Improved Homology Stability of the Mapping Class Group

Ph.d. Thesis

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Contents

Summary

My ph.d. thesis consists of my two articles:

(I) S. Boldsen, Improved homological stability for the mapping class group with integral or twisted coefficients, (59 pages), submitted for publication to Journal of Topology and available at arXiv:0904.3269.

(II) S. Boldsen, Different versions of mapping class groups of surfaces, (18 pages), will soon be available at arXiv.

Both papers investigate the properties of the mapping class group of surfaces. Mapping class groups are central to many areas of mathematics; most prominently to algebraic geometry, differential geometry and topology. It also plays a role in various field theories from mathematical physic, and in geometric group theory.

Let $F_{g,r}$ denote the compact oriented surface of genus g with r boundary circles, then the associated mapping class group, $\Gamma_{g,r}$, is

$$\Gamma_{g,r} = \pi_0 \mathrm{Diff}_+(F_{g,r}; \partial),$$

the components of the group of orientation-preserving diffeomorphisms of $F_{g,r}$ keeping the boundary pointwise fixed.

The paper (I) has as its starting point a never published manuscript of J. Harer, [Harer2], from 1993. This manuscript states an improved stability range for the homology of the mapping class group, but it rests upon certain unproven statements. My goal from the outset was to prove these statements.

We compare different mapping class groups using the maps induced by gluing a pair of pants onto one or two boundary circles, and extending the diffeomorphism by the identity on the pair of pants,

$$\Sigma_{0,1} : \Gamma_{g,r} \longrightarrow \Gamma_{g,r+1}, \quad \Sigma_{1,-1} : \Gamma_{g,r} \longrightarrow \Gamma_{g+1,r-1}$$

Homology stability means these maps induce isomorphism on homology in certain degrees. We now state our main results. The first result is:

Theorem 1. *The map $H_n(\Gamma_{g,r}) \longrightarrow H_n(\Gamma_{g+l,r+m})$ induced by $\Sigma_{l,m}$ satisfies:*

(i) $\Sigma_{0,1}$ *is an isomorphism for $2g \geq 3n$, when $r \geq 1$*

(ii) $\Sigma_{1,-1}$ *is surjective for $2g \geq 3n-1$, and an isomorphism for $2g \geq 3n+2$, when $r \geq 2$.*

While Harer got his result only for homology with rational coefficients, we have integer coefficients. Theorem 1 only holds for surfaces with boundary. To get a result for closed surfaces, we use the map induced by gluing on a disk to a boundary component, and obtain

Theorem 2. *The map $H_k(\Gamma_{g,1}) \longrightarrow H_k(\Gamma_g)$ is surjective for $2g \geq 3k-1$, and an isomorphism for $2g \geq 3k+2$.*

This was not considered by Harer, but N. Ivanov has shown how to deduce such a result from the one for surfaces with boundary.

We wish to obtain such a stability result, not only for trivial coefficients but also for so-called coefficients systems of a finite degree. A coefficient system is a functor V from $\mathfrak{C}$ to the category of abelian groups without infinite division. If the functor is constant, we say V has degree 0. We then define a coefficient system of degree k inductively, by requiring that the maps $V(F) \longrightarrow V(\Sigma_{i,j}F)$ are split injective and their cokernels are coefficient systems of degree $k-1$, see Definition 4.4. As an example, the functor $H_1(F;\mathbb{Z})$ is a coefficients system of degree 1, and its kth exterior power $\Lambda^k H_1(F;\mathbb{Z})$, considered in [Morita1], has degree k.

Theorem 3. *Let F be a surface of genus g, and let V be a coefficient system of degree k. Then the map*

$$H_n(\Gamma(F); V(F)) \longrightarrow H_n(\Gamma(\Sigma_{l,m}F); V(\Sigma_{l,m}F))$$

induced by $\Sigma_{l,m}$ satisfies:

(i) *$\Sigma_{0,1}$ is an isomorphism for $2g \geq 3n+k$.*

(ii) *$\Sigma_{1,-1}$ is surjective for $2g \geq 3n+k-1$, and an isomorphism for $2g \geq 3n+k+2$.*

Note that for the result for the integers is a special case of this. One reason to study coefficient systems is that we can then calculate the homology of the space of surfaces mapping into a background space X from [Cohen-Madsen]:

$$\mathcal{S}_{g,r}(X,\gamma) \;=\; \big\{(F_{g,r},\varphi,f) \mid F_{g,r} \subseteq \mathbb{R}^\infty \times [a,b], \varphi : \sqcup S^1 \longrightarrow \partial F_{g,r} \text{ is a parametrization}, f : F_{g,r} \longrightarrow X \text{ is continuous with } f\circ\varphi = \gamma\big\}$$

Define the coefficient system $V_n^X(F) = H_n(\mathrm{Map}(F/\partial F, X))$. Let $\mathcal{S}_{g,r}(X,\gamma)_\bullet$ denote the connected path component corresponding to the trivial class $0 \in \pi_2(X)$, and similarly for $\Omega^\infty(\mathbb{CP}^\infty_{-1} \wedge X_+)_\bullet$. Then

Theorem 4. *Let X be a simply connected space such that V_m^X is without infinite division for all m. Then for $2g \geq 3n+3$ and $r \geq 1$ we get an isomorphism*

$$H_n(\mathcal{S}_{g,r}(X,\gamma)_\bullet) \cong H_n(\Omega^\infty(\mathbb{CP}^\infty_{-1} \wedge X_+)_\bullet).$$

In this paper, we first prove Theorem 1 for constant integral coefficients, $V = \mathbb{Z}$. Our proof of Theorem 1 in this case is much inspired by Harer's manuscript [Harer2]. The rational stability results claimed by Harer are "one degree better" than what is obtained here with integral coefficients. Before discussing the discrepancy it is convenient to compare the stability with Faber's conjecture.

Let $\mathcal{M}_g$ be Riemann's moduli space; recall that $H^*(\mathcal{M}_g;\mathbb{Q}) \cong H^*(\Gamma_g;\mathbb{Q})$. From above we have maps

$$H^*(\Gamma_g;\mathbb{Q}) \longrightarrow H^*(\Gamma_{g,1};\mathbb{Q}) \longleftarrow H^*(\Gamma_{\infty,1};\mathbb{Q})$$

and by [Madsen-Weiss],

$$H^*(\Gamma_{\infty,1};\mathbb{Q}) = \mathbb{Q}[\kappa_1,\kappa_2,\dots]. \tag{1}$$

The classes $\kappa_i \in H^{2i}(\Gamma_{g,r})$ for $r \geq 0$ are the standard classes defined by Miller, Morita and Mumford (κ_i is denoted e_i by Morita).

The tautological algebra $R^*(\mathcal{M}_g)$ is the subring of $H^*(\Gamma_g;\mathbb{Q})$ generated multiplicatively by the classes κ_i. Faber conjectured in [Faber] the complete algebraic structure of $R^*(\mathcal{M}_g)$. Part of the conjecture asserts that it is a Poincaré duality algebra (Gorenstein) of formal dimension $2g-4$, and that it is generated by $\kappa_1,\dots,\kappa_{[g/3]}$, where $[g/3]$ denotes $g/3$ rounded down. The latter statement was proved by Morita (cf. [Morita1] prop 3.4).

It follows from our theorems above that $\kappa_1,\dots,\kappa_{[g/3]}$ are non-zero in $H^*(\Gamma_g;\mathbb{Q})$ when $* \leq 2[\frac{g}{3}]-2$. More precisely, if $g \equiv 1,2 \pmod 3$ then our results show that

$$H^*(\Gamma_g;\mathbb{Q}) \cong H^*(\Gamma_{\infty,1};\mathbb{Q}) \quad \text{for } * \leq 2[\tfrac{g}{3}], \tag{2}$$

but if $g \equiv 0 \pmod 3$, our result only show the isomorphism for $* \leq 2[\frac{g}{3}]-1$. In contrast, [Harer2] asserts the isomorphism for $* \leq 2[\frac{g}{3}]$ for all g. We note that is follows from (3) and Morita's result that the best possible stability range for $H^*(\Gamma_g;\mathbb{Q})$ is $* \leq 2[\frac{g}{3}]$. We are "one degree off" when $g \equiv 0 \pmod 3$.

The stability of [Harer2] is based on three unproven assertions that I have not been able to verify. I will discuss two of them below, and the third in section 3.1.

Boundary connected sum of surfaces with non-empty boundary defines a group homomorphism $\Gamma_{g,r} \times \Gamma_{h,s} \longrightarrow \Gamma_{g+h,r+s-1}$, and hence a product in homology

$$H_*(\Gamma_{g,r}) \otimes H_*(\Gamma_{h,s}) \longrightarrow H_*(\Gamma_{g+h,r+s-1}), \quad r,s>0.$$

The classes κ_i are primitive with respect to this homology product, in the sense that $\langle \kappa_i, a \cdot b\rangle = 0$ if both a and b have positive degree [Morita2]. Harer proves in [Harer3] that $H^2(\Gamma_{3,1};\mathbb{Q}) = \mathbb{Q}\{\kappa_1\}$. Let $\check{\kappa}_1 \in H_2(\Gamma_{3,1};\mathbb{Q})$ be the dual to κ_1, and let $\check{\kappa}_1^n$ be the n'th power under the multiplication

$$H_2(\Gamma_{3,1})^{\otimes n} \longrightarrow H_{2n}(\Gamma_{3n,1}).$$

Then $\langle \kappa_1^n, \check{\kappa}_1^n\rangle = n!$, so $\check{\kappa}_1^n \neq 0$ in $H^{2n}(\Gamma_{3n,1};\mathbb{Q})$, cf. part (i) of Theorem 1. Dehn twist around the $(r+1)$st boundary circle yields a group homomorphism $\mathbb{Z} \longrightarrow \Gamma_{1,r+1}$, and hence a class $\tau_{r+1} \in H_1(\Gamma_{1,r+1})$.

We can now formulate two of Harer's three assertions one needs in order to improve the rational stability result by "one degree" when $g \equiv 0 \pmod 3$, i.e. from $* \leq 2[\frac{g}{3}] - 1$ to $* \leq 2[\frac{g}{3}]$. The assertions are:

(i) $\check{\kappa}_1^n = 0$ in $H_{2n}(\Gamma_{g,r};\mathbb{Q})$ for $g < 3n$.

(ii) $\tau_{r+1}\cdot\check{\kappa}_1^n$ is non-zero in $\mathrm{Coker}(H_{2n+1}(\Gamma_{3n+1,r};\mathbb{Q}) \longrightarrow H_{2n+1}(\Gamma_{3n+1,r+1};\mathbb{Q})$.

The short paper (II) is about the connection between the topological groups of either diffeomorphisms, homeomorphisms or homotopy equivalences of a surface. The main result is that these groups have the same connected components. This is basically a result that dates back to Baer in the 1920ies, but it is hard to find in the written literature; there is no good reference. This paper gives a short, self-contained exposition of this result and its proof.

As defined above, the mapping class group of a surface F is $\Gamma(F) = \pi_0(\mathrm{Diff}_+(F, \partial F))$. We now also consider the group $\mathrm{Diff}(F, \{\partial F\})$ of diffeomorphisms mapping ∂F to itself as a set. We compare the groups of diffeomorphisms to the corresponding groups of homeomorphisms, $\mathrm{Top}(F, \partial F)$, and homotopy equivalences, $\mathrm{hAut}(F, \partial F)$. Part (4) of the Theorem below shows that it does not matter whether one considers diffeomorphisms, homeomorphisms, or even homotopy equivalences, when working in the mapping class group.

Theorem 5. *Let F be a compact surface and not a sphere, a disk, a cylinder, a Möbius band, a torus, a Klein bottle, or $\mathbb{R}P^2$. Then there are bijections*

(1) $\pi_0(\mathrm{Diff}(F, \{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{Top}(F, \{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{hAut}(F, \{\partial F\}))$

(2) $\pi_0(\mathrm{Diff}(F,\partial F)) \xrightarrow{\cong} \pi_0(\mathrm{Top}(F,\partial F)) \xrightarrow{\cong} \pi_0(\mathrm{hAut}(F,\partial F))$,

(3) $\pi_0(\mathrm{Diff}_+(F,\{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{Top}_+(F,\{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{hAut}_+(F,\{\partial F\}))$,

(4) $\pi_0(\mathrm{Diff}_+(F,\partial F)) \xrightarrow{\cong} \pi_0(\mathrm{Top}_+(F,\partial F)) \xrightarrow{\cong} \pi_0(\mathrm{hAut}_+(F,\partial F))$.

The proof uses mostly elementary topological tools, such as covering spaces, tubular neighborhoods, and transversality. The main method is cutting up the surface in elementary pieces, proving the results for those, and carefully gluing them back together. This requires a few heavier tools, most importantly the classification of surfaces, and a result of Smale that any diffeomorphism of the disc, which is identity on the boundary, is isotopic to the identity relative to the boundary.

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Part I
Improved homological stability for the mapping class group with integral or twisted coefficients

Introduction

Let $F_{g,r}$ denote the compact oriented surface of genus g with r boundary circles, and let $\Gamma_{g,r}$ be the associated mapping class group,

$$\Gamma_{g,r} = \pi_0 \mathrm{Diff}_+(F_{g,r}, \partial F_{g,r}),$$

the components of the group of orientation-preserving diffeomorphisms of $F_{g,r}$ keeping the boundary pointwise fixed. Gluing a pair of pants onto one or two boundary circles induce maps

$$\Sigma_{0,1} : \Gamma_{g,r} \longrightarrow \Gamma_{g,r+1}, \quad \Sigma_{1,-1} : \Gamma_{g,r} \longrightarrow \Gamma_{g+1,r-1}$$

whose composite $\Sigma_{1,0} := \Sigma_{1,-1} \circ \Sigma_{0,1}$ corresponds to adding to $F_{g,r}$ a genus one surface with two boundary circles. Using the mapping cone of $\Sigma_{i,j}$, $(i,j) = (0,1), (1,-1)$ or $(1,0)$ we get a relative homology group, which fits into the exact sequence

$$\ldots \longrightarrow H_n(\Sigma_{i,j}\Gamma_{g,r}) \longrightarrow H_n(\Sigma_{i,j}\Gamma_{g,r}, \Gamma_{g,r}) \longrightarrow H_{n-1}(\Gamma_{g,r}) \longrightarrow \ldots$$

Homology stability results for the mapping class group can then be derived from the vanishing the relative group (in some range).

We wish to show such a stability result for not only for trivial coefficients but also for so-called coefficients systems of a finite degree. For this, we work in Ivanov's category $\mathfrak{C}$ of marked surfaces, cf. [Ivanov1] and §4.1 below for details. The maps $\Sigma_{1,0}$ and $\Sigma_{0,1}$ are functors on $\mathfrak{C}$, and $\Sigma_{1,-1}$ is a functor on a subcategory.

A coefficient system is a functor V from $\mathfrak{C}$ to the category of abelian groups without infinite division. If the functor is constant, we say V has degree 0. We then define a coefficient system of degree k inductively, by requiring that the maps $V(F) \longrightarrow V(\Sigma_{i,j}F)$ are split injective and their cokernels are coefficient systems of degree $k-1$, see Definition 4.4. As an example, the functor $H_1(F;\mathbb{Z})$ is a coefficients system of degree 1, and its kth exterior power $\Lambda^k H_1(F;\mathbb{Z})$, considered in [Morita1], has degree k. To formulate our stability result, we consider relative homology group with coefficients in V,

$$Rel_n^V(\Sigma_{l,m}F, F) = H_n(\Sigma_{l,m}\Gamma(F), \Gamma(F); V(\Sigma_{l,m}F), V(F)).$$

These groups again fit into a long exact sequence. Our main result is

Theorem 1. *For F a surface of genus g with at least 1 boundary component, and V a coefficient system of degree k_V, we have*

$$Rel_n^V(\Sigma_{1,0}F, F) = 0 \textit{ for } 3n \le 2g - k_V,$$

$$Rel_n^V(\Sigma_{0,1}F, F) = 0 \textit{ for } 3n \leq 2g - k_V.$$

Moreover, if F has at least 2 boundary components, we have

$$Rel_q^V(\Sigma_{1,-1}F, F) = 0 \textit{ for } 3q \leq 2g - k_V + 1.$$

As a corollary, we obtain that $H_n(\Gamma_{g,r}; V(F_{g,r}))$ is independent of g and r for $3n \leq 2g - k_V - 2$ and $r \geq 1$. For a more precise statement, see Theorem 4.17. This uses that $\Sigma_{0,1}$ is always injective, since the composition $\Gamma_{g,r} \xrightarrow{\Sigma_{0,1}} \Gamma_{g,r+1} \xrightarrow{\Sigma_{0,-1}} \Gamma_{g,r}$ is an isomorphism, where $\Sigma_{0,-1}$ is the map gluing a disk onto a boundary component.

The proof of Theorem 1 with twisted coefficients uses the setup from [Ivanov1]. His category of marked surfaces is slightly different from ours, since we also consider surfaces with more than one boundary component and thus get results for $\Sigma_{0,1}$ and $\Sigma_{1,-1}$.

For constant coefficients, $V = \mathbb{Z}$, we also consider the map $\Sigma_{0,-1} : \Gamma_{g,1} \longrightarrow \Gamma_g$ induced by gluing a disk onto the boundary circle, where our result is:

Theorem 2. *The map*

$$\Sigma_{0,-1} : H_k(\Gamma_{g,1}; \mathbb{Z}) \longrightarrow H_k(\Gamma_g; \mathbb{Z})$$

is surjective for $2g \geq 3k - 1$, and an isomorphism for $2g \geq 3k + 2$.

The proof of Theorem 2 follows [Ivanov1], where a stability result for closed surfaces is deduced from a stability theorem on surfaces with boundary. We get an improved result, because Theorem 1 has a better bound than Ivanov's stability theorem (which has isomorphism for $g > 2k$).

In this paper, we first prove Theorem 1 for constant integral coefficients, $V = \mathbb{Z}$. Our proof of Theorem 1 in this case is much inspired by Harer's manuscript [Harer2], which was never published. Harer's manuscript is about rational homology stability. The rational stability results claimed in [Harer2] are "one degree better" than what is obtained here with integral coefficients. Before discussing the discrepancy it is convenient to compare the stability with Faber's conjecture.

Let $\mathcal{M}_g$ be Riemann's moduli space; recall that $H^*(\mathcal{M}_g; \mathbb{Q}) \cong H^*(\Gamma_g; \mathbb{Q})$. From above we have maps

$$H^*(\Gamma_g; \mathbb{Q}) \longrightarrow H^*(\Gamma_{g,1}; \mathbb{Q}) \longleftarrow H^*(\Gamma_{\infty,1}; \mathbb{Q})$$

and by [Madsen-Weiss],

$$H^*(\Gamma_{\infty,1}; \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots]. \tag{3}$$

The classes $\kappa_i \in H^{2i}(\Gamma_{g,r})$ for $r \geq 0$ are the standard classes defined by Miller, Morita and Mumford (κ_i is denoted e_i by Morita).

The tautological algebra $R^*(\mathcal{M}_g)$ is the subring of $H^*(\Gamma_g;\mathbb{Q})$ generated multiplicatively by the classes κ_i. Faber conjectured in [Faber] the complete algebraic structure of $R^*(\mathcal{M}_g)$. Part of the conjecture asserts that it is a Poincaré duality algebra (Gorenstein) of formal dimension $2g-4$, and that it is generated by $\kappa_1, \ldots, \kappa_{[g/3]}$, where $[g/3]$ denotes $g/3$ rounded down. The latter statement was proved by Morita (cf. [Morita1] prop 3.4).

It follows from our theorems above that $\kappa_1, \ldots, \kappa_{[g/3]}$ are non-zero in $H^*(\Gamma_g;\mathbb{Q})$ when $* \leq 2[\frac{g}{3}] - 2$. More precisely, if $g \equiv 1, 2 \pmod 3$ then our results show that

$$H^*(\Gamma_g;\mathbb{Q}) \cong H^*(\Gamma_{\infty,1};\mathbb{Q}) \quad \text{for } * \leq 2[\tfrac{g}{3}], \tag{4}$$

but if $g \equiv 0 \pmod 3$, our result only show the isomorphism for $* \leq 2[\frac{g}{3}] - 1$. In contrast, [Harer2] asserts the isomorphism for $* \leq 2[\frac{g}{3}]$ for all g. We note that is follows from (3) and Morita's result that the best possible stability range for $H^*(\Gamma_g;\mathbb{Q})$ is $* \leq 2[\frac{g}{3}]$. We are "one degree off" when $g \equiv 0 \pmod 3$.

The stability of [Harer2] is based on three unproven assertions that I have not been able to verify. I will discuss two of them below, and the third in section 3.1.

Boundary connected sum of surfaces with non-empty boundary defines a group homomorphism $\Gamma_{g,r} \times \Gamma_{h,s} \longrightarrow \Gamma_{g+h,r+s-1}$, and hence a product in homology

$$H_*(\Gamma_{g,r}) \otimes H_*(\Gamma_{h,s}) \longrightarrow H_*(\Gamma_{g+h,r+s-1}), \quad r, s > 0.$$

The classes κ_i are primitive with respect to this homology product, in the sense that $\langle \kappa_i, a \cdot b\rangle = 0$ if both a and b have positive degree [Morita2]. Harer proves in [Harer3] that $H^2(\Gamma_{3,1};\mathbb{Q}) = \mathbb{Q}\{\kappa_1\}$. Let $\check{\kappa}_1 \in H_2(\Gamma_{3,1};\mathbb{Q})$ be the dual to κ_1, and let $\check{\kappa}_1^{\,n}$ be the n'th power under the multiplication

$$H_2(\Gamma_{3,1})^{\otimes n} \longrightarrow H_{2n}(\Gamma_{3n,1}).$$

Then $\langle \kappa_1^{\,n}, \check{\kappa}_1^{\,n}\rangle = n!$, so $\check{\kappa}_1^{\,n} \neq 0$ in $H^{2n}(\Gamma_{3n,1};\mathbb{Q})$, cf. part (i) of Theorem 1. Dehn twist around the $(r+1)$st boundary circle yields a group homomorphism $\mathbb{Z} \longrightarrow \Gamma_{1,r+1}$, and hence a class $\tau_{r+1} \in H_1(\Gamma_{1,r+1})$.

We can now formulate two of Harer's three assertions one needs in order to improve the rational stability result by "one degree" when $g \equiv 0 \pmod 3$, i.e. from $* \leq 2[\frac{g}{3}] - 1$ to $* \leq 2[\frac{g}{3}]$. The assertions are:

(i) $\check{\kappa}_1^{\,n} = 0$ in $H_{2n}(\Gamma_{g,r};\mathbb{Q})$ for $g < 3n$.

(ii) $\tau_{r+1} \cdot \check{\kappa}_1^{\,n}$ is non-zero in $\mathrm{Coker}(H_{2n+1}(\Gamma_{3n+1,r};\mathbb{Q}) \longrightarrow H_{2n+1}(\Gamma_{3n+1,r+1};\mathbb{Q})$.

The third assertion one needs is stated in Remark 3.5.

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1 Homology of groups and spectral sequences

1.1 Relative homology of groups

For a group G, and $\mathbb{Z}[G]$-modules M and M', left and right modules, respectively, we have the bar construction:

$$B_n(M', G, M) = M' \otimes (\mathbb{Z}[G])^{\otimes n} \otimes M,$$

with the differential

$$\begin{aligned} d_n(m' \otimes g_1 \otimes \cdots \otimes g_n \otimes m) &= (m'g_1) \otimes g_2 \otimes \cdots \otimes g_n \otimes m \\ &+ \sum_{i=1}^{n-1} (-1)^i m' \otimes g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n \otimes m \\ &+ (-1)^n m' \otimes g_1 \otimes \cdots \otimes g_{n-1} \otimes (g_n m). \end{aligned}$$

If either M or M' are free $\mathbb{Z}[G]$-modules, $B_*(M', G, M)$ is contractible. If $M' = \mathbb{Z}$ with trivial G-action, we write $B_*(G, M)$. Then the nth homology group of G with coefficients in M is defined to be

$$H_n(G; M) = H_n(B_*(G, M)) \cong \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M).$$

There is a relative version of this. Suppose $f : G \longrightarrow H$ is a group homomorphism and $\varphi : M \longrightarrow N$ is an f-equivariant map of $\mathbb{Z}[G]$-modules. One defines the relative homology $H_*(H, G; N, M)$ to be the homology of the algebraic mapping cone of

$$(f, \varphi)_* : B_*(G, M) \longrightarrow B_*(H, N),$$

so that there is a long exact sequence

$$\cdots \to H_n(G; M) \to H_n(H; N) \to H_n(H, G; M, N) \to H_{n-1}(G; M) \to \cdots$$

1.2 Spectral sequences of group actions

Suppose next that X is a connected simplicial complex with a simplicial action of G. Let $C_*(X)$ be the cellular chain complex of X. Given a $\mathbb{Z}[G]$-module M, define the chain complex

$$C_n^{\dagger}(X; M) = \begin{cases} 0, & n < 0; \\ M, & n = 0; \\ C_{n-1}(X) \otimes_{\mathbb{Z}} M, & n \geq 1; \end{cases} \tag{5}$$

with differential $\partial_n^\dagger$ defined to be $\partial_{n-1} \otimes \mathrm{id}_M$ for $n > 1$, and equal to the augmentation $\varepsilon \otimes \mathrm{id}_M$ for $n = 1$. Note if X is d-connected for some $d \geq 1$, or more generally, if the homology $H_i(X) = 0$ for $1 \leq i \leq d$, then $C_*^\dagger(X; M)$ is exact for $* \leq d+1$. This is used below in the spectral sequence.

Again there is a relative version. Let $f : G \longrightarrow H$, $\varphi : M \longrightarrow N$ be as above, and let $X \subseteq Y$ be a pair of simplicial complexes with a simplicial action of G and H, respectively, compatible with f in the sense that the inclusion $i : X \longrightarrow Y$ is f-equivariant. Assume in addition that the induced map on orbits,

$$i_\sharp : X/G \xrightarrow{\cong} Y/H \tag{6}$$

is a bijection.

Definition 1.1. With G, M and X as above, let σ be a p-cell of X. Let G_σ denote the stabiliser of σ, and let $M_\sigma = M$, but with a twisted G_σ-action, namely

$$g * m = \begin{cases} gm, & \text{if } g \text{ acts orientation preservingly on } \sigma; \\ -gm, & \text{otherwise.} \end{cases}$$

Theorem 1.2. *Suppose X and Y are d- connected and that the orbit map (6) is a bijection. Then there is a spectral sequence $\left\{E^n_{r,s}\right\}_n$ converging to zero for $r + s \leq d + 1$, with*

$$E^1_{r,s} \cong \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} H_s(H_\sigma, G_\sigma; N_\sigma, M_\sigma).$$

Here $\bar{\Delta}_p = \bar{\Delta}_p(X)$ denotes a set of representatives for the G-orbits of the p-simplices in X.

Proof. Consider the double complex with chain groups

$$C_{n,m} = F_n(H) \otimes_{\mathbb{Z}[H]} C_m^\dagger(Y, N) \oplus F_{n-1}(G) \otimes_{\mathbb{Z}[G]} C_m^\dagger(X, M),$$

where $F_n(G) = B_n(G, \mathbb{Z}[G])$, and differentials (superscripts indicate horizontal and vertical directions)

$$\begin{aligned} d^h_m &= \mathrm{id} \otimes \partial^Y_m \oplus \mathrm{id} \otimes \partial^X_m \\ d^v_n &= \partial^H_n \otimes \mathrm{id} \oplus \left(f_* \otimes (i, \varphi)_* + \partial^G_{n-1} \otimes \mathrm{id}\right). \end{aligned} \tag{7}$$

Standard spectral sequence constructions give two spectral sequences both converging to $H_*(\mathrm{Tot}\, C)$, where $\mathrm{Tot}\, C$ is the total complex of $C_{*,*}$,

$(\operatorname{Tot} C)_k = \bigoplus_{n+m=k} C_{n,m}$ and $d^{\mathrm{Tot}} = d^h + d^v$. The vertical spectral sequence (induced by d^v) has E^1 page:

$$\begin{aligned} E^1_{r,s} &= H_r(C_{s,*}) \\ &= H_r\left(F_s(H) \otimes_{\mathbb{Z}[H]} C^\dagger_*(Y;N)\right) \oplus H_r\left(F_{s-1}(G) \otimes_{\mathbb{Z}[G]} C^\dagger_*(X;M)\right). \end{aligned}$$

Since the resolutions F_* are free, this is zero where $C^\dagger_*(X;M)$ and $C^\dagger_*(Y;N)$ are exact, i.e. for $r \leq d+1$. So this spectral sequence converges to zero where $r+s \leq d+1$, and we conclude that $H_*(\operatorname{Tot} C) = 0$ for $* \leq d+1$.

The horizontal spectral sequence, which consequently also converges to zero in total degrees $\leq d+1$, has E^1 page

$$E^1_{r,s} = H_s\left(F_*(H) \otimes_{\mathbb{Z}[H]} C^\dagger_r(Y,N) \oplus F_{*-1}(G) \otimes_{\mathbb{Z}[G]} C^\dagger_r(X,M)\right). \tag{8}$$

For $r \geq 1$ we have

$$\begin{aligned} C^\dagger_r(X,M) &= C_{r-1}(X) \otimes_{\mathbb{Z}[G]} M \cong \bigoplus_{\sigma \in \Delta_{r-1}(X)} \mathbb{Z}[G \cdot \sigma] \otimes_{\mathbb{Z}[G]} M \\ &\cong \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_\sigma]} M_\sigma = \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} \operatorname{Ind}^G_{G_\sigma} M_\sigma, \end{aligned} \tag{9}$$

where $\Delta_p(X)$ denotes the p-cells in X, and where $\bar{\Delta}_p \subseteq \Delta_p(X)$ is a set of representatives for the G-orbits. Finally, $\operatorname{Ind}^G_{G_\sigma} M_\sigma = \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_\sigma]} M_\sigma$.

By assumption (6), the image of $\bar{\Delta}_{r-1}$ under i also works as representatives for the H-orbits of $(r-1)$-cells in Y. Therefore we also have:

$$C^\dagger_r(Y,N) \cong \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} \operatorname{Ind}^H_{H_\sigma} N_\sigma. \tag{10}$$

We insert (9) and (10) into the formula (8) to get for $r \geq 1$:

$$\begin{aligned} E^1_{r,s} &= H_s\left(F_*(H) \otimes_{\mathbb{Z}[H]} C^\dagger_r(Y,N) \oplus F_{*-1}(G) \otimes_{\mathbb{Z}[G]} C^\dagger_r(X,M)\right) \\ &\cong H_s\left(F_*(H) \otimes_{\mathbb{Z}[H]} \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} \operatorname{Ind}^H_{H_\sigma} N_\sigma \oplus F_{*-1}(G) \otimes_{\mathbb{Z}[G]} \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} \operatorname{Ind}^G_{G_\sigma} M_\sigma\right) \\ &\cong \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} H_s\left(F_*(H) \otimes_{\mathbb{Z}[H]} \operatorname{Ind}^H_{H_\sigma} N_\sigma \oplus F_{*-1}(G) \otimes_{\mathbb{Z}[G]} \operatorname{Ind}^G_{G_\sigma} M_\sigma\right) \\ &\cong \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} H_s\left(F_*(H) \otimes_{\mathbb{Z}[H_\sigma]} N_\sigma \oplus F_{*-1}(G) \otimes_{\mathbb{Z}[G_\sigma]} M_\sigma\right) \\ &\cong \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} H_s(H_\sigma, G_\sigma, N_\sigma, M_\sigma). \end{aligned} \tag{11}$$

The final isomorphism above uses that $F_*(H)$ is also a $\mathbb{Z}[H_\sigma]$-module. For $r = 0$,

$$E^1_{0,s} = H_s(H, G; N, M).$$

Thus we set $H_\sigma = H$ when $\sigma \in \bar{\Delta}_{-1} = \{\emptyset\}$. □

For application in the proof of Theorem 4.15, we need to relax the condition (6) to the situation where $i_\sharp$ is only injective:

Theorem 1.3. *With the assumptions of Theorem 1.2, but with $i_\sharp : X/G \longrightarrow Y/H$ is only* injective*, there is a spectral sequence $\left\{E^n_{r,s}\right\}_n$ converging to zero for $r + s \leq d + 1$, and*

$$E^1_{r,s} \cong \bigoplus_{\sigma \in \Sigma_{r-1}(X)} H_s(H_\sigma, G_\sigma; N_\sigma, M_\sigma) \oplus \bigoplus_{\sigma \in \Gamma_{r-1}(Y)} H_s(H_\sigma, N_\sigma).$$

Here $\Sigma_p(X)$ denotes a set of representatives for the G-orbits of the p-cells in X, and $\Gamma_n(Y)$ denotes a set of representatives for those H-orbits which do not come from n-cells in X under $i_\sharp$.

Proof. We can choose $\Sigma_n(Y) = i(\Sigma_n(X)) \cup \Gamma_n(Y)$. In this case we obtain:

$$E^1_{r,s} \cong \bigoplus_{\sigma \in \Sigma_{r-1}} H_s(H_\sigma, G_\sigma, N_\sigma, M_\sigma) \oplus \bigoplus_{\sigma \in \Gamma_{r-1}(Y)} H_s(H_\sigma, N_\sigma).$$

The first direct sum is obtained in the same way as in the bijective case. The second consists of absolute homology, since the cells of $\Gamma_n(Y)$ are not in orbit with cells from X. □

We are primarily going to use the absolute case, $Y = \emptyset$:

Corollary 1.4. *For a group G acting on a d-connected simplicial complex X, and a G-module M, there is a spectral sequence converging to zero for $r + s \leq d + 1$, with*

$$E^1_{r,s} = \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} H_s(G_\sigma, M_\sigma),$$

where $\bar{\Delta}_{r-1}$ is a set of representatives of the G-orbits of $(r-1)$-cells in X.

In our applications, we often have a rotation-free group action, in the following sense:

Definition 1.5. A simplicial group action of G on X is rotation-free if for each simplex σ of X, the elements of G_σ fixes σ pointwise.

Corollary 1.6. *For rotation-free actions, the spectral sequence of Thm. 1.2 takes the form:*

$$E^1_{r,s} \cong \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} H_s(H_\sigma, G_\sigma, N, M)$$

in the relative case, and

$$E^1_{r,s} \cong \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} H_s(G_\sigma, M)$$

in the absolute case.

Proof. The extra assumption implies that each $g \in G_\sigma$ preserves the orientation of σ. Thus g acts on M_σ in the same way as on M, so M_σ and M are identical as G_σ-modules. The same applies to N. □

Remark 1.7. In some of our applications of the absolute version of the spectral sequence, G acts both transitively and rotation-freely on the n-simplices of X. In this case there is only one G-orbit, so we get

$$E^1_{r,s} \cong H_s(G_\sigma; M),$$

where σ is any $(r-1)$-cell in X.

1.3 The first differential

We will need a formula for the first differential $d^1_{r,s} : E^1_{r,s} \longrightarrow E^1_{r-1,s}$. From the construction of the spectral sequences of a double complex, d^1 is induced from the vertical differentials d^v on homology. In the absolute version of the spectral sequence, assuming that G acts rotation-freely on X,

$$E^1_{r,s} \cong \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} H_s(G_\sigma, M).$$

and it is not hard to se that the differential

$$d^1_{r,s} : \bigoplus_{\sigma \in \bar{\Delta}_{r-1}} H_s(G_\sigma, M) \longrightarrow \bigoplus_{\tau \in \bar{\Delta}_{r-2}} H_s(G_\tau, M).$$

has the following description (see e.g. [Brown], Chapter VII, Prop 8.1.) Let σ be an $(r-1)$-simplex of X and τ an $(r-2)$-dimensional face of σ. We have the boundary operator

$$\partial : C_{r-1}(X, M) \longrightarrow C_{r-2}(X, M)$$

and we denote its (σ,τ)th component by $\partial_{\sigma\tau} : M \longrightarrow M$. This is a G_σ-map, so together with the inclusion $G_\sigma \longrightarrow G_\tau$ it induces a map

$$u_{\sigma\tau} : H_*(G_\sigma, M) \longrightarrow H_*(G_\tau, M).$$

Up to a sign $u_{\sigma\tau}$ is the inclusion, because X is a simplicial complex. Consequently

$$\partial(\sigma) = \sum_{j=0}^{r-1} (-1)^j (j\text{th face of } \sigma).$$

So if τ is the ith face of σ, then $u_{\sigma\tau} = (-1)^i$. For $\sigma \in \bar{\Delta}_{r-1}$, we cannot be sure that $\tau \in \bar{\Delta}_{r-2}$, but there is a $g(\tau) \in G$ such that $g(\tau)\tau = \tau_0 \in \bar{\Delta}_{r-2}$. The conjugation, $g \mapsto g(\tau)gg(\tau)^{-1}$, induces a map from G_τ to G_{τ_0} and hence an isomorphism,

$$c_{g(\tau)} : H_*(G_\tau, M) \xrightarrow{\cong} H_*(G_{\tau_0}, M).$$

Now d^1 is given by

$$d^1 \mid_{H_*(G_\sigma,M)} = \sum_{\tau \text{ face of } \sigma} u_{\sigma\tau} c_{g(\tau)}. \tag{12}$$

Denoting the ith face of σ by τ_i, this can be written:

$$d^1|_{H_*(G_\sigma,M)} = \sum_{i=0}^{r-1} (-1)^i c_{g(\tau_i)}. \tag{13}$$

2 Arc complexes and permutations

We write $F_{g,r}$ for a compact oriented surface of genus g with r boundary components.

Definition 2.1. Let F be a surface with boundary. The mapping class group

$$\Gamma(F) = \pi_0(\mathrm{Diff}_+(F, \partial F))$$

is the connected components of the group of orientation-preserving diffeomorphisms which are the identity on a small collar neighborhood of the boundary. We write $\Gamma_{g,r} = \Gamma(F_{g,r})$.

To establish stability results about the homology of $\Gamma_{g,r}$, we will make extensive use of cutting along arcs in $F_{g,r}$. These arcs will be the vertices in simplicial complexes, the so-called arc complexes. The mapping class group act on these arc complexes, and we can use the spectral sequences of section 1.2. The differentials in the spectral sequences are closely related to the homomorphisms of Theorem 1 and Theorem 2 from the introduction.

2.1 Definitions and basic properties

Let F be a surface with boundary. To define the ordering of the vertices used in the arc complexes, we will need the orientation of ∂F. An orientation at a point $p \in \partial F$ is determined by a tangent vector v_p to the boundary circle at p. Let w_p be tangent to F at p, perpendicular to v_p and pointing into F. We call the orientation of ∂F at p determined by v_p *incoming* if the pair (v_p, w_p) is positively oriented, and *outgoing* if (v_p, w_p) is negatively oriented, and use the same terminology for the connected component of ∂F that contains p.

Definition 2.2. Given a surface F with non-empty boundary. Fix two points b_0 and b_1 in ∂F. If b_0 and b_1 are on the same boundary component, the arc complex we define is denoted $C_*(F; 1)$. If b_0 and b_1 are on two different boundary components of F, the resulting arc complex is denoted $C_*(F; 2)$.

- A *vertex* of $C_*(F; i)$ is the isotopy class rel endpoints of an arc (image of a curve) in F starting in b_0 and ending in b_1, which has a representative that meets ∂F transversally and only in b_0 and b_1.

- An n*-simplex* α in $C_*(F; i)$ (called an arc simplex) is set of $n+1$ vertices, such that there are representatives meeting each other transversally in b_0 and b_1 and not intersecting each other away from these two points. We further require that the complement of the $n+1$ arcs be connected.

  The set of arcs is ordered by using the incoming orientation of ∂F at the starting point b_0, and we write $\alpha = (\alpha_0, \dots, \alpha_n)$.

- Let $\Delta_n(F; i)$ denote the set of n-simplices, and let $C_*(F, i)$ be the chain complex with chain groups $C_n(F; i) = \mathbb{Z}\Delta_n(F; i)$ and differentials $d : C_n(F; i) \longrightarrow C_{n-1}(F; i)$ given by:

$$d(\alpha) = \sum_{j=1}^{n} (-1)^j \partial_j(\alpha), \text{ where } \partial_j(\alpha) = (\alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_n).$$

The mapping class group $\Gamma(F)$ acts on $\Delta_n(F; i)$ (by acting on the $n+1$ arcs representing an n-simplex), and thus on $C_n(F; i)$. This action is obviously compatible with the differentials $d : C_n(F; i) \longrightarrow C_{n-1}(F; i)$, so we can consider the quotient complex with chain groups $C_n(F; i)/\Gamma(F)$.

To apply the spectral sequence of the action of $\Gamma_{g,r}$ on $C_*(F_{g,r}; i)$, we need to know that the complex is highly-connected:

Theorem 2.3 ([Harer1])**.** *The chain complex $C_*(F_{g,r}; i)$ is $(2g-3+i)$-connected.*

Definition 2.4. Given an arc simplex α in $C_*(F;i)$, we denote by $N(\alpha)$ the union of a small, open normal neighborhood of α with an open collar neighborhood of the boundary component(s) of F containing b_0 and b_1. Then the cut surface F_α is given by

$$F_\alpha = F \setminus N(\alpha).$$

For a surface S, let $\sharp\partial S$ denote the number of boundary components of S. Then we have the following

$$\sharp\partial(F_\alpha) = \sharp\partial N(\alpha) + r - 2i. \tag{14}$$

Lemma 2.5. *Given an n-simplex α in $C_*(F;i)$, the Euler characteristic of the cut surface F_α is*

$$\chi(F_\alpha) = \chi(F) + n + 1$$

Proof. We prove the formula inductively by removing one arc α_0 at a time, so it suffices to show that $\chi(F_{\alpha_0}) = \chi(F) + 1$. Give F the structure of a CW complex with α_0 as a 1-cell (glued onto the 0-cells b_0 and b_1). When we cut along α_0, we get two copies of α_0; that is, an additional 1-cell and two additional 0-cells. Using the standard formula for the Euler characteristic of a CW complex, we see that it increases by 1. □

2.2 Permutations

Let Σ_{n+1} denote the group of permutations of the set $\{0, 1, \ldots, n\}$. I will write a permutation $\sigma \in \Sigma_n$ as $\sigma = [\sigma(0)\,\sigma(1)\,\ldots\,\sigma(n)]$; e.g. $[0\,2\,1]$ in Σ_3 is the permutation fixing 0 and interchanging 1 and 2.

To each n-arc simplex α in one of the arc complexes $C_*(F;i)$ we assign a permutation $P(\alpha)$ in Σ_{n+1} as follows: Recall that the arcs in $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ are ordered using the incoming orientation of ∂F at the starting point b_0. We use the *outgoing* orientation in the end point b_1 to read off the positions of the $n+1$ arcs at b_1: α_j is the $\sigma(j)$'th arc at b_1, for $j = 0, \ldots, n$. In other words, the arcs at b_1 will be ordered $(\alpha_{\sigma^{-1}(0)}, \alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})$. This gives the permutation $\sigma = P(\alpha)$. See Example 2.6 below.

So we have a map $P : \Delta_n(F;i) \longrightarrow \Sigma_{n+1}$. Since $\gamma \in \Gamma(F)$ keeps a small neighborhood of ∂F fixed, this induces a well-defined map

$$P : \Delta_n(F;i)/\Gamma(F) \longrightarrow \Sigma_{n+1}.$$

There are several reasons why it is useful to look at the permutation $P(\alpha)$ of an arc simplex α. One is that $P(\alpha)$ determines the number of boundary components of the cut surface F_α, as we shall see below. Before explaining this, we will need a few preliminary remarks.

Let α be an arc in $C_*(F;i)$. We orient it from b_0 to b_1, and let $t_p(\alpha)$ be the (positive) tangent vector at $p \in \alpha$. A normal vector v_p to α at p is called *positive* if $(v_p, t_p(\alpha))$ is a positive basis of T_pF. We say that the right-hand side of α is the part of the normal tube given by the positive normal vectors.

When drawing pictures to aid the geometric intuition, we always indicate the orientation of F and ∂F (with arrows). Also, the orientation of F will always be the same, namely the orientation induced by the standard orientation of this paper. This has the advantage that orientation-depending properties like the right-hand side will be consistent throughout the picture, even if we draw two different areas of one surface.

Example 2.6. Let $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ be a 2-simplex in $C_*(F_{g,r};1)$, with permutation $P(\alpha) = [1\,2\,0]$. Close to b_0 and b_1 we see the situation depicted on Figure 1, with the orientations of ∂F at b_0 and b_1 used for determining the permutation as indicated.

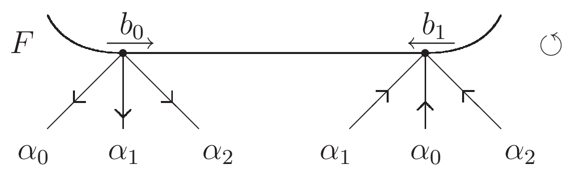


Figure 1: An arc with permutation $[1\,2\,0]$ in $C_*(F;1)$.

We want to find the number of boundary components of F_α. This goes as follows. Pick an arc, say α_0, at b_0 and start coloring the right-hand side of it (here, we color it dark grey), following the arc all the way to b_1. See Figure 2. Here, continue to the left-hand side of the next arc; in our case it is α_2. Note that in general this means going from $\alpha_{\sigma^{-1}(j)}$ to $\alpha_{\sigma^{-1}(j-1)}$ (see the definition); in this example $j = 1$. Color the left-hand side of α_2, reaching b_0 again and continuing to the right-hand side of the arc next to α_2. In this algorithm the boundary component(s) containing b_0 and b_1 also counts as arcs, as shown in the figure. Continue in this fashion until you get back where you started (i.e. the right-hand side of α_0). This closed, dark grey loop constitutes one boundary component of F_α. Start over again with a different color (here light grey) at another arc, and you get a picture as in Figure 2. So there are $2 + (r-1) = r+1$ boundary components of $(F_{g,r})_\alpha$ for $\alpha \in C_*(F;1)$ with $P(\alpha) = [1\,2\,0]$.

We could consider the same permutation in $C_*(F_{g,r};2)$, and we would get a different picture (Figure 3). So there are $3 + (r-2) = r+1$ boundary

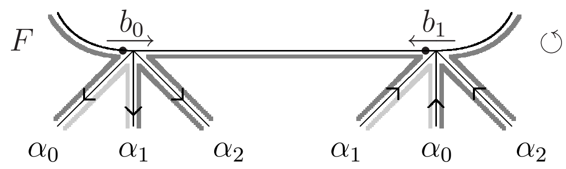


Figure 2: Boundary components of F_α for α in $C_*(F;1)$.

components of $(F_{g,r})_\alpha$ for $\alpha \in C_*(F;2)$ with $P(\alpha) = [1\,2\,0]$.

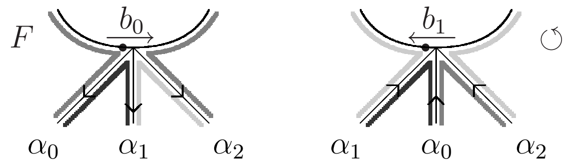


Figure 3: Boundary components of F_α for α in $C_*(F;2)$.

The method of the above example gives a formula – albeit a rather cumbersome one – for $\sharp\partial N(\alpha)$, and thus by (14) for the number of boundary components of F_α in terms of $P(\alpha)$:

Proposition 2.7. *Let $\sharp\partial S$ denote the number of boundary components in S, and let $\sigma_k \in \Sigma_k$ be given by $\sigma_k = [1\,2 \cdots k{-}1\,0]$. Then*

(*i*) *If $\alpha \in C_{n-1}(F;1)$ then* $\sharp\partial N(\alpha) = \mathrm{Cyc}\Big(\sigma_{n+1}\widehat{P(\alpha)}^{-1}\sigma_{n+1}^{-1}\widehat{P(\alpha)}\Big) + 1.$

(*ii*) *If $\alpha \in C_{n-1}(F;2)$ then* $\sharp\partial N(\alpha) = \mathrm{Cyc}\Big(\sigma_n P(\alpha)^{-1}\sigma_n^{-1}P(\alpha)\Big) + 2,$

Here $\mathrm{Cyc} : \Sigma_k \to \mathbb{N}$ denotes the number of disjoint cycles in the given permutation, and for $\tau \in \Sigma_k$, $\widehat{\tau} \in \Sigma_{k+1}$ is given by $\widehat{\tau} = [0, \tau+1]$, that is

$$\widehat{\tau}(j) = \begin{cases} 0, & j = 0, \\ \tau(j-1)+1, & i = 1, \dots, k. \end{cases}$$

In particular, $\sharp\partial N(\alpha)$ depends only on $P(\alpha)$.

Proof. This is simply a way to formulate the method described in Example 2.6. Let us look at $C_*(F;2)$ first, so b_0 and b_1 are in different boundary components. As in the example, we start on the right-hand side of one of the arcs at b_0, follow it (using $P(\alpha)$), then at b_1 we go left to the next arc (using σ^{-1}). Now we follow the right side of that arc (using $P(\alpha)^{-1}$)

ending at b_0, and we must now go left to the next arc (using σ). Thus the permutation $P(\alpha)\sigma^{-1}P(\alpha)^{-1}\sigma$ captures how the boundary of $N(\alpha)$ behaves, and a boundary component in $\partial N(\alpha)$ clearly corresponds to a cycle in the permutation. Remembering the two extra components corresponding to the components of $\partial N(\alpha)$ containing b_0 and b_1, this proves (ii).

For $C_*(F;1)$, b_0 and b_1 lie on the same boundary component. We wish to use (ii), so we consider a new surface $\hat{F}$ and a new arc simplex, $\hat{\alpha} = (\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_n)$ in $C_*(\hat{F}, 2)$, which are constructed from F and α as follows.

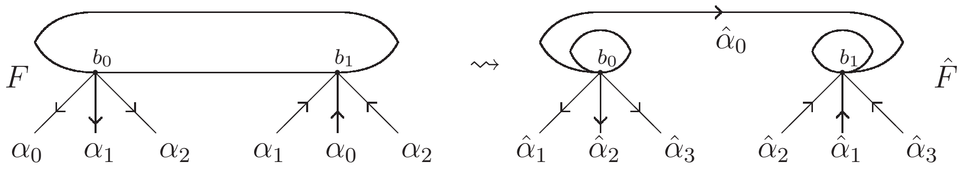


Figure 4: Constructing $\hat{F}$ and $\hat{\alpha}$ from F and α.

We take the boundary component of F containing b_0 and b_1, and close up part of it between b_0 and b_1 so we get two boundary components, cf. Figure 4. Then $\hat{\alpha}_0$ will be the arc from b_0 to b_1 consisting of the part of the old boundary component which was first (i.e. right-most) in the incoming ordering at b_0 (cf. Figure 4), and $\hat{\alpha}_j = \alpha_{j-1}$ for $1 \leq j \leq n$. By this construction, $\sharp\partial N(\alpha) = \sharp\partial N(\hat{\alpha}) - 1$, since we count two boundary components for $\hat{\alpha} \in C_*(\hat{F}; 2)$, and we should count only one. Clearly $P(\hat{\alpha}) = \widehat{P(\alpha)}$, and the result now follows from (ii). $\square$

I would like to thank my brother, Jens Boldsen, for help with the above proposition.

Proposition 2.8. *The permutation map*

$$P : \Delta_n(F; i)/\Gamma(F) \longrightarrow \Sigma_{n+1}$$

is injective.

Proof. We have to show that given two n-arc simplices α and β with $P(\alpha) = P(\beta)$, there exists $\gamma \in \Gamma$ such that $\gamma\alpha = \beta$. Consider the cut surfaces F_α and F_β. Since the permutations are the same, F_α and F_β have the same number of boundary components, by Prop. 2.7 above. Now since we have parameterizations of the boundary components and the curves $\alpha_0, \ldots, \alpha_n$ this gives a diffeomorphism $\varphi : \partial(F_\alpha) \longrightarrow \partial(F_\beta)$. The Euler characteristic of F_α and F_β are also the same, according to Lemma 2.5. This implies that F_α and F_β have the same genus. By the classification of surfaces with boundary, $F_\alpha \cong F_\beta$ via an orientation preserving diffeomorphism Φ extending φ. Gluing both F_α and F_β up again gives a diffeomorphism $\bar{\Phi} : F \longrightarrow F$ taking α to β. Thus α and β are conjugate under $\gamma = \left[\bar{\Phi}\right]$ in the mapping class group $\Gamma(F)$. $\square$

Whether P is surjective depends on the genus g, cf. Corollary 2.17 below.

Remark 2.9. The proof of this proposition also shows that the action of $G(F)$ on $C_*(F;i)$ is rotation-free, cf. Def. 1.5. For given $\alpha \in \Delta_n(F;i)$ and $\gamma = [\varphi] \in \Gamma_\alpha$,

2.3 Genus

Definition 2.10 (Genus)**.** To an arc simplex α we associate the number $S(\alpha) = \text{genus}(N(\alpha))$, cf. Def. 2.4. We call $S(\alpha)$ the genus of α.

Note that Harer calls this quantity the *species* of α.

Lemma 2.11. *For $\alpha \in \Delta_n(F;i)$, we have*

$$\chi(N(\alpha)) = -(n+1)$$

Proof. In $C_*(F;1)$, $N(\alpha)$ has $\alpha \cup_{b_0,b_1} S^1$ as a retract. Now there is a homotopy taking b_1 to b_0 along S^1, so up to homotopy, this is a wedge of $n+2$ copies of S^1 coming from $\alpha_0, \dots, \alpha_n$ and from the boundary component. This gives the result. For $C_*(F;2)$ the argument is similar. $\square$

Proposition 2.12. *Let $\sharp\partial S$ denote the number of boundary components in a surface S. Let $i = 1, 2$. Then for any $\alpha \in \Delta_n(F_{g,r};i)$, the following relations hold:*

(*i*) $S(\alpha) = \frac{1}{2}\big(n + 3 - \sharp\partial N(\alpha)\big)$,

(*ii*) $\sharp\partial(F_\alpha) = r + n - S(\alpha) + 3 - 2i$,

(*iii*) $\text{genus}(F_\alpha) = g + S(\alpha) - (n + 2 - i)$,

Proof. (*i*) As $S(\alpha)$ is the genus of $N(\alpha)$, we can derive this from the Euler characteristic of $N(\alpha)$, which by Lemma 2.11 is $-(n+1)$. Using the formula $\chi(N(\alpha)) = 2 - 2S(\alpha) - \sharp\partial N(\alpha)$ gives the result.

(*ii*) This follows from (*i*) and (14).

(*iii*) As in (*i*) we use the connection between Euler characteristic, genus and number of boundary components, together with (*i*) and (*ii*):

$$\begin{aligned}
\mathrm{genus}(F_\alpha) &= \tfrac{1}{2}\big(-\chi(F_\alpha) - \sharp\partial(F_\alpha) + 2\big) \\
&= \tfrac{1}{2}\big(-(2-2g-r) - (n+1) - (\sharp\partial N(\alpha) + r - 2i) + 2\big) \\
&= \tfrac{1}{2}\big(2g + (n+1-\sharp\partial N(\alpha)+2) + 2i - 2 - 2(n+1)\big) \\
&= g + S(\alpha) - (n+2-i)
\end{aligned}$$

□

Consequently all information about F_α can be extracted from $\sharp\partial(F_\alpha)$, so it is important that we can compute this quantity:

Lemma 2.13. *Given $\alpha \in \Delta_n(F;i)$ be given, and let $\nu \in \Delta_0(F;i)$ be an arc such that $\alpha' = \alpha \cup \nu$ is an $(n+1)$-simplex. Consider $\alpha' \in C_*(F_\alpha;i)$. Then:*

$$\sharp\partial(F_{\alpha'}) = \begin{cases} \sharp\partial(F_\alpha) + 1, & \text{if } \nu \in \Delta_0(F_\alpha;1); \\ \sharp\partial(F_\alpha) - 1, & \text{if } \nu \in \Delta_0(F_\alpha;2). \end{cases}$$

Proof. Let $k = \sharp\partial(F_\alpha)$. Since all boundary components in $F_{\alpha'}$ not intersecting ν correspond to boundary components in F_α, it is enough to consider the situation close to ν. There are two possibilities: Either ν will start and end on two different boundary components of F_α, so $\nu \in \Delta_0(F_\alpha;2)$, or ν will start and end on the same boundary component of F_α, so $\nu \in \Delta_0(F_\alpha;1)$. Cf. Figure 5, where the boundary components of F_α are indicated as in Example 2.6.

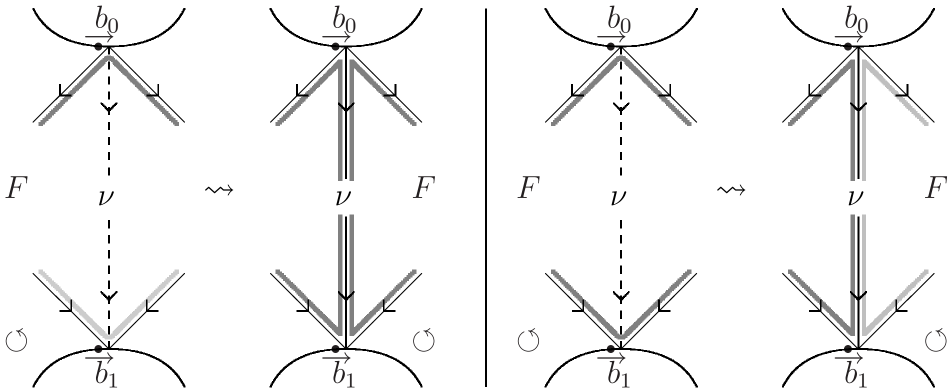


Figure 5: Before and after cutting along the arc ν – the two cases.

Taking the case $\nu \in \Delta_0(F_\alpha;2)$ (left-hand side of Figure 5), when we cut along ν we get one boundary component instead of two. So we get $k-1$ boundary components in this case. In the case $\nu \in \Delta_0(F_\alpha;1)$ (right-hand side of Figure 5) cutting along ν splits the boundary component into two, so we get $k+1$ boundary components. □

Combining Lemma 2.13 and Prop. 2.12, we have proved,

Corollary 2.14. *For $\alpha \in \Delta_0(F;i)$, let $\alpha' = \alpha \cup \nu$ as in Lemma 2.13. Then:*

$$S(\alpha') = \begin{cases} S(\alpha), & \text{if } \nu \in \Delta_0(F_\alpha;1); \\ S(\alpha)+1, & \text{if } \nu \in \Delta_0(F_\alpha;2). \end{cases}$$

and

$$\mathrm{genus}(F_{\alpha'}) = \begin{cases} \mathrm{genus}(F_\alpha)-1, & \text{if } \nu \in \Delta_0(F_\alpha;1); \\ \mathrm{genus}(F_\alpha), & \text{if } \nu \in \Delta_0(F_\alpha;2). \end{cases}$$

□

Lemma 2.15. *Let $\alpha \in \Delta_0(F;i)$. Then $S(\alpha) = 0$ if and only if*

(*i*) *for $i = 1$, $P(\alpha) = \mathrm{id}$.*

(*ii*) *for $i = 2$, $P(\alpha)$ is a cyclic permutation, i.e. one of the following:*

$$\mathrm{id}, [1\,2 \cdots n\,0], [2\,3 \cdots n\,0\,1], \cdots, [n\,0\,1 \cdots n\text{–}1].$$

Proof. We prove "only if". The converse is clear, e.g. by Prop. 2.7 and Prop. 2.12 (*i*).

By Cor. 2.14, any subsimplex of α has genus equal to or lower than $S(\alpha) = 0$, so any subsimplex of α must have genus 0. If $\alpha \in \Delta_n(F;1)$, this means all 1-subsimplices must have permutation equal to the identity, and this forces $P(\alpha) = \mathrm{id}$. If $\alpha \in \Delta_n(F;2)$ the condition on 1-subsimplices is vacuous, but for a 2-subsimplex β of α, we see by Cor. 2.14 that $S(\beta) = 0$ implies that $P(\beta)$ is either id, $[1\,2\,0]$, or $[2\,0\,1]$. For this to hold for any 2-subsimplex of α, $P(\alpha)$ must be as stated in (*ii*). □

2.4 More about permutations

By Prop. 2.7, given $\alpha \in \Delta_n(F;i)$, the number $\sharp\partial N(\alpha)$ is a function only of $P(\alpha)$ and i. By Prop. 2.12(*i*), the same is true for $S(\alpha)$. Thus, given a permutation $\sigma \in \Sigma_{n+1}$, we can calculate these quantities and simply define the numbers $\sharp\partial N(\sigma)$ and $S(\sigma)$ by the formulas of Prop. 2.7 and 2.12(*i*).

Now we are going to see that given a permutation $\sigma \in \Sigma_{n+1}$, there exists $\alpha \in \Delta_n(F_{g,r};i)$ with $P(\alpha) = \sigma$ if at all possible, that is, provided the formula (*iii*) of Prop. 2.12 for the genus of F_α gives a non-negative result. Rearranging this conditions we have the following lemma, also stated in [Harer2]:

Lemma 2.16. *Given a permutation $\sigma \in \Sigma_{n+1}$, let $s = S(\sigma)$ as above. There exists $\alpha \in \Delta_0(F;i)$ with $P(\alpha) = \sigma$ if and only if*

$$s \geq n - g + 2 - i. \tag{15}$$

Proof. Given a permutation σ, one can try to construct an arc simplex α inductively with $P(\alpha) = \sigma$ by first choosing an arc $\alpha_0 \in \Delta_0(F;i)$ from b_0 to b_1, and cutting F up along it. This will give us two copies of b_0 and b_1, respectively, one to the left of our arc and one to the right. The permutation determines from which copy of b_0 and b_1 a new arc will join.

Suppose we have constructed $k+1 \leq n+1$ arcs as above, i.e. a k-simplex $\beta = (\alpha_0, \dots, \alpha_k)$, and consider the cut surface F_β. Inductively we assume that F_β is connected. Now we must verify that when adding a new arc, ν, as in Lemma 2.13, the cut surface $(F_\beta)_\nu$ is connected. If this holds, $\beta \cup \nu$ is a $(k+1)$-simplex, and we have completed the induction step.

There are two cases. First assume that ν must join two different boundary components of F_β. Then $(F_\beta)_\nu$ is connected, no matter how we choose ν, since F_β is connected.

Secondly, if ν connects two points on the same boundary component of F_β, we choose ν so that it winds around a genus-hole in F_β. This ensures that $(F_\beta)_\nu$ is connected, so we must prove that $\text{genus}(F_\beta) \geq 1$. From Prop. 2.12, we know that $\text{genus}(F_\beta) = g + S(\beta) - (k+2-i)$, and we want to prove

$$S(\beta) - k \geq s - n + 1. \tag{16}$$

Using this, we can complete the induction step:

$$\text{genus}(F_\beta) = g + S(\beta) - k - 2 + i \geq g + s - n - 1 + i \geq 1$$

by assumption (15).

To prove (16), recall that $S(\beta)$ only depends on $P(\beta)$, not on the surface F. So consider another surface F' with genus $g' > n$. We can construct $\beta' \in \Delta_k(F', i)$ with $P(\beta') = P(\beta)$, as above. We can further construct $\alpha' \in \Delta_n(F', i)$ with β' as a subsimplex and $P(\alpha') = \sigma$, simply by adding $n-k$ new arcs to β' which each wind around a genus-hole in F'. This is possible because $g' > n$. We claim

$$S(\alpha') \leq S(\beta') + n - k - 1. \tag{17}$$

Applying Cor. 2.14 $n-k$ times to β', we obviously get $S(\alpha') \leq S(\beta') + n - k$. We get the extra -1, because the first time we add an arc ν' to β' we have $\nu' \in \Delta_0(F'_{\beta'}; 1)$, since $\nu \in \Delta_0(F_\beta, 1)$ by assumption. This proves (17). Since $P(\beta') = P(\beta)$ and $P(\alpha') = \sigma$, (17) implies $s = S(\sigma) \leq S(\beta) + n - k - 1$. This proves (16). □

Combining Prop. 2.8 and Lemma 2.16 we have proved,

Corollary 2.17. *The permutation map*

$$P : \Delta_n(F;i)/\Gamma(F) \longrightarrow \Sigma_{n+1}$$

is bijective if $n \leq g-2+i$. $\square$

Lemma 2.18 ([Harer4])**.** *For* $F = F_{g,b}$ *with* $g \geq 2$*, the sequence*

$$C_{p+1}(F;i)/\Gamma(F) \xrightarrow{d^1} C_p(F;i)/\Gamma(F) \xrightarrow{d^1} C_{p-1}(F;i)/\Gamma(F)$$

is split exact for $1 \leq p \leq g-2+i$.

Proof. Let $\mathbb{Z}\Sigma_*$ denote the chain complex with chain groups $\mathbb{Z}\Sigma_n$, $n \geq 1$, and differentials

$$\partial : \mathbb{Z}\Sigma_{n+1} \longrightarrow \mathbb{Z}\Sigma_n$$

given as follows: For $\sigma = [\sigma(0)\cdots\sigma(n)] \in \Sigma_{n+1}$, let

$$\partial_j(\sigma) = [\sigma(0)\cdots\sigma(j-1)\,\sigma(j+1)\ldots\sigma(n)],$$

where the set $\{0,1,\ldots,n\} \setminus \{\sigma(j)\}$ is identified with $\{0,1,\ldots,n-1\}$ by subtracting 1 from all numbers exceeding $\sigma(j)$. Then we define $\partial(\sigma) = \sum_{j=0}^n (-1)^j \partial_j(\sigma)$ and extend linearly. Extending the permutation map P linearly leads to the commutative diagram

$$\begin{tikzcd} C_n(F;i)/\Gamma(F) \arrow[r, "d"] \arrow[d, "P"] & C_{n-1}(F;i)/\Gamma(F) \arrow[d, "P"] \\ \mathbb{Z}\Sigma_{n+1} \arrow[r, "\partial"] & \mathbb{Z}\Sigma_n \end{tikzcd} \tag{18}$$

i.e. a chain map $C_*(F;i)/\Gamma(F) \longrightarrow \mathbb{Z}\Sigma_*$. By Prop. 2.8, P is injective, so $C_*(F;i)/\Gamma(F)$ is isomorphic to a subcomplex of $\mathbb{Z}\Sigma_*$, namely the subcomplex generated by permutations $\sigma \in \Sigma_{n+1}$ with $S(\sigma)$ satisfying the requirements of Lemma 2.16. In particular, for $n \leq g-2+i$, the chain groups of $\mathbb{Z}\Sigma_*$ and of $C_*(F;i)/\Gamma(F)$ are identified.

Define $D : \mathbb{Z}\Sigma_n \longrightarrow \mathbb{Z}\Sigma_{n+1}$ by

$$D(\sigma) = \hat{\sigma} = [0 \quad \sigma(0){+}1 \quad \sigma(1){+}1 \quad \cdots \quad \sigma(n){+}1]. \tag{19}$$

It is an easy consequence of the definitions that $D\partial + \partial D = 1$, so D is a contracting homotopy and $\mathbb{Z}\Sigma_*$ is split exact. By the diagram (18), $C_*(F;i)/\Gamma(F)$ is also split exact in the range where

$$D \circ P\Big(C_n(F;i)/\Gamma(F)\Big) \subseteq P\Big(C_{n+1}(F;i)/\Gamma(F)\Big), \tag{20}$$

since D lifts to a contracting homotopy $\bar{D}$ of $C_*(F;i)/\Gamma(F)$.

We will first consider $C_*(F;1)/\Gamma(F)$. By Cor. 2.17, P is bijective for $n \leq g-1$, so (20) is satisfied for $n \leq g-2$. It remains to consider the degree $n = g-1$. We have the commutative diagram,

$$\begin{tikzcd}
C_g(F;i)/\Gamma(F) \arrow[r, "d"] \arrow[d, hook, "P"] & C_{g-1}(F;i)/\Gamma(F) \arrow[r, "d"] \arrow[d, "P", "\cong"'] & C_{g-2}(F;i)/\Gamma(F) \arrow[d, "P", "\cong"'] \\
\mathbb{Z}\Sigma_{g+1} \arrow[r, "\partial"] & \mathbb{Z}\Sigma_g \arrow[r, "\partial"] & \mathbb{Z}\Sigma_{g-1}
\end{tikzcd}$$

with the bottom sequence exact. We must show that

$$P \circ d(C_g(F;i)/\Gamma(F)) = \partial(\mathbb{Z}\Sigma_{g+1}).$$

According to Cor. 2.17, $P : C_g(F;1)/\Gamma(F) \longrightarrow \mathbb{Z}\Sigma_{g+1}$ hits everything except what is generated by permutations σ with $S(\sigma) = 0$. Thus we must show $\partial(\sigma) \in \mathrm{Im}(P \circ d) = \mathrm{Im}(\partial \circ P)$ for all $\sigma \in \Sigma_{g+1}$ with $S(\sigma) = 0$. From Lemma 2.15 we know that the only such permutation is the identity. As

$$\partial([0\,1\,\cdots\,g]) = \sum_{j=0}^{g}(-1)^j[0\,1\,\cdots\,g{-}1] = \begin{cases} 0, & \text{if } g \text{ is odd,} \\ \mathrm{id}, & \text{if } g \text{ is even,} \end{cases}$$

we are done if g is odd, and the desired contracting homotopy $\bar{D}$ is obtained by lifting D when $S(\alpha) > 0$ and setting by $\bar{D}(\alpha) = 0$ when $S(\alpha) = 0$.

If g is even, consider $\tau = [2\,0\,1\,3\,4\,\cdots\,g] \in \Sigma_{g+1}$. Then by Lemma 2.15 $S(\tau) > 0$, and

$$\begin{aligned}
\partial(\tau) &= [0\,1\,2\,\cdots\,g{-}1] - [1\,0\,2\,3\,\cdots\,g{-}1] + [1\,0\,2\,3\,\cdots\,g{-}1] \\
&\quad + \sum_{j=3}^{g}(-1)^j[2\,0\,1\,3\,4\,\cdots\,g{-}1] = [0\,1\,2\,\cdots\,g{-}1] = \partial[0\,1\,2\,\cdots\,g].
\end{aligned}$$

Thus we can obtain a contracting homotopy $\bar{D}$ by taking $\bar{D}(\alpha) = P^{-1}(\tau)$ when $S(\alpha) = 0$.

For $C_*(F;2)/\Gamma(F)$, Cor. 2.17 gives that P is bijective for $n \leq g$, so we are left with $j = g$, where we use exactly the same method as above. We must show that $\partial(\sigma) \in \mathrm{Im}(\partial \circ P)$ for all $\sigma \in \Sigma_{g+2}$ with $S(\sigma) = 0$. We only need to consider $\sigma \in \mathrm{Im}(D)$, because $\mathrm{Im}\partial = \mathrm{Im}(\partial \circ D)$ by the equation $\partial D + D\partial = 1$. The only $\sigma \in \Sigma_{g+2}$ with $S(\sigma) = 0$ and $P \in \mathrm{Im}D$ is the identity, according to Lemma 2.15. Now we are in the same situation as above, so we can use $\tau = [2\,0\,1\,3\,4\,\cdots\,g\,g{+}1] \in \Sigma_{g+2}$ which has genus $S(\tau) > 0$ in $C_*(F;2)$, since $g \geq 2$. □

3 Homology stability of the mapping class group

Let F be a surface with boundary. Given F we can glue on a "pair of pants", $F_{0,3}$, to one or two boundary components. We denote the resulting surface by $\Sigma_{i,j}F$, the subscripts indicating the change in genus and number of boundary components, respectively.

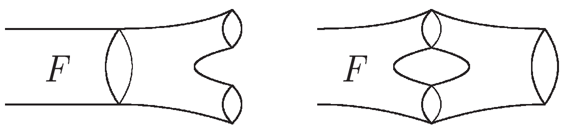


Figure 6: $\Sigma_{0,1}F$ and $\Sigma_{1,-1}F$.

These two operations induce homomorphisms between the mapping class groups after extending a mapping class by the identity on the pair of pants;

$$\Sigma_{i,j} : \Gamma(F) \longrightarrow \Gamma(\Sigma_{i,j}F).$$

Given a surface F, applying $\Sigma_{0,1}$ and then adding a disk at one of the pant legs gives a surface diffeomorphic to F (with a cylinder glued onto a boundary component). It is easily seen that the induced composition

$$\Gamma(F) \longrightarrow \Gamma(\Sigma_{0,1}F) \longrightarrow \Gamma(F)$$

is the identity, so $\Sigma_{0,1}$ induces an injection on homology

$$H_n(\Gamma(F)) \hookrightarrow H_n(\Gamma(\Sigma_{0,1}F)). \tag{21}$$

For the proof of the stability theorems, the opposite operation is essential: One expresses the surface F as the result of cutting $\Sigma_{0,1}F$ or $\Sigma_{1,-1}F$ along an arc representing a 0-simplex in one of the arc complexes of definition 2.2:

$$F \cong (\Sigma_{0,1}F)_\alpha, \quad \text{and} \quad F \cong (\Sigma_{1,-1}F)_\beta,$$

for $\alpha \in \Delta_0(\Sigma_{0,1}F, 2)$ and $\beta \in \Delta_0(\Sigma_{1,-1}F, 1)$ as indicated below

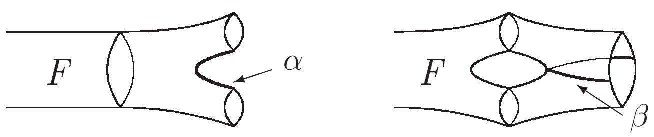


Figure 7: α and β.

A diffeomorphism of F_α that fixes the points on the boundary pointwise extends to a diffeomorphism of F by adding the identity on $N(\alpha)$, and this defines an inclusion $\Gamma(F_\alpha) \longrightarrow \Gamma$ whose image is the stabilizer Γ_α.

3.1 The spectral sequence for the action of the mapping class group

In this section, $F = F_{g,r}$ with $g \geq 2$ and $\Gamma = \Gamma(F)$. We shall consider the spectral sequences $E^n_{p,q} = E^n_{p,q}(F;i)$ from section 1.2 associated to the action of Γ on the arc complexes $C_*(F;i)$ for $i = 1, 2$. By Cor. 1.6 and Thm. 2.3, we have $E^1_{0,q} = H_q(\Gamma)$ and

$$E^1_{p,q} = \bigoplus_{\alpha \in \bar{\Delta}_{p-1}} H_q(\Gamma_\alpha) \Rightarrow 0, \quad \text{for } p + q \leq 2g - 2 + i, \tag{22}$$

where $\overline{\Delta}_{p-1} \subseteq \Delta_{p-1}(F;1)$ is a set of representatives of the Γ-orbits of $\Delta_{p-1}(F;i)$ in $C_*(F;i)$.

The permutation map

$$P : \Delta_{p-1}(F;i)/\Gamma \longrightarrow \Sigma_p$$

is injective by Prop. 2.8. Let $\overline{\Sigma}_p$ be the image, and $T : \overline{\Sigma}_p \xrightarrow{\sim} \overline{\Delta}_{p-1} \hookrightarrow \Delta_{p-1}(F;i)$ a section, $P \circ T = \text{id}$. Then

$$E^1_{p,q} = \bigoplus_{\sigma \in \overline{\Sigma}_p} E^1_{p,q}(\sigma), \quad E^1_{p,q}(\sigma) = H_q(\Gamma_{T(\sigma)}). \tag{23}$$

The first differential, $d^1_{p,q} : E^1_{p,q} \longrightarrow E^1_{p-1,q}$, is described in section 1.3. The diagrams

$$\begin{array}{ccc} \Delta_p(F;i) & \xrightarrow{\partial_j} & \Delta_p(F;i) \\ \downarrow & & \downarrow \\ \overline{\Sigma}_{p+1} & \xrightarrow{\partial_j} & \overline{\Sigma}_p \end{array} \qquad j = 0, \ldots, p$$

commute, where ∂_j omits entry j as in Def. 2.2 and the vertical arrows divide out the Γ action and compose with P. Thus for each $\sigma \in \overline{\Sigma}_{p+1}$, there is $g_j \in \Gamma$ such that

$$g_j \cdot \partial_j T(\sigma) = T(\partial_j \sigma), \tag{24}$$

and conjugation by g_j induces an isomophism $c_{g_j} : \Gamma_{\partial_j T(\sigma)} \longrightarrow \Gamma_{T(\partial_j \sigma)}$. The induced map on homology is denoted ∂_j again, i.e.

$$\partial_j : H_q(\Gamma_{T(\sigma)}) \xrightarrow{\text{incl}_*} H_q(\Gamma_{\partial_j T(\sigma)}) \xrightarrow{(c_{g_j})_*} H_q(\Gamma_{T(\partial_j \sigma)}) \,. \tag{25}$$

Note that $(c_{g_j})_*$ does not depend on the choice of g_j in (46): Another choice g_j' gives $c_{g_j'} = c_{g_j' g_j^{-1}} c_{g_j}$, and $g_j' g_j^{-1} \in \Gamma_{T(\partial_j \sigma)}$ so $c_{g_j' g_j^{-1}}$ induces the identity on $H_q(\Gamma_{T(\partial_j\sigma)})$. Then

$$d^1 = \sum_{j=0}^{p-1} (-1)^j \partial_j. \tag{26}$$

The proof of the main stability Theorem depends on a partial calculation of the spectral sequence (22). More specifically, the first differential $d^1 : E^1_{1,q} \longrightarrow E^1_{0,q}$ is equivalent to a stability map $H_q(\Gamma_\alpha) \longrightarrow H_q(\Gamma)$, so the question becomes whether d^1 is an isomorphism resp. an epimorphism. In a range of dimensions the spectral sequence converges to zero, so that d^1 must be an isomorphism unless other (higher) differentials interfere. The next three lemma are the key elements that give sufficient hold of the spectral sequence. The first lemma gives the general induction step. The next two lemmas about $d^1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$ for $p = 3, 4$ are necessary for the improved stability.

Lemma 3.1. *Let $i = 1, 2$, and let $k, j \in \mathbb{N}$ with $k \le g - 3 + i$. For any $\alpha \in \Delta_{p-1}(F; i)$ and all $q \le k - j$, assume that*

$$H_q(\Gamma_\alpha) \xrightarrow{\cong} H_q(\Gamma) \text{ is an isomorphism} \quad \text{if } p + q \le k + 1, \tag{27}$$

$$H_q(\Gamma_\alpha) \twoheadrightarrow H_q(\Gamma) \text{ is surjective} \quad \text{if } p + q = k + 2. \tag{28}$$

Then $E^2_{p,q}(F; i) = 0$ for all p, q with $p + q = k + 1$ and $q \le k - j$.

Proof. Let $\overline{C}_n(F; i) = C_n(F; i)/\Gamma$. By (22) and the assumptions, we get for $q \le k - j$:

$$\begin{aligned} E^1_{p,q} &\cong \overline{C}_{p-1}(F; i) \otimes H_q(\Gamma) \quad \text{if } p + q \le k + 1, \\ E^1_{p,q} &\twoheadrightarrow \overline{C}_{p-1}(F; i) \otimes H_q(\Gamma) \quad \text{if } p + q = k + 2. \end{aligned} \tag{29}$$

Now we have the following commutative diagram, for a fixed pair p, q with $q \le k - j$ and $p + q = k + 1$:

$$\begin{tikzcd} E^1_{p-1,q} \arrow[d, "\cong"] & E^1_{p,q} \arrow[l, "d^1"'] \arrow[d, "\cong"] & E^1_{p+1,q} \arrow[l, "d^1"'] \arrow[d] \\ \overline{C}_{p-2}(F; i) \otimes H_q(\Gamma) & \overline{C}_{p-1}(F; i) \otimes H_q(\Gamma) \arrow[l, "\bar{d}^1"'] & \overline{C}_p(F; i) \otimes H_q(\Gamma) \arrow[l, "\bar{d}^1"'] \end{tikzcd} \tag{30}$$

Using the formula (48) for $\bar{d}^1$, $(c_{g_j})_*(\omega) = \omega$ for $\omega \in H_*(\Gamma)$, since conjugation induces the identity in $H_*(\Gamma)$. Thus the bottom row of diagram (30) is just the sequence from Lemma 2.18, tensored with $H_q(\Gamma)$. Since $p \leq k+1 \leq g-2+i$ that sequence is split exact, so the bottom row of (30) is exact. We conclude that $E^2_{p,q} = 0$ for all p, q with $q \leq k-j$ and $p+q = k+1$, as desired. □

We next examine the chain complex

$$\cdots \xrightarrow{d^1} E^1_{3,q}(F,i) \xrightarrow{d^1} E^1_{2,q}(F,i) \xrightarrow{d^1} E^1_{1,q}(F,i) \xrightarrow{d^1} E^1_{0,q}(F,i)$$

associated with $C(F;i)$, but first we need an easy geometric proposition. Recall from definition 2.4, that for $\alpha \in \Delta_p(F;i)$ we write $F_\alpha = F \setminus N(\alpha)$ for the surface cut along the arcs of α.

Proposition 3.2. *Let $\alpha \in \Delta_n(F;i)$ with permutation $P(\alpha) = \sigma$, and assume there is $k, l < n$ such that $\sigma(k) = l+1$ and $\sigma(k+1) = l$. Then there exists $f \in \Gamma(F)$ with $f(\alpha_{k+1}) = \alpha_k$, $f(\alpha_i) = \alpha_i$ for $i \notin \{k, k+1\}$ and $f|_{F_\alpha} = \mathrm{id}_{F_\alpha}$.*

Proof. A (right) *Dehn twist* in an annulus in F is an element of $\Gamma(F)$ given by performing a full twist to the right inside the annulus, and extending by the identity outside the annulus. Figure 8 shows a Dehn twist γ in an annulus, and its effect on a curve β intersecting the annulus.

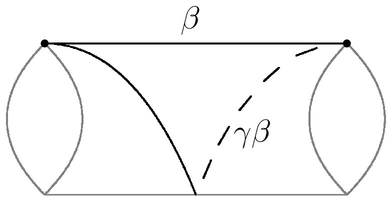


Figure 8: A Dehn twist γ in an annulus.

Consider the curves α_k and α_{k+1}. Take an annulus as depicted on Figure 9 below (in grey). By the requirements of the proposition it is easy to construct the annulus so that it only intersects α in α_k and α_{k+1}. Let f be the Dehn twist in this annulus. Since f is the identity outside the annulus, we have $f(\alpha_i) = \alpha_i$ for all $i \notin \{k, k+1\}$ and $f|_{F_\alpha} = \mathrm{id}_{F_\alpha}$. By Figure 9 it is easy to see that $f(\alpha_{k+1}) = \alpha_k$. □

The stabilizer Γ_α of $\alpha \in \Delta_p(F;i)$ depends up to conjugation only on the orbit $\Gamma\alpha$, i.e. on $P(\alpha) \in \Sigma_{p+1}$. So when conjugation is of no importance we shall for $\sigma \in \overline{\Sigma}_{p+1}$ write Γ_σ for any of the conjugate subgroups Γ_α with $P(\alpha) = \sigma$. If $\tau \in \overline{\Sigma}_p$ is a face of $\sigma \in \overline{\Sigma}_{p+1}$ then Γ_σ is conjugate to a subgroup of Γ_τ, and there is a homomorphism

$$H_q(\Gamma_\sigma) \longrightarrow H_q(\Gamma_\tau),$$

well-determined up to isomorphism of source and target.

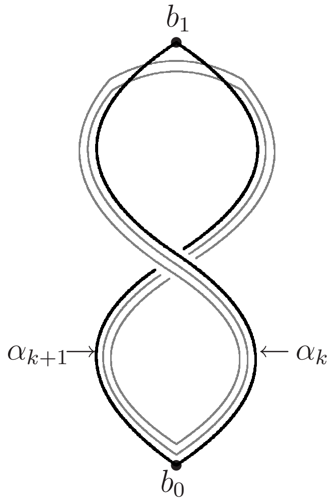


Figure 9: The Dehn twist f.

Lemma 3.3. *Let c_1 and c_2 be the isomorphism classes*

$$c_1 : H_q(\Gamma_{[0\,2\,1]}) \longrightarrow H_q(\Gamma_{[1\,0]}), \quad c_2 : H_q(\Gamma_{[1\,2\,0]}) \longrightarrow H_q(\Gamma_{[0\,1]})$$

(i) If c_1 and c_2 are surjective, then $d^1_{3,q} : E^1_{3,q} \longrightarrow E^1_{2,q}$ is surjective, and $E^2_{2,q} = 0$.

(ii) If c_1 and c_2 are injective, then

$$d^1_{3,q} : E^1_{3,q}([0\,2\,1]) \oplus E^1_{3,q}([1\,2\,0]) \longrightarrow E^1_{2,q}$$

is injective.

Proof. The target of d^1 is $E^1_{2,q} = E^1_{2,q}([0\,1]) \oplus E^1_{2,q}([1\,0])$, and we first examine the component

$$d^1_{3,q} : E^1_{3,q}([0\,2\,1]) \longrightarrow E^1_{2,q}([0\,1]). \tag{31}$$

If $\beta = T([0\,2\,1])$ with $\beta = (\beta_0, \beta_1, \beta_2)$, let $\gamma \in \Gamma$ satisfy $(\gamma\beta_0, \gamma\beta_1) = T([0\,1])$, and write $\alpha = \gamma\beta$. Then

$$(c_g)_* : E^1_{3,q}([0\,2\,1]) \xrightarrow{\cong} H_q(\Gamma_\alpha),$$

and the $E^1_{2,q}([0\,1])$-component of $d^1_{3,q} \circ (c_g)_*$ is the difference of

$$\begin{aligned} \partial_2 : H_q(\Gamma_\alpha) &\longrightarrow H_q(\Gamma_{(\alpha_0,\alpha_1)}) \\ \partial_1 : H_q(\Gamma_\alpha) &\longrightarrow H_q(\Gamma_{(\alpha_0,\alpha_2)}) \longrightarrow H_q(\Gamma_{(\alpha_0,\alpha_1)}) \end{aligned} \tag{32}$$

where $f \cdot (\alpha_0, \alpha_2) = (\alpha_0, \alpha_1)$. By the previous proposition 3.2 we may choose f such that $f|_{F_\alpha} = \mathrm{id}_{F_\alpha}$. It follows that $c_f : \Gamma \longrightarrow \Gamma$ restricts to the identity on Γ_α, and hence that the two maps in (32) are equal. Thus the component of $d^1_{3,q}$ in (31) is zero. On the other hand, the component

$$d^1_{3,q} : E^1_{3,q}([0\,2\,1]) \longrightarrow E^1_{2,q}([1\,0])$$

is equal to ∂_0, so it belongs to the isomorphism class c_1. Thus it is surjective resp. injective under the assumptions (i) resp. (ii).

The restriction of $d^1_{3,q}$ to $E^1_{3,q}([1\,2\,0])$,

$$d^1_{3,q} : E^1_{3,q}([1\,2\,0]) \longrightarrow E^1_{2,q}([0\,1]) \oplus E^1_{2,q}([1\,0]),$$

is treated in a similar fashion. This time there are two terms with opposite signs in $E^1_{2,q}([1\,0])$ which cancel by Prop. 3.2, and the component

$$d^1_{3,q} : E^1_{3,q}([1\,2\,0]) \longrightarrow E^1_{2,q}([0\,1])$$

is in the isomorphism class of c_2. This proves the lemma. $\square$

We next consider the situation of Lemma 3.3(ii) where c_1 and c_2 are injective. If we further assume that $g(F) \geq 3$, then $\overline{\Sigma}_3 = \Sigma_3$ and $\overline{\Sigma}_4 = \Sigma_4 \setminus \{\mathrm{id}\}$. We consider the maps

$$\begin{aligned} c_3 &: H_q(\Gamma_{[1\,2\,3\,0]}) \longrightarrow H_q(\Gamma_{[1\,2\,0]}) \\ c_4 &: H_q(\Gamma_{[0\,3\,2\,1]}) \longrightarrow H_q(\Gamma_{[2\,1\,0]}) \\ c_5 &: H_q(\Gamma_{[0\,2\,1\,3]}) \longrightarrow H_q(\Gamma_{[1\,0\,2]}) \\ c_6 &: H_q(\Gamma_{[0\,3\,1\,2]}) \longrightarrow H_q(\Gamma_{[2\,0\,1]}) \end{aligned} \tag{33}$$

Lemma 3.4. *Let $g \geq 3$ and assume that c_1 and c_2 of Lemma 3.3 are injective and that the four maps in* (33) *are surjective. Then $E^2_{3,q}(F; i) = 0$ for $i = 1, 2$.*

Proof. The group $E^1_{3,q}$ decomposes into six summands since $\overline{\Sigma}_3 = \Sigma_3$. By Lemma 3.3, to show that $E^2_{3,q} = 0$ under the above conditions, it suffices to check that $d^1_{4,q}$ maps onto the four components not considered in Lemma 3.3. More precisely, let

$$\tilde{E}^1_{3,q} = E^1_{3,q}([0\,1\,2]) \oplus E^1_{3,q}([2\,1\,0]) \oplus E^1_{3,q}([1\,0\,2]) \oplus E^1_{3,q}([2\,0\,1]).$$

We must show that the composition

$$\bar{d}^1 : E^1_{4,q} \xrightarrow{d^1} E^1_{3,q} \xrightarrow{\mathrm{proj}} \tilde{E}^1_{3,q}$$

is surjective. the argument is quite similar to the proof of Lemma 3.3, using Prop. 3.2 to cancel out elements. Then the components of $\bar{d}^1$ can be described as follows:

$$\begin{aligned}
\bar{d}^1 = -\partial_3 \quad &: \quad E^1_{4,q}([1\,2\,3\,0]) \longrightarrow E^1_{3,q}([0\,1\,2]),\\
\bar{d}^1 = \partial_0 \quad &: \quad E^1_{4,q}([0\,3\,2\,1]) \longrightarrow E^1_{3,q}([2\,1\,0]),\\
\bar{d}^1 = \partial_0 \quad &: \quad E^1_{4,q}([0\,2\,1\,3]) \longrightarrow E^1_{3,q}([1\,0\,2]),\\
\bar{d}^1 = (\partial_0, -\partial_3) \quad &: \quad E^1_{4,q}([0\,3\,1\,2]) \longrightarrow E^1_{3,q}([2\,0\,1]) \oplus E^1_{3,q}([0\,1\,2]).
\end{aligned}$$

It follows from the surjections in (33) that $\bar{d}^1$ is surjective, and hence that $E^1_{3,q}(F;i) = 0$. □

Remark 3.5. Now we can state Harer's third assertion needed to improve our main stability Theorem by "one degree" (cf. the Introduction). It is easy to show that $d^1_{2,2n}[1\,0]$ is the zero map for all n. Then the homology class $[\breve{\kappa}_1{}^n]$ of $\breve{\kappa}_1{}^n$ with respect to d^1 is an element of $E^2_{2,2n}$. The assertion is

(iii) $d^2_{2,2n}([\breve{\kappa}_1{}^n]) = x \cdot [\breve{\kappa}_1{}^n]$ for some Dehn twist x around a simple closed curve in F. Here, $\cdot$ denotes the Pontryagin product in group homology.

3.2 The stability theorem for surfaces with boundary

In this section we prove the first of the two stability theorems listed in the introduction. Our proof is strongly inspired by the 15 year old manuscript [Harer2], but with two changes. We work with integral coefficients, and we avoid the assertions made in [Harer2] discussed in the introduction. The theorem we prove is

Theorem 3.6 (Main Theorem)**.** *Let $F_{g,r}$ be a surface of genus g with r boundary components.*

(i) Let $r \geq 1$ and let $i = \Sigma_{0,1} : \Gamma_{g,r} \longrightarrow \Gamma_{g,r+1}$. Then

$$i_* : H_k(\Gamma_{g,r}) \longrightarrow H_k(\Gamma_{g,r+1})$$

is an isomorphism for $2g \geq 3k$.

(ii) Let $r \geq 2$ and let $j = \Sigma_{1,-1} : \Gamma_{g,r} \longrightarrow \Gamma_{g+1,r-1}$. Then

$$j_* : H_k(\Gamma_{g,r}) \longrightarrow H_k(\Gamma_{g+1,r-1})$$

is surjective for $2g \geq 3k - 1$, and an isomorphism for $2g \geq 3k + 2$.

Proof. The proof is by induction in the homology degree k. For $k = 0$ the results are obvious, since $H_0(G, \mathbb{Z}) = \mathbb{Z}$ for any group G. So assume now $k > 0$ and that the theorem holds for homology degrees less than k.

The case $\Sigma_{0,1}$

In this case we know from (21) that $\Sigma_{0,1}$ is injective, so to prove that it is an isomorphism it is enough to show surjectivity.

Assume $2g \geq 3k$ and write $\Gamma = \Gamma_{g,r+1}$. We use that $\Gamma_{g,r}$ is the stabilizer Γ_α for $\alpha \in \Delta_0(F_{g,r+1;2}$ as on Figure 7, $\Gamma_{g,r} = \Gamma_\alpha$. Now we use the spectral sequence (22) associated with the action of Γ on $C_*(F_{g,r+1};2)$, and we recognize the map $i_* : H_k(\Gamma_\alpha) \longrightarrow H_k(\Gamma)$ as the differential $d^1 : E^1_{1,k} \longrightarrow E^1_{0,k}$. The spectral sequence converges to zero at $E^n_{0,k}$. So it suffices to show that $E^2_{p,k+1-p}$ is zero for all $p \geq 2$.

We begin by proving $E^2_{2,k-1} = 0$ using Lemma 3.3 (i), noting that $g \geq 2$, since $k \geq 1$. We must verify that c_1 and c_2 are surjective, and we will do this inductively. Prop. 2.7 (or Example 2.6) and Prop. 2.12 calculate the genus and the number of boundary components of Γ_σ. The figures below show the relevant simplices $\sigma \in \Delta_*(F_{g,r+1};2)$ so that the method in Example 2.6 can easily be applied. The circles are the boundary components containing b_0 and b_1.

$\Gamma_{[1\,0]} = \Gamma_{g-1,r+1}$,  $\Gamma_{[0\,2\,1]} = \Gamma_{g-1,r}$, 

$\Gamma_{[0\,1]} = \Gamma_{g-1,r+1}$,  $\Gamma_{[1\,2\,0]} = \Gamma_{g-2,r+2}$. 

We see that

$$\begin{aligned} c_1 = (\Sigma_{0,1})_* :&\quad H_{k-1}(\Gamma_{g-1,r}) \longrightarrow H_{k-1}(\Gamma_{g-1,r+1}), \qquad \text{and} \\ c_2 = (\Sigma_{1,-1})_* :&\quad H_{k-1}(\Gamma_{g-2,r+2}) \longrightarrow H_{k-1}(\Gamma_{g-1,r+1}) \end{aligned}$$

are both surjective by induction. So $E^2_{2,k-1} = 0$.

We now show that $E^2_{p,q} = 0$ for $p + q = k + 1$ and $p > 2$, i.e. $q \leq k - 2$, using Lemma 3.1, so we must verify (27) and (24). By Prop. 2.12 we have $\Gamma_\alpha = \Gamma_{g-p+s+1,r+p-2s-1}$, for $\alpha \in \overline{\Delta}_{p-1}$ of genus s. So for $q \leq k-2$, we will show by induction:

$$H_q(\Gamma_{g-p+s+1,r+p-2s-1}) \cong H_q(\Gamma_{g,r+1}), \quad \text{for} \quad p+q \leq k+1 \tag{34}$$
$$H_q(\Gamma_{g-p+s+1,r+p-2s-1}) \twoheadrightarrow H_q(\Gamma_{g,r+1}), \quad \text{for} \quad p+q = k+2. \tag{35}$$

The maps in (34) and (30) are induced from the composition

$$\Gamma_{g-p+s+1,r+p-2s-1} \xrightarrow{(\Sigma_{0,1})^{s+1}} \Gamma_{g-p+s+1,r+p-s} \xrightarrow{(\Sigma_{1,-1})^{p-s-1}} \Gamma_{g,r+1}\ .$$

The result follows by induction if

$$2(g-p+s+1) \geq 3q \quad \text{and} \quad 2(g-p+s+1) \geq 3q+2; \quad \text{for } q \leq k-2.$$

Let us prove (34). We know that $2g \geq 3k$, and we have $p+q \leq k+1$. Let q be fixed. Since more arcs (greater p) and smaller genus of α implies a smaller genus of the cut surface F_α, it suffices to show the inequality for $p+q=k+1$ and $s=0$. In this case

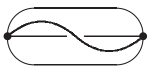
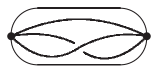
$$2(g-p+1) = 2(g-k-1+q+1) \geq 3k-2k+2q = 2q+k \geq 3q+2.$$

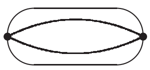
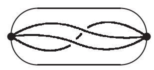
where in the last inequality we have used the assumption $q \leq k-2$. The proof of (31) is similar. Now by Lemma 3.1, $E^2_{p,q} = 0$ for all $p+q=k+1$ with $q \leq k-2$. This proves that $d^1_{1,k} = (\Sigma_{0,1})_*$ is surjective.

Surjectivity in the case $\Sigma_{1,-1}$

Assume $2g \geq 3k-1$, and write $\Gamma = \Gamma_{g+1,r-1}$. Then $\Gamma(F_{g,r}) = \Gamma_\beta$ for $\beta \in \Delta_0(F_{g+1,r-1};1)$ as on Figure 7. In the spectral sequence (22) associated with the action of Γ on $C_*(F_{g+1,r-1};1)$, we recognize the map $(\Sigma_{1,-1})_* : H_k(\Gamma_{g,r}) \longrightarrow H_k(\Gamma_{g+1,r-1})$ as the differential $d^1_{1,k} : E^1_{1,k} \longrightarrow E^1_{0,k}$. It suffices to show that $E^2_{p,q} = 0$ for $p+q = k+1$ and $q \leq k-1$.

We first show that $E^2_{2,k-1} = 0$ using Lemma 3.3. As before, the figures below show the relevant simplices in $\Delta_*(F_{g+1,r-1};1)$, and the oval is the boundary component containing b_0 and b_1.

$\Gamma_{[1\,0]} = \Gamma_{g,r-1}$,  $\Gamma_{[0\,2\,1]} = \Gamma_{g-1,r}$, 

$\Gamma_{[0\,1]} = \Gamma_{g-1,r+1}$,  $\Gamma_{[1\,2\,0]} = \Gamma_{g-1,r}$. 

We see that

$$\begin{aligned} c_1 = (\Sigma_{1,-1})_* :\ & H_{k-1}(\Gamma_{g-1,r}) \longrightarrow H_{k-1}(\Gamma_{g,r-1}), \quad \text{and} \\ c_2 = (\Sigma_{0,1})_* :\ & H_{k-1}(\Gamma_{g-1,r}) \longrightarrow H_{k-1}(\Gamma_{g-1,r+1}) \end{aligned} \tag{36}$$

are both surjective by induction. So $E^2_{2,k-1} = 0$.

Next we show that $E^2_{3,k-2} = 0$ using Lemma 3.4. To verify the conditions, we calculate as before,

$$\begin{array}{lcll} \Gamma_{[0\,1\,2]} & = & \Gamma_{g-2,r+2}, & \\ \Gamma_\sigma & = & \Gamma_{g-1,r} & \text{for } \sigma \in \Sigma_3 \text{ the remaining 3 permutations in (33)} \\ \Gamma_\sigma & = & \Gamma_{g-2,r+1} & \text{for } \sigma \in \Sigma_4 \text{ the remaining 4 permutations in (33).} \end{array}$$

We see that

$$\begin{aligned} c_3 = (\Sigma_{0,1})_* :\ & H_{k-2}(\Gamma_{g-2,r+1}) \longrightarrow H_{k-2}(\Gamma_{g-2,r+2}), \quad \text{and} \\ c_j = (\Sigma_{1,-1})_* :\ & H_{k-2}(\Gamma_{g-2,r+1}) \longrightarrow H_{k-2}(\Gamma_{g-1,r}) \qquad \text{for } j = 4,5,6. \end{aligned} \tag{37}$$

Inductively we can verify that these four maps are surjective. The maps c_1 and c_2 we calculated in (36), and we see by induction that they are injective in homology degree $k-2$. So by Lemma 3.4, $E^2_{3,k-2} = 0$.

Finally we prove that $E^2_{p,q} = 0$ for $p + q = k + 1$ and $q \leq k - 3$ using Lemma 3.1. This is done as in **The case** $\Sigma_{0,1}$ so we'll skip the calculations, and just show the final inequality:

$$\begin{aligned} 2(g-p+1) &= 2g-2(k+1-q)+2 \quad \geq \quad 3k-1-2k+2q \\ &= k+2q-1 \geq q+3+2q-1 \quad = \quad 3q+2. \end{aligned}$$

So by Lemma 3.1, $E^2_{p,q} = 0$ for $p + 1 = k + 1$ and $q \leq k - 3$. We conclude that $(\Sigma_{1,-1})_* = d^1_{1,k}$ is surjective.

Injectivity in the case $\Sigma_{1,-1}$

Assume $2g \geq 3k + 2$ and let as in the above case $\Gamma = \Gamma_{g+1,r-1}$ and $E^n_{p,q} = E^n_{p,q}(F_{g+1,r-1}; 1)$. We will show that $(\Sigma_{1,-1})_* = d^1_{1,k}$ is injective. Since $E^n_{1,k}$ converges to 0, it suffices to show that all differentials with target $E^n_{1,k}$ are trivial. This holds if we can show that $E^2_{p,q} = 0$ for all $p + q = k + 2$ with $q \leq k - 1$ and that $d^1_{2,k} : E^1_{2,k} \longrightarrow E^1_{1,k}$ is trivial.

We first prove that $d^1_{2,k} : E^1_{2,k} \longrightarrow E^1_{1,k}$ is trivial by proving that $d^1_{3,k} : E^1_{3,k} \longrightarrow E^1_{2,k}$ is surjective, using Lemma 3.3. We have already calculated c_1 and c_2, cf. (36):

$$\begin{aligned} c_1 = (\Sigma_{1,-1})_* : &\quad H_k(\Gamma_{g-1,r}) \longrightarrow H_k(\Gamma_{g,r-1}), \qquad \text{and} \\ c_2 = (\Sigma_{0,1})_* : &\quad H_k(\Gamma_{g-1,r}) \longrightarrow H_k(\Gamma_{g-1,r+1}) \end{aligned}$$

In this case we cannot use induction, since the homology degree is k, but we can use the surjectivity result for $\Sigma_{0,1}$ and $\Sigma_{1,-1}$ since we have already proved this. So by Theorem 3.6 (ii), c_1 and c_2 are surjective.

Next we prove that $E^2_{3,k-1} = 0$, using Lemma 3.4. We have already calculated c_j for $j = 1, 2, 3, 4, 5, 6$ in the proof of surjectivity of $(\Sigma_{1,-1})_*$, cf. (36) and (37), and in this case we get

$$\begin{aligned} c_1 = (\Sigma_{1,-1})_* : &\quad H_{k-1}(\Gamma_{g-1,r}) \longrightarrow H_{k-1}(\Gamma_{g,r-1}), \\ c_2 = (\Sigma_{0,1})_* : &\quad H_{k-1}(\Gamma_{g-1,r}) \longrightarrow H_{k-1}(\Gamma_{g-1,r+1}) \\ c_3 = (\Sigma_{0,1})_* : &\quad H_{k-1}(\Gamma_{g-2,r+1}) \longrightarrow H_{k-1}(\Gamma_{g-2,r+2}), \quad \text{and} \\ c_j = (\Sigma_{1,-1})_* : &\quad H_{k-1}(\Gamma_{g-2,r+1}) \longrightarrow H_{k-1}(\Gamma_{g-1,r}) \qquad \text{for } j = 4, 5, 6. \end{aligned}$$

Inductively we can verify that c_1 and c_2 are injective, and that c_j for $j = 3, 4, 5, 6$ are surjective. So by Lemma 3.4, $E^2_{3,k-1} = 0$.

Finally we prove that $E^2_{p,q} = 0$ for $p + q = k + 1$ and $q \leq k - 2$ using Lemma 3.1. As before we skip the calculations, and the final inequality is the same as in **Surjectivity in the case** $\Sigma_{1,-1}$. □

Remark 3.7. Another possibility for proving the above result is to use another arc complex. Inspired by [Ivanov1] we consider a subcomplex of $C(F;i)$ consisting of all n-simplices with a given permutation σ_n, $n \geq 0$. Ivanov takes $\sigma = \mathrm{id}$, which means the cut surfaces F_α have minimal genus. For the inductive assumption, it would be better to have maximal genus, which can be achieved by taking $\sigma_n = [n\ n{-}1\ \cdots\ 1\ 0]$. Potentially, this could give a better stability range, but it is not known how connected this subcomplex is, which means that the proof above cannot be carried through.

3.3 The stability theorem for closed surfaces

In this section we study $l = \Sigma_{0,-1} : \Gamma_{g,1} \longrightarrow \Gamma_g$, the homomorphism induced by gluing on a disk to the boundary circle. The main result is

Theorem 3.8.

$$l_* : H_k(\Gamma_{g,1}) \longrightarrow H_k(\Gamma_g)$$

is surjective for $2g \geq 3k - 1$, and an isomorphism for $2g \geq 3k + 2$.

The proof we give is modelled on [Ivanov1]. See also [Cohen-Madsen].

Definition 3.9. Let F be a surface, possibly with boundary. The arc complex $D_*(F)$ has isotopy classes of closed, non-trivial, oriented, embedded circles as vertices, and $n+1$ distinct vertices ($n \geq 0$) form an n-simplex if they have representatives $(\alpha_0, \dots \alpha_n)$ such that:

(*i*) $\alpha_i \cap \alpha_j = \emptyset$ and $\alpha_i \cap \partial(F) = \emptyset$,

(*ii*) $F \setminus (\bigcup_{i=0}^n \alpha_i)$ is connected.

We note that

$$(F_{g,r})_\alpha \cong F_{g-1,r+2}, \quad \text{for each vertex } \alpha \text{ in } D(F_{g,r}). \tag{38}$$

Indeed, for a vertex α, $F_\alpha := F \setminus N(\alpha)$ has two more boundary components than F, but the same Euler characteristic, since $F = F \setminus N(\alpha) \cup_{\partial N(\alpha)} N(\alpha)$, and $\chi(N(\alpha)) = 0 = \chi(\partial N(\alpha))$. Then (38) follows from $\chi(F_{g,r}) = 2 - 2g - r$.

We need the following connectivity result, which we state without proof:

Theorem 3.10 ([Harer1]). *The arc complex $D_*(F_{g,r})$ is $(g-2)$-connected, and $\Gamma_{g,r}$ acts transitively in each dimension.*

We can now prove the stability theorem for closed surfaces:

Proof of Theorem 3.8. We use the unaugmented spectral sequences associated with the action of $\Gamma(F_i)$ on $D_*(F_i)$, where $F_i = F_{g,i}$ for $i = 0, 1$. They converge to the homology of $\Gamma(F_i)$ in degrees less than or equal to $g-2$. Since $\Gamma(F_i)$ acts transitively on the set of n-simplices,

$$E^1_{p,q}(F_i) \cong H_q(\Gamma(F_i)_\alpha, \mathbb{Z}_\alpha) \Rightarrow H_{p+q}(\Gamma(F_i)), \quad \text{for } i = 0, 1; \tag{39}$$

where α is p-simplex in $D_p(F_1)$, by identifying α with its image in $D_p(F_0)$ under the inclusion $l : F_1 \longrightarrow F_0$.

We use Moore's comparison theorem for spectral sequences, cf. [Cartan]: If $l_* : H_q(\Gamma(F_1)_\alpha, \mathbb{Z}_\alpha) \longrightarrow H_q(\Gamma(F_0)_\alpha, \mathbb{Z}_\alpha)$ is an isomorphism for $p + q \leq m$ and surjective for $p + q \leq m + 1$, then $l_* : H_k(\Gamma(F_1)) \longrightarrow H_k(\Gamma(F_0))$ is a isomorphism for $k \leq m$ and surjective for $k \leq m + 1$. To apply this, we will compare $H_q(\Gamma(F_i)_\alpha, \mathbb{Z}_\alpha)$ and $H_q(\Gamma((F_i)_\alpha))$ for a fixed p-simplex α.

First we need to analyse $\Gamma(F_i)_\alpha$ for $i = 0, 1$, and to ease the notation we call the surface F and write $\Gamma = \Gamma(F)$. Unlike for $C_*(F; i)$, the stabilizer Γ_α is not $\Gamma(F_\alpha)$. For $\gamma \in \Gamma_\alpha$,

(*i*) γ need not stabilize α pointwise and can thus permute the circles of α;

(*ii*) γ can change the orientation of any circle in α;

(*iii*) γ can rotate each circle α in α.

In order to take care of (*i*) and (*ii*), consider the exact sequence,

$$1 \longrightarrow \widetilde{\Gamma_\alpha} \longrightarrow \Gamma_\alpha \longrightarrow (\mathbb{Z}/2)^{p+1} \ltimes \Sigma_{p+1} \longrightarrow 1. \tag{40}$$

Here $\widetilde{\Gamma_\alpha} \subseteq \Gamma_\alpha$ consists of the mapping classes in Γ_α fixing each vertex of α and its orientation. We now compare $\widetilde{\Gamma_\alpha}$ and $\Gamma(F_\alpha)$,

$$0 \longrightarrow \mathbb{Z}^{p+1} \longrightarrow \Gamma(F_\alpha) \longrightarrow \widetilde{\Gamma_\alpha} \longrightarrow 1. \tag{41}$$

We must explain the map $\mathbb{Z}^{p+1} \longrightarrow \Gamma(F_\alpha)$. Let $\alpha = (\alpha_0, \ldots, \alpha_p)$, then the cut surface F_α has two boundary components, α_i^+ and α_i^-, for each circle α_i. Then the standard generator $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{p+1}$, $j = 0, \ldots, p$, maps to the mapping class making a right Dehn twist on α_j^+ and a left Dehn twist on α_j^-, and identity everywhere else. This is extended to a group homomorphism, i.e. $-e_j$ makes a left Dehn twist on α_j^+ and a right Dehn twist on α_j^-.

Let us see that (41) is exact. The hard part is injectivity of $\mathbb{Z}^{p+1} \longrightarrow \Gamma(F_\alpha)$, so we only show this. Assume $m \neq n \in \mathbb{Z}^{p+1}$, and say $m_0 \neq n_0$. For $p \geq 1$, the surface F_α has at least four boundary components. Two of them come from cutting up along the circle α_0, call one of these S. If $p = 0$, then $\alpha = \alpha_0$, and F_α has genus $g - 1 \geq 2$ by (38), since $2g \geq 3k + 3 \geq 6$. In both cases, there is a non-trivial loop γ in F_α starting on S which does not commute with the Dehn twist f around S in $\pi_1(F_\alpha)$. Since F_α has boundary, $\pi_1(F_\alpha)$ is a free group, so the subgroup $\langle \gamma, f \rangle$ is also free. The action of $m \in \mathbb{Z}^{p+1}$ on γ is $f^{m_0}\gamma f^{-m_0}$, and since f and γ does not commute, $f^{m_0}\gamma f^{-m_0} \neq f^{n_0}\gamma f^{-n_0}$ when $n_0 \neq m_0$.

Consider $l_* : \Gamma((F_1)_\alpha) \longrightarrow \Gamma((F_0)_\alpha)$. Both surfaces $(F_i)_\alpha$ have non-empty boundary, so we can use Main Theorem 3.6. We must relate l_* to the maps $\Sigma_{0,1}$ and $\Sigma_{1,-1}$, so let $\hat{F}$ denote a surface such that $\Sigma_{0,1}(\hat{F}) = (F_1)_\alpha$. Then $\hat{F}$ has one less boundary components than $(F_1)_\alpha$, so $\hat{F}$ and $(F_0)_\alpha$ are isomorphic. This gives the diagram:

$$\begin{tikzcd}
H_*(\Gamma(\hat{F})) \arrow[rr, "\cong"] \arrow[dr, "(\Sigma_{0,1})_*"'] & & H_*(\Gamma((F_0)_\alpha)) \\
& H_*(\Gamma((F_1)_\alpha)) \arrow[ur, "l_*"'] &
\end{tikzcd}$$

We see that l_* is always surjective. By Theorem 3.6, $(\Sigma_{0,1})_* : H_s(\Gamma(\hat{F})) \longrightarrow H_s(\Gamma((F_1)_\alpha))$ is an isomorphism for $3s \leq 2(g - p - 1)$, so the same holds for l_*.

The Lynden-Serre spectral sequence of (41) for F is

$$\bar{E}^2_{s,t}(F) \cong H_s(\widetilde{\Gamma_\alpha}, H_t(\mathbb{Z}^{p+1})) \Rightarrow H_{s+t}(\Gamma(F_\alpha)). \tag{42}$$

We showed above that $l_* : H_{s+t}(\Gamma((F_1)_\alpha)) \longrightarrow H_{s+t}(\Gamma((F_0)_\alpha))$ is an isomorphism for $3(s + t) \leq 2(g - p - 1)$ and surjective always. Note that $\mathbb{Z}^{p+1}$ lies in the center of $\Gamma(F_\alpha)$, since the Dehn twists can take place as close to the boundary of F_α as desired. By the Künneth formula, we have an isomorphism

$$\bar{E}^2_{s,t}(F) \cong \bar{E}^2_{s,0}(F) \otimes \bar{E}^2_{0,t}(F) = H_s(\widetilde{\Gamma_\alpha}) \otimes H_t(\mathbb{Z}^{p+1})$$

Now since $l_* : H_{s+t}(\Gamma((F_1)_\alpha)) \longrightarrow H_{s+t}(\Gamma((F_0)_\alpha))$ is an isomorphism for $3(s+t) \leq 2(g-p-1)$ and always surjective, it follows by an easy inductive argument that $l_* : H_s(\widetilde{\Gamma(F_0)_\alpha}) \longrightarrow H_s(\widetilde{\Gamma(F_1)_\alpha})$ is an isomorphism for $3s \leq 2(g - p - 1)$ and surjective for $3s \leq 2(g - p - 1) + 3$.

The Lynden-Serre spectral sequence of (40) is

$$\tilde{E}^2_{r,s}(F) \cong H_r\left((\mathbb{Z}/2)^{p+1} \ltimes \Sigma_{p+1}; H_s(\widetilde{\Gamma_\alpha}; \mathbb{Z}_\alpha)\right) \Rightarrow H_{r+s}(\Gamma_\alpha; \mathbb{Z}_\alpha). \tag{43}$$

Since $\widetilde{\Gamma_\alpha}$ preserves the orientation of the simplices, we can drop the local coordinates to obtain

$$\tilde{E}^2_{r,s}(F) \cong H_r\left((\mathbb{Z}/2)^{p+1} \times \Sigma_{p+1}, H_s(\widetilde{\Gamma_\alpha}) \otimes \mathbb{Z}_\alpha\right).$$

It follows from the above that $l_* : \tilde{E}^2_{r,s}(F_1) \longrightarrow \tilde{E}^2_{r,s}(F_0)$ is an isomorphism for $3s \leq 2(g-p-1)$ and surjective for $3s \leq 2(g-p-1)+3$. Then by Moore's comparison theorem,

$$l_* : H_q(\Gamma(F_1)_\alpha; \mathbb{Z}_\alpha) \longrightarrow H_q(\Gamma(F_0)_\alpha; \mathbb{Z}_\alpha)$$

is an isomorphism for $3q \leq 2(g-p-1)$ and surjective for $3q \leq 2(g-p-1)+3$. Then in particular, it is an isomorphism for $3(p+q) \leq 2g-2$ and surjective for $3(p+q) \leq 2g-2+3$. Now a final application of Moore's comparison theorem on the spectral sequence in (39) gives the desired result, as explained in the beginning of the proof. $\square$

4 Stability with twisted coefficients

4.1 The category of marked surfaces

Definition 4.1. The category of marked surfaces $\mathfrak{C}$ is defined as follows: The objects are triples $F, x_0, (\partial_1 F, \partial_2 F, \dots, \partial_r F)$, where F is a compact connected orientable surface with non-empty boundary $\partial F = \partial_1 F \cup \cdots \partial_r F$, with a numbering $(\partial_1 F, \dots, \partial_r F)$ of the boundary components of F, and $x_0 \in \partial_1 F$ is a marked point.

A morphism (ψ, σ) between marked surfaces (F, x_0) and (G, y_0) is an ambient isotopy class of an embedding $\psi : F \longrightarrow G$, where each boundary component of F is either mapped to the inside of G or to a boundary component of G. If $\psi(x_0) \in \partial G$ then $\psi(x_0) = y_0$, else there is a embedded arc σ in G connecting x_0 and y_0.

The objects of $\mathfrak{C}$ is can be grouped

$$\operatorname{Ob} \mathfrak{C} = \coprod_{g,r} \operatorname{Ob} \mathfrak{C}_{g,r},$$

where $\mathfrak{C}_{g,r}$ consists of the surfaces with genus g and r boundary components.

Definition 4.2. The morphisms $\Sigma_{1,0}$, $\Sigma_{0,1}$ in $\mathfrak{C}$ are the embeddings $\Sigma_{i,j} : F \longrightarrow \Sigma_{i,j} F$ given by gluing onto $\partial_1 F$ a torus with 2 disks cut out, or a pair of pants, respectively, as on Figure 10. The embedded arc σ is also shown here. The boundary components of $\Sigma_{0,1} F$ are numbered such that the new boundary component from the pair of pants is $\partial_{r+1}(\Sigma_{0,1} F)$.

The morphism $\Sigma_{1,-1}$ in the subcategory of $\coprod_{r \geq 2} \operatorname{Ob} \mathfrak{C}_{g,r}$ is the embedding given by gluing a pair of pants onto $\partial_1(F)$ and $\partial_2(F)$, as on Figure 10. The numbering is that $\partial_j(\Sigma_{1,-1} F) = \partial_{j-1} F$ for $j > 1$.

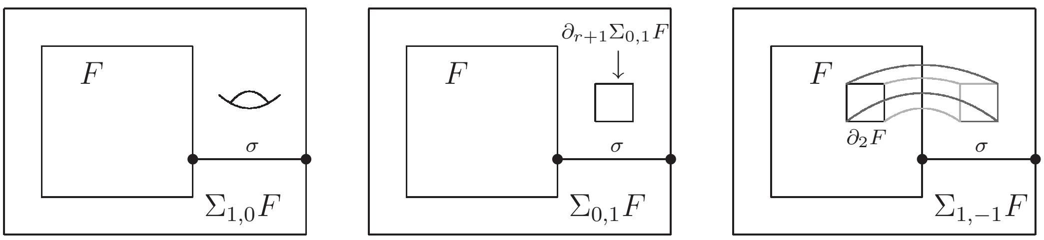


Figure 10: The morphisms $\Sigma_{1,0}$, $\Sigma_{0,1} F$, and $\Sigma_{1,-1} F$.

In the figure, the black rectangles are boundary components of F or $\Sigma_{i,j} F$, and the outer boundary component is always $\partial_1 F$ with the marked point indicated. On the figure of $\Sigma_{1,-1}F$ the grey "tube" is a cylinder glued onto $\partial_2 F$.

Now we will see how $\Sigma_{i,j}$ can be made into functors. First we define the subcategory $\mathfrak{C}(2)$ of $\mathfrak{C}$ to be the category with objects $\coprod_{r\geq 2} \operatorname{Ob} \mathfrak{C}_{g,r}$ and whose morphisms $\varphi : F \longrightarrow S$ must restrict to an orientation-preserving diffeomorphism $\varphi : \partial_2 F \longrightarrow \partial_2 S$. Note that $\Sigma_{1,0}$ and $\Sigma_{0,1}$ are morphisms in this category.

$\Sigma_{1,0}$ and $\Sigma_{0,1}$ are functors from $\mathfrak{C}$ to itself, and $\Sigma_{1,-1}$ is a functor from $\mathfrak{C}(2)$ to $\mathfrak{C}$ in the following way: Given a morphism $\varphi : F \longrightarrow S$ we must specify the morphism $\Sigma_{i,j}(\varphi)$, and this is done on the following diagram (drawn in the case of $\Sigma_{1,0}$). Here, the grey line shows how $\Sigma_{1,0}$ is embedded in $\Sigma_{0,1}S$ by $\Sigma_{1,0}(\varphi)$. Notice how the arc σ determines the embedding.

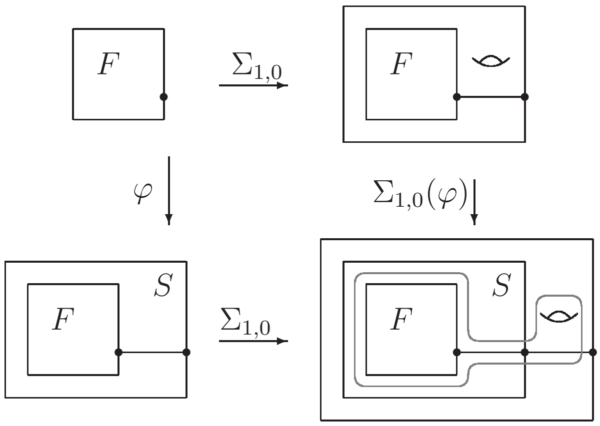


Figure 11: The functor $\Sigma_{1,0}$.

Similar diagrams can be drawn for $\Sigma_{0,1}$ and $\Sigma_{1,-1}$. In the latter case $\Sigma_{1,-1}(\varphi)$ exists because when $\varphi \in \mathfrak{C}(2)$, $\varphi : F \longrightarrow S$ has not done anything to $\partial_2(F)$, so that $\Sigma_{1,-1}F$ can be embedded in $\Sigma_{1,-1}S$ just as on Figure 11.

4.2 Coefficient systems

We now define the coefficient systems we are interested in. We say that an abelian group G is *without infinite division* if the following holds for all $g \in G$: If $n \mid g$ for all $n \in \mathbb{Z}$, then $g = 0$. By $n \mid g$ we mean $g = nh$ for some $h \in G$. Note that finitely generated abelian groups are without infinite division.

Definition 4.3. A coefficient system is a functor from $\mathfrak{C}$ to $\mathrm{Ab}_{\mathrm{wid}}$, the category of abelian groups without infinite division.

We say that a constant coefficient system has degree 0 and make the general

Definition 4.4. [Ivanov1] A coefficient system V has degree $\leq k$ if the map $V(F) \longrightarrow V(\Sigma_{i,j}F)$ is split injective for $(i,j) \in \{(1,0),(0,1),(1,-1)\}$, and the cokernel $\Delta_{i,j}V$ is a coefficient system of degree $\leq k-1$ for $(i,j) \in \{(1,0),(0,1)\}$. The degree of V is the smallest such k.

Example 4.5. (i) $V(F) = H_1(F, \partial F)$ is a coefficient system of degree 1.

(ii) $V_k^*(F) = H_k(\mathrm{Map}((F/\partial F), X)$. This is the coefficient system used in [Cohen-Madsen]. It has degree $\leq \lfloor \frac{k}{d} \rfloor$ if X is d-connected, which will be proved in Theorem 5.3.

We write $\Sigma_{i,j}V$ for the functor $F \rightsquigarrow V(\Sigma_{i,j}F)$, where $(i,j) \in \{(1,0),(0,1)\}$.

Lemma 4.6 (Ivanov)**.** *Let V be a coefficient system of degree $\leq k$. Then $\Sigma_{1,0}V$ and $\Sigma_{0,1}V$ are coefficient systems of degree $\leq k$.*

Proof. See [Ivanov1] for $\Sigma_{1,0}V$. The case $\Sigma_{0,1}V$ can be handled similarly. ☐

4.3 The inductive assumption

Below I will use the following notational conventions: F denotes a surface in $\mathfrak{C}$, and unless otherwise specified, g is the genus of F. $\Sigma_{l,m}$ refers to any of $\Sigma_{1,0}$, $\Sigma_{0,1}$, $\Sigma_{1,-1}$.

Definition 4.7. Given a morphism $\psi : F \longrightarrow S$, Φ will denote a finite composition of $\Sigma_{0,1}$ and $\Sigma_{1,-1}$ such that $\Phi(\psi)$ is defined, i.e. makes the following diagram comutative

$$\begin{tikzcd} F \arrow[r, "\Phi"] \arrow[d, "\psi"] & \Phi(F) \arrow[d, dashed, "\Phi(\psi)"] \\ S \arrow[r, "\Phi"] & \Phi(S) \end{tikzcd}$$

By a finite composition we mean $\Phi = \Sigma_{i_1,j_1} \circ \cdots \circ \Sigma_{i_s,j_s}$ for some $s \geq 0$, where $(i_k, j_k) \in \{(0,1),(1,-1)\}$ for each $k = 1,\ldots,s$. We say that such a Φ is *compatible* with $\psi : F \longrightarrow S$.

To prove our main stability result for twisted coefficients, we will study certain relative homology groups:

Definition 4.8. Let $\psi : F \longrightarrow S$ be a morphism of surfaces, and let Φ be compatible. Let V be a coefficient system. Then we define

$$\mathrm{Rel}_n^{V,\Phi}(S,F) = H_n(\Gamma(S),\Gamma(F);V(\Phi(S)),V(\Phi(F))).$$

If $\Phi = \mathrm{id}$, we write $\mathrm{Rel}_n^V(G,F)$ for $\mathrm{Rel}_n^{V,\mathrm{id}}(G,F)$.

Theorem 4.9 (Ivanov, Madsen-Cohen)**.** *For sufficiently large g:*

(i) $\mathrm{Rel}_q^V(\Sigma_{1,0}F,F) = 0$.

(ii) $\mathrm{Rel}_q^V(\Sigma_{0,1}F,F) = 0$.

(iii) $\mathrm{Rel}_q^V(\Sigma_{1,-1}F,F) = 0$.

Proof. For (i), see [Ivanov1]. For (ii), see [Cohen-Madsen]. Their proof only requires that the groups $V(\cdot)$ are without infinite division.

To prove (iii), we use the following long exact sequence,

$$\begin{aligned} H_q(F,V(F)) &\longrightarrow H_q(\Sigma_{1,-1}F,V(\Sigma_{1,-1}F)) \longrightarrow \mathrm{Rel}_q^V(\Sigma_{1,-1}F,F) \longrightarrow \\ H_{q-1}(F,V(F)) &\longrightarrow H_{q-1}(\Sigma_{1,-1}F,V(\Sigma_{1,-1}F)) \end{aligned}$$

Thus to see that $\mathrm{Rel}_q^V(\Sigma_{1,-1}F,F) = 0$ all we have to do is to see that the first map is surjective and that the last map is injective. Both of these maps are $\Sigma_{1,-1}$, so they fit into the following diagram, for $k \in \{q,q-1\}$:

$$\begin{tikzcd} H_k(F,V(F)) \arrow[r, "\Sigma_{1,-1}"] & H_k(F,V(F)) \\ H_k(S,V) \arrow[u, "\Sigma_{0,1}"] \arrow[ur, "\Sigma_{1,0}"'] & \end{tikzcd}$$

where S is a surface with $\Sigma_{0,1}S = F$. Now by (i) and (ii), if g is sufficiently large, both the diagonal and the vertical map is an isomorphism, so $\Sigma_{1,-1}$ is also an isomorphism. □

Define $\varepsilon_{l,m}$ by

$$\varepsilon_{l,m} = \begin{cases} 1, & \text{if } (l,m) = (1,-1); \\ 0, & \text{if } (l,m) = (1,0) \text{ or } (0,1). \end{cases}$$

Inductive Assumption 4.10. *The inductive assumption $I_{k,n}$ is the following: For any coefficient system W of degree k_W, any surface F of genus g, and any Φ compatible with $\Sigma_{l,m} : F \longrightarrow \Sigma_{l,m}F$, we have*

$$\mathrm{Rel}_q^{W,\Phi}(\Sigma_{l,m}F,F) = 0 \quad \textit{for} \quad 2g \geq 3q + k_W - \varepsilon_{l,m},$$

if either $k_W < k$, or $k_W = k$ and $q < n$.

In the rest of this section I am going to assume $I_{k,n}$. Note that $I_{k,m}$ for all $m \in \mathbb{N}$ is equivalent to $I_{k+1,0}$. Thus the goal is to prove $I_{k,n+1}$. Let V be a given coefficient system of degree k.

Lemma 4.11 (Ivanov)**.** *Let F be a surface of genus g. If $2g \geq 3q+k-1-\varepsilon_{l,m}$ then for $(i,j) \in \{(1,0),(0,1)\}$*

$$\mathrm{Rel}_q^{V,\Phi}(\Sigma_{l,m}F,F) \longrightarrow \mathrm{Rel}_q^{V,\Sigma_{i,j}\Phi}(\Sigma_{l,m}F,F)$$

is surjective.

Proof. Since $\mathrm{Rel}_q^{V,\Sigma_{i,j}\Phi}(\Sigma_{l,m}F,F) = \mathrm{Rel}_q^{\Sigma_{i,j}V,\Phi}(\Sigma_{l,m}F,F)$ we have the following long exact sequence :

$$\mathrm{Rel}_q^{V,\Phi}(\Sigma_{l,m}F,F) \longrightarrow \mathrm{Rel}_q^{V,\Sigma_{i,j}\Phi}(\Sigma_{l,m}F,F) \longrightarrow \mathrm{Rel}_q^{\Delta_{i,j}V,\Phi}(\Sigma_{l,m}F,F)$$

Since $\Delta_{i,j}V$ is a coefficient system of degree $k-1$, the assumption $I_{k,n}$ implies that $\mathrm{Rel}_q^{\Delta_{i,j}V,\Phi}(\Sigma_{l,m}F,F) = 0$, and the result follows. $\square$

Theorem 4.12. *Assume that h satisfies $2h \geq 3n+k-1-\varepsilon_{l,m}$ and that the maps below are injective for all surfaces F of genus $g \geq h$ and Φ compatible with $\Sigma_{l,m} : F \longrightarrow \Sigma_{l,m}F$,*

$$\begin{aligned} \mathrm{Rel}_n^{V,\Phi\Sigma_{1,-1}}(\Sigma_{l,m}F,F) &\longrightarrow \mathrm{Rel}_n^{V,\Phi}(\Sigma_{l,m}\Sigma_{1,-1}F,\Sigma_{1,-1}F), \\ \mathrm{Rel}_n^{\Sigma_{0,1}V}(\Sigma_{l,m}F,F) &\longrightarrow \mathrm{Rel}_n^{V}(\Sigma_{l,m}\Sigma_{0,1}F,\Sigma_{0,1}F). \end{aligned}$$

Then for any compatible Φ, $\mathrm{Rel}_n^{V,\Phi}(\Sigma_{l,m}F,F) = 0$ for $g \geq h$.

Proof. Assume $2g \geq 3n+k-1-\varepsilon_{l,m}$. Write $\Phi = \Sigma_{i_1,j_1} \circ \cdots \circ \Sigma_{i_s,j_s}$, where $(i_k,j_k) \in \{(1,-1),(0,1)\}$. Observe that we can write $\Phi = \Phi' \circ (\Sigma_{1,-1})^d$ for some d, where $\Phi' = \Sigma_{\lambda_1,\mu_1} \circ \cdots \circ \Sigma_{\lambda_t,\mu_t}$ with $(\lambda_k,\mu_k) \in \{(1,0),(0,1)\}$. Then by the first assumption in the theorem, we get by induction in d:

$$\mathrm{Rel}_n^{V,\Phi}(\Sigma_{l,m}F,F) \longrightarrow \mathrm{Rel}_n^{V,\Phi'}(\Sigma_{l,m}(\Sigma_{1,-1})^dF,(\Sigma_{1,-1})^dF)$$

is injective. Thus it suffices to show $\mathrm{Rel}_n^{V,\Phi'}(\Sigma_{l,m}(\Sigma_{1,-1})^dF,(\Sigma_{1,-1})^dF) = 0$. Since $\mathrm{genus}((\Sigma_{1,-1})^dF) \geq g \geq h$, it is certainly enough to show $\mathrm{Rel}_n^{V,\Phi'}(\Sigma_{l,m}F,F) = 0$, where Φ' is a finite composition of $\Sigma_{1,0}$ and $\Sigma_{0,1}$. By Lemma 4.11, we get inductively that

$$\mathrm{Rel}_n^{V}(\Sigma_{l,m}F,F) \longrightarrow \mathrm{Rel}_n^{V,\Phi'}(\Sigma_{l,m}F,F)$$

is surjective, so it suffices to show that $\mathrm{Rel}_n^{V}(\Sigma_{l,m}F,F) = 0$. Now by the second assumption in the Theorem, we know

$$\mathrm{Rel}_n^{\Sigma_{0,1}V}(\Sigma_{l,m}F,F) \longrightarrow \mathrm{Rel}_n^{V}(\Sigma_{l,m}\Sigma_{0,1}F,\Sigma_{0,1}F)$$

is injective. Since V is a coefficient system of degree k, $V(F) \longrightarrow V(\Sigma_{0,1}F)$ and $V(F) \longrightarrow V(\Sigma_{1,-1}F)$ are split injective, so the composition,

$$\begin{aligned}\mathrm{Rel}_n^V(\Sigma_{l,m}F, F) &\longrightarrow \mathrm{Rel}_n^{\Sigma_{0,1}V}(\Sigma_{l,m}F, F) \longrightarrow \mathrm{Rel}_n^V(\Sigma_{l,m}\Sigma_{0,1}F, \Sigma_{0,1}F)\\ &\longrightarrow \mathrm{Rel}_n^{\Sigma_{1,-1}V}(\Sigma_{l,m}\Sigma_{0,1}F, \Sigma_{0,1}F) \longrightarrow \mathrm{Rel}_n^V(\Sigma_{l,m}\Sigma_{1,0}F, \Sigma_{1,0}F)\end{aligned}$$

is injective, where the second and the last maps are the maps in the assumption and thus injective. Iterating this, we get an injective map

$$\mathrm{Rel}_n^V(\Sigma_{l,m}F, F) \longrightarrow \mathrm{Rel}_n^V(\Sigma_{l,m}(\Sigma_{1,0})^d F, (\Sigma_{1,0})^d F)$$

for any $d \in \mathbb{N}$. But $\mathrm{genus}((\Sigma_{1,0})^d F) = g+d$, so by Theorem 4.9, $\mathrm{Rel}_n^V(\Sigma_{l,m}F, F)$ injects into zero. This proves $\mathrm{Rel}_n^{V,\Phi}(\Sigma_{l,m}F, F) = 0$. $\square$

4.4 The main theorem for twisted coefficients

In the proof of stability for relative homology groups, we will use the relative version of the spectral sequence, cf. Theorem 1.2, $E^1_{p,q} = E^1_{p,q}(\Sigma_{i,j}F; 2-i)$ associated with the action of $\Gamma(\Sigma_{i,j}F)$ on the arc complex $C_*(\Sigma_{i,j}F; 2-i)$ and the action of $\Gamma(\Sigma_{l,m}\Sigma_{i,j}F)$ on the arc complex $C_*(\Sigma_{l,m}\Sigma_{i,j}F; 2-i)$. Let b_0, b_1 be the points in the definition of $C_*(\Sigma_{i,j}F; 2-i)$; and $\tilde{b}_0, \tilde{b}_1$ be the corresponding points for $C_*(\Sigma_{l,m}\Sigma_{i,j}F; 2-i)$. We demand that b_0, $\tilde{b}_0$ lie in the 1st boundary component, but is different from the marked point. To define the spectral sequence, $\Sigma_{l,m}$ must induce a map

$$\Sigma_{l,m} : C_*(\Sigma_{i,j}F; 2-i) \longrightarrow C_*(\Sigma_{l,m}\Sigma_{i,j}F; 2-i), \tag{44}$$

which we now define: If $i = 0$, b_0 and b_1 lie in different boundary components, and the map is given on $\alpha \in \Delta_k(\Sigma_{i,j}F)$ by a simple path γ from $\tilde{b}_0 \in \Sigma_{l,m}\Sigma_{i,j}F$ to $b_0 \in \Sigma_{i,j}F$ inside $\Sigma_{l,m}\Sigma_{i,j}F \setminus \Sigma_{i,j}F$. Then the arcs of α are extended by parallel copies of γ that all start in $\tilde{b}_0$. Note that in this case $\tilde{b}_1 = b_1$, so no extension is necessary here. If $i = 1$, b_0 and b_1 lie on the same boundary component, and we choose disjoint paths for them to the new marked boundary component, and extend as for $i = 0$.

Now the spectral sequence (typically) has E^1 page:

$$\begin{aligned}E^1_{p,q} &= \bigoplus_{\sigma \in \overline{\Sigma}_p} E^1_{p,q}(\sigma)\\ E^1_{p,q}(\sigma) &= H_q(\Gamma(\Sigma_{i,j}\Sigma_{l,m}F)_{\Sigma_{l,m}T(\sigma)}, \Gamma(\Sigma_{i,j}F)_{T(\sigma)};\\ &\qquad V(\Phi\Sigma_{i,j}\Sigma_{l,m}\Sigma_{s,t}(F)), V(\Phi\Sigma_{i,j}\Sigma_{s,t}(F)))\\ &= \mathrm{Rel}_q^{V,\Phi_\sigma}((\Sigma_{i,j}\Sigma_{l,m}F)_{\Sigma_{l,m}T(\sigma)}, (\Sigma_{i,j}F)_{T(\sigma)})\end{aligned} \tag{45}$$

Here, $\Phi_\sigma : (\Sigma_{i,j}F)_{T(\sigma)} \hookrightarrow \Sigma_{i,j}F$ is the inclusion, which is a finite composition of $\Sigma_{0,1}$ and $\Sigma_{1,-1}$. Furthermore, Γ_σ denotes the stabilizer of the $(p-1)$-simplex σ in Γ. The direct sum is over the orbits of $(p-1)$-simplices σ in $C_*(\Sigma_{i,j}F; 2-i)$, whose images under $\Sigma_{l,m}$ are also $(p-1)$-simplices in $C_*(\Sigma_{l,m}\Sigma_{i,j}F; 2-i)$. In most cases, $\Sigma_{l,m}$ induces a bijection on the representatives of orbits of $(p-1)$-simplices. Also recall that the set of orbits are in $1-1$ correspondence with a subset $\overline{\Sigma}_p$ of the permutation group Σ_p. Lemma 2.16 characterizes $\overline{\Sigma}_p$. As a general remark, note that if a permutation is represented in $C_*(F; 2-i)$, then it is also represented in $C_*(\Sigma_{l,m}F; 2-i)$, since $\operatorname{genus}(\Sigma_{l,m}F) \geq \operatorname{genus}(F)$. So we will only check the condition for $C_*(F, 2-i)$.

In certain cases we will either not have $\Sigma_{l,m}$ inducing bijection on the representatives of orbits of $(p-1)$-simplices, or they will not include the permutation used in the standard proof. All such cases will be found in Lemma 4.13 below and taken care of in the *Inductive start* section at the end of the proof.

The first differential, $d^1_{p,q} : E^1_{p,q} \longrightarrow E^1_{p-1,q}$, is described in section 1.3. The diagrams

$$\begin{tikzcd} \Delta_p(F;i) \arrow[r, "\partial_j"] \arrow[d] & \Delta_p(F;i) \arrow[d] \\ \overline{\Sigma}_{p+1} \arrow[r, "\partial_j"] & \overline{\Sigma}_p \end{tikzcd} \qquad j = 0, \dots, p$$

commute, where ∂_j omits entry j as in Def. 2.2 and the vertical arrows divide out the Γ action and compose with P. Thus for each $\sigma \in \overline{\Sigma}_{p+1}$, there is $g_j \in \Gamma$ such that

$$g_j \cdot \partial_j T(\sigma) = T(\partial_j \sigma), \tag{46}$$

and conjugation by g_j induces an injection $c_{g_j} : \Gamma_{T(\sigma)} \hookrightarrow \Gamma_{T(\partial_j\sigma)}$. The induced map on homology is denoted ∂_j again, i.e.

$$\begin{aligned} \partial_j : H_q(\Gamma(\Sigma_{i,j}\Sigma_{l,m}F)_{\Sigma_{l,m}T(\sigma)}, \Gamma(\Sigma_{i,j}F)_{T(\sigma)}; \mathbf{V}) \hookrightarrow \\ H_q(\Gamma(\Sigma_{i,j}\Sigma_{l,m}F)_{\Sigma_{l,m}\partial_j T(\sigma)}, \Gamma(\Sigma_{i,j}F)_{\partial_j T(\sigma)}; \mathbf{V}) \xrightarrow{(c_{g_j})_*} \\ H_q(\Gamma(\Sigma_{i,j}\Sigma_{l,m}F)_{\Sigma_{l,m}T\partial_j(\sigma)}, \Gamma(\Sigma_{i,j}F)_{T\partial_j(\sigma)}; \mathbf{V}) \end{aligned} \tag{47}$$

Note that $(c_{g_j})_*$ does not depend on the choice of g_j in (46): Another choice g'_j gives $c_{g'_j} = c_{g'_j g_j^{-1}} c_{g_j}$, and $g'_j g_j^{-1} \in \Gamma_{T(\partial_j\sigma)}$ so $c_{g'_j g_j^{-1}}$ induces the identity on the homology. Then

$$d^1 = \sum_{j=0}^{p-1} (-1)^j \partial_j. \tag{48}$$

Lemma 4.13. *Let $n \geq 1$. The subset $\overline{\Sigma}_p \subseteq \Sigma_p$, which is in $1-1$ correspondence with a set of representatives of the orbits of $\Delta_{p-1}(\Sigma_{i,j}F; 2-i)$, has the following properties:*

Surjectivity of $\Sigma_{0,1}$: *Assume $2g \geq 3n + k - 2 - \varepsilon_{l,m}$. Then $\overline{\Sigma}_p = \Sigma_p$ for $2 \leq p \leq n+1$ and for $p = n+2 = 3$, unless:*

- $(l,m) \neq (1,-1)$, $\quad n = 1$, $\quad g = 1$, $\quad k = 0, 1$, $\quad$ *or*
- $(l,m) = (1,-1)$, $\quad n = 1$, $\quad g = 0$, $\quad k = 0$, $\quad$ *or*
- $(l,m) = (1,-1)$, $\quad n = 1$, $\quad g = 1$, $\quad k = 0, 1, 2$.

Surjectivity of $\Sigma_{1,-1}$: *Assume $2g \geq 3n + k - 3 - \varepsilon_{l,m}$. Then $\overline{\Sigma}_p = \Sigma_p$ for $2 \leq p \leq n+1$, and $\sigma \in \overline{\Sigma}_p$ if $S(\sigma) \geq 1$ for $p = n+2 \leq 4$, unless:*

- $(l,m) \neq (1,-1)$, $\quad n = 1$, $\quad g = 0$, $\quad k = 0$, $\quad$ *or*
- $(l,m) = (1,-1)$, $\quad n = 1$, $\quad g = 0$, $\quad k = 0, 1$, $\quad$ *or*
- $(l,m) = (1,-1)$, $\quad n = 2$, $\quad g = 1$, $\quad k = 0$.

Injectivity of $\Sigma_{1,-1}$: *Assume $2g \geq 3n + k - \varepsilon_{l,m}$. Then $\overline{\Sigma}_p = \Sigma_p$ for $2 \leq p \leq n+2$, and $\sigma \in \overline{\Sigma}_p$ if $S(\sigma) \geq 1$ for $p = n+3 = 4$, unless:*

- $(l,m) = (1,-1)$, $\quad n = 1$, $\quad g = 1$, $\quad k = 0$.

Proof. We only prove the first of the three cases, as the other two are completely analogous. So assume $2g \geq 3n + k - 2 - \varepsilon_{l,m}$, and let $\sigma \in \Sigma_p$ be a given permutation of genus s. Let $2 \leq p \leq n+1$. By Lemma 2.16, $\sigma \in \overline{\Sigma}_p$ if and only if $s \geq p-1-g$. This inequality is certainly satisfied if $p-1-g \leq 0$. The hardest case is $p = n+1$, so we must show $n - g \leq 0$. By assumption,

$$2(n-g) \leq 2n - (3n + k - 2 + \varepsilon_{l,m}) = -n - k + 2 + \varepsilon_{l,m} \overset{?}{\leq} 0,$$

For $n \geq 3$ this holds. If $n = 2$, the assumption $2g \geq 3n + k - 2 - \varepsilon_{l,m}$ forces $g \geq 2$, so $n - g \leq 0$. For $n = 1$ and $(l,m) \neq (1,-1)$, we have $\varepsilon_{l,m} = 0$, so $g \geq 1$, which means $n - g \leq 0$. Last for $n = 1$ and $(l,m) = (1,-1)$, we have $\varepsilon_{l,m} = 1$, so we get one exception, $g = k = 0$.

Now let $p = n+2 = 3$, so $n = 1$. The requirement in Lemma 2.16 is $p - 1 - g \leq 0$, i.e. $g \geq 2$. By assumption $2g \geq 3n + k - 2 - \varepsilon_{l,m}$, so if $g = 1$, we have $k - \varepsilon_{l,m} - 1 \leq 0$. Now for $(l,m) \neq (1,-1)$, the only exceptions are $k = 0, 1$, and for $(l,m) = (1,-1)$, the only exceptions are $k = 0, 1, 2$. If $g = 0$, we have $k - \varepsilon_{l,m} + 1 \leq 0$, so the only exception is $(l,m) = (1,-1)$ and $k = 0$. This finishes the proof. $\square$

Proposition 4.14. *Let α denote a simplex either in $\Delta_1(F;1)$ with $P(\alpha) = [1\,0]$, or in $\Delta_2(F;2)$ with $P(\alpha) = [2\,1\,0]$. Let g be the genus of F_α, and let Φ be compatible with $\Sigma_{l,m} : F \longrightarrow \Sigma_{l,m}F$. Then if $2g \geq 3q + k_W - 1 - \varepsilon_{l,m}$, the maps $\partial_0 = \partial_1$ are equal as maps from*

$$Rel_n^{V,\Phi_\alpha}((\Sigma_{l,m}F)_{\Sigma_{l,m}\alpha}, F_\alpha).$$

Proof. Write $\sigma = P(\alpha)$. First note that ∂_0 and ∂_1 have the same target, since $\partial_0(\sigma) = \partial_1(\sigma) =: \tau$ by assumption. We can assume $T(\sigma) = \alpha$ and $T(\tau) = \partial_0\alpha$. Then we can choose the element $g = g_1$ from (46), which must satisfy $g \cdot \partial_1\alpha = \partial_0\alpha$, to be as in Prop. 3.2. Then g commutes with the stabilizers $\Gamma(\Sigma_{l,m}F)_{\alpha_0\cup\alpha_1}$, $\Gamma(F)_{\alpha_0\cup\alpha_1}$ and thus also with $\Gamma(\Sigma_{l,m}F)_\alpha$ and $\Gamma(F)_\alpha$.

We now extend the arcs of α to arcs in ΦF as follows: If $\alpha \in \Delta_1(F;1)$ we use (44) to obtain $\tilde{\alpha} = \Phi(\alpha) \in \Delta_1(\Phi F;1)$. If $\alpha \in \Delta_2(F;2)$, we extend, if possible, the 1-simplex $\alpha_0 \cup \alpha_1$ to a 1-simplex $\tilde{\alpha} \in \Delta_1(\Phi F;1)$, i.e. the extended arcs start and end on the same boundary component in ΦF. If this is not possible, we extend α to $\tilde{\alpha} \in \Delta_2(\Phi F;2)$. These extensions must satisfy the same requirements as (44) does. Then we make the same extensions for $\beta := \Sigma_{l,m}\alpha$ to $\tilde{\beta}$ in $\Phi\Sigma_{l,m}F$. Now the conjugation $(c_g)_*$ acts as the identity on

$$H_n(\Gamma(\Sigma_{l,m}F)_\beta, \Gamma(F)_\alpha; V((\Phi\Sigma_{l,m}F)_{\tilde{\beta}}), V((\Phi F)_{\tilde{\alpha}}))$$

If we are in the case $\tilde{\alpha}\Delta_1(\Phi F;1)$, then the inclusion map on the coefficients,

$$\begin{aligned} i_* \;:\; & H_n(\Gamma(\Sigma_{l,m}F)_\beta, \Gamma(F)_\alpha; V((\Phi\Sigma_{l,m}F)_{\tilde{\beta}}), V((\Phi F)_{\tilde{\alpha}})) \longrightarrow \\ & H_n(\Gamma(\Sigma_{l,m}F)_\beta, \Gamma(F)_\alpha; V(\Phi\Sigma_{l,m}F), V(\Phi F)) = \mathrm{Rel}_n^{V,\Phi_\alpha}((\Sigma_{l,m}F)_{\Sigma_{l,m}\alpha}, F_\alpha) \end{aligned} \tag{49}$$

equals $\Sigma_{1,0}$ on the coefficient systems, and by Lemma 4.11 it is surjective since $2g \geq 3n + k - 1 - \varepsilon_{l,m}$ by assumption. Now as i_* is surjective and $(c_g)_* \circ i_* = i_*$ we see that $(c_g)_*$ is the identity on $\mathrm{Rel}_n^{V,\Phi_\alpha}(\Sigma_{l,m}F_\alpha, F_\alpha)$, and thus $\partial_1 = (c_g)_*\partial_0 = \partial_0$. For $\tilde{\alpha} \in \Delta_2(\Phi F;2)$ we do the same, except that we use α instead of only $\alpha_0 \cup \alpha_1$. In this case i_* in (49) is going to be $\Sigma_{1,0}\Sigma_{0,1}$ on the coefficient systems, which again by Lemma 4.11 is surjective. □

By Theorem 4.12, to prove $I_{k,n+1}$ it is enough to prove:

Theorem 4.15. *The map induced by $\Sigma_{i,j}$,*

$$\mathrm{Rel}_n^{V,\Phi\Sigma_{i,j}}(\Sigma_{l,m}F, F) \longrightarrow \mathrm{Rel}_n^{V,\Phi}(\Sigma_{i,j}\Sigma_{l,m}F, \Sigma_{i,j}F)$$

satisfies:

(*i*) *For* $\Sigma_{i,j} = \Sigma_{0,1}$, *it is surjective for* $2g \geq 3n + k - 2 - \varepsilon_{l,m}$, *and if* $\Phi = \mathrm{id}$ *it is an isomorphism for* $2g \geq 3n + k - 1 - \varepsilon_{l,m}$. *For* $k = 0$ *it is always injective.*

(*ii*) *For* $\Sigma_{i,j} = \Sigma_{1,-1}$, *it is surjective for* $2g \geq 3n + k - 3 - \varepsilon_{l,m}$, *and an isomorphism for* $2g \geq 3n + k - \varepsilon_{l,m}$.

Proof. We prove the theorem by induction in the homology degree n. Assume $n \geq 1$. The induction start $n = 0$ will be handled separately below, along with all exceptional cases from Lemma 4.13. This means that in the main proof, any permutation is represented by an arc simplex (in some special cases only if its genus is ≥ 1).

Surjectivity for $\Sigma_{0,1}$:

Assume $2g \geq 3n + k - 2 - \varepsilon_{l,m}$. We use the spectral sequence $E^1_{p,q} = E^1_{p,q}(\Sigma_{0,1}F; 2)$, and claim that $E^1_{p,q} = 0$ for $p + q = n + 1$ with $p \geq 3$. Note that $\Gamma(\Sigma_{0,1}F)_\sigma = \Gamma(\Sigma_{0,1}F_\sigma)$, and $\mathrm{genus}(\Sigma_{0,1}F_\sigma) = g-p+1+S(\sigma) \geq g-p+1$. We will use the assumption $I_{k,n}$, and must show $2(g-p+1) \geq 3q + k - \varepsilon_{l,m}$ for $p \geq 3$. These inequalities follows from the one for $p = 3$, which is $2(g-2) \geq 3(n-2) + k - \varepsilon_{l,m}$, and this holds by assumption.

Now all we need is to show that $E^2_{2,n-1} = 0$. We consider

$$E^1_{2,n-1} = E^1_{2,n-1}([0\,1]) \oplus E^1_{2,n-1}([1\,0])$$

We wish to show that $d_1 : E^1_{3,n-1} \longrightarrow E^1_{2,n-1}$ is surjective and thus $E^1_{2,n-1} = 0$. We look at $E^1_{3,n-1}(\tau)$ indexed by the permutation $\tau = [2\,1\,0]$. We will show that d^1 restricted to $E^1_{3,n-1}(\tau)$ surjects onto $E^1_{2,n-1}([1\,0])$ without hitting $E^1_{2,n-1}([0\,1])$. Since $S(\tau) = 1$, $\Sigma_{0,1}F_\tau$ is $F_{g-1,r}$, and thus by Proposition 4.14, $\partial_0 = \partial_1$. We then see

$$d_1 = \partial_0 - \partial_1 + \partial_2 = \partial_2$$

and $\partial_2 : E^1_{3,n-1}(\tau) \longrightarrow E^1_{2,n-1}[1\,0]$ equals $\Sigma_{0,1}$ and so is surjective by induction, since $2(g-1) \geq 3(n-1) + k - 2 - \varepsilon_{m,l}$. All that remains is to hit $E^1_{2,n-1}([0\,1])$ surjectively, regardless of $E^1_{2,n-1}([1\,0])$. Consider the following component of d^1:

$$\partial_0 : E^1_{3,n-1}([2\,0\,1]) \longrightarrow E^1_{2,n-1}([0\,1]).$$

This is the map induced by $\Sigma_{1,-1}$. By induction this map is surjective, since $2(g-2) \geq 3(n-1)+k-3-\varepsilon_{l,m}$ by assumption. This proves that $E^2_{2,n-1} = 0$.

Injectivity for $\Sigma_{0,1}$:

Assume $2g \geq 3n + k - 1 - \varepsilon_{l,m}$. For this proof we take another approach. Consider the following composite map,

$$\begin{aligned}
\mathrm{Rel}_q^V(\Sigma_{l,m}F, F) &\longrightarrow \mathrm{Rel}_q^{\Sigma_{0,1}V}(\Sigma_{l,m}F, F) \\
&\xrightarrow{\Sigma_{0,1}} \mathrm{Rel}_q^V(\Sigma_{l,m}\Sigma_{0,1}F, \Sigma_{0,1}F) \\
&\xrightarrow{p_*} \mathrm{Rel}_q^V(\Sigma_{0,-1}\Sigma_{l,m}\Sigma_{0,1}F, \Sigma_{0,-1}\Sigma_{0,1}F) \\
&= \mathrm{Rel}_q^V(\Sigma_{l,m}F, F) \qquad (50)
\end{aligned}$$

Here $p : F_{g,r} \longrightarrow F_{g,r-1}$ is the map that glues a disk onto a the unmarked boundary circle created by $\Sigma_{0,1}$. Since the composite map (50) is induced by gluing on a cylinder to the marked boundary circle of $\Sigma_{l,m}F$ and F, it is an isomorphism. Now by Lemma 4.11, since $2g \geq 3n+k-1-\varepsilon_{l,m}$, the first map is surjective, so $\Sigma_{0,1}$ is forced to be injective. Note with constant coefficients ($k = 0$), the first map is the identity, so here $\Sigma_{0,1}$ is always injective.

Surjectivity for $\Sigma_{1,-1}$:

Assume $2g \geq 3n + k - 3 - \varepsilon_{l,m}$. We use the spectral sequence $E^1_{p,q} = E^1_{p,q}(\Sigma_{1,-1}F; 1)$. We show $E^1_{p,q} = 0$ if $p + q = n + 1$ and $p \geq 4$, using assumption $I_{k,n}$. We know $\Gamma(\Sigma_{1,-1}F)_\sigma = \Gamma((\Sigma_{1,-1}F)_\sigma)$, and $\mathrm{genus}((\Sigma_{1,-1}F)_\sigma) = g-p+1+S(\sigma) \geq g-p+1$. So we must show $2(g-p+1) \geq 3q+k-\varepsilon_{l,m}$ for all $p + q = n + 1$, $p \geq 4$. This follows if we show it for $p = 4$, which is easy:

$$2(g-3) = 2g-6 \geq 3n+k-3-\varepsilon_{m,l}-6 = 3(n-3)+k-\varepsilon_{m,l}.$$

To show that the map $d_1 : E^1_{1,n} \longrightarrow E^1_{1,n}$ is surjective, we thus only need to show that $E^2_{2,n-1} = 0$ and $E^2_{3,n-2} = 0$. Consider $E^1_{2,n-1}$:

$$E^1_{2,n-1} = E^1_{2,n-1}([0\,1]) \oplus E^1_{2,n-1}([1\,0]).$$

For $\sigma = [1\,0]$, since $S(\sigma) = 1$, we have $\mathrm{genus}((\Sigma_{1,-1}F)_\sigma) = g-p+1+S(\sigma) = g$. Thus by $I_{k,n}$, $E^1_{2,n-1}([1\,0]) = 0$, since $2g \geq 3n+k-1-\varepsilon_{m,l} = 3(n-1)+k+2-\varepsilon_{l,m}$. Now consider the summand in $E^1_{3,n-1}$ indexed by $\tau = [2\,0\,1]$ which has genus 1. Then $(\Sigma_{1,-1}F)_\tau = F_{g-1,r}$, so d_1 on this summand is exactly the map induced by $\Sigma_{0,1}$ (since d_1 has 3 terms, only one of which hit $E^1_{2,n-1}([0\,1])$). To show this is surjective onto $E^1_{2,n-1}$, we use induction, and must check that $2(g-1) \geq 3(n-1)+k-\varepsilon_{l,m}$, which follows by assumption. So d^1 is surjective onto $E^1_{2,n-1}$, which implies that $E^2_{2,n-1} = 0$.

Consider $E^1_{3,n-2}$. As above, by $I_{k,n}$, all summands are zero, except for the one indexed by $\mathrm{id} = [0\,1\,2]$. Consider $E^1_{4,n-2}(\tau')$ indexed by $\tau' = [3\,0\,1\,2]$, which has genus 1. Restricting d^1 to this summand, only one term hits $E^1_{3,n-2}([0\,1\,2])$. As above, one checks that this restriction of d^1 is exactly the map induced by $\Sigma_{0,1}$, so by induction it is surjective.

Injectivity for $\Sigma_{1,-1}$:

Assume $2g \geq 3n + k + 2 - \varepsilon_{l,m}$. We use the same spectral sequence as in the surjectivity of $\Sigma_{1,-1}$. We claim $E^1_{p,q} = 0$ if $p + q = n + 2$ and $p \geq 4$. Again, $\Gamma(\Sigma_{1,-1}F)_\sigma = \Gamma(\Sigma_{1,-1}F_\sigma)$, and $\text{genus}(\Sigma_{1,-1}F_\sigma) = g-p+1+S(\sigma) \geq g-p+1$. So we must show $2(g-p+1) \geq 3q+k+2-\varepsilon_{m,l}$ for all $p+q = n+2$, $p \geq 4$, and this follows from $2g \geq 3n + k + 2 - \varepsilon_{m,l}$, as above.

To show that the map $d_1 : E^1_{1,n} \longrightarrow E^1_{0,n}$ is injective, we thus only need to show that $E^2_{3,n-1} = 0$ and $d^1 : E^1_{2,n} \longrightarrow E^1_{1,n}$ is the zero-map. That $E^2_{3,n-1} = 0$ is proved precisely as for $E^2_{3,n-2}$ in surjectivity for $\Sigma_{1,-1}$, so we omit it. To show $d^1 : E^1_{2,n} \longrightarrow E^1_{1,n}$ is the zero-map, note that $E^1_{2,n}$ has two summands, $E^1_{2,n}([0\,1])$ and $E^1_{2,n}([1\,0])$. We get that d^1 is zero on $E^1_{2,n}([1\,0])$, since $d_1 = \partial_0 - \partial_1 = 0$ by Proposition 4.14. Next we consider $d^1 : E^1_{3,n} \longrightarrow E^1_{2,n}$. If we can show this is surjective onto $E^1_{2,n}([0\,1])$, we are done. Again we use the summand $E^1_{3,n}(\tau)$, where $\tau = [2\,0\,1]$. The restricted differential $d^1 : E^1_{3,n}(\tau) \longrightarrow E^1_{2,n}([0\,1])$ is exactly the map induced by $\Sigma_{0,1}$, so we can show it is surjective, since we have already proved the Theorem for $\Sigma_{0,1}$. The relevant inequality is $2(g-1) \geq 3n+k-\varepsilon_{l,m}$, which holds by assumption. So $d^1 : E^1_{2,n} \longrightarrow E^1_{1,n}$ is the zero-map, and we have shown that $d_1 : E^1_{1,n} \longrightarrow E^1_{1,n}$ is injective.

Induction start and special cases:

Here we handle the the inductive start $n = 0$, along with the cases missing in the general argument above, namely the exceptions from Lemma 4.13.

The induction start $n = 0$. For $n = 0$ and $k = 0$, we always get $\text{Rel}_0^{V,\Phi}(\Sigma_{l,m}F, F) = 0$ since $H_0(F, V(F)) \longrightarrow H_0(\Sigma_{l,m}F, V(\Sigma_{l,m}F))$ is an isomorphism when the coefficients are constant. So the theorem holds in this case. Now let $n = 0$ and let k be arbitrary. By considering the spectral sequence, see Figure 12, we see that $\Sigma_{i,j}$ is automatically surjective, since the spectral sequence always converges to zero at $(0, 0)$.

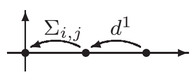


Figure 12: The spectral sequence for $n = 0$.

For the sake of the case $n = 1$, note that the surjectivity argument for $\Sigma_{0,1}$ when $n = 0$ also works for any k when using the spectral sequence for *absolute* homology for the action of $\Gamma(F_{0,r+1})$ on $C_*(F_{0,r+1}; 2)$.

For $\Sigma_{0,1}$, the injectivity argument used above holds for all n. So we must show that $\Sigma_{1,-1}$ is injective. For $g \geq 1$, the argument from above works, since there are arc simplices representing all the permutations used above. The problem is thus $g = 0$, which means $k = 0, 1$, but we will also show the result for $k = 2$ since we will need in the case $n = 1$ below.

As the complex we use, $C_*(F_{1,r-1}; 1)$, is connected, the spectral sequence converges to 0 for $p + q \leq 1$, so we can apply that spectral sequence. We must show that $d^1 = d^1_{2,0}$ in Figure 12 is the zero map. We consider $(l, m) \in \{(1, 0), (1, -1)\}$ and $(l, m) = (0, 1)$ separately. For $\Sigma_{0,1}$, $E^1_{2,0} = E^1_{2,0}([1\,0])$, since the permutation $[0\,1]$ has genus 0 and is by Lemma 2.16 neither represented in $C_*(F_{1,r-1}; 1)$ nor $C_*(\Sigma_{0,1}F_{1,r-1}; 1)$. Now the argument used to show injectivity of $\Sigma_{1,-1}$ in general works here, too.

For $\Sigma_{1,0}$ or $\Sigma_{1,-1}$, $E^1_{2,0} = E^1_{2,0}([1\,0]) \oplus \tilde{E}^1_{2,0}([0\,1])$ where $\tilde{E}^1_{2,0}([0\,1])$ is the *absolute* homology group,

$$\tilde{E}^1_{2,0}([0\,1]) = H_0(\Gamma(\Sigma_{l,m}F_{1,r-1})_{T([0\,1])}; V(\Sigma_{l,m}F_{1,r-1})),$$

since $[0\,1]$ is represented in $C_*(\Sigma_{1,-1}F_{1,r-1}; 1)$ and $C_*(\Sigma_{1,0}F_{1,r-1}; 1)$, but not in $C_*(F_{1,r-1}; 1)$, see Theorem 1.3. For $E^1_{2,0}([1\,0])$, the general argument for injectivity of $\Sigma_{1,-1}$ shows that $d^1_{2,0}([1\,0])$ is zero. That $d^1 : \tilde{E}^1_{2,0}([0\,1])$ is the zero map will follow if we show that $\tilde{E}^1_{3,0}$ hits $\tilde{E}^1_{2,0}([0\,1])$ surjectively. But the d^1-component $\tilde{E}^1_{3,0}([2\,0\,1]) \longrightarrow \tilde{E}^1_{2,0}([0\,1])$ is just $\Sigma_{0,1}$ in the absolute case for $n = 0$, $g = 0$ and $k \leq 2$. This d^1-component is surjective onto $\tilde{E}^1_{2,0}([0\,1])$, by the remark on surjectivity for $n = 0$.

Surjectivity when $n = 1$. Now let $n = 1$ and $k \leq 2$. Consider the relative spectral sequence, as depicted in Figure 13. If we show that the map $d^2_{2,0} : E^2_{2,0} \longrightarrow E^2_{0,1}$ is zero, we have shown surjectivity. We will show that $E^1_{2,0} = 0$. Recall by Theorem 1.3, $E^1_{2,0} = E^1_{2,0}([0\,1]) \oplus E^1_{2,0}([1\,0])$, where

$$E^1_{2,0}(\sigma) = \begin{cases} \mathrm{Rel}_0^{V,\Phi_\sigma}(\Gamma(F_{g+i+l,r+j+m})_{\Sigma_{m,l}\sigma}, \Gamma(F_{g+i,r+j})_\sigma), & \text{if } \sigma \in \overline{\Sigma}_1^{l,m} \cap \overline{\Sigma}_1; \\ H_0(\Gamma(F_{g+i+l,r+j+m})_{\Sigma_{m,l}\sigma}; V(\Gamma(F_{g+i+l,r+j+m}))), & \text{if } \sigma \in \overline{\Sigma}_1^{l,m} \setminus \overline{\Sigma}_1; \\ 0, & \text{if } \sigma \notin \overline{\Sigma}_1^{l,m}. \end{cases} \tag{51}$$

and $\overline{\Sigma}_1$, $\overline{\Sigma}_1^{l,m}$ are the subsets of Σ_1 in $1-1$ correspondence with the orbits of $\Delta_1(\Sigma_{i,j}F; 2-i)$ and $\Delta_1(\Sigma_{l,m}\Sigma_{i,j}F; 2-i)$, respectively.

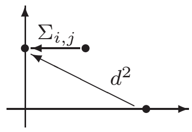


Figure 13: The spectral sequence for $n = 1$.

Surjectivity of $\Sigma_{1,-1}$ when $n = 1$. Assume $(l, m) = (0, 1)$, $g = 0$ and $k = 0$. Then by Lemma 2.16 only $[1\,0]$ is represented as an arc simplex, and by (51) above, $E^1_{2,0}$ is a relative homology group of degree 0 with constant coefficients, so $E^1_{2,0} = 0$.

The remaining exceptions are $(l, m) \neq (0, 1)$, $g = 0$ and $k \leq 1$. By Lemma 2.16, $[1\,0]$ is represented as an arc simplex in both $F_{1+l,r+m}$ and $F_{1,r-1}$, so $E^1_{2,0}([1\,0]) = 0$ by Theorem 4.12. Now $[0\,1]$ is only represented in $F_{1+l,r+m}$, so by (51), $E^1_{2,0}([1\,0])$ is an absolute homology group. To kill it, consider $E^1_{3,0}([2\,0\,1])$,. which is also an absolute homology group. The restricted differential and $d^1 : E^1_{3,0}([2\,0\,1]) \longrightarrow E^1_{2,0}([0\,1])$ equals $\Sigma_{0,1}$, so it is surjective by the case $n = 0$, which as remarked also holds for absolute homology group.

Surjectivity of $\Sigma_{0,1}$ when $n = 1$. First assume $g = 1$. The possible permutations $[0\,1]$ and $[1\,0]$ are by Lemma 2.16 represented as 1-simplices in both arc complexes. Thus $E^1_{2,0}$ is a direct sum of two relative homology groups in degree 0 with coefficients of degree $k \leq 2$. Then by the *Induction start* $n = 0$, $\Sigma_{0,1}$ and $\Sigma_{1,-1}$ are injective for $g \geq 0$, so by Theorem 4.12, $E^1_{2,0} = 0$.

For $(m, l) = (1, -1)$, we have the special case $g = k = 0$. We will show $H_1(\Gamma_{1,r}, \Gamma_{0,r+1}) = 0$, by showing $\Sigma_{1,-1} : H_1(\Gamma_{0,r+1}; \mathbb{Z}) \longrightarrow H_1(\Gamma_{1,r}; \mathbb{Z})$ is surjective, and thus that any map into $H_1(\Gamma_{1,r}, \Gamma_{0,r+1})$ is surjective. We use [Harer3], Lemma 1.1 and 1.2, which give sets of generators for $H_1(\Gamma_{0,r+1}; \mathbb{Z})$ and $H_1(\Gamma_{1,r}; \mathbb{Z})$, as follows. Let τ_i be the Dehn twist around each boundary component $\partial_i F_{1,r}$, for $i = 1, \ldots, r$, and let x be the Dehn twist on any non-separating simple closed curve γ in $F_{1,r}$. Then $H_1(\Gamma_{1,r}; \mathbb{Z})$ is generated by $\tau_2, \ldots, \tau_r, x$. We remark that Harer states this for $\mathbb{Q}$-coefficients, but in H_1 his proof also holds for $\mathbb{Z}$-coefficients. We can choose the curve γ as the image of $\partial_2 F_{0,r+1}$ under $\Sigma_{1,-1}$. Similarly in $\Gamma_{0,r+1}$, we have Dehn twists τ'_i around each boundary component $\partial_i F_{0,r+1}$, and these are among the generators for $H_1(\Gamma_{0,r+1}; \mathbb{Z})$. Then $\Sigma_{1,-1}$ maps $\tau'_{i+1} \mapsto \tau_i$ for $i = 2, \ldots, r$ by construction of $\Sigma_{1,-1}$, and $\tau'_2 \mapsto x$ by the choice of γ. So $\Sigma_{1,-1} : H_1(\Gamma_{0,r+1}; \mathbb{Z}) \longrightarrow H_1(\Gamma_{1,r}; \mathbb{Z})$ is surjective.

Injectivity of $\Sigma_{1,-1}$ when $n = 1$. The only exception is $(l, m) = (1, -1)$, $g = 1$ and $k = 0$. For this we will use a different argument, drawing on the stability Theorem for $\mathbb{Z}$-coefficients. Consider the following exact sequence:

$$\begin{aligned} & H_1(\Gamma_{1,r}; V) \twoheadrightarrow H_1(\Gamma_{2,r-1}; V) \longrightarrow \mathrm{Rel}_1^V(\Gamma_{2,r-1}, \Gamma_{1,r}) \\ \longrightarrow\ & H_0(\Gamma_{1,r}; V) \xrightarrow{\cong} H_0(\Gamma_{2,r-1}; V) \end{aligned} \tag{52}$$

Since $k = 0$ we have constant coefficients, so we can use Theorem 3.6. Since $2 \cdot 1 \geq 3 \cdot 1 - 1$, the first map in (52) is surjective, and the last map is an isomorphism. Thus $\mathrm{Rel}_1^V(\Gamma_{2,r-1}, \Gamma_{1,r}) = 0$ and any map from it is thus injective. This finishes the special cases when $n = 1$.

Surjectivity of $\Sigma_{1,-1}$ when $n = 2$. Again we have only one exception, namely $(l, m) = (1, -1)$, $g = 1$ and $k = 0$. It suffices to show $E^2_{2,1} = 0$ and $E^2_{3,0} = 0$. For $E^2_{2,1}$ the argument in *Surjectivity of* $\Sigma_{1,-1}$ works since all the permutations used there are in $\overline{\Sigma}_2$. So consider $E^2_{3,0}$. Here for all permutations τ except $[0\,1\,2]$ we have $\tau \in \overline{\Sigma}_3 \cap \Sigma_3^{l,m}$ (for this notation, see (51). Thus for these τ we know that $E^1_{3,0}(\tau) = 0$, since it is a relative homology group in degree 0 with constant coefficients. But $[0\,1\,2] \in \overline{\Sigma}_3^{1,-1} \backslash \overline{\Sigma}_3$, so $E^1_{3,0}([0\,1\,2])$ is an absolute homology group. However, this group is hit surjectively by $E^1_{4,0}[3\,0\,1\,2]$, since the restricted differential equals $\Sigma_{0,1}$ (see the remark for $n = 0$). Thus $E^2_{3,0} = 0$, as desired. $\square$

Remark 4.16. As a Corollary to this result, we can be a bit more specific about what happens when stability with $\mathbb{Z}$-coefficients fails, cf. Theorem 3.6. More precisely,

(*i*) The cokernels of the maps

$$\begin{aligned} \Sigma_{0,1} &: H_{2n+1}(\Gamma_{3n+1,r}) \longrightarrow H_k(\Gamma_{3n+1,r+1}) \\ \Sigma_{0,1} &: H_{2n+2}(\Gamma_{3n+2,r}) \longrightarrow H_k(\Gamma_{3n+2,r+1}) \end{aligned}$$

are independent of $r \geq 1$.

(*ii*) Let $r \geq 2$. Then the cokernel of the map

$$\Sigma_{1,-1} : H_{2n+1}(\Gamma_{3n,r}) \longrightarrow H_k(\Gamma_{3n+1,r-1})$$

is independent of r.

Proof. Since $\Sigma_{0,1}$ is always injective, it fits into the following long exact sequence,

$$H_{2n+1}(\Gamma_{3n+1,r}) \longrightarrow H_{2n+1}(\Gamma_{3n+1,r+1}) \longrightarrow \mathrm{Rel}^{\mathbb{Z}}_{2n+1}(F_{3n+1,r+1}, F_{3n+1,r}) \longrightarrow 0.$$

Since $2(3n+2) \geq 3(2n+2)-2$, we get by Theorem 4.15 that the cokernel is independent of r. The other case is similar. For (ii) we get

$$\begin{tikzcd} H_q(\Gamma_{3n,r}) \arrow[r, "\Sigma_{1,-1}"] \arrow[d] & H_q(\Gamma_{3n+1,r-1}) \arrow[r] \arrow[d] & \mathrm{Rel}_q^{\mathbb{Z}}(F_{3n+1,r-1}, F_{3n,r}) \arrow[r] \arrow[d, "\cong"] & H_{q-1}(\Gamma_{3n,r}) \arrow[d, "\cong"] \\ H_q(\Gamma_{3n,r+1}) \arrow[r, "\Sigma_{1,-1}"] & H_q(\Gamma_{3n+1,r}) \arrow[r] & \mathrm{Rel}_q^{\mathbb{Z}}(F_{3n+1,r}, F_{3n,r+1}) \arrow[r] & H_{q-1}(\Gamma_{3n,r+1}) \end{tikzcd}$$

(We have written $q = 2n+1$ to save space.) As the last two vertical maps are isomorphisms, the cokernels of the first map in the top and bottom rows are equal. □

The above Theorem finishes the inductive proof of the assumption $I_{n,k}$. The reason for proving the inductive assumption is that we now get the following Main Theorem for homology stability with twisted coefficients:

Theorem 4.17. *Let F be a surface of genus g, and let V be a coefficient system of degree k. Let $(l,m) = (1,0)$, $(0,1)$ or $(1,-1)$. Then the map*

$$H_n(F; V(F)) \longrightarrow H_n(\Sigma_{l,m}F; V(\Sigma_{l,m}F))$$

induced by $\Sigma_{l,m}$ satisfies:

- (i) *For $\Sigma_{l,m} = \Sigma_{0,1}$, it is an isomorphism for $2g \geq 3n+k$.*
- (ii) *For $\Sigma_{l,m} = \Sigma_{1,0}$ or $\Sigma_{1,-1}$, it is surjective for $2g \geq 3n+k-\varepsilon_{l,m}$, and an isomorphism for $2g \geq 3n+k+2$.*

Proof. Consider the following exact sequence

$$\mathrm{Rel}_{n+1}^V(\Sigma_{l,m}F, F) \longrightarrow H_n(F; V) \longrightarrow H_n(\Sigma_{l,m}F; \Sigma_{l,m}V) \longrightarrow \mathrm{Rel}_n^V(\Sigma_{l,m}F, F).$$

To show surjectivity, we must prove that $\mathrm{Rel}_n^V(\Sigma_{l,m}F, F) = 0$. By $I_{k,n+1}$ this is the case when $2g \geq 3n+k$. To show injectivity, we first note that as usual, $\Sigma_{0,1}$ is always injective. For $\Sigma_{1,-1}$, we get by $I_{k,n+2}$ that $\mathrm{Rel}_{n+1}^V(\Sigma_{l,m}F, F) = 0$ when $2g \geq 3(n+1)+k+2$. Finally, $\Sigma_{1,0} = \Sigma_{1,-1}\Sigma_{0,1}$ and thus also injective when $2g \geq 3(n+1)+k+2$. □

5 Stability of the space of surfaces

In [Cohen-Madsen], Cohen and Madsen consider the following type of coefficients

$$V_n^X(F) := H_n(\mathrm{Map}(F/\partial F, X))$$

for X a fixed topological space.

Lemma 5.1. *Let $K = K(G;k)$ be an Eilenberg-MacLane space with $k \geq 2$. Assume $H_*(K)$ is without infinite division. Then V_n^K is a coefficient system of degree $\leq \lfloor \frac{n}{k-1} \rfloor$.*

Proof. To prove V_n^K is a coefficient system of degree $\leq \lfloor \frac{n}{k-1} \rfloor$, we must prove that the groups $V_n^K(F)$ are without infinite division, and that V_n^K has the right degree.

We consider the degree first, and the proof is by induction on n. Take $\Sigma = \Sigma_{1,0}$, the other cases are similar. We have the following homotopy cofibration:

$$S^1 \wedge S^1 \longrightarrow \Sigma F/\partial \Sigma F \longrightarrow F/\partial F$$

Taking $\mathrm{Map}(-, K)$ leads to the following fibration:

$$\mathrm{Map}(F/\partial F, K) \longrightarrow \mathrm{Map}(\Sigma F/\partial \Sigma F, K) \longrightarrow \Omega(K) \times \Omega(K) \tag{53}$$

Since $K = K(G, k)$ is an infinite loop space it has a multiplication, and consequently so has each space in the fibration (64) above. Thus the total space is up to homotopy the product of the base and the fiber. Using Künneth's formula, we get:

$$V_n^K(\Sigma F) = \bigoplus_{i=0}^{n} V_{n-i}^K(F) \otimes H_i(\Omega(K) \times \Omega(K)) \tag{54}$$

Note for $n = 0$ this says that Σ induces an isomorphism, so $V_0^K(F)$ has degree 0. This was the induction start.

Now since $\Omega(K) = K(G, k-1)$ is $(k-2)$-connected and $k \geq 2$, $H_0(\Omega(K) \times \Omega(K)) = \mathbb{Z}$ and $H_j(\Omega(K) \times \Omega(K)) = 0$ for $j \leq k-2$. This means that the cokernel of Σ is:

$$\Delta(V_n^K(F)) = \bigoplus_{i=k-1}^{n} V_{n-i}^K(F) \otimes H_i(\Omega(K) \times \Omega(K))$$

Since the degree of a direct sum is the maximum of the degrees of its components, we get by induction that the degree of $\Delta(V_n^K(F))$ is $\leq \lfloor \frac{n-(k-1)}{k-1} \rfloor = \lfloor \frac{n}{k-1} \rfloor - 1$. This shows that the degree of V_n^K is $\leq \lfloor \frac{n}{k-1} \rfloor$.

It remains to show that $V_n^K(F)$ is an abelian group without infinite division for any surface F. To prove this, we use a double induction in n and F. There are two base cases.

First consider $n=0$, F any surface. From (54) we see that V_0^K does not depend on the surface F. So we can calculate $V_0^K(F)$ using $F=D$ a disk:

$$V_0^K(F) = H_0(\mathrm{Map}(D/\partial D, K)) = \mathbb{Z}[\pi_2(K)] = \begin{cases} \mathbb{Z}, & k>2; \\ \mathbb{Z}[G], & k=2. \end{cases}$$

This is an abelian group without infinite division.

Secondly, let $F=D$ be a disk, and n any natural number. We see

$$\begin{aligned} V_n^K(D) &= H_n(\mathrm{Map}(D/\partial D, K)) = H_n(\mathrm{Map}(S^2, K)) \\ &= H_n(\mathrm{Map}(S^0, \Omega^2(K)) = H_n(\Omega^2(K)) \end{aligned}$$

and according to our assumptions on $H_*(K)$, this is without infinite division.

The general case now follows from induction using (54) and its counterpart for $\Sigma = \Sigma_{0,1}$, along with the fact that any surface F with boundary can be obtained from a disk D using $\Sigma_{1,0}$ and $\Sigma_{0,1}$ finitely many times. □

To prove the next theorem we need a couple of lemmas:

Lemma 5.2. *Let V and W be coefficient systems of degrees $\leq s$ and $\leq t$, respectively. Then $V \otimes W$ is a coefficient system of degree $\leq s+t$, and $V \oplus W$ is a coefficient system of degree $\leq \max(s,t)$.*

Proof. Since V is a coefficient system, we have the split exact sequence:

$$0 \longrightarrow V(F) \longrightarrow V(\Sigma F) \longrightarrow \Delta(V(F)) \longrightarrow 0.$$

Likewise for W. Then for the tensor product we get the split exact sequence:

$$\begin{aligned} 0 &\longrightarrow V(F)\otimes W(F) \longrightarrow V(\Sigma F)\otimes W(\Sigma F) \\ &\longrightarrow \Delta(V(F))\otimes W(F) \oplus V(F)\otimes \Delta(W(F)) \longrightarrow 0. \end{aligned}$$

□

Theorem 5.3. *Let X be a k-connected space, $k \geq 1$. If $V_n^X(F)$ is without infinite division for any surface F, then V_n^X is a coefficient system of degree $\leq \lfloor \frac{n}{k} \rfloor$.*

Proof. First note: If we prove the assertion concerning the degree as in Def. 4.4 (not including without infinite division), then since V_n^X is assumed without infinite division, the cokernels $\Delta_{i,j}(V_n^X)$ (and their cokernels, etc) are automatically without infinite division, since they are direct summands of V_n^X.

The proof uses Postnikov towers and Lemma 5.1 above. The Postnikov tower of X is a sequence $\{X_m \longrightarrow X_{m-1}\}_{m\geq k}$ with each term a fibration

$$K(\pi_m(X), m) \longrightarrow X_m \longrightarrow X_{m-1}. \tag{55}$$

The proof is by induction in m, so assume for $l < m$ that $V_n^{X_l}$ is a coefficient system of degree $\leq \lfloor \frac{n}{k} \rfloor$. To make the induction work, we also assume inductively that the splitting s_l we then have by definition,

$$0 \longrightarrow V_n^{X_l} \longrightarrow \Sigma V_n^{X_l} \overset{s_l}{\underset{}{\rightleftarrows}} \Delta(V_n^{X_l}) \longrightarrow 0$$

is a natural transformation from $\Delta(V_n^{X_l})$ to $\Sigma V_n^{X_l}$.

Now we take the induction step. Let F be a surface. Then using $\mathrm{Map}(F, -)$ on (55) yields a new fibration

$$\mathrm{Map}(F, K(\pi_m(X), m)) \longrightarrow \mathrm{Map}(F, X_m) \longrightarrow \mathrm{Map}(F, X_{m-1}).$$

Serre's spectral sequence for this fibration has E^2-term:

$$\begin{aligned} E^2_{s,t}(F) &= H_s(\mathrm{Map}(F, X_{m-1})) \otimes H_t(\mathrm{Map}(F, K(\pi_m(X), m)) \\ &= V_s^{X_{m-1}}(F) \otimes V_t^{K(\pi_m(X),m)}(F). \end{aligned} \tag{56}$$

Now X_{m-1} is k-connected, since X is, and $K(\pi_m(X), m)$ is at least k-connected. Then by induction and Lemma 5.2, $E^2_{s,t}$ is a coefficient system of degree $\leq \lfloor \frac{s}{k} \rfloor + \lfloor \frac{t}{k} \rfloor \leq \lfloor \frac{s+t}{k} \rfloor$.

We now want to prove that $E^r_{s,t}$ is a coefficient system of degree $\leq \lfloor \frac{s+t}{k} \rfloor$ for all $r \geq 2$, by induction in r. Let $V_1 \xrightarrow{d} V \xrightarrow{d} V_2$ be groups in the E^r term of the spectral sequence, where d denotes the rth differential, and say V has degree $\leq q$. We assume by induction in r that the splittings for V, V_1 and V_2 (see (57)) are natural transformations. For $r = 2$ this holds according to (56) by induction in m and by (54) (the Eilenberg-MacLane space case). We want to show that the homology of V with respect to d, $H(V)$, is a coefficient system of degree $\leq q$, and that the splitting for $H(V)$ is also natural. Suppose by another induction that this holds for coefficient systems of degrees $< q$.

Then consider the following diagram, where Σ as usual denotes either

$\Sigma_{1,0}$ or $\Sigma_{0,1}$.

$$\begin{tikzcd}
0 \arrow[r] & V_1 \arrow[r, "\Sigma"] \arrow[d, "d"] & \Sigma V_1 \arrow[r] \arrow[d, "d"] & \Delta_1 \arrow[r] \arrow[d, "d"] \arrow[l, bend right] & 0 \\
0 \arrow[r] & V \arrow[r, "\Sigma"] \arrow[d, "d"] & \Sigma V \arrow[r] \arrow[d, "d"] & \Delta \arrow[r] \arrow[d, "d"] \arrow[l, bend right] & 0 \\
0 \arrow[r] & V_2 \arrow[r, "\Sigma"] & \Sigma V_2 \arrow[r] & \Delta_2 \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd} \tag{57}$$

We know $\Sigma V = V \oplus \Delta$, and similarly for V_1 and V_2. By our induction hypothesis in r we get that the splittings in the right-most squares above commute with d. Then the homology with respect to d satisfies $H(\Sigma V) = H(V) \oplus H(\Delta)$, and the splitting for $H(V)$ is again natural. This shows that the cokernel $\Delta(H(V))$ of Σ is $H(\Delta)$. Since Δ is a coefficient system of degree $\leq q-1$, we get by induction in the degree that $H(V)$ is a coefficient system of degree $\leq q$. For the degree-induction start, if V is constant, $H(V)$ is also constant.

To finish the induction in m we must prove that the splitting $s_m : \Delta(V_n^{X_m}) \longrightarrow \Sigma V_n^{X_m}$ is a natural transformation. By the above, $E^r_{s,t}$ is a coefficient system of degree $\leq \lfloor \frac{s+t}{k} \rfloor$ for all r, so the same is true for $E^\infty_{s,t}$. Since the spectral sequence converges to $V_n^{X_m}(F)$ for $n = s+t$, we get that $V_n^{X_m}(F)$ is a coefficient system of degree $\leq \lfloor \frac{n}{k} \rfloor$.

The inverse limit of the Postnikov tower $\lim_{\leftarrow} X_m$ is weakly homotopy equivalent to X, and the result follows. □

The space of surfaces mapping into a background space X with boundary conditions γ is defined as follows: Let X be a space with base point $x_0 \in X$, and let $\gamma : \coprod S^1 \longrightarrow X$ be r loops in X. Then

$$\begin{aligned}
\mathcal{S}_{g,r}(X,\gamma) \quad = \quad & \big\{ (F_{g,r}, \varphi, f) \mid F_{g,r} \subseteq \mathbb{R}^\infty \times [a,b], \varphi : \sqcup S^1 \longrightarrow \partial F_{g,r} \text{ is a para-} \\
& \text{metrization}, f : F_{g,r} \longrightarrow X \text{ is continuous with } f \circ \varphi = \gamma \big\}
\end{aligned}$$

Assume now X is simply-connected. Then we observe that the homotopy type of $\mathcal{S}_{g,r}(X,\gamma)$ does not depend on γ: For consider the space of surfaces with no boundary conditions, call it $\overline{\mathcal{S}_{g,r}(X)}$. The restriction map to the boundary of the surfaces,

$$\mathcal{S}_{g,r}(X,\gamma) \longrightarrow \overline{\mathcal{S}_{g,r}(X)} \longrightarrow (LX)^r$$

is a Serre fibration. Here, $LX = \mathrm{Map}(S^1, X)$ is the free loop space, so as X is simply-connected, $(LX)^r$ is connected, so the fiber is independent of the choice of $\gamma \in (LX)^r$. So when X is simply-connected, we use the abbreviated notation $\mathcal{S}_{g,r}(X) = \mathcal{S}_{g,r}(X,\gamma)$ for any choice of γ.

Theorem 5.4. *Let X be a simply-connected space such that V_m^X is without infinite division for all $m \leq n$. Then*

$$H_n(\mathcal{S}_{g,r}(X))$$

is independent of g and r for $2g \geq 3n+3$ and $r \geq 1$.

Proof. Let Σ be either $\Sigma_{1,0}$ or $\Sigma_{0,1}$. From the definition we observe that

$$\mathcal{S}_{g,r}(X) \cong \operatorname{Emb}(F_{g,r}, \mathbb{R}^\infty) \times_{\operatorname{Diff}(F_{g,r},\partial)} \operatorname{Map}(F_{g,r}, X),$$

and since $\operatorname{Emb}(F_{g,r}, \mathbb{R}^\infty)$ is contractible, we get

$$\mathcal{S}_{g,r}(X) \cong E(\operatorname{Diff}(F_{g,r}, \partial)) \times_{\operatorname{Diff}(F_{g,r},\partial)} \operatorname{Map}(F_{g,r}, X).$$

So there is an obvious fibration sequence

$$\operatorname{Map}(F_{g,r}, X) \longrightarrow \mathcal{S}_{g,r}(X) \longrightarrow B(\operatorname{Diff}(F_{g,r}, \partial),$$

and thus we can apply Serre's spectral sequence, which has E^2 term:

$$E^2_{s,t} = H_s(B(\operatorname{Diff}(F_{g,r}, \partial); H_t(\operatorname{Map}(F_{g,r}, X)))$$

where the coefficients are local. The path components of $\operatorname{Diff}(F_{g,r}, \partial)$ are contractible, so we get an isomorphism

$$E^2_{s,t} \cong H_s(\Gamma(F_{g,r}); H_t(\operatorname{Map}(F_{g,r}, X))) \tag{58}$$

Consider the map induced by Σ on this spectral sequence

$$\Sigma_* : H_s(\Gamma(F_{g,r}); H_t(\operatorname{Map}(F_{g,r}, X))) \longrightarrow H_s(\Gamma(\Sigma F_{g,r}); H_t(\operatorname{Map}(\Sigma F_{g,r}, X)))$$

By Theorem 5.3 and 4.17, we know that this map is surjective for $2g \geq 3s+t$, and an isomorphism for $2g \geq 3s+t+2$. We use Zeeman's comparison theorem to carry the result to E^∞. To get the optimum stability range, we must find the maximal $N = N(g) \in \mathbb{Z}$ such that for $t \geq 1$,

$$\begin{aligned} s+t \leq N \quad &\Rightarrow \quad 2g \geq 3s+t+2 \quad \text{(isomorphism)} \\ s+t = N+1 \quad &\Rightarrow \quad 2g \geq 3s+t \quad \text{(surjectivity)} \end{aligned}$$

Zeeman's comparison theorem then says that Σ_* induces isomorphism on $E^\infty_{s,t}$ for $s+t \leq N(g)$ and a surjection for $s+t = N(g)+1$. Since the spectral sequence converges to $H_n(\mathcal{S}_{g,r}(X))$, we get stability for $n \leq N(g)$.

Clearly, the hardest requirement is $t = 0$ (surjectivity), where we get the inequality $2g \geq 3N+3$. One checks that this satisfies all the other cases. So $H_n(\mathcal{S}_{g,r}(X))$ is independent of g, r for $2g \geq 3n+3$. □

Using this we can improve the stability range in Cohen-Madsen's stability result for the homology of the space of surfaces to the following, cf [Cohen-Madsen] Theorem 0.1:

Theorem 5.5. *Let X be a simply connected space such that V_m^X is without infinite division for all m. Then for $2g \geq 3n+3$ and $r \geq 1$ we get an isomorphism*

$$H_n(\mathcal{S}_{g,r}(X)_\bullet) \cong H_n(\Omega^\infty(\mathbb{CP}^\infty_{-1} \wedge X_+)_\bullet).$$

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Part II
Different versions of mapping class groups of surfaces

6 Different versions of mapping class groups of surfaces

6.1 Introduction

Let F be a compact connected smooth surface, possibly with boundary and not necessarily oriented. The objects of study in this paper are

$$\begin{aligned}
\mathrm{Diff}(F,\{\partial F\}) &= \{\varphi : (F,\partial F) \longrightarrow (F,\partial F) \mid \varphi \text{ is a diffeomorphism}\}\,,\\
\mathrm{Top}(F,\{\partial F\}) &= \{\varphi : (F,\partial F) \longrightarrow (F,\partial F) \mid \varphi \text{ is a homeomorphism}\}\,,\\
\mathrm{hAut}(F,\{\partial F\}) &= \{\varphi : (F,\partial F) \longrightarrow (F,\partial F) \mid \varphi \text{ is a homotopy equivalence}\}\,.
\end{aligned}$$

The main theorem of this paper is the following:

Theorem 6.1. *Let F be a compact surface and not a sphere, a disk, a cylinder, a Möbius band, a torus, a Klein bottle, or $\mathbb{R}P^2$. Then*

$$\pi_0(\mathrm{Diff}(F,\{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{Top}(F,\{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{hAut}(F,\{\partial F\}))$$

are bijections.

This result is far from new, but this paper will present a thorough and self-contained proof of the following bijection

$$\pi_0(\mathrm{Diff}(F,\{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{hAut}(F,\{\partial F\})). \tag{59}$$

To get the Main Theorem from this result, we will use the result of [Epstein] Thm 6.4 without proof.

We consider slightly different versions of the groups, where we assume F is oriented in the last two groups:

$$\begin{aligned}
\mathrm{Diff}(F,\partial F) &= \{\varphi \in \mathrm{Diff}(F,\{\partial F\}) \mid \varphi|_{\partial F} = \mathrm{id}\}\,,\\
\mathrm{Diff}_+(F,\{\partial F\}) &= \{\varphi \in \mathrm{Diff}(F,\{\partial F\}) \mid \varphi \text{ is orientation-preserving}\}\,,\\
\mathrm{Diff}_+(F,\partial F) &= \mathrm{Diff}(F,\partial F) \cap \mathrm{Diff}_+(F,\{\partial F\}),
\end{aligned}$$

and similar for Top and hAut. By orientation-preserving we mean that the orientation class $[F,\partial F] \in H_2(F,\partial F)$ is preserved by φ_*. From the Main Theorem we easily deduce

Theorem 6.2. *Let F be a compact surface and not a sphere, a disk, a cylinder, a Möbius band, a torus, a Klein bottle, or $\mathbb{R}P^2$. Then there are bijections*

(1) $\pi_0(\mathrm{Diff}(F,\partial F)) \xrightarrow{\cong} \pi_0(\mathrm{Top}(F,\partial F)) \xrightarrow{\cong} \pi_0(\mathrm{hAut}(F,\partial F))$,

(2) $\pi_0(\mathrm{Diff}_+(F,\{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{Top}_+(F,\{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{hAut}_+(F,\{\partial F\}))$,

(3) $\pi_0(\mathrm{Diff}_+(F,\partial F)) \xrightarrow{\cong} \pi_0(\mathrm{Top}_+(F,\partial F)) \xrightarrow{\cong} \pi_0(\mathrm{hAut}_+(F,\partial F))$.

The standard definition of the mapping class group of a surface F is $\Gamma(F) = \pi_0(\mathrm{Diff}_+(F,\partial F))$. The last part of Theorem 6.2 shows that it does not matter whether one considers diffeomorphisms, homeomorphisms, or even homotopy equivalences, when working in the mapping class group.

It is a pleasure to thank Jørgen Tornehave for many fruitful discussions and help during my work on this paper.

6.2 Preliminaries

Definition 6.3. An isotopy ψ of F is a path in $\mathrm{Top}(F,\{\partial F\})$, i.e. $\psi : F \times I \longrightarrow F$ is continuous map such that $\psi_t = \psi(-,t) : F \longrightarrow F$ is a homeomorphism for all $t \in I$, and we say that ψ_0 and ψ_1 are isotopic.
An isotopy is smooth if we can exchange homeomorphism with diffeomorphism in the above. We then say that ψ_0 and ψ_1 are smoothly isotopic.

Lemma 6.4. *Let $f : S^1 \longrightarrow S^1$ an orientation preserving diffeomorphism. Then f is smoothly isotopic to the identity via a smooth isotopy $f_t : S^1 \times I \longrightarrow S^1$ such that the function $F : S^1 \times I \longrightarrow S^1 \times I$ given by $F(z,t) = (f_t(z),t)$ is a diffeomorphism, and*

$$f_t(z) = \begin{cases} f(z), & \text{for } 0 \le t < \varepsilon, \\ z, & \text{for } 1-\varepsilon < t \le 1. \end{cases}$$

Proof. Since f is smooth it defines a smooth function $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ by lifting f under the universal covering $\exp : \mathbb{R} \longrightarrow S^1$. Now take a smooth bump function $\rho : I \longrightarrow I$ satisfying

$$\rho(t) = \begin{cases} 0, & t \le \varepsilon, \\ 1, & t \ge 1-\varepsilon. \end{cases}$$

Let $\tilde{F} : \mathbb{R} \times I \longrightarrow \mathbb{R}$ be given by $\tilde{F}(x,t) = \rho(t)\tilde{f}(x) + (1-\rho(t))x$. This now defines an isotopy from $\tilde{f}$ to the identity, and $F(\exp(x),t) = (\exp(\tilde{F}(x,t)),t)$ is a diffeomorphism. $\square$

The idea of the following proof is due to J. Alexander.

Lemma 6.5. *Let D be a disk and N a collar neighborhood of the boundary. Suppose $f : D \longrightarrow D$ is a homotopy equivalence which restricts to an orientation preserving diffeomorphism of N of the form $f(z,t) = (f(z),t)$ for*

$(z,t) \in N$. Then f is homotopic to a diffeomorphism relative to a smaller collar neighborhood.

Proof. We can assume $f : D \longrightarrow D$, where $D = \{z \in \mathbb{R}^2 \mid |z| \leq 1+\varepsilon\}$, and $N = \{z \in D \mid 1-\varepsilon < |z| \leq 1+\varepsilon\}$. The tubular coordinates on N are $s \in [0, 2\pi]$ and $t \in (-\varepsilon, \varepsilon]$. We first construct a homotopy φ_x, $x \in [0,1]$, which is constant in x outside N, from f to a function g such that $g(s,0) = (s,0)$ in tubular coordinates. We use the isotopy $f_x(s)$ from Lemma 6.4, and set

$$\varphi_x(s,t) = (f_{x(1-\frac{1}{\varepsilon}|t|)}(s), t), \quad t \in (-\varepsilon, \varepsilon]$$

in tubular coordinates. Then $\varphi_0 = f$ and $\varphi_1(s,0) = (s,0)$, and φ_x is the identity on a collar neighborhood of ∂D by Lemma 6.4.

We now make a homotopy ψ_x, $x \in [0,1]$, from g to the function h satisfying

$$h(z) = \begin{cases} g(z), & |z| > 1; \\ z, & |z| \leq 1. \end{cases}$$

Let $D' = \{z \in D \mid |z| \leq 1\}$, and define the solid cone

$$C = \{(z,x) \subseteq D \times I | \, |z| \leq 1 - x\}$$

with bottom $D' \times \{0\}$ and top $(0,1)$, and set

$$\psi_x(z) = \begin{cases} (1-x) f(\frac{z}{1-x}), & (z,x) \in C, \\ z, & (z,x) \in (D' \times I) \setminus C, \\ g(z), & (z,x) \in (D \setminus D') \times I. \end{cases}$$

This is clearly continuous and constitutes a homotopy from g to h through maps which are the identity on a collar neighborhood of ∂D, since g is. We claim $h : D \longrightarrow D$ is a diffeomorphism. Clearly, $h : D' \longrightarrow D'$ is a diffeomorphism, and by Lemma 6.4, h is smooth on D, and for $|z| > 1$, $h = g$ is a diffeomorphism $D \setminus D' \longrightarrow D \setminus D'$. □

A result we will use repeatedly is the following smooth version of the Schönflies curve theorem.

Lemma 6.6. *Let $f : S^1 \longrightarrow F$ be a smoothly embedded simple closed curve homotopic to zero in a surface F. Then the closure of the interior of $f(S^1)$ is a smoothly embedded disk in F.*

Proof. By Thm 1.7 in [Epstein] we know that f separates F into two components, and that one of them (call it D') is homeomorphic to a disk D^2. Thus D' is a connected orientated smooth 2-manifold with 1 boundary component and with Euler characteristic $\chi(D') = 1$. Now by the classification of smooth surfaces, D' is a smooth disk. □

Definition 6.7. Let α be a smoothly embedded 1-submanifold in a surface F. By the surface cut up along α, denoted $F \setminus \alpha$, we will mean the surface with boundary $F \setminus N(\alpha)$, where $N(\alpha)$ is a tubular neighborhood of α in F.

Lemma 6.8. *Let $\alpha : (I, \partial I) \longrightarrow (F, \partial F)$ be a simple curve in a surface F. If the cut-up surface $F \setminus \alpha(I)$ is disconnected, then the induced map $\alpha_* : H_1(I, \partial I) \longrightarrow H_1(F, \partial F)$ is the zero map.*

Proof. Let $\bar{\alpha} = \alpha(I) \subseteq F$, and consider the long exact sequence for the triple $(\partial F, \bar{\alpha} \cup \partial F, F)$:

$$H_1(\bar{\alpha} \cup \partial F, \partial F) \xrightarrow{i_*} H_1(F, \partial F) \xrightarrow{j_*} H_1(F, \bar{\alpha} \cup \partial F) \longrightarrow H_0(\bar{\alpha} \cup \partial F, \partial F)$$

Here $H_0(\alpha \cup \partial F, \partial F) = 0$, so j_* is surjective. Also $H_1(F, \partial F) \cong \mathbb{Z}^{2g+r-1}$ for $F = F_{g,r}$. Since $F \setminus \bar{\alpha}$ is not connected, we can write $F \setminus \bar{\alpha} = F_1 \sqcup F_2$, and by excision,

$$\begin{aligned} H_1(F, \bar{\alpha} \cup \partial F) &\cong H_1(F_1 \sqcup F_2, \partial F_1 \sqcup \partial F_2) \cong H_1(F_1, \partial F_1) \oplus H_1(F_2, \partial F_2) \\ &\cong \mathbb{Z}^{2g_1+r_1-1} \oplus \mathbb{Z}^{2g_2+r_2-1}. \end{aligned}$$

Here $g = g_1 + g_2$ and $r + 1 = r_1 + r_2$, so since j_* is surjective, we conclude that j_* is an isomorphism. Thus $i_* = 0$, and the following diagram shows that $\alpha_* = 0$:

$$\begin{tikzcd} H_1(I, \partial I) \arrow[r, "\alpha_*"] \arrow[d, "\alpha_*"'] & H_1(F, \partial F) \\ H_1(\bar{\alpha} \cup \partial F, \partial F) \arrow[ur, "i_*"'] & \end{tikzcd}$$

□

6.3 Surjectivity

In this section we will prove that the map in (59) is surjective, i.e. a homotopy equivalence of a surface F is homotopic to a diffeomorphism. We first prove this for surfaces with non-empty boundary, and then use this to obtain the proof for closed surfaces. The result for surfaces with non-empty boundary is strongly inspired by [Hempel].

Theorem 6.9. *Let F and G be compact surfaces with non-empty boundaries. Suppose $\pi_1(F)$ is non-trivial. Let $f : (F, \partial F) \longrightarrow (G, \partial G)$ be a map such that $f_* : \pi_1(F) \longrightarrow \pi_1(G)$ is injective and $f|_{\partial F} : \partial F \longrightarrow \partial G$ is a smooth embedding. Then there is a homotopy $f_t : (F, \partial F) \longrightarrow (G, \partial G)$ with $f_0 = f$ and $f_1 : F \longrightarrow G$ a diffeomorphism.*

Proof. First consider each boundary component J of F, and K of G where $f(J) \subseteq K$. We can assume each J and K has a collar neighborhood of the form $J \times [0, \varepsilon]$ and $K \times [0, \varepsilon]$, where the map f has the form $f(x,t) = (f|_J(x), t)$, by gluing on small cylinders, extending f as desired, and smoothing out. Since f is continuous, it is homotopic to a smooth map, and we can choose the homotopy to be constant on the collar neighborhoods, so we can assume that f is smooth an embedding on a neighborhood of ∂F.

We are going to cut up G by a non-separating arc α (i.e. an embedded connected 1-manifold with boundary) connecting two boundary components of G in the image of f. We would like to cut up F by $f^{-1}(\alpha)$. To do this we must ensure that $f^{-1}(\alpha)$ is also an embedded 1-manifold. This holds if f is transverse to α. By Thom's transversality theorem, f can be approximated by a smooth map g transverse to α arbitrarily close to f. Even better, g can be chosen such that $g|_A = f|_A$ for a closed subset $A \subseteq F$ on which the transversality condition on f is already satisfied. If we choose the arc α to have the form $\alpha = (x_0, t)$, $t \in [0, \varepsilon]$ on the collars $K \times [0, \varepsilon]$ for some $x_0 \in K$, then clearly we can take $A = \bigcup_{J \in \pi_0(F)} J \times [0, \varepsilon]$ in the above. Since the transverse map g is arbitrarily close to f, they are homotopic, and we can assume f is transverse to α.

Since $f|_{\partial F} : \partial F \longrightarrow \partial G$ is an embedding we can see that $f^{-1}(\alpha)$ must consist of one arc in F and possibly a number of embedded circles, and as F is compact, there is a finite number of circles. Since f_* is injective, the circles must be null-homotopic in F, thus they must each bound a disk D_0 in F. Taking a slightly larger disk $D \supseteq D_0$, then $f(\partial D)$ must be contained in a tubular neighborhood of α. Since ∂D is disjoint from $f^{-1}(\alpha)$, all of $f(\partial D)$ is to the same side of α in the tubular neighborhood.

Now D is a disk and $f(\partial D)$ is contained in a disk $E \subseteq G$ on one side of α in the tubular neighborhood. Thus we can make a map $h : D \longrightarrow G$ with $h(D) \subseteq E$ and such that $f|_{\partial D} = h|_{\partial D}$. This gives a map $H : S^2 \longrightarrow G$ by mapping the lower hemisphere by f and the upper hemisphere by h. Since G is not S^2 or $\mathbb{R}P^2$, we know $\pi_2(G) = 0$, so the map H can be extended to a map $D^3 \longrightarrow G$, thus giving a homotopy from f to h. This will reduce the number of circles in the preimage, and we can thus assume that $f^{-1}(\alpha)$ is just an arc in F. By transversality we can assume that we have a tubular neighborhood of $f^{-1}(\alpha)$ mapping to a tubular neighborhood of α.

We can now cut F along $f^{-1}(\alpha)$ and G along α, to obtain $\hat{F}$ and $\hat{G}$. After cutting up F and G along an arc, we will actually have manifolds with corners, $\hat{F}$ and $\hat{G}$. But clearly we can smooth out these corners inside the collar neighborhoods where $f : \hat{F} \longrightarrow \hat{G}$ is smooth.

Now we would like to show that the process will not separate F. Consider the situation when we cut up along a non-separating arc α in G. We can parametrise α and think of it as a function $\alpha : (I, \partial I) \longrightarrow (G, \partial G)$. This induces a map $\alpha_* : H_1(I, \partial I; \mathbb{Z}_2) \longrightarrow H_1(G, \partial G; \mathbb{Z}_2)$. The condition that α is nonseparating translates as $\alpha_* \neq 0$. By the above we can assume that $f^{-1}(\alpha)$ is a single arc, which we parametrize as $\tilde{\alpha} : (I, \partial I) \longrightarrow (F, \partial F)$:

$$\begin{tikzcd} & (I, \partial I) \arrow[dl, "\tilde{\alpha}"'] \arrow[dr, "\alpha"] & \\ (F, \partial F) \arrow[rr, "f"] & & (G, \partial G) \end{tikzcd}$$

On homology this induces the commutative diagram

$$\begin{tikzcd} & H_1(I, \partial I; \mathbb{Z}_2) \arrow[dl, "\tilde{\alpha}_*"'] \arrow[dr, "\alpha_*"] & \\ H_1(F, \partial F; \mathbb{Z}_2) \arrow[rr, "f_*"] & & H_1(G, \partial G; \mathbb{Z}_2) \end{tikzcd}$$

But since $\alpha_* \neq 0$ we get $\tilde{\alpha}_* \neq 0$ and thus by Lemma 6.8, $\tilde{\alpha} \subseteq F$ is nonseparating.

Now we show that $f_* : \pi_1(\hat{F}) \to \pi_1(\hat{G})$ is still injective after cutting up. We use that F is homotopic to $\hat{F} \cup I$, where I is a small interval connecting two points $b_0, b_1 \in \partial \hat{F}$. Using that $\hat{F}$ is connected we choose a path J in $\hat{F}$ from b_0 to b_1, such that $I \cup J$ form a loop. Now $F \simeq \hat{F} \vee S^1$ (by contracting J in $\hat{F}$ to a point). Then $i_* : \pi_1(\hat{F}) \longrightarrow \pi_1(F)$ is injective, since $i_* : \pi_1(\hat{F}) \longrightarrow \pi_1(F) = \pi_1(\hat{F}) * \mathbb{Z}$ is just the inclusion in the first factor by van Kampen's theorem. Now it follows from the commutative diagram

$$\begin{tikzcd} \pi_1(\hat{F}) \arrow[r, hook, "i_*"] \arrow[d, "\hat{f}_*"] & \pi_1(F) \arrow[d, hook, "f_*"] \\ \pi_1(\hat{G}) \arrow[r, "i_*"] & \pi_1(G) \end{tikzcd}$$

that $\hat{f}_* : \pi_1(\hat{F}) \longrightarrow \pi_1(\hat{G})$ is also injective.

It remains to show that by cutting up F and G they have to become disks at the same time. Firstly if G is a disk, then $f_* : \pi_1(F) \longrightarrow \{1\}$ is injective, so $\pi_1(F) = \{1\}$, and this implies that F is also a disk (since F is a surface with boundary). Conversely, if G is not a disk then neither is F, since given a non-separating arc α in G we have shown above that there exists a non-separating arc in F.

We are down to the case where f is a map from a disk to a disk that is smooth in a collar of the boundary, and this case is handled by Lemma 6.6. We can glue the resulting smooth maps on the pieces together again, since the collar neighborhoods of the boundary of each piece (where the map is smooth) are fixed by the homotopy from Lemma 6.6. So we are done. □

Corollary 6.10. *Let F and G be compact surfaces with non-empty boundaries. Suppose $\pi_1(F)$ is non-trivial. Let $f : (F, \partial F) \longrightarrow (G, \partial G)$ be a map such that $f_* : \pi_1(F) \longrightarrow \pi_1(G)$ is injective and $f|_{N(\partial F)} : N(\partial F) \longrightarrow N(\partial G)$ is a smooth embedding, where $N(-)$ denotes a neighborhood. Then there is a homotopy $f_t : (F, \partial F) \longrightarrow (G, \partial G)$ with $f_0 = f$ and $f_1 : F \longrightarrow G$ a diffeomorphism, such that $f_t = f_0$ on a neighborhood of ∂F.*

Proof. Use the proof above, but skip the first part which proves that $f|_{N(\partial F)} : N(\partial F) \longrightarrow N(\partial G)$ can be made into a smooth embedding. □

Lemma 6.11. *Let $f_0, f_1 : S^1 \longrightarrow F$ be disjoint non-trivial two-sided embeddings in the surface F. Assume there exist $m, n \in \mathbb{Z}_+$ such that f_0^n and f_1^m represent the same free homotopy class in F. Then there is an embedding $\varphi : S^1 \times I \longrightarrow F$ such that $\varphi|_{S^1 \times \{i\}} = f_i$ for $i = 0, 1$, so f_0 and f_1 bound a cylinder.*

Proof. This is a special case of [Epstein], Lemma 2.4.

We start by cutting F up along f_0 and then gluing a disk onto each of the two new boundary components; let M be the connected component containing f_1 in the resulting surface. Since f_0 is null-homotopic in M, then so is f_0^n and thus f_1^m. Now we will show that f_1 is null-homotopic in M, so that it bounds a disk in M. First if $\partial M \neq \emptyset$, then $\pi_1(M)$ is a free group and thus if $f_1^m = 1$ then $f_1 = 1$. Else $\pi_1(M)$ is a free group modulo the relation $\partial = \Pi_{i=1}^g [a_i, b_i] \in \pi_1(M)$ (oriented case) or $\partial = \Pi_{i=1}^g a_i^2 \in \pi_1(M)$ (unoriented case). If $f_1^m = 1$ but $f_1 \neq 1$, $\pi_1(M)$ will have torsion, and by [Lyndon-Schupp] Prop. 5.18, the only case that allows for torsion is the unoriented case with $g = 1$. Then the component of M containing f_1 is an $\mathbb{R}P^2$, but then there are no non-trivial two-sided embeddings of S^1. So there can be no torsion, and $f_1 = 1$ in $\pi_1(M)$.

The disk in M bounded by f_1 contains either one or two of the disks glued onto f_0 to form M, since f_1 was non-trivial in F. If the disk bounded by f_1 in M contains just one glued-on disk, then f_0 and f_1 together bound a disk blown up at one point; a cylinder in F. In particular, if f_0 is separating, then the disk bounded by f_1 in M contains just one glued-on disk, so we are done. Now if the disk bounded by f_1 in M contains two of the glued-on disks, then f_1 was separating in F, since we obtain F from M by removing the glued-on disks and gluing up along their boundaries. The cylinder can thus be obtained if we interchange f_0 and f_1. □

The condition in the preceding Theorem 6.9 about the map f being an embedding on the boundary is not essential if f is a homotopy equivalence, as we show next:

Lemma 6.12. *Suppose $f : (F, \partial F) \longrightarrow (G, \partial G)$ induces an isomorphism $f_* : \pi_1(F) \longrightarrow \pi_1(G)$, and suppose F is compact with $\partial F \neq \emptyset$ and is neither a disk, a cylinder nor a Möbius band. Then the following holds:*

(*i*) *For all boundary components $J \subseteq \partial F$ and $K \subseteq \partial G$ such that $f(J) \subseteq K$, the composite $\mathbb{Z} \cong \pi_1(J) \xrightarrow{f} \pi_1(K) \cong \mathbb{Z}$ is multiplication by ± 1, and no two different boundary components in F are taken to the same boundary component in G.*

(*ii*) *f is homotopic to a map $g : (F, \partial F) \longrightarrow (G, \partial G)$ with $g|_{\partial F} : \partial F \longrightarrow \partial G$ an embedding.*

Proof. Let $J \subseteq F$ be a boundary component, and let $K \subseteq G$ be the boundary component with $f(J) \subseteq K$. We have a commutative diagram,

$$\begin{tikzcd} \pi_1(J) \arrow[r, hook, "(f|_J)_*"'] \arrow[d, hook] & \pi_1(K) \arrow[d] \\ \pi_1(F) \arrow[r, "\cong", "f_*"'] & \pi_1(G) \end{tikzcd} \tag{60}$$

Here, the vertical map $\pi_1(J) \longrightarrow \pi_1(F)$ is injective, since it is a non-zero map (as F is not a disk) from $\pi_1(J) \cong \mathbb{Z}$ into the free group $\pi_1(F)$. Then $(f|_J)_*$ is multiplication by an integer $n \neq 0$.

If F has more than 1 boundary component, we can choose generators for $\pi_1(F)$ such that the generator of $\pi_1(J)$ goes to a generator of $\pi_1(F)$ under the left vertical map in (60). Since f_* is an isomorphism, it takes generators to generators, and thus it follows by commutativity that $n = \pm 1$.

If F only has the one boundary component J, then the generator α of $\pi_1(J)$ maps to either $\partial = \Pi_{i=1}^{g}[a_i, b_i] \in \pi_1(F)$ (oriented case) or $\partial = \Pi_{i=1}^{g} a_i^2 \in \pi_1(F)$ (unoriented case). If $f_*(\alpha) = x^n$ for a generator x of $\pi_1(K)$, we get by commutativity that $f_*(\partial) \in \pi_1(G)$ would be an nth power of something. Since $f_* : \pi_1(F) \longrightarrow \pi_1(G)$ is an isomorphism, ∂ itself would be an nth power of some element. In case $\partial = a_1^2$, F is a Möbius band, so this cannot happen. In all other cases we get $n = \pm 1$.

We have shown that $(f|_J)_* : \pi_1(J) \longrightarrow \pi_1(K)$ is an isomorphism. Thus we can homotope f in a collar neighborhood around J such that $f|_J : J \longrightarrow K$ is a diffeomorphism. We do this for every boundary component of F.

All that is left is to check that no two boundary components J_1, J_2 of F map to the same boundary component K in G. If that were the case, the elements of $\pi_1(F)$ generating $\pi_1(J_1)$ and $\pi_1(J_2)$ would both map to a generator of $\pi_1(K)$, i.e. would coincide up to a sign, since $f_* : \pi_1(F) \longrightarrow \pi_1(G)$ is an isomorphism. Then by Lemma 6.11, F would be a cylinder, which it is not. □

Theorem 6.13. *Let F and G be compact surfaces, and let $f : F \longrightarrow G$ be a homotopy equivalence. Assume neither F nor G is a disk, a sphere, a cylinder, a Möbius band, a torus, a Klein bottle, or $\mathbb{R}P^2$. Then f is homotopic to a diffeomorphism.*

Proof. If F and G have non-empty boundary, Lemma 6.12 and Theorem 6.9 give the result. So assume that F and G are closed surfaces.

Let $B \subseteq G$ be a non-separating, 2-sided simple closed curve in G. Since f is homotopic to a smooth map which is transverse to B, we can assume that f is smooth and transverse to B. Consider the components of $f^{-1}(B)$. By transversality and compactness, this is a finite set of disjoint 1-submanifolds of F. As in the proof of Theorem 6.9, we can homotope f so that no component in $f^{-1}(B)$ bounds a disk. For any 1-sided simple closed curve γ in $f^{-1}(B)$, take a small tubular neighborhood M of γ such that $f(M) \subseteq N$, where N is a tubular neighborhood of B. Since $M \setminus \gamma$ is connected and $f(M \setminus \gamma) \subseteq N \setminus B$, it follows that $M \setminus \gamma$ maps to the same side of the 2-sided curve B under f. This implies that we can homotope f in M to a function not hitting B. So we can assume that no component of $f^{-1}(B)$ is a 1-sided simple closed curve.

Now let H_0, H_1 be two components of $f^{-1}(B)$, and let $h_0, h_1 : S^1 \longrightarrow F$ be parametrizations of H_0 and H_1, respectively. Then

$$\mathbb{Z} \cong \pi_1(H_i) \xrightarrow{f} \pi_1(B) \cong \mathbb{Z}$$

is multiplication by some $m_i \in \mathbb{Z}$. Note that $m_i \neq 0$ since h_i is nontrivial in F and f is injective on $\pi_1(F)$. This gives that $f_*(h_0^{m_1}) = f_*(h_1^{m_0}) \in \pi_1(G)$, and since f is injective on $\pi_1(F)$, $h_0^{m_1} = h_1^{m_0} \in \pi_1(F)$. Then by Lemma 6.11 they bound a cylinder (if h_0 and $\bar{h}_1$ bound a cylinder then so do h_0 and h_1). This cylinder might contain components of $f^{-1}(B)$, but since there are finitely many such components, we can take a cylinder whose intersection with $f^{-1}(B)$ is precisely its ends, call them h_0 and h_1 again.

Now the cylinder gives a homotopy $c : S^1 \times I \longrightarrow F$ from h_0 to h_1, and thus $f \circ c : S^1 \times I \longrightarrow G$ is a homotopy in G, with $f(c(S^1 \times]0,1[)) \cap B = \emptyset$. Thus we get a continuous map $\widetilde{f \circ c} : S^1 \times I \longrightarrow G \backslash B$ into the cut-up surface $G \backslash B$. This is a homotopy between non-zero powers of boundary components of $G \backslash B$. Now by Lemma 6.11, if these two boundary components are distinct, $G \setminus B$ would be a cylinder. But this is impossible, since G is neither a torus nor a Klein bottle. This implies that both ends of the cylinder is mapped to the same boundary component in $G \setminus B$, and thus we can change f by a homotopy to remove h_0 and h_1 from $f^{-1}(B)$ without changing $f^{-1}(B)$ otherwise. We can now assume that $f^{-1}(B)$ is a single closed curve, since $f^{-1}(B) = \emptyset$ implies that $f_* : \pi_1(F) \longrightarrow \pi_1(G)$ factors through $\pi_1(G \setminus B)$ but $\pi_1(G \setminus B) \longrightarrow \pi_1(G)$ is not surjective. We can finally see that the curve $f^{-1}(B)$ is non-separating by Lemma 6.8, since B is non-separating and $f_* : H_1(F) \longrightarrow H_1(G)$ is a group homomorphism.

Consider $f| : N(f^{-1}(B)) \longrightarrow N(B)$, where $N(-)$ denotes a tubular neighborhood. Then, using a method as in the proof of Lemma 6.4 on $f^{-1}(B)$ and a bump function to extend to $N(f^{-1}(B))$, one can see that f is homotopic to a map g with $g^{-1}(B) = f^{-1}(B)$, such that $g|_{N(g^{-1}(B))}$ is a smooth covering map (the number of sheets will be the degree of $f : f^{-1}(B) \longrightarrow B$). So now we assume that f is a smooth covering map on a neighborhood of $f^{-1}(B)$.

Since $f_* : \pi_1(F \setminus f^{-1}(B)) \longrightarrow \pi_1(G \setminus B)$ is injective ([Lyndon-Schupp] prop 5.1), we can choose a covering $\rho : \widetilde{G \setminus B} \longrightarrow G \setminus B$ and lift f as in the diagram,

$$\begin{tikzcd} & \widetilde{G \setminus B} \arrow[d, "\rho"] \\ F \setminus f^{-1}(B) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & G \setminus B \end{tikzcd} \tag{61}$$

such that $\tilde{f}_* : \pi_1(F \setminus f^{-1}(B)) \cong \pi_1(\widetilde{G \setminus B})$. Moreover, ρ is a finite-sheet covering, since f maps (a parametrization of) $f^{-1}(B)$ to a non-zero multiple of (a parametrization of) B, and the number of sheets is locally constant. So $\widetilde{G \setminus B}$ is compact.

Now in a neighborhood of the boundary of $F \setminus f^{-1}(B)$, $\tilde{f}$ is a covering map, and $\tilde{f}_*$ is an isomorphism on π_1. So $\tilde{f}$ is an embedding on a neighborhood of the boundary. By Corollary 6.10 on $\tilde{f} : F \setminus f^{-1}(B) \longrightarrow \widetilde{G \setminus B}$, $\tilde{f}$ is homotopic to a diffeomorphism, relative to a neighborhood of the boundary. Glue up this diffeomorphism to a map $g : F \longrightarrow G$ which will be homotopic to f, and be both a homotopy equivalence and a smooth covering map. The last two imply that g is a diffeomorphism $F \longrightarrow G$. $\square$

6.4 Injectivity

In this section we will prove that the map in (59) is injective, i.e. if a diffeomorphism is homotopic to the identity, it is smoothly isotopic to the identity.

Definition 6.14. Let $f, g : I \longrightarrow F$ be smooth embeddings into a surface F. We say that f and g *form an "eye"* if the following is satisfied:

(*i*) $f(I) \cup g(I)$ bounds a disk in F.

(*ii*) $f|_{[0,\varepsilon[} = g|_{[0,\varepsilon[}$, $f|_{]1-\varepsilon,1]} = g|_{]1-\varepsilon,1]}$, and f is disjoint from g on $]\varepsilon, 1-\varepsilon[$.

Lemma 6.15. *Let $f, g : I \longrightarrow F$ be two smooth embeddings into a surface F which form an "eye". Then there is a smooth isotopy φ_t of F with $\varphi_0 = \mathrm{id}_F$ and $\varphi_1 \circ g = f$. Furthermore, there is a small neighborhood N of the disk bounded by f and g for which φ_t is the identity outside N for all t.*

Proof. Let N_f be a tubular neighborhood of $f(I)$, given by a normal vector field ξ_f to N_f. Let also N_g be a tubular neighborhood of $g(I)$ given by a normal vector field ξ_g, such that $N_f \cup N_g$ is an annulus. This is possible since $f(I) \cup g(I)$ bounds a disk in F.

There is a diffeomorphism $\psi_f : N_f \longrightarrow V_f \subseteq \mathbb{R}^2$ such that $\psi_f \circ f$ is the standard embedding $I \longrightarrow \mathbb{R} \times \{0\}$. We can take $V_f = I \times]-\varepsilon, \varepsilon[$. We want to extend ψ_f to a diffeomorphism $\psi_{fg} : N_f \cup N_g \hookrightarrow \mathbb{R}^2$, i.e. $\psi_{fg}|_{N_f} = \psi_f$.

First we note that inside $V_f = I \times]-\varepsilon, \varepsilon[$ we have the image

$$G = \psi_f(g(I) \cap N_f).$$

By taking ε small, we can ensure that G is the graph $\{(t, h(t))\}$ of a smooth function $h : [0, \delta[\cup]1-\delta, 1] \longrightarrow [0, \infty[$. We can extend ψ_f to a map $\tilde{\psi}_{fg}$ defined on $N_f \cup g(I)$ such that $\tilde{\psi}_{fg} \circ g : I \longrightarrow \mathbb{R}^2$ is smooth, using bump functions etc as usual, such that the image $G_I = \tilde{\psi}_{fg} \circ g(I)$ is the graph $\{(t, h(t)\}$ of a function $h : I \longrightarrow [0, \infty[$, see Figure 6.4.

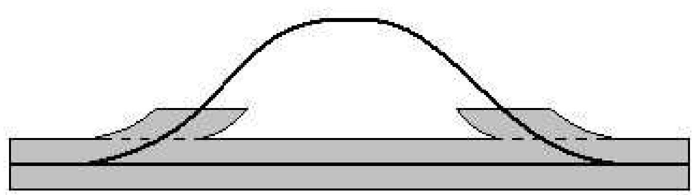

Figure 14: The tubular neighborhood V_f and the graph G_I of h in $\mathbb{R}^2$.

We define a tubular neighborhood of G using the vector field $\eta_G = (\psi_f)_*(\xi_g|_{N_f \cap N_g})$. Since ψ_f is a diffeomorphism, η_G is a transverse vector field, and so defines a tubular neighborhood N_G of G inside V_f. Now we shrink V_f

to $I\times]-\varepsilon', \varepsilon'[$ where $\varepsilon' < \varepsilon$ (thus also shrinking N_f). Then we cover G_I by two open sets in $\mathbb{R}^2$, U_1 covering $G_I \setminus G$, and U_2 whose intersection with U_1 lie in N_G and outside V_f, see Figure 6.4. Then we take a partition of unity ρ_1, ρ_2 with respect to U_1, U_2.

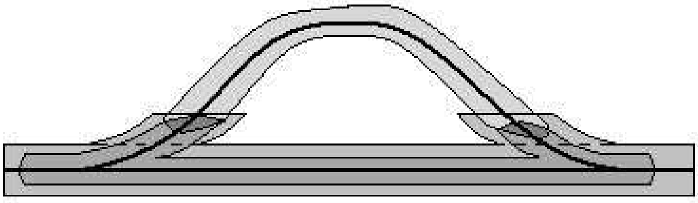

Figure 15: Neighborhoods U_1 and U_2 of G_I.

Let η_I be the standard normal vector field to G_I, defined on $G_I \setminus G$. Then we make a new vector field $\rho_1\eta_I + \rho_2\eta_G$. Since $\rho_1\eta_I + \rho_2\eta_G$ is never 0 or tangent to G, this defines a tubular neighborhood V_g of G_I. This tubular neighborhood coincides with N_G on V_f, and thus gives a diffeomorphism $\psi_{fg} : N_f \cup N_g \longrightarrow V_f \cup V_g$ which extends $\tilde{\psi}_{fg}$.

The inner boundary circle C of the annulus $N_f \cup N_g$ bounds a disk $D' \subseteq F$, and so the image $\psi_{fg}(C)$ also bounds a disk $D_{\mathbb{R}^2} \subseteq \mathbb{R}^2$. Then we can extend $\psi_{fg}|_C$ to a map $D' \longrightarrow D_{\mathbb{R}^2}$, which is necessarily a homotopy equivalence, so by Lemma 6.6 we can replace it by a diffeomorphism $\psi_{D'} : D' \longrightarrow D_{\mathbb{R}^2}$, such that $\psi_{D'}|_C = \psi_{fg}|_C$. The we can glue $\psi_{D'}$ and ψ_{fg} along C to obtain a diffeomorphism Ψ from $D = D' \cup N_f \cup N_g$ onto a disk in $\mathbb{R}^2$.

Now we can use a vertical flow in $D' \cup N_g \cup N_f$ (i.e. a pullback under Ψ of the obvious vertical flow in $\mathbb{R}^2$) to make $\mathrm{Im}(g) = \mathrm{Im}(f)$, and lastly a horizontal flow in N_f to make $g = f$. $\square$

Lemma 6.16. *Given two smoothly embedded arcs $f, g : I \longrightarrow F$ satisfying $f(\{0,1\}) \cap g(I) = f(I) \cap g(\{0,1\}) = \emptyset$. Then there is a smooth isotopy φ_t of $\mathrm{id}|_F$ such that $\varphi_1 \circ f$ and g intersect transversally. Moreover φ_t is the identity outside a tubular neighborhood of f.*

Proof. Take an open tubular neighborhood of f, N_f, of constant radius, where $r : N_f \longrightarrow f(I)$ is the retraction. Inside N_f take a closed tubular neighborhood of f of constant radius, N_f^c. We cover $g(I) \cap N_f^c$ with sets of the form $N_f(a,b) = \{x \in N_f \mid f^{-1}(r(x)) \in]a,b[\}$, where $a < b \in I$, and $f(a), f(b)$ is outside $g(I)$. Since $g(I) \cap N_f^c$ is compact, we can assume that it is a finite covering, $N_f(a_i, b_i)$, $i = 1, \ldots, N$, where $a_1 < a_2 < \cdots < a_N$.

For each $x \in F$ where f and g intersect non-transversally, $x \in N_f(a_i, b_i)$ for some i. Now take the first such i. Then we can choose another arc $\tilde{g} : I \longrightarrow N_f(a_i, b_i)$ such that g and $\tilde{g}$ form an "eye" and $\tilde{g}$ and f intersect transversally for all $x \in \tilde{g}(I) \cap f(I) \subseteq N_f(a_i, b_i)$. Now by Lemma 6.15 there is an isotopy from g to $\tilde{g}$ in $N_f(a_i, b_i)$, which is the identity outside $N_f(a_i, b_i)$. Doing this for each i, we obtain in finitely many steps an isotopy which is the identity outside N_f, making f and g intersect transversally. □

Lemma 6.17. *Let F be a compact surface with $F \neq \mathbb{R}P^2$, S^2, and let f be a diffeomorphism of F.*

- *(i) Let $\alpha_i : S^1 \longrightarrow F \backslash \partial F$ be a finite family of disjoint, non-trivial, pairwise non-homotopic two-sided simple closed curves, with $f \circ \alpha_i \simeq \alpha_i$ for all i. Then there is an smooth isotopy f_t of F such that $f_0 = f$ and $f_1 \circ \alpha_i = \alpha_i$ and the identity extends to tubular neighborhoods.*
- *(ii) Let $\alpha_i : I \longrightarrow F$ be a finite family of simple curves, disjoint except possibly at endpoints, with $f \circ \alpha_i \simeq \alpha_i$ and $f \circ \alpha_i = \alpha_i$ near the endpoints for all i. Let $A \subseteq F$ be a union of disjoint non-trivial closed curves, with $f|_A = \mathrm{id}$ and $\alpha_i(I) \cap A = \alpha_i(\partial I)$ for all i. Then there is an smooth isotopy f_t of F, such that $f_0 = f$, $f_1 \circ \alpha_i = \alpha_i$ and the identity extends to tubular neighborhoods. Furthermore $f_t|_A = \mathrm{id}$ for all t.*

Proof. (i) and (ii) can be proved by the same methods, so we handle the two cases as one initially. But we will also use (i) to prove (ii). First, in both cases we have a closed subset $A \subseteq F$ with $f|_A = \mathrm{id}$ (in case (i), A starts as $\emptyset$). Consider a single curve $\alpha = \alpha_1$. We will make an isotopy f_t of F such that $f_0 = f$, $f_1 \circ \alpha = \alpha$, and $f_t|_A = \mathrm{id}$ for all t. Then we can let $A_1 = A \cup \alpha(I)$, and use the result for f_1 and A_1 on α_2, completing the proof in a finite number of steps. So consider a curve α as in (i) or (ii), and let $\beta = f \circ \alpha$ be the image curve. By assumption, $\beta \simeq \alpha$.

In case (ii), there are small neighborhoods N_0 and N_1 of the start and end points where α and β agree. Inside N_0 and N_1 we can make an isotopy of f which perturbs β slightly, so that α and β agree near the start/end point, and then become disjoint. By shrinking N_0 and N_1 we can assume that α and β are disjoint on ∂N_0 and ∂N_1. Our goal is now to make α and β disjoint outside N_0 and N_1. From now on, we will ignore N_0 and N_1 in the proof, and only work with α and β outside them.

By Lemma 6.16 we can assume α and β are transverse to each other. Then α and β have finitely many intersection points by compactness. To get an isotopy of F taking β to α, we will first ensure that α and β have no intersection points. To do this, consider the universal covering $\pi : \tilde{F} \longrightarrow F$. We can model $\tilde{F}$ as an open disk in $\mathbb{R}^2$. Take a fixed lift $\tilde{\beta}$ of β.

We consider all the connected components of $\pi^{-1}(\alpha)$ that intersect $\tilde{\beta}$. There are finitely many such components, call them $\tilde{\alpha}_k$, since α and β have finitely many intersection points. The $\tilde{\alpha}_k$ are also transverse to $\tilde{\beta}$. Now we look for a pair of intersection points between $\tilde{\beta}$ and an $\tilde{\alpha}_i$, such that the part of the two curves between these points (a closed curve, call it σ) bounds a disk whose interior does not contain any points on $\tilde{\beta}$ or $\tilde{\alpha}_k$ for any k. So σ is a simple closed curve in $\tilde{F}$ bounding a disk. Projecting onto F, we get $\pi \circ \sigma$ (the parts of α and of β between two intersection points) also a simple closed curve, which is null-homotopic, so according to Lemma 6.6, $\pi \circ \sigma$ bounds a disk in F. We can choose a curve β' which form an "eye" with β and which does not intersect α in a neighborhood of the disk bounded by $\pi \circ \sigma$. Then by lemma 6.15 we can isotope β to β', so that there are two fewer intersection points between α and β'. Since there are finitely many intersection points, this procedure terminates.

But we must show why we can always find such a σ in $\tilde{F}$. Since $\tilde{\alpha}_k$ is a connected component of $\pi^{-1}(\alpha)$, each $\tilde{\alpha}_k$ separates $\tilde{F}$. So if $\tilde{\beta}$ crosses $\tilde{\alpha}_i$ once, it must cross it again (let us choose the first time it does so), as it is transverse to $\tilde{\alpha}_i$. Now $\tilde{F} \subseteq \mathbb{R}^2$, so the part of $\tilde{\beta}$ and $\tilde{\alpha}_i$ between these two intersection points will bound a disk. If this disk contains parts of $\tilde{\beta}$ or $\tilde{\alpha}_k$'s, there will be a smaller disk inside which satisfies the requirements, since there are finitely many intersection points. In this way we can isotope β to a curve which does not intersect α (in case (ii), except in N_0 and N_1).

In case (i), we now have two homotopic disjoint simple closed curves α and β. Then according to Lemma 1.4, they bound a cylinder. Recall that the set A (fixed by f) consists of the curves already handled, i.e. a union of non-trivial closed curves, none of which are homotopic to α, and thus not to β, either. Thus A cannot intersect the cylinder bounded by α and β (in fact, A cannot intersect a small open neighborhood of the cylinder). Then clearly there is an isotopy f_t of F, which is the identity on A, taking β to α.

In case (ii), the two curves α and β are homotopic and form a simple closed curve, so again they bound a disk. Recall that A originally consisted of non-trivial closed curves, so none of these can be inside the disk. As we add curves to A, the circles get connected by arcs. None of these can intersect α, since they were assumed to be disjoint from the start. As f is the identity on A, they cannot intersect $\beta = f \circ \alpha$, either. Thus A cannot cross the boundary of the disk, so A and the disk are disjoint. Thus by lemma 6.15 we can make an isotopy f_t of F, which is the identity on A, so that $f_1 \circ \alpha = \alpha$.

Now we extend the result to tubular neighborhoods of the curves. We make a tubular neighborhood M_0 of α, and by compactness identify it with $S^1 \times]-\varepsilon, \varepsilon[$ in case (i) and $I \times]-\varepsilon, \varepsilon[$ in case (ii). Now for (x, t) in a smaller neighborhood $M_1 \subset M_0$ of α, the projection the second coordinate $\mathrm{pr}_t f_x(t) :=$

$\mathrm{pr}_t(f(x,t))$ has positive differential, and thus for all x the image of $f_x(t)$, $\{(x',t') \mid (x',t') = f_x(t) \text{ for some } t \in]-\varepsilon,\varepsilon[\}$ is the graph of a function $h_x(t') = x'$. Now we can make tubular neighborhood M_2 such that $M_2 \subset f(M_1)$ and by possibly shrinking it assume that $M_2 = I\times]-\delta,\delta[$ or $M_2 = S^1\times]-\delta,\delta[$. For definiteness, say $M_2 = I\times]-\delta,\delta[$. Choose a smooth bump function $\rho(t)$ with $\rho(t) = 1$ for $|t| \leq \frac{1}{2}\delta$ and $\rho(t) = 0$ for $|t| = \delta$. Let

$$g_s(x,t) = \begin{cases} \big((1-s)h_x(t') + s\big(\rho(t')x + (1-\rho(t'))h_x(t')\big), t'\big) & \text{for } (x,t) \in f^{-1}(M_2) \\ f(x,t) & \text{otherwise.} \end{cases}$$

where t' is the second coordinate of $f(x,t)$ as above. Then g_s defines an isotopy from f to a function g_1 with the property that $g_1(x,t) = (x,t')$ for $t' \in]-\frac{\delta}{2},\frac{\delta}{2}[$. Now by stretching the parameter t' in each interval $\{x\}\times]-\delta,\delta[$, we can assume that f is the identity on a (smaller) neighborhood. □

Corollary 6.18. *If we in addition to the requirements in lemma 6.17 require that f is the identity on ∂F, then the isotopy can be assumed also to be the identity on ∂F.*

Proof. All the steps in the proof can be done away from the boundary. □

Theorem 6.19. *Let $F \neq S^2, \mathbb{R}P^2$ and let $f, g \in \mathrm{Diff}(F, \partial F)$ be homotopic. Then f and g are smoothly isotopic.*

To prove this I use the following result from [Smale] without proof.

Theorem 6.20 (Smale)**.** *Let $f \in \mathrm{Diff}(D^2, \partial D^2)$. Then f is smoothly isotopic to the identity, and if f is the identity on the boundary then so is the isotopy.*

Proof of Theorem 6.19. If we prove that $f^{-1}g$ is smoothly isotopic to the identity, we will have a smooth isotopy from g to f. Thus we can restrict our attention to the case $f \simeq \mathrm{id}$.

Choose a pair of pants/annular decomposition of the surface F, i.e. a collection of disjoint simple closed curves $\alpha_i : I \longrightarrow F, i = 1,\ldots,n$, in F. By Lemma 6.17 (i), f is smoothly isotopic to a map g, which is the identity on a tubular neighborhood of the α_i. In each pair of pants P, chose two curves that cut P up into a disk (for each annulus, choose one curve). By Lemma 6.17 (ii), there is an isotopy of F, which is the identity on the α_i, from g to a map h fixing a tubular neighborhood of the two curves in each pair of pants. Then we can use Smale's Theorem 6.20 on each disk, getting an isotopy to the identity. □

Corollary 6.21. *In addition to the requirements of Theorem 6.19, assume that f and g are the identity on ∂F. Then we can choose the isotopy to be the identity on ∂F.*

Proof. This is done as in theorem 6.19, except that we use Corollary 6.18 instead of Lemma 6.17, and in addition use that the isotopy in theorem 6.20 can be chosen to be the identity on boundary. □

6.5 Proof of the Main Theorem

As explained in the introduction, we will use a result of Epstein to prove the statement about $\mathrm{Top}(F, \{\partial F\})$:

Theorem 6.22 (Epstein)**.** *Let F a compact surface and let $f : F \longrightarrow F$ be a homeomorphism homotopic to the identity. Then f is isotopic to the identity.*

Proof. This is a part of [Epstein] Thm 6.4, which states exactly this result, but for maps preserving a basepoint. And clearly, by an isotopy we can assume that f preserves any given point x_0, and then f will be homotopic to the identity through maps preserving x_0. □

Now we are ready to prove the bijections of the Main Theorem 1.1:

$$\pi_0(\mathrm{Diff}(F, \{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{Top}(F, \{\partial F\})) \xrightarrow{\cong} \pi_0(\mathrm{hAut}(F, \{\partial F\}))$$

Proof of Theorem 6.1. Suppose F is not a sphere, a disk, a cylinder, a Möbius band, a torus, a Klein bottle, or $\mathbb{R}P^2$. Consider the composite map from (59),

$$\pi_0(\mathrm{Diff}(F, \{\partial F\})) \longrightarrow \pi_0(\mathrm{hAut}(F, \{\partial F\})). \tag{62}$$

According to Theorem 6.13, the map is surjective, and by Theorem 6.19, it is injective. Now all that is left is to show that

$$\pi_0(\mathrm{Top}(F, \{\partial F\})) \longrightarrow \pi_0(\mathrm{hAut}(F, \{\partial F\}))$$

is injective. But that is Theorem 6.22. □

We now deduce Theorem 6.2:

Proof of Theorem 6.2. Suppose F is not a sphere, a disk, a cylinder, a Möbius band, a torus, a Klein bottle, or $\mathbb{R}P^2$.

Similar to the proof of Theorem 6.1, we consider the composite

$$\pi_0(\mathrm{Diff}(F, \partial F)) \longrightarrow \pi_0(\mathrm{hAut}(F, \partial F)).$$

We can assume $\partial F \neq \emptyset$, otherwise this is the Main Theorem. By Cor. 6.10, it is surjective, and by Cor. 6.21 it is injective. To prove the result, it suffices to show that

$$\pi_0(\mathrm{Diff}(F, \partial F)) \longrightarrow \pi_0(\mathrm{Top}(F, \partial F)) \tag{63}$$

is surjective. Consider the following fibration,

$$\mathrm{Diff}(F, \partial F)) \longrightarrow \mathrm{Diff}(F, \{\partial F\}) \longrightarrow \mathrm{Diff}(\partial F). \tag{64}$$

Here, $\mathrm{Diff}(\partial F)$ is a semidirect product $\Sigma_n \ltimes \mathrm{Diff}(S^1)$, where Σ_n denotes the symmetric group of permutations of n elements, and n is the number of boundary components of F. We have of course a similar fibration for Top. We use $\mathrm{Diff}(S^1) \xrightarrow{\cong} \mathrm{Top}(S^1)$, and this implies

$$\mathrm{Diff}(\partial F) \xrightarrow{\cong} \mathrm{Top}(\partial F). \tag{65}$$

Now apply the long exact sequence of homotopy groups for the fibration (64) and its counterpart for Top. Using (65) and the Main Theorem, we get by the 5-lemma that the map (63) is surjective.

Now assume F is oriented. We can write $\mathrm{Diff}(F, \{\partial F\})$ as the disjoint union

$$\mathrm{Diff}(F, \{\partial F\}) = \mathrm{Diff}_+(F, \{\partial F\}) \sqcup \mathrm{Diff}_-(F, \{\partial F\}),$$

where the latter denotes the orientation-reversing maps. Similarly for Top and hAut. Since the maps in the Main Theorem respect this disjoint union, we immediately get the second part of 6.2.

By the same argument we can deduce the last part of 6.2 from the first part. $\square$

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