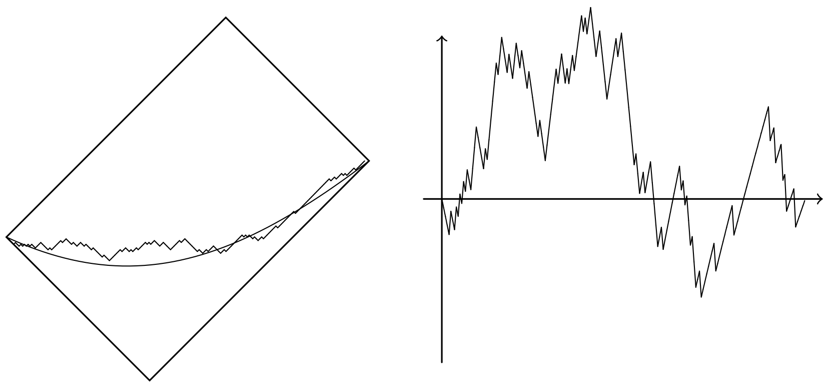
Limit Shapes and Fluctuations of Bounded Random Partitions



Dan Beltoft

Limit Shapes and Fluctuations of Bounded Random Partitions

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PhD Dissertation
July 2010

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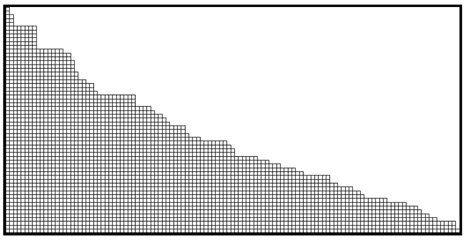
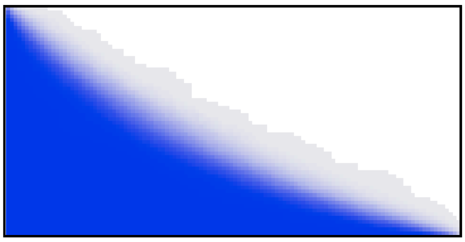




Introduction

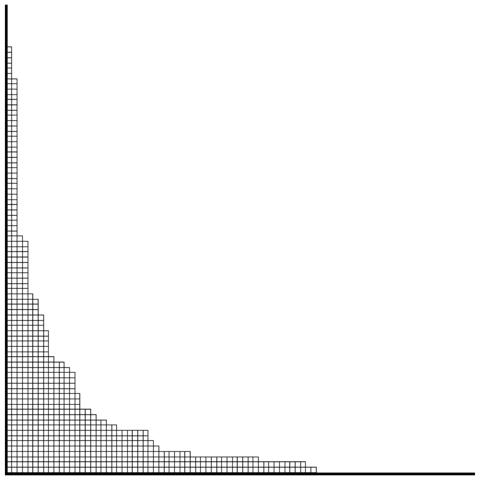
This thesis presents the results of my four years as a PhD student at the Department of Mathematical Sciences at Aarhus University. The topic of the thesis is limit phenomena of random partitions of integers, specifically the viewpoint of statistical mechanics introduced by Vershik. The probability distributions under consideration depend only on the size of a partition, so that asymptotic properties are governed by entropy. A variety of models for such distributions exist, differing in the restrictions placed on partitions. In keeping with the tradition in the field, our focus is on the scaling regimes of limit shapes and fluctuation processes.

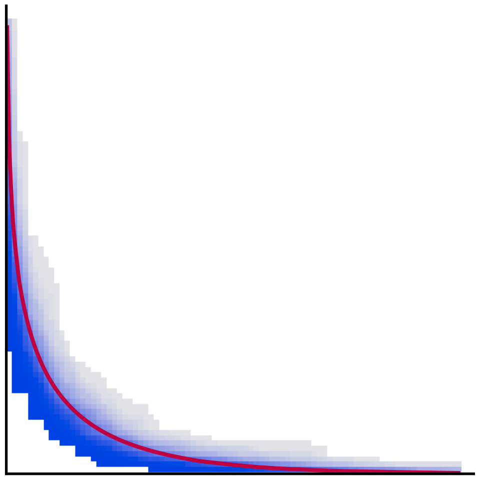
Chapter 1 reviews the basic theory of the statistical mechanics of partitions, and introduces the models of unbounded, semibounded and bounded partitions. Our main focus is the latter model, where partitions are bounded both in the number of summands and the size of summands, corresponding to confining Young diagrams to a rectangle. This is the topic of Chapter 2, where we prove limit shape and fluctuation theorems for such partitions. The limit shape is known in the literature, while the fluctuations have not previously been studied. The fluctuation process is shown to converge to an Ornstein-Uhlenbeck bridge. We give detailed proofs of both, using asymptotics of the Gaussian binomials describing the partition function. In Chapter 3 we compare the results from three models, which turn out to be very closely related. The connection between the limit shapes has been noted elsewhere, but the connections between the fluctuations were, as far as the author is aware, not described earlier. Partitions are identified with continuous piecewise linear curves, which seems to provide the right setting for identifying the connections between the models. The appendix contains a short overview of the theory of weak convergence of probability measures, which is the framework for our convergence results, as well as a section detailing the various Ornstein-Uhlenbeck processes that play a role in the exposition.

Working with a subject having both a simple graphical representation and random elements involved, it was natural to do some numerical experiments. The effort resulted in a simple piece of software to sample Young diagrams from various distributions. Several figures in the text were created using the program.

I wish to thank my advisors Nicolai Reshetikhin and Henning Haahr Andersen for their support on the mathematical and personal level. I am grateful to Nicolai Reshetikhin for introducing me to the fascinating subject of random partitions. Also, a warm thanks to Cedric Boutillier and Nathanaël Enriquez for sharing their time and wisdom and for taking good care of me on my visits to Paris. Thanks to the Department of Mathematical Sciences for providing an inspiring work environment over the past eight years, and thanks to Niels and Magnus for making life at A4.32 so very enjoyable. Finally, thanks to my family and friends for being there when I need them and a special thanks to my girlfriend for her patience with me during the writing of this thesis.

Dan Beltoft
Århus, July 2010





CHAPTER 1

Random Partitions

This chapter contains a brief exposition of Vershik's theory of multiplicative statistics of partitions. The concept of a limit shape is introduced and some classical limit shape results are presented. For the reader who needs to refresh his knowledge of weak convergence of probability measures, a brief tour of the subject can be found in the appendix (p. 49).

1.1 · Partitions and Young Diagrams

A *partition* of a non-negative integer n is a finite non-increasing sequence of positive integers

$$\omega = (\omega_1, \omega_2, \ldots, \omega_b),$$

with $\omega_1 + \cdots + \omega_b = n$. The notation $\omega \vdash n$ or $|\omega| = n$ indicates that ω is a partition of n. The number of summands b is called the *length* of the partition and is denoted $\ell(\omega)$. We denote the set of partitions of n by $\mathcal{P}^n$, while $\mathcal{P}^n_b$ is the subset of $\mathcal{P}^n$ of partitions into exactly b summands. Let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}^n$ be the set of all partitions. The number of partitions of n is denoted $p(n)$, while $p(n,b) = \#\mathcal{P}^n_b$ denotes the number of partitions of n with exactly b summands. Note that $p(0) = 1$: the empty sequence is a partition of zero, and the only one at that.

The study of partitions goes back to Euler in the 18th century. He discovered, among many other things, the classical generating function for the numbers $p(n)$:

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k} = \sum_{n=0}^{\infty} p(n) q^n. \tag{1.1}$$

An argument for the validity of this equality goes as follows. Expand each factor on the left as a geometric series:

$$(1 + q + q^2 + q^3 + \cdots)(1 + q^2 + q^4 + q^6 + \cdots)(1 + q^3 + q^6 + q^9 + \cdots)\cdots$$

The coefficient of q^n in this product is the number of ways to pick a term from each parenthesis such that their product is q^n. This is exactly the number of ways to partition n into a sum of positive integers, with the term from the first parenthesis representing the 1's, the term from the second representing the 2's and so on.

A convenient graphical representation of a partition is its *Young diagram*. The Young diagram associated to a partition $\omega = (\omega_1, \omega_2, \ldots, \omega_b)$ is the (closure of the) subset of the (x, y)-plane defined by $0 \le y \le \omega_{\lceil x \rceil}$, $0 < x \le b$. The Young diagram obviously has area n. The 'jagged' part of the boundary connecting $(0, \omega_1)$ and $(b, 0)$ is called the *interface* of the diagram. See figure 1.1. There are varying traditions regarding the preferred rotation and reflection of a Young diagram. Our choice is apparently a mix of the 'French' and 'Russian' styles.

In the present text, we are concerned with asymptotic aspects of random partitions, and no matter the convention, Young diagrams are very useful in this regard. Convergence theorems are naturally expressed in terms of some form of convergence of a suitable rescaling of the interface. Suppose that we have for each $n \in \mathbb{N}$ a probability measure μ^n on $\mathcal{P}^n$. The question is: Is it possible to normalize the Young diagrams in such a way that the sequence (μ^n), seen as a measure on some suitable space of generalized Young diagrams, has a weak limit as $n \to \infty$?

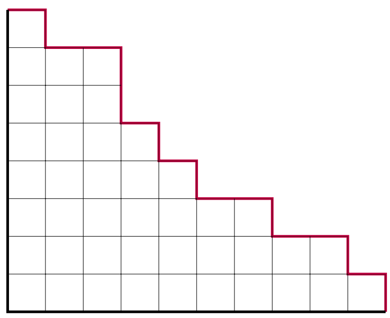


Figure 1.1: The Young diagram for the partition $(8, 7, 7, 5, 4, 3, 3, 2, 2, 1)$ of $n = 42$. The interface of the diagram is the colored curve.

1.2 · Multiplicative Statistics and Limit Shapes

In [19], Vershik introduced a family of probability measures on partitions known as *multiplicative statistics*. These measures are inspired by the statistical physics of a quantum ideal gas: a partition of n represents a collection of particles (summands) with total energy n. The number of particles with a specific energy is

$$r_k(\omega) = \#\{j \mid \omega_j = k\}.$$

These are called the *occupation numbers*. The idea is to assign weights to each value of each occupation number, in such a way that the occupation numbers become independent variables in a sense to be made precise below. The reader is referred to [18] for an in-depth discussion of the connection to statistical physics.

Vershik's Young diagrams are transposed compared to the definition above: the horizontal part of the interface is the graph of the function

$$\varphi_\omega(t) = \sum_{k \ge t} r_k(\omega). \tag{1.2}$$

Diagrams are rescaled along the ordinate axis to have unit area, and a scaling parameter $\gamma_n > 0$ is introduced to compensate:

$$\bar{\varphi}_\omega(t) = \frac{\gamma_n}{n}\varphi_\omega(\gamma_n t), \qquad \gamma_n > 0. \tag{1.3}$$

The map $\mathcal{P}^n \ni \omega \mapsto \bar{\varphi}_\omega$ is denoted τ_{γ_n}. Obviously $\bar{\varphi}_\omega$ is a non-negative non-increasing function on $(0,\infty)$ with $\int_0^\infty \bar{\varphi}_\omega(t)\,dt = 1$. Let $\mathcal{D}$ denote the space of all such functions, and endow $\mathcal{D}$ with the topology of uniform convergence on compacts. The space of generalized diagrams considered in [19] and [18] is enlarged to include atoms at 0 and ∞, allowing for escape of probability mass related to Bose-Einstein condensation, but that will not be needed here.

If μ^n is a probability measure on $\mathcal{P}^n$, the question is whether there exists a sequence $\gamma = (\gamma_n)$ such that the pushforward measures $\tau_\gamma^*\mu^n = \mu^n \circ \tau_{\gamma_n}^{-1}$ converge weakly to a limit measure on $\mathcal{D}$. If such a limit measure exists and is *singular*, ie. is a δ-measure on some element $L \in \mathcal{D}$, then L is called the *limit shape*. From the physics point of view, the limit shape describes a limit distribution of the energies of particles, answering questions like, what fraction of the total energy is bound in particles with energies exceeding some given level.

The class of probability measures under consideration is that of multiplicative statistics. Such a measure is defined by a sequence of functions $s = (s_k)$, with $s_k \colon \mathbb{N}_0 \to \mathbb{R}_+$. Define $F_s \colon \mathcal{P} \to \mathbb{R}_+$ by

$$F_s(\omega) = \prod_{k=1}^{\infty} s_k(r_k(\omega))$$

and let

$$\mu^n(\omega) = \frac{F_s(\omega)}{Z_n},$$

where $Z_n = \sum_{\omega \vdash n} F_s(\omega)$ is the normalizing factor known as the *partition function*. The trick is to combine all the measures μ^n into a single measure on $\mathcal{P}$, depending on a parameter $q > 0$. In a way, this corresponds to regarding the number being partitioned as a random variable itself. Generalizing the right hand side of (1.1), we set $\mathcal{F}(q) = \sum_{n=0}^\infty Z_n q^n$, assuming that the sum is convergent when $0 \le q < q_0$, and define

$$\mu_q(\omega) = \frac{q^{|\omega|}F_s(\omega)}{\mathcal{F}(q)}.$$

A mathematician will say that the measure has been *poissonized*, while a physicist will speak of it as the measure on the *macrocanonical ensemble*, ie. the set $\mathcal{P}$ (as compared to the *canonical ensemble* $\mathcal{P}^n$). Once the scaling $\gamma = (\gamma_n)$ is chosen, τ_γ maps $\mathcal{P}$ into $\mathcal{D}$, and we can ask whether the measures $\tau_\gamma^*\mu_q$ have a weak limit as $q \uparrow q_0$. This question will be answered shortly. The measure μ_q is a convex combination of the measures μ^n:

$$\mu_q = \frac{1}{\mathcal{F}(q)}\sum_{n=0}^{\infty} q^n Z_n \mu^n,$$

so restricting μ_q to $\mathcal{P}^n$ returns the measure μ^n (scaled by $\mu_q(\mathcal{P}^n)$). The generalization of the left hand side of (1.1) involves the functions $\mathcal{F}_k(y) = \sum_{r=0}^\infty s_k(r)y^r$. By the same argument that proved (1.1), we have

$$\begin{aligned}\mathcal{F}(q) &= \sum_{n=0}^{\infty}\sum_{\omega\vdash n} q^n \prod_{k=1}^{\infty} s_k(r_k(\omega)) \\ &= \sum_{n=0}^{\infty}\sum_{\omega\vdash n} \prod_{k=1}^{\infty} q^{kr_k(\omega)} s_k(r_k(\omega)) \\ &= \prod_{k=1}^{\infty}\sum_{r=0}^{\infty} s_k(r) q^{rk} = \prod_{k=1}^{\infty} \mathcal{F}_k(q^k),\end{aligned} \tag{1.4}$$

using the fact that $n = |\omega| = \sum_k kr_k(\omega)$. The generating function $\mathcal{F}(q)$ together with the decomposition (1.4) completely determines the family μ_q. Restricting the sum over $\omega \vdash n$ in (1.4) to partitions with a fixed number of summands equal to some k', ie. $r_{k'}(\omega) = r'$, has the effect of replacing the factor $\mathcal{F}_{k'}(q^{k'})$ in the product on the right by $s_{k'}(r')q^{r'k'}$ (pick the term $q^{r'k'}$ from parenthesis number k'). Thus, the total probability of partitions with $r_{k'} = r'$ is

$$\mu_q(r_{k'}(\omega) = r') = \frac{s_{k'}(r')q^{r'k'}}{\mathcal{F}_{k'}(q^{k'})}. \tag{1.5}$$

Restricting several of the r_k's at the same time, we get the product of expressions as above, showing that the occupation numbers $r_k : \mathcal{P} \to \mathbb{N}_0$ are independent random variables. In fact, any family of measures μ_q on $\mathcal{P}$ satisfying (i) the occupation numbers are independent, and (ii) the restriction to $\mathcal{P}^n$ does not depend on q, is of the form (1.4) (see [18]). In [4], where a uniform distribution on $\mathcal{P}^n$ is considered, the independence of the occupation numbers is the starting point for the derivation of a large deviation principle, to which we briefly return in Section 3.4.

Vershik provides several examples of multiplicative statistics where $\mathcal{F}(q)$ has one of the forms

$$\mathcal{F}(q) = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^{\lfloor k^\beta \rfloor}} \tag{1.6}$$

or

$$\mathcal{F}(q) = \prod_{k=1}^{\infty} (1+q^k)^{\lfloor k^\beta \rfloor}, \tag{1.7}$$

with $\beta \geq 0$ a constant related to the dimension of the physical problem (ideal gas) under consideration. For such measures, he proves a trinity of convergence theorems:

(i) Adopting the scaling $\gamma_n = n^{1/(2+\beta)}$, the measures $\tau_\gamma^* \mu_q$ have a nontrivial weak limit as $q \to 1^-$.

(ii) The limit measure is singular.

(iii) The sequence $\tau_\gamma^* \mu^n$ is weakly convergent as $n \to \infty$ and

$$\lim_{n\to\infty} \tau_\gamma^* \mu^n = \lim_{q\to 1^-} \tau_\gamma^* \mu_q.$$

This last property is traditionally known in statistical physics as the *equivalence of the canonical and the macrocanonical ensemble.* Note that since μ_q is a convex combination of the μ^n, it is clear that if the limit on the left exists, then so does the limit on the right, and the two agree.

A limit shape theorem for a sequence of measures is an equivalent of the weak law of large numbers: weak convergence with a deterministic limit. Once a limit shape has been found, one can ask about the fluctuations, in an intermediate scaling, of the random interfaces around the limit shape: if the rescaled interfaces $\bar{\varphi}_\omega$ converge to a limit shape L, consider a sequence of the form $\gamma'_n(\bar{\varphi}_\omega - L)$, for some suitably chosen scaling (γ'_n). One can then hope to prove weak convergence of this sequence to some stochastic process. This is an equivalent of the central limit theorem. We will return to fluctuations in the next chapters and content ourselves with limit shape phenomena for now.

1.3 · Examples of Multiplicative Statistics

A number of limit shape formulas given in [19] have been derived in greater detail in other papers. In this section we introduce some of these models and their limit shapes. The simplest case is when $\mathcal{F}(q)$ is given by (1.1), ie. $\beta = 0$ in (1.6) or $s_k \equiv 1$ for all k, so that μ^n is the uniform measure on $\mathcal{P}^n$. This is known to physicists as the Bose-Einstein statistics. The scaling is $\gamma_n = \sqrt{n}$ and the limit shape is

$$\exp\bigl(-\tfrac{\pi}{\sqrt{6}}x\bigr) + \exp\bigl(-\tfrac{\pi}{\sqrt{6}}y\bigr) = 1, \tag{1.8}$$

originally presented in [21]. We will call this *Vershik's curve*. The figure opposite the beginning of this chapter shows samples from this distribution and a plot of the limit shape. We have been unable to find a treatment of the fluctuations of this process in the literature. In Section 3.1 we rederive the limit shape and calculate the covariance of the fluctuations in a slightly different coordinate system, using the methods developed in Chapter 2.

The case $\beta = 0$ in (1.7) corresponds to the uniform measure on so-called *strict* partitions of n, ie. partitions where all summands are different. The function (1.7) is the generating function for the numbers of strict partitions of n. To physicists, this model is known as the Fermi-Dirac statistics. It applies to particles satisfying the Pauli exclusion principle: no two particles can occupy the same state at the same time. The limit shape is the curve

$$\exp\bigl(\tfrac{\pi}{\sqrt{12}}y\bigr) - \exp\bigl(-\tfrac{\pi}{\sqrt{12}}x\bigr) = 1.$$

The papers [20] and [24] deal with this case in more detail, including the fluctuations around the limit shape.

Another type of restriction is to bound the number of summands $\ell(\omega)$, the basic case being the uniform measure on partitions of n into exactly b positive summands. In this case, a second variable z is introduced in the generating function to count the number of summands:

$$\mathcal{F}_k(q,z) = \frac{1}{1 - zq^k}$$

so that

$$\mathcal{F}(q,z) = \prod_{k=1}^{\infty} \mathcal{F}_k(q,z) = \sum_{n,b} p(n,b) q^n z^b.$$

When q and z are such that the sum converges, we get the measure $\mu_{q,z}$ on $\mathcal{P}$ given by

$$\mu_{q,z}(\omega) = \frac{q^{|\omega|} z^{\ell(\omega)}}{\mathcal{F}(q,z)}.$$

A limit result for $n, b \to \infty$ depends on a choice of the growth of b relative to n. In [23], Vershik and Yakubovich consider the case $b = \alpha\sqrt{n}$ as well as sub-squareroot power growth. Also, one must specify how to rescale the diagrams, and there are two natural options: scale by $\ell(\omega)$ so that $\bar{\varphi}_\omega(0) = 1$ for all ω, or take a uniform scaling by $\sqrt{n}$. We will focus on the latter case, since this is the one related to the bounded partitions of Chapter 2. In this case, the limit shape is the curve

$$e^{-y\sqrt{\mathrm{Li}_2(z_\alpha)}} + z_\alpha e^{-x\sqrt{\mathrm{Li}_2(z_\alpha)}} = 1, \tag{1.9}$$

where $\mathrm{Li}_2(z) = \sum_{k=1}^\infty \frac{z^k}{k^2}$ is the dilogarithm function and $z_\alpha \in (0,1)$ is chosen depending on the scaling factor α to ensure that the subgraph of the curve has unit area. The proof uses the technique of finding the limit as $q \to 1$ of $\mu_{q,z}$, while fixing z at a cleverly chosen value to ensure that the expected length of a diagram agrees with that prescribed by α. Also, since the scaling is based on the value of n, relations between q and n, b are needed to ensure that the scalings coincide. We return to this model in Section 3.1.

A last variant is when both the number of summands and the size of summands is bounded. Say $\ell(\omega) \le b$ and all $\omega_i \le a$, then the Young diagram is contained in the box $[0, b] \times [0, a]$. The model was presented and the limit shape given in [19]. In [15], Petrov proves that the limit shapes arising in this way are simply restrictions of the curve (1.8), chosen according to the geometry of the bounding box and the prescribed area of diagrams. This model is the topic of Chapter 2, where we derive the limit shape and fluctuations in full detail using asymptotic estimates of the partition function.

1.4 · The Plancherel Measure

Let us give an example of a limit shape theorem for random partitions which is not a multiplicative statistic. The complex irreducible representations of the symmetric group S_n are indexed by partitions of n. Let $\dim\omega$ denote the dimension of the representation corresponding to ω. The *Plancherel measure* on $\mathcal{P}^n$ is the measure

$$\mu_{\mathrm{Pl}}(\omega) = \frac{(\dim\omega)^2}{n!}. \tag{1.10}$$

The number $\dim\omega$ is also the number of chains of partitions

$$\varnothing = \omega^{(0)} \lhd \omega^{(1)} \lhd \cdots \lhd \omega^{(n-1)} \lhd \omega^{(n)} = \omega \tag{1.11}$$

where the notation $\omega \lhd \lambda$ means that λ is obtained by adding one cell to ω. One can think of such a chain as a path in the Young graph (Figure 1.3) from $\varnothing$ to ω, or as a *standard tableau* on the Young diagram of ω, ie. a numbering of the cells by $1, 2, \ldots, n$ that is increasing to the right and up. The classic *hook formula* ([8], see also [9] and [10]) is a convenient way to calculate $\dim\omega$:

$$\dim\omega = \frac{n!}{\prod_{c\in\omega} h_c},$$

where the product extends over the cells of ω and h_c is the *hook length* of the cell c, the number of cells to the right of and above c, including c itself.

Consider the space of inifinite chains as in (1.11), and equip it with the inverse limit $\bar{\mu}$ of the measures μ_n. In [22] and [21] (see also [13]), Vershik and Kerov prove that for almost all infinite chains with respect to $\bar{\mu}$, the rotated and rescaled interfaces

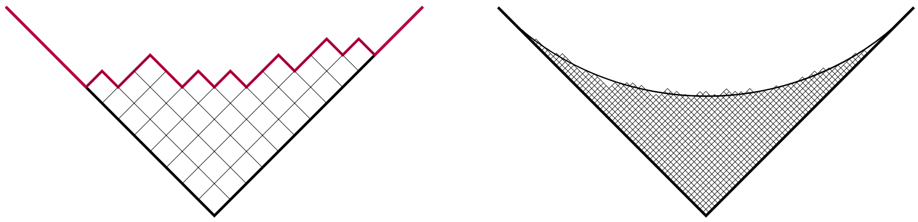


Figure 1.2: On the left, a Young diagram rotated by 45°. The interface is considered to extend left and right as the graph of $|x|$. On the right, a diagram with 500 cells sampled from the Plancherel distribution, together with a plot of the limit shape Ω.

of the diagrams (by an angle of 45° and a factor of $\sqrt{n}$) converge uniformly to the curve

$$\Omega(t) = \begin{cases} \frac{2}{\pi}(t \arcsin t + \sqrt{1-t^2}) & |t| \leq 1 \\ |t| & |t| > 1 \end{cases} \tag{1.12}$$

See Figure 1.2.

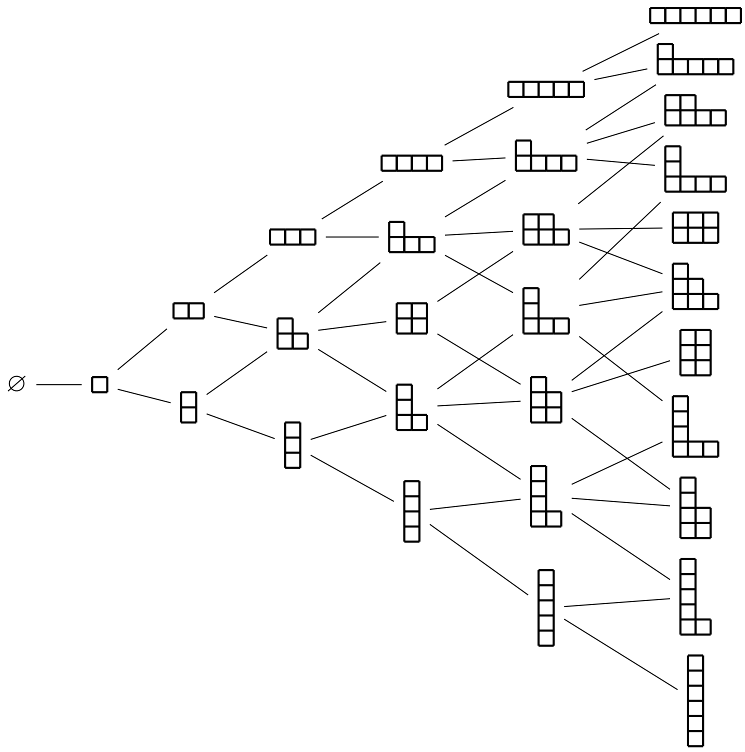
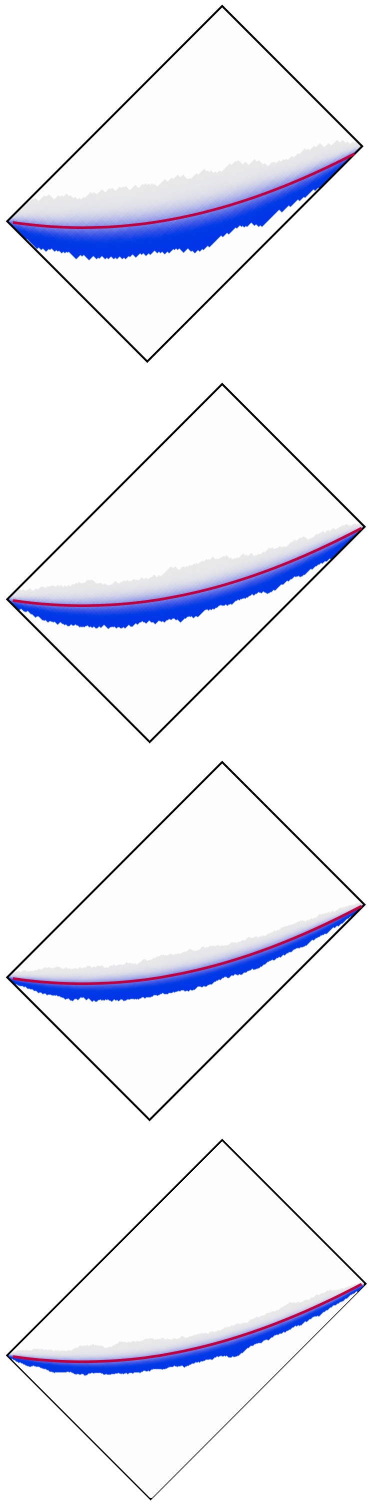


Figure 1.3: The first few levels of the Young graph



Chapter 2

Diagrams in a Box

This chapter contains the material of the paper [2] by C. Boutillier, N. Enriquez and the author of the present text. We study the probability measure on the set of partitions with at most b parts, each part no greater then a, where a partition of n has a probablility proportional to some parameter q to the nth power. This model is different from the multiplicative statistics of Chapter 1, in that the occupation numbers are no longer independent. On the other hand, the partition function can be expressed as a Gaussian binomial coefficient, and the asymptotics of these lead to limit shape and fluctuation results.

It is easy to see that if the parameter q remains fixed while the box grows, the system degenerates in the limit. Thus, the first task is to choose a suitable sequence of q's, depending on the size of the box.

The limit shapes we find are restrictions of Vershik's curve, the limit shape of unbounded partitions under the same distribution. The fluctuations of the interfaces around the limit shape turn out to be an Ornstein-Uhlenbeck bridge, which is also related to the fluctuations in the unbounded case. This angle is explored in Chapter 3.

We make use of big-O and little-o notation. When $\alpha \in \mathbb{R}$, the equality $f(n) = O(n^\alpha)$ signifies that $\frac{f(n)}{n^\alpha}$ is bounded as $n \to \infty$, while $g(n) = o(n^\alpha)$ signifies that $\frac{g(n)}{n^\alpha} \to 0$ as $n \to \infty$.

2.1 · Young Diagrams in Boxes

Given positive integers a, b we let $\mathcal{P}_{a,b}$ be the set of all partitions with at most b parts, each no greater than a. To each such partition corresponds a unique Young diagram contained in the rectangle $[0,b] \times [0,a]$. We will make no effort to distinguish between a partition and its Young diagram.

Fixing a real number $q > 0$, we define a probability measure on $\mathcal{P}_{a,b}$ by assigning to each partition ω the probability

$$\mathsf{P}^q_{a,b}[\omega] = \frac{q^{|\omega|}}{Z_{a,b}(q)}, \tag{2.1}$$

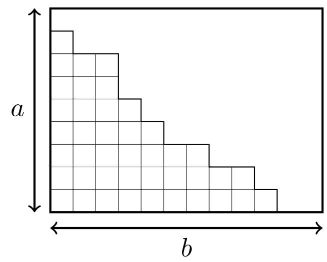


Figure 2.1: A Young diagram in a box.

where $|\omega|$ is the number of cells of ω and $Z_{a,b}(q)$ is the *partition function*, the sum of $q^{|\omega|}$ over $\omega \in \mathcal{P}_{a,b}$. It is a classical result that the partition function is a Gaussian binomial coefficient (or q-binomial coefficient).

Definition 2.1. *Fix $q > 0$. If $n, m \in \mathbb{N}_0$, the* q-integer $(n)_q$ *is given by*

$$(n)_q = \frac{1-q^n}{1-q},$$

the q-factorial $n!_q$ *is given by*

$$n!_q = \prod_{j=1}^{n} (j)_q = \prod_{j=1}^{n} \frac{1-q^j}{1-q},$$

and the q-binomial coefficient $\binom{n}{m}_q$ *is given by*

$$\binom{n}{m}_q = \frac{n!_q}{(n-m)!_q m!_q} = \prod_{j=1}^{m} \frac{1-q^{n-m+j}}{1-q^j}. \tag{2.2}$$

Lemma 2.2. *For all integers $a, b \geq 1$, and all $q > 0$,*

$$Z_{a,b}(q) = \binom{a+b}{a}_q = \binom{a+b}{b}_q. \tag{2.3}$$

Proof. With the natural convention that $Z_{a,b}(q)$ equals 1 if either of a or b is zero (only the empty diagram fits in the empty box), (2.3) holds in this case. When $a, b \geq 1$, we observe that the diagrams in $\mathcal{P}_{a,b}$ fall into two subsets: those with at least one summand equal to a, and those with all summands smaller than a. Hence $Z_{a,b}(q)$ satisfies

$$Z_{a,b}(q) = q^a Z_{a,b-1}(q) + Z_{a-1,b}(q). \tag{2.4}$$

Using induction in $a + b$, we need only show that the q-binomial coefficients also satisfy this recurrence relation, ie. that

$$\binom{a+b}{b}_q = q^a \binom{a+b-1}{b-1}_q + \binom{a-1+b}{b}_q. \tag{2.5}$$

Factoring out common factors, the right hand side equals $\frac{a+b-1!_q}{b!_q a!_q}(q^a(b)_q + (a)_q)$, and since $(a)_q + q^a(b)_q = 1 + q + \cdots + q^{a-1} + q^a(1 + q + \cdots + q^{b-1}) = (a+b)_q$, we arrive at the left hand side of the equation. □

Forgetting for the moment all we know about multiplicative statistics, this formula for the partition function gives information on the scaling properties of the problem. Assume $0 < q < 1$ and let $c = -\log q$. Consider the expected area of a Young diagram:

$$\mathsf{E}^q_{a,b}\big[|\omega|\big] = \frac{1}{Z_{a,b}(q)} \sum_{\omega} q^{|\omega|}|\omega| = \frac{q}{Z_{a,b}(q)} \frac{\partial Z_{a,b}(q)}{\partial q} = -\frac{\partial \log Z_{a,b}(e^{-c})}{\partial c}.$$

It is not difficult to show that this quantity remains bounded as $a, b \to \infty$, when q is fixed. Hardy and Ramanujan's asymptotic formula for $p(n)$,

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\Big(\pi\sqrt{\tfrac{2n}{3}}\Big) \qquad \text{as } n \to \infty \tag{2.6}$$

does the trick, for example. If such a tool is considered too advanced, one can apply the lemma together with the Euler-Maclaurin summation formula

$$\sum_{k=a}^{b} f(k) = \int_a^b f(t)\ dt + \tfrac{1}{2}\big[f(b) + f(a)\big] + \int_a^b B_1(t) f'(t)\ dt, \tag{2.7}$$

when f is a C^1 function and $a < b$ are integers. The function $B_1(t) = t - \lfloor t \rfloor - 1/2$ is the first periodic Bernoulli polynomial. The formula is easily proved by applying integration by parts to the integral

$$\int_{k-1}^{k} f(t) \frac{d}{dt}(t - k + \tfrac{1}{2})\ dt,$$

rearranging a few terms and summing over k. Using the lemma,

$$\log Z_{a,b}(q) = \log \prod_{i=1}^{a} \frac{1 - q^{b+i}}{1 - q^i} = \sum_{i=1}^{a} \log(1 - e^{-c(b+i)}) - \log(1 - e^{-ci}),$$

so if $f_c(t) = \frac{d}{dc}\log(1 - e^{-ct}) = \frac{t}{e^{ct}-1}$, we have

$$\mathsf{E}^q_{a,b}\big[|\omega|\big] = \sum_{i=1}^{a} f_c(i) - f_c(b+i), \tag{2.8}$$

and after applying (2.7), the limit as $a, b \to \infty$ is easily seen to be finite. Thus, if we rescale the diagrams to keep the bounding box a fixed size, we would get a degenerate limit in this case when q is fixed. With this pedestrian approach, we arrive at the recipe from the previous chapter: let $q \to 1$ while rescaling the box.

We consider a sequence of boxes and q's. Fix $c \in \mathbb{R}$ and define for $n \in \mathbb{N}$

$$q = q_n = e^{-\frac{c}{n}}. \tag{2.9}$$

This is the scaling of [23], the case of diagrams bounded on one side. The limiting behavior of the bounding box is governed by a parameter $\rho \in (0,1)$: For each $n \in \mathbb{N}$, pick positive integral dimensions a_n, b_n such that

$$a_n + b_n = 2n \qquad \text{and} \qquad \rho_n = \frac{a_n}{2n} \to \rho \quad \text{as} \quad n \to \infty. \tag{2.10}$$

It turns out to be useful to assume that $\rho_n = \rho + O(\frac{1}{n})$, so we adopt this additional assumption. Then, for each $n \in \mathbb{N}$, (2.1) defines a probability measure $\mathsf{P}^{q_n}_{a_n,b_n}$ on $\mathcal{P}_{a_n,b_n}$. Note that $c = 0$ corresponds to the uniform probability measure. The results

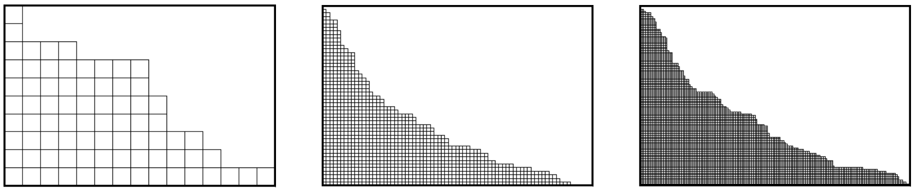


Figure 2.2: Samples of random diagrams in a 10×15, 50×75 and 100×150 box, respectively.

we present essentially don't depend on the sign of c, but since almost all formulas do, we will assume $c > 0$ for convenience.

A Young diagram is determined by its *interface*, the jagged part of the boundary, and our approach is to view the interface as a random continuous function by tilting the bounding box by $45°$. In the following, by a *box* we will mean any rectangle in the (s, x)-plane having sides of slope ± 1. The *shape parameter* of a box B is the ratio $\rho = \frac{a}{a+b}$, where a and b are the lengths of the sides of B. We say that a box is *spanned* by its leftmost and righmost corners.

We take the nth bounding box to be the box spanned by the origin and the point $(2n, b_n - a_n)$, ie. scaled by $\sqrt{2}$. A partition $\omega \in \mathcal{P}_{a_n,b_n}$ corresponds to a Young diagram contained in the bounding box, which we encode as a lattice path $X^{(n)}(\omega) = \big(X^{(n)}_m(\omega)\big)_{0\le m\le 2n}$ with $X_0 = 0$, $X_{2n} = b_n - a_n$ and $X_{m+1} = X_m \pm 1$ for all $m = 0, 1, \ldots, 2n-1$. See Figure 2.3. The probability measure on such lattice paths induced by $\mathsf{P}^{q_n}_{a_n,b_n}$ is denoted $\mathbb{P}_n = \mathbb{P}^{\rho_n,c}_n$.

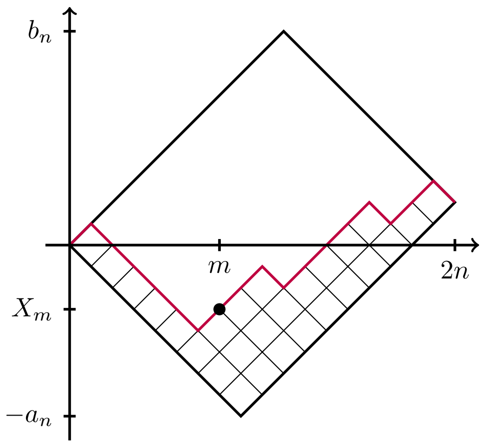


Figure 2.3: A partition encoded as a lattice path.

2.2 · Computing Probabilities

Fortunately, this is straightforward. A lattice path passing through (m, X_m) is composed of a path from $(0, 0)$ to (m, X_m) and a path from (m, X_m) to $(2n, b_n - a_n)$. See Figure 2.4. Since the lattice path has steps of ± 1, it is suficient from the perspective of limit phenomena to consider the behavior at even times, and since it will simplify a few formulas later on we give specific formulas for the marginal probability in case of even and odd times.

Proposition 2.3. *The* 1*-dimensional marginal of* $(X_m)_{0\le m\le 2n}$ *under* $\mathbb{P}_n$ *is given by*

$$\mathbb{P}_n\big[X_{2k}=2i\big]=\frac{q^{(k+i)(a_n-k+i)}}{Z_{a_n,b_n}(q)}Z_{k-i,k+i}(q)Z_{a_n-k+i,b_n-k-i}(q) \tag{2.11}$$

and

$$\mathbb{P}_n\big[X_{2k+1}=2i+1\big]=\frac{q^{(k+i+1)(a_n-k+i)}}{Z_{a_n,b_n}(q)}Z_{k-i,k+i+1}(q)Z_{a_n-k+i,b_n-k-i-1}(q) \tag{2.12}$$

The proof is by picture: Figure 2.4 explains (2.11). The 2-dimensional marginal at even times follows a formula similar to the 1-dimensional marginal. See Figure 2.5.

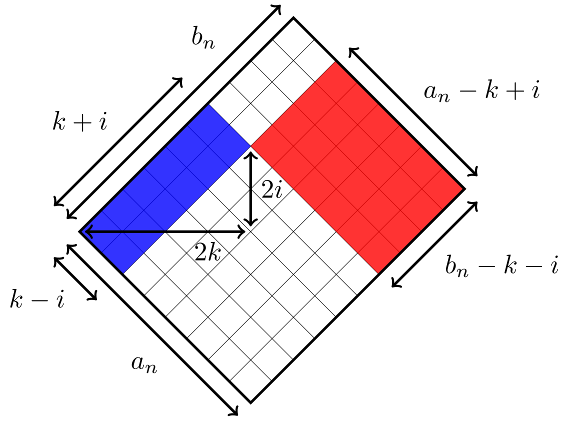


Figure 2.4: Illustration of (2.11).

Proposition 2.4. *The 2-dimensional marginal of* $(X_{2k})_{0\le k\le n}$ *under* $\mathbb{P}_n$ *is given by*

$$\begin{aligned}\mathbb{P}_n\big[X_{2k}=2i,\ X_{2\ell}=2j\big]&=Z_{k+i,k-i}(q)Z_{j+\ell-i-k,\ell+i-k-j}(q)Z_{a_n-\ell+j,b_n-\ell-j}(q)\\&\qquad\cdot\frac{q^{(a_n-k+i)(k+i)+(\ell+j-k-i)(a_n-\ell+j)}}{Z_{a_n,b_n}(q)}\end{aligned} \tag{2.13}$$

for all $0\le k\le \ell\le n$.

At this point it is also clear that (X_m) is a Markov chain with transition probability

$$\begin{aligned}\mathbb{P}_n\big[X_{m+1}=p+1\,\big|\,X_m=p\big]&=q^{a_n-\frac{m-p}{2}}\frac{Z_{a_n-\frac{m-p}{2},b_n-\frac{m+p}{2}-1}(q)}{Z_{a_n-\frac{m-p}{2},b_n-\frac{m+p}{2}}(q)}\\&=\frac{q^{a_n-\frac{m-p}{2}}-q^{2n-m}}{1-q^{2n-m}}.\end{aligned} \tag{2.14}$$

Our next observation is the following unimodality result for the 1-dimensional marginal, which will be useful throughout the chapter.

Lemma 2.5. *The function* $i\mapsto\mathbb{P}_n\big[X_{2k}=2i\big]$ *is unimodal: there exists an integer* $L_n^\sharp(k)$ *such that*

$$\frac{\mathbb{P}_n\big[X_{2k}=2(i+1)\big]}{\mathbb{P}_n\big[X_{2k}=2i\big]}\le 1\iff i\ge L_n^\sharp(k). \tag{2.15}$$

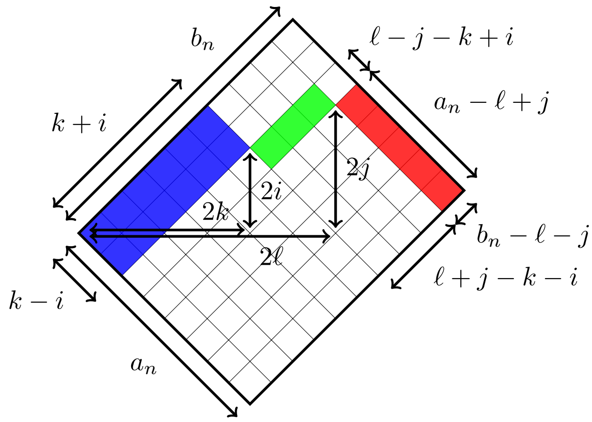


Figure 2.5: Illustration of (2.13).

Proof. Using (2.11) we find (after much cancellation)

$$\frac{\mathbb{P}_n\big[X_{2k} = 2(i+1)\big]}{\mathbb{P}_n\big[X_{2k} = 2i\big]} = q^{a_n+2i+1}\frac{(1-q^{k-i})(1-q^{b_n-k-i})}{(1-q^{k+i+1})(1-q^{a_n-k+i+1})} \tag{2.16}$$

which is smaller than 1 if and only if

$$(1-q^{k-i})(1-q^{b_n-k-i})q^{a_n+2i+1} \leq (1-q^{k+i+1})(1-q^{a_n-k+i+1}).$$

Rearranging a few terms, this is equivalent to

$$q^{2n+1} - 1 \leq q^{a_n+1}(q-1)q^{2i} + \big(q^{k+1}(q^{a_n}-1) + q^{a_n-k+1}(q^{b_n}-1)\big)q^i,$$

where both terms on the right hand side are increasing continuous functions of i. This proves the existence of an integer $L_n^\sharp(k)$ with the asserted property. $\square$

2.3 · Asymptotics of q-Factorials

To study the limiting behavior of the random interface $X^{(n)}$, we need to find the asymptotic behavior of the q-factorial. Define the function

$$S_c(\alpha) = \int_0^\alpha \log\Big(\tfrac{1-e^{-cx}}{c}\Big)\,dx,$$

defined for $\alpha, c \geq 0$, with $S_0(\alpha) = \alpha(\log\alpha - 1)$. This is related to the dilogarithm function $\mathrm{Li}_2(z) = \sum_{k=1}^\infty \frac{z^k}{k^2}$, by $S_c(\alpha) = \frac{1}{c}\,\mathrm{Li}_2(e^{-\alpha c}) - \frac{\pi^2}{6c} - \alpha\log c$.

Proposition 2.6 (q-Stirling's Formula). *Let $c > 0$ and fix $\varepsilon > 0$. In the limit when n goes to infinity, with $q = e^{-\frac{c}{n}}$, the following asymptotics hold for all $\ell \geq \varepsilon n$*

$$\ell!_q = \sqrt{2\pi n}\sqrt{\tfrac{1}{c}(e^{\frac{c\ell}{n}}-1)}\,n^\ell \exp(nS_c(\tfrac{\ell}{n}))\big(1+O(\tfrac{1}{n})\big). \tag{2.17}$$

In particular, there exist constants $m, M > 0$ such that for all n, and all ℓ between 1 and $2n$,

$$m < \frac{\ell!_q}{\sqrt{2\pi n}\sqrt{\frac{1}{c}(e^{\frac{c\ell}{n}}-1)}\,n^\ell\exp(nS_c(\frac{\ell}{n}))} < M.$$

Proof. Consider the logarithm

$$\log \ell!_q = \sum_{k=1}^{\ell} \log\Big(\tfrac{1-e^{-\frac{ck}{n}}}{c}\Big) - \ell \log\Big(\tfrac{1-e^{-\frac{c}{n}}}{c}\Big).$$

By the Euler-Maclaurin formula, we have

$$\begin{aligned} \log \ell!_q = &\int_1^{\ell} \log\Big(\tfrac{1-e^{-\frac{c}{n}t}}{c}\Big)\,dt + \frac{1}{2}\Big[\log\Big(\tfrac{1-e^{-\frac{c\ell}{n}}}{c}\Big) + \log\Big(\tfrac{1-e^{-\frac{c}{n}}}{c}\Big)\Big] \\ &+ \int_1^{\ell} B_1(t)\frac{c}{n}\frac{1}{e^{\frac{c}{n}t}-1}\,dt - \ell\log\Big(\tfrac{1-e^{-\frac{c}{n}}}{c}\Big). \end{aligned} \tag{2.18}$$

The first term is

$$\begin{aligned} \int_1^{\ell} \log\Big(\tfrac{1-e^{-\frac{c}{n}t}}{c}\Big)\,dt &= nS_c(\tfrac{\ell}{n}) - \int_0^1 \log\Big(\tfrac{1-e^{-\frac{c}{n}t}}{c}\Big)\,dt \\ &= nS_c(\tfrac{\ell}{n}) + 1 + \log n + O(\tfrac{1}{n}). \end{aligned} \tag{2.19}$$

Next, the terms involving $\log\Big(\tfrac{1-e^{-\frac{c}{n}}}{c}\Big)$:

$$-(\ell - \tfrac{1}{2})\log\Big(\tfrac{1-e^{-\frac{c}{n}}}{c}\Big) = (\ell - \tfrac{1}{2})\log n + \tfrac{c\ell}{2n} + O(\tfrac{1}{n}) \tag{2.20}$$

For the second integral in (2.18), we add and subtract $\frac{1}{t}$ to get

$$\int_1^{\ell} B_1(t)\frac{c}{n}\frac{1}{e^{\frac{c}{n}t}-1}\,dt = \int_1^{\ell} B_1(t)\tfrac{1}{t}\,dt + c\int_{\frac{1}{n}}^{\frac{\ell}{n}} B_1(nt)\left(\frac{1}{e^{ct}-1} - \frac{1}{ct}\right)dt \tag{2.21}$$

The first term is

$$\begin{aligned} \int_1^{\ell} B_1(t)\tfrac{1}{t}\,dt &= \ell - 1 - (\ell + \tfrac{1}{2})\log \ell + \log \ell! \\ &= -1 + \tfrac{1}{2}\log 2\pi + O(\tfrac{1}{\ell}) \end{aligned} \tag{2.22}$$

by the classical Stirling approximation $\ell! = \sqrt{2\pi\ell}\big(\frac{\ell}{e}\big)^{\ell}\big(1 + O(\frac{1}{\ell})\big)$. Since $\ell > \varepsilon n$, the $O(\frac{1}{\ell})$ term is of the order $O(\frac{1}{n})$.

Turning to the second term of (2.21), we note first that the derivative of the function $f(t) = \frac{1}{e^t-1} - \frac{1}{t}$ is positive, bounded above by 1, and of the order $O(\frac{1}{t^2})$ as $t \to \infty$. On the interval $[\frac{k-1}{n}, \frac{k}{n}]$, the function $B_1(nt)$ is given by $B_1(nt) = nt - k + \frac{1}{2}$ and has the antiderivative

$$\bar{B}_{n,k}(t) = \frac{n}{2}t^2 - (k - \tfrac{1}{2})t + \frac{k(k-1)}{2n}$$

with $\bar{B}_{n,k}(\frac{k-1}{n}) = \bar{B}_{n,k}(\frac{k}{n}) = 0$. The integral of this is over $[\frac{k-1}{n}, \frac{k}{n}]$ is some constant times $\frac{1}{n^2}$. Assuming $\ell > n$, we apply partial integration:

$$c\left|\int_{\frac{1}{n}}^{\frac{\ell}{n}} B_1(nt)f(ct)\,dt\right| \le c^2\sum_{k=1}^{n}\int_{\frac{k}{n}}^{\frac{k+1}{n}} |\bar{B}_{n,k}(t)|\,dt + c^2\sum_{k=n+1}^{\ell-1}\frac{n^2}{k^2}\int_{\frac{k}{n}}^{\frac{k+1}{n}} |\bar{B}_{n,k}(t)|\,dt,$$

and note that both sums are of the order $O(\frac{1}{n})$.

Combining (2.19), (2.20) and (2.22) and the term $\frac{1}{2}\log\Big(\frac{1-e^{-\frac{c\ell}{n}}}{c}\Big)$ from (2.18), we get

$$\begin{aligned}\log \ell!_q &= nS_c(\tfrac{\ell}{n}) + (\ell+\tfrac{1}{2})\log n - (\ell+\tfrac{1}{2})\log c\\ &\quad + \tfrac{1}{2}\log\Big(\tfrac{1-e^{-\frac{c\ell}{n}}}{c}\Big) + \tfrac{c\ell}{2n} + \tfrac{1}{2}\log 2\pi + O(\tfrac{1}{n}),\end{aligned}$$

which yields (2.17) when exponentiated. □

From the proof it is clear that if $A > 0$ is fixed, the constant hidden in the $O(\frac{1}{n})$ error term can be chosen to be uniformly valid when c ranges over $[0, A]$.

We can now derive the asymptotics of the 1- and 2-dimensional marginals of X under $\mathbb{P}_n$. Define

$$f_c(u,v) = S_c(u+v) - S_c(u) - S_c(v), \qquad u, v \geq 0$$

and

$$h_c(u,v) = \sqrt{\frac{c(e^{c(u+v)}-1)}{(e^{cu}-1)(e^{cv}-1)}}, \qquad u, v > 0.$$

Letting $u = \frac{\ell}{n}$ and $v = \frac{k}{n}$, and substituting (2.17) into (2.11), we get the asymptotics of the q-binomial coefficient:

$$\binom{\ell+k}{\ell}_q = \frac{1}{\sqrt{2\pi n}} h_c(u,v)\exp(nf_c(u,v))\big(1+O(\tfrac{1}{n})\big), \tag{2.23}$$

with the caveat that $u, v \geq \varepsilon$ for some fixed ε. From this we can find the asymptotics of the 1-dimensional marginal distributions using Proposition 2.3. Define the functions

$$\begin{aligned}F^{(1)}_{\rho,c}(s,x) &= -c(2\rho - s + x)(s+x) + f_c(s-x, s+x)\\ &\quad + f_c(2\rho - s + x, 2 - 2\rho - s - x) - f_c(2\rho, 2-2\rho)\end{aligned} \tag{2.24}$$

and

$$H^{(1)}_{\rho,c}(s,x) = \frac{h_c(s+x, s-x)h_c(2\rho - s + x, 2 - 2\rho - s - x)}{h_c(2\rho, 2-2\rho)}.$$

The domain of $F^{(1)}_{\rho,c}$ is the box B_ρ spanned by the origin and the point $(1, 1-2\rho)$, while $H^{(1)}_{\rho,c}$ is defined on the interior of B_ρ and tends to infinity near the boundary. The minimum value of $H^{(1)}$ is attained at the center of B_ρ, $(s,x) = (\frac{1}{2}, \frac{1}{2} - \rho)$, and $H^{(1)}$ is bounded away from zero uniformly in ρ, c. From now on, we will not attempt to specify the exact ranges of the arguments when these functions are involved – it is always implicitly assumed that (s,x) is in B_ρ or its interior as needed.

The condition $u, v \geq \varepsilon$ in (2.23) translates to the condition that each of $s+x$, $s-x$, $2\rho - s + x$ and $2 - 2\rho - s - x$ are greater than ε, which has the geometric interpretation that the point (s,x) is at a distance of at least $\frac{\varepsilon}{\sqrt{2}}$ from the boundary of B_ρ:

$$d\big((s,x), \partial B_\rho\big) \geq \tfrac{\varepsilon}{\sqrt{2}}. \tag{2.25}$$

In particular, ρ is bounded away from 0 and 1. If $A > 0$ is fixed, then the partial derivatives of $F^{(1)}_{\rho,c}$ and $H^{(1)}_{\rho,c}$ with respect to ρ, s and x are bounded on the compact set described by (2.25), uniformly for $c \in [0, A]$. By the mean value theorem, we conclude that if $s' = s + O(n^\alpha)$, $x' = x + O(n^\alpha)$ and $\rho' = \rho + O(n^\alpha)$ as $n \to \infty$, where $\alpha < 0$ and s', x' and ρ' also satisfy (2.25), then

$$F^{(1)}_{\rho',c}(s', x') = F^{(1)}_{\rho,c}(s,x) + O(n^\alpha) \tag{2.26}$$

and similarly

$$H^{(1)}_{\rho',c}(s',x') = H^{(1)}_{\rho,c}(s,x)\big(1+O(n^{\alpha})\big), \tag{2.27}$$

where we have used the fact that $H^{(1)}$ is bounded away from zero.

Corollary 2.7. *Fix $\varepsilon > 0$. Then, if $s = \frac{k}{n}$ and $x = \frac{i}{n}$,*

$$\mathbb{P}_n\big[X_{2k} = 2i\big] = \frac{1}{\sqrt{2\pi n}} H^{(1)}_{\rho,c}(s,x)\exp\Big(nF^{(1)}_{\frac{a_n}{2n},c}(s,x)\Big)\big(1+O(\tfrac{1}{n})\big),$$

whenever (s,x) satisfies (2.25) both for ρ and $\rho_n = \frac{a_n}{2n}$. Furthermore, there exists a constant $M > 0$ such that for all n, k and i,

$$\mathbb{P}_n\big[X_{2k} = 2i\big] \le M\frac{1}{\sqrt{2\pi n}} H^{(1)}_{\rho,c}(s,x)\exp\big(nF^{(1)}_{\rho,c}(s,x)\big).$$

Proof. This is an immediate consequence of Proposition 2.3 and (2.23), except that $\frac{a_n}{2n}$ has been replaced with ρ in the indices of $H^{(1)}$ in the first equality, and in both $H^{(1)}$ and $F^{(1)}$ in the second. According to (2.27), the error coming from the substitution in $H^{(1)}$ is absorbed in the factor $(1+O(\frac{1}{n}))$, since we are assuming $\frac{a_n}{2n} = \rho+O(\frac{1}{n})$. Making the same substitution in the indices of $F^{(1)}$ changes the multiplicative constant in the asymptotics, which is not a problem for the second statement. □

The same arguments lead to the asymptotics of the 2-dimensional marginals, which involve the functions

$$\begin{aligned} F^{(2)}_{\rho,c}(s,t,x,y) = &-c\big((2\rho-s+x)(s+x)+(t-s+y-x)(2\rho-t+y)\big)\\ &+f_c(s-x,s+x)+f_c(t-s+y-x,t-s-y+x)\\ &+f_c(2\rho-t+y,2-2\rho-t-y)-f_c(2\rho,2-2\rho)\end{aligned}$$

and

$$\begin{aligned} &H^{(2)}_{\rho,c}(s,t,x,y)\\ &\quad=\frac{h_c(s+x,s-x)h_c(t-s+y-x,t-s-y+x)h_c(2\rho-t+y,2-2\rho-t-y)}{h_c(2\rho,2-2\rho)},\end{aligned}$$

derived from Proposition 2.4. These functions have properties similar to (2.26) and (2.27) whenever s, t, x, y and ρ satisfy a condition similar to (2.25).

Corollary 2.8. *Fix $\varepsilon > 0$. Then, if $s = \frac{k}{n}$, $t = \frac{\ell}{n}$, $x = \frac{i}{n}$ and $y = \frac{j}{n}$, where $s < t$, we have*

$$\mathbb{P}_n\big[X_{2k} = 2i, X_{2\ell} = 2j\big] = \frac{1}{2\pi n} H^{(2)}_{\rho,c}(s,t,x,y)\exp\Big(nF^{(2)}_{\frac{a_n}{2n},c}(s,t,x,y)\Big)\big(1+O(\tfrac{1}{n})\big),$$

whenever all of $s+x$, $s-x$, $t-s+y-x$, $t-s-y+x$, $2\rho-t+y$ and $2-2\rho-t-y$ as well as $\frac{a_n}{n}-t+y$ and $2-\frac{a_n}{n}-t-y$ are greater than ε.

2.4 · The Limit Shape

Obviously the probability measures on lattice paths we have been working with so far are not very well suited to make limit statements – we need to rescale. First, we associate to the lattice path $(X_m)_{0\leq m\leq 2n}$ the continuous piecewise linear function $u \mapsto X_u$ defined on $[0, 2n]$, which coincides with X_m when $u = m$. The graph of the function X is the interface of the random Young diagram. We can then rescale to the unit interval: define for $t \in [0, 1]$

$$\bar{X}_t = \bar{X}_t^{(n)} = \tfrac{1}{2n} X_{2nt}. \tag{2.28}$$

Let $\mathcal{C} = C[0, 1]$, the space of continuous functions on the unit interval equipped with the uniform metric, then $\bar{X}^{(n)} : \mathcal{P}_{a_n,b_n} \to \mathcal{C}$ is a random function. Our aim is to prove that $\bar{X}^{(n)}$ converges in probability to a deterministic curve, the *limit shape*. The limit is the curve $L_{\rho,c}$, defined on $[0, 1]$ by

$$L_{\rho,c}(t) = 1 - 2\rho + \frac{1}{c} \log \frac{\sinh(ct) + e^{c(2\rho-1)} \sinh(c(1-t))}{\sinh c}. \tag{2.29}$$

When $c \to 0$, $L_{\rho,c}$ becomes the straight line $t \mapsto (1 - 2\rho)t$. Note that in the case of a square box ($\rho = \frac{1}{2}$), the expression for $L_{\rho,c}$ boils down to

$$L_{\frac{1}{2},c}(t) = \frac{1}{c} \log \frac{\cosh(c(t - \frac{1}{2}))}{\cosh \frac{c}{2}}.$$

The term limit shape will be used freely to designate the function or its graph. Figure 2.6 shows a few plots of the limit shape for different values of c.

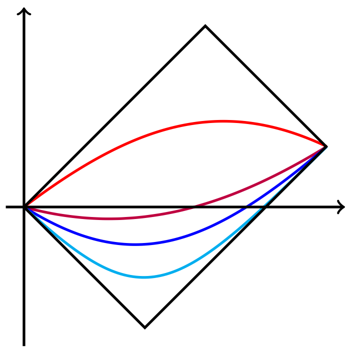


Figure 2.6: Limit shapes for various values of c. The concave curve corresponds to a negative value of c.

Before we get to the proof of convergence, we need a couple of lemmas. The first gives the properties of the function $F^{(1)}$, which are obviously key to determining the limiting behavior of the random Young diagrams.

Lemma 2.9. *Let $c > 0$. The function $F^{(1)}_{\rho,c}$ has the following properties:*

(a) The partial derivatives of $F^{(1)}_{\rho,c}$ with respect to ρ, s and x all vanish when $x = L_{\rho,c}(s)$.

(b) For all $s, \rho \in (0, 1)$, $F^{(1)}_{\rho,c}(s, L_{\rho,c}(s)) = 0$.

(c) For all $s, \rho \in (0, 1)$, the function $x \mapsto F^{(1)}_{\rho,c}(s, x)$ is concave.

(d) For all s, x, the function $\rho \mapsto F^{(1)}_{\rho,c}(s,x)$ is concave.

(e) Given $A > 0$, there exists a constant $C > 0$ such that

$$F^{(1)}_{\rho,c}(s, L_{\rho,c}(s) + y) \leq \frac{-Cy^2}{s(1-s)} \tag{2.30}$$

for all $s, \rho \in (0,1)$, all $c \in [0, A]$ and all y.

Proof. For (a), we have for example

$$\frac{\partial F^{(1)}_{\rho,c}}{\partial x} = -c + \log \frac{\sinh(\frac{c}{2}(s-x))\sinh(\frac{c}{2}(2-2\rho-s-x))}{\sinh(\frac{c}{2}(s+x))\sinh(\frac{c}{2}(2\rho-s+x))}, \tag{2.31}$$

and substituting $x = L_{\rho,c}(s)$ yields zero after much cancellation. The other two statements are proved by similar computations. For (b), we apply the chain rule and (a) to get

$$\frac{\partial F^{(1)}_{\rho,c}(s, L_{\rho,c}(s))}{\partial s} = 0,$$

and since $F^{(1)}_{\rho,c}(0,0) = 0$ this implies (b). For (c), we have $S''_c(u) = \frac{c}{e^{cu}-1}$ and

$$\frac{\partial^2 F^{(1)}_{\rho,c}}{\partial x^2} = -\frac{c}{e^{c(s+x)}-1} - \frac{c}{e^{c(s-x)}-1} - \frac{c}{e^{c(2\rho-s+x)}-1} - \frac{c}{e^{c(2-2\rho-s-x)}-1} - 2c, \tag{2.32}$$

which is clearly negative. For (d), note that

$$\frac{\partial^2 F^{(1)}_{\rho,c}(s,x)}{\partial \rho^2} = \frac{4c}{e^{2c\rho}-1} - \frac{4c}{e^{c(2\rho-s+x)}-1} + \frac{4c}{e^{c(2-2\rho)}-1} - \frac{4c}{e^{c(2-2\rho-s-x)}-1},$$

and the sum of the first and second terms is non-positive since $s - x \geq 0$ and similarly with the third and fourth term since $s + x \geq 0$. For (e), we note that, by (a) and (b), it is enough to bound (2.32) away from 0 for all s, x, ρ and c. For the first two terms we have

$$-\frac{c}{e^{c(s+x)}-1} - \frac{c}{e^{c(s-x)}-1} \leq \frac{-c}{e^{cs}-1} \leq -\frac{C}{s},$$

since the first term on the left obeys the bound in case $x \leq 0$, and otherwise the second term does. By the same argument,

$$-\frac{c}{e^{c(2\rho-s+x)}-1} - \frac{c}{e^{c(2-2\rho-s-x)}-1} \leq \frac{-c}{e^{c(1-s)}-1} \leq -\frac{C}{1-s},$$

where the distinction is whether $x \leq 1 - 2\rho$ or not. □

The next lemma estimates the proximity between the value of the function $L_{\rho,c}$ and the most probable value for X_{2k} which we denoted $L^\sharp_n(k)$ in Lemma 2.5.

Lemma 2.10. *For all n and all $k \leq n$,*

$$\left|\tfrac{1}{n}L^\sharp_n(k) - L_{\rho,c}(\tfrac{k}{n})\right| \leq \tfrac{1}{n} + 2\left|\tfrac{a_n}{2n} - \rho\right|.$$

Consequently, in the limit when $n \to \infty$ and $\frac{a_n}{2n} \to \rho$,

$$\left|\tfrac{1}{n}L^\sharp_n(k) - L_{\rho,c}(\tfrac{k}{n})\right| \to 0, \tag{2.33}$$

uniformly in k.

Proof. Consider the ratio of probabilities (2.16) which was used to define $L_n^\sharp(k)$:

$$Q_{a,b,c}(k,\ell) = \frac{\mathbb{P}_n\big[X_{2k} = 2(\ell+1)\big]}{\mathbb{P}_n\big[X_{2k} = 2\ell\big]} = \frac{(1-q^{k-\ell})(1-q^{b-k-\ell})}{(1-q^{k+\ell+1})(1-q^{a-k+\ell+1})} q^{a+2\ell+1}. \tag{2.34}$$

Were it not for the '+1's in the denominator, $Q_{a,b,c}(k,\ell)$ could be rewritten in terms of the function

$$R_{\rho,c}(t,x) = \exp\bigg(\frac{\partial F_{\rho,c}(t,x)}{\partial x}\bigg) = e^{-c}\frac{\sinh(\frac{c}{2}(t-x))\sinh(\frac{c}{2}(2-2\rho-t-x))}{\sinh(\frac{c}{2}(t+x))\sinh(\frac{c}{2}(2\rho-t+x))}, \tag{2.35}$$

namely as $R_{\frac{a}{2n},c}(\frac{k}{n},\frac{\ell}{n})$. Instead we observe that

$$R_{\frac{a_n}{2n},c}\big(\tfrac{k}{n},\tfrac{\ell+1}{n}\big) \le Q_{a_n,b_n,c}(k,\ell) \le Q_{a_n,b_n,c}(k,\ell-1) \le R_{\frac{a_n}{2n},c}\big(\tfrac{k}{n},\tfrac{\ell-1}{n}\big) \tag{2.36}$$

for all k and ℓ. By definition of $L_n^\sharp(k)$, we have

$$Q_{a_n,b_n,c}(k,L_n^\sharp(k)) \le 1 \le Q_{a_n,b_n,c}(k,L_n^\sharp(k)-1), \tag{2.37}$$

so from (2.36) we conclude that

$$R_{\frac{a_n}{2n},c}\Big(\tfrac{k}{n},\tfrac{L_n^\sharp(k)+1}{n}\Big) \le 1 \le R_{\frac{a_n}{2n},c}\Big(\tfrac{k}{n},\tfrac{L_n^\sharp(k)-1}{n}\Big) \tag{2.38}$$

On the other hand, from Lemma 2.9 (b) we know that for each $t\in[0,1]$, the equation

$$R_{\rho,c}(t,x) = 1$$

has the solution $x = L_{\rho,c}(t)$. Since $x\mapsto R_{\rho,c}(t,x)$ is decreasing, we conclude that

$$\Big|\tfrac{1}{n}L_n^\sharp(k) - L_{\frac{a_n}{2n},c}\big(\tfrac{k}{n}\big)\Big| \le \tfrac{1}{n} \tag{2.39}$$

for all n and k. Differentiating $L_{\rho,c}$ with respect to ρ, we find that

$$\bigg|\frac{\partial L_{\rho,c}}{\partial\rho}(t)\bigg| \le 2$$

for all t, and so by the mean value theorem,

$$\Big|L_{\frac{a_n}{2n},c}\big(\tfrac{k}{n}\big) - L_{\rho,c}\big(\tfrac{k}{n}\big)\Big| \le 2\,\Big|\frac{a_n}{2n}-\rho\Big| \tag{2.40}$$

for all $k\le n$. From (2.39), (2.40) and the triangle equality, we get (2.33). $\square$

Theorem 1. *Let $c>0$ and $\rho\in(0,1)$. The sequence $\big(\bar X^{(n)}\big)$ describing the boundary of a random Young diagram converges in probability in $\mathcal{C}$ to the curve $L_{\rho,c}$: for every $\varepsilon>0$,*

$$\lim_{n\to 0}\mathbb{P}_n\bigg[\sup_{t\in[0,1]}\big|\bar X_t^{(n)} - L_{\rho,c}(t)\big| > \varepsilon\bigg] = 0.$$

Proof. Fix $\varepsilon>0$. For $t\le\frac{\varepsilon}{2}$ or $t\ge 1-\frac{\varepsilon}{2}$, the difference $\big|\bar X_t - L_{\rho,c}(t)\big|$ is always smaller than ε. We have to control what happens for $t\in\big(\frac{\varepsilon}{2},1-\frac{\varepsilon}{2}\big)$. The slope of $L_{\rho,c}(t) = L(t)$ is between ±1, so

$$\big|L\big(\tfrac{1}{n}\lfloor tn\rfloor\big) - L(t)\big| \le \frac{1}{n},$$

and the same is true of $\left|\bar{X}_t - \frac{1}{2n}X_{2\lfloor tn\rfloor}\right|$. Thus, to control the supremum over $[0,1]$, it is sufficient to control what happens at points of the form $t = \frac{k}{n}$:

$$\mathbb{P}_n\left[\sup_{t\in(\frac{\varepsilon}{2},1-\frac{\varepsilon}{2})}\left|\bar{X}_t - L(t)\right| > \varepsilon\right] \\ \leq \sum_{k\in\mathbb{Z}\cap n(\frac{\varepsilon}{2},1-\frac{\varepsilon}{2})} \mathbb{P}_n\Big[X_{2k} > 2n\big(L(\tfrac{k}{n})+\varepsilon\big)\Big] + \mathbb{P}_n\Big[X_{2k} < 2n\big(L(\tfrac{k}{n})-\varepsilon\big)\Big] \tag{2.41}$$

From (2.33) it follows that for n sufficiently large, $2n\big(L(\frac{k}{n})+\varepsilon\big) \geq 2L_n^\sharp(k)$ for all $k \leq n$, and we can appeal to unimodality of the law of X_{2k} (Lemma 2.5) to get

$$\mathbb{P}_n\Big[X_{2k} > 2n\big(L(\tfrac{k}{n})+\varepsilon\big)\Big] = \sum_{i>n(L(\frac{k}{n})+\varepsilon)} \mathbb{P}_n\left[X_{2k} = 2i\right] \\ \leq n\cdot\mathbb{P}_n\Big[X_{2k} = 2\Big\lceil n\big(L(\tfrac{k}{n})+\varepsilon\big)\Big\rceil\Big].$$

By decreasing ε we can ensure that, for all sufficiently large n and all $k \in \mathbb{Z}\cap n(\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2})$, the point $\Big(\frac{k}{n}, \frac{1}{n}\big\lceil n\big(L(\frac{k}{n})+\varepsilon\big)\big\rceil\Big)$ satisfies the conditions of Corollary 2.7. We conclude that the above probability is bounded above by

$$\frac{nM}{\sqrt{2\pi n}}H^{(1)}_{\rho,c}\Big(\tfrac{k}{n}, \tfrac{1}{n}\big\lceil n\big(L(\tfrac{k}{n})+\varepsilon\big)\big\rceil\Big)\exp\Big(nF^{(1)}_{\rho,c}\big(\tfrac{k}{n}, L(\tfrac{k}{n})+\varepsilon\big)\Big).$$

Since $H^{(1)}_{\rho,c}(s,x)$ is bounded above by, say, K when (s,x) is bounded away from the boundary of the bounding box, we get from Lemma 2.9 (e) the bound

$$\frac{nMK}{\sqrt{2\pi n}}\exp(-nC\varepsilon^2),$$

and the theorem is proved. □

From the proof and Lemma 2.10, we see that the theorem holds uniformly for a family of sequences $\{(a_n,b_n)\}$, so long as the convergence $\frac{a_n}{2n} = \rho + O(\frac{1}{n})$ is uniform for the family.

2.5 · Fluctuations

We now turn to the fluctuations of the interface around the limit shape. The goal of this and the last two sections is to prove that the fluctuation process converges weakly to the stochastic process known as the *Ornstein-Uhlenbeck bridge.* The appendix (p. 49) contains the definition and some properties of this process, as well as a quick overview of those parts of the theory of weak convergence of probability measures which are needed for the statement and proof of the theorem. The rather technical proof of tightness occupies a major part of the remainder of the chapter.

The fluctuations of the random interface are described by the process defined for $n \in \mathbb{N}$ and $t \in [0,1]$ by

$$\tilde{X}_t = \tilde{X}^{(n)}_t = \sqrt{n}\left(\bar{X}_t - L_{\rho,c}(t)\right) = \frac{\frac{1}{2}X_{2nt} - nL_{\rho,c}(t)}{\sqrt{n}}. \tag{2.42}$$

Then $\tilde{X}^{(n)}\colon \mathcal{P}_{a_n,b_n} \to \mathcal{C}$ is a random function. Figure 2.7 shows a few samples.

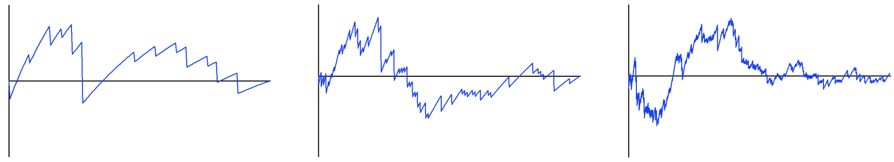


Figure 2.7: Samples of fluctuations in a 25×25, a 100×100 and a 400×400 box, respectively.

To get the exact covariance structure of the Ornstein-Uhlenbeck bridge, it turns out we must tweak the process by dividing by the function

$$g(s) = g_{\rho,c}(s) = \frac{1}{\sqrt{2}} \frac{\sqrt{(e^{2c\rho}-1)(1-e^{-2c(1-\rho)})}}{\sinh cs + e^{c(2\rho-1)}\sinh(c(1-s))}, \qquad s \in [0,1]. \tag{2.43}$$

Note that in the case of a square ($\rho = \frac{1}{2}$), the expression of g simplifies drastically to

$$g(s) = \frac{1}{\sqrt{2}\cosh\bigl(c(s-\frac{1}{2})\bigr)}.$$

In Section 3.2, we provide some explanation of the role of this function. For now, we state our second main result:

Theorem 2. *The sequence of random functions $\bigl(\frac{1}{g}\tilde{X}^{(n)}\bigr)$ converges weakly in $\mathcal{C}$ to the Ornstein-Uhlenbeck bridge $(Y_t)_{0\le t\le 1}$, which is the Gaussian process on $[0,1]$ with mean zero and covariance*

$$\mathrm{E}[Y_sY_t] = \frac{\sinh(cs)\sinh(c(1-t))}{c\sinh(c)}$$

for $0 \le s < t \le 1$.

Note that of our two parameters c and ρ, the limiting Ornstein-Uhlenbeck bridge depends only on c. The 'drift' controlled by ρ becomes deterministic in the limit.

The proof of the theorem consists of checking the criterion on p. 51: prove convergence of the finite-dimensional marginals, and prove tightness. But first we need to get a few technical points out of the way. The following lemma describes the relation of the limit shape in a subbox to the full limit shape.

Lemma 2.11. *Let $0 \le r < s < t \le 1$. The limit shape L satisfies the relations*

$$L_{\rho,c}(r) = sL_{\rho',c'}\left(\tfrac{r}{s}\right) \tag{2.44}$$

and

$$L_{\rho,c}(t) - L_{\rho,c}(s) = (1-s)L_{\rho'',c''}\left(\tfrac{t-s}{1-s}\right), \tag{2.45}$$

where

$$\rho' = \frac{s - L_{\rho,c}(s)}{2s}, \qquad c' = sc \tag{2.46}$$

and

$$\rho'' = \frac{2\rho - s + L_{\rho,c}(s)}{2(1-s)}, \qquad c'' = (1-s)c \tag{2.47}$$

The situation is illustrated in Figure 2.8.

Proof. The relations can be checked analytically. A less computational argument for (2.44) goes as follows (the same strategy applies to (2.45)). Fix $s \in (0,1)$ and consider the subbox B_{left} spanned by the origin and the point $(s, L_{\rho,c}(s))$, having shape parameter ρ'. Theorem 1 states that, given sequences (a_n), (b_n) as in (2.10), the random interface $\bar{X}^{(n)}$ converges in probability to $L_{\rho,c}$. As a consequence, the restriction of $\bar{X}^{(n)}$ to B_{left}, must converge in probability to the restriction of $L_{\rho,c}$. However, if we rescale B_{left} by a factor of $\frac{1}{s}$, we get a version of the original problem with appropriate sequences (a'_n) and (b'_n) satisfying $a'_n + b'_n = 2ns$ and $\frac{a'_n}{2ns} = \rho'$ (for the sake of clarity, we ignore the fact that ns may not be an integer). Since the value of q remains the same and is coupled to the value of n, we have $q = e^{-c/n} = e^{-\frac{c'}{ns}}$, from which we get $c' = sc$. Theorem 1 then asserts the convergence in probability of the rescaling of $\bar{X}^{(n)}$ to $L_{\rho',c'}$, which must therefore coincide with the restriction of $L_{\rho,c}$ in the sense of (2.44). □

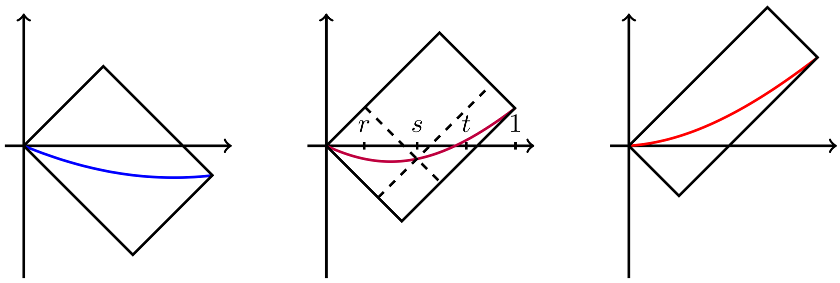


Figure 2.8: On the left and right are the 'full scale' versions of the limit shapes in the subboxes, cf. Lemma 2.11.

Lemma 2.12. *Let $0 \le r < s < t \le 1$. The functions $F^{(1)}$ and $F^{(2)}$ satisfy the relations*

$$F^{(2)}_{\rho,c}(r,s,x,y) = F^{(1)}_{\rho,c}(s,y) + sF^{(1)}_{\rho',c'}(\tfrac{r}{s}, \tfrac{x}{s}) \tag{2.48}$$

and

$$F^{(2)}_{\rho,c}(s,t,y,z) = F^{(1)}_{\rho,c}(s,y) + (1-s)F^{(1)}_{\rho'',c''}(\tfrac{t-s}{1-s}, \tfrac{z-y}{1-s}) \tag{2.49}$$

where ρ', c', ρ'' and c'' are given by (2.46) and (2.47) with y in place of $L_{\rho,c}(s)$.

Proof. Again, the relations can be checked analytically. It is easier, however, to note that (2.48) follows from the identity

$$\mathbb{P}\big[X_{2k} = 2i, X_{2\ell} = 2j\big] = \mathbb{P}\big[X_{2\ell} = 2j\big]\,\mathbb{P}\big[X_{2k} = 2i \;\big|\; X_{2\ell} = 2j\big]$$

and Corollary 2.7 by scaling arguments akin to those used in the proof of the previous lemma. For (2.49), condition instead on X_{2k}. □

Corollary 2.13. *Let $c > 0$. The function $F^{(2)}_{\rho,c}$ has the following properties:*

(a) The partial derivatives of $F^{(2)}_{\rho,c}$ with respect to ρ, s, t, x and y all vanish at $x = L_{\rho,c}(s)$ and $y = L_{\rho,c}(t)$.

(b) For all $0 < s < t < 1$ and all $\rho \in (0,1)$, $F^{(2)}_{\rho,c}(s,t,L_{\rho,c}(s),L_{\rho,c}(t)) = 0$.

Proof. The statements are immediate consequences of Lemma 2.9, upon application of Lemma 2.12 and Lemma 2.11. For example, to see that $\frac{\partial F^{(2)}_{\rho,c}}{\partial t} = 0$, we apply the chain rule to (2.49), substitute (2.45) and invoke Lemma 2.9 (a). $\square$

2.6 · Convergence of Marginals

To prove convergence of the marginal distribution to a Gaussian variable, we apply a saddle-point method. The previous corollary shows that $F^{(2)}$ has a critical point 'on the limit shape', and that it attains its maximal value of 0 at this point. The Taylor expansion of $F^{(2)}$ will provide the covariance matrix for the limiting Gaussian distribution.

Proposition 2.14. *Let $m \in \mathbb{N}$ and let $0 < t_1 < \cdots < t_m < 1$. The random vector $(\tilde{X}^{(n)}_{t_1}, \ldots, \tilde{X}^{(n)}_{t_m})$ converges weakly to*

$$\big(g(t_1)Y_{t_1}, \ldots, g(t_m)Y_{t_m}\big), \tag{2.50}$$

where $(Y_t)_{0 \le t \le 1}$ is the Ornstein-Uhlenbeck bridge (see p. 52).

Proof. Write L for $L_{\rho,c}$ and let $\rho_n = \frac{a_n}{2n}$. Set $t_i' = \frac{\lfloor nt_i \rfloor}{n}$, $i = 1, \ldots, m$. We apply the two results mentioned at the end of Section A.1. Since $|\tilde{X}_u - \tilde{X}_v| \le 2\sqrt{n}|u - v|$ for all $u, v \in [0, 1]$, we have

$$\left\| (\tilde{X}_{t_1'}, \ldots, \tilde{X}_{t_m'}) - (\tilde{X}_{t_1}, \ldots, \tilde{X}_{t_m}) \right\| < 2\sqrt{\frac{m}{n}}, \tag{2.51}$$

and it is sufficient to prove that $\big(\tilde{X}_{t_1'}, \ldots, \tilde{X}_{t_m'}\big)$ converges weakly to (2.50). This random vector takes values on the lattice

$$\Lambda_n = \Big(\tfrac{1}{\sqrt{n}}\mathbb{Z} - \sqrt{n}L(t_1')\Big) \times \cdots \times \Big(\tfrac{1}{\sqrt{n}}\mathbb{Z} - \sqrt{n}L(t_m')\Big), \tag{2.52}$$

and our task is to show that if $(x^{(n)})$ is any sequence with $x^{(n)} \in \Lambda_n$ for all n and $x^{(n)} \to x$ for $n \to \infty$, then

$$\sqrt{n}^m \mathbb{P}_n\Big[\big(\tilde{X}_{t_1'}, \ldots, \tilde{X}_{t_m'}\big) = x^{(n)}\Big] \to \mathsf{P}_{t_1 \cdots t_m}(x), \tag{2.53}$$

where $\mathsf{P}_{t_1 \cdots t_m}$ is the density function of (2.50). We begin with the case $m = 2$, with fixed times $0 < s < t < 1$. Then $\mathsf{P}_{s,t}$ is given by

$$\mathsf{P}_{s,t}(x_1, x_2) = \frac{1}{2\pi\sqrt{\det \Sigma}} \exp\left(-\tfrac{1}{2}(x_1, x_2)\Sigma^{-1}(x_1, x_2)^\mathsf{T}\right), \tag{2.54}$$

where Σ is the covariance matrix

$$\Sigma = \frac{1}{c \sinh c} G(s,t) \begin{pmatrix} \sinh cs \sinh c(1-s) & \sinh cs \sinh c(1-t) \\ \sinh cs \sinh c(1-t) & \sinh ct \sinh c(1-t) \end{pmatrix} G(s,t) \tag{2.55}$$

with $G(s,t) = \operatorname{diag}(g(s), g(t))$. The matrix in the middle together with the factor $\frac{1}{c \sinh c}$ is the covariance matrix of the Ornstein-Uhlenbeck bridge (cf. (A.12)). The left hand side of (2.53) is

$$n\mathbb{P}_n\Big[X_{2ns'} = 2nL(s') + 2\sqrt{n}x_1^{(n)},\ X_{2nt'} = 2nL(t') + 2\sqrt{n}x_2^{(n)}\Big]. \tag{2.56}$$

Since $x^{(n)}$ is convergent and the slope of the limit shape L is bounded away from ± 1 and $\rho_n = \rho + O(\frac{1}{n})$, we can pick $\varepsilon > 0$ such that for n sufficiently large, the restrictions on the sizes of the subboxes in Corollary 2.8 are satisfied. Using the equivalent of (2.27) for $H^{(2)}$, the corollary shows that (2.56) is equal to

$$\frac{1}{2\pi} H^{(2)}_{\rho,c}\big(s,t,L(s),L(t)\big) \exp\Big(nF^{(2)}_{\rho_n,c}\Big(s',t',L(s')+\tfrac{x_1^{(n)}}{\sqrt{n}},L(t')+\tfrac{x_2^{(n)}}{\sqrt{n}}\Big)\Big)\big(1+o(1)\big).$$

A computation reveals that

$$H^{(2)}_{\rho,c}(s,t,L(s),L(t)) = \frac{1}{\sqrt{\det(\Sigma)}} \tag{2.57}$$

since both sides equal

$$\frac{c}{g(s)g(t)}\sqrt{\frac{\sinh c}{\sinh cs \sinh c(t-s)\sinh c(1-t)}},$$

and it is left to show that the exponential factor converges. To this end, we expand the function $(x,y) \mapsto F^{(2)}_{\rho_n,c}\big(s',t',x,y\big)$ around the point $(L(s'),L(t'))$:

$$\begin{aligned}
&nF^{(2)}_{\rho_n,c}\Big(s',t',L(s')+\tfrac{x_1^{(n)}}{\sqrt{n}},L(t')+\tfrac{x_2^{(n)}}{\sqrt{n}}\Big)\\
&\quad= nF^{(2)}_{\rho_n,c} + \frac{\partial F^{(2)}_{\rho_n,c}}{\partial x}\sqrt{n}x_1^{(n)} + \frac{\partial F^{(2)}_{\rho_n,c}}{\partial y}\sqrt{n}x_2^{(n)}\\
&\qquad+\frac{1}{2}\frac{\partial^2 F^{(2)}_{\rho_n,c}}{\partial x^2}\big(x_1^{(n)}\big)^2 + \frac{1}{2}\frac{\partial^2 F^{(2)}_{\rho_n,c}}{\partial y^2}\big(x_2^{(n)}\big)^2 + \frac{\partial^2 F^{(2)}_{\rho_n,c}}{\partial x \partial y}x_1^{(n)}x_2^{(n)}\\
&\qquad+O(\tfrac{1}{\sqrt{n}})
\end{aligned} \tag{2.58}$$

where everything is evaluated at $(s',t',L(s'),L(t'))$. By Corollary 2.13, the first term on the right has limit zero. Since the components of $(s',t',L(s'),L(t'))$ deviate only on the scale $O(\frac{1}{n})$ from the critical point of $F^{(2)}$, and the partial derivatives of $F^{(2)}$ satisfy a condition similar to (2.26), the second and third terms on the right also tend to zero as $n \to \infty$. This shows that the left hand side of (2.53) converges, and it remains only to check that the limit is as claimed in (2.54) and (2.55). The second partial derivatives at $(s,t,L(s),L(t))$ are given by

$$\begin{aligned}
\frac{\partial^2 F^{(2)}_{\rho,c}}{\partial x^2}(s,t,L(s),L(t)) &= \frac{-c\sinh ct}{g(s)^2 \sinh cs \sinh c(t-s)},\\
\frac{\partial^2 F^{(2)}_{\rho,c}}{\partial y^2}(s,t,L(s),L(t)) &= \frac{-c\sinh c(1-s)}{g(t)^2 \sinh c(1-t) \sinh c(t-s)}\\
\frac{\partial^2 F^{(2)}_{\rho,c}}{\partial x \partial y}(s,t,L(s),L(t)) &= \frac{c}{g(s)g(t)\sinh c(t-s)}
\end{aligned}$$

The first two can be calculated by using Lemma 2.12 to express $\frac{\partial F^{(2)}}{\partial x}$ and $\frac{\partial F^{(2)}}{\partial y}$ in terms of $\frac{\partial F^{(1)}}{\partial x}$ and the limit shapes in subboxes from Lemma 2.11. This way we only need to plug the expression (2.29) for $L_{\rho,c}$ into (2.32) and reduce. The covariance matrix Σ is the negative of the inverse of the Hessian matrix containing the derivatives above. This completes the proof of the case $m = 2$.

The general case follows from the above by using the Markov property of the proces $\tilde{X}$: If $0 \le r < s < t \le 1$, then conditional on the value of $\tilde{X}_s$, the variables $\tilde{X}_r$

and $\tilde{X}_t$ are independent. If $P_{s,t}^{(n)}(x,y)$ denotes the transition kernel of this Markov chain, ie.

$$P_{s,t}^{(n)}(y,z) = \mathbb{P}_n\big[\tilde{X}_t = z \mid \tilde{X}_s = y\big],$$

then the left hand side of (2.53) equals

$$\sqrt{n}^m P_{0,t_1'}^{(n)}\big(0, x_1^{(n)}\big) P_{t_1',t_2'}^{(n)}\big(x_1^{(n)}, x_2^{(n)}\big) \cdots P_{t_{m-1}',t_m'}^{(n)}\big(x_{m-1}^{(n)}, x_m^{(n)}\big). \tag{2.59}$$

The case $m = 2$ implies the convergence of each $\sqrt{n}P_{t_i',t_{i+1}'}^{(n)}\big(x_i^{(n)}, x_{i+1}^{(n)}\big)$ to the conditional density of $g(t_{i+1})Y_{t_{i+1}}$ given $g(t_i)Y_{t_i}$, and so (2.59) converges to the value of the density of (2.50) at x. □

2.7 · Tightness

We need to show that the sequence of probability measures $(\mathbb{P}_n)$ is tight, which amounts to verifying condition (ii) of the criterion on p. 51. The proof builds on a few technical lemmas, which all revolve around the following concept.

Definition 2.15 (ε-parallelogram). *Let B be a box with shape parameter ρ and let $0 < \varepsilon < 2\rho \wedge (2 - 2\rho)$. The* ε-parallelogram *of B is the unique parallelogram that shares a diagonal with B and has sides of slope $1 - \varepsilon$ and $\varepsilon - 1$. The interior of the ε-parallelogram is called the* ε-interior *and is denoted B^ε. The complement of the ε-interior in B is called the* ε-boundary.

See Figure 2.9. Recall that B_ρ is the box spanned by the origin and $(1, 1 - 2\rho)$. For $s_0 \in (0,1)$, the sides of the ε-parallelogram of the box B_ρ intersect the straight line $s = s_0$ in two points. We denote the ordinates of these two points by $g_\varepsilon^+(s_0)$, respectively $g_\varepsilon^-(s_0)$.

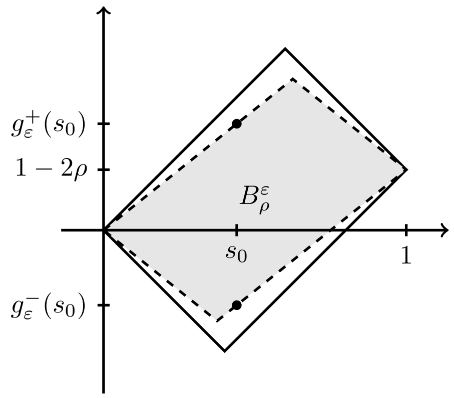


Figure 2.9: The ε-parallelogram of the box B_ρ.

The ε-parallelograms have the following useful property:

Lemma 2.16. *Let $\delta \in (0,1)$ and $A > 0$. There exists an $\varepsilon > 0$, such that for all $\rho \in [\delta, 1-\delta]$ and all $c \in [0, A]$, the limit shape $L_{\rho,c}$ lies in the ε-interior of B_ρ (except the endpoints).*

Proof. It is enough to bound the slope at both ends of the curve. Here we have

$$\left.\frac{dL_{\rho,c}(t)}{dt}\right|_{t=0} = e^{-2c\rho} - 2e^{-c\rho}\frac{\cosh(c)\sinh(c\rho)}{\sinh(c)}$$

and

$$\left.\frac{dL_{\rho,c}(t)}{dt}\right|_{t=1} = \frac{\cosh(c) - e^{-c(1-2\rho)}}{\sinh(c)},$$

which are bounded away from ± 1 for $c \in [0, A]$ and $\rho \in [\delta, 1-\delta]$. □

The ε-parallelogram provides the following bounds on the functions $F^{(1)}$ and $H^{(1)}$.

Lemma 2.17. *Let $\delta \in (0,1)$ and $A > 0$. Take ε as in Lemma 2.16. Then there exists a positive constant C such that for all $\rho \in [\delta, 1-\delta]$ and all $c \in [0, A]$,*

$$F^{(1)}_{\rho,c}(s,x) \le -Cs(1-s) \qquad \textit{for all } (s,x) \textit{ in the } \varepsilon\textit{-boundary of } B_\rho$$

and

$$H^{(1)}_{\rho,c}(s,x) \le \frac{C}{\sqrt{s(1-s)}} \qquad \textit{for all } (s,x) \textit{ in the } \varepsilon\textit{-interior of } B_\rho.$$

Proof. For fixed ρ, c and s, the function $x \mapsto F^{(1)}_{\rho,c}(s,x)$ is concave with maximum at $x = L_{\rho,c}(s)$, and the point $(s, L_{\rho,c}(s))$ is inside the ε-interior of the box B_ρ. Therefore

$$F^{(1)}_{\rho,c}(s,x) \le \max\left\{F^{(1)}_{\rho,c}(s, g^+_\varepsilon(s)), F^{(1)}_{\rho,c}(s, g^-_\varepsilon(s))\right\}.$$

In a neighborhood of 0, we have $g^-_\varepsilon(s) = (\varepsilon - 1)s$. We must show that $\frac{F^{(1)}_{\rho,c}(s,(\varepsilon-1)s)}{s}$ is bounded away from zero as ρ, c and s vary. This is increasing as a function of c, and by Lemma 2.9 (d) and our choice of ε, it is also increasing as a function of ρ. By compactness, it is then left to check that the limit

$$\begin{aligned}\lim_{s\to 0^+}\frac{F^{(1)}_{1-\delta,A}(s,(\varepsilon-1)s)}{s} &= -2c\varepsilon\rho + \log 4 - \varepsilon\log\varepsilon - (2-\varepsilon)\log(2-\varepsilon)\\ &\quad - 2\log(1-e^{-2c}) + \varepsilon\log(1-e^{-2c(1-\rho)})\\ &\quad + (2-\varepsilon)\log(1-e^{-2c\rho})\end{aligned} \tag{2.60}$$

is negative. It turns out the above expression vanishes exactly when $\varepsilon - 1$ is equal to the slope of $L_{\rho,c}$ at $t = 0$. This is excluded by our choice of ε, and we conclude that (2.60) must be negative. The same arguments yield similar bounds in a neighbourhood of $s = 1$ and for g^+_ε.

For the bound on $H^{(1)}$, note that there is a $K > 0$ such that $h_c(x,y) \le K\sqrt{\frac{x+y}{xy}}$ for all x and y and all $c \in [0, A]$. Therefore

$$H_{\rho,c}(s,x) \le K'\sqrt{\rho(1-\rho)}\sqrt{\frac{s}{(s+x)(s-x)}}\sqrt{\frac{1-s}{(2\rho-s+x)(2-2\rho-s-x)}}. \tag{2.61}$$

The factor $\sqrt{\rho(1-\rho)}$ is bounded. The next factor is of order $\frac{1}{\sqrt{s}}$ in a neighborhood of $(0,0)$ in the ε-interior of the box. A simple change of variable $(s,x) \mapsto (1-s, 1-2\rho-x)$ interchanges this and the last factor, which shows that the last factor is of order $\frac{1}{\sqrt{1-s}}$ in a neigborhood of $(1, 1-2\rho)$ in the ε-interior of the box. □

The next lemma controls the probability of large values of $\tilde{X}$ while the path remains inside the ε-interior.

Lemma 2.18. *Let $\delta \in (0,1)$ and $A > 0$. Take ε as in Lemma 2.16. There exists a constant $M > 0$ such that for all $n \geq 1$, all $\rho \in \frac{1}{2n}\mathbb{Z} \cap [\delta, 1-\delta]$, all $c \in [0, A]$, all $s \in \frac{1}{n}\mathbb{Z} \cap (0,1)$ and all $\lambda > 0$,*

$$\mathbb{P}_n\Big[\tilde{X}_s \notin [-\lambda, \lambda],\ (s, \bar{X}_s) \in B^\varepsilon\Big] \leq M \frac{s^2(1-s)^2}{\lambda^4} \tag{2.62}$$

Proof. Combining the bound in Corollary 2.7 with Lemma 2.9 (e) and the bound on $H^{(1)}$ in Lemma 2.17 gives positive constants K and κ such that

$$\mathbb{P}_n\Big[X_{2ns} = 2\lfloor nL_{\rho,c}(s) + y\sqrt{n}\rfloor\Big] \leq \frac{K}{\sqrt{n}} \frac{1}{\sqrt{2\pi\kappa s(1-s)}} \exp\Big(-\frac{y^2}{2\kappa s(1-s)}\Big) \tag{2.63}$$

whenever $\big(s, \frac{1}{n}\lfloor nL_{\rho,c}(s) + y\sqrt{n}\rfloor\big) \in B^\varepsilon_\rho$. Note that the integer part introduces a deviation of the order $O(\frac{1}{n})$ from $nL(s) + y\sqrt{n}$, which, after squaring and multiplying by n, is of the order $\frac{y}{\sqrt{n}}$, which is at most $s(1-s)$ (the height of the bounding box at time s), and so contributes only to the constant K. The probability on the left in (2.62) is bounded by the sum over $y \in \frac{1}{\sqrt{n}}\mathbb{Z} - \sqrt{n}L_{\rho,c}(s)$ with $|y| > \lambda$ of the right hand side of (2.63). This constitutes a Riemann sum for the integral

$$\int_{\mathbb{R}\setminus[-\lambda,\lambda]} \frac{1}{\sqrt{2\pi\kappa s(1-s)}} \exp\Big(-\frac{y^2}{2\kappa s(1-s)}\Big)\, dy = P\Big[|N| > \tfrac{\lambda}{\sqrt{\kappa s(1-s)}}\Big], \tag{2.64}$$

where N is a standard Gaussian variable. By Markov's inequality for the fourth moment $P[|X| > \alpha] \leq \frac{E[X^4]}{\alpha^4}$, (2.64) is bounded above by the right hand side of (2.62). □

It remains to bound the probability that the path leaves the ε-interior. The last lemma takes care of this.

Lemma 2.19. *Let $\delta \in (0,1)$ and $A > 0$, and take $\varepsilon \in (0,1)$ as in Lemma 2.16. There exists a constant $M > 0$ such that: for all $c \in [0, A]$, all $n \geq 1$, all $\rho \in \frac{1}{2n}\mathbb{Z} \cap [\delta, 1-\delta]$, and all $s \in \frac{1}{n}\mathbb{Z} \cap (0,1)$,*

$$\mathbb{P}_n\big[(s, \bar{X}_s) \notin B^\varepsilon_\rho\big] \leq \frac{M}{(ns(1-s))^2} \tag{2.65}$$

Proof. We need to apply both bounds from Lemma 2.17, one concerning points in the ε-boundary, and the other concerning points in the ε-interior. Fix $\varepsilon' \in (0, \varepsilon)$. Then, as long as $ns(1-s) \geq \frac{1}{\varepsilon - \varepsilon'}$, there are possible positions for the lattice path $(s, x^\pm_{s,\varepsilon}) \in (B_\rho \setminus B^\varepsilon_\rho) \cap B^{\varepsilon'}_\rho$ immediately above, respectively below, the ε-interior. In the opposite case, (2.65) is automatically satisfied, as long as $M \geq \frac{1}{(\varepsilon-\varepsilon')^2}$.

Since there are at most $16ns(1-s)$ possible locations for the lattice path at time s, unimodality of the distribution of $\bar{X}_s$ together with Corollary 2.7 and Lemma 2.17 shows that

$$\begin{aligned}
\mathbb{P}_n\big[(s, \bar{X}_s) \notin B^\varepsilon\big] &\leq 16ns(1-s)\mathbb{P}_n\big[\bar{X}_s = x^\pm_{s,\varepsilon}\big] \\
&\leq M' \frac{ns(1-s)}{\sqrt{n}} H_{\rho,c}(s, x^\pm_{s,\varepsilon}) \exp(nF_{\rho,c}(s, x^\pm_{s,\varepsilon})) \\
&\leq M'' \sqrt{ns(1-s)} \exp(-nCs(1-s)).
\end{aligned}$$

Since the function $u \mapsto \sqrt{u}\exp(-Cu)$ is bounded on $(0,\infty)$ by $\frac{1}{u^2 C^{5/2}}$, we get the bound in (2.65). □

We are now set to verify the tightness criterion in the setup of Section 2.1: $c > 0$ and $0 < \rho < 1$ are fixed, (a_n) and (b_n) are sequences of positive integers such that $a_n + b_n = 2n$ and $\rho_n = \frac{a_n}{2n} = \rho + O(\frac{1}{n})$ as $n \to \infty$.

Lemma 2.20. *There exists a constant $K > 0$ such that for all $n > 0$, for all $0 \le r \le s \le t \le 1$, for all $\lambda > 0$,*

$$\mathbb{P}_n\big[|\tilde{X}_s - \tilde{X}_r| \ge \lambda,\ |\tilde{X}_t - \tilde{X}_s| \ge \lambda\big] \le \frac{K(t-r)^2}{\lambda^4}. \tag{2.66}$$

Proof. The proof proceeds in a number of steps, according to a decomposition of the event on the left of (2.66). The basic idea is to condition on the value of $\tilde{X}_s$ and apply Lemma 2.18 to the subpaths on the left and right, respectively. There are two main complications: to verify the extra assumption $(s, \bar{X}_s) \in B^\varepsilon_{\rho_n}$ needed in Lemma 2.18 and to translate the 'tolerance' λ in (2.66) into the corresponding tolerance for the subpaths. To begin with, we will assume that $r, s, t \in \frac{1}{n}\mathbb{Z} \cap [0,1]$. Once we have proved (2.66) for such r, s, t, we can tackle generic $r, s, t \in [0,1]$.

Step 1: Subboxes and conditional probability. Fix $\delta > 0$ such that $\rho_n \in [\delta, 1-\delta]$ for all n. From Lemma 2.16 with $A = c$ and this δ, we get $\varepsilon > 0$ such that all the lemmas hold for all ρ_n. For every $n \ge 1$, every $s \in \frac{1}{n}\mathbb{Z} \cap (0,1)$ and every value of the random variable $\bar{X}_s$, we have two subboxes: B_{left} is the box spanned by the origin and the point $(s, \bar{X}_s)$, while B_{right} is the box spanned by the points $(s, \bar{X}_s)$ and $(1, 1-2\rho_n)$. The conditional probability measure on paths in the left subbox is equivalent to the probability measure $\mathbb{P}_{\text{left}} = \mathbb{P}^{\rho',c'}_{n'}$ with $n' = ns$, $\rho' = \frac{1}{2}\big(1 - \frac{\bar{X}_s}{s}\big)$ and $c' = sc$:

$$\mathbb{P}_n\big[\bar{X}_r = x \mid \bar{X}_s\big] = \mathbb{P}_{\text{left}}\big[\bar{X}'_{\frac{r}{s}} = \tfrac{x}{s}\big],$$

where $\bar{X}'_u = \frac{1}{2n'}X'_{2n'u}$ is the random interface defined by the parameters c', ρ' and n'. In the right subbox we have $\mathbb{P}_{\text{right}} = \mathbb{P}^{\rho'',c''}_{n''}$ with $n'' = n(1-s)$, $\rho'' = \frac{2\rho_n - s + \bar{X}_s}{2(1-s)}$ and $c'' = (1-s)c$:

$$\mathbb{P}_n\big[\bar{X}_t = z \mid \bar{X}_s\big] = \mathbb{P}_{\text{right}}\Big[\bar{X}''_{\frac{t-s}{1-s}} = \tfrac{z - \bar{X}_s}{1-s}\Big],$$

where $\bar{X}''_u = \frac{1}{2n''}X''_{2n''u}$.

As long as $(s, \bar{X}_s) \in B^\varepsilon_{\rho_n}$, the boxes B_{left} and B_{right} have shape parameters in the interval $(\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2})$. From Lemma 2.16 with $\delta = \frac{\varepsilon}{2}$ and $A = c$, we get $\eta(\varepsilon) > 0$ such that the limit shapes in B_{left} and B_{right} stay within the $\eta(\varepsilon)$-interiors. See Figure 2.10. We denote by $E^\varepsilon_{r,s,t}$ the event $\big\{(s, \bar{X}_s) \in B^\varepsilon_{\rho_n}\big\} \cap \big\{(r, \bar{X}_r) \in B^{\eta(\varepsilon)}_{\text{left}}\big\} \cap \big\{(t, X_t) \in B^{\eta(\varepsilon)}_{\text{right}}\big\}$.

Step 2: The sticking condition. Next, we look at the conditional probability measure for the fluctuations of the subpaths on the left and right. Let $u \mapsto L_{\text{left}}(u) = L_{\rho',c'}(u)$ be the limit shape corresponding to the parameters in the left subbox, and define L_{right} similarly. Then a quick calculation shows that

$$\mathbb{P}_n\big[\tilde{X}_r = x \mid \tilde{X}_s\big] = \mathbb{P}_{\text{left}}\Big[\tilde{X}'_{\frac{r}{s}} = \tfrac{x}{\sqrt{s}} + \tfrac{\sqrt{n}}{\sqrt{s}}\big(L_{\rho,c}(r) - sL_{\text{left}}(\tfrac{r}{s})\big)\Big],$$

where $\tilde{X}'_{\frac{r}{s}}$ on the right hand side is defined with respect to the parameters in the left subbox, ie. $\tilde{X}'_u = \sqrt{ns}\big(\bar{X}'_u - L_{\text{left}}(u)\big)$. Similarly we have in B_{right}:

$$\mathbb{P}_n\big[\tilde{X}_t = z \mid \tilde{X}_s\big] = \mathbb{P}_{\text{right}}\Big[\tilde{X}''_{\frac{t-s}{1-s}} = \tfrac{z}{\sqrt{1-s}} + \tfrac{\sqrt{n}}{\sqrt{1-s}}\big(L_{\rho,c}(t) - \bar{X}_s - (1-s)L_{\text{right}}(\tfrac{t-s}{1-s})\big)\Big],$$

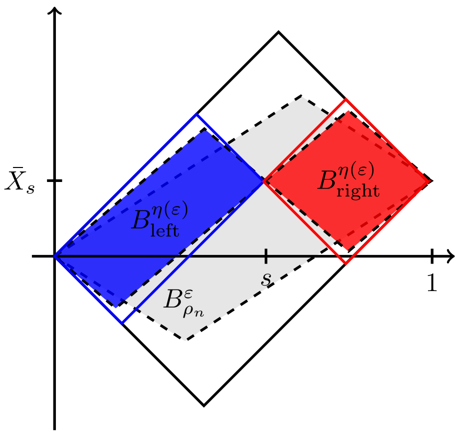


Figure 2.10: The ε-interiors $B^{\varepsilon}_{\rho_n}$, $B^{\eta(\varepsilon)}_{\mathrm{left}}$ and $B^{\eta(\varepsilon)}_{\mathrm{right}}$.

where $\tilde{X}''_u = \sqrt{n(1-s)}(\bar{X}''_u - L_{\mathrm{right}}(u))$ is defined by the parameters in B_{right}. With these observations, we have

$$\begin{aligned}\mathbb{P}_n\Big[\big|\tilde{X}_s - \tilde{X}_r\big| \geq \lambda \;\Big|\; \tilde{X}_s\Big] \\ &= \mathbb{P}_n\Big[\tilde{X}_r \notin \big(\tilde{X}_s - \lambda, \tilde{X}_s + \lambda\big) \;\Big|\; \tilde{X}_s\Big] \\ &= \mathbb{P}_{\mathrm{left}}\Big[\tilde{X}'_{\frac{r}{s}} \notin \tfrac{1}{\sqrt{s}}(-\lambda,\lambda) + \tfrac{1}{\sqrt{s}}\tilde{X}_s + \tfrac{\sqrt{n}}{\sqrt{s}}\big(L_{\rho,c}(r) - sL_{\mathrm{left}}(\tfrac{r}{s})\big)\Big],\end{aligned}$$

and if the value of $\tilde{X}_s$ is close to $\sqrt{n}\big(sL_{\mathrm{left}}(\frac{r}{s}) - L_{\rho,c}(r)\big)$ (note that the signs are necessarily the same – see Figure 2.11), this probability can be bounded using Lemma 2.18. We call this the 'sticking condition' on $\tilde{X}_s$ relative to L_{left}:

$$\Big|\tilde{X}_s + \sqrt{n}\big(L_{\rho,c}(r) - sL_{\mathrm{left}}(\tfrac{r}{s})\big)\Big| \leq \frac{\lambda}{2}. \tag{2.67}$$

Similarly, we have in B_{right}:

$$\begin{aligned}&\mathbb{P}_n\Big[\big|\tilde{X}_s - \tilde{X}_t\big| \geq \lambda \;\Big|\; \tilde{X}_s\Big] \\ &= \mathbb{P}_n\Big[\tilde{X}_t \notin \big(\tilde{X}_s - \lambda, \tilde{X}_s + \lambda\big) \;\Big|\; \tilde{X}_s\Big] \\ &= \mathbb{P}_{\mathrm{right}}\Big[\tilde{X}''_{\frac{t-s}{1-s}} \notin \tfrac{1}{\sqrt{1-s}}(-\lambda,\lambda) + \tfrac{\sqrt{n}}{\sqrt{1-s}}\big(L_{\rho,c}(t) - L_{\rho,c}(s) - (1-s)L_{\mathrm{right}}(\tfrac{t-s}{1-s})\big)\Big],\end{aligned}$$

which gives rise to the sticking condition on $\tilde{X}_s$ relative to L_{right}:

$$\sqrt{n}\,\Big|L_{\rho,c}(t) - L_{\rho,c}(s) - (1-s)L_{\mathrm{right}}(\tfrac{t-s}{1-s})\Big| \leq \frac{\lambda}{2}. \tag{2.68}$$

Note that the apparent absence of $\tilde{X}_s$ from the above expression is a trick of the notation: L_{right} depends on $\tilde{X}_s$ through ρ''.

Denote by S_{left} the event that (2.67) is satisfied, and by S_{right} the event that (2.68) is satisfied. If $A^{\lambda}_{r,s} = \big\{|\tilde{X}_s - \tilde{X}_r| \geq \lambda\big\}$, then $A^{\lambda}_{r,s} \cap A^{\lambda}_{s,t}$ is the event in the tightness condition (2.66) and we have the following decomposition:

$$\begin{aligned}\mathbb{P}_n\big[A^{\lambda}_{r,s} \cap A^{\lambda}_{s,t}\big] \leq \mathbb{P}_n\big[A^{\lambda}_{r,s} \cap E \cap S_{\mathrm{left}}\big] + \mathbb{P}_n\big[A^{\lambda}_{s,t} \cap E \cap S_{\mathrm{right}}\big] \\ + \mathbb{P}_n\big[E \cap S^{\mathsf{c}}_{\mathrm{left}} \cap S^{\mathsf{c}}_{\mathrm{right}}\big] + \mathbb{P}_n\big[E^{\mathsf{c}}\big],\end{aligned} \tag{2.69}$$

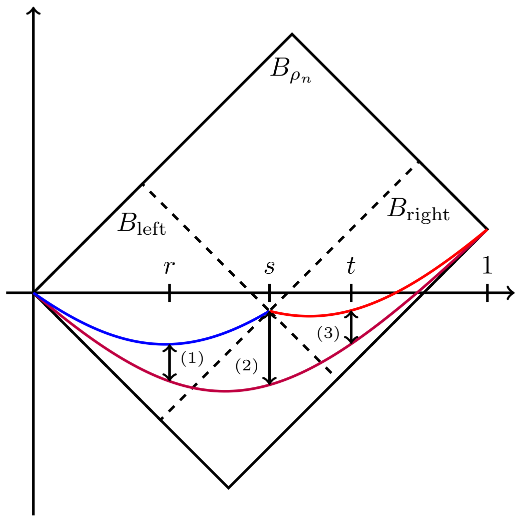


Figure 2.11: Limit shapes in the box B_{ρ_n} and the subboxes B_{left} and B_{right}. The double arrow (2) represents the deviation $\bar{X}_s - L_{\rho,c}(s)$, while (1) represents the difference of $L_{\rho,c}$ and the limit shape in B_{left} at time r, ie. $sL_{\text{left}}(\frac{r}{s}) - L_{\rho,c}(r)$. The difference of (1) and (2) appears in the left sticking condition (2.67). The arrow (3) represents the quantity $\bar{X}_s + (1-s)L_{\text{right}}(\frac{t-s}{1-s}) - L_{\rho,c}(t)$. The difference of (1) and (3) appears in the right sticking condition (2.68).

where $E = E^{\varepsilon}_{r,s,t}$. We need to show that each term in (2.69) is bounded by $\frac{(t-r)^2}{\lambda^4}$ times some constant depending only on c, δ and ε.

Step 3: Fluctuations of the subpaths. The first term of (2.69) can be rewritten by conditioning on $\tilde{X}_s$:

$$\mathbb{P}_n\big[A^{\lambda}_{r,s} \cap E \cap S_{\text{left}}\big] = \sum_y \mathbb{P}_n\big[\tilde{X}_s = y\big]\mathbb{P}_n\Big[\big\{\big|\tilde{X}_s - \tilde{X}_r\big| \geq \lambda\big\} \cap E \;\Big|\; \tilde{X}_s = y\Big], \quad (2.70)$$

where y runs over those possible values of $\tilde{X}_s$ which satisfy the sticking condition (2.67). When (2.67) is satisfied,

$$\tfrac{1}{\sqrt{s}}(-\lambda,\lambda) + \tfrac{1}{\sqrt{s}}\tilde{X}_s + \tfrac{\sqrt{n}}{\sqrt{s}}\big(L_{\rho,c}(r) - sL_{\text{left}}(\tfrac{r}{s})\big) \supseteq \Big(-\tfrac{\lambda}{2\sqrt{s}}, \tfrac{\lambda}{2\sqrt{s}}\Big)$$

and Lemma 2.18 yields

$$\begin{aligned}\mathbb{P}_n\Big[\big\{\big|\tilde{X}_s - \tilde{X}_r\big| \geq \lambda\big\} \cap E \;\Big|\; \tilde{X}_s\Big] &\leq \mathbb{P}_{\text{left}}\Big[\tilde{X}_{\frac{r}{s}} \notin \Big(-\tfrac{\lambda}{2\sqrt{s}}, \tfrac{\lambda}{2\sqrt{s}}\Big),\ \big(\tfrac{r}{s}, \bar{X}_{\frac{r}{s}}\big) \in B^{\eta(\varepsilon)}_{\text{left}}\Big] \\ &\leq M\frac{\big(\frac{r}{s}(1-\frac{r}{s})\big)^2}{\big(\frac{\lambda}{2\sqrt{s}}\big)^4} \leq 2^4 M\frac{(t-r)^2}{\lambda^4}.\end{aligned}$$

The same arguments show that, when S_{right} is satisfied,

$$\mathbb{P}_n\Big[\big\{\big|\tilde{X}_s - \tilde{X}_t\big| \geq \lambda\big\} \cap E \;\Big|\; \tilde{X}_s\Big] \leq M\frac{\big(\frac{t}{1-s}(1-\frac{t}{1-s})\big)^2}{\big(\frac{\lambda}{2\sqrt{1-s}}\big)^4} \leq 2^4 M\frac{(t-r)^2}{\lambda^4}.$$

Substituting these bounds into (2.70) gives the bound on the first two terms in (2.69).

Step 4: When the sticking conditions fail. It is enough to bound the probability that (2.67) fails. Using (2.44) in Lemma 2.11, we replace $L_{\rho,c}(r)$ in (2.67) by

$$sL_{\frac{1}{2}-\frac{1}{2s}L_{\rho,c}(s),sc}\big(\tfrac{r}{s}\big)$$

and get the condition

$$\left|\tilde{X}_s + s\sqrt{n}\Big(L_{\frac{1}{2}-\frac{1}{2s}L_{\rho,c}(s),sc}\big(\tfrac{r}{s}\big) - L_{\rho',sc}\big(\tfrac{r}{s}\big)\Big)\right| \leq \frac{\lambda}{2} \tag{2.71}$$

Applying the mean value theorem to the function $\rho \mapsto L_{\rho,sc}(\frac{r}{s})$ yields a ρ_0 between $\rho' = \frac{1}{2} - \frac{1}{2s}\bar{X}_s$ and $\frac{1}{2} - \frac{1}{2s}L_{\rho,c}(s)$ such that

$$\begin{aligned} s\sqrt{n}\Big(L_{\frac{1}{2}-\frac{1}{2s}L_{\rho,c}(s),sc}\big(\tfrac{r}{s}\big) - L_{\rho',sc}\big(\tfrac{r}{s}\big)\Big) &= s\sqrt{n}\tfrac{1}{2s}\big(\bar{X}_s - L_{\rho,c}(s)\big)\frac{\partial L_{\rho,sc}(\frac{r}{s})}{\partial\rho}\bigg|_{\rho=\rho_0} \\ &= \frac{\tilde{X}_s}{2}\frac{\partial L_{\rho,sc}(\frac{r}{s})}{\partial\rho}\bigg|_{\rho=\rho_0} \end{aligned}$$

Differentiating $L_{\rho,c}(u)$ with respect to ρ, for generic values of ρ, c, u yields

$$\begin{aligned} 0 \leq 1 + \frac{1}{2}\frac{\partial L_{\rho,c}(u)}{\partial\rho} &= \frac{e^{c(2\rho-1)}\sinh(c(1-u))}{\sinh(cu) + e^{c(2\rho-1)}\sinh(c(1-u))} \\ &\leq 1 \wedge e^{c(2\rho-1)}\frac{\sinh(c(1-u))}{\sinh(cu)} \\ &\leq 1 \wedge K\frac{1-u}{u} \end{aligned}$$

with a constant K that works for all $c \in [0, A]$, all $\rho \in (0, 1)$, and all $u \in (0, 1)$. Therefore, the probability that (2.67) is not satisfied is less than

$$\mathbb{P}_n\Big[\big|\tilde{X}_s\big|\big(1 \wedge K\tfrac{s-r}{r}\big) > \tfrac{\lambda}{2}\Big]$$

which, by Lemma 2.18 (the additional condition $(s, \bar{X}_s) \in B^{\varepsilon}_{\rho_n}$ is assumed to be satisfied in the present case, according to (2.69)) is bounded by

$$M\frac{s^2(1-s)^2}{\lambda^4}\big(1 \wedge K^4(\tfrac{s-r}{r})^4\big). \tag{2.72}$$

Finally, we can show that this has the upper bound $K'\frac{(s-r)^2}{\lambda^4} \leq K'\frac{(t-r)^2}{\lambda^4}$. Suppose first that $K\frac{s-r}{r} > 1$. Then $r \leq \frac{K}{K+1}s$ and

$$\frac{s(1-s)}{s-r} \leq \frac{s(1-s)}{s - \frac{K}{K+1}s} = (K+1)(1-s) \leq K+1$$

so that $s^2(1-s)^2 \leq (K+1)^2(s-r)^2$. In the opposite case $r \geq \frac{K}{K+1}s$ and since (2.72) is a decreasing function of r, we can again substitute $\frac{K}{K+1}s$ for r and obtain the bound $(K+1)^2(s-r)^2$. We have proved the bound on the third term in (2.69).

Step 5: When the path leaves the ε-interior. It remains to bound the probability that $E^{\varepsilon}_{r,s,t}$ fails. Note that we have for all n and all u, v:

$$|\tilde{X}_u - \tilde{X}_v| \leq 2\sqrt{n}|u-v|.$$

Hence we can immediately assume that

$$\lambda \leq 2\sqrt{n}\left((s-r)\wedge(t-s)\right). \tag{2.73}$$

Isolating $\sqrt{n}$ and substituting into the bound from Lemma 2.19,

$$\begin{aligned}
\mathbb{P}_n\left[(s,\bar{X}_s)\notin B^{\varepsilon}_{\rho_n}\right] &\leq \frac{M}{(ns(1-s))^2} \\
&\leq 16M\frac{(s-r)^4\wedge(t-s)^4}{\lambda^4 s^2(1-s)^2} \\
&\leq 64M\frac{(s-r)^2\wedge(t-s)^2}{\lambda^4} \\
&\leq 64M\frac{(t-r)^2}{\lambda^4},
\end{aligned} \tag{2.74}$$

where we used the observation $(s-r)^2\wedge(t-s)^2 \leq s^2\wedge(1-s)^2 \leq 4s^2(1-s)^2$. The weaker condition $n \geq \frac{\lambda^2}{s^2\wedge(1-s)^2}$ yields

$$\mathbb{P}_n\left[(s,\bar{X}_s)\notin B^{\varepsilon}_{\rho_n}\right] \leq M\frac{s^2\wedge(1-s)^2}{\lambda^4}. \tag{2.75}$$

As noted above, we have $\rho',\rho''\in(\frac{\varepsilon}{2},1-\frac{\varepsilon}{2})$, and invoking Lemma 2.19 with $\delta=\frac{\varepsilon}{2}$ and $A=c$, we get a constant $M'>0$ that is valid in both subboxes. To repeat the above calculation we need also an assumption similar to (2.73):

$$r \geq \frac{\lambda}{4\sqrt{n}} \qquad \text{and} \qquad 1-t \geq \frac{\lambda}{4\sqrt{n}}. \tag{2.76}$$

If (2.76) is satisfied, we combine it with (2.73) and find that

$$ns \geq \frac{1}{4^2}\frac{\left(\lambda/\sqrt{s}\right)^2}{\left(\frac{r}{s}\wedge\frac{s-r}{s}\right)^2} \qquad \text{and} \qquad n(1-s) \geq \frac{1}{4^2}\frac{\left(\lambda/\sqrt{s}\right)^2}{\left(\frac{t-s}{1-s}\wedge\frac{1-t}{1-s}\right)^2}. \tag{2.77}$$

Then, by the same argument that yielded (2.75), we have

$$\begin{aligned}
\mathbb{P}_n\left[(r,\bar{X}_r)\notin B^{\eta(\varepsilon)}_{\text{left}}\;\middle|\;\bar{X}_s\right] &= \mathbb{P}_{\text{left}}\left[\left(\tfrac{r}{s},\bar{X}'_{\frac{r}{s}}\right)\notin B^{\eta(\varepsilon)}_{\rho'}\right] \\
&\leq M'\frac{\left(\frac{r}{s}\right)^2\wedge\left(\frac{s-r}{s}\right)^2}{\left(\lambda/\sqrt{s}\right)^4} \\
&\leq M'\frac{(s-r)^2}{\lambda^4}
\end{aligned} \tag{2.78}$$

and

$$\begin{aligned}
\mathbb{P}_n\left[(t,\bar{X}_t)\notin B^{\eta(\varepsilon)}_{\text{right}}\;\middle|\;\bar{X}_s\right] &= \mathbb{P}_{\text{right}}\left[\left(\tfrac{t-s}{1-s},\bar{X}''_{\frac{t-s}{1-s}}\right)\notin B^{\eta(\varepsilon)}_{\rho''}\right] \\
&\leq M'\frac{\left(\frac{t-s}{1-s}\right)^2\wedge\left(\frac{1-t}{1-s}\right)^2}{\left(\lambda/\sqrt{1-s}\right)^4} \\
&\leq M'\frac{(t-s)^2}{\lambda^4}.
\end{aligned} \tag{2.79}$$

If the first inequality in (2.76) fails, ie. $r < \frac{\lambda}{4\sqrt{n}}$, then $\left|\tilde{X}_r\right| \le 2r\sqrt{n} < \frac{\lambda}{2}$ and

$$\begin{aligned}\mathbb{P}_n\Big[\big|\tilde{X}_s - \tilde{X}_r\big| \ge \lambda,\ \big|\tilde{X}_t - \tilde{X}_s\big| \ge \lambda\Big]& \\ &\le \mathbb{P}_n\Big[\big|\tilde{X}_s - \tilde{X}_r\big| \ge \lambda\Big] \\ &\le \mathbb{P}_n\Big[\big|\tilde{X}_s\big| \ge \tfrac{\lambda}{2}\Big] \\ &\le \mathbb{P}_n\Big[\big|\tilde{X}_s\big| \ge \tfrac{\lambda}{2},\ (s, \bar{X}_s) \in B^{\varepsilon}_{\rho_n}\Big] + \mathbb{P}_n\Big[(s, \bar{X}_s) \notin B^{\varepsilon}_{\rho_n}\Big],\end{aligned} \tag{2.80}$$

where the last term is bounded by $M\frac{(t-r)^2}{\lambda^4}$ according to (2.74). Moreover, according to Lemma 2.18, there is a constant $M'' > 0$ such that

$$\mathbb{P}_n\Big[\big|\tilde{X}_s\big| \ge \tfrac{\lambda}{2},\ (s, \bar{X}_s) \in B^{\varepsilon}_{\rho_n}\Big] \le M''\frac{s^2(1-s)^2}{\lambda^4} \le M''\frac{s^2}{\lambda^4},$$

and using $r < \frac{\lambda}{4\sqrt{n}} < \frac{s}{2}$, we get $s < 2(s-r)$ and

$$\mathbb{P}_n\Big[\big|\tilde{X}_s\big| \ge \tfrac{\lambda}{2},\ (s, \bar{X}_s) \in B^{\varepsilon}_{\rho_n}\Big] \le 4M''\frac{(s-r)^2}{\lambda^4} \le 4M''\frac{(t-r)^2}{\lambda^4}.$$

Thus both terms on the right in (2.80) obey the required bound, and the tightness condition (2.66) is satisfied. By completely analogous arguments we can show tightness when the second inequality in (2.76) fails, ie. in the case $1 - t < \frac{\lambda}{4\sqrt{n}}$. This completes the proof of the bound on the last term in (2.69).

Step 6: Generic values of r, s, t. Assume that $K \ge 256$. From (2.73) we see that the right hand side of the tightness condition (2.66) is greater than $\frac{K}{2^4 n^2}\frac{1}{(s-r)^2 \wedge (t-s)^2}$, which is greater than 1 in case $(s-r) \wedge (t-s) \le \frac{4}{n}$. We can therefore assume that

$$(s-r) \wedge (t-s) \ge \frac{4}{n}. \tag{2.81}$$

In turn, this shows that the right hand side of (2.66) is greater than $\frac{16K}{n^2\lambda^4}$, which is greater than 1 if $\lambda \le \frac{8}{\sqrt{n}}$. We can therefore assume that

$$\lambda \ge \frac{8}{\sqrt{n}} \tag{2.82}$$

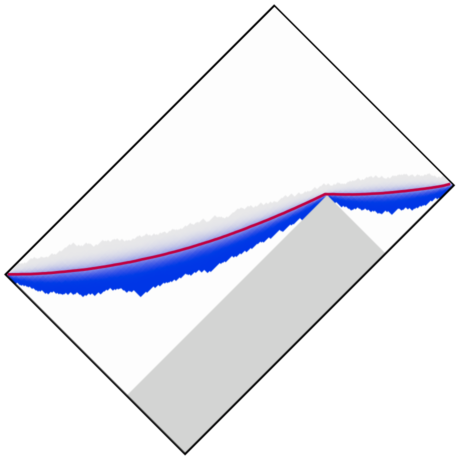
Defining $u' = \frac{1}{n}\lfloor nu \rfloor$, we apply the triangle inequality:

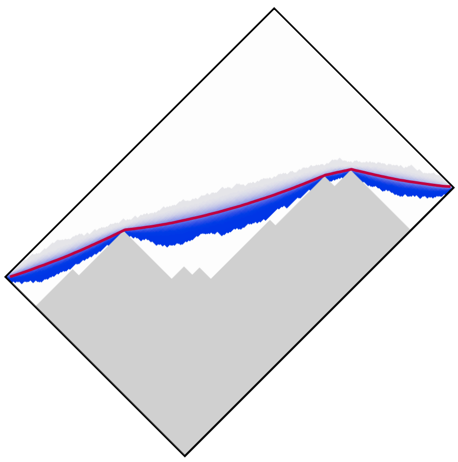
$$\begin{aligned}\big|\tilde{X}_s - \tilde{X}_r\big| &\le \big|\tilde{X}_s - \tilde{X}_{s'}\big| + \big|\tilde{X}_{s'} - \tilde{X}_{r'}\big| + \big|\tilde{X}_{r'} - \tilde{X}_r\big| \\ &\le 2\sqrt{n}(s - s') + \big|\tilde{X}_{s'} - \tilde{X}_{r'}\big| + 2\sqrt{n}(r - r') \\ &\le \big|\tilde{X}_{s'} - \tilde{X}_{r'}\big| + \frac{4}{\sqrt{n}}.\end{aligned}$$

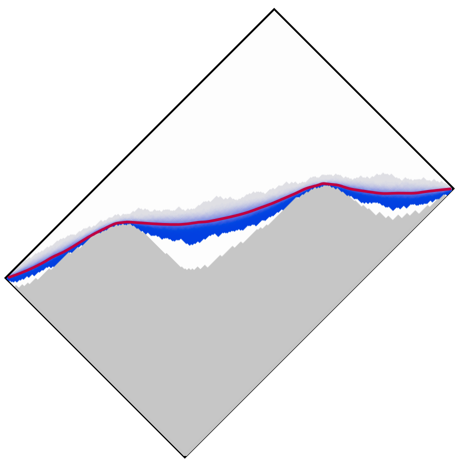
Since $\lambda - \frac{4}{\sqrt{n}} \ge \frac{\lambda}{2}$ according to (2.82), we get the bound

$$\begin{aligned}\mathbb{P}_n\big[\big|\tilde{X}_s - \tilde{X}_r\big| \wedge \big|\tilde{X}_s - \tilde{X}_t\big| \ge \lambda\big] &\le \mathbb{P}_n\big[\big|\tilde{X}_{s'} - \tilde{X}_{r'}\big| \wedge \big|\tilde{X}_{s'} - \tilde{X}_{t'}\big| \ge \tfrac{\lambda}{2}\big] \\ &\le 2^4K\frac{(t'-r')^2}{\lambda^4} \le 2^4K\frac{4(t-r)^2}{\lambda^4}.\end{aligned}$$

This completes the proof of Lemma 2.20. □







Chapter 3

Links with Related Models

In the first two sections of this chapter we take a new look at Vershik's unbounded partitions and partitions bounded on one side, and obtain results connecting them to the bounded partitions of Chapter 2. Section 3.3 covers another variant of the bounded model, where we restrict to diagrams containing some predefined part of the bounding box. The last section briefly covers the large deviation principle in the case of random partitions, a possible refinement of the limit shape theorem.

3.1 · Vershik's Curve Revisited

We sketch a proof of the limit shape theorem for the macrocanonical ensemble $\mathcal{P}$ using the techniques of the previous chapter. See [19, Theorem 4.4] for Vershik's original statement and proof. We keep the parameter $c > 0$ from Chapter 2 as well as the convention $q = e^{-\frac{c}{n}}$. The probability measure P_n^∞ on $\mathcal{P}$ is defined by $\mathsf{P}_n^\infty[\omega] = \frac{q^{|\omega|}}{Z_\infty(q)}$, where

$$Z_\infty(q) = \sum_{\omega \in \mathcal{P}} q^{|\omega|} = \prod_{i=1}^{\infty} \frac{1}{1-q^i} \tag{3.1}$$

is the partition function. We will refer to this case as the *unbounded* model. A partition is encoded as an infinite lattice path (or piecewise linear function) $(X_u)_{u\in\mathbb{Z}}$, with $X_u = |u|$ for $|u|$ sufficiently large. To compute probabilities, we need the 'one-dimensional' limit of the Gaussian binomial coefficient. Define

$$Z_{a,\infty}(q) = \lim_{b\to\infty} \binom{a+b}{a}_q = \prod_{i=1}^{a} \frac{1}{1-q^i}. \tag{3.2}$$

We then have the formula

$$\mathbb{P}_n^\infty\left[X_{2k} = 2i\right] = \frac{q^{(i-k)(i+k)} Z_{i-k,\infty}(q) Z_{i+k,\infty}(q)}{Z_\infty(q)}, \tag{3.3}$$

see Figure 3.1. From the relation $S_c(\alpha) = \frac{1}{c}\operatorname{Li}_2(e^{-\alpha c}) - \frac{\pi^2}{6c} - \alpha \log c$ we have

$$\lim_{v\to\infty} f_c(u,v) = -S_c(u) - u\log c. \tag{3.4}$$

This gives the asymptotics of both $Z_{a,\infty}(q)$ and $Z_\infty(q)$ using (2.23):

$$Z_{a,\infty}(q) = \sqrt{\frac{c}{2\pi n}} \frac{1}{\sqrt{1-e^{-ca/n}}} \exp\bigl(-n(S_c(\tfrac{a}{n}) + \tfrac{a}{n}\log c)\bigr)(1+O(\tfrac{1}{n})) \tag{3.5}$$

$$Z_\infty(q) = \sqrt{\frac{c}{2\pi n}} \exp\bigl(n\tfrac{\pi^2}{6c}\bigr)(1+O(\tfrac{1}{n})), \tag{3.6}$$

and thereby that of the marginal probability in terms of the functions

$$F^{(1)}_{\infty,c}(s,x) = -c(x-s)(x+s) - S_c(x-s) - S_c(x+s) - 2x\log c - \tfrac{\pi^2}{6c}, \tag{3.7}$$

and

$$H^{(1)}_{\infty,c}(s,x) = \frac{1}{\sqrt{(1-e^{-c(x-s)})(1-e^{-c(x+s)})}}, \tag{3.8}$$

namely

$$\mathbb{P}^\infty_n\left[X_{2k} = 2i\right] = \sqrt{\frac{c}{2\pi n}} H^{(1)}_{\infty,c}(s,x)\exp(nF^{(1)}_{\infty,c}(s,x))(1+O(\tfrac{1}{n})), \tag{3.9}$$

where $s = \frac{k}{n}$ and $x = \frac{i}{n}$.

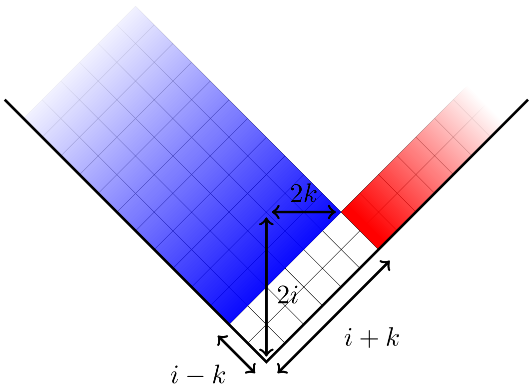


Figure 3.1: Illustration of (3.3).

The limit shape of the rescaled process $s \mapsto \bar{X}^{(n)}_s = \frac{1}{2n}X_{2ns}$ is Vershik's curve in the rotated coordinates, given by

$$L^\infty_c(s) = \tfrac{1}{c}\log(2\cosh cs), \tag{3.10}$$

and satisfying the equation $e^{-c(x-s)} + e^{-c(x+s)} = 1$. Convergence in probability of $\bar{X}^{(n)}$ to Vershik's curve can be proved as follows. First note that since

$$L^\infty_c - |t| \to 0 \qquad \text{when} \qquad |t| \to \infty, \tag{3.11}$$

we can pick $K > 0$ such that $L^\infty_c(t) - |t| < \varepsilon$ for $|t| \geq K$. Then, for $|t| > K$, the interface can only deviate by ε when it is above the limit shape, and if this happens, then $\frac{1}{2n}X_{2nK}$ also deviates by ε, since the interface has slope ± 1. Hence it is enough to look at $|t| < K$, where the proof of Theorem 1 can be copied, with the following minor modification to take into account the infinitely many possible positions of the lattice path. Statements analogous to Lemma 2.5 and Lemma 2.10 hold in this case as well: The 1-dimensional marginal distribution is unimodal and the mode is within a distance of $\frac{1}{n}$ from the limit shape. The function $x \mapsto F^{(1)}_{\infty,c}(s,x)$ has a critical point and a value of zero at $L^{\infty}_c(s)$, and its second derivative is bounded above by $-2c$. Thus, $F_{\infty,c}(s, L^{\infty}_c(s)+y) \leq -2cy^2$, and for example

$$\begin{aligned}\mathbb{P}^{\infty}_n[X_{2nt} > 2n(L_{\infty,c}(t)+\varepsilon)] &\leq \sum_{i\geq 0} M\sqrt{\tfrac{c}{2\pi n}}\exp\big(-2nc(\varepsilon+\tfrac{i}{n})^2\big)\\ &= M\sqrt{\frac{c}{2\pi n}}\frac{e^{-2nc\varepsilon^2}}{1-e^{-4c\varepsilon}},\end{aligned}$$

and after summing over the the $O(n)$ integer times in $[-2nK, 2nK]$, this can still be made arbitrarily small by taking n large.

In a similar fashion, we can derive the limit shape in the model with diagrams bounded only on one side, say $\omega_i \leq a_n$ for all i. Here (a_n) is a sequence of positive integers with $\frac{a_n}{2n} \to \rho > 0$ for $n \to \infty$. This model will be referred to as the *semi-bounded* case. For each n we have the probability measure $\mathsf{P}^{\circ}_n = \mu_q$ on the set of partitions with summands no greater than a_n, given explicitly by $\mathsf{P}^{\circ}_n[\omega] = \frac{q^{|\omega|}}{Z_{a_n,\infty}(q)}$. We place the infinitely long bounding box with its left corner at the origin in the (s,x)-plane, see Figure 3.2 and Figure 3.4. The rescaling is as usual by a factor of $2n$. The limit shape $L^{\circ}_{\rho,c}$ is the curve (1.9) from [23], and using methods as above, it is straightforward to rederive it, for instance $L^{\circ}_{\rho,c}(s)$ is the critical point of the function $F^{(1)}$ in this setup. We find that it is given by

$$L^{\circ}_{\rho,c}(s) = s - 2\rho + \tfrac{1}{c}\log(1+e^{-2cs}(e^{2c\rho}-1)). \tag{3.12}$$

Figure 3.2 shows a few plots of these curves for different values of c. The curve satisfies the equation

$$e^{-c(2\rho-s+x)} + (1-e^{-2c\rho})e^{-c(s+x)} = 1.$$

Thus, if $\alpha = 2\rho$ and $z_\alpha = 1-e^{-2c\rho}$, where c satisfies the equation $\sqrt{\mathrm{Li}_2(1-e^{-2c\rho})} = c$, then this is exactly Vershik and Yakubovich's curve (1.9) in the rotated coordinates.

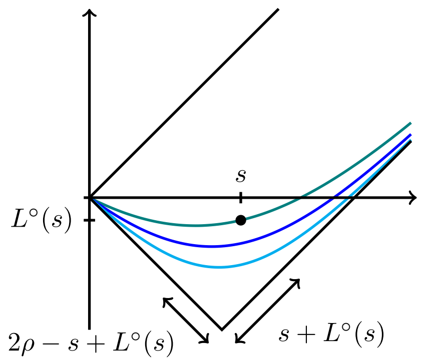


Figure 3.2: Limit shapes for different values of c in the semibounded case.

3.2 · Relations Between the Models

In this section we look at the relations between the models of unbounded, semibounded and bounded partitions, both on limit shape scale and for fluctuations. The basic idea is illustrated in Figure 3.3: Two fixed times $s_0 < s_1$ define a (macroscopic) bounding box spanned by $(s_0, L_c^\infty(s_0))$ and $(s_1, L_c^\infty(s_1))$. As noted by Petrov in [15], it turns out that the limit shape for the bounded problem in this box (for a suitably modified value of c) is just the restriction of Vershik's curve. This fits well with the observation in Lemma 2.11 that the limit shapes in subboxes agree with the full limit shape. The unbounded limit shape L_c^∞ is related to the bounded limit shape $L_{\rho,c}$ by the relation

$$L_{\bar\rho,\bar c}(t) = \frac{L_c^\infty(ts_1 + (1-t)s_0) - L_c^\infty(s_0)}{s_1 - s_0}, \tag{3.13}$$

where

$$\bar\rho = \frac{1}{2}\left(1 - \frac{L_c^\infty(s_1) - L_c^\infty(s_0)}{s_1 - s_0}\right) \qquad \text{and} \qquad \bar c = c(s_1 - s_0). \tag{3.14}$$

Similar relations hold with the semibounded limit shape:

$$L^\circ_{\rho_0,c}(t) = L_c^\infty(s_0 + t) - L_c^\infty(s_0), \tag{3.15}$$

where $\rho_0 = \frac{1}{2}(L_c^\infty(s_0) - s_0)$ defines the value of ρ for the semibounded box defined by the point $(s_0, L_c^\infty(s_0))$ and the diagonal line $x = s$. The relation between the semibounded curve and the bounded curve is defined by a fixed $s_2 > 0$, and is given by

$$L_{\rho',c'}(t) = \frac{L^\circ_{\rho,c}(ts_2)}{s_2}, \tag{3.16}$$

where $\rho' = \frac{1}{2}(1 - \frac{1}{s_2}L^\circ_{\rho,c}(s_2))$ and $c' = cs_2$. Figure 3.3 shows a graphical representation of the relations.

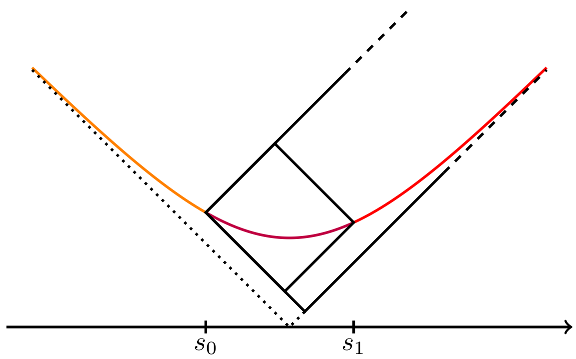


Figure 3.3: A fixed $s_0 \in \mathbb{R}$ defines an infinite bounding box for the semibounded model. Fixing another point $s_1 > s_0$ defines a bounding box for the bounded case. The limit shape in each case is simply the restriction of the unbounded limit shape, when c is chosen properly.

The similarities between the models extend also to the fluctuations, which turn out to be given essentially by Ornstein-Uhlenbeck processes in both the unbounded and semibounded cases. Note that our results here are not fully compatible with those obtained in [23] – we study fluctuations 'along the diagonal' compared to the coordinates in Chapter 1.

The fluctuations can be calculated exactly as in Section 2.6, and the proof of convergence of the marginal distributions translates directly to the present cases.

The covariance matrix for the Gaussian limit of the joint distribution at two times $s < t$ is given by the negative of the inverse of the Hessian matrix of the function $F^{(2)}$, evaluated at $(s, t, L(s), L(t))$. In the unbounded case, this function is given by

$$\begin{aligned} F^{(2)}_{\infty,c}(s,t,x,y) = &-c(x-s)(x+s) - c(t+y-x-s)(y-t) - S_c(x+s) \\ &- (x+s)\log c - S_c(y-t) - (y-t)\log c \\ &+ f_c(x-s-y+t, t+y-x-s) - \tfrac{\pi^2}{6c}, \end{aligned}$$

while in the semibounded case we have

$$\begin{aligned} F^{(2)}_{\circ,\rho,c}(s,t,x,y) = &-c(2\rho-s+x)(s+x) - c(t+y-s-x)(2\rho-t+y) \\ &+ f_c(s+x, s-x) + f_c(y+t-s-x, t-y-s+x) \\ &- S_c(2\rho-t+y) - (2\rho-t+y)\log c + S_c(2\rho) + 2\rho\log c. \end{aligned}$$

Figure 3.4 shows the microscopic justification for the latter function.

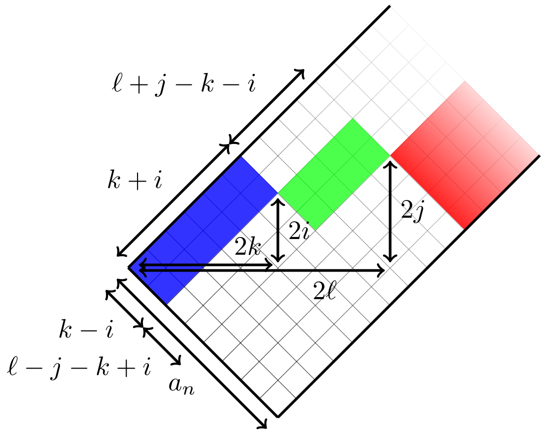


Figure 3.4: The two-dimensional marginal in the semibounded case.

Calculating the second partial derivatives and inverting the Hessian, we get the covariance matrix in the unbounded model:

$$\frac{1}{2c}\begin{pmatrix} g_c^\infty(s) & \\ & g_c^\infty(t)\end{pmatrix}\begin{pmatrix} 1 & e^{-c(t-s)} \\ e^{-c(t-s)} & 1\end{pmatrix}\begin{pmatrix} g_c^\infty(s) & \\ & g_c^\infty(t)\end{pmatrix}, \tag{3.17}$$

where the central matrix together with the factor $\frac{1}{2c}$ is the covariance matrix for the Ornstein-Uhlenbeck process on $(-\infty, \infty)$, while the correcting factor is the function

$$g_c^\infty(s) = \frac{1}{\sqrt{2}\cosh cs}. \tag{3.18}$$

Apparently, the fluctuations in the unbounded model are an Ornstein-Uhlenbeck process ‘dampened’ by hyperbolic cosine. In the semibounded case, the correction function is

$$g^\circ_{\rho,c}(s) = \frac{\sqrt{2(e^{2c\rho}-1)}}{e^{cs} - e^{-cs} + e^{c(2\rho-s)}}, \tag{3.19}$$

and the two-point covariance matrix is

$$\frac{1}{2c}\begin{pmatrix} g^\circ_{\rho,c}(s) & \\ & g^\circ_{\rho,c}(t)\end{pmatrix}\begin{pmatrix} 1-e^{-2cs} & 2e^{-ct}\sinh cs \\ 2e^{-ct}\sinh cs & 1-e^{-2ct}\end{pmatrix}\begin{pmatrix} g^\circ_{\rho,c}(s) & \\ & g^\circ_{\rho,c}(t)\end{pmatrix} \tag{3.20}$$

The central matrix together with the factor $\frac{1}{2c}$ is the covariance matrix for the Ornstein-Uhlenbeck process on $[0, \infty)$ with initial value 0. From the description of the Ornstein-Uhlenbeck processes in the appendix, we can say that one gets the fluctuations of the semibounded and bounded models by conditioning the process on $(-\infty, \infty)$ to be zero at the appropriate times. We have not attempted a proof of tightness in the unbounded and semibounded models.

Analogous to the limit shape relations (3.13), (3.15) and (3.16), there exist relations between the correction factors g, g° and g^∞:

$$g_{\bar{\rho},\bar{c}}(t) = g_c^\infty(s_0 + t(s_1 - s_0)) \tag{3.21}$$
$$g^\circ_{\rho_0,c}(t) = g_c^\infty(s_0 + t) \tag{3.22}$$
$$g_{\rho',c'}(t) = g^\circ_{\rho,c}(ts_2) \tag{3.23}$$

The first relation reveals the nature of the somewhat complicated correction factor we encountered in Chapter 2. It is simply the dampening factor of hyperbolic cosine from the unbounded model, expressed in the coordinates of the bounded case.

3.3 · Cutting a Corner Off

We consider now a variation on the model of bounded Young diagrams where we restrict to diagrams containing a predetermined rectangular diagram. Alternatively one can say that we cut a corner off of the bounding box. Keep the parameters ρ, c, (a_n), (b_n) and $\rho_n = \frac{a_n}{2n}$ from before, and introduce new parameters as follows. Fix a point $(s_1, x_1) \in B_\rho$ and let $k_n = \lfloor s_1 n \rfloor$ and $i_n = \lfloor x_1 n \rfloor$ for each $n \geq 1$. We assume that $(\frac{k_n}{n}, \frac{i_n}{n}) \in B_{\rho_n}$, ie. that $2i_n$ is a possible location for $X^{(n)}$ at time $2k_n$. The corner we are cutting off is the rectangular Young diagram $\hat{\lambda}_n = [0, a_n - k_n + i_n] \times [0, k_n + i_n]$. The top figure opposite the beginning of this chapter shows a sample of diagrams from this distribution. We will get to the other two figures in a moment.

Let $\hat{\mathcal{P}}_{a_n,b_n}$ denote the subset of $\mathcal{P}_{a_n,b_n}$ of partitions whose Young diagrams contain $\hat{\lambda}_n$, ie. those with $X^{(n)}_{2k_n} \geq 2i_n$. Let $\hat{\mathbb{P}}_n$ denote be the probability measure on $\mathcal{C}$ that is the rescaled restriction of $\mathbb{P}_n$, and denote the partition function by $\hat{Z}_n(q)$, ie.

$$\hat{Z}_n(q) = \sum_{\omega \in \hat{\mathcal{P}}_{a_n,b_n}} q^{|\omega|}.$$

Note that ratios of probabilities are unchanged compared to $\mathbb{P}_n$. The partition function could be computed as a sum of terms resembling (2.11) over all possible values of X_{2k_n}. However, the asymptotics of such an expression is not easily computed, and as it turns out, we can get some way with Lemma 2.5 and Theorem 1: The marginal distribution is unimodal and the probability mass concentrates in diagrams close to the limit shape. These two facts are all we need to derive a limit shape result in this model.

The point is that if the 'free' limit shape $L_{\rho,c}$ does not touch the corner we cut off, the new process has the same limit shape: the probability concentrates in diagrams that contain the corner anyway. On the other hand, if $L_{\rho,c}(s_1) \leq x_1$, we are removing the principal part of the probability mass, and of the remaining lattice paths, those passing close to the point (s_1, x_1) contribute most to the partition function. Passing to the limit, the interfaces will converge to a limit shape passing through the point (s_1, x_1) and on either side of this point, consists of a copy of $L_{\rho,c}$ with the parameters of the subboxes, given in this case by

$$\rho_j = \frac{1}{2}\left(1 - \frac{x_{j+1} - x_j}{s_{j+1} - s_j}\right) \tag{3.24}$$

and

$$c_j = c(s_{j+1} - s_j), \tag{3.25}$$

for $j = 0, 1$, where $s_0 = 0$, $s_2 = 1$ and $x_0 = 0$, $x_2 = 1 - 2\rho = L_{\rho,c}(1)$.

Proposition 3.1. *Under the distribution $\hat{\mathbb{P}}_n$, the process $\bar{X}^{(n)}$ converges in probability in $\mathcal{C}$. The limit shape depends on the value of $L_{\rho,c}(s_1)$:*

(a) If $L_{\rho,c}(s_1) > x_1$, the limit shape is $L_{\rho,c}$.

(b) If $L_{\rho,c}(s_1) < x_1$, the limit shape is the curve

$$t \mapsto \hat{L}(t) = \begin{cases} s_1 L_{\rho_0,c_0}(\frac{t}{s_1}) & 0 \le t \le s_1 \\ (1 - s_1) L_{\rho_1,c_1}(\frac{t-s_1}{1-s_1}) + x_1 & s_1 < t \le 1 \end{cases} \tag{3.26}$$

Proof. Take $0 < \varepsilon < L(s_1) - x_1$. If $\sup_{t\in[0,1]} \left|\bar{X}_t - L_{\rho,c}(t)\right| \le \varepsilon$, then in particular $X_{2k_n} \ge 2i_n$ (at least when n is sufficiently large). Therefore,

$$\hat{\mathbb{P}}_n\Big[\sup_t\left|\bar{X}_t - L_{\rho,c}(t)\right| \le \varepsilon\Big] = \frac{Z_n(q)}{\hat{Z}_n(q)}\,\mathbb{P}_n\Big[\sup_t\left|\bar{X}_t - L_{\rho,c}(t)\right| \le \varepsilon\Big] \tag{3.27}$$

since the two sides represent the same sum over partitions. Since $Z_n(q) \ge \hat{Z}_n(q)$ and, according to Theorem 1, the probability on the right tends to 1 as $n \to \infty$, the same holds for the left hand side.

To prove case (b), we first need to prove that the lattice path stays close to the point $(2k_n, 2i_n)$ with a high probability. Let $0 < \gamma < 1$. We will prove that

$$\lim_{n\to\infty} \hat{\mathbb{P}}_n\Big[\tfrac{1}{2n} X_{2k_n} \ge x_1 + n^{\gamma-1}\Big] = 0. \tag{3.28}$$

As remarked above, the ratios of probabilities are unchanged from the original case, so (2.34) holds for $\hat{\mathbb{P}}_n$. Setting $i = i_n$ and taking the limit $n \to \infty$, (2.36) shows that

$$\lim_{n\to\infty} \frac{\hat{\mathbb{P}}_n\big[X_{2k_n} = 2(i_n+1)\big]}{\hat{\mathbb{P}}_n\big[X_{2k_n} = 2i_n\big]} = R_{\rho,c}(s_1, x_1) < 1,$$

where $R_{\rho,c}$ is the function from (2.35). The last bound comes from the fact that $x_1 > L_{\rho,c}(s_1)$ and $x \mapsto R_{\rho,c}(s_1, x)$ is decreasing and takes the value 1 at $x = L_{\rho,c}(s_1)$. We conclude that there is a positive constant $K < 1$ such that

$$\hat{\mathbb{P}}_n\big[X_{2k_n} = 2(i+1)\big] \le K \cdot \hat{\mathbb{P}}_n\big[Y_{2k_n} = 2i\big]$$

for all sufficiently large n and all $i \ge i_n$. By iteration, this implies that

$$\begin{aligned} \hat{\mathbb{P}}_n\Big[\tfrac{1}{2n} X_{2k_n} \ge x_1 + n^{\gamma-1}\Big] &= \sum_{i \ge nx_1 + n^\gamma} \hat{\mathbb{P}}_n\big[X_{2k_n} = 2i\big] \\ &\le n \cdot \hat{\mathbb{P}}_n\big[X_{2k_n} = 2\lceil nx_1 + n^\gamma\rceil\big] \\ &\le nK^{2\lceil nx_1+n^\gamma\rceil - 2i_n}\hat{\mathbb{P}}_n\big[X_{2k_n} = 2i_n\big] \\ &\le nK^{2n^\gamma+4}, \end{aligned}$$

proving (3.28). We rewrite the probability that the path deviates by more than ε by conditioning on the value of X_{2k_n}:

$$\hat{\mathbb{P}}_n\left[\sup_t \left|\bar{X}_t - \hat{L}(t)\right| > \varepsilon\right] = \sum_{j\geq 0} \hat{\mathbb{P}}_n\left[\sup_t \left|\bar{X}_t - \hat{L}(t)\right| > \varepsilon,\ X_{2k_n} = 2i_n + 2j\right],$$

Now (3.28) allows us to ignore the terms with $j \geq n^{\gamma-1}$. A typical remaing term is bounded above by the corresponding conditional probability

$$\hat{\mathbb{P}}_n\left[\sup_t \left|\bar{X}_t - \hat{L}(t)\right| > \varepsilon \;\middle|\; X_{2k_n} = 2i_n + 2j\right]. \tag{3.29}$$

The conditioning implies that events in the left and right subbox are independent. In each subbox we have a version of the original problem with the appropriately modified parameters in (3.24) and (3.25). The bound $j \leq n^{\gamma-1}$ ensures that the side lengths of the subboxes for the various j converge uniformly, as specified in the remark following Theorem 1. We conclude that (3.29) decays exponentially, uniformly for j smaller than $n^{\gamma-1}$, which proves case (b). □

A generalization of this model is to consider m points $(s_1, x_1), \ldots, (s_m, x_m) \in B_\rho$ and require the lattice path to have $X_{2ns_j} \geq 2nx_j$ for all j. In this case we conjecture that the limit shape is given as follows (see Figure 3.5). For every $0 < s < 1$, the function $\mathbb{R} \ni c \mapsto L_{\rho,c}(s)$ is continuous and decreasing. Hence, for each j there is a unique c_j such that $L_{\rho,c_j}(s_j) = x_j$. If $c_j > c$ for all j, then the 'free' limit shape $L_{\rho,c}$ evades the region that we cut out, and the limit shape in the restricted model is the same. Otherwise, there is a j_0 with minimal c_{j_0}. The limit shape will pass through the point (s_{j_0}, x_{j_0}), and the limit shapes on either side can be constructed by repeating the above with the new values of the parameters ρ and c given by (3.24) and (3.25). The result is a piecewise smooth curve as depicted on the right in Figure 3.5.

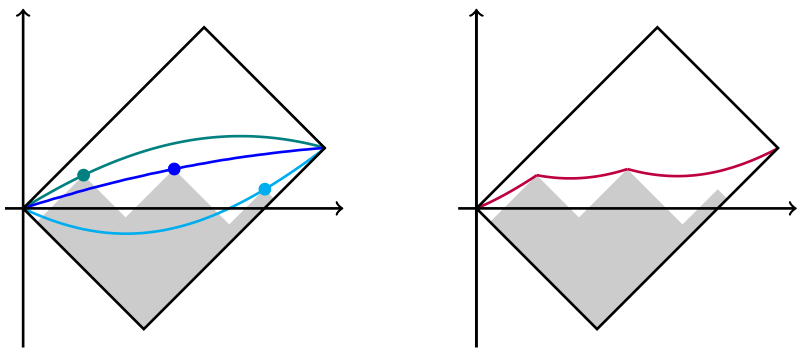


Figure 3.5: Construction of the limit shape when a macroscopic diagram is cut out of the bounding box.

As a further generalization, we take a sequence of partitions (λ_n) with $\lambda_n \in \mathcal{P}_{a_n,b_n}$, such that the interfaces $\bar{X}^{(n)}(\lambda_n)$ converge uniformly to some smooth curve $s \mapsto \Gamma(s)$, with $|\Gamma'(s)| < 1$ for all s, connecting the origin and the right corner of the bounding box. At each step n, we consider the usual probability measure restricted to diagrams containing the diagram of λ_n, and once again we can ask for a description of the limit shape (and a proof of its existence). A natural conjecture is that it follows a description similar to case above. Consider the family of limit shapes $L_{\rho,c}$ parametrized by $c \in \mathbb{R}$. If we assume there is a c such that $L_{\rho,c}(s) > \Gamma(s)$ for all

$s \in (0,1)$, then we can find a minimal c such that the two curves are tangent at some time s_0. As c increases, the limit shape will agree with Γ on some interval containing s_0, and this interval will grow. At the endpoints of the interval, the two curves will be tangent. Other such intervals will emerge and grow as c tends towards $+\infty$. See Figure 3.6. If we allow Γ to be only piecewise smooth, cusps as in Figure 3.5 would correspond to 1-point intervals that are fixed as $c \to \infty$.

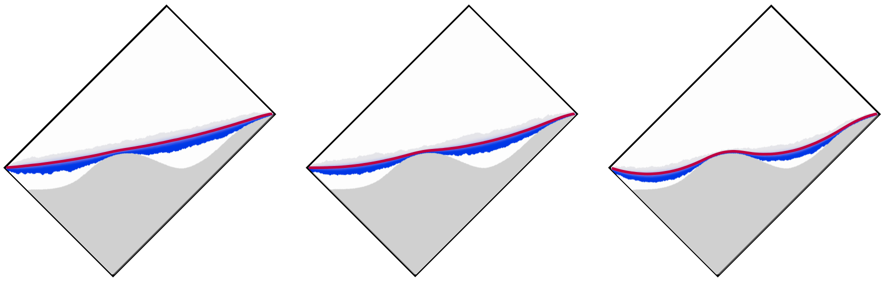


Figure 3.6: Simulation of partitions for a large value of n. A diagram approximating a smooth curve has been cut out of the bounding box. The parameter c increases from the left to the right picture.

3.4 · The Large Deviation Principle

A possible refinement of theorem 1 on the convergence to the limit shape is to prove a *large deviation principle* for the process $\bar{X}^{(n)}$. Let S be a complete and separable metric space. The large deviation principle describes the limiting behavior of a sequence of probability measures on the Borel σ-algebra of S in terms of a *rate function*, a map $I\colon S \to [0,\infty]$ such that for all $M \geq 0$, the preimage $I^{-1}\big([0,M]\big)$ is closed. If all these preimages are in fact compact, I is called a *good rate function*, and it attains its infimum on any nonempty closed set.

Let (α_n) be a sequence of positive real numbers with $\lim_n \alpha_n = \infty$. A sequence of probability measures $(\mathbb{P}_n)$ on S is said to satisfy the large deviation principle with rate function I and speed (α_n) if

(i) for each closed subset $F \subseteq S$,

$$\limsup_{n\to\infty} \frac{1}{\alpha_n} \log \mathbb{P}_n[F] \leq -\inf_{x\in F} I(x) \tag{3.30}$$

(ii) for each open subset $G \subseteq S$,

$$\liminf_{n\to\infty} \frac{1}{\alpha_n} \log \mathbb{P}_n[G] \geq -\inf_{x\in G} I(x). \tag{3.31}$$

These conditions can be compared to those for weak convergence in the Portmanteau lemma, see p. 50. Taking $F = S$ in (3.30) shows that $\inf_{x\in S} I(x) = 0$, so if I is a good rate function, there is at least one $x_0 \in S$ with $I(x_0) = 0$.

Suppose the measure $\mathbb{P}_n$ is the distribution of a random element X^n. In this case, the large deviation principle is equivalent to a condition known as the *Laplace principle* (see [7]): for every bounded, continuous function $h\colon S \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{\alpha_n} \log \mathbb{E}\big[\exp(-\alpha_n h(X^n))\big] = -\inf_{x\in S}\{h(x) + I(x)\}. \tag{3.32}$$

This formula should be compared with the classical Laplace method for studying the asymptotics of certain integrals on $\mathbb{R}$. Say h is a bounded continuous real function on $[0,1]$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^1 \exp(-nh(x))\ dx = -\min_{x\in[0,1]} h(x). \tag{3.33}$$

Analogous to the concept of tightness we used to prove weak convergence of the fluctuations of the interface of random Young diagrams, there is the notion of *exponential tightness* related to the large deviation principle. The sequence of probability measures $(\mathbb{P}_n)$ is said to be exponentially tight if for each $M > 0$ there is a compact subset $K \subseteq S$ such that

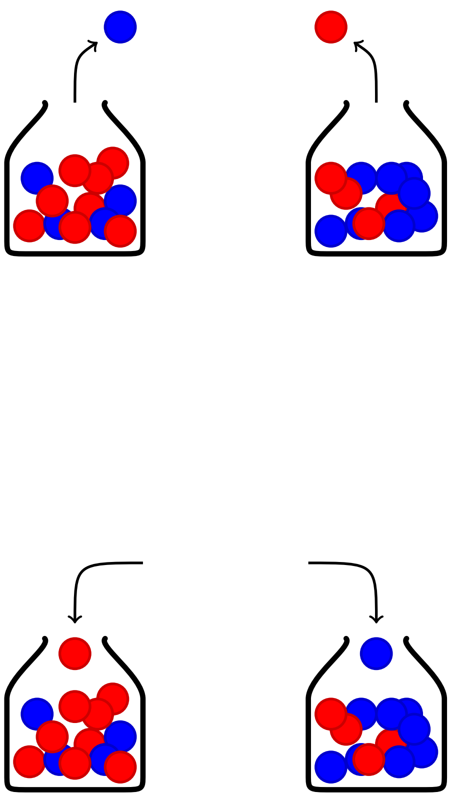
$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n[K^{\mathrm{c}}] \geq -M. \tag{3.34}$$

Exponential tightness, together with the existence of the limit on the left of (3.32) for all bounded continuous h, implies the large deviation principle for $\mathbb{P}_n$, with an explicitly given rate function ([7, Theorem 1.3.8]).

In the world of random partitions, a large deviation principle for the cases of uniformly distributed partitions, respectively strict partitions, of n has been proved by Vershik, Dembo and Zeitouni in [4]. Let $D(0,\infty)$ denote the space of real functions on $(0,\infty)$ that are left-continuous and have right limits, and let DF be the subspace of non-increasing functions with $\lim_{t\to\infty} f(t) = 0$. Let $\widehat{DF}$ denote the collection of all non-increasing functions g that agree almost everywhere with some function in DF, and equip $\widehat{DF}$ with the topology of pointwise convergence. The uniform probability measure on partitions of n induces a probability measure $\mathbb{P}_n$ on $\widehat{DF}$ via the map τ_γ taking a partition ω to the rescaled Young diagram $\bar{\varphi}_\omega$ in (1.3). As the scaling γ, take $\gamma_n = \sqrt{n}$. The sequence $\mathbb{P}_n$ satisfies the large deviation principle in $\widehat{DF}$ with speed $\sqrt{n}$ and good rate function I defined as follows. A function $f \in DF$ corresponds to a measure μ_f on $(0,\infty)$ by the relation $f(t) = \mu_f([t,\infty))$. The Lebesgue decomposition of this measure into an absolutely continuous measure and a singular measure (with respect to Lebesgue measure) yields the decomposition $f = f_{\mathrm{ac}} + f_{\mathrm{s}}$, which can be extended to $f \in \widehat{DF}$. The rate function I is then given by

$$I(f) = \tfrac{2\pi}{\sqrt{6}} - \int_0^\infty f'_{\mathrm{ac}}(t) \log(-f'_{\mathrm{ac}}(t)) + (1 - f'_{\mathrm{ac}}(t)) \log(1 - f'_{\mathrm{ac}}(t))\, dt \tag{3.35}$$

if $\int_0^\infty -t\, d\mu_f \leq 1$ and $I(f) = \infty$ otherwise. A similar theorem for the bounded partitions of Chapter 2 would be of great interest. Given the close relation between the unbounded and bounded models exhibited in Section 3.2, it is natural to conjecture that the large deviations are described by the same rate function, but proving this will be a job for someone else.



Appendix

This short appendix consists of two sections, the first containing the definition and a few results from the theory of weak convergence of probability measures, based on the exposition in [3]. In particular, the concept of tightness and the criterion used to prove weak convergence of the fluctuations of bounded partitions around the limit shape can be found here. The second section introduces the various Ornstein-Uhlenbeck processes referenced in the text.

A.1 · Weak Convergence of Probability Measures

Let (S, d) be a metric space. A probability measure on S is a non-negative countably additive set function $\mathbb{P}$ on the Borel σ-algebra of S, with $\mathbb{P}[S] = 1$. A sequence $(\mathbb{P}_n)$ of probability measures on S is said to *converge weakly* to $\mathbb{P}$, written $\mathbb{P}_n \Rightarrow_n \mathbb{P}$, if for every bounded, continuous function $f\colon S \to \mathbb{R}$,

$$\int_S f \, d\mathbb{P}_n \to \int_S f \, d\mathbb{P} \qquad \text{as } n \to \infty. \tag{A.1}$$

The concept of weak convergence can also be expressed in terms of random elements of S: A (measurable) map X from a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ to S is called a *random element* of S. It induces the pushforward probability measure $\mathbb{P}_X = \mathsf{P} \circ X^{-1}$ on S, called the *distribution* of X. We say then that a sequence of random elements (X^n) converges weakly to X if $\mathbb{P}_{X^n} \Rightarrow_n \mathbb{P}_X$. If the limit X happens to be constant, then this is equivalent to the concept of *convergence in probability*:

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}[d(X^n, X) > \varepsilon] = 0. \tag{A.2}$$

Example. Two well-known examples of weak convergence are the weak law of large numbers and the central limit theorem. The weak law of large numbers states that if (X_i) are independent, identically distributed random variables with $E(X_i) = \mu$ and finite variance $\sigma^2 > 0$, then the sample average $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ converges in probability to the constant variable μ. Centering the sequence and rescaling by $\sqrt{n}$, the central limit theorem states that the sequence $\tilde{X}_n = \sqrt{n}(\bar{X}_n - \mu)$, converges weakly to a centered Gaussian variable with variance σ^2. $\square$

Example. Let (X^n) be a sequence in S converging to some element X^0. Denote by δ_X the *unit mass at* X, the probability measure given by $\delta_X(A) = \mathbb{1}_A(X)$. Then, if f is continuous on S,

$$\int_S f\, d(\delta_{X^n}) = f(X^n) \to f(X^0) = \int_S f\, d(\delta_{X^0}),$$

and $\delta_{X^n} \Rightarrow_n \delta_{X^0}$. If $X^n \not\to X^0$, then, if $\varepsilon > 0$ is such that $d(X^n, X^0) > \varepsilon$ for infinitely many n, (A.1) fails when f is the positive part of the function $y \mapsto (1 - \frac{1}{\varepsilon} d(y, X^0))$. Therefore, $X^n \to X^0$ if and only if $\delta_{X^n} \Rightarrow_n \delta_{X^0}$. □

The Portmanteau lemma lists conditions equivalent to weak convergence:

(i) $\limsup_n \mathbb{P}_n[F] \leq \mathbb{P}[F]$ for all closed subsets $F \subseteq S$.

(ii) $\liminf_n \mathbb{P}_n[G] \geq \mathbb{P}[G]$ for all open subsets $G \subseteq S$.

(iii) $\mathbb{P}_n[A] \to \mathbb{P}[A]$ for all Borel sets $A \subseteq S$ with $\mathbb{P}[\partial A] = 0$ (these are called $\mathbb{P}$-*continuity sets*).

To prove weak convergence it is often enough to prove $\mathbb{P}_n[A] \to \mathbb{P}[A]$ for some nice subclass of Borel sets. For example, if $S = \mathbb{R}^k$, it is enough to show convergence on closed rectangles ([3, Example 2.3]).

The random elements X^n need not be defined on the same probability space. This is the case for the random Young diagrams in Chapter 2, where the underlying probability spaces are the sets $\mathcal{P}_{a,b}$ of partitions with at most b parts, each at most a, while S is the space $\mathcal{C} = C[0,1]$ of continuous functions on the unit interval, equipped with the uniform metric

$$d(X, Y) = \sup_{t \in [0,1]} |X_t - Y_t|.$$

Each k-tuple $0 \leq t_1 < t_2 < \cdots < t_k \leq 1$ defines a (continuous) natural projection $\pi_{t_1 \cdots t_k}$ from $\mathcal{C}$ to $\mathbb{R}^k$ by $\pi_{t_1 \cdots t_k}(X) = (X_{t_1}, \ldots, X_{t_k})$. The random vector $(X_{t_1}, \ldots, X_{t_k})$ is called a *finite dimensional marginal* of X. Knowing the distribution of all marginals of X is enough to specify the distribution over $\mathcal{C}$ of X itself (in the terminology of [3], the family $\{\pi^{-1}_{t_1 \cdots t_k} U\}$ with $k \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_k \leq 1$ and U ranging over Borel subsets of $\mathbb{R}^k$, constitutes a *separating class* of subsets of $\mathcal{C}$).

A necessary condition for the weak convergence of a sequence (X^n) in $\mathcal{C}$ is that all finite-dimensional marginals converge. This is a consequence of the continuous mapping theorem: a continuous image of a weakly convergent sequence is weakly convergent. However, as the following simple example shows, marginal convergence is not sufficient – weak convergence in $\mathcal{C}$ is stronger than convergence of all marginal distributions.

Example. Let (X^n) be a sequence of real functions on $[0,1]$ converging pointwise but not uniformly to X^0, and let $\mathbb{P}_n = \delta_{X^n}$. The typical marginal distribution is the unit mass at the point $(X^n_{t_1}, \ldots, X^n_{t_k}) \in \mathbb{R}^k$, and pointwise convergence and Example 2 show that the finite dimensional marginals of X^n converge weakly to those of X^0. However, the opposite implication in Example 2 shows that $\delta_{X^n} \not\Rightarrow_n \delta_{X^0}$, since $X^n \not\to X^0$ in $\mathcal{C}$. □

Fortunately, there exists an additional condition to ensure that convergence of the marginal distributions implies weak convergence. A sequence of probability measures $(\mathbb{P}_n)$ is said to be *tight* if for every $\varepsilon > 0$ there exists a compact subset $K \subseteq S$ such that $\mathbb{P}_n[K] > 1 - \varepsilon$ for all n. While this condition may not be particularly easy to apply as is, it can be reworked into a more useful criterion: [3, Theorem 13.5]

A sequence of random functions (X^n) converges weakly in $\mathcal{C}$ to X provided that

(i) $(X^n_{t_1}, \dots, X^n_{t_k}) \Rightarrow_n (X_{t_1}, \dots, X_{t_k})$ *for all k and all k-tuples $t_1, \dots, t_k \in [0,1]$.*

(ii) There exist constants $\alpha > \frac{1}{2}$, $\beta \geq 0$ and a non-decreasing, continuous function F on $[0,1]$ such that

$$\mathbb{P}\big[|X^n_s - X^n_r| \wedge |X^n_s - X^n_t| \geq \lambda\big] \leq \frac{1}{\lambda^{4\beta}}[F(t) - F(r)]^{2\alpha} \tag{A.3}$$

for all $n \in \mathbb{N}$, all $0 \leq r \leq s \leq t \leq 1$ and all $\lambda > 0$.

The main points of the translation from tightness of probability measures on $\mathcal{C}$ to condition (ii) above are as follows. Compactness in $\mathcal{C}$ is characterized by the Arzelà-Ascoli theorem, which states that a subset $A \subseteq \mathcal{C}$ is relatively compact (ie. its closure is compact) if and only if (1) the set $\big\{|X(0)| \;\big|\; X \in A\big\}$ is bounded, and (2) the functions in A are uniformly equicontinuous, ie. $\lim_{\delta \to 0} \sup_{X \in A} w_X(\delta) = 0$, where $w_X(d) = \sup_{|s-t|<\delta}|X(s) - X(t)|$ is the modulus of continuity. This translates tightness of a sequence of probability measures into conditions on the probability of events of the types $\big\{|X(0)| > a\big\}$ and $\big\{w_X(\delta) > \varepsilon\big\}$. The assumption of weak convergence of X^n_0 takes care of the first condition. The probability of the latter event is related to that in (A.3) by a so-called *maximal inequality*, ie. a universal inequality giving an upper bound for the probability of an event of the type $\big\{\sup_t |X_t| \geq \lambda\big\}$ in terms of the probability of an event similar to the one in (A.3).

Apart from the above criterion, we make use of two other results that we include here for completeness. The first ([3, Theorem 3.1]) says that if (X^n, Y^n) is a random element of $S \times S$, and if $X^n \Rightarrow_n X$ and $d(X^n, Y^n) \Rightarrow_n 0$, then $Y^n \Rightarrow_n X$.

The other ([3, Theorem 3.3]) states conditions under which a *local limit*, ie. a limit of density functions, can be translated into an *integral limit*, ie. weak convergence. Let $n \in \mathbb{N}$, let $\delta(n) = (\delta_1(n), \dots, \delta_k(n)) \in \mathbb{R}^k$ be a vector with positive coordinates and let $\alpha(n) = (\alpha_1(n), \dots, \alpha_k(n))$ be any point of $\mathbb{R}^k$. Define the lattice

$$\Lambda_n = (\mathbb{Z}\delta_1 - \alpha_1) \times \cdots \times (\mathbb{Z}\delta_k - \alpha_k).$$

A point $x = (x_1, \dots, x_k) \in \Lambda_n$ defines the cell

$$\big\{y \in \mathbb{R}^k \;\big|\; x_1 - \delta_i(n) < y_i \leq x_i,\, i = 1, \dots, k\big\},$$

which has volume $v_n = \delta_1(n) \cdots \delta_k(n)$, and $\mathbb{R}^k$ is the countable union of these cells. Now suppose $(\mathbb{P}_n)$ is a sequence of probability measures on $\mathbb{R}^k$ such that $\mathbb{P}_n$ is supported on Λ_n, and that $\mathbb{P}$ is a probability measure on $\mathbb{R}^k$ which has density p with respect to Lebesgue measure. For $x \in \Lambda_n$, let $p_n(x)$ denote the $\mathbb{P}_n$-mass at x. If the conditions

(i) $\max\{\delta_1(n), \dots, \delta_k(n)\} \to 0$ for $n \to \infty$, and

(ii) for any sequence (x_n) with $x_n \in \Lambda_n$ for all n and $x_n \to x$,

$$\frac{p_n(x_n)}{v_n} \to p(x),$$

are satisfied, then $\mathbb{P}_n \Rightarrow_n \mathbb{P}$.

A.2 · The Ornstein-Uhlenbeck Bridge

The Ornstein-Uhlenbeck bridge is an example of a continuous *Gaussian random process*, ie. an element X of $\mathcal{C}$ whose finite-dimensional marginal distributions are Gaussian. The distribution over $\mathcal{C}$ of such an element is completely specified by the mean values $\mathbb{E}\big[X_t\big]$ together with all $\mathbb{E}\big[X_s X_t\big]$, since these are enough to determine the finite-dimensional marginal distributions, which in turn determine the distribution of X. Perhaps the best known example of a Gaussian process is *Brownian motion* $(B_t)_{t\geq 0}$, with $\mathbb{E}\big[B_t\big] = 0$ and $\mathbb{E}\big[B_s B_t\big] = s \wedge t$.

The *Ornstein-Uhlenbeck process* (from now on O-U process) was originally introduced in [17] as a modification of the model of Brownian motion for describing the movement of a particle subjected to random pushes from the molecules of a surrounding fluid. In [6], Doob characterizes an O-U process $(U_t)_{t\in\mathbb{R}}$ as a stationary Gaussian process with continuous sample paths defined by the expectation and covariance

$$\mathbb{E}[U_t] = \mu \qquad \text{and} \qquad \mathbb{E}[(U_s - \mu)(U_t - \mu)] = \sigma_0^2 e^{-c|t-s|} \tag{A.4}$$

for all $s, t \in \mathbb{R}$. Stationary means that the distributional properties of the process are unchanged under translations of time. This process comes up as the fluctuations of the interfaces of unbounded partitions in Section 3.1.

Restricting to the positive axis $[0, \infty)$, the O-U process can also be defined as the solution to the stochastic differential equation

$$dU_t = c(\mu - U_t)\, dt + \sigma\, dB_t, \tag{A.5}$$

where $c, \sigma > 0$ and $\mu \in \mathbb{R}$ are parameters. We can safely assume that $\mu = 0$, so let us adopt this assumption. Solving the equation requires that an initial value U_0 be given. Using Itō's lemma, the equation is not difficult to solve, and the solution can be represented as a stochastic integral

$$U_t = e^{-ct} U_0 + \sigma e^{-ct} \int_0^t e^{cs}\, dB_s, \tag{A.6}$$

or as a scaled and time-transformed Brownian motion:

$$U_t = e^{-ct} U_0 + \sigma e^{-ct} B_{\int_0^t e^{cs}\, ds} = e^{-ct} U_0 + \sigma e^{-ct} B_{\frac{e^{2ct}-1}{2c}} \tag{A.7}$$

(see e.g. [14, Corollary 8.16] for the step from (A.6) to (A.7)). This representation shows that the expectation is

$$\mathbb{E}\big[U_t\big] = e^{-ct} U_0 \tag{A.8}$$

and, assuming that U_0 is stochastically independent of B, the covariance is

$$\operatorname{Cov}\big[U_s U_t\big] = e^{-c(s+t)} \Big(\operatorname{Var}[U_0] + \sigma^2 \frac{e^{2cs} - 1}{2c} \Big), \qquad s < t. \tag{A.9}$$

From this we see first that if U_0 has a centered Gaussian distribution with variance $\frac{\sigma^2}{2c}$, then (U_t) has the covariance of the double-sided version in (A.4) with $\sigma_0^2 = \frac{\sigma^2}{2c}$. On the other hand, if $U_0 = 0$, then

$$\operatorname{Cov}\big[U_s U_t\big] = \sigma^2 e^{-ct} \frac{\sinh(cs)}{c}. \tag{A.10}$$

This O-U process is the weak limit of the so-called *Bernoulli-Laplace urn model*: Take $2n$ balls, n red and n blue, and distribute them randomly in two urns A and B, with n balls in each urn. Now draw a ball from each urn at random, and put the ball from urn A in urn B and vice versa. Let W_k^n denote the number of red balls in urn A after k draws, and define the process $(U_t^{(n)})$ by $W_{\lfloor nt \rfloor}^n = \frac{1}{2}(n + U_t^{(n)}\sqrt{n})$. In [11], Jacobsen proves that if $m_0 \in \mathbb{R}$ is fixed, and the initial value of W^n is given by

$$W_0^n = \left\lfloor \tfrac{1}{2}(n + m_0\sqrt{n}) \right\rfloor$$

(for n sufficiently large so that this is a legal value of W_0^n), then the sequence $(U^{(n)})$ converges weakly in the Skorohod space $D[0,\infty)$ of right-continuous paths with left limits to the O-U process with parameters $c = 2$, $\sigma = \sqrt{2}$ and initial value $U_0 = m_0$.

The limit process found in Chapter 2 is a modification of the O-U process with initial value zero and 'tied down' to also have a fixed value of zero at time $t = 1$. It is appropriately named the *Ornstein-Uhlenbeck bridge.* The process can be constructed as follows. Set

$$h(t) = \frac{\sinh(ct)}{\sinh c}$$

and define

$$Y_t = U_t - h(t)U_1. \tag{A.11}$$

Observe that, for $0 \le t \le 1$,

$$\begin{aligned}\mathbb{E}\big[Y_t U_1\big] &= \mathbb{E}\big[U_t U_1 - h(t)U_1^2\big] \\ &= e^{-c}\frac{\sinh(ct)}{c} - \frac{\sinh(ct)}{\sinh c}e^{-c}\frac{\sinh c}{c} \\ &= 0,\end{aligned}$$

showing that Y_t is independent of U_1. Now let f be any test function and consider

$$\begin{aligned}\mathbb{E}\big[f(U_t) \bigm| U_1 = 0\big] &= \mathbb{E}\big[f(Y_t + h(t)U_1) \bigm| U_1 = 0\big] \\ &= \mathbb{E}\big[f(Y_t) \bigm| U_1 = 0\big] \\ &= \mathbb{E}\big[f(Y_t)\big],\end{aligned}$$

where the last equality is a consequence of the independence just proved. Thus, the process (Y_t) has the probability law of the O-U process conditional on $U_1 = 0$. Its covariance is

$$\begin{aligned}\mathbb{E}[Y_s Y_t] &= \mathbb{E}\left[Y_s(U_t - h(t)U_1)\right] = \mathbb{E}[Y_s U_t] \\ &= \mathbb{E}[U_s U_t] - h(s)\mathbb{E}[U_1 U_t] = \frac{\sinh(cs)}{c\sinh(c)}\left(e^{-ct}\sinh c - e^{-c}\sinh(ct)\right) \\ &= \frac{\sinh(cs)\sinh(c(1-t))}{c\sinh(c)},\end{aligned} \tag{A.12}$$

cf. the middle matrix in (2.55).

Bibliography

[1] G. E. Andrews. *The Theory of Partitions*. Cambridge Univ Press, 1998.

[2] D. Beltoft, C. Boutillier, and N. Enriquez. "Random Young Diagrams in a Rectangular Box". In: *(to appear)* (2010).

[3] P. Billingsley. *Convergence of Probability Measures*. Second Edition. Wiley Series in Probability and Statistics. New York: John Wiley & Sons Inc., 1999.

[4] A. Dembo, A. M. Vershik, and O. Zeitouni. "Large Deviations for Integer Partitions". In: *Markov Process. Related Fields* 6.2 (2000), pp. 147–179.

[5] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Second Edition. Vol. 38. Applications of Mathematics (New York). New York: Springer-Verlag, 1998.

[6] J. L. Doob. "The Brownian Movement and Stochastic Equations". In: *Ann. of Math. (2)* 43 (1942), pp. 351–369.

[7] P. Dupuis and R. S. Ellis. *A Weak Convergence Approach to the Theory of Large Deviations*. Wiley Series in Probability and Statistics. New York: John Wiley & Sons Inc., 1997.

[8] J. S. Frame, G. de B. Robinson, and R. M. Thrall. "The Hook Graphs of the Symmetric Groups". In: *Canadian J. Math.* 6 (1954), pp. 316–324.

[9] C. Greene, A. Nijenhuis, and H. S. Wilf. "A Probabilistic Proof of a Formula for the Number of Young Tableaux of a Given Shape". In: *Adv. in Math.* 31.1 (1979), pp. 104–109.

[10] C. Greene, A. Nijenhuis, and H. S. Wilf. "Another Probabilistic Method in the Theory of Young Tableaux". In: *J. Comb. Theory, Ser. A* 37.2 (1984), pp. 127–135.

[11] M. Jacobsen. "Laplace and the Origin of the Ornstein-Uhlenbeck Process". In: *Bernoulli* 2.3 (1996), pp. 271–286.

[12] S. V. Kerov. "A q-Analog of the Hook Walk Algorithm for Random Young Tableaux". In: *J. Alg. Comb.* 2 (1993), pp. 383–396.

[13] B. F. Logan and L. A. Shepp. "A Variational Problem for Random Young Tableaux". In: *Advances in Mathematics* 26.2 (1977), pp. 206–222.

[14] B. Oksendal. *Stochastic Differential Equations. An Introduction With Applications.* Springer-Verlag, 1989.

[15] F. Petrov. "Limit Shapes of Young Diagrams. Two elementary approaches." In: *Journal of Mathematical Sciences* 166.1 (2010), pp. 63–74.

[16] H. N. V. Temperley. "Statistical Mechanics and the Partition of Numbers. II. The Form of Crystal Surfaces". In: *Proc. Cambridge Philos. Soc.* 48 (1952), pp. 683–697.

[17] G. E. Uhlenbeck and L. S. Ornstein. "On the Theory of the Brownian motion". In: *Physical Review* 36.5 (1930), pp. 823–841.

[18] A. M. Vershik. "Limit Distribution of the Energy of a Quantum Ideal Gas from the Viewpoint of the Theory of Partitions of Natural Numbers". In: *Russian Math. Surveys* 52 (1997), p. 379.

[19] A. M. Vershik. "Statistical Mechanics of Combinatorial Partitions, and Their Limit Shapes". In: *Functional Anal. Appl.* 30.2 (1996), pp. 90–105.

[20] A. M. Vershik, G. Freiman, and Y. Yakubovich. "A Local Limit Theorem for Random Partitions of Natural Numbers". In: *Theory Probab. Appl* 44 (2000), pp. 453–468.

[21] A. M. Vershik and S. V. Kerov. "Asymptotic of the Largest and the typical Dimensions of Irreducible Representations of a Symmetric Group". In: *Functional Analysis and its Applications* 19.1 (1985), pp. 21–31.

[22] A. M. Vershik and S. V. Kerov. "Asymptotics of the Plancherel measure of the Symmetric Group and the Limiting Form of Young Tableaux". In: *Soviet Math. Dokl.* Vol. 18. 1977, pp. 527–531.

[23] A. M. Vershik and Y. Yakubovich. "The Limit Shape and Fluctuations of Random Partitions of Naturals with Fixed Number of Summands". In: *Mosc. Math. J.* 1.3 (2001), pp. 457–468.

[24] Y. Yakubovich. "Central Limit Theorem for Random Strict Partitions". In: *Journal of Mathematical Sciences* 107.5 (2001), pp. 4296–4304.