_r)} G(c_r).$$

If $\overline{G}$ denotes the composition of G with the forgetful functor from s.Ab to $\mathrm{s.Set}_*$, there is a map $\underset{\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, G \to \underset{\mathcal{C}}{\underrightarrow{\operatorname{hocolim}}}\, \overline{G}$. The proof of lemma 2.5.1 can easily be modified to prove the following lemma:

Lemma 2.5.3. *Let $G : \mathcal{C} \to$ s.Ab be a functor defined on a filtered monoid in* cat*. If the unit for the multiplication in $\mathcal{C}$ is an initial object, and if $\lambda : \mathbb{N} \to \mathbb{N}$ is a function such that every morphism $c_1 \to c_2$ in $F_i\mathcal{C}$ induces a $\lambda(i)$-connected map $G(c_1) \to G(c_2)$, then the map*

$$G(c) \to \operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}} G$$

induced by inclusion in the 0-skeleton is $(\lambda(i) - 1)$-connected for every $c \in F_i\mathcal{C}$.

3. Functor with stabilization

3.1. Definitions.

Definition 3.1.1. A k-functor with stabilization is a pointed simplicial functor $M : (\underline{\text{s.Set}_*})^k \to \underline{\text{s.Set}_*}$ such that if X_i is n_i-connected and $n = n_1 + \cdots + n_k$, then

1. $M(X_1, \ldots, X_k)$ is n-connected.
2. The map $Y \wedge M(X_1, \ldots, X_k) \to M(X_1, \ldots, X_{i-1}, Y \wedge X_i, X_{i+1}, \ldots, X_k)$ induced from the simplicial structure is $(n + n_i - c)$ connected for some constant number c not depending on $Y, X_1, \ldots, X_k$ or i.

A map of k-functors with stabilization is simply a natural transformation of functors. We shall say functor with stabilization instead of 1-functor with stabilization. We shall let $\mathcal{FS}$ denote the category of functors with stabilization and $\mathcal{FS}_k$ the category of k-functors with stabilization.

A spectrum is a sequence X^m with maps $S^1 \wedge X^m \xrightarrow{f^m} X^{m+1}$, and an Ω-spectrum is a spectrum where the adjoints of the structure maps yield homotopy equivalences $X^m \simeq \Omega X^{m+1}$. A map of spectra is a stable equivalence if it induces an isomorphism on all homotopy groups. A functor with stabilization M gives rise to a spectrum $M(S^n)$ with structure maps $S^1 \wedge M(S^n) \to M(S^1 \wedge S^n) = M(S^{n+1})$ (here $S^n = S^1 \wedge \cdots \wedge S^1$). The homotopy groups of M are the homotopy groups of this spectrum. We shall say that M is n-connected if $\pi_i(M) = 0$ for $i \leq n$. A map of functors with stabilization is a stable equivalence if it induces a stable equivalence on spectra.

For $f : M_1 \to M_2$ a map in $\mathcal{FS}$, let $C(f)$ be the functor with $C(f)(X)$ the mapping cone of the map $f_X : M_1(X) \to M_2(X)$. It follows from the glueing lemma that $C(f)$ is a functor with stabilization. There is an obvious map $i_f : M_2 \to C(f)$.

Definition 3.1.2. A sequence $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ of maps in $\mathcal{FS}$ is a cofibration sequence if there is a stable equivalence $C(f) \to M_3$ making the diagram

$$\begin{tikzcd} M_2 \arrow[r, "i_f"] \arrow[d, equal] & C(f) \arrow[d] \\ M_2 \arrow[r, "g"] & M_3 \end{tikzcd}$$

commutative.

A cofibration sequence of functors with stabilization induces a long exact sequence on homotopy groups as is seen by the Blakers Massey theorem.

For a functor G to $\underline{\text{s.Ab}}$ we shall write $\overline{G}$ for the composition of G with the forgetful functor $\underline{\text{s.Ab}} \to \underline{\text{s.Set}_*}$.

Definition 3.1.3. A k-functor with stabilization in $\underline{\text{s.Ab}}$ is a functor

$$M : (\underline{\text{s.Set}_*})^k \to \underline{\text{s.Ab}}$$

with $\overline{M}$ a k-functor with stabilization. A map of k-functors with stabilization in $\underline{\text{s.Ab}}$ is simply a natural transformation. We shall denote the category of k-functors with stabilization in $\underline{\text{s.Ab}}$ by $\mathcal{FSA}_k$, and we shall let $\mathcal{FSA} = \mathcal{FSA}_1$, the category of functors with stabilization in $\underline{\text{s.Ab}}$. A map in $\mathcal{FSA}$ is a stable equivalence if the induced map in $\mathcal{FS}$ is a stable equivalence, and a sequence of maps in $\mathcal{FSA}$ is a cofibration sequence if the induced sequence in $\mathcal{FS}$ is a cofibration sequence.

Example 3.1.4. Let us make some basic observations:

- A simplicial set Z defines a k-functor with stabilization $\Sigma^\infty Z$ with
$$\Sigma^\infty Z(X_1, \dots, X_k) = Z \wedge X_1 \wedge \dots \wedge X_k.$$
- If $X_1 \to X_2 \to X_3$ is a cofibration sequence of simplicial sets, then the sequence $\Sigma^\infty X_1 \to \Sigma^\infty X_2 \to \Sigma^\infty X_3$ is a cofibration sequence in $\mathcal{FS}$.
- If M is a functor with stabilization and Z is a simplicial set, then the map $\Sigma^\infty Z \circ M \to M \circ \Sigma^\infty Z$ induced by the stabilization $Z \wedge M(X) \to M(Z \wedge X)$ is a stable equivalence in $\mathcal{FS}$. (This is not true if we restrict condition 2 in 3.1.1 to the case where Y is a sphere as is common.)
- If $M \in \mathcal{FSA}$ and Z is a simplicial set, then the simplicial structure of M induces a stable equivalence $\mathbb{Z}(Z) \otimes M \to M \circ \Sigma^\infty Z$ in $\mathcal{FSA}$.
- If M is a functor with stabilization, and $X_1 \to X_2 \to X_3$ is a cofibration sequence of simplicial sets, then the sequence $M \circ \Sigma^\infty X_1 \to M \circ \Sigma^\infty X_2 \to M \circ \Sigma^\infty X_3$ is a cofibration sequence in $\mathcal{FS}$.
- If $M_1 \to M_2 \to M_3$ is a sequence of maps in $\mathcal{FSA}$ such that for each X the sequence $M_1(X) \to M_2(X) \to M_3(X)$ is a short exact sequence of abelian groups, then it is a cofibration sequence in $\mathcal{FSA}$. This is a consequence of the Blakers Massey theorem.

Example 3.1.5. A simplicial abelian group M gives rise to a functor with stabilization in $\underline{\text{s.Ab}}$, let us denote it by M too, with

$$M(X)_n = M_n[X_n]/M_n[*] = \bigoplus_{x \in X_n} M_n \cdot x / M_n \cdot *.$$

To see that $\overline{M}$ is a functor with stabilization, suppose that X is n-connected. Since the homotopy groups of $\overline{M}(X)$ are the homology groups of X with trivial coefficients in M, it follows from the Hurewicz theorem that $\overline{M}(X)$ is n-connected. For any pointed set U the map

$$U \wedge \overline{M}(X) \cong \bigvee_{u \in U - \{*\}} \overline{M}(X) \to \bigoplus_{u \in U - \{*\}} \overline{M}(X) \cong \overline{M}(U \wedge X)$$

is a $2n-1$ equivalence by the Blakers Massey theorem. Therefore for any simplicial set Y the map $Y \wedge \overline{M}(X) \to \overline{M}(Y \wedge X)$ is a $2n-1$ equivalence. It follows that $\overline{M}$ is a functor with stabilization.

Note that for simplicial abelian groups M and N there is an isomorphism

$$(M \otimes N)(X \wedge Y) \cong M(X) \otimes N(Y),$$

natural in all factors. In particular there is an isomorphism $M(X) \cong \mathbb{Z}(X) \otimes M$.

Lemma 3.1.6. *(1) If M is a functor with stabilization satisfying that $M(X)$ is $(n+k)$-connected for X n-connected, then the functor $X \mapsto F(S^k, M(X))$ is a functor with stabilization.*

(2) If $M_1 \to M_2 \to M_3$ is a cofibration sequence in $\mathcal{FS}$, and $M_i(X)$ is $(n+m)$-connected for X n -connected, then the sequence

$$F(S^m, M_1(-)) \to F(S^m, M_2(-)) \to F(S^m, M_3(-))$$

is a cofibration sequence.

Proof. (1) It is enough to show that if Z is $(n+k)$-connected, then the map

$$X \wedge F(S^k, Z) \to F(S^k, X \wedge Z)$$

is $(2n-c)$-connected. This map can be written as the composition

$$\begin{aligned} X \wedge F(S^k, Z) &\to X \wedge F(S^k, S^k \wedge F(S^k, Z)) \\ &\to F(S^k, X \wedge S^k \wedge F(S^k, Z)) \\ &\to F(S^k, X \wedge Z). \end{aligned}$$

The first of these maps is $(2n-1)$-connected by Freudental. The last map is $(2n-2)$-connected since the composition

$$F(S^k, Z) \to F(S^k, S^k \wedge F(S^k, Z)) \to F(S^k, Z)$$

is the identity. From the commutative diagram

$$\begin{tikzcd} X \wedge F(S^k, Z) \arrow[r, "="] \arrow[d] & X \wedge F(S^k, Z) \arrow[d] \\ X \wedge F(S^k, S^k \wedge F(S^k, Z)) \arrow[r] & F(S^k, X \wedge S^k \wedge F(S^k, Z)) \end{tikzcd}$$

with $(2n-1)$-connected vertical maps it follows that the lower horizontal map is $2n-2$-connected. Hence the composition above is $(2n-2)$-connected.

(2) follows from the five lemma applied to the map of exact sequences induced by the map $C(F(S^m, f)) \to F(S^m, Cf)$. $\square$

3.2. **Diagonalization.** Let $\mathcal{J}$ denote the category of finite subsets of $\{1, 2, \dots\}$ with morphisms injective maps. A simplicial functor $M : \underline{\text{s.Set}}_* \to \underline{\text{s.Set}}_*$ induces a functor $V(M) : \mathcal{J} \to \text{s.Set}_*$ with $V(M)(n) = F(S^n, M(S^n))$. (Here S^n is a quotient of $\underline{\text{s.Set}}_*(n_+, \Delta(1))$.) A morphism $\alpha : n \to m$ induces a map $\alpha_* = V(M)(\alpha) : V(M)(n) \to V(M)(m)$ as follows:

$$\begin{aligned} F(S^n, M(S^n)) &\xrightarrow{\approx} F(S^{\alpha(n)}, M(S^{\alpha(n)})) \\ &\to F(S^{\alpha(n)}, F(S^{m-\alpha(n)}, S^{m-\alpha(n)} \wedge M(S^{\alpha(n)}))) \\ &\to F(S^{\alpha(n)} \wedge S^{m-\alpha(n)}, M(S^{m-\alpha(n)} \wedge S^{\alpha(n)})) \\ &\xrightarrow{\approx} F(S^m, M(S^m)). \end{aligned}$$

Here the first and the last maps are induced by identifications of spheres. The second map is induced by the simplicial structure of $F(S^{m-\alpha(n)}, S^{m-\alpha(n)} \wedge -)$, and the third map is the adjunction map for the functors $F(X, -)$ and $- \wedge X$ together with the simplicial structure of M. For $n \in \mathcal{J}$ we shall denote the cardinality of n by $|n|$. If M is a functor with stabilization, then α_* is $(|n| - c)$ - connected.

A k-functor with stabilization M induces a functor $V(M) : \mathcal{J}^k \to \underline{\text{s.Set}}_*$ in a similar way. More precisely,

$$V(M)(n) = F(S^n, M(S^n)),$$

with the convention that for $n = (n_1, \dots, n_k) \in \mathcal{J}^k$, $S^n = S^{n_1} \wedge \cdots \wedge S^{n_k}$ and $M(S^n) = M(S^{n_1}, \dots, S^{n_k})$. Given a morphism $n \to m$ in $\mathcal{J}^k$, the map $\alpha_* : V(M)(n) \to V(M)(m)$ is $(\min\{|n_i|\} - c - 1)$-connected for $i = 1, \dots, k$.

Given a $(k+1)$-functor with stabilization M and a simplicial set X, the functor with value $M(X, X_1, \dots, X_k)$ on $(X_1, \dots X_k)$ is a k-functor with stabilization. Let $\mathcal{J}(k) \subseteq \mathcal{J}^k$ denote the full subcategory of tuples of disjoint subsets of $\{1, 2, \dots\}$, and for $n = (n_1, \dots, n_k) \in \mathcal{J}(k)$ let $|n| = |n_1| + \cdots + |n_k|$.

Definition 3.2.1. Given a $(k+1)$-functor with stabilization M, the diagonalization of M is the functor $DM : \underline{\text{s.Set}}_* \to \underline{\text{s.Set}}_*$ with

$$DM(X) = \underset{\substack{\longrightarrow \\ n \in \mathcal{J}(k)}}{\text{hocolim}}\, F(S^n, M(X, S^n))$$

with the convention that $S^n = S^{n_1} \wedge \cdots \wedge S^{n_k}$ and $M(X, S^n) = M(X, S^{n_1}, \dots, S^{n_k})$ for $n = (n_1, \dots n_k) \in \mathcal{J}(k)$.

Proposition 3.2.2. *For a $(k+1)$-functor with stabilization M, the functor DM is a functor with stabilization.*

Proof. The functor $X \mapsto F(S^n, M(X, S^n))$ is a functor with stabilization by lemma 3.1.6. It follows from the approximation lemma 2.5.1 that $DM(X)$ is a functor with stabilization. $\square$

Remark 3.2.3. *We have actually defined functors $D : \mathcal{FS}_{k+1} \to \mathcal{FS}$. Another functorial property of D is the following: Given a $(k+1)$-functor with stabilization M and any map $\phi : \{1, \dots, l\} \to \{1, \dots, k\}$, we obtain a $(l+1)$-functor with stabilization $\phi^* M$ with $\phi^* M(X, X_1, \dots, X_l) = M(X, Y_1, \dots, Y_k)$ where $Y_i = \bigwedge_{j \in \phi^{-1}(i)} X_j$. ϕ induces a functor $\phi_* : \mathcal{J}(l) \to \mathcal{J}(k)$ with $\phi_*(n_1, \dots, n_l) = (m_1, \dots, m_k)$ where $m_i = \bigcup_{j \in \phi^{-1}(i)} n_j$. ϕ_* in turn induces a map $D(\phi^* M)(X) \to DM(X)$.*

For a k-functor with stabilization in $\underline{\text{s.Ab}}$ M we can make similar constructions. There is a functor $W(M) : \mathcal{J}^k \to \underline{\text{s.Ab}}$ with

$$W(M)(n) = \underline{\text{s.Ab}}(\mathbb{Z}(S^n), M(S^n)),$$

where $S^n = S^{n_1} \wedge \cdots \wedge S^{n_k}$ and $M(S^n) = M(S^{n_1}, \dots, S^{n_k})$ for $n = (n_1, \dots, n_k) \in \mathcal{J}^k$.

Definition 3.2.4. Given a $(k+1)$-functor with stabilization in $\underline{\text{s.Ab}}$ M, the diagonalization of M is the functor $DM : \underline{\text{s.Set}}_* \to \underline{\text{s.Ab}}$ with

$$DM(X) = \underset{\substack{\longrightarrow \\ n \in \mathcal{J}(k)}}{\text{hocolim}}\, \underline{\text{s.Ab}}(\mathbb{Z}(S^n), M(X, S^n)).$$

Since there are natural weak equivalences

$$D\overline{M}(X) \leftarrow \underset{\substack{\longrightarrow \\ \mathcal{J}(k)}}{\text{hocolim}}\, \underline{\text{s.Set}}_*(S^n, M(X, S^n)) \to \overline{DM}(X),$$

DM is a functor with stabilization in $\underline{\text{s.Ab}}$.

4. Functor with smash product

4.1. Definitions.

Definition 4.1.1. A functor with smash product (FSP) is a functor with stabilization L together with natural transformations

$$1_X : X \to L(X), \qquad \mu_{X,Y} : L(X) \wedge L(Y) \to L(X \wedge Y)$$

such that the following relations are satisfied:

$$\begin{aligned} \mu_{X\wedge Y,Z}(\mu_{X,Y} \wedge \mathrm{id}_Z) &= \mu_{X,Y\wedge Z}(\mathrm{id}_X \wedge \mu_{Y,Z}), \\ \mu_{X,Y}(1_X \wedge 1_Y) &= 1_{X\wedge Y}, \\ \mu_{X,Y}(1_X \wedge \mathrm{id}_{L(Y)}) &= \lambda_{X,Y}, \\ \mu_{X,Y}(\mathrm{id}_{L(X)} \wedge 1_Y) &= \rho_{X,Y}, \end{aligned}$$

where $\rho_{X,Y} : L(X) \wedge Y \to L(X \wedge Y)$ and $\lambda_{X,Y} : X \wedge L(Y) \to L(X \wedge Y)$ are induced by the simplicial structure of L. We say that L is commutative if $L(\mathrm{tw}_{X,Y}) \circ \mu_{X,Y} = \mu_{Y,X} \circ \mathrm{tw}_{L(X),L(Y)}$. Here tw denotes twist of factors in a smash product.

A morphism $L \to L_1$ of functors with smash product is a natural transformation commuting with 1 and μ. We shall denote the category of functors with smash product by $\mathcal{FSP}$.

Definition 4.1.2. A left module over a FSP L is a functor with stabilization M together with natural maps

$$l_{X,Y} : L(X) \wedge M(Y) \to M(X \wedge Y)$$

such that

$$l_{X\wedge Y,Z}(\mu_{X,Y} \wedge \mathrm{id}_{M(Z)}) = l_{X,Y\wedge Z}(\mathrm{id}_{L(X)} \wedge l_{Y,Z}), \qquad \lambda_{X,Y} = l_{X,Y}(1_X \wedge \mathrm{id}_{M(Y)}).$$

Similarly we can define right L-modules.

Definition 4.1.3. Let L_1 and L_2 be FSP's. A $L_1 - L_2$- bimodule is a functor M with stabilization together with structure of left L_1- and right L_2-module such that

$$l_{X,Y\wedge Z}(\mathrm{id}_{L_1(X)} \wedge r_{Y,Z}) = r_{X\wedge Y,Z}(l_{X,Y} \wedge \mathrm{id}_{L_2(Z)}),$$

where l (resp. r) is the structure of left (resp. right) module over L_1 (resp. L_2).

In stead of $L - L$-bimodule we shall simply say L-bimodule or bimodule over L.

Definition 4.1.4. An algebra over a commutative FSP L is a L-bimodule A together with natural transformations

$$m_{X,Y} : A(X) \wedge A(Y) \to A(X \wedge Y)$$

satisfying the relations

$$\begin{aligned} \mu_{X\wedge Y,Z}(\mu_{X,Y} \wedge \mathrm{id}_Z) &= \mu_{X,Y\wedge Z}(\mathrm{id}_X \wedge \mu_{Y,Z}), \\ m_{X\wedge Y,Z}(l_{X,Y} \wedge \mathrm{id}_{A(Z)}) &= l_{X,Y\wedge Z}(\mathrm{id}_{L(X)} \wedge m_{Y,Z}) \\ &= m_{Y,X\wedge Z}(\mathrm{id}_{A(Y)} \wedge l_{X,Z})(\mathrm{tw}_{L(X),A(Y)} \wedge \mathrm{id}_{A(Z)}) \end{aligned}$$

and

$$\begin{aligned} m_{X,Y\wedge Z}(\mathrm{id}_{A(X)} \wedge r_{Y,Z}) &= r_{X\wedge Y,Z}(m_{X,Y} \wedge \mathrm{id}_{L(Z)}) \\ &= m_{X\wedge Z,Y}(r_{X,Z} \wedge \mathrm{id}_{A(Y)})(\mathrm{id}_{A(X)} \wedge \mathrm{tw}_{A(Y),L(Z)}), \end{aligned}$$

where l (resp. r) denotes left (resp. right) module structure of A. (On elements these relations can be written as $(xa_1)a_2 = x(a_1a_2) = a_1(xa_2)$ and $a_1(a_2x) = (a_1a_2)x = (a_1x)a_2$.)

A is unital if there is a L-bilinear map $1 : L \to A$ such that $\mu_{X,Y}(1_X \wedge 1_Y) = 1_{X\wedge Y}$.

Note that a unital algebra A is a FSP with two morphisms $r, l : L \to A$ in $\mathcal{FSP}$ satisfying that $\mu_{X,Y}(l_X \wedge r_Y) = A(\mathrm{tw}_{Y,X})\mu_{Y,X}(r_Y \wedge l_X)\,\mathrm{tw}_{L(X),L(Y)}$.

Definition 4.1.5. A functor with tensor product (FTP) is a functor with stabilization in $\underline{\text{s.Ab}}$, say A, together with natural linear maps

$$\mu_{X,Y} : A(X) \otimes A(Y) \to A(X \wedge Y), \qquad 1_X : \mathbb{Z}(X) \to A(X)$$

making $\overline{A}$ a unital algebra over $\overline{\mathbb{Z}}$ by precomposition with the map $A(X) \wedge A(Y) \to A(X) \otimes A(Y)$.

A morphism $A \to A_1$ of functors with tensor product is a natural transformation commuting with 1 and μ. We shall denote the category of functors with tensor product $\mathcal{FTP}$.

Modules and algebras over a FTP can be defined by substituting $\otimes$ for $\wedge$ in the definitions of modules and algebras over a FSP, and demanding that all maps are linear.

Note that for $M \in \mathcal{FSA}$, and X n-connected, the map $M(X)\wedge Y \to M(X)\otimes\mathbb{Z}(Y)$ is $(2n-1)$-connected by the Blakers Massey theorem.

Example 4.1.6. Let us restrict ourselves to some very basic examples and constructions of functors with smash product.

- For a pointed monoid Π the functor $\Sigma^\infty\Pi$ becomes a FSP when given the unit and multiplication maps $X \to \Pi \wedge X$ and $(\Pi \wedge X) \wedge (\Pi \wedge Y) \to (\Pi \wedge \Pi) \wedge X \wedge Y \to \Pi \wedge X \wedge Y$ induced by the unit and multiplication in Π. One particularly important example is the pointed monoid $\{0, 1\} \subseteq \mathbb{Z}$, and we shall write $\mathbb{S}$ insted of $\Sigma^\infty\{0, 1\}$. Note that $\mathcal{FS}$ is isomorphic to the category of $\mathbb{S}$-modules, and that the category of FSP's is isomorphic to the category of unital $\mathbb{S}$-algebras.
- Let L and L_1 be FSP's and assume that $L \circ L_1$ is a functor with stabilization. Then $L \circ L_1$ is a FSP. If L is commutative, then $L \circ L_1$ is a L-algebra. In particular if Π is a pointed monoid, then $L[\Pi] = \Sigma^\infty\Pi \circ L$ is a FSP.
- For a commutative pointed monoid Π, the functor $\Sigma^\infty(\Pi\wedge\Pi)$ is a unital $\mathbb{S}[\Pi] = \Sigma^\infty\Pi$-algebra.
- For a ring R the underlying abelian group defines a functor with stabilization in $\underline{\text{s.Ab}}$. The multiplication in R gives a group homomorphism $R \otimes R \to R$, and hence it defines homomorphisms

  $$\mu_{X,Y} : R(X) \otimes R(Y) \cong (R \otimes R)(X \wedge Y) \to R(X \wedge Y).$$

  The homomorphism $\mathbb{Z} \to R$ induced by $1 \in R$ induces a map $1_X : \mathbb{Z}(X) \to R(X)$. These maps give R the structure of a FTP. Note that $\mathcal{FSA}$ is isomorphic to the category of modules over $\mathbb{Z}$, and that the category of FTP's is isomorphic to the category of unital $\mathbb{Z}$-algebras.
- If A is a FTP, L is a FSP and $A \circ L$ is in $\mathcal{FSA}$, then $A \circ L$ a FTP. If A is commutative, then $A \circ L$ is a unital A-algebra. In particular if Π is a pointed monoid, then $A[\Pi] = \mathbb{Z}(\Pi) \otimes A$ is a FTP.

4.2. **Topological Hochschild homology.**

Definition 4.2.1. (1) Let Π be a pointed monoid. The cyclic nerve of Π is the cyclic set with k-simplices $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_k = \Pi^{\wedge(k+1)}$ with face, degeneracy and cyclic operation given by

$$d_i(x_0 \wedge \cdots \wedge x_k) = \begin{cases} x_0 \wedge \cdots \wedge x_i x_{i+1} \wedge x_k & 0 \le i < k \\ x_k x_0 \wedge \cdots \wedge x_{k-1} & i = k, \end{cases}$$
$$s_i(x_0 \wedge \cdots \wedge x_k) = x_0 \wedge \cdots \wedge x_i \wedge 1 \wedge x_{i+1} \wedge \cdots \wedge x_k \quad \text{and}$$
$$t(x_0 \wedge \cdots \wedge x_k) = x_k \wedge x_0 \wedge \cdots \wedge x_{k-1}.$$

(2) Let Π be a pointed monoid and let M be a simplicial set with commuting left- and right actions of Π. $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi, M)$ is the simplicial set with k-simplices $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi, M)_k = M \wedge \Pi^{\wedge k}$ with face and degeneracy maps given by the same formulas as above.

Let S^1 be the simplicial model of the circle with k-simplices $S^1_k = \{0, \dots, k\}$ and face maps given by $(d_i(0), \dots, d_i(k)) = (0, \dots, i-1, i, i, i+1, \dots, k-1)$ for $0 \le i < k$ and $(d_k(0), \dots, d_k(k)) = (0, 1, \dots, k-1, 0)$ and degeneracy maps given by $(s_i(0), \dots, s_i(k)) = (0, \dots, i-1, i, i+2, i+3, \dots, k+1)$ for $0 \le i \le k$. The cyclic operator t given by $(t(0), \dots, t(k)) = (k, 0, 1, \dots, k-1)$ gives S^1 the structure of a cyclic set. From a FSP L and a bimodule M over L we can form a $(k+1)$-functor with stabilization, $\mathrm{thh}(L, M)_k = M \wedge L^{\wedge k}$, with

$$\mathrm{thh}(L, M)_k(X_0, \dots, X_k) = M(X_0) \wedge L(X_1) \wedge \cdots \wedge L(X_k).$$

There are maps

$$d_i : \mathrm{thh}(L, M)_k \to d_i^* \, \mathrm{thh}(L, M)_{k-1},$$
$$s_i : \mathrm{thh}(L, M)_k \to s_i^* \, \mathrm{thh}(L, M)_{k+1} \quad \text{and}$$
$$t : \mathrm{thh}(L, L)_k \to t^* \, \mathrm{thh}(L, L)_k$$

in $\mathcal{FS}_{k+1}$ given by the same formula as for the cyclic nerve. Here we use the notation of remark 3.2.3. We shall let $\mathbb{S} \wedge \mathrm{thh}(L, M)_k$ and $\mathrm{thh}(L, M \circ (\mathbb{S} \wedge \mathbb{S}))_k$ denote the $(k+2)$-functors with stabilization with

$$\mathbb{S} \wedge \mathrm{thh}(L, M)_k(X, X_0, \dots, X_k) = X \wedge \mathrm{thh}(L, M)_k(X_0, \dots, X_k)$$
$$\mathrm{thh}(L, M \circ (\mathbb{S} \wedge \mathbb{S}))_k(X, X_0, \dots, X_k) = \mathrm{thh}(L, M)_k(X \wedge X_0, \dots, X_k).$$

Definition 4.2.2. (1) Let L be a FSP. Topological Hochschild homology of L is the cyclic object $\mathrm{THH}(L)$ in $\mathcal{FS}$ with $\mathrm{THH}(L)_k = D(\mathbb{S} \wedge \mathrm{thh}(L)_k)$ and face, degeneracy and cyclic operation induced by the maps above and the functorial properties of D described in remark 3.2.3.

(2) Let L be a FSP and let M be a bimodule over L. $\mathrm{THH}(L, M)$ is the simplicial object in $\mathcal{FS}$ with $\mathrm{THH}(L, M)_k = D \, \mathrm{thh}(L, M \circ (\mathbb{S} \circ \mathbb{S}))_k$. Face and degeneracy maps are given by the maps above and the functorial properties of D.

More explicitly

$$\mathrm{THH}(L)_k(X) = \underset{n \in \mathcal{J}(k+1)}{\underrightarrow{\mathrm{hocolim}}} \, F(S^n, X \wedge L(S^{n_0}) \wedge \cdots \wedge L(S^{n_k})) \quad \text{and}$$
$$\mathrm{THH}(L, M)_k(X) = \underset{n \in \mathcal{J}(k+1)}{\underrightarrow{\mathrm{hocolim}}} \, F(S^n, X \wedge M(X \wedge S^{n_0}) \wedge L(S^{n_1}) \wedge \cdots \wedge L(S^{n_k})),$$

where $S^n = S^{n_0} \wedge \cdots \wedge S^{n_k}$ for $n = (n_0, \dots, n_k) \in \mathcal{J}(k+1)$. Note that $\mathrm{THH}(L)$ and $\mathrm{THH}(L, M)$ are functors with stabilization. The simplicial structure of L induces a homotopy equivalence $\mathrm{THH}(L)(X) \to \mathrm{THH}(L, L)(X)$. This is a natural transformation of functors from $\mathcal{FSP}$ to $\mathcal{FS}$.

Definition 4.2.3. (1) Let R be a simplicial ring and let M be a R-bimodule. $Z(R, M)$ is the simplicial abelian group with k-simplices $Z_k(R, M) = M \otimes R^{\otimes k}$, with face maps given by

$$d_i(m \otimes r_1 \otimes \cdots \otimes r_k) = \begin{cases} (mr_1 \otimes r_2 \otimes \cdots \otimes r_k) & i = 0 \\ (m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_k) & 0 < i < k \\ (r_k m \otimes r_1 \otimes \cdots \otimes r_{k-1}) & i = k \end{cases}$$

and with degeneracy maps given by

$$s_i(m \otimes r_1 \otimes \cdots \otimes r_k) = (m \otimes r_1 \otimes \cdots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_k).$$

(2) Let R be a simplicial ring. $Z(R)$ is the cyclic abelian group with $Z_k(R) = Z_k(R, R)$ and with simplicial structure induced by that of $Z(R, R)$. The cyclic operator is given by

$$t(r_0 \otimes r_1 \otimes \cdots \otimes r_k) = (r_k \otimes r_0 \otimes \cdots \otimes r_{k-1}).$$

From a FTP A and a bimodule M over A we can form $\mathrm{hh}(A, M)_k = M \otimes A^{\otimes k} \in \mathcal{FSA}_{k+1}$ with

$$\mathrm{hh}(A, M)_k(X_0, \dots, X_k) = M(X_0) \otimes A(X_1) \otimes \cdots \otimes A(X_k).$$

There are maps

$$\begin{aligned} d_i &: \mathrm{hh}(A, M)_k \to d_i^* \,\mathrm{hh}(A, M)_{k-1}, \\ s_i &: \mathrm{hh}(A, M)_k \to s_i^* \,\mathrm{hh}(A, M)_{k+1} \quad \text{and} \\ t &: \mathrm{hh}(A, A)_k \to t^* \,\mathrm{hh}(A, A)_k \end{aligned}$$

in $\mathcal{FSA}_{k+1}$ given by the same formula as for $Z(R)$. We shall let $\mathbb{Z} \circ \mathbb{S} \otimes \mathrm{hh}(A, M)_k$ and $\mathrm{hh}(A, M \circ (\mathbb{S} \wedge \mathbb{S}))_k$ denote the objects in $\mathcal{FSA}_{k+2}$ with

$$\begin{aligned} \mathbb{Z} \circ \mathbb{S} \otimes \mathrm{hh}(A, M)_k(X, X_0, \dots, X_k) &= \mathbb{Z}(X) \otimes \mathrm{hh}(A, M)_k(X_0, \dots, X_k) \\ \mathrm{hh}(A, M \circ (\mathbb{S} \wedge \mathbb{S}))_k(X, X_0, \dots, X_k) &= \mathrm{hh}(A, M)_k(X \wedge X_0, \dots, X_k). \end{aligned}$$

Definition 4.2.4. (1) Let A be a FTP. Hochschild homology of A is the cyclic object $\mathrm{HH}(A)$ in $\mathcal{FSA}$ with $\mathrm{HH}(L)_k = D(\mathbb{Z} \circ \mathbb{S} \otimes \mathrm{hh}(A)_k)$ and face, degeneracy and cyclic operation induced by the maps above and the functorial properties of D.

(2) Let A be a FTP and let M be a bimodule over A. $\mathrm{HH}(A, M)$ is the simplicial object in $\mathcal{FSA}$ with $\mathrm{HH}(A, M)_k = D\,\mathrm{hh}(A, M \circ (\mathbb{S} \wedge \mathbb{S}))_k$. Face and degeneracy maps are induced by the maps above and the functorial properties of D.

The simplicial structure of A induces a homotopy equivalence $\mathrm{HH}(A)(X) \to \mathrm{HH}(A, A)(X)$. This is a natural transformation of functors from $\mathcal{FTP}$ to $\mathcal{FSA}$.

Proposition 4.2.5. *Let L be a FSP and let M be a bimodule over the FTP $\mathbb{Z} \circ L$. Then $\mathrm{THH}(L, \overline{M})$ and $\overline{\mathrm{HH}(\mathbb{Z} \circ L, M)}$ are weakly equivalent.*

Proof. Let $\mathrm{HH}'(\mathbb{Z}\circ L, M)$ be the simplicial object in $\mathcal{FS}$ with k-simplices

$$\mathrm{HH}'(\mathbb{Z}\circ L, M)_k(X) = \underset{n\in\mathcal{J}(k+1)}{\underrightarrow{\mathrm{hocolim}}}\, F(S^n, M(X\wedge S^{n_0})\otimes \mathbb{Z}(L(S^{n_1}))\otimes\cdots\otimes\mathbb{Z}(L(S^{n_k}))),$$

and with face and degeneracy maps similar to those of $\mathrm{THH}(L)$. Then there are homotopy equivalences $\mathrm{THH}(L,\overline{M})(X)\to \mathrm{HH}'(\mathbb{Z}\circ L, M)(X)$ and $\mathrm{HH}(\mathbb{Z}\circ L, M)(X)\to \mathrm{HH}'(\mathbb{Z}\circ L, M)(X)$. To see this apply the approximation lemma and the Blakers Massey theorem. □

Proposition 4.2.6. *Let R be a simplicial ring and let M be a bimodule over R. There is a homotopy equivalence*

$$\mathbb{Z}(X)\otimes Z(R,M)\to \mathrm{HH}(R,M)(X).$$

Proof. Use the natural homeomorphism

$$\mathrm{HH}(R,M)_k(X)\approx \underset{\mathcal{J}(k+1)}{\underrightarrow{\mathrm{hocolim}}}\, \underline{\mathrm{s.Ab}}(\mathbb{Z}(S^n), (\mathbb{Z}(X)\otimes Z_k(R,M))(S^n)).$$

□

Proposition 4.2.7. *Let A be a FTP and let M be a bimodule over A. The map*

$$\mathrm{HH}(A,M)(X)\otimes\mathbb{Z}(Y)\to \mathrm{HH}(A,M)(X\wedge Y)$$

is a homotopy equivalence.

In particular $\mathrm{HH}(A,M)(X)$ *is homotopy equivalent to* $\mathrm{HH}(A,M)(S^0)\otimes\mathbb{Z}(X)$, *and thus* $n\mapsto \mathrm{HH}(A,M)(S^n)$ *is an* Ω*-spectrum.*

Proof. Note that for Q a n-connected simplicial abelian group the map

$$\underline{\mathrm{s.Ab}}(\mathbb{Z}(S^n), Q)\otimes\mathbb{Z}(Y)\to \underline{\mathrm{s.Ab}}(\mathbb{Z}(S^n), Q\otimes\mathbb{Z}(Y))$$

is a homotopy equivalence. (This is a consequence of proposition II.6.1 in [10].) By the approximation lemma for a functor $N\in\mathcal{FSA}_{k+1}$ the map

$$(\underset{\mathcal{J}(k+1)}{\underrightarrow{\mathrm{hocolim}}}\, F(S^n, N(S^n)))\otimes\mathbb{Z}(Y)\to \underset{\mathcal{J}(k+1)}{\underrightarrow{\mathrm{hocolim}}}\, F(S^n, N(S^n)\otimes\mathbb{Z}(Y))$$

is a homotopy equivalence. We obtain a homotopy equivalence

$$\begin{aligned}
&\mathrm{HH}(A,M)_k(X)\otimes\mathbb{Z}(Y)\\
&\quad\to \underset{\mathcal{J}(k+1)}{\underrightarrow{\mathrm{hocolim}}}(\mathbb{Z}(S^n), M(X\wedge S^{n_0})\otimes\mathbb{Z}(Y)\otimes A(S^{n_1})\otimes\cdots\otimes A(S^{n_k}))\\
&\quad\to \underset{\mathcal{J}(k+1)}{\underrightarrow{\mathrm{hocolim}}}(\mathbb{Z}(S^n), M(X\wedge Y\wedge S^{n_0})\otimes A(S^{n_1})\otimes\cdots\otimes A(S^{n_k}))\\
&\quad= \mathrm{HH}(A,M)_k(X\wedge Y)
\end{aligned}$$

by the approximation lemma. □

Let R be a simplicial ring and let M be a bimodule over R. There is a map $\mathbb{Z}\circ\overline{R}\to R$ of FTP's given by evaluation of formal linear combinations. Hence there is an induced map

$$l_R : \mathrm{HH}(\mathbb{Z}\circ\overline{R}, M)\to \mathrm{HH}(R,M).$$

Let $\mathbb{Z}[\overline{R}] = \mathbb{Z}[\overline{R}(S^0)]$ denote the integral group ring on the pointed monoid with underlying set R. There is a homomorphism $\oplus\mathbb{Z}(\overline{R}) \to \mathbb{Z}(\oplus R)$ inducing a map $\mathbb{Z}[\overline{R}] \to \mathbb{Z} \circ \overline{R}$ of FTP's. There is an induced map

$$s_R : \mathrm{HH}(\mathbb{Z}[\overline{R}], M) \to \mathrm{HH}(\mathbb{Z} \circ \overline{R}, M).$$

4.3. **THH of monoid rings.** Let L be a FSP, let M be a bimodule over L and let Π be a pointed monoid. We shall denote the FSP $\Sigma^\infty\Pi \circ L$ by $L[\Pi]$, and we shall denote $\Sigma^\infty\Pi \circ M$ by $M[\Pi]$. Note that $M[\Pi]$ is a $L[\Pi]$-bimodule. There is a homotopy equivalence

$$\begin{aligned}\mathrm{THH}(L[\Pi], M[\Pi])(X) &\approx \mathrm{THH}(L, \Sigma^\infty \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi) \circ M)(X)\\ &\to \mathrm{THH}(L, M) \circ \Sigma^\infty \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi)(X),\end{aligned}$$

natural in all factors. Here the first map is given by permutation of factors of Π. (When $M = \Sigma^\infty Z$ for a space Z, this is a homeomorphism.) Hence there is a natural weak equivalence

$$\mathrm{THH}(L[\Pi], M[\Pi]) \to \mathrm{THH}(L, M) \circ \Sigma^\infty \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi)$$

in $\mathcal{FS}$.

Similarly for a FTP A, an A-bimodule N and a pointed monoid Π, let $A[\Pi] = \mathbb{Z}(\Pi) \otimes A$ and let $N[\Pi] = \mathbb{Z}(\Pi) \otimes N$. Then there is a natural weak equivalence

$$\mathrm{HH}(A[\Pi], N[\Pi]) \to \mathrm{HH}(A, N) \circ \Sigma^\infty \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi)$$

in $\mathcal{FSA}$. When N is an abelian group, this is actually an isomorphism. Let us spell out the naturality when the morphisms are $\mathbb{Z} \circ \overline{R} \to R$, $\mathbb{Z} \circ \overline{R[\Pi]} \to R[\Pi]$ and $(\mathbb{Z} \circ \overline{R})[\Pi] \to \mathbb{Z} \circ \overline{R[\Pi]}$ in the commutative diagram

$$\begin{tikzcd}
\mathrm{HH}(\mathbb{Z} \circ \overline{R}, R) \circ \Sigma^\infty \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi) \arrow[r, "l_R"] \arrow[d, "\approx"'] & \mathrm{HH}(R, R) \circ \Sigma^\infty \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi) \arrow[d, "\approx"'] \\
\mathrm{HH}((\mathbb{Z} \circ \overline{R})[\Pi], R[\Pi]) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{HH}(R[\Pi], R[\Pi]) \arrow[d, "="'] \\
\mathrm{HH}(\mathbb{Z} \circ \overline{R[\Pi]}, R[\Pi]) \arrow[r, "l_{R[\Pi]}"] & \mathrm{HH}(R[\Pi], R[\Pi]).
\end{tikzcd}$$

4.4. **THH of a commutative FSP.**

Proposition 4.4.1. *For a commutative FSP L,* $\mathrm{THH}(L)$ *is a cyclic object in the category of FSP's.*

Proof. For $n \subseteq \mathbb{N}$ let $\max(n)$ denote the maximum of the set n with the convention that the maximum of the empty set is 0. The maps

$$X \to F(S^0, X \wedge \mathrm{thh}(L)_k(S^0, \dots, S^0))$$

induced by the unit in L give a natural map $1^k_X : X \to \mathrm{THH}(L)_k(X)$. We can construct a product

$$\mu^k_{X,Y} : \mathrm{THH}(L)_k(X) \wedge \mathrm{THH}(L)_k(Y) \to \mathrm{THH}(L)_k(X \wedge Y)$$

that makes $\mathrm{THH}(L)_k$ a FSP as follows: First consider the case when $k=0$. There is a functor $\mu : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ with $\mu(n,m) = n \cup (m + \max(n))$ and natural homeomorphisms $S^n \wedge S^m \approx S^{\mu(n,m)}$. Consider the composition

$$\begin{aligned}\mathrm{THH}(L)_0(X) \wedge \mathrm{THH}(L)_0(Y) &\to \underset{(n,m)\in\mathcal{J}\times\mathcal{J}}{\underrightarrow{\mathrm{hocolim}}}\; F(S^n, X \wedge L(S^n)) \wedge F(S^m, Y \wedge L(S^m)) \\ &\to \underset{(n,m)\in\mathcal{J}\times\mathcal{J}}{\underrightarrow{\mathrm{hocolim}}}\; F(S^n \wedge S^m, X \wedge Y \wedge L(S^n \wedge S^m)) \\ &\to \underset{(n,m)\in\mathcal{J}\times\mathcal{J}}{\underrightarrow{\mathrm{hocolim}}}\; F(S^{\mu(n,m)}, X \wedge Y \wedge L(S^{\mu(n,m)})) \\ &\to \underset{l\in\mathcal{J}}{\underrightarrow{\mathrm{hocolim}}}\, F(S^l, X \wedge Y \wedge L(S^l)) = \mathrm{THH}(L)_0(X \wedge Y),\end{aligned}$$

where the first map is induced by the map

$$\underset{\mathcal{C}_1}{\underrightarrow{\mathrm{hocolim}}}\, G_1 \wedge \underset{\mathcal{C}_2}{\underrightarrow{\mathrm{hocolim}}}\, G_2 \to \underset{\mathcal{C}_1\times\mathcal{C}_2}{\underrightarrow{\mathrm{hocolim}}}\, G_1 \wedge G_2,$$

the second map is induced by multiplication in L, and the rest of the maps are induced from the structure discussed above. We leave it to the reader to check that 1^0 and μ^0 give $\mathrm{THH}(L)_0$ the structure of a FSP.

For $k > 0$ there is a functor $\mu : \mathcal{J}(k+1) \times \mathcal{J}(k+1) \to \mathcal{J}(k+1)$ with $\mu(n,m)_i = n_i \cup (m_i + \max(n_0 \cup \cdots \cup n_k))$ for $n = (n_0, \ldots, n_k)$ and $m = (m_0, \ldots, m_k)$, and there are natural homeomorphisms $S^{n_i} \wedge S^{m_i} \approx S^{\mu(n,m)_i}$. The product for $\mathrm{THH}(L)_k$ is now constructed analogously to the product for $\mathrm{THH}(L)_0$, and again we leave it to the reader to check that we have given $\mathrm{THH}(L)_k$ the structure of a FSP. It is a simple matter to check that

$$\mu^{k+1}_{X,Y} \circ (s_i, s_i) = s_i \circ \mu^k_{X,Y} : \mathrm{THH}(L)_k(X) \wedge \mathrm{THH}(L)_k(Y) \to \mathrm{THH}(L)_{k+1}(X \wedge Y)$$

for $i = 0, \ldots, k$ and that $\mu^k_{X,Y} \circ (t,t) = t \circ \mu^k_{X,Y}$. When L is commutative we also have the identity

$$\mu^{k-1}_{X,Y} \circ (d_i, d_i) = d_i \circ \mu^k_{X,Y} : \mathrm{THH}(L)_k(X) \wedge \mathrm{THH}(L)_k(Y) \to \mathrm{THH}(L)_{k-1}(X \wedge Y)$$

for $i = 0, \ldots, k$. On the index set level this identity boils down to the identity $\mu \circ (d_{0*}, d_{0*}) = d_{0*} \circ \mu : \mathcal{J}(2) \times \mathcal{J}(2) \to \mathcal{J}$. □

The following proposition is obtained by a variation of the above argument:

Proposition 4.4.2. *Let L be a commutative FSP and let A be an algebra over L. Then* $\mathrm{THH}(L, A)$ *is an algebra over L. If A is commutative and unital, then* $\mathrm{THH}(L, A)$ *is a unital algebra over A.*

Proposition 4.4.3. *Let A be a commutative FTP. Then* $\mathrm{HH}(A)$ *is a cyclic object in the category of FTP's.*

Proof. For fixed X and $n \in \mathcal{J}(k+1)$ let

$$W^X(n) = \underline{\mathrm{s.Ab}}(\mathbb{Z}(S^n), \mathbb{Z}(X) \otimes A(S^{n_0}) \otimes \cdots \otimes A(S^{n_k})).$$

Multiplication in A induces a linear map

$$W^X(n) \otimes W^Y(m) \to W^{X\wedge Y}(\mu(n,m)).$$

Composing with the linear map

$$\underset{\substack{\longrightarrow \\ n\in\mathcal{J}(k+1)}}{\mathrm{hocolim}}\, W^X(n) \otimes \underset{\substack{\longrightarrow \\ m\in\mathcal{J}(k+1)}}{\mathrm{hocolim}}\, W^Y(m) \to \underset{\substack{\longrightarrow \\ (n,m)\in\mathcal{J}(k+1)\times\mathcal{J}(k+1)}}{\mathrm{hocolim}} W^X(n)\otimes W^Y(m)$$

we get a linear map

$$m_{X,Y} : \mathrm{HH}(A)_k(X) \otimes \mathrm{HH}(A)_k(Y) \to \mathrm{HH}(A)_k(X\wedge Y).$$

We leave it to the reader to check that we obtain a map of cyclic objects in this way, and that it satisfies the required relations. □

Proposition 4.4.4. *Let A be a commutative FTP and let B be an algebra over A. Then $\mathrm{HH}(A,B)$ is an algebra over A. If B is commutative and unital, then $\mathrm{HH}(A,B)$ is a unital algebra over B.*

Note that since $n \mapsto \mathrm{HH}(A,B)(S^n)$ is an Ω-spectrum, $\mathrm{HH}(A,B)(S^0)$ is a simplicial ring with $\pi_i \,\mathrm{HH}(A,B)(S^0) = \pi_i \,\mathrm{HH}(A,B)$.

4.5. **Bökstedts definition of** THH. Let $I \subseteq J$ denote the full subcategory with objects $|n| = \{1,2,\dots,n\}$ for $n \geq 1$ and the empty set $|0|$. There is a functor $I^k \xrightarrow{\phi} \mathcal{J}(k)$ sending $(|n_1|,\dots,|n_k|)$ to $(\{1,\dots,n_1\},\{n_1+1,\dots,n_1+n_2\},\dots,\{n_1+\cdots+n_{k-1}+1,\dots,n_1+\cdots+n_k\})$. Therefore for a FSP L we can define

$$\mathrm{THH}'(L)_k(X) = \underset{\substack{\longrightarrow \\ x\in I^{k+1}}}{\mathrm{hocolim}}\, F(S^{\phi(x)}, X\wedge L(S^{\phi(x)_0})\wedge\cdots\wedge L(S^{\phi(x)_k})).$$

The functors $d_i : I^{k+1} \to I^k$ and $s_i : I^{k+1} \to I^{k+2}$ given by

$$d_i(|n_0|,\dots,|n_k|) = \begin{cases} (|n_0|,\dots,|n_i+n_{i+1}|,\dots,|n_k|) & \text{for } 0 \leq i < k, \\ (|n_k+n_0|,|n_1|,\dots,|n_{k-1}|) & \text{for } i = k \end{cases}$$

and

$$s_i(|n_0|,\dots,|n_k|) = (|n_0|,\dots,|n_i|,|0|,\dots,|n_k|) \quad \text{for } 0 \leq i \leq k,$$

together with the unit and multiplication in L define a simplicial structure on $\mathrm{THH}'(L)(X)$. This is Bökstedts original definition of topological Hochschild homology [1]. Similarly we can define a simplical set $\mathrm{THH}''(L)(X)$ by

$$\mathrm{THH}''(L)_k(X) = \underset{\substack{\longrightarrow \\ (n,x)\in\mathcal{J}(k+1)\times I^{k+1}}}{\mathrm{hocolim}} F(S^n\wedge S^{\phi(x)}, X\wedge L(S^{n_0}\wedge S^{\phi(x)_0})\wedge\cdots\wedge L(S^{n_k}\wedge S^{\phi(x)_k})).$$

The initial objects in $\mathcal{J}(k+1)$ and I^{k+1} induce maps

$$\mathrm{THH}(L)_k(X) \to \mathrm{THH}''(L)_k(X) \leftarrow \mathrm{THH}'(L)_k(X).$$

These maps are seen to be homotopy equivalences by the approximation lemma. Since these maps are simplicial we have obtained a weak equivalence between $\mathrm{THH}(L)(X)$ and $\mathrm{THH}'(L)(X)$.

5. Filtered FSP

5.1. Definitions.

Definition 5.1.1. A filtered pointed monoid is a pointed monoid Π together with pointed subsets $\Pi(n) \subseteq \Pi$ for $n = 0, 1, \dots$ satisfying that $\Pi(n) \subseteq \Pi(n-1)$, and that the product of Π maps $\Pi(m) \wedge \Pi(n)$ into $\Pi(m+n)$.

For $a = (a_0, \dots, a_k) \in \mathbb{N}^{k+1}$ let $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_a = \Pi(a_0) \wedge \cdots \wedge \Pi(a_k)$ and let

$$\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_{a,a+1} = \frac{\Pi(a_0)}{\Pi(a_0+1)} \wedge \cdots \wedge \frac{\Pi(a_k)}{\Pi(a_k+1)}.$$

For $s \in \mathbb{Z}$ we shall use the notation

$$\mathbb{N}(k,s) = \{(a_0, \dots, a_k) \in \mathbb{N}^{k+1} \mid a_0 + \cdots + a_k \geq -s\}$$
$$\mathbb{N}(k,s,s-1) = \{(a_0, \dots, a_k) \in \mathbb{N}^{k+1} \mid a_0 + \cdots + a_k = -s\}.$$

Definition 5.1.2. Let Π be a filtered pointed monoid, and let $s \in \mathbb{Z}$. $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi, s)$ is the cyclic subset of $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)$ with k-simplices

$$\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi, s)_k = \bigcup_{a \in \mathbb{N}(k,s)} \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_a.$$

and $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi, s, s-1)$ is the cyclic set with k-simplices

$$\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi, s, s-1)_k = \bigvee_{a \in \mathbb{N}(k,s,s-1)} \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_{a,a+1},$$

and with cyclic structure given by the same formula as for $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)$.

Definition 5.1.3. A filtered FSP L is a FSP $L = L(0)$ together with L-bimodules $L(n)$ for $n \in \mathbb{N}$ satisfying that $L(n)(X) \subseteq L(n-1)(X)$, and that the multiplication $\mu_{X,Y} : L(X) \wedge L(Y) \to L(X \wedge Y)$ maps $L(n)(X) \wedge L(m)(Y)$ into $L(n+m)(X \wedge Y)$.

Definition 5.1.4. A left module over a filtered FSP L is a left module $M = M(0)$ over L together with left L-modules $M(n)$ for $n \in \mathbb{N}$ satisfying that $M(n)(X) \subseteq M(n-1)(X)$, and that the multiplication $l_{X,Y} : L(X) \wedge M(Y) \to M(X \wedge Y)$ maps $L(n)(X) \wedge M(m)(Y)$ into $M(n+m)(X \wedge Y)$. Similarly we define right L -modules and L-bimodules.

Let L be a filtered FSP and let M be a bimodule over L. For $a = (a_0, \dots, a_k) \in \mathbb{N}^{k+1}$ we shall let $\mathrm{thh}(L, M)_a \in \mathcal{FS}_{k+1}$ be the functor with

$$\mathrm{thh}(L, M)_a(X) = M(a_0)(X_0) \wedge L(a_1)(X_1) \wedge \cdots \wedge L(a_k)(X_k),$$

for $X = (X_0, \dots, X_k)$, and we shall let $\mathrm{thh}(L, M)_{a,a+1}$ be the $(k+1)$-functor with

$$\mathrm{thh}(L, M)_{a,a+1}(X) = \frac{M(a_0)(X_0)}{M(a_0+1)(X_0)} \wedge \frac{L(a_1)(X_1)}{L(a_1+1)(X_1)} \wedge \cdots \wedge \frac{L(a_k)(X_k)}{L(a_k+1)(X_k)}.$$

We shall also use the notation

$$\mathrm{thh}(L, M, s)_k(X) = \bigcup_{a \in \mathbb{N}(k,s)} \mathrm{thh}(L, M)_a(X),$$

$$\mathrm{thh}(L, M, s, s-1)_k(X) = \bigvee_{a \in \mathbb{N}(k,s,s-1)} \mathrm{thh}(L, M)_{a,a+1}(X),$$

for $k \geq 0$, $s \in \mathbb{Z}$ and $X = (X_0, \dots, X_k)$.

Definition 5.1.5. (1) Let L be a filtered FSP and let $s \in \mathbb{Z}$. $\mathrm{THH}(L,s)$ and $\mathrm{THH}(L,s,s-1)$ are the cyclic object in $\mathcal{FS}$ with k-simplices given by

$$\mathrm{THH}(L,s)_k = D(\mathbb{S} \wedge \mathrm{thh}(L,L,s)),$$
$$\mathrm{THH}(L,s,s-1)_k = D(\mathbb{S} \wedge \mathrm{thh}(L,L,s,s-1)),$$

and cyclic structure similar to that of $\mathrm{THH}(L)$.

(2) Let L be a filtered FSP and let M be a bimodule over L. For $s \in \mathbb{Z}$, $\mathrm{THH}(L,M,s)$ and $\mathrm{THH}(L,M,s,s-1)$ are the simplicial sets with k-simplices

$$\mathrm{THH}(L,M,s)_k = D\,\mathrm{thh}(L, M \circ (\mathbb{S} \wedge \mathbb{S}), s)_k,$$
$$\mathrm{THH}(L,M,s,s-1)_k = D\,\mathrm{thh}(L, M \circ (\mathbb{S} \wedge \mathbb{S}), s, s-1)_k,$$

and simplicial structure similar to $\mathrm{THH}(L,M)$.

Definition 5.1.6. A filtered FTP is a FTP $A = A(0)$ together with A-bimodules $A(n) \subseteq A$ such that $\overline{A}$ becomes a filtered FSP. Modules over filtered FTP's are defined in the obvious way similar to modules over filtered FSP's.

Let A be a filtered FTP and let M be bimodule over A. For $a \in \mathbb{N}^{k+1}$ we shall let $\mathrm{hh}_a(A,M)$ and $\mathrm{hh}_{a,a+1}(A,M)$ denote the functors with

$$\begin{aligned}\mathrm{hh}_a(A,M)(X) = \mathrm{Im}(M(a_0)(X_0) \otimes A(a_1)(X_1) \otimes \cdots \otimes A(a_k)(X_k)& \\ \to M(X_0) \otimes A(X_1) \otimes \cdots \otimes A(X_k)&),\end{aligned}$$

and

$$\mathrm{hh}(A)_{a,a+1}(X) = \frac{M(a_0)(X_0)}{M(a_0+1)(X_0)} \otimes \frac{A(a_1)(X_1)}{A(a_1+1)(X_1)} \otimes \cdots \otimes \frac{A(a_k)(X_k)}{A(a_k+1)(X_k)}$$

for $X = (X_0, \ldots, X_k)$. We shall also use the notation

$$\mathrm{hh}(A,M,s)_k(X) = \sum_{a \in \mathbb{N}(k,s)} \mathrm{hh}_a(A,M)(X) \subseteq \mathrm{hh}(A,M)_k(X),$$

and

$$\mathrm{hh}(A,M,s,s-1)_k(X) = \bigoplus_{a \in \mathbb{N}(k,s,s-1)} \mathrm{hh}_{a,a+1}(A,M)(X)$$

for $s \in \mathbb{Z}$ and $k \geq 0$.

Definition 5.1.7. (1) Let A be a filtered FTP and let $s \in \mathbb{Z}$. $\mathrm{HH}(A,s)$ and $\mathrm{HH}(A,s,s-1)$ are the cyclic objects in $\mathcal{FSA}$ with k-simplices given by

$$\mathrm{HH}(A,s)_k = D(\mathbb{Z} \circ \mathbb{S} \otimes \mathrm{hh}(A,A,s)_k)$$
$$\mathrm{HH}(A,s,s-1)_k = D(\mathbb{Z} \circ \mathbb{S} \otimes \mathrm{hh}(A,A,s,s-1)_k)$$

and cyclic structure similar to that of $\mathrm{HH}(A)$.

(2) Let A be a filtered FTP and let M be a bimodule over A. For $s \in \mathbb{Z}$, $\mathrm{HH}(A,M,s)$ and $\mathrm{HH}(A,M,s,s-1)$ are the simplicial sets with

$$\mathrm{HH}(A,M,s)_k = D\,\mathrm{hh}(A, M \circ (\mathbb{S} \wedge \mathbb{S}), s)_k$$
$$\mathrm{HH}(A,M,s,s-1)_k = D\,\mathrm{hh}(A, M \circ (\mathbb{S} \wedge \mathbb{S}), s, s-1)_k$$

and simplicial structure similar to that of $\mathrm{HH}(A,M)$.

Definition 5.1.8. A filtered simplicial ring is a simplicial ring $R = R(0)$ together with R-bimodules $R(n) \subset R(n-1)$ satisfying that the multiplication in R maps $R(n) \otimes R(m)$ into $R(n+m)$.

Note that all the constructions we have made in this section are functorial.

Example 5.1.9.
- A filtered pointed monoid Π gives rise to a filtered FSP $\Sigma^\infty \Pi$.
- The pointed monoid $\Pi = \{0, 1, t, \dots, t^N\} \subseteq \mathbb{Z}[t]/(t^{N+1})$ together with the sets $\Pi(n) = \{0, t^n, t^{n+1}, \dots, t^N\}$ is a filtered pointed monoid.
- A filtered simplicial ring gives rise to a filtered FTP.
- Let Π be a filtered pointed monoid and let A be a FTP. Then $A[\Pi](n) = \mathbb{Z}(\Pi(n)) \otimes A(X)$ gives $A[\Pi]$ the structure of a filtered FTP.
- Let R be a simplicial ring, and let I be a two-sided ideal in R. Then $R(n) = I^n$ gives R the structure of a filtered ring.
- A graded ring $R = \bigoplus_{t \in \mathbb{N}} R_t$ gives rise to a filtered ring with $R(n) = \bigoplus_{t \geq n} R_t$. Conversely a filtered ring gives R rise to a graded $\mathcal{G}R$ ring with $\mathcal{G}R_t = R(t)/R(t+1)$.
- A filtered FSP L gives rise to another filtered FSP $\mathcal{G}(L)$ with
$$\mathcal{G}(L)(n)(X) = \bigvee_{i=n}^{\infty} \frac{L(i)(X)}{L(i+1)(X)}.$$
We shall call $\mathcal{G}(L)$ the graded FSP associated to L. Note that there is a homeomorphism
$$\mathrm{THH}(L, s, s-1) \approx \mathrm{THH}(\mathcal{G}(L), s, s-1)$$
for $s \in \mathbb{Z}$.
- A filtered FTP A gives rise to another FTP $\mathcal{G}(A)$ with
$$\mathcal{G}(A)(n)(X) = \bigoplus_{i=n}^{\infty} \frac{A(i)(X)}{A(i+1)(X)}.$$
Note that there is an isomorphism
$$\mathrm{HH}(A, s, s-1) \cong \mathrm{HH}(\mathcal{G}(A), s, s-1).$$
If M is a bimodule over A we obtain a bimodule $\mathcal{G}(M)$ over $\mathcal{G}(A)$ with
$$\mathcal{G}(M)(n)(X) = \bigoplus_{i=n}^{\infty} \frac{M(i)(X)}{M(i+1)(X)},$$
and there is an isomorphism
$$\mathrm{HH}(A, M, s, s-1) \approx \mathrm{HH}(\mathcal{G}(A), \mathcal{G}(M), s, s-1)$$

5.2. **Cofibration sequences.**

Proposition 5.2.1. *Let Π be a filtered pointed monoid. There are cofibration sequences*
$$\mathrm{N}^{\mathrm{cy}}_\wedge(\Pi, s-1) \to \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi, s) \to \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi, s, s-1)$$
for $s \in \mathbb{Z}$.

Proof. There is a splitting
$$\frac{\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi, s)_k}{\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi, s-1)_k} \approx \bigvee_{a\in\mathbb{N}(r,s,s-1)} \frac{\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_a}{\bigcup_{i=0}^{k} \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_{a^i}},$$
where $a^i = (a_0, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_k)$. On the other hand there is a homeomorphism
$$\frac{\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_a}{\bigcup_{i=0}^{k} \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_{a^i}} \approx \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi)_{a,a+1}.$$
□

Proposition 5.2.2. *Let L be a filtered FSP and let M be a bimodule over L. There are cofibration sequences*
$$\mathrm{THH}(L, M, s-1) \to \mathrm{THH}(L, M, s) \to \mathrm{THH}(L, M, s, s-1).$$
in $\mathcal{FS}$ for $s \in \mathbb{Z}$.

Proof. As in 5.2.1 we see that there is a cofibration sequence
$$\mathrm{thh}(L, M, s-1)_k(X) \to \mathrm{thh}(L, M, s)_k(X) \to \mathrm{thh}(L, M, s, s-1)_k(X)$$
for $X = (X_0, \dots, X_k)$. Applying lemma 3.1.6 and the approximation lemma we obtain the result. □

Proposition 5.2.3. *Let A be a filtered FTP and let M be a A-bimodule. If the simplicial abelian groups $A(n)(X)$, $M(n)(X)$, $A(n)(X)/A(n+1)(X)$ and $M(n)(X)/M(n+1)(X)$ are free, then there is a cofibration sequence*
$$\mathrm{HH}(A, M, s-1) \to \mathrm{HH}(A, M, s) \to \mathrm{HH}(A, M, s, s-1)$$
in $\mathcal{FSA}$ for $s \in \mathbb{Z}$.

Proof. By expanding in direct summands we can see that there is a splitting
$$\frac{\mathrm{HH}(A, M, s)_k(X)}{\mathrm{HH}(A, M, s-1)_k(X)} \cong \bigoplus_{a\in\mathbb{N}(r,s,s-1)} \frac{Z_a(A, M)(X)}{\sum_{i=0}^{k} Z_{a^i}(A, M)(X)}$$
where $a^i = (a_0, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots a_k)$ and $X = (X_0, \dots, X_k)$. On the other hand there is an isomorphism
$$\frac{Z_a(A, M)(X)}{\sum_{i=0}^{k} Z_{a^i}(A, M)(X)} \cong Z_{a,a+1}(A, M)(X).$$
□

5.3. **Monoid rings.** Let Π be a filtered pointed monoid and let R be a simplicial ring that is free as an abelian group. Let A be a FTP such that R is a A-bimodule. There are commutative diagrams of natural maps,
$$\begin{array}{ccc}
\mathrm{HH}(A, R, s) \circ \Sigma^\infty \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi) & \xrightarrow{\cong} & \mathrm{HH}(A[\Pi], R[\Pi], s) \\
\downarrow & & \downarrow \\
\mathrm{HH}(A, R, s, s-1) \circ \Sigma^\infty \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi) & \xrightarrow{\cong} & \mathrm{HH}(A[\Pi], R[\Pi], s, s-1),
\end{array}$$
where the horizontal maps are isomorphisms similar to those of section 4.3. We shall be particularly interested in the case where A is R or $\mathbb{Z} \circ \overline{R}$, and the diagrams similar the one in section 4.3.

6. Spectral sequences

6.1. **Filtered objects.** Let $M(s)$ and $M(s, s-1)$ be functors with stabilization together with cofibration sequences

$$M(s-1) \to M(s) \to M(s, s-1)$$

for $s \in \mathbb{Z}$. The homotopy groups build up an exact couple, and there is an associated spectral sequence with E^1-terms $E^1_{s,t} = \pi_t M(s, s-1)$. Let $J_{s,t}$ be the image of the map $\pi_{s+t} M(s) \to \pi_{s+t} M$, where $M(X) = \operatorname{hocolim}_{\overrightarrow{\mathbb{Z}}} M(s)$. If for fixed s and t there is a N such that $\pi_{s+t} M(s-r) = 0$ for $r \geq N$, then there are isomorphisms $J_{s,t}/J_{s-1,t+1} \cong E^\infty_{s,t}$. A proof of this fact is given in Whitehead [12, XIII.3].

Let L be a filtered FSP and let M be a bimodule over L. It follows from proposition 5.2.2 that if there is a number k such that $M(n)$ and $L(n)$ are $(n-k)$-connected, then there is a spectral sequence $E^r_{s,t}(\mathrm{THH}(L, M))$ with E^1-terms $\pi_{s+t} \mathrm{THH}(L, M, s, s-1)$ converging towards $\pi_{s+t} \mathrm{THH}(L, M)$.

Let A be a filtered FTP and let M be a bimodule over A. It follows from proposition 5.2.3 that if there is a number k such that $M(n)$ and $A(n)$ are $(n-k)$-connected, then there is a spectral sequence $E^r_{s,t}(\mathrm{HH}(A, M))$ with E^1-terms $\pi_{s+t} \mathrm{HH}(A, M, s, s-1)$ converging towards $\pi_{s+t} \mathrm{HH}(A, M)$.

We shall use this spectral sequence in the case where the filtered FTP is on the form $A[\Pi]$ for a filtered pointed monoid Π and where $M = M[\Pi]$. From the weak equivalence of section 5.3 and the stable equivalence $\mathrm{HH}(A, M) \wedge Z \to \mathrm{HH}(A, M) \circ \Sigma^\infty Z$ we see that there are isomorphisms

$$\begin{aligned} E^1_{s,t}(\mathrm{HH}(A[\Pi], M[\Pi])) &\cong \pi_{s+t}(\mathrm{HH}(A, M) \wedge \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi, s, s-1)) \quad \text{and} \\ \pi_{s+t} \mathrm{HH}(A[\Pi], M[\Pi]) &\cong \pi_{s+t}(\mathrm{HH}(A, M) \wedge \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi)). \end{aligned}$$

These isomorphisms are natural in A and M. In particular in the case of a simplicial ring R and a bimodule M over R there are commutative diagrams:

$$\begin{array}{ccc} \pi_{s+t}(\mathrm{HH}(\mathbb{Z} \circ \overline{R}, M) \wedge \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi, s, s-1)) & \xrightarrow{l_R} & \pi_{s+t}(\mathrm{HH}(R, M) \wedge \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi, s, s-1)) \\ \cong \downarrow & & \cong \downarrow \\ E^1_{s,t}(\mathrm{HH}(\mathbb{Z} \circ \overline{R[\Pi]}, M[\Pi])) & \xrightarrow{l_{R[\Pi]}} & E^1_{s,t}(\mathrm{HH}(R[\Pi], M[\Pi])) \end{array}$$

and

$$\begin{array}{ccc} \pi_{s+t}(\mathrm{HH}(\mathbb{Z} \circ \overline{R}, M) \wedge \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi)) & \xrightarrow{l_R} & \pi_{s+t}(\mathrm{HH}(R, M) \wedge \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi)) \\ \cong \downarrow & & \cong \downarrow \\ \pi_{s+t}(\mathrm{HH}(\mathbb{Z} \circ \overline{R[\Pi]}, M[\Pi])) & \xrightarrow{l_{R[\Pi]}} & \pi_{s+t}(\mathrm{HH}(R[\Pi], M[\Pi])). \end{array}$$

6.2. **Homological algebra.**

Definition 6.2.1. Let A be a FTP, let M be a right- and let N be a left A-module. The bar-construction $B(M, A, N)$ is the simplicial object in $\mathcal{FSA}$ with k-simplices

$$B_k(M, A, N) = D(M \circ (\mathbb{S} \wedge \mathbb{S}) \otimes A^{\otimes k} \otimes N),$$

and with face and degeneracy maps induced by the formulas

$$d_i(x_0 \otimes \cdots \otimes x_{k+1}) = x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{k+1},$$
$$s_i(x_0 \otimes \cdots \otimes x_{k+1}) = x_0 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_{k+1},$$

$0 \leq i \leq k$, in the same way as the face and degeneracy maps for THH are induced by similar formulas.

Lemma 6.2.2. *(1) Let A be a FTP and let M be a right A-module. There is a homotopy equivalence $M \xrightarrow{\simeq} B(M, A, A)$.*

(2) Let A_1 and A_2 be FTP's, let M be a $A_1 - A_2$-bimodule and let N be a $A_2 - A_1$-bimodule. There is a weak equivalence

$$\mathrm{HH}(A_1, B(M, A_2, N)) \simeq \mathrm{HH}(A_2, B(N, A_1, M)).$$

Proof. (1) There is a simplicial contraction of $B(M, A, A)$ given by the formula

$$h_i(x_0 \otimes \cdots \otimes x_{k+1}) = x_0 \otimes \cdots \otimes x_i \otimes x_{i+1}x_{i+2} \ldots x_{k+1} \otimes 1 \otimes \cdots \otimes 1$$

for $0 \leq i \leq k$.

(2) Let $W(A_1, M, A_2, N)$ be the bisimplicial object in $\mathcal{FSA}$ with (k,l)-simplices given by

$$W(A_1, M, A_2, N)_{k,l}(X) = \\ \underset{(n,m)\in \mathcal{J}(l+2)\times\mathcal{J}(k+2)}{\underrightarrow{\mathrm{hocolim}}} \underline{\mathrm{s.Ab}}(\mathbb{Z}(S^n \wedge S^m), M(S^{m_{k+1}} \wedge X \wedge S^{n_0}) \otimes A_2(S^{n_1}) \otimes \cdots \otimes \\ A_2(S^{n_l}) \otimes N(S^{n_{l+1}} \wedge S^{m_0}) \otimes A_1(S^{m_1}) \otimes \cdots \otimes A_1(S^{n_k}))$$

and with face and degeneracy maps similar to those of $\mathrm{HH}(A_1, B(M, A_2, N))$. Clearly the diagonal of $W(A_1, M, A_2, N)$ and of $W(A_2, N, A_1, M)$ are isomorphic. For a functor $G : \mathcal{J}(l+2) \to \underline{\mathrm{s.Ab}}$ there is a map

$$\underset{n\in\mathcal{J}(l+2)}{\underrightarrow{\mathrm{hocolim}}}\, \underline{\mathrm{s.Ab}}(\mathbb{Z}(S^m), G(n)) \to \underline{\mathrm{s.Ab}}(\mathbb{Z}(S^m), \underset{n\in\mathcal{J}(l+2)}{\underrightarrow{\mathrm{hocolim}}}\, G(n)).$$

Together with the adjunction of mapping space and $\otimes$ this map induces a weak equivalence

$$W(A_1, M, A_2, N)_{k,l}(X) \simeq \mathrm{HH}_k(A_1, B_l(M, A_2, N))(X).$$

by the approximation lemma. □

Lemma 6.2.3. *Let R be a simplicial ring, let M be a right R-module and let $N' \to N$ be a homotopy equivalence of left R-modules.*

(1) There are weak equivalences

$$B(M, \mathbb{Z} \circ R, \mathbb{Z} \circ N') \xrightarrow{\simeq} B(M, \mathbb{Z} \circ R, \mathbb{Z} \circ N),$$
$$B(M, \mathbb{Z} \circ R, \mathbb{Z} \circ N) \simeq B(\mathbb{Z} \circ M, \mathbb{Z} \circ R, N).$$

(2) If N' is free over R there is a weak equivalence

$$M \otimes_R N' \simeq B(\mathbb{Z} \circ M, \mathbb{Z} \circ R, N').$$

Proof. (1) A weak equivalence $N' \to N$ induces a weak equivalence $\mathbb{Z} \circ N' \to \mathbb{Z} \circ N$ by the Hurewicz theorem. By the Künneth formula the map

$$(M \otimes (\mathbb{Z} \circ R)^{\otimes k} \otimes \mathbb{Z} \circ N')(X_0, \ldots, X_{k+1}) \to (M \otimes (\mathbb{Z} \circ R)^{\otimes k} \otimes \mathbb{Z} \circ N)(X_0, \ldots, X_{k+1})$$

is a weak equivalence. The first of the weak equivalences follows. From the stable equivalences

$$N \otimes \mathbb{Z} \circ (R^{\wedge k} \wedge M) \leftarrow N \wedge R^{\wedge k} \wedge M \to \mathbb{Z} \circ (N \wedge R^{\wedge k}) \otimes M,$$

the other weak equivalence follows.

(2) Follows from the weak equivalences

$$B(\mathbb{Z} \circ M, \mathbb{Z} \circ R, R) \simeq B(M, \mathbb{Z} \circ R, \mathbb{Z} \circ R) \simeq M$$

and additivity of $B(\mathbb{Z} \circ M, \mathbb{Z} \otimes R, -)$. □

Corollary 6.2.4. *Let R be a simplicial ring and let M be a bimodule over R. If R is free as an abelian group then there are weak equivalences*

$$\mathrm{HH}(R, \mathrm{HH}(\mathbb{Z} \circ R, R \otimes M)) \simeq \mathrm{HH}(\mathbb{Z} \circ R, M) \quad \textit{and}$$
$$\mathrm{HH}(\mathbb{Z} \circ R, R \otimes M) \simeq \mathrm{HH}(\mathbb{Z} \circ \mathbb{Z}, M).$$

Proof. There are chains of weak equivalences

$$\begin{aligned}\mathrm{HH}(R, \mathrm{HH}(\mathbb{Z} \circ R, R \otimes M)) &\simeq \mathrm{HH}(R, \mathrm{HH}(\mathbb{Z} \circ R, B(R, \mathbb{Z}, M)) \\ &\simeq \mathrm{HH}(R, \mathrm{HH}(\mathbb{Z}, B(M, \mathbb{Z} \circ R, R)) \\ &\simeq \mathrm{HH}(R, B(M, \mathbb{Z} \circ R, R)) \\ &\simeq \mathrm{HH}(\mathbb{Z} \circ R, B(R, R, M)) \simeq \mathrm{HH}(\mathbb{Z} \circ R, M)\end{aligned}$$

and

$$\begin{aligned}\mathrm{HH}(\mathbb{Z} \circ R, M \otimes R) &\simeq \mathrm{HH}(\mathbb{Z} \circ R, B(\mathbb{Z} \circ R, \mathbb{Z} \circ \mathbb{Z}, M)) \\ &\simeq \mathrm{HH}(\mathbb{Z} \circ \mathbb{Z}, B(M, \mathbb{Z} \circ R, \mathbb{Z} \circ R)) \\ &\simeq \mathrm{HH}(\mathbb{Z} \circ \mathbb{Z}, M).\end{aligned}$$

□

Before we can state the next theorem we need to note that for a simplicial ring R there is a homomorphism $R \to \pi_0 R$, where $\pi_0 R$ is a constant simplicial ring.

Theorem 6.2.5. *Let R be a simplicial ring that is free as an abelian group and let M be a simplicial bimodule over $\pi_0 R$. There is a spectral sequence with E^2-terms*

$$E^2_{s,t} = \pi_s \,\mathrm{HH}(R, \pi_t \,\mathrm{HH}(\mathbb{Z} \circ \mathbb{Z}, M))$$

converging towards $\pi_{s+t} \,\mathrm{HH}(\mathbb{Z} \circ R, M)$. In the case where R is commutative and M is an algebra over $\pi_0 R$, this is a spectral sequence of graded algebras over $\pi_0 R$.

This spectral sequence was first considered in [9].

Proof. Let G denote the bi-simplicial abelian group with (k, l)-simplices

$$G_{k,l} = \mathrm{HH}_k(R_k, \mathrm{HH}_l(\mathbb{Z} \circ R_k, R_k \otimes M_l))(S^0).$$

There is a bicomplex $C_{*,*}$ associated to G with $C_{k,l} = G_{k,l}$ and differentials given by $d^1 g = \sum_{i=0}^{k} (-1)^i d^1_i g$ and $d^2 g = \sum_{i=0}^{l} (-1)^{k+i} d^2_i g$ for $g \in G_{k,l}$. Here d^1_i and d^2_i denote the face maps in the two simplicial directions of G. There is a spectral sequence with E^1-terms $E^1_{s,t} = H_t(C_{*,s})$ converging towards $\pi_{s+t} G$. Since R_s is free as an abelian group we have from corollary 6.2.4 that

$$\begin{aligned}E^1_{s,t} &= \mathrm{HH}_s(R_s, \pi_t \,\mathrm{HH}(\mathbb{Z} \circ R_s, R_s \otimes M))(S^0) \\ &\cong \mathrm{HH}_s(R_s, \pi_t \,\mathrm{HH}(\mathbb{Z} \circ \mathbb{Z}, M))(S^0).\end{aligned}$$

It follows that the E^2-terms of the spectral sequence have the asserted form.

In the case where R is commutative and M is a commutative algebra over $\pi_0 R$, proposition 4.4.4 says that $\mathrm{HH}(R, \mathrm{HH}(\mathbb{Z} \circ R \otimes M))$ is a FTP, and hence that $C_{*,*}$ is a differential bigraded algebra over $\pi_0 R$. Therefore the spectral sequence is a spectral sequence of graded algebras over $\pi_0 R$. □

Remark 6.2.6. *The structure of $\pi_t \mathrm{HH}(\mathbb{Z} \circ \mathbb{Z}, M)$ as a module over $R \otimes R$ is not given explicitly in the above proof. But since $\pi_t \mathrm{HH}(\mathbb{Z} \circ \mathbb{Z}, M)$ is a constant simplicial abelian group we know that the action of $R \otimes R$ factors through $\pi_0 R \otimes \pi_0 R$. Hence there is an isomorphism*

$$E^1_{s,t} \cong \mathrm{HH}_s(R_s, \pi_0 R \otimes \pi_0 R)(S^0) \otimes_{\pi_0 R \otimes \pi_0 R} \pi_t \mathrm{HH}(\mathbb{Z} \circ \mathbb{Z}, M),$$

and the algebra structure of $E^1_{*,*}$ is the one induced by $\pi_* \mathrm{HH}(R, \pi_0 R \otimes \pi_0 R)$ and $\pi_* \mathrm{HH}(\mathbb{Z} \circ \mathbb{Z}, M)$.

Theorem 6.2.7. *Let R be a commutative ring, let S be an algebra over R and let M be a bimodule over S. There is a spectral sequence with E^2-terms*

$$E^2_{s,t} = \pi_s \mathrm{HH}(\mathbb{Z} \circ S, \mathrm{Tor}^R_t(S, M))$$

converging towards $\pi_{s+t} \mathrm{HH}(\mathbb{Z} \circ R, M)$.

If S is commutative and if M is an algebra over S, then this is a spectral sequence of graded rings.

Proof. From the chain of weak equivalences

$$\begin{aligned} \mathrm{HH}(\mathbb{Z} \circ R, M) &\simeq \mathrm{HH}(\mathbb{Z} \circ R, B(M, \mathbb{Z} \circ S, \mathbb{Z} \circ S)) \\ &\simeq \mathrm{HH}(\mathbb{Z} \circ S, B(\mathbb{Z} \circ S, \mathbb{Z} \circ R, M)) \end{aligned}$$

we obtain a bi-simplicial abelian group. We can proceed as in the proof of theorem 6.2.5 □

Proposition 6.2.8. *Let R be a simplicial commutative ring, let S be an algebra over R and let M be a bimodule over S. There is a spectral sequence with E^2-terms*

$$E^2_{s,t} = \pi_s \mathrm{HH}(S, \mathrm{Tor}^R_t(S, M))$$

converging towards $\pi_{s+t} \mathrm{HH}(R, M)$.

Proof. This is a corollary to theorem II.6.6 in [10], since $\pi_* \mathrm{HH}(R, M) = \pi_*(R \overset{L}{\otimes}_{R \otimes R} M)$ in the terminology of [10]. □

Remark 6.2.9. *For simplicity we have chosen to work with $\mathbb{Z}$ as ground ring in this paper. But we might as well choose another ground ring k and define FTP's and* HH *with respect to tensor products over k. We then have the following version of theorem 6.2.5:*

If R is a simplicial k-algebra that is free as a module over k and if M is a bimodule over $\pi_0 R$, there is a spectral sequence with E^2-terms

$$E^2_{s,t} = \pi_s \mathrm{HH}^k(R, \pi_t \mathrm{THH}(k, M))$$

converging towards $\pi_{s+t} \mathrm{THH}(R, M)$.

Example 6.2.10. Let R be a simplicial ring that is free as an abelian group, with a homotopy equivalence $R \xrightarrow{\simeq} \mathbb{Z}/n$. By proposition 6.2.8 there is a spectral sequence with E^2-terms

$$E^2_{s,t} = \pi_s \operatorname{HH}(R, \operatorname{Tor}^{\mathbb{Z}}_t(R, \mathbb{Z}/n))$$

converging towards $\pi_{s+t} \operatorname{HH}(\mathbb{Z}, \mathbb{Z}/n)$. An easy induction argument shows that $\pi_{2k} \operatorname{HH}(R, \mathbb{Z}/n) \cong \mathbb{Z}/n$ and $\pi_{2k+1} \operatorname{HH}(R, \mathbb{Z}/n) = 0$ for $k \geq 0$.

Theorem 6.2.5 gives a spectral sequence with E^2-terms

$$E^2_{s,t} = \pi_s \operatorname{HH}(R, \pi_t \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}, \mathbb{Z}/n))$$

converging towards $\pi_{s+t} \operatorname{HH}(\mathbb{Z} \circ R, \mathbb{Z}/n)$. As an algebra over $\mathbb{Z}/n$ the E^1-terms of this spectral sequence are

$$E^1_{s,t} \cong \operatorname{HH}_s(R_s, \mathbb{Z}/n)(S^0) \otimes_{\mathbb{Z}/n} \pi_t \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}, \mathbb{Z}/n).$$

By the Künneth formula there are isomorphisms

$$\pi_*(\operatorname{HH}(R, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} M) \cong \pi_* \operatorname{HH}(R, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} M$$

for any module M over $\mathbb{Z}/n$. Therefore as a graded ring the E^2-terms can be written

$$E^2_{s,t} \cong \pi_s \operatorname{HH}(R, \mathbb{Z}/n) \otimes \pi_t \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}, \mathbb{Z}/n).$$

Example 6.2.11. This computation is due to T.Pirashvili [8]. Theorem 6.2.7 gives a map of spectral sequences $E^r \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}, \mathbb{Z}/p) \to E^r \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}/p^2, \mathbb{Z}/p)$ converging towards the map $\pi_* \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}, \mathbb{Z}/p) \to \pi_* \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}/p^2, \mathbb{Z}/p)$ induced by the map $\mathbb{Z} \to \mathbb{Z}/p^2$. On E^2-terms this is the map

$$\pi_s \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}/p, \operatorname{Tor}^{\mathbb{Z}}_t(\mathbb{Z}/p, \mathbb{Z}/p)) \to \pi_s \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}/p, \operatorname{Tor}^{\mathbb{Z}/p^2}_t(\mathbb{Z}/p, \mathbb{Z}/p)),$$

induced by the map

$$\operatorname{Tor}^{\mathbb{Z}}_*(\mathbb{Z}/p, \mathbb{Z}/p) \to \operatorname{Tor}^{\mathbb{Z}/p^2}_*(\mathbb{Z}/p, \mathbb{Z}/p).$$

It is not hard to see that there is a commutative diagram

$$\begin{tikzcd} \operatorname{Tor}^{\mathbb{Z}}_*(\mathbb{Z}/p, \mathbb{Z}/p)) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Tor}^{\mathbb{Z}/p^2}_*(\mathbb{Z}/p, \mathbb{Z}/p)) \arrow[d, "\cong"'] \\ E(x) \arrow[r] & E(x) \otimes S(y) \end{tikzcd}$$

where $E(x)$ is an exterior algebra on a generator x of degree 1, $S(y)$ is a symmetric algebra on a generator y of degree 2, and the lower horizontal map is the canonical inclusion. Bökstedt has computed that as an algebra $\pi_* \operatorname{THH}(\mathbb{Z}/p)$ is isomorphic to a symmetric algebra $S(\sigma)$ on a generator σ of degree 2 [2]. We obtain a commutative diagram of bigraded algebras:

$$\begin{tikzcd} \pi_* \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}/p, \operatorname{Tor}^{\mathbb{Z}}_*(\mathbb{Z}/p, \mathbb{Z}/p)) \arrow[r] \arrow[d, "\cong"'] & \pi_* \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}/p, \operatorname{Tor}^{\mathbb{Z}/p^2}_*(\mathbb{Z}/p, \mathbb{Z}/p)) \arrow[d, "\cong"'] \\ S(\sigma) \otimes E(x) \arrow[r] & S(\sigma) \otimes E(x) \otimes S(y) \end{tikzcd}$$

where σ has bidegree $(2, 0)$, x has bidegree $(0, 1)$, y bas bidegree $(0, 2)$, and the lower horizontal map is the canonical inclusion. Hence to determine the spectral sequences it suffices to compute $d^2\sigma$ in $E^2 \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}, \mathbb{Z}/p)$. Since $\pi_1 \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p) = 0$, we must have $d^2\sigma = x$. We can now deduce that for $r > 2$

$$E^r_{*,*} \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}/p^2, \mathbb{Z}/p) \cong S(\sigma^p) \cdot (S(y) + \sigma^{p-1} x S(y)).$$

7. THH FOR $\mathbb{Z}/p^2$

7.1. **The main result.** In this section we shall prove

Theorem 7.1.1. *Let p be an odd prime. For $i \geq 0$ there is an isomorphism*

$$\pi_i \operatorname{THH}(\mathbb{Z}/p^2) \cong \bigoplus_{k\geq 0} \pi_{i-2k} \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p^2)$$

The underlying graded abelian group of the graded ring $\pi_* \operatorname{THH}(\mathbb{Z})$ has been computed by Bökstedt [2].

Theorem 7.1.2. *$\pi_0 \operatorname{THH}(\mathbb{Z}) \approx \mathbb{Z}$ and $\pi_{2i-1} \operatorname{THH}(\mathbb{Z}) \approx \mathbb{Z}/i$ for $i > 0$, and these are the only non-zero homotopy groups for $\operatorname{THH}(\mathbb{Z})$.*

The short exact sequence $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p^2$ induces a cofibration sequence

$$\operatorname{THH}(\mathbb{Z}) \to \operatorname{THH}(\mathbb{Z}) \to \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p^2),$$

and we can deduce that for $k > 0$

$$\pi_{2k} \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p^2) \cong \pi_{2k-1} \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p^2) \cong \mathbb{Z}/(k, p^2),$$

where (k, p^2) denotes the greatest common divisor of p^2 and k, and that $\pi_0 \operatorname{THH}(\mathbb{Z}, \mathbb{Z}/p^2) \cong \mathbb{Z}/p^2$.

In this section and in the next section p will denote a fixed odd prime. The main step in our proof of theorem 7.1.1 is the following proposition.

Proposition 7.1.3. *Let R be a simplicial ring, that is free as an abelian group, with a homotopy equivalence $R \xrightarrow{\simeq} \mathbb{Z}/p^2$. The map $\mathbb{Z} \circ R \to R$ induces a surjection*

$$\pi_* \operatorname{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2) \to \pi_* \operatorname{HH}(R, \mathbb{Z}/p^2)$$

of graded rings.

Before we prove proposition 7.1.3, let us show how theorem 7.1.1 follows. In example 6.2.10 we saw that there is a spectral sequence with E^2-terms

$$\begin{aligned} E^2_{s,t} &= \pi_s \operatorname{HH}(R, \pi_t \operatorname{HH}(\mathbb{Z} \circ R, R \otimes \mathbb{Z}/p^2)) \\ &\cong \pi_s \operatorname{HH}(R, \pi_t \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}, \mathbb{Z}/p^2)) \\ &\cong \pi_s \operatorname{HH}(R, \mathbb{Z}/p^2) \otimes \pi_t \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}, \mathbb{Z}/p^2) \end{aligned}$$

converging towards $\pi_{s+t} \operatorname{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2)$. From the commutative diagram

$$\begin{tikzcd} \operatorname{HH}(R, \operatorname{HH}(\mathbb{Z} \circ R, R \otimes \mathbb{Z}/p^2)) \arrow[r, "\simeq"] \arrow[d] & \operatorname{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2) \arrow[d, "l_R"'] \\ \operatorname{HH}(R, \pi_0 \operatorname{HH}(\mathbb{Z} \circ R, R \otimes \mathbb{Z}/p^2)) \arrow[r, "\simeq"] & \operatorname{HH}(R, \mathbb{Z}/p^2) \end{tikzcd}$$

it follows that the map

$$l_R : \pi_* \operatorname{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2) \to \pi_* \operatorname{HH}(R, \mathbb{Z}/p^2)$$

is an edge homomorphism for this spectral sequence. Since this map is onto and since we have a spectral sequence of graded rings, we can conclude that

$$E^r_{s,t} \cong \pi_s \operatorname{HH}(R, \mathbb{Z}/p^2) \otimes \pi_t \operatorname{HH}(\mathbb{Z} \circ \mathbb{Z}, \mathbb{Z}/p^2)$$

for all r. To find $\pi_* \operatorname{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2)$ from the E^∞-terms we have to solve an extension problem. But there is no room for non-trivial extensions since the dimension of

$\pi_* \operatorname{HH}(\mathbb{Z}\circ R, \mathbb{Z}/p^2)\otimes\mathbb{Z}/p$ as a $\mathbb{Z}/p$-vector space is equal to the dimension of $\pi_* \operatorname{HH}(\mathbb{Z}\circ R, \mathbb{Z}/p)$, and by the computation in example 6.2.11 this is equal to the dimension of

$$\pi_* \operatorname{HH}(R, \mathbb{Z}/p^2) \otimes \pi_* \operatorname{HH}(\mathbb{Z}\circ\mathbb{Z}, \mathbb{Z}/p^2) \otimes \mathbb{Z}/p.$$

7.2. **Surjectivity of the map from** THH **to** HH **for** $\mathbb{Z}/p^2$**.** The main idea in the prof of proposition 7.1.3 is to compare the filtered rings $0 \subseteq p\mathbb{Z}/p^2 \subseteq \mathbb{Z}/p^2$ and $0 \subseteq t\mathbb{Z}/p[\Pi_2] \subseteq \mathbb{Z}/p[\Pi_2]$ where Π_2 is the pointed monoid $\Pi_2 = \{0, 1, t\} \subseteq \mathbb{Z}[t]/t^2$. Since $\mathbb{Z}/p[\Pi_2]$ is isomorphic to the graded ring associated to $\mathbb{Z}/p^2$ (cf. example 5.1.9), we obtain an isomorphism

$$E^1_{s,t} \operatorname{THH}(\mathbb{Z}/p^2) \xrightarrow{\cong} E^1_{s,t} \operatorname{THH}(\mathbb{Z}/p[\Pi_2])$$

of E^1-terms in spectral sequences converging towards $\pi_{s+t} \operatorname{THH}(\mathbb{Z}/p^2)$ and $\pi_{s+t} \operatorname{THH}(\mathbb{Z}/p[\Pi_2])$ respectively. (Warning: this map does not extend to a map of spectral sequences.) In section 6.1 we found isomorphisms

$$E^1_{s,t} \operatorname{THH}(\mathbb{Z}/p[\Pi_2]) \cong \pi_{s+t}(\operatorname{THH}(\mathbb{Z}/p) \wedge \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi_2, s, s-1))$$

and

$$\pi_{s+t} \operatorname{THH}(\mathbb{Z}/p[\Pi_2]) \cong \pi_{s+t}(\operatorname{THH}(\mathbb{Z}/p) \wedge \mathrm{N}^{\mathrm{cy}}_\wedge(\Pi_2)).$$

To proceed we need to know $\pi_* \operatorname{THH}(\mathbb{Z}/p)$ and the reduced homology of $\mathrm{N}^{\mathrm{cy}}_\wedge(\Pi_2, s, s-1)$ with coefficients in $\mathbb{Z}/p$.

Theorem 7.2.1 ([2])**.** *There is an isomorphism $\pi_* \operatorname{THH}(\mathbb{Z}/p) \cong S_{\mathbb{Z}/p}(\sigma)$ of graded rings, where $S_{\mathbb{Z}/p}(\sigma)$ denotes the symmetric algebra over $\mathbb{Z}/p$ on a generator σ of degree* 2.

Lemma 7.2.2. *For odd $s < 0$ the only non-zero reduced homology groups of $\mathrm{N}^{\mathrm{cy}}_\wedge(\Pi_2, s, s-1)$ with coefficients in $\mathbb{Z}/p$ are*

$$\widetilde{H}_{-s-1}(\mathrm{N}^{\mathrm{cy}}_\wedge(\Pi_2, s, s-1); \mathbb{Z}/p) \cong \widetilde{H}_{-s}(\mathrm{N}^{\mathrm{cy}}_\wedge(\Pi_2, s, s-1); \mathbb{Z}/p) \cong \mathbb{Z}/p.$$

For even $s < 0$, $\widetilde{H}_*(\mathrm{N}^{\mathrm{cy}}_\wedge(\Pi_2, s, s-1); \mathbb{Z}/p) = 0$, and $\mathrm{N}^{\mathrm{cy}}_\wedge(\Pi_2, 0, -1) \approx S^0$.

Proof. For $s \le 0$, $\mathrm{N}^{\mathrm{cy}}_\wedge(\Pi_2, s, s-1)$ only has non-degenerate simplices $t^{\wedge(-s)}$ and $1 \wedge t^{\wedge(-s)}$. It is easy to compute the differential in the cellular chain complex. □

The next step in the proof of proposition 7.1.3 is to compare with the analogous linear situation. To do this we need to choose a filtered simplicial ring R that is free as an abelian group, fits into the commutative diagram

$$\begin{tikzcd} 0 = R(2) \arrow[r] \arrow[d, "="'] & R(1) \arrow[r] \arrow[d, "\simeq"'] & R(0) = R \arrow[d, "\simeq"'] \\ 0 \arrow[r] & p\mathbb{Z}/p^2 \arrow[r] & \mathbb{Z}/p^2 \end{tikzcd}$$

with vertical maps homotopy equivalences, and that satisfies that $R(0)/R(1)$ is free as an abelian group. Then there must be a homotopy equivalence

$$R(0)/R(1) \xrightarrow{\simeq} (\mathbb{Z}/p^2)/(p\mathbb{Z}/p^2) \cong \mathbb{Z}/p.$$

One possible choice of R is the following: Let $I \subseteq \mathbb{Z}[\Pi_2]$ be the two-sided ideal generated by $(t-p)$. Note that $[1] = \{0, 1\}$ is a pointed monoid, and that it induces the structure of pointed monoid on $\Delta(1)$. Therefore we can form the monoid ring $(\mathbb{Z}[\Pi_2])[\Delta(1)]$. It is easy to check that $R = \mathbb{Z}[\Pi_2] + I[\Delta(1)] \subseteq (\mathbb{Z}[\Pi_2])[\Delta(1)]$ is a subring, and clearly R is free as an abelian group. R inherits a filtration from Π_2 with $R(n) = t^n R$. The short exact sequence

$$t^n I[\Delta(1)] \to t^n R \to t^n \mathbb{Z}[\Pi_2]/t^n I$$

shows that there are homotopy equivalences $R(n) \xrightarrow{\simeq} p^n\mathbb{Z}/p^2$, since $t^n\mathbb{Z}[\Pi_2]/t^n I \cong p^n\mathbb{Z}/p^2$, and clearly these homotopy equivalences fit into the commutative diagram above. Note that if we replace Π_2 by $\Pi_n = \{0, 1, t, \ldots, t^{n-1}\} \subseteq \mathbb{Z}[t]/t^n$ we obtain a similar model for $\mathbb{Z}/p^n$.

If we let $S = R(0)/R(1)$, then the graded ring associated to R is isomorphic to $S[\Pi_2]$. We obtain a commutative diagram

$$\begin{array}{ccc}
E^1_{s,t}\,\mathrm{HH}(\mathbb{Z}\circ R, \mathbb{Z}/p^2) & \xrightarrow{l_R} & E^1_{s,t}\,\mathrm{HH}(R, \mathbb{Z}/p^2) \\
\cong\downarrow & & \cong\downarrow \\
E^1_{s,t}(\mathbb{Z}\circ S[\Pi_2], \mathbb{Z}/p[\Pi_2]) & \xrightarrow{l_{S[\Pi_2]}} & E^1_{s,t}(S[\Pi_2], \mathbb{Z}/p[\Pi_2]) \\
\cong\downarrow & & \cong\downarrow \\
\pi_{s+t}(\mathrm{HH}(\mathbb{Z}\circ S, \mathbb{Z}/p)\wedge \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi_2, s, s-1)) & \xrightarrow{l_S\wedge \mathrm{id}} & \pi_{s+t}(\mathrm{HH}(S, \mathbb{Z}/p)\wedge \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi_2, s, s-1)).
\end{array}$$

This diagram is the main ingredient in the proof of the following lemma.

Lemma 7.2.3. *The map*

$$l_R : E^1_{s,t}\,\mathrm{HH}(\mathbb{Z}\circ R, \mathbb{Z}/p^2) \to E^1_{s,t}\,\mathrm{HH}(R, \mathbb{Z}/p^2)$$

is an isomorphism for $s + t < 2p - s - 2$.

Proof. The map $l_S : \mathrm{HH}(\mathbb{Z}\circ S, \mathbb{Z}/p) \to \mathrm{HH}(S, \mathbb{Z}/p)$ is $(2p-1)$-connected. One way to see this is to consider the spectral sequence with E^2-terms

$$E^2_{s,t} = \pi_s\,\mathrm{HH}(S, \pi_t\,\mathrm{THH}(\mathbb{Z}\circ\mathbb{Z}, \mathbb{Z}/p))$$

converging towards $\pi_{s+t}\,\mathrm{HH}(\mathbb{Z}\circ S, \mathbb{Z}/p)$. Since $\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi_2, s, s-1)_k = *$ for $k < -s-1$, the map

$$l_S\wedge \mathrm{id} : \mathrm{HH}(\mathbb{Z}\circ S, \mathbb{Z}/p)\wedge \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi_2, s, s-1) \to \mathrm{HH}(S, \mathbb{Z}/p)\wedge \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi_2, s, s-1)$$

is $(2p - s - 2)$-connected, and the lemma follows from the commutative diagram above. □

Let $A \subseteq \mathbb{Z}^2$ consist of the points (s, t) for which either ($s = 0$, $t \geq 0$ and t is even) or ($s < 0$, s is odd and $t \geq -2s - 1$). Then

$$\begin{aligned}
E^1_{s,t}\,\mathrm{HH}(\mathbb{Z}\circ R, \mathbb{Z}/p^2) &\cong \pi_{s+t}(\mathrm{HH}(\mathbb{Z}\circ S, \mathbb{Z}/p)\wedge \mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi_2, s, s-1)) \\
&\cong \bigoplus_{k=0}^{\infty} \widetilde{H}_{s+t-2k}(\mathrm{N}^{\mathrm{cy}}_{\wedge}(\Pi_2, s, s-1); \mathbb{Z}/p) \\
&\cong \begin{cases} \mathbb{Z}/p & \text{for } (s,t)\in A \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

by theorem 7.2.1, lemma 7.2.2 and the fact that $\mathrm{HH}(\mathbb{Z} \circ S, \mathbb{Z}/p)(S^0)$ is a product of Eilenberg-MacLane spaces. Similarly by the computation of $\mathrm{HH}(R, \mathbb{Z}/p)$ in example 6.2.10 we see that

$$E^1_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2) \cong \begin{cases} \mathbb{Z}/p & \text{for } (s,t) \in A \\ 0 & \text{otherwise.} \end{cases}$$

Let $B \subseteq \mathbb{Z}^2$ consist of the points (s,t) for which either $(s = t = 0)$ or ($s < 0$, s is odd and $t \in \{-2s-1, -2s+1\}$). Note that $B \subseteq A$.

Lemma 7.2.4. *$E^r_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2) \cong \mathbb{Z}/p$ for $(s,t) \in B$ and $r \geq 1$, and if $(s,t) \notin B$ then $E^\infty_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2) = 0$.*

Proof. In example 6.2.10 we have computed $\pi_* \mathrm{HH}(R, \mathbb{Z}/p^2)$. Since the spectral sequence converges we must have

$$\dim_{\mathbb{Z}/p} \left(\bigoplus_{s+t=n} E^\infty_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2) \right) = \begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd,} \end{cases}$$

and from the formula for $E^1_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2)$ given above we can conclude that

$$\dim_{\mathbb{Z}/p} \left(\bigoplus_{s+t=n} E^1_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2) \right) - \dim_{\mathbb{Z}/p} \left(\bigoplus_{s+t=n} E^\infty_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2) \right) = k$$

if $n = 2k$ or $n = 2k-1$. By induction on the total degree the only non-zero differentials start in even total degrees. By the geometry of the spectral sequence no non-zero differentials can start in $E^r_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2)$ for $(s,t) \in B$. Since $s+t$ is even for $(s,t) \in B$ we can conclude that $E^\infty_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2) \cong \mathbb{Z}/p$ for $(s,t) \in B$, and that $E^\infty_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2) = 0$ if $(s,t) \notin B$. □

Lemma 7.2.5. *The map*

$$l_R = l^r_R : E^r_{s,t} \mathrm{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2) \to E^r_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2)$$

is an isomorphism for $(s,t) \in B$ and $r \geq 1$.

Proof. The case $r = 1$ follows from lemma 7.2.3. Assume by induction that l^r_R is an isomorphism. For $(s,t) \in B$ there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[d] & E^r_{s,t} \mathrm{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2) \arrow[l, "d^r"'] \arrow[d, "l^r_R"', "\cong"] & E^r_{s+r,t-r+1} \mathrm{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2) \arrow[l, "d^r"'] \arrow[d, "l^r_R"'] \\ 0 & E^r_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2) \arrow[l, "d^r"'] & E^r_{s+r,t-r+1} \mathrm{HH}(R, \mathbb{Z}/p^2), \arrow[l, "d^r=0"'] \end{tikzcd}$$

where the differentials in the lower horizontal sequence are zero by lemma 7.2.4, and the middle vertical map is an isomorphism by assumption. It follows that the differentials in the upper horizontal sequence are zero, and hence the map l^{r+1}_R is also an isomorphism. □

It follows from lemma 7.2.4 and lemma 7.2.5 that the map

$$l_R : E^\infty_{s,t} \mathrm{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2) \to E^\infty_{s,t} \mathrm{HH}(R, \mathbb{Z}/p^2)$$

is onto for all (s,t), and since these spectral sequences converge towards $\pi_{s+t} \mathrm{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2)$ and $\pi_{s+t} \mathrm{HH}(R, \mathbb{Z}/p^2)$ respectively, we can conclude that the map

$$l_R : \pi_* \mathrm{HH}(\mathbb{Z} \circ R, \mathbb{Z}/p^2) \to \pi_* \mathrm{HH}(R, \mathbb{Z}/p^2)$$

is surjective as stated in proposition 7.1.3

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