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On lower bounded orbits of the times q-map

Jonas Lindstrøm Jensen*

Abstract

In this paper we consider the times-q map on the unit interval as a subshift of finite type by identifying each number with its base q expansion, and we study certain non-dense orbits of this system where no element of the orbit is smaller than some fixed parameter c.

The Hausdorff dimension of these orbits can be calculated using the spectral radius of the transition matrix of the corresponding subshift, and using simple methods based on Euclidean division in the integers, we completely characterize the characteristic polynomials of these matrices as well as give the value of the spectral radius for certain values of c. It is known through work of Urbanski and Nilsson that the Hausdorff dimension of the orbits mentioned above as a map of c is continuous and constant almost everywhere, and as a new result we give some asymptotic results on how this map behaves as $q \to \infty$.

1 Introduction

In this paper we study the set

$$F_c^q = \{x \in [0,1) \mid q^n x \geq c \text{ for all } n \geq 0\}$$

where $q \geq 2$ is an integer. This set is related to badly approximable numbers in diophantine approximation, and has been studied by Nilsson [2], who studied the Hausdorff dimension of the set as a map of c, and in more generality by Urbanski [4] who considered the orbit of an expanding map on the circle.

As Nilsson did we will consider F_c^q as a subshift of finite type which enables us to see it as a problem in dynamical systems. When studied as a subshift of finite type we can find the dimension of F_c^q using the spectral radius of the corresponding transition matrix, and this motivates the theorem of this paper which characterizes the characteristic polynomial of this matrix.

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2 Basic definitions

We fix an integer $q \geq 2$ and begin with the definition of part and residue which comes from elementary integer division with residue.

Proposition 1. *For integers $n \in \mathbb{N}$ and $m \geq 0$ there are unique integers $\langle n, m \rangle \in \mathbb{N}$* (part) *and $0 \leq [n, m] < q^m$* (residue) *such that*

$$n = q^m \langle n, m \rangle + [n, m].$$

We note that if we write $n = n_k \cdots n_1$ in base q it is easy to find the part and the residue, since $[n, m] = n_m \cdots n_1$ and $\langle n, m \rangle = n_k \cdots n_{m+1}$.

The matrix defined below will be of great importance in this paper, since it and its submatrices turns out to be the transition matrices of the dynamical systems we consider.

Definition 2. For $m \geq 1$ we define a 0-1 matrix A_m of size $q^m \times q^m$ by

$$(A_m)_{ij} = 1 \iff [i - 1, m - 1] = \langle j - 1, 1 \rangle.$$

We let $A_m(P)$ with $P \subseteq \{1, 2, \ldots, q^m\}$ be the $\#P \times \#P$ matrix made from picking only the rows and columns from A_m corresponding to the elements in P and for $0 \leq k \leq m$ we let $A_m(k)$ be the $m - k \times m - k$ matrix where we have removed the first k rows and columns from A_m.

We will often omit the dependency on m when it is not confusing. Considering $i - 1$ and $j - 1$ in base q we see that $(A_m)_{ij} = 1$ if and only if the first $m - 1$ digits of $j - 1$ are equal to the last $m - 1$ digits of $i - 1$. So when $c = \frac{i}{q^m}$ we see that the base q^m expansions of the numbers in F_c^q can be seen as a subshift of finite type with transition matrix $A_m(i)^m$. The metric of the subshift and the unit interval are equivalent so the dimensional properties are the same. In particular, finding the Hausdorff dimension of F_c^q now boils down to finding the spectral radius $\rho(A_m(k))$, since

$$\dim_H F(c) = \frac{\rho(A_m(i)^m)}{\log q^m} = \frac{\rho(A_m(i))}{\log q}. \tag{1}$$

The second equality is simple arithmetic, and for a proof of the first see [3]. This is the main reason we were interested in finding the characteristic polynomials of $A_m(i)$. The main theorem of this paper is a complete characterization of these polynomials, and to formulate this theorem we need the following definition.

Definition 3. For $n, m \geq 1$ with $0 \leq n < q^m$ we define

$$l_m(n) = \min\{1 \leq j \leq m \mid \langle n, j \rangle \geq [n, m - j]\}.$$

Using this definition we let

$$\overline{n}_m = n - [n, m - l_m(n)] = q^{m - l_m(n)} \langle n, m - l_m(n) \rangle$$

be the *minimal prefix* of n.

This is well defined since $[n, 0] = \langle n, m\rangle = 0$ for any n with $0 \leq n < q^m$. The notion of minimal prefix is taken from Nilsson [2], but is here defined somewhat different since we only consider finite sequences.

Let us consider some examples.

Example 4. Let $q = 3, m = 3$. Then

$$\langle 11, 1\rangle = 3 \geq 2 = [11, 2]$$

so $l_3(11) = 1$ and

$$\overline{11}_3 = 11 - [11, 2] = 9.$$

If we let $n = 7$ we have

$$\langle 7, 1\rangle = 2 < 7 = [7, 2]$$

and

$$\langle 7, 2\rangle = 0 < 1 = [7, 1]$$

but

$$\langle 7, 3\rangle = 0 = [7, 0]$$

so $l_3(7) = 3$ and $\overline{7}_3 = 7$.

We are now ready to state the main theorem.

Theorem 5. *Let $0 < i < q^m$ and let $f_i^m(x)$ be the characteristic polynomial of $A_m(i)$. Then*

$$f_i^m(x) = g_i^m(x)x^{q^m - m - i}$$

where

$$g_i^m(x) = x^m - a_1x^{m-1} - \cdots - a_m$$

and $a_1a_2 \ldots a_m$ is the base q expansion of $q^m - \bar{i}_m$.

Notice that this implies the nice equality

$$g_i^m(q) = \bar{i}_m.$$

3 Proof outline

First recall that we can find the characteristic polynomial $f(x) = x^{q^m-i} - a_1x^{q^m-i-1} - \cdots - a_{q^m-i}$ of $A_m(i)$ as

$$a_k = (-1)^k \sum_{\#P=k,\ \min P>i} \det A_m(P), \tag{2}$$

or as

$$a_k = \frac{1}{k}\big(\operatorname{trace} A_m(i)^k + a_1 \operatorname{trace} A_m(i)^{k-1} + \cdots + a_{k-1} \operatorname{trace} A_m(i)\big). \tag{3}$$

The first formula is sometimes used as the definition of the characteristic polynomial, and for a proof of the latter see [1]. We now try to outline the proof that essentially is the construction of an algorithm that calculates both the characteristic polynomial of $A_m(i)$ and $\bar{i}_m$.

- We prove that all the submatrices $A(P)$ that gives non-zero principal minors are permutations, so when removing rows and columns from the first to the last, we only change the characteristic polynomial when removing rows and columns corresponding to the smallest element of a cycle.

- If $l_m(i) = m$ then i is the smallest element of an m-cycle and this is the only permutation of size $\leq m$ that has i as an element. So removing i decreases the m'th coefficient of the characteristic polynomial by 1 and leaves all the preceding coefficients unchanged. On the other hand, if $l_m(i) = n < m$, then the nontrivial part of the characteristic polynomial, $g_i^m(x)$, can be found as $x^{m-n} g^n_{\langle i,m-n\rangle}(x)$ since we have (3) and can prove that

  $$\operatorname{trace} A_m(i)^k = \operatorname{trace} A_n(\langle i, m-n\rangle)^k$$

  for all $k \leq m$.

- If $l_m(i) = m$, then $\bar{i}_m = \overline{i+1}_m - 1$, and if $l_m(i) = n < m$ then $\bar{i}_m = \overline{q^{m-n}\langle i, m-n\rangle}_n$, so we see that $\bar{i}$ and the characteristic polynomials follow the same pattern.

- Since the theorem is true for $m = 1$, we can now use induction if $l_m(i) < m$. If not, we increase i until we have $l_m(i) < m$, which happens at some point since $l_m(q^m - 1) = 1$.

- The $m+1$'st, $m+2$'nd, $\ldots$, q^m'th coefficient of $f_i^m(x)$ are all zero, because we have found the first M coefficients of the characteristic polynomial for any M, so we pick K such that $l_M(K) = m$ and $\langle K, M-m\rangle = k$, then we see that $g_K^M(x)$ has its $m+1$'th, $m+2$'th, $\ldots$, M'th coefficients equal to zero, which will then also be true for $g_k^m(x)$. This finishes the proof of the theorem.

4 Part and residue

The results in this sections explain some properties of the part and residue functions and gives a characterization of the powers of A. We will use these results throughout the paper, often without specifically stating so. The proofs in this section are rather straightforward and may be skipped on a first read.

Proposition 6. *1. For $j, k, n \geq 0$ we have $[[n, j], k] = [n, \min\{j, k\}]$ and*

$$\big\langle \langle n, k\rangle, j\big\rangle = \langle n, k+j\rangle.$$

2. For $j > k$ we have

$$\langle [n, j], k\rangle = [\langle n, k\rangle, j-k].$$

Proof. Let us first prove the two equalities in 1. Since $[n, k]$ is the same as $n \pmod{q}^k$ we have the first equality. Now assume that $j + k \leq m$. Now $\langle n, k\rangle = q^j \big\langle \langle n, k\rangle, j\big\rangle + [\langle n, k\rangle, j]$, so

$$n = q^k \langle n, k\rangle + [n, k] = q^{k+j} \big\langle \langle n, k\rangle, j\big\rangle + q^k [\langle n, k\rangle, j] + [n, k],$$

Table 1: Calculation of the characteristic polynomials of $A_3(i)$ when $q = 3$. We let $g_i(x) = x^3 - a_1x^2 - a_2x - a_3$. We also give the minimal prefix and the length of the minimal prefix. The numbers in bold indicates that we consider a minimal number (see Definition 10) with non-maximal prefix length $l(i) < 3$.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$\bar{i}$	0	1	2	3	4	5	6	7	8	9	9	9	9	9	14	15	15	17	18	18	18	18	18	18	18	18	18
a_1	3	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	0
a_2	0	2	2	2	1	1	1	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0
a_3	0	2	1	0	2	1	0	2	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0
$l(i)$	1	3	3	2	3	3	2	3	3	1	1	1	1	1	3	2	2	3	1	1	1	1	1	1	1	1	1
$A_{i+1,i+1}$	**1**	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	1
$A^2_{i+1,i+1}$	1	0	0	**1**	0	0	**1**	0	0	0	1	0	0	1	0	0	**1**	0	0	0	1	0	0	1	0	0	**1**

but since $[\langle n,k\rangle,j] < q^j$ and $[n,k] < q^k$ we have

$$q^k[\langle n,k\rangle,j] + [n,k] \leq q^k(q^j-1) + q^k - 1 = q^{k+j} - 1 < q^{k+j},$$

and by the uniqueness of the residue and parts we see that $\langle\langle n,k\rangle,j\rangle = \langle n,k+j\rangle$. Now consider 2., so let $j > k$. From 1. we have

$$\langle n,k\rangle = q^{j-k}\big\langle\langle n,k\rangle,j-k\big\rangle + [\langle n,k\rangle,j-k] = q^{j-k}\langle n,j\rangle + [\langle n,k\rangle,j-k]$$

and

$$[n,j] = q^k\langle[n,j],k\rangle + \big[[n,j],k\big] = q^k\langle[n,j],k\rangle + [n,k].$$

So

$$\begin{aligned} n &= q^k\langle n,k\rangle + [n,k] \\ &= q^j\langle n,j\rangle + q^k[\langle n,k\rangle,j-k] - q^k\langle[n,j],k\rangle + [n,j] \\ &= q^j\langle n,j\rangle + [n,j] + q^k([\langle n,k\rangle,j-k] - \langle[n,j],k\rangle) \end{aligned}$$

and since $n = q^j\langle n,j\rangle + [n,j]$ this implies that

$$[\langle n,k\rangle,j-k] = \langle[n,j],k\rangle.$$

□

Lemma 7. *Let $1 \leq k \leq m$. Then $A^k_{ij} = 1$ if and only if*

$$[i-1,m-k] = \langle j-1,k\rangle.$$

Proof. We will prove this by induction. For $k=1$ it is the definition of A, so assume that $1 < k \leq m$. We assume that the lemma is true for all smaller k. If $A^k_{ij} = 1$ there must exist some n with $0 \leq n < q^m$ and $A_{nj} = 1$ and $A^{k-1}_{in} = 1$. Using the induction hypothesis we get

$$[i-1,m-k+1] = \langle n-1,k-1\rangle \quad \text{and} \quad [n-1,m-1] = \langle j-1,1\rangle \tag{4}$$

for this n. Now by part 2. of the above proposition we have

$$[\langle n-1,k-1\rangle,m-k] = \langle[n-1,m-1],k-1\rangle,$$

and using (4) we get

$$\big[[i-1,m-k+1],m-k\big] = \big\langle\langle j-1,1\rangle,k-1\big\rangle,$$

and using part 1. of the proposition we get

$$[i-1,m-k] = \langle j-1,k\rangle$$

as desired.

Now assume that $[i-1,m-k] = \langle j-1,k\rangle$. Let

$$n-1 = q^{k-1}[i-1,m-k+1] + [\langle j-1,1\rangle,k-1].$$

This is a positive integer smaller than q^m. By the uniqueness of the residue and parts we see that

$$[i-1, m-k+1] = \langle n-1, k-1\rangle \tag{5}$$

and

$$[\langle j-1, 1\rangle, k-1] = \langle n-1, k-1\rangle. \tag{6}$$

From (5) and the induction hypothesis we see that $A_{in}^{k-1} = 1$. We now want to prove that $A_{nj} = 1$. Recall that we assume $[i-1, m-k] = \langle j-1, k\rangle$, so

$$\begin{aligned}\langle [n-1, m-1], k-1\rangle &= [\langle n-1, k-1\rangle, m-k] \\ &= \big[[i-1, m-k+1], m-k\big] \\ &= [i-1, m-k] \\ &= \langle j-1, k\rangle.\end{aligned}$$

Using this and (6) we see that

$$\begin{aligned}[n-1, m-1] &= q^{k-1}\langle [n-1, m-1], k-1\rangle + \big[[n-1, m-1], k-1\big] \\ &= q^{k-1}\langle j-1, k\rangle + [n-1, k-1] \\ &= q^{k-1}\langle j-1, k\rangle + [\langle j-1, 1\rangle, k-1] \\ &= q^{k-1}\big\langle \langle j-1, 1\rangle, k-1\big\rangle + [\langle j-1, 1\rangle, k-1] \\ &= \langle j-1, 1\rangle.\end{aligned}$$

This proves that $A_{in}^{k-1} = 1$ and $A_{nj} = 1$ which implies that $A_{ij}^k > 0$. Now assume that there is another n' such that $A_{in'}^{k-1} = 1$ and $A_{n'j} = 1$. Then

$$[i-1, m-k+1] = \langle n'-1, k-1\rangle$$

and

$$[\langle j-1, 1\rangle, k-1] = \langle n'-1, k-1\rangle$$

so

$$\begin{aligned}n'-1 &= q^{k-1}\langle n'-1, k-1\rangle + [n'-1, k-1] \\ &= q^{k-1}[i-1, m-k+1] + \big[[n'-1, m-1], k-1\big] \\ &= q^{k-1}[i-1, m-k+1] + [\langle j-1, 1\rangle, k-1] \\ &= n-1,\end{aligned}$$

which proves that there can be only one such n, so $A_{ij}^k = 1$. $\square$

Lemma 8. *If a, b, k is such that $[a, k] < [b, k]$ and $\langle a, k\rangle = \langle b, k\rangle$, then*

$$[a, k+j] < [b, k+j]$$

for all $0 \leq j \leq m-k$.

Proof. If $\langle a,k\rangle = \langle b,k\rangle$ then

$$\big\langle \langle a,k\rangle, j\big\rangle = \big\langle \langle b,k\rangle, j\big\rangle,$$

and hence

$$\langle a, k+j\rangle = \langle b, k+j\rangle.$$

Since $a < b$ we thus have

$$[a, k+j] < [b, k+j]$$

as desired. □

5 Minimality

We now prove the following rather simple lemma which states that the only non-zero principal minors can be found as submatrices of A who are permutations.

Lemma 9. *If* $\det A(P) \neq 0$ *then the corresponding matrix is a permutation matrix.*

Proof. Assume that we choose P such that one of the rows of $A(P)$ has two ones. In other words there are $i, j_1, j_2 \in P$ such that

$$A_{ij_1} = A_{ij_2} = 1.$$

Using the definition of A this implies that

$$\langle j_1 - 1, 1\rangle = [i-1, m-1] = \langle j_2 - 1, 1\rangle.$$

Now let $k \in P$ be arbitrary. Then $A_{kj_1} = 1$ if and only if $[k-1, m-1] = \langle j_1 - 1, 1\rangle$, which is true if and only if

$$[k-1, m-1] = \langle j_2 - 1, 1\rangle,$$

so $A_{kj_1} = A_{kj_2}$ for all $k \in P$, so the j_1'th and j_2'nd column are equal and so $\det A(P) = 0$. The proof is similar when we assume that there are two ones in one column. □

Recall that if $A(P)$ is a permutation, then $P = P_1 \cup \cdots \cup P_n$ where $\cap_i P_i = \emptyset$ and $A(P_i)$'s are all cycles. This motivates the following two theorems, where we characterize the subsets P where $A(P)$ is a cycle. We are interested in the smallest elements of cycles, since the whole cycle is removed when we remove this element, which we will prove is exactly the numbers that are *minimal*.

Definition 10. We say that $0 \le n \le q^m$ is m-minimal if

$$A^{l(n)}_{n+1,n+1} = 1,$$

or equivalently using lemma 7 if

$$[n, m - l(n)] = \langle n, l(n)\rangle.$$

Theorem 11. *Let $P \subset \{1, 2, \dots, q^m\}$ be such that $A(P)$ is a k-cycle for some $1 \le k \le m$. Then $\min P - 1$ is minimal with $l_m(\min P - 1) = k$.*

Proof. Let $P = \{i_1, i_2, \dots, i_k\}$ be a k-cycle with $A^j_{i_1 i_{j+1}} = 1$ for $1 \le j < k$ and $A^k_{i_1 i_1} = 1$. Without loss of generality we can assume that $\min P = i_1$. Using lemma 7 we get that

$$[i_1 - 1, m - j] = \langle i_{j+1} - 1, j \rangle,$$

for $1 \le j < k$ and

$$[i_1 - 1, m - k] = \langle i_1 - 1, k \rangle$$

so we need to prove that $\langle i_{j+1} - 1, j \rangle > \langle i_1 - 1, j \rangle$ for $j = 1, 2, k - 1$. We have the non-strict inequality since $i_1 < i_j$. So assume for contradiction that

$$\langle i_1 - 1, j \rangle = \langle i_{j+1} - 1, j \rangle.$$

Now since $i_1 < i_{j+1}$ we have

$$[i_1 - 1, j] < [i_{j+1} - 1, j],$$

and due to lemma 8 we have

$$[i_1 - 1, m - k + j] < [i_{j+1} - 1, m - k + j] \tag{7}$$

since $k \le m$. Since $A^{k-j}_{i_{j+1} i_1} = 1$ we have $[i_{j+1} - 1, m - k + j] = \langle i_1 - 1, k - j \rangle$. Using (7) we get

$$[i_1 - 1, m - k + j] < \langle i_1 - 1, k - j \rangle.$$

Now consider i_{k-j+1}. Since $j < k$ we have $A^{k-j}_{i_1 i_{k-j+1}} = 1$ so

$$[i_1 - 1, m - k + j] = \langle i_{k-j+1} - 1, k - j \rangle,$$

and hence

$$\langle i_{k-j+1} - 1, k - j \rangle < \langle i_1 - 1, k - j \rangle.$$

This implies that $i_{k-j+1} < i_1$ which is a contradiction against i_1 being the least element in P. $\square$

Theorem 12. *Assume that $i - 1$ is minimal. Then there is a unique $P \subseteq \{1, 2, \dots, q^m\}$ such that $\min P = i$ and $A(P)$ is a $l(i - 1)$-cycle.*

Proof. We let $P = \{i, i_2, i_3, \dots, i_k\}$ where

$$\begin{aligned} i_2 - 1 &= q[i - 1, m - 1] + \langle i - 1, m - 1 \rangle \\ i_3 - 1 &= q^2[i - 1, m - 2] + \langle i - 1, m - 2 \rangle \\ &\vdots \\ i_k - 1 &= q^{k-1}[i - 1, m - k + 1] + \langle i - 1, m - k + 1 \rangle. \end{aligned}$$

We now need to prove that $A^{n-1}_{i i_n} = 1$ and that $i < i_n$ for all $n = 2, 3, \dots, k$. Using the uniqueness of the part and residue we see that

$$\langle i_n - 1, n - 1 \rangle = [i - 1, m - n + 1]$$

and

$$[i_n - 1, n - 1] = \langle i - 1, m - n + 1\rangle$$

for $n = 2, 3, \ldots, k$. The first of these equations implies that $A_{ii_n}^{n-1} = 1$.

Since $l_m(i-1) = k$ we know that

$$\langle i - 1, n\rangle < [i - 1, m - n]$$

for $n = 1, 2, \ldots, k - 1$. This implies that

$$i_{n+1} - 1 = q^n[i - 1, m - n] + \langle i - 1, m - n\rangle > q^n\langle i - 1, n\rangle + [i - 1, n] = i - 1$$

since both $\langle i - 1, m - n\rangle$ and $[i - 1, n]$ are smaller than q^n.

We now need to prove that this P is unique. Assume that we have $P' = \{i, i'_2, \ldots, i'_k\}$, where we order the elements such that $A_{ii'_n}^{n-1} = 1$. This implies that

$$[i - 1, m - n + 1] = \langle i'_n - 1, n - 1\rangle$$

for all $n = 2, 3, \ldots, k$. Since $A(P)$ is a k-cycle, we furthermore know that $A_{i'_n i}^{k-n+1} = 1$, so

$$[i'_n - 1, m - k + n - 1] = \langle i - 1, k - n + 1\rangle.$$

Now we want to prove that $i'_n = i_n$, so let $2 \leq n \leq k$ be given. We have

$$i'_n - 1 = q^{n-1}\langle i'_n - 1, n - 1\rangle + [i'_n - 1, n - 1]$$

and $\langle i'_n - 1, n - 1\rangle = [i - 1, m - n + 1]$, so we just need to prove that

$$[i'_n - 1, n - 1] = \langle i - 1, m - n + 1\rangle.$$

We have

$$\begin{aligned}
[i'_n - 1, n - 1] &= \big[[i'_n - 1, m - k + n - 1], n - 1\big] \\
&= [\langle i - 1, k - n + 1\rangle, n - 1] \\
&= \big[[i_n - 1, m - k + n - 1], n - 1\big] \\
&= [i_n - 1, n - 1] \\
&= \langle i - 1, m - n + 1\rangle
\end{aligned}$$

so $i_n = i'_n$ for all n, and so $P = P'$. □

Corollary 13. *If $l_m(i-1) = m$ then there is exactly one $P \subseteq \{1, 2, \ldots, q^m\}$ such that* $\min P = i$ *and $A(P)$ is a m-cycle.*

Proof. This follows from the fact that $A_{ij}^m = 1$ for all i, j. In particular we have $A_{ii}^m = 1$ for all i. □

Now compare this corollary with the following lemma.

Lemma 14. *If $l_m(i-1) = m$, then $\bar{i}_m = \overline{i-1}_m + 1$.*

Proof. It is enough to prove that $\bar{i} = i$, since we certainly have $\overline{i-1} = i - 1$. Using the definition we see that this is equivalent with $[i, m - l(i)] = 0$. If $l(i) = m$ we are done, so assume that $l(i) < m$. Now either $[i, m - l(i)] = 0$, in which case we are done, or $[i, m - l(i)] = [i - 1, m - l(i)] + 1$. Now since $l(i - 1) = m$ we have

$$[i - 1, m - l(i)] < \langle i - 1, l(i) \rangle,$$

since $l(i) < m = l(i - 1)$, but

$$[i - 1, m - l(i)] = [i, m - l(i)] - 1 \leq \langle i, l(i) \rangle - 1 \leq \langle i - 1, m - l(i) \rangle,$$

which is a contradiction. $\square$

Recalling the idea of the proof we here see that if $l_m(i - 1) = m$ and we remove the i'th row and column of A_m, then we remove exactly one permutation of size $\leq m$, namely a m-cycle, which increases the m'th coefficient of the characteristic polynomial by one, and we also see that it increases the m'th digit of the base q expansion of $\bar{i}$ by one.

6 Induction mapping

In the following chapter we will no longer suppress the dependency on m, since we are interested in mapping permutations between matrices of different sizes while preserving cycles. We will illustrate the idea with an example. If $q = 3$, and we write all numbers in base 3 we see that

$$012, 120, 201 \tag{8}$$

is a 3-cycle in $A_3(012)$. We now map this up to

$$0120, 1201, 2012$$

which is a 3-cycle in $A_4(0120)$. On the other hand we could also map (8) down to

$$01, 12, 20$$

which is a 3-permutation in $A_2(01)$. In this section we will formally define these maps, and also prove that they map cycles to cycles. We begin with the 'down' map which is defined in the following way.

Definition 15. If $0 \leq i < q^{m+1}$ then we define

$$D_m(i) = \langle i, 1 \rangle.$$

For $M > m$ and $0 \leq i \leq q^M$ we let

$$D_{m,M}(i) = D_m \circ \cdots \circ D_{M-1}(i) = \langle i, M - m \rangle.$$

We now prove the following lemma.

Lemma 16. *If $l_M(i) = m < M$ we have*

$$l_m(D_{m,M}(i)) = m.$$

Proof. We have $[i, M-m] \geq \langle i, m\rangle$ and $[i, M-j] < \langle i, j\rangle$ for all $1 \leq j < m$, and we need to prove that $[i, m-j] < \langle i, j\rangle$ for all $1 \leq j \leq m$. But this is clearly the case since $m < M$, so

$$[i, m-j] < [i, M-j] < \langle i, j\rangle$$

for all $1 \leq j < m$. $\square$

Corollary 17. *Let $0 \leq i < q^M$. If $l_M(i) = m < M$, then*

$$\bar{i}_M = q^{M-m}\overline{D_{m,M}(i)}_m.$$

Proof. This follows from the definition of the minimal prefix. $\square$

We saw earlier that the characteristic polynomial of a matrix can be found by considering the trace of the powers of the matrix. So if we can map permutations bijectively between two transition matrices we must have the same characteristic polynomials. As before we only need to consider cycles as all permutations are products of cycles.

First we formally define what we mean by a cycle in a matrix.

Definition 18. An ordered k-tuple of distinct elements, $(i_1, \ldots, i_k)$ with $0 \leq i_j \leq q^m$ for all $j = 1, 2, \ldots, k$ is a k-cycle in $A_m(c)$ if $A_m(c)_{i_j, i_{j+1}} = 1$ for all $j = 1, 2, \ldots, k-1$, and $A_m(c)_{i_k, i_1} = 1$. In other words, if we have

$$[i_j, m-1] = \langle i_j, 1\rangle$$

for $j = 1, 2, \ldots, k-1$ and $[i_k, m-1] = \langle i_1, 1\rangle$ and $i_j \geq c$ for all $j = 1, 2, \ldots, k$.

We have a 'down' map, mapping from large matrices to smaller and we now define an 'up' map, mapping from smaller to larger.

Definition 19. Let $P = (i_1, \ldots, i_k)$ be a k-cycle in $A_m(c)$. Then we let

$$U_m(P) = (qi_1 + [i_2, 1], \cdots, qi_k + [i_1, 1]),$$

and for $M > m$ we let $U_{m,M} = U_{M-1} \circ U_{M-2} \circ \cdots \circ U_m$.

Lemma 20. *Let $m = l_M(c)$ and let $P = (i_1, i_2, \ldots, i_k)$ be a k-cycle in $A_M(c)$. Then*

$$D_{m,M}(P) = (D_{m,M}(i_1), \cdots, D_{m,M}(i_k))$$

is a k-cycle in $A_m(D_{m,M}(c))$. Furthermore, if $Q = (j_1, \ldots, j_k)$ is a k-cycle in $A_m(D_{m,M}(c))$, then $U_{m,M}(Q)$ is a k-cycle in $A_M(c)$.

Proof. To prove that $D_{m,M}(P)$ is a k-cycle in $A_m(D_{m,M}(c))$ can be done by straightforward calculations. We also get that $U_{m,M}(Q)$ is a k-cycle in $A_M(q^{M-m}\langle c, M-m\rangle)$ rather straightforward. The problem is to prove that it actually is a k-cycle in $A_M(c)$, or in other words that there are no k-cycles with its smallest element in the interval between $q^{M-m}\langle c, M-m\rangle$ and c. Recalling the definition of $\overline{c}_M$ and that the least element of a cycle always is minimal we thus need to prove that if we have $\overline{c}_M \leq n < c$, then n cannot be minimal.

We get that $\overline{n}_M = \overline{c}_M$ and $l_M(n) = l_M(c)$ so

$$[c, M-m] - [n, M-m] = c - n$$

so if we assume that n is minimal we get

$$\langle c, m\rangle \geq [c, M-m] = [n, M-m] + c - n = \langle n, m\rangle + c - n$$

which is a contradiction. This finishes the proof of the theorem. $\square$

These two lemmas now lead to the following theorem regarding the invariance of the traces.

Theorem 21. *Let $m, k \leq M$. Then*

$$\operatorname{trace} A_m(c)^k = \operatorname{trace} A_M(q^{M-m}c)^k.$$

More generally we have

$$\operatorname{trace} A_m(\langle c, M-m\rangle)^k = \operatorname{trace} A_M(c)^k$$

whenever $l_M(c) \geq m$.

Proof. Each k-cycle contributes to the trace, and since the maps used in the lemmas maps all k-cycles injectively, we get the theorem. $\square$

Newton's formula for the characteristic polynomial gives us, that if

$$f_i^m(x) = x^n - a_1 x^{n-1} - \cdots - a_n = \det(xI - A_m(k))$$

is the characteristic polynomial of $A_m(i)$ where $n = q^m - k$, then

$$a_k = \frac{1}{k}\big(\operatorname{trace} A_m(i)^k - a_1 \operatorname{trace} A_m(i)^{k-1} - \cdots - a_{t-1} \operatorname{trace} A_m(i)\big)$$

so the above theorem gives us that

$$f_i^M(x) = x^{M-m} f^m_{q^{M-m}i}(x).$$

Combining this with the simple lemma below gives us the proof of the main theorem.

Lemma 22. *Let $0 \leq n < q^m$. Then*

$$q\overline{n}_m = \overline{qn}_{m+1}.$$

Proof. We see that

$$q\overline{n}_m = q(n - [n, m - l_m(n)]) = qn - [qn, m + 1 - l_m(n)],$$

so we just need to prove that $l_{m+1}(qn) = l_m(n)$. Assume that $j = l_m(n)$. Then

$$\langle qn, j\rangle \geq q\big\langle \langle qn, j\rangle, 1\big\rangle = q\langle qn, j+1\rangle = q\langle n, j\rangle \geq q[n, m-j] = [qn, m].$$

Now assume that $\langle qn, j\rangle \geq [qn, m+1-j]$ for some $j > l_m(n)$. Then

$$q[n, j] = [qn, j] \leq \langle qn, m+1-j\rangle$$

so

$$[n, j] \leq \langle\langle qn, m+1-j\rangle, 1\rangle = \langle n, m-j\rangle$$

which is a contradiction. □

We are now ready to prove the main theorem, so let us restate it.

Theorem 23. *Let* $1 \leq i \leq q^m$ *and let* $f_i^m(x)$ *be the characteristic polynomial of* $A_m(i)$*. Then*

$$f_i^m(x) = g_i^m(x)x^{q^m - m - i}$$

where

$$g_i^m(x) = x^m - a_1x^{m-1} - \cdots - a_m$$

and $a_1a_2\ldots a_m$ *is the base* q *expansion of* $q^m - \overline{i}_m$*.*

Proof. We prove this theorem using induction. If $m = 1$ it is certainly true since $\overline{i}_1 = i$ for all $0 \leq i < q$ and A_1 is the all one matrix of size $q \times q$.

We see that when choosing m and $i > 0$ we have two possibilities: Either we have $l(i-1) = m$ or $l(i-1) < m$. In the first case removing the i'th column and row only removes one non-zero minor, namely the unique m-cycle with i as its minimal element given in theorem 12. In this case we also have that the last digit of $\overline{i-1}_m$ is $[i-1, 1]$ which must be non-zero, so here we just decrease a_m with 1, so the first m coefficients of the characteristic polynomial changes in the right way due to lemma 14.

If we have $l(i-1) = n < m$ we see that we can find the characteristic polynomial of the smaller matrix of size q^n instead and multiply it by x^{m-n}. As we see in Corrolary 17 this is also the case for $\overline{i}$. So by induction we are done.

Now we need to prove that the remaining coefficients are all zero. To prove this we once again use lemma 21 to see that the M'th coefficient of f_i^m must be equal to the M'th coefficient of $f^M_{q^{M-m}i}$ for any $M > m$. And here we see that the $m+1$'th, $m+2$'th, $\ldots$, and M'th coefficient all are zero, since the M'th digit of the base q expansion of

$$q^M - \overline{q^{M-m}i}_M = q^{M-m}(q^m - \overline{i}_m)$$

is zero. This finishes the proof of the theorem. □

7 Constant dimension

Now define $\phi : c \mapsto \dim_H F(c)$. Recall from (1) that when c has finite base q expansion we can calculate $\phi(c)$. Nilsson [2] proved that this function is continuous and constant almost everywhere. Using the theorem we see that if we have $0 \leq i < j < q^m$ such that $\bar{i}_m = \bar{j}_m$ then

$$\phi\left(\frac{i}{q^m}\right) = \phi\left(\frac{j}{q^m}\right)$$

and since ϕ is a decreasing function it must be constant on the interval

$$\left[\frac{i}{q^m}, \frac{j}{q^m}\right].$$

Now let $0 \leq i < q$ be given and let

$$j(m) = \sum_{n=1}^{m} iq^{n-1}.$$

We now claim that

$$\overline{q^{m-1}i}_m = \overline{j(m)}_m.$$

To prove this we see that $l_m(q^{m-1}i) = 1$ and so $\overline{q^{m-1}i}_m = q^{m-1}i$. Now $l_m(j(m)) = 1$ and

$$\overline{j(m)}_m = iq^{m-1}$$

which proves the claim. This gives us

$$\phi\left(\frac{i}{q}\right) = \phi\left(\frac{j(m)}{q^m}\right)$$

for all m and letting $m \to \infty$ we get that ϕ is constant on the interval

$$\left[\frac{i}{q}, \frac{i}{q-1}\right].$$

Now letting $m = 1$ we find

$$g_i^1(x) = x - \bar{i}_1 = x - i$$

which has one root, $x = i$, so we get

$$\phi\left(\frac{i}{q}\right) = \frac{\log i}{\log q}$$

on this interval.

A bit more work allows us to calculate $\phi(x)$ for $x = \frac{i}{q^n}$ for larger n since we here need to solve polynomial equations of degree n.

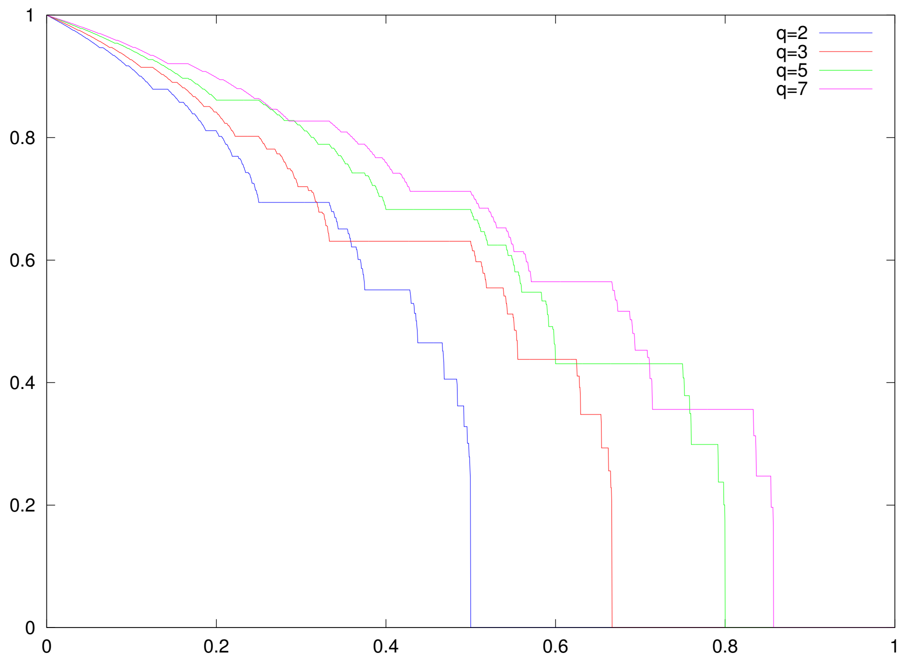

Figure 1: Numerical plots of ϕ for $q \in \{2, 3, 5, 7\}$.

8 Numerical plots

Calculating the spectral radii of $A_m(i)$, we can make numerical plots of the function ϕ. The plots in figure 1–3 was made using GNU Octave.

9 Asymptotics

We now want to consider ϕ as $q \to \infty$. We consider the function $\psi : [0, 1) \to [0, 1)$ where

$$\psi(c) = \begin{cases} 1 + \frac{\log(1-c)}{\log q} & 0 \leq c < \frac{q-1}{q} \\ 0 & \text{otherwise.} \end{cases}$$

and wish to prove that ϕ and ψ are somewhat asymptotically similar. This can also be expressed by saying that $\rho(A_c)$ behaves somewhat like $q - qc$, which is true in the starting point of the intervals where ϕ is constant, so we get the following theorem.

Theorem 24. *For all $c \in [0, 1)$ we have*

$$\frac{\phi(c)}{\psi(c)} \to 1$$

as $q \to \infty$.

Proof. Let $c \in [0, 1)$ be given. Then if we let $i = \lfloor qc \rfloor$ we have

$$\frac{i}{q} \leq c \leq \frac{i+1}{q}.$$

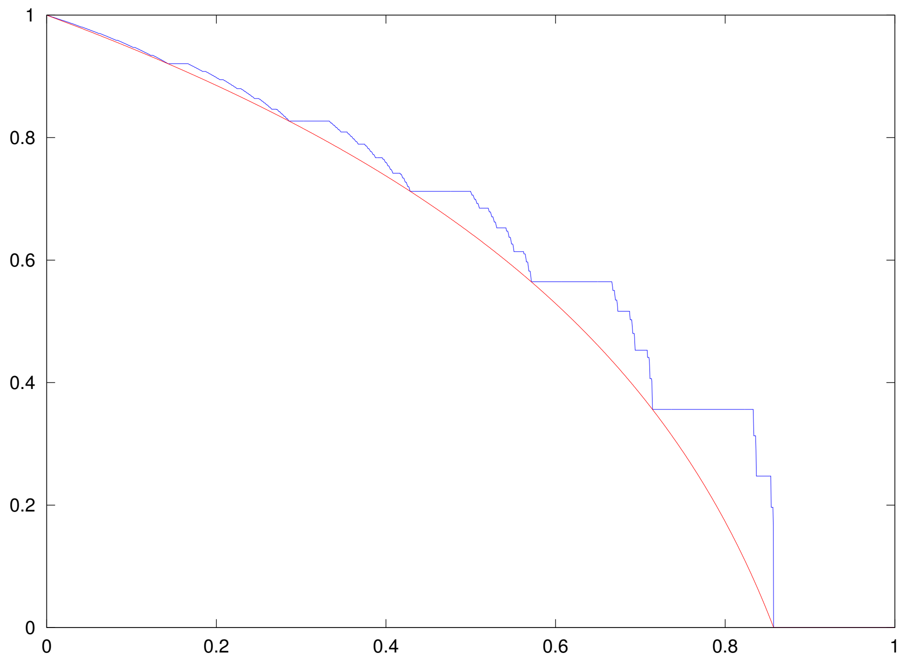

Figure 2: Plots of ϕ and ψ when $q = 7$.

Now

$$\phi\left(\frac{i}{q}\right) \geq \phi(c) \geq \phi\left(\frac{i+1}{q}\right)$$

and likewise for ψ since both functions are decreasing. Due to the result we got earlier on constant intervals we have

$$\frac{\log(q-i)}{\log q} \geq \phi(c), \psi(c) \geq \frac{\log(q+1-i)}{\log q}$$

so recalling the definition of i we have

$$\frac{\log(q-i)}{\log(q-i+1)} \geq \frac{\phi(c)}{\psi(c)} \geq \frac{\log(q-i+1)}{\log(q-i)}$$

and since $i \to \infty$ as $q \to \infty$, both the lower and upper bound converges to 1. This finishes the proof. $\square$

Since we also see that $\psi(c) \to 1$ as $q \to \infty$, we also have the following corollary.

Corollary 25. *For all $c \in [0, 1)$ we have*

$$\phi(c) \to 1 \text{ as } q \to \infty.$$

The convergence is very slow though – since ϕ and ψ are equal on q points we can just look at the convergence of

$$\frac{\log(1-c)}{\log q}$$

to zero which is easy to calculate.

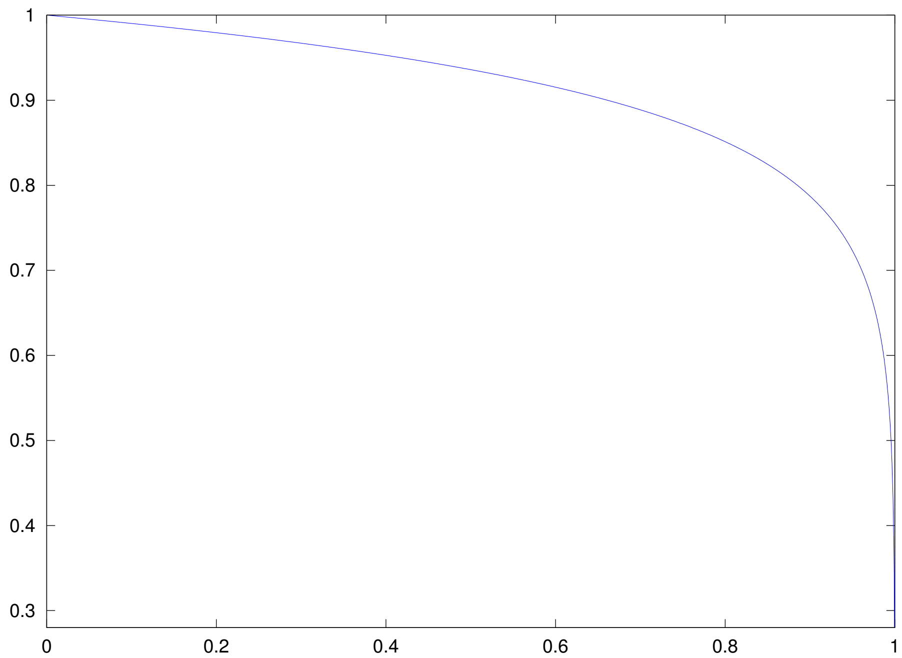

Figure 3: Plot of ϕ when $q = 50000$.

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