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Abstract

We consider an M/M/1 queueing system with inventory under the (r, Q) policy and with lost sales, in which demands occur according to a Poisson process and service times are exponentially distributed. All arriving customers during stockout are lost. We derive the stationary distributions of the joint queue length (number of customers in the system) and on-hand inventory when lead times are random variables and can take various distributions. The derived stationary distributions are used to formulate long-run average performance measures and cost functions in some numerical examples.

Keywords: Queueing, Inventory, Stationary distribution, Lost sale, Regenerative process.

1 Introduction

In classical inventory models, arriving demands are satisfied immediately if there is enough on-hand inventory. Most of these models consider optimization problems which chooses the optimal policy or optimal value of decision variables without computing the stationary distribution of inventory states. Nevertheless there are some studies that have derived stationary distributions to formulate long-run average cost functions which are used for optimization. Sahin [6] considered (s, S) inventory system and general random demand process with fixed lead time and backordering. He derived time dependent and stationary distribution to derive approximations for optimal control policy.

Focusing on lost sale problems, let us take a brief look at some of these studies. Mohebbi and Posner [4] considered a continuous-review inventory system with compound Poisson demand, Erlang as well as hyper-exponentially distributed lead time and lost sales. They derived the stationary distribution of inventory level for the purpose of formulating long-run average cost functions with/without a service level constraint. Mohebbi and Hao [5] considered inventory system with compound Poisson demand, Erlang-distributed lead times, random supply interruptions and derived the stationary distribution of the inventory level under an (r, Q)-type control policy.

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Many recent studies deal with complex integrated production-inventory systems or service-inventory systems. In these models satisfying each demand needs on hand-inventory and involves a process or service that takes some time. Production-inventory or service-inventory systems can be discussed in connection with integrated supply chain management. In these systems, an important issue is the reaction of inventory management to queueing of demands. Interaction of production/service processes with attached inventories can be described by queueing systems with inventories. Queueing systems with inventory in comparison with previous inventory models are more general and realistic. Also in a supply chain, every segment can be adequately described by a queueing system with inventory.

An early contribution is [9] where a light traffic heuristic was derived for an M/G/1 queue with an attached inventory. A few analytical models in this field have been developed up to now. Berman and Kim [2] considered a service system with an attached inventory, with Poisson customer arrival process, exponential service times and Erlang distribution of replenishment. Their formulation was a Markov decision problem to characterize an optimal inventory policy as a monotonic threshold structure which minimizes system costs. Berman and Kim [3] presented an extension where revenue is generated upon the service. They found an optimal policy which maximizes the profit.

Schwarz and Daduna [7] considered an M/M/1 queue with inventory under continuous-review with backordering when lead times are exponentially distributed. They computed performance measures and derived optimality conditions under different order policies. For evaluating performance measures and steady state probabilities they presented an approximation scheme. Also Schwarz et al. [8] considered a similar model with lost sales of customers that arrive during stockout and different inventory management policies. They derived stationary distributions of the joint queue length and inventory processes in explicit product form.

In the last two models mentioned above, lead times are supposed to be exponentially distributed. It is obvious that removing this assumption would provide more generality. In this paper, we consider an M/M/1 queueing system with inventory with infinite waiting room and lost sales when lead times can take any probability distribution or can have fixed values. The aim of our research is to derive joint stationary distributions of joint queue length and inventory processes. These distributions are used to provide long-run average cost functions which can be used for optimization problem to find optimal values of the decision variables.

2 Model description

2.1 The M/M/1 queue with inventory and lost sale

In a queueing system with an attached inventory, customers arrive one by one and require service. In the $M/M/1/\infty$ queue with inventory, customers arrive according to a Poisson process with arrival rate λ. There is a single server with unlimited waiting room and an inventory with unlimited capacity of items. Service times are exponentially distributed with parameter μ.

In our present model, during the period that there are some on-hand inventory, arriving demands join the queue and if there is no customer in the system the arriving demand directly takes service. But customers, who arrive during a period when the server waits for the replenishment order, are rejected and lost. Each customer needs exactly one item from the inventory for service, and the on-hand inventory and number of customers in system decreases by one at the moment of service completion. When the on-hand inventory reaches zero, customers in the queue wait for the next replenishment and the server will serve the customer which is at the head of the line at the instant when the next replenishment arrives at the inventory. Replenishment lead times are random variables with known probability distribution (which could be degenerate at one point) and independent of on-hand inventory and number of customers in system.

2.2 The inventory management policy

We consider a version of continuous review inventory management policy which is called (r, Q). It corresponds to an $M/M/1$ queue with inventory and lost sale where a fixed order quantity (Q) is ordered each time the on hand-inventory reaches the reorder point (r). Inventory management policy follows first come, first served (FCFS) regime. We assume that $r < Q$; this excludes degenerate cycles in which no demand occurs.

3 Stationary distribution

Our objective in this section is to derive the stationary joint distribution of on-hand inventory and number of customers in the system.

Let $X(t)$ denote the number of customers present at the system at time $t \geq 0$, either waiting or in process, and let $Y(t)$ denote the on-hand inventory at time $t \geq 0$. Then we can present the joint number of customers and on-hand inventory by $Z = ((X(t), Y(t)), t \geq 0)$ and the state space of Z is $E_Z = \{(n, k) : n \in N_0, 0 \leq k \leq Q + r\}$. Also we write limiting and stationary distribution $P(n, k)$ for the limiting probabilities of the number n of customers in the system and on-hand inventory k.

For computing the joint stationary distribution of the number X of customers in the system and the on-hand inventory Y, we give two theorems which reveal that in stationarity X and Y are independent random variables.

Definition 1 (Assumption on emergency supplying policy)**.** *For the queueing system with inventory which is described in Section* 2*, suppose that it is possible to supply from a second source with zero lead time. When the on-hand inventory reaches zero,the management triggers an order of size* Q *to the second supplier and cancels the previous order. We call this system an* $M/M/1/\infty$ *queueing system with inventory and emergency supplying policy.*

Theorem 2. *In the stationary $M/M/1/\infty$ queueing system with inventory and emergency supplying policy, the queue length process (number of customers in the system) is independent of the on-hand inventory process and identical to the queue length process in a classical $M/M/1/\infty$ system.*

Proof. When the on-hand inventory reaches zero, an order is triggered to the second supplier and because of the assumption of zero lead time, replenishment is received immediately from the second supplier. Thus the queueing system works without any interruption just like a standard $M/M/1/\infty$ queue. In particular, the marginal probability distribution of the queue length is identical to the probability distribution of the queue length in a classical $M/M/1/\infty$ system. It just remains to prove the independence. It is easily seen that on-hand inventory at time t (denoted by $Y(t)$) only depends on lead times and departures of the queueing system prior to t . Lead times are assumed to be independent of the queue length process, and it is also well known that in stationarity the departure process of an $M/M/1/\infty$ prior to t is a Poisson process and independent of number of customers in the system at time t (denoted by $X(t)$; see Asmussen [1] p.116. Therefore the on-hand inventory will be independent of number of customers. □

Corollary 3. *In an $M/M/1/\infty$ queueing system with inventory and emergency supplying policy, the conditional stationary distribution of queue length can be written as:*

$$P(n \mid k) = P_n(n) = \left(\frac{\mu-\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \tag{1}$$

Note that this is just the stationary distribution of the queue length in a classical $M/M/1/\infty$ system.

Theorem 4. *In the $M/M/1$ queueing system with inventory, unlimited waiting room and lost sale as described in Section 2, the conditional stationary distribution of the queue length is identical to the stationary distribution of the queue length in a classical $M/M/1/\infty$ system and can be written as in (1).*

Proof. When on-hand inventory reaches zero, the system freezes because no changes of inventory and number of customers occur. Omitting these freezing periods, it is obvious that remained working periods form a system identical to $M/M/1$ queueing system with inventory and emergency supplying policy. Therefore we just need to check the stationary distribution of queue length at the moment that on-hand inventory reaches zero. It can be concluded directly from Corollary 3 that this probability distribution is identical to stationary probability distribution of the queue length in a classical $M/M/1/\infty$ system and independent of the inventory level at that time which is zero. □

Consider now the $M/M/1$ queueing system with inventory and lost sale as described in Section 2. Based on Theorem 4. the joint stationary distribution of on-hand inventory and number of customers in the system can be written as follows:

$$P(n,k) = P_n(n)\cdot P_k(k) = \left(\frac{\mu-\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \cdot P_k(k) \tag{2}$$

where P_k is the stationary distribution of on-hand inventory process and P_n is the stationary distribution of the number of customers in the classical $M/M/1/\infty$ system. To derive the stationary joint distribution of on-hand inventory and number of customers in the system, we will compute stationary distribution of on-hand inventory in the next section.

3.1 Stationary distribution of on-hand inventory for general form of lead times

The departure process of the queue system is identical to the output process of inventory. As mentioned in Theorem 2, this process has the same distribution as the customer arrival process which is Poisson process with intensity λ during the periods where the inventory is not depleted, and no departure occurs during stockout. Considering the (r, Q) inventory policy, it can be seen that the inventory process in this case is a regenerative process and inventory levels of r can be considered as regeneration point. Thus each cycle starts at the moment that on-hand inventory reaches r and ends when the next cycle starts (see Asmussen [1] p.168). Denote by τ the cycle length. As a property of regenerative processes, the following formula can be applied to compute the stationary distribution of the inventory level:

$$P_k(k) \equiv \lim_{t\to\infty} P\{Y(t) = k\} = \frac{E[t_k]}{E[\tau]} \tag{3}$$

Here t_i is amount of time during a cycle that on-hand inventory spends in state i .

Conditioning on the lead time we have

$$E(\tau) = \int\limits_L E(\tau \mid L = \ell) f(\ell)\, d\ell$$

Each cycle can be divided into two periods. During the first the inventory level is equal or less than r (thus the length is the lead time) and the second period starts when an order is received and ends when inventory drops to r. If the inventory level is j when the order is received, it is obvious that this second period has an Erlang distribution with parameters $Q + j - r$, λ. Thus its mean is $\frac{Q+j-r}{\lambda}$ and so

$$E(\tau \mid L = \ell) = \ell + \sum_{j=0}^{r} \left(\frac{Q + j - r}{\lambda}\right) \cdot P(X_{\tau-} = j \mid L = \ell)$$

where $X_{\tau-}$ is the inventory level just before the order is received. The mean cycle time can be written as

$$E(\tau) = E(L) + \frac{Q - r}{\lambda} + \int\limits_L \sum_{j=0}^{r} \left(\frac{j}{\lambda}\right) \cdot P(X_{\tau-} = j \mid L = \ell) f(\ell)\, d\ell\,, \tag{4}$$

where

$$P(X_{\tau-} = j \mid L = \ell) = \frac{e^{-\lambda\ell}(\lambda\ell)^{r-j}}{(r - j)!} \qquad \text{for } j > 0$$

$$P(X_{\tau-} = 0 \mid L = \ell) = \sum_{j=r}^{\infty} \frac{e^{-\lambda\ell}(\lambda\ell)^j}{j!}$$

It just remains to compute $E(t_i \mid L = \ell)$:

$$E(t_i \mid L = \ell) = \frac{1}{\lambda} \qquad \text{for } r < i \le Q,$$

$$E(t_i \mid L = \ell) = \int_0^\ell \frac{\lambda^{r-i} t^{r-i-1} e^{-\lambda t}}{(r-i-1)!} \cdot E[Y_\lambda \wedge (\ell - t)]\, dt \qquad \text{for } 0 < i < r$$

where

$$E[Y_\lambda \wedge a] = \int_a^\infty a\lambda e^{-Y\lambda}\, dY + \int_0^a Y\lambda e^{-Y\lambda} dY = \frac{1}{\lambda}(1 - e^{-a\lambda}),$$

$$E(t_i \mid L = \ell) = E[Y_\lambda \wedge \ell] \qquad \text{for } i = r$$

$$E(t_i \mid L = \ell) = \sum_{j=i-Q}^{r} \frac{1}{\lambda} P(X_{\tau-} = j) \qquad \text{for } Q < i \le Q + r$$

$$E(t_0 \mid L = \ell) = \int_0^\ell \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!} (\ell - t)\, dt$$

Easy algebra gives

$$E(t_0 \mid L = \ell) = \ell \overline{\pi}_{r-1}(\ell) - \frac{r}{\lambda} \overline{\pi}_r(\ell) \tag{5}$$

Also the following equation is always true for the expected time until inventory is depleted :

$$E(t_0) = E(\tau) - \frac{Q}{\lambda} \tag{6}$$

$$E(t_i \mid L = \ell) = \frac{1}{\lambda} \overline{\pi}_{r-i-1}(\ell) - \frac{(\lambda t)^{r-i} e^{-\lambda \ell}}{(r-i)!} = \frac{1}{\lambda} \overline{\pi}_{r-i}(\ell) \qquad \text{for } 0 < i < r \tag{7}$$

$$E(t_r \mid L = \ell) = \frac{1}{\lambda}(1 - e^{-\lambda \ell}) \tag{8}$$

$$E(t_i \mid L = \ell) = \frac{1}{\lambda} \qquad \text{for } r < i \le Q \tag{9}$$

$$E(t_i \mid L = \ell) = \frac{1}{\lambda}[1 - \overline{\pi}_{Q+r-i}(\ell)] \qquad \text{for } Q < i \le Q + r. \tag{10}$$

Here $\pi_k(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$ and $\sum_{j=k+1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$

4 Optimization: numerical examples

We consider the following cost function giving mean costs per time unit in steady state :

$$C(r, Q) = H \cdot \sum_{k=0}^{Q+1} k \cdot P_k^{(r,Q)}(k) + A \cdot \frac{1}{E^{(r,Q)}(\tau)}$$
$$+ S \cdot \lambda P_k^{(r,Q)}(0) + W \cdot L \cdot P_k^{(r,Q)}(0)$$

Here H is the holding cost of each on-hand inventory unit per unit time, A is a fixed cost of each replenishment order, S is the shortage costs for each lost sale demand and W is the waiting cost of each customer during the stockout per unit time. $L = \frac{\lambda}{\mu-\lambda}$ is the average number of customers in system in the steady state.

The aim is to find the optimal $Q^\star = Q^\star(r)$ minimizing $C(r,Q)$ for a fixed r. For each lead time distribution and each fixed r, the cost function has the following general form:

$$C(Q) = \frac{a_1 Q^2 + b_1 Q + c_1}{b_2 Q + c_2}$$

Knowing that $a_1, b_1, c_1, b_2, c_2 \geq 0$, the cost function has a local minimum Q^*. If $Q^* > r \geq 0$, it is also the global minimum point of $C(r,Q)$. Otherwise the minimum cost occurs at $Q^* = r + 1$.

In our numerical examples summarized in Table 1, we assume that $\lambda = 20$ and $\mu = 50$. We computed the minimum cost function for $r = 100, 75, 50, 25$. We considered 5 well known probability distribution for lead times with the same mean 2.5, the exponential distribution with $\nu = 0.4$, the Erlang distribution with $m = 5, \nu = 2$, the hyperexponential distribution with $p_i = 0.2$, $\nu_1 = 2$, $\nu_2 = 1$, $\nu_3 = 0.5$, $\nu_4 = 0.25$, $\nu_5 = 0.2$, the uniform$(0,5)$ distribution and finally a fixed lead time equal to 2.5. We took $H = 1$, $A = 200$, $S = 50$, $W = 25$. The stationary distribution and performance measures of systems with those lead time distributions are derived using results in Appendix (stationary distributions and optimal batch orders are computed in Matlab).

Table 1: Optimal reorder point and order size

	Exponential	Erlang	Hyperexponential	Uniform	fixed
$Q^*(r=25)$	235	221	242	229	219
$C(25,Q^*)$	241.08	222.56	250.52	232.46	219.29
$Q^*(r=50)$	195	154	211	171	114
$C(50,Q^*)$	213.59	163.64	235.26	184.21	117.61
$Q^*(r=75)$	163	111	187	119	76
$C(75,Q^*)$	199.68	138.77	229.69	147.69	116.14
$Q^*(r=100)$	140	101	167	101	101
$C(100,Q^*)$	196.41	145.68	230.56	142.88	140.60

For the same examples, minimum cost functions and their relative minimum points (r^*, Q^*) (obtained applying an trial and error scheme) are presented in Table 2. As was to be expected, increased variation in the lead time increases the minimum total cost, in fact quite substantially.

Table 2: Optimization results for various reorder points

	$Q^\star$ & Total cost				
	Exponential	Erlang	Hyperexponential	Uniform	fixed
r^*	96	80	83	90	60
Q^*	142	106	180	98	92
Minimum Total Cost	196.30	138.15	229.36	139.40	103.08

Appendix: Stationary distributions for some distributions of lead times

Exponentially distributed lead times

Let the lead time be exponentially distributed with rate parameter ν. We first compute the expected cycle time:

$$E(\tau) = E(L) + \frac{Q-r}{\lambda} + \int_L \sum_{j=1}^{r} \left(\frac{j}{\lambda}\right) .P(X_{\tau-} = j \mid L = \ell) \cdot \nu e^{-\nu \ell}\, d\ell$$

$$\begin{aligned} E(\tau) &= \frac{1}{\nu} + \frac{Q-r}{\lambda} + \sum_{j=0}^{j-1} \frac{(r-j)\nu\lambda^{j-1}}{(\lambda+\nu)^{j+1}} \int_0^\infty \frac{\ell^j (\lambda+\nu)^{j+1} e^{-(\lambda+\nu)\ell}}{j!}\, d\ell \\ &= \frac{1}{\nu} + \frac{Q-r}{\lambda} + \sum_{j=0}^{r-1} \frac{(r-j)\nu\lambda^{j-1}}{(\lambda+\nu)^{j+1}}. \end{aligned}$$

Therefore after some algebra,

$$E(\tau) = \frac{1}{\lambda}\left(\frac{\lambda}{\lambda+\nu}\right)^r \left[Q\left(\frac{\lambda+\nu}{\lambda}\right)^r + \frac{\lambda}{\nu}\right].$$

Now we must compute the expected time that system remains in each state in a cycle:

$$\begin{aligned} E(t_0) &= \int_0^\infty \int_0^\ell \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!} (\ell - t)\nu e^{-\nu\ell}\, dt\, d\ell \\ &= \int_0^\infty [\ell \overline{\pi}_{r-1}(\ell) - \frac{r}{\lambda}\overline{\pi}_r(\ell)] \cdot \nu e^{-\nu\ell}\, d\ell \\ &= \sum_{j=r}^{\infty} \frac{(j+1)\lambda^j}{(\lambda+\nu)^{j+2}} - \frac{r}{\lambda} \sum_{j=r+1}^{\infty} \frac{\lambda^j \nu}{(\lambda+\nu)^{j+1}} \\ &= \cdots = \frac{1}{\nu}\left(\frac{\lambda}{\lambda+\nu}\right)^r \end{aligned}$$

$$E(t_i) = \int_0^\infty \frac{1}{\lambda}\overline{\pi}_{r-i}(\ell).\nu e^{-\nu\ell}\, d\ell = \cdots = \frac{1}{\lambda}\left(\frac{\lambda}{\lambda+\nu}\right)^{r-i+1} \qquad \text{for } 0 < i < r$$

$$E(t_r) = \int_0^\infty \frac{1}{\lambda}(1-e^{-\lambda\ell})\nu e^{-\nu\ell}\, d\ell = \frac{1}{\lambda}\left(\frac{\lambda}{\lambda+\nu}\right)$$

$$E(t_i) = \frac{1}{\lambda} \qquad \text{for } r < i \le Q$$

$$E(t_i) = \int_0^\infty \sum_{j=0}^{Q+r-i} \frac{1}{\lambda}.\frac{e^{-\lambda l}(\lambda\ell)^j}{j!}\nu e^{-\nu\ell}\, d\ell = \sum_{j=0}^{Q+r-i} \frac{\nu\lambda^{j-1}}{(\lambda+\nu)^{j+1}}$$

$$= \frac{1}{\lambda}\left[1 - \left(\frac{\lambda}{\lambda+\nu}\right)^{Q+r-i+1}\right] \qquad \text{for } Q < i \le Q+r.$$

Lead times with Erlang distribution

Let the lead times be Erlang random variables with parameter ν, m. We first compute the expected cycle time:

$$E(\tau) = \frac{m}{\nu} + \frac{Q-r}{\lambda} + \int_L \sum_{j=1}^{r} \left(\frac{j}{\lambda}\right) \cdot P(X_{\tau-} = j \mid L = \ell).\frac{\ell^{m-1}\nu^m e^{-\nu\ell}}{(m-1)!}\, d\ell$$

$$E(\tau) = \frac{m}{\nu} + \frac{Q-r}{\lambda}$$
$$+ \sum_{j=0}^{r-1}\left[\frac{(r-j)\nu^m\lambda^{j-1}}{(\lambda+\nu)^{m+j}} \cdot \frac{(m+j-1)!}{j!(m-1)!} \int_0^\infty \frac{\ell^{m+j-1}(\lambda+\nu)^{m+j}e^{-(\lambda+\nu)\ell}}{(m+j-1)!}\, d\ell\right]$$

$$E(\tau) = \frac{m}{\nu} + \frac{Q-r}{\lambda}$$
$$+ \left(\frac{\nu^m}{\lambda(\lambda+\nu)^m(m-1)!}\right)\sum_{j=0}^{r-1}(r-j)\left(\frac{\lambda}{\lambda+\nu}\right)^j \frac{(m+j-1)!}{j!}$$

Similarly aftersome algebra, the following results for expected time in each state can be inferred:

$$E(t_0) = \frac{m}{\nu} - \frac{r}{\lambda} + \left(\frac{\nu^m}{\lambda(\lambda+\nu)^m(m-1)!}\right)\sum_{j=0}^{r-1}(r-j)\left(\frac{\lambda}{\lambda+\nu}\right)^j \frac{(m+j-1)!}{j!}$$

$$E(t_i) = \frac{1}{\lambda} - \left(\frac{\nu^m}{\lambda(\lambda+\nu)^m(m-1)!}\right)\sum_{j=0}^{r-i}\left(\frac{\lambda}{\lambda+\nu}\right)^j \frac{(m+j-1)!}{j!} \qquad \text{for } 0 < i < r.$$

$$E(t_r) = \frac{1}{\lambda}\left(1 - \left(\frac{\nu}{\lambda+\nu}\right)^m\right)$$
$$E(t_i) = \frac{1}{\lambda} \qquad \text{for } r < i \le Q.$$

$$E(t_i) = \frac{1}{(m-1)!}\left(\frac{\nu}{\lambda+\nu}\right)^m \sum_{j=0}^{Q+r-i} \frac{\lambda^{j-1}(m+j-1)!}{j!(\lambda+\nu)^j} \qquad \text{for } Q < i \leq Q+r.$$

Fixed lead time

In this section we compute the stationary distribution, supposing that the lead times take a known fixed value ℓ. It can be directly concluded from (4)-(10) that the expected cycle time and expected time in each state is:

$$\begin{aligned}
E(\tau) &= \ell + \frac{Q-r}{\lambda} + \sum_{j=0}^{r-1} \frac{(r-j)\lambda^{j-1}\ell^j e^{-\lambda\ell}}{j!},\\
E(t_0) &= \ell\overline{\pi}_{r-1}(\ell) - \frac{r}{\lambda}\overline{\pi}_r(\ell),\\
E(t_i) &= \frac{1}{\lambda}\overline{\pi}_{r-i}(\ell) \qquad \text{for } 0 < i < r,\\
E(t_r) &= \frac{1}{\lambda}(1 - e^{-\lambda\ell}),\\
E(t_i) &= \frac{1}{\lambda} \qquad \text{for } r < i \leq Q,\\
E(t_i) &= \frac{1}{\lambda}[1 - \overline{\pi}_{Q+r-i}(\ell)] \qquad \text{for } Q < i \leq Q+r.
\end{aligned}$$

Lead times with uniform distribution

Let the lead time be uniformly distributed over the interval$[0, a]$. Using results of Section 3.1 and some algebra, the expected cycle time and the expected time in each state can be computed as followed:

$$\begin{aligned}
E(\tau) &= E(L) + \frac{Q-r}{\lambda} + \sum_{j=0}^{r-1} \int_0^a \frac{(r-j)\lambda^{j-1}\ell^j e^{-\lambda\ell}}{a \cdot j!}\, d\ell\\
&= \frac{a}{2} + \frac{Q-r}{\lambda} + \frac{1}{a\lambda^2}\sum_{j=0}^{r-1}(r-j)\overline{\pi}_j(a)\\
E(t_0) &= \frac{a}{2} - \frac{r}{\lambda} + \frac{1}{a\lambda^2}\sum_{j=0}^{r-1}(r-j)\overline{\pi}_j(a)\\
E(t_i) &= \frac{1}{\lambda} - \frac{1}{a\lambda^2}\sum_{j=0}^{r-i}\overline{\pi}_j(a) = \frac{1}{a\lambda^2}\sum_{j=r-i+1}^{\infty}\overline{\pi}_j(a) \qquad \text{for } 0 < i < r\\
E(t_r) &= \frac{1}{\lambda}(1 - \frac{1}{a\lambda}(1 - e^{-\lambda a}))\\
E(t_i) &= \frac{1}{\lambda} \qquad \text{for } r < i \leq Q\\
E(t_i) &= \sum_{j=0}^{Q+r-i} \frac{1}{a\lambda^2}(\overline{\pi}_j(a)) \qquad \text{for } Q < i \leq Q+r.
\end{aligned}$$

Lead times with hyperexponential distribution

Let the lead time L be a random variables with hyperexponential distribution,

$$f_L(\ell) = \sum_{k=1}^{n} f_{L_k}(\ell) p_k$$

where L_k is an exponentially distributed random variable with rate ν_k, and p_k is the probability that $L = L_k$.

Using the results of Section 3.1 and the stationary distribution of system with exponential distribution of lead times, it can be easily shown that the expected cycle time and expected time in each state is:

$$\begin{aligned}
E(\tau) &= \frac{Q}{\lambda} + \sum_{k=1}^{n} \frac{p_k}{\nu_k} \left(\frac{\lambda}{\lambda + \nu_k} \right)^r \\
E(t_0) &= \sum_{k=1}^{n} \frac{p_k}{\nu_k} \left(\frac{\lambda}{\lambda + \nu_k} \right)^r \\
E(t_i) &= \sum_{k=1}^{n} \frac{p_k}{\lambda} \left(\frac{\lambda}{\lambda + \nu_k} \right)^{r-i+1} \qquad \text{for } 0 < i < r \\
E(t_r) &= \sum_{k=1}^{n} \frac{p_k}{\lambda} \left(\frac{\lambda}{\lambda + \nu_k} \right) \\
E(t_i) &= \frac{1}{\lambda} \qquad \text{for } r < i \leq Q \\
E(t_i) &= \sum_{k=1}^{n} \frac{p_k}{\lambda} \left[1 - \left(\frac{\lambda}{\lambda + \nu_k} \right)^{Q+r-i+1} \right] \qquad \text{for } Q < i \leq Q + r.
\end{aligned}$$

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