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Flag varieties, Toric varieties and characteristic p geometry

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SUMMARY

This thesis contains the material which I have been studying during my time as a Ph.D. student at the University of Aarhus, Denmark, in the period 1994-98. The thesis is divided into 5 separated parts with the following titles in order of appearance

(1) "The Steinberg module and Frobenius splitting of varieties related to flag varieties"
(2) "The Frobenius morphism on a Toric variety"
(3) "$\mathcal{D}$-affinity and Toric varieties"
(4) "Frobenius direct images of line bundles on Toric varieties"
(5) "Irreducibility of $\overline{M}_{0,n}(G/P, \beta)$"

As one might guess from the titles I have been particularly interested in the study of flag varieties and toric varieties, and mostly from a positive characteristic viewpoint. I will now try to give an overview over the material mentioned above and relate it to other results in the literature. Hopefully this will be of help to the reader. I have divided the overview into 3 parts. The first part concerns (1), the second part (2)-(4) and the third part is a description of (5).

1. Part 1 : Characteristic p methods on flag varieties

The study of varieties over fields of characteristic $p > 0$ is in many cases completely different from the characteristic 0 case. Many statements about varieties in characteristic 0 is simply not true when they are formulated in the positive characteristic situation. The question therefore arises when results in characteristic 0 remains true in the positive characteristic case. This is one of the basic question which faces one, when working with varieties over fields of positive characteristic. Another question is whether or not results in characteristic 0 may be proved by using methods or results from the positive characteristic case.

One of the differences between the characteristic 0 and the the positive characteristic case, is the existence of a Frobenius morphism. The (absolute) Frobenius morphism F on a variety X is a the map, which is the identity on points and the p'th power map on the level of functions $\mathcal{O}_X \to F_*\mathcal{O}_X$. One of the most usable properties of the Frobenius morphism is the fact that the pull back $F^*\mathcal{L}$ of a line bundle $\mathcal{L}$ on X equals $\mathcal{L}^p$. In fact this is one of the main reasons why Frobenius splitting, as defined by V. Mehta and A. Ramanathan in their fundamental paper [18], turns out to be so powerful. Remember that a variety is said to be Frobenius split if there exist a section to the map $\mathcal{O}_X \to F_*\mathcal{O}_X$. One of the main classes of examples (which was also of main interest in [18]) of Frobenius split varieties, are the flag varieties. That Frobenius splitting works so well for flag varieties as it does, is in some sense more surprising than the implications of Frobenius splitting.

The method for proving Frobenius splitting of flag varieties (or more precisely in proving compatibly Frobenius splitting of the Schubert varieties in the flag variety) in [18], was by considering a Demazure desingularisation of the Schubert varieties, and using the knowledge of the canonical bundle on this desingularisation. As an application of this V. Mehta and A. Ramanathan (besides other things) proved the cohomology vanishing of ample line bundles (coming from the flag variety) on Schubert varieties. Later many other results, centered among varieties related to flag varieties such as Schubert varieties and Nilpotent varieties, were proven. With respect to the material in this thesis let us just mention the papers [19], [23] and [22]. In [23] and [22] several applications of Frobenius splitting was given. In particular, the concept of diagonal Frobenius splitting was introduced and consequences, such as projective normality of Schubert varieties, was proven from this. In [19] Frobenius splitting of closures of conjugacy classes in the nilpotent variety of a group of type $\mathbb{A}_n$ was proved. From this it was possible to prove normality of the closures of the conjugacy classes.

In all of the papers mentioned above, the proofs relied mainly on algebraic geometric arguments. In [20] and [14] it however became clear that representation theory, and especially the Steinberg module (denoted by St in the following), should play a central role in Frobenius splitting of flag varieties. This was the starting point of [17], which was a joint work with my advisor Niels Lauritzen. In here it is proven that there is a map ($X = G/B$ a flag variety)

$$\varphi : St \otimes St \to \mathcal{H}om_{O_X}(F_*\mathcal{O}_X, \mathcal{O}_X),$$

such that $\varphi(v \otimes w)$ (essentially) is a Frobenius splitting of X if and only if $< v, w > \neq 0$. Here $<,>$ is a G-invariant form on St. A criteria for certain subvarieties of X to be compatibly Frobenius split was also given. From this we were able to obtain new proofs of the compatibly Frobenius splitting of the Schubert varieties and the diagonal Frobenius splitting of X. One of the most essential ingredients in the proof, is the existence of a line bundle $\mathcal{L}$ on G/B such that $F_*\mathcal{L}$ is a direct sum of $\mathcal{O}_X$'s. This was a result proved by H.H. Andersen in [2] and W. Haboush in [12]. All of this is contained in (1). Besides this I have in (1) included material from a joint work with Shrawan Kumar and Niels Lauritzen. It concerns the Frobenius splitting of the unipotent variety of G, and the result is very much similar to the result on the Frobenius splitting of G/B. More precisely what we prove is that there exist a map (Y the unipotent variety)

$$\phi : St \otimes St \to \mathcal{H}om_{O_Y}(F_*\mathcal{O}_Y, \mathcal{O}_Y),$$

such that $\phi(v \otimes w)$ is a Frobenius splitting of Y if and only if $< v, w > \neq 0$. In particular we get a (characteristic independent) proof of the Frobenius splitting of the unipotent variety. It is a complete surprise that there is this similar description of Frobenius splittings of G/B and the unipotent variety. It should be noted that by using the isomorphism (in good characteristics) between the unipotent variety and the nilpotent variety, we in particular get exactly the Frobenius splitting of the nilpotent variety which were considered in [19]. As a side result of the above, we get the vanishing result

$$\mathrm{H}^i(\mathrm{S}^n\,\mathfrak{u}^* \otimes \lambda) = 0\ ,\ i > 0,\ \lambda \text{ strictly dominant},$$

where $\mathfrak{u}$ is the Lie algebra of the unipotent radical of the Borel subgroup B of G. Result in this direction (in positive characteristic) has earlier been found by H.H. Andersen and J.C. Jantzen in [3]. In characteristic zero results of this type has been proved by B. Broer [4].

2. Part 2: Toric varieties

The class of toric varieties constitute a non trivial class of examples where one can get acquainted with the nature of a problem. This is how the material on toric varieties in this thesis arose. Usually the work started with a similar problem on flag varieties, but whereas the flag variety case usually remained unsuccessful, we were sometimes able to prove non trivial results on toric varieties.

The material in (2) is a reprint of an article published in [6] (a shorter version of this paper was published in [5]). It was a joint work with Anders Buch, Niels Lauritzen and Vikram Mehta. If X is a variety defined over a field k of characteristic $p > 0$, we say that $X^{(2)}$ is a lifting of the variety X to the Witt vectors $W_2(k)$ of length 2, if $X^{(2)}$ is a flat scheme over $W_2(k)$ which module p reduces to X. A lifting of the Frobenius morphism of X is a compatibly lifting of F to the scheme $X^{(2)}$. Using results appearing in [9], it was clear that varieties, which had a lifting of the Frobenius morphism to the Witt vectors of length 2, would have nice homological properties. In fact, we prove that for a smooth variety X on which the Frobenius morphism lifts to the Witt vectors of length 2, we have the vanishing (Bott vanishing)

$$\mathrm{H}^i(X, \Omega^j_{X/k} \otimes \mathcal{L}) = 0 \ , i > 0,$$

for every ample line bundle $\mathcal{L}$ on X. A similar result is true if we only assume X to be normal. In (2) it is proven that on any toric variety there is a lifting of the Frobenius morphism, proving Bott vanishing for every normal toric variety. Without proof Danilov had earlier stated this result in [8]. Besides this the Bott vanishing implies indirectly that for a general flag varieties there is no lifting of the Frobenius morphism. This is in good correspondence with the results in [21]. Here it is proved that if the Frobenius morphism on a flag variety X has a lift to the p-adic numbers, then X is a product of projective spaces.

Let $\mathcal{D}$ denote the sheaf of differential operators on a variety X. Then X is said to be $\mathcal{D}$-affine, if every $\mathcal{D}$-module is generated by global sections and has vanishing higher cohomology. Beilinson and Berstein have shown [1] that every flag variety over a field of characteristic 0 is $\mathcal{D}$-affine. They used this to prove a conjecture by Kazhdan and Lusztig about the multiplicity of irreducible representations in a Jordan-Hölder serie of a Verma module. The question therefore arises whether or not a flag varieties X over a field of positive characteristic is $\mathcal{D}$-affine. B. Haastert has in [11] shown that every $\mathcal{D}$-module over X is generated by global sections. This is what B. Haastert calls $\mathcal{D}$-quasiaffine, and it implies that $\mathcal{D}$-affinity of X is equivalent to the vanishing of the higher cohomology groups of $\mathcal{D}$. The question of cohomology vanishing of $\mathcal{D}$ does however not seem to be easy to answer. In [11] the vanishing of the higher cohomology groups of $\mathcal{D}$ is proven in case X is a projective space or SL_3/B. Besides these examples (and products of them) I do not know of other projective varieties over fields of positive characteristic which are $\mathcal{D}$-affine. A natural place to look for such examples would be in the set of toric varieties. This is the subject in (3) which is a reprint of the paper [24]. In here it is proven (for any characteristic of the field) that the only $\mathcal{D}$-affine projective toric varieties are products of projective spaces. A result similar to B. Haastert result of $\mathcal{D}$-qausiaffinity of flag varieties, is however true for smooth toric varieties. This result, which is proven by J. Cheah and P. Sin [7], states that $\mathrm{H}^0(\mathcal{F})$ is nonzero when $\mathcal{F}$ is a nonzero $\mathcal{D}$-module. Notice that one may replace the global generations condition in the definition of $\mathcal{D}$-affinity with this condition.

Let $X = \mathbb{P}^n$ be a projective space, and F the Frobenius morphism on X. Then R. Hartshorne [13] has shown, that for any line bundle $\mathcal{L}$ on X, the vector bundle $F_*\mathcal{L}$ splits into a direct sum of line bundles. This is the main ingredient in B. Haastert proof of the $\mathcal{D}$-affinity of X over fields of positive characteristic. Before I knew the result of the work in (3), I tried to generalize B. Haastert result to toric varieties. This of course required a generalization of R. Hartshorne result to the class of toric varieties. It turns out that R. Hartshornes result remains true for all smooth toric varieties. This is the subject in (4) which is a copy of a paper which is going to appear in *Journal of Algebra*. The method used for proving R. Hartshornes result for toric varieties, is by explicit calculations on the level of the fans connected to toric varieties. The calculation are quite brutal, but they have the advantage of giving an constructive way of finding the decomposition of $F_*\mathcal{L}$ into line bundles. A non constructive, but nicer proof, has later been given by R. Bøgvad [10]. R. Bøgvad is furthermore able to generalize the result to T-linearized vector bundles. His proof uses results on Grothendieck differential operators and T-linearized sheaves.

3. Irreducibility of $\overline{M}_{0,n}(G/P, \beta)$

The material in (5) is a copy of a paper which is going to appear in *International Journal of Mathematics*. The material in here differs from the other part of the thesis, in that it is completely concerned with varieties over the complex numbers. Still it centers around generalized flag varieties G/P. The setup is the following. An n-pointed stable curve C of genus 0, is a connected at most nodal curve C with arithmetic genus 0 together with n marked points on it, such that each component contains at least 3 special points. A special point is either a nodal point or a marked point. The set of n-pointed stable curves of genus 0, can be given a structure of a variety $\overline{M}_{0,n}$, the so called moduli space of stable n-pointed genus 0 curves. F. Knudsen [16] has earlier proved projectivity (and irreducibility) of these moduli spaces.

Generalization of the moduli spaces $\overline{M}_{0,n}$ have recently become very important when defining the quantum cohomology of a complex variety. Let X be a complex variety and β be a 1-cycle on X. A map $\mu : C \to X$ from an n-pointed connected at most nodal curve of arithmetic genus 0 to X, is called an n-pointed genus 0 stable map representing β, if $\mu_*[C] = \beta$ and each component of C which maps to a point contains at least 3 special points. The set of n-pointed genus 0 stable maps representing β can be given a structure of a variety denoted by $\overline{M}_{0,n}(X, \beta)$, and called the moduli space of n-pointed genus 0 stable maps representing β. These moduli spaces was first defined by M. Kontsevich. In case X is a point these moduli spaces coincide with $\overline{M}_{0,n}$. Quantum cohomology of a variety X is defined by degrees of certain intersections on these moduli spaces. The theory of quantum cohomology has recently showed useful with respect to enumerative question. The simplest example of this is the determination of a formula for the number of plane smooth curves of degree d passing through $3d - 1$ points. The material in (5) does however not deal with enumerative questions, but concentrates completely on the moduli spaces $\overline{M}_{0,n}(G/P, \beta)$ for generalized flag varieties G/P. In (5) it is proved that the moduli spaces $\overline{M}_{0,n}(G/P, \beta)$ are irreducible. For this Borel's fixed point theorem turns out to be extremely useful. The observation is that if the image $\mu : C \to G/P$ is invariant under left translation of a Borel subgroup B in P, then the image of μ is a union of Schubert varieties of dimension 1. This is the key point in (5). It is possible to generalize the result to higher genus cases, proving connectedness of $\overline{M}_{g,n}(G/P, n)$. This has been done by B. Kim and R. Pandharipande in [15]. They have furthermore also obtained the irreducibility mentioned above in the genus 0 case.

4. Acknowledgements

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The Steinberg module and Frobenius splitting of varieties related to flag varieties

Contents

0.1 Introduction

The notion of a Frobenius split variety was first introduced by V.B. Mehta and A. Ramanathan in [15]. The definition is simple, but the consequences turns out to be enormous. Among other things, vanishing theorems and normality is among the applications. Some varieties turns out to be particular nice in respect to Frobenius splitting, and this is the flag varieties. In this thesis we will concentrate on flag varieties and related varieties. In particular we will prove that every flag variety is Frobenius split. In fact, this was already contained in [15], but here we will use a different and more representation theoretical approach. Surprisingly this representation theoretical approach generalizes naturally to the cotangent bundle over a flag variety. This we will also cover here. It should be pointed out that the new material in this thesis, is not alone due to the author of this thesis. In particular Niels Lauritzen is coauthor to all of the material, and Shrawan Kumar to everything except the material taken from [12]. Besides this I am grateful to H. H. Andersen, J. Jantzen, V.B. Mehta and T.R. Ramadas for very useful comments and help.

0.2 Notation

Throughout this note we will use the following notation and conventions. First of all k will denote an algebraically closed field. The characteristic of k we will denote by $p \geq 0$. By a scheme we will mean a scheme in the sense of [8]. In particular a scheme need not to be reduced and separated. The reduced and separated schemes of finite type over k will be called varieties.

Chapter 1

The Cartier operator

We start this thesis by a review of the Cartier operator, which was first defined by Cartier in [5].

1.1 The Frobenius morphism

Let $\pi : X \to \mathrm{Spec}(k)$ be a scheme over k. The absolute Frobenius morphism on X, is the map of schemes $F_{abs} : X \to X$, which on the level on points is the identity, and on the level of functions is the p'th power map. Notice that F_{abs} is not a morphism of schemes over k. In this thesis we therefore prefer to work with the relative Frobenius morphism $F : X \to X'$, which is defined by the absolute Frobenius morphism and the following fiber product diagram

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[dr, "\pi"'] \arrow[rr, bend left, "F_{abs}"] & X' \arrow[r, "F'"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"'] \\
& \mathrm{Spec}(k) \arrow[r, "F_{abs}"] & \mathrm{Spec}(k)
\end{tikzcd}$$

Notice that if we forget the k-scheme structure, then X' is isomorphic to X and F and F_{abs} coincide.

Lemma 1.1 *Let L be a line bundle on X. If $L' = F'^{*}L$ denote the corresponding line bundle on X' then $F^{*}(L') \backsimeq L^p$.*

Proof Locally the isomorphism is given by sending an element $l \otimes f$ in $F^{*}(L') = L' \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X$ to fl^p in L^p. $\square$

1.2 The Cartier operator

We will now restrict our attention to a smooth N-dimensional variety X over an algebraically closed field k of characteristic $p > 0$. By (Ω_X, d_X) we denote the sheaf of differentials on X.

Definition 1.1 *Let Ω_X^n denote the n'th exterior power of the sheaf of differentials on X. Then we define a complex of sheaves of k-vector spaces*

$$\Omega_X^\bullet : 0 \xrightarrow{d^{-1}} \mathcal{O}_X \xrightarrow{d^0} \Omega_X^1 \xrightarrow{d^1} \Omega_X^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{N-1}} \Omega^N \xrightarrow{d^N} 0.$$

by

(1) $d^0 = d_X$

(2) $d^{i+j}(\omega \wedge \tau) = d^i\omega \wedge \tau + (-1)^i \omega \wedge d^j\tau \ , \ \omega \in \Omega_X^i, \tau \in \Omega_X^j.$

In general the differentials d^i are not $\mathcal{O}_X$-linear, which means that the complex $\Omega_X^\bullet$ is not a complex of $\mathcal{O}_X$-modules. But, taking direct images via the Frobenius morphism F, it is easily seen that we get a complex $F_*\Omega_X^\bullet$ of $\mathcal{O}_{X'}$-modules. In the following result we will regard the $\mathcal{O}_{X'}$-modules $\oplus_i \Omega_{X'}^i$ and $\oplus_i \mathrm{H}^i(F_*\Omega_X^\bullet)$ as graded $\mathcal{O}_{X'}$-algebras through the $\wedge$-product. Then

Theorem 1.1 *There exists a unique isomorphism of graded $\mathcal{O}_{X'}$-algebras*

$$C^{-1} : \oplus_i \Omega_{X'}^i \to \oplus_i \mathrm{H}^i(F_*\Omega_X^\bullet)$$

which in degree 1 is given by $C^{-1}(d_{X'}(x)) = x^{p-1}d_{X'}(x)$, where x is an element in $\mathcal{O}_{X'}$.

Proof See Theorem 7.2. in [11]. □

The inverse C of the map in Theorem 1.1 is called the Cartier operator. In this thesis we will not use the existence of the Cartier operator in this strength. What will be important for us is that the Cartier operator induces a surjective map

$$F_*\omega_X \to \mathrm{H}^N(F_*\Omega_X^\bullet) \xrightarrow{C} \omega_{X'}. \tag{1.1}$$

Tensorising this map with $\omega_{X'}^{-1}$ and using the projection formula and Lemma 1.1, we get a surjective map of $\mathcal{O}_{X'}$-modules

$$F_*\omega_X^{1-p} \to \mathcal{O}_{X'}.$$

By abuse of notation we will in the following also denote this map by C, and we will furthermore also call it the Cartier operator.

Remark 1.1 *By the description of the inverse Cartier operator in Theorem 1.1, it follows that the Cartier operator $C_X : F_*\omega_X^{1-p} \to \mathcal{O}_{X'}$ is a functorial operator. More precisely let $f : X \to Y$ be a morphism of smooth varieties of the same dimension. Then the following diagram is commutative*

$$\begin{tikzcd} F_*\omega_Y^{1-p} \arrow[r, "C_Y"] \arrow[d] & \mathcal{O}_{Y'} \arrow[d] \\ F_*f_*\omega_X^{1-p} \arrow[r, "f'_*C_X"] & f'_*\mathcal{O}_{X'} \end{tikzcd}$$

The vertical map on the left, is the canonical map coming from the functoriality of the sheaf of differentials.

Lemma 1.2 *Assume that $X = \mathrm{Spec}(A)$ is affine and that there exist $a_1, \dots, a_N$ in A such that $da_1, \dots, da_N$ is a basis for $\Omega_{A/k}$. Let $\alpha_1, \dots \alpha_N$ be non negative integers strictly less than p. Then $C : F_* \omega_X^{1-p} \to \mathcal{O}_{X'}$ satisfies that*

$$C(a_1^{\alpha_1} \cdots a_N^{\alpha_N} (da_1 \wedge \cdots \wedge da_N)^{1-p}) = \begin{cases} 1 & \text{if } \alpha_i = p-1 \text{ for all } i, \\ 0 & \text{else.} \end{cases}$$

Proof To ease the notation let us by da denote the element $da_1 \wedge \cdots da_N$ and by a^α denote the element $a_1^{\alpha_1} \cdots a_N^{\alpha_N}$. Notice first of all that if $\alpha_i \neq p-1$ then

$$d^{N-1}\left(\frac{(-1)^{i-1}}{\alpha_i + 1} a_i a^\alpha (da_1 \wedge \cdots \wedge da_{i-1} \wedge da_{i+1} \wedge \cdots \wedge da_N)\right) = a^\alpha da.$$

This means that the map (1.1) above maps $a^\alpha da$ to 0 unless $\alpha_i = p-1$ for all i. In case $\alpha_i = p-1$ for every i, it follows from the description of C^{-1} in Theorem 1.1 that $a^\alpha da$ is mapped to da. The result now follows by explicit tracing up the isomorphisms given by the projection formula and the proof of Lemma 1.1. This is left to the reader. □

Chapter 2

Frobenius splitting

In this chapter we will review the basic definitions and consequences of Frobenius splitting, as it was first done in [15]. Throughout the chapter k will denote an algebraically closed field of positive characteristic $p > 0$.

2.1 Frobenius splitting

Definition 2.1 *A scheme X over k is said to be Frobenius split (or F-split), if the map of $\mathcal{O}_{X'}$-modules*

$$F^{\#} : \mathcal{O}_{X'} \to F_* \mathcal{O}_X,$$

induced by F, is split. In other words, X is Frobenius split if there exist a map $s : F_* \mathcal{O}_X \to \mathcal{O}_{X'}$ of $\mathcal{O}_{X'}$-modules, such that the composite $s \circ F^{\#}$ is the identity. If so we say that s is a Frobenius splitting of X.

Definition 2.2 *Let Y be a closed subscheme of X given by an ideal $\mathcal{I}_Y$, and let $\mathcal{I}_{Y'}$ be the ideal of Y' inside X'. If s is a Frobenius splitting of X, then we say that Y is compatibly s-split (or split) if $s(F_* \mathcal{I}_Y) \subseteq \mathcal{I}_{Y'}$.*

Remark 2.1 *The following remarks are immediately consequences of the definitions above.*

(1) As $F^{\#}$ is injective if and only if X is reduced, we see that X can only be Frobenius split if X is reduced.

(2) Let $s : F_* \mathcal{O}_X \to \mathcal{O}_{X'}$ be a map of $\mathcal{O}_{X'}$-modules. Then s correspond to a Frobenius splitting of X if and only if $s(1) = 1$.

(3) If Y is compatibly s-split inside X, then Y is Frobenius split. The Frobenius splitting of Y induced by s, will in the following, by abuse of notation, also be denoted by s.

Lemma 2.1 *Let s be a Frobenius splitting of X.*

(1) Let U be an open subscheme of X and Y be a closed irreducible subscheme of X. If $U \cap Y \neq \varnothing$ then Y is compatibly s split if and only if $U \cap Y$ is compatibly $s_{|U}$-split in U.

(2) *If Y_1 and Y_2 are compatibly s-split, then the scheme theoretical intersection $Y_1 \cap Y_2$ is compatibly s-split.*

(3) *Assume that X is Noetherian and Y is a closed compatibly s-split subscheme of X. Then every irreducible component of Y is compatibly s-split.*

Proof For (1) we may use the proof of Lemma 1 in [15], and (2) is immediate by definition. Finally (3) follows from (1), as X is assumed to be Noetherian. $\square$

2.2 Frobenius splitting of smooth varieties

Let X be a smooth variety over k of dimension N. In this section we will examine, when X is Frobenius split. The definition of Frobenius splitting tells us to look at the $\mathcal{O}_{X'}$- module

$$\mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}).$$

We will however not only consider this sheaf of abelian groups as a $\mathcal{O}_{X'}$-module, but also as the $\mathcal{O}_X$-module $F^!O_{X'}$ as defined in [8] Exercise III.6.10. The $\mathcal{O}_X$-module structure is in other words given by

$$(f \cdot \phi)(g) = \phi(gf)$$

when $f, g \in \mathcal{O}_X$ and $\phi \in \mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$. With this O_X-module structure it is clear that

$$F_*F^!O_{X'} = \mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$$

as $\mathcal{O}_{X'}$-modules. The $\mathcal{O}_{X'}$- module $\mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$ comes with a natural evaluation map

$$ev : \mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}) \to \mathcal{O}_{X'}.$$

This map is defined by $ev(\phi) = \phi(1)$, and is very much related to the Cartier operator on X. Remember that the Cartier operator (as we defined it), was a map of $\mathcal{O}_{X'}$-modules

$$C : F_*\omega_X^{1-p} \to \mathcal{O}_{X'}.$$

The relation between C and ev will follow from the next wonderful result taken from [15].

Proposition 2.1 *There exist a functorial isomorphism of $\mathcal{O}_{X'}$-modules*

$$D' : F_*\omega_X^{1-p} \to \mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$$

with the following properties

(1) *Let P be a closed point on X, and let $\mathcal{O}_P$ denote the regular local ring of P in X. Choose a system of regular parameters $x_1, \dots, x_N$ in $\mathcal{O}_P$. If $\alpha = (\alpha_1, \dots, \alpha_N)$ is an N-tuple of rational numbers, we use the notation*

$$x^\alpha = \begin{cases} x_1^{\alpha_1} \dots x_N^{\alpha_N} & \text{if } \alpha_i \in \mathbb{N} \text{ for all } i, \\ 0 & \text{else.} \end{cases}$$

Then D' is locally given and determined by

$$D'(x^\alpha/(dx)^{p-1})(x^\beta) = x^{(\alpha+\beta+1-p)/p}.$$

where $dx = dx_1 \wedge \cdots \wedge dx_N$.

(2) The following diagram is commutative

$$\begin{tikzcd} F_*\omega_X^{1-p} \arrow[r, "C"] \arrow[d, "D'"'] & \mathcal{O}_{X'} \\ \mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}). \arrow[ur, "ev"'] & \end{tikzcd}$$

Proof Let D' be the map defined by

$$D'(\tau)(f) = C(f\tau) \ , \ \tau \in F_*\omega_X^{1-p} \ , \ f \in F_*\mathcal{O}_X.$$

In case X is projective (1) is just Proposition 5 in [15]. The essential ingredient in the proof of Proposition 5 in [15], is the existence of the Cartier operator C. As the Cartier operator not only exists for projective varieties, but for smooth varieties in general (as we have seen above), the proof of Proposition 5 in [15] goes through without changes for a general smooth X. This proves (1), and (2) is an easy consequences of the definition of D'. □

Remark 2.2 *In Chapter 1 we saw that the Cartier operator $C : F_*\omega_X^{1-p} \to \mathcal{O}_{X'}$ was a surjective map of $\mathcal{O}_{X'}$- modules. By Proposition 2.1 we therefore conclude that the evaluation map ev is a surjective map of $\mathcal{O}_{X'}$-modules. In particular if X is smooth and affine, then $\mathrm{H}^0(ev)$ is surjective and X must be Frobenius split by Remark 2.1(2).*

Addendum 2.1 *Let $X = \mathrm{Spec}(A)$ be affine variety and assume that $x_1, \ldots, x_N$ are elements in A such that $dx_1, \ldots, dx_N$ is a basis for $\Omega_{A/k}$. With a similar multinomial notation as in Proposition 2.1, we have that D' on global sections satisfies*

$$D'(x^\alpha)(x^\beta(dx)^{1-p}) = x^{(\alpha+\beta+1-p)/p}.$$

Proof By the proof of Proposition 2.1 we know that

$$D'(\tau)(f) = C(f\tau) \ , \ \tau \in F_*\omega_X^{1-p} \ , \ f \in F_*\mathcal{O}_X.$$

Therefore

$$D'(x^\alpha)(x^\beta(dx)^{1-p}) = C(x^{\alpha+\beta}(dx)^{1-p}),$$

and the result follows from Lemma 1.2. □

In Proposition 2.1 above we claimed that D' locally in $\mathcal{O}_P$ was determined by its values on $x^\alpha/(dx)^{p-1}$. This also follows from the following lemma.

Lemma 2.2 *Let (A, m, k) be a regular local ring which is a localization of a finitely generated k algebra, and let $x_1, \ldots, x_N$ be a system of regular parameters. Let further M be an A-module and $\mu : A \to M$ be a A^p-linear map. In other words*

$$\mu(a^p b) = a^p \mu(b) \ , \ a, b \in A.$$

With multinomial notation as in Proposition 2.1 we have

(1) *If* $\mu(x^\alpha) = 0$ *for all* N*-tuples* α*, then* $\mu = 0$.

(2) *If* $\mu(x^\alpha) = x^\alpha \mu(1)$*, then* μ *is* A*-linear.*

Proof By assumption $m = (x_1, \ldots, x_N)$. For every positive integer s, we therefore have

$$m^{Nps} \subseteq (x_1^{ps}, \ldots, x_N^{ps}) \subseteq m^{ps}.$$

Let $\hat{A}$ be the abelian subgroup of A, which is the k-span of the elements x^α. Let a be an element in A, and choose a sequence $a_1, \ldots, a_i, \ldots$ of elements in $\hat{A}$ such that

$$a - a_s \in m^{Nps} \subseteq (x_1^{ps}, \ldots, x_N^{ps}).$$

Then

$$\mu(a - a_s) \in (x_1^{ps}, \ldots, x_N^{ps})M \subseteq m^{ps}M.$$

To prove (1) assume that $\mu(x^\alpha) = 0$ for all N-tuples α. Then

$$\mu(a) = \mu(a - a_s) + \mu(a_s) = \mu(a - a_s) \in m^{ps}M.$$

As $\cap_s m^s M = 0$ by Theorem 8.9 in [14], we conclude that $\mu(a) = 0$ and (1) follows. Now (2) follows by using (1) on the function $\mu'(a) = \mu(a) - a\mu(1)$. $\Box$

We can now prove a stronger version of Proposition 2.1 which concerns the $\mathcal{O}_X$-module $F^!\mathcal{O}_{X'}$.

Corollary 2.1 *There exist an isomorphism of* $\mathcal{O}_X$*-modules*

$$D : \omega_X^{1-p} \to F^!\mathcal{O}_{X'},$$

such that $F_*D = D'$*. If we want to emphasize that* X *is the underlying variety, we will in the following write* D_X *in place of* D*.*

Proof As sheaves of abelian groups we have the following identities

$$F_*\omega_X^{1-p} = \omega_X^{1-p}$$

$$\mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}) = F^!\mathcal{O}_{X'}.$$

Therefore D', of Proposition 2.1, induces an isomorphism of sheaves of abelian groups

$$D : \omega_X^{1-p} \to F^!\mathcal{O}_{X'}.$$

Notice that D is a map between two $\mathcal{O}_X$-modules, and that it is enough to prove that D is $\mathcal{O}_X$-linear. This can be done locally at a point P in X. Choose therefore a system of regular parameters $x_1, \ldots, x_N$ in $\mathcal{O}_P$. By Lemma 2.2 it is enough to show that

$$D(x^\alpha/(dx)^{p-1}) = x^\alpha D(1/(dx)^{p-1}).$$

But by the definition of the $\mathcal{O}_X$-module structure on $F^!\mathcal{O}_{X'}$ this is clearly the case, as D by Proposition 2.1 satisfies

$$D(x^\alpha/(dx)^{p-1})(x^\beta) = x^{(\alpha+\beta+1-p)/p}.$$

$\Box$

Definition 2.3 *Let X be a smooth variety. If s is a global section of ω_X^{1-p} such that $D(s)$ is a Frobenius splitting, then we say that s is a Frobenius splitting.*

Lemma 2.3 *Let X be an irreducible smooth variety and s be a global section of ω_X^{1-p}. Let P be a closed point of X and let $x_1, \dots, x_N$ be a system of regular parameters in the local ring $\mathcal{O}_{X,P}$. Write $s(dx)^{p-1}$ as a power series in $x_1, \cdots, x_N$*

$$s(dx)^{p-1} = \sum_{I=(i_1,\cdots,i_N)} a_I x_1^{i_1} \cdots x_N^{i_N}.$$

Then s is a Frobenius splitting of X if and only if the following holds

(1) $a_{(p-1,\dots,p-1)} = 1$.

(2) *Let $I = (i_1, \dots, i_N)$ be a nonzero vector with non negative integer entries. Then $a_{(p-1+pi_1,\dots,p-1+pi_N)} = 0$.*

Proof By Remark 2.1(2) the element s is a Frobenius splitting of X if and only if $D(s)(1) = 1$. This may be checked locally at the point P in X. Let $\underline{\text{p-1}}$ denote the vector $(p-1, \dots, p-1)$. By Proposition 2.1 we then have

$$D(s)(1) = \sum_I \sqrt[p]{a_{\underline{\text{p-1}}+pI}}\, x^I.$$

This is easily seen to imply the lemma. □

2.3 Criteria for Frobenius splitting

In this section we will state a few ways to check whether a variety X is Frobenius split. Let us first consider the case when X is smooth.

Proposition 2.2 *Let X be a smooth variety over k. Then X is Frobenius split if and only if the Cartier operator $C : F_* \omega_X^{1-p} \to \mathcal{O}_{X'}$ is surjective on global sections. If X is projective it is enough for the Cartier operator to be non-zero on global sections.*

Proof The first claim follows from Proposition 2.1 and Remark 2.1(2). The statement about the case when X is projective follows from the fact that in this case the global sections of $\mathcal{O}_{X'}$ is k. □

In case X is not smooth one may try to find a variety Y and a map $f : Y \to X$, such that Y is Frobenius split and $f_* \mathcal{O}_Y = \mathcal{O}_X$. Taking the direct image by f_* of any Frobenius splitting of Y :

$$s : F_* \mathcal{O}_Y \to \mathcal{O}_{Y'},$$

then gives a Frobenius splitting of X

$$f_* s : F_* \mathcal{O}_X \to \mathcal{O}_{X'}.$$

In most cases one will choose Y to be smooth, as this enables one to use Proposition 2.2 in proving that Y is Frobenius split. Along the same ideas one has the following result.

Proposition 2.3 *([15]) Let $f : Y \to X$ be a morphism of algebraic varieties such that $f_*\mathcal{O}_Y = \mathcal{O}_X$. Then*

(1) If Y is Frobenius split then X is Frobenius split.

(2) Assume that f is proper and that Z is a closed compatibly split subvariety of Y. Then the image $f(Z)$ is compatibly Frobenius split in X. Here we take the reduced scheme structure on $f(Z)$

In view of this the following is useful

Corollary 2.2 *Let X be a normal variety and let Y be an irreducible Frobenius split variety. If $f : Y \to X$ is a surjective birational proper morphism, then X is Frobenius split.*

Proof We claim that $f_*\mathcal{O}_Y = \mathcal{O}_X$. This is seen as follows. First of all we may assume that $X = \mathrm{Spec}(A)$ is affine. Consider the Stein factorizations of f

$$f : Y \to Z \to X.$$

Here $Z = \mathrm{Spec}(B)$, where B is the ring of global regular functions on Y. Then B is finite over A, and as f is birational, B is contained in the quotient field of A. As A is normal we conclude that $B = A$. This proves the claim and the corollary now follows from Proposition 2.3. □

Consider a morphism of varieties $f : Y \to X$. Above we gave conditions under which X was Frobenius split if Y was. Next we will state conditions for the opposite conclusion. This statement only works for smooth varieties.

Definition 2.4 *Let $f : Y \to X$ be a map of smooth varieties such that $f_*\mathcal{O}_Y = \mathcal{O}_X$. We define D_f to be the map*

$$D_f : f_*\omega_Y^{1-p} \to \omega_X^{1-p}.$$

which is given by the identification in Corollary 2.1 and the natural map

$$\begin{gathered} f_*F^!\mathcal{O}_{Y'} \to F^!\mathcal{O}_{X'} \\ (\phi : F_*\mathcal{O}_Y \to \mathcal{O}_{Y'}) \mapsto (f_*\phi : F_*\mathcal{O}_X \to \mathcal{O}_{X'}). \end{gathered}$$

Proposition 2.4 *Let $f : Y \to X$ be a morphism of smooth varieties such that $f_*\mathcal{O}_Y = \mathcal{O}_X$. A global section s of ω_Y^{1-p} is a Frobenius splitting of Y if and only if $D_f(s)$ is a Frobenius splitting of X. In particular if X is Frobenius split and D_f is surjective on global sections, then Y is Frobenius split.*

Proof As sheaves of abelian groups we have the following commutative diagram

$$\begin{tikzcd} f_*\omega_Y^{1-p} \arrow[r, "f_*D_Y"] \arrow[d, "D_f"'] & f_*F^!\mathcal{O}_{Y'} = f_*\mathcal{H}om_{\mathcal{O}_{Y'}}(F_*\mathcal{O}_Y, \mathcal{O}_{Y'}) \arrow[r, "ev"] \arrow[d, "f_*"'] & f_*\mathcal{O}_Y \arrow[d, "\cong"'] \\ \omega_X^{1-p} \arrow[r, "D_X"] & F^!\mathcal{O}_{X'} = \mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}) \arrow[r, "ev"] & \mathcal{O}_X. \end{tikzcd}$$

And By Remark 2.1(2) we know that a global section s of say ω_Y^{1-p} is a Frobenius splitting exactly when $ev(D_y(s)) = 1$. This implies the result. □

2.4 Applications of Frobenius splitting

For completeness we will in this section state a few consequences of Frobenius splitting.

Proposition 2.5 *[15] Let X be a projective Frobenius split variety, and let L be an ample line bundle on X. Then*

(1) $H^i(X, L) = 0$ when $i > 0$.

(2) If X is smooth then Kodaira vanishing holds

$$H^i(X, L^{-1}) = 0, \ i < \dim(X).$$

(3) If Y is compatibly Frobenius split in X, then the restriction map

$$H^0(X, L) \to H^0(Y, L)$$

is surjective.

Proof To see the structure of the arguments, let us prove the first statement. As X is Frobenius split we have an injective map

$$L' \to L' \otimes F_* \mathcal{O}_X$$

which splits. By the projection formula and Lemma 1.1 we have that $L' \otimes F_* \mathcal{O}_X \simeq F_* L^p$. As F is an affine morphism, $H^i(X', L')$ therefore injects into $H^i(X', (L')^p)$. Iterating we see that $H^i(X', L')$ injects into $H^i(X', (L')^{p^r})$ for every $r > 0$, but for r large this is zero. □

A definition related to Frobenius splitting is the following taken from [18].

Definition 2.5 *A separated variety X is diagonal Frobenius split if the diagonal is compatibly Frobenius split in $X \times_{\mathrm{Spec}(k)} X$.*

Corollary 2.3 *[18] Assume that X is a diagonal Frobenius split normal projective variety and L is very ample. Then X is projectively normal with respect to L.*

Proof Use statement (3) in Proposition 2.5. □

Chapter 3

Frobenius splitting of projective bundles

In this chapter we have the following setup. Let X denote an N-dimensional smooth variety over an algebraically closed field k of characteristic $p > 0$, and $\mathcal{V}$ be a locally free sheaf of rank n on X. The projective bundle variety corresponding to $\mathcal{V}$ is by definition

$$Y = \mathbb{P}(\mathcal{V}) = \mathbf{Proj}(\mathrm{S}(\mathcal{V})), \ \ \mathrm{S}(\mathcal{V}) = \bigoplus_{l \geq 0} \mathrm{S}^l \, \mathcal{V}.$$

The projection map from Y to X is denoted by $\pi : Y \to X$. In this section we will examine the map $D_\pi : \pi_* \omega_Y^{1-p} \to \omega_X^{1-p}$, which was defined in Definition 2.4. It turns out that D_π is a surjective map of sheaves, and that we locally can describe it quite explicitly.

Lemma 3.1 *The canonical sheaf on Y is naturally isomorphic to*

$$\omega_Y \backsimeq \pi^*(\omega_X \otimes \det(\mathcal{V})) \otimes \mathcal{O}_Y(-n),$$

where $det(\mathcal{V})$ is the top wedge product of $\mathcal{V}$.

Proof As π is a smooth map, we first of all have the following fundamental short exact sequence of sheaves on Y

$$0 \to \pi^*(\Omega_X) \to \Omega_Y \to \Omega_{Y/X} \to 0.$$

Taking top wedge product this gives us the expression $\omega_Y \backsimeq \pi^* \omega_X \otimes \omega_{Y/X}$. Now $\omega_{Y/X}$ can be found from the following short exact sequence (see [8] Exercise III 8.4)

$$0 \to \Omega_{Y/X} \to (\pi^*(\mathcal{V}))(-1) \to \mathcal{O}_Y \to 0,$$

which yields $\omega_{Y/X} \backsimeq \pi^*(det(\mathcal{V})) \otimes \mathcal{O}_Y(-n)$. □

Lemma 3.2 *Let $X = \mathrm{Spec}(A)$ be an affine variety such that $\Omega_{A/k}$ is a free A-module with basis $da_1, \ldots, da_N$, and let $\mathcal{V}$ be a free $\mathcal{O}_X$-module of rank n with basis $v_1, \ldots, v_n$. Let $Y_i = X \times \mathrm{Spec}(k[v_1/v_i, \ldots, v_n/v_i])$ be one of the standard open affine coverings of $Y = \mathbb{P}(\mathcal{V})$. Then the restriction to Y_i of the functorial map of Lemma 3.1 is given by*

(1) *The dualizing sheaf ω_{Y_i} is free with basis $(\bigwedge_j da_j) \wedge (\bigwedge_{j\neq i} d(v_j/v_i))$, where the wedge product is assumed to be chosen in order of increasing j. In the following the same assumption is true for the order of all wedge products.*

(2) *The sheaf $\pi^*(\omega_X)$ is free with basis $\bigwedge_j da_j$.*

(3) *The sheaf $\pi^*(\det(\mathcal{V}))$ is free with basis $\bigwedge_j v_j$.*

(4) *The sheaf $\mathcal{O}_{Y_i}(-n)$ is free with basis v_i^{-n}.*

(5) *The map is given by*

$$(\bigwedge_j da_j) \wedge (\bigwedge_{j\neq i} d(v_j/v_i)) \mapsto (-1)^{n-i}(\bigwedge_j da_j \otimes \bigwedge_j v_j \otimes v_i^{-n}).$$

Proof The proof is simple but troublesome. We have to track down explicitly the isomorphism constructed in Lemma 3.1. First notice that $\omega_{Y/X}$ is free with basis $\bigwedge_{j\neq i} d(v_j/v_i)$. The isomorphism $\omega_Y \backsimeq \pi^*(\omega_X) \otimes \omega_{Y/X}$ is then easily seen to be given by

$$(\bigwedge_j da_j) \wedge (\bigwedge_{j\neq i} d(v_j/v_i)) \mapsto \bigwedge_j da_j \otimes \bigwedge_{j\neq i} d(v_j/v_i).$$

It is a bit more difficult to find out how the isomorphism $\omega_Y \backsimeq \mathcal{O}_Y(-n) \otimes \omega_{Y/X}$ looks. For this we have to describe the short exact sequence

$$0 \to \Omega_{Y/X}(Y_i) \to (\pi^*(\mathcal{V}) \otimes \mathcal{O}_Y(-1))(Y_i) \to \mathcal{O}_Y(Y_i) \to 0.$$

As $\pi^*(\mathcal{V})$ is free with basis $v_1, \dots, v_n$ this is given by

$$0 \to \bigoplus_{j\neq i} \mathcal{O}_Y(Y_i) d(v_j/v_i) \xrightarrow{\iota} \bigoplus_j \mathcal{O}_Y(Y_i)(v_j \otimes v_i^{-1}) \xrightarrow{p} \mathcal{O}_Y(Y_i) \to 0$$

where $\iota(d(v_j/v_i)) = ((v_j \otimes v_i^{-1}) - v_j/v_i(v_i \otimes v_i^{-1}))$ and $p(v_j \otimes v_i^{-1}) = v_j/v_i$. We conclude that the isomorphism between $\omega_{Y/X}$ and $\pi^*(\omega_X) \otimes \mathcal{O}_Y(-n)$ is given by

$$\bigwedge_{j\neq i} d(v_j/v_i) \mapsto (-1)^{n-i}(\bigwedge_j v_j \otimes v_i^{-n})$$

which ends the proof. □

By the projection formula and Lemma 3.1 and as $\pi_*\mathcal{O}_Y((p-1)n) = \mathrm{S}^{(p-1)n}\,\mathcal{V}$, we see that there is a natural isomorphism

$$\omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}^{n(p-1)}\,\mathcal{V} \backsimeq \pi_*\omega_Y^{1-p}.$$

Together with D_π this induces a map

$$\omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}^{(p-1)n}\,\mathcal{V} \to \omega_X^{1-p}.$$

By abuse of notation we will also denote this map by D_π. The good thing about writing D_π in this form is, that we may give a nice local explicit description of it, but first we need the following observation.

Lemma 3.3 *Notation as in Lemma 3.2. The isomorphism*

$$\omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}^{n(p-1)}\,\mathcal{V} \to \pi_* \omega_Y^{1-p}$$

is given by

(1) *The sheaf* $\mathrm{S}^{n(1-p)}\,\mathcal{V}$ *is free with basis consisting of the set monomials* $P(v_1, \ldots, v_n)$ *of degree* $n(p-1)$.

(2) *If* $P(v_1, \ldots, v_n)$ *is a monomial of degree* $n(p-1)$, *then the restriction to* Y_i *of the image of* $(\bigwedge_j da_j)^{1-p} \otimes (\bigwedge_j v_j)^{1-p} \otimes P(v_1, \ldots, v_n)$ *is*

$$P(v_1/v_i, \ldots, v_n/v_i)((\bigwedge_j da_j) \wedge (\bigwedge_{j \neq i} d(v_j/v_i)))^{1-p}.$$

Proof Let $s = (\bigwedge_j da_j)^{1-p} \otimes (\bigwedge_j v_j)^{1-p} \otimes P(v_1, \ldots, v_n)$ be a basis element in $\omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}^{n(p-1)}\,\mathcal{V}$. The corresponding element in $\pi^*(\omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p}) \otimes \mathcal{O}_Y((p-1)n)$ over the affine subset Y_i is given by

$$P(v_1/v_i, \ldots, v_n/v_i)((\bigwedge_j da_j)^{1-p} \otimes (\bigwedge_j v_j)^{1-p} \otimes v_i^{n(p-1)}).$$

We conclude by Lemma 3.2. □

By Proposition 2.4 we finally arrive at the following result.

Theorem 3.1 *Let* X *be a smooth variety over* k, *and let* $Y = \mathbb{P}(\mathcal{V})$ *be a projective bundle over* X *with projection map* π *onto* X. *There exist a map of* $\mathcal{O}_X$*-modules*

$$D_\pi : \omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}^{n(p-1)}\,\mathcal{V} \to \omega_X^{1-p}$$

with the following properties

(1) *Let* $U = \operatorname{Spec}(A)$ *be an open affine subset of* X *such that* $\mathcal{V}_{|U}$ *is free. Choose a basis* $v_1, \ldots, v_n$ *for* $\mathcal{V}$ *over* U. *Then* D_π *over* U

$$D_\pi(U) : \omega_X^{1-p}(U) \otimes \det(\mathcal{V})^{1-p}(U) \otimes (\mathrm{S}^{n(p-1)}\,\mathcal{V})(U) \to \omega_X^{1-p}(U)$$

is given in the following way

$$\eta \otimes (\bigwedge_j v_j)^{1-p} \otimes P(v_1, \ldots, v_n) \mapsto P_{p-1}\eta \quad , \ \eta \in \omega_X^{1-p}(U),$$

where $P(v_1, \ldots, v_n)$ *is a polynomial of homogeneous degree* $n(p-1)$ *in* $v_1, \ldots, v_n$ *with coefficients in* k. *Furthermore,* P_{p-1} *denotes the coefficient of* $(v_1 \ldots v_m)^{p-1}$ *in* $P(v_1, \ldots, v_n)$.

(2) D_π *is surjective.*

(3) Y *is Frobenius split if and only if the image of* $H^0(D_\pi)$ *contains a Frobenius splitting of* X.

(4) *If* X *is Frobenius split and the map on global sections*

$$\mathrm{H}^0(D_\pi) : \mathrm{H}^0(\omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}^{n(p-1)}\,\mathcal{V}) \to \mathrm{H}^0(\omega_X^{1-p})$$

is surjective, then Y *is Frobenius split.*

Proof The map D_π in the theorem is of course the one which we have already defined above. Therefore if we can prove the first statement then (3) and (4) will follow from Proposition 2.4, and (2) will follow from the description of D_π in (1). So let us concentrate on proving (1). We may assume that $X = \mathrm{Spec}(A)$ satisfies that $\Omega_{A/k}$ is free with basis $da_1, \ldots, da_N$. Let $P(v_1, \ldots, v_n)$ be a monomial of degree $(p-1)n$, and let

$$s = (\bigwedge_j da_j)^{1-p} \otimes (\bigwedge_j v_j)^{1-p} \otimes P(v_1, \ldots, v_n).$$

The restriction $\tilde{s}_i$, to the affine subset $X \times \mathrm{Spec}(k[v_1/v_i, \ldots, v_n/v_i])$, of the corresponding element $\tilde{s}$ in $\pi_* \omega_Y^{(1-p)}$, is by Lemma 3.3 given by

$$P(v_1/v_i, \ldots, v_n/v_i)((\bigwedge_j da_j) \wedge (\bigwedge_{j \neq i} d(v_j/v_i)))^{1-p}.$$

By Corollary 2.1 the element $\tilde{s}$ correspond to a $\mathcal{O}_{Y'}$-linear map : $F_* \mathcal{O}_Y \to \mathcal{O}_{Y'}$, which is given by Addendum 2.1. If we take direct image of this map via π_* we get an $\mathcal{O}_{X'}$-linear map : $F_* \mathcal{O}_X \to \mathcal{O}_{X'}$, which by Addendum 2.1 and the local description of $\tilde{s}$ above, is seen to be given by

$$a_1^{\alpha_1} \ldots a_N^{\alpha_N} \mapsto \begin{cases} a^{(\alpha+1-p)/p} \text{ if } P(v_1, \ldots, v_n) = (v_1 \ldots v_n)^{p-1} \ , \\ 0 \text{ else.} \end{cases}$$

The image of this map by the isomorphism in Corollary 2.1, is by definition the image $D_\pi(s)$. This ends the proof. □

Remark 3.1 *If we tensorise the map D_π in Theorem 3.1 above with ω_X^{p-1}, we get a map*

$$\det(\mathcal{V})^{1-p} \otimes \mathrm{S}^{n(p-1)} \mathcal{V} \to \mathcal{O}_X. \tag{3.1}$$

With the same notation as in Theorem 3.1 this is locally described by

$$(\bigwedge_j v_j)^{1-p} \otimes P(v_1, \ldots, v_n) \mapsto P_{p-1}.$$

This map may also be described by using a relative Cartier operator. In fact,

$$\det(\mathcal{V})^{1-p} \otimes \mathrm{S}^{n(p-1)} \mathcal{V} \simeq \pi_* \omega_{Y/X}^{1-p},$$

and under this isomorphism the map above correspond to the direct image (by the map $Y'_X \to X$) of the relative Cartier operator $(F_{Y/X})_* \omega_{Y/X}^{1-p} \to \mathcal{O}_{Y'_X}$. Here the scheme Y'_X is defined relative to X in the same way as Y' was defined relative to Spec(k)*. If the map (3.1) on global sections is surjective, one may think of the map $\pi : Y \to X$ as being Frobenius split. Notice also that the relative Cartier operator makes sense even when X is not smooth. This proves that the map above can been defined for every locally free sheaf $\mathcal{V}$ on any variety X. As we will not need this we will not go into details.*

Example 3.1 *The projective cotangent bundle on X is by definition the projective bundle corresponding to the tangent bundle T_X. In this case D_π looks especially nice*

$$D_\pi : \mathrm{S}^{N(p-1)} T_X \to \omega_X^{1-p}.$$

Example 3.2 *Let $X = \mathrm{Spec}(A)$ be a smooth affine variety and $Y = \mathbb{P}(\mathcal{V})$ be a projective bundle over X. By Theorem 3.1 the $\mathcal{O}_X$-linear map D_π is surjective. As X is affine $\mathrm{H}^0(D_\pi)$ is therefore also surjective. By Remark 2.2 any smooth affine variety is Frobenius split, so we conclude that Y is Frobenius split. This shows that any projective bundle over an affine variety is Frobenius split, and that any Frobenius splitting of the affine base can be lifted to a Frobenius splitting of the bundle.*

Chapter 4

Frobenius splitting of vector bundles

In this chapter we have the following setup. Let X denote an N-dimensional smooth variety over an algebraically closed field k of characteristic $p > 0$, and $\mathcal{V}$ be a locally free sheaf of rank n on X. The vector bundle variety corresponding to V is by definition

$$Y = \mathbb{A}(\mathcal{V}) = \mathbf{Spec}(\mathrm{S}(\mathcal{V})).$$

The projection map from Y to X is denoted by $\pi : Y \to X$. In this section we will examine a map similar to the map D_π in the previous chapter. It turns out that the Frobenius splitting of $\mathbb{A}(\mathcal{V})$ is very much related to the Frobenius splitting of $\mathbb{P}(\mathcal{V})$.

Lemma 4.1 *The canonical sheaf ω_Y on Y is naturally isomorphic to*

$$\omega_Y \simeq \pi^*(\omega_X \otimes \det(\mathcal{V})).$$

Proof Consider the projective bundle $\pi_Z : Z = \mathbb{P}(\mathcal{V} \oplus \mathcal{O}_X) \to X$. By Lemma 3.1 the canonical sheaf on Z is isomorphic to $\pi_Z^*(\omega_X \otimes \det(\mathcal{V})) \otimes \mathcal{O}_Z(-n-1)$. The vector bundle Y can be regarded as an open subset of Z in such a way that π is compatible with π_Z, and such that the restriction of $\mathcal{O}_Z(-n-1)$ to Y is trivial. This ends the proof. □

Let $\mathcal{W}$ be an $\mathcal{O}_X$-module. By the projection formula we have that

$$\pi_*(\pi^*(\mathcal{W})) = \mathcal{W} \otimes \pi_*\mathcal{O}_X = \mathcal{W} \otimes \mathrm{S}(\mathcal{V}).$$

As $\mathrm{S}(\mathcal{V})$ is a graded $\mathcal{O}_X$-module the sheaf $\pi_*(\pi^*(\mathcal{W}))$ becomes a graded $\mathcal{O}_X$-module in a naturally way. When we in the following will speak about the m-graded part of an element in $\pi_*(\pi^*(\mathcal{W}))$, it will be with respect to this grading. As the global sections of $\pi_*(\pi^*(\mathcal{W}))$ and $\pi^*(\mathcal{W})$ coincide, we will also speak about the graded parts of the global sections of $\pi^*(\mathcal{W})$. In particular, we see by Lemma 4.1 that the global sections of ω_Y are graded. What will be more important to us, is that the global sections of $\pi_*\omega_Y^{1-p}$ are graded. In fact by Lemma 4.1 we have the following isomorphism

$$\pi_*\omega_Y^{1-p} \simeq \omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}(\mathcal{V}).$$

Inspired by Theorem 3.1 in the preceding chapter we define

Definition 4.1 *With notation as above we define*

$$G_m : \omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}(\mathcal{V}) \to \omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes S^m(\mathcal{V})$$

to be the projection onto the m-graded piece. The inclusion map in the opposite direction will be denoted by I_m.

Our main result in this chapter is the following theorem.

Theorem 4.1 *Let*

$$D_\pi : \omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes S^{n(p-1)}(\mathcal{V}) \to \omega_X^{1-p}$$

be the map defined in Theorem 3.1. Let s denote a global section of the sheaf $\omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes S^{n(p-1)}(\mathcal{V})$ and $\tilde{s}$ denote a global section of the sheaf $\omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}(\mathcal{V})$. Then

(1) If $\tilde{s}$ (considered as a global section of ω_Y^{1-p}) is a Frobenius splitting of Y, then $\mathrm{H}^0(D_\pi)(G_{n(p-1)}(\tilde{s}))$ is a Frobenius splitting of X.

(2) Assume that $s = G_{n(p-1)}(\tilde{s})$. If $\tilde{s}$ has no graded terms of degree strictly larger than $(n+1)(p-1)$, then $\tilde{s}$ is a Frobenius splitting of $Y = \mathbb{A}(\mathcal{V})$ if and only if $H^0(D_\pi)(s)$ is a Frobenius splitting of X.

(3) $H^0(D_\pi)(s)$ is a Frobenius splitting of X if and only if $H^0(I_{n(p-1)})(s)$ (regarded as a global section of ω_Y^{1-p}) is a Frobenius splitting of $Y = \mathbb{A}(\mathcal{V})$.

(4) The vector bundle $\mathbb{A}(\mathcal{V})$ is Frobenius split if and only if the corresponding projective bundle $\mathbb{P}(\mathcal{V})$ is Frobenius split.

Proof Let us first concentrate on proving (1) and (2). By Lemma 2.1 we may assume that $X = \mathrm{Spec}(A)$ is irreducible and affine, and that there exists $x_1, \dots, x_N$ in A such that $dx_1, \dots, dx_N$ is a basis for $\Omega_{A/k}$. Let P be a closed point of X. By adding constants to $x_1, \dots, x_N$ we may assume that every x_i vanishes at P. We may also assume that $\mathcal{V}$ is a free $\mathcal{O}_X$-module with basis $v_1, \dots, v_n$. Then $Y = X \times \mathrm{Spec}\, k[v_1, \dots, v_n]$ and ω_Y is a free $\mathcal{O}_Y$-module with generator

$$dx \wedge dv = dx_1 \wedge \cdots \wedge dx_N \wedge dv_1 \wedge \cdots \wedge dv_n.$$

Let $P(v_1, \dots, v_n) \in A[v_1, \dots, v_n]$ be the global regular function on Y such that $\tilde{s} = P(v_1, \dots, v_n)(dx \wedge dv)^{1-p}$. If $I = (i_1, \dots, i_n)$ is a vector with non negative integer entries we let v^I denote the element $v_1^{i_1} \cdots v_n^{i_n}$. Then we may write

$$P(v_1, \dots, v_n) = \sum_I P_I v^I \ , \ P_I \in A.$$

By Theorem 3.1(1) we see that $\mathrm{H}^0(D_\pi)(G_{n(p-1)}(\tilde{s}))$ is equal to $P_{\underline{\text{p-1}}}(dx)^{1-p}$. We would now like to use the criteria in Lemma 2.3 to conclude (1). Let P be the point chosen above. Then $x_1, \dots, x_N$ is a regular system of parameters in the local ring $\mathcal{O}_{X,P}$. Let P' be the element $(P, 0)$ in Y. Then $x_1, \dots, x_N, v_1, \dots, v_n$ is a system of regular parameters in the local ring $\mathcal{O}_{Y,P'}$. If $J = (j_1, \dots, j_N)$ is a vector with non negative integer entries we write x^J for the element $x_1^{j_1} \dots x_N^{j_N}$.

Consider now the power series expression of $P(v_1, \ldots, v_n)$ in the chosen system of regular parameters at P'

$$P(v_1, \ldots, v_n) = \sum_{I,J} a_{I,J} v^I x^J \ , \ a_{I,J} \in k.$$

Then $P_{\underline{\text{p-1}}}$ written as a power series in $x_1, \ldots, x_N$ at P is equal to

$$P_{\underline{\text{p-1}}} = \sum_J a_{\underline{\text{p-1}},J} x^J.$$

Assume that $\tilde{s}$ is a Frobenius splitting of Y. Then by Lemma 2.3 we know that $a_{\underline{\text{p-1}},\underline{\text{p-1}}} = 1$. Furthermore we in particular know that $a_{\underline{\text{p-1}},\underline{\text{p-1}}+pJ'} = 0$ if J' is non zero. This implies by Lemma 2.3 that $\mathrm{H}^0(D_\pi)(G_{n(p-1)}(\tilde{s})) = P_{\underline{\text{p-1}}}(dx)^{1-p}$ is a Frobenius splitting of X, which ends the proof of (1).

Let us turn to the proof of (2). We have already proved one of the inclusions. Assume therefore that $\mathrm{H}^0(D_\pi)(s) = P_{\underline{\text{p-1}}}(dx)^{1-p}$ is a Frobenius splitting of X. Then by Lemma 2.3 and the above we know that

$$a_{\underline{\text{p-1}},\underline{\text{p-1}}} = 1 \ \text{ and } \ a_{\underline{\text{p-1}},\underline{\text{p-1}}+pJ'} = 0 \ , \ J' \neq 0.$$

On the other hand $\tilde{s}$ does (by assumption) not contain terms of degree strictly larger than $(p-1)(n+1)$. Therefore $a_{\underline{\text{p-1}}+pI',J} = 0$ if I' is non zero. By Lemma 2.3 we therefore conclude that $\tilde{s}$ is a Frobenius splitting of Y.

With $\tilde{s} = \mathrm{H}^0(I_{n(p-1)})$ statement (3) now follows from (2). Finally (4) follows from (3), (1) and Theorem 3.1(3). $\square$

Example 4.1 *By Theorem 4.1 and Lemma 3.2 we see that any vector bundle over an affine smooth variety is Frobenius split.*

Chapter 5

Flag varieties

In this chapter we will introduce the varieties which will be of special interest to us. These are the flag varieties. Other related varieties such as the Schubert varieties, will also be introduced here. The varieties we consider will be defined over a fixed algebraically closed field k of (unless otherwise mentioned) arbitrary characteristic.

5.1 Basic notation

Let G be a semisimple simply connected linear algebraic group. Let B be a Borel subgroup, and T be a maximal torus contained in B. The Borel group opposite to B with respect to T is denoted by B^+. By U and U^+ we denote the unipotent radicals of B and B^+ respectively. The roots with respect to T is denoted by R. To each root α in R we fix non zero group root homomorphism $X_\alpha : \mathbb{A}^1 \to G$ such that

$$tX_\alpha(s)t^{-1} = X_\alpha(\alpha(t)s).$$

The set of roots α where the image of X_α is contained in B will be denoted by R^-, and will be called the negative roots. The set of positive roots is by definition the complement to R^- in R, and will be denoted by R^+. We fix the notation $\alpha_1, \ldots, \alpha_N$ for the elements in R^+. The simple roots Δ is the set of positive roots which cannot be written as a sum of other roots. We will assume that indexes is chosen such that $\alpha_1, \ldots, \alpha_l$ is the set of simple roots. By $W = N_G(T)/T$ we denote the Weyl group corresponding to T. Then W is finite and to each root α in R there correspond an reflection s_α in W. Furthermore W is generated by the set of simple reflections $s_{\alpha_1}, \ldots, s_{\alpha_l}$. For a given element w in W we may therefore write

$$w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_{l'}} \ , \ \beta_i \in \Delta.$$

If l' is minimal we say that w has length $l(w) := l'$. There is a unique element of longest length in W. It is denoted by w_0 and has length $l(w_0) = N$.

The set of (algebraic) group homomorphisms form T to k^* is called the set of weights. The set of weights is denoted by $\chi(T)$, and has a group structure isomorphic to $\mathbb{Z}^l$. Let $E = \chi(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Then W acts naturally on E, and as W

is finite we may choose a positive definite W-invariant symmetric bilinear form $(,)$ on E. If $\beta \in E$ and α is a roots we define

$$< \beta, \alpha^\vee > := 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}.$$

If β is an element in $\chi(T)$ then $< \beta, \alpha^\vee >$ is an integer. In fact, as G is assumed to be simply connected, $\chi(T)$ is exactly the elements β in E such that $< \beta, \alpha^\vee >$ is integral. The elements β in $\chi(T)$ such $< \beta, \alpha_i^\vee > \geq 0$ for $i = 1, 2, \ldots, N$ is called the dominant weights. If $< \beta, \alpha_i^\vee > > 0$ for $i = 1, \ldots, N$, then β is called strictly dominant. The elements $\lambda_1, \ldots, \lambda_l$ in $\chi(T)$ such that ($\delta_{i,j}$ is Kronecker's delta) :

$$< \lambda_i, \alpha_j^\vee > = \delta_{i,j} \ , 0 \leq i, j \leq l,$$

is called the fundamental dominant weights. By ρ we denote the sum of the fundamental dominant weights. Then ρ is also equal to half the sum of the positive roots.

5.2 Schubert varieties

With notation as in the section above we let X denote the full flag variety G/B. Then X is a smooth projective variety of dimension N. The Borel subgroup B acts by left multiplication on X. The orbits of this action is in one to one correspondence with the elements in the Weyl group W by

$$w \in W \mapsto C(w) := (Bw)B \subseteq G/B.$$

Every orbit $C(w)$ is a locally closed subvariety of X isomorphic to $\mathbb{A}^{l(w)}$. In particular, the "big cell" $C(w_0)$ is an open subvariety of X isomorphic to U by the map

$$U \to C(w_0).$$
$$u \mapsto uw_0B.$$

This implies that the canonical map $\pi_G : G \to X$ is a locally trivial B-bundle. The closures of the orbits $C(w)$ in X is called the Schubert varieties and will be denoted by X_w. In general Schubert varieties are not smooth.

It is clear that $X = X_{w_0}$, as w_0 has length N. The Schubert varieties of codimension 1 inside X is given by the Weyl group elements $w_0 s_{\alpha_1}, \ldots, w_0 s_{\alpha_l}$. Every Schubert variety of codimension $d \geq 1$ is a component of an intersection of Schubert varieties of codimension $\geq d - 1$.

5.3 Induced representations and vector bundles

Let H be an linear algebraic group, and V be a finite dimensional vectorspace over k. A (finite dimensional) rational representation of H on V is a algebraic group homomorphism

$$\varphi : H \to \mathrm{GL}(V).$$

More generally, an arbitrary (i.e. not necessarily finite dimensional) vectorspace V is said to be a rational representation of H, if V is a (compatible) union of finite dimensional rational representations.

Example 5.1 *Let V be a rational representation of T. For every T-character $\lambda \in \chi(T)$ we define the weight space corresponding to λ to be*

$$V_\lambda = \{v \in V \mid tv = \lambda(t)v,\ t \in T\}.$$

If $V_\lambda \neq 0$ we call λ a weight of the representation V. An element in V_λ is said to be T semi-invariant. As T is a torus

$$V = \bigoplus_{\lambda \in \chi(T)} V_\lambda.$$

In other words, every element in V can be written as a unique sum of semi-invariant elements.

Assume that V is a rational representation of H, and that K is a linear algebraic group containing H as a closed subgroup. Then V induces a rational representation $\mathrm{Ind}_H^K(V)$ of K by

$$\mathrm{Ind}_H^K(V) = \{f : K \to V | f(gh) = h^{-1}f(g)\ , h \in H, g \in K\}.$$

In this definition f is implicit assumed to be a morphism of varieties. The action of K on $\mathrm{Ind}_H^K(V)$ is defined by

$$(gf)(\tilde{g}) = f(g^{-1}\tilde{g})\ , g, \tilde{g} \in G, f \in \mathrm{Ind}_H^K(V).$$

This induced module has a very nice functorial property called Frobenius reciprocity

Proposition 5.1 *Let H, K and V be as above, and let W be a representation of K. Then the map*

$$\mathrm{Hom}_K(W, Ind_H^K(V)) \to \mathrm{Hom}_H(W, V),$$

given by

$$\phi \mapsto (w \mapsto \phi(w)(1)),$$

is an isomorphism.

5.3.1 Homogeneous Vector bundles

Remember that a vector bundle of relative dimension n over X is a variety Y and a map $\pi : Y \to X$ such that the following holds

(1) There exist an cover $\{U_i\}$ of X such that $\pi^{-1}(U_i)$ as a variety is isomorphic to $U_i \times \mathbb{A}^n$.

(2) The transition maps between the identifications $\pi^{-1}(U_i) \simeq U_i \times \mathbb{A}^n$ are linear.

If G furthermore acts on Y as a vector bundle over X, and if this action under π is compatible with the natural G-action on X, we say that $\pi : Y \to X$ is a homogeneous vector bundle.

Example 5.2 *Let V be a rational B representation. Then B acts on the variety $G \times V$ by*

$$b(g,v) = (gb^{-1}, bv) \ , b \in B, g \in G, v \in V.$$

As remarked above the canonical map $\pi_G : G \to G/B$ is locally trivial. Therefore the quotient $(G \times V)/B$ exist as a variety, and we will denote this by $G \times^B V$. Notice that there is a natural map

$$\pi : G \times^B V \to G/B.$$

$$(g,b) \mapsto gB.$$

As V is a rational representation of B it is clear that $\pi : G \times^B V \to X$ is a vector bundle over X. Let now G act on $G \times^B V$ by

$$g(\tilde{g}, v) = (g\tilde{g}, v) \ , \ g, \tilde{g} \in G, v \in V.$$

Then $\pi : G \times^B V \to X$ is a homogeneous vector bundle over X.

This example shows that given an n-dimensional B-representation V, we may construct a homogeneous vector bundle $\pi : G \times^B V \to X$ of relative dimension n. On the other hand if $\pi : Y \to X$ is a homogeneous vector bundle of relative dimension n, then the fiber $\pi^{-1}(eB)$ is clearly an n-dimensional rational B-representation. In this way we get a one to one correspondence between n dimensional rational B-representations and homogeneous vector bundles over X of relative dimension n.

5.3.2 Locally free sheaves

Let $\pi : Y \to X$ be a vector bundle over X of relative dimension n. By $\mathcal{L}(Y)$ we denote the sheaf of sections of π. In other words, if U is an open subset of X, then

$$\mathcal{L}(Y)(U) = \{ \ s : U \to Y \mid \pi \circ s = Id_U \ \}.$$

Then $\mathcal{L}(Y)$ is a locally free $\mathcal{O}_X$-module of rank n. Notice that with the notation as in Chapter 4 we have that $Y = \mathbb{A}(\mathcal{L}(Y)^*)$, where $\mathcal{L}(Y)^*$ is the dual $\mathcal{O}_X$-module of $\mathcal{L}(Y)$.

In case Y is equal to $G \times^B V$ for an n dimensional rational B-representation, we will also denote $\mathcal{L}(Y)$ by $\mathcal{L}(V)$. Furthermore, we will think of the sections of $\mathcal{L}(V)$ to be

$$\mathcal{L}(V)(U) = \{f : \pi_G^{-1}(U) \to V \mid f(ub) = b^{-1}f(u) \ , u \in \pi_G^{-1}(U), b \in B\}.$$

The two descriptions of the sections of $\mathcal{L}(V)$ is connected in the following way. Let $f : \pi_G^{-1}(U) \to V$ be a map such that

$$f(ub) = b^{-1}f(u), \ u \in \pi_G^{-1}(U), b \in B.$$

Define $\tilde{f} : \pi_G^{-1}(U) \to G \times^B V$ by $\tilde{f}(u) = (u, f(u))$, $u \in \pi_G^{-1}(U)$. Then $\tilde{f}$ factors through U and gives us a section $s : U \to G \times^B V$ to π. It is easy to see that this correspondence is bijective.

Remark 5.1 *If V is a finite dimensional rational B representation, then the global sections of $\mathcal{L}(V)$ is equal to $\mathrm{Ind}_B^G(V)$. This justifies the notation $\mathrm{H}^0(V)$ for $\mathrm{Ind}_B^G(V)$. As G/B is a projective variety, the rational G representation $\mathrm{H}^0(V)$ is finite dimensional.*

5.3.3 Line bundles

A line bundle is a vector bundle of rank 1. Every line bundle over X is isomorphic (as vector bundles) to an unique homogeneous line bundle, and does therefore correspond to a unique 1-dimensional rational B representation, i.e. to the set $\chi(B)$ of algebraic group homomorphism from B to k^*. The elements in $\chi(B)$ are the so called characters of B. As U is unipotent and as $B = UT$ the restriction map $\chi(B) \to \chi(T)$ is an isomorphism. If $\lambda : T \to k^*$ is a T-character we let $\mathcal{L}(\lambda)$ denote the homogeneous line bundle on X given by the identifications above. If we think of λ as a character on B, the global sections of $\mathcal{L}(\lambda)$ is given by

$$\mathrm{H}^0(\lambda) := \mathcal{L}(\lambda)(X) = \{f : G \to k \mid f(gb) = \lambda(b^{-1})f(g),\ b \in B, g \in G\} \subseteq k[G].$$

This set is non zero exactly when λ is dominant. If so, the T representation $\mathrm{H}^0(\lambda)$ has highest weight λ and lowest weight $w_0\lambda$. The weight spaces corresponding to λ and $w_0\lambda$ are both of dimension 1. A line bundle $\mathcal{L}(\lambda)$ is ample if and only if λ is strictly dominant. The line bundles corresponding to dominant weights are generated by global sections.

Example 5.3 *Let $G = \mathrm{SL_n(k)}$, the $n \times n$ matrices with entries in k and determinant 1. Let B be the upper triangular matrices and T be the diagonal matrices. For each $s = 1, \dots, n-1$ let ϵ_s be the element in $\chi(T)$ given by*

$$\epsilon_s(A) = a_{s,s}\ , A = (a_{i,j})_{0 \le i,j \le n} \in T.$$

Then $\alpha_s = \epsilon_{s+1} - \epsilon_s$, $s = 1, \dots, n-1$, is the set of positive roots. The corresponding fundamental dominant weights are given by

$$\lambda_s = -\sum_{i=1}^{s} \epsilon_i.$$

For each $s = 1, \dots, n-1$ define $u_s : G \to k$ to be the function

$$u_s(A) = \mathrm{Det}((a_{i,j})_{0 \le i,j \le s})\ , A = (a_{i,j})_{0 \le i,j \le n} \in G.$$

A small calculation shows that u_s is an element in $\mathrm{H}^0(\lambda_s)$ of weight λ_s. In other words u_s is a generator for the highest weight space in $\mathrm{H}^0(\lambda_s)$. The product $u = u_1 \cdots u_{n-1}$ is therefore a generator for the highest weight space in $\mathrm{H}^0(\rho)$.

For each $s = 1, \dots, n-1$ let $l_s : G \to k$ be the function

$$l_s(A) = \mathrm{Det}((a_{i,j})_{n-s+1 \le i \le n, 0 \le j \le s})\ , A = (a_{i,j})_{0 \le i,j \le n} \in G.$$

Again a small calculation shows that l_s is an element in $\mathrm{H}^0(\lambda_s)$ with weight $-\lambda_{n-s} = w_0\lambda_s$. This shows that l_s is a generator for the lowest weight space in $\mathrm{H}^0(\lambda_s)$. Therefore the product $l = l_1 \cdots l_{n-1}$ is a generator for the lowest weight space (with weight $-\rho$) in $\mathrm{H}^0(\rho)$.

Let f be a non zero global section of $\mathcal{L}(\lambda)$. The divisor of zeroes $Z(f)$ is a finite positive linear combination of irreducible codimension 1 subvarieties $D_1, \dots, D_s$ of X. The union of these subvarieties is clearly equal to the image of $\{g \in G \mid f(g) = 0\}$ under π_G.

Assume now that f is a lowest weight vector in $\mathrm{H}^0(\lambda)$. Then f is B semi-invariant, and $\{g \in G \mid f(g) = 0\}$ is thus invariant under left multiplication by B. As B is irreducible we conclude that each of the divisors $D_1, \ldots, D_s$ must be B-invariant under left multiplication. In other words, each of $D_1, \ldots, D_s$ must be a Schubert variety of codimension 1 in X.

Example 5.4 *Let λ be equal to a fundamental dominant weight λ_s, and let f be a non zero lowest weight vector in $\mathrm{H}^0(\lambda)$. Then as we saw above*

$$Z(f) = \sum_{i=1}^{l} a_i X_{w_0 s_{\alpha_i}} \ , a_i \geq 0.$$

Let $\mathcal{L}_i$ denote the line bundle $\mathcal{L}(X_{w_0 s_{\alpha_i}})$. As the set of global sections of $\mathcal{L}_i$ is non zero, there exist (as remarked above) dominant weights ω_i such that $\mathcal{L}_i = \mathcal{L}(\omega_i)$. We conclude that

$$\lambda_s = \sum_i a_i \omega_i \ , a_i \geq 0.$$

But as λ_s was a fundamental dominant weight there exist an integer i such that $a_j = \delta_{i,j}$ (Kronecker' delta). In other words

$$Z(f) = X_{w_0 s_{\alpha_j}}.$$

As the line bundles $\mathcal{L}(\lambda_s)$, $s = 1, \ldots, l$ mutually are non isomorphic, we conclude that the zero divisor of a non zero lowest weight vector v^- in $\mathcal{L}(\rho)$ (remember that ρ was the sum of the fundamental weights) is the sum of the codimension 1 Schubert varieties

$$Z(v^-) = \sum_{i=1}^{l} X_{w_0 s_{\alpha_i}}.$$

By this we also conclude that the zero divisor of a non zero highest weight vector v^+ in $\mathcal{L}(\rho)$ is

$$Z(v^+) = \sum_{i=1}^{l} w_0 X_{w_0 s_{\alpha_i}},$$

i.e. the sum of the opposite codimension 1 Schubert varieties.

5.4 The Steinberg module

In this section we will assume that the characteristic p of the field k is strictly positive. In this situation the B character $(p-1)\rho$ turns out to be very useful. The corresponding induced G-representation $\mathrm{H}^0((p-1)\rho)$ is denoted by St, and is called the Steinberg module. The Steinberg module is known [6] to be a simple and selfdual G-representation. Fix therefore an isomorphism $\gamma : St \to St^*$ of G-representations. Up to a none zero constant there is only one such isomorphism. This follows from the following result.

Lemma 5.1 $\mathrm{Hom}_G(St, St) = k$.

Proof By Frobenius reciprocity we have

$$\mathrm{Hom}_G(St, St) = \mathrm{Hom}_B(St, k_{(p-1)\rho}).$$

But the weight space in St with weight $(p-1)\rho$ is 1-dimensional, so that the rightside of this equation is also of dimension 1. $\square$

Definition 5.1 *Let $<,>$ denote the none zero G-invariant bilinear form on St given by*

$$< v, w >= \gamma(v)(w) \ , v, w \in St.$$

Remark 5.2 *It is clear that there is a one to one correspondence between the set of nonzero G-invariant bilinear forms on St, and the set of nonzero isomorphisms between St and St^*. By the above we therefore conclude that up to nonzero constant, there is only one nonzero G-invariant bilinear form on St.*

The following theorem is due to Andersen [1] and Haboush [7]. In some sense the result implies that flag varieties G/B are Frobenius split in a very strong way.

Theorem 5.1 *Let X denote G/B. Then the natural map*

$$\mathcal{O}_{X'} \otimes_k St \to F_*\mathcal{L}((p-1)\rho),$$

is an isomorphism.

In general it is true that if there exist a line bundle $\mathcal{L}$ on any variety X, such that the vector bundle $F_*\mathcal{L}$ has a line bundle sitting as a direct summand, then X is Frobenius split. The theorem above shows that on G/B there exist a line bundle $\mathcal{L}((p-1)\rho)$, such that $F_*\mathcal{L}((p-1)\rho)$ not only has a line bundle sitting as a direct summand, but it splits completely into a sum of line bundles. This is a strong result, and happens rarely. For toric varieties it is however true for every line bundle, which is the subject in [22]. However we do not know of other examples of varieties X (not even toric varieties), where there exist a line bundle $\mathcal{L}$ such that $F_*\mathcal{L}$ splits into a direct sum of $\mathcal{O}_{X'}$.

Chapter 6

Frobenius splitting of flag varieties

In this chapter we will concentrate on proving that every full flag variety $X = G/B$ is Frobenius split. The material in this chapter is mainly taken from [12]. Unless otherwise mentioned, we will consider G/B to be defined over an algebraically closed field k of strictly positive characteristic.

Recall that by Corollary 2.1 there is an isomorphism of k-vectorspaces

$$\mathrm{H}^0(X, \omega_X^{1-p}) \simeq \mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}). \tag{6.1}$$

As ω_X^{1-p} is a homogeneous line bundle this isomorphism induces a structure on $\mathrm{Hom}_{\mathcal{O}'_X}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$ as a rational G-representation. The next result tells us explicit how this representation is given.

Lemma 6.1 *Let $g \in G$ and s be an element of $\mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$. The translation $g.s$ of s by g, induced by the isomorphism above, is given by*

$$(g.s)(f) = g.(s(g^{-1}.f)) \ , f \in \mathcal{O}_X.$$

Here we think of g as acting on $\mathcal{O}_X$ in the way induced by the action of G on X. More precisely

$$(g.f)(x) = f(g^{-1}x) \ , f \in \mathcal{O}_X, x \in X.$$

Proof For any element h in G let f_h denote the automorphism of X, which is given by left multiplication with h. Let ω be a global section of ω_X^{1-p}. Then $g.\omega$ is the image of ω under the functorial map

$$\omega_X^{1-p} \to (f_{g^{-1}})_*\omega_X^{1-p}.$$

Recall that by the proof the Proposition 2.1 the isomorphism

$$\phi : \mathrm{H}^0(X, \omega_X^{1-p}) \simeq \mathrm{Hom}_{\mathcal{O}'_X}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$$

is given by

$$\phi(\omega)(f) = C(f\omega) \ , f \in \mathcal{O}_X,$$

where $C : F_*\omega_X^{1-p} \to \mathcal{O}_{X'}$ is the Cartier operator on X. Choose ω such that $s = \phi(\omega)$. Then by definition $g.s = \phi(g.\omega)$. Therefore

$$(g.s)(f) = C(f \cdot (g.\omega)) = C(g.((g^{-1}.f) \cdot \omega)) \ , f \in \mathcal{O}_X.$$

Finally using Remark 1.1 on the function $f_{g^{-1}}$ we see that

$$C(g.((g^{-1}.f) \cdot \omega)) = g.C((g^{-1}.f) \cdot \omega) = g.s(g^{-1}.f),$$

which ends the proof. □

Corollary 6.1 *Regard k as a trivial G-representation. Then the evaluation map*

$$ev : \mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}) \to k,$$

given by $ev(s) = s(1)$, is G-equivariant. Notice that this evaluation map is the evaluation map from Chapter 2 taken on global sections.

Proof As G acts trivial on the constant functions 1, this result follows from the Lemma 6.1 above. □

With the description above of the G-representation $\mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$, we will now define a G-equivariant map

$$\varphi : St \otimes St \to \mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}).$$

Let $v \otimes w$ be an element in $St \otimes St$. Then we define $\varphi(v \otimes w)$ in the following way. As v by definition is a global section of $\mathcal{L}((p-1)\rho)$, it correspond to a morphism of $\mathcal{O}_X$-modules :

$$v : \mathcal{O}_X \to \mathcal{L}((p-1)\rho).$$

Using the functor F_* this induces a map

$$F_*v : F_*\mathcal{O}_X \to F_*\mathcal{L}((p-1)\rho).$$

By Theorem 5.1 we know that $St \otimes \mathcal{O}_{X'} \simeq F_*\mathcal{L}((p-1)\rho)$ under the natural multiplication map. By using the function $< *, w >$ on the left factor of $St \otimes \mathcal{O}_{X'}$, we see that w gives us a map

$$< *, w >: F_*\mathcal{L}((p-1)\rho) \to \mathcal{O}_{X'}.$$

We then define $\varphi(v \otimes w) =< *, w > \circ F_*v$. Above we claimed that φ is a G-equivariant map. This is proved now.

Lemma 6.2 *The map φ defined above is G-equivariant.*

Proof Let $v_1, \ldots, v_n$ be a k-basis for St, and let v and w be any two elements in St. By Theorem 5.1 there exist maps $\varphi_i : \mathcal{O}_X \to \mathcal{O}_{X'}$ such that

$$F_*v(f) = \sum_{i=1}^{n} v_i \otimes \varphi_i(f) \ , f \in \mathcal{O}_X.$$

This means in other words that we have the relation

$$fv = \sum_{i=1}^{n} (\varphi_i(f))^p v_i \ ,$$

inside $\mathcal{L}((p-1)\rho)$. Furthermore

$$\varphi(v \otimes w)(f) = \sum_{i=1}^{n} < v_i, w > \varphi_i(f).$$

We will now compare this with $\varphi(gv \otimes gw)$, when g is an element in G. First of all notice that

$$f \cdot (gv) = g((g^{-1}f)v) = g(\sum_{i=1}^{n} (\varphi_i(g^{-1}f))^p v_i).$$

Therefore

$$(F_*(gv))(f) = \sum_{i=1}^{n} gv_i \otimes g\varphi(g^{-1}f).$$

Using that $<,>$ is G-invariant we finally conclude that

$$\begin{aligned} \varphi(gv \otimes gw)(f) &= \sum_{i=1}^{n} < gv_i, gw > (g\varphi_i(g^{-1}f)) \\ &= g(\sum_{i=1}^{n} < v_i, w > \varphi(g^{-1}f)) \\ &= (g\varphi(v \otimes w))(f), \end{aligned}$$

which ends the proof. □

It is an easy exercise to check the relation $\phi(v \otimes w)(1) =< v, w >$. By Remark 2.1 we therefore conclude the following result.

Proposition 6.1 *Consider the G-equivariant map defined above*

$$\varphi : St \otimes St \to \mathrm{Hom}_{\mathcal{O}_{X'}}(F_* \mathcal{O}_X, \mathcal{O}_{X'}).$$

Then $\varphi(v \otimes w)$ is a Frobenius splitting (up to a nonzero constant) of $X = G/B$ if and only if $< v, w > \neq 0$. In particular, as $<,>$ is nonzero X is Frobenius split.

Remark 6.1 *A Frobenius splitting of X of the form $\varphi(v \otimes w)$ factorises, by definition, through $F_*\mathcal{L}((p-1)\rho)$. If D is the zero divisor of the element v in St, then this amounts to saying that $\varphi(v \otimes w)$ is a Frobenius D-splitting, in the sense of Ramanan and Ramanathan [17].*

As a first application of Frobenius splitting of G/B let us note the following.

Proposition 6.2 *(Kempf vanishing) Let λ be a dominant weight. Then*

$$\mathrm{H}^i(G/B, \mathcal{L}(\lambda)) = 0 \ , i > 0.$$

Proof Strictly speaking this does not follow from the Frobenius splitting alone, but from the stronger form of Frobenius splitting stated in Remark 6.1. An easy exercise shows that $(p-1)\rho + p\lambda$ is a strictly dominant weight. Therefore $\mathcal{L}((p-1)\rho) \otimes \mathcal{L}(p\lambda)$ is ample and we conclude by Proposition 2.5, that this line bundle has vanishing higher cohomology. Using the stronger Frobenius splitting stated in Remark 6.1, the desired result follows by using Proposition 1.12 in [18]. □

6.1 Compatibly Frobenius splitting

We now turn to problem of compatibly splitting subvarieties of $X = G/B$. Eventually we will see, that the zero scheme of every element in $\mathrm{H}^0(\rho)$ is compatibly split inside X. In particular, this will imply that every Schubert variety in X is compatibly Frobenius split. It also follows the diagonal Δ_X inside $X \times X$ is compatibly split, proving that X i diagonal split.

An easy argument, which can be found on page 229 in [10], shows that $\omega_X = \mathcal{L}(2\rho)$. This means that $\omega_X^{1-p} = \mathcal{L}(2(p-1)\rho)$, and therefore that we have a canonical G-equivariant multiplication map

$$St \otimes St \to \mathrm{H}^0(\omega_X^{1-p}). \tag{6.2}$$

We claim that up to a constant there is only one G-equivariant map from $St \otimes St$ to $\mathrm{H}^0(\omega_X^{1-p})$. This follows from Frobenius reciprocity by the following calculation

$$\mathrm{Hom}_G(St \otimes St, \mathrm{H}^0(2(p-1)\rho)) = \mathrm{Hom}_B(St \otimes St, k_{2(p-1)\rho}) = k.$$

The last equality follows from the fact that the highest weight of $St \otimes St$ is $2(p-1)\rho$, and that the corresponding weight space has dimension 1. Under the isomorphism (6.1) the map φ also defines a map from $St \otimes St$ to $\mathrm{H}^0(\omega_X^{1-p})$. By multiplying $<,>$, if necessary, by a non zero constant, we may therefore assume that φ correspond to the product map (6.2) above. This essentially proves the following result.

Lemma 6.3 *Let v and w be elements of St. Then*

$$\varphi(v \otimes w) = \varphi(w \otimes v).$$

Proof This follows from the corresponding result for the product map (6.2). □

Remark 6.2 *Lemma 6.3 also proves that $<,>$ is symmetric as*

$$\varphi(v \otimes w)(1) = < v, w > .$$

We can now state and prove our main theorem for Frobenius splitting of $X = G/B$.

Theorem 6.1 *Consider the map*

$$\varphi : St \otimes St \to \mathrm{Hom}_{\mathcal{O}_{X'}}(F_* \mathcal{O}_X, \mathcal{O}_{X'}).$$

Let v and w be two elements in St. Then

(1) *$\varphi(v \otimes w)$ is a Frobenius splitting of X (up to a nonzero constant) if and only if $< v, w > \neq 0$.*

(2) *Assume that there exist an element s in $\mathrm{H}^0(\rho)$ such that $v = s^{p-1}$. If $< v, w >= 1$, then the zero scheme $Z(s)$ is compatibly Frobenius split inside X.*

(3) *Assume that there exist an element s in $\mathrm{H}^0(\rho)$ such that $w = s^{p-1}$. If $< v, w >= 1$, then the zero scheme $Z(s)$ is compatibly Frobenius split inside X.*

Proof As $\varphi(v \otimes w)(1) =< v, w >$ the first statement follows from Remark 2.1. By Lemma 6.3, (3) will follow if we can prove (2). We therefore concentrate on proving (2). Assume that $< v, w >= 1$, and that there exist an element s in $\mathrm{H}^0(\rho)$ such that $v = s^{p-1}$. Then by (1) we know that $\varphi(v \otimes w)$ is a Frobenius splitting of X. We would like to show that $\varphi(v \otimes w)$ compatibly splits $D = Z(s)$. For this consider the following diagram.

$$\begin{tikzcd}
& \mathcal{O}_{X'} \arrow[d] & \\
F_*\mathcal{L}(-D) \arrow[r, "F_*s^\vee"] \arrow[d, "\theta\otimes 1_{\mathcal{L}(-D')}"] & F_*\mathcal{O}_X \arrow[r, "F_*s^{p-1}"] & F_*\mathcal{L}((p-1)\rho) \arrow[d, "\theta"] \\
St \otimes \mathcal{L}(-D') \arrow[rr, "1\otimes s^\vee"] \arrow[d, "<*{,}w>"] & & St \otimes \mathcal{O}_{X'} \arrow[d, "<*{,}w>"] \\
\mathcal{L}(-D') \arrow[rr, "s^\vee"] & & \mathcal{O}_{X'}
\end{tikzcd}$$

Here θ denote the isomorphism from Theorem 5.1. A local calculation shows that this diagram is commutative. By definition of φ we know that

$$\varphi(v \otimes w) =< *, w > \circ \theta \circ F_* s^{p-1}.$$

Noticing that $\mathcal{L}(-D)$ is the ideal sheaf of $D = Z(s)$ inside X, the result follows by the commutativity of the diagram. □

Corollary 6.2 *Let s be a nonzero element of $\mathrm{H}^0(\rho)$. Then there exist a Frobenius splitting of X, which compatibly splits the zero scheme $Z(s)$ of s. In particular, the zero scheme $Z(s)$ is reduced.*

Proof Choose an element w in St such that $< s^{p-1}, w >= 1$. Then $\varphi(s^{p-1} \otimes w)$ does the job (see also Remark 2.1(1)). □

6.1.1 Frobenius splitting of Schubert varieties

Recall the result of Example 5.4. Here we showed that if v^- denoted a nonzero lowest weight vector in $\mathrm{H}^0(\rho)$, then $Z(v^-)$ was the the union of the codimension 1 Schubert varieties. Similarly if v^+ was a nonzero highest weight vector in $\mathrm{H}^0(\rho)$, then $Z(v^+)$ was the union of the opposite codimension 1 Schubert varieties.

Lemma 6.4 $< (v^-)^{p-1}, (v^+)^{p-1} >\neq 0$.

Proof By definition $<,>$ is non degenerated. Therefore there exist a T semi-invariant element w in St such that $< (v^-)^{p-1}, w >\neq 0$. As $<,>$ is G-invariant it is in particular T-invariant. Let λ denote the T-weight of w. Then

$$< (v^-)^{p-1}, w >=< t(v^-)^{p-1}, tw >= (\lambda - (p-1)\rho)(t) < (v^-)^{p-1}, w > \ , t \in T.$$

We conclude that $\lambda = (p-1)\rho$, so that w is a highest weight vector in St. But the $(p-1)\rho$-weight space in St is of dimension 1, and therefore $(v^+)^{p-1}$ is a nonzero constant multiplum of w. This ends the proof. $\square$

By multiplying with a nonzero constants we may from now on assume that $< (v^-)^{p-1}, (v^+)^{p-1} >= 1$.

Corollary 6.3 *The flag variety $X = G/B$ is Frobenius split compatibly with it's Schubert varieties X_w and opposite Schubert varieties $w_0 X_w$.*

Proof Consider the Frobenius splitting $\varphi((v^-)^{p-1} \otimes (v^+)^{p-1})$ of X. By Theorem 6.1 and Lemma 6.4 this is a Frobenius splitting of X, which compatibly split $Z(v^-)$ and $Z(v^+)$. By Lemma 2.1(3) and Example 5.4 we conclude that every codimension 1 Schubert variety, and every opposite codimension 1 Schubert variety, is compatibly Frobenius split. As noted in Section 5.2, every Schubert variety of codimension $d \geq 1$ is a component of an intersection of Schubert varieties of codimension $\geq d-1$. The result now follows from consecutive use of this and Lemma 2.1. $\square$

6.1.2 Diagonal splitting

The next application of Theorem 6.1 will be to show that $X = G/B$ is diagonal Frobenius split. Remember that this means that the diagonal Δ_X inside $X \times X$ is compatibly Frobenius split. In the following we will think of G as acting on $X \times X$ through the diagonal action.

Remember that if B acts on a variety Y, then we define $G \times^B Y$ to be the quotient variety of $G \times Y$ under the B-action

$$b(g, y) = (gb^{-1}, by) \ , b \in B.$$

If the B-action on Y can be extended to a G-action then the map

$$G \times^B Y \xrightarrow{\simeq} G/B \times Y$$

$$(g, y) \mapsto (gB, gy),$$

is clearly an isomorphism. In particular we may regard $X \times X$ as $G \times^B X$. In this way $G \times^B X_w \ , w \in W$ may be regarded as closed subvarieties of $X \times X$, and we claim

Lemma 6.5 *The subvarieties $G \times^B X_w \ , w \in W$, are exactly the closed G-invariant irreducible subvarieties of $X \times X$.*

Proof First of all it is clear that the subvarieties $G \times^B X_w$ of $X \times X$ are closed, irreducible and G-invariant. On the other hand, assume that Y is a closed G-invariant irreducible subvariety of $X \times X$. Consider

$$Z = \{gB \in G/B \mid (1B, gB) \in Y\} \subseteq G/B.$$

As Y is G-invariant it is in particular B-invariant, and therefore Z is B-invariant. As Z is also closed we conclude that Z is a union of Schubert varieties. Let $w_1, \dots, w_s$ be a set of Weyl group elements such that

$$Z = \bigcup_{i=1}^{s} X_{w_i}.$$

As every G-orbit in Y contains an element of the form $(1B, gB)$ we must have

$$Y = \bigcup_{i=1}^{s} (G \times^B X_{w_i}).$$

But Y was by assumption irreducible, and therefore $Y = G \times^B X_{w_i}$ for some i. □

The variety $X \times X$ is itself a flag variety. In fact, we may regard $X \times X$ as $(G \times G)/(B \times B)$ in the obvious way. In particular, we may regard the Picard group of $X \times X$ as $\chi(T \times T) = \chi(T) \times \chi(T)$.

Lemma 6.6 *Let λ_j be a fundamental dominant weight of T, and let $\mathcal{L}$ denote the line bundle $\mathcal{L}(\lambda_i, -w_0\lambda_i)$ on $X \times X$. Then*

(1) There exist (up to a nonzero constant) a unique nonzero G-invariant global section v_j of $\mathcal{L}$.

(2) The zero scheme $Z(v_j)$, of the G-invariant element in (1), is of the form $G \times^B X_{w_0 s_\alpha}$, for some simple reflection s_α.

Proof By Frobenius reciprocity we have

$$\begin{aligned} \mathrm{Hom}_G(k, \mathrm{H}^0(\lambda_j) \otimes \mathrm{H}^0(-w_0\lambda_j)) &= \mathrm{Hom}_G(\mathrm{H}^0(-w_0\lambda_j)^*, \mathrm{H}^0(\lambda_j)) \\ &= \mathrm{Hom}_B(\mathrm{H}^0(-w_0\lambda_j)^*, k_{\lambda_j}). \end{aligned} \tag{6.3}$$

As the highest weight of $\mathrm{H}^0(-w_0\lambda_j)^*$ is λ_j, and as the corresponding weight space is of dimension 1, the first statement now follows from (6.3). Let now v_j denote a nonzero G-invariant global section of $\mathcal{L}$. Then the zero scheme $Z(v_j)$ is G-invariant, and by Lemma 6.5 we conclude that there exist $w_1, \dots, w_s$ in W such that

$$Z(v_j) = \sum_{i=1}^{s} (G \times^B X_{w_i}).$$

By dimension reasoning it follows that each of the elements w_i is of the form $w_0 s_\alpha$, where s_α is a simple reflection. Then

$$\mathcal{L} = \bigotimes_{i=1}^{s} \mathcal{O}(G \times^B X_{w_i}).$$

Choose now $\mu_i, \theta_i \in \chi(T)$, $i = 1, \dots s$, such that

$$\mathcal{O}(G \times^B X_{w_i}) \simeq \mathcal{L}(\mu_i, \theta_i).$$

By the results stated in Section 5.3.3, we we conclude first of all that μ_i and θ_i are dominant for all i. At the same time

$$\lambda_j = \sum_{i=1}^{s} \mu_i \text{ and } - w_0 \lambda_j = \sum_{i=1}^{s} \theta_i.$$

But λ_j (and therefore also) $-w_0\lambda_j$ are fundamental dominant weight, which implies that we may assume ($\delta_{*,*}$ denotes Kronecker's delta)

$$\mu_i = \delta_{1,i}\lambda \text{ and } \theta_i = \delta_{s,i}(-w_0\lambda_j).$$

If $s = 1$ we are done, so assume that $s > 1$. Then

$$\mathcal{O}(G \times^B X_{w_1}) \simeq \mathcal{L}(\mu_1, 0),$$

But $\mathcal{L}(\mu_1, 0)$ can clearly not conatin global sections with zero scheme $G \times^B X_{w_1}$, which is a contradiction. □

Corollary 6.4 *The $G \times G$ representation $\mathrm{H}^0(\rho) \otimes \mathrm{H}^0(\rho)$ contains (up to a nonzero constant) a unique nonzero G-invariant element v. Any such v has the following properties*

(1) The zero scheme $Z(v)$ is equal to $G \times^B Z(v^-)$, where v^- is a non zero lowest weight vector in $\mathrm{H}^0(\rho)$. Here $Z(v^-)$ is the zero scheme of v^-, which by Example 5.4 is the union of the codimension 1 Schubert varieties.

(2) Let v^- (resp. v^+) denote a nonzero lowest (resp. highest) weight vector in $\mathrm{H}^0(\rho)$, Then v, expressed in a basis of weight vectors, contains $v^+ \otimes v^-$ and $v^- \otimes v^+$ with nonzero coefficient.

Proof By Frobenius reciprocity we have

$$\begin{aligned} \mathrm{Hom}_G(k, \mathrm{H}^0(\rho) \otimes \mathrm{H}^0(\rho)) &= \mathrm{Hom}_G(\mathrm{H}^0(\rho)^*, \mathrm{H}^0(\rho)) \\ &= \mathrm{Hom}_B(\mathrm{H}^0(\rho)^*, k_\rho). \end{aligned} \tag{6.4}$$

As the highest weight space in $\mathrm{H}^0(\rho)^*$ has dimension 1 and weight ρ, the first statement follows. For each $j = 1, \dots, l$, let v'_j be nonzero G-invariant global sections of the line bundle $\mathcal{L}(\lambda_j, -w_0\lambda_j)$. By Lemma 6.6 such elements exist. Then the product $v = v'_1 \dots v'_l$ is a nonzero G-invariant element in $\mathrm{H}^0(\rho) \otimes \mathrm{H}^0(\rho)$. Using that the line bundles $\mathcal{L}(\lambda_j, -w_0\lambda_j)$, $j = 1, \dots, s$ are non-isomorphic, we by Lemma 6.6 then conclude statement (1). Now we turn to the proof of (2). Let $v_1, \dots, v_n$ be a T semi-invariant basis of $\mathrm{H}^0(\rho)$, such that $v^- = v_1$ and $v^+ = v_n$. Write

$$v = \sum_{i=1}^{n} a_{ij}(v_i \otimes v_j) \ , \ a_{ij} \in k.$$

By Equation (6.4) above, v then correspond to a nonzero B-equivariant map

$$\eta : \mathrm{H}^0(\rho)^* \to k_\rho,$$

given by

$$\eta(f) = \sum_{i,j=1}^{n} a_{ij} f(v_i) v_j.$$

Let f^+ be a highest weight vector in $\mathrm{H}^0(\rho)^*$. Then $\eta(f^+)$ must be nonzero, and as f^+ is T semi-invariant with weight ρ we have that $f^+(v_i) = 0$ unless $i = 1$. Therefore

$$0 \neq \eta(f^+) = \sum_{j=1}^{n} a_{1j} f^+(v_1) v_j.$$

As v was G-invariant a_{ij} must be zero unless the sum of the T-weight of v_i and v_j are zero. Therefore $\eta(f^+) = a_{1n} f^+(v_1) v_n$. We conclude that a_{1n}, the coefficient to $v^- \otimes v^+$, is nonzero. By using that $w_0 v = v$, the coefficient to $v^+ \otimes v^-$ must also be nonzero. This ends the proof. □

Corollary 6.5 *The flag variety $X = G/B$ is diagonal Frobenius split. More precisely $X \times X$ is Frobenius split compatibly with*

(1) $X_w \times X$ for every $w \in W$.

(2) $X \times w_0 X_w$ for every $w \in W$.

(3) $G \times^B X_w$ for every $w \in W$.

Notice that $\Delta_X = G \times^B X_1$.

Proof Let v be a nonzero G-invariant element of $\mathrm{H}^0(\rho) \otimes \mathrm{H}^0(\rho)$ (which exist by Corollary 6.4) , and v^- and v^+ be a nonzero lowest and highest weight vector in $\mathrm{H}^0(\rho)$ respectively. Then v^{p-1} and $(v^- \otimes v^+)^{p-1}$ are elements in the Steinberg module $St \otimes St$ of $G \times G$. Let $<,>$ be a nonzero G invariant form on St. Then the tensor product of $<,>$ with itself, is clearly a nonzero $G \times G$ invariant form $<,>$ on $St \otimes St$. By Corollary 6.4(2) and Lemma 6.4 we see that

$$< v^{p-1}, (v^- \otimes v^+)^{p-1} > \neq 0.$$

Therefore Theorem 6.1 tells us that there exist a Frobenius splitting of $X \times X$, which compatibly splits the zero schemes $Z(v^- \otimes v^+) = Z(v^-) \times X + X \times Z(v^+)$, and $Z(v) = G \times^B Z(v^-)$ (By Corollary 6.4). By using the description of $Z(v^-)$ and $Z(v^+)$ in Example 5.4, the statements (1),(2) and (3) follows when $l(w) = l(w_0) - 1$. With a similar argument as in Corollary 6.3, this is enough to conclude the rest of the cases. □

Corollary 6.5 above is a weak version of the following result proved in [12]. This result is related to the fact that a flag variety is defined by quadrics in any projective embedding given by a very ample line bundle. On more on this subject we refer to [18].

Proposition 6.3 *The product $(G/B)^n = G/B \times G/B \times \cdots \times G/B$ is compatibly Frobenius split with the subvarieties*

(1) $X_w \times G/B \times \ldots G/B$.

(2) $G/B \times \cdots \times (G \times^B X_w) \times \cdots \times G/B$.

Proof We challenge the reader to prove this using techniques similar to the ones used in Corollary 6.5. □

As an immediate consequence of Corollary 6.5 we get

Corollary 6.6 *([18]) Let $\mathcal{L}$ be an ample line bundle on G/B. Then*

(1) G/B is projectively normal with respect to $\mathcal{L}$

(2) The multiplication map : $\mathrm{H}^0(\mathcal{L}) \otimes \mathrm{H}^0(\mathcal{L}) \to \mathrm{H}^0(\mathcal{L}^2)$, is surjective.

Proof (1) follows by using Corollary 2.3, and (2) follows by using Proposition 2.5(3). □

6.2 Splittings of homogeneous bundles

Let V be a rational B-representation of dimension n. V determines both a projective bundle and a vector bundle. The vector bundle is just $G \times^B V$ with the natural projection map onto $X = G/B$. To describe the projective bundle corresponding to V, let k^* act on $G \times^B V$ by

$$a(g, v) = (g, av) \ , a \in k^*.$$

Then the quotient of of this action is a projective bundle over G/B which will be denoted by $(G \times^B V)/\backsim$. We would like to know when $G \times^B V$ and $(G \times^B V)/\backsim$ are Frobenius split. In Chapter 3 and Chapter 4 we saw that this was determined by a map of locally free $\mathcal{O}_X$-modules. First we would like to describe this map in the flag variety case.

Let $\mathcal{L}(V)$ be the sheaf of sections of the projection map

$$G \times^B V \to G/B,$$

and let $\mathcal{V}$ be the dual $\mathcal{L}(V)^*$ of $\mathcal{L}(V)$. With notation as in Chapter 3 and Chapter 4, we then have

$$G \times^B V = \mathbb{A}(\mathcal{V}) \ \text{ and } \ (G \times^B V/\backsim) = \mathbb{P}(\mathcal{V}).$$

Consider the map defined in Theorem 3.1

$$D_\pi : \omega_X^{1-p} \otimes \det(\mathcal{V})^{1-p} \otimes \mathrm{S}^{n(p-1)}\,\mathcal{V} \to \omega_X^{1-p}.$$

We claim that D_π is a map of homogeneous G-bundles. It should be possible to check this directly, using the functorial property of the Cartier operator stated in Remark 1.1. But we will not do this. Instead we choose a different approach.

Definition 6.1 *Let W be a rational B representation of dimension n, and choose a basis $w_1, \dots w_n$ of W. Define D'_W to be the map of B-representations*

$$D'_W : S^{n(p-1)}W \otimes det(W)^{1-p} \to k$$

given by

$$D'_W((w_1^{\beta_1} \dots w_n^{\beta_n})/(\wedge_i w_i)^{p-1}) = \begin{cases} 1 & \text{if } \beta_1 = \dots = \beta_n = p-1, \\ 0 & \text{else.} \end{cases}$$

Here $\beta_1, \dots, \beta_n$ is a set of positive integers with sum $n(p-1)$.

Remark 6.3 *Using Lemma 1.2 on the variety* $W^* = \mathrm{Spec}(\mathrm{S}^\bullet W)$*, tells us that* D'_W *is independent of the choice of basis* $w_1, \dots, w_n$ *of* W*. In particular,* D'_W *is really an* B*-equivariant map.*

By tensorising D'_{V^*} with $k_{2(p-1)\rho}$, we get in this way a B-equivariant map

$$D_V : \mathrm{S}^{n(p-1)}\, V^* \otimes det(V^*)^{1-p} \otimes k_{2(p-1)\rho} \to k_{2(p-1)\rho}.$$

The induced map between the corresponding locally free sheaves, is furthermore easily seen to be equal to the map D_π. This shows that D_π is a map of homogeneous G-bundles as claimed. By Theorem 3.1 we also see

Proposition 6.4 *Let* V *be a rational* B*-representation, and* $v_1, \dots, v_n$ *be a basis of* V^**. Consider the* B*-equivariant map*

$$D_V : \mathrm{S}^{n(p-1)}\, V^* \otimes det(V^*)^{1-p} \otimes k_{2(p-1)\rho} \to k_{2(p-1)\rho},$$

given by

$$D_V((v_1^{\beta_1} \dots v_n^{\beta_n})/(\wedge_i v_i)^{p-1}) = \begin{cases} 1 & \text{if } \beta_1 = \dots = \beta_n = p-1, \\ 0 & \text{else,} \end{cases}$$

for every set $\beta_1, \dots, \beta_n$ *of positive integers with sum* $n(p-1)$*.*
Then $G \times^B V/ \backsim$ *(resp.* $G \times^B V$*) is Frobenius split if and only the image of* $\mathrm{H}^0(D_V)$ *inside*

$$\mathrm{H}^0(2(p-1)\rho) = \mathrm{H}^0(G/B, \omega_{G/B}^{1-p}),$$

contains a Frobenius splitting of G/B*.*

Lemma 6.7 *Let* K *be the kernel of the* G*-equivariant evaluation map (see Corollary 6.1) :*

$$\mathrm{H}^0(2(p-1)\rho) \overset{\simeq}{\to} \mathrm{H}^0(\omega_{G/B}^{1-p}) \overset{\simeq}{\to} \mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}) \overset{ev}{\to} k.$$

If $p \geq 2h-2$*, where* h *is the Coxeter number of* G*, then* K *contains every non trivial subrepresentation of* $\mathrm{H}^0(2(p-1)\rho)$*.*

Proof Dualizing the evaluation map we get a G-equivariant embedding

$$k \hookrightarrow \mathrm{H}^0(2(p-1)\rho)^*.$$

The socle $\mathrm{Soc}_G(\mathrm{H}^0(2(p-1)\rho)^*)$ (i.e. the sum of the simple subrepresentations) of $\mathrm{H}^0(2(p-1)\rho)^*$, does therefore contain k. We claim that $\mathrm{Soc}_G(\mathrm{H}^0(2(p-1)\rho)^*)$ is equal to k. Notice that this is equivalent to the statement in the lemma. To prove this claim notice first of all that

$$\mathrm{H}^0(2(p-1)\rho)^* = \mathrm{H}^0(-w_0(2(p-1)\rho))^* = V(2(p-1)\rho),$$

where $V(2(p-1)\rho)$ is the Weyl module corresponding to the weight $2(p-1)\rho$. As pointed out to me by Jens C. Jantzen, one may (when $p \geq 2h-2$) calculate the socle of $V(2(p-1)\rho)$, by using Corollar 6.3. in [9] on the values $\nu = 0$, $n = 1$ and $\lambda = (p-1)\rho$. This implies the claim and ends the proof. □

Corollary 6.7 *Let $p \geq 2h-2$, where h is the Coxeter number of G. Then $G \times^B V/\backsim$ (resp. $G \times^B V$) is Frobenius split if and only if $\mathrm{H}^0(D_V)$ is surjective. In particular, if $G \times^B V/\backsim$ is Frobenius split, then every Frobenius splitting of G/B can be lifted to $G \times^B V/\backsim$.*

Proof If $\mathrm{H}^0(D_V)$ is surjective then $G \times^B V/\backsim$ and $G \times^B V$ are Frobenius split by Proposition 6.4. Assume therefore that $G \times^B V$ or $G \times^B V/\backsim$ is Frobenius split. Let I be the image of $\mathrm{H}^0(D_V)$ in $\mathrm{H}^0(2(p-1)\rho)$. Then I is a rational G-representation in $\mathrm{H}^0(2(p-1)\rho)$, which by Proposition 6.4 contains a Frobenius splitting of G/B. Consider the composed map

$$\eta : I \hookrightarrow \mathrm{H}^0(2(p-1)\rho) \xrightarrow{\simeq} \mathrm{H}^0(\omega_{G/B}^{1-p}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}) \xrightarrow{ev} k,$$

where ev is the G-equivariant map defined in Corollary 6.1. That I contains a Frobenius splitting is then equivalent to η being surjective. Therefore I is not contained in K (K defined as in Lemma 6.7 above), and we conclude by Lemma 6.7 that $I = \mathrm{H}^0((2(p-1)\rho)$. $\square$

Example 6.1 *Let $\mathfrak{u}$ be the Lie-algebra of the unipotent radical U of B. Then $\mathfrak{u}$ is a rational B-representation for which $D_{\mathfrak{u}}$ looks particular nice*

$$D_{\mathfrak{u}} : \mathrm{S}^{N(p-1)}\,\mathfrak{u}^* \to k_{2(p-1)\rho}.$$

In the next chapter we will aim at proving that $\mathrm{H}^0(D_{\mathfrak{u}})$ is surjective when p is good. By Proposition 6.4, this will imply that $G \times^B \mathfrak{u}$ and $G \times^B \mathfrak{u}/\backsim$ are Frobenius split.

Chapter 7

Unipotent and nilpotent varieties

In this chapter we will be concerned with the unipotent and nilpotent variety of a semisimple simply connected linear algebraic group G. We will in particular be interested in whether or not these varieties are Frobenius split. Eventually we will see that at least for p (i.e. the characteristic of our ground field) big, both the unipotent and the nilpotent variety turns out to be Frobenius split. Throughout this chapter we will use the notation introduced in Chapter 6.

7.1 The unipotent variety

An element x in the ring $GL_n(k)$ is called unipotent if $x - I$, where I is the identity element in $GL_n(k)$, is nilpotent. More generally, an element x in the linear algebraic group G is called unipotent, if there exist an embedding $\iota : G \hookrightarrow GL_n(k)$ of G as a closed subgroup of $GL_n(k)$, such that $\iota(x)$ is unipotent. This definition turns out to be independent of the embedding ι. The set of unipotent elements in G will, following T. A. Springer [19], be denoted by $V(G)$. Then $V(G)$ is closed in the Zariski topology and we regard $V(G)$ as a closed subvariety of G, by putting the reduced variety structure on it. Then as G is simply connected $V(G)$ is a normal variety ([20], [19]) of dimension $\dim(V(G)) = \dim(\mathrm{G}) - \dim(T)$.

In general $V(G)$ is not smooth, but there exist a nice desingularisation of it. The observation is, that every unipotent element in $V(G)$ is conjugated to an element inside the unipotent radical U of B. Letting B act on U by conjugation it is therefore natural to consider the map

$$\pi : G \times^B U \to V(G).$$

$$(g, x) \mapsto gxg^{-1}.$$

This map is called the Springer resolution of $V(G)$. In [21] it is proved that π really is a desingularisation of $V(G)$. By Corollary 2.2, Frobenius splitting of $V(G)$ will then follow, if we can prove that $G \times^B U$ is Frobenius split. In a later chapter we will in fact prove that $G \times^B U$ is Frobenius split for any characteristic p of the field k. More precisely we will later prove that there is a morphism

$$\phi : St \otimes St \to \mathrm{H}^0(\omega_{G\times^B U}^{1-p}),$$

such that $\phi(v \otimes w)$ is a Frobenius splitting of $G \times^B U$ (up to a nonzero constant) if and only if $< v, w > \neq 0$. This should be compared with the description of Frobenius splittings of G/B given in Proposition 6.1.

7.2 The nilpotent variety

Whereas the unipotent variety was a subset of G, the nilpotent variety is a subset of the Lie algebra $\mathfrak{g}$ of G.

An element x in the Lie algebra $\mathfrak{gl}(n,k)$ is called nilpotent if it is nilpotent regarded as a linear map. More generally, an element x in $\mathfrak{g}$ is called nilpotent if there exist a Lie algebra embedding $\iota : \mathfrak{g} \to \mathfrak{gl}(n,k)$ such that $\iota(x)$ is nilpotent. This definition turns out to be independent of the embedding ι. The set of nilpotent elements in $\mathfrak{g}$ will, following T. A. Springer [19], be denoted by $\mathfrak{B}(G)$. The set $\mathfrak{B}(G)$ is closed and irreducible in the Zariski topology, and we regard it as a subvariety of $\mathfrak{g}$ with reduced scheme structure. When the characteristic p of the field k is good, then $\mathfrak{B}(G)$ is known to be normal ([19], [3]). Here good means

Definition 7.1 *Let G be an almost simple linear algebraic group over a field k of characteristic p. If p does not divide any of the coefficients of the highest root written as a sum of simple roots, then we say that p is good. More precisely, with respect to the type of G the good primes are*

$$\boldsymbol{A}_n : p \geq 0 \ ; \ \boldsymbol{B}_n, \boldsymbol{C}_n, \boldsymbol{D}_n : p \neq 2 \ ; \ \boldsymbol{E}_6, \boldsymbol{E}_7, \boldsymbol{F}_4, \boldsymbol{G}_2 : p \neq 2, 3 \ ; \ \boldsymbol{E}_8 : p \neq 2, 3, 5.$$

If G is arbitrary, then p is said to be good if p is good for every almost simple normal subgroup of G.

Even more is true as Springer has showed ([19], Theorem 3.1) that for p good, the nilpotent variety and the unipotent variety are isomorphic. There is also a natural Springer resolution of the nilpotent variety

$$\pi : G \times^B \mathfrak{u} \to \mathfrak{B}(G).$$

$$(g, x) \mapsto gx.$$

Here we think of B (resp. G) as acting on $\mathfrak{u}$ (resp. $\mathfrak{B}(G)$) through the adjoint action. As expected in case p is good, the resolution of the nilpotent variety is closely related to the resolution of the unipotent variety. In fact we have the following important result

Theorem 7.1 *([19] Prop 3.5.) Let p be good. Then there exist a B-equivariant isomorphism of varieties between $\mathfrak{u}$ and U. In particular, the varieties $G \times^B \mathfrak{u}$ and $G \times^B U$ are isomorphic.*

Chapter 8

Frobenius splitting of the cotangent bundle on G/B

Throughout this chapter we will use the notation introduced in Chapter 6. The cotangent bundle on G/B is by definition the vector bundle $G \times^B \mathfrak{u}$, where $\mathfrak{u}$ is the Lie algebra of the unipotent radical U of B, and where B acts on $\mathfrak{u}$ by the adjoint action. We will prove that the cotangent bundle is Frobenius split under the assumption that p is good. Notice that this, by Theorem 7.1, implies that $G \times^B U$ is Frobenius split. By the discussion in Chapter 7, this then implies that the unipotent variety $V(G)$ is Frobenius split, and finally therefore also that the nilpotent variety $\mathfrak{B}(G)$ is Frobenius split, still under the assumption that p is good.

8.1 Preliminary definitions

Consider G as acting on G by conjugation. The corresponding action of G on the global regular functions :

$$(g.f)(x) = f(g^{-1}xg) \ , g \in G, f \in k[G].$$

makes $k[G]$ into a (infinite) dimensional rational G-representation. Let now ϕ_G be the G-equivariant map

$$\phi_G : St \otimes St \to k[G]$$

given by

$$\phi(v \otimes w)(g) = < v, gw > \ \ , v, w \in St.$$

Notice that it is the G-invariance of the form $<,>$, which makes this map G-equivariant. Let B act on $k[U]$ by

$$(bf)(u) = f(b^{-1}xb) \ , b \in B, \ f \in k[U].$$

Then the restriction map from $k[G]$ to $k[U]$ is B-equivariant. The composition of ϕ_G and this restriction map is a B-equivariant map, which in the following will be denoted by ϕ_U.

Definition 8.1 *The unipotent radical U is isomorphic to $\mathbb{A}^N$. Therefore ω_U^{1-p} is isomorphic to $\mathcal{O}_U$. Fix such an isomorphism $\tau : \mathcal{O}_U \to \omega_U^{1-p}$. Then we say that a global function f on U is a Frobenius splitting of U, if $\tau(f)$ is a Frobenius splitting. More generally we say that f is a Frobenius splitting of U if $\tau(f)$, up to a nonzero constant, is a Frobenius splitting of U. This notion clearly has the advantage that it does not depend on τ.*

8.2 Splitting of the cotangent bundle

Lemma 8.1 *Let v^- (resp. v^+) be a nonzero lowest (resp. highest) weight vector in St. Then the global regular function on G given by*

$$g \mapsto < v^-, gv^- >< v^+, gv^- > \quad , g \in G$$

is an element in $\mathrm{H}^0(2(p-1)\rho)$, which (up to a nonzero constant) correspond to a Frobenius splitting of G/B.

Proof Consider the global regular function $g \mapsto < v^-, gv^- >$ on G. As St is an irreducible G-module, it is clear that this function represent a nonzero lowest weight vector in St. Similarly the global regular function $g \mapsto < v^+, gv^- >$ on G, represent a highest weight vector in St. By the discussion in Section 6.1 and Theorem 6.1, we therefore conclude that the product

$$g \mapsto < v^-, gv^- >< v^+, gv^- >,$$

up to a nonzero constant, is a Frobenius splitting of G/B $\square$

Keeping the notation from Lemma 8.1 we now claim.

Proposition 8.1 *Consider the B-equivariant map*

$$\phi_U : St \otimes St \to k[U],$$

defined in the previous section. Then $\phi_U(v^+ \otimes v^+)$ is a Frobenius splitting of U.

Proof Reformulating the statement, we see that we have to show that the function

$$u \mapsto < v^+, uv^+ >,$$

is a Frobenius splitting of U. Consider U as an open subset of G/B through the map

$$u \mapsto (uw_0)B \ , u \in U.$$

By Lemma 8.1 we then conclude that the function

$$u \mapsto < v^-, uw_0v^- >< v^+, uw_0v^- >,$$

is a Frobenius splitting of U. As v^- is invariant under U, the first factor in this function is constant. Therefore the function

$$u \mapsto < v^+, uw_0v^- >,$$

is a Frobenius splitting of U. Notice finally that w_0v^-, as a highest weight vector in St, must be proportional to v^+. $\square$

From now on and in the rest of this chapter we assume that p is good relative to G. Fix an B-equivariant isomorphism between $\mathfrak{u}$ and U (see Theorem 7.1). Then the ring $\mathrm{S}(\mathfrak{u}^*)$ of global regular functions on $\mathfrak{u}$ is B-equivariant isomorphic to $k[U]$. In this way we get a B-equivariant map (induced by ϕ_U)

$$\phi_{\mathfrak{u}} : St \otimes St \to \mathrm{S}(\mathfrak{u}^*).$$

By Proposition 8.1 we see that $\phi_{\mathfrak{u}}(v^+ \otimes v^+)$ is a Frobenius splitting of $\mathfrak{u}$. This implies

Lemma 8.2 *The composed B-equivariant map*

$$St \otimes St \xrightarrow{\phi_{\mathfrak{u}}} \mathrm{S}(\mathfrak{u}^*) \to S^{N(p-1)}\mathfrak{u}^* \xrightarrow{D_{\mathfrak{u}}} k_{2(p-1)\rho},$$

is nonzero.

Proof We claim that the image of $v^+ \otimes v^+$ is nonzero. To see this let $u_1, \ldots, u_N$ be a basis of $\mathfrak{u}$. Then $\mathrm{S}(\mathfrak{u}^*) = k[u_1, \ldots, u_N]$. We may therefore write $\phi_{\mathfrak{u}}(v^+ \otimes v^+)$ as linear combination of monomials in $u_1, \ldots, u_N$. As $\phi_{\mathfrak{u}}(v^+ \otimes v^+)$, as mentioned above, is a Frobenius splitting of $\mathfrak{u}$, it follows from Proposition 2.1 that the coefficient c to $(u_1 \ldots u_N)^{p-1}$ is nonzero. By the description of $D_{\mathfrak{u}}$ in Proposition 6.4, we on the other hand see that $v^+ \otimes v^+$ maps to c. □

Theorem 8.1 *The map $\mathrm{H}^0(D_{\mathfrak{u}})$ defined in Proposition 6.4 is surjective. In particular, the cotangent bundle over G/B is Frobenius split. (when p is good)*

Proof Above we have constructed a nonzero B-equivariant map

$$\phi_k : St \otimes St \to k_{2(p-1)\rho},$$

As mentioned in section 6.1 there is, up to a nonzero constant, only one such map. Therefore

$$\mathrm{H}^0(\phi_k) : St \otimes St \to \mathrm{H}^0(2(p-1)\rho)$$

must be the multiplication map. But this is surjective by Corollary 6.6. Finally we notice that $\mathrm{H}^0(\phi_k)$, by definition, factors through $\mathrm{H}^0(D_{\mathfrak{u}})$. □

Let ϕ be the composed map

$$\phi : St \otimes St \to \mathrm{S}(\mathfrak{u}^*) \to \mathrm{S}^{(p-1)N}\,\mathfrak{u}^*.$$

Using the results in Chapter 3, we see that $\mathrm{H}^0(\mathrm{S}^{(p-1)N}\,\mathfrak{u}^*)$ correspond to the global sections of the line bundle ω_Y^{1-p} on $Y = G \times^B \mathfrak{u}/\backsim$. By Chapter 4 we may also regard $\mathrm{H}^0(\mathrm{S}^{(p-1)N}\,\mathfrak{u}^*)$ as the $N(p-1)$'th graded part of the global sections of the line bundle $\omega_{G\times^B\mathfrak{u}}^{1-p}$ on $G \times^B \mathfrak{u}$. Looking closer at the proof of the Frobenius splitting of the cotangent bundle above, we see that we in fact have proven.

Addendum 8.1 *(p good) Let Y denote either the projectivised cotangent bundle $G \times^B \mathfrak{u}/\backsim$, or the cotangent bundle $G \times^B \mathfrak{u}$. Consider the map*

$$\mathrm{H}^0(\phi) : St \otimes St \to \mathrm{H}^0(Y, \omega_Y^{1-p})$$

defined above. Then $\mathrm{H}^0(\phi)(v \otimes w)$ is (up to a nonzero constant) a Frobenius splitting of Y if and only if $< v, w > \neq 0$. Furthermore any Frobenius splitting of G/B can be lifted to $G \times^B \mathfrak{u}/\backsim$.

Example 8.1 *Let $G = \mathrm{SL_n(k)}$, the $n \times n$ matrices with entries in k and determinant 1. Let B be the upper triangular matrices and T be the diagonal matrices. Then U is the upper triangular matrices with 1's on the diagonal, and we can regard $\mathfrak{u}$ as the set of strictly upper triangular matrices. The map $\mathfrak{u} \to U$ which maps x to $I + x$, is then a B-equivariant isomorphism between $\mathfrak{u}$ and U.*

We will now describe the element $\mathrm{H}^0(\phi)(v^+ \otimes v^-)$ inside $\mathrm{Ind}_B^G(\mathrm{S}^{N(p-1)}\,\mathfrak{u}^*)$. By Addendum 8.1 we know that this element, up to a nonzero constant, correspond to a Frobenius splitting of Y. Notice first of all that the map

$$\mathrm{H}^0(\phi_{\mathfrak{u}}) : St \otimes St \to \mathrm{Ind}_B^G(\mathrm{S}\,\mathfrak{u}^*) = \mathrm{Ind}_B^G(k[\mathfrak{u}]),$$

is given by

$$(\mathrm{H}^0(\phi_{\mathfrak{u}})(v \otimes w))(g) \to \phi_{\mathfrak{u}}(g^{-1}(v \otimes w)) \ , v, w \in St, \ g \in G.$$

And that $\phi_{\mathfrak{u}}(g^{-1}(v \otimes w))$ is the function on $\mathfrak{u}$ given by

$$\phi_{\mathfrak{u}}(g^{-1}(v \otimes w))(x) = < g^{-1}v, (I+x)g^{-1}w > = < v, g(I+x)g^{-1}w > \ \ , x \in \mathfrak{u}.$$

Let $\eta = \mathrm{H}^0(\phi_{\mathfrak{u}})(v^+ \otimes v^-)$. Then in particular we have

$$(\eta(g))(x) = < v^+, g(x+I)g^{-1}v^- > \ \ , g \in G, \ x \in \mathfrak{u}.$$

As mentioned in the proof of Lemma 8.1 the function : $g \mapsto < v^+, gv^- >$ on G is a highest weight vector in St, and in Example 5.3 these was described. Keeping the notation of Example 5 (and neglecting nonzero constants) we see

$$(\eta(g))(x) = u(g(I+x)g^{-1}),$$

where u is a product of certain determinant functions. The Frobenius splitting $\mathrm{H}^0(\phi)(v^+ \otimes v^-)$ is the $N(p-1)$ graded part of η, and a small calculation shows that in fact

$$((\mathrm{H}^0(\phi)(v^+ \otimes v^-))(g))(x) = u(gxg^{-1}),$$

where u in the naturally way is extended to a function on the set of $n \times n$ matrices. In particular we see that $\mathrm{H}^0(\phi)(v^+ \otimes v^-)$ is exactly the Frobenius splitting of the cotangent bundle, which V.B.Mehta and W. van der Kallen considers in [16].

8.3 A vanishing Theorem

We will now give a short application of the fact that the projectivised cotangent bundle $Y = G \times^B \mathfrak{u}/ \backsim$ is Frobenius split. With notation as in Section 3, we will think of Y as constructed as $\mathbb{P}(T_{G/B})$, with projection map $\pi : Y \to G/B$. Here $T_{G/B}$ denotes the tangent bundle on G/B. Using that G act transitively on G/B, it easily follows that $T_{G/B}$ is generated by global sections. The line bundle $\mathcal{O}_Y(1)$ does therefore define a closed morphism ([8] proof of Proposition II.7.10.)

$$\eta : Y \hookrightarrow \mathbb{P}^N \times X,$$

such that $\eta(\mathcal{O}_{\mathbb{P}^N}(1) \times \mathcal{O}_X) = \mathcal{O}_Y(1)$. We therefore conclude that $\mathcal{O}_Y(n) \otimes \pi^*(\mathcal{L})$ is ample on Y, for every $n > 0$ and for every ample line bundle $\mathcal{L}$ on G/B. By Proposition 2.5 we are therefore ready to prove

Theorem 8.2 *(p good) Let $\mathcal{L}$ be an ample line bundle on G/B, and n be an non negative integer. Then*

$$\mathrm{H}^i(G/B, \mathrm{S}^n\, T_{G/B} \otimes \mathcal{L}) = 0,\ \ i > 0.$$

Proof If $n = 0$ the result follows from Kempf vanishing (Proposition 6.2). Assume therefore that $n > 0$. Then by the above and Proposition 2.5 we have

$$\mathrm{H}^i(Y, \mathcal{O}_Y(n) \otimes \pi^*(\mathcal{L})) = 0,\ \ i > 0.$$

Now use the projection formula to conclude

$$\pi_*(\mathcal{O}_Y(n) \otimes \pi^*(\mathcal{L})) = \mathrm{S}^n\, T_{G/B} \otimes \mathcal{L}$$

and

$$R^i(\pi_*)(\mathcal{O}_Y(n) \otimes \pi^*(\mathcal{L})) = 0,\ \ i > 0.$$

By the Leray spectral sequence we conclude

$$\mathrm{H}^i(G/B, \mathrm{S}^n\, T_{G/B} \otimes \mathcal{L}) = \mathrm{H}^i(Y, \mathcal{O}_Y(n) \otimes \pi^*(\mathcal{L})),$$

which ends the proof. $\square$

Remark 8.1 *In the representation theoretical contest Theorem 8.2 says that*

$$\mathrm{H}^i(\mathrm{S}^n\, \mathfrak{u}^* \otimes \lambda) = 0\ , i > 0, n \geq 0, \tag{8.1}$$

when λ is strictly dominant and p is good. Results in this direction has been proved before. In positive characteristic, H.H.Andersen and J.C.Jantzen [2] has proved (8.1), when $< \lambda, \alpha^\vee > \geq h - 1$ (h is the Coxeter number of G) for every simple root α. For classical groups they furthermore have a proof for λ dominant and $p \geq h-1$. In characteristic 0 more in known. In fact B. Broer [4] has given a necessary and sufficient condition on λ for (8.1) to be true for every n. This includes all dominant weights.

Chapter 9

Frobenius splitting of $G \times^B U$ and $G \times^B B$

Think of B as acting on U and B by conjugation. In this chapter we will consider Frobenius splittings of the varieties $G\times^B U$ and $G\times^B B$. When the characteristic of the ground field k is good, we have (Chapter 8) already seen that $G \times^B U$ is Frobenius split. In this chapter we will see that $G \times^B B$ is Frobenius split compatibly with $G \times^B U$ for every p. It should be noted that the proof of this, is independent of many of the results stated in the previous chapters. In fact it only depends on the material in Chapter 1, 2 and 6. In particular, most of the results in Chapter 8 could be derived from the results in this chapter. In spite of that, we have chosen to to present Chapter 8 as it is, as the method used there is more natural that the method in this chapter. It should also be noted than if we were only interested in Frobenius splitting $G \times^B U$, then the proof in this chapter could be simplified.

Throughout this chapter we will use the notation introduced in Chapter 5.

9.1 Coordinate rings and Volume forms

We fix the following notation for the coordinate ring of U, U^+, T and B.

(1) Let $e_1, \dots, e_N$ denote regular non zero functions on U such that e_i has T-weight α_i and such that $k[U] = k[e_1, \dots, e_N]$.

(2) Let $f_1, \dots, f_N$ denote regular non zero functions on U^+ such that f_i has T-weight $-\alpha_i$ and such that $k[U^+] = k[f_1, \dots, f_N]$.

(3) Let t_i be the fundamental dominant weights of T, corresponding to α_i. The coordinate ring of T is the given by $k[t_1, t_1^{-1}, \dots, t_l, t_l^{-1}]$

(4) We think of B as $U \times T$ through the multiplication map. The coordinate ring of B will therefore be considered as $k[e_1, \dots, e_N, t_1, t_1^{-1}, \dots, t_l, t_l^{-1}]$.

If $I = (i_1, \dots, i_N)$ is a vector with non negative integer coordinates, we will use the multinomial notation

$$e^I := e_1^{i_1} \dots e_N^{i_N}.$$

If $K = (k_1, \dots, k_N)$ is a vector with non negative integer coordinates, we will use the multinomial notation

$$f^K := f_1^{k_1} \dots f_N^{k_N}.$$

Finally, if $J = (j_1, \dots, j_l)$ is a vector with integer coordinates, we will use the multinomial notation

$$t^J := t_1^{j_1} \dots t_N^{j_l}.$$

A vector with all entries equal to n will in the following be denoted by $\underline{n}$. With this notation we let $M(I,J) = e^I t^J$ denote the natural basis for the coordinate ring of B.

Definition 9.1 *Let X be a smooth variety such that ω_X is trivial, and let dX be a volume form, that is a nowhere vanishing global section of ω_X. Then a function f on X is said to be a Frobenius splitting of X if $f dX^{1-p}$ (up to a non zero constant) is a Frobenius splitting of X. In the following we will assume that volume forms on U, U^+ and B has been fixed as follows*

(1) $dU = de_1 \wedge \dots \wedge de_N$.

(2) $dU^+ = df_1 \wedge \dots \wedge df_N$.

(3) $dB = de_1 \wedge \dots de_N \wedge dt_1 \wedge \dots \wedge dt_l$.

9.2 Frobenius splitting $G \times^B B$

We start by defining a map which will correspond to the map φ in Theorem 6.1 and the map $\mathrm{H}^0(\phi)$ in Addendum 8.1.

Definition 9.2 *Let $\phi : St \times St \to H^0(\mathcal{O}_{G \times^B B})$ be the map defined by*

$$\phi(v \otimes w)(g,x) = < v, gxg^{-1}w > .$$

The map that we now define should be considered as a map similarly to the map D_π in Theorem 3.1.

Definition 9.3 *Define $\pi : H^0(\mathcal{O}_{G \times^B B}) \to k[G]$ as follows. Let h be a global function on $G \times^B B$. Composing h with the natural map : $G \times B \to G \times^B B$ we get a global function $\tilde{h}$ on $G \times B$. Write $\tilde{h}$ on the form*

$$\tilde{h} = \sum_{I,J} f(I,J) \otimes M(I,J) \ , \ f(I,J) \in k[G].$$

Then $\pi(h) = f(\underline{p\text{-}1}, \underline{p\text{-}1})$.

9.2.1 The canonical sheaf

In this section we will show that the canonical sheaf $\omega_{G\times^B B}$ is trivial.

Lemma 9.1 *Let $U \times U$ act on U by : $(u_1,u_2)u = u_1uu_2^{-1}$. The induced action on $\omega_U = k[U](de_1 \wedge \cdots \wedge de_N)$ satisfies that :*

$$(u_1,u_2)(de_1 \wedge \cdots \wedge de_N) = (de_1 \wedge \cdots \wedge de_N).$$

Proof As the map : $u \mapsto u_1^{-1}uu_2$ is an automorphism of U, it is first of all clear that there exist a unique unit $a = a(u_1,u_2)$ in $k[U]$ such that

$$(u_1,u_2)(de_1 \wedge \cdots \wedge de_N) = a(de_1 \wedge \cdots \wedge de_N).$$

The units in $k[U]$ are k^*, and we may therefore define a morphism $a : U\times U \to k^*$, such that

$$(u_1,u_2)(de_1 \wedge \cdots \wedge de_N) = a(u_1,u_2)(de_1 \wedge \cdots \wedge de_N).$$

Then a is a group homomorphism, and as $U \times U$ is unipotent the map a must be trivial. □

Lemma 9.2 *Let b be an element in B and γ be the automorphism of $k[B]$ given by*

$$\gamma(f)(x) = f(b^{-1}xb).$$

Then the induced map $\tilde{\gamma}$ on ω_B, satisfies

$$\tilde{\gamma}(de_1 \wedge \cdots \wedge de_N \wedge dt_1 \wedge \cdots \wedge dt_l) = b^{2\rho}(de_1 \wedge \cdots \wedge de_N \wedge dt_1 \wedge \cdots \wedge dt_l),$$

where 2ρ is regarded as a B-character.

Proof We may assume that $b \in T$ or $b \in U$. If $b \in T$ then $\gamma(t_i) = t_i$ and $\gamma(e_i) = \alpha_i(b)e_i$ and the result follows. Assume therefore that $b \in U$. Under the isomorphism $B \backsimeq U \times T$ the automorphism : $x \mapsto b^{-1}xb$ of B corresponds to

$$(u,t) \mapsto (b^{-1}ub', t) \ , \ b' = tbt^{-1} \in U.$$

By this and Lemma 9.1 (and obvious abuse of notation) we conclude that

$$\tilde{\gamma}(de_1 \wedge \cdots \wedge de_N) = de_1 \wedge \cdots \wedge de_N + \sum_{i=1}^{l} dt_i \wedge \omega_i,$$

for some $\omega_i \in \Omega_B^{N-1}$. Furthermore, $\gamma(t_i) = t_i$ so that

$$\tilde{\gamma}(de_1 \wedge \cdots \wedge de_N \wedge dt_1 \wedge \cdots \wedge dt_l) = (de_1 \wedge \cdots \wedge de_N \wedge dt_1 \wedge \cdots \wedge dt_l),$$

which ends the proof. □

Corollary 9.1 *The canonical sheaf $\omega_{G\times^B B}$ is trivial.*

Proof This follows from Lemma 12 in [13] and Lemma 9.2 above. □

We conclude that there exist a volume form (See Definition 9.1) on $G \times^B B$, and we claim

Lemma 9.3 *There exists an volume form on $G \times^B B$, such that its restriction to the open subset $U^+ \times B$ is*

$$df_1 \wedge \dots df_N \wedge de_1 \wedge \dots de_N \wedge dt_1 \wedge \dots \wedge dt_l.$$

Proof By Corollary 9.1 there exist a volume form ω. The restriction of ω to the open subset $U^+ \times B$ is a volume form on $U^+ \times B$, and it must be of the form

$$h(df_1 \wedge \dots df_N \wedge de_1 \wedge \dots de_N \wedge dt_1 \wedge \dots \wedge dt_l),$$

where $h \in k[U^+ \times B]$ is a global unit. But the only global units in $k[U^+ \times B]$ are elements of the form

$$at_1^{n_1} \dots t_l^{n_l} \ , \ a \in k^*.$$

All of these functions extends to global units on $G \times^B B$. This is so as any regular function on T lifts to a regular function on $G \times^B B$ via the composed map

$$G \times^B B \to G \times^B (B/U) \simeq G \times^B T \simeq G/B \times T \to T.$$

If then h' is an extension of h to $G \times^B B$, the volume form $h'^{-1}\omega$ satisfies the desired property. □

Definition 9.4 *In connection with Definition 9.1, we let $d(G \times^B B)$ be the volume form on $G \times^B B$ whose restriction to $U^+ \times B$ is equal to :*

$$df_1 \wedge \dots df_N \wedge de_1 \wedge \dots de_N \wedge dt_1 \wedge \dots \wedge dt_l.$$

(By Lemma 9.3 above, such an element exists.)

9.2.2 Moving on

With notation as in Definition 9.3 we have

Lemma 9.4 *Let $h = \phi(w_1 \otimes w_2)$ and I and J be integral vectors. Then*

$$f(\underline{p\text{-}1} + pI, J) = 0,$$

unless $I = \underline{0}$ and $J = \underline{p\text{-}1}$.

Proof Assume that $f(\underline{\text{p-1}} + pI, J) \neq 0$ and choose an element $g \in G$ such that $f(\underline{\text{p-1}} + pI, J)(g) \neq 0$. Then

$$\mu = \sum_{I,J} f(I,J)(g) M(I,J)$$

is a function on B such that the coefficient to $e^{\underline{\text{p-1}} + pI} t^J$ is nonzero. Define $\tilde{w_1} = g^{-1} w_1$ and $\tilde{w_2} = g w_2$. Then μ is given by

$$\mu(x) =< \tilde{w_1}, x\tilde{w_2} > .$$

As we only have to show that $I = 0$ and $J = \underline{\text{p-1}}$, we may assume that $\tilde{w_1}$ and $\tilde{w_2}$ are non zero and T semi-invariant with weights μ_1 and μ_2 respectively. Then

$$(t\mu)(x) = \mu(t^{-1}xt) =< t\tilde{w_1}, xt\tilde{w_2} >= (\mu_1 + \mu_2)(t)\mu(x).$$

So the weight of μ is $\mu_1+\mu_2$, and as $\tilde{w_1}$ and $\tilde{w_2}$ are elements in St, we conclude that the weight of μ is less that $2(p-1)\rho$. As the weight of $e^{\underline{\text{p-1}}}$ is $2(p-1)\rho$ we conclude from this that $I=0$ and $\mu_1=\mu_2=(p-1)\rho$. Then

$$\mu(ut) =< \tilde{w_1}, ut\tilde{w_2} >= ((p-1)\rho)(t)\mu(u) \ , u \in U, \ t \in T.$$

So μ as a function on $B = U \times T$ is given by $\mu = \mu_{|U} \otimes t^{\underline{\text{p-1}}}$. This proves the lemma. $\square$

Remark 9.1 *Let X be a smooth variety over k of dimension n. Consider the Cartier operator $C : F_*\omega_X^{1-p} \to \mathcal{O}_X$ on X. Assume that $X = \operatorname{Spec}(A)$ is affine and that there exist $x_1,\ldots,x_n$ in A such that $dx_1,\ldots,dx_n$ is a basis for $\Omega_{A/k}$. Then C, as we have seen in Lemma 1.2, is described by*

$$C(x^\alpha dx^{1-p}) = x^{(\alpha+\underline{1\text{-}p})/p}. \tag{9.1}$$

Here we use multinomial notation, with the extra condition that when non integer exponents occur the expression is regarded as zero. It is clear that $C(x^\alpha dx^{1-p})$ is non zero exactly when $\alpha = \underline{p\text{-}1} + p\beta$, for some β.

Assume that γ is an automorphism of X. Then $y_i = \gamma^{\#}(x_i)(= x_i \circ \gamma)$ is another set of elements in A such that $dy_1,\ldots,dy_n$ is a basis for $\Omega_{A/k}$. We may therefore also describe C relative to $y_1,\ldots,y_n$ by

$$C(y^\alpha dy^{1-p}) = y^{(\alpha+\underline{1\text{-}p})/p}. \tag{9.2}$$

Again this will be non zero exactly when $\alpha = \underline{p\text{-}1} + p\beta$. Let a be the unit in A such that $dx = a \cdot dy$. Then C satisfies by combining (9.1) and (9.2)

$$C(y^\alpha a^{p-1} dx^{1-p}) = y^{(\alpha+\underline{1\text{-}p})/p}.$$

Lemma 9.5 *Let b be an element in B and γ be the automorphism of $k[B]$ defined in Lemma 9.2. Let I and J be vectors as above, and assume that the coefficient to $M(\underline{p\text{-}1},\underline{p\text{-}1})$ in $\gamma^{\#}M(I,J)$ is non zero. Then there exist I' and J' such that*

$$I = \underline{p\text{-}1} + pI' \text{ and } J = \underline{p\text{-}1} + pJ'.$$

Proof This follows from Remark 9.1 as follows. We let $e_1,\ldots,e_n,t_1\ldots,t_l$ correspond to $x_1,\ldots,x_n$, and let y_i be the images of x_i under $\gamma^{\#}$. By Lemma 9.2 the unit a in Remark 9.1 is an element in k^*. We let a' denote the p'th root of a^{p-1}. Then the Cartier operator C satisfies

$$C(y^\alpha dx^{1-p}) = a' y^{(\alpha+\underline{\text{1-p}})/p}. \tag{9.3}$$

Express $\gamma^{\#}(M(I,J))$ in the basis $M(I',J')$ as

$$\gamma^{\#}(M(I,J)) = \sum_{I',J'} a(I',J')M(I',J') \ , \ a(I',J') \in k.$$

By assumption $a(\underline{\text{p-1}},\underline{\text{p-1}}) \neq 0$, and we conclude from the description

$$C(x^\alpha dx^{1-p}) = x^{(\alpha+\underline{\text{1-p}})/p}$$

of C, that $C(\gamma^{\#}(M(I,J))dx^{p-1}) \neq 0$. But by Equation (9.3) above we also know (as in Remark 9.1) that $C(\gamma^{\#}(M(I,J))dx^{p-1})$ is non zero exactly when there exists I' and J' such that

$$I = \underline{\text{p-1}} + pI' \text{ and } J = \underline{\text{p-1}} + pJ'.$$

This ends the proof. □

Proposition 9.1 *Let v and w be elements in St and $h = \phi(v \otimes w)$. Then*

$$\pi(h)(yb) = b^{-2(p-1)\rho}\pi(h)(y) \ , \ b \in B, \ y \in G.$$

In other words $\pi(h) \in H^0(G/B, 2(p-1)\rho)$.

Proof Notation as in Defn. 9.3. Let B act on $G \times B$ by $b(g,x) = (gb^{-1}, bxb^{-1})$. The corresponding action on $k[G \times B]$ is given by

$$(b.\eta)(g,x) = \eta(gb, b^{-1}xb) \ , \eta \in k[G \times B], b, x \in B, g \in G.$$

The element $\tilde{h}$ in $k[G \times B]$ is clearly invariant under this action. On the other hand

$$\tilde{h} = b.\tilde{h} = \sum_{I,J} (b.f(I,J)) \otimes (b.M(I,J)) \tag{9.4}$$

where B acts on $k[G]$ (resp. $k[B]$) by right (resp. conjugation) translation. We will now calculate the coefficient to $M(\underline{\text{p-1}}, \underline{\text{p-1}})$ on the right side of Equation (9.4), and compare it with the known value $\pi(h)$. The result will follow from this. By Lemma 9.5 we know that the coefficient to $M(\underline{\text{p-1}}, \underline{\text{p-1}})$ in $b.M(I,J)$ is zero, unless there exists I' and J' such that

$$I = \underline{\text{p-1}} + pI' \text{ and } J = \underline{\text{p-1}} + pJ'.$$

But by Lemma 9.4 we know that $f(\underline{\text{p-1}} + pI', \underline{\text{p-1}} + pJ') = 0$, unless $I' = \underline{0}$ and $J' = \underline{0}$. Non zero coefficients to $M(\underline{\text{p-1}}, \underline{\text{p-1}})$ can therefore only come from the term

$$(b.f(\underline{\text{p-1}}, \underline{\text{p-1}})) \otimes (b.M(\underline{\text{p-1}}, \underline{\text{p-1}})) = b.\pi(h) \otimes (b.M(\underline{\text{p-1}}, \underline{\text{p-1}})).$$

Let therefore $c(b) \in k$ denote the coefficient to $M(\underline{\text{p-1}}, \underline{\text{p-1}})$ in $b.M(\underline{\text{p-1}}, \underline{\text{p-1}})$. In this way we get a function $c : B \to k$ such that

$$c(b)(b.\pi(h)) = \pi(h). \tag{9.5}$$

As the result is trivial in case $\pi(h) = 0$, we will from now assume that $\pi(h) \neq 0$. Then $c(b) \in k^*$ and

$$\frac{c(b_1)c(b_2)}{c(b_1 b_2)}\pi(h) = c(b_1)c(b_2)((b_1 b_2).\pi(h)) = c(b_1)(b_1.\pi(h)) = \pi(h).$$

In other word $c : B \to k^*$ is a group homomorphism. Therefore c must be trivial on the unipotent radical U of B. On T we may calculate it by hand. If $b \in T$ then

$$b.t_i = t_i \text{ and } b.e_i = \alpha_i(b)e_i.$$

Therefore

$$b.M(\underline{\text{p-1}}, \underline{\text{p-1}}) = (\prod_i \alpha_i(b))^{p-1} M(\underline{\text{p-1}}, \underline{\text{p-1}}) = b^{2(p-1)\rho} M(\underline{\text{p-1}}, \underline{\text{p-1}}).$$

which implies $c = 2(p-1)\rho$. The result now follows from Equation (9.5). □

9.2.3 Final stage

Definition 9.5 *By Proposition 9.1 the composed map $(\pi \circ \phi) : St \otimes St \to k[G]$ factors through $H^0(G/B, 2(p-1)\rho)$. The resulting map is denoted by*

$$\mu : St \otimes St \to H^0(G/B, 2(p-1)\rho).$$

Notice that we now have the following commutative diagram

$$\begin{tikzcd} St \otimes St \arrow[r, "\mu"] \arrow[d, "\phi"'] & H^0(G/B, 2(p-1)\rho) \arrow[d, hook] \\ H^0(\mathcal{O}_{G\times^B B}) \arrow[r, "\pi"] & k[G] \end{tikzcd}$$

In the following we will let G act on $k[G]$ by left translation, and on $H^0(\mathcal{O}_{G\times^B B})$ by

$$(gh)(x,y) = h(g^{-1}x, y)\ ,\ h \in H^0(\mathcal{O}_{G\times^B B}).$$

With these G-actions it is straight forward by definition to check that π ,ϕ and μ are G-equivariant.

Proposition 9.2 *The maps π,ϕ and μ are G-equivariant.*

Proof Notice that it is enough to show that π and ϕ are G-equivariant. We will first concentrate on ϕ. Let v and w be elements in St. Then

$$\begin{aligned}(g\phi(v\otimes w))(x,y) &= \phi(v\otimes w)(g^{-1}x, y)\\ &=< v, g^{-1}xyx^{-1}gw >\\ &=< gv, xyx^{-1}gw >\\ &= \phi((gv)\otimes(gw))(x,y).\end{aligned}$$

This implies that ϕ is G-equivariant. To show that π is G-equivariant, let h be an element in $H^0(\mathcal{O}_{G\times^B B})$. Compose h with the natural map : $G \times B \to G\times^B B$ and denote the induced function on $G \times B$ with $\tilde{h}$. Write

$$\tilde{h} = \sum_{I,J} f(I,J) \otimes M(I,J).$$

By definition $\pi(h) = f(\underline{\text{p-1}}, \underline{\text{p-1}})$. Let now g be an element in G. Then gh is an an element in $H^0(\mathcal{O}_{G\times^B B})$, and induces as above a function $\widetilde{gh}$ on $G \times B$ given by : $\widetilde{gh}(x,y) = \tilde{h}(g^{-1}x, y)$. Therefore

$$\widetilde{gh} = \sum_{I,J} (gf(I,J)) \otimes M(I,J),$$

which implies that $\pi(gh) = g\pi(h)$. $\square$

By Frobenius reciprocity there is (up to a constant) only one G-equivariant map from $St \otimes St$ to $H^0(G/B, 2(p-1)\rho)$. By Proposition 9.2 the map μ is therefore determined up to a constant. We claim that it is non zero. Before we prove this we need the following result.

Lemma 9.6 *The global regular function $x \mapsto < v^+, xv^+ >$ on U is a Frobenius splitting of U.*

Proof This follows from Proposition 8.1, as $\phi_U(v^+ \otimes v^+)$ in Proposition 8.1 is equal to the function $x \mapsto < v^+, xv^+ >$ above. $\square$

Proposition 9.3 *The map μ is non zero.*

Proof We claim that $f = \mu(v^+ \otimes v^+)$ is non zero. To see this consider the global function $h = \phi(v^+ \otimes v^+)$ on $G \times^B B$. By composing h with the natural map : $G \times B \to G \times^B B$ we get a global function $\tilde{h}$ on $G \times B$. Write

$$\tilde{h} = \sum_{I,J} f(I,J) \otimes M(I,J) \ , \ f(I,J) \in k[G].$$

Then f is by definition equal to $f(\underline{\text{p-1}}, \underline{\text{p-1}})$. By definition we also know

$$\tilde{h}(g,x) = < v^+, gxg^{-1}v^+ > = < g^{-1}v^+, xg^{-1}v^+ > .$$

As v^+ is invariant under U^+ it follows from this, that the restriction $\tilde{h}^+$ of $\tilde{h}$ to $U^+ \times B$ is given by

$$\tilde{h}^+(g,x) = < v^+, xv^+ > .$$

Consider B as $U \times T$. Then as v^+ has T-weight $(p-1)\rho$ it is clear that

$$\tilde{h}^+ = \tilde{h}^+_U \otimes t^{\underline{\text{p-1}}},$$

where $\tilde{h}^+_U$ is the function on U given by $\tilde{h}^+_U(x) = < v^+, xv^+ >$. By Lemma 9.6 we know that $\tilde{h}^+_U$ is a Frobenius splitting of U. Therefore, if we write $\tilde{h}^+_U$ in the basis e^I, the coefficient to $e^{\underline{\text{p-1}}}$ is non zero. But then the coefficient to $M(\underline{\text{p-1}}, \underline{\text{p-1}})$ in $\tilde{h}^+$ (written in the basis $M(I,J)$) is non zero, and this implies that f is nonzero. $\square$

Lemma 9.7 *Let v and w be elements in St. Then $\mu(v \otimes w)$ is (up to a non zero constant) a Frobenius splitting of G/B if and only if $< v, w > \neq 0$.*

Proof As μ is non zero by Proposition 9.3, this follows from Theorem 6.1. $\square$

Now we have arrived at our main result.

Theorem 9.1 *Let v and w be elements in St. Then $\phi(v \otimes w)$ is a Frobenius splitting (with respect to the volume form in Definition 9.4) of $G \times^B B$ if and only if $< v, w > \neq 0$.*

Proof Let $h = \phi(v \otimes w)$. The restriction of h to the open $U^+ \times B$ of $G \times^B B$ is denoted by h^+. We will use the identification

$$k[U^+ \times B] = k[f_1, \ldots, f_N, e_1, \ldots, e_N, t_1, t_1^{-1}, \ldots, t_l, t_l^{-1}].$$

It is then clear that $M(K,I,J) = f^K e^I t^J$ is a k-basis for $k[U^+ \times B]$. Write

$$h^+ = \sum_{K,I,J} a(K,I,J) M(K,I,J) \ , \ a(K,I,J) \in k.$$

Remember (See Lemma 1.2) that h^+ is a Frobenius splitting of $U^+ \times B$ if and only if

$$a(\underline{\text{p-1}}, \underline{\text{p-1}}, \underline{\text{p-1}}) \neq 0$$

and

$$a(\underline{\text{p-1}} + pK', \underline{\text{p-1}} + pI', \underline{\text{p-1}} + pJ') = 0, \text{ if } (K', I', J') \neq (\underline{0}, \underline{0}, \underline{0}).$$

As h is a Frobenius splitting of $G \times^B B$ if and only if h^+ is a Frobenius splitting of $U^+ \times B$, we may restrict our attention to these two conditions. Let $\tilde{h}$ be the regular function on $G \times B$ which is the composition of h and the natural map $G \times B \to G \times^B B$. Write

$$\tilde{h} = \sum_{I,J} f(I,J) \otimes M(I,J) \ , \ f(I,J) \in k[G].$$

If $f(I,J)_{|U^+}$ denotes the restriction of $f(I,J)$ to U^+ we have the following relation

$$f(I,J)_{|U^+} = \sum_K a(K,I,J) f^K. \tag{9.6}$$

Assume that there exist K',I' and J' such that

$$a(\underline{\text{p-1}} + pK', \underline{\text{p-1}} + pI', \underline{\text{p-1}} + pJ') \neq 0.$$

By Lemma 9.4 and Equation 9.6, we immediately conclude that $I' = \underline{0}$ and $J' = \underline{0}$. Consider therefore

$$\pi(h)_{|U^+} = f(\underline{\text{p-1}}, \underline{\text{p-1}})_{|U^+} = \sum_K a(K, \underline{\text{p-1}}, \underline{\text{p-1}}) f^K.$$

By Lemma 9.7 this is a Frobenius splitting of U^+ if and only if $< v, w > \neq 0$. Let us first consider the case $< v, w > \neq 0$. Then

$$a(\underline{\text{p-1}} + pK', \underline{\text{p-1}}, \underline{\text{p-1}}) \neq 0$$

exactly when $K' = \underline{0}$. This implies both of our conditions for h^+ to be a Frobenius splitting. Assume therefore that $< v, w >= 0$. Then $\pi(h)_{|U^+}$ is not a Frobenius splitting of U^+ and either there exist a $K' \neq \underline{0}$ such that $a(\underline{\text{p-1}} + pK', \underline{\text{p-1}}, \underline{\text{p-1}}) \neq 0$, or $a(\underline{\text{p-1}}, \underline{\text{p-1}}, \underline{\text{p-1}}) = 0$. Both cases implies that h^+ is not a Frobenius splitting of $U^+ \times B$. $\Box$

9.3 Frobenius splitting of $G \times^B U$

In this section we will show that the restriction of $\phi(v \otimes w)$ to $G \times^B U$ is a Frobenius splitting of $G \times^B U$ if and only if $< v, w > \neq 0$. One way of doing this is to give a similar proof as the one for $G \times^B B$. In fact, this turns out to be much easier than the case $G \times^B B$. Nevertheless we will here try to use what we already have proven in the section above. In particular, we will keep the notation from the section above.

Definition 9.6 *Let $\bar{\phi} : St \times St \to H^0(\mathcal{O}_{G \times^B U})$ be the map defined by*

$$\bar{\phi}(v \otimes w)(g, x) =< v, gxg^{-1}w > .$$

Definition 9.7 *Define $\bar{\pi} : H^0(\mathcal{O}_{G\times^B U}) \to k[G]$ as follows. Let h be a global function on $G \times^B U$. Composing h with the natural map : $G \times U \to G \times^B U$ we get a global function $\tilde{h}$ on $G \times U$. Write $\tilde{h}$ on the form*

$$\tilde{h} = \sum_I f(I) \otimes e^I \ , \ f(I) \in k[G].$$

Then $\bar{\pi}(h) = f(\underline{p\text{-}1})$.

9.3.1 The canonical sheaf

We need the know that the set of regular functions on $G \times^B U$, is the place to look for Frobenius splittings of $G \times^B U$. In other words we would like to know that the canonical sheaf $\omega_{G\times^B U}$ is trivial.

Lemma 9.8 *Let b be an element in B and γ be the automorphism of $k[U]$ given by*

$$\gamma(f)(x) = f(b^{-1}xb).$$

Then the induced map $\tilde{\gamma}$ on ω_U, satisfies

$$\tilde{\gamma}(de_1 \wedge \cdots \wedge de_N) = b^{2\rho}(de_1 \wedge \cdots \wedge de_N),$$

where 2ρ is regarded as a B-character.

Proof We may assume that $b \in T$ or $b \in U$. If $b \in T$ then $\gamma(e_i) = \alpha_i(b)e_i$ and the result follows. The case $b \in U$ follows from Lemma 9.1. $\square$

Corollary 9.2 *The canonical sheaf $\omega_{G\times^B U}$ is trivial.*

Proof This follows from Lemma 12 in [13] and Lemma 9.8 above. $\square$

Definition 9.8 *Any restriction to $U^+ \times U$ of a volume form on $G \times^B U$ is of the form*

$$a(df_1 \wedge \cdots \wedge f_N \wedge de_1 \wedge \cdots \wedge df_N) \ , \ a \in k^*.$$

Define $d(G \times^B U)$ to be the volume form with $a = 1$. It is with respect to this volume form that regular function are thought of being Frobenius splittings.

9.3.2 Moving on

Corollary 9.3 *The following relation is true*

$$\bar{\pi} \circ \bar{\phi} = \pi \circ \phi.$$

Proof Let v and w be elements in St and $h = \phi(v \otimes w)$. Denote the composition of h and the natural map $G \times B \to G \times^B B$ by $\tilde{h}$, and write

$$\tilde{h} = \sum_{I,J} f(I,J) \otimes M(I,J) \ , \ f(I,J) \in k[G].$$

Then clearly

$$\bar{\pi}(\bar{\phi}(v \otimes w)) = \sum_J f(\underline{\text{p-1}}, J),$$

which by Lemma 9.4 is equal to $f(\underline{\text{p-1}}, \underline{\text{p-1}}) = \pi(\phi(v \otimes w))$. $\square$

Lemma 9.9 *Let v and w be elements in St and let $h = \bar{\phi}(v \otimes w)$. Composing h with the natural map : $G \times U \to G \times^B U$ we get a global function $\tilde{h}$ on $G \times U$. Write $\tilde{h}$ on the form*

$$\tilde{h} = \sum_I f(I) \otimes e^I \ , \ f(I) \in k[G].$$

Then $f(\underline{p\text{-}1} + pI') = 0$ if $I' \neq 0$.

Proof Let $h' = \phi(v \otimes w)$. Composing h' with the natural map : $G \times B \to G \times^B B$ we get a global function $\tilde{h}'$ on $G \times B$. Write $\tilde{h}'$ on the form

$$\tilde{h}' = \sum_{I,J} f(I,J) \otimes M(I,J) \ , \ f(I,J) \in k[G].$$

Then clearly

$$f(I) = \sum_J f(I,J),$$

and the result follows from Lemma 9.4. □

9.3.3 And finally...

Corollary 9.4 *Let v and w be elements in St. Then $\bar{\phi}(v \otimes w)$ is a Frobenius splitting (with respect to the volume form in Definition 9.8) of $G \times^B U$ if and only if $< v, w > \neq 0$.*

Proof Let $h = \phi(v \otimes w)$. The restriction of h to the open $U^+ \times U$ of $G \times^B U$ is denoted by h^+. We will use the identification

$$k[U^+ \times U] = k[f_1, \ldots, f_N, e_1, \ldots, e_N].$$

It is then clear that $\bar{M}(K, I) = f^K e^I$ is a natural k-basis for $k[U^+ \times U]$. Write

$$h^+ = \sum_{K,I} a(K,I) \bar{M}(K,I) \ , \ a(K,I) \in k.$$

Remember (see Lemma 1.2) that h^+ is a Frobenius splitting of $U^+ \times U$ if and only if

$$a(\underline{\text{p-1}}, \underline{\text{p-1}}) \neq 0$$

and

$$a(\underline{\text{p-1}} + pK', \underline{\text{p-1}} + pI') = 0, \text{ if } (K', I') \neq (\underline{0}, \underline{0}).$$

As h is a Frobenius splitting of $G \times^B U$ if and only if h^+ is a Frobenius splitting of $U^+ \times U$, we may restrict our attention to these two conditions. Let $\tilde{h}$ be the regular function on $G \times U$ which is the composition of h and the natural map $G \times U \to G \times^B U$. Write

$$h = \sum_I f(I) \otimes e^I \ , \ f(I) \in k[G].$$

If $f(I)_{|U^+}$ denotes the restriction of $f(I)$ to U^+ we have the following relation

$$f(I)_{|U^+} = \sum_K a(K,I) f^K. \tag{9.7}$$

Assume that there exist K' and I' such that

$$a(\underline{\text{p-1}} + pK', \underline{\text{p-1}} + pI') \neq 0.$$

By Lemma 9.9 and Equation (9.7) we immediately conclude that $I' = \underline{0}$. Consider therefore

$$\pi(h)_{|U^+} = f(\underline{\text{p-1}})_{|U^+} = \sum_K a(K, \underline{\text{p-1}}) f^K.$$

By Lemma 9.7 and Corollary 9.3 this is a Frobenius splitting of U^+ if and only if $< v, w > \neq 0$. Let us first consider the case $< v, w > \neq 0$. Then

$$a(\underline{\text{p-1}} + pK', \underline{\text{p-1}}) \neq 0$$

exactly when $K' = \underline{0}$. This implies both of our conditions for h^+ to be a Frobenius splitting. Assume therefore that $< v, w >= 0$. Then $\pi(h)_{|U^+}$ is not a Frobenius splitting of U^+ and either there exist a $K' \neq \underline{0}$ such that $a(\underline{\text{p-1}} + pK', \underline{\text{p-1}}) \neq 0$, or $a(\underline{\text{p-1}}, \underline{\text{p-1}}) = 0$. Both cases implies that h^+ is not a Frobenius splitting of $U^+ \times B$. □

9.4 Compatibly splitting

Let us start this section with an example which shows that not everything is as nice as one could hope.

Example 9.1 *Let $V = \operatorname{Spec} k[X, Y]$ and $Z = \operatorname{Spec} k[X]$ be the closed subscheme given by the ideal (Y). The global regular function $f = X^{p-1} + X^{p-1}Y^{p-1}$ is clearly a Frobenius splitting of V, with respect to the volume form $dX \wedge dY$. Furthermore, the restriction $f_{|Z} = X^{p-1}$ of f to Z is a Frobenius splitting of Z, with respect to the volume form dY. But this does not mean that f compatible splits Z, as Y^{p-1} (up to a non zero constant) maps to 1 under the Frobenius splitting f.*

We may therefore not conclude that functions of the form $\phi(v \otimes w)$ with $< v, w > \neq 0$ is a Frobenius splitting of $G \times^B B$ which compatibly splits $G \times^B U$. In fact, we do not know of any examples of this.

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THE FROBENIUS MORPHISM ON A TORIC VARIETY

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Abstract. We give a characteristic p proof of the Bott vanishing theorem [4] for projective toric varieties using that the Frobenius morphism on a toric variety lifts to characteristic p^2. A proof of the Bott vanishing theorem was previously known only in the simplicial case [2]. We also generalize the work of Paranjape and Srinivas [14] about non-liftability to characteristic zero of the Frobenius morphism on flag varieties by showing that Bott vanishing fails for a large class of flag varieties not isomorphic to a product of projective spaces.

Let X be a projective toric variety over a field k. In [4] Danilov states the Bott vanishing theorem

$$\mathrm{H}^i(X, \tilde{\Omega}^j_{X/k} \otimes L) = 0$$

where $\tilde{\Omega}^j_{X/k}$ denotes the Zariski differentials, L is an ample line bundle on X and $i > 0$. Batyrev and Cox proves this theorem in the simplicial case in [2]. The purpose of this paper is to show that the Bott vanishing theorem is a simple consequence of a very specific condition on the Frobenius morphism in prime characteristic p.

Assume now that $k = \mathbb{Z}/p\mathbb{Z}$, where $p > 0$ and let X be any smooth variety over k. Recall that the absolute Frobenius morphism $F : X \to X$ on X is the identity on point spaces and the p-th power map locally on functions. Assume that there is a flat scheme $X^{(2)}$ over $\mathbb{Z}/p^2\mathbb{Z}$, such that $X \cong X^{(2)} \times_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. The condition on F is that there should be a morphism $F^{(2)} : X^{(2)} \to X^{(2)}$ which gives F by reduction mod p. In this case we will say that the Frobenius morphism lifts to $\mathbb{Z}/p^2\mathbb{Z}$. It is known that a lift of the Frobenius morphism to $\mathbb{Z}/p^2\mathbb{Z}$ leads to a quasi-isomorphism

$$\sigma : \bigoplus_{0 \le i} \Omega^i_X[-i] \to F_* \Omega^\bullet_X$$

where the complex on the left has zero differentials and $\Omega^\bullet_X$ denotes the de Rham complex of X [5, Remarques 2.2(ii)]. Using duality we prove that σ is in fact a split quasi-isomorphism.

In general it is very difficult to decide when Frobenius lifts to $\mathbb{Z}/p^2\mathbb{Z}$. However for varieties which are glued together by monomial automorphisms it is easy. This is the case for toric varieties, where we show that the Frobenius morphism lifts to $\mathbb{Z}/p^2\mathbb{Z}$. This places the Bott vanishing theorem for (singular and smooth) toric varieties and the degeneration of the Danilov spectral sequence [4, Theorem 7.5.2, Theorem 12.5] in a natural characteristic p framework.

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In the second half of this paper we study the Frobenius morphism on flag varieties. This is related to the work of Paranjape and Srinivas [14]. They have proved using complex algebraic geometry that if Frobenius for a flag variety X over k lifts to the p-adic numbers $\mathbb{Z}_p = \operatorname{proj\,lim}_n \mathbb{Z}/p^n\mathbb{Z}$, then X is a product of projective spaces. We generalize this result by showing that Frobenius for a large class of flag varieties admits no lift to $\mathbb{Z}/p^2\mathbb{Z}$. This is done using a lemma on fibrations linking non-lifting of Frobenius to Bott non-vanishing cohomology groups for flag varieties of Hermitian symmetric type over the complex numbers. These cohomology groups have been studied thoroughly by M.-H. Saito and D. Snow. It seems likely that if X is a flag variety over $\mathbb{C}$ for which the Bott vanishing theorem holds, then X is a product of projective spaces.

Part of these results have been announced in [1]. We are grateful to D. Cox for his interest in this work and for pointing out the paper [2]. We thank the referee for pointing out several inaccuracies and for carefully reading the manuscript.

1. Preliminaries

Throughout this paper k will denote a perfect field of characteristic $p > 0$ and X a smooth k-variety unless otherwise stated.

Let $n = \dim X$. By Ω_X we denote the sheaf of k-differentials on X and $\Omega_X^j = \wedge^j \Omega_X$. The absolute Frobenius morphism $F : X \to X$ is the morphism on X, which is the identity on the level of points and given by $F^{\#} : \mathcal{O}_X(U) \to F_*\mathcal{O}_X(U)$, $F^{\#}(f) = f^p$ on the level of functions. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $F_*\mathcal{F} = \mathcal{F}$ as sheaves of abelian groups, but the $\mathcal{O}_X$-module structure is changed according to the homomorphism $\mathcal{O}_X \to F_*\mathcal{O}_X$.

1.1. The Cartier operator.

The universal derivation $d : \mathcal{O}_X \to \Omega_X$ gives rise to a family of k-homomorphisms $d^j : \Omega_X^j \to \Omega_X^{j+1}$ making $\Omega_X^\bullet$ into a complex of k-modules which is called the de Rham complex of X. By applying F_* to the de Rham complex, we obtain a complex $F_*\Omega_X^\bullet$ of $\mathcal{O}_X$-modules. Let $B_X^i \subseteq Z_X^i \subseteq F_*\Omega_X^i$ denote the coboundaries and cocycles in degree i. There is the following very nice description of the cohomology of $F_*\Omega_X^\bullet$ due to Cartier [3]

Theorem 1. *There is a uniquely determined graded $\mathcal{O}_X$-algebra isomorphism*

$$C^{-1} : \Omega_X^\bullet \to \mathcal{H}^\bullet(F_*\Omega_X^\bullet)$$

which in degree 1 *is given locally as*

$$C^{-1}(da) = a^{p-1}da.$$

Proof [3] and [9, Theorem 7.2]. □

With some abuse of notation, we let C denote the natural homomorphism $Z_X^i \to \Omega_X^i$, which after reduction modulo B_X^i gives the inverse isomorphism to C^{-1}. The isomorphism $\bar{C} : Z_X^i/B_X^i \to \Omega_X^i$ is called the Cartier operator.

2. Liftings of the Frobenius to $W_2(k)$

There is a very interesting connection [13, §5.3] between the Cartier operator and liftings of the Frobenius morphism to flat schemes of characteristic p^2 due to Mazur. We go on to explore this next.

2.1. **Witt vectors of length two.** The Witt vectors $W_2(k)$ (cf., e.g., [11, Lecture 26]) of length 2 over k can be interpreted as the set $k \times k$, where multiplication and addition for $a = (a_0, a_1)$ and $b = (b_0, b_1)$ in $W_2(k)$ are defined by

$$a\,b = (a_0\,b_0, a_0^p b_1 + b_0^p a_1)$$

and

$$a + b = (a_0 + b_0, a_1 + b_1 + \sum_{j=1}^{p-1} p^{-1} \binom{p}{j} a_0^j\, b_0^{p-j}).$$

In the case $k = \mathbb{Z}/p\mathbb{Z}$, one can prove that $W_2(k) \cong \mathbb{Z}/p^2\mathbb{Z}$. The projection on the first coordinate $W_2(k) \to k$ corresponds to the reduction $W_2(k) \to W_2(k)/pW_2(k) \cong k$ modulo p. The ring homomorphism $F^{(2)}$: $W_2(k) \to W_2(k)$ given by $F^{(2)}(a_0, a_1) = (a_0^p, a_1^p)$ reduces to the Frobenius homomorphism F on k modulo p.

2.2. **Splittings of the de Rham complex.** The previous section shows that there is a canonical morphism $\operatorname{Spec} k \to \operatorname{Spec} W_2(k)$. Assume that there is a flat scheme $X^{(2)}$ over $\operatorname{Spec} W_2(k)$ such that

$$X \cong X^{(2)} \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k. \tag{1}$$

We shall say that the Frobenius morphism F lifts to $W_2(k)$ if there exists a morphism $F^{(2)} : X^{(2)} \to X^{(2)}$ covering the Frobenius homomorphism $F^{(2)}$ on $W_2(k)$, which reduces to F via the isomorphism (1). When we use the statement that Frobenius lifts to $W_2(k)$ we will always implicitly assume the existence of the flat lift $X^{(2)}$.

Theorem 2. *If the Frobenius morphism on X lifts to $W_2(k)$ then there is a split quasi-isomorphism*

$$0 \to \bigoplus_{0 \le i} \Omega_X^i[-i] \xrightarrow{\sigma} F_* \Omega_X^\bullet$$

Proof For an affine open subset $\operatorname{Spec} A^{(2)} \subseteq X^{(2)}$ there is a ring homomorphism $F^{(2)} : A^{(2)} \to A^{(2)}$ such that

$$F^{(2)}(b) = b^p + p \cdot \varphi(b)$$

where $\varphi : A^{(2)} \to A = A^{(2)}/pA^{(2)}$ is some function and $p\,\cdot : A \to A^{(2)}$ is the $A^{(2)}$-homomorphism derived from tensoring the short exact sequence of $W_2(k)$-modules

$$0 \to p\,W_2(k) \to W_2(k) \xrightarrow{p\,\cdot} p\,W_2(k) \to 0$$

with the flat $W_2(k)$-module $A^{(2)}$ identifying $A \cong A^{(2)}/pA^{(2)}$ with $p\,A^{(2)}$. We get the following properties of φ:

$$\varphi(a+b) = \varphi(a) + \varphi(b) - \sum_{j=1}^{p-1} p^{-1}\binom{p}{j}\bar{a}^j\bar{b}^{p-j}$$
$$\varphi(a\,b) = \bar{a}^p\varphi(b) + \bar{b}^p\varphi(a)$$

where the bar means reduction modulo p. Now it follows that

$$a \mapsto a^{p-1}da + d\varphi(\tilde{a})$$

where $\tilde{a}$ is any lift of a, is a well defined derivation $\delta : A \to Z^1_{\mathrm{Spec}\,A} \subset F_*\Omega^1_{\mathrm{Spec}\,A}$, which gives a homomorphism $\varphi : \Omega^1_{\mathrm{Spec}\,A} \to Z^1_{\mathrm{Spec}\,A} \subset F_*\Omega^1_{\mathrm{Spec}\,A}$. This homomorphism can be extended via the algebra structure to give an A-algebra homomorphism $\sigma : \oplus_i\Omega^i_{\mathrm{Spec}\,A} \to Z^\bullet_{\mathrm{Spec}\,A} \subseteq F_*\Omega^\bullet_{\mathrm{Spec}\,A}$, which composed with the canonical homomorphism $Z^\bullet_{\mathrm{Spec}\,A} \to \mathcal{H}^\bullet(F_*\Omega^\bullet_{\mathrm{Spec}\,A})$ gives the inverse Cartier operator. Since an affine open covering $\{\mathrm{Spec}\,A^{(2)}\}$ of $X^{(2)}$ gives rise to an affine open covering $\{\mathrm{Spec}\,A^{(2)}/pA^{(2)}\}$ of X, we have proved that σ is a quasi-isomorphism of complexes inducing the inverse Cartier operator on cohomology.

Now we give a splitting homomorphism of each component $\sigma_i : \Omega^i_X \to F_*\Omega^i_X$. Notice that $\sigma_0 : \mathcal{O}_X \to F_*\mathcal{O}_X$ is the Frobenius homomorphism and that σ_i $(i > 0)$ splits C in the exact sequence

$$0 \to B^i_X \to Z^i_X \xrightarrow{C} \Omega^i_X \to 0.$$

Noting that $\oplus_i Z_i$ is an ideal in the $\mathcal{O}_X$-algebra $F_*\Omega^{\cdot}_X$ there is a well defined homomorphism

$$\varphi : F_*\Omega^i_X \to \mathcal{H}om_X(\Omega^{n-i}_X, \Omega^n_X)$$

given by $\omega \mapsto \varphi(\omega)$, where $\varphi(\omega)(\eta) = C(\sigma_{n-i}(\eta) \wedge \omega)$. Evaluating φ on $\sigma_i(z)$, where z is an i-form, one gets

$$\varphi(\sigma_i(z))(\eta) = C(\sigma_{n-i}(\eta) \wedge \sigma_i(z)) = C(\sigma_n(\eta \wedge z)) = \eta \wedge z.$$

Now using the perfect duality between Ω^{n-i}_X and Ω^i_X given by the wedge product one obatins the desired splitting of σ_i. □

2.3. **Bott vanishing.** Let X be a normal variety and let j denote the inclusion of the smooth locus $U \subseteq X$. If the Frobenius morphism lifts to $W_2(k)$ on X, then the Frobenius morphism on U also lifts to $W_2(k)$. Define the Zariski sheaf $\tilde{\Omega}^i_X$ of i-forms on X as $j_*\Omega^i_U$. Since $\mathrm{codim}(X-U) \geq 2$ it follows (cf., e.g., [7, Proposition 5.10]) that $\tilde{\Omega}^i_X$ is a coherent sheaf on X.

Theorem 3. *Let X be a projective normal variety such that F lifts to $W_2(k)$. Then*

$$\mathrm{H}^s(X, \tilde{\Omega}^r_X \otimes L) = 0$$

for $s > 0$ and L an ample line bundle.

Proof Let U be the smooth locus of X and let j denote the inclusion of U into X. On U we have by Theorem 2 a split sequence

$$0 \to \Omega^r_U \to F_*\Omega^r_U$$

which pushes down to the split sequence (F commutes with j)

$$0 \to \tilde{\Omega}^r_X \to F_*\tilde{\Omega}^r_X$$

Now tensoring with L and using the projection formula we get injections for $s > 0$

$$\mathrm{H}^s(X, \tilde{\Omega}^r_X \otimes L) \hookrightarrow \mathrm{H}^s(X, \tilde{\Omega}^r_X \otimes L^{\otimes p})$$

Iterating these injections and noting that the Zariski sheaves are coherent one gets the desired vanishing theorem by Serre's cohomological ampleness criterion [8, Proposition III.5.3]. □

2.4. **Degeneration of the Hodge to de Rham spectral sequence.** Let X be a projective normal variety with smooth locus U. Associated with the complex $\tilde{\Omega}^\bullet_X$ there is a spectral sequence

$$E_1^{pq} = \mathrm{H}^q(X, \tilde{\Omega}^p_X) \implies \mathrm{H}^{p+q}(X, \tilde{\Omega}^\bullet_X)$$

where $\mathrm{H}^\bullet(X, \tilde{\Omega}^\bullet_X)$ denotes the hypercohomology of the complex $\tilde{\Omega}^\bullet_X$. This is the Hodge to de Rham spectral sequence for Zariski sheaves.

Theorem 4. *If the Frobenius morphism on X lifts to $W_2(k)$, then the spectral sequence degenerates at the E_1-term.*

Proof As complexes of sheaves of abelian groups $\tilde{\Omega}^\bullet$ and $F_*\tilde{\Omega}^\bullet$ are the same so their hypercohomology agree. Applying hypercohomology to the split injection (Theorem 2)

$$\sigma : \bigoplus_{0 \le i} \tilde{\Omega}^i_{X/k}[-i] \to F_*\tilde{\Omega}^\bullet_X$$

we get

$$\begin{aligned} \sum_{p+q=n} \dim_k E_\infty^{pq} = \dim_k \mathrm{H}^n(X, \tilde{\Omega}^\bullet_X) &= \dim_k \mathrm{H}^n(X, F_*\tilde{\Omega}^\bullet_X) \geq \\ \sum_{p+q=n} \dim_k \mathrm{H}^q(X, \tilde{\Omega}^p_X) &= \sum_{p+q=n} \dim_k E_1^{pq} \end{aligned}$$

Since E_∞^{pq} is a subquotient of E_1^{pq}, it follows that $\sum_{p+q=n} \dim_k E_\infty^{p,q} \leq \sum_{p+q=n} \dim_k E_1^{p,q}$ holds, hence $E_\infty^{pq} \cong E_1^{pq}$, so that the spectral sequence degenerates at E_1. □

3. Toric varieties

In this section we briefly sketch the definition of toric varieties [12, 6] and demonstrate how the results of Section 2 may be applied.

3.1. **Convex geometry.** Let N be a lattice, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ the dual lattice, and let V be the real vector space $V = N \otimes_{\mathbb{Z}} \mathbb{R}$. It is natural to identify the dual space V^* of V with $M \otimes_{\mathbb{Z}} \mathbb{R}$, and we think of $N \subset V$ and $M \subset V^*$ as the subsets of integer points.

By a cone in N we will mean a subset $\sigma \subset V$ taking the form $\sigma = \{r_1 v_1 + \cdots + r_s v_s \mid r_i \geq 0\}$ for some $v_i \in N$. The vectors $v_1, \ldots, v_s$ are called generators of σ. We define the dual cone to be $\sigma^\vee = \{u \in V^* | \langle u, v\rangle \geq 0, \forall v \in \sigma\}$. One may show that $\sigma^\vee$ is a cone in M. A face of σ is any set $\sigma \cap u^\perp$ for some $u \in \sigma^\vee$. Any face of σ is clearly a cone in N, generated by the v_i for which $\langle u, v_i\rangle = 0$.

Now let σ be a strongly convex cone in N, which means that $\{0\}$ is a face of σ or equivalently that no nontrivial subspace of V is contained in σ.

We define S_σ to be the semigroup $\sigma^\vee \cap M$. Since $\sigma^\vee$ is a cone in M, S_σ is finitely generated.

3.2. **Affine toric schemes.** If k is any commutative ring the semigroup ring $k[S_\sigma]$ is a finitely generated commutative k-algebra, and $U_\sigma = \operatorname{Spec} k[S_\sigma]$ is an affine scheme of finite type over k. Schemes of this form are called affine toric schemes.

3.3. **Glueing affine toric schemes.** Let $\tau = \sigma \cap u^\perp$ be a face of σ. One may assume that $u \in S_\sigma$. Then it follows that $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u)$, so that $k[S_\tau] = k[S_\sigma]_u$. In this way U_τ becomes a principal open subscheme of U_σ. This may be used to glue affine toric schemes together. We define a fan in N to be a nonempty set Δ of strongly convex cones in N satisfying that the faces of any cone in Δ are also in Δ and the intersection of two cones in Δ is a face of each. The affine schemes arising from cones in Δ may be glued together to form a scheme $X_k(\Delta)$ as follows. If $\sigma, \tau \in \Delta$, then $\sigma \cap \tau \in \Delta$ is a face of both τ and σ, so $U_{\sigma\cap\tau}$ is isomorphic to open subsets $U_{\sigma\tau}$ in U_σ and $U_{\tau\sigma}$ in U_τ. Take the transition morphism $\phi_{\sigma\tau} : U_{\sigma\tau} \to U_{\tau\sigma}$ to be the one going through $U_{\sigma\cap\tau}$. A scheme $X_k(\Delta)$ arising from a fan Δ in some lattice is called a toric scheme.

3.4. **Liftings of the Frobenius morphism on toric varieties.** Let $X = X_k(\Delta)$ be a toric scheme over the commutative ring k of characteristic $p > 0$. We are going to construct explicitly a lifting of the absolute Frobenius morphism on X to $W = W_2(k)$. Define $X^{(2)}$ to be $X_W(\Delta)$. Since all the rings $W[S_\sigma]$ are free W-modules, this is clearly a flat scheme over $W_2(k)$. Moreover, the identities $W[S_\sigma] \otimes_W k \cong k[S_\sigma]$ immediately give an isomorphism $X^{(2)} \times_{\operatorname{Spec} W} \operatorname{Spec} k \cong X$.

For $\sigma \in \Delta$, let $F_\sigma^{(2)} : W[S_\sigma] \to W[S_\sigma]$ be the ring homomorphism extending $F^{(2)} : W \to W$ and mapping $u \in S_\sigma$ to u^p. It is easy to see that these maps are compatible with the transition morphisms, so we may take $F^{(2)} : X^{(2)} \to X^{(2)}$ to be the morphism which is defined by $F_\sigma^{(2)}$ locally on $\operatorname{Spec} W[S_\sigma]$. This gives the lift of F to $W_2(k)$ and completes the construction.

3.5. **Bott vanishing and the Danilov spectral sequence.** Since toric varieties are normal we get the following corollary of Section 2:

Theorem 5. *Let X be a projective toric variety over over a perfect field k of characteristic $p > 0$. Then*

$$\mathrm{H}^q(X, \tilde{\Omega}_X^p \otimes L) = 0$$

where $q > 0$ and L is an ample line bundle. Furthermore the Danilov spectral sequence

$$E_1^{pq} = \mathrm{H}^q(X, \tilde{\Omega}_X^p) \implies \mathrm{H}^{p+q}(X, \tilde{\Omega}_X^\bullet)$$

degenerates at the E_1-term.

Remark 1. *One may use the above to prove similar results in characteristic zero. The key issue is that we have proved that Bott vanishing and degeneration of the Danilov spectral sequence holds in any prime characteristic. Also using the Poincaré residue map on the weight filtration of the logarithmic* *de Rham complex* [4, §15.7], *one may prove that the Bott vanishing theorem implies the vanishing theorem of Batyrev and Cox* [2, Theorem 7.2] *for general projective toric varieties.*

4. Flag varieties

In this section we generalize a result due to Paranjape and Srinivas on the non-lifting of Frobenius on flag varieties not isomorphic to products of projective spaces. The key issue is that one can reduce to flag varieties with rank 1 Picard group. In many of these cases one can exhibit ample line bundles with Bott non-vanishing.

We now set up notation. In this section k will denote an algebraically closed field of characteristic $p > 0$ and varieties are k-varieties. Let G be a semisimple algebraic group and fix a Borel subgroup B in G. Recall that (reduced) parabolic subgroups $P \supseteq B$ are given by subsets of the simple root subgroups of B. These correspond bijectively to subsets of nodes in the Dynkin diagram associated with G. A parabolic subgroup Q is contained in P if and only if the simple root subgroups in Q is a subset of the simple root subgroups in P. A maximal parabolic subgroup is the maximal parabolic subgroup not containing a specific simple root subgroup.

We shall need the following result from the appendix to [10]:

Proposition 1. *Let X be a smooth variety. If the sequence*

$$0 \to B^1_X \to Z^1_X \xrightarrow{C} \Omega^1_X \to 0$$

splits, then the Frobenius morphism on X lifts to $W_2(k)$.

We also need the following fact derived from, for instance, [8, Proposition II.8.12 and Exercise II.5.16(d)].

Proposition 2. *Let $f : X \to Y$ be a smooth morphism between smooth varieties X and Y. Then for every $n \in \mathbb{N}$ there is a filtration $F^0 \supseteq F^1 \supseteq \dots$ of Ω^n_X such that*

$$F^i/F^{i+1} \cong f^*\Omega^i_Y \otimes \Omega^{n-i}_{X/Y}.$$

Lemma 1. *Let $f : X \to Y$ be a surjective, smooth and projective morphism between smooth varieties X and Y such that the fibers have no non-zero global n-forms, where $n > 0$. Then there is a canonical isomorphism*

$$\Omega^\bullet_Y \to f_*\Omega^\bullet_X$$

and a splitting $\sigma : \Omega^1_X \to Z^1_X$ of the Cartier operator $C : Z^1_X \to \Omega^1_X$ induces a splitting $f_*\sigma : \Omega^1_Y \to Z^1_Y$ of $C : Z^1_Y \to \Omega^1_Y$.

Proof Notice first that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism of rings as f is projective and smooth. The assumption on the fibers translates into $f_*\Omega^n_{X/Y} \otimes k(y) \cong \mathrm{H}^0(X_y, \Omega^n_{X_y}) = 0$ for geometric points $y \in Y$, when $n > 0$. So we get $f_*\Omega^n_{X/Y} = 0$ for $n > 0$. By Proposition 2 this means that all of the natural homomorphisms $\Omega^n_Y \to f_*\Omega^n_X$ induced by $\mathcal{O}_Y \to f_*\mathcal{O}_X \to f_*\Omega^1_X$ are isomorphisms giving an isomorphism of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \Omega^1_Y & \longrightarrow & \Omega^2_Y & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f_*\mathcal{O}_X & \longrightarrow & f_*\Omega^1_X & \longrightarrow & f_*\Omega^2_X & \longrightarrow & \dots
\end{array}$$

This means that the middle arrow in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B^1_Y \arrow[r] \arrow[d] & Z^1_Y \arrow[r, "C"] \arrow[d] & \Omega_Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & f_*B^1_X \arrow[r] & f_*Z^1_X \arrow[r, "f_*C"] & f_*\Omega_X \arrow[r] & 0 \end{tikzcd}$$

is an isomorphism and the result follows. □

Corollary 1. *Let $Q \subseteq P$ be two parabolic subgroups of G. If the Frobenius morphism on G/Q lifts to $W_2(k)$, then the Frobenius morphism on G/P lifts to $W_2(k)$.*

Proof It is well known that $G/Q \to G/P$ is a smooth projective fibration, where the fibers are isomorphic to $Z = P/Q$. Since Z is a rational projective smooth variety it follows from [8, Exercise II.8.8] that $\mathrm{H}^0(Z, \Omega^n_Z) = 0$ for $n > 0$. Now the result follows from Lemma 1 and Proposition 1. □

In specific cases one can prove using the "standard" exact sequences that certain flag varieties do not have Bott vanishing. We go on to do this next.

Let Y be a smooth divisor in a smooth variety X. Suppose that Y is defined by the sheaf of ideals $I \subseteq \mathcal{O}_X$. Then [8, Proposition II.8.17(2) and Exercise II.5.16(d)], for instance, gives for $n \in \mathbb{N}$ an exact sequence of $\mathcal{O}_Y$-modules

$$0 \to \Omega^{n-1}_Y \otimes I/I^2 \to \Omega^n_X \otimes \mathcal{O}_Y \to \Omega^n_Y \to 0.$$

From this exact sequence and induction on n it follows that $\mathrm{H}^0(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n} \otimes \mathcal{O}(m)) = 0$, when $m \leq j$ and $j > 0$.

4.1. **Quadric hypersurfaces in $\mathbb{P}^n$.** Let Y be a smooth quadric hypersurface in $\mathbb{P}^n$, where $n \geq 4$. There is an exact sequence

$$0 \to \mathcal{O}_Y(1-n) \to \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}(3-n) \otimes \mathcal{O}_Y \to \Omega^1_Y \otimes \mathcal{O}_Y(3-n) \to 0.$$

From this it is easy to deduce that

$$\mathrm{H}^{n-2}(Y, \Omega^1_Y \otimes \mathcal{O}_Y(3-n)) \cong \mathrm{H}^1(Y, \Omega^{n-2}_Y \otimes \mathcal{O}_Y(n-3)) \cong k$$

using that $\mathrm{H}^0(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n} \otimes \mathcal{O}(m)) = 0$, when $m \leq j$ and $j > 0$.

4.2. **The incidence variety in $\mathbb{P}^n \times \mathbb{P}^n$.** Let X be the incidence variety of lines and hyperplanes in $\mathbb{P}^n \times \mathbb{P}^n$, where $n \geq 2$. Recall that X is the zero set of $x_0y_0 + \cdots + x_ny_n$, so that there is an exact sequence

$$0 \to \mathcal{O}(-1) \times \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^n} \times \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0.$$

Using the Künneth formula it is easy to deduce that

$$\mathrm{H}^{2n-2}(X, \Omega^1_X \otimes \mathcal{O}(1-n) \times \mathcal{O}(1-n)) \cong \mathrm{H}^1(X, \Omega^{2n-2} \otimes \mathcal{O}(n-1) \times \mathcal{O}(n-1)) \cong k.$$

4.3. **Bott non-vanishing for flag varieties.** In this section we search for specific maximal parabolic subgroups P and ample line bundles L on $Y = G/P$, such that

$$\mathrm{H}^i(Y, \Omega^j_Y \otimes L) \neq 0$$

where $i > 0$. These are instances of Bott non-vanishing. This will be used in Section 4.4 to prove non-lifting of Frobenius for a large class of flag varieties.

Let $\mathcal{O}(1)$ be the ample generator of $\operatorname{Pic} Y$. By flat base change one may produce examples of Bott non-vanishing for Y for fields of arbitrary prime characteristic by restricting to the field of the complex numbers. This has been done in the setting of Hermitian symmetric spaces, where the cohomology groups $\mathrm{H}^p(Y, \Omega^q \otimes \mathcal{O}(n))$ have been thoroughly investigated by Saito [15] and Snow [16, 17]. We now show that these examples exist. In each of the following subsections Y will denote G/P, where P is the maximal parabolic subgroup not containing the root subgroup corresponding to the marked simple root in the Dynkin diagram in Figure 4.3. These flag varieties are the irreducible Hermitian symmetric spaces.

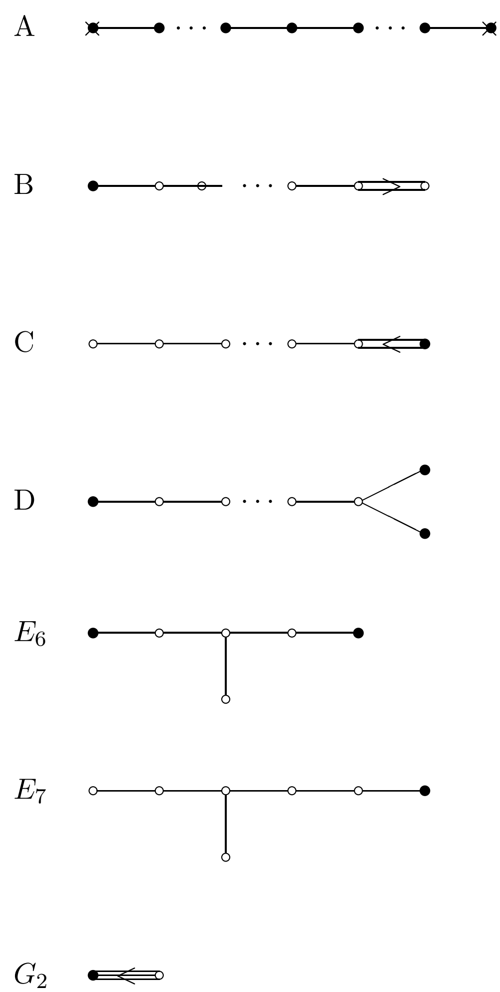

Figure 4.3

4.3.1. *Type A.* If Y is a Grassmann variety not isomorphic to projective space ($Y = G/P$, where P corresponds to leaving out a simple root which is not the left or right most one), one may prove [16, Theorem 3.3] that

$$\mathrm{H}^1(Y, \Omega_Y^3 \otimes \mathcal{O}(2)) \neq 0.$$

4.3.2. *Type B.* Here Y is a smooth quadric hypersurface in $\mathbb{P}^n$, where $n \geq 4$ and the Bott non-vanishing follows from Section 4.1.

4.3.3. *Type C.* By [17, Theorem 2.2] it follows that

$$\mathrm{H}^1(Y, \Omega_Y^2 \otimes \mathcal{O}(1)) \neq 0.$$

4.3.4. *Type D.* For the maximal parabolic P corresponding to the leftmost marked simple root, Y=G/P is a smooth quadric hypersurface in $\mathbb{P}^n$, where $n \geq 4$ and Bott non-vanishing follows from Section 4.1. For the maximal parabolic subgroup corresponding to one of the two rightmost marked simple roots we get by [17, Theorem 3.2] that

$$\mathrm{H}^2(Y, \Omega_Y^4 \otimes \mathcal{O}(2)) \neq 0.$$

4.3.5. *Type E_6.* By [17, Table 4.4] it follows that

$$\mathrm{H}^3(Y, \Omega^5 \otimes \mathcal{O}(2)) \neq 0.$$

4.3.6. *Type E_7.* By [17, Table 4.5] it follows that

$$\mathrm{H}^4(Y, \Omega^6 \otimes \mathcal{O}(2)) \neq 0.$$

4.3.7. *Type G_2.* Here Y is a smooth quadric hypersurface in $\mathbb{P}^6$ and the Bott non-vanishing follows from Section 4.1.

4.4. **Non-lifting of Frobenius for flag varieties.** We now get the following:

Theorem 6. *Let Q be a parabolic subgroup contained in a maximal parabolic subgroup P in the list 4.3.1 - 4.3.7. Then the Frobenius morphism on G/Q does not lift to $W_2(k)$. Furthermore if G is of type A, then the Frobenius morphism on any flag variety $G/Q \ncong \mathbb{P}^m$ does not lift to $W_2(k)$.*

Proof If P is a maximal parabolic subgroup in the list 4.3.1-4.3.7, then the Frobenius morphism on G/P does not lift to $W_2(k)$. By Corollary 1 we get that the Frobenius morphism on G/Q does not lift to $W_2(k)$. In type A the only flag variety not admitting a fibration to a Grassmann variety $\ncong \mathbb{P}^m$ is the incidence variety. The non-lifting of Frobenius in this case follows from Section 4.2. □

Remark 2. *The above case by case proof can be generalized to include projective homogeneous G-spaces with non-reduced stabilizers. It would be nice to prove in general that the only flag varieties enjoying the Bott vanishing property are products of projective spaces. We know of no other visible obstruction to lifting Frobenius to $W_2(k)$ for flag varieties than the non-vanishing Bott cohomology groups.*

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$\mathcal{D}$ -AFFINITY AND TORIC VARIETIES

JESPER FUNCH THOMSEN

1. Introduction

Let k be an algebraically closed field of any characteristic. A toric variety over k is a normal variety X containing the algebraic group $T = (k^*)^n$ as an open dense subset, with a group action $T \times X \to X$ extending the group law of T.

On any smooth variety X over a field k we can define the sheaf of differential operators $\mathcal{D}$, which carries a natural structure as a $\mathcal{O}_X$- bisubalgebra of $\operatorname{End}_k(\mathcal{O}_X)$. A $\mathcal{D}$-module on X is a sheaf $\mathcal{F}$ of abelian groups having a structure as a left $\mathcal{D}$-module, such that $\mathcal{F}$ is quasi-coherent as an $\mathcal{O}_X$-module. A smooth variety X is called $\mathcal{D}$-affine if for every $\mathcal{D}$-module $\mathcal{F}$ we have

- $\mathcal{F}$ is generated by global sections over $\mathcal{D}$
- $H^i(X, \mathcal{F}) = 0, i > 0$

Beilinson and Bernstein have shown [1] that every flag variety over a field of characteristic zero is $\mathcal{D}$-affine, from which they deduced a conjecture of Kazhdan and Lusztig. In fact flag varieties are the only known examples of $\mathcal{D}$-affine projective varieties. In this paper we prove that the $\mathcal{D}$-affinity of a smooth complete toric variety implies that it is a product of projective spaces. Part of the method will be to translate a proof of the non $\mathcal{D}$-affinity of a 2-dimensional Schubert variety, given by Haastert in [3], into the language of toric varieties.

I would like to thank my advisor Niels Lauritzen for introducing this problem to me.

2. Toric Varieties

Toric varieties are given by convex bodies called fans. In this section we review the definitions following [2].

Let $N = \mathbb{Z}^n$ and $M = N^\vee$ the dual of N. By a rational convex polyhedral cone σ in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ we understand a set $\sigma = \{r_1 v_1 + r_2 v_2 + \cdots + r_l v_l \mid r_i \geq 0\} \subseteq N_\mathbb{R}$ where $v_i \in N$ (in the following we will use the notation $\operatorname{Span}_{\mathbb{R}_{\geq 0}}\{v_1, \ldots, v_l\}$ for $\{r_1 v_1 + r_2 v_2 + \cdots + r_l v_l \mid r_i \geq 0\}$). If σ does not contain any $\mathbb{R}$-linear subspace we say that σ is strongly convex. In the following we only consider strongly convex cones. A face of σ is a subset of the form $\sigma \cap u^\perp = \{v \in \sigma | < v, u >= 0\}$ for $u \in \sigma^\vee = \{w \in M_\mathbb{R} \mid \ < v, w > \geq 0 \ \forall v \in \sigma\}$. A face $\sigma \cap u^\perp$ is also a rational strongly convex polyhedral cone. We use the notation $\tau \prec \sigma$ to denote that τ is a face of σ. Notice that if $v_1, \ldots v_l$ is a minimal set of generators for σ then $\{r v_i \mid r \geq 0\} \prec \sigma$. When σ is a rational strongly convex cone, the semi-group $\sigma^\vee \cap M$ is finitely generated, and we can form the affine variety $U_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$. If Δ is a fan, i.e. a finite collection of rational strongly convex polyhedral cones with the properties

- $\tau \prec \sigma \ \wedge \ \sigma \in \Delta \Rightarrow \tau \in \Delta$
- $\sigma \cap \tau \prec \tau$ for $\tau, \sigma \in \Delta$

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we can form a variety $X(\Delta)$ by patching the U_σ's, $\sigma \in \Delta$, together over common faces. Notice that the two properties of a fan listed above ensures that this can be done in a natural way. It is known that a variety is toric exactly when it can be constructed from a fan Δ in this way.

Example 1. *Projective n-space $\mathbb{P}^n$ is constructed from the fan*

$$\Delta = \{\mathrm{Span}_{\mathbb{R}_{\geq 0}}(E) \mid E \subseteq \{e_1, e_2, \dots, e_n, -e_1 - e_2 - \cdots - e_n\}\ ,\ \#E \leq n\},$$

where $e_1, e_2, \dots e_n$ denotes the standard basis for $N = \mathbb{Z}^n$.

Example 2. *Affine n-space $\mathbb{A}^n$ can be constructed from the fan*

$$\Delta = \{\mathrm{Span}_{\mathbb{R}_{\geq 0}}(E) \mid E \subseteq \{e_1, e_2, \dots, e_n\}\},$$

where $e_1, e_2, \dots e_n$, as in Example 1, denotes the standard basis for $N = \mathbb{Z}^n$.

Example 3. *The torus $T = (k^*)^n$ can be constructed from the fan $\Delta = \{\{0\}\}$ in $N = \mathbb{Z}^n$.*

Example 4. *If $X_1 = X(\Delta_1)$ and $X_2 = X(\Delta_2)$ then $X_1 \times X_2 \cong X(\Delta_1 \times \Delta_2)$.*

The toric variety $X(\Delta)$ is smooth exactly when every element σ of Δ is generated by part of a $\mathbb{Z}$-basis for N.

2.1. **Orbits.** Let $X(\Delta)$ be a toric variety of dimension n. Now $X(\Delta)$ is a T-space and there is 1-1 correspondence between elements of Δ and T- orbits : $\sigma \in \Delta \leftrightarrow \mathcal{O}_\sigma$. The correspondence has the following properties

- $\mathcal{O}_\sigma \cong (k^*)^{n-d}\ ,\ d = dim(\sigma)$
- $U_\sigma = \bigcup_{\tau \prec \sigma} \mathcal{O}_\tau$
- $\overline{\mathcal{O}_\sigma} = \bigcup_{\substack{\tau \in \Delta \\ \sigma \prec \tau}} \mathcal{O}_\tau$

In the following we will use the notation $V(\sigma)$ for $\overline{\mathcal{O}_\sigma}$. From the definition of a toric variety given in the introduction it is clear that $X_\sigma = X(\Delta) \setminus V(\sigma)$ is a toric variety for all $\sigma \in \Delta \setminus \{\{0\}\}$. In fact X_σ can be constructed from the fan $\Delta_\sigma = \{\tau \in \Delta \mid \sigma \not\prec \tau\}$.

3. D-Affinity

Let X be a smooth algebraic variety over an algebraically closed field k. The sheaf $\mathcal{D}$ of differential operators on X is an $\mathcal{O}_X$-bisubalgebra of $\mathrm{End}_k(\mathcal{O}_X)$. Over an open affine subset $V = \mathrm{Spec}(A)$ of X the sheaf $\mathcal{D}$ is given by

$$\mathcal{D}(V) = \{\phi \in \mathrm{End}_k(A) \mid \exists n > 0\ :\ I^n\phi = 0\},$$

where I is the kernel of the product map $A \otimes_k A \to A$, and $\mathrm{End}_k(A)$ is considered as an $A \otimes_k A$-module given by the A-bialgebra structure on $\mathrm{End}_k(A)$. A $\mathcal{D}$-module is a left $\mathcal{D}$-module which, considered as an $\mathcal{O}_X$-module, is quasi-coherent. Notice that $\mathcal{O}_X$ is a $\mathcal{D}$-module.

Definition 1. *A smooth variety X is $\mathcal{D}$-affine when all $\mathcal{D}$-modules $\mathcal{F}$ satisfy*

1. *$\mathcal{F}$ is generated by global sections as a $\mathcal{D}$-module*
2. $H^i(X, \mathcal{F}) = 0,\ i > 0$

Lemma 1. *Let X be a smooth $\mathcal{D}$-affine variety and let $D_1, D_2, \dots D_l$ be a collection of closed subvarieties of codimension 1. Then the open subvariety $U = X \setminus \bigcup_i D_i$ is affine if and only if it is quasi-affine.*

Proof. Consider the inclusion morphism $j : U \to X$. As the D_i's have codimension 1 in X, j is an affine morphism. From the local definition of $\mathcal{D}$ it is clear that $j^*\mathcal{D}_X \cong \mathcal{D}_U$, from which we get a homomorphism of $\mathcal{O}_X$-algebras $\mathcal{D}_X \to j_*\mathcal{D}_U$. Therefore $j_*\mathcal{O}_U$ is a $\mathcal{D}_X$-module and $H^i(U, \mathcal{O}_U) = H^i(X, j_*\mathcal{O}_U) = 0$, $i > 0$ by definition of $\mathcal{D}$-affinity. If U is quasi-affine every $\mathcal{O}_U$-module is generated by global sections (see [4, 5.1.2, p.94]), so by Serre's theorem we conclude that U is affine. □

4. D-affinity and Toric Varieties

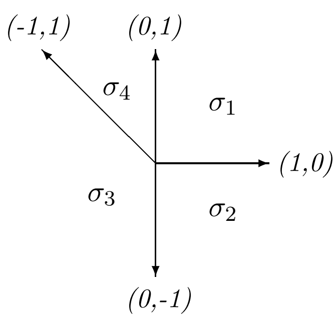
Let $X(\Delta)$ be a smooth toric variety, and let $\sigma_1, \sigma_2, \ldots, \sigma_l$ be the one dimensional cones in Δ. For each i there exist a unique $v_i \in N$ part of a basis for N, so that $\sigma_i = \{rv_i \mid r \geq 0\}$. Let $G(\Delta)$ denote the set $\{v_1, v_2, \ldots, v_l\}$.

Lemma 2. *Let $X(\Delta)$ be a $\mathcal{D}$-affine toric variety. If $w_1, w_2, \ldots, w_s \in G(\Delta)$ is a collection of linearly independent vectors, then $\sigma = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{w_1, w_2 \ldots w_s\} \in \Delta$.*
Proof.
We may assume that $w_i = v_i$, $i \leq s$. Consider the toric variety

$$Y = X \setminus \bigcup_{j=s+1}^{l} V(\sigma_j).$$

From section 2.1 it follows that Y is a smooth toric variety represented by the fan $\Delta_Y = \{\tau \in \Delta \mid \sigma_j \not\prec \tau \ \forall j > s\}$. The affine toric variety U_σ is constructed from the fan $\widetilde{\Delta} = \{\mathrm{Span}_{\mathbb{R}_{\geq 0}}(E) \mid E \subseteq \{w_1, w_2, \ldots, w_s\}\}$ (Notice that since the elements $w_1, w_2, \ldots, w_s$ are linearly independent, $\widetilde{\Delta}$ consist of rational strongly convex polyhedral cones). Since $\Delta_Y \subseteq \widetilde{\Delta}$ we can consider Y to be an open subset of U_σ, so Y is quasi-affine. If $\sigma \notin \Delta$ then $\widetilde{\Delta} \neq \Delta_Y$. Therefore $Y \neq U_\sigma$ and section 2.1 tells us that $\mathrm{codim}(U_\sigma \setminus Y, U_\sigma) \geq 2$. As X is a normal variety $\Gamma(Y, \mathcal{O}_Y) \cong \Gamma(U_\sigma, \mathcal{O}_{U_\sigma})$, and Y can not be affine, which contradicts the result of Lemma 1. □

Example 5. A Hirzebruch Surface : *Consider the toric variety $X(\Delta)$ constructed from the fan*



$$\sigma_1 = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{(0,1),(1,0)\}$$
$$\sigma_2 = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{(1,0),(0,-1)\}$$
$$\sigma_3 = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{(0,-1),(-1,1)\}$$
$$\sigma_4 = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{(-1,1),(0,1)\}$$
$$G(\Delta) = \{(0,1),(1,0),(0,-1),(-1,1)\}$$

i.e Δ consist of the cones $\sigma_1, \sigma_2, \sigma_3$ and σ_4 together with their faces. Here $(1,0)$ and $(-1,1)$ are linearly independent but they do not generate a cone in Δ. So $X(\Delta)$ is not $\mathcal{D}$-affine. This toric variety has also been shown not to be $\mathcal{D}$-affine by Haastert [3, p.133]. *Haastert considers $X(\Delta)$ to be a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ (and a 2-dimensional Schubert variety). His proof, translated into the language of fans, is essentially the same as the proof of Lemma 2.*

In Example 5 we saw that non $\mathcal{D}$-affinity follows easily from Lemma 2, and in fact Lemma 2 is the key to our main result.

Theorem 1. *Let $X(\Delta)$ be a smooth $\mathcal{D}$-affine toric variety. Then $X(\Delta)$ is isomorphic (as toric varieties) to a product of projective n-spaces $\mathbb{P}^n$, k^* and an affine n-space $\mathbb{A}^n$. In particular if $X(\Delta)$ is complete it will be a product of projective spaces.*

Proof. *Let $\sigma \in \Delta$ be a cone of maximal dimension. Since $X(\Delta)$ is smooth we may assume that $\sigma = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_1, e_2 \dots e_s\}$, where $e_1, e_2, \dots, e_n$ is the standard basis for $N = \mathbb{Z}^n$, and that $v_i = e_i$, $1 \leq i \leq s$. From Lemma 2 it is clear that $G(\Delta) \subseteq \mathrm{Span}_{\mathbb{Z}}\{e_1, \dots, e_s\}$. Let $v_i = (v_{i1}, v_{i2}, \dots, v_{in})$ be the coordinates of v_i relative to the standard basis. If $v_{it} \neq 0$, $i > s$, $1 \leq t \leq s$ then $e_1, \dots e_{t-1}, e_{t+1}, \dots, e_s, v_i$ are linearly independent elements of $G(\Delta)$, and they must be part of a $\mathbb{Z}$-basis for N according to Lemma 2. But then v_{it} must be equal to 1 or -1. Using that Δ is a fan we can exclude the case $v_{it} = 1$ (otherwise $\mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_1, \dots e_{t-1}, e_{t+1}, \dots e_s, v_i\} \cap \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_1, e_2, \dots, e_s\}$ contains an element of the interior of $\mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_1, e_2, \dots, e_s\}$, and thus can not be a face). This means that all coordinates of v_i , $i > s$ must be 0 or -1. We may now assume that v_{s+1} has the maximal numbers of -1 as coordinates among v_i , $i > s$, and that the -1's appears on the first m_1 coordinates. Suppose now that there exists a $t > s+1$ so that v_t is not perpendicular to v_{s+1}. We may then assume that $v_{t1} = -1$ and $v_{t2} = 0$. It now readily follows that $\mathrm{Span}_{\mathbb{R}_{\geq 0}}\{v_{s+1}, v_t, e_3, e_4, \dots e_s\} \cap \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{v_{s+1}, e_2, e_3, e_4, \dots e_s\}$ cannot be a face of $\mathrm{Span}_{\mathbb{R}_{\geq 0}}\{v_{s+1}, e_2, e_3, e_4, \dots e_s\}$. This contradicts Lemma 2. So every element v_i , $i > s+1$ is perpendicular to v_{s+1}. We may now assume that v_{s+2} has the maximal number of -1 as coordinates among v_i , $i > s+1$, and that they appear from coordinate $m_1 + 1$ to m_2. With the same procedure as above we can then show that every element v_i , $i > s+2$ are perpendicular to v_{s+2}. Continuing in this way we find integer $m_0 = 0 \leq m_1 \leq m_2 \leq \dots \leq m_{l-s}$ so that*

$$v_{tj} = \begin{cases} -1 & \text{for } j \in \{m_{t-s-1}+1, \dots, m_{t-s}\} \\ 0 & \text{else} \end{cases}$$

for $t > s$. By using Lemma 2 (and Example 1-4) it follows that

$$(*) \qquad X(\Delta) = (k^*)^{n-s} \times \mathbb{A}^{s-m_{l-s}} \times \mathbb{P}^{m_1} \times \mathbb{P}^{m_2-m_1} \times \dots \times \mathbb{P}^{m_{l-s}-m_{l-s-1}}$$

from which the proposition follows. □

On the other hand if X is a variety of the form (*) it follows, by Haastert [3] (in positive characteristic) and Beilinson and Bernstein [1] (in characteristic zero), that X is $\mathcal{D}$-affine.

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FROBENIUS DIRECT IMAGES OF LINE BUNDLES ON TORIC VARIETIES

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1. Introduction

A toric variety over an algebraically closed field k is a normal variety X containing the algebraic group $T = (k^*)^n$ as an open dense subset, and with a group action $T \times X \to X$ extending the group law of T. If k has characteristic $p > 0$ we can define the absolute Frobenius morphism on X. Remember that the absolute Frobenius morphism on a variety X, over a field of characteristic $p > 0$, is the morphism of schemes $F : X \to X$, which is the identity on the underlying set of X and the p'th power map on sheaf level. An old result of Hartshorne ([3],Cor.6.4., p.138) tells us that if $X = \mathbb{P}^n_k$ and L is a line bundle on X, then the direct image of L via F_*, that is F_*L, splits into a direct sum of line bundles. In this paper we generalize this result to smooth toric varieties. Our proof will be constructive and will give us an algorithm for computing a decomposition explicitly. We will consider toric varieties as constructed from convex bodies called fans as described in [2], in particular we will assume that the reader is familiar with the first chapter of [2].

Using results on Grothendieck differential operators and T-linearized sheaves R. Bøgvad ([1]) has later given nonconstructive generalizations of the above. One generalization is that F_*M, when M is a T-linearized vector bundle on X, splits into a sum of bundles of the same rank as M.

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2. Idea behind the proof

This section should be regarded as an example illustrating the idea behind the proof.

Consider the absolute Frobenius morphism $F : \mathbb{P}^1 \to \mathbb{P}^1$ on the projective space $\mathbb{P}^1$ over an algebraically closed field k of characteristic p. By ([3],Cor.6.4.,p.138) we know that $F_*\mathcal{O}_{\mathbb{P}^1}$ splits into a direct sum of p line bundles. In the following we will show how we may choose such line bundles $L_1, L_2, \dots L_p$ and an isomorphism

$$\phi : L_1 \oplus L_2 \oplus \cdots \oplus L_p \to F_*\mathcal{O}_{\mathbb{P}^1}.$$

For this we first need to define some notation. Cover $\mathbb{P}^1$ by the standard open affine varieties $U_0 = \mathrm{Spec}(k[X])$ and $U_1 = \mathrm{Spec}(k[Y])$ (so with this notation we should regard X as being equal to Y^{-1}). Let $F_*k[X]$ (resp. $F_*k[Y]$) denote the $k[X]$-module (resp. $k[Y]$-module) which as an abelian group is $k[X]$ (resp. $k[Y]$) but where the $k[X]$-module (resp. $k[Y]$-module) structure is twisted by the p'th power map. Then $F_*\mathcal{O}_{\mathbb{P}^1}(U_0) = F_*k[X]$ and $F_*\mathcal{O}_{\mathbb{P}^1}(U_1) = F_*k[Y]$. A basis for $F_*\mathcal{O}_{\mathbb{P}^1}(U_0)$ (resp. $F_*\mathcal{O}_{\mathbb{P}^1}(U_1)$) as a $\mathcal{O}_{\mathbb{P}^1}(U_0)$-module (resp. $\mathcal{O}_{\mathbb{P}^1}(U_1)$-module) is : $e_i = X^i$, $i = 0, 1, \dots, p-1$ (resp. $f_j = Y^j$, $j = 0, 1, \dots, p-1$).

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The line bundles L_r will be constructed as subsheaves of the constant sheaf $\mathcal{K}$ corresponding to the field of rational functions on $\mathbb{P}^1$. More precise L_r will be regarded as the subsheaf of $\mathcal{K}$ corresponding to a Cartier divisors $\{(U_0, 1), (U_1, Y^{h(r)})\}$ determined by an integer $h(r)$. Clearly $h(r)$ is the unique integer such that L_r is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(h(r))$.

The choices above enables us to put some simple conditions on ϕ, which will turn out to determine ϕ and the integers $h(r)$ (up to permutation) uniquely. More precise we want ϕ to be a map such that the image of $1 \in L_r(U_0)$ (resp. $Y^{-h(r)} \in L_r(U_1)$) is some $e_{i(r)}$ (resp. $f_{j(r)}$). Notice that this condition forces $i(\cdot)$ and $j(\cdot)$ to be permutations of the set $\{0, 1, \ldots, p-1\}$.

The fact that the local describing conditions on ϕ should coincide on $U_0 \cap U_1$ forces $i(r), j(r)$ and $h(r)$ to satisfy

$$Y^{-i(r)} = (Y^{h(r)})^p Y^{j(r)} \Leftrightarrow -i(r) = h(r)p + j(r).$$

As $j(r)$ is an integer between 0 and $p-1$ this means that $h(r)$ and $j(r)$ is determined from $i(r)$ simply by division by p with remainder. Let us permute the indices such that $i(r) = r$. Then

$$(j(0), h(0)) = (0, 0) \text{ and } (j(r), h(r)) = (p - r, -1), \ r = 1, 2, \ldots, p-1.$$

The uniqueness of the line bundles L_r follows as they only depend on the integers $h(r)$, and also the map ϕ is determined as it only depend on $i(r)$ and $j(r)$.

The conditions which we putted on ϕ has determined ϕ for us. That a ϕ with the given conditions exists is now an easy exercise. In fact this only amounts to saying that $i(\cdot)$ and $j(\cdot)$ are permutations. We conclude that

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(p-1)} \cong F_* \mathcal{O}_{\mathbb{P}^1}.$$

This ends our example.

3. Basic definitions and notations

In this section we will introduce the notation that will be used throughout the paper. The varieties we consider will be defined over a fixed algebraically closed field k of positive or zero characteristic.

Let $N = \mathbb{Z}^n$ be a lattice of rank n and M be its dual. A fan $\Delta = \{\sigma_0, \sigma_1, \ldots, \sigma_q\}$ consisting of rational strongly convex polyhedral cones in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ determines an n-dimensional toric variety $X(\Delta)$. In the following we will assume that such an n-dimensional smooth toric variety $X(\Delta)$ has been chosen. Choose an ordered $\mathbb{Z}$-basis $(e_1, e_2, \ldots, e_n)$ for N and let $(\hat{e_1}, \hat{e_2}, \ldots, \hat{e_n})$ be the dual basis in M, that is $\hat{e_i}(e_j) = \delta_{ij}$. Since $X(\Delta)$ is smooth every cone $\sigma_i \in \Delta$ is generated by part of a $\mathbb{Z}$-basis ([2],p.29) for N,

$$\sigma_i = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{v_{i1}, v_{i2}, \ldots, v_{id_i}\} \ , \ d_i = \dim(\sigma_i).$$

For each i expand the set $\{v_{i1}, v_{i2}, \ldots, v_{id_i}\}$ to a $\mathbb{Z}$-basis for N : $\{v_{i1}, v_{i2}, \ldots, v_{in}\}$, and form the matrix $A_i \in \mathrm{GL}_n(\mathbb{Z})$, having as the j'th row the coordinates of v_{ij} expressed in the basis $(e_1, e_2, \ldots, e_n)$. Let $B_i = A_i^{-1} \in \mathrm{GL}_n(\mathbb{Z})$ and denote the j'th column vector in B_i by w_{ij}. Introducing the symbols $X^{\hat{e}_1}, X^{\hat{e}_2}, \ldots X^{\hat{e}_n}$ we can form the ring $R = k[(X^{\hat{e}_1})^{\pm 1}, (X^{\hat{e}_2})^{\pm 1}, \ldots (X^{\hat{e}_n})^{\pm 1}]$, which is the affine coordinate ring of the torus $T \subset X(\Delta)$. The coordinate ring of the open affine subvariety U_{σ_i} of $X(\Delta)$, corresponding to the cone σ_i in Δ, is then equal to the subring

$$R_i = k[X^{w_{i1}}, \ldots, X^{w_{id_i}}, X^{\pm w_{i(d_i+1)}}, \ldots, X^{\pm w_{in}}] \subseteq R$$

where we use the multinomial notation $X^w = (X^{\hat{e}_1})^{w_1} \dots (X^{\hat{e}_n})^{w_n}$, when w is a vector of the form $w = (w_1, \dots w_n)$. For convenience we will in the following also write $X_{ij} = X^{w_{ij}}$ such that

$$R_i = k[X_{i1}, \dots, X_{id_i}, X_{i(d_i+1)}^{\pm 1}, \dots, X_{in}^{\pm 1}].$$

For each i and j we have that $\sigma_i \cap \sigma_j$ is a face of σ_i. This implies ([2],Prop.2,p.13) that there exist a monomial M_{ij} in $X_{i1}, \dots, X_{in}$ such that the localized ring $(R_i)_{M_{ij}}$ is the coordinate ring of $\sigma_i \cap \sigma_j$. Choose such monomials for all i and j, and notice that $(R_i)_{M_{ij}} = (R_j)_{M_{ji}}$. For every i and j define

$$I_{ij} = \{v \in \mathrm{Mat}_{n,1}(\mathbb{Z}) \mid X_i^v \text{ is a unit in } (R_i)_{M_{ij}}\}$$

where we use the multinomial notation $X_i^v = (X_{i1})^{v_1} \dots (X_{in})^{v_n}$ when v is a column vector with entries $v_1, \dots, v_n$. Defining $C_{ij} = B_j^{-1} B_i \in \mathrm{GL}_n(\mathbb{Z})$, and letting f_s , $s = 1, \dots, n$, denote the element in $\mathrm{Mat}_{n,1}(\mathbb{Z})$ with a single 1 in position s and zeroes elsewhere, we can formulate and prove our first lemma.

Lemma 1. *With the notations above we have*

1. $X_i^v = X_j^{C_{ij}v}$, $v \in \mathrm{Mat}_{n,1}(\mathbb{Z})$.
2. I_{ij} *is a subgroup of* $\mathrm{Mat}_{n,1}(\mathbb{Z})$.
3. $C_{ij} I_{ij} = I_{ji}$.
4. $v \in \mathrm{Mat}_{n,1}(\mathbb{Z}) : X_i^v \in (R_i)_{M_{ij}}$ *and* $v_l < 0 \Rightarrow f_l \in I_{ij}$.
5. $v \in I_{ij}$ *and* $v_l \neq 0 \Rightarrow f_l \in I_{ij}$.
6. $(C_{ji})_{kl} < 0 \Rightarrow f_k \in I_{ij}$.

Proof. Points (1) to (5) are straight forward by definition. Point (6) follows from (1) and (4) using that $X_i^{C_{ji} f_l} = X_{jl} \in (R_j)_{M_{ji}} = (R_i)_{M_{ij}}$. □

For every positive integer m define $P_m = \{v \in \mathrm{Mat}_{n,1}(\mathbb{Z}) \mid 0 \leq v_i < m\}$. Our main lemma then says.

Lemma 2. *Let* $v \in P_m$ *and* $w \in I_{ji}$. *If* $h \in \mathrm{Mat}_{n,1}(\mathbb{Z})$ *and* $r \in P_m$ *satisfies*

$$C_{ij} v + w = mh + r$$

then $h \in I_{ji}$.

Proof. Using Lemma 1(2) it is enough to show that $f_l \in I_{ji}$ whenever the l'th entry h_l in h is nonzero. So suppose $h_l \neq 0$. By Lemma 1(5) we may assume that $w_l = 0$. Furthermore Lemma 1(6) tells us that we may assume $(C_{ij})_{lk} \geq 0$ for all k.

Now look at the matrix C_{ji}. If $(C_{ji})_{st} < 0$, then Lemma 1(6) implies that $f_s \in I_{ij}$ and further Lemma 1(3) gives $C_{ij} f_s \in I_{ji}$. Lemma 1(5) now tells us that we may assume that the l'th entry of $C_{ij} f_s$ is zero, that is $(C_{ij})_{ls} = 0$. The conclusion is that we may assume the following implication to be true

$$(C_{ji})_{st} < 0 \Rightarrow (C_{ij})_{ls} = 0 \ , \ s, t \in \{1, 2, \dots, n\}.$$

As C_{ij} and C_{ji} are inverse to each other

$$\sum_{s=1}^{n} (C_{ij})_{ls} (C_{ji})_{sl} = 1.$$

Our assumptions now implies that there exist an integer a such that

$$(C_{ij})_{la} = (C_{ji})_{al} = 1.$$

$$(C_{ij})_{ls} (C_{ji})_{sl} = 0 \ , \ s \neq a.$$

Looking at the equality $C_{ij}v+w=mh+r$ tells us that there must exist an integer $b\neq a$ such that $(C_{ij})_{lb}\neq 0$ (else $h_l=0$). Now consider the relation

$$\sum_{s=1}^{n}(C_{ij})_{ls}(C_{ji})_{sz}=0,\ z\neq l.$$

This implies

$$(C_{ij})_{ls}(C_{ji})_{sz}=0\ ,\ z\neq l\ ,\ s\in\{1,2,\dots,n\}.$$

As $(C_{ij})_{lb}\neq 0$ we get $(C_{ji})_{bz}=0, z\neq l$. But then the b'th row of C_{ji} is zero, which is a contradiction. □

Using Lemma 2 we can now, for every $m\in\mathbb{N}$ and $w\in I_{ji}$, define maps

$$h^w_{ijm}\ :\ P_m\to I_{ji}.$$

$$r^w_{ijm}\ :\ P_m\to P_m.$$

which are determined by the equality

$$v\in P_m\ :\ C_{ij}v+w=mh^w_{ijm}(v)+r^w_{ijm}(v).$$

Lemma 3. *The map* $r^w_{ijm}\ :\ P_m\to P_m$ *is a bijection.*

Proof. Suppose that $r^w_{ijm}(v_1)=r^w_{ijm}(v_2)$ for $v_1,v_2\in P_m$. From the definitions it then follows that

$$v_1-v_2=m(C_{ji}(h^w_{ijm}(v_1)-h^w_{ijm}(v_2)))\in m\cdot\mathrm{Mat}_{n,1}(\mathbb{Z}).$$

which can only be the case if $v_1=v_2$. □

From now on we will by a toric variety understand a smooth toric variety with notations and choices fixed as above.

4. Constructions of certain vector bundles

Given a line bundle L on an n-dimensional toric variety $X=X(\Delta)$ and a positive integer m, we will in this section construct a vector bundle F_mL of rank m^n. The vector bundle F_mL will by definition be a direct sum of line bundles.

Let notation and choices be fixed as in Sections 3. Let K denote the field of rational functions on X, and let $\mathcal{K}$ be the constant sheaf corresponding to K. Recall that a Cartier divisor on X can be represented by a collection of pairs $\{(U_i,f_i)\}$ where

- The U_i's form an open affine cover of X.
- $f_i\in K^*$ and f_i/f_j is a unit in $\Gamma(U_i\cap U_j,\mathcal{O}_X)$.

Denote the group of Cartier divisors on X by $\mathrm{Div}(X)$. Given a Cartier divisor D on X we can form the associated line bundle $\mathcal{O}(D)$, which is the subsheaf of $\mathcal{K}$ generated by $1/f_i$ over U_i. In this way we get a surjective map from $\mathrm{Div}(X)$ onto the group $\mathrm{Pic}(X)$ of line bundles on X modulo isomorphism. The kernel of this map is the group of principal Cartier divisors, that is Cartier divisors which can be represented by a set $\{(U_i,f)\}, f\in K^*$. This way we get the well known short exact sequence

$$0\to K^*\to\mathrm{Div}(X)\to\mathrm{Pic}(X)\to 0.$$

Now the group $\mathrm{Div}(X)$ contains the subgroup $\mathrm{Div}_T(X)$ of T-Cartier divisors. By a T-Cartier divisor we will understand a Cartier divisor D which can be represented in the form ([2],Chapter 3.3)

$$\{(U_{\sigma_i},X_i^{u_i})\}_{\sigma_i\in\Delta}\ ,\ u_i\in\mathrm{Mat}_{n,1}(\mathbb{Z}).$$

Notice that the condition for such a set to represent a Cartier divisor can be expressed as

$$u_j - C_{ij}u_i \in I_{ji} \text{ , for all } i \text{ and } j.$$

A general fact about toric varieties ([2],p.63) tells us that the induced map from $\mathrm{Div}_T(X)$ to $\mathrm{Pic}(X)$ is surjective. We arrive at the short exact sequence

$$0 \to K^* \cap \mathrm{Div}_T(X) \to \mathrm{Div}_T(X) \to \mathrm{Pic}(X) \to 0.$$

Notice that an element $D \in \mathrm{Div}_T(X)$ is in $K^* \cap \mathrm{Div}_T(X)$ exactly when it is represented by a set $\{(U_{\sigma_i}, X_i^{u_i})\}$ with $u_j - C_{ij}u_i = 0$.

Let now D be a T-Cartier divisor and $m \in \mathbb{N}$. Fix a set $\{(U_{\sigma_i}, X_i^{u_i})\}$ which represents D and define $u_{ij} = u_j - C_{ij}u_i$. Then

Lemma 4. *Fix $v \in P_m$ and a cone $\sigma_l \in \Delta$. If $v_i = h^{u_{li}}_{lim}(v)$ then $\{(u_{\sigma_i}, X_i^{v_i})\}$ represents a T-Cartier divisor.*

Proof. As noticed above it is enough to show that $v_j - C_{ij}v_i \in I_{ji}$ for all i and j. For this purpose consider the equality $C_{li}v + u_{li} = mv_i + r^{u_{li}}_{lim}(v)$. Multiplying by C_{ij} and using that $C_{ij}u_{li} = u_{lj} - u_{ij}$ and $C_{lj}v + u_{lj} = mv_j + r^{u_{lj}}_{ljm}(v)$ tells us

$$C_{ij}r^{u_{li}}_{lim}(v) + u_{ij} = m(v_j - C_{ij}v_i) + r^{u_{lj}}_{ljm}(v).$$

This means

$$r^{u_{lj}}_{ljm}(v) = r^{u_{ij}}_{ijm}(r^{u_{li}}_{lim}(v)).$$

$$v_j - C_{ij}v_i = h^{u_{ij}}_{ijm}(r^{u_{li}}_{lim}(v)) \in I_{ji}.$$

by Lemma 2. □

Addendum 1. *With the notation above we have*

$$r^{u_{lj}}_{ljm}(v) = r^{u_{ij}}_{ijm}(r^{u_{li}}_{lim}(v)).$$

$$h^{u_{lj}}_{ljm}(v) - C_{ij}h^{u_{li}}_{lim}(v) = h^{u_{ij}}_{ijm}(r^{u_{li}}_{lim}(v)).$$

for all i,j and l.□

Lemma 4 enables us for each $m \in \mathbb{Z}$ to define a map

$$F_m \;:\; \mathrm{Pic}(X) \to \mathrm{Vect}^m(X)$$

where we by $\mathrm{Vect}^m(X)$ denote the set of vector bundles of rank m^n modulo isomorphism. Given $[L] \in \mathrm{Pic}(X)$ we define the map in the following way :
Let $\{(U_{\sigma_i}, X_i^{u_i})\}$ represent a T-Cartier divisor D such that $\mathcal{O}(D) \cong L$, and choose an element $l \in \{0, 1, \ldots, q\}$ (remember that $q + 1$ is the number of cones in Δ). Let D_v,$v \in P_m$, denote the Cartier divisor corresponding to $\{(U_{\sigma_i}, X_i^{v_i})\}$ where $v_i = h^{u_{li}}_{lim}(v)$. We define

$$F_m([L]) = [\bigoplus_{v \in P_m} \mathcal{O}(D_v)].$$

4.1. F_m **is well defined.** We will now show that F_m is well defined. Let us first show that F_m is independent of the choice of l. So suppose $[L]$, D and $\{(U_{\sigma_i}, X_i^{u_i})\}$ are chosen as in the definition of F_m, and let l and l' be two elements in $\{0, 1, \ldots, q\}$. If $v \in P_m$ we denote the Cartier divisors corresponding to l and l' by D_v and D'_v respectively. Referring to Lemma 3 it is enough to show that

$$\forall v \in P_m \;:\; \mathcal{O}(D_v) \cong \mathcal{O}(D'_w) \text{ , } w = r^{u_{ll'}}_{ll'm}(v).$$

which is equivalent to

$$\{(U_{\sigma_i}, X_i^{z_i})\} \text{ , } z_i = h^{u_{li}}_{lim}(v) - h^{u_{l'i}}_{l'im}(w)$$

being a principal Cartier divisor. But by Addendum 1 we know that

$$z_i = C_{l'i} h_{ll'm}^{u_{ll'}}(v).$$

from which we conclude that

$$z_{ij} = z_j - C_{ij} z_i = 0.$$

It follows that $\{(U_{\sigma_i}, X_i^{z_i})\}$ is principal as desired.

Let us next show that the definition is independent of the choice of $\{(U_{\sigma_i}, X_i^{u_i})\}$. Suppose $\{(U_{\sigma_i}, X_i^{u'_i})\}$ represent a Cartier divisor D' such that $\mathcal{O}(D') \cong L$. Then $u_{ij} = u'_{ij}$ for all i and j. But then

$$v_i = h_{lim}^{u_{li}}(v) = h_{lim}^{u'_{li}}(v) = v'_i.$$

We conclude that $\mathcal{O}(D_v)$ are equal to $\mathcal{O}(D_{v'})$. It is thereby shown that F_m is well defined. Notice however that we do not claim that F_m is independent of the definitions and choices made in Section 3 (even though it might be).

5. Splitting of $F_* L$

Let X be a variety over an algebraically closed field k of characteristic $p > 0$. The absolute Frobenius morphism on X is the morphism $F : X \to X$ of schemes which is the identity on the underlying set of X, and the p'th power map on sheaf level $F^{\#} : \mathcal{O}_X \to \mathcal{O}_X$. When X is a smooth variety and E is a locally free sheaf on X of rank d, the direct image $F_* E$ is also a locally free sheaf, but of rank $dp^{\dim(X)}$. We will now prove our main result.

Theorem 1. *Let $X = X(\Delta)$ be a smooth toric variety over a field k of positive characteristic and $F : X \to X$ be the absolute Frobenius morphism. Taking direct images via F_* induces for each $r \in \mathbb{N}$ a map*

$$(F_*)^r \ : \ \mathrm{Pic}(X) \to \mathrm{Vect}^m(X) \ , \ m = p^r$$

which coincide with F_m. In particular $F_* L$ splits into a direct sum of line bundles, when L is a line bundle on X.

Proof. Let $[L] \in \mathrm{Pic}(X)$ and choose a T-Cartier divisor D such that $\mathcal{O}(D) \cong L$ and a set $\{(U_{\sigma_i}, X_i^{u_i})\}$ representing D. For each $v \in P_m$ let D_v be the Cartier divisor associated to $\{(U_{\sigma_i}, X_i^{v_i})\}$ where $v_i = h_{0im}^{u_{0i}}(v)$. For a given $v \in P_m$ we will now construct a map

$$\pi^v \ : \ \mathcal{O}(D_v) \to (F_*)^r(\mathcal{O}(D)).$$

This will be done by defining it locally over U_{σ_i} for each $i \in \{0, 1, \dots q\}$

$$\pi_i^v \ : \ \mathcal{O}(D_v)(U_{\sigma_i}) \to (F_*^r(\mathcal{O}(D)))(U_{\sigma_i}).$$

For this purpose denote by $F_*^r K$ the $\mathcal{O}_X(U_{\sigma_i})$-module which as an abelian group is the field of rational functions K on X, but where the $\mathcal{O}_X(U_{\sigma_i})$-module structure is twisted by the p^r'th power map on $\mathcal{O}_X(U_{\sigma_i})$. Then $(F_*^r(\mathcal{O}(D)))(U_{\sigma_i})$ is the free submodule of $F_*^r K$ with basis $\{X_i^w X_i^{-u_i}\}_{w \in P_m}$. On the other hand $\mathcal{O}(D_v)(U_{\sigma_i})$ is the free submodule of K with generator $X_i^{-v_i}$. We define the map π_i^v to be the unique $\mathcal{O}_X(U_{\sigma_i})$-linear map satisfying

$$\pi_i^v(1 \cdot X_i^{-v_i}) = X_i^{r_{0im}^{u_{0i}}(v)} \cdot X_i^{-u_i}.$$

We claim that these local morphisms glue together. This is so because over $U_{\sigma_i} \cap U_{\sigma_j}$ we have by Lemma 1(1) that

$$1 \cdot X_i^{-v_i} = X_j^{v_j - C_{ij} v_i} \cdot X_j^{-v_j} = X_j^{v_{ij}} \cdot X_j^{-v_j}$$

which by π_j^v maps to

$$X_j^{mv_{ij}+r_{0jm}^{u_{0j}}(v)} \cdot X_j^{-u_j}.$$

Further this last expression is by Lemma 1 and Addendum 1 easily seen to be equal to $\pi_i^v(1 \cdot X_i^{-v_i})$. As the local morphisms glue together we get the desired morphism $\pi^v : \mathcal{O}(D_v) \to (F_*)^r(\mathcal{O}(D))$. The collection of maps π^v now induces a morphism

$$\pi : \bigoplus_{v \in P_m} \mathcal{O}(D_v) \to (F_*)^r(\mathcal{O}(D))$$

and Lemma 3 tells us that this is an isomorphism. $\square$

6. Results

In this section we will state and prove a few results about F_m. In the characteristic $p > 0$ situation these results translate into statements about F_*^r, referring to Theorem 1.

Proposition 1. *Let L be a fixed line bundle on a smooth toric variety. Then the set of line bundles occurring in the definition of F_mL, as m runs through $\mathbb{N}$, is finite.*

Proof. We only have to notice that $h_{ijm}^{u_{ij}}(v)$ in the definition of F_mL is bounded for $m \in \mathbb{Z}$. This is an easy exercise. $\square$

Proposition 2. *Let L be a line bundle on a smooth toric variety. The line bundle* $\det(F_mL)$ *(i.e. the top exterior power of F_mL) is isomorphic to*

$$\det(F_mL) \cong \omega_X^{m^{n-1}(m-1)/2} \otimes L^{m^{n-1}}$$

where ω_X is the dualizing sheaf on X.

Proof. Choose a T-Cartier divisor D so that $\mathcal{O}(D) \cong L$ and a set $\{(U_{\sigma_i}, X_i^{u_i})\}$ representing D. As $D_v, v \in P_m$, is represented by $\{(U_{\sigma_i}, X_i^{h_{0im}^{u_{0i}}(v)})\}$ (we use $l = 0$ in the definition of F_mL) we get that $\det(\sum_{v \in P_m} D_v)$ is represented by $\{(U_{\sigma_i}, X_i^{h_i})\}$, where $h_i = \sum_{v \in P_m} h_{0im}^{u_{0i}}(v)$. Using the equality

$$C_{0i}v + u_{0i} = mh_{0im}^{u_{0i}}(v) + r_{0im}^{u_{0i}}(v)$$

and summing over $v \in P_m$ we find

$$C_{0i}(\sum_{v \in P_M} v) + m^n u_{0i} = mh_i + \sum_{v \in P_m} r_{0im}^{u_{0i}}(v).$$

Now Lemma 3 tells us that

$$\sum_{v \in P_m} r_{0im}^{u_{0i}}(v) = \sum_{v \in P_m} v = \frac{1}{2}m^n(m-1)e$$

where $e = (1, 1, \ldots, 1)$, so that

$$h_i = m^{n-1}u_{0i} + \frac{1}{2}m^{n-1}(m-1)(C_{0i}e - e).$$

Recognizing $\{(U_{\sigma_i}, X_i^{-e})\}$ as representing the canonical divisor ([2] p.85-86) the lemma follows. $\square$

In the characteristic $p > 0$ situation the absolute Frobenius morphism F is an affine morphism. This means that the cohomology groups of L and F_*L are isomorphic. One can therefore ask if a similar result is true when F_*L is replaced by F_mL. The answer will follow from the next proposition.

Proposition 3. *Let L be a line bundle on a smooth toric variety and $m \in \mathbb{N}$. As sheaves of abelian groups L and $F_m L$ are isomorphic. In particular their cohomology agree.*

Proof. Regard L as the sheaf $\mathcal{O}(D)$ where D is represented by $\{(U_{\sigma_i}, X_i^{u_i})\}$. Let $v \in P_m$ and define $v_i = h_{0im}^{u_{0i}}(v)$. We will then construct a map of sheaves of abelian groups

$$\psi^v : \mathcal{O}(D_v) \to L.$$

This will be done by defining it locally over U_{σ_i}

$$\psi_i^v : \mathcal{O}(D_v)(U_{\sigma_i}) \to L(U_{\sigma_i}).$$

As submodules of K we have that $\mathcal{O}(D_v)(U_{\sigma_i})$ and $L(U_{\sigma_i})$ are the free $\mathcal{O}_X(U_{\sigma_i})$-modules with generators $1 \cdot X_i^{-v_i}$ and $1 \cdot X_i^{-u_i}$ respectively. If $w \in \mathrm{Mat}_{n,1}(\mathbb{Z})$ is a vector with $X_i^w \in R_i$ we define

$$\psi_i^v(aX_i^w \cdot X_i^{-v_i}) = aX_i^{mw + r_{0im}^{u_{0i}}(v)} \cdot X_i^{-u_i} \ , \ a \in k$$

and expand this to a morphism of groups. This defines the map ψ_i^v of abelian groups. It is now an easy exercise to check that these local morphism glue together and gives us the desired map $\psi^v : \mathcal{O}(D_v) \to L$. The collection of maps ψ^v now induces a map of sheaves of abelian groups

$$\psi : \bigoplus_{v \in P_m} \mathcal{O}(D_v) \to L$$

which by Lemma 3 can be seen to be an isomorphism. □

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IRREDUCIBILITY OF $\overline{M}_{0,n}(G/P, \beta)$

JESPER FUNCH THOMSEN

1. Introduction

Let G be a complex connected linear algebraic group, P be a parabolic subgroup of G and $\beta \in A_1(G/P)$ be a 1-cycle class in the Chow group of G/P. An n-pointed genus 0 stable map into G/P representing the class β, consists of data $(\mu : C \to X; p_1, \ldots, p_n)$, where C is a connected, at most nodal, complex projective curve of arithmetic genus 0, and μ is a complex morphism such that $\mu_*[C] = \beta$ in $A_1(G/P)$. In addition p_i, $i = 1, \ldots, n$ denote n nonsingular marked points on C such that every component of C, which by μ maps to a point, has at least 3 points which is either nodal or among the marked points (this we will refer to as every component of C being stable). The set of n-pointed genus 0 stable maps into G/P representing the class β, is parameterized by a coarse moduli space $\overline{M}_{0,n}(G/P, \beta)$. In general it is known that $\overline{M}_{0,n}(G/P, \beta)$ is a normal complex projective scheme with finite quotient singularities. In this paper we will prove that $\overline{M}_{0,n}(G/P, \beta)$ is irreducible. It should also be noted that we in addition will prove that the boundary divisors in $\overline{M}_{0,n}(G/P, \beta)$, usually denoted by $D(A, B, \beta_1, \beta_2)$ ($\beta = \beta_1 + \beta_2$, $A \cup B$ a partition of $\{1, \ldots, n\}$) , are irreducible.

After this work was carried out we learned that B. Kim and R. Pandharipande [7] had proven the same results, and proved connectedness of the corresponding moduli spaces in higher genus. Our methods however differ in many ways. For example in this paper we consider the action of a Borel subgroup of G on $\overline{M}_{0,n}(G/P, \beta)$, while Kim and Pandharipande mainly concentrate on maximal torus action. Another important difference is that we in this presentation proceed by induction on β. This means that the question of $\overline{M}_{0,n}(G/P, \beta)$ being irreducible, can be reduced to simple cases.

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2. Summary on $\overline{M}_{0,n}(G/P, \beta)$

In this section we will summarize the properties of the coarse moduli space $\overline{M}_{0,n}(G/P, \beta)$ which we will make use of. The notes on quantum cohomology by W. Fulton and R. Pandharipande [3] will serve as our main reference.

As mentioned in the introduction the moduli space $\overline{M}_{0,n}(G/P, \beta)$ parameterizes n-pointed genus 0 stable maps into G/P representing the class β. By definition β is effective if it is represented by some n-pointed genus 0 stable map. In the following

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we will only consider values of n and β where $\overline{M}_{0,n}(G/P,\beta)$ is non-empty. This means β must be effective and $n \geq 0$, and if $\beta = 0$ we must have $n \geq 3$.

The moduli space $\overline{M}_{0,n}(G/P,\beta)$ is known to be a normal projective scheme (see [3]). This implies that $\overline{M}_{0,n}(G/P,\beta)$ splits up into a finite disjoint union of its components. This we will use several times.

2.1. **Contraction morphism.** On $\overline{M}_{0,n+1}(G/P,\beta)$ we have a contraction morphism

$$\overline{M}_{0,n+1}(G/P,\beta) \to \overline{M}_{0,n}(G/P,\beta)$$

which "forget" the $(n+1)$'th marked point. The contraction morphisms value on a closed point in $\overline{M}_{0,n+1}(G/P,\beta)$, represented by $(\mu : C \to G/P; p_1, \ldots, p_{n+1})$, is the point in $\overline{M}_{0,n}(G/P,\beta)$ represented by $(\mu^\circ : C^\circ \to G/P; p_1, \ldots, p_n)$, where C° denote C with the unstable components collapsed, and μ° is the map induced from μ. From the construction of $\overline{M}_{0,n}(G/P,\beta)$ it follows, that the contraction map is a surjective map with connected fibres.

2.2. **Evaluation map.** For each element $a \in \{1, \ldots n\}$ we have an evaluation map

$$\delta_a : \overline{M}_{0,n}(G/P,\beta) \to G/P.$$

Its value on a closed point in $\overline{M}_{0,n}(G/P,\beta)$ represented by the element $(\mu : C \to G/P; p_1, \ldots, p_n)$ is defined to be $\mu(p_a)$.

2.3. **Boundary.** By a boundary point in $\overline{M}_{0,n}(G/P,\beta)$ we will mean a point which correspond to a reducible curve. Let $A \cup B = \{1, \ldots n\}$ be a partition of $\{1, \ldots n\}$ in disjoint sets, and let $\beta_1, \beta_2 \in A_1(X)$ be effective classes such that $\beta = \beta_1 + \beta_2$. We will only consider the cases when $\beta_1 \neq 0$ (resp. $\beta_2 \neq 0$) or $|A| \geq 2$ (resp. $|B| \geq 2$). With these conditions on β_1, β_2, A and B we let $D(A, B, \beta_1, \beta_2)$ denote the set of elements in $\overline{M}_{0,n}(G/P,\beta)$ where the corresponding curve C is of the following form

- C is the union of (at most nodal) curves C_A and C_B meeting in a point.
- The markings of A and B lie on C_A and C_B respectively.
- C_A and C_B represent the classes β_1 and β_2 respectively.

Notice here that our restrictions on A,B,β_1 and β_2 is the stability conditions on C_A and C_B.

It is clear that every boundary element lies in at least one of these $D(A, B, \beta_1, \beta_2)$. The sets $D(A, B, \beta_1, \beta_2)$ are in fact closed, and we will regard them as subschemes of $\overline{M}_{0,n}(G/P,\beta)$ by giving them the reduced scheme structure. Closely related to $D(A, B, \beta_1, \beta_2)$ is the scheme $M(A, B, \beta_1, \beta_2)$ defined by the fibre square

$$\begin{array}{ccc} M(A,B,\beta_1,\beta_2) & \xrightarrow{p_2} & \overline{M}_{0,A\cup\{\bullet\}}(G/P,\beta_1) \times \overline{M}_{0,B\cup\{\bullet\}}(G/P,\beta_2) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \delta_\bullet^A \times \delta_\bullet^B} \\ G/P & \xrightarrow{\Delta} & G/P \times G/P \end{array}$$

Here Δ is the diagonal embedding and $\delta_\bullet^A$ and $\delta_\bullet^B$ denotes the evaluation maps with respect to the point $\{\bullet\}$. In [3] it is proved that $M(A, B, \beta_1, \beta_2)$ is a normal projective variety and that we have a canonical map

$$M(A, B, \beta_1, \beta_2) \longrightarrow D(A, B, \beta_1, \beta_2).$$

This map is clearly surjective. As $M(A, B, \beta_1, \beta_2)$ is a closed subscheme of the product $\overline{M}_{0,A\cup\{\bullet\}}(G/P,\beta_1) \times \overline{M}_{0,B\cup\{\bullet\}}(G/P,\beta_2)$, we can regard the closed points of $M(A, B, \beta_1, \beta_2)$ as elements of the form (z_1, z_2), where $z_1 \in \overline{M}_{0,A\cup\{\bullet\}}(G/P,\beta_1)$ and $z_2 \in \overline{M}_{0,B\cup\{\bullet\}}(G/P,\beta_2)$. The image of (z_1, z_2) in $D(A, B, \beta_1, \beta_2)$ will then be denoted by $z_1 \sqcup z_2$. Given $z_1 \in \overline{M}_{0,A\cup\{.\}}(G/P,\beta_1)$, $z_2 \in \overline{M}_{0,B\cup\{.\}\cup\{*\}}(G/P,\beta_2)$ and $z_3 \in \overline{M}_{0,C\cup\{*\}}(G/P,\beta_3)$, with $\delta_.(z_1) = \delta_.(z_2)$ and $\delta_*(z_2) = \delta_*(z_3)$, we then have the identity $(z_1 \sqcup z_2) \sqcup z_3 = z_1 \sqcup (z_2 \sqcup z_3)$ inside $\overline{M}_{0,A\cup B\cup C}(G/P,\beta_1+\beta_2+\beta_3)$.

2.4. **G-action.** As mentioned in the introduction we have a G-action

$$G \times \overline{M}_{0,n}(G/P,\beta) \to \overline{M}_{0,n}(G/P,\beta).$$

On closed points we can describe the action in the following way. Let x be the closed point in $\overline{M}_{0,n}(G/P,\beta)$ corresponding to the data $(\mu : C \to G/P; p_1,\ldots,p_n)$, and let g be a closed point in G. Then $g\cdot x$ is the point in $\overline{M}_{0,n}(G/P,\beta)$ corresponding to $(\mu_g : C \to G/P; p_1,\ldots,p_n)$, where $\mu_g = (g\cdot)\circ\mu$. Here $g\cdot$ denotes multiplication with g on G/P.

2.5. **Special cases.** The following special cases of our main result follows from the construction and formal properties of our moduli spaces.
$\beta = 0$: Here the moduli space $\overline{M}_{0,n}(G/P,\beta)$ is canonical isomorphic to $\overline{M}_{0,n}\times G/P$, where $\overline{M}_{0,n}$ denote the moduli space of stable n-pointed curves of genus 0. As $\overline{M}_{0,n}$ is known to be irreducible [8] we get that $\overline{M}_{0,n}(G/P,0)$ is irreducible.
$G/P = \mathbb{P}^1$: The irreducibility of $\overline{M}_{0,n}(\mathbb{P}^1,d)$ follows from the construction of the moduli space in [3]. First of all $\overline{M}_{0,0}(\mathbb{P}^1,1) \cong \mathrm{Spec}(\mathbb{C})$ so we may assume that $(n,d) \neq (0,1)$. With this assumption $\overline{M}_{0,n}(\mathbb{P}^1,d)$ is the quotient of a variety M by a finite group. Now M is glued together by the moduli spaces $\overline{M}_{0,n}(\mathbb{P}^1,d,\bar{t})$ of $\bar{t}$-maps spaces (here $\bar{t} = (t_0,t_1)$ is a basis of $\mathcal{O}_{\mathbb{P}^1}(1)$). See section 3 in [3] for a definition of $\overline{M}_{0,n}(\mathbb{P}^1,d,\bar{t})$. The moduli spaces $\overline{M}_{0,n}(\mathbb{P}^1,d,\bar{t})$ are irreducible (in fact they are $\mathbb{C}^*$-bundles over an open subscheme of $\overline{M}_{0,m}$ for a suitable m). This follows from the proof of Proposition 3.3 in [3]. It is furthermore clear that $\overline{M}_{0,n}(\mathbb{P}^1,d,\bar{t})$, and $\overline{M}_{0,n}(\mathbb{P}^1,d,\bar{t}')$ intersect non-trivially for different choices of bases $\bar{t}$ and $\bar{t}'$. This imply that $\overline{M}_{0,n}(\mathbb{P}^1,d)$ is connected, and as it is locally normal it must be irreducible.

3. Flag varieties

In this section we will give a short review on flag varieties. Main references will be [9], [1] and [6]. In [9] one can find the general theory on the structure of linear algebraic groups. The Chow group of G/B, where B is a Borel subgroup, can be found in [1]. From this one easily recovers the Chow group for a general flag variety G/P (e.g. [6] Section 1).

3.1. **Schubert varieties.** Let G be a complex connected linear algebraic group and P be a parabolic subgroup of G. As we will only be interested in the quotient G/P, we may assume that G is semisimple. Fix a maximal torus T and a Borel subgroup B such that

$$T \subseteq B \subseteq P \subseteq G.$$

Let W (resp. R) denote the Weyl group (resp. roots) associated to T and let R^+ denote the positive roots with respect to B. Let further $D \subseteq R^+$ denote the simple roots. Given $\alpha \in R$ we let $s_\alpha \in W$ denote the corresponding reflection.

From general theory on algebraic groups we know that P is associated to a unique subset $I \subseteq D$, such that $P = BW_IB$, where W_I is the subgroup of W generated by the reflections s_α with $\alpha \in I$. The flag variety G/P is then the disjoint union of a finite number of B-invariant subsets $C(w) = BwP/P$ with $w \in W^I$, where

$$W^I = \{w \in W | w\alpha \in R^+ \text{ for all } \alpha \in I\}.$$

Each $C(w)$, $w \in W^I$ is isomorphic to $\mathbb{A}^{l(w)}$. Here $l(w)$ denotes the length of a shortest expression of w as a product of simple reflections s_α, $\alpha \in D$. The closures of $C(w)$, $w \in W$, inside G/P is called the generalized Schubert varieties. We will denote them by X_w, $w \in W$, respectively. In case $l(w) = 1$ we have $X_w \cong \mathbb{P}^1$.

3.2. **Chow group.** The Chow group $A_*(G/P)$ is freely generated. As a basis we can pick $[X_w]$, $w \in W^I$. In [6] it is proved that this basis is orthogonal. Using that positive classes intersect in positive classes on G/P (Cor. 12.2 in [2]), we conclude that a class in $A_*(G/P)$ is positive (or zero) if and only if it is of the form

$$\sum_{w \in W^I} a_w [X_w] \text{ with } a_w \geq 0.$$

3.3. **Effective classes.** Let $\beta \in A_1(G/P)$. From above it is clear that β can only be effective (in the sense of Section 2), if β is a positive linear combination of $[X_{s_\alpha}]$ with $\alpha \in D \cap W^I = D \setminus I$. Noticing that $X_{s_\alpha} \cong \mathbb{P}^1$, $\alpha \in D \setminus I$, implies the inverse, that is, a positive linear combination of $[X_{s_\alpha}]$, $\alpha \in D \setminus I$ is effective.

Using the above we can introduce a partial ordering on the set of effective classes in $A_1(G/P)$.

Definition 1. *Let β_1 and β_2 be effective classes. If there exist an effective class β_3 such that $\beta_2 = \beta_1 + \beta_3$ we write $\beta_1 \prec \beta_2$. If β is an effective class with the property*

$$\beta' \prec \beta \Rightarrow \beta' = 0 \textit{ or } \beta' = \beta$$

we say that β is irreducible. An effective class β is reducible if it is not irreducible.

Notice that a non-zero effective class β is irreducible if and only if $\beta = [X_{s_\alpha}]$ for some $\alpha \in D \setminus I$.

In the proof of the irreducibility of $\overline{M}_{0,n}(G/P, \beta)$ we will use induction on β with respect to this ordering. This is possible because given an effective class $\beta \in A_1(G/P)$, there is only finitely many other effective classes β' with $\beta' \prec \beta$.

3.4. **Summary.** We are ready to summarize what will be important for us

- The set of effective classes in $A_1(G/P)$ has a $\mathbb{Z}_{\geq 0}$-basis represented by B-invariant closed subvarieties X_{s_α}, $\alpha \in D \setminus I$, of G/P.
- The subsets X_{s_α}, $\alpha \in D \setminus I$ are the only B-invariant irreducible 1-dimensional closed subsets of G/P.
- $X_{s_\alpha} \cong \mathbb{P}^1$, $\alpha \in D \setminus I$.

4. Boundary of $\overline{M}_{0,n}(X, \beta)$

In this section we begin the proof of our main result. Remember that our convention is that whenever we write $\overline{M}_{0,n}(G/P, \beta)$, $D(A, B, \beta_1, \beta_2)$ or $M(A, B, \beta_1, \beta_2)$, we assume that these are well defined and non-empty. From now on we will assume that G, a semisimple linear algebraic group, and a parabolic subgroup P have been fixed. We let X denote G/P.

We will need to know when $D(A, B, \beta_1, \beta_2)$ is irreducible and for this purpose we have the following proposition.

Proposition 1. *Assume that $\overline{M}_{0,A\cup\{\bullet\}}(X, \beta_1)$ and $\overline{M}_{0,B\cup\{\bullet\}}(X, \beta_2)$ are irreducible. Then the scheme $M(A, B, \beta_1, \beta_2)$ is also irreducible. In particular $D(A, B, \beta_1, \beta_2)$ will be irreducible in this case.*

Proof As $M(A,B,\beta_1,\beta_2)$ is a normal scheme it splits up into a disjoint union of irreducible components $C_1, C_2, \dots, C_l$. Our task is to show that $l=1$. Consider the natural map $\pi : G \to G/P$. Locally (in the Zariski topology) this map has a section ([5] p.183) , i.e. there exists an open cover $\{U_i\}_{i\in I}$ of X (we assume $U_i \neq \emptyset$) and morphisms $s_i : U_i \to G$ such that $\pi \circ s_i$ is the identity map. By pulling back the covering $\{U_i\}_{i\in I}$ of X, by the evaluation maps $\delta^A_{\bullet}$ and $\delta^B_{\bullet}$, we get open coverings $\{V_i^A\}_{i\in I}$ and $\{V_i^B\}_{i\in I}$ of $\overline{M}_{0,A\cup\{\bullet\}}(X,\beta_1)$ and $\overline{M}_{0,B\cup\{\bullet\}}(X,\beta_2)$ respectively. Finally an open cover $\{W_i\}_{i\in I}$ of $M(A,B,\beta_1,\beta_2)$ is obtained by setting $W_i = p_1^{-1}(U_i) = p_2^{-1}(V_i^A \times V_i^B)$. We claim

$$\forall i,j \in I : W_i \cap W_j \neq \emptyset.$$

To see this consider $U_i, U_j \subseteq X$. As X is irreducible there exists a closed point $x \in U_i \cap U_j$. Using that G acts transitively on X we can choose elements $z_1 \in \overline{M}_{0,A\cup\{\bullet\}}(X,\beta_1)$ and $z_2 \in \overline{M}_{0,B\cup\{\bullet\}}(X,\beta_2)$, with $\delta^A_{\bullet}(z_1) = \delta^B_{\bullet}(z_2) = x$. With these choices it is clear that (z_1,z_2) correspond to a point in $W_i \cap W_j$.

Next we want to show that W_i is irreducible. For this consider the map

$$\begin{array}{rccl} \psi_i: & V_i^A \times V_i^B & \to & \overline{M}_{0,A\cup\{\bullet\}}(X,\beta_1) \times \overline{M}_{0,B\cup\{\bullet\}}(X,\beta_2) \\ & (z_1,z_2) & \mapsto & (z_1, ((s_i\circ\delta^A_{\bullet})(z_1))((s_i\circ\delta^B_{\bullet})(z_2))^{-1}z_2) \end{array}$$

where we use the group action of G on $\overline{M}_{0,B\cup\{\bullet\}}(X,\beta_2)$. By definition ψ_i factors through W_i. We therefore have an induced map

$$\psi_i' : V_i^A \times V_i^B \to W_i.$$

Clearly $\psi_i' \circ p_2$ is the identity map. This implies that ψ_i' is surjective, and as $V_i^A \times V_i^B$ is irreducible, we get that W_i is irreducible.

At last we notice that as W_i is irreducible it must be contained in one of the components $C_1, C_2, \dots, C_l$ of $M(A,B,\beta_1,\beta_2)$. On the other hand the W_i's intersect non-trivially so all of them must be contained in the same component. But $\{W_i\}_{i\in I}$ was an open cover of $M(A,B,\beta_1,\beta_2)$. We conclude that $l=1$, as desired. Being a surjective image of $M(A,B,\beta_1,\beta_2)$ this implies that $D(A,B,\beta_1,\beta_2)$ is also irreducible. □

5. Properties of the components of $\overline{M}_{0,n}(X,\beta)$

In this section we study the behaviour of the components of $\overline{M}_{0,n}(X,\beta)$. Let $K_1, K_2, \dots, K_l$ denote the components of $\overline{M}_{0,n}(X,\beta)$. As $\overline{M}_{0,n}(X,\beta)$ is normal, the K_i's are disjoint. Remember that we had a group action of G on $\overline{M}_{0,n}(X,\beta)$ which was introduced in Section 2. We claim

Lemma 1. *Let K be a component of $\overline{M}_{0,n}(X,\beta)$. Then K is invariant under the group action of G on $\overline{M}_{0,n}(X,\beta)$.*

Proof *Let $\eta : G \times \overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,n}(X,\beta)$ denote the group action, and consider the image $\eta(G\times K)$ of $G \times K$. As $G\times K$ is irreducible $\eta(G\times K)$ will also be irreducible. This means that $\eta(G\times K)$ is contained in a component, say K_1, of $\overline{M}_{0,n}(X,\beta)$. On the other hand $\eta(\{e\}\times K) \subseteq K$ (here e denotes the identity element in G) so we conclude that $K = K_1$.* □

The next lemma concerns the boundary of components of $\overline{M}_{0,0}(X,\beta)$ when β is reducible.

Lemma 2. *Let β be a reducible effective class and K be a component of $\overline{M}_{0,0}(X,\beta)$. Then there exist boundary elements in K.*

Proof Assume K do not have boundary elements. Then by definition of boundary points, each element in K would correspond to an irreducible curve. Using Lemma 1 we have an induced B-action on K. As K is projective, and B is a connected solvable linear algebraic group, we can use Borel's fixed point Theorem (see [9] p.159) to conclude that this action has a fixed point. This means that there exist $z \in K$ such that $bz = z$, for all $b \in B$. Let $\mathbb{P}^1 \xrightarrow{\mu} X$ be the stable curve with its morphism to X which correspond to z. By definition of the group action of B on z we conclude that $\mu(\mathbb{P}^1)$ must be a B-invariant subset of X. On the other hand $\mu(\mathbb{P}^1)$ is closed, irreducible and of dimension 1. By the properties stated in Section 3 $\mu(\mathbb{P}^1)$ must be equal to a 1-dimensional Schubert variety X_{s_α} of X. From this we conclude that $\mu_*[\mathbb{P}^1] = m[X_{s_\alpha}]$, where m is a positive integer. As β is reducible $m \geqq 2$. The closed embedding $i : X_{s_\alpha} \to X$ induces a map $i_* : \overline{M}_{0,0}(X_{s_\alpha}, m[X_{s_\alpha}]) \to \overline{M}_{0,0}(X, \beta)$, where an element $(C \xrightarrow{f} X_{s_\alpha}) \in \overline{M}_{0,0}(X_{s_\alpha}, m[X_{s_\alpha}])$ goes to $i_*(C \xrightarrow{f} X_{s_\alpha}) = (C \xrightarrow{i \circ f} X)$. As X_{s_α} is isomorphic to $\mathbb{P}^1$ we know that $\overline{M}_{0,0}(X_{s_\alpha}, m[X_{s_\alpha}])$ is irreducible. On the other hand z is in the image of i_* so we conclude that $i_*(\overline{M}_{0,0}(X_{s_\alpha}, m[X_{s_\alpha}])) \subseteq K$. But a boundary element in $\overline{M}_{0,0}(X_{s_\alpha}, m[X_{s_\alpha}])$ is easy to construct by hand (as $m \geq 2$), which gives us the desired contradiction. □

The following will also be useful.

Lemma 3. *Let β be an effective reducible element in $A_1(X)$, and assume that $\overline{M}_{0,0}(X, \beta')$ is irreducible for $\beta' \prec \beta$. Furthermore let K be a component of $\overline{M}_{0,0}(X, \beta)$. Then there exists a non-zero irreducible class β', with $\beta - \beta'$ effective, such that $D(\emptyset, \emptyset, \beta', \beta - \beta') \cap K \neq \emptyset$.*

Proof By Lemma 2 we can choose a boundary point $z \in K$. There exists effective classes β_1 and β_2 such that $z \in D(\emptyset, \emptyset, \beta_1, \beta_2)$. We may assume that β_1 is reducible. Choose an effective non-zero irreducible class β' and an effective class β'' such that $\beta_1 = \beta' + \beta''$. Choose also $z_1 \in \overline{M}_{0,\{Q_1\}}(X, \beta')$, $z_2 \in \overline{M}_{0,\{Q_1\} \cup \{Q_2\}}(X, \beta'')$ and $z_3 \in \overline{M}_{0,\{Q_2\}}(X, \beta_2)$, such that $\delta_{Q_1}(z_1) = \delta_{Q_1}(z_2)$ and $\delta_{Q_2}(z_2) = \delta_{Q_2}(z_3)$. Then $z_1 \sqcup z_2 \in \overline{M}_{0,\{Q_2\}}(X, \beta_1)$ from which we conclude $(z_1 \sqcup z_2) \sqcup z_3 \in D(\emptyset, \emptyset, \beta_1, \beta_2)$. On the other hand $z_2 \sqcup z_3 \in \overline{M}_{0,\{Q_1\}}(X, \beta - \beta')$ by which we conclude $z_1 \sqcup (z_2 \sqcup z_3) \in D(\emptyset, \emptyset, \beta', \beta - \beta')$. Using Proposition 1 we know that $D(\emptyset, \emptyset, \beta_1, \beta_2)$ is irreducible and as $z \in D(\emptyset, \emptyset, \beta_1, \beta_2) \cap K$, we must have $D(\emptyset, \emptyset, \beta_1, \beta_2) \subseteq K$, in particular $(z_1 \sqcup z_2) \sqcup z_3 \in K$. On the other hand

$$(z_1 \sqcup z_2) \sqcup z_3 = z_1 \sqcup (z_2 \sqcup z_3) \in D(\emptyset, \emptyset, \beta', \beta - \beta').$$

This proves the lemma. □

6. Irreducibility of $\overline{M}_{0,n}(X, \beta)$

In this section we will prove that the moduli spaces $\overline{M}_{0,n}(X, \beta)$ are irreducible. First we notice that for $\beta \neq 0$ we can restrict our attention to a fixed n.

Lemma 4. *Let $n_1, n_2 \geqq 0$ be integers and $\beta \in A_1(X) \setminus \{0\}$ be an effective class. Then $\overline{M}_{0,n_1}(X, \beta)$ is irreducible if and only if $\overline{M}_{0,n_2}(X, \beta)$ is irreducible.*

Proof It is enough to consider the case $n_2 = n + 1$ and $n_1 = n$ for a positive integer n. The contraction morphism $f : \overline{M}_{0,n+1}(X, \beta) \to \overline{M}_{0,n}(X, \beta)$ which forgets the $(n+1)$'th point is a surjective map with connected fibres. Let $K_1, K_2, \dots, K_s$ (resp. $C_1, C_2, \dots, C_t$) be the components of $\overline{M}_{0,n+1}(X, \beta)$ (resp. $\overline{M}_{0,n}(X, \beta)$). As the components are mutually disjoint and f is surjective we must have $s \geqq t$. Let us now restrict our attention to one of the components of $\overline{M}_{0,n}(X,\beta)$, say C_1. Assume that $K_1, K_2, \dots, K_r$ ($r \leqq s$) are the components which by f maps to C_1. It will be enough to show that $r = 1$. Assume $r \geqq 2$. As

$$C_1 = \bigcup_{i=1}^{r} f(K_i)$$

and as C_1 is irreducible, at least one of the components $K_1, K_2, \dots, K_r$ maps surjectively onto C_1. So there must exist a point x in C_1 which is in the image of at least 2 of the components in $\overline{M}_{0,n+1}(X,\beta)$. But then the fibre of f over x is not connected, which is a contradiction. □

The idea in proving the irreducibility of $\overline{M}_{0,n}(X,\beta)$ is to use induction on the class $\beta \in A_1(X)$. By this we mean that we will prove that $\overline{M}_{0,n}(X,\beta)$ is irreducible assuming the same condition is true for $\beta' \prec \beta$. The first step in the induction procedure will be to show that $\overline{M}_{0,0}(X,\beta)$ is irreducible, when β is a non-zero irreducible class.

Lemma 5. *Let β be a non-zero irreducible class. Then $\overline{M}_{0,0}(X,\beta)$ is irreducible.*

Proof As β is a non-zero irreducible class, β must be the class of a 1-dimensional Schubert variety X_{s_α}. Let K be a component of $\overline{M}_{0,0}(X,\beta)$. As in the proof of Lemma 2 we have a B-action on K, which by Borel's fixed point theorem is forced to have a fixed point. Let $x \in K$ be a fixed point. As β is irreducible x must correspond to an irreducible curve, i.e. x correspond to a map of the form $\mathbb{P}^1 \xrightarrow{\mu} X$. The image $\mu(\mathbb{P}^1)$ is a closed 1-dimensional B-invariant irreducible subset of X. As by assumption $\mu_*[\mathbb{P}^1] = [\overline{X}_{s_\alpha}]$, we conclude that $\mu(\mathbb{P}^1) = X_{s_\alpha}$. Now X_{s_α} is isomorphic to $\mathbb{P}^1$, so μ must be an isomorphism onto its image. But clearly every map $\mathbb{P}^1 \xrightarrow{f} X$ with $f(\mathbb{P}^1) = X_{s_\alpha}$, which is an isomorphism onto its image, represent the same point in $\overline{M}_{0,0}(X,\beta)$. Above we have shown that this point belongs to every component of $\overline{M}_{0,0}(X,\beta)$. Using that the components of $\overline{M}_{0,0}(X,\beta)$ are disjoint the lemma follows. □

Now we are ready for the general case.

Theorem 1. *Let $\beta \in A_1(X)$ be an effective class and $X = G/P$ be a flag variety. Then $\overline{M}_{0,n}(X,\beta)$ is irreducible for every positive integer n.*

Proof The case $\beta = 0$ is trivial as noted in Section 2. By Lemma 4 we may therefore assume that $n = 0$. As remarked above we will proceed by induction. Assume that the theorem has been proven for β' with $\beta' \prec \beta$. Referring to Lemma 5 we may assume that β is reducible. Write $\beta = \sum_{i=1}^{m} \beta_i$ as a sum of non-zero irreducible effective classes β_i. Then $m \geq 2$. We divide into 2 cases.

Assume first that $m = 2$. So $\beta = \beta' + \beta''$, where β' and β'' are effective irreducible classes. In this case every boundary element lie in $D(\emptyset, \emptyset, \beta', \beta'')$, which we by induction know is irreducible (Proposition 1). On the other hand do every component of $\overline{M}_{0,0}(X,\beta)$ contain a boundary point (by Lemma 2). Using that the components of $\overline{M}_{0,0}(X,\beta)$ are disjoint, the theorem follows in this case.

Assume therefore that $m \geq 3$. For each $i = 1, \dots, m$ choose $z_i \in \overline{M}_{0,\{Q_i\}}(X,\beta_i)$, a point such that $\delta_{Q_i}(z_i) = eP$, where δ_{Q_i} is the evaluation map onto X. Let $Q = \{Q_1, Q_2, \dots, Q_m\}$, and choose a point $z_0 \in \overline{M}_{0,Q}(X,0)$ corresponding to a curve $C \cong \mathbb{P}^1$ and a map $\mu : C \to X$ such that $\mu(C) = eP$. Define

$$z = z_0 \sqcup (\sqcup_{i=1}^{m} z_i) \in \overline{M}_{0,0}(X,\beta).$$

Then clearly $z \in D(\emptyset, \emptyset, \beta_i, \beta - \beta_i)$ for all i. Let K be the component of $\overline{M}_{0,0}(X, \beta)$ which contains z. By the induction hypothesis and Proposition 1, $D(\emptyset, \emptyset, \beta_i, \beta - \beta_i)$ is irreducible for all i, which implies $D(\emptyset, \emptyset, \beta_i, \beta - \beta_i) \subseteq K$ for all i. On the other hand, by Lemma 3, every component of $\overline{M}_{0,0}(X, \beta)$ will intersect at least one of the sets $D(\emptyset, \emptyset, \beta_i, \beta - \beta_i)$. Using, and now for the last time, that the components of $\overline{M}_{0,0}(X, \beta)$ are disjoint, the theorem follows. □

Corollary 1. *The boundary divisors $D(A, B, \beta_1, \beta_2)$ of $\overline{M}_{0,n}(G/P, \beta)$ are irreducible.*

Proof Use Proposition 1 and Theorem 1. □

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