Cobordism Obstructions to Vector Fields and a Generalization of Lin's Theorem

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PhD Thesis

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Contents

Preface

This thesis is written for my final examn concluding four years of studies as a PhD student at the Department of Mathematical Sciences at Aarhus University. It presents the results obtained in the reaserch part of my project.

The initial topic for my project was the classical vector field problem. The idea of my first advisor, Johan Dupont, was to take up some of his early work with Atiyah, see [3] and [11]. He realized that recent work by Galatius, Madsen, Tillmann and Weiss in [14] could shed new light on his approach. Indeed, his ideas applied to give slight improvements of the old results.

Dupont retired in 2009, and Marcel Bökstedt took over as my advisor. As we got stuck with low-dimensional calculations, the project drifted in various directions in the attempt to get more general results. We started exploring the relations to cobordism theory. This resulted in a nice geometric interpretation of the theory. At some point, Bökstedt discovered an inverse limit of spectra resembling a spectrum studied by Lin in [26], and we began a study of this.

Throughout the process, my advisors have been a great help and I would like to take this opportunity to thank them. In particular, I wish to thank Marcel Bökstedt for the enormous amount of time he has spent working on the inverse limit spectrum and for encouraging me not to give up this part. I also wish to thank Ulrike Tillmann and the University of Oxford for kindly hosting my visit back in 2008.

Summary

The first chapter begins with an introduction to the vector field problem and some of the best known results. The main results in this direction obtained in the thesis are also stated. Next, some of the classical approaches to the problem are sketched and we outline our strategy. We then define the invariants we are going to consider. These are homotopy classes in the homotopy groups of certain spectra. We prove some basic properties and show how they relate to classical obstruction theory and the work of Atiyah and Dupont.

The spectra from Chapter 1 are actually linked to cobordism theory via the paper [14]. In Chapter 2, this relation is studied further. We prove that the invariants are in fact obstructions to vector fields 'up to cobordism'. We also show that the homotopy groups in which they lie have an interpretation as cobordism groups with vector fields. Finally we give another description of these cobordism groups in terms of generators and relations. The generators will be manifolds with vector fields, and the relations are given by certain cutting and glueing operations.

In Chapter 3 we study the cohomology structure of the cobordism spectra. This allows us to apply the Adams spectral sequence to perform some low-dimensional calculations of the homotopy groups in which our invariants lie. We also show that the spectra are in some sense periodic. From this, we obtain an identification of the invariant with the top obstruction to the existence of independent vector fields in certain cases and a description in terms of well-known invariants. We also apply the calculations to determine the 'vector field cobordism groups' from Chapter 2 for a small number of vector fields.

In Chapter 4 we summarize the computations from the previous chapter in a spectral sequence. The attempt to stabilize the spectral sequence using the periodicity from Chapter 3 leads to an inverse limit system of spectra. It resembles an inverse system studied by Lin. We shall prove by a topological induction argument that this is actually a generalization of his spectrum to all Thom spaces of vector bundles over a compact space.

In the last chapter we consider the inverse limit spectrum from a more algebraic viewpoint. This yields a new proof of the result from Chapter 4. In the universal situation, however, the cohomology of the inverse limit behaves completely differently. We obtain a complete description of the limit in the unoriented situation and partial results in the oriented situation. Surprisingly, it turns out that we no longer get the expected generalization of Lin's theorem.

I should mention that I worked with Dupont on Chapter 1 and parts of Chapter 3. This was also more or less the content of my progress report written for my qualifying examn in June 2009. The rest is carried out in close cooperation with Bökstedt.

Anne Marie Svane
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Chapter 1

Invariants for the Vector Field Problem

We begin this chapter by a brief introduction to the classical vector field problem and recall some of the best results known. We also state the results we are going to obtain in this thesis. Then in Section 1.2, we outline our strategy for dealing with the problem. The rest of the chapter is concerned with the set-up of the invariants we are going to consider. This is described in more detail in Section 1.2.

1.1 Introduction to the Vector Field Problem

Let M be a smooth compact d-dimensional manifold with tangent bundle TM. Then the vector field problem is the following:

Problem. Does there exist r continuous vector fields $s_1, \ldots, s_r : M \to TM$ such that $s_1(p), \ldots, s_r(p)$ are linearly independent in T_pM for all $p \in M$?

We will call such r vector fields independent. As an example, the 2-torus allows two independent vector fields:

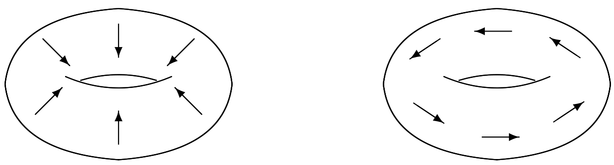

Figure 1.1: Two independent vector fields on a torus.

Of course, r independent vector fields span an r-dimensional trivial subbundle of TM, so an equivalent formulation of the problem is whether there exists a splitting $TM \cong \mathbb{R}^r \oplus E$ for some $(d-r)$-dimensional vector bundle E. A special case is when $d = r$. In this case, r independent vector fields define a trivialization of TM, and M is then said to be parallelizable. So, in some sense, the maximal number of independent vector fields says something about how trivial or non-trivial the tangent bundle is.

Though easy to state, the vector field problem turns out to be very difficult to solve. The general solution is only known for small values of r and for certain nice classes of manifolds.

As an example of how to approach this problem, let us look at the case $r = 1$. A single independent vector field is just a vector field without any zeros. In this case, the answer is known.

Theorem 1.1 (Poincaré–Hopf)**.** *A closed connected manfold M allows a vector field without zeros if and only if the Euler characteristic $\chi(M)$ is zero.*

This classical theorem was first proved by Poincaré in the 1880's for surfaces and later generalized to all manifolds by Heinz Hopf in [21].

Sketch of proof. Step 1: Suppose s is a vector field with only finitely many zeros $x_1, \ldots, x_m$. Choose small disjoint disks $D_1, \ldots, D_m$ around each x_i. Then the tangent bundle TM restricted to D_i is trivial. This means that the restriction of s to D_i may be viewed as a map $D_i \to \mathbb{R}^d$. Since this map has no zeros on the boundary, there is a map $s_i : \partial D_i \to S^{d-1}$ given by $s_i(x) = \frac{s(x)}{|s(x)|}$. Then one may define the index of s

$$\mathrm{Ind}(s) = \sum_{i=1}^{m} \deg s_i \in \mathbb{Z}. \tag{1.1}$$

One can prove, e.g. using obstruction theory, that this index is independent of the chosen sections, see Theorem 1.9.

Step 2: Assume that $\mathrm{Ind}(s) = 0$. One may choose a vector field s with only one zero x contained in a disk D. By Step 1, $s_{|\partial D} : \partial D \to S^{d-1}$ is homotopic to a constant map. But this means that $s_{|\partial D}$ extends to a non-zero map on all of D. This defines a new vector field without any zeros. Alternatively, one could refer to obstruction theory, c.f. Theorem 1.9.

Together Step 1 and 2 show that a vector field without singularities exists if and only if the index vanishes.

Step 3: $\mathrm{Ind}(s) = \chi(M)$. By Step 1, one way of proving this is by constructing a vector field with a zero of degree $(-1)^k$ at the interior of each k-simplex in a triangulation. Figure 1.2 shows how to construct such a vector field on each 2-simplex when $d = 2$. The barycentric subdivision is applied once to the triangulation, and the vector field is defined as shown on each simplex in the new triangulation. □

Example 1.2. If M is an oriented genus g surface, then $\chi(M) = 2 - 2g$. Thus M allows a nowhere zero vector field if and only if $g = 1$. But the torus has two independent vector fields, as indicated in Figure 1.1. This solves the problem completely in the oriented $d = 2$ case.

By Poincaré duality, $\chi(M) = 0$ for all closed odd dimensional manifolds. Thus every closed odd dimensional manifold has a nowhere vanishing vector field.

When $M = S^d$ is a sphere, the Poincaré–Hopf theorem is the well-known "Hairy ball theorem".

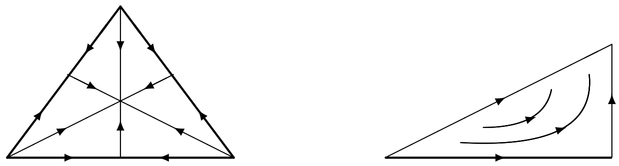

Figure 1.2: A vector field on a simplex in a triangulation

To deal with the general case of r independent vector fields, we shall try a similar approach. The existence of $r > 1$ vector fields with only finitely many singularities is no longer trivial. However, we will assume that our manifold M allows such vector fields $s_1, \dots, s_r$. In this case, obstruction theory provides a definition of an index $\mathrm{Ind}(s_1, \dots, s_r)$ as in Step 1, but now it may depend on the choice of $s_1, \dots, s_r$ so that a non-zero index does not necessarily imply the non-existence of a zero-free vector field. Step 2 still works in the sense that r vector fields without singularities do exist if and only if the index vanishes for some choice of $s_1, \dots, s_r$. Finally, the index in itself is not so easy to understand, so we would like an identification with a well-known invariant as in Step 3. The index is defined more formally in Section 1.2

The initial goal for this thesis was to find conditions under which the index is an invariant of M and in this case identify it with some more familiar invariants. Atiyah and Dupont tried this in [3]. They proved that the index is always an invariant for $r \leq 3$, and they identified the index for $r = 3$ as follows:

d mod 4	$\mathrm{Ind}(s)$
0	$\chi \oplus \frac{1}{2}(\sigma + \chi) \in \mathbb{Z} \oplus \mathbb{Z}/4$
1	$\chi_{\mathbb{R}} \in \mathbb{Z}/2$
2	$\chi \in \mathbb{Z}$
3	0

Table 1.1: $\mathrm{Ind}(s)$ for $r = 3$.

Here $\chi_{\mathbb{R}}(M) \in \mathbb{Z}/2$ is the real Kervaire semi-characteristic given by

$$\sum_k \dim_{\mathbb{R}} H^{2k}(M; \mathbb{R}) \mod 2,$$

and the signature $\sigma(M)$ is the signature of the quadratic form on $H^{\frac{d}{2}}(M; \mathbb{R})$ defined by Poincaré duality.

We are going to try a similar approach. Our main result in this direction is the following:

Theorem 1.3. *Assume $r < \frac{d}{2}$. If M is oriented of even dimension d and $r = 4, 5$ or 6, then the index is an invariant of M. For $d \equiv 2 \mod 4$,*

$$\mathrm{Ind}(s_1, \dots, s_r) = \chi(M) \in \mathbb{Z},$$

and for $d \equiv 0 \mod 4$,

$$\operatorname{Ind}(s_1, \dots, s_r) = \chi(M) \oplus \frac{1}{2}(\chi(M) + \sigma(M)) \in \mathbb{Z} \oplus \mathbb{Z}/8.$$

If M *has a spin structure, the index is an invariant for* $r \leq 6$ *and, if* d *is even, also for* $r = 7$.

This is proved in Theorem 1.31, 3.23, 3.51, and Corollary 3.33 below.

In [11], Dupont studied the obstruction to r vector fields with finitely many singularities and found the complete conditions for up to 3 independent vector fields on a manifold. These are some of the strongest general results known.

We conclude this introduction by a few examples of classes of manifolds for which the vector field problem has been solved.

The vector field problem was solved completely for the spheres by Adams in [1]. Define the function $\rho(d)$ as follows. Let 2^a be the largest power of 2 dividing $d+1$. Write $a = 4b + c$ where $0 \leq c \leq 3$. Then $\rho(d) = 2^c + 8b - 1$.

Theorem 1.4. *The maximal number of independent vector fields on* S^d *is* $\rho(d)$.

This should indicate that the general solution to the problem is complicated. The number $\rho(d)$ is related to representations of Clifford algebras, see Section 3.4.

Another class of manifolds for which the vector field problem has been solved is the class of π-manifolds. These are d-dimensional manifolds for which the stable tangent bundle is trivial. When d is odd, define the mod 2 semi-characteristic by

$$\chi_2(M) = \sum_k \dim H^{2k}(M; \mathbb{Z}/2) \mod 2.$$

Then the following result is due to Bredon and Kosinski, c.f. [9].

Theorem 1.5. *The maximal number of independent vector fields on a* d*-dimensional* π*-manifold* M *is* $\rho(d)$ *if* d *is odd and* $\chi_2(M) \neq 0$ *or if* d *is even and* $\chi(M) \neq 0$. *Otherwise* M *is parallelizable.*

For more results and conjectures about the vector field problem, see [43].

1.2 Our Approach to the Problem

As explained in the introduction, we are going to study the vector field problem by means of the index. This comes from classical obstruction theory, so we begin this section by recalling the necessary theory. The standard reference for this is [40]. We then explain the ideas of Atiyah and Dupont and outline how we are going to generalize these.

In order to introduce obstruction theory, consider the more general case of a d-dimensional oriented vector bundle $p : E \to X$ over a compact CW-complex X. We ask for r sections $s_1, \dots, s_r : X \to E$ such that $s_1(x), \dots, s_r(x)$ are linearly independent in the vector space $p^{-1}(x)$ for every $x \in X$.

Let $V_{d,r}$ denote the Stiefel manifold consisting of all ordered r-tuples $(v_1, \ldots, v_r)$ of orthonormal vectors $v_i \in \mathbb{R}^d$. If E has an inner product, there is an associated fiber bundle

$$V_{d,r} \to V_r(E) \to X.$$

The fiber over some $x \in X$ consists of r-tuples of orthonormal vectors in $p^{-1}(x)$. A section $s : X \to V_r(E)$ is the same as r orthonormal sections in E.

The idea of obstruction theory is to try to construct a section $s : X \to V_r(E)$ inductively on the skeleta X^k. A 0-cell is just a point, so at each 0-cell we may certainly choose r orthonormal vectors in the fiber. Now assume that a section is given on the k-skeleton. Let $\varphi : D^{k+1} \to X$ be the inclusion of some $(k+1)$-cell e^{k+1}. Since D^{k+1} is contractible, the pullback $\varphi^*(E)$ is isomorphic to a trivial bundle $D^{k+1} \times \mathbb{R}^d$. Hence a section on D^{k+1} is the same as a map $f : D^{k+1} \to V_{d,r}$. We already have r independent sections on ∂D^{k+1}; that is, f is already defined on the boundary. Thus $f_{|\partial D^{k+1}}$ defines class in $\pi_k(V_{d,r})$ called the local obstruction at e^{k+1}. It is possible to extend $f_{|\partial D^{k+1}}$ to all of D^{k+1} if and only if $f_{|\partial D^{k+1}}$ is null-homotopic.

The local obstructions define an element $\bar{c}^{k+1}$ in the cellular cochain group $C^{k+1}(X, \pi_k(V_{d,r}))$, namely the cochain that on each $(k+1)$-cell takes the value of the local obstruction at this cell. As explained above, the section extends to all of X^{k+1} if and only if this cochain vanishes.

Theorem 1.6. *$\bar{c}^{k+1}$ is a cocycle, and therefore it represents an element c^{k+1} in $H^{k+1}(X; \pi_k(V_{d,r}))$. The restriction of the given section to the $(k-1)$-skeleton extends to a section of the $(k+1)$-skeleton if and only if c^{k+1} is zero.*

Similarly, if the sections are already defined on a subcomplex $Y \subseteq X$ and on X^k, there is a relative obstruction class c^{k+1} in $H^{k+1}(X, Y; \pi_k(V_{d,r}))$. This is the obstruction to an extension of the sections to all of $X^{k+1} \cup Y$.

It is important to note that in general c^{k+1} is not an invariant of E. It may depend on the particular choice of sections made in the inductive construction. So $c^{k+1} \neq 0$ does not necessarily mean that r independent sections do not exist on the $(k+1)$-skeleton.

Note that, since $V_{d,r}$ is $(d-r-1)$-connected, a map $S^k \to V_{d,r}$ always extends to all of D^{k+1} when $k+1 \leq d-r$. Thus it is always possible to construct a section on the $(d-r)$-skeleton in this way. The first non-trivial obstruction occurs for the extension to X^{d-r+1}. This is called the primary obstruction, and it is independent of the choices made:

Proposition 1.7. *A section exists on the $(d-r+1)$-skeleton if and only if c^{d-r+1} is zero.*

For a compact oriented d-dimensional manifold M, let $[M] \in H^d(M, \partial M; \mathbb{Z})$ denote the fundamental class. Then there is a Poincaré isomorphism

$$\cdot \frown [M] : H^d(M, \partial M; \pi_{d-1}(V_{d,r})) \to H_0(M; \pi_{d-1}(V_{d,r})) \cong \pi_{d-1}(V_{d,r}).$$

Assume that r independent sections $s = \{s_1, \ldots, s_r\}$ are given on the boundary and the $(d-1)$-skeleton.

Definition 1.8. *Define the index* $\operatorname{Ind}(s)$ *to be the image of the top obstruction* c^d *under the Poincaré isomorphism. Then*

$$\operatorname{Ind}(s) = \sum_{i=1}^{m} O_i(s) \in \pi_{d-1}(V_{d,r})$$

where $O_i(s)$ *denotes the local obstruction at the* i*th* d*-cell.*

In the special case $r = 1$, the index is given by the formula (1.1).

Theorem 1.9. *For* $r = 1$*, the top obstruction is primary and hence an invariant of* M*. A vector field without zeros exists if and only if the index is zero.*

Remark 1.10. In general, the index is only well-defined for an oriented manifold, since a change of orientation may act non-trivially on $\pi_{d-1}(V_{d,r})$.

The index is only defined if the lower obstructions vanish. The first obstruction c^{d-r+1} on a closed manifold M is identified in [40] to be

$$c^{d-r+1} = \begin{cases} w_{d-r+1}(TM) & \text{if } d-r \text{ odd,} \\ \delta^* w_{d-r}(TM) & \text{if } d-r \text{ even.} \end{cases}$$

Here $w_i(TM)$ is the ith Stiefel–Whitney class of the tangent bundle, see Section 3.2, and $\delta^* : H^{d-r}(M;\mathbb{Z}/2) \to H^{d-r+1}(M;\mathbb{Z})$ is the Bockstein boundary map.

By Poincaré duality, the higher obstruction classes must vanish if M is $(r-2)$-connected. So for the class of closed $(r-2)$-connected oriented manifolds with $c^{d-r+1} = 0$, the index may certainly be defined.

Having defined the index, we turn to the question of whether it is an invariant of M. The idea of Atiyah and Dupont in [3] and [11] was to define a homomorphism

$$\tilde{\theta}_r^t : \pi_{d-1}(V_{d,r}) \to KR^t(tH_r)$$

such that $\tilde{\theta}_r^t(\operatorname{Ind}(s))$ is an invariant of M for a closed oriented manifold M. Hence injectivity of $\tilde{\theta}_r^t$ would imply that also $\operatorname{Ind}(s)$ were an invariant. Then they proved that $\tilde{\theta}_r^t$ is indeed injective for $r \leq 3$.

More precisely they defined a characteristic class

$$\tilde{\alpha}_{d,r}^t(E, s) \in KR^t(iE_{|X-Y} \times tH_r)$$

for a vector bundle $E \to X$ with r sections $s = \{s_1, \dots, s_r\}$ given on a subcomplex $Y \subseteq X$. Here KR-theory is K-theory of spaces with involution, and iE denotes E with the involution given in each fiber by $x \mapsto -x$. Moreover, tH_r denotes the sum of t copies of the Hopf bundle over the real projective space $\mathbb{R}P^{r-1}$ where t is any number such that 4 divides $d + t$.

For $X = M$ an oriented manifold with boundary $Y = \partial M$, the image $\tilde{\theta}_r^t(\operatorname{Ind}(s))$ is the index $\operatorname{Ind}(\tilde{\alpha}_{d,r}^t(TM, s))$ of this characteristic class. This depends only on the sections restricted to the boundary. In particular, it is the desired invariant if the boundary is empty. The Atiyah–Singer index theorem applied to give the identification of this invariant displayed in Table 1.1.

Analogously, we are going to define a characteristic class

$$\alpha^r(E,s) \in MT^n_{d,r}(\mathrm{Th}(N), \mathrm{Th}(N_{|Y}))$$

in Section 1.4. Here N is an n-dimensional complement of E and $MT^*_{d,r}$ denotes the generalized cohomology theory defined by a certain spectrum $MT(d,r)$. This spectrum will be introduced in Section 1.3.

In the special case where $M \subseteq \mathbb{R}^{n+d}$ is a compact manifold, $E = TM$ is the tangent bundle, and $Y = \partial M$ is the boundary of M, we can pull this characteristic class back by the Thom element $[M, \partial M] \in \pi_{n+d}(\mathrm{Th}(N)/\mathrm{Th}(N_{|\partial(M)}))$ and get an invariant

$$\beta^r(M,s) = \langle \alpha^r(TM,s), [M,\partial M]\rangle \in MT^{-d}_{d,r}(S^0) \cong \pi_d(MT(d,r)).$$

If M is a closed manifold, there is no dependence on s, so this defines a global invariant on M.

In Section 1.5 we define a homomorphism

$$\theta^r : \pi_{d-1}(V_{d,r}) \to \pi_d(MT(d,r))$$

such that $\theta^r(\mathrm{Ind}(s)) = \beta^r(M,s)$.

Finally, in Section 1.6, we shall see that there is actually a factorization $\tilde{\theta}^t_r = \Psi \circ \theta^r$ for a suitable map Ψ. Thus there is a hope that our invariants carry a bit more information about the index than the Atiyah–Dupont invariants do. We will see later in this thesis that indeed they do. The downside is that the groups $\pi_d(MT(d,r))$ are much harder to compute.

1.3 A Suitable Spectrum

Definition 1.11. *Let $G(d,n)$ be the Grassmann manifold. This is the set of all d-dimensional subspaces of $\mathbb{R}^{d+n}$ equipped with the topology of $O(d+n)/O(d)\times O(n)$. The vector bundle $U_{d,n} \to G(d,n)$ is the vector bundle with fiber over a plane in $G(d,n)$ consisting of all points in that plane, and $U^\perp_{d,n}$ denotes its n-dimensional orthogonal complement.*

Given a d-dimensional vector bundle over a compact space $E \to X$, there is a map $f : X \to G(d,n)$ for some large n such that E is isomorphic to the pullback $f^*U_{d,n}$. The map f is called a classifying map for E. If two maps $f_0, f_1 : X \to G(d,n)$ are homotopic, the induced pullback bundles are isomorphic. Conversely, any two maps $f_0, f_1 : X \to G(d,n)$ inducing isomophic bundles are homotopic when they are composed with the inclusion $G(d,n) \to G(d,N)$ for some sufficiently large N. Here the inclusion comes from the identification $\mathbb{R}^{N+d} = \mathbb{R}^{N-n} \oplus \mathbb{R}^{n+d}$. With this convention we define:

Definition 1.12. *Let $BO(d) = \varinjlim_n G(d,n)$. This is the classifying space for $O(d)$. Let $U_d \to BO(d)$ be the universal vector bundle $U_d = \varinjlim_n U_{d,n}$.*

Consider the spectrum $MTO(d)$ with nth space[1]

$$MTO(d)_n = \mathrm{Th}(U_{d,n}^{\perp}).$$

The spectrum maps $\Sigma MTO(d)_n \to MTO(d)_{n+1}$ are defined as follows. The inclusion $G(d,n) \to G(d,n+1)$ maps a d-plane $P \subseteq \mathbb{R}^{n+d}$ to the plane $0 \oplus P \subseteq \mathbb{R}^{1+n+d}$. The restriction of $U_{d,n+1}^{\perp}$ to $G(d,n)$ is just $\mathbb{R} \oplus U_{d,n}^{\perp}$, so there is an inclusion

$$\Sigma\,\mathrm{Th}(U_{d,n}^{\perp}) = \mathrm{Th}(\mathbb{R} \oplus U_{d,n}^{\perp}) \to \mathrm{Th}(U_{d,n+1}^{\perp}). \tag{1.2}$$

There is an inclusion of $G(d-r,n)$ into $G(d,n)$ mapping a $(d-r)$-dimensional plane P to the plane $P \oplus \mathbb{R}^r \subseteq \mathbb{R}^{n+d-r} \oplus \mathbb{R}^r$. The restriction of $U_{d,n}^{\perp}$ to $G(d-r,n)$ is exactly $U_{d-r,n}^{\perp}$. Thus there is a map

$$MTO(d-r)_n \to MTO(d)_n \tag{1.3}$$

commuting with the map in (1.2), so it defines a map of spectra. This is actually an inclusion of a closed subspectrum, so we can make the following definition:

Definition 1.13. *Denote by $MTO(d,r)$ the cofiber of* (1.3)*, i.e. the spectrum with nth space*

$$MTO(d,r)_n = \mathrm{Th}(U_{d,n}^{\perp}) / \mathrm{Th}(U_{d-r,n}^{\perp}).$$

Proposition 1.14. *$MTO(d,r)$ is a connective spectrum of finite type. It has no cells in dimensions less than or equal to $d-r$.*

Proof. With the cell structures given in [35], $G(d,n)$ and $G(d,n+1)$ share the same n-skeleton, and, since they are compact, all skeleta are finite. This shows the finite type. Also, $G(d-r,n)$ and $G(d,n)$ share the same $(d-r)$-skeleton for n large, so the quotient has no cells in dimensions less than or equal to $d-r$. □

Remark 1.15. We may replace the Grassmann manifolds in the above construction with oriented Grassmannians $G^{SO}(d,n)$. These are the simply connected double covers of $G(d,n)$ with points consisting of a subspace $V \subseteq \mathbb{R}^{d+n}$ together with an orientation. This yields an oriented version of the spectrum, which we denote by $MTSO(d,r)$.

Another possibility is to look at a spin version of the spectrum. The simply connected double cover $Spin(d) \to SO(d)$ induces a fibration of classifying spaces $p : BSpin(d) \to BSO(d)$. We get a filtration of $BSpin(d)$ by the subspaces $G^{Spin}(d,n) = p^{-1}(G(d,n))$. Now we may pull back the universal bundle to get a bundle $U_{d,n} \to G^{Spin}(d,n)$. The construction also works in this case, and the resulting spectra will be denoted $MTSpin(d,r)$.

Most results in the following work for all three spectra, so we will just write $MT(d,r)$ for the spectrum and $G(d,n)$ for the corresponding filtration of the classifying space $B(d)$ whenever there is no difference between the three cases. Only in the spin case one has to be a little careful since all constructions will be done in the oriented universal bundle first and then pulled back.

[1]This grading differs from the one in [14], but it seems more natural for our purpose.

We now give an alternative construction of the spectrum. For a d-dimensional vector bundle $p : E \to X$ with inner product, the fiber bundle $V_r(E) \to X$ was defined in Section 1.2. In particular for the universal bundle, a point in $V_r(U_{d,n})$ consists of a d-plane in $\mathbb{R}^{n+d}$ and r orthonormal vectors in that plane.

Similarly, there is a bundle $W_r(E)$ with fiber the cone on $V_{d,r}$. A point in the fiber over $x \in X$ consists of r orthogonal vectors $v_1, \ldots, v_r$ in $p^{-1}(x)$ of length $|v_1| = \cdots = |v_r| \leq 1$.

There is an inclusion $\eta^r : G(d-r,n) \to V_r(U_{d,n})$ mapping the $(d-r)$-plane P to the plane $P \oplus \mathbb{R}^r$ with the last r standard basis vectors $e_{n+d-r+1}, \ldots, e_{n+d} \in \mathbb{R}^{d+n}$ as the r-frame.

Proposition 1.16. *The map η^r can be extended to a section $\eta : G(d,n) \to W_r(U_{d,n})$ such that the following diagram commutes*

$$\begin{tikzcd} V_r(U_{d,n}) \arrow[r] & W_r(U_{d,n}) \\ G(d-r,n) \arrow[r] \arrow[u, "\eta^r"] & G(d,n). \arrow[u, "\eta"'] \end{tikzcd} \tag{1.4}$$

Proof. Observe that every point $P\oplus\mathbb{R}^r$ in $G(d-r,n)$ has a neighbourhood in $G(d,n)$ where the projection of $\mathbb{R}^{n+d}$ onto $P \oplus \mathbb{R}^r$ is an isomorphism when restricted to a plane in the neighbourhood. See [35] for a suitable description of the topology on $G(d,n)$. The union of all such sets is a neighbourhood U of $G(d-r,n)$ in $G(d,n)$. For $Q \in U$, let pr_Q denote the projection onto Q. Then $\mathrm{pr}_Q(e_{n+d-r+1}), \ldots, \mathrm{pr}_Q(e_{n+d})$ are linearly independent vectors in Q. A section on U is given by applying the Gram–Schmidt process to these vectors. Finally, multiplication by a bump function that takes the value 1 on $G(d-r,n)$ and 0 outside U yields the desired section. $\square$

The projections $p_{W_r} : W_r(U_{d,n}^\perp) \to G(d,n)$ and $p_{V_r} : W_r(U_{d,n}^\perp) \to G(d,n)$ induce a commutative diagram of bundle maps covering the diagram (1.4)

$$\begin{tikzcd} p_{V_r}^* U_{d,n}^\perp \arrow[r] & p_{W_r}^* U_{d,n}^\perp \\ U_{d-r,n}^\perp \arrow[r] \arrow[u, "\eta^r"] & U_{d,n}^\perp. \arrow[u, "\eta"'] \end{tikzcd}$$

The Thom spaces $\mathrm{Th}(p_{W_r}^* U_{d,n}^\perp)$ and $\mathrm{Th}(p_{V_r}^* U_{d,n}^\perp)$ give rise to the spectra $MT(d)_{W_r}$ and $MT(d)_{V_r}$, respectively. Let $MTV(d,r)$ denote the cofiber of the inclusion $MT(d)_{V_r} \to MT(d)_{W_r}$.

Theorem 1.17. *In the map of cofibration sequences defined by η,*

$$\begin{tikzcd} MT(d-r) \arrow[r] \arrow[d, "\eta^r"] & MT(d) \arrow[r] \arrow[d, "\eta"] & MT(d,r) \arrow[d, "\bar{\eta}"] \\ MT(d-r)_{V_r} \arrow[r] & MT(d)_{W_r} \arrow[r] & MTV(d,r), \end{tikzcd} \tag{1.5}$$

all vertical maps are homotopy equivalences.

Proof. The map $\eta : G(d,n) \to W_r(U_{d,n})$ is a homotopy equivalence, so it induces a homotopy equivalence $MT(d) \to MT(d)_{W_r}$.

There is a fibration

$$G(d-r,n) \xrightarrow{\eta^r} V_r(U_{d,n}) \to V_{n+d,r}$$

where the last map takes a plane $P \subseteq \mathbb{R}^{n+d}$ with r orthonormal vectors $v_1,\dots,v_r$ in P to the point $(v_1,\dots,v_r) \in V_{n+d,r}$. Since $V_{n+d,r}$ is $(n+d-r-1)$-connected, the pair $(G(d-r,n), V_r(U_{d,n}))$ must also be $(n+d-r-1)$-connected.

We may replace the pair $(V_r(U_{d,n}), G(d-r,n))$ by a pair $(Z, G(d-r,n))$ that is homotopy equivalent relative to $G(d-r,n)$ and such that Z and $G(d-r,n)$ have the same $(n+d-r-1)$-skeleton, c.f. [18], Corollary 4.16. Then the Thom spaces $\mathrm{Th}(U^{\perp}_{d-r,n})$ and $\mathrm{Th}(p^*_{V_r}U^{\perp}_{d,n} \to Z)$ have the same $(2n+d-r-1)$-cells, and thus they are $(2n+d-r-1)$-connected. Letting n tend to infinity, we get isomorphisms of homotopy groups

$$\eta^r_* : \pi_*(MT(d-r)) \to \pi_*(MT(d)_{V_r}).$$

But then η^r is a homotopy equivalence of spectra, see [19], Proposition 2.1.

From the long exact sequences of homotopy groups for cofibrations of spectra applied to (1.5), we see that also $\bar{\eta}$ induces an isomorphism on homotopy groups and thus is a homotopy equivalence. □

Given two spectra X and Y, let $[X,Y]$ denote the abelian group of homotopy classes of maps $f : X \to Y$ of spectra.

Corollary 1.18. *If X is any spectrum, there is an isomorphism*

$$\bar{\eta}_* : [X, MT(d,r)] \to [X, MTV(d,r)].$$

In the case $r = 1$, the spectrum has a particularly nice description. There are homeomorphisms

$$\begin{aligned} MTV(d,1)_n &= \mathrm{Th}(p^*_{W_1}U^{\perp}_{d,n} \to BU_{d,n}) / \mathrm{Th}(p^*_{V_1}U^{\perp}_{d,n} \to SU_{d,n}) \\ &\cong \mathrm{Th}(U^{\perp}_{d,n} \oplus U_{d,n}) \\ &\cong \mathrm{Th}(G(d,n) \times \mathbb{R}^{n+d}). \end{aligned}$$

Here BE and SE denote the disk and sphere bundles, respectively, of the vector bundle E. This allows us to give the following description of the homotopy groups of $MT(d,1)$:

Proposition 1.19. *There is an isomorphism*

$$\pi_q(MTV(d,1)) \cong \pi^s_q(S^d) \oplus \pi^s_{q-d}(B(d)).$$

The map induced by

$$c : \mathrm{Th}(G(d,n) \times \mathbb{R}^{n+d}) \to \mathrm{Th}(pt \times \mathbb{R}^{n+d})$$

is the projection onto the first direct summand. In particular, c induces an isomorphism

$$c_* : \pi_q(MTO(d,1)) \to \pi^s_q(S^d)$$

for $q \leq d$.

For $MTSO(d,1)$*, the last statement holds for* $q \leq d+1$*, and for* $MTSpin(d,1)$*, it is true for* $q \leq d+3$.

Proof. Note that $\mathrm{Th}(G(d,n)\times\mathbb{R}^{n+d}) = \Sigma^{n+d}(G(d,n)_+)$ where X_+ denotes the disjoint union of X and point. This is homotopy equivalent to $\Sigma^{n+d}G(d,n)\vee S^{d+n}$, and under this equivalence, c corresponds to the map that collapses $\Sigma^{n+d}G(d,n)$. Finally,

$$\varinjlim_n \pi_{n+q}(\Sigma^{n+d}G(d,n)\vee S^{n+d}) \cong \pi_q^s(S^d)\oplus\pi_{q-d}^s(BO(d))$$

where the map into the homotopy group of one summand in the wedge sum is induced by collapsing the other summand, see e.g. [19], Section 2.1.

Since $BO(d)$ is connected, $BSO(d)$ is simply connected, and $BSpin(d)$ is 3-connected, c_* is an isomorphism in the dimensions claimed. $\square$

1.4 The Global Invariants

We are now ready to intoduce the invariants promised in Section 1.2. We shall keep the notation $MT(d,r)$ for the spectrum, $B(d)$ for the corresponding classifying space, and $G(d,n)$ for the spaces in the filtration of $B(d)$. When we work with $MTSO(d,r)$ or $MTSpin(d,r)$, we implicitly assume that all bundles involved have an orientation or a spin structure, respectively.

Suppose we are given a d-dimensional vector bundle $E\to X$ over a compact q-dimensional CW complex X and r independent sections $s=\{s_1,\dots,s_r\}$ over a subcomplex $Y\subseteq X$.

A classifying map $\xi : X\to G(d,n)$ for E defines an inner product on E, and one can apply the Gram–Schmidt process to make the given sections orthonormal. Now extend the sections to r orthogonal sections on all of X, possibly with zeros. This yields a map

$$s:(X,Y)\to(W_r(E),V_r(E)). \tag{1.6}$$

Furthermore, ξ defines a map

$$(W_r(E),V_r(E))\to(W_r(U_{d,n}),V_r(U_{d,n})). \tag{1.7}$$

Let N be an n-dimensional normal bundle of E and let $(p^*_{W_r(E)}N, p^*_{V_r(E)}N)$ denote the pullbacks of N by the projections $p_{W_r(E)} : W_r(E)\to X$ and $p_{V_r(E)} : V_r(E)\to X$, respectively. Then there are bundle maps over the maps (1.6) and (1.7)

$$(N,N_{|Y}) \xrightarrow{\bar{s}} (p^*_{W_r(E)}N, p^*_{V_r(E)}N) \xrightarrow{\bar{\xi}} (p^*_{W_r}U^\perp_{d,n}, p^*_{V_r}U^\perp_{d,n}), \tag{1.8}$$

inducing a map of Thom spaces

$$(\mathrm{Th}(N),\mathrm{Th}(N_{|Y}))\to(MT(d)_{W_r,n}, MT(d)_{V_r,n}).$$

This map represents a class

$$\bar{\alpha}^r(E,s)\in[\Sigma^{\infty-n}\,\mathrm{Th}(N)/\,\mathrm{Th}(N_{|Y}), MTV(d,r)].$$

Definition 1.20. *For E with the section $s : Y \to V_r(E_{|Y})$, there is a characteristic class*

$$\alpha^r(E,s) \in [\Sigma^{\infty-n}\operatorname{Th}(N)/\operatorname{Th}(N_{|Y}), MT(d,r)] = MT^n_{d,r}(\operatorname{Th}(N), \operatorname{Th}(N_{|Y}))$$

given by $(\bar{\eta}_*)^{-1}(\bar{\alpha}^r(E,s))$ where $\bar{\eta}_*$ is the isomorphism from Corollary 1.18. Here $MT^*_{d,r}(-)$ is the generalized cohomology theory with coefficients in $MT(d,r)$.

Consider the case where $E = TM$ is the tangent bundle of a smooth manifold M with boundary $Y = \partial M$. In the following, a fundamental class for $(M, \partial M)$ will be a class $[M, \partial M] \in \pi^s_{n+d}(\operatorname{Th}(N)/\operatorname{Th}(N_{|\partial M}))$. This class is defined by embedding ∂M in $\mathbb{R}^{n+d-1}$, extending to an embedding of a collar $\partial M \times [0,1]$ in $\mathbb{R}^{n+d-1} \times [0,1]$, and finally extending this to an embedding of M in $\mathbb{R}^{n+d}$. Identify N with a tubular neighbourhood $N_\varepsilon \subseteq \mathbb{R}^{n+d}$ of M. Then the fundamental class is represented by the Pontryagin–Thom map $S^{n+d} \to \operatorname{Th}(N)/\operatorname{Th}(N_{|\partial M})$ that collapses everything not in the interior of N_ϵ to a point and then applies the identification with N.

Definition 1.21. *If M is a compact manifold with boundary ∂M and a given section $s : \partial M \to V_r(TM)_{|\partial M}$, then $\alpha(TM, s)$ can be evaluated on the fundamental class. This defines*

$$\beta^r(M,s) = \langle \alpha^r(TM,s), [M,\partial M]\rangle \in MT^{-d}_{d,r}(S^0) \cong \pi_d(MT(d,r)).$$

Generally, the evaluation of two classes is given by choosing representing maps $f : S^{l+n+d} \to \Sigma^l \operatorname{Th}(N)/\operatorname{Th}(N_{|\partial M})$ and $g : \Sigma^k \operatorname{Th}(N)/\operatorname{Th}(N_{|\partial M}) \to MT(d,r)_{n+k}$. Then the evaluation of the classes is represented by the composite map

$$\begin{aligned} S^k \wedge S^l \wedge S^{d+n} &\xrightarrow{\Sigma^k f} S^k \wedge S^l \wedge \operatorname{Th}(N)/\operatorname{Th}(N_{|\partial M}) \\ &\xrightarrow{(\Sigma^l g)\circ\sigma_{k,l}} S^l \wedge MT(d,r)_{n+k} \to MT(d,r)_{n+k+l}. \end{aligned}$$

Here $\sigma_{k,l} : S^k \wedge S^l \to S^l \wedge S^k$ permutes the factors. In our situation, we may take $k, l = 0$.

Lemma 1.22. *$\alpha^r(E,s)$ and $\beta^r(M,s)$ are independent of the choices made and depend only on $s_{|Y}$ up to homotopy through independent sections.*

Proof. First of all, $\alpha^r(E,s)$ does not depend on the extension of the section given on Y. Suppose given two extensions $s = \{s_1, \dots, s_r\}$ and $s' = \{s'_1, \dots, s'_r\}$ that agree on Y. Then there are homotopies $S_i : X \times I \to E$ between s_i and s'_i given by linear combinations. $S_1, \dots, S_r$ are fixed on $Y \times I$ and independent on an open set U containing $Y \times I$. Apply Gram–Schmidt to the sections on U and multiply by a bump function φ that takes the value 1 on $Y \times I$ and 0 outside U. This yields a homotopy $S : X \times I \to W_r(E)$ that is fixed on Y. On $X \times \{0,1\}$, the sections are already orthogonal, so Gram–Schmidt and the bump function only change them by multiplication by some functions $\varphi_1, \varphi_2 : X \to [0,\infty)$ with value 1 on Y. So S is a homotopy between $\varphi_1 \cdot s$ and $\varphi_2 \cdot s'$ that is fixed on Y. These maps are again homotopic relative to Y to s and s', respectively, so s and s' are homotopic relative to Y. Thus they define the same characteristic class.

Moreover, $\alpha^r(E,s)$ only depends on the section given on Y up to homotopy through independent sections, since one may apply Gram–Schmidt to such a homotopy to get a homotopy $Y \times I \to V_r(E)$. Then choose an extension of one of the sections and get a homotopy $(X \times I, Y \times I) \to (W_r(E), V_r(E))$ by applying the homotopy extension property. This defines a homotopy between the maps defining $\alpha^r(E,s)$.

For any two choices of classifying maps $\xi_0, \xi_1 : X \to G(d,n)$, there is a homotopy $F : X \times I \to G(d, n+k)$ between them. But then $F^*(U^\perp_{d,n+k}) \cong \mathbb{R}^k \oplus N \times I$, and thus $\mathbb{R}^k \oplus N \times I \to F^*(U^\perp_{d,n+k}) \to U^\perp_{d,n+k}$ is a homotopy through bundle maps between the maps used to define $\alpha^r(E,s)$, so it induces a homotopy of maps on Thom spaces. Furthermore, F defines a homotopy between the inner products induced on E and thus a homotopy between the orthonormalizations of the given section. So two different choices of classifying map define the same element in stable homotopy.

Finally, two different embeddings $i_0, i_1 : M \to \mathbb{R}^{n+d}$ define the same fundamental class when n is large enough. As in the proof of the Pontryagin–Thom theorem given in [41], we may choose an embedding $H : M \times I \to \mathbb{R}^{n+d} \times I$ such that $H_{|M\times 0} = i_0 \times 0$ and $H_{|M \times 1} = i_1 \times 1$. Then there is an embedding $H \times \pi_I : M \times I \to \mathbb{R}^{n+d+1} \times I$ where π_I is the projection onto I. Now apply the Thom construction to the embedding $M \times t$ for each t to get a homotopy $S^{n+d+1} \times I \to \mathrm{Th}(N)/\mathrm{Th}(N_Y) \wedge S^1$. $\square$

The sections $s_1, \dots, s_r : Y \to E$ define an isomorphism $E_{|Y} \cong E' \oplus \mathbb{R}^r$. Choose a classifying map $\xi_{|Y} : Y \to G(d-r, n)$ for E'. This extends to a classifying map $\xi : X \to G(d,n)$ for E. To see this, choose any classifying map ξ' for E. Then $\xi'_{|Y}$ is homotopic to $\xi_{|Y}$, and the homotopy extension property yields the desired map. For such a classifying map,

$$\begin{tikzcd} V_r(E) \arrow[r, "\bar{\xi}"] & V_r(U_{d,n}) \\ Y \arrow[u, "s"] \arrow[r, "\xi"] & G(d-r,n) \arrow[u, "\eta^r"'] \end{tikzcd} \tag{1.9}$$

commutes. This gives an equivalent definition of the characteristic class.

Proposition 1.23. *For a classifying map $\xi : X \to G(d,n)$ such that* (1.9) *commutes up to homotopy, $\alpha^r(E,s)$ is represented in $MT^n_{d,r}(\mathrm{Th}(N), \mathrm{Th}(N_{|Y}))$ by the map*

$$\bar{\xi} : (\mathrm{Th}(N), \mathrm{Th}(N_{|Y})) \to (\mathrm{Th}(U^\perp_{d,n}), \mathrm{Th}(U^\perp_{d-r,n}))$$

induced by $\xi : (X,Y) \to (G(d,n), G(d-r,n))$.

The following desirable properties for the characteristic class are immediate from the definition:

Proposition 1.24. *If s extends to a set of r independent sections on all of X, then $\alpha^r(E,s)$ is zero. The characteristic class is natural with respect to maps of pairs $f : (X,Y) \to (X',Y')$ in the sense that $f^*(\alpha^r(E,s)) = \alpha^r(f^*E, f^*s)$ for a bundle $E \to X'$.*

For the remainder of the thesis, we are mainly concerned with the case where M is a closed manifold, TM is the tangent bundle, and ∂M is empty. Since there is no dependence on s, $\beta^r(M,s)$ is an invariant of M that vanishes if M allows r independent sections.

Definition 1.25. *Let M be a closed manifold.*

(i) Let $\beta^r(M) = \langle \alpha^r(TM), [M] \rangle \in \pi_d(MT(d,r))$ denote the global invariant for M.

(ii) Let $\beta(M) \in \pi_d(MT(d))$ be the class represented by the map

$$S^{n+d} \to \mathrm{Th}(N) \to MT(d)_n$$

where the first map is the Pontryagin–Thom collapse and the second is the classifying map.

The cofibration sequence $MT(d-r) \to MT(d) \xrightarrow{j_r} MT(d,r)$ induces an exact sequence

$$\pi_d(MT(d-r)) \to \pi_d(MT(d)) \xrightarrow{j_{r*}} \pi_d(MT(d,r)).$$

Proposition 1.26. *$\beta^r(M) = j_{r*}(\beta(M))$ and $\beta^r(M) = 0$ if and only if $\beta(M)$ lifts to $\pi_d(MT(d-r))$.*

1.5 The Local Situation

Look at the bundle $(D^q \times \mathbb{R}^d, S^{q-1} \times \mathbb{R}^d)$. Given a section $s : S^{q-1} \to V_{d,r}$, the above construction of the characteristic class $\alpha^r(D^q \times \mathbb{R}^d, s)$ depends only on the homotopy class of s, so we can make the following definition:

Definition 1.27. *Let θ^r be the map*

$$\theta^r : \pi_{q-1}(V_{d,r}) \to MT^0_{d,r}(D^q, S^{q-1}) \cong \pi_q(MT(d,r))$$

given by

$$\theta^r([s]) = \alpha^r(D^q \times \mathbb{R}^d, s).$$

Proposition 1.28. *The map θ^r factors as the composition*

$$\pi_{q-1}(V_{d,r}) \to \pi^s_q(\Sigma V_{d,r}) \xrightarrow{f_{\theta *}} \pi_q(MTV(d,r))$$

where the first map is the map into the direct limit. This is an isomorphism for $q \leq 2(d-r)$. The second map is induced by the map of Thom spaces over the inclusion of a fiber

$$f_\theta : (W_{d,r}, V_{d,r}) \to (W_r(U_{d,n}), V_r(U_{d,n})).$$

In particular, θ^r is a homomorphism.

Proof. Let $s : S^{q-1} \to V_{d,r}$ be given and extend it to $s : D^q \to V_{d,r}$ by letting $s(x) = |x|s(\frac{x}{|x|})$. Note that the map in (1.8) defining $\alpha^r(E,s)$ is given on base spaces by

$$(D^q, S^{q-1}) \xrightarrow{id\times s} (D^q \times W_r(\mathbb{R}^d), D^q \times V_r(\mathbb{R}^d)) \xrightarrow{\xi} (W_r(U_{d,n}), V_r(U_{d,n})).$$

The classifying map ξ is constant, so this is the same as the composition

$$(D^q, S^{q-1}) \xrightarrow{s} (W_r(\mathbb{R}^d), V_r(\mathbb{R}^d)) \xrightarrow{f_\theta} (W_r(U_{d,n}), V_r(U_{d,n})).$$

On quotients, the first map is the suspension $\Sigma s : \Sigma S^{q-1} \to \Sigma V_{d,r}$, while the second map is the inclusion of a fiber. Since $V_{d,r}$ is $(d-r-1)$-connected, the first map is an isomorphism for $q \leq 2(d-r)$ by the Freudenthal suspension theorem. □

Proposition 1.29. *When $r = 1$,*

$$\theta^1 : \pi_{q-1}(S^{d-1}) \to \pi_q(MT(d,1)) \cong \pi_q^s(S^d) \oplus \pi_q^s(\Sigma^d B(d))$$

is the map into the first direct summand. Here the last isomorphism is the one from Proposition 1.19.

Proof. Since θ^1 factors as in Proposition 1.28 and the composition

$$\Sigma^{n+1}(S^{d-1}) \xrightarrow{f_\theta} MTV(d,1)_n \cong \mathrm{Th}(G(d,n) \times \mathbb{R}^{d+n}) \to S^{d+n}$$

is the identity, the result follows from Proposition 1.19. □

Proposition 1.30. *There is a cofibration sequence*

$$MT(d-r+k,k) \to MT(d,r) \to MT(d,r-k).$$

For $0 \leq k \leq r$, the following diagram of long exact sequences is commutative for $q < 2(d-r)$:

$$\begin{tikzcd}
{} \arrow[r] & \pi_q(MT(d-r+k,k)) \arrow[r] & \pi_q(MT(d,r)) \arrow[r] & \pi_q(MT(d,r-k)) \arrow[r] & {} \\
{} \arrow[r] & \pi_{q-1}(V_{d-r+k,k}) \arrow[r] \arrow[u, "\theta^k"] & \pi_{q-1}(V_{d,r}) \arrow[r] \arrow[u, "\theta^r"] & \pi_{q-1}(V_{d,r-k}) \arrow[r] \arrow[u, "\theta^{r-k}"] & {}.
\end{tikzcd}$$

The lower row is the exact sequence for the fibration

$$V_{d-r+k,k} \to V_{d,r} \to V_{d,r-k}.$$

The first map adds the last $r-k$ standard basis vectors to a k-frame as the last vectors in the frame, and the second map forgets the first k vectors of an r-frame.

Proof. The cofibration is obvious from the definitions.

By Proposition 1.28, the groups in the lower row may be replaced by the stable homotopy groups when $q \leq 2(d-r)$. Also note that $\bar{\eta}$ defines a map of pairs

$$(MT(d,r), MT(d-r+k,k)) \to (MTV(d,r), MTV(d-r+k,k))$$

so that the upper row can be replaced by the long exact sequence for the cofibration $MTV(d-r+k,k) \to MTV(d,r)$.

To see this, it is enough to check that the diagram

$$\begin{tikzcd}
(G(d-r+k,n), G(d-r,n)) \arrow[r, "\eta"] \arrow[d] & (W_k(U_{d-r+k,n}), V_k(U_{d-r+k,n})) \arrow[d] \\
(G(d,n), G(d-r,n)) \arrow[r, "\eta"] & (W_r(U_{d,n}), V_r(U_{d,n}))
\end{tikzcd} \tag{1.10}$$

commutes when η is defined as in the proof of Proposition 1.16. Going down and then right in this diagram, a plane $P \in G(d-r+k,n)$ is mapped to

$$(P \oplus \mathbb{R}^{r-k}, \psi(P \oplus \mathbb{R}^{r-k})\mathrm{pr}_{P\oplus\mathbb{R}^{r-k}}(e_{n+d-r+1}), \dots, \psi(P \oplus \mathbb{R}^{r-k})\mathrm{pr}_{P\oplus\mathbb{R}^{r-k}}(e_{n+d})).$$

Here ψ is a certain bump function on a neighbourhood of $G(d-r,n)$. The Gram–Schmidt process has been applied to the vectors before multiplying with ψ, starting with the last vector. The restriction of ψ to $G(d-r+k,n)$ is again a bump function with the right properties, so going right and then down in the diagram maps P to $P \oplus \mathbb{R}^{r-k}$ with the sections

$$(\psi(P)\mathrm{pr}_P(e_{n+d-r+1}), \dots, \psi(P)\mathrm{pr}_P(e_{n+d-r+k}), \psi(P)e_{n+d-r+k+1}, \dots, \psi(P)e_{n+d}).$$

This shows the commutativity of (1.10).

The vertical maps in the following diagram are induced by f_θ, so the diagram certainly commutes.

$$\begin{tikzcd}
\pi_q(MTV(d-r+k,k)) \arrow[r] & \pi_q(MTV(d,r)) \arrow[r] & \pi_q(MTV(d,r)/MTV(d-r+k,k)) \\
\pi_q^s(\Sigma V_{d-r+k,k}) \arrow[r] \arrow[u, "f_{\theta *}"] & \pi_q^s(\Sigma V_{d,r}) \arrow[r] \arrow[u, "f_{\theta *}"] & \pi_q^s(\Sigma V_{d,r}/\Sigma V_{d-r+k,k}) \arrow[u]
\end{tikzcd}$$

By the above, it is enough to see that the last vertical map is actually θ^{r-k} under the isomorphism for $q < 2(d-r)$,

$$\pi_q^s(\Sigma V_{d,r}/\Sigma V_{d-r+k,k}) \to \pi_q^s(\Sigma V_{d,r-k}),$$

that forgets the first k vectors. But this is true because the following diagram commutes

$$\begin{tikzcd}
 & MT(d,r-k) \arrow[r, equal] \arrow[dl, "\bar\eta"'] \arrow[d] & MT(d,r)/MT(d-r+k,k) \arrow[d, "\bar\eta"] \\
MTV(d,r-k) \arrow[r] & MTV(d,r-k)/\mathrm{Th}(U^{\perp}_{d|I\times Z}) & MTV(d,r)/MTV(d-r+k,k) \arrow[l] \\
\Sigma^\infty(\Sigma V_{d,r-k}) \arrow[r, equal] \arrow[u, "\theta"] & \Sigma^\infty(\Sigma V_{d,r-k}) \arrow[u] & \Sigma^\infty(\Sigma V_{d,r}/\Sigma V_{d-r+k,k}). \arrow[l] \arrow[u]
\end{tikzcd}$$

The upper right square is the map of Thom spaces over

$$\begin{tikzcd}
(G(d,n), G(d-r+k,n)) \arrow[r, equal] \arrow[d, "\eta"] & (G(d,n), G(d-r,n) \cup G(d-r+k,n)) \arrow[d, "\eta"] \\
(W_{r-k}(U_{d,n}), V_{r-k}(U_{d,n}) \cup Z) & (W_r(U_{d,n}), V_r(U_{d,n}) \cup W_k(U_{d-r+k,n})). \arrow[l]
\end{tikzcd}$$

This commutes up to homotopy. Note that we must divide out by

$$Z = \{e_{n+d-r+k+1}, \ldots, e_{n+d}\} \times I \times G(d-r+k,n) \subseteq W_{r-k}(U_{d,n})$$

to make the lower horizontal map well-defined. Obviously, this does not change the homotopy type.

The lower horizontal map in the large diagram comes from forgetting the first k vectors in an r-frame. The lower vertical maps come from inclusion of the fibers, so the lower squares are easily seen to commute. ☐

Theorem 1.31. *θ^1 is injective for all $q < 2(d-1)$. Assume $q < 2(d-r)-1$. Then*

$$\theta^r : \pi_{q-1}(V_{d,r}) \to \pi_q(MTO(d,r))$$

is an isomorphism for $q \leq d-r+1$. For $MTSO(d,r)$, it is an isomorphism for $q \leq d-r+2$, and for $MTSpin(d,r)$, it is an isomorphism for $q \leq d-r+4$ and injective for $q \leq d-r+6$.

Proof. The injectiveness of θ^1 follows from Proposition 1.29, and the fact that it is an isomorphism in the dimensions claimed also follows from Proposition 1.29 and Proposition 1.19.

The case of a general r follows from the diagram in Proposition 1.30 for $k=1$,

$$\begin{tikzcd}
{} \arrow[r] & \pi_q(MT(d-r+1,1)) \arrow[r] & \pi_q(MT(d,r)) \arrow[r] & \pi_q(MT(d,r-1)) \arrow[r] & {} \\
{} \arrow[r] & \pi_{q-1}(S^{d-r}) \arrow[r] \arrow[u, "\theta^1"] & \pi_{q-1}(V_{d,r}) \arrow[r] \arrow[u, "\theta^r"] & \pi_{q-1}(V_{d,r-1}) \arrow[r] \arrow[u, "\theta^{r-1}"] & {},
\end{tikzcd}$$

using the result for $r=1$, induction on r, and the 5-lemma.

Since $\pi_{q-1}(S^{d-r}) = 0$ when $q = d-r+5$ or $q = d-r+6$, the above diagram becomes

$$\begin{tikzcd}
 & \pi_q(MTSpin(d,r)) \arrow[r] & \pi_q(MTSpin(d,r-1)) \\
0 \arrow[r] & \pi_{q-1}(V_{d,r}) \arrow[r] \arrow[u, "\theta^r"] & \pi_{q-1}(V_{d,r-1}). \arrow[u, "\theta^{r-1}"]
\end{tikzcd} \tag{1.11}$$

If θ^{r-1} is injective, then θ^r must also be injective. ☐

Let us return to the vector field problem. Consider a compact connected oriented d-dimensional manifold M with tangent bundle TM. Suppose r sections $s = \{s_1, \ldots, s_r\}$ are given on M such that they are independent except at finitely many points $x_1, \ldots, x_m$ in the interior of M. Choose small disjoint disks D_j around each point x_j. The restriction of the tangent bundle to each D_j is the trivial bundle $D_j \times \mathbb{R}^d$. Thus s restricted to the boundary of D_j defines an element $[s_{|\partial D_j}] \in \pi_{d-1}(V_{d,r})$. Recall from Section 1.2 that

$$\operatorname{Ind}(s) = \sum_{j=1}^{m} [s_{|\partial D_j}] \in \pi_{d-1}(V_{d,r}).$$

Theorem 1.32. *The following formula holds in* $\pi_d(MT(d,r))$

$$\beta^r(M,s) = \sum_{j=1}^{m} \theta^r([s_{|\partial D_j}]) = \theta^r(\operatorname{Ind}(s)).$$

In particular when M is closed, the right hand side is independent of the section s.

Proof. We may assume that the r sections are orthonormal on $Y = M - \bigcup_j \operatorname{int}(D_j)$. The map in (1.8) defining $\alpha^r(TM, s_{|\partial M})$ factors as

$$\operatorname{Th}(N)/\operatorname{Th}(N_{|\partial M}) \to \operatorname{Th}(N)/\operatorname{Th}(N_{|Y}) = \bigvee_j S_j^{n+d} \to MT(d,r)_n.$$

On each S_j^{n+d}, the last map is just $f_\theta \circ \Sigma^{n+1} s_{|\partial D_j}$. The Pontryagin–Thom map

$$S^{n+d} \to \operatorname{Th}(N)/\operatorname{Th}(N_{|\partial M}) \to \operatorname{Th}(N)/\operatorname{Th}(N_{|Y}) = \bigvee_j S_j^{n+d}$$

is just the pinching map, so the evaluation is the sum of the $\theta^r([s_{|\partial D_j}])$. Thus

$$\langle \alpha(TM, s_{|\partial M}), [M, \partial M] \rangle = \sum_{j=1}^{m} \theta^r([s_{|\partial D_j}]).$$

$\square$

Remark 1.33. For $r = 1$, this is a classical result. It is shown in [4] that the composite map

$$S^{n+d} \to \operatorname{Th}(N) \to \operatorname{Th}(M \times \mathbb{R}^{d+n}) \to S^{n+d}$$

has degree equal to the Euler characteristic of M. This is also true when M is not oriented. In the above, the last map is just factored through $\operatorname{Th}(G(d,n) \times \mathbb{R}^{n+d})$.

1.6 Relation to KR-theory

In Section 1.2 we claimed that our global invariants for oriented vector bundles are refinements of the Atiyah–Dupont invariants. We shall now see how they are related and extract a few more injectivity results for θ^r.

Let $E \to X$ be an oriented vector bundle with a section $s : Y \to V_r(E)$ given on some $Y \subseteq X$. Recall that Atiyah and Dupont defined a characteristic class

$$\tilde{\alpha}^t_{d,r}(E,s) \in KR^t(iE_{|(X-Y)} \times tH_r).$$

This class is natural with respect to bundle maps covering maps of pairs. In particular, there is a universal class

$$\tilde{\alpha}^t_{d,r} \in \varprojlim_n KR^t(iU_{d,n|(G(d,n)-G(d-r,n))} \times tH_r)$$

such that the characteristic class for any other bundle is the pullback of this class by the classifying map.

Let N be an n-dimensional complement of E. We want to define a map

$$\Psi : MTSO^{n-q}_{d,r}(\mathrm{Th}(N), \mathrm{Th}(N_{|Y})) \to KR^{t-q}(iE_{|(X-Y)} \times tH_r). \tag{1.12}$$

For this, let

$$f : \Sigma^q \,\mathrm{Th}(N)/\,\mathrm{Th}(N_{|Y}) \to MT(d,r)_n$$

be any map. Let $i_{d,n} : U_{d,n} \to U_d$ be the inclusion. Then

$$i^*_{d,n}(\tilde{\alpha}^t_{d,r}) \in KR^t(iU_{d,n|(G(d,n)-G(d-r,n))} \times tH_r).$$

But there is a Thom isomorphism

$$\begin{aligned} KR^t(iE_{|(X-Y)} \times tH_r) &\cong KR^t((iE \oplus iN \oplus N)_{|(X-Y)} \times tH_r) \\ &= KR^{t+n+d}(N_{|(X-Y)} \times tH_r), \end{aligned}$$

so we have a diagram where ϕ_1, ϕ_2 are Thom isomorphisms

$$\begin{tikzcd} KR^{t-q}(iE_{|(X-Y)} \times tH_r) \arrow[r, "\phi_2"] & KR^{t+n+d-q}(N_{|(X-Y)} \times tH_r) \\ KR^t(iU_{d,n|(G(d,n)-G(d-r,n))} \times tH_r) \arrow[r, "\phi_1"] & KR^{t+n+d}(U^{\perp}_{d,n|(G(d,n)-G(d-r,n))} \times tH_r). \arrow[u, "f^*"] \end{tikzcd}$$

Now define

$$\Psi([f]) = \phi_2^{-1} \circ f^* \circ \phi_1(i^*_{d,n}(\tilde{\alpha}^t_{d,r})) \in KR^{t-q}(iE_{|(X-Y)} \times tH_r).$$

Obviously, if f comes from a classifying map $\xi : (X,Y) \to (G(d,n), G(d-r,n))$ for E, then by naturality of the Thom isomorphism, we get:

Proposition 1.34.

$$\Psi(\alpha^r(E,s)) = \xi^*(i^*_{d,n}(\tilde{\alpha}^t_{d,r})) = \tilde{\alpha}^t_{d,r}(E,s).$$

Thus Ψ reduces the invariants of Section 1.4 to the Atiyah–Dupont invariants. In particular, considering $(X,Y) = (D^q, S^{q-1})$ yields:

Theorem 1.35. *The map $\tilde{\theta}^t_r$ constructed in [3] factors as the composition*

$$\pi_{q-1}(V_{d,r}) \xrightarrow{\theta^r} \pi_q(MTSO(d,r)) \xrightarrow{\Psi} KR^{t-q}(tH_r).$$

Thus our θ^r may have a better chance of being injective than the one defined by Atiyah and Dupont. The injectivity results for $\tilde{\theta}^t_r$ given in [3], Proposition 5.6, imply:

Corollary 1.36. $\theta^r : \pi_{q-1}(V_{d,r}) \to \pi_q(MTSO(d,r))$ *is injective for* $q \leq d-r+3$ *and* $d \geq r+3$. *Moreover,* $\theta^5 : \pi_{d-1}(V_{d,5}) \to \pi_d(MTSO(d,5))$ *is injective when* $8 \mid d$.

Chapter 2

Cobordism Categories with Vector Fields

The spectra $MTO(d)$ introduced in the previous chapter are closely related to the Thom cobordism spectrum MO, and in fact, recent work [14] of Galatius, Madsen, Tillmann and Weiss shows that they do have a geometric interpretation as classifying spaces of embedded cobordism categories. We recall the classical cobordism theory in Section 2.1. In Section 2.2 we describe the embedded cobordism category and relate it to our situation. Section 2.3 discusses the fundamental group of classifying spaces of cobordism categories in general. We apply this theory to give a geometric interpretation of the higher homotopy groups of MTO as cobordism groups with vector fields. This is done in Section 2.4. As a corollary, we obtain a geometric interpretation of the invariants introduced in Chapter 1. The main theorem of the section is the following:

Theorem 2.1. *If d is odd or $r < \frac{d}{2}$, $\beta^r(M)$ vanishes if and only if M is Reinhart cobordant to a manifold that allows r independent vector fields.*

This is the content of Corollary 2.26 below. A Reinhart cobordism between M and N is a cobordism that allows a nowhere zero vector field which is inward normal at M and outward normal at N. It is proved in [38] that two manifolds are Reinhart cobordant if and only if they are cobordant and have the same Euler characteristic. If we are in the oriented category and $d \equiv 1 \bmod 4$, they must also have the same (real or mod 2) semi-characteristic.

In the last two sections, we obtain a geometric description of generators and relations for $\pi_d(MTO(d))$. The main result is the following:

Theorem 2.2. *$\pi_d(MTO(d))$ is the abelian group generated by the diffeomorphism classes $[M]$ of closed manifolds. The only relations are as follows. If W_1 and W_2 are cobordisms from $\emptyset$ to M and W_3 and W_4 are cobordisms from M to $\emptyset$, then*

$$[W_1 \cup_M W_3] + [W_2 \cup_M W_4] = [W_1 \cup_M W_4] + [W_2 \cup_M W_4].$$

The class $[M]$ corresponds to the invariant $\beta(M)$.

There is also a version of the theorem for manifolds with tangential structures under certain conditions.

2.1 Classical Cobordism Theory

We start out by recalling the classical cobordism theory. This is good to have in mind in the following. The section also serves as an introduction of notation. A good reference is [41].

Definition 2.3. *A cobordism between two closed $(d-1)$-dimensional manifolds M and N is a compact d-dimensional manifold W with boundary $\partial W = M \sqcup N$ the disjoint union of M and N. Cobordism defines an equivalence relation on the set of diffeomorphism classes of closed $(d-1)$-dimensional manifolds. The set of equivalence classes is denoted Ω^O_{d-1}.*

Disjoint union defines a commutative addition on Ω^O_{d-1}. The empty set acts as the identity, and every manifold is its own inverse, so Ω^O_{d-1} becomes an abelian group. The Cartesian product of manifolds even defines a ring structure on Ω^O_*.

More generally, let $\theta : X \to BO$ be a fibration. For a $(d-1)$-dimensional manifold M, an embedding $M \subseteq \mathbb{R}^{n+d-1}$ defines a classifying map $M \to G(d-1, n) \to BO(n)$. A θ-structure on M is now a choice of a lift of the classifying map $M \to BO(n)$ to a map $M \to \theta^{-1}(BO(n))$ up to homotopy through such lifts. The specific choice of embedding is not part of the structure, as a regular homotopy of embeddings will define a 1-1 correspondance of θ-structures.

Given two θ-manifolds M and N, a cobordism from M to N is a θ-manifold W such that the θ-structure on the boundary ∂W agrees with the structures on M and N in the following sense. Embed $W \subseteq \mathbb{R}^{n+d-1} \times [0, 1]$ in such a way that

$$W \cap \mathbb{R}^{n+d-1} \times [0, \varepsilon) = M \times [0, \varepsilon)$$
$$W \cap \mathbb{R}^{n+d-1} \times (1-\varepsilon, 1] = N \times (1-\varepsilon, 1].$$

Then the restrictions of the classifying map $W \to G(d, n) \to BO(n)$ to the boundaries define classifying maps $M, N \to G(d-1, n) \to BO(n)$. The condition for W to be a cobordism is that the lift of $W \to BO(n)$ restricts to the given θ-structures on the boundary for some n sufficiently large.

Definition 2.4. *Cobordism induces an equivalence relation on the isomorphism classes of closed $(d-1)$-dimensional manifolds with θ-structure. The set of equivalence classes is denoted $\Omega_{d-1}(X, \theta)$. Disjoint union defines a commutative sum operation. The empty set acts as the identity, making $\Omega_{d-1}(X, \theta)$ into an abelian group.*

In particular, an inverse of some $[M]$ is given by assigning to $M \times [0, 1]$ the θ-structure defined by M. Then embed $M \times [0, 1] \subseteq \mathbb{R}^{n+d-1} \times [0, 1]$ such that $M \times \{0, 1\} \subseteq \mathbb{R}^{n+d-1} \times \{0\}$. This defines a θ-structure on $M \times \{1\}$, and this represents an inverse of $[M]$.

Remark 2.5. In fact, a cobordism W has an opposite structure defined in a similar way, making cobordism reflexive and hence an actual equivalence relation (transitivity given by glueing cobordisms together). Note that this only works because we consider the classifying map $W \to BO(n)$, rather than $W \to G(d, n)$.

Example 2.6. We are particularly interested in the cases where X is BO, BSO, or $BSpin$. In these cases, $\Omega_*(X, \theta)$ will also be denoted Ω_*^O, Ω_*^{SO}, or Ω_*^{Spin}, respectively. The Cartesian product of two manifolds induces a ring structure on $\Omega_*(X, \theta)$ in this case.

The main theorem of cobordism theory is the Pontryagin–Thom theorem. This expresses the cobordism groups as the homotopy groups of Thom spectra as follows. Let $M \subseteq \mathbb{R}^{n+d-1}$ be a manifold with a lift $M \to \theta^{-1}(BO(n))$ of the classifying map. Then the collapse of everything outside a tubular neighbourhood N followed by the lift of the classifying map defines a map

$$(\mathbb{R}^{n+d-1})^+ \to \mathrm{Th}(N) \to \mathrm{Th}(\theta^* U_n).$$

One can show that this defines a map

$$\Omega_{d-1}(X, \theta) \to \pi_{d-1}(\theta^* MO) \tag{2.1}$$

where $\theta^* MO$ is the spectrum formed by the Thom spaces $\mathrm{Th}(\theta^* U_n)$. In particular, the pullbacks to BSO and $BSpin$ are denoted MSO and $MSpin$, respectively.

Theorem 2.7 (Pontryagin–Thom)**.** *The map* (2.1) *is an isomorphism.*

Remark 2.8. Note how this construction is closely related to the construction of the invariant $\beta(M)$. In fact, $\mathrm{Th}(U_n) = \varinjlim_d \mathrm{Th}(U_{d,n}^\perp)$ so $MO = \varinjlim_d MTO(d)$. We will often use the notation $MTO = \varinjlim_d MTO(d)$, rather than MO, to emphasize that we think of U_n as the complement of the universal bundle.

The inclusion $MTO(d) \to MTO$ is d-connected since the map $G(d, n) \to BO(n)$ is d-connected for n sufficiently large. The induced map $\pi_d(MTO(d)) \to \pi_d(MTO)$ takes $\beta(M)$ to the cobordism class of M.

Thus we would expect some relation between our invariants and cobordism theory. To investigate this, we need a slightly different definition of cobordism.

2.2 The Embedded Cobordism Category

We now introduce the topological category of embedded cobordisms and state some main results. This is the suitable cobordism category for our purpose. We follow the definition given in [15].

Definition 2.9. *Let $\theta : X \to BO(d+l)$ be a fibration. As a set, $\Psi_{\theta_d}(\mathbb{R}^{n+d})$ consists of all pairs $(M, \bar{\xi})$ where $M \subseteq \mathbb{R}^{n+d}$ is an embedded d-dimensional manifold without boundary such that M is a closed subset of $\mathbb{R}^{n+d}$ and $\bar{\xi}$ is a lift under θ of the classifying map $\xi : M \to G(d, n) \to G(d+l, n)$. A suitable topology is given in [15].*

Let $\psi_{\theta_d}(n+d, k)$ denote the subspace consisting of those M that are contained in $(-1, 1)^{n+d-k} \times \mathbb{R}^k$.

Definition 2.10. *Let $\theta : X \to BO(d)$ be a fibration. The cobordism category $\mathcal{C}_{d,n+d}^\theta$ has objects $\psi_{\theta_{d-1}}(n+d-1, 0)$. The space of morphisms is the disjoint union of the identity morphisms and a subspace of $\psi_{\theta_d}(n+d, 1) \times (0, \infty)$. A morphism*

$(W,a) \in \mathcal{C}^\theta_{d,n+d}(M_0,M_1)$ from M_0 to M_1 is a $W \in \psi_{\theta_d}(n+d,1)$ such that for some $\varepsilon > 0$,

$$W \cap (\mathbb{R}^{n+d-1} \times (-\infty,\varepsilon)) = M_0 \times (-\infty,\varepsilon)$$
$$W \cap (\mathbb{R}^{n+d-1} \times (a-\varepsilon,\infty)) = M_1 \times (a-\varepsilon,\infty)$$

such that the θ-structures agrees. Composition of the morphisms (W,a) and (W',a') is given by $(W \circ W', a+a')$ where $W \circ W'$ is obtained from $W \cap (\mathbb{R}^{n+d-1} \times (-\infty,a])$ and $W' \cap (\mathbb{R}^{n+d-1} \times [0,\infty)) + ae_{n+d}$ by glueing together.

We will often leave the a out of the notation for the morphisms when it plays no significant role. Usually we let n tend to infinity and denote the resulting category by $\mathcal{C}^\theta_d$ with obejcts $\mathrm{Ob}(\mathcal{C}^\theta_d)$ and morphisms $\mathrm{Mor}(\mathcal{C}^\theta_d)$.

For a topological category, the classifying space $B\mathcal{C}^\theta_d$ is defined as follows. Let $N_k(\mathcal{C}^\theta_d)$ be the subspace of $\mathrm{Mor}(\mathcal{C}^\theta_d)^k$ consisting of k-tuples of composable morphisms. This is called the kth nerve of the category. In particular, $N_0(\mathcal{C}^\theta_d)$ is the space of objects. The set of nerves form a simplicial set with continuous face and degeneracy operators given by composing morphisms and inserting identity morphisms, respectively. See e.g. [16] for details.

Definition 2.11. *$B\mathcal{C}^\theta_d$ is the topological space*

$$\bigsqcup N_k(\mathcal{C}^\theta_d) \times \Delta^k / \sim .$$

Here $\bigsqcup$ is disjoint union, Δ^k is the standard k-simplex, and the equivalence relation is given by the face and degeneracy operators, see [31] for the precise relations.

The main theorem about the cobordism category, proved in [14] and, in our set-up, in [15], is the following:

Theorem 2.12. *There is a weak homotopy equivalence*

$$\alpha_{d,\theta} : B\mathcal{C}^\theta_d \to \Omega^{\infty+d-1}\theta^* MTO(d).$$

We will be particularly interested in the fibration

$$i_r : V_r(U_d) \to BO(d).$$

The corresponding category will be denoted $\mathcal{C}^r_d$. The objects are embedded compact $(d-1)$-dimensional manifolds M with r vector fields in $TM \oplus \mathbb{R}$. The morphisms are cobordisms with r tangent vector fields extending the ones given on the boundary. Of course, there is also an oriented version of this theory. We will use the same notation when there is no essential difference.

Reading through the definition of the map $\alpha_{d,\theta}$, one sees that there is a commutative diagram

$$\begin{tikzcd} B\mathcal{C}^r_d \arrow[r, "BF"] \arrow[d, "\alpha_{d,r}"] & B\mathcal{C}_d \arrow[d, "\alpha_d"] \\ \Omega^{\infty+d-1} i_r^* MT(d) \arrow[r, "i_r"] & \Omega^{\infty+d-1} MT(d). \end{tikzcd} \tag{2.2}$$

Here BF is induced by the functor F that forgets the tangential structure.

By Theorem 1.17, $i_r^* MT(d) = MT(d)_{V_r}$ is homotopy equivalent to $MT(d-r)$. Recall from Proposition 1.26 that in the exact sequence

$$\pi_d(MT(d-r)) \xrightarrow{i_r} \pi_d(MT(d)) \xrightarrow{j_r} \pi_d(MT(d,r)),$$

the invariant $\beta(M) \in \pi_d(MT(d))$ reduces to $\beta^r(M) \in \pi_d(MT(d,r))$. Thus $\beta^r(M)$ vanishes if and only if $\beta(M)$ lifts to $\pi_d(MT(d-r))$. The diagram (2.2) translates this into a study of the map

$$\pi_1(B\mathcal{C}_d^r) \to \pi_1(B\mathcal{C}_d).$$

The study of these fundamental groups is the topic for the rest of this chapter.

2.3 Representing Classes in $\pi_1(B\mathcal{C}_d^\theta)$ by Morphisms

In this section we consider a general cobordism category corresponding to a fibration $\theta : X \to BO(d)$.

Definition 2.13. *Let (W, a) be a morphism in $\mathcal{C}_d^\theta$ from M_0 to M_1. The 1-simplex $\{(W, a)\} \times \Delta^1$ inside $B\mathcal{C}_d^\theta$ defines a path $\gamma_{(W,a)}$ between the objects M_0 and M_1. A composition of such paths and their inverses (denoted $\bar{\gamma}_{(W,a)}$) will be called a zigzag of morphism paths.*

The goal of this section is to find conditions on the category $\mathcal{C}_d^\theta$ such that all elements of $\pi_1(B\mathcal{C}_d^\theta)$ can be represented by a morphism path $\gamma_{(W,a)}$ of some closed manifold W, considered as a morphism from the empty manifold to itself.

We start out by showing that elements of $\pi_1(B\mathcal{C}_d^\theta)$ are represented by zigzags.

First some notation. A path $\alpha : [0,1] \to \mathrm{Ob}(\mathcal{C}_d^\theta)$ that is smooth in the sense of [15] and constant near the endpoints determines a morphism $(W_\alpha, 1)$ such that $W_\alpha \subseteq \mathbb{R}^{\infty-1} \times \mathbb{R}$ with $W_\alpha \cap (\mathbb{R}^{\infty-1} \times \{t\}) = \alpha(t)$ for all $t \in [0,1]$ and such that the projection $W_\alpha \to \{0\} \times \mathbb{R}$ is a submersion. We want to see that $\alpha \simeq \gamma_{W_\alpha}$ in $B\mathcal{C}_d^\theta$.

Lemma 2.14. *A non-identity morphism path $\gamma_{(W,a)}$ followed by a smooth path that is constant near endpoints, $\alpha : I \to \mathrm{Ob}(\mathcal{C}_d^\theta)$, is homotopic relative to endpoints inside $\mathrm{Ob}(\mathcal{C}_d^\theta) \cup (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1)$ to the morphism path $\gamma_{(W \circ W_\alpha, a+1)}$.*

Proof. Recall from [15], Theorem 3.9, that there are homotopy equivalences of categories

$$\mathcal{C}_d^\theta \xleftarrow{c} D_\theta^\perp \xrightarrow{i} D_\theta. \tag{2.3}$$

The precise definitions of the categories $D_\theta^\perp$ and D_θ are given in [15], Definition 3.8.

The morphism $(W \circ W_\alpha, a+1)$ corresponds to the morphism $(W \circ W_\alpha, 0 \leq a+1)$ in the category D_θ. In $\mathrm{Mor}(D_\theta)$, there is a path

$$t \mapsto (W \circ W_\alpha, 0 \leq a+t)$$

from $(W \circ W_\alpha, 0 \leq a)$ to $(W \cup W_\alpha, 0 \leq a+1)$ because all $a+t \in [a, a+1]$ are regular values for the projection $W \circ W_\alpha \to \{0\} \times \mathbb{R}$.

This lifts to a path $(W_t, 0 \leq a+t)$ in $\mathrm{Mor}(D_\theta^\perp)$ under the inverse of the homotopy equivalence i. This just stretches $W \circ W_\alpha$ near $W \circ W_\alpha \cap \mathbb{R}^{\infty-1} \times \{a+t\}$ and leaves the rest fixed. In particular $W_t \cap (\mathbb{R}^{\infty-1} \times \{a+t\}) = \alpha(t)$.

Now, $c(W_t, 0 \leq a+t) = (W'_t, a+t)$ defines a path from (W, a) to $(W \circ W_\alpha, a+1)$ with $W'_t \cap (\mathbb{R}^{\infty-1} \times \{a+t\}) = \alpha(t)$. Thus $\gamma_{(W'_t, a+t)} \cdot \alpha_{|[t,1]}$ is a homotopy from $\gamma_W \cdot \alpha$ to $\gamma_{W \circ W_\alpha}$. □

Lemma 2.15. *A smooth path $\alpha : [0,1] \to \mathrm{Ob}(\mathcal{C}_d^\theta)$ is homotopic relative to endpoints to the morphism path γ_{W_α} inside $B\mathcal{C}_d^\theta$.*

Proof. Let $M = \alpha(0)$. Then $M \times \mathbb{R}$ is a morphism. By Lemma 2.14, $\gamma_{M \times \mathbb{R}} \cdot \alpha$ is homotopic to $\gamma_{(M \times \mathbb{R}) \circ W_\alpha}$. Furthermore, there is a 2-simplex in the classifying space making $\gamma_{(M \times \mathbb{R}) \circ W_\alpha}$ homotopic to $\gamma_{M \times \mathbb{R}} \cdot \gamma_{W_\alpha}$. Composition with $\bar{\gamma}_{M \times \mathbb{R}}$ shows that α is homotopic to γ_{W_α}. □

Theorem 2.16. *Any path between two objects in $B\mathcal{C}_d^\theta$ is homotopic relative to endpoints to a zigzag of morphism paths.*

Proof. Let $f : [0,1] \to B\mathcal{C}_d^\theta$ be given such that $f(\{0,1\}) \subseteq \mathrm{Ob}(\mathcal{C}_d^\theta)$. We may deform f to have image in $\mathrm{Ob}(\mathcal{C}_d^\theta) \cup (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1)$, so assume this is the case.

Since $f^{-1}(\mathrm{Mor}(\mathcal{C}_d^\theta) \times \mathrm{int}(\Delta^1))$ is open, it can be written as the disjoint union $\bigsqcup_{l \in J} I_l$ of open intervals $I_l \subseteq (0,1)$. Write $f_l : \bar{I}_l \to \mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1$ for the restriction of f to the closure of I_l. This has the form

$$f_l(t) = ((W_t, a(t)), g(t)).$$

Let $\bar{I}_l = [t_1, t_2]$. Then $g(t_1), g(t_2) \in \{0,1\} = \partial(\Delta^1)$.

If $g(t_1) = g(t_2)$, then g is homotopic relative to the endpoints to a constant map. Thus f_l is homotopic to a map into the object space, and we may remove all such l.

Now assume $g(t_1) = 0$ and $g(t_2) = 1$. Then g is homotopic relative to endpoints to a map which is linear on $[t_1, \frac{t_1+t_2}{2}]$ and constant on $[\frac{t_1+t_2}{2}, t_2]$. The function $t \mapsto (W_t, a(t))$ is homotopic to a map which is constant equal to $(W_0, a(0))$ on $[t_1, \frac{t_1+t_2}{2}]$ and given by $t \mapsto (W_{2t-t_2}, a(2t-t_2))$ on $[\frac{t_1+t_2}{2}, t_2]$. That is, f_l is homotopic to $\gamma_{(W_0, a(0))}$ followed by a path in the object space. We replace I_l by $(t_1, \frac{t_1+t_2}{2})$.

The case $g(t_1) = 1$ and $g(t_2) = 0$ is similar except that γ_{W_0} is travelled backwards.

Thus we may assume that $f_l(t) = ((W_l, a_l), g(t))$ where g is a linear homeomorphism taking $\bar{I}_l$ to $[0,1]$ for all $l \in J$. The index set J must be finite. To see this, choose $t_l \in I_l$ such that $f(t_l) = ((W_l, a_l), \frac{1}{2})$. Then $\{t_l, l \in J\}$ is a closed discrete subset of $[0,1]$ and hence finite.

It follows that $f^{-1}(\mathrm{Ob}(\mathcal{C}_d^\theta))$ is a finite disjoint union of closed intervals. It is now enough to show that a map $\alpha : [0,1] \to \mathrm{Ob}(\mathcal{C}_d^\theta)$ is homotopic to a morphism path. But by [15], Lemma 2.18, α is homotopic to a smooth path α' which is constant near the endpoints. Finally, by Lemma 2.15, this is homotopic to the morphism path $\gamma_{W_{\alpha'}}$. □

We are now ready to give conditions under which every element of $\pi_1(B\mathcal{C}_d^\theta)$ may be represented by a single morphism path of a closed manifold.

Theorem 2.17. *Assume:*

(i) Any morphism $W \in \mathcal{C}_d^\theta(\emptyset, \emptyset)$ has an inverse $W^- \in \mathcal{C}_d^\theta(\emptyset, \emptyset)$ such that the disjoint union $W \sqcup W^-$ defines a null-homotopic loop in $B\mathcal{C}_d^\theta$.

(ii) If $W \in \mathcal{C}_d^\theta(M_0, M_1)$, then there exists a morphism $\overline{W} \in \mathcal{C}_d^\theta(M_1, M_0)$ in the opposite direction.

Then any element of $\pi_1(B\mathcal{C}_d^\theta)$ can be represented by a path corresponding to a morphism in $\mathcal{C}_d^\theta(\emptyset, \emptyset)$.

From Theorem 2.16 we know that we can always represent an element of $\pi_1(B\mathcal{C}_d^\theta)$ as a zigzag of morphisms. In general, the closed manifold cannot be chosen diffeomorphic to the one defined by glueing together the morphisms in the zigzag, since this may not allow a θ-structure.

Proof. Let $\gamma : I \to B\mathcal{C}_d^\theta$ be a path from the empty manifold to itself. By Theorem 2.16, we can assume that γ has the form $g_1 \cdot g_2 \cdots g_n$, where g_i is the path γ_{W_i} or the inverse path for some W_i's such that the composition makes sense.

First look at the path $\gamma_{W_i} \cdot \gamma_{W_{i+1}}$ for a pair of morphisms $W_i \in \mathcal{C}_d^\theta(M_i, M_{i+1})$ and $W_{i+1} \in \mathcal{C}_d^\theta(M_{i+1}, M_{i+2})$. The composition forms a new morphism $W_i \circ W_{i+1}$ in $\mathcal{C}_d^\theta(M_i, M_{i+2})$. The 2-simplex $\{(W_i, W_{i+1})\} \times \Delta^2$ inside $B\mathcal{C}_d^\theta$ defines a homotopy

$$\gamma_{W_i} \cdot \gamma_{W_{i+1}} \simeq \gamma_{W_i \circ W_{i+1}}. \tag{2.4}$$

Thus we may assume that γ is an alternating zigzag of morphism paths

$$\gamma = \gamma_{W_1} \cdot \bar{\gamma}_{W_2} \cdot \gamma_{W_3} \cdots \bar{\gamma}_{W_n}$$

where $W_i \in \mathcal{C}_d^\theta(M_i, M_{i+1})$ for i odd and $W_i \in \mathcal{C}_d^\theta(M_{i+1}, M_i)$ for i even. Of course, it could also happen that the first path is an inverse path or that n is odd. These cases are similar.

For each i, choose an opposite $\overline{W}_i$ of W_i as in assumption (ii). Then

$$\begin{aligned}
\gamma =& \gamma_{W_1} \cdot \bar{\gamma}_{W_2} \cdot \gamma_{W_3} \cdots \bar{\gamma}_{W_n} \\
\simeq& \gamma_{W_1} \cdot (\gamma_{\overline{W}_2} \cdot \gamma_{W_3} \cdots \gamma_{\overline{W}_n}) \cdot \overline{(\gamma_{\overline{W}_2} \cdot \gamma_{W_3} \cdots \gamma_{\overline{W}_n})} \cdot \bar{\gamma}_{W_2} \cdot (\bar{\gamma}_{\overline{W}_3} \cdots \bar{\gamma}_{W_n}) \\
& \cdot \overline{(\bar{\gamma}_{\overline{W}_3} \cdots \bar{\gamma}_{W_n})} \cdots \gamma_{W_{n-1}} \cdot (\gamma_{\overline{W}_n}) \cdot \overline{(\gamma_{\overline{W}_n})} \cdot \bar{\gamma}_{W_n} \\
=& (\gamma_{W_1} \cdot \gamma_{\overline{W}_2} \cdot \gamma_{W_3} \cdots \gamma_{\overline{W}_n}) \cdot (\bar{\gamma}_{\overline{W}_n} \cdots \bar{\gamma}_{W_3} \cdot \bar{\gamma}_{\overline{W}_2} \cdot \bar{\gamma}_{W_2} \cdot \bar{\gamma}_{\overline{W}_3} \cdots \bar{\gamma}_{W_n}) \\
& \cdot (\gamma_{W_n} \cdots \gamma_{\overline{W}_3} \cdots) \cdots (\cdots \gamma_{W_{n-1}} \cdot \gamma_{\overline{W}_n}) \cdot (\bar{\gamma}_{\overline{W}_n} \cdot \bar{\gamma}_{W_n}) \\
\simeq& (\gamma_{W_1 \circ \overline{W}_2 \circ W_3 \circ \cdots \circ \overline{W}_n}) \cdot (\bar{\gamma}_{\overline{W}_n \circ \cdots \circ W_3 \circ \overline{W}_2 \circ W_2 \circ \overline{W}_3 \circ \cdots \circ W_n}) \cdot (\gamma_{W_n \circ \cdots \circ \overline{W}_3 \cdots}) \cdots \\
& \cdot (\gamma_{\cdots \circ W_{n-1} \circ \overline{W}_n}) \cdot (\bar{\gamma}_{\overline{W}_n \circ W_n}).
\end{aligned}$$

The idea is here that we first run along γ_{W_1} as we are supposed to. Then we follow morphism paths in the positive direction all the way to the empty manifold and go back again, now following paths in the inverse direction. We run γ_{W_2} backwards as we are supposed to and then follow paths in the inverse direction back to the base point and go back again. Continuing this way, we end up with a path that is homotopic to the original one. But this new path has the property that we always run from the base point to itself along paths that are either all positively directed or all negatively directed. Thus we may glue the manifolds together by (2.4).

This yields an expression for γ involving only paths of closed manifolds. We now apply assumption (i) to all the paths that are still travelled in the wrong direction. Finally we apply (2.4) again to write γ as a path corresponding to a single morphism.

$$\begin{aligned}\gamma \simeq &\gamma_{(W_1 \circ \overline{W}_2 \circ W_3 \circ \cdots \circ \overline{W}_n)} \cdot \gamma_{(\overline{W}_n \circ \cdots \circ W_3 \circ \overline{W}_2 \circ W_2 \circ \overline{W}_3 \circ \cdots \circ W_n)^-} \cdot \gamma_{(W_n \circ \cdots \circ \overline{W}_3 \cdots)} \cdots \\ &\cdot \gamma_{(\cdots \circ W_{n-1} \circ \overline{W}_n)} \cdot \gamma_{(\overline{W}_n \circ W_n)^-} \\ \simeq &\gamma_{(W_1 \circ \overline{W}_2 \circ W_3 \circ \cdots \circ \overline{W}_n) \circ (\overline{W}_n \circ \cdots \circ W_3 \circ \overline{W}_2 \circ W_2 \circ \overline{W}_3 \circ \cdots \circ W_n)^- \circ (W_n \circ \cdots \circ \overline{W}_3 \cdots) \circ \cdots} \\ &{}_{\circ (\cdots \circ W_{n-1} \circ \overline{W}_n) \circ (\overline{W}_n \circ W_n)^-}\end{aligned}$$

□

2.4 Geometric Interpretation of the Invariants

We now return to the cobordism category with vector fields. We start out by showing that $\mathcal{C}_d^r$ satisfies the conditions (i) and (ii) of Theorem 2.17. From this we shall obtain a geometric interpretation of the invariants from Chapter 1.

Theorem 2.18. *Let d be odd or $r < \frac{d}{2}$. Let $W \in \mathcal{C}_d^r(M_0, M_1)$. Then there exists a $\overline{W} \in \mathcal{C}_d^r(M_1, M_0)$.*

Proof. Suppose $(W, a) \in \mathcal{C}_d^r(M_0, M_1)$ is given. That is, $W \subseteq (-1,1)^n \times \mathbb{R}$ is a d-dimensional manifold with r vector fields $v_1, \ldots, v_r : W \to TW$. Then the reflection $t \mapsto a - t$ in the $\mathbb{R}$ direction takes W to a manifold which is a morphism in $\mathcal{C}_d(M_1, M_0)$. In the oriented case, the orientation must be reversed. However, it is not an element of $\mathcal{C}_d^r(M_1, M_0)$ yet, since the vector fields on M_0 and M_1 have been reflected in the normal direction. They must be reflected once more to get the correct vector fields on the objects. But there is no obvious way to extend this reflection to the vector fields on the rest of W.

First we choose a normal vector field on ∂W and extend this to the $(d-1)$-skeleton $W^{(d-1)}$. This is always possible, see Section 1.2. Denote this vector field by

$$V : W^{(d-1)} \to TW_{|W^{(d-1)}}.$$

This defines a map

$$\sigma_V : W^{(d-1)} \to O(TW)_{|W^{(d-1)}}.$$

Here $O(TW)$ is the bundle over W with fiber over x the orthogonal group $O(T_x W)$, and $\sigma_V(x)$ is defined to be the reflection of $T_x W$ that takes $V(x)$ to $-V(x)$ and leaves $V(x)^\perp$ fixed.

With this definition, σ_V acts on the given vector fields by multiplication

$$\sigma_V(x)(v_1(x), \ldots, v_r(x)) = (w_1(x), \ldots, w_r(x))$$

for all $x \in W^{(d-1)}$. On the boundary, σ_V is the reflection of the normal direction, so the w_i's yield an extension of the vector fields on the boundary to the $(d-1)$-skeleton.

We still need to extend these new vector fields over each d-cell $D \subseteq W$. This may not be possible. The idea is to take the connected sum with a suitable manifold in the interior of D such that the vector fields extend.

Choose a trivialization $TW_{|D} \cong D \times \mathbb{R}^d$. On the boundary, the v_i's define a map $f : S^{d-1} \to V_{d,r}$ which is homotopic to a constant map because the vector fields extend to all of D. So we may assume that f is constant with value $e_1, \dots, e_r$. Then the w_i's are given on ∂D by

$$g : S^{d-1} \xrightarrow{\sigma_V} O(d) \xrightarrow{h} V_{d,r},$$

where h comes from the standard fibration $O(d-r) \to O(d) \to V_{d,r}$.

In the trivialization, $V_{|\partial D}$ becomes a map $V_{|\partial D} : S^{d-1} \to S^{d-1}$. There is a map $\rho_d : S^{d-1} \to O(d)$ that takes $x \in S^{d-1}$ to the reflection of the line spanned by x. Then

$$\sigma_V = \rho_d \circ V_{|\partial D} \text{ and } g = h \circ \rho_d \circ V_{|\partial D}.$$

By possibly dividing D into smaller cells, we may assume that the degree of $V_{|\partial D}$ is either 0 or ± 1.

If the degree of $V_{|\partial D}$ is zero, then σ_V is homotopic to constant map. Hence, so is g, and the vector fields extend to all of D.

If the degree is $+1$, then V is homotopic to the identity map. This means that σ_V is the reflection in the normal direction. Choose r vector fields on the torus T^d. Cut out a disk. The vector fields on the boundary of this disk are now homotopic to the v_i's on ∂D, since both extend over a disk. Thus, after reflecting the v_i's in the normal direction, the vector fields on the torus fit with the w_i's so that we can form the connected sum of W and T^d in the interior of D.

If the degree is -1 and d is odd, V is homotopic to minus the identity. Thus σ_V is again the reflection in the normal direction, and we do as in the degree $+1$ case.

We are now left with the case where $d = 2k$ is even and the degree is -1. In this case we would like to take the connected sum with a product of two spheres, rather than a torus.

First look at what happens to the vector fields when they are reflected in a map of degree -1. Consider the diagram

$$\begin{tikzcd}
\pi_{d-1}(S^{d-1}) \arrow[d, "V_{\partial D}"] & \pi_d(S^d) \arrow[d, "\delta"] \arrow[dr, "\delta_d"] \arrow[drr, "\cdot 2"] & & \\
\pi_{d-1}(S^{d-1}) \arrow[r, "\rho_d"] \arrow[d] & \pi_{d-1}(O(d)) \arrow[r, "h"] \arrow[d] & \pi_{d-1}(V_{d,r}) \arrow[r, "p"] & \pi_{d-1}(S^{d-1}) \\
\pi_{d-1}(S^d) = 0 \arrow[r, "\rho_{d+1}"] & \pi_{d-1}(O(d+1)). & &
\end{tikzcd}$$

It follows that ρ_d maps into the image of δ. But the composition $p \circ h \circ \rho_d$ maps $x \in S^{d-1}$ to the reflection of the first basis vector in the x direction. This is a map of degree 2, being the obstruction to a vector field on S^{d-1}. Thus $\rho_d([1]) = \delta([1])$, and therefore

$$[g] = h \circ \rho_d \circ V_{|\partial D}([1]) = \delta_d([-1]).$$

Here and in the following, $[m] \in \pi_l(S^l)$ denotes the class of degree m maps.

Consider

$$\begin{array}{ll} S^k \times S^k & \text{if } k \text{ is even,} \\ S^{k-1} \times S^{k+1} & \text{if } k \text{ is odd.} \end{array}$$

For simplicity, we write this product as $S^i \times S^j$ in the following. Choose r vector fields with one singularity on each sphere. This is possible because $r < k$ by assumption. These are given by maps

$$\begin{aligned} u_1 &: S^i \to W_r(TS^i) \\ u_2 &: S^j \to W_r(TS^j). \end{aligned}$$

Assume that these vector fields have length one outside small open disks D^i and D^j, respectively. This defines a map $u : S^i \times S^j \to W_r(T(S^i \times S^j))$ by the formula

$$u(x, y) = \frac{u_1(x) + u_2(y)}{\sqrt{2}}$$

outside $D^i \times D^j$ and

$$u(x, y) = \frac{|x|u_1(x) + |y|u_2(y)}{\sqrt{|u_1(x)|^2 + |u_2(y)|^2}}$$

inside $D^i \times D^j$. On the boundary of $D^i \times D^j$, this is the join

$$u_{1|S^i} \star u_{2|S^j} : S^{i-1} \star S^{j-1} \to V_{d,r}.$$

This map represents the obstruction to r vector fields on $S^i \times S^j$.

Now look at the diagram

$$\begin{tikzcd} \pi_i(S^i) \times \pi_j(S^j) \arrow[d, "\delta_i \times \delta_j"] & \pi_d(S^d) \arrow[d, "\delta_d"] \arrow[dr, "\cdot 2"] & \\ \pi_{i-1}(V_{i,r}) \times \pi_{j-1}(V_{j,r}) \arrow[r, "\star"] \arrow[d, "\eta_i \times \eta_j"] & \pi_{d-1}(V_{d,r}) \arrow[r, "p"] \arrow[d, "\eta_d"] & \pi_{d-1}(S^{d-1}) \\ \pi_{i-1}(V_{i+1,r+1}) \times \pi_{j-1}(V_{j+1,r+1}) & \pi_{d-1}(V_{d+1,r+1}). & \end{tikzcd}$$

For $l = i, j, d$, the class $\delta_l([1])$ is the obstruction to r vector fields on S^l. Thus the homotopy class of $u_{1|S^i} \star u_{2|S^j}$ is $\delta_i([1]) \star \delta_j([1])$. By [22], formula (2.12 b),

$$\eta_d(\delta_i([1]) \star \delta_j([1])) = \eta_i(\delta_i([1])) \star \delta'_j([1]).$$

Here $\delta'_j : \pi_j(S^j) \to \pi_{j-1}(V_{j,r+1})$ is the boundary map. This is well-defined when $r < k$. But $\eta_i \circ \delta_i = 0$, so $\delta_i([1]) \star \delta_j([1])$ is in the image of δ_d. Furthermore,

$$p(\delta_i([1]) \star \delta_j([1])) = [4]$$

since it is the obstruction to a single vector field, which is the Euler characteristic. Thus $\delta_i([1]) \star \delta_j([1]) = \delta_d([2])$.

Summarizing the above, there are vector fields on $S^i \times S^j \backslash D^i \times D^j$ and $W \backslash \operatorname{int}(D)$ given on the boundaries of the removed disks by $\delta_d([2])$ and $\delta_d([-1])$, respectively. We can take the connected sum if the vector fields agree after reflecting the vector fields on $\partial(D^i \times D^j)$. That is, it remains to show that $\sigma_1(\delta_d([2])) = \delta_d([-1])$ where σ_1 is the normal reflection.

Note that

$$p(\sigma_1(\delta_d([m]))) = \sigma_1(p(\delta_d([m]))) = \sigma_1([2m]) = [2 - 2m]. \tag{2.5}$$

The last equality follows because a degree $2m$ map $S^{d-1} \to S^{d-1}$ defines a vector field on S^d with two singularities, one of degree $2m$ and one of degree $\sigma_1([2m])$. The sum of these must be $\chi(S^d) = 2$.

Also note that

$$\eta_d(\sigma_1(\delta_d([2]))) = \sigma_1'(\eta_d \circ \delta_d([2])) = 0$$

where σ_1' is the image of σ_1 under $\pi_{d-1}(O(d)) \to \pi_{d-1}(O(d+1))$.

Thus $\sigma_1(\delta_d([2]))$ is in the image of δ_d. According to (2.5),

$$p(\sigma_1(\delta_d([2]))) = p(\delta_d([-1])) = [-2].$$

But $p \circ \delta_d$ is injective, so $\sigma_1(\delta_d([2])) = \delta_d([-1])$. □

This proof was inspired by Proposition 4.23 in [15].

Remark 2.19. In the case where S^d allows r vector fields, it is possible to choose $\overline{W}$ diffeomorphic to W. This is because we may glue in disks, rather than tori, when constructing $\overline{W}$.

If S^d does not allow r vector fields, the disk D^d with r vector fields is an example of a morphism such $\overline{D}^d$ cannot be chosen diffeomorphic to D^d. Otherwise they would glue together to a sphere with r vector fields.

The above proof would work more generally for any θ satisfying that S^{d-1} with any θ-structure bounds a θ-manifold. For d odd, it would also suffice that S^d allows a θ-structure.

Example 2.20. Consider the case $d = 2$ and $r = 1$. A surface of genus $g > 1$ allows a vector field with only one singularity. Cut out a disk containing the singularity. This defines a morphism from the $\emptyset$ to S^1 with a vector field. If there were a morphism in the opposite direction, they would glue together to a closed surface of genus at least $g > 1$ with a zero-free vector field, which is impossible.

This shows that the condition $r < \frac{d}{2}$ is not always redundant. Whether it can be refined is not clear. However, it appears naturally as a condition in many of our applications anyway.

Theorem 2.21. *There are weak homotopy equivalences*

$$BC_{d+k}^{r+k} \to \Omega^k BC_d^r \to \Omega^{\infty+d+k-1} MT(d-r).$$

In the case $k = 1$, assume $M \in \operatorname{Ob}(\mathcal{C}_{d+1}^{r+1})$ with the $(r+1)$th vector field equal to the positively directed normal. Then the component of M in $\pi_0(BC_{d+1}^{r+1})$ is mapped to the morphism path in $\pi_1(BC_d^r)$ corresponding to M, now considered as a morphism in $\mathcal{C}_d^r(\emptyset, \emptyset)$ with the first r vector fields. Both correspond to the Pontryagin–Thom element in $\pi_d(MT(d-r))$.

Proof. We have the following commutative diagram for each n

$$\begin{tikzcd} \Omega^k B\mathcal{C}^r_{d,n} \arrow[r] & \Omega^k \psi^r_d(n,1) \arrow[r] \arrow[d] & \Omega^{n+k-1}\psi^r_d(n,n) \arrow[d] \\ B\mathcal{C}^{r+k}_{d+k,n+k} \arrow[r] & \Omega^k \psi^{r+k}_{d+k}(n+k,k+1) \arrow[r] & \Omega^{n+k-1}\psi^{r+k}_{d+k}(n+k,n+k). \end{tikzcd} \tag{2.6}$$

The vertical maps take a manifold $M \subseteq \mathbb{R}^n$ with r vector fields to the manifold $M \times \mathbb{R}^k \subseteq \mathbb{R}^n \times \mathbb{R}^k$ with the r vector fields from M together with the k standard vector fields in the $\mathbb{R}^k$ direction. The horizontal maps are the homotopy equivalences from [15].

The diagram

$$\begin{tikzcd} \psi^r_d(n,n) \arrow[r] \arrow[d] & \mathrm{Th}(U^\perp_{d,n-d} \to V_r(U_{d,n-d})) \arrow[d] \\ \psi^{r+k}_{d+k}(n+k,n+k) \arrow[r] & \mathrm{Th}(U^\perp_{d+k,n-d} \to V_{r+k}(U_{d+k,n-d})) \end{tikzcd}$$

also commutes. The horizontal maps take a manifold M to $-p$ in the fiber over T_pM with the vector fields at this point, where p is the point on M closest to the identity (whenever this is defined). Thus commutativity is obvious. Some details have been omitted, see [15] for the precise definition of the maps.

If we let n tend to infinity, the right vertical map in (2.6) is a homotopy equivalence, since the diagram

$$\begin{tikzcd} \mathrm{Th}(U^\perp_{d-r,n-d} \to G(d-r,n-d)) \arrow[r] \arrow[dr] & \mathrm{Th}(U^\perp_{d,n-d} \to V_r(U_{d,n-d})) \arrow[d] \\ & \mathrm{Th}(U^\perp_{d+k,n-d} \to V_{r+k}(U_{d+k,n-d})) \end{tikzcd}$$

commutes and the two maps to the left are homotopy equivalences of spectra.

Now look at the case $k = 1$. Let $M \subseteq (-1,1)^n$ be an object in $\mathcal{C}^{r+1}_{d+1,n+1}$ with the positively directed normal in $TM \oplus \mathbb{R}$ as its last vector field. This corresponds to the manifold $M \times \mathbb{R}$ in $\psi^{r+1}_{d+1}(n+1,1)$ with the vector fields defined by the ones on M. Under the first lower horizontal map in (2.6), this is mapped to the loop $\mathbb{R}^+ \to \psi^{r+1}_{d+1}(n+1,2)$ given by $t \mapsto M \times \mathbb{R} - (0,\dots,0,t,0)$. But this is in the image of the vertical map. More precisely, it is the image of the map $\gamma_1 : \mathbb{R}^+ \to \psi^r_d(n,1)$ given by $t \mapsto M - (0,\dots,0,t)$. But this map is the map corresponding to the morphism path of M in $B\mathcal{C}^r_{d,n}$. To see this, recall the factorization

$$B\mathcal{C}^r_{d,n} \to BD_r \to \psi^r_d(n+1,1)$$

where D_r is the category defined in [15], Definition 3.8.

Indeed, γ_1 lifts to $\mathbb{R} \to BD_r$ to the path

$$t \mapsto ((M - te_n, \min\{-1-t,-1\} \le \max\{1-t,1\}), \varphi(t)) \in \mathrm{Mor}(D_r) \times \Delta^1$$

where $\varphi(t) = \frac{1}{2}(t+1)$ for $t \in [-1, 1]$ and φ is constant on $[1, \infty)$ and $(-\infty, -1]$. This defines a path in BD_r between the objects $(\emptyset, -1)$ and $(\emptyset, 1)$. We may close this to a loop inside $\{\emptyset\} \times \mathbb{R}$ and get a loop $\mathbb{R}^+ \to BD_r$ whose image in $\psi_d^r(n+1, 1)$ is homotopic to γ_1.

This new loop again lifts to a map homotopic to $\gamma : \mathbb{R}^+ \to B\mathcal{C}_{d,n}^r$ given by $\gamma(t) = ((W + e_n, 2), \varphi(t)) \in \mathrm{Mor}(\mathcal{C}_{d,n}^r) \times \Delta^1$ for $t \in [-1, 1]$ and $\gamma(t) = (\{\emptyset\}, \varphi(t))$ in $\mathrm{Ob}(\mathcal{C}_{d,n}^r) \times \partial\Delta^1$ otherwise. This is the morphism loop corresponding to M.

Going in the other direction in the upper part of (2.6) shows that M corresponds to the map $(\mathbb{R}^n)^+ \to \psi(n, n)$ given by $t \mapsto M - t$. If $M - t$ has a unique point p closest to the zero, $M - t$ is sent to the point $-p$ in the fiber over $T_p(M - t)$ in $U_{d,n-d}^\perp \to V_r(U_{d,n-d})$. Otherwise, $M - t$ is mapped to ∞. The set of t's such that $M - t$ has a unique point closest to the identity deformation retracts onto a tubular neighbourhood of M consisting of those vectors at distance at most ε from M. Collapsing all vectors in $U_{d,n-d}^\perp \to V_r(U_{d,n-d})$ of length greater than or equal to ε makes the map well-defined, and this is certainly the Pontryagin–Thom collapsing map. Again some details are omitted. $\square$

Theorem 2.22. *For $r \geq 0$, all morphisms in $\mathcal{C}_d^r(\emptyset, \emptyset)$ have inverses in the sense of Theorem 2.17 (i).*

Proof. Let W be a closed d-dimensional manifold with r vector fields

$$v_1, \ldots, v_r : W \to TW.$$

Assume d is odd or $r \geq 1$. We consider W as an object of $\mathcal{C}_{d+1}^{r+1}$ with the positive normal vector field ε as the $(r+1)$th vector field. Then $W \times \mathbb{R}$ has $r + 1$ vector fields given on $W \times [0, 1]$ as follows. Choose a vector field $v : W \to TW$. If $r \geq 1$, we simply choose this to be v_r. This defines the $r + 1$ vector fields $w_1, \ldots, w_{r+1}$ on $W \times [0, 1]$ by

$$\begin{aligned} w_i(x,t) =& v_i(x) \\ w_r(x,t) =& \cos(\pi t)v(x) + \sin(\pi t)\varepsilon(x) \\ w_{r+1}(x,t) =& -\sin(\pi t)v(x) + \cos(\pi t)\varepsilon(x). \end{aligned}$$

Extend these trivially to $W \times \mathbb{R}$. Embed $W \times \mathbb{R}$ a cobordism from $W \times \{0, 1\}$ to $\emptyset$ in $\mathcal{C}_{d+1}^{r+1}$. Let $W^- = W \times \{1\}$ with the induced vector fields and, in the oriented case, orientation. Then $W \sqcup W^-$ belongs to the base point component of $B\mathcal{C}_{d+1}^{r+1}$.

Under the isomorphism from Theorem 2.21

$$\pi_1(B\mathcal{C}_d^r) \to \pi_0(B\mathcal{C}_{d+1}^{r+1}),$$

$W \sqcup W^-$ lifts to $\gamma_{W \sqcup W^-}$, so this must be null-homotopic.

In the remaining case where d is even and there are no vector fields, we may still view W as an object in $\mathcal{C}_{d+1}^1$ with the positive normal vector field. As before, we seek another manifold such that the disjoint union with W is Reinhart cobordant to the empty manifold. By [38] it is enough to find a manifold W^- which is a cobordism inverse to W and has Euler characteristic

$$\chi(W^-) = -\chi(W). \tag{2.7}$$

Let W' be a copy of W. In the oriented category, give it the opposite orientation. Then W' is a cobordism inverse of W. Taking the disjoint union with a bounding manifold does not change the cobordism class. Taking the disjoint union with a sphere increases the Euler characteristic by 2, and taking disjoint union with a connected sum of two tori decreases the Euler characteristic by 2. Thus, defining W^- to be the disjoint union of W' with a suitable bounding manifold, (2.7) is satisfied. □

Corollary 2.23. *For d odd or $r < \frac{d}{2}$, any class in $\pi_1(B\mathcal{C}_d^r)$ can be represented by a morphism path.*

Proof. This follows from Theorem 2.17, 2.18 and 2.22. □

Definition 2.24. *Let M and N be two $(d-1)$-dimensional manifolds with r vector fields in $TM \oplus \mathbb{R}$ and $TN \oplus \mathbb{R}$, respectively. We say that M and N are vector field cobordant if there is a cobordism W from M to N with r vector fields in the tangent bundle extending the ones already given on the boundary.*

The above yields the following interpretation of the groups $\pi_{d-1}(MT(d-r))$:

Corollary 2.25. *For d odd or $r < \frac{d}{2}$, vector field cobordism is an equivalence relation, and $\pi_0(B\mathcal{C}_d^r) \cong \pi_{d-1}(MT(d-r))$ is the set of vector field cobordism classes.*

For $r < \frac{d}{2}$, each of these classes contains a manifold with one of the vector fields equal to the normal vector field, corresponding to a morphism path in $\pi_1(B\mathcal{C}_{d-1}^{r-1})$.

In particular, $\pi_d(MT(d)) \cong \pi_0(B\mathcal{C}_{d+1}^1)$ is the group of Reinhart cobordism classes of d-dimensional manifolds represented by the invariants $\beta(M)$.

Proof. For the equivalence statement, symmetry follows from Theorem 2.18 and transitivity is given by glueing cobordisms together.

If two manifolds are vector field cobordant, they obviously belong to the same path component of $B\mathcal{C}_d^r$. By Theorem 2.16, M and N belong to the same path component if and only if there is a zigzag of morphisms relating them. Turning some of the morphisms around, if necessary, and glueing them together yields a vector field cobordism from M to N.

By Theorem 2.23, any element of $\pi_1(B\mathcal{C}_d^r)$ is represented by a morphism path, which corresponds to an object with the last vector field equal to the normal vector field. □

Corollary 2.26. *Assume that d is odd or $r < \frac{d}{2}$. Then $\beta^r(M^d) = 0$ if and only if M is Reinhart cobordant to a manifold that allows r tangent vector fields.*

Proof. Let M be given. Then $\beta^r(M) = 0$ if and only if $\beta(M)$ lifts to a class $\alpha \in \pi_d(MT(d-r))$. But this is represented by a morphism loop in $\pi_1(B\mathcal{C}_d^r)$ by Corollary 2.23. This means that there is a closed d-dimensional manifold N with r tangent vector fields representing $\alpha \in \pi_d(MT(d-r))$. This maps to $\beta(N) = \beta(M)$ in $\pi_d(MT(d))$ by (2.2). By Corollary 2.25, this is the case if and only if M and N are Reinhart cobordant □

Moreover, we may deduce the following geometric interpretation of the maps

$$\pi_{d-1}(MT(d-r-k)) \to \pi_{d-1}(MT(d-r))$$

for d odd or $r+k<\frac{d}{2}$.

Corollary 2.27. *Under the map $\pi_0(B\mathcal{C}_d^{r+k}) \to \pi_0(B\mathcal{C}_d^r)$, a component containing some manifold M with r vector fields in $TM \oplus \mathbb{R}$ is in the image if and only if there is a cobordism with r vector fields from M to some M' such that the r vector fields in $TM' \oplus \mathbb{R}$ extend to $r+k$ vector fields.*

The image of $\pi_0(B\mathcal{C}_d^{r+1}) \to \pi_0(B\mathcal{C}_d^1)$ is the set of Reinhart cobordism classes containing a manifold with r tangent vector fields.

The image of $\pi_0(B\mathcal{C}_d^{r+1}) \to \pi_0(B\mathcal{C}_d^r)$ is the subgroup of the vector field cobordism group containing all manifolds with r tangent vector fields.

2.5 Equivalence of Zigzags

We saw in Theorem 2.16 that all elements of $\pi_1(B\mathcal{C}_d^\theta)$ are represented by zigzags of morphism paths. In this section we find necessary and sufficient conditions for two such zigzags to be homotopic.

First some notation. We picture a zigzag of morphisms as

$$\cdots \to M_i \xleftarrow{W_i} M_{i+1} \xrightarrow{W_{i+1}} M_{i+2} \leftarrow \cdots . \tag{2.8}$$

The sequence should be read from left to right. An arrow pointing in this direction corresponds to a morphism path, while an arrow pointing in the opposite direction represents the inverse of a morphism path. Moreover, $\partial_i : \mathrm{Mor}(\mathcal{C}_d^\theta) \to \mathrm{Ob}(\mathcal{C}_d^\theta)$ will denote the boundary maps given on $W \in \mathcal{C}_d^\theta(M_0, M_1)$ by $\partial_i(W) = M_i$ for $i = 0, 1$.

The main theorem of this section is:

Theorem 2.28. *Two zigzags are homotopic relative to endpoints if and only if they are related by a finite sequence of moves of the following two types:*

(I) Any sequence of arrows from M_i to M_j in the diagram

$$\begin{tikzcd} M_0 \arrow[d, "W_1"'] \arrow[dr, "W_1 \circ W_2"] & \\ M_1 \arrow[r, "W_2"'] & M_2, \end{tikzcd}$$

may be replaced by any other such sequence in a diagram like (2.8).

(II) A morphism path

$$\cdots M_0 \xrightarrow{W} M_1 \cdots$$

may be replaced by

$$\cdots M_0 \xrightarrow{W'} M_1 \cdots$$

if there is a path $\gamma : I \to \mathrm{Mor}(\mathcal{C}_d^\theta)$ from W to W' such that the boundary paths $\partial_i \circ \gamma : I \to \mathrm{Ob}(\mathcal{C}_d^\theta)$ are constant for $i = 0, 1$.

In particular, (I) allows us to insert and remove an identity morphism or a morphism path followed by its inverse.

Here is an interpretation of the relation (II):

Lemma 2.29. *Two morphisms W_0 and W_1 may be joined by a path $\gamma : I \to \mathrm{Mor}(\mathcal{C}_d^\theta)$ with $\partial_i \circ \gamma$ constant if and only if there is a diffeomorphism between them that fixes $\partial_i W_j$ pointwise for $i, j = 0, 1$ and preserves the equivalence class of θ-structures.*

The proof uses the topology on $\mathrm{Mor}(\mathcal{C}_d^\theta)$ constructed in [15]. The reader is referred to this paper for notation and precise definitions in the proof below. An equivalence class of θ-structures means an element of $\pi_0(\mathrm{Bun}(TW, \theta^* U_d))$ where $\mathrm{Bun}(TW, \theta^* U_d)$ is the space of bundle maps $TW \to \theta^* U_d$. We shall ignore θ-structures below, but the proof immediately generalizes.

Proof. Let $\gamma(t) = (W_t, a(t))$ be a path of morphisms W_t between the fixed objects M_0 and M_1. We may assume that $W_t \subseteq \mathbb{R}^n$ for all $t \in I$ and n sufficiently large. The continuous map $a : I \to \mathbb{R}$ is bounded on I by some A. Let $K = [0,1]^n \times [0, A]$.

Let $t_0 \in I$ be given and choose a small open neighbourhood $V \subseteq \Gamma_c(NW_{t_0})$ of the zero section that maps diffeomorphically onto an open neighbourhood $c_{W_{t_0}}(V) = V'$ of W_{t_0} in $\Psi(\mathbb{R}^n)^{cs}$ by embedding. Here $\Psi(\mathbb{R}^n)^{cs}$ is $\Psi(\mathbb{R}^n)$ with a certain topology.

There is a map $\pi_K : \Psi(\mathbb{R}^n)^{cs} \to \Psi(K \subseteq \mathbb{R}^n)$ that identifies manifolds that agree inside an open set containing K. By construction of the topologies, this map is open, see [15], Lemma 2.5. Furthermore, by definition of the toplogy of the morphism space in [15],

$$\gamma' : I \to \mathrm{Mor}(\mathcal{C}_d^\theta) \to \Psi(\mathbb{R}^n) = \varprojlim_L \Psi(\mathbb{R}^n)^L \to \Psi(\mathbb{R}^n)^K \to \Psi(K \subseteq \mathbb{R}^n)$$

is continuous. Since $\pi_K(V') \subseteq \Psi(K \subseteq \mathbb{R}^n)$ is open, ${\gamma'}^{-1}(\pi_K(V'))$ is also open.

Let $t \in {\gamma'}^{-1}(\pi_K(V'))$. Then $W_t \cap U = s(W_{t_0}) \cap U$ for some open set U containing K and some $s \in V$. Since W_t and W_{t_0} agree outside K, s defines a diffeomorphism between them which is the identity outside K.

Covering I by finitely many such open sets shows that there is a diffeomorphism between W_0 and W_1 fixing the boundary.

Conversely, if there is a diffeomorphism between W_0 and W_1 fixing the boundary, then there is a smooth isotopy between the embeddings fixing the boundary. This defines the desired path. $\square$

Lemma 2.30. *Let $\gamma : [0,1] \to \mathrm{Mor}(\mathcal{C}_d^\theta)$ be a smooth path from W_0 to W_1 that is constant near $0, 1$. Let $W_{\partial_i \gamma}$ be the morphisms determined by $\partial_i \circ \gamma : [0,1] \to \mathrm{Ob}(\mathcal{C}_d^\theta)$ for $i = 0, 1$. Then there are Type (I) and (II) moves relating the following zigzags:*

$$\begin{gathered} \cdots \xrightarrow{W_0} \cdot \xrightarrow{W_{\partial_1 \gamma}} \cdots \\ \cdots \xrightarrow{W_{\partial_0 \gamma}} \cdot \xrightarrow{W_1} \cdots \end{gathered} \tag{2.9}$$

Proof. First compose the morphisms in (2.9) by a Type (I) move. We must see that $W_0 \circ W_{\partial_1 \gamma}$ and $W_{\partial_0 \gamma} \circ W_1$ are related by a Type (II) move.

Look at the morphisms $(W_{\partial_0\gamma}, 0 \le 1)$ and $(W_{\partial_1\gamma}, 0 \le 1)$ in the category D_d^θ. There are paths in $\mathrm{Mor}(D_d^\theta)$ given by $(W_{\partial_0\gamma}, 0 \le t)$ and $(W_{\partial_1\gamma}, t \le 1)$ for $t \in (0,1)$. These lift to paths γ_0 and γ_1 in $\mathrm{Mor}(\mathcal{C}_d^\theta)$ under the homotopy equivalences (2.3) satisfying

$$\gamma_0(t) \in \mathcal{C}_d^\theta(M_0, \partial_0(\gamma(t)))$$
$$\gamma_1(t) \in \mathcal{C}_d^\theta(\partial_1(\gamma(t)), M_1).$$

Thus the composition of morphisms $\gamma_0(t) \circ \gamma(t) \circ \gamma_1(t) \in \mathcal{C}_d^\theta(M_0, M_1)$ is a well-defined path in the morphism space for $t \in (0,1)$. This naturally extends to all $t \in [0,1]$, and this is the desired path from $W_0 \circ W_{\partial_1\gamma}$ to $W_{\partial_0\gamma} \circ W_1$. □

The next two proofs consider homotopy groups with multiple base points. If X is a topological space and X_0 is a discrete subset, $\pi_1(X, X_0)$ denotes set of homotopy classes of paths in X starting and ending in X_0. The path composition makes this into a groupoid where the identity elements correspond to the constant paths.

Theorem 2.31. *Any two zigzags that are homotopic inside* $\mathrm{Ob}(\mathcal{C}_d^\theta) \cup (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1)$ *are related by a sequence of Type (I) and (II) moves.*

Proof. First choose a set of objects M_i for $i \in I$, one in each path component of $\mathrm{Ob}(\mathcal{C}_d^\theta)$. Then choose a W_j, $j \in J$, in each component of $\mathrm{Mor}(\mathcal{C}_d^\theta)$ such that $\partial_\varepsilon(W_j) \in \{M_i, i \in I\}$ for all $j \in J$ and $\varepsilon = 0, 1$. These will serve as the base point sets. By construction, the source and target maps are base point preserving.

To describe $\pi_1(\mathrm{Ob}(\mathcal{C}_d^\theta) \cup (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1))$, we need a generalized version of the van Kampen theorem. This is the main theorem of [10]. For this, let

$$\begin{aligned}
U_1 &= \mathrm{Ob}(\mathcal{C}_d^\theta) \cup (\mathrm{Mor}(\mathcal{C}_d^\theta) \times (\Delta^1 \backslash \{pt\})) \\
U_2 &= \mathrm{Mor}(\mathcal{C}_d^\theta) \times \mathrm{int}(\Delta^1) \\
U_1 \cap U_2 &= \mathrm{Mor}(\mathcal{C}_d^\theta) \times (\mathrm{int}(\Delta^1) \backslash \{pt\}) \\
X_0 &= \{(W_j, \varepsilon) \mid j \in J, \varepsilon = 0, 1\} \\
X_0' &= \{M_i \mid i \in I\}.
\end{aligned}$$

According to [10], $\pi_1(\mathrm{Ob}(\mathcal{C}_d^\theta) \cup (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1), X_0) = \pi_1(U_1 \cup U_2, X_0)$ is the coequalizer in the category of groupoids of the diagram

$$\pi_1(U_1 \cap U_2, X_0) \rightrightarrows \pi_1(U_1, X_0) \sqcup \pi_1(U_2, X_0) \to \pi_1(U_1 \cup U_2, X_0).$$

We begin by describing the first three groupoids in the diagram.

The map $\pi_1(U_1, X_0) \to \pi_1(U_1, X_0')$ induced by $(W_j, \varepsilon) \mapsto \partial_\varepsilon(W_j)$ is a vertex and piecewise surjection in the sense of [20], and thus it is a quotient map, according to [20], Proposition 25. The kernel is the inverse mage of the identity elements, i.e. the set

$$N = \bigcup_{i \in I} \{((W_{j_1}, \varepsilon_1), (W_{j_2}, \varepsilon_2)) \mid j_1, j_2 \in J, \varepsilon_1, \varepsilon_2 \in \{0,1\}, \partial_{\varepsilon_1} W_{j_1} = \partial_{\varepsilon_2} W_{j_2}\}$$

with multiplication

$$((W_{j_1}, \varepsilon_1), (W_{j_2}, \varepsilon_2))((W_{j_2}, \varepsilon_2), (W_{j_3}, \varepsilon_3)) = ((W_{j_1}, \varepsilon_1), (W_{j_3}, \varepsilon_3)).$$

If $\pi_1(U_1, X_0)$ is replaced by $\pi_1(U_1, X_0')$ in the coequalizer diagram, $\pi_1(U_1 \cup U_2, X_0)$ must be replaced by the quotient of this with the normal subgroupoid generated by N, c.f. [20], Proposition 27. But N is also the kernel of the quotient map

$$\pi_1(U_1 \cup U_2, X_0) \to \pi_1(U_1 \cup U_2, X_0'),$$

so there is a new coequalizer diagram

$$\pi_1(U_1 \cap U_2, X_0) \rightrightarrows \pi_1(U_1, X_0') \sqcup \pi_1(U_2, X_0) \to \pi_1(U_1 \cup U_2, X_0').$$

We compute:

$$\begin{aligned}
\pi_1(U_1 \cap U_2, X_0) &= \bigsqcup_{j \in J} \pi_1(\mathrm{Mor}(\mathcal{C}_d^\theta), W_j) \times \{0,1\} \\
\pi_1(U_1, X_0') &= \bigsqcup_{i \in I} \pi_1(\mathrm{Ob}(\mathcal{C}_d^\theta), M_i) \\
\pi_1(U_2, X_0) &= \bigsqcup_{j \in J} \pi_1(\mathrm{Mor}(\mathcal{C}_d^\theta), W_j) \times G.
\end{aligned}$$

Here $G = \{(i,j) \mid i,j = 0,1\}$ is the groupoid with multiplication $(i,j)(j,k) = (i,k)$.

By [20] the coequalizer, viewed as a category, is given as follows. The object set is just the set of base points X_0'. A morphism is represented by a sequence $x_1 \cdots x_n$ where each x_i is an element of either $\pi_1(U_1, X_0')$ or $\pi_1(U_2, X_0)$ such that the target of x_i coincides with the source of x_{i+1} in X_0'. Two such sequences are equivalent if and only if they are related by a sequence of relations of the following three types:

(i) If e is an identity element in either $\pi_1(U_1, X_0')$ or $\pi_1(U_2, X_0)$, then

$$\cdots x_i e x_{i+1} \cdots \simeq \cdots x_i x_{i+1} \cdots .$$

(ii) If the product $x_i x_{i+1} = x$ makes sense in either $\pi_1(U_1, X_0')$ or $\pi_1(U_2, X_0)$, then

$$\cdots x_i x_{i+1} \cdots \simeq \cdots x \cdots .$$

(iii) Let $i_1 : \pi_1(U_1 \cap U_2, X_0) \to \pi_1(U_1, X_0')$ and $i_2 : \pi_1(U_1 \cap U_2, X_0) \to \pi_1(U_2, X_0)$ denote the inclusions. Then for $x \in \pi_1(U_1 \cap U_2, X_0)$,

$$\cdots i_1(x) \cdots \simeq \cdots i_2(x) \cdots .$$

The next step is to canonically identify such a sequence $x_1 \cdots x_n$ with a zigzag representing the same homotopy class. To each x_i we associate a part of a zigzag in the following way. If $x_i \in \pi_1(\mathrm{Ob}(\mathcal{C}_d^\theta), M_l)$, let $\alpha : I \to \mathrm{Ob}(\mathcal{C}_d^\theta)$ be a smooth representative. This corresponds to a morphism W_α. Otherwise x_i has the form $([\gamma], (k,l))$ for some $[\gamma] \in \pi_1(\mathrm{Mor}(\mathcal{C}_d^\theta), W_j)$ and $k, l \in \{0,1\}$. We choose the following assignments:

$$\begin{aligned}
[\alpha] &\rightsquigarrow \cdot \xrightarrow{W_\alpha} \cdot \\
([\gamma], (0,0)) &\rightsquigarrow \cdot \xrightarrow{W_{\partial_0 \gamma}} \cdot \\
([\gamma], (1,1)) &\rightsquigarrow \cdot \xrightarrow{W_{\partial_1 \gamma}} \cdot \\
([\gamma], (0,1)) &\rightsquigarrow \cdot \xrightarrow{W_{\partial_0 \gamma}} \cdot \xrightarrow{W_j} \cdot \\
([\gamma], (1,0)) &\rightsquigarrow \cdot \xleftarrow{W_j} \cdot \xrightarrow{W_{\partial_0 \gamma}} \cdot
\end{aligned}$$

Note that the manifolds W_α depend on the choice of representative α. A different choice of representative yields a morphism that differs from W_α by a Type (II) move. Hence the assignment is canonical up to Type (II) moves.

To each sequence $x_1 \cdots x_n$ this associates a zigzag. We need to see that the relations (i)-(iii) on sequences correspond to performing Type (I) and (II) moves on the associated zigzags. This is a straightforward check, and we will only show some of the relations.

(i) If e is the identity element in $\pi_1(\mathrm{Ob}(\mathcal{C}_d^\theta), M_i)$, it is assigned the morphism path of $W_e = M_i \times \mathbb{R}$. Hence this relation just removes a

$$\cdot \xrightarrow{M_i \times \mathbb{R}} \cdot$$

from the zigzag. This can be done by Type (I) and (II) moves.

(ii) If $x_i, x_{i+1} \in \pi_1(\mathrm{Ob}(\mathcal{C}_d^\theta), M_i)$ are represented by smooth loops α_i and α_{i+1}, the relation becomes

$$\cdot \xrightarrow{W_{\alpha_i}} \cdot \xrightarrow{W_{\alpha_{i+1}}} \cdot \simeq \cdot \xrightarrow{W_{\alpha_i \cdot \alpha_{i+1}}} \cdot .$$

But this last morphism is equal to $W_{\alpha_i} \circ W_{\alpha_{i+1}}$, so the zigzags differ only by a Type (I) move.

If $x_i, x_{i+1} \in \pi_1(\mathrm{Mor}(\mathcal{C}_d^\theta), W_j) \times G$, there are various special cases to check. We shall check only two of them here.

Assume $x_i = ([\gamma_i], (0,0))$ and $x_{i+1} = ([\gamma_{i+1}], (0,0))$. Then their product is given by $x_i x_{i+1} = ([\gamma_i \cdot \gamma_{i+1}], (0,0))$. On the zigzag, this means that

$$\cdot \xrightarrow{W_{\partial_0 \gamma_i}} \cdot \xrightarrow{W_{\partial_0 \gamma_{i+1}}} \cdot \simeq \cdot \xrightarrow{W_{\partial_0 (\gamma_i \cdot \gamma_{i+1})}} \cdot .$$

But since $(\partial_0 \circ \gamma_i) \cdot (\partial_0 \circ \gamma_{i+1}) = \partial_0(\gamma_i \cdot \gamma_{i+1})$, the claim follows as in the previous case.

The case $x_i = ([\gamma_i], (1,0))$ and $x_{i+1} = ([\gamma_{i+1}], (0,1))$ is one of the more complicated. Now $x_i x_{i+1}$ defines the following part of a zigzag

$$\cdot \xleftarrow{W_j} \cdot \xrightarrow{W_{\partial_0 \gamma_i}} \cdot \xrightarrow{W_{\partial_0 \gamma_{i+1}}} \cdot \xrightarrow{W_j} \cdot .$$

By a Type (I) move, this is equivalent to

$$\cdot \xrightarrow{W_{\partial_1 \gamma_i}} \cdot \xleftarrow{W_{\partial_1 \gamma_i}} \cdot \xleftarrow{W_j} \cdot \xrightarrow{W_{\partial_0 \gamma_i}} \cdot \xrightarrow{W_{\partial_0 \gamma_{i+1}}} \cdot \xrightarrow{W_j} \cdot .$$

By Lemma 2.30, this is again equivalent to

$$\cdot \xrightarrow{W_{\partial_1 \gamma_i}} \cdot \xleftarrow{W_j} \cdot \xleftarrow{W_{\partial_0 \gamma_i}} \cdot \xrightarrow{W_{\partial_0 \gamma_i}} \cdot \xrightarrow{W_j} \cdot \xrightarrow{W_{\partial_1 \gamma_{i+1}}} \cdot .$$

Removing the middle part by Type (I) moves yields

$$\cdot \xrightarrow{W_{\partial_1 (\gamma_i \cdot \gamma_{i+1})}} \cdot .$$

This corresponds to the product $x_i x_{i+1} = ([\gamma_i \cdot \gamma_{i+1}], (1,1))$.

(iii) This is obvious from the definitions.

We are now ready to prove the theorem. Let a zigzag be given. For each morphism

$$\cdots \xrightarrow{W} \cdots \tag{2.10}$$

we do as follows. First choose a smooth path γ from W to the base point W_j in the W component of $\mathrm{Mor}(\mathcal{C}_d^\theta)$. Then (2.10) is equivalent to

$$\cdots \xleftarrow{W_{\partial_0\gamma}} \cdot \xrightarrow{W_j} \cdot \xrightarrow{W_{\partial_1\gamma}} \cdots$$

by Lemma 2.30. Finally, $W_{\partial_0\gamma} \circ W_{\overline{\partial_0\gamma}}$ is related to W_e by a Type (I) move. Thus (2.10) is also equivalent to

$$\cdots \xrightarrow{W_{\overline{\partial_0\gamma}}} \cdot \xrightarrow{W_j} \cdot \xrightarrow{W_{\partial_1\gamma}} \cdots .$$

But this zigzag is associated to a sequence $x_1 \cdots x_n$. Given another zigzag homotopic to this one, it is also equivalent to a zigzag coming from a sequence $x_1' \cdots x_{n'}'$. We know that these sequences are related by the operations (i)-(iii), and this corresponds to doing Type (I) and (II) moves on the zigzags. □

Proof of Theorem 2.28. The relation (I) certainly holds, since there is a 2-simplex in the classifying space having γ_{W_1}, γ_{W_2} and $\gamma_{W_1\circ W_2}$ as its sides. The relation (II) holds because the path γ determines a homotopy between the two zigzags.

To see that these are the only relations, we apply the generalized van Kampen theorem once again. Note that the inclusion

$$\pi_1(\mathrm{Ob}(\mathcal{C}_d^\theta) \cup (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1) \cup (N_2(\mathcal{C}_d^\theta) \times \Delta^2)) \to \pi_1(B\mathcal{C}_d^\theta)$$

is an isomorphism. This time, let

$$\begin{aligned} U_1 =& \mathrm{Ob}(\mathcal{C}_d^\theta) \cup (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1) \cup (N_2(\mathcal{C}_d^\theta) \times \Delta^2 \backslash \{pt\}) \\ U_2 =& (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1) \cup (N_2(\mathcal{C}_d^\theta) \times \Delta^2) \\ U_1 \cap U_2 =& (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1) \cup (N_2(\mathcal{C}_d^\theta) \times \Delta^2 \backslash \{pt\}). \end{aligned}$$

As base point set X_0, choose one representative $x_l = (W_1^l, W_2^l)$ for each element in $\pi_0(N_2(\mathcal{C}_d^\theta))$ such that $\partial_0(W_1^l) \in X_0'$ where X_0' is as in the proof of Theorem 2.31. Then

$$\begin{aligned} \pi_1(U_1, X_0) =& \pi_1(\mathrm{Ob}(\mathcal{C}_d^\theta) \cup (\mathrm{Mor}(\mathcal{C}_d^\theta) \times \Delta^1), X_0) \\ \pi_1(U_2, X_0) =& \bigsqcup_{l \in L} \pi_1(N_2(\mathcal{C}_d^\theta), x_l) \\ \pi_1(U_1 \cap U_2, X_0) =& \bigsqcup_{l \in L} \pi_1(N_2(\mathcal{C}_d^\theta), x_l) \times \mathbb{Z}. \end{aligned}$$

There is a map $X_0 \to X_0'$ given by $(W_1^l, W_2^l) \mapsto \partial_0(W_1^l)$. Again, this allows us to replace $\pi_1(U_1, X_0)$ by $\pi_1(U_1, X_0')$ and $\pi_1(U_1 \cup U_2, X_0)$ by $\pi_1(U_1 \cup U_2, X_0')$.

Let $K = \bigcup_{l \in L} \{x_l\} \times \mathbb{Z}$ be the kernel of $i_2 : \pi_1(U_1 \cap U_2, X_0) \to \pi_1(U_2, X_0)$. Since i_2 is vetex surjective and piecewise surjective in the sense of [20], Chapter 12, it is a quotient map. Thus $i_2 : \pi_1(U_1 \cap U_2, X_0)/K \to \pi_1(U_2, X_0)$ is an isomorphism.

Now we want to apply Proposition 27 of [20] to compute the coequalizer of the diagram. Let $N_1(K)$ denote the normal subgroupoid of $\pi_1(U_1, X_0')$ generated by the image of K, and let $N_2(K) = \bigcup_{l \in L} \{x_l\}$ be the trivial normal subgroupoid of $\pi_1(U_2, X_0)$. Then there is a diagram

$$K \rightrightarrows N_1(K) \sqcup N_2(K).$$

The coequalizer is the trivial normal subgroupoid, so by the proposition, there is a new coequalizer diagram

$$\pi_1(U_1 \cap U_2, X_0)/K \rightrightarrows \pi_1(U_1, X_0')/N_1(K) \sqcup \pi_1(U_2, X_0) \to \pi_1(U_1 \cup U_2, X_0').$$

But since $i_2 : \pi_1(U_1 \cap U_2, X_0)/K \to \pi_1(U_2, X_0)$ is an isomorphism, the coequalizer simply becomes $\pi_1(U_1, X_0')/N_1(K)$. This means that $\pi_1(U_1 \cup U_2, X_0')$ is $\pi_1(U_1, X_0')$, which we computed in Theorem 2.31, with the only new relations being the Type (I) relations determined by the $x_l \in N_2(\mathcal{C}_d^\theta)$. □

2.6 The Chimera Relations

In this section we give another description of $\pi_1(B\mathcal{C}_d^\theta)$ in terms of generators and relations.

Let F denote the free abelian group generated by diffeomorphism classes of d-dimensional manifolds with an equivalence class of θ-structures. Let $[W]$ denote the class of W. Since $B\mathcal{C}_d^\theta$ is a loop space by [14], its fundamental group is abelian. Hence the homomorphism

$$F \to \pi_1(B\mathcal{C}_d^\theta) \tag{2.11}$$

taking $[W]$ to the homotopy class of γ_W is well-defined by Lemma 2.29.

Let $W_1, W_2 \in \mathcal{C}_d^\theta(\emptyset, M)$ and $W_3, W_4 \in \mathcal{C}_d^\theta(M, \emptyset)$. The following loops are clearly homotopic in $B\mathcal{C}_d^\theta$:

$$\begin{aligned}
\gamma_{W_1 \circ W_3} \simeq& \gamma_{W_1} \cdot \gamma_{W_3} \\
\simeq& \gamma_{W_1} \cdot \gamma_{W_4} \cdot \bar{\gamma}_{W_4} \cdot \bar{\gamma}_{W_2} \cdot \gamma_{W_2} \cdot \gamma_{W_3} \\
\simeq& \gamma_{W_1 \circ W_4} \cdot \bar{\gamma}_{W_2 \circ W_4} \cdot \gamma_{W_2 \circ W_3}.
\end{aligned}$$

This implies:

Proposition 2.32. *For $W_1, W_2 \in \mathcal{C}_d^\theta(\emptyset, M)$ and $W_3, W_4 \in \mathcal{C}_d^\theta(M, \emptyset)$, the identity*

$$[W_1 \circ W_3] + [W_2 \circ W_4] = [W_1 \circ W_4] + [W_2 \circ W_3] \tag{2.12}$$

holds in $\pi_1(B\mathcal{C}_d^\theta)$.

We will refer to (2.12) as the chimera relations. Let C be the subgroup of F generated by the chimera relations. Then (2.11) induces a homomorphism

$$F/C \to \pi_1(B\mathcal{C}_d^\theta). \tag{2.13}$$

We can now state the main theorem of this section.

Theorem 2.33. *Assume* (*ii*) *of Theorem 2.17. Then* (2.13) *is an isomorphism.*

Assuming (ii) in Theorem 2.17, (2.13) is surjective. Indeed, the alternating zigzag

$$\cdot \xrightarrow{W_0} \cdot \xleftarrow{W_1} \cdots \xrightarrow{W_n} \cdot$$

is homotopic to the image of

$$\begin{aligned}[][W_0 \circ \overline{W}_1 \circ \cdots \circ W_n] + &\sum_{\substack{i=1 \\ i \text{ even}}}^{n} [W_n \circ \overline{W}_{n-1} \circ \cdots \circ W_i \circ \overline{W}_i \circ \cdots \circ \overline{W}_n] \\ - &\sum_{\substack{i=1 \\ i \text{ odd}}}^{n} [W_n \circ \overline{W}_{n-1} \circ \cdots \circ \overline{W}_i \circ W_i \circ \cdots \circ \overline{W}_n],\end{aligned} \tag{2.14}$$

see the proof of Theorem 2.17. Similarly, if the zigzag starts with a morphism path in the opposite direction, just switch all signs in the sum. If n is odd, the bars over the W_n's should be switched. If the zigzag is not alternating, insert identity morphisms to make it alternating and apply the formula.

We want to see that the formula (2.14) defines an inverse of (2.13). We break the proof up in lemmas.

Lemma 2.34. *The formula* (2.14) *obtained from an alternating zigzag only depends on the choice of opposite morphisms* $\overline{W}_i$ *up to chimera relations.*

Proof. Let an alternating zigzag

$$\cdot \xrightarrow{W_0} \cdot \xleftarrow{W_1} \cdots \xleftarrow{W_k} \cdots \xleftarrow{W_n} \cdot$$

be given. We choose opposites $\overline{W}_i$ of W_i for all i. Assume $\overline{W}'_k$ is a different choice of opposite to W_k. Define

$$\begin{aligned} L_0 &= W_0 \circ \overline{W}_1 \circ \cdots \circ \overline{W}_k \circ \cdots \circ \overline{W}_n \\ N_0 &= W_0 \circ \overline{W}_1 \circ \cdots \circ \overline{W}'_k \circ \cdots \circ \overline{W}_n, \end{aligned}$$

and for $1 \le i \le n$ and i odd,

$$\begin{aligned} L_i &= W_n \circ \overline{W}_{n-1} \circ \cdots \circ W_k \circ \cdots \circ W_i \circ \overline{W}_i \circ \cdots \circ \overline{W}_k \circ \cdots \circ \overline{W}_n \\ N_i &= W_n \circ \overline{W}_{n-1} \circ \cdots \circ W_k \circ \cdots \circ W_i \circ \overline{W}_i \circ \cdots \circ \overline{W}'_k \circ \cdots \circ \overline{W}_n. \end{aligned}$$

For i even, L_i and N_i are defined by the same formulas except the bars over the middle W_i's should be switched.

Using $\overline{W}_k$ as opposite, we see that (2.14) is given by

$$\sum_{i=0}^{n} (-1)^i [L_i], \tag{2.15}$$

while using $\overline{W}'_k$, it is

$$\sum_{i=0}^{n} (-1)^i [N_i]. \tag{2.16}$$

Note that $N_i = L_i$ for $i > k$. For $i \leq k$, we cut all the L_i's and N_i's between W_{k-1} and, respectively, $\overline{W}_k$ and $\overline{W}'_k$. Denote the parts by $L_i^{(j)}$ and $N_i^{(j)}$ for $j = 1, 2$ such that the $j = 2$ parts contain $\overline{W}_k$ or $\overline{W}'_k$. Then for $i, l \leq k$,

$$\begin{aligned} L_i^{(2)} &= L_l^{(2)} \\ N_i^{(2)} &= N_l^{(2)} \\ L_i^{(1)} &= N_i^{(1)}. \end{aligned}$$

Thus there is a chimera relation

$$\begin{aligned} [L_i] - [L_{i+1}] =& [L_i^{(1)} \circ L_i^{(2)}] - [L_{i+1}^{(1)} \circ L_{i+1}^{(2)}] \\ \sim& [L_i^{(1)} \circ N_i^{(2)}] - [L_{i+1}^{(1)} \circ N_{i+1}^{(2)}] \\ =& [N_i] - [N_{i+1}]. \end{aligned}$$

Since k is odd, the two sums (2.15) and (2.16) differ by $\frac{k+1}{2}$ applications of this relation. This takes care of the case where k is odd, n is odd, and the first path is travelled in the positive direction.

Changing the direction of all arrows in the zigzag only changes the signs in (2.15) and (2.16). If n is increased by one, an extra $L_{n+1} = N_{n+1}$ is added. This does not change the argument. Finally, if k is even, then $L_0 = N_0$ and the remaining L_i and N_i are as before. There is now an even number of $1 \leq i \leq k$, so the L_i and N_i still pair up. $\square$

Lemma 2.35. *If two zigzags differ only by the relation (I), the corresponding sums* (2.14) *are related by chimera relations.*

Proof. Let a zigzag be given. After inserting identity morphisms if necessary, we assume that it is alternating of the form

$$\cdot \xrightarrow{W_0} \cdot \xleftarrow{W_1} \cdots \xrightarrow{W_n} \cdot. \tag{2.17}$$

We choose opposites of all W_i and define

$$\begin{aligned} L_0 =& W_0 \circ \overline{W}_1 \circ \cdots \circ \overline{W}_k \circ \cdots \circ \overline{W}_n \\ L_i =& W_n \circ \overline{W}_{n-1} \circ \cdots \circ W_k \circ \cdots \circ W_i \circ \overline{W}_i \circ \cdots \circ \overline{W}_k \circ \cdots \circ \overline{W}_n. \end{aligned}$$

Then the zigzag (2.17) corresponds to the class

$$\sum_{i=0}^{n} (-1)^i [L_i]. \tag{2.18}$$

We first consider a special case of how apply the relation (I). Let k be odd and $W_k \circ U = W_{k+1}$ for some U. After inserting identity morphisms, the original zigzag is equivalent to

$$\cdot \xrightarrow{W_0} \cdot \xleftarrow{W_1} \cdots \xrightarrow{W_{k-1}} \cdot \xleftarrow{1_{\partial_1(W_{k-1})}} \cdot \xrightarrow{U} \cdot \xleftarrow{W_{k+2}} \cdots \xrightarrow{W_n} \cdot.$$

(If W_{k-1} is an inserted identity morphism, we should really remove two identity morphisms, but this does not change (2.14).) Let

$$\begin{aligned} N_0 =& W_0 \circ \overline{W}_1 \circ \cdots \circ W_{k-1} \circ U \circ \overline{W}_{k+2} \circ \cdots \circ \overline{W}_n \\ N_i =& W_n \circ \overline{W}_{n-1} \circ \cdots \circ W_{k+2} \circ \overline{U} \circ \overline{W}_{k-1} \circ W_{k-2} \circ \\ & \cdots \circ W_i \circ \overline{W}_i \circ \cdots \circ W_{k-1} \circ U \circ \overline{W}_{k+2} \circ \cdots \circ \overline{W}_n \end{aligned}$$

for $0 < i \leq k-1$ and

$$N_k = N_{k+1} = W_n \circ \bar{W}_{n-1} \circ \cdots \circ W_{k+2} \circ \bar{U} \circ U \circ \bar{W}_{k+2} \circ \cdots \circ \bar{W}_n.$$

For $k+2 \leq i \leq n$, let $N_i = L_i$. Then the new zigzag corresponds to

$$\sum_{i=0}^{n} (-1)^i [N_i]. \tag{2.19}$$

We may choose $\overline{U} = \overline{W}_{k+1} \circ W_k$. For $0 \leq i \leq k-1$, we cut all L_i between W_{k-1} and $\overline{W}_k$ and all N_i between W_{k-1} and U. Then there are chimera relations

$$[L_i] + [N_{i+1}] \sim [L_{i+1}] + [N_i]$$

for all $0 \leq i \leq k-2$. There is an even number of $i \leq k-2$. Moreover,

$$[N_{k-1}] + [L_k] \sim [L_{k-1}] + [L_{k+1}]$$

by another chimera relation. Finally, $N_k = N_{k+1}$ and $N_i = L_i$ for $i \geq k+2$. Hence the two sums (2.18) and (2.19) are equivalent under the chimera relations.

If we consider the case $W_k = W_{k+1} \circ U$ instead, $[L_0] = [N_0]$ and there is a chimera relation

$$[L_k] + [N_{k+1}] \sim [L_{k+1}] + [N_k].$$

From these cases, the statement is easily deduced for n odd, k even, and the case where all arrows are switched.

We could also apply (I) to replace

$$\cdot \xrightarrow{W_k} \cdot \xleftarrow{1_{\partial_1(W_k)}} \cdot \xrightarrow{W_{k+1}} \cdot$$

by

$$\cdot \xrightarrow{W_k \circ W_{k+1}} \cdot.$$

This only removes two identical terms with opposite signs from the sum (2.14).

All other applications of (I) may be given as a sequence of the moves considered above. □

Lemma 2.36. *A Type (II) move does not change the sum* (2.14).

Proof. Let a zigzag

$$\cdot \xrightarrow{W_0} \cdots \xrightarrow{W_k} \cdots \xrightarrow{W_n} \cdot$$

be given.

If W_k is replaced by some W_k' by a Type (II) move, we may choose $\overline{W}_k'$ equal to $\overline{W}_k$. The manifolds in the sum (2.14) coming from the first zigzag are of the forms

$$W_0 \circ \overline{W}_1 \circ \cdots \circ W_k \circ \cdots \circ W_n$$
$$W_n \circ \overline{W}_{n-1} \circ \cdots \circ W_k \circ \cdots W_i \circ \overline{W}_i \circ \cdots \circ \overline{W}_k \circ \cdots \circ W_{n-1} \circ \overline{W}_n.$$

By Lemma 2.29, replacing W_k by W_k' does not change the diffeomorphism classes, so (2.14) is unchanged. □

Proof of Theorem 2.33. We need to see that the surjection

$$F/C \to \pi_1(B\mathcal{C}_d^\theta)$$

is injective. Consider the composition

$$Z \xrightarrow{\pi} F/C \to \pi_1(B\mathcal{C}_d^\theta)$$

where Z is the set of all zigzags and π is defined by the formula (2.14). This is well-defined by Lemma 2.34. By Lemma 2.35 and 2.36, π induces a well-defined map $\bar{\pi}$ on the equivalence classes $\bar{Z}$ for the equivalence relation on Z generated by the relations (I) and (II). But by Theorem 2.28, the composition

$$\bar{Z} \xrightarrow{\bar{\pi}} F/C \to \pi_1(B\mathcal{C}_d^\theta)$$

is bijective. Thus it is enough to see that π is surjective.

Let $x \in F/C$. The chimera relations imply that $[W_1] + [W_2] = [W_1 \sqcup W_2]$. Thus we can represent x by an element $[W] - [W'] \in F$. This is $\pi(\gamma_W \cdot \bar{\gamma}_{W'})$, since we may choose $\overline{W}' = \emptyset$. □

Chapter 3

Low-Dimensional Calculations of Homotopy Groups

Recall that we are interested in the homomorphisms

$$\theta^r : \pi_{q-1}(V_{d,r}) \to \pi_q(MT(d,r))$$

for $q \leq d$. In particular, we would like to find some conditions under which this map is injective. The groups $\pi_{q-1}(V_{d,r})$ have been calculated in [36] for $r \leq 6$. The purpose of this chapter is to compute the homotopy groups $\pi_q(MT(d,r))$ and use this to show injectivity of θ^r for certain values of q, d, and r.

The computations are carried out by means of the Adams spectral sequence. Therefore we recall this and the basic properties we need in Section 3.1. The input for the spectral sequence is the cohomology of $MT(d,r)$, so we determine this in Section 3.3. To do so, we first need some notation and facts about the cohomology structure of Grassmannians given in Section 3.2. In Section 3.4, we prove a periodicity property for the spectra $MT(d,r)$ that will come in handy. It also plays an important role in the last two chapters of the thesis. The actual computations are performed for the oriented spectrum in Section 3.5, in the spin case in Section 3.6 and 3.7, and in the unoriented case in Section 3.8 .

From the computations of $\pi_q(MT(d,r))$ and the long exact sequence

$$\cdots \to \pi_q(MT(d-r)) \to \pi_q(MT(d)) \to \pi_q(MT(d,r)) \to \cdots ,$$

the groups $\pi_q(MT(d-r))$ may be determined. This also yields an identification of the invariant $\beta^4(M)$. This is worked out in Section 3.9. Together, the results of this chapter prove the claim of Theorem 1.3.

3.1 The Adams Spectral Sequence

If X is a spectrum of finite type, $H^*(X;\mathbb{Z}/p) \cong \varprojlim_n \widetilde{H}^{*+n}(X_n;\mathbb{Z}/p)$ is a module over the mod p Steenrod algebra $\mathcal{A}_p$ in the obvious way. If one knows the cohomology structure of X, there is spectral sequence converging to the p-primary part of the homotopy groups of X, i.e. $\pi_*(X)$ divided out by torsion elements of order prime to p. This is known as the Adams spectral sequence:

Theorem 3.1. *For a connective spectrum of finite type, there is a natural spectral sequence* $\{E_k^{s,t}, d_k\}$ *with differentials* $d_k : E_k^{s,t} \to E_k^{s+r,t+r-1}$ *such that:*

(i) $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p}^{s,t}(H^*(X;\mathbb{Z}/p), \mathbb{Z}/p)$.

(ii) There is a filtration

$$\cdots \subseteq F^{s+1,t+1} \subseteq F^{s,t} \subseteq \cdots \subseteq F^{0,t-s} = \pi_{t-s}(X)$$

such that $E_\infty^{s,t} = F^{s,t}/F^{s+1,t+1}$.

(iii) $\bigcap_k F^{s+k,t+k}$ *is the subgroup of* $\pi_{t-s}(X)$ *consisting of elements of finite order prime to* p.

See e.g. [19] for the construction.

$\mathrm{Ext}_{\mathcal{A}_p}^{s,t}(H^*(X;\mathbb{Z}/p), \mathbb{Z}/p)$ is defined as follows. Take a resolution of $H^*(X;\mathbb{Z}/p)$, i.e. an exact sequence

$$0 \leftarrow H^*(X;\mathbb{Z}/p) \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$$

where each F_s is a free $\mathcal{A}_p$-module. Then apply the functor $\mathrm{Hom}_{\mathcal{A}_p}^t(-, \mathbb{Z}/p)$, i.e. homomorphisms to the $\mathcal{A}_p$-module $\mathbb{Z}/p$ that lower degree by t. The homology of this dual complex at F_s is $\mathrm{Ext}_{\mathcal{A}_p}^{s,t}(H^*(X;\mathbb{Z}/p), \mathbb{Z}/p)$.

Naturality means that a map $f : X \to Y$ induces a filtration preserving map $f^* : H^*(Y;\mathbb{Z}/p) \to H^*(X;\mathbb{Z}/p)$. The extension to a map of resolutions defines a map of spectral sequences.

There is a pairing

$$\mathrm{Ext}_{\mathcal{A}_p}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes \mathrm{Ext}_{\mathcal{A}_p}^{s',t'}(H^*(X;\mathbb{Z}/p), \mathbb{Z}/p) \to \mathrm{Ext}_{\mathcal{A}_p}^{s+s',t+t'}(H^*(X;\mathbb{Z}/p), \mathbb{Z}/p)$$

defined algebraically using resolutions. This induces a product

$$\bar{E}_k^{s,t} \otimes E_k^{s',t'} \to E_k^{s+s',t+t'}$$

where $\bar{E}$ denotes the spectral sequence for the sphere spectrum. The product converges to the composition product

$$\pi_*^s(S^0) \otimes \pi_*(X) \to \pi_*(X).$$

That is, the composition product respects the filtrations given by the spectral sequences, and the induced product on E_∞ agrees with the one coming from the product on the spectral sequences. The differentials behave nicely on products:

$$d^k(xy) = x d_k(y) + (-1)^{t-s} d_k(x) y.$$

See [32] for a more detailed explanation of this product.

We are mainly interested in the elements $h_0 \in \mathrm{Ext}_{\mathcal{A}_p}^{1,1}(\mathbb{Z}/p, \mathbb{Z}/p)$ corresponding to a degree p map and $h_1 \in \mathrm{Ext}_{\mathcal{A}_p}^{1,2}(\mathbb{Z}/2, \mathbb{Z}/2)$ corresponding to the Hopf map. If $x \in E_2^{s,t}$ represents some map in $\pi_{t-s}(X)$, $h_0 x$ represents p times this map if it survives to E_∞. Similarly, $h_1 x$ corresponds to the map composed with the Hopf map. Thus the product yields some information about extensions.

3.2 Cohomology of Grassmannians

In this section we shall recall the cohomology structure of the Grassmann manifolds. It also serves as an introdution of notation. A good reference for this is [35].

For a d-dimensional vector bundle $p : E \to X$, there is a Thom isomorphism $\phi : H^*(X; \mathbb{Z}/2) \to \widetilde{H}^{*+d}(\mathrm{Th}(E); \mathbb{Z}/2)$ with Thom class $u \in H^d(\mathrm{Th}(E); \mathbb{Z}/2)$.

Definition 3.2. *Let $w_i(E) \in H^i(X; \mathbb{Z}/2)$ be the class $\phi^{-1}(\mathrm{Sq}^i(u))$. This is called the ith Stiefel–Whitney class of E.*

Proposition 3.3. *The Stiefel–Whitney classes have the following properties:*

(i) $w_i(E) = 0$ *for* $i > d$.

(ii) If $f : Y \to X$ is a map and $E \to X$ is a vector bundle, then

$$f^*(w_i(E)) = w_i(f^*E).$$

(iii) For a sum $E \oplus F$ of vector bundles,

$$w_i(E \oplus F) = \sum_j w_j(E) w_{i-j}(F).$$

(iv) For the trivial vector bundle $X \times \mathbb{R}^d$, $w_i(X \times \mathbb{R}^d) = 0$ for all $i \geq 1$.

If a vector bundle E splits as a sum $E \cong E' \oplus \mathbb{R}^r$ for some $(d-r)$-dimensional bundle E', Proposition 3.3 (iii) and (iv) imply that $w_i(E) = w_i(E')$ for all i. In particular, $w_{d-r+1}(E), \ldots, w_d(E)$ must vanish. However, this is not a sufficient condition for the existence of r independent sections. For instance, the tangent bundle of S^2 satisfies $TS^2 \oplus \mathbb{R} \cong S^2 \times \mathbb{R}^3$. Thus $w_i(TS^2) = 0$ for all $i \geq 1$, even though TS^2 has no vector field without zeros.

Let w_i denote the ith Stiefel–Whitney class for the universal bundle $U_d \to BO(d)$, and let $R[x_1, \ldots, x_k]$ denote the polynomial algebra over R on generators x_i in degree i.

Theorem 3.4.

$$H^*(BO(d); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \ldots, w_d]$$
$$H^*(BSO(d); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, \ldots, w_d].$$

The restriction $H^*(B(d); \mathbb{Z}/2) \to H^*(G(d,n); \mathbb{Z}/2)$ is surjective in both the oriented and unoriented case with kernel the ideal generated by

$$\bar{w}_{n+1}, \bar{w}_{n+2}, \ldots$$

The classes $\bar{w}_i$ are characterized by $\bar{w}_0 = 1$ and

$$\sum_j w_j \bar{w}_{i-j} = 1.$$

By Proposition 3.3 (iii), the direct sum map $BO \times BO \to BO$ induces a comultiplication $\Delta^* : H^*(BO;\mathbb{Z}/2) \to H^*(BO;\mathbb{Z}/2) \otimes H^*(BO;\mathbb{Z}/2)$ given by

$$\Delta^*(w_i) = \sum_j w_j \otimes w_{i-j}. \tag{3.1}$$

If $E \to X$ is oriented, there is a Thom isomorphism with coefficients in any ring R.

Definition 3.5. *Let $E \to X$ be an oriented vector bundle of dimension d. The Euler class $e(E) \in H^d(X;\mathbb{R})$ is the pullback $s^*(u)$ of the Thom class $u \in H^d(\mathrm{Th}(E);R)$ by the inclusion $s : X \to \mathrm{Th}(E)$ of the zero section.*

The Euler class vanishes if E allows a nowhere vanishing section.

Theorem 3.6. *Let F denote one of the fields $\mathbb{Q}$ and $\mathbb{Z}/p$ for p an odd prime. Then*

$$H^*(BO(d);F) \cong F[p_1,\dots,p_{[\frac{d}{2}]}].$$

Here $p_i \in H^{4i}(BO(d);F)$ are the Pontryagin classes for the universal bundle. For $x \in \mathbb{R}$, $[x]$ denotes the integer part of x.

When d is odd, $BSO(d) \to BO(d)$ induces an isomorphism on cohomology with coefficients in F. For d even,

$$H^*(BSO(d);F) \cong F[p_1,...,p_{\frac{d}{2}},e_d]/\langle e_d^2 - p_{\frac{d}{2}}\rangle$$

where the p_i's are the Pontryagin classes for the oriented universal bundle and e_d is the Euler class in $H^d(BSO(d);F)$ for the oriented universal bundle with coefficients in F. It satisfies the relation $e_d^2 = p_{\frac{d}{2}}$.

3.3 Cohomology of the Spectrum

We are now ready to describe the cohomology groups of the spectra $MT(d,r)$. Unless otherwise specified, everything works for both the oriented and unoriented spectrum.

We first consider cohomology groups with $\mathbb{Z}/2$ coefficients understood. The cohomology group $H^k(MT(d))$ is given by $\varprojlim_n \widetilde{H}^{k+n}(\mathrm{Th}(U_{d,n}^\perp))$. The bundle $U_{d,n}^\perp$ has a Thom class $\bar{u}_n \in H^n(\mathrm{Th}(U_{d,n}^\perp))$. The $\bar{u}_n$'s define a stable Thom class $\bar{u}$ in $H^0(MT(d))$, and there is a Thom isomorphism given by cup product with $\bar{u}$

$$\phi : H^*(B(d)) \to H^*(MT(d)).$$

That is, $H^*(MT(d))$ is the free module over $H^*(B(d))$ generated by $\bar{u}$.

Theorem 3.7. *The map*

$$H^*(MT(d,r)) \to H^*(MT(d))$$

is injective with image the $H^*(B(d))$-submodule generated by $\phi(w_{d-r+1}),\dots,\phi(w_d)$.

Proof. There is a long exact sequence

$$\cdots \to H^*(BO(d), BO(d-r)) \to H^*(BO(d)) \to H^*(BO(d-r)) \to \cdots .$$

By Theorem 3.4 and naturality of the Stiefel–Whitney classes, the map

$$H^*(BO(d)) \to H^*(BO(d-r))$$

is the surjection

$$\mathbb{Z}/2[w_1, \dots, w_d] \to \mathbb{Z}/2[w_1, \dots, w_{d-r}].$$

Thus $H^*(BO(d), BO(d-r))$ is the exactly the kernel, i.e. the ideal generated by $w_{d-r+1}, \dots, w_d$.

For any n, the Thom isomorphism yields a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & H^k(\mathrm{Th}(U_{d,n}^\perp), \mathrm{Th}(U_{d-r,n}^\perp)) \arrow[r] & \widetilde{H}^k(\mathrm{Th}(U_{d,n}^\perp)) \arrow[r] & \widetilde{H}^k(\mathrm{Th}(U_{d-r,n}^\perp)) \arrow[r] & {} \\
{} \arrow[r] & H^k(G(d,n), G(d-r,n)) \arrow[r] \arrow[u, "\cong"] & H^k(G(d,n)) \arrow[r] \arrow[u, "\cong"] & H^k(G(d-r,n)) \arrow[r] \arrow[u, "\cong"] & {}.
\end{tikzcd}$$

Letting n tend to infinity, the claim follows. The oriented case is similar. □

The multiplication

$$H^*(B(d)) \otimes H^*(MT(d)) \to H^*(MT(d))$$

is also induced by the diagonal $MT(d) \to B(d)_+ \wedge MT(d)$. This takes $MT(d-r)$ to $B(d)_+ \wedge MT(d-r)$. Hence there is a diagonal $MT(d,r) \to B(d)_+ \wedge MT(d,r)$ inducing the $H^*(B(d))$-module structure on $H^*(MT(d,r))$. The diagram

$$\begin{tikzcd}
MT(d,r) \arrow[r] & B(d)_+ \wedge MT(d,r) \\
\Sigma^\infty \Sigma V_{d,r} \arrow[r] \arrow[u, "f_\theta"] & S^0 \wedge \Sigma^\infty \Sigma V_{d,r} \arrow[u, "i \wedge f_\theta"']
\end{tikzcd}$$

shows that $f_\theta^* : H^*(MT(d,r)) \to H^*(\Sigma^\infty \Sigma V_{d,r})$ is a homomorphism of $H^*(B(d))$-modules when $H^*(\Sigma^\infty \Sigma V_{d,r})$ is given the trivial module structure.

Theorem 3.8. *Assume $k < 2(d-r)$. Then*

$$\widetilde{H}^k(V_{d,r}) = \begin{cases} \mathbb{Z}/2 & \text{if } d-r \le k \le d-1, \\ 0 & \text{if } k < d-r, \\ 0 & \text{if } d \le k. \end{cases}$$

The map

$$H^*(MT(d,r)) \xrightarrow{f_\theta^*} H^*(\Sigma^\infty \Sigma V_{d,r})$$

is the $H^*(B(d))$-module homomorphism that takes $\phi(w_k) \in H^k(MT(d,r))$ to the generator in $H^k(\Sigma V_{d,r})$.

Proof. There is a $2(d-r)$-connected map $\mathbb{R}P^{d-1}/\mathbb{R}P^{d-r-1} \to V_{d,r}$, see [23]. The first claim follows from this.

Thus we just need to determine $f_\theta^*(\phi(w_k))$. For $r = 1$, f_θ is the inclusion

$$\Sigma^{\infty+d} S^0 \to \Sigma^{\infty+d} B(d)_+.$$

This clearly induces an isomorphism on H^d, and the claim follows. For general r, the claim follows from the diagram

$$\begin{tikzcd} {} \arrow[r] & H^*(MT(d,1)) \arrow[r] \arrow[d, "f_\theta^*"] & H^*(MT(d,r)) \arrow[r] \arrow[d, "f_\theta^*"] & H^*(MT(d-1,r-1)) \arrow[r] \arrow[d, "f_\theta^*"] & {} \\ {} \arrow[r] & H^*(\Sigma V_{d,1}) \arrow[r] & H^*(\Sigma V_{d,r}) \arrow[r] & H^*(\Sigma V_{d-1,r-1}) \arrow[r] & {} \end{tikzcd}$$

by induction. □

Let C_θ denote the cofiber of the inclusion of spectra

$$f_\theta : \Sigma^{\infty+1} V_{d,r} \to MT(d,r). \tag{3.2}$$

Corollary 3.9. *The cofibration* (3.2) *induces an injective map*

$$H^k(C_\theta) \to H^k(MT(d,r))$$

in dimensions $k \leq 2(d-r)+1$ with image the $H^*(B(d))$-submodule

$$H^{>0}(B(d)) \cdot H^*(MT(d,r))$$

in dimensions $k \leq 2(d-r)$.

Proof. This follows from the long exact sequence in cohomology for the cofibration (3.2) and Theorem 3.8. □

For the remainder of this section, let F be either $\mathbb{Q}$ or $\mathbb{Z}/p$ for p an odd prime. For $MTSO(d)$, there is again a Thom isomorphism on cohomology with coefficients in F. This allows us to compute the cohomology groups with coefficients in F

Theorem 3.10. *In dimensions $* \leq 2(d-r)+1$, $H^*(MTSO(d,r);F)$ is isomorphic to the free $H^*(BSO(d);F)$-module on generators*

$$\phi(\delta(e_{d-r})) \in H^{d-r+1}(MTSO(d,r);F)$$
$$\phi(e'_d) \in H^d(MTSO(d,r);F).$$

Here δ is the coboundary map and ϕ is the Thom isomorphism. Moreover, e_{d-r} is the Euler class in $H^{d-r}(BSO(d-r);F)$ and e'_d maps to the Euler class in $H^d(BSO(d);F)$. These are zero exactly when $d-r$ and d, respectively, are odd.

Proof. In the following, all cohomology groups have coefficients in F.

Look at the exact sequence

$$\cdots \to H^*(BSO(d), BSO(d-r)) \to H^*(BSO(d)) \to H^*(BSO(d-r)) \to \cdots .$$

The Pontryagin classes $p_i \in H^{4i}(BSO(d))$ map to the corresponding classes in $H^{4i}(BSO(d-r))$. The Euler class $e_d \in H^d(BSO(d))$ maps to the Euler class for the bundle $U_{d-r} \oplus \mathbb{R}^r$, which is zero. Thus there must be a non-zero class in $H^d(BSO(d), BSO(d-r))$ that maps to e_d, and similarly there must be something hit all of $H^*(BSO(d)) \cdot e_d$.

On the other hand, the Euler class $e_{d-r} \in H^{d-r}(BSO(d-r))$ is not hit by anything, so $\delta(e_{d-r})$ is non-zero, and similarly $\delta(H^*(BSO(d-r)) \cdot e_{d-r})$ must be non-zero. But this is equal to $H^*(BSO(d)) \cdot \delta(e_{d-r})$ in the relevant dimensions by [8], Chapter VI, Theorem. 4.5. This combined with a Thom isomorphism yields the result. □

Theorem 3.11. *Assume that $k < 2(d-r)$ and F is as in Theorem 3.10.*

$$\widetilde{H}^k(V_{d,r}; F) = \begin{cases} F & \text{if } k = d-r \text{ and } d-r \text{ is even,} \\ F & \text{if } k = d-1 \text{ and } d \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The map $f_\theta^* : H^*(MTSO(d,r); F) \to H^*(\Sigma V_{d,r}; F)$ is a $H^*(BSO(d); F)$-module homomorphism that maps $\delta(e_{d-r})$ to a generator of $H^{d-r+1}(\Sigma V_{d,r}; F)$ and e'_d to a generator of $H^d(\Sigma V_{d,r}; F)$.

Proof. The proof goes more or less as in the $\mathbb{Z}/2$ case. □

3.4 A Periodicity Map

Let a_r be the number given by the table

r	1	2	3	4	5	6	7	8
a_r	1	2	4	4	8	8	8	8

(3.3)

for $r \leq 8$, and in general by $a_{r+8} = 16a_r$. The number a_{r+1} is the least integer such that $\mathbb{R}^{a_{r+1}}$ is a module over the Clifford algebra Cl_r. See [25] for details.

Theorem 1.4 can be stated as follows:

Theorem 3.12. *S^{d-1} has r independent vector fields if and only if d is divisible by a_{r+1}.*

The vector fields may be constructed in the following way. Assume that $a_{r+1} \mid d$. Then the Clifford algebra Cl_r has a d-dimensional representation V. There is a monomorphism $\mathbb{R}^{r+1} \to Cl_r$ such that, for the standard orthonormal basis $e_0, \ldots, e_r$ for $\mathbb{R}^{r+1}$, e_0 is mapped to the identity element and $\text{span}\{e_1, \ldots, e_r\}$ is mapped to the copy of $\mathbb{R}^r$ in the degree one part of Cl_r. The Clifford multiplication restricts to a bilinear multiplication $\mathbb{R}^{r+1} \times V \to V$, and one can choose an inner product on

V such that for all $x \in V$, the inner product $\langle e_i x, e_j x\rangle$ is zero for $i \neq j$ and $\langle x, x\rangle$ for $i = j$. In particular for $x \in S^{d-1}$, this means that $e_0 x, \dots, e_r x$ are orthonormal. Since $e_0 x = x$, r orthonormal vector fields on S^{d-1} are given by $e_1 x, \dots, e_r x$.

Let $\mathbb{R}^{d+k} = \mathbb{R}^d \oplus \mathbb{R}^k$ and assume $a_r \mid k$. Then by the above, there exists an orthogonal bilinear multiplication $\mathbb{R}^r \times \mathbb{R}^k \to \mathbb{R}^k$. This defines a map

$$f_0 : (W_{d,r}, V_{d,r}) \times (D^k, S^{k-1}) \to (W_{d+k,r}, V_{d+k,r}) \tag{3.4}$$

by

$$(v_0, \dots, v_{r-1}, x) \mapsto (\sqrt{1-|x|^2}v_0 + e_0 x, \dots, \sqrt{1-|x|^2}v_{r-1} + e_{r-1}x).$$

Lemma 3.13. *The map $f_0 : \Sigma^k \Sigma V_{d,r} \to \Sigma V_{d+k,r}$ from* (3.4) *is a* $(2(d-r)+k+1)$*-equivalence.*

Proof. The proof will proceed by induction on r. For $r = 1$, the map

$$(D^d, S^{d-1}) \times (D^k, S^{k-1}) \to (D^{d+k}, S^{d+k-1})$$

is given by

$$(x, y) \mapsto (\sqrt{1-|x|^2}y + x),$$

which is a homeomorphism $S^d \wedge S^k \to S^{d+k}$.

More generally, look at the map $g : S^{d-r} \to V_{d,r}$ mapping $v_0 \in S^{d-r}$ to the frame $(v_0, u_1, \dots, u_{r-1})$ where $u_1, \dots, u_{r-1}$ are the first $r-1$ standard basis vectors in $\mathbb{R}^d$ and $v_0 \in \mathbb{R}^{d-r+1}$ is included in $\mathbb{R}^d = \mathbb{R}^{r-1} \oplus \mathbb{R}^{d-r+1}$. The following diagram commutes up to homotopy

$$\begin{tikzcd}
\Sigma^k \Sigma S^{d-r} \arrow[r, "\Sigma^k \Sigma g"] \arrow[d, "f_0"] & \Sigma^k \Sigma V_{d,r} \arrow[d, "f_0"] \\
\Sigma S^{d+k-r} \arrow[r, "\Sigma g"] & \Sigma V_{d+k,r}.
\end{tikzcd}$$

A homotopy is given by applying to the $(i+1)$th vector the homotopy

$$F(x, v_0, t) = \sqrt{(1-t^2)|x|^2 + (1-|x|^2)|v_0|}u_i + te_i x. \tag{3.5}$$

Both g's in the diagram induce isomorphisms on cohomology with $\mathbb{Z}/2$ coefficients in dimension $d-r+k+1$. Furthermore, the right hand f_0 induces an isomorphism in this dimension, so the left hand f_0 must also be an isomorphism in dimension $d-r+k+1$. Similarly, one can see that f_0 induces an isomorphism on cohomology with $\mathbb{Q}$ and $\mathbb{Z}/p$ coefficients in this dimension.

Finally consider the map $h : V_{d,r} \to V_{d,r-1}$ that forgets the last vector. This is an isomorphism on cohomology in dimension $d-r+1, \dots, 2(d-r)$. This follows e.g. from the Serre spectral sequence for the fibration $S^{d-r} \to V_{d,r} \to V_{d,r-1}$. The diagram

$$\begin{tikzcd}
\Sigma^{k+1} V_{d,r} \arrow[r, "h"] \arrow[d, "f_0"] & \Sigma^{k+1} V_{d,r-1} \arrow[d, "f_0"] \\
\Sigma V_{d+k,r} \arrow[r, "h"] & \Sigma V_{d+k,r-1}
\end{tikzcd}$$

commutes. Thus the left vertical map induces an isomorphism on cohomology in dimensions $d+k-r+2, \dots, 2(d-r)+k+1$, since the other three maps do so by induction. Similarly with $\mathbb{Q}$ and $\mathbb{Z}/p$ coefficients.

By the universal coefficient theorem for homology, f_0 also induces an isomorphism on homology with integer coefficients, see e.g. [18], Corollary A3.6. So by the Whitehead theorem, the map is a $(2(d-r)+k+1)$-equivalence. $\square$

Since f_0 is independent of the coordinates in $\mathbb{R}^d$, it extends to a map

$$f : (W_r(U_{d,n}), V_r(U_{d,n})) \times (D^k, S^{k-1}) \to (W_r(U_{d+k,n}), V_r(U_{d+k,n}))$$

given by

$$f(V, v_1, \dots, v_r, x) = (V \oplus \mathbb{R}^k, \sqrt{1-|x|^2}v_0 + e_0 x, \dots, \sqrt{1-|x|^2}v_{r-1} + e_{r-1}x).$$

This defines a map $f : MTO(d,r) \wedge S^k \to MTO(d+k,r)$. For $MTSO$ and $MTSpin$, the map can be constructed similarly.

Theorem 3.14. *For MTO and $MTSO$, the map f induces an isomorphism of $H^*(B(d+k);\mathbb{Z}/2)$-modules*

$$H^{*+k}(MT(d+k,r);\mathbb{Z}/2) \to H^*(MT(d,r);\mathbb{Z}/2)$$

in dimensions $* \leq 2(d-r+1)$. It takes the generators $\phi(w_{d+k-r+1}), \dots, \phi(w_{d+k})$ to $\phi(w_{d_r+1}), \dots, \phi(w_d)$. In the $MTSO(d,r)$ case, f is a $(2(d-r+1)+k)$-equivalence.

In the following, $\phi(w_i) \in H^i(MT(d,r);\mathbb{Z}/2)$ will just be denoted by w_i for simplicity.

Proof. Consider the diagram

$$\begin{tikzcd}
\Sigma^{\infty+k+1}V_{d,r} \arrow[r, "f_\theta"] \arrow[d, "f_0"] & \Sigma^k MT(d,r) \arrow[r] \arrow[d, "f"] & \Sigma^k MT(d,r) \wedge B(d) \arrow[d, "f \wedge i"] \\
\Sigma^{\infty+1}V_{d+k,r} \arrow[r, "f_\theta"] & MT(d+k,r) \arrow[r] & MT(d+k,r) \wedge B(d+k).
\end{tikzcd} \tag{3.6}$$

By Theorem 3.7, $H^*(MT(d+k,r);\mathbb{Z}/2)$ is isomorphic to the free $H^*(B(d+k);\mathbb{Z}/2)$-module on generators $w_{d+k-r+1}, \dots, w_{d+k}$ in dimensions $* < 2(d+k-r+1)$ and $H^*(MT(d,r);\mathbb{Z}/2)$ is the free $H^*(B(d);\mathbb{Z}/2)$-module generated by $w_{d-r+1}, \dots, w_d$ up to dimension $2(d-r+1)$. Moreover, $i^* : H^*(B(d+k);\mathbb{Z}/2) \to H^*(B(d);\mathbb{Z}/2)$ is an isomorphism up to dimension d, and f^* respects the module structure by the right hand side of the diagram. Thus we just need to check that f^* maps $w_{d+k-r+1}, \dots, w_{d+k}$ to $w_{d-r+1}, \dots, w_d$.

It obviously takes $w_{d+k-r+1}$ to w_{d-r+1} by the left hand side of the diagram. In particular, this proves the claim for $r=1$. For general r, it follows by induction and the diagram

$$\begin{tikzcd}
H^*(\Sigma^k MT(d,r-1);\mathbb{Z}/2) \arrow[r] & H^*(\Sigma^k MT(d,r);\mathbb{Z}/2) \\
H^*(MT(d+k,r-1);\mathbb{Z}/2) \arrow[r] \arrow[u, "f^*"] & H^*(MT(d+k,r);\mathbb{Z}/2). \arrow[u, "f^*"']
\end{tikzcd}$$

In the oriented case, a similar argument shows that f^* is a $(2(d-r+1)+k)$-equivalence on homology with $\mathbb{Q}$ and $\mathbb{Z}/p$ coefficients for $p>2$. So by the universal coefficient theorem, the same holds with integer coefficients. This implies that f is a $(2(d-r+1)+k)$-equivalence. $\square$

Theorem 3.14 and the diagram (3.6) implies:

Corollary 3.15. *There is a commutative diagram*

$$\begin{tikzcd} \pi_q(\Sigma V_{d,r}) \arrow[r, "\theta^r"] \arrow[d, "f_{0*}"] & \pi_q(MTSO(d,r)) \arrow[d, "f_*"] \\ \pi_{q+k}(\Sigma V_{d+k,r}) \arrow[r, "\theta^r"] & \pi_{q+k}(MTSO(d+k,r)). \end{tikzcd}$$

For all $q<2(d-r+1)$, the vertical maps are isomorphisms.

3.5 Calculation of Homotopy Groups

We are now ready to calculate the homotopy groups of $MTSO(d,r)$ in low dimensions. We want to make use of the Adams spectral sequence described in Section 3.1, so we need to compute $\operatorname{Ext}^{s,t}_{\mathcal{A}_p}(H^*(MTSO(d,r);\mathbb{Z}/p),\mathbb{Z}/p)$. Thus we start out by describing the structure of $H^*(MTSO(d,r);\mathbb{Z}/2)$ as a module over the mod 2 Steenrod algebra $\mathcal{A}_2$. Throughout this section, all cohomology groups have coefficients in $\mathbb{Z}/2$ unless explicitly indicated.

By Exercise 8-A in [35], the $\mathcal{A}_2$-action on $w_i \in H^i(BO(d))$ is given by the formula

$$Sq^k(w_i) = \sum_{j=0}^{k} \binom{i-k+j-1}{j} w_{i+j}w_{k-j}. \tag{3.7}$$

Together with the Cartan formula, this yields formulas for the $\mathcal{A}_2$-action on all of $H^*(BSO(d))$. Consider the Thom isomorphism

$$\phi : H^i(BSO(d)) \to H^i(MT(d)).$$

It follows from the definition of the Stiefel–Whitney classes that

$$Sq(\phi(x)) = \phi(Sq(x) \smile w(U_d^\perp))$$

where $Sq = \sum \mathrm{Sq}^i$ denotes the total square, $w=\sum_i w_i$ is the total Stiefel–Whitney class, and

$$w(U_d^\perp) = \varprojlim_n w(U_{d,n}^\perp) \in \varprojlim_n \widetilde{H}^{*+n}(\mathrm{Th}(U_{d,n}^\perp)) \cong H^*(MT(d)).$$

By Proposition 3.3, $w(U_d^\perp)$ must satisfy

$$w(U_d) \smile w(U_d^\perp) = 1,$$

so the $w_i(U_d^\perp) = \bar{w}_i$ can be computed from $w(U_d)$ inductively.

In this way we obtain formulas for the Steenrod action on $H^*(MTSO(d))$, and these formulas hold in $H^*(MTSO(d,r))$ as well by naturality of the Steenrod squares.

The table below shows how $\mathcal{A}_2$ acts on $H^*(MTSO(d,r))$ in low dimensions. The middle column displays a $\mathbb{Z}/2$-basis for the cohomology in each dimension, and the right column shows how the Steenrod algebra acts on these basis elements.

t	$H^t(MTSO(d,r))$	
$d-r+1$	w_{d-r+1}	
$d-r+2$	w_{d-r+2}	$Sq^1(w_{d-r+1}) = (d-r)w_{d-r+2}$
$d-r+3$	w_{d-r+3} w_2w_{d-r+1}	$Sq^2(w_{d-r+1}) = \binom{d-r}{2}w_{d-r+3}$ $Sq^1(w_{d-r+2}) = (d-r+1)w_{d-r+2}$
$d-r+4$	w_3w_{d-r+1} w_2w_{d-r+2} w_{d-r+4}	$Sq^{1,2}(w_{d-r+1}) = (d-r)\binom{d-r}{2}w_{d-r+4}$ $Sq^{2,1}(w_{d-r+1}) = (d-r+1)\binom{d-r+1}{2}w_{d-r+4}$ $Sq^2(w_{d-r+2}) = \binom{d-r+1}{2}w_{d-r+4}$ $Sq^1(w_{d-r+3}) = (d-r)w_{d-r+4}$ $Sq^1(w_2w_{d-r+1}) = w_3w_{d-r+1} + (d-r)w_2w_{d-r+2}$

We notice that the formulas depend on d and r, so it is necessary to consider the special cases seperately to get a clear picture.

In each case, the Adams spectral sequence can be applied. Recall from Section 3.1 that we must construct a resolution

$$H^*(MTSO(d,r)) \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \cdots .$$

In Figure 3.1, such a resolution is constructed in the special case $d \equiv 3 \bmod 4$ and $r = 4$. First a basis for $H^{d-3}(MTSO(d,4))$ is chosen. Then all squares have been computed and, if necessary, a minimal number of classes in $H^{d-2}(MTSO(d,4))$ have been added to get a basis for $H^{d-2}(MTSO(d,4))$. Again all squares are computed and classes have been added to get a basis for $H^{d-1}(MTSO(d,4))$ and so on. These added basis elements form a generating set for $H^*(MTSO(d,4))$ as an $\mathcal{A}_2$-module. Let F_0 be the free module with a copy of $\mathcal{A}_2$ for each of these generators and $F_0 \to H^*(MTSO(d,4))$ the obvious surjection. Now the process is repeated with $H^*(MTSO(d,4))$ replaced by $\mathrm{Ker}(F_0 \to H^*(MTSO(d,4)))$ and F_0 replaced by F_1. When $\mathrm{Hom}^t_{\mathcal{A}_2}(-,\mathbb{Z}/2)$ is applied to the resolution, we get a space dual to the space of generators of the free modules in dimension t. Since the chosen resolution is minimal in the sense of [19], Lemma 2.8, the sequence is now exact, i.e.

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2}(H^*(MTSO(d,4)),\mathbb{Z}/2) = \mathrm{Hom}^t(F_s,\mathbb{Z}/2).$$

So each generator of the free modules corresponds to a copy of $\mathbb{Z}/2$ in E_2.

In Figure 3.1, the grading of the modules is shown vertically and an element in F_s is displayed next to its image in F_{s-1}. When $s \geq 3$, F_s is zero in dimensions less than or equal to $d+s$. From this, the E_2-term is determined, see Figure 3.5. Each dot represents a $\mathbb{Z}/2$ summand. A vertical line represents multiplication by h_0, while a sloped line indicates multiplication by h_1, see Section 3.1. This multiplication can be read off from the resolution, e.g. $\beta_1^* = h_1\alpha_1^*$ because β_1 maps to $Sq^2(\alpha_1)$.

t	$H^t(MTSO(d,r))$	F_0	F_1	F_2
$d-3$	w_{d-3}	x_1		
$d-2$	$Sq^1(w_{d-3}) = w_{d-2}$	$Sq^1(x_1)$		
$d-1$	$Sq^2(w_{d-3}) = w_{d-1}$ w_2w_{d-3}	$Sq^2(x_1)$ x_2		
d	$Sq^{1,2}(w_{d-3}) = w_d$ $Sq^{2,1}(w_{d-3}) = 0$ $Sq^1(w_2w_{d-3}) = w_3w_{d-3} + w_2w_{d-2}$ w_3w_{d-3}	$Sq^{1,2}(x_1)$ $Sq^{2,1}(x_1)$ $Sq^1(x_2)$ x_3	α_1	
$d+1$	$Sq^4(w_{d-3}) = w_3w_{d-2} + w_2w_{d-1}$ $Sq^{1,2,1}(w_{d-3}) = 0$ $Sq^2(w_2w_{d-3}) = w_2^2w_{d-3} + w_3w_{d-2}$ $+w_2w_{d-1}$ $Sq^1(w_3w_{d-3}) = w_3w_{d-2}$ w_4w_{d-3}	$Sq^4(x_1)$ $Sq^{1,2,1}(x_1)$ $Sq^2(x_2)$ $Sq^1(x_3)$ x_4	$Sq^1(\alpha_1)$	
$d+2$			$Sq^2(\alpha_1)$	β_1
$d+3$				$Sq^1(\beta_1)$

Figure 3.1: Resolution of $H^*(MTSO(d,4))$ for $d \equiv 3 \bmod 4$.

The constructions of the resolutions in the remaining special cases have been omitted here since they would take up too much space. The E_2-terms are shown in the following figures, and in each case, the differentials are determined and the extension problem is solved.

To do this, we will need naturality of the Adams spectral sequence. One needs to know the resolutions in order to see what the maps between the spectral sequences look like. Hopefully, the reader will believe the naturality claims made in the following.

Another fact that is helpful when trying to solve extension problems is

$$H^*(X,\mathbb{Q}) \cong \pi_*^s(X) \oplus \mathbb{Q} \tag{3.8}$$

for all spaces X, see [19], Chapter 1. This means that we already know the free part of the homotopy groups from the cohomology structure. We also need to know the homotopy groups $\pi_q(V_{d,r})$, see [36].

Recall that $\pi_q(MTSO(d,1)) \cong \pi_q^s(S^d) \oplus \pi_q^s(\Sigma^d BSO(d))$.

Theorem 3.16. *$\pi_q(MTSO(d,1))$ is given by the following table for $q \leq 2d$:*

q	d	$d+1$	$d+2$	$d+3$	$d+4$	$d+5$
$\pi_q^s(S^d)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0
$\pi_q^s(\Sigma^d BSO(d))$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2$

Proof. The first line is well-known. Since $MTSO(d,1) \cong S^{\infty+d} \vee \Sigma^{\infty+d}BSO(d)$, the Adams spectral sequence splits as the sum of the sequence for $S^{\infty+d}$ and the

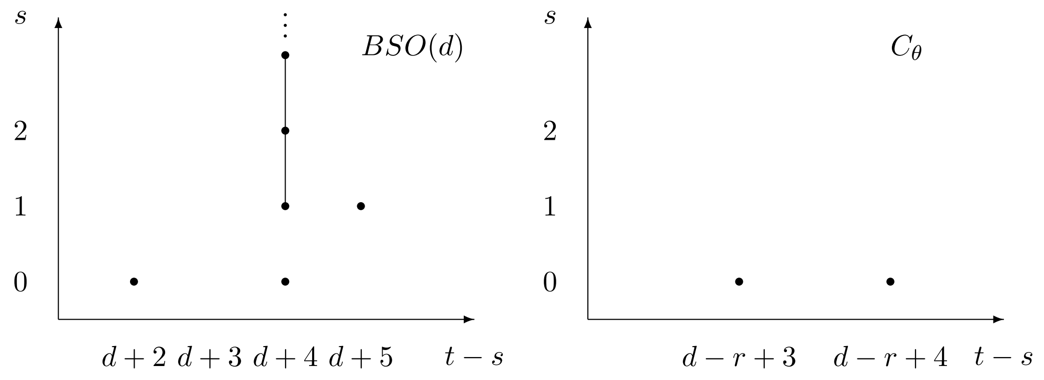

Figure 3.2: The Adams spectral sequence for $BSO(d)$ and C_θ.

sequence for $\Sigma^{\infty+d}BSO(d)$. The E_2-term of the $BSO(d)$ part is shown in Figure 3.2. It follows from the formula $d_k(h_0 x) = h_0 d_k(x)$ that there can be no differentials in this part of the sequence. This yields the second line. $\square$

Corollary 3.17. *The map*

$$\theta^2 : \pi_{d+1}(V_{d,2}) \to \pi_{d+2}(MTSO(d,2))$$

is injective for $d \geq 5$.

Proof. Consider the diagram from Proposition 1.30

$$\begin{tikzcd}
\pi_{d+3}(MT(d,1)) \arrow[r] & \pi_{d+2}(MT(d-1,1)) \arrow[r] & \pi_{d+2}(MT(d,2)) \arrow[r] & \pi_{d+2}(MT(d,1)) \\
\pi_{d+2}(V_{d,1}) \arrow[r] \arrow[u, "\theta^1"] & \pi_{d+1}(V_{d-1,1}) \arrow[r] \arrow[u, "\theta^1"] & \pi_{d+1}(V_{d,2}) \arrow[r] \arrow[u, "\theta^2"] & \pi_{d+1}(V_{d,1}). \arrow[u, "\theta^1"]
\end{tikzcd}$$

The first two vertical maps are isomorphisms by Theorem 3.16 and the fourth is injective by Theorem 1.31, so the third is injective by the 5-lemma. $\square$

Lemma 3.18. *For $r > 1$ and $q \leq 2(d-r)-1$ and C_θ the cofiber of* (3.2)*, the homotopy groups $\pi_q(C_\theta)$ are given by:*

q	$q \leq d-r+2$	$d-r+3$	$d-r+4$
$\pi_q(C_\theta)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$

Proof. The cohomology structure with $\mathbb{Z}/p$ coefficients is given in Corollary 3.9 for $p = 2$, and otherwise it is the kernel of the map in Theorem 3.11. Only in the $p = 2$ case, anything interesting happens in dimensions $q \leq d-r+4$. The relevant part of the spectral sequence is shown in Figure 3.2. Obviously, there can be no differentials, and the claims follow. $\square$

Corollary 3.19. *For $d - r \geq 2$,*

$$\theta^r : \pi_{d-r+2}(V_{d,r}) \to \pi_{d-r+3}(MTSO(d,r))$$

is injective with cokernel $\mathbb{Z}/2$. In particular, it is not an isomorphism.

This is contrary to the $\tilde{\theta}_r^t$ defined by Atiyah and Dupont, which was an isomorphism in this dimension.

Proof. It follows from the long exact sequence

$$\cdots \to \pi_q^s(\Sigma V_{d,r}) \xrightarrow{f_{\theta *}} \pi_q(MT(d,r)) \to \pi_q(C_\theta) \to \pi_{q-1}^s(\Sigma V_{d,r}) \to \cdots \tag{3.9}$$

combined with Lemma 3.18 and Corollary 1.36 that the cokernel must be $\mathbb{Z}/2$. □

Theorem 3.20. *For $q < 2(d-1)$, $\pi_q(MTSO(d,2))$ is given by the table:*

q	$d-1$	d	$d+1$	$d+2$
d even	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/48 \oplus \mathbb{Z}/2$
d odd	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4 \oplus \mathbb{Z}/2$	$\mathbb{Z}/4 \oplus \mathbb{Z}/2$

Proof. Again we consider the Adams spectral sequence for the 2-primary part. The E_2-term is shown in Figure 3.3. We immediately see that the $(d-1)$th and the dth column must survive to E_∞ in both cases. In the even case, this is true by (3.8) since we know the cohomology with $\mathbb{Q}$ coefficients has a $\mathbb{Q}$ in each of the two dimensions.

For d even, consider the long exact sequence for the pair $(\Sigma^\infty \Sigma V_{d,2}, MTSO(d,2))$. The homotopy groups of $V_{d,2}$ are known from [36], and the homotopy groups of the cofiber are calculated in Lemma 3.18. Inserting the known groups in the exact sequence (3.9) yields the following:

$$\begin{aligned} \cdots\ \mathbb{Z}/24 \oplus \mathbb{Z}/2 &\to \pi_{d+2}(MTSO(d,2)) \to \mathbb{Z}/2 \\ \to\ \mathbb{Z}/2 \oplus \mathbb{Z}/2\ &\to \pi_{d+1}(MTSO(d,2)) \to \mathbb{Z}/2 \end{aligned}$$

It follows from Theorem 1.31 that the last map is surjective. By Corollary 3.17, the first map is injective and by Corollary 1.36, the third map is zero. Thus the third and fourth column in the spectral sequence survive to E_∞, otherwise the groups would be too small for the exact sequence. The odd case is similar.

Most of the necessary information about extensions is given by the vertical lines. In the even case, there may still be an extension problem in dimension $d+1$. Note, however, that under the map induced by $MTSO(d-1,1) \to MTSO(d,2)$ on spectral sequences, the generator of $E_\infty^{d+1,0}$ is hit by something representing a map of order two. But the map is induced by a filtration preserving map of homotopy groups. Thus, the generator cannot represent a map of order greater than two.

When d is odd and $p > 2$, $H^*(MTSO(d,2);\mathbb{Z}/p) = 0$ in the relevant dimensions, so there is no p-torsion.

When d is even,

$$H^*(MTSO(d,2);\mathbb{Z}/p) \cong \delta(e_{d-2}) \cdot H^*(MTSO(d);\mathbb{Z}/p) \oplus e_d \cdot H^*(MTSO(d);\mathbb{Z}/p)$$

in low dimensions. Since $\delta(e_{d-2})$ maps to zero under the map

$$H^{d-1}(MTSO(d,2);\mathbb{Z}/p) \to H^{d-1}(MTSO(d);\mathbb{Z}/p),$$

so must all Steenrod powers of $\delta(e_{d-2})$. That is, they all lie in the kernel, which is

$$\delta(e_{d-2}) \cdot H^*(MTSO(d);\mathbb{Z}/p).$$

On the other hand, the map

$$H^*(MTSO(d,2);\mathbb{Z}/p) \to H^*(MTSO(d-1,1);\mathbb{Z}/p)$$

is injective on $\delta(e_{d-2}) \cdot H^*(MTSO(d);\mathbb{Z}/p)$. Since $\delta(e_{d-2})$ maps to the generator of the $H^{d-1}(S^{d-1};\mathbb{Z}/p)$ summand and all Steenrod powers of this are zero, all powers of $\delta(e_{d-2})$ must be zero in $H^*(MTSO(d,2),\mathbb{Z}/p)$ as well. Using this, the Adams spectral sequence immediately shows that the only p-torsion is a $\mathbb{Z}/3$ summand in dimension $d+2$. □

Theorem 3.21. *For $q < 2(d-2)$, $\pi_q(MTSO(d,3))$ is given by the following table:*

q	$d-2$	$d-1$	d	$d+1$
$d \equiv 0 \bmod 4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$
$d \equiv 1 \bmod 4$	$\mathbb{Z}$	$\mathbb{Z}/4$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/24\oplus\mathbb{Z}/2$
$d \equiv 2 \bmod 4$	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$d \equiv 3 \bmod 4$	$\mathbb{Z}$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/48\oplus\mathbb{Z}/4$

Proof. Consider the spectral sequence for the 2-primary part shown in Figure 3.4.

For $d \equiv 1,2 \bmod 4$, we immediately see that the first three columns survive to E_∞. Now, if $d \equiv 1 \bmod 4$, we fill in the already computed groups in the exact sequence for the pair $(MTSO(d-1,2), MTSO(d,3))$. This yields

$$\mathbb{Z}/2\oplus\mathbb{Z}/2 \to \mathbb{Z}/16\oplus\mathbb{Z}/2 \to \pi_{d+1}(MTSO(d,3)) \to \mathbb{Z}/2,$$

so $\pi_{d+1}(MTSO(d,3))$ must contain elements of order at least 8. Thus there can be no differentials hitting $h_0^2 y$ where $y \in E_2^{d+1,0}$ is the generator. The other dot in $E_2^{d+1,2}$ is $h_1 x$ for some non-zero $x \in E_2^{d,1}$. But under the map

$$g : MTSO(d,3) \to MTSO(d,2),$$

this x is mapped to $g_*(x)$, which is non-zero in the spectral sequence for $MTSO(d,2)$, and $h_1 g_*(x)$ survives to E_∞. From the resolutions we see that $g_*(h_1 x) = h_1 g_*(x)$. Therefore, since $g_* d_2 = d_2 g_*$, $h_1 x$ is not hit by any differential. Also, $g_*(h_0^2 y) = 0$. We conclude that $d_2 : E_2^{d+2,0} \to E_2^{d+1,2}$ is zero.

For $d \equiv 2 \bmod 4$, the only unknown differential is $d_2 : E_2^{d+2,0} \to E_2^{d+1,2}$. However, we may apply naturality of the Adams spectral sequence to the map $g : MTSO(d-2,1) \to MTSO(d,3)$. There is a $y \in E_2'^{d+2,0}$ in the spectral sequence E' for $MTSO(d-2,1)$ that maps to the non-zero element $x \in E_2^{d+2,0}$. We know $d_2(y) = 0$ from the $r = 1$ case, so

$$d_2(x) = d_2(g_*(y)) = g_*(d_2(y)) = 0.$$

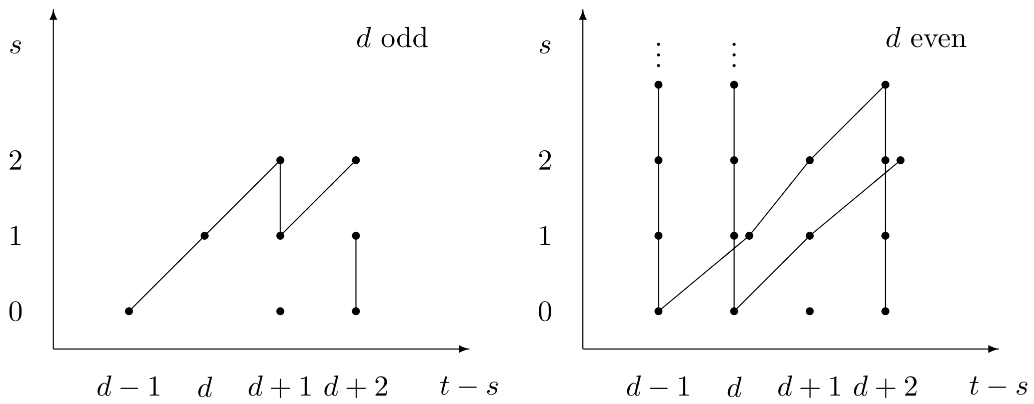

Figure 3.3: The Adams spectral sequence, $r = 2$

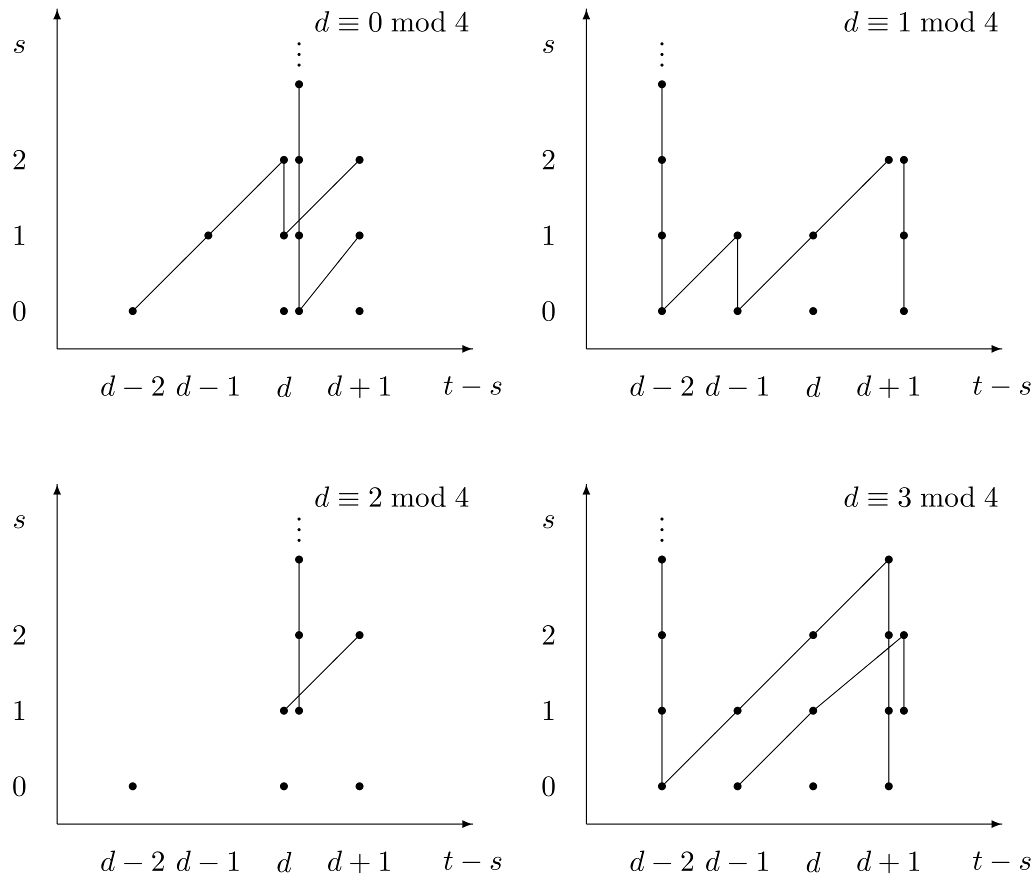

Figure 3.4: The Adams spectral sequence, $r = 3$.

In the case $d \equiv 0 \bmod 4$, there is a possible differential $d_2 : E_2^{d+1,0} \to E_2^{d,2}$. However, we have a map $MTSO(d-1,2) \to MTSO(d,3)$. By comparing the spectral sequences, we see that this differential must be zero. There is also a differential $d_2 : E_2^{d+2,0} \to E_2^{d+1,2}$, but a comparison with the spectral sequence for $MTSO(d-2,1)$ shows that this is zero. Both are similar to the $d \equiv 2 \bmod 4$ case.

In the case $d \equiv 3 \bmod 4$, we see that the differential $d_2 : E_2^{d+1,0} \to E_2^{d,2}$ must be zero, again by comparing with the spectral sequence for $MTSO(d-1,2)$. There could be an extension problem in dimension d, but this is solved exactly as in the $r = 2$ case.

The p-primary part for $p > 2$ is determined precisely as in the $r = 2$ case.

Finally there are extension problems in the even cases in dimension $d+1$. Insert the groups found in the exact sequence for the cofibration

$$MTSO(d-2,1) \to MTSO(d,3) \to MTSO(d,2).$$

For $d \equiv 0 \mod 4$, this yields:

$$\begin{array}{cccc} \cdots & \mathbb{Z}/24 & \to \pi_{d+1}(MTSO(d,3)) \to & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\ \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to & \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & \to & \mathbb{Z} \oplus \mathbb{Z}/2 \end{array}$$

We see that the third map must be zero, so $\pi_{d+1}(MTSO(d,3)) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is surjective, and thus an isomorphism. The case $d \equiv 2 \bmod 4$ is similar.

The only remaining problem is when $d \equiv 3 \bmod 4$ and $q = d + 1$. The result in this case will follow from the cases $r = 4, 5$ in the next theorem. ☐

Theorem 3.22. *For $q < 2(d-3)$, $\pi_q(MTSO(d,4))$ is given by:*

q	$d-3$	$d-2$	$d-1$	d
$d \equiv 0 \bmod 4$	$\mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/48 \oplus \mathbb{Z}/4$
$d \equiv 1 \bmod 4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$d \equiv 2 \bmod 4$	$\mathbb{Z}$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/24$
$d \equiv 3 \bmod 4$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

For d even and $q < 2(d-4)$, $\pi_q(MTSO(d,5))$ is given by:

q	$d-4$	$d-3$	$d-2$	$d-1$	d
$d \equiv 0 \bmod 4$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$d \equiv 2 \bmod 4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/8$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$

Proof. The calculation of the p-primary part for $p > 2$ is again similar to the case $r = 2$. The spectral sequences for $p = 2$ are shown in Figure 3.5 and 3.6. For $r = 4$, the first two columns in all four diagrams must survive to E_∞.

For $d \equiv 1 \bmod 4$, the only possibly non-zero differential is $d_2 : E_2^{d,0} \to E_2^{d-1,2}$, but the generator of $E_2^{d,0}$ in the spectral sequence for $MTSO(d-1,3)$ is mapped to the generator of $E_2^{d,0}$ in the spectral sequence for $MTSO(d,4)$ under the map of spectral sequences induced by $MTSO(d-1,3) \to MTSO(d,4)$. This forces the differential to be zero.

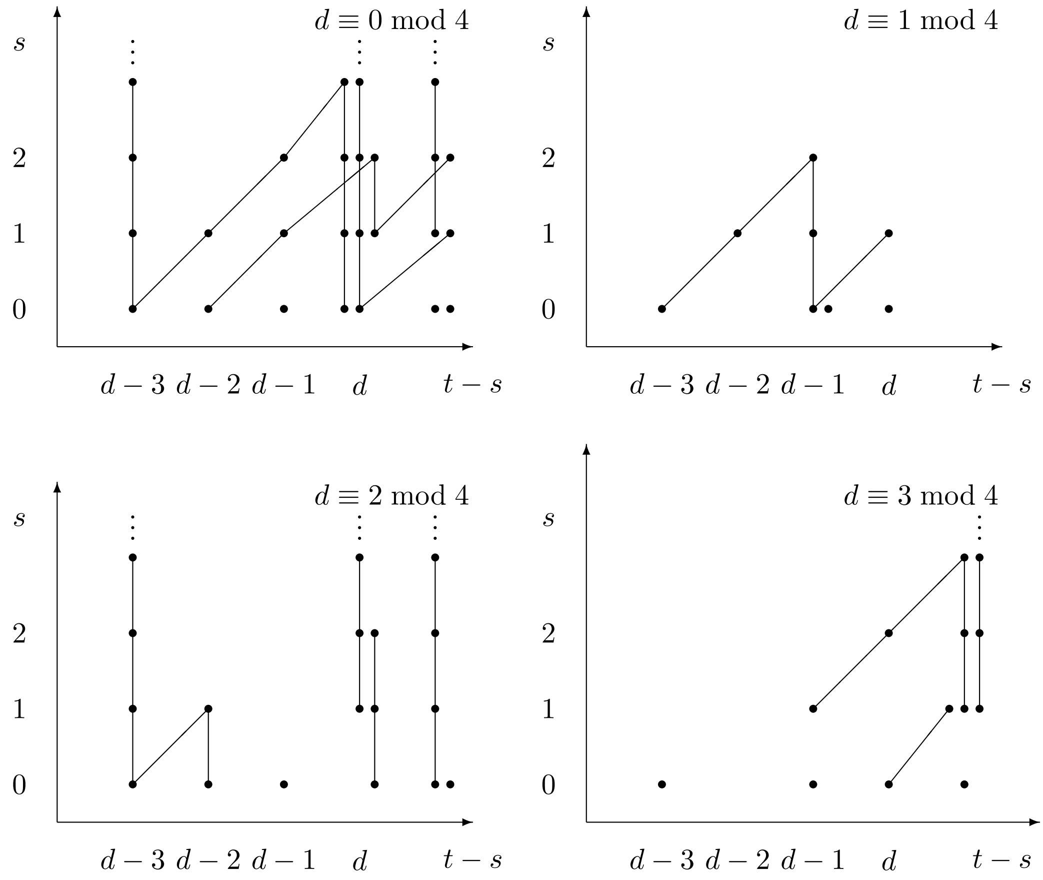

Figure 3.5: The Adams spectral sequence, $r = 4$.

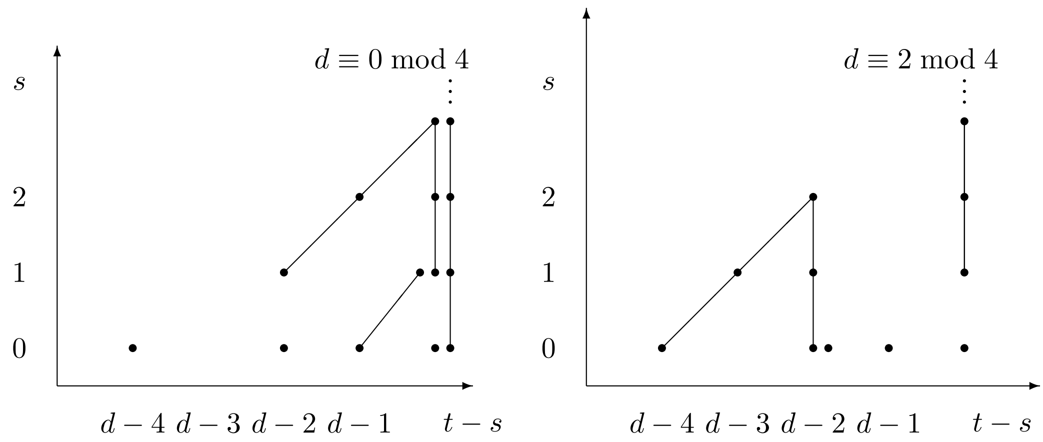

Figure 3.6: The Adams spectral sequence, $r = 5$.

When $d \equiv 3 \bmod 4$, there is a differential $d_2 : E_2^{d+1,0} \to E_2^{d,2}$. Comparing with the spectral sequence for $MTSO(d-3,1)$, one sees that this differential must be zero. The extension problem in dimension d is solved using the case $MTSO(d-1,3)$.

When $d \equiv 2 \bmod 4$, there may be a non-zero $d_2 : E_2^{d+1,0} \to E_2^{d,2}$. But one of the generators of $E_2^{d+1,0}$ is in the image of the spectral sequence for $MTSO(d-3,1)$, so the earlier results imply that the differential must be zero on this element. The other generator is in the image of spectral sequence for $MTSO(d-1,3)$ where we also know the corresponding differential to be zero.

Finally, for $d \equiv 0 \bmod 4$, the differential $d_2 : E_2^{d,0} \to E_2^{d-1,2}$ must be zero. One of the generators of $E_2^{d,0}$ is in the image of the spectral sequence for $MTSO(d-1,3)$, and the other one is in the image of the spectral sequence for $\Sigma^\infty \Sigma V_{d,4}$ under the map induced by f_θ. Using our knowledge of the homotopy groups of $V_{d,4}$ and thus the spectral sequence, the differentials must be zero. The extension problem in dimension $d-1$ is solved as in the case $r = 2$.

The dth column is a bit more complicated. One of the generators in $E_2^{d+1,1}$ is in the image of the spectral sequence for $MTSO(d-3,1)$. The other one is $h_1 x$ for some $x \in E_2^{d,0}$. Thus $d_2(h_1 x) = h_1 d_2(x) + d_2(h_1)x = 0$. In $E_2^{d+1,0}$, one of the generators is in the image of the spectral sequence for $MTSO(d-3,1)$, while the other one is in the image of the spectral sequence for $MTSO(d,5)$. Thus, we just need the corresponding differential to be zero in the sequence for $MTSO(d,5)$.

The results for $MTSO(d,5)$ for d even again follow by comparing the spectral sequences with those for lower r. The only problem is when $d \equiv 0 \bmod 4$. By Corollary 1.36,

$$\theta^5 : \pi_{d-1}(V_{d,5}) \to \pi_d(MTSO(d,5))$$

is injective when d is divisible by 8. Since $\pi_{d-1}(V_{d,5}) \cong \mathbb{Z} \oplus \mathbb{Z}/8$, also $\pi_d(MTSO(d,5))$ must contain torsion of order eight. Thus there cannot be any non-zero differential $d_k : E_k^{d+1,0} \to E_k^{d,k}$. This implies that the remaining differential for $MTSO(d,4)$ must be zero when $8 \mid d$, and by periodicity, this holds for all $d \equiv 0 \bmod 4$.

In the next theorem we shall see that θ^5 is also injective when $d \equiv 4 \bmod 8$, proving the claim also for $\pi_d(MTSO(d,5))$ when $8 \nmid d$ by a similar argument. □

The above computation allows us to improve Theorem 1.31 further.

Theorem 3.23. *For* $2r \leq d+1$

$$\theta^r : \pi_{d-1}(V_{d,r}) \to \pi_d(MTSO(d,r))$$

is injective for d even and $r \leq 6$. For d odd and $r = 4$, it is not injective.

Proof. For $d \equiv 2 \bmod 4$, the exact sequence for the pair $(MTSO(d,4), \Sigma^\infty \Sigma V_{d,4})$ in dimension d becomes

$$\to \mathbb{Z} \oplus \mathbb{Z}/12 \xrightarrow{\theta^4} \mathbb{Z} \oplus \mathbb{Z}/24 \to \mathbb{Z}/2 \to 0.$$

Since this is exact, θ^4 must be injective.

For $d \equiv 0 \bmod 4$, the exact sequence for the pair $(MTSO(d,4), \Sigma^\infty \Sigma V_{d,4})$ in dimension d becomes

$$\to \mathbb{Z} \oplus \mathbb{Z}/24 \oplus \mathbb{Z}/4 \xrightarrow{\theta^4} \mathbb{Z} \oplus \mathbb{Z}/48 \oplus \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

Again this is exact, so θ^4 must be injective.

For $r = 5, 6$, the claim follows as in the proof of Theorem 1.31 by a diagram similar to (1.11).

For d odd, the long exact sequence for the pair $(MTSO(d,4), \Sigma^\infty \Sigma V_{d,4})$ in dimension d yields

$$\to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\theta^4} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to$$

where the last map is surjective. We see that θ^4 cannot be injective. □

Remark 3.24. Consider the case $r = 4$ and $d \equiv 3 \bmod 8$. Let $x \in \pi_d(S^d) \cong \mathbb{Z}$ be the canonical generator, and consider the long exact sequence for the fibration

$$V_{d,4} \to V_{d+1,5} \to S^d.$$

Then $\partial(x) \in \pi_{d-1}(V_{d,4})$ is the index of four vector fields on S^d with one singularity. If we insert the computed groups in the diagram from Proposition 1.30 for $k = r-1$, we see that

$$\beta^4(S^d) = \theta^4(\partial(x)) = 0,$$

even though S^d does not allow four independent vector fields according to Theorem 3.12.

3.6 The Spin Case

The cohomology of $BSpin(d)$ is more complicated. The map $BSpin(d) \to BSO(d)$ induces a map

$$H^*(BSO(d); \mathbb{Z}/2) \to H^*(BSpin(d); \mathbb{Z}/2).$$

Let J_d denote the ideal in $H^*(BSO(d); \mathbb{Z}/2)$ generated by

$$w_2, Sq^1(w_2), Sq^2 Sq^1(w_2), \dots, Sq^{\frac{1}{2}a_d} Sq^{\frac{1}{4}a_d} \cdots Sq^1(w_2). \tag{3.10}$$

Here a_d is the power of 2 given in the table (3.3). Quillen showed in [37]:

Theorem 3.25. *The kernel of $H^*(BSO(d); \mathbb{Z}/2) \to H^*(BSpin(d); \mathbb{Z}/2)$ is exactly J_d and*

$$H^*(BSpin(d); \mathbb{Z}/2) \cong H^*(BSO(d); \mathbb{Z}/2)/J_d \oplus \mathbb{Z}/2[v_{a_d}].$$

Here v_{a_d} is a class in dimension a_d.

Lemma 3.26. *The generators of J_d are of the form*

$$Sq^{2^{k-1}} \cdots Sq^1(w_2) = w_{2^k+1} + \sum_{j=2}^{2^{k-1}} w_j w_{2^k-j+1} + \cdots . \tag{3.11}$$

The dots indicate terms that are products of more than two w_i's. In particular, $w_i \in J_d$ if and only if $i = 2, 3, 5, 9$.

Proof. The formula (3.11) is obviously true for $k = 1$. Now suppose that it is true for $k - 1$. Then

$$\begin{aligned} Sq^{2^k} \cdots Sq^1(w_2) &= Sq^{2^k}\Big(w_{2^k+1} + \sum_{j=2}^{2^{k-1}} w_j w_{2^k-j+1} + \cdots\Big) \\ &= Sq^{2^k}(w_{2^k+1}) + \cdots \\ &= \sum_{j=0}^{2^k} \binom{(2^k+1) - 2^k + j - 1}{j} w_{2^k+j+1} w_{2^k-j} + \cdots \\ &= w_{2^{k+1}+1} + \sum_{j=2}^{2^k} w_j w_{2^{k+1}-j+1} + \cdots . \end{aligned}$$

The first equality follows by induction. The second equality follows from the Cartan fomula

$$\begin{aligned} Sq^{2^k}(w_j w_{2^k-j+1}) &= Sq^j(w_j) Sq^{2^k-j}(w_{2^k-j+1}) + Sq^{j-1}(w_j) Sq^{2^k-j+1}(w_{2^k-j+1}) \\ &= w_j^2 Sq^{2^k-j}(w_{2^k-j+1}) + w_{2^k-j+1}^2 Sq^{j-1}(w_j). \end{aligned}$$

Here each term is a product of more than two Stiefel–Whitney classes. The third equality is the formula (3.7).

For the last statement we calculate

$$\begin{aligned} Sq^1(w_2) &= w_3 \\ Sq^2(w_3) &= w_5 + w_2 w_3 \\ Sq^4(w_5 + w_2 w_3) &= w_9 + w_7 w_2 + w_6 w_3 + w_5 w_4 + w_3^3 + w_2^3 w_3 + w_2^2 w_5. \end{aligned}$$

We see inductively that w_2, w_3, w_5 and w_9 belong to J_d. However, for $k > 4$, the relation becomes

$$w_{2^k+1} = \sum_{j=0}^{2^{k-1}} w_j w_{2^k-j+1} + \cdots$$

and here the right hand side does not belong to J_d. □

Theorem 3.27. *$H^q(BSpin(d), BSpin(d-r); \mathbb{Z}/2)$ is isomorphic to the kernel of*

$$H^q(BSpin(d); \mathbb{Z}/2) \to H^q(BSpin(d-r); \mathbb{Z}/2)$$

for $q < a_{d-r}$, and the map

$$p^* : H^q(BSO(d), BSO(d-r); \mathbb{Z}/2) \to H^q(BSpin(d), BSpin(d-r); \mathbb{Z}/2)$$

is surjective with kernel $J_d \cap H^q(BSO(d), BSO(d-r); \mathbb{Z}/2)$.

Proof. In the following, coefficients in $\mathbb{Z}/2$ are understood. There is a map of long exact sequences

$$\begin{tikzcd} H^*(BSO(d), BSO(d-r)) \arrow[r, "j^*"] \arrow[d, "p^*"] & H^*(BSO(d)) \arrow[r, "i^*"] \arrow[d] & H^*(BSO(d-r)) \arrow[d, "p'^*"] \\ H^*(BSpin(d), BSpin(d-r)) \arrow[r, "j'^*"] & H^*(BSpin(d)) \arrow[r, "i'^*"] & H^*(BSpin(d-r)). \end{tikzcd}$$

Since both i^* and p'^* are surjective in dimensions less than a_{d-r}, we see that i'^* is also surjective, so $H^q(BSpin(d), BSpin(d-r))$ is just the kernel of i'^* when $q \leq a_{d-r}$.

Thus we just need to describe

$$\mathrm{Ker}(\mathbb{Z}/2[w_2,\dots,w_d]/J_d \to \mathbb{Z}/2[w_2,\dots,w_d]/J'_{d-r}) \tag{3.12}$$

where J'_{d-r} is the ideal generated by J_{d-r} and $w_{d-r+1},\dots,w_d$. We must show that this is

$$\mathrm{Ker}(\mathbb{Z}/2[w_2,\dots,w_d] \to \mathbb{Z}/2[w_2,\dots,w_{d-r}])/J \tag{3.13}$$

where $J = J_d \cap \mathrm{Ker}(\mathbb{Z}/2[w_2,\dots,w_d] \to \mathbb{Z}/2[w_2,\dots,w_{d-r}])$.

Clearly, there is an injective map from (3.13) to (3.12). Now, let P be some polynomial in the Stiefel–Whitney classes representing an element in (3.12). Then $P \in J'_{d-r}$. Let $g_1,\dots,g_m$ denote the generators of J_d in dimensions up to a_{d-r} given by (3.10) and $g'_1,\dots,g'_m$ the generators of J_{d-r} given by (3.10). Under the map $H^*(BSO(d)) \to H^*(BSO(d-r))$, g_i maps to g'_i by naturality of the Steenrod squares. Thus, g_i and g'_i differ only by an element of the ideal generated by $w_{d-r+1},\dots,w_d$. This means that $\{g_1,\dots,g_m,w_{d-r+1},\dots,w_d\}$ is also a set of generators for J'_{d-r}. So for suitable polynomials $\lambda_1,\dots,\lambda_m$ and $\mu_1,\dots,\mu_r$,

$$P = \lambda_1 g_1 + \cdots + \lambda_m g_m + \mu_1 w_{d-r+1} + \cdots + \mu_r w_d.$$

But $\lambda_1 g_1 + \cdots + \lambda_m g_m \in J_d$ so P represents the same element as $\mu_1 w_{d-r+1} + \cdots + \mu_r w_d$ in $\mathbb{Z}/2[w_2,\dots,w_d]/J_d$. This lies in (3.13), so the map is also surjective. □

Corollary 3.28. *Assume $2r < d$ and $9 \leq d-r$. In dimensions $* < 2(d-r)$, $H^*(MTSpin(d,r);\mathbb{Z}/2)$ is isomorphic to the free $H^*(BSpin;\mathbb{Z}/2)$-module on generators*

$$p^*(w_{d-r+1}),\dots,p^*(w_d)$$

where p^* is as in Theorem 3.27.

Proof. By Theorem 3.27, it is enough to show that

$$J_d \cap H^q(BSO(d), BSO(d-r);\mathbb{Z}/2) \subseteq J_d \cdot H^*(BSO(d), BSO(d-r);\mathbb{Z}/2)$$

for all $q \leq 2(d-r)$. Let g_k denote the generator of J_d of degree 2^k+1 given in (3.10). Suppose

$$\lambda_0 g_0 + \cdots + \lambda_m g_m \in J_d \cap H^q(BSO(d), BSO(d-r);\mathbb{Z}/2)$$

where m is unique with the property $d-r < 2^m+1 < 2(d-r)$. Then the image in $H^q(BSO(d-r);\mathbb{Z}/2)$ must be zero. But this is also $\sum_k \lambda'_k g'_k \in J_{d-r}$ where the g'_k are the generators of J_{d-r} and λ'_k is the image of λ_k. The g'_k form a regular sequence in the sense of [37]. Hence we conclude that λ'_m belongs to the ideal generated by $g'_0,\dots,g'_{m-1}$. Since $\deg(\lambda_m) < d-r$, also λ_m lies in the ideal generated by $g_0,\dots,g_{m-1}$. Rearranging the terms, we may assume $\lambda_m = 0$. But none of the g_k with $k < m$ contains terms involving $w_{d-r+1},\dots,w_d$ for degree reasons. Thus we can choose all $\lambda_0,\dots,\lambda_{m-1}$ in $H^*(BSO(d), BSO(d-r);\mathbb{Z}/2)$. This proves the claim. □

Theorem 3.29. *Let F be either $\mathbb{Q}$ or $\mathbb{Z}/p$ for p an odd prime. Then*

$$H^*(MTSO(d,r);F) \to H^*(MTSpin(d,r);F)$$

is an isomorphism.

Proof. Consider the fibration

$$B\mathbb{Z}/2 \to BSpin(d) \to BSO(d).$$

Since $\widetilde{H}^*(B\mathbb{Z}/2, F) = 0$, it follows from the Serre spectral sequence that

$$H^*(BSO(d);F) \to H^*(BSpin(d);F)$$

is an isomorphism. By the 5-lemma,

$$H^*(BSO(d), BSO(d-r);F) \to H^*(BSpin(d), BSpin(d-r);F)$$

is an isomorphism. A Thom isomorphism yields the result. □

Corollary 3.30. *The periodicity map $\Sigma^{a_r} MTSpin(d,r) \to MTSpin(d+a_r, r)$ is a $(2(d-r)+a_r-1)$-equivalence for $2r < d$ and $9 \leq d-r$.*

Proof. It is enough to see that

$$H^{q+a_r}(MTSpin(d+a_r,r);\mathbb{Z}/2) \to H^q(MTSpin(d,r);\mathbb{Z}/2)$$

is an isomorphism for $q < 2(d-r)$. This follows because the periodicity map for $MTSO(d,r)$ takes $J_{d+a_r} \cdot H^*(MTSO(d+a_r,r))$ to $J_d \cdot H^*(MTSO(d,r))$. □

We will also need some information about the $\mathcal{A}_3$-action on $H^*(MTSO(d);\mathbb{Z}/3)$. Let $\mathcal{P}^1$ denote the first Steenrod power. It is shown in [6] that

$$\mathcal{P}^1(c_{j-2}) = c_1^2 c_{j-1} - 2c_2 c_{j-2} - c_1 c_{j-1} + j c_j$$

where $c_i \in H^{2i}(BU(d);\mathbb{Z}/3)$ is the ith Chern class. Since $p_j = g^*((-1)^j c_{2j})$ under the map $g : BSO(d) \to BU(d)$, we get

$$\mathcal{P}^1(p_{j-1}) = 2p_1 p_{j-1} - 2j p_j. \tag{3.14}$$

See [35] for the relation between Chern and Pontryagin classes.

3.7 Calculations in the Spin Case

The above considerations in cohomology allow us to calculate the homotopy groups $\pi_q(MTSpin(d,r))$. We will only consider dimensions $q \leq d-r+9$ and assume $q < a_{d-r}$. Then the cohomology with $\mathbb{Z}/2$ coefficients is just $H^q(MTSO(d,r);\mathbb{Z}/2)$ with the relations

$$\begin{aligned} &w_2 = w_3 = w_5 = w_9 = 0 \\ &w_{17} + w_4 w_{13} + w_6 w_{11} + w_7 w_{10} = 0 \end{aligned} \tag{3.15}$$

as the only relevant ones.

In the following table, a basis of $H^q(MTSpin(d,r);\mathbb{Z}/2)$ is shown for low values of q. Of course, when $d-r+k>d$, then $w_{d-r+k}=0$ is understood. To avoid the relations (3.15), we will assume $d-r+1>9$, and when we deal with cohomology in dimensions $q>d-r+7$, we also assume $q-7>10$.

q	$H^q(V_{d,r})$	$H^q(C_\theta)$				
$d-r+1$	w_{d-r+1}					
$d-r+2$	w_{d-r+2}					
$d-r+3$	w_{d-r+3}					
$d-r+4$	w_{d-r+4}					
$d-r+5$	w_{d-r+5}	w_4w_{d-r+1}				
$d-r+6$	w_{d-r+6}	w_4w_{d-r+2}				
$d-r+7$	w_{d-r+7}	w_4w_{d-r+3}	w_6w_{d-r+1}			
$d-r+8$	w_{d-r+8}	w_4w_{d-r+4}	w_6w_{d-r+2}	w_7w_{d-r+1}		
$d-r+9$	w_{d-r+9}	w_4w_{d-r+5}	w_6w_{d-r+3}	w_7w_{d-r+2}	w_8w_{d-r+1}	$w_4^2w_{d-r+1}$

Projection onto the first column yields $H^*(V_{d,r};\mathbb{Z}/2)$, while the second column is the cohomology of the cofiber C_θ of the inclusion $\Sigma^\infty\Sigma V_{d,r}\to MTSpin(d,r)$. This follows because we know the composite map

$$H^*(MTSO(d,r);\mathbb{Z}/2)\to H^*(MTSpin(d,r);\mathbb{Z}/2)\to H^*(\Sigma V_{d,r};\mathbb{Z}/2)$$

is surjective, so the cohomology of C_θ is the kernel of the last map.

We may now determine the action of the Steenrod algebra from the action on $H^*(MTSO(d,r);\mathbb{Z}/2)$. In dimensions up to $d-r+8$, the Steenrod action respects the two columns. However, $Sq^8(w_{d-r+1})$ contains mixed terms. Applying the Adams spectral sequence, we get:

Theorem 3.31. *For $q<2d$, $d>9$, and $q<a_{d-1}$, $\pi_q(MTSpin(d,1))$ is given by the table*

q	d	$d+1$	$d+2$	$d+3$	$d+4$	$d+5$	$d+6$	$d+7$
$\pi_q^s(S^d)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$
$\pi_q^s(\Sigma^d BSpin(d))$	0	0	0	0	$\mathbb{Z}$	0	0	0

For the last two columns, we must assume $q>17$.

Proof. This is exactly similar to the oriented case. To calculate the 3-torsion, note that

$$H^{d+4}(\Sigma^{d+\infty}BSpin(d);\mathbb{Z}/3)\cong H^4(BSO(d);\mathbb{Z}/3)\cong\mathbb{Z}/3,$$

generated by the first Pontryagin class p_1. But

$$\mathcal{P}^1(p_1)=2p_1^2-p_2\neq 0$$

by (3.14). The Adams spectral sequence then shows that there is no 3-torsion in $\pi_{d+7}^s(\Sigma^d BSpin(d))$. □

Theorem 3.32. *$\pi_q(C_\theta)$ is given by the following table for $r > 4$, $q < 2(d-r)-1$, and $d-r > 10$:*

q	$q \leq d-r+4$	$d-r+5$	$d-r+6$	$d-r+7$	$d-r+8$
$d-r$ *even*	*0*	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
$d-r$ *odd*	*0*	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	*0*

Proof. Again this follows from the spectral sequence. For $d-r$ even, there could be some 3-torsion since there is a $\delta(e_{d-r}p_1) \in H^{d-r+5}(C_\theta;\mathbb{Z}/3)$. But under the composite map $g : MTSpin(d-r+1,1) \to MTSpin(d,r) \to C_\theta$,

$$g^*(\mathcal{P}^1(\delta(e_{d-r}p_1))) = \mathcal{P}^1(p_1) = 2p_1^2 - p_2 \neq 0$$

by the $r = 1$ case. Using this, the Adams spectral sequence shows that there is no 3-torsion. □

Corollary 3.33. *When $d-r > 10$ is odd and $q < 2(d-r)-1$,*

$$\theta^r : \pi_{q-1}(V_{d,r}) \to \pi_q(MTSpin(d,r))$$

is an isomorphism for $q = d-r+6$, and it is injective for $q = d-r+7$. In particular,

$$\theta^7 : \pi_{d-1}(V_{d,7}) \to \pi_d(MTSpin(d,7))$$

is injective for d even.

According to Theorem 1.31, $\pi_{q-1}(V_{d,r}) \to \pi_q(MTSpin(d,r))$ is an isomorphism for $q \leq d-r+4$, so we know the homotopy groups in these dimensions. Here are some more:

Theorem 3.34. *$\pi_q(MTSpin(d,r))$ is given by*

r	d	q	$\pi_q(MTSpin(d,r))$
$5,6$	0 mod 4	d	$\mathbb{Z}\oplus\mathbb{Z}/8\oplus\mathbb{Z}/2$
5	1 mod 4	d	$\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$
$5,6$	2 mod 4	d	$\mathbb{Z}\oplus\mathbb{Z}/2$
5	3 mod 4	d	$\mathbb{Z}\oplus\mathbb{Z}/2$
6	1 mod 4	d	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
6	3 mod 4	d	$\mathbb{Z}/2$

for $q < 2(d-r)-1$, $9 \leq d-r$, and $q < a_{d-r}$.

3.8 The Unoriented Case

By similar calculations of the Steenrod action on $H^*(MTO(d,r);\mathbb{Z}/2)$, we obtain the E_2-terms of Adams spectral sequences shown in Figure 3.7 and 3.8. This immediately yields the first three homotopy groups. For the fourth, the only problem is to determine the differential $d_2 : E_2^{0,d-r+4} \to E_2^{2,d-r+5}$. So far, we have not found a way to do that.

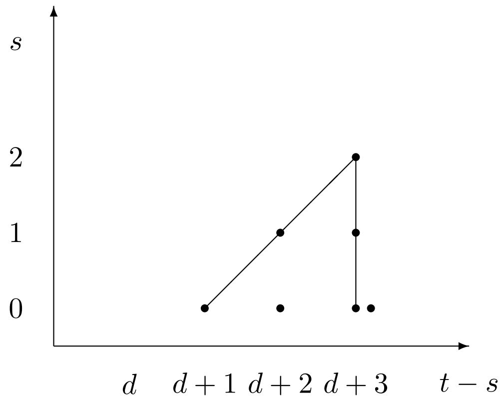

Figure 3.7: The Adams spectral sequence for $\Sigma^{d+\infty}BO$.

Remark 3.35. Note how the spectral sequences look like they depend only on d mod 2 even though the computations depend on d mod 4. Similarly, the spectral sequences for $\pi_d(MTSO(d,5))$ only depended on d mod 4 even though the computations depended on d mod 8. Whether or not this is a general thing is not clear. The periodicity map only explains the 4- and 8-periodicities, respectively. However, in Chapter 5 we show a certain 2- and 4-periodicity, respectively, when we let r tend to infinity.

3.9 Computation of Oriented Vector Field Cobordism Groups

The above calculations of the groups $\pi_q(MTSO(d,r))$ allow us to compute the homotopy groups $\pi_{d+r}(MTSO(d))$, for which we gave an interpretation as vector field cobordism groups in Chapter 2. For the remainder of this chapter, $MTSO$ is abbreviated to MT and Ω_* denotes the oriented cobordism ring.

Consider the map

$$\pi_{d+r}(MT(d)) \to \pi_{d+r}(MT(d+r+1)) \cong \Omega_{d+r}. \tag{3.16}$$

The last map is an isomorphism because the inclusion $MT(d+r+1) \to MT$ is $(d+r+1)$-connected. We can interpret this geometrically as the map that takes a vector field cobordism class to its ordinary oriented cobordism class. Since the groups Ω_d are well-known, we want to describe $\pi_{d+r}(MT(d))$ in terms of the map (3.16). The following geometric interpretation is nice to have in mind.

Proposition 3.36. *The image of* (3.16) *is the subgroup of cobordism classes that contain a manifold with r tangent vector fields. The kernel is the group of vector field cobordism classes of boundary manifolds.*

If the map is injective, it means that any two cobordant manifolds allowing r vector fields are vector field cbordant. In particular, a boundary manifold with r vector fields always bounds a manifold with an extension of the vector fields.

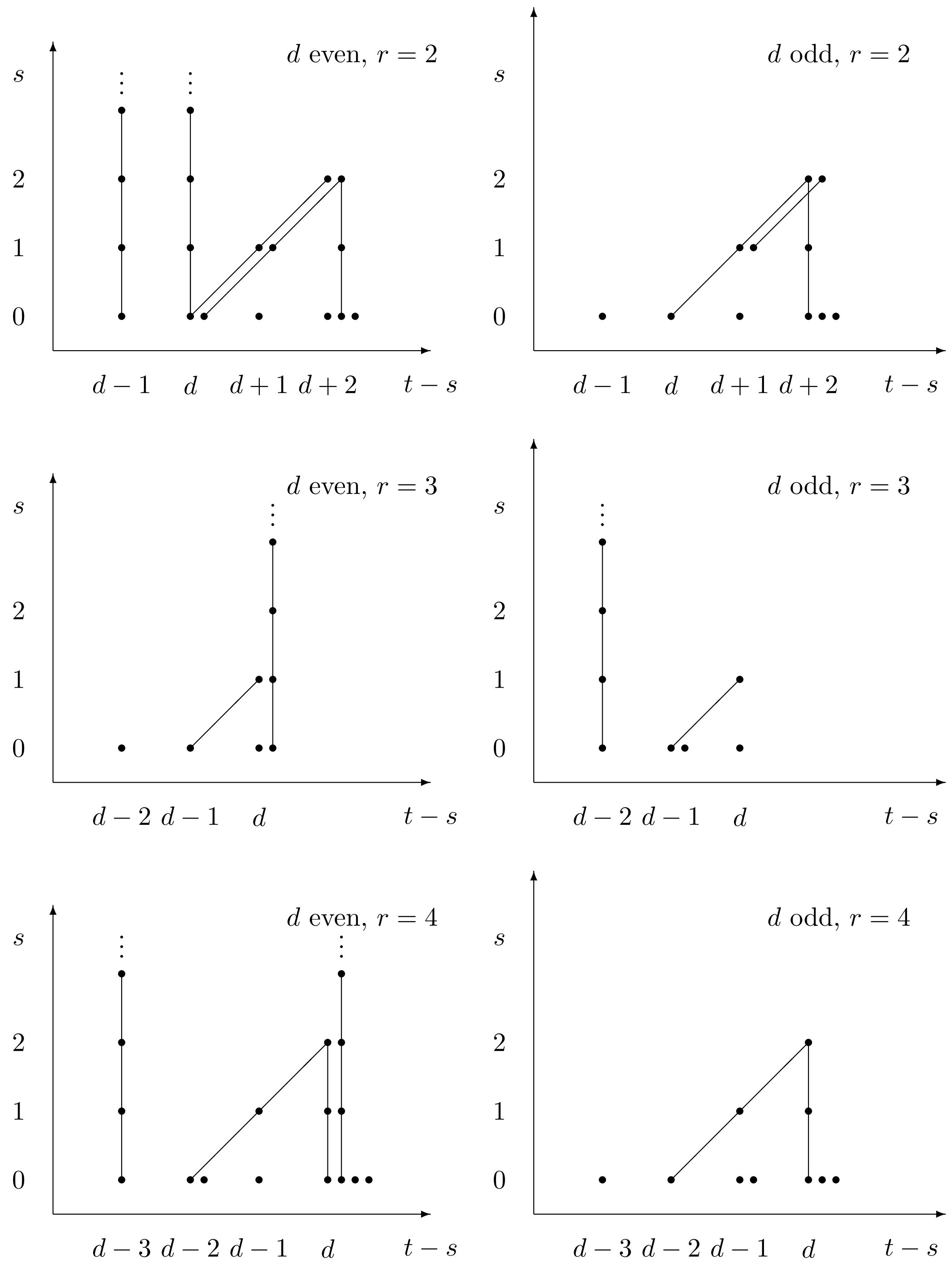

Figure 3.8: The Adams spectral sequence, the unoriented case.

The idea is to look at the long exact sequence for the cofibration

$$MT(d) \to MT(d+r) \to MT(d+r,r)$$

in order to determine $\pi_{d+r-1}(MT(d))$:

$$\begin{array}{cccc} \cdots & \to \pi_{d+r}(MT(d+r)) & \xrightarrow{\beta^r} & \pi_{d+r}(MT(d+r,r)) \\ \to \pi_{d+r-1}(MT(d)) \to & \Omega_{d+r-1} & \to & \pi_{d+r-1}(MT(d+r,r)) \\ \cdots & \cdots & & \cdots \\ \to \quad \pi_{d+1}(MT(d)) \quad \to & \Omega_{d+1} & \to & \pi_{d+1}(MT(d+r,r)) \\ \to \quad \pi_d(MT(d)) \quad \to & \Omega_d & \to & 0 \end{array} \tag{3.17}$$

The maps to the right of $\pi_{d+r-1}(MT(d))$ are easily described by induction on r. Thus the main problem is to determine the image of β^r. For this, recall that we have the map

$$\pi_{d+r}(MT(d+r)) \xrightarrow{\beta^r} \pi_{d+r}(MT(d+r,r)) \xrightarrow{\Psi} KR^{t-d}(tH_r). \tag{3.18}$$

It follows from Theorem 1.35 and [3] that Ψ is an isomorphism for $r = 1, 2$ and a surjection for $r = 3$. We saw in Corollary 2.23 that every element in $\pi_d(MT(d))$ is $\beta(M)$ for some compact oriented d-manifold M, and $\Psi \circ \beta^r$ computes the Atiyah–Dupont invariant of this M.

For $r = 1$, consider the map $MT(d) \to MT(d+1)$. There is a diagram with exact rows

$$\begin{tikzcd} \pi_{d+1}(MT(d+1,1)) \arrow[r] \arrow[d] & \pi_d(MT(d)) \arrow[r] \arrow[d, "\beta^1"] & \pi_d(MT(d+1)) \arrow[r] \arrow[d] & 0 \\ \pi_{d+1}(MT(d+1,1)) \arrow[r] & \pi_d(MT(d,1)) \arrow[r] & \pi_d(MT(d+1,2)) \arrow[r] & 0. \end{tikzcd} \tag{3.19}$$

Here $\beta^1(M) = \chi(M)$ by Theorem 1.32. From this, $\pi_d(MT(d))$ can be computed. This is done in e.g. [14] or [12]. We include the statements and proofs here for completeness and later reference.

Proposition 3.37. *Let d be even. There is a split short exact sequence*

$$0 \to \mathbb{Z} \to \pi_d(MT(d)) \to \Omega_d \to 0. \tag{3.20}$$

A splitting $\pi_d(MT(d)) \to \mathbb{Z}$ is given by $\frac{1}{2}\chi$ when $d \equiv 2 \bmod 4$ and $\frac{1}{2}(\sigma + \chi)$ when $d \equiv 0 \bmod 4$. Here σ is the signature.

Proof. If we fill in the known groups, (3.19) becomes

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \pi_d(MT(d)) \arrow[r] \arrow[d, "\chi"] & \Omega_d \arrow[r] \arrow[d] & 0 \\ \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0. \end{tikzcd}$$

For $d \equiv 2 \bmod 4$, all manifolds have even Euler characteristic, see [30], Exercise 13.2. Thus the image of χ is $2\mathbb{Z}$. By the diagram, $\frac{1}{2}\chi$ defines a splitting of the upper row in the diagram.

For $d \equiv 0 \bmod 4$, χ is surjective. For instance, $\chi(S^d) = 2$ and $\chi(\mathbb{C}P^{2n}) = n+1$. Now,

$$\begin{aligned} \sigma &: \pi_d(MT(d)) \to \Omega_d \to \mathbb{Z} \\ \chi &: \pi_d(MT(d)) \to \pi_d(MT(d,1)) \cong \mathbb{Z} \end{aligned}$$

are both well-defined homomorphisms and they always have the same parity. Therefore, $\frac{1}{2}(\sigma + \chi) : \pi_d(MT(d)) \to \mathbb{Z}$ is well-defined. This clearly defines a splitting. $\square$

Proposition 3.38. *For $d \equiv 3 \bmod 4$, $\pi_d(MT(d)) \cong \Omega_d$. When $d \equiv 1 \bmod 4$, there is a short exact sequence*

$$0 \to \mathbb{Z}/2 \to \pi_d(MT(d)) \to \Omega_d \to 0 \tag{3.21}$$

which is split by the real semi-characteristic $\chi_{\mathbb{R}}$.

Proof. The long exact sequence for the pair $(MT(d), MT(d+1))$ is

$$\pi_{d+1}(MT(d+1)) \xrightarrow{\chi} \pi_{d+1}(MT(d+1,1)) \to \pi_d(MT(d)) \to \Omega_d \to 0.$$

Here χ is surjective when $d+1 \equiv 0 \bmod 4$ and has image $2\mathbb{Z}$ when $d+1 \equiv 2 \bmod 4$. This yields the isomorphism and the short exact sequence. To see that the sequence splits, consider the diagram

$$\begin{tikzcd} \pi_{d+1}(MT(d+1,1)) \arrow[r] \arrow[d] & \pi_d(MT(d)) \arrow[r] \arrow[d] & \pi_d(MT(d+1)) \arrow[r] \arrow[d] & 0 \\ \pi_{d+1}(MT(d+1,1)) \arrow[r] & \pi_d(MT(d,2)) \arrow[r] & \pi_d(MT(d+1,3)) \arrow[r] & 0. \end{tikzcd}$$

With the known groups inserted, this becomes

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \pi_d(MT(d)) \arrow[r] \arrow[d, "\beta^2"] & \Omega_d \arrow[r] \arrow[d] & 0 \\ \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd}$$

Thus β^2 defines a splitting. By (3.18) and [3], $\beta^2(M) = \chi_{\mathbb{R}}(M)$ $\square$

We now proceed to the higher homotopy groups $\pi_{d+r}(MT(d))$.

Theorem 3.39. *For $d \equiv 0 \bmod 4$, there is a split short exact sequence*

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \pi_{d+1}(MT(d)) \to \Omega_{d+1} \to 0.$$

One of the $\mathbb{Z}/2$ summands is $\chi_{\mathbb{R}}$, while the other one is the generator of the $\mathbb{Z}$ in (3.20) *composed with a Hopf map.*

Proof. By Theorem 3.20, the sequence (3.17) is as follows:

$$\begin{array}{cccccc} \cdots & \to & \pi_{d+2}(MT(d+2)) & \xrightarrow{\beta^2} & \mathbb{Z} \oplus \mathbb{Z}/2 \\ \to \pi_{d+1}(MT(d)) & \to & \Omega_{d+1} & \to & \mathbb{Z} \\ \to \pi_d(MT(d)) & \to & \Omega_d & \to & 0 \end{array}$$

This only requires $d \geq 4$. It follows from Proposition 3.37 that

$$\pi_{d+1}(MT(d)) \to \Omega_{d+1}$$

is surjective.

By (3.18) and [3], $\beta^2(M^{d+2}) = (\chi(M), 0)$. Since $d+2 \equiv 2 \bmod 4$, the Euler characteristic is even, so the cokernel of β^2 is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, i.e. we get the short exact sequence

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \pi_{d+1}(MT(d)) \to \Omega_{d+1} \to 0.$$

This means that the long exact sequence

$$\cdots \to \pi_{d+2}(MT(d+1,1)) \to \pi_{d+1}(MT(d)) \to \pi_{d+1}(MT(d+1)) \to \cdots$$

becomes a short exact sequence

$$0 \to \mathbb{Z}/2 \to \pi_{d+1}(MT(d)) \to \Omega_{d+1} \oplus \mathbb{Z}/2 \to 0. \tag{3.22}$$

Together with the fact that $\pi_{d+2}(MT(d+1,1)) \cong \mathbb{Z}/2$ is generated by the composition of the generator in $\pi_{d+1}(MT(d+1,1)) \cong \mathbb{Z}$ with a Hopf map, this yields the interpretation of the $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

To see that (3.22) splits, consider the diagram with exact rows

$$\begin{tikzcd}
{} \arrow[r] & \pi_{d+2}(MT(d+1,1)) \arrow[r] \arrow[d] & \pi_{d+1}(MT(d)) \arrow[r] \arrow[d] & \pi_{d+1}(MT(d+1)) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi_{d+2}(MT(d+1,1)) \arrow[r] & \pi_{d+1}(MT(d,2)) \arrow[r] & \pi_{d+1}(MT(d+1,3)) \arrow[r] & {}.
\end{tikzcd}$$

Inserting the known groups, we get

$$\begin{tikzcd}
{} \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d, "\cong"] & \pi_{d+1}(MT(d)) \arrow[r] \arrow[d] & \pi_{d+1}(MT(d+1)) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r] & {}.
\end{tikzcd}$$

The calculation of these homotopy groups only requires $d \geq 4$. It follows from the diagram that the upper sequence must split.

For $d = 0$, the theorem follows by direct computations. □

Theorem 3.40. *For $d \equiv 1 \bmod 4$, there is a split short exact sequence*

$$0 \to \mathbb{Z}/2 \to \pi_{d+1}(MT(d)) \to \Omega_{d+1} \to 0.$$

The $\mathbb{Z}/2$ summand is the $\mathbb{Z}/2$ from (3.21) *composed with a Hopf map.*

Proof. In this case, the sequence (3.17) becomes:

$$\begin{array}{cccccc}
 & \cdots & \to & \pi_{d+2}(MT(d+2)) & \xrightarrow{\beta^2} & \mathbb{Z}/2 \\
\to & \pi_{d+1}(MT(d)) & \to & \Omega_{d+1} & \to & \mathbb{Z}/2 \\
\to & \pi_d(MT(d)) & \to & \Omega_d & \to & 0
\end{array}$$

All manifolds of dimension $d+2 \equiv 3 \bmod 4$ have two independent vector fields, so $\beta^2(M) = 0$ for all M. Together with (3.21), this yields a short exact sequence

$$0 \to \mathbb{Z}/2 \to \pi_{d+1}(MT(d)) \to \Omega_{d+1} \to 0$$

for all $d > 0$. Since $\pi_{d+2}(MT(d+2,2)) \cong \eta \cdot \pi_{d+1}(MT(d+2,2))$ where η is the Hopf map, the interpretation of the $\mathbb{Z}/2$ immediately follows.

To solve the extension problem, consider the diagram

$$\begin{tikzcd}
{} \arrow[r] & \pi_{d+2}(MT(d+1,1)) \arrow[r] \arrow[d] & \pi_{d+1}(MT(d)) \arrow[r] \arrow[d] & \pi_{d+1}(MT(d+1)) \arrow[r] \arrow[d, "\cong"] & {} \\
{} \arrow[r] & \pi_{d+2}(MT(d+1,1)) \arrow[r] & \pi_{d+1}(MT(d,3)) \arrow[r] & \pi_{d+1}(MT(d,4)) \arrow[r] & {} .
\end{tikzcd}$$

For $d > 6$, the groups are

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[d, "\cong"] & \pi_{d+1}(MT(d)) \arrow[r] \arrow[d] & \pi_{d+1}(MT(d+1)) \arrow[r] \arrow[d] & \mathbb{Z} \\
\mathbb{Z}/2 \arrow[r] & \mathbb{Z}/24 \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}/24 \arrow[r] & \mathbb{Z}.
\end{tikzcd}$$

The first map in the lower row must be the inclusion of a direct summand, and thus the sequence splits.

For $d = 1$ and $d = 5$, $\Omega_2 = \Omega_6 = 0$ so the extension problem is trivial. $\square$

Theorem 3.41. *For $d \equiv 2 \bmod 4$, $\pi_{d+1}(MT(d)) \cong \Omega_{d+1}$.*

Proof. In this case, the exact sequence is:

$$\begin{array}{ccccc}
\dots & \to & \pi_{d+2}(MT(d+2)) & \xrightarrow{\beta^2} & \mathbb{Z} \oplus \mathbb{Z}/2 \\
\to \pi_{d+1}(MT(d)) & \to & \Omega_{d+1} & \to & \mathbb{Z} \\
\to \pi_d(MT(d)) & \to & \Omega_d & \to & 0
\end{array}$$

By (3.18) and [3],

$$\beta^2(M) = (\chi(M), \frac{1}{2}(\chi(M) + \sigma(M))) \in \mathbb{Z} \oplus \mathbb{Z}/2$$

when $d > 2$. This is surjective because

$$(\chi, \frac{1}{2}(\chi + \sigma)) : \pi_{d+2}(MT(d+2)) \to \mathbb{Z} \oplus \mathbb{Z} \tag{3.23}$$

is surjective by Proposition 3.37 and the fact that $\sigma : \Omega_{d+2} \to \mathbb{Z}$ is surjective.

When $d = 2$, the claim follows by a direct computation. $\square$

Theorem 3.42. *For $d \equiv 3 \bmod 4$, there is a short exact sequence*

$$0 \to \pi_{d+1}(MT(d)) \to \Omega_{d+1} \to \mathbb{Z}/2 \to 0.$$

The map $\Omega_d \to \mathbb{Z}/2$ is χ (or equivalently σ) mod 2.

Proof. In this case, the exact sequence becomes:

$$\begin{array}{ccccc} \cdots & \to & \pi_{d+2}(MT(d+2)) & \xrightarrow{\beta^2} & \mathbb{Z}/2 \\ \to \pi_{d+1}(MT(d)) & \to & \Omega_{d+1} & \to & \mathbb{Z}/2 \\ \to \pi_d(MT(d)) & \to & \Omega_d & \to & 0 \end{array}$$

By (3.18) and [3], $\beta^2(M) = \chi_{\mathbb{R}}(M)$ so β^2 is surjective. Thus we get a short exact sequence

$$0 \to \pi_{d+1}(MT(d)) \to \Omega_{d+1} \to \mathbb{Z}/2 \to 0.$$

The last map is the Euler characteristic mod 2 because the diagram

$$\begin{tikzcd} \pi_{d+1}(MT(d)) \arrow[r] & \pi_{d+1}(MT(d+1)) \arrow[r, "\chi"] \arrow[d] & \pi_{d+1}(MT(d+1,1)) \cong \mathbb{Z} \arrow[d] \\ & \pi_{d+1}(MT(d+2)) \arrow[r] & \pi_{d+1}(MT(d+2,2)) \cong \mathbb{Z}/2 \end{tikzcd}$$

commutes. □

Next we consider the groups $\pi_{d+2}(MT(d))$. Again the main problem is to determine the map β^3 in (3.17).

Theorem 3.43. *For $d \equiv 0 \bmod 4$ and $d \geq 4$, there is a short exact sequence*

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \pi_{d+2}(MT(d)) \to \Omega_{d+2} \to 0.$$

Proof. The sequence (3.17) together with Theorem 3.39 yields

$$\pi_{d+3}(MT(d+3)) \xrightarrow{\beta^3} \pi_{d+3}(MT(d+3,3)) \to \pi_{d+2}(MT(d)) \to \Omega_{d+2} \to 0.$$

Since all manifolds of dimension $d+3$ allow three vector fields, β^3 vanishes on all of $\pi_{d+3}(MT(d+3))$. Thus the result follows from Theorem 3.21. □

Theorem 3.44. *For $d \equiv 1 \bmod 4$ and $d \geq 5$, there is a short exact sequence*

$$0 \to \mathbb{Z}/2 \to \pi_{d+2}(MT(d)) \to \Omega_{d+2} \to 0.$$

Proof. In this case, $\pi_{d+3}(MT(d+3),3) \cong \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$. The composition

$$\pi_{d+3}(MT(d+3)) \xrightarrow{\beta^3} \pi_{d+3}(MT(d+3,3)) \xrightarrow{\Psi} KR(P_2) \cong \mathbb{Z} \oplus \mathbb{Z}/4$$

is surjective since (3.23) is surjective. Thus the cokernel of β^3 can be at most $\mathbb{Z}/2$.

In the Adams spectral sequence, $E_2^{0,d+3}(MT(d+3,3)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The generators are represented by the fact that w_{d+3} and $w_2 w_{d+1}$ are not decomposable over $\mathcal{A}_2$ in $H^*(MT(d+3,3);\mathbb{Z}/2)$.

For $d+1 \equiv 2 \bmod 8$, there is a relation

$$w_2 w_{d+1} + w_{d+3} = \mathrm{Sq}^4(w_{d-1}) + \mathrm{Sq}^1(w_2 w_d),$$

while for $d+1 \equiv 6 \bmod 8$,

$$w_2 w_{d+1} = \mathrm{Sq}^4(w_{d-1}) + \mathrm{Sq}^1(w_2 w_d)$$

in $H^*(MT(d+3))$. Thus the map $E_2^{0,d+3}(MT(d+3)) \to E_2^{0,d+3}(MT(d+3,3))$ is not surjective. Since there are no differentials in either of the spectral sequences in the relevant range, the map on $E_\infty^{0,d+3}$ is also not surjective. The element not hit cannot come from an element of $\pi_{d+3}(MT(d+3))$ of higher filtration, so the map $\pi_{d+3}(MT(d+3)) \to \pi_{d+3}(MT(d+3,3))$ cannot be surjective. Hence the cokernel of β^3 must be $\mathbb{Z}/2$. □

Theorem 3.45. *For all $d \equiv 2 \bmod 4$, there is a short exact sequence*

$$0 \to \pi_{d+2}(MT(d)) \to \Omega_{d+2} \xrightarrow{\sigma} \mathbb{Z}/4 \to 0$$

where σ is the signature.

Proof. In this case, $\pi_{d+3}(MT(d+3,3)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The generators come from the fact that $w_2 w_{d+1}$ and w_{d+2} are indecomposable in $H^*(MT(d+3,3);\mathbb{Z}/2)$ and $\mathrm{Sq}^2(w_{d+2}) = 0$. These are also indecomposable in $H^*(MT(d+3);\mathbb{Z}/2)$ as one can see by computing

$$\xi_5 \xi_4^{\frac{d-2}{4}}(w_2 w_{d+1}) = 1$$
$$\xi_4^{\frac{d+2}{4}}(w_{d+2}) = 1.$$

Here ξ_i is the unique $\mathcal{A}_2$-homomorphism $\xi_i : H^i(MTSO;\mathbb{Z}/2) \to \mathbb{Z}/2$ for $i = 4, 5$. The products are induced by the direct sum map $BSO \times BSO \to BSO$.

This means that we get a surjection on E_2-terms of Adams spectral sequences that both collapse in the relevant range. Hence β^3 must be surjective. The existence of the short exact sequence then follows from previous results.

By the results of [3], a manifold must satisfy $\sigma \equiv 0 \bmod 4$ if it has two independent vector fields. Since the image of $\pi_{d+2}(MT(d)) \to \Omega_{d+2}$ is the set of cobordism classes containing manifolds with two independent vector fields, this lies in the kernel of $\sigma : \Omega_{d+2} \to \mathbb{Z}/4$. By the short exact sequence, the kernel can be no larger, and the last claim follows. □

Theorem 3.46. *For $d \equiv 3 \bmod 4$ and $d \geq 7$, there is a short exact sequence*

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \pi_{d+2}(MT(d)) \to \Omega_{d+2} \to 0.$$

Proof. Now $\pi_{d+3}(MT(d+3,3)) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Note that β^3 factors as

$$\pi_{d+3}(MT(d+3)) \xrightarrow{\beta^5} \pi_{d+3}(MT(d+3,5)) \to \pi_{d+3}(MT(d+3,3)).$$

By Theorem 3.22, $\pi_{d+3}(MT(d+3,5)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$. The Adams spectral sequences show that the $\mathbb{Z}/2$ summand is in the image of $\pi_{d+3}(MT(d,2))$, so it is mapped to 0 in $\pi_{d+3}(MT(d+3,3))$. From the spectral sequences, one also sees that the generator of the $\mathbb{Z}$ summand is mapped surjectively onto the generator of the $\mathbb{Z}$ summand in $\pi_*(MT(d+3,3))$. This yields the short exact sequence claimed. □

Finally we consider the groups $\pi_{d+3}(MT(d))$. Using similar techniques we obtain the following results:

Theorem 3.47. *For $d \equiv 0 \bmod 4$ and $d \geq 8$, there is a short exact sequence*

$$0 \to \mathbb{Z}/24 \to \pi_{d+3}(MT(d)) \to \Omega_{d+3} \to 0.$$

Proof. A very careful study of the the spectral sequences for the cofibration

$$MT(d,1) \to MT(d+4,5) \to MT(d+4,4)$$

shows that the image of β^4 is a subgroup of $\pi_{d+4}(MT(d+4,4)) \cong \mathbb{Z} \oplus \mathbb{Z}/48 \oplus \mathbb{Z}/4$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/8$ with cokernel $\mathbb{Z}/24$. $\square$

Theorem 3.48. *For $d \equiv 1 \bmod 4$ and $d \geq 5$, there is a short exact sequence*

$$0 \to \pi_{d+3}(MT(d)) \to \Omega_{d+3} \xrightarrow{\sigma} \mathbb{Z}/8 \to 0.$$

Proof. Comparing the Adams spectral sequences, one finds that β^4 is surjective and that the image of $\Omega_{d+3} \to \mathbb{Z}/8 \oplus \mathbb{Z}/2$ contains an element of order eight. The identification of the map $\Omega_{d+3} \to \mathbb{Z}/8$ again follows from [3] where it is shown that a manifold with three independent vector fields must have $\sigma \equiv 0 \bmod 8$. $\square$

For an oriented d-dimensional manifold M and $w \in H^d(M;\mathbb{Z}/2)$ a product of Stiefel–Whitney classes for TM, the Stiefel–Whitney number $w[M]$ is given by evaluating w on the fundamental class.

Theorem 3.49. *For $d \equiv 2 \bmod 4$ and $d \geq 6$, there is a short exact sequence*

$$0 \to \mathbb{Z}/24 \to \pi_{d+3}(MT(d)) \to \Omega_{d+3} \to \mathbb{Z}/2 \to 0.$$

The last map is the Stiefel–Whitney number $w_2 w_{d+1}[M]$.

Proof. This sequence $\pi_{d+4}(MT(d,1)) \to \pi_{d+4}(MT(d+4,5)) \to \pi_{d+4}(MT(d+4,4))$ becomes

$$\cdots \to \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z} \oplus \mathbb{Z}/24 \to \cdots.$$

The Adams spectral sequences show that the first two $\mathbb{Z}/2$ and the last two $\mathbb{Z}$ summands map isomorphically to each other, so the $\mathbb{Z}/24$ summand is not hit, thus it is not hit by β^4 either. The $\mathbb{Z}$ summand is hit by β^4 because it maps onto $2\mathbb{Z} \in \pi_{d+4}(MT(d+4,1)) \cong \mathbb{Z}$. This yields the short exact sequence. $\square$

Theorem 3.50. *For $d \equiv 3 \bmod 4$ and $d \geq 7$, there is a short exact sequence*

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \pi_{d+3}(MT(d)) \to \Omega_{d+3} \to 0.$$

Proof. Look at the exact sequence for the cofibration

$$MT(d+1,1) \to MT(d+4,4) \to MT(d+4,3).$$

Filling in the known groups, we see that $\pi_{d+4}(MT(d+4,4)) \to \pi_{d+4}(MT(d+4,3))$ is injective, so since $\beta^3 : \pi_{d+4}(MT(d+4)) \to \pi_{d+4}(MT(d+4,3))$ is zero, β^4 must also be zero. $\square$

In the above computations, we found descriptions of the invariants $\beta^4(M)$ which we summarize in the next theorem.

Theorem 3.51. *The following table displays the image* $\operatorname{Im}\beta^4$ *of*

$$\beta^4 : \pi_d(MT(d)) \to \pi_d(MT(d,4))$$

and the interpretation of β^4*:*

d mod 4	$\operatorname{Im}\beta^4$	β^4
0	$\mathbb{Z}\oplus\mathbb{Z}/8$	$\chi\oplus\frac{1}{2}(\chi+\sigma)$
1	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\chi_2\oplus\chi_{\mathbb{R}}$
2	$\mathbb{Z}$	χ
3	0	0

In particular, $\beta^4(M)$ *is the top obstruction to* 4, 5 *and* 6 *vector fields when* d *is even.*

Proof. The groups $\operatorname{Im}\beta^4$ were determined in the proofs above. For $d \equiv 0 \bmod 4$, we still need to identify $\beta^4(M)$. We saw that the image of β^4 was $\mathbb{Z}\oplus\mathbb{Z}/8$. On the other hand, $\chi\oplus\frac{1}{2}(\chi+\sigma) : \pi_d(MT(d)) \to \mathbb{Z}\oplus\mathbb{Z}/8$ vanishes for a manifold that allows four vector fields because of a theorem due to Mayer and Frank, see e.g. [3], Corollary 6.6. Hence the kernel of β^4 is contained in the kernel of $\frac{1}{2}(\chi+\sigma)$ by Corollary 2.26. Thus there is a factorization

$$\chi\oplus\frac{1}{2}(\chi+\sigma) : \pi_d(MT(d)) \to \operatorname{Im}(\beta^4) \to \mathbb{Z}\oplus\mathbb{Z}/8.$$

The composition is surjective, so the last map is forced to be an isomorphism.

For $d \equiv 1 \bmod 4$, it follows from the spectral sequences that

$$\beta^4(M) = w_2 w_{d-2}[M] \oplus \chi_{\mathbb{R}}(M).$$

By [29]

$$w_2 w_{d-2}[M] = \chi_2(M) + \chi_{\mathbb{R}}(M).$$

For the last statement, it follows from the proof of Theorem 3.23 that

$$\pi_{d-1}(V_{d,r}) \to \pi_d(MT(d,4))$$

is injective for $r = 4, 5, 6$ and d even. It maps the index $\operatorname{Ind}(s)$ to $\beta^4(M)$ by construction. □

Remark 3.52. For $d \equiv 0 \bmod 4$, the Atiyah–Dupont invariant is

$$\chi\oplus\frac{1}{2}(\chi\pm\sigma) \in \mathbb{Z}\oplus\mathbb{Z}/4.$$

Hence our invariant carries strictly more information in this case.

Chapter 4

A Spectral Sequence Converging to the Homotopy Groups

We begin this chapter by setting up a spectral sequence converging to $\pi_*(MT(d,r))$. This sequence does not immediately give any new information about the homotopy groups, but the calculations in Chapter 3 provide some information about the differentials in the spectral sequence.

In order to make things easier to work with, we stabilize the spectra by the periodicity map. These stable spectra form an inverse system. The remainder of the thesis is devoted to the study of the inverse limit.

The necessary background on direct and inverse limits of groups and spectra is given in Section 4.2. In Section 4.3 we recall a classical inverse limit spectrum studied by Lin in [26]. We give a more convenient construction of his spectrum in Section 4.4, and in Section 4.5, this definition is generalized to an inverse limit of the spectra $MT(d,r)$. In Section 4.6 we show how this generalizes Lin's theorem to all vector bundles over a compact space. Finally, in Section 4.7, we construct a stable version of the spectral sequence we began with and show that it converges to the homotopy groups of the inverse limit spectrum. In the last Section 4.8, we attempt to relate the constructions back to the vector field problem.

Throughout this chapter, all constructions and results hold for both the unoriented, oriented, and spin spectra, unless otherwise specified. The spectra will just be denoted by MT and $B(d)$ denotes the corresponding classifying space filtered by the Grassmannians $G(d,n)$.

4.1 The Spectral Sequence

Consider the map $MT(d-r) \to MT(d)$. This is filtered as the composition

$$MT(d-r) \to MT(d-r+1) \to \cdots \to MT(d-1) \to MT(d). \tag{4.1}$$

Each map fits into a cofibration sequence

$$MT(d-r+k) \to MT(d-r+k+1) \to MT(d-r+k+1,1).$$

The corresponding long exact sequences of homotopy groups form a spectral sequence with

$$E^1_{s,t} = \pi_{d-s+t}(MT(d-s,1))$$

for $0 \le s < r$ and $E^1_{s,t} = 0$ otherwise. The differentials are

$$d^k : E^k_{s,t} \to E^k_{s+k,t+k-1}.$$

There is a filtration

$$MT(d-r,0) \to MT(d-r+1,1) \to \cdots \to MT(d-1,r-1) \to MT(d,r) \quad (4.2)$$

defining a spectral sequence with the same E^1-term. The obvious map from (4.1) to (4.2) induces an isomorphism on E^1 and thus an isomorphism of spectral sequences.

Proposition 4.1. *This spectral sequence converges to $\pi_{d+t-s}(MT(d,r))$ filtered by the subgroups*

$$F_{s,t} = \mathrm{Im}(\pi_{d+t-s}(MT(d-s,r-s)) \to \pi_{d+t-s}(MT(d,r)))$$

such that $E^\infty_{s,t} = F_{s,t}/F_{s+1,t+1}$.

Proof. Using the construction (4.2), the result follows from standard convergence theorems, since $\pi_*(MT(d-r,0)) = 0$, see e.g. [19], Proposition 1.2. □

The first differential $d^1 : E^1_{s,t} \to E^1_{s+1,t}$ in this spectral sequence is induced by the composite map

$$\tau : MT(d-s,1) \xrightarrow{\partial} \Sigma MT(d-s-1) \to \Sigma MT(d-s-1,1).$$

This is the boundary map in the cofibration sequence

$$MT(d-s,2) \to MT(d-s,1) \xrightarrow{\tau} \Sigma MT(d-s-1,1). \quad (4.3)$$

It turns out that this is a familiar map:

Proposition 4.2. *The map $\tau : MT(d,1) \to \Sigma MT(d-1,1)$ is the Becker–Gottlieb transfer associated to the fibration*

$$S^{d-1} \to SU_d \xrightarrow{p} B(d).$$

Proof. Recall how the Becker–Gottlieb transfer

$$\Sigma^{n+d} G(d,n) \to \Sigma^{n+d} SU_{d,n}$$

is defined. We may think of $SU_{d,n}$ as a subset of $U_{d,n}$. This extends to an embedding of the normal bundle $\mathbb{R} \times SU_{d,n} \to U_{d,n}$ in the obvious way. A point $(t, v \in V)$ with $V \in G(d,n)$ is mapped to $e^t v \in V$. Hence there is a Thom map

$$\gamma_1 : \mathrm{Th}(U_{d,n}) \to \mathrm{Th}(\mathbb{R} \times SU_{d,n} \to SU_{d,n}) \quad (4.4)$$

given by collapsing $G(d,n)$. Let $\gamma_2 : \mathbb{R} \times SU_{d,n} \to p^* U_{d,n}$ be the inclusion of a subbundle over $SU_{d,n}$ that takes a point $(t, v \in V)$ to tv in the fiber over $v \in V$. The transfer is then defined to be the composition

$$\mathrm{Th}(U_{d,n}^{\perp} \oplus U_{d,n}) \xrightarrow{\gamma_1 \oplus i} \mathrm{Th}(\mathbb{R} \oplus p^* U_{d,n}^{\perp}) \xrightarrow{\gamma_2 \oplus 1} \mathrm{Th}(p^* U_{d,n}^{\perp} \oplus p^* U_{d,n})$$

where i is the natural inclusion of fibers.

Now,

$$MT(d,1)_n \cong \mathrm{Th}(U_{d,n}^{\perp} \to BU_{d,n}) / \mathrm{Th}(U_{d,n}^{\perp} \to SU_{d,n}).$$

The map $\partial : MT(d,1)_n \to \Sigma MT(d-1)_{n-1}$ is given by collapsing the subset $\mathrm{Th}(U_{d,n}^{\perp} \to G(d,n))$ over the zero section $G(d,n) \to BU_{d,n}$. This is exactly what the map $\gamma_1 \oplus i$ does.

Since τ factors as

$$MT(d,1) \xrightarrow{\partial} \Sigma MT(d-1) \xrightarrow{q} \Sigma MT(d-1,1),$$

we just need to see that $\gamma_2 \oplus 1$ is homotopic to q. But q can be thought of as the inclusion $\mathrm{Th}(\mathbb{R} \oplus U_{d-1,n}^{\perp}) \to \mathrm{Th}(\mathbb{R} \oplus U_{d-1,n}^{\perp} \oplus U_{d-1,n})$, while $\gamma_2 \oplus 1$ was the inclusion $\mathrm{Th}(\mathbb{R} \oplus p^* U_{d,n}^{\perp}) \to \mathrm{Th}(p^*(U_{d,n}^{\perp} \oplus U_{d,n}))$. The result now follows by commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Th}(\mathbb{R} \oplus U_{d-1,n}^{\perp}) & \xrightarrow{q} & \mathrm{Th}(\mathbb{R} \oplus U_{d-1,n}^{\perp} \oplus U_{d-1,n}) \\
\downarrow & & \downarrow \\
\mathrm{Th}(\mathbb{R} \oplus p^* U_{d,n}^{\perp}) & \xrightarrow{\gamma_2 \oplus 1} & \mathrm{Th}(p^*(U_{d,n}^{\perp} \oplus U_{d,n}))
\end{array}$$

where the vertical maps are the homotopy equivalences of spectra induced by the inclusion $G(d-1,n) \to SU_{d,n}$. □

Corollary 4.3. *The composition*

$$\Sigma^\infty S^0 \to \Sigma^\infty BSO(d)_+ \to \Sigma^\infty BSO(d-1)_+ \to \Sigma^\infty S^0$$

has degree $\chi(S^d)$, which is 2 for d even and 0 for d odd. Thus on the $\pi_*(S^0)$ summands of E^1, d^1 is multiplication by $\chi(S^d)$.

Proof. This follows from [4], Property (3.2) and (3.4). □

Corollary 4.4. *The Becker–Gottlieb transfer*

$$\tau_* : \pi_*(\Sigma^\infty BSO(d)_+)) \to \pi_*(\Sigma^\infty BSO(d-1)_+)$$

depends only on d mod 2 in dimensions $* < d-1$.

Proof. We are going to show in Corollary 4.23 below that the periodicity map $\Sigma^2 MT(d,2) \to MT(d+2,2)$ induces an isomorphism of the long exact sequences for the cofibrations (4.3). □

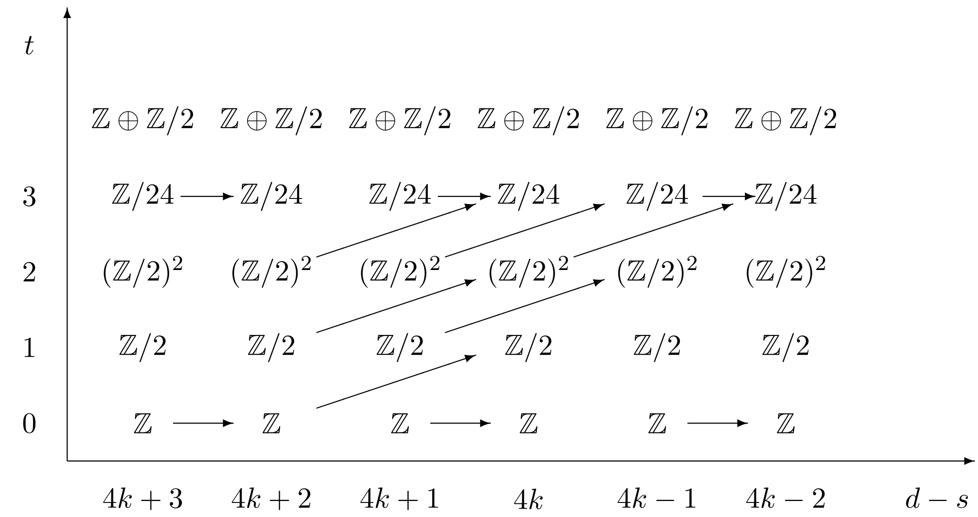

Figure 4.1: The first differentials in the spectral sequence.

The computations of the groups $\pi_q(MTSO(d,r))$ from Chapter 3 allow us to determine all differentials in the spectral sequence for $MTSO(d,r)$ that enter the first four rows, at least in the stable area $t-s<d-2r$. This is displayed in Figure 4.1. An arrow indicates a non-zero differential. The horizontal differentials are multiplication by 2 according to Corollary 4.3. The picture is repeated horizontally with a period of 4.

There is a similar filtration of Stiefel manifolds

$$V_{d-r,0} \to V_{d-r+1,1} \to \cdots \to V_{d-1,r-1} \to V_{d,r}.$$

The long exact sequences for the fibrations $V_{d-k,r-k} \to V_{d-k+1,r-k+1} \to S^{d-k}$ define a spectral sequence with $E^1_{s,t} \cong \pi_{d-s+t}(S^{d-s})$. When d is large compared to r, all homotopy groups involved are stable for low values of t and it is possible to determine the differentials from the known stable homotopy groups. The result is shown in Figure 4.2.

The map f_θ takes the E^1 term of this spectral sequence to a direct summand in the spectral sequence for $MT(d,r)$. Note that there are differentials in the spectral sequence for $MT(d,r)$ entering the S^0 summands but not coming from the spectral sequence for $V_{d,r}$, as expected from the non-injectivity of θ.

From Figure 4.1 it is tempting to extend the spectral sequence in all directions by periodicity. This was the original motivation for the rest of this chapter.

4.2 Background on Direct and Inverse Limits

The purpose of this chapter is mainly to introduce notation and to list some facts about direct and inverse limits for later reference.

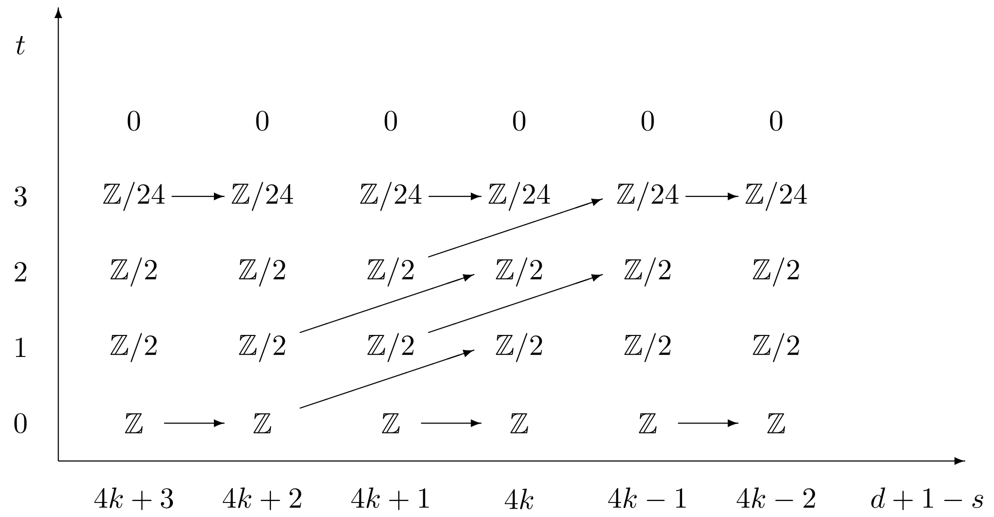

Figure 4.2: The spectral sequence for $V_{d,r}$.

We first consider inverse limits of groups. Throughout this chapter, a 2-group will mean an abelian torsion group in which every element has order some power of 2. A 2-profinite group will mean an inverse limit of finite 2-groups. An inverse limit $\varprojlim_n G_n$ of finite groups is topologized by declaring G_n discrete and giving it the coarsest topology making all maps $\varprojlim_n G_n \to G_n$ continuous.

In general, the inverse limit functor is not exact. It takes an inverse system of short exact sequences $0 \to A_n \to B_n \to C_n \to 0$ to an exact sequence

$$0 \to \varprojlim_n A_n \to \varprojlim_n B_n \to \varprojlim_n C_n \to \varprojlim\nolimits^1_n A_n \to \varprojlim\nolimits^1_n B_n \to \varprojlim\nolimits^1_n C_n \to 0.$$

A convenient criterion for the vanishing of $\varprojlim^1_n G_n$ is the Mittag–Leffler condition, see [42], Chapter 7. This applies for instance if infinitely many of the G_n are finite. It follows that the inverse limit functor is exact on systems of 2-profinite groups with continuous maps between them.

Let

$$X_1 \to \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

be a direct system of spectra. Then the direct limit $\varinjlim_n X_n$ is defined to be the cofiber of the map

$$\bigvee_n X_n \xrightarrow{\bigvee_n (1-f_n)} \bigvee_n X_n.$$

This ensures that for any spectrum Y there is an exact sequence

$$\cdots \to [Y, \bigvee_n X_n] \xrightarrow{\bigvee(1-f_n)} [Y, \bigvee_n X_n] \to [Y, \varinjlim_n X_n] \to \cdots .$$

Since $[Y, \bigvee_n X_n] \cong \bigoplus_n [Y, X_n]$, this sequence is actually short exact with

$$[Y, \varinjlim_n X_n] \cong \varinjlim_n [Y, X_n].$$

Similarly, the exact sequence

$$\cdots \leftarrow [\bigvee_n X_n, Y] \xleftarrow{\bigvee(1-f_n)} [\bigvee_n X_n, Y] \leftarrow [\varinjlim_n X_n, Y] \leftarrow \cdots$$

yields a short exact sequence

$$0 \to \varprojlim_n^1 [\Sigma X_n, Y] \to [\varinjlim_n X_n, Y] \to \varprojlim_n [X_n, Y] \to 0. \tag{4.5}$$

Given an inverse system of spectra,

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1,$$

there is a functorial construction of the product $\prod_n X_n$ with the property that $[Y, \prod_n X_n] \cong \prod_n [Y, X_n]$. If $X_{n,N}$ denotes the Nth space of X_n, then $\prod_n X_n$ may be defined as the spectrum with mth space

$$\prod_n \varinjlim_N \Omega^{N-m} X_{n,N}.$$

To define an inverse limit spectrum, consider the map

$$\prod_n X_n \xrightarrow{1-\prod f_n} \prod_n X_n.$$

If C denotes the cofiber of this map, the inverse limit is defined to be $\Sigma^{-1}C$. For any spectrum Y, this ensures that there is an exact sequence

$$0 \to \varprojlim_n^1 [\Sigma Y, X_n] \to [Y, \varprojlim_n X_n] \to \varprojlim_n [Y, X_n] \to 0. \tag{4.6}$$

Note that the constructions above only define homotopy limits and colimits in the sense that the properties one would want from a limit or colimit only hold up to homotopy.

Recall that for a group G, the 2-completion is given by $G_2^\wedge = \varprojlim_n G/2^n G$. Similarly, there is a functorial construction of the 2-completion of a spectrum X, see e.g. [17] or [7] for the definition and basic properties. This is a spectrum $X_2^\wedge$ with a map $f : X \to X_2^\wedge$. We will need the following facts:

(i) If $\pi_*(X)$ is finitely generated, then $\pi_*(X_2^\wedge) \cong \pi_*(X)_2^\wedge$ and f induces the obvious map $\pi_*(X) \to \pi_*(X)_2^\wedge$.

(ii) If $\pi_*(X)$ is 2-profinite in all dimensions $* < d$, then $\pi_*(X_2^\wedge) \cong \pi_*(X)$ for $* < d$.

See [7], Proposition 2.5. We will also need the fact that 2-completion is exact with respect to sequences of finitely generated groups, see e.g. [13].

4.3 Inverse Limits of Stunted Projective Spaces

Let $P_{d,r}$ denote the stunted projective space $\mathbb{R}P^{d-1}/\mathbb{R}P^{d-r-1}$. James constructed homeomorphisms $\Sigma^{a_r}P_{d,r} \to P_{d+a_r,r}$ using Clifford multiplication, see e.g. [24]. This makes the suspension spectra $\Sigma^\infty P_{d,r}$ well-defined even for $d < 0$ and $r \geq d$ by interpreting $\Sigma^{ka_r}P_{d,r}$ as $P_{d+ka_r,r}$ for k sufficiently large.

The natural maps $P_{d+ka_r,r+1} \to P_{d+ka_r,r}$ define an inverse system of spectra

$$\cdots \to \Sigma^\infty P_{d,r+1} \to \Sigma^\infty P_{d,r} \to \cdots \to \Sigma^\infty P_{d,1}.$$

Recall that $H^*(\mathbb{R}P^{d-1};\mathbb{Z}/2) \cong \mathbb{Z}/2[t]/(t^d)$, i.e. the truncated polynomial algebra on the generator t. On $\mathbb{Z}/2$ cohomology, the inverse system of spectra induces a direct system

$$\mathbb{Z}/2\{t^{d-r},\ldots,t^{d-1}\} \to \mathbb{Z}/2\{t^{d-r-1},\ldots,t^{d-1}\} \to \cdots.$$

Here $\mathbb{Z}/2\{x^1,\ldots,x^k\}$ denotes the graded $\mathbb{Z}/2$-vector space with basis elements x^i in dimension i. The direct limit of these cohomology groups is $\mathbb{Z}/2\{t^l, l < d\}$, i.e. the Laurent polynomials in the variable t of degree at most $d-1$.

There are maps $S^{ka_r-1} \to \Sigma^{ka_r}P_{d,r}$ commuting with the maps in the inverse system. Hence there is a map of spectra $\varphi_0 : S^{-1} \to \varprojlim_r \Sigma^\infty P_{d,r}$. Observing that the induced map on cohomology

$$\varphi_0^* : \mathbb{Z}/2\{t^l, l < d\} \to \Sigma^{-1}\mathbb{Z}/2$$

induces an isomorphism on $\mathrm{Ext}_{\mathcal{A}}^{s,t}$ in degrees $t-s < d$ and a surjection for $t-s = d$, Lin managed to prove the following conjecture by Mahowald in [26]:

Theorem 4.5.

(i) $\varprojlim_r \pi_q(\Sigma^\infty P_{d,r}) = 0$ *when* $q < -1$ *and* $q < d-1$.

(ii) For $0 < d$,

$$\varprojlim_r \pi_{-1}(\Sigma^\infty P_{d,r}) \cong \mathbb{Z}_2^\wedge$$

as topological groups where $\mathbb{Z}_2^\wedge = \varprojlim_r \mathbb{Z}/2^r$ *denotes the* 2*-adic integers. The inclusion*

$$\varphi_{0*} : \mathbb{Z} = \pi_{-1}(S^{-1}) \to \varprojlim_r \pi_{-1}(\Sigma^\infty P_{d,r})$$

is non-zero mod 2.

(iii) When $q > -1$ *and* $d > 0$, *the map induced by* φ_0

$$\varphi_{0*} : \pi_q(S^{-1})_2^\wedge \to \varprojlim_r \pi_q(\Sigma^\infty P_{d,r})$$

is an isomorphism when $q-2 < d$ *and surjective when* $q = d-2$.

4.4 An Alternative Construction of Lin's Spectrum

In this section we construct a version of the inverse limit spectrum from the previous section using Stiefel manifolds and the periodicity maps from Section 3.4.

Recall that we defined maps

$$f_0 : \Sigma^{a_r}\Sigma V_{d,r} \to \Sigma V_{d+a_r,r}$$

and showed that these were $(2(d-r)+a_r+1)$-equivalences. They form a direct system of spectra

$$\Sigma^\infty \Sigma V_{d,r} \to \Sigma^{\infty - a_r}\Sigma V_{d+a_r,r} \to \Sigma^{\infty-2a_r}\Sigma V_{d+2a_r,r} \to \cdots .$$

We may take the direct limit of these spectra in the sense of Section 4.2. The resulting spectrum is denoted $\mathcal{V}_{d,r}$ and satisfies

$$\pi_q(\mathcal{V}_{d,r}) \cong \varinjlim_k \pi^s_{q+ka_r}(\Sigma V_{d+ka_r,r}).$$

Note that this is defined even when d is negative or $r > d$ just by starting the direct sequence at $\Sigma^{\infty-ka_r}\Sigma V_{d+ka_r,r}$ for some large k.

By [23] there is $2(d-r)$-equivalence $P_{d,r} \to V_{d,r}$.

Theorem 4.6. *The composite map*

$$\Sigma^\infty \Sigma P_{d,r} \to \Sigma^\infty\Sigma V_{d,r} \to \mathcal{V}_{d,r}$$

is a homotopy equivalence if $2r < d$. More generally, $\mathcal{V}_{d,r}$ is homotopy equivalent to $\Sigma^\infty\Sigma P_{d,r}$ for all d and r.

Proof. Let $2r < d$. The map $P_{d,r} \to V_{d,r}$ is $2(d-r)$-connected. Thus it induces an isomorphism on homology in dimensions less than $2(d-r) > d$. The map $\Sigma^{ka_r}\Sigma V_{d,r} \to \Sigma V_{d+ka_r,r}$ is $(2(d-r+1)+ka_r)$-connected. Thus the composite map

$$\Sigma^{ka_r}\Sigma P_{d,r} \to \Sigma^{ka_r}\Sigma V_{d,r} \to \Sigma V_{d+ka_r,r} \tag{4.7}$$

induces an isomorphism on homology in dimensions up to $d+ka_r$. Both $\Sigma^{ka_r}\Sigma P_{d,r}$ and $\Sigma V_{d+ka_r,r}$ have homology zero in dimensions between $d+ka_r$ and $2(d-r+ka_r)$, since they have no cells in these dimensions, see e.g. [18], Chapter 3.D. Thus (4.7) is $(2(d-r)+ka_r)$-connected. Letting k tend to infinity, we get an isomorphism

$$\pi^s_*(\Sigma P_{d,r}) \to \varinjlim_r \pi^s_{*+ka_r}(\Sigma V_{d+ka_r,r}) = \pi_*(\mathcal{V}_{d,r}).$$

Thus the map inducing it must be a homotopy equivalence.

For general d and r, the above argument shows that the composite map

$$\Sigma^{\infty+1}P_{d,r} \cong \Sigma^{\infty-ka_r}\Sigma P_{d+ka_r,r} \to \Sigma^{\infty-ka_r}\Sigma V_{d+ka_r,r} \to \mathcal{V}_{d,r}$$

is a homotopy equivalence some k sufficiently large. $\square$

The following diagram of Stiefel manifolds

$$\begin{tikzcd} V_{d,r+1} \arrow[r] \arrow[d] & V_{d,r} \arrow[d] \\ V_{d+1,r+1} \arrow[r] & V_{d+1,r} \end{tikzcd} \tag{4.8}$$

commutes. We want to see that this induces a well-defined commutative diagram on direct limits

$$\begin{tikzcd} \mathcal{V}_{d,r+1} \arrow[r] \arrow[d] & \mathcal{V}_{d,r} \arrow[d] \\ \mathcal{V}_{d+1,r+1} \arrow[r] & \mathcal{V}_{d+1,r}. \end{tikzcd} \tag{4.9}$$

So far, we have made no assumptions on the actual choice of Clifford multiplication in the construction of the periodicity maps. This becomes important if we want to make the diagram strictly commutative.

Lemma 4.7. *Let $\mu : \mathbb{R}^{r+1} \times \mathbb{R}^{a_{r+1}} \to \mathbb{R}^{a_{r+1}}$ be an orthogonal Clifford multiplication and define $\mathcal{V}_{d,r} = \varinjlim_k \Sigma^{\infty - ka_{r+1}} \Sigma V_{d+ka_{r+1},r}$ using the periodicity maps defined by the restriction of μ to $\mathbb{R}^r \times \mathbb{R}^{a_{r+1}}$. Then the diagram (4.9) is well-defined and commutative.*

Proof. It is easy to see from the formulas that the diagrams

$$\begin{tikzcd} \Sigma^{a_r}\Sigma V_{d,r} \arrow[r] \arrow[d] & \Sigma V_{d+a_r,r} \arrow[d] \\ \Sigma^{a_r}\Sigma V_{d+1,r} \arrow[r] & \Sigma V_{d+1+a_r,r} \end{tikzcd} \qquad \begin{tikzcd} \Sigma^{a_{r+1}}\Sigma V_{d,r+1} \arrow[r] \arrow[d] & \Sigma V_{d+a_{r+1},r+1} \arrow[d] \\ \Sigma^{a_{r+1}}\Sigma V_{d,r} \arrow[r] & \Sigma V_{d+a_{r+1},r} \end{tikzcd}$$

commute. Here the vertical maps in the first diagram come from the inclusion $\mathbb{R}^d \oplus \mathbb{R}^{a_r} \subseteq \mathbb{R}^d \oplus \mathbb{R} \oplus \mathbb{R}^{a_r}$ and in the second diagram they come from forgetting the last vector. Hence the maps $\mathcal{V}_{d,r} \to \mathcal{V}_{d+1,r}$ and $\mathcal{V}_{d,r+1} \to \mathcal{V}_{d,r}$ are well-defined.

Commutativity of (4.9) now follows because the diagram

$$\begin{tikzcd} V_{d+ka_{r+1},r+1} \arrow[r] \arrow[d] & V_{d+ka_{r+1},r} \arrow[d] \\ V_{d+1+ka_{r+1},r+1} \arrow[r] & V_{d+1+ka_{r+1},r} \end{tikzcd}$$

commutes for all k. ☐

Lemma 4.8. *For each d, there is an inverse system of spectra*

$$\cdots \to \mathcal{V}_{d,r+1} \to \mathcal{V}_{d,r}. \tag{4.10}$$

The inclusions $\mathcal{V}_{d,r} \to \mathcal{V}_{d+1,r}$ define a map of inverse systems.

Proof. For each r, choose once and for all an orthogonal Clifford representation $Cl_{r-1} \times \mathbb{R}^{ka_r} \to \mathbb{R}^{ka_r}$. Here $k = 1$ except when $4|r$. In this case (i.e. when irreducible representations are not unique) $k = 2$ and we choose the representation to be the restriction of the chosen map $Cl_r \times \mathbb{R}^{a_{r+1}} \to \mathbb{R}^{a_{r+1}}$.

In general, the Cl_r-action on $\mathbb{R}^{a_{r+1}}$ may not restrict to an orthogonal sum of the chosen Cl_{r-1} representations. However, it does up to conjugation by an isomorphism $A : \mathbb{R}^{a_{r+1}} \to \mathbb{R}^{a_{r+1}}$, which we may assume to be orientation preserving. Let $\mathcal{V}_{d,r}$ be defined by the chosen Cl_{r-1} representation and $\mathcal{V}'_{d,r}$ by the restricted Cl_r-action.

It is easy to see that a double application of a periodicity map

$$\Sigma^{a_r}\Sigma^{a_r} V_{d+la_{r+1},r} \to V_{d+(l+2)a_{r+1},r}$$

corresponding to two Clifford representation is equal to the one periodicity map defined by the direct sum of two of these representations. Hence we can assume that the chosen Cl_{r-1} and Cl_r representations have the same dimension.

Choose a path A_t from A to the identity. Then for each t and $x \in S^{a_{r+1}-1}$, the vectors

$$A_t^{-1} e_0 A_t x, \ldots, A_t^{-1} e_{r-1} A_t x$$

are linearly indepedent. Apply Gram–Schmidt to make them orthonormal. This defines a homotopy of sections

$$(B^{a_{r+1}}, S^{a_{r+1}}) \times I \to (W_{a_{r+1},r}, V_{a_{r+1},r}).$$

Apply this to the periodicity maps $\Sigma^{ka_{r+1}} V_{d+la_{r+1},r} \to V_{d+(l+k)a_{r+1},r}$ defined by the respective multiplications. This defines a homotopy equivalence $\mathcal{V}'_{d,r} \to \mathcal{V}_{d,r}$. It is clear that this commutes with the inclusion in $\mathcal{V}_{d+1,r}$.

The conclusion is that there is a well-defined map

$$\mathcal{V}_{d,r+1} \to \mathcal{V}'_{d,r} \to \mathcal{V}_{d,r}$$

and this commutes with the inclusions $\mathcal{V}_{d,r} \to \mathcal{V}_{d+1,r}$. □

Definition 4.9.

$$\widehat{V}_d = \varprojlim_r \mathcal{V}_{d,r}$$
$$\widehat{V} = \varinjlim_d \widehat{V}_d.$$

One could even define $\widehat{V}_d$ for any $d \in \mathbb{Z}_2^\wedge = \varprojlim_r \mathbb{Z}/a_r$, but we shall not consider this.

For $d \geq 0$, Clifford multiplication defines a map $\varphi_0 : S^{ka_r-1} \to V_{d+ka_r,r}$ by

$$\varphi_0(x) = (e_0 x, \ldots, e_{r-1} x) \in 0 \oplus \mathbb{R}^{ka_r} \subseteq \mathbb{R}^{d+ka_r}.$$

This yields a map of spectra $\varphi_0 : S^0 \to \mathcal{V}_{d,r}$ for each d and r such that

$$\begin{tikzcd}
\mathcal{V}_{d,r+1} \arrow[r] & \mathcal{V}_{d+1,r+1} \arrow[d] \\
S^0 \arrow[u, "\varphi_0"] \arrow[ur, "\varphi_0"] \arrow[r, "\varphi_0"] & \mathcal{V}_{d+1,r}
\end{tikzcd}$$

commutes. Thus there are maps

$$\hat{\varphi}_0 : S^0 \to \widehat{V}_d$$
$$\hat{\varphi}_0 : S^0 \to \widehat{V}.$$

The inverse system (4.10) satisfies the following version of Lin's theorem:

Theorem 4.10.

(i) $\varprojlim_r \pi_q(\mathcal{V}_{d,r}) = 0$ *when* $q < 0$ *and* $q < d$.

(ii) For $0 < d-1$, $\varprojlim_r \pi_0(\mathcal{V}_{d,r}) \cong \mathbb{Z}_2^\wedge$ *as topological groups. The inclusion*

$$\hat{\varphi}_{0*} : \mathbb{Z} = \pi_0(S^0) \to \varprojlim_r \pi_0(\mathcal{V}_{d,r})$$

is non-zero mod 2.

(iii) When $q > 0$ *and* $d > 0$, *the map induced by* $\hat{\varphi}_0$

$$\hat{\varphi}_{0*} : \pi_q(S^0)_2^\wedge \to \varprojlim_r \pi_q(\mathcal{V}_{d,r})$$

is an isomorphism when $q < d-1$ *and surjective when* $q = d-1$.

Proof. In [26] Lin considers the inverse system

$$\cdots \to \Sigma^\infty P_{d,r} \to \Sigma^\infty P_{d,r-1} \to \cdots \to \Sigma^\infty P_{d,1}.$$

Since $\mathcal{V}_{d,r}$ is equivalent to $\Sigma^\infty P_{d,r}$, we are also considering such an inverse system of spectra, but we have not been able to construct an explicit map between these two inverse systems. However, the proof of Lin's theorem only rely on the induced maps on $\mathbb{Z}/2$ cohomology and the fact that each spectrum in the inverse system has an S-dual, so the arguments apply to our situation as well.

To see that φ_{0*} is as claimed, we need $\varphi_0^* : \varinjlim_r H^0(\mathcal{V}_{d,r}; \mathbb{Z}/2) \to H^0(S^0; \mathbb{Z}/2)$ to be non-zero. But φ_0 factors as

$$S^{ka_r-1} \to V_{ka_r,r} \to V_{d+ka_r,r}.$$

The composition $S^{ka_r-1} \to V_{ka_r,r} \to S^{ka_r-1}$ is the identity. Thus the first map is an isomorphism on H^{ka_r-1}. So is the second map if $r > d$.

Alternatively, one could prove the theorem by referring to [28], Proposition 2.2. □

Remark 4.11. Actually, since the composition

$$\hat{\varphi}_{0*} : \pi_q(S^0)_2^\wedge \to \varprojlim_r \pi_q(\mathcal{V}_{d,r}) \to \varprojlim_r \pi_q(\mathcal{V}_{d+1,r})$$

is injective, the first map is an isomorphism for $q < d$.

For $d-r$ and p odd, $H^q(\Sigma V_{d,r}; \mathbb{Z}/p) = 0$ for $q < d$ or $q = d$ odd, see Section 3.3. Thus $\pi_q^s(V_{d,r})$ must be a finite 2-group. Since $\pi_q(\mathcal{V}_{d,r}) \cong \pi^s_{q+ka_r}(V_{d+ka_r,r})$ for some even ka_r, we conclude that $\pi_q(\mathcal{V}_{d,r})$ is a finite 2-group for $d-r$ odd.

It follows that $\varprojlim^1_r \pi_q(\mathcal{V}_{d,r}) = 0$ by the Mittag–Leffler condition, so by (4.6), $\pi_q(\widehat{V}_d) \cong \varprojlim_r \pi_q(\mathcal{V}_{d,r})$ for $q < d-1$. Furthermore, by Section 4.2, $\pi_q(\widehat{V}_d) \cong \pi_q((\widehat{V}_d)_2^\wedge)$ for $q < d$.

4.5 Stabilization of the Spectra $MT(d,r)$

In this section, we shall generalize the construction of the direct and inverse limit spectra in the previous section to the spectra $MT(d,r)$ and, even more generally, to any vector bundle over a compact CW complex.

Recall that we defined the periodicity map

$$\Sigma^{a_r} MT(d,r) \to MT(d+a_r,r).$$

As before, this allows us to define define the direct limit

$$\mathcal{MT}(d,r) = \varinjlim_k \Sigma^{-ka_r} MT(d+ka_r,r).$$

Moreover, the construction is a fiberwise application of the periodicity map for the Stiefel manifolds. Thus the commutative diagram (4.9) yields a well-defined commutative diagram

$$\begin{tikzcd}
\mathcal{MT}(d,r+1) \arrow[r] \arrow[d] & \mathcal{MT}(d,r) \arrow[d] \\
\mathcal{MT}(d+1,r+1) \arrow[r] & \mathcal{MT}(d+1,r).
\end{tikzcd}$$

Here the vertical maps use the inclusion $G(d+ka_r,n) \to G(d+1+ka_r,n)$ coming from $\mathbb{R}^d \oplus \mathbb{R}^{ka_r} \subseteq \mathbb{R}^d \oplus \mathbb{R} \oplus \mathbb{R}^{ka_r}$. Again we may define:

Definition 4.12.

$$\widehat{MT}(d) = \varprojlim_r \mathcal{MT}(d,r)$$
$$\widehat{MT} = \varinjlim_d \widehat{MT}(d).$$

The maps $f_\theta : \Sigma^\infty \Sigma V_{d,r} \to MT(d,r)$ induce maps $\mathcal{V}_{d,r} \to \mathcal{MT}(d,r)$ and thus also maps $\hat{\theta} : \widehat{V}_d \to \widehat{MT}(d)$ and $\hat{\theta} : \widehat{V} \to \widehat{MT}$.

There is a map $\hat{\varphi} : MT(d) \to \widehat{MT}(d)$ defined in such a way that the diagram

$$\begin{tikzcd}
S^0 \arrow[r, "\hat{\varphi}_0"] \arrow[d] & \widehat{V}_d \arrow[d] \\
MT(d) \arrow[r, "\hat{\varphi}"] & \widehat{MT}(d)
\end{tikzcd} \tag{4.11}$$

commutes. The map $S^0 \to MT(d)$ is the inclusion of a fiber in the Thom space.

The map $\hat{\varphi}$ is defined as follows. First we define $\varphi : MT(d) \to \mathcal{MT}(d,r)$. That is, we need to define $\Sigma^{ka_r} MT(d) \to MT(d+ka_r,r)$. Recall that $\Sigma^{ka_r} MT(d)$ is the quotient of the pair

$$(B^{ka_r}, S^{ka_r-1}) \times (BU^\perp_{d,n}, SU^\perp_{d,n}).$$

If $p : W_r(U_{d+ka_r,n}) \to G(d+ka_r,n)$, we can think of $MT(d+ka_r,r)$ as the quotient of

$$(p^* BU^\perp_{d+ka_r,n}, p^* SU^\perp_{d+ka_r,n} \cup (p^* BU^\perp_{d+ka_r,n})_{|V_r(U_{d+ka_r,n})}).$$

Let $(x, v, P) \in B^{ka_r} \times BU^{\perp}_{d,n}$ where $x \in B^{ka_r}$, $P \in G(d,n)$, and $v \in P^{\perp}$. This should be mapped to $(e_0x, \dots, e_{r-1}x, v, P \oplus \mathbb{R}^{ka_r})$ where $P \oplus \mathbb{R}^{ka_r} \in G(d + ka_r, n)$, $(e_0x, \dots, e_{r-1}x)$ is a frame in $0 \oplus \mathbb{R}^{ka_r} \subseteq P \oplus \mathbb{R}^{ka_r}$ and $v \in (P \oplus \mathbb{R}^{ka_r})^{\perp}$. It is easy to see that this map commutes with all the relevant maps in the limit systems and thus defines the desired map $\hat{\varphi}$.

Theorem 4.13. *The map*

$$\hat{\theta} : \pi_q(\widehat{V}_d) \to \pi_q(\widehat{MTSO}(d))$$

is zero for $q < d$ and $q \neq 0$.

Proof. The map $\pi_q(S^0) \to \pi_q(MTSO(d))$ is zero in the relevant dimensions, being the inclusion of framed cobordism into the oriented cobordism group. So since $\pi_q(S^0) \to \pi_q(\widehat{V}_d)$ is surjective in these dimensions, the claim follows from the diagram (4.11). ☐

Even more generally, there is a similar construction for any vector bundle $E \to X$ over a compact CW complex X. Let $f : X \to BO$ be a classifying map. For some large k and n, $f(X) \subseteq G(d + ka_r, n)$ and then $E \oplus \mathbb{R}^m \cong f^*(U_{d+ka_r,n})$ for an appropriate m. Let $N^n_{W_r}, N^n_{V_r}$ be the pullbacks of the bundle $N = f^*U^{\perp}_{d+ka_r,n} \to X$ to the spaces

$$W_r(E \oplus \mathbb{R}^m), V_r(E \oplus \mathbb{R}^m),$$

respectively. Let $f^*MT(d + ka_r, r)$ be the spectrum with nth space

$$f^*MT(d + ka_r, r)_n = \mathrm{Th}(N^n_{W_r}) / \mathrm{Th}(N^n_{V_r}).$$

The periodicity map $\Sigma^{a_r} MT(d+ka_r, r)_n \to MT(d+(k+1)a_r, r)_n$ takes the fiber over the point $P \in G(d + ka_r, n)$ to the fiber over $P \oplus \mathbb{R}^{a_r} \in G(d + (k + 1)a_r, n)$. Thus it naturally pulls back to a periodicity map

$$\Sigma^{a_r} f^*MT(d + ka_r, r) \to f^*MT(d + (k + 1)a_r, r).$$

Once again we get direct and inverse systems of spectra:

Definition 4.14. *For $f : X \to BO$ defined on the compact complex X, define*

$$f^*\mathcal{MT}(d, r) = \varinjlim_k \Sigma^{-ka_r} f^*MT(d + ka_r, r)$$
$$\widehat{f^*MT}(d) = \varprojlim_r f^*\mathcal{MT}(d, r).$$

When f is an inclusion $X \to BO$, we sometimes use the notation $\mathcal{MT}(d, r)_{|X}$ and $\widehat{MT}(d)_{|X}$.

Proposition 4.15. *This construction is natural, i.e. a composition $X \xrightarrow{f} Y \xrightarrow{g} BO$ induces a map $f_* : (g \circ f)^*\mathcal{MT}(d, r) \to g^*\mathcal{MT}(d, r)$. Furthermore,*

$$\mathcal{V}_{d,r} = \mathcal{MT}(d, r)_{|pt}$$
$$\mathcal{MT}(d, r) = \varinjlim_{X \subseteq BO} \mathcal{MT}(d, r)_{|X}.$$

However, direct and inverse limits do not commute in general, so we cannot expect

$$\varinjlim_{X\subseteq BO} \widehat{MT}(d)_{|X} \cong \varinjlim_{X\subseteq BO} \varprojlim_{r} \mathcal{MT}(d,r)_{|X} \cong \varprojlim_{r} \varinjlim_{X\subseteq BO} \mathcal{MT}(d,r)_{|X} \cong \widehat{MT}(d).$$

In fact, we shall see in Chapter 5 that this is not at all the case.

Remark 4.16. For a completely general map $f : X \to BO$, one could get a similar construction from filtering X by the subspaces $X_{d,n} = f^{-1}(G(d,n))$ and letting $f^*MT(d,r)_n = f^*_{|X_{d,n}} MT(d,r)_n$.

Let $f^*MT(d)$ be the spectrum with nth space $\mathrm{Th}(f^*U^{\perp}_{d,n})$. This is nothing but the spectrum $\Sigma^{\infty-l}\,\mathrm{Th}(N)$ where N is an l-dimensional complement of E. The map $\hat{\varphi} : MT(d) \to \mathcal{MT}(d,r)$ pulls back to a map

$$\widehat{f^*\varphi} : f^*MT(d) \to f^*\mathcal{MT}(d,r). \tag{4.12}$$

4.6 The Compact Case

In this section, we only consider the pullback spectra $f^*MT(d,r)$ where f is defined on a compact complex. If $f : pt \to BO$ is the inclusion of a point, Lin's theorem tells us that in dimensions $q < d$, the map $\widehat{f^*\varphi}_2 : (S^0)^\wedge_2 \to (\widehat{V}_d)^\wedge_2$ induces an isomorphism

$$\widehat{f^*\varphi}_{2*} : \pi_q((S^0)^\wedge_2) \to \pi_q((\widehat{V}_d)^\wedge_2) \cong \pi_q(\widehat{V}_d).$$

The goal is to generalize Lin's theorem to the map (4.12). The proof is a topological induction argument on the cell structure of X starting with Lin's case.

We first need a few lemmas:

Lemma 4.17. *If $g : X \to Y$ is a homotopy equivalence and the diagram*

$$\begin{tikzcd} X \arrow[r, "f_X"] \arrow[d, "g"'] & BO \\ Y \arrow[ur, "f_Y"'] & \end{tikzcd}$$

commutes, then the induced map

$$g_* : \varprojlim_{r} \pi_*(f_X^*\mathcal{MT}(d,r)) \to \varprojlim_{r} \pi_*(f_Y^*\mathcal{MT}(d,r))$$

is an isomorphism.

Proof. Since $g_* : f_X^*MT(d+ka_r,r) \to f_Y^*MT(d+ka_r,r)$ is a homotopy equivalence, it induces an isomorphism on homotopy groups. Thus it also induces an isomorphism in the limit. □

Lemma 4.18. *Assume $q < d$ and let $f : X \to BO$ be defined on a finite CW complex X. Then $\pi_q(f^*\mathcal{MT}(d,r))$ is a finite 2-group when $d-r$ is odd, and thus $\varprojlim_r \pi_q(f^*\mathcal{MT}(d,r))$ is 2-profinite. For d odd, this also holds when $d = q$. Moreover, the periodicity map*

$$\Sigma^{ka_r} f^*MT(d,r) \to f^*MT(d+ka_r, r)$$

is a $(2(d-r)+ka_r+1)$-equivalence.

Proof. In the case where X is a point, i.e. $f^*\mathcal{MT}(d,r) = \mathcal{V}_{d,r}$, the first statement is true by Remark 4.11.

Now suppose that X is obtained from Y by glueing on an n-cell D^n such that $Y \cap D^n = S^{n-1}$. Then there is a Mayer–Vietoris sequence

$$\begin{aligned} \to \pi_q(f^*MT(d,r)_{|S^{n-1}}) \to \pi_q(f^*MT(d,r)_{|D^n}) \oplus \pi_q(f^*MT(d,r)_{|Y}) \\ \to \pi_q(f^*MT(d,r)_{|X}) \to \pi_{q-1}(f^*MT(d,r)_{|S^{n-1}}) \to . \end{aligned} \tag{4.13}$$

The periodicity map defines a map of these exact sequences, and since direct limits preserve exactness, there is also an exact sequence

$$\begin{aligned} \to \pi_q(f^*\mathcal{MT}(d,r)_{|S^{n-1}}) \to \pi_q(f^*\mathcal{MT}(d,r)_{|D^n}) \oplus \pi_q(f^*\mathcal{MT}(d,r)_{|Y}) \\ \to \pi_q(f^*\mathcal{MT}(d,r)_{|X}) \to \pi_{q-1}(f^*\mathcal{MT}(d,r)_{|S^{n-1}}) \to . \end{aligned}$$

Since $D^n \simeq pt$, the theorem is known for $X = D^n$ by Lemma 4.17.

S^n can be built up from two disks by glueing their boundaries along a copy of S^{n-1}, so by induction on n, $\pi_q(f^*\mathcal{MT}(d,r)_{|S^n})$ must be a finite 2-group in order to fit into the exact sequence.

For a general simplicial complex X, it now follows by induction on the number of cells in X that $\pi_q(f^*\mathcal{MT}(d,r)_{|X})$ is a finite 2-group when $d-r$ is odd, by applying the Mayer–Vietoris sequence and the sphere case. Finally we get the result for a general CW complex by Lemma 4.17.

The last statement was shown for $X = pt$ in Lemma 3.13. In general, the claim follows by an induction argument similar to the above applied to the periodicity map between the sequences (4.13). □

Lemma 4.19. *Let $f : X \to BO$ be given and assume that X is the union of two finite subcomplexes Y_1 and Y_2. For $q < d$, there is an exact Mayer–Vietoris sequence*

$$\begin{aligned} \to \varprojlim_r \pi_q(f^*\mathcal{MT}(d,r)_{|Y_1\cap Y_2}) \to \varprojlim_q \pi_*(f^*\mathcal{MT}(d,r)_{|Y_1}) \oplus \varprojlim_r \pi_q(f^*\mathcal{MT}(d,r)_{|Y_2}) \\ \to \varprojlim_r \pi_q(f^*\mathcal{MT}(d,r)_{|Y_1\cup Y_2}) \to \varprojlim_r \pi_{q-1}(f^*\mathcal{MT}(d,r)_{|Y_1\cap Y_2}) \to . \end{aligned}$$

Proof. As in the proof of Lemma 4.18, there is a Mayer–Vietoris sequence

$$\begin{aligned} \to \pi_q(f^*\mathcal{MT}(d,r)_{|Y_1\cap Y_2}) \to \pi_q(f^*\mathcal{MT}(d,r)_{|Y_1}) \oplus \pi_q(f^*\mathcal{MT}(d,r)_{|Y_2}) \\ \to \pi_q(f^*\mathcal{MT}(d,r)_{|Y_1\cup Y_2}) \to . \end{aligned}$$

Taking the inverse limit over r yields the desired sequence. Generally, inverse limits do not preserve exactness. However, for $q < d$ and $d-r$ odd, all groups involved are finite, so the Mittag–Leffler condition ensures that the inverse limit of these sequences is again exact. □

Proposition 4.20. *Let X be a finite complex and $f : X \to BO$ a map. Then for $q < d$,*

$$\pi_q(\widehat{f^*MT}(d)) \cong \varprojlim_r \pi_q(f^*\mathcal{MT}(d,r)).$$

Proof. By (4.6) we need to see that $\varprojlim^1_r \pi_{q+1}(f^*\mathcal{MT}(d,r))$ vanishes. For $q < d-1$ or $q = d-1$ and d odd, it follows from Lemma 4.18 that the Mittag–Leffler condition is satisfied. For $q = d-1$ and d even, we apply the long exact sequence constructed in the next section in Corollary 4.23. This yields short exact sequences

$$0 \to \mathbb{Z} \to \pi_d(f^*\mathcal{MT}(d,r)) \to \pi_d(f^*\mathcal{MT}(d+1,r+1)) \to 0$$

that map to each other. Thus we get the exact sequence

$$\cdots \to \varprojlim^1_r \mathbb{Z} \to \varprojlim^1_r \pi_d(f^*\mathcal{MT}(d,r)) \to \varprojlim^1_r \pi_d(f^*\mathcal{MT}(d+1,r+1)) \to 0.$$

The $\mathbb{Z}$'s map isomorphically onto each other, so the first term vanishes, and we already explained that the last term is zero. $\square$

We are now ready to prove the main theorem of this section.

Theorem 4.21. *Let $f : X \to BO(d)$ with X a finite complex. Then for $q < d$,*

$$\widehat{f^*\varphi}_{2*} : \pi_q(f^*MT(d)^\wedge_2) \to \pi_q(\widehat{f^*MT}(d)^\wedge_2) \cong \pi_q(\widehat{f^*MT}(d))$$

is an isomorphism. In particular, for $X \subseteq B(d)$ we get

$$\varinjlim_{X \subseteq B(d)} \pi_q(\widehat{MT}(d)_{|X}) \cong \pi_q(MT(d))^\wedge_2$$

for $q < d$. Taking the direct limit over d,

$$\varinjlim_d \varinjlim_{X \subseteq B(d)} \pi_q(\widehat{MT}_{|X}) \cong \pi_q(MT)^\wedge_2.$$

for all q.

Proof. We want to do an induction on the cells in X as in the proof of Lemma 4.18. Assume that X is built from Y by glueing on a disk D^n such that $D^n \cap Y = S^{n-1}$. There is a map of Mayer–Vietoris sequences for $q < d$

$$\begin{tikzcd}
\pi_q(\widehat{f^*MT}(d)_{|S^{n-1}}) \arrow[r] & \pi_q(\widehat{f^*MT}(d)_{|D^n}) \oplus \pi_q(\widehat{f^*MT}(d)_{|Y}) \arrow[r] & \pi_q(\widehat{f^*MT}(d)_{|X}) \\
\pi_q(f^*MT(d)_{|S^{n-1}}) \arrow[r] \arrow[u] & \pi_q(f^*MT(d)_{|D^n}) \oplus \pi_q(f^*MT(d)_{|Y}) \arrow[r] \arrow[u] & \pi_q(f^*MT(d)_{|X}). \arrow[u]
\end{tikzcd}$$

The upper row is exact by Lemma 4.19 and Proposition 4.20. This diagram maps into the 2-completed diagram

$$\begin{tikzcd}
\pi_q(\widehat{f^*MT}(d)_{|S^{n-1}})^\wedge_2 \arrow[r] & \pi_q(\widehat{f^*MT}(d)_{|D^n})^\wedge_2 \oplus \pi_q(\widehat{f^*MT}(d)_{|Y})^\wedge_2 \arrow[r] & \pi_q(\widehat{f^*MT}(d)_{|X})^\wedge_2 \\
\pi_q(f^*MT(d)_{|S^{n-1}})^\wedge_2 \arrow[r] \arrow[u] & \pi_q(f^*MT(d)_{|D^n})^\wedge_2 \oplus \pi_q(f^*MT(d)_{|Y})^\wedge_2 \arrow[r] \arrow[u] & \pi_q(f^*MT(d)_{|X})^\wedge_2. \arrow[u]
\end{tikzcd}$$

The lower row is still exact because 2-completion is exact with respect to finitely generated groups. All the groups in the upper row were already 2-profinite by Lemma 4.18. Therefore, the 2-completion does not change this row. For $X = D^n$, the vertical map is an isomorphism by Lin's theorem. The same is true for a general X by induction on the cells. This is exactly the map induced by $\widehat{f^*\varphi}$, see Section 4.2.

When we take the direct limit over $X \subseteq B(d)$, we need to see that

$$\varinjlim_{X \subseteq B(d)} \pi_q(MT(d)_{|X})^\wedge_2 \to \pi_q(MT(d))^\wedge_2$$

is an isomorphism. This is true because $\pi_q(MT(d)_{|X}) \to \pi_q(MT(d))$ is an isomorphism for $X = G(d,n)$ and n sufficiently large. In the spin case, $G(d,n)$ is not compact, but we can choose X to be the $(q+1)$-skeleton instead. □

4.7 Stabilization of the Spectral Sequence

We now introduce a stable version of the spectral sequence from Section 4.1. We also form an inverse limit of these and show that this defines a new spectral sequence that converges strongly.

In this section, we consider maps $f : X \to BO$ where X is either a compact CW complex, BO, BSO, or $BSpin$. In all these cases, we use the notation $f^*\mathcal{MT}(d,r)$ etc. for the corresponding spectra.

Lemma 4.22. *For $f : X \to G(d + ma_r, n)$ and $N = ka_{r+l}$, there are long exact sequences*

$$\to \pi_*(f^*MT(d+N,r)) \to \pi_*(f^*MT(d+l+N,r+l)) \to \pi_*(f^*MT(d+l+N,l)) \to$$

for all N sufficiently large and $* < 2(d+N-1)-r$.

Proof. The maps in the sequence are the maps of Thom spaces over the maps of pairs

$$\begin{aligned}(W_r(f^*U_{d+N,n}), V_r(f^*U_{d+N,n})) \to &(W_{r+l}(f^*U_{d+l+N,n}), V_{r+l}(f^*U_{d+l+N,n})) \\ \to &(W_l(f^*U_{d+l+N,n}), V_l(f^*U_{d+l+N,n})).\end{aligned}$$

Let $f^*(U_{d+N,n}) = F$ and $f^*(U_{d+l+N,n}) = E$ for simplicity. These are both vector bundles over X and $E \cong F \oplus \mathbb{R}^l$. We need to see that the map of pairs

$$(W_{r+l}(E), V_{r+l}(E) \cup W_r(F)) \to (W_l(E), V_l(E) \cup X \times I) \tag{4.14}$$

is highly connected, since the first pair corresponds to the cofiber of

$$f^*MT(d+N,r) \to f^*MT(d+N+l, r+l),$$

while the second pair corresponds to $f^*MT(d+N+l,l)$. Here a point (x,s) in $X \times I$ should be interpreted as $(su_1, \dots, su_l) \in W_l(F \oplus \mathbb{R}^l)$ where $(u_1,\dots,u_l)$ is the standard frame in $0 \oplus \mathbb{R}^l$. All the spaces in (4.14) are fiber bundles over X, so it is enough to see that the fibers are highly connected. Now, $W_{d+l+N,r+l}$ and $W_{d+l+N,l}$ are both contractible. The fibers $V_{d+l+N,r+l} \cup W_{d+N,r}$ and $V_{d+l+N,l}$ are $(2(d+N-1)-r)$-connected since the first is the mapping cone of the fiber inclusion $V_{d+N,r} \to V_{d+l+N,r+l}$ and the other one is the base space. □

Corollary 4.23. *For $f : X \to BO$, there are long exact sequences*

$$\to \pi_*(f^*\mathcal{MT}(d,r)) \to \pi_*(f^*\mathcal{MT}(d+l,r+l)) \to \pi_*(f^*\mathcal{MT}(d+l,l)) \to . \quad (4.15)$$

Proof. First consider the compact case. We may assume $f(X) \subseteq G(d+N,n)$ for all $N = ka_{r+l}$ with k sufficiently large. The map

$$f^*MT(d+N,r) \to f^*MT(d+N+l,r+l)$$

only commutes with the periodicity map up to homotopy. We need to see that the periodicity map still defines a map between the long exact sequences of homotopy groups. Then the claim follows because direct limits preserve exactness.

The diagram

$$\begin{tikzcd} \Sigma^{a_{r+l}} f^*MT(d+N,r) \arrow[r, "i_1"] \arrow[d, "j_2"] & \Sigma^{a_{r+l}} f^*MT(d+l+N,r+l) \arrow[d, "j_1"] \\ f^*MT(d+N+a_{r+l},r) \arrow[r, "i_2"] & f^*MT(d+l+N+a_{r+l},r+l) \end{tikzcd}$$

only commutes up to homotopy. There is a homotopy between $j_1 \circ i_1$ and $i_2 \circ j_2$ defined by fiberwise application of the homotopy from the proof of Lemma 3.13. Let H be an extension of this to $\Sigma^{a_{r+l}} f^*MT(d+l+N,r+l) \times I$ which is j_1 on $\Sigma^{a_{r+l}} f^*MT(d+l+N,r+l) \times \{0\}$. Then there is a strictly commutative and homotopy equivalent diagram

$$\begin{tikzcd} \Sigma^{a_{r+l}} f^*MT(d+N,r) \arrow[r, "i_1 \times 1"] \arrow[d, "j_2"] & \Sigma^{a_r+l} f^*MT(d+l+N,r+l) \times I \arrow[d, "H"] \\ f^*MT(d+N+a_{r+l},r) \arrow[r, "i_2"] & f^*MT(d+l+N+a_{r+l},r+l). \end{tikzcd}$$

This certainly defines a map of the exact sequences from Lemma 4.22. We need see that the map of cofibers is actually the periodicity map. Introduce the abbreviations $E = f^*U_{d+l+N,n}$, $F = f^*U_{d+N,n}$, $E' = f^*U_{d+l+N+a_{r+l},n}$, and $F' = f^*U_{d+N+a_{r+l},n}$. On base spaces, the induced map on cofibers

$$\begin{array}{c} (D^{a_{r+l}}, S^{a_{r+l}-1}) \times (W_{r+l}(E) \times I, V_{r+l}(E) \times I \cup W_r(F) \times I) \\ \big\downarrow \\ (W_{r+l}(E'), V_{r+l}(E') \cup W_r(F')) \end{array}$$

corresponds to a map

$$\begin{array}{c} (D^{a_{r+l}}, S^{a_{r+l}-1}) \times (W_l(E) \times I, V_l(E) \times I \cup X \times I \times I) \\ \big\downarrow \\ (W_l(E'), V_l(E') \cup X \times I) \end{array}$$

under the homotopy equivalences defined in the proof of Lemma 4.22. Again, the point $(x,s) \in X \times I \subseteq W_l(E')$ should be thought of as the frame $(su_1, \dots, su_l)$ in $E'_x = F'_x \oplus \mathbb{R}^l$ where $(u_1, \dots, u_l)$ is the standard basis in $\mathbb{R}^l$.

The restriction of the last map to the homotopy equivalent pairs

$$(D^{a_{r+l}}, S^{a_{r+l}-1}) \times (W_l(E) \times \{0\}, V_l(E) \times \{0\}) \to (W_l(E'), V_l(E'))$$

yields exactly the periodicity map except we multiply by $e_{r+1}, \dots, e_{r+l}$ rather than $e_1, \dots, e_l$. But a path between these frames in $V_{r+l,l}$ defines a homotopy between the maps, so the map induced on homotopy groups is the desired map.

For $X = BO$, BSO, or $BSpin$, there are long exact sequences

$$\to \pi_*(f^*MT(d+N,r)) \to \pi_*(f^*MT(d+l+N,r+l)) \to \pi_*(f^*MT(d+l+N,l)) \to .$$

The check that the periodicity map defines a map of these sequences is similar. □

The sequences (4.15) fit together to form a spectral sequence:

Theorem 4.24. *There is a spectral sequence with*

$$E^1_{s,t} = \pi_{d-s+t}(f^*\mathcal{MT}(d-s,1)) \cong \pi^s_{t-s}(X) \oplus \pi^s_{t-s}(S^0)$$

for $0 \le s < r$ *and* $E^1_{s,t} = 0$ *otherwise, converging to* $\pi_*(f^*\mathcal{MT}(d,r))$.

This is the spectral sequence from Section 4.1 with all homotopy groups replaced by their stable version.

Replacing all groups in (4.15) by their 2-completions yields

$$\to \pi_*(f^*\mathcal{MT}(d,r))^\wedge_2 \to \pi_*(f^*\mathcal{MT}(d+1,r+1))^\wedge_2 \to \pi_*(f^*\mathcal{MT}(d+1,1))^\wedge_2 \to .$$

This is again an exact sequence because all the groups involved are finitely generated. Since the inverse limit functor is exact with respect to sequences of profinite groups, there are long exact sequences

$$\to \varprojlim_r \pi_*(f^*\mathcal{MT}(d,r))^\wedge_2 \to \varprojlim_r \pi_*(f^*\mathcal{MT}(d+1,r+1))^\wedge_2 \to \pi_*(f^*\mathcal{MT}(d+1,1))^\wedge_2 \to .$$

These sequences form a spectral sequence where $\varprojlim_r \pi_*(f^*\mathcal{MT}(d,r))^\wedge_2$ is filtered by the image of the groups $\varprojlim_r \pi_*(f^*\mathcal{MT}(d-l,r-l))^\wedge_2$ and with

$$E^1_{s,t} = \pi_{d-s+t}(f^*\mathcal{MT}(d-s,1))^\wedge_2$$

and differentials

$$d^k : E^k_{s,t} \to E^k_{s+r,t+r-1}.$$

Theorem 4.25. *This spectral sequence converges strongly in the sense of [5] to* $\varprojlim_r \pi_*(f^*\mathcal{MT}(d,r))^\wedge_2$.

Strong convergence basically means that the E^∞-terms are the filtration quotients of a complete Hausdorff filtration of $\varprojlim_r \pi_*(f^*\mathcal{MT}(d,r))^\wedge_2$.

Proof. Since $E^1_{s,t} = 0$ for $t < 0$, we have a half-plane spectral sequence with entering differentials in the sense of [5]. Thus, by Theorem 7.3 of this paper, it is enough to check that the following three conditions are satisfied:

(i)

$$\varprojlim_l \varprojlim_r \pi_*(f^*\mathcal{MT}(d-l,r-l))^\wedge_2 \cong \varprojlim_r \varprojlim_l \pi_*(f^*\mathcal{MT}(d-l,r-l))^\wedge_2 = 0,$$

since $\pi_*(f^*\mathcal{MT}(d-l,r-l))$ is zero for $k > r$.

(ii)

$$\varprojlim^1_l \varprojlim_r \pi_*(f^*\mathcal{MT}(d-l,r-l))^\wedge_2 = 0.$$

This follows from the fact that

$$\varprojlim_l \pi_*(f^*\mathcal{MT}(d-l,r-l))^\wedge_2 = \varprojlim^1_l \pi_*(f^*\mathcal{MT}(d-l,r-l))^\wedge_2 = 0$$

and diagram chasing in the diagram defining the double limit.

(iii)

$$\varprojlim^1_k Z^k_{s,t} = 0.$$

Here $Z^k_{s,t}$ denotes the cycles on the kth page, i.e. the inverse image of

$$\mathrm{Im}(\varprojlim_r \pi_{*-1}(f^*\mathcal{MT}(d-s-k,r-k))^\wedge_2 \to \varprojlim_r \pi_{*-1}(f^*\mathcal{MT}(d-s-1,r))^\wedge_2)$$

under the map

$$\pi_*(f^*\mathcal{MT}(d-s,1))^\wedge_2 \to \varprojlim_r \pi_{*-1}(f^*\mathcal{MT}(d-s-1,r))^\wedge_2.$$

But the sequence

$$\begin{aligned}\to \varprojlim_r \pi_*(f^*\mathcal{MT}(d-k-s,r-k))^\wedge_2 \to \varprojlim_r \pi_*(f^*\mathcal{MT}(d-s-1,r))^\wedge_2 \\ \to \pi_*(f^*\mathcal{MT}(d-s-1,k))^\wedge_2 \to\end{aligned}$$

is exact, so $Z^k_{s,t}$ is actually the kernel of the composite map

$$\begin{aligned}\pi_*(f^*\mathcal{MT}(d-s,1))^\wedge_2 \to \varprojlim_r \pi_{*-1}(f^*\mathcal{MT}(d-s-1,r))^\wedge_2 \\ \to \pi_{*-1}(f^*\mathcal{MT}(d-s-1,k))^\wedge_2,\end{aligned}$$

which is continuous. Thus the $Z^k_{s,t}$ are closed subgroups of a 2-profinite group. But $\varprojlim^1$ vanishes for any inverse system of closed subgroups of a profinite group because these are again profinite, the quotients are profinite, and $\varprojlim$ is exact on sequences of profinite groups with continuous maps.

□

Proposition 4.26. *For $q < d$ and $f : X \to BO$ where X is a finite complex, BSO, or $BSpin$,*

$$\varprojlim_r \pi_q(f^*\mathcal{MT}(d,r))^\wedge_2 \cong \pi_q(f^*\widehat{MT(d)}).$$

Proof. The groups $\pi_q(f^*\mathcal{MT}(d,r))$ are finite 2-groups for infinitely many r, therefore

$$\varprojlim_r \pi_q(f^*\mathcal{MT}(d,r))^\wedge_2 = \varprojlim_r \pi_q(f^*\mathcal{MT}(d,r)) \cong \pi_q(f^*\widehat{MT(d)}).$$

□

4.8 Geometric Interpretation

This last section is an attempt to relate the inverse limit spectrum back to the vector field problem.

Let M be a closed d-dimensional manifold. Then $M \times D^{ka_r}$ is a manifold with boundary $M \times S^{ka_r-1}$. This naturally has r vector fields on the boundary given by Clifford multiplication.

Let $\xi : M \to G(d,n)$ be a classifying map for TM for some n so large that $\pi_d(MT(d)) \cong \pi_d(MT(d)_{|G(d,n)})$. Then the trivial extension $\xi : M \times D^{ka_r} \to G(d,n)$ is a classifying map for $T(M \times D^{ka_r})$. The construction (1.8) of the invariant in Section 1.4 restricts to

$$\beta^r(M \times D^{ka_r}, M \times S^{ka_r-1}; \xi) \in \pi_{d+ka_r}(MT(d+ka_r, r)_{|G(d,n)}).$$

Then, by construction, we have:

Proposition 4.27. *The map $\pi_d(MT(d)_{|G(d,n)}) \to \pi_d(\mathcal{MT}(d,r)_{|G(d,n)})$ takes $\beta(M)$ to $\varinjlim_k \beta^r(M \times D^{ka_r}, M \times S^{ka_r-1}; \xi)$.*

The image of $\beta(M)$ under the map

$$\pi_d(MT(d)) \to \varinjlim_{X \subseteq BO(d)} \varprojlim_r \pi_d(\mathcal{MT}(d,r)_{|X})$$

vanishes if and and only if for some classifying map $\xi : M \to X \subseteq BO(d)$ and all r, $\beta^r(M \times D^{ka_r}, M \times S^{ka_r-1}; \xi) = 0$.

Proposition 4.28. *For a map $f : X \to BO(d)$ and X compact, the map*

$$\pi_d(f^*MT(d)) \to \varprojlim_r \pi_d(f^*\mathcal{MT}(d,r))$$

is injective.

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
\varprojlim_r \pi_d(f^*\mathcal{MT}(d,r))^\wedge_2 \arrow[r] & \varprojlim_r \pi_d(f^*\mathcal{MT}(d+1,r))^\wedge_2 \\
\pi_d(f^*MT(d))^\wedge_2 \arrow[u] \arrow[r, "="] & \pi_d(f^*MT(d+1))^\wedge_2 . \arrow[u, "\cong"']
\end{tikzcd}$$

□

Corollary 4.29. *If for all r there is a k such that M allows an extension of the Clifford vector fields on $M \times S^{ka_r-1}$ to $M \times D^{ka_r}$, then M must be Reinhart cobordant to the empty manifold.*

Proof. If $M \times D^{ka_r}$ allows an extension of the Clifford vector fields, then

$$\varinjlim_k \beta^r(M \times D^{ka_r}, M \times S^{ka_r-1}; \xi) \in \pi_d(\mathcal{MT}(d,r)_{|G(d,n)})$$

vanishes. Thus $\beta(M)$ is mapped to zero in $\varprojlim_r \pi_d(\mathcal{MT}(d,r)_{|G(d,n)})$. By Proposition 4.28, also $\beta(M) = 0$. □

Finally we shall see that the injectivity of θ we are looking for is far from true in the limit:

Proposition 4.30. *The map $\pi_d^s(S^0)_2^\wedge \to \pi_d(\mathcal{V}_d)$ is injective, and the image lies in the kernel of the map*

$$\hat{\theta} : \pi_d(\mathcal{V}_d) \to \varinjlim_{X \subseteq BO} \pi_d(\widehat{MT}(d)_{|X}).$$

In particular, for most d, $\hat{\theta}$ is not injective.

Proof. The diagram

$$\begin{tikzcd} \pi_d(\mathcal{V}_d) \arrow[r] & \pi_d(\mathcal{V}_{d+1}) \\ \pi_d^s(S^0)_2^\wedge \arrow[u] \arrow[ur, "\cong"'] & \end{tikzcd}$$

commutes, proving the first claim.

Recall from Proposition 3.37 and 3.38 that

$$\pi_d(MT(d)) \cong \pi_d(MT(d+1)) \oplus G$$

for a suitable G depending on d. The map $\pi_d^s(S^0) \to \pi_d(MT(d+1))$ is zero, see the proof of Theorem 4.13. Therefore, the image of $\pi_d^s(S^0) \to \pi_d(MT(d))$ lies in G. Since $\pi_d^s(S^0)$ is torsion, this can only be non-zero when $d \equiv 1 \bmod 4$. However, the map factors as

$$\pi_d^s(S^0) \to \pi_d(MT(d-2)) \to \pi_d(MT(d)),$$

and thus $\pi_d^s(S^0) \to \pi_d(MT(d,2)) \cong G$ is zero. Hence $\pi_d^s(S^0) \to \pi_d(MT(d))$ is the zero map.

The last claim now follows because the diagram

$$\begin{tikzcd} \pi_d(MT(d)) \arrow[r] & \varinjlim_{X \subseteq BO} \pi_d(\mathcal{MT}(d)_{|X}) \\ \pi_d(S^0) \arrow[u, "0"] \arrow[r] & \pi_d(\mathcal{V}_d) \arrow[u] \end{tikzcd}$$

commutes. ∎

Chapter 5

The Inverse Limit Spectrum

In this chapter we are going to study the inverse limit spectra from Chapter 4 by more algebraic methods. The idea is to apply a version of the Adams spectral sequence for inverse limit spectra. Thus the main problem becomes the computation of certain Ext-groups. The original motivation for the study of the inverse limit spectrum was that the cohomology of the inverse limit spectrum resembled the Singer construction. This would immediately yield the desired Ext-groups. We recall the Singer construction in Section 5.1, and in Section 5.2 we show how this applies in the compact case.

In Section 5.3 we consider the non-compact case for the unoriented spectra. By a direct computation we determine the desired Ext-groups. Quite surprisingly, it turns out that we get a completely different result than in the compact case. For instance, the inverse limit spectrum is no longer connected. Then in Section 5.4, a deeper study of the homology and cohomology structures allows us to give a more explicit description of the Ext-groups. Finally, in Section 5.5, we compute the Ext-groups for the oriented version of the spectrum.

5.1 The Singer Construction

In this chapter we shall only consider cohomology with $\mathbb{Z}/2$ coefficients understood. Hence the mod 2 Steenrod algebra will just be denoted by $\mathcal{A}$.

Let $\mathbb{Z}/2[t, t^{-1}]$ be the graded ring of Laurent polynomials in the variable t. This is an $\mathcal{A}$-module with $\mathrm{Sq}^k(t^l) = \binom{l}{k} t^{k+l}$. For l negative, the binomial coefficient should be interpreted as

$$\binom{l}{k} = \frac{l \cdot (l-1) \cdots (l-k+1)}{k!}.$$

Then as $\mathcal{A}$-modules,

$$\varprojlim_{d} \varinjlim_{r} H^*(\mathcal{V}_{d,r}) \cong \Sigma\mathbb{Z}/2[t, t^{-1}].$$

Lin's theorem is based on the observation that the map

$$\Sigma\mathbb{Z}/2[t, t^{-1}] \to \mathbb{Z}/2$$

induced by $\hat{\varphi}_0 : S^0 \to \widehat{V}$ is an isomorphism on Ext-groups

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(\Sigma\mathbb{Z}/2[t, t^{-1}], \mathbb{Z}/2).$$

See e.g. [27] for a proof.

This was later generalized by Singer in [39] as follows. Let M be an $\mathcal{A}$-module. Then the Singer construction R_+M is the graded vector space $\Sigma(\mathbb{Z}/2[t, t^{-1}] \otimes M)$, but not with the Cartan action of $\mathcal{A}$. Rather, this is given by the formula

$$\mathrm{Sq}^a(t^b \otimes x) = \sum_j \binom{b-j}{a-2j} t^{a+b-j} \otimes \mathrm{Sq}^j(x). \tag{5.1}$$

The advantage of this module structure is that it makes the map $\epsilon : R_+M \to M$ given by $t^k \otimes x \mapsto \mathrm{Sq}^{k+1}(x)$ into an $\mathcal{A}$-homomorphism.

It was proved by Gunawardena and Miller in [2] that ϵ is a Tor-equivalence, so in particular it induces isomorphisms

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(M, \mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}}^{s,t}(R_+M, \mathbb{Z}/2).$$

5.2 The Singer Construction in the Compact Case

We wish to compute the homotopy groups of $\widehat{MT}(d) = \varprojlim_r \mathcal{MT}(d,r)$. In [28], Proposition 2.2, the following version of the Adams spectral sequence for an inverse system $\cdots \to Y_{n+1} \to Y_n$ of spectra is constructed.

Theorem 5.1. *Assume that the spectra Y_n have finite $\mathbb{Z}/2$ cohomology in each dimension and each $\pi_*(Y_n)$ is bounded below. Then there is a spectral sequence with E_2-term $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\varinjlim_n H^*(Y_n), \mathbb{Z}/2)$ converging strongly to the homotopy groups $\pi_{t-s}((\varprojlim_n Y_n)_2^\wedge)$.*

For each k, $E_k^{s,t}$ is the inverse limit of the Adams spectral sequences $E_k^{s,t}(Y_n)$ for the Y_n.

With $Y_n = \mathcal{MT}(d,r)$, the spectral sequences $E_k^{s,t}(Y_n)$ are the ones considered in Chapter 3 and $(\varprojlim_n Y_n)_2^\wedge = \widehat{MT}(d)_2^\wedge$. Thus we must investigate $\varinjlim_r H^*(\mathcal{MT}(d,r))$ in order to apply the theorem. For simplicity, we first introduce some notation.

Definition 5.2. *Let $B(d)$ denote denote either $BO(d)$ or $BSO(d)$. Let MT denote the corresponding spectra. Define*

$$\begin{aligned} H(d)^* &= H^*(B(d)) \\ H^* &= H^*(\varinjlim_d B(d)) = \varprojlim_d H^*(B(d)) \\ WH(d)^* &= \varinjlim_r H^*(\mathcal{MT}(d,r)) \\ WH^* &= \varprojlim_d WH(d). \end{aligned}$$

Theorem 5.3. *$WH(d)^*$ is isomorphic as an H^*-module to*

$$H^* \otimes \mathbb{Z}/2\{\tilde{w}_l, l \leq d\}$$

where $\mathbb{Z}_2\{\tilde{w}_l, l \leq d\}$ is the graded vector space with basis $\tilde{w}_l$ in dimension l. The Steenrod algebra $\mathcal{A}$ acts by the Cartan formula. The action on H^* is the usual one, while the action on $\tilde{w}_l$ is given by the formula

$$\mathrm{Sq}^k(\tilde{w}_l) = \sum_{j=0}^{k}\sum_{i=0}^{j}\binom{j-l}{i} w_{j-i}\bar{w}_{k-j}\tilde{w}_{l+i}. \tag{5.2}$$

Proof. By Theorem 3.14, the periodicity map $f : MT(d,r) \to \Sigma^{-a_r}MT(d+a_r,r)$ induces an isomorphism

$$f^* : H^*(\Sigma^{-a_r}MT(d+a_r,r)) \to H^*(MT(d,r)) \tag{5.3}$$

in dimensions $* \leq 2(d-r+1)$. Thus, the inverse system of cohomology groups

$$\cdots \to H^*(\Sigma^{-(k+1)a_r}MT(d+(k+1)a_r,r)) \to H^*(\Sigma^{-ka_r}MT(d+ka_r,r))$$

stabilizes in each dimension. Hence by (4.5)

$$H^*(\mathcal{MT}(d,r)) \cong \varprojlim_k H^*(\Sigma^{-ka_r}MT(d+ka_r,r)).$$

It was also shown in Theorem 3.14 that the map (5.3) takes the generators $w_{d+a_r-r+1},\dots,w_{d+a_r}$ to $w_{d-r+1},\dots,w_d$ and commutes with the $H^*(d+a_r)$-action. Furthermore, both are isomorphic to the free $H^*(d+a_r)$-module on these generators up to dimension $2(d-r)$.

Thus, in the limit the cohomology groups become

$$H^*(\mathcal{MT}(d,r)) \cong H^* \otimes \mathbb{Z}_2\{\tilde{w}_l, d-r+1 \leq l \leq d\}$$

where $\tilde{w}_l$ corresponds to the generator $w_{l+ka_r} \in H^l(\Sigma^{-ka_r}\mathcal{MT}(d+ka_r,r))$. Taking the direct limit over r proves the claim.

The formula for the $\mathcal{A}$-action on $\tilde{w}_l$ follows from the considerations in the beginning of Section 3.5. $\square$

Remark 5.4. The proof also works for $MTSpin$ by referring to Corollary 3.28 and the proof of Corollary 3.30.

The inclusion $\mathcal{V}_{d,r} \to \mathcal{MT}(d,r)$ induces an $\mathcal{A}$-linear projection

$$WH^* = H^* \otimes \mathbb{Z}/2\{\tilde{w}_l, l \in \mathbb{Z}\} \to \Sigma\mathbb{Z}/2[t,t^{-1}]$$

with kernel $H^{>0} \otimes \mathbb{Z}/2\{\tilde{w}_l, l \in \mathbb{Z}\}$. Thus WH^* looks like the Singer construction applied to H^*. We shall see that this is indeed the case when we restrict to a finite quotient of H^* and twist the action by a Thom class.

Definition 5.5. *Let $\mathbb{Z}/2[t,t^{-1}]$ act on WH^* by the formula $t^l(x\tilde{w}_k) = x\tilde{w}_{k+l}$. For any M, $\mathbb{Z}/2[t,t^{-1}]$ acts on $R_+(M) = \Sigma(\mathbb{Z}/2[t,t^{-1}] \otimes M)$ in the obvious way.*

Theorem 5.6. *Let I be an ideal in H^* that is preserved by the Steenrod action and such that H^*/I is finite. There is a $\mathbb{Z}/2$-linear map Φ such that the diagram*

$$\begin{tikzcd} WH^*/IWH^* \arrow[r, "\Phi"] \arrow[dr, "\hat{\varphi}^*"'] & R_+(H^*(MT)/IH^*(MT)) \arrow[d, "\epsilon"] \\ & H^*(MT)/IH^*(MT) \end{tikzcd} \tag{5.4}$$

commutes and Φ commutes with the $\mathbb{Z}/2[t,t^{-1}]$-actions. Here $\hat{\varphi}^*$ is induced by the inclusion $\hat{\varphi}: MT \to \widehat{MT}$.

Proof. We must construct a map of the form $\Phi(x\tilde{w}_l) = t^{l-1}\phi(x)$ where $\phi(x)$ is linear of the form $\sum_i \phi_i(x)t^{-i}$ for $x \in H^*(MT)/IH^*(MT)$. Then it will automatically commute with the $\mathbb{Z}/2[t,t^{-1}]$-actions. For the diagram (5.4) to commute, these ϕ_i must satisfy

$$xw_l = \hat{\varphi}^*(x\tilde{w}_l) = \epsilon\Phi(x\tilde{w}_l) = \sum_{i=0}^{N} Sq^{l-i}\phi_i(x)$$

for all l. This can be written as a matrix equation:

$$\begin{pmatrix} 1 & \mathrm{Sq}^1 & \cdots & \mathrm{Sq}^l \\ 0 & 1 & & \mathrm{Sq}^{l-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_l(x) \\ \vdots \\ \phi_1(x) \\ \phi_0(x) \end{pmatrix} = \begin{pmatrix} xw_l \\ \vdots \\ xw_1 \\ x \end{pmatrix} \tag{5.5}$$

Let $\chi(\mathrm{Sq}^k) \in \mathcal{A}$ denote the dual squares defined inductively by $\chi(\mathrm{Sq}^0) = 1$ and

$$\sum_i \chi(\mathrm{Sq}^i)\,\mathrm{Sq}^{k-i} = 0.$$

Then the matrix

$$\begin{pmatrix} 1 & \chi(\mathrm{Sq}^1) & \cdots & \chi(\mathrm{Sq}^l) \\ 0 & 1 & & \chi(\mathrm{Sq}^{l-1}) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \tag{5.6}$$

is a right inverse for the matrix in equation (5.5), and multiplication on both sides yields a formula defining the ϕ_i. Note that only finitely many ϕ_i can be non-zero, since H^*/I is finite and ϕ_i has degree $i + \deg(x)$. Clearly, Φ is linear because each ϕ_i is. □

Lemma 5.7. *The formula*

$$\mathrm{Sq}^k(tx) = t\,\mathrm{Sq}^k(x) + t^2\,\mathrm{Sq}^{k-1}(x)$$

holds in both WH^* and $R_+(H^*(MT))$.

Proof. This is a straightforward check using the formulas (5.1) and (5.2). □

Theorem 5.8. *The map Φ in Theorem 5.6 is an isomorphism of Steenrod modules. In particular, $\hat{\varphi}^*$ induces an isomorphism on Ext-groups.*

Proof. Note that WH^*/IWH^* is an $\mathcal{A}$-module by Theorem 5.3.

First we show that Φ is an $\mathcal{A}$-homomorphism. Assume that for some k,

$$\Phi(\mathrm{Sq}^{k-1}(x)) = \mathrm{Sq}^{k-1}(\Phi(x))$$

for all x. It is clearly satisfied for $k = 1$. Then

$$\begin{aligned}\Phi(\mathrm{Sq}^k(tx)) =& t\Phi(\mathrm{Sq}^k(x)) + t^2\Phi(\mathrm{Sq}^{k-1}(x))\\ =& t\Phi(\mathrm{Sq}^k(x)) + t^2\,\mathrm{Sq}^{k-1}(\Phi(x)).\end{aligned}$$

On the other hand,

$$\mathrm{Sq}^k(\Phi(tx)) = t\,\mathrm{Sq}^k(\Phi(x)) + t^2\,\mathrm{Sq}^{k-1}(\Phi(x))$$

since Φ commutes with t. Introducing the notation

$$\delta(x) = \Phi(\mathrm{Sq}^k(x)) - \mathrm{Sq}^k(\Phi(x)),$$

the above implies that

$$\delta(tx) = t\delta(x).$$

Iterating this yields $\delta(t^l x) = t^l\delta(x)$ for all $l \in \mathbb{Z}$.

We must show that $\delta(x) = 0$. Write $\delta(x) = \sum_{i=-N}^{N}\delta_i t^i$ and note that

$$\begin{aligned}\epsilon(\delta(x)) &= \epsilon(\Phi(\mathrm{Sq}^k(x))) - \epsilon(\mathrm{Sq}^k(\Phi(x)))\\ &= \hat{\varphi}^*(\mathrm{Sq}^k(x)) - \mathrm{Sq}^k(\hat{\varphi}^*(x))\\ &= 0\end{aligned}$$

by commutativity of (5.4). This means that for all k,

$$\epsilon(\delta(t^k x)) = \epsilon(\sum_{i=-N}^{N}\delta_i t^{i+k}) = \sum_{i=-N}^{N}\mathrm{Sq}^{i+k+1}(\delta_i) = 0.$$

This yields the following matrix equation:

$$\begin{pmatrix}1 & \mathrm{Sq}^1 & \cdots & \mathrm{Sq}^{2N}\\ 0 & 1 & & \mathrm{Sq}^{2N-1}\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & 1\end{pmatrix}\begin{pmatrix}\delta_N\\ \delta_{N-1}\\ \vdots\\ \delta_{-N}\end{pmatrix} = \begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}$$

As in the proof of Theorem 5.6, we multiply by the matrix (5.6) and obtain the unique solution $\delta_i = 0$ for all i.

It is clear that Φ is injective because $\Phi(\sum_i x_i\tilde{w}_{k-i})$ is of the form $\sum_i x_i t^{k-i-1}$ plus terms involving only powers of t that are strictly smaller than the largest power occuring in this sum. Thus it is non-zero. Since Φ is a map between vector spaces that are finite and of the same dimension in each degree, it is also an isomorphism.

Since ϵ is an Ext-isomorphism, so is $\hat{\varphi}^*$. $\square$

Lemma 5.9. *The map $H^*(\mathcal{MT}(d,r)) \to H^*(\mathcal{MT}(d,r)_{|G(m,l)})$ is a surjection with kernel $I \cdot \mathbb{Z}/2\{\tilde{w}_{d-r+1}, \dots, \tilde{w}_d\}$ where I is the ideal in H^* generated by the Stiefel–Whitney classes $w_{m+1}, w_{m+2}, \dots$ and the dual classes $\bar{w}_{l+1}, \bar{w}_{l+2}, \dots$*

Proof. Look at the Serre spectral sequence for the fibration

$$V_{d+ka_r,r} \to V_r(U_{d+ka_r,n})_{|G(m,l)} \to G(m,l)$$

for n large. This has E^2-term

$$E_2^{p,q} \cong H^p(G(m,l)) \otimes H^q(V_{d+ka_r,r}).$$

For k large compared to m, there can be no differentials in the lower left corner of the spectral sequence. Hence

$$H^*(V_r(U_{d+ka_r,n})_{|G(m,l)}) \cong H^*(V_{d+ka_r,r}) \otimes H^*(G(m,l))$$

for $* < 2(d-r) + ka_r$. In the exact sequence

$$H^*(G(m,l), V_r(U_{d+ka_r,n})_{|G(m,l)}) \to H^*(G(m,l)) \to H^*(V_r(U_{d+ka_r,n})_{|G(m,l)}),$$

the last map is an injection by the spectral sequence. Thus for $* < 2(d-r) + ka_r$,

$$H^*(G(m,l), V_r(U_{d+ka_r,n})_{|G(m,l)}) \cong H^{>0}(\Sigma V_{d+ka_r,r}) \otimes H^*(G(m,l)).$$

To see that the map is actually as claimed, consider

$$\begin{tikzcd}
H^*(G(m,l), V_r(U_{d+ka_r,n})_{|G(m,l)}) \arrow[r] & H^*(\Sigma V_{d+ka_r,r}) \\
H^*(G(d+ka_r,n), V_r(U_{d+ka_r,n})) \arrow[r] \arrow[u] & H^*(\Sigma V_{d+ka_r,r}). \arrow[u, "\cong"']
\end{tikzcd}$$

The vertical map to the left takes the generators $w_{d+ka_r-r+1}, \dots, w_{d+ka_r}$ to the generators of $H^{>0}(V_{d+ka_r,r}) \otimes H^0(G(m,l))$, which can be proved by an induction on r as in the proof of Theorem 5.3. It commutes with the $H^*(G(d+ka_r,n))$-module structure, so the map is as expected. $\square$

Theorem 5.10. $\hat{\varphi}_2 : (MT(d)_{|G(m,l)})_2^\wedge \to (\widehat{MT}(d)_{|G(m,l)})_2^\wedge$ *is a $(d-1)$-equivalence.*

Proof. It follows from Lemma 5.9 that

$$\varinjlim_r H^*(\mathcal{MT}(d,r)_{|G(m,l)}) \cong WH(d)^*/IWH(d)^*$$

where $I = \langle \bar{w}_k, w_n, k > l, n > m\rangle$ is an ideal in H^* preserved by $\mathcal{A}$. Now, according to Theorem 5.1 there is a spectral sequence with E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(WH(d)^*/IWH(d)^*, \mathbb{Z}/2)$$

converging strongly to $\pi_{t-s}((\widehat{MT}(d)_{|G(m,l)})_2^\wedge)$. Similarly, there is a spectral sequence converging strongly to $\pi_{t-s}((MT(d)_{|G(m,l)})_2^\wedge)$ with

$$\bar{E}_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(MT(d))/IH^*(MT(d)), \mathbb{Z}/2)$$

and a map $\hat{\varphi}_{2*} : \bar{E}_2^{s,t} \to E_2^{s,t}$ between them.

From the long exact sequence of Ext-groups associated to the short exact sequence $0 \to K \to WH^*/IWH^* \to WH(d)^*/IWH(d)^* \to 0$, it follows that

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(WH(d)^*/IWH(d)^*, \mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}}^{s,t}(WH^*/IWH^*, \mathbb{Z}/2)$$

is an isomorphism for $t - s < d$. Similarly for

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(MT(d))/IH^*(MT(d)), \mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(MT)/IH^*(MT), \mathbb{Z}/2).$$

Combined with Theorem 5.8, this shows that also $\hat{\varphi}_{2*} : \bar{E}_2^{s,t} \to E_2^{s,t}$ is an isomorphism for $t-s < d$. Thus it it a $(d-1)$-equivalence on $E_\infty^{s,t}$. Since both spectral sequences converge strongly, we conclude by [5], Theorem 2.6, that

$$\hat{\varphi}_{2*} : \pi_{t-s}((MT(d)_{|G(m,l)})_2^\wedge) \to \pi_{t-s}((\widehat{MT}(d)_{|G(m,l)})_2^\wedge) \tag{5.7}$$

is an isomorphism for $t-s < d-1$. Since all the Ext-groups are finite, a modification of Boardman's proof also shows that $\hat{\varphi}_{2*}$ is a surjection for $t - s = d - 1$. □

Remark 5.11. The above proofs generalize to the situation where $f : X \to BO(d)$ is given on a finite complex X to show that the map

$$f^*MT(d)_2^\wedge \to \widehat{f^*MT}(d)_2^\wedge$$

is a $(d-1)$-equivalence, yielding a new proof of Theorem 4.21. The Serre spectral sequence shows, exactly as in the proof of Lemma 5.9, that $\varinjlim_r H^*(f^*\mathcal{MT}(d,r))$ is the free $H^*(X)$-module on one generator $\tilde{v}_l$ in each dimension $l \le d$ and

$$WH(d)^* \to \varinjlim_r H^*(f^*\mathcal{MT}(d,r))$$

is an $\mathcal{A}$-homomorphism taking $\tilde{w}_l$ to $\tilde{v}_l$. The proof that this is the Singer construction on $H^*(f^*MT)$ goes as before. The map Φ can be constructed as in Theorem 5.6. The crucial point is to reprove Lemma 5.7. Observing that it is enough to check the formula on $\tilde{v}_l$, it follows from the formulas for $\tilde{w}_l \in WH(d)$. The proof of Theorem 5.8 then carries over.

Note, however, that compactness of X is essential in the proof of Theorem 5.6 to make Φ well-defined and show that it is an isomorphism. The rest of this chapter is devoted to the non-compact case, in which the spectra behave completely differently.

5.3 The Non-Compact Case

We now turn to the non-compact case. In this section we only consider the unoriented spectra MTO. Now H^* will denote $H^*(MTO)$, while WH^* is as in Definition 5.2. WH^* might still look like the Singer construction applied to H^*. However, we shall see that the algebraic behavior is very different. In fact, WH^* has non-zero $\mathrm{Ext}_{\mathcal{A}}^{0,*}$-groups in all dimensions.

It is well-known that H^* is a free $\mathcal{A}$-module, see e.g. [41]. The set $\mathrm{Hom}_{\mathcal{A}}^*(H^*, \mathbb{Z}/2)$ is known to be the polynomial algebra $\mathbb{Z}/2[\xi_k, k \neq 2^s - 1]$. The multiplication comes from the map $\Delta^* : H^* \to H^* \otimes H^*$ induced by the direct sum map $BO \times BO \to BO$. This is the $H^*(BO)$-linear map given by the formula (3.1). The classical way to prove that H^* is free is by an algebraic study of the map Δ^*, see [41], Chapter VI. In this section we try to generalize this to WH^*. However, some difficulties occur because WH^* is not bounded below.

The map $MT(d) \to \widehat{MT}(d)$ induces a surjective $\mathcal{A}$-homomorphism $WH^* \to H^*$. Thus there is an injection $\mathrm{Hom}^*_{\mathcal{A}}(H^*, \mathbb{Z}/2) \to \mathrm{Hom}^*_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$. We even have:

Theorem 5.12. *There is an $\mathcal{A}$-homomorphism $\Delta : WH^* \to H^* \hat{\otimes} WH^*$ where $\mathcal{A}$ acts on the right hand side by the Cartan formula. It is also a map of $H^*(BO)$-modules where $H^*(BO)$ acts on the right hand side by the formula* (3.1)*. This makes* $\mathrm{Hom}^*_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$ *into a module over* $\mathrm{Hom}^*_{\mathcal{A}}(H^*, \mathbb{Z}/2)$*.*

Here $H^* \hat{\otimes} WH^*$ denotes the inverse limit $\varprojlim_d H(d)^* \otimes WH^*$. One can think of this as a submodule of $\prod_k H^k \otimes WH^{*-k}$.

Proof. The inclusion $BO(d') \times BO(d + ka_r) \to BO(d' + d + ka_r)$ induces a map

$$H^*(MT(d' + d + ka_r)) \to H^*(MT(d')) \otimes H^*(MT(d + ka_r)).$$

Then the inclusion $BO(d' + d - r + ka_r) \to BO(d' + d + ka_r)$ yields a commutative diagram

$$\begin{array}{ccc} H^*(MT(d' + d + ka_r)) & \longrightarrow & H^*(MT(d')) \otimes H^*(MT(d + ka_r)) \\ \downarrow & & \downarrow \\ H^*(MT(d' + d - r + ka_r)) & \longrightarrow & H^*(MT(d')) \otimes H^*(MT(d - r + ka_r)). \end{array}$$

This defines an $\mathcal{A}$-map on the kernels

$$H^*(MT(d' + d + ka_r, r)) \to H^*(MT(d')) \otimes H^*(MT(d + ka_r, r)).$$

This commutes with the periodicity maps and the maps in the inverse system, inducing an $\mathcal{A}$-homomorphism

$$WH(d' + d)^* \to H(d')^* \otimes WH(d)^*.$$

Taking the inverse limit over d, we get a map $WH^* \to H(d')^* \otimes WH^*$, and the inverse limit over d' yields the desired map $WH^* \to H^* \hat{\otimes} WH^*$.

The $H^*(BO)$-module structure comes from the commutative diagram

$$\begin{array}{ccc} MT(d') \wedge MT(d + ka_r) & \longrightarrow & MT(d') \wedge MT(d + ka_r) \wedge (BO(d') \times BO(d + ka_r))_+ \\ \downarrow & & \downarrow \\ MT(d' + d + ka_r) & \longrightarrow & MT(d' + d + ka_r) \wedge BO(d' + d + ka_r)_+. \end{array}$$

$\square$

As promised, we are now going to construct an infinite collection of elements in $\mathrm{Hom}^t_{\mathcal{A}}(WH^*, \mathbb{Z}/2) = \mathrm{Ext}^{0,t}_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$ in each dimension t. In H^* one can compute $\mathrm{Sq}^2(1) = \bar{w}_2 = w_2 + w_1^2$. Thus there is an $\mathcal{A}$-homomorphism $\xi_2 : H^2 \to \mathbb{Z}/2$ taking the value 1 on both w_2 and w_1^2. This element plays a special role.

Theorem 5.13. *ξ_2 is invertible in $\mathrm{Hom}^*_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$, i.e there are elements ξ_2^{-n} satisfying $\xi_2^m \cdot \xi_2^{-n} = \xi_2^{m-n}$ for all $n, m \in \mathbb{N}$. The monomials $\xi_2^n \xi_I$ for $n \in \mathbb{Z}$ and $\xi_I \in \mathbb{Z}/2[\xi_4, \xi_5, \dots]$ are linearly independent in $\mathrm{Hom}_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$. Moreover, the $\mathrm{Hom}^*_{\mathcal{A}}(H^*, \mathbb{Z}/2)$-module structure of $\mathrm{Hom}^*_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$ extends to a module structure over $\mathrm{Hom}^*_{\mathcal{A}}(H^*, \mathbb{Z}/2)[\xi_2^{-1}]$.*

Proof. We must determine the value of $\xi_2^{-n}(p_i \tilde{w}_{-i-2n})$ for every $p_i \in H^i(BO)$. This comes from some $p_i w_{ka_r - i - 2n} \in H^{ka_r - 2n}(MT(d + ka_r, r))$ in the direct system and here we have the $\mathcal{A}$-homomorphism

$$H^{ka_r-2n}(MT(d+ka_r,r)) \to H^{ka_r-2n}(MT(d+ka_r)) \xrightarrow{\xi_2^{(k\frac{a_r}{2}-n)}} \mathbb{Z}/2. \tag{5.8}$$

Define $\xi_2^{-n}(p_i \tilde{w}_{-i-2n}) = \xi_2^{(k\frac{a_r}{2}-n)}(p_i w_{ka_r-i-2n})$. We must check that this definition does not depend on the choice of ka_r for r large enough.

The $(k\frac{a_r}{2}-n)$-fold direct sum map $BO \times \cdots \times BO \to BO$ induces

$$\Delta^* : H^{ka_r-2n}(MT(d+ka_r)) \cong H^{ka_r-2n}(MTO) \to H^2(MTO)^{\otimes(k\frac{a_r}{2}-n)}.$$

To evaluate $\xi_2^{(k\frac{a_r}{2}-n)}(p_i w_{ka_r-i-2n})$, we must apply ξ_2 to each factor of

$$\Delta^*(p_i w_{ka_r-i-2n}) = \Delta^*(p_i)\Delta^*(w_{ka_r-i-2n}).$$

This yields the formula

$$\xi_2^{(k\frac{a_r}{2}-n)}(p_i \tilde{w}_{ka_r-i-2n}) = \sum_{a+2b=i} \binom{k\frac{a_r}{2}-n}{a, b} \xi_1^a \xi_2^b(p_i). \tag{5.9}$$

Here

$$\xi_1^a\xi_2^b : H^i(BO) \xrightarrow{\Delta} H^0(BO)^{\otimes(k\frac{a_r}{2}-n-a-b)} \otimes H^1(BO)^{\otimes a} \otimes H^2(BO)^{\otimes b} \to \mathbb{Z}/2$$

is the evaluation in each factor of the $\mathbb{Z}/2$-linear maps $\xi_1 : H^1(BO) \to \mathbb{Z}/2$ and $\xi_2 : H^2(BO) \to \mathbb{Z}/2$, given by $\xi_1(w_1) = 1$, $\xi_2(w_1^2) = 1$, and $\xi_2(w_2) = 1$.

The multinomial coefficient in (5.9) only depends on $(k\frac{a_r}{2} - n) \bmod 2^N$ where N is the smallest number such that $i < 2^N$. But 2^N divides $\frac{a_r}{2}$ for all r sufficiently large. Also, $\xi_1^a\xi_2^b(p_i)$ is independent of ka_r when this is larger than i, since a larger ka_r only will add more copies of $1 \in H^0(BO)$ to the formula for $\Delta(p_i)$. Hence (5.9) does not depend on ka_r for r sufficiently large, and we have a well-defined $\mathbb{Z}/2$-linear map.

We must see that this map is actually an $\mathcal{A}$-homomorphism. Let $x \in WH^{-2n-j}$ be given. We must see that $\xi_2^{-n}(Sq^j(x)) = 0$. For this, choose d, r and k so large that x comes from some $x' \in H^*(MT(d+ka_r, r))$ and so that we may compute

$$\xi_2^{-n}(\mathrm{Sq}^j(x)) = \xi_2^{(k\frac{a_r}{2}-n)}(\mathrm{Sq}^j(x')).$$

This is zero because $\xi_2^{(k\frac{a_r}{2}-n)}$ is an $\mathcal{A}$-homomorphism.

Now we look at how the ξ_2^{-n} multiply. This is given by the map

$$WH^* \to H^* \hat{\otimes} WH^* \to H^m \otimes WH^{-2n} \xrightarrow{\xi_2^m \otimes \xi_2^{-n}} \mathbb{Z}/2.$$

For sufficiently large d', d, r and N, we have the following diagram

$$\begin{tikzcd}
H^*(MT(d'+d+2^N,r)) \arrow[r] \arrow[d] & H^*(MT(d')) \otimes H^*(MT(d+2^N,r)) \arrow[d] \\
H^*(MT(d'+d+2^N)) \arrow[r] \arrow[dr, "\xi_2^{(m+2^{N-1}-n)}"'] & H^*(MT(d')) \otimes H^*(MT(d+2^N)) \arrow[d, "\xi_2^m \otimes \xi_2^{(2^{N-1}-n)}"] \\
 & \mathbb{Z}/2.
\end{tikzcd}$$

But $\xi_2^{(2^{N-1}+m-n)}(p_i w_{2^N+k}) = \xi_2^{(m-n)}(p_i w_k)$. This is by definition when $m-n$ is negative and because (5.9) also holds for $m-n$ positive. Thus the result follows by commutativity of the diagram.

Finally, the linear independence follows from this multiplicative structure. Suppose a finite sum of monomials $\sum a_{n,I}\xi_2^{-n}\xi_I = 0$ is given. We can multiply this by a large power of ξ_2 so that the sum only contains positive powers of ξ_2. Since all such monomials are linearly independent, all $a_{n,I}$ must be 0.

Let $\xi \in \mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)[\xi_2^{-1}]$ and $\eta \in \mathrm{Hom}_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$. Then for $p_i\tilde{w}_l \in WH^*$ we define

$$\xi \cdot \eta(p_i\tilde{w}_l) = \xi_2^{2^{N-1}}\xi \cdot \eta(p_i\tilde{w}_{l+2^N})$$

for N large. As before, one can write down the formulas to see that this is an $\mathcal{A}$-homomorphism and that, for a given $p_i\tilde{w}_l$, it is independent of N for N large. $\square$

Let $\mathcal{A}(n)$ denote the Hopf subalgebra of $\mathcal{A}$ generated by the elements Sq^{2^i} for $i \le n$. This is finite by [33], and $\mathcal{A}$ is free over each $\mathcal{A}(n)$ by general results on Hopf algebras given in [34]. In the following, $\mathcal{A}^+$ and $\mathcal{A}(n)^+$ will denote elements of positive degree in the respective algebras.

We want to generalize the proof that H^* is free over $\mathcal{A}$ to WH^*. However, since WH^* is not bounded below, the proof only works over the finite subalgebras $\mathcal{A}(n)$. It turns out that this is enough to compute the Ext-groups we are after.

Lemma 5.14. *WH^* is free over $\mathcal{A}(n)$.*

Proof. Let $D_n^t = \mathcal{A}(n)WH^{<t} + WH^{>t}$. This is a submodule over $\mathcal{A}(n)$. Therefore, $M_n^t = WH^*/D_n^t$ is again an $A(n)$-module with trivial action, and the projection $WH^* \to M_n^t$ is $\mathcal{A}(n)$-linear. Since $\mathcal{A}$ is free over $\mathcal{A}(n)$, H^* is also free over $\mathcal{A}(n)$, so we may choose an $\mathcal{A}(n)$-linear projection $H^* \to \mathcal{A}(n)$. Thus there is a well-defined $\mathcal{A}(n)$-homomorphism

$$WH^* \to H^*\hat{\otimes}WH^* \to \mathcal{A}(n)\otimes M_n^t. \tag{5.10}$$

This is clearly surjective, since a lift $M_n^t \to WH^t$ of the projection defines an $\mathcal{A}(n)$-linear map $i_t : \mathcal{A}(n) \otimes M_n^t \to WH^*$ whose composition with (5.10) is clearly the identity. Then also

$$WH^* \to \bigoplus_{t=N}^{\infty} \mathcal{A}(n) \otimes M_n^t \tag{5.11}$$

must be surjective. This can be seen using the splitting since the $\mathcal{A}(n) \otimes M_n^N$ summand is hit by $i_N(\mathcal{A}(n) \otimes M_n^N)$. Then $i_{N+1}(\mathcal{A}(n) \otimes M_n^{N+1})$ hits the $\mathcal{A}(n) \otimes M_n^{N+1}$ summand mod $\mathcal{A}(n) \otimes M_n^N$. Continuing this way, we see that it is surjective in each dimension. Since $\mathcal{A}(n)$ is finite, the right hand side of (5.11) is a finite sum in each dimension. Letting $N \to \infty$, this implies that

$$WH^* \to \bigoplus_{t=-\infty}^{\infty} \mathcal{A}(n) \otimes M_n^t$$

is a surjection. Let K denote the kernel. Then WH^* splits as a sum

$$K \oplus \bigoplus_{t=\infty}^{\infty} i_t(\mathcal{A}(n) \otimes M_n^t)$$

of $\mathcal{A}(n)$-modules. Every element in K must be decomposable, otherwise they would be non-zero in some M_n^t. Because of the splitting, we must have $\mathcal{A}(n)^+K = K$. Iterating this, we get $(\mathcal{A}(n)^+)^l K = K$ for all l. But $\mathcal{A}(n)$ is finite, so K must be zero. □

Define $M^* = \varinjlim_n M_n^* = WH^*/\mathcal{A}^+WH^*$. In the following, the $\mathbb{Z}/2$-dual of a $\mathbb{Z}/2$-vector space V will be denoted by $V^\vee$.

Theorem 5.15.

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(WH^*, \mathbb{Z}/2) = \begin{cases} (M^t)^\vee & \textit{for } s = 0, \\ 0 & \textit{for } s > 0. \end{cases}$$

Proof. It follows from Lemma 5.14 that

$$\mathrm{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*) = \begin{cases} WH^t/\mathcal{A}(n)^+WH^* & \text{for } s = 0, \\ 0 & \text{for } s > 0. \end{cases}$$

There is an isomorphism

$$\varinjlim_n \mathrm{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*) \to \mathrm{Tor}_{s,t}^{\mathcal{A}}(\mathbb{Z}/2^\vee, WH^*),$$

see e.g. [27]. This allows us to calculate

$$\mathrm{Tor}_{0,t}^{\mathcal{A}}(\mathbb{Z}/2^\vee, WH^*) \cong \varinjlim_n WH^t/\mathcal{A}(n)^+WH^* \cong WH^t/\mathcal{A}WH^* = M^t.$$

But again by [27], Lemma 4.3,

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(WH^*, \mathbb{Z}/2) \cong \mathrm{Hom}(\mathrm{Tor}_{s,t}^{\mathcal{A}}(\mathbb{Z}/2^\vee, WH^*), \mathbb{Z}/2),$$

and the claim follows. □

Corollary 5.16. *For all $t < d$, $\pi_t(\widehat{MT}(d)_2^\wedge) \cong (M^t)^\vee$.*

Proof. We have that $\mathrm{Ext}_{\mathcal{A}}^{s,t}(WH(d)^*, \mathbb{Z}/2) \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(WH^*, \mathbb{Z}/2)$ whenever $t - s < d$, so in these dimensions the spectral sequence of Theorem 5.1 is concentrated on the line $s = 0$. Hence there can be no non-trivial differentials in this part of the sequence. □

5.4 More on the Structure of WH^*

In this section we shall investigate WH^* further. Most of this section is due to, and partially written by, Bökstedt.

We saw in Section 5.3 that WH^* has the Ext-groups of a free $\mathcal{A}$-module. To see that it is in fact a free module and find an explicit description of the generators requires a more convenient basis for H^*.

Let $v_i \in H_i(BO)$ be the image of the generator in $H_i(BO(1)) \cong H^i(\mathbb{R}P^\infty)$. The direct sum map $BO \times BO \to BO$ induces a multiplication in $H_*(BO)$ dual to the comultiplication Δ^* in $H^*(BO)$, and the dual of the cup product in $H^*(BO)$ defines a comultiplication Δ_* on $H_*(BO)$.

Lemma 5.17. *$H_*(BO)$ is the polynomial algebra on the generators v_i. The comultiplication is given by $\Delta_*(v_i) = \sum_j v_j \otimes v_{i-j}$.*

See e.g. [42], Chapter 16. We will need the following relations to the Stiefel-Whitney classes:

Lemma 5.18.

(i)

$$w_i(v_i) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $j \geq 2$ and $x \in H_*(BO)$, then $w_n(v_j x) = 0$.

(iii) If $x \in H_*(BO)$, $a = \prod_{1 \leq k \leq n} w_{i_k}$, and $j \geq n$, then $a(v_j x) = 0$.

(iv) Assume that $2^N > i_j$ for all $j > 1$. Then

$$w_{i_1+2^N} w_{i_2} \cdots w_{i_n}(v_1^{2^N} x) = w_{i_1} w_{i_2} \cdots w_{i_n}(x)$$

and

$$w_{i_1+k2^N} w_{i_2} \ldots w_{i_n}(v_k^{2^N} x) = 0.$$

Proof. The first formula follows from the definition of v_i because the restriction of w_i to $H^i(BO(1))$ is non-trivial if and only if $i = 1$.

Formula (ii) follows from (i) using the comultiplication in $H^*(BO)$:

$$w_n(v_j x) = \Delta^*(w_n)(v_j x) = w_j(v_j) \cdot w_{n-j}(x) = 0.$$

Formula (iii) follows from (ii) since

$$a(v_j x) = (\prod_{1 \leq k \leq n} w_{i_k})(\Delta_* v_j \Delta_* x)$$

is a sum of terms of the form $\prod_{1 \leq k \leq n}(w_{i_k}(v_{i_k} x_k))$ with $\sum_{1 \leq k \leq n} i_k = j$. But $j > n$ implies $i_k \geq 2$ for some k, so these terms all equal 0.

Finally, (iv) follows from (ii) and the formula $\Delta_*(v_k^{2^N} x) = \Delta_*(v_k)^{2^N} \Delta_*(x)$ where only the term $(v_k^{2^N} \otimes 1)\Delta_*(x)$ contributes when we evaluate $w_{i_1+k2^N} w_{i_2} \cdots w_{i_n}$. □

We may filter H^* by length of monomials

$$H^*(n) = \mathbb{Z}/2\{w_{i_1} \cdots w_{i_n} \in H^*\}.$$

The inclusion $i_n : H^*(n) \to H^*$ is not compatible with the action of $\mathcal{A}$, but the projection $p_n : H^* \to H^*(n)$ is. That is, if a monomial a in the w_i has length at least $n+1$, then so has every monomial in $\mathrm{Sq}^i(a)$. This follows from Formula (3.7) and the Cartan formula.

Lemma 5.19. *Evaluation induces a perfect pairing in each dimension*

$$\mu_n : \mathbb{Z}/2[v_1, v_2, \ldots, v_n] \otimes H^*(n) \to \mathbb{Z}/2.$$

Proof. Evaluation defines a perfect pairing

$$\mu : \mathbb{Z}/2[v_1, v_2, \ldots] \otimes H^* \to \mathbb{Z}/2.$$

The restriction to $H^*(n) \subset H^*$ yields a map

$$\mu_n : \mathbb{Z}/2[v_1, v_2, \ldots, v_n] \otimes H^*(n) \to \mathbb{Z}/2.$$

According to in Lemma 5.18 (iii), the adjoint map

$$\mu_n^* : H^*(n) \to \mathrm{Hom}(\mathbb{Z}/2[v_1, v_2, \ldots, v_n], \mathbb{Z}/2)$$

must be injective, since the isomorphism μ^* is. In each degree, $\mathbb{Z}/2[v_1, v_2, \ldots, v_n]$ and $H^*(n)$ are finite of the same dimension, so μ_n^* is an isomorphism, i.e. μ_n is a perfect pairing. $\square$

In the following, let $H_* = H_*(MTO)$.

Lemma 5.20. *The Steenrod algebra acts on $H_*(MTO)$ by*

$$\mathrm{Sq}_*^k\, v_i = \binom{i-k-1}{k} v_{i-k}.$$

Proof. In $H^*(BO(1))$ the formula $\mathrm{Sq}^k(w_1^{i-k}) = \binom{i-k}{k} w_1^i$ holds. In H^* the formula becomes

$$\mathrm{Sq}^k(w_1^{i-k}) = \sum_{l=0}^{k} \binom{i-k}{l} w_1^{i-k+l} \bar{w}_{l-k}.$$

This restricts to

$$\mathrm{Sq}^k(w_1^{i-k}) = \sum_{l=0}^{k} \binom{i-k}{l} w_1^{i} = \binom{i-k-1}{k} w_1^i$$

in $H^*(MTO(1))$ because $\bar{w}_{l-k}$ restricts to w_1^{l-k}. Dualizing yields the formula. $\square$

Let $S = \mathbb{Z}/2[v_2, v_3, \ldots]$ be the subalgebra in H_*.

Corollary 5.21. *S is an $\mathcal{A}$-submodule of H_*. As algebras with an $\mathcal{A}$-action,*

$$\mathbb{Z}/2[v_1, v_2, \ldots, v_n] \cong \mathbb{Z}/2[v_1] \otimes \mathbb{Z}/2[v_2, v_3, \ldots, v_n].$$

Proof. Clearly, $\mathbb{Z}/2[v_1]$ is a subalgebra of $\mathbb{Z}/2[v_1, v_2, \dots, v_n]$ which is closed under the action of $\mathcal{A}$. It is not quite as obvious that the subalgebra $\mathbb{Z}/2[v_2, v_3, \dots]$ is closed, but it follows from the preceeding formula. If $\mathrm{Sq}_*^k(v_i) = v_j$ we have that $\binom{j-1}{k} = 1$. So if $j = 1$, we must have $k = 0$. The inclusions combine to an isomorphism

$$\mathbb{Z}/2[v_1] \otimes \mathbb{Z}/2[v_2, v_3, \dots, v_n] \to \mathbb{Z}/2[v_1, v_2, \dots, v_n]$$

of algebras compatible with the $\mathcal{A}$-action. $\square$

We now want to a give a similar description of WH^* and its dual. By Theorem 5.3, WH^* has a $\mathbb{Z}/2$-basis

$$\{\tilde{w}_{i_1} w_{i_2} \cdots w_{i_n} \mid i_1 \in \mathbb{Z},\ i_l \geq 0 \text{ for } l \geq 2\}.$$

Again there is a filtration of WH^* by length of monomials

$$WH^*(n) = \mathbb{Z}/2\{\tilde{w}_{i_1} \cdots w_{i_n} \in WH^*\}.$$

The inclusion $i_n : WH^*(n) \to WH^*$ is not compatible with the action of $\mathcal{A}$, but the projection $WH^* \to WH^*(n)$ is by Formula (3.7) and (5.2).

Let $\xi_2 \in H_2(BO)$ be as in the previous section. Then $\xi_2 = v_1^2 + v_2$.

Definition 5.22.

$$R(n) = \mathbb{Z}/2[v_1, v_2, \dots, v_n][\xi_2^{-1}]$$
$$R = \varinjlim_n R(n).$$

The Steenrod algebra acts on $R(n)$, and the inclusion maps $R(n) \to R(n+1)$ are compatible with this action. Thus R has a natural action of the Steenrod algebra.

The pairing μ_n extends to a pairing

$$\tilde{\mu}_n : R(n) \otimes WH^*(n) \to \mathbb{Z}/2$$

by the formula

$$\tilde{\mu}_n(\xi_2^k x, \tilde{w}_{i_1} w_{i_2} \cdots w_{i_n}) = \mu_n(\xi_2^{2^{N-1}+k} x, w_{i_1+2^N} w_{i_2} \cdots w_{i_n})$$

where $x \in H_*(MTO)$ and N is sufficiently big. Repeated use of Lemma 5.18 (iv) shows that this definition is independent of N. The pairing induces a natural map $\tilde{\mu}^* : WH^* \to \mathrm{Hom}(R, \mathbb{Z}/2)$ compatible with the Steenrod action.

Lemma 5.23. *For $a \in WH^*$, there are natural numbers m, n such that*

$$\tilde{\mu}^*(a) \in \mathrm{Hom}(R(m,n), \mathbb{Z}/2) \subseteq \mathrm{Hom}(R, \mathbb{Z}/2)$$

where $R(m,n) = R/I(m,n)$ and $I(m,n)$ is the ideal in R generated by

$$v_2^m, v_3^m, \dots, v_n^m, v_{n+1}, v_{n+2}, \dots$$

Proof. Choose n such that $a \in WH^*(n)$. Then by Lemma 5.18 (iii), $\tilde{\mu}^*(a)(v_j x) = 0$ for $j > n$.

Assume without restriction that $a = \tilde{w}_{i_1} w_{i_2} \cdots w_{i_n}$. Choose $m = 2^N > i_j$ for all $j > 1$. Then $\tilde{\mu}^*(a)(v_i^m x) = 0$ by Lemma 5.18 (iv). □

Definition 5.24. *Let $R^\wedge = \lim_n \lim_m R(m,n)$ with the inverse limit topology. If $x \in I(m,n)$, then $\mathrm{Sq}^k(x) \in I(m-k, n-k)$. This defines a continuous action of the Steenrod algebra on $R^\wedge$.*

We first give two equivalent filtrations of $R^\wedge$. It follows from Corollary 5.21 that $S\{1, v_1\} = S \oplus S v_1$ is an $\mathcal{A}$-submodule of H_*.

Proposition 5.25. *$R^\wedge$ is isomorphic as a topological $\mathcal{A}$-module to the completion of R with respect to the ideals*

$$J_r = \{\xi_2^m p_0 + \cdots + \xi_2^{m-k} p_k \mid p_i \in S\{1, v_1\},\ \deg(p_i) \geq r\}.$$

One can think of $R^\wedge$ as the space of power series $\xi_2^m p_0 + \xi_2^{m-1} p_1 + \cdots$ with coefficients $p_i \in S\{1, v_1\}$ and $\mathcal{A}$ acting on the coefficients.

Note, however, that it is not clear from this description how these power series multiply.

Proof. The topologies agree because

$$\begin{aligned} J_r \subset I(m,n) &\quad \text{if } r > \frac{n(n+1)}{2} m, \\ I(m,n) \subset J_r &\quad \text{if } \min\{m,n\} > r. \end{aligned}$$

Furthermore, if $x \in J_r$, then $\mathrm{Sq}_*^k(x) \in J_{r-k}$. Thus we get the same $\mathcal{A}$-action.

Observe that

$$R \cong \mathbb{Z}/2[\xi_2, \xi_2^{-1}] \otimes S\{1, v_1\}$$

as $\mathcal{A}$-modules. Thus any element of R has a unique representation of the form $\xi_2^m p_0 + \cdots + \xi_2^{m-k} p_k$. It is then clear that the completion of R in the J_r's is the space of power series of the form $\xi_2^m p_0 + \xi_2^{m-1} p_1 + \cdots$ with $p_i \in S\{1, v_1\}$ and $\mathcal{A}$ acting on coefficients, since it acts trivially on ξ_2. □

Proposition 5.26. *$R^\wedge$ is isomorphic as a topological ring to the ring $R_1^\wedge$ of power series*

$$v_1^m p_0 + v_1^{m-1} p_1 + \cdots + v_1^{m-i} p_i + \cdots$$

with coefficients $p_i \in S$.

Proof. $R_1^\wedge$ is its own completion in the ideals

$$L_r = \{v_1^m p_0 + v_1^{m-1} p_1 + \cdots \mid p_i \in S,\ \deg(p_i) \geq r\},$$

while $R^\wedge$ is its own completion in the ideals

$$J_r = \{\xi_2^m p_0 + \xi_2^{m-1} p_1 + \cdots \mid p_i \in S\{1, v_1\},\ \deg(p_i) \geq r\}.$$

But v_1 is invertible in $R^\wedge$ by the formula

$$v_1^{-1} = v_1\xi_2^{-1}(1 + v_2\xi_2^{-1} + v_2^2\xi_2^{-2} + \cdots)$$

and ξ_2 is invertible in $R_1^\wedge$ via the formula

$$\xi_2^{-1} = v_1^{-2} + v_1^{-4}v_2 + v_1^{-6}v_2^2 + \cdots .$$

This defines isomorphisms $J_r \to L_{r-1}$ and $L_{r-1} \to J_r$ inducing a continuous isomorphism $R_1^\wedge \cong R^\wedge$. □

Let $\mathrm{Hom}^{\mathrm{top}}(R, \mathbb{Z}/2)$ denote the vector space of continuous homomorphisms in either of the three equivalent topologies. Then $\mathrm{Hom}^{\mathrm{top}}(R, \mathbb{Z}/2) \subset \mathrm{Hom}(R, \mathbb{Z}/2)$, and according to Lemma 5.23, the image of the map $\tilde{\mu}^* : WH^* \to \mathrm{Hom}(R, \mathbb{Z}/2)$ is contained in $\mathrm{Hom}^{\mathrm{top}}(R, \mathbb{Z}/2) = \varinjlim_{m,n} \mathrm{Hom}(R(m,n), \mathbb{Z}/2)$.

Lemma 5.27. *The map $\tilde{\mu}^* : WH^* \to \mathrm{Hom}^{\mathrm{top}}(R, \mathbb{Z}/2)$ is an isomorphism.*

Proof. We consider $R^\wedge$ with the topology given in Proposition 5.26.

Let $\mathcal{I}_n$ be the set of sequences $i_1, i_2, \ldots, i_n$ with $i_1 \in \mathbb{Z}$ and $i_k \geq 0$ for $k \geq 2$ such that $i_2 \geq i_3 \geq \cdots \geq i_n$. For $I \in \mathcal{I}_n$ define $w_I = \tilde{w}_{i_1} w_{i_2} \cdots w_{i_n}$. Then the w_I for $I \in \mathcal{I}_n$ form a basis for $WH^*(n)$. Let $l_s(I) = \sum_{k=s}^n i_k$. Then the degree of w_I is $l_1(I)$. We can define a partial order on $\mathcal{I}_n$ by saying that $I \leq I'$ if $l_1(I) = l_1(I')$ and $l_s(I) \leq l_s(I')$ for $2 \leq s \leq n$.

Let $\mathcal{J}_n$ be the set of sequences $j_1, j_2, \ldots, j_n$ where $j_1 \in \mathbb{Z}$ and $j_k \geq 0$ for $k \geq 2$. Let $v^J = v_1^{j_1} v_2^{j_2} \cdots v_n^{j_n}$. The set of v^J for $J \in \mathcal{J}_n$ constitutes a basis for $\mathbb{Z}/2[v_1, v_2, \ldots, v_n][{v_1}^{-1}]$. Let $l_s(J) = \sum_{k=s}^n (k - s + 1)j_k$. The degree of v^J is $l_1(J)$.

There is a bijection $\alpha : \mathcal{J}_n \to \mathcal{I}_n$ given by $\alpha(j_1, j_2, \ldots, j_n) = (i_1, i_2, \ldots, i_n)$ such that $i_k = \sum_{k \leq m \leq n} j_m$. Note that $l_s(\alpha(J)) = l_s(J)$. Give $\mathcal{J}$ the partial order $J \leq J'$ if $l_s(J) \leq l_s(J')$ for $2 \leq s \leq n$ and $l_1(J) = l_1(J')$. This is compatible with α.

We claim that

$$w_I(v^J) = \begin{cases} 1 & \text{if } \alpha(J) = I, \\ 0 & \text{unless } \alpha(J) \leq I. \end{cases} \tag{5.12}$$

To prove this formula, we first compute using Lemma 5.18:

$$\tilde{w}_{i_1} w_{i_2} \cdots w_{i_n}(v_1^{j_1} v_2^{j_2} \cdots v_n^{j_n}) = \begin{cases} \tilde{w}_{i_1 - j_n} w_{i_2 - j_n} \cdots w_{i_n - j_n}(v_1^{j_1} v_2^{j_2} \cdots v_{n-1}^{j_{n-1}}) & j_n \leq i_n, \\ 0 & j_n > i_n. \end{cases}$$

The claim (5.12) follows for $\alpha(J) = I$ by induction on n.

Now assume $w_I(v^J) \neq 0$. We claim that for all $1 \leq s \leq n$,

$$\tilde{w}_{i_1} w_{i_2} \cdots w_{i_n}(v_1^{j_1} v_2^{j_2} \cdots v_n^{j_n}) = \sum \tilde{w}_{i'_1} w_{i'_2} \cdots w_{i'_n}(v_1^{j_1} v_2^{j_2} \cdots v_s^{j_s}) \tag{5.13}$$

where the sum is over some non-empty set of $I' \in \mathcal{J}_n$ satisfying for $2 \leq t \leq s + 1$

$$\sum_{k \geq t} i'_k \leq \sum_{k \geq t} i_k - \sum_{k \geq s+1} (k - t + 1) j_k. \tag{5.14}$$

This is clearly true for $s = n$. Assume that this is true for some s. To compute (5.13), we must evaluate on $\Delta_*(x)\Delta_*(v_s)^{j_s}$ where $x = v_1^{j_1}v_2^{j_2}\cdots v_{s-1}^{j_{s-1}}$. In $\Delta_*(v_s)^{j_s}$, only terms of the form $\sum v_1^{l_1}\otimes\cdots\otimes v_1^{l_n}$ contribute. The l_k must satisfy $l_k \le j_s$ and $\sum l_k = sj_s$. In particular,

$$\sum_{k\ge t} l_k \ge (s-t+1)j_s. \tag{5.15}$$

Thus by Lemma 5.18,

$$\begin{aligned}\tilde{w}_{i'_1}w_{i'_2}\cdots w_{i'_n}(v_1^{j_1}v_2^{j_2}\cdots v_s^{j_s}) &= \sum_l \tilde{w}_{i'_1}w_{i'_2}\cdots w_{i'_n}(\Delta_*(x)v_1^{l_1}\otimes\cdots\otimes v_1^{l_n})\\ &= \sum_l \tilde{w}_{i'_1-l_1}w_{i'_2-l_2}\cdots w_{i'_n-l_n}(x)\end{aligned}$$

where only terms with $i'_k - l_k \ge 0$ for all $k \ge 2$ contribute to the sum. By (5.15) and the induction hypothesis,

$$\sum_{k\ge t}(i'_k - l_k) \le \sum_{k\ge t} i'_k - (s-t+1)j_s \le \sum_{k\ge t} i_k - \sum_{k\ge s}(k-t+1)j_k.$$

Thus the claim follows for $s-1$.

For each $s \ge 1$, put $t = s+1$ in (5.14). This yields

$$0 \le \sum_{k\ge t} i'_k \le l_{s+1}(I) - l_{s+1}(J) = l_{s+1}(I) - l_{s+1}(\alpha(J))$$

where the first inequality follows because at least one term in the sum (5.13) is non-zero. But this means that $\alpha(J) \le I$.

Let $\mathcal{J}_{n,J} = \{J' \in \mathcal{J}_n \mid J' < J\}$ and note that this is a finite set. Let

$$R_{n,J} = \mathbb{Z}/2[v_1, v_2, \dots][v_1^{-1}]/\mathbb{Z}/2\{v^{J'} \mid J' \notin \mathcal{J}_{n,J}\}.$$

By Proposition 5.26, there is an isomorphism of topological rings

$$R^\wedge \cong \varprojlim_{n,J} R_{n,J}.$$

For each $J \in \mathcal{J}_n$, define the homomorphism $v^{J*} : \mathbb{Z}/2[v_1, v_2, \dots][{v_1}^{-1}] \to \mathbb{Z}/2$ by $v^{J*}(v^{J'}) = 1$ if and only if $J = J'$. Then the $v^{J'*}$ for $J' \in \mathcal{J}_{n,J}$ form a basis for the vector space $\mathrm{Hom}(R_{n,J}, \mathbb{Z}/2)$. Thus the collection of all the v^{J*} constitutes a basis for

$$\mathrm{Hom}^{\mathrm{top}}(R^\wedge, \mathbb{Z}/2) = \varinjlim_{n,J} \mathrm{Hom}(R_{n,J}, \mathbb{Z}/2).$$

We claim that $\tilde{\mu}^*(w_{\alpha(J')})$ for $J' \in \mathcal{J}_{n,J}$ also form a basis for $\mathrm{Hom}(R_{n,J}, \mathbb{Z}/2)$. This would obviously complete the proof that $\tilde{\mu}^*$ is an isomorphism.

According to (5.12), $\tilde{\mu}^*$ restricts to a map

$$\tilde{\mu}^* : \mathbb{Z}/2\{w_{\alpha(J')} \mid J' \in \mathcal{J}_{n,J}\} \to \mathbb{Z}/2\{v^{J'*} \mid J' \in \mathcal{J}_{n,J}\}.$$

Using (5.12), an induction on J shows that $\tilde{\mu}^*$ is a surjection and hence an isomorphism, which finishes the proof of the lemma. $\square$

Since H^* is free over $\mathcal{A}$, $H_* \cong \mathbb{Z}/2[\xi_2, \xi_4, \xi_5, \dots] \otimes \mathcal{A}_*$ where $\mathcal{A}_*$ is the $\mathbb{Z}/2$-dual of $\mathcal{A}$.

Lemma 5.28. *There is an isomorphism of $\mathcal{A}$-modules*

$$\phi : \mathbb{Z}/2[\xi_4, \xi_5, \dots] \otimes \mathcal{A}_* \cong S\{1, v_1\}.$$

Note that the left hand side is not closed under multiplication. Even the generators $\phi(\xi_i)$ do not form a polynomial algebra. The lemma only yields a bijection between elements of $\mathbb{Z}/2[\xi_4, \xi_5, \dots]$ and elements of $\mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)$ lying in $S\{1, v_1\}$.

Proof. Since

$$\mathbb{Z}/2[\xi_2] \otimes S\{1, v_1\} \cong H_* \cong \mathbb{Z}/2[\xi_2] \otimes \mathbb{Z}/2[\xi_4, \xi_5, \dots] \otimes \mathcal{A}_*$$

as $\mathcal{A}$-modules, dividing out by the $\mathcal{A}$-submodule $\xi_2 \cdot H_*$ yields

$$S\{1, v_1\} \cong H_*/\xi_2 \cdot H_* \cong \mathbb{Z}/2[\xi_4, \xi_5, \dots] \otimes \mathcal{A}_*$$

as $\mathcal{A}$-modules. □

Let $S_{\mathcal{A}} = \phi(\mathbb{Z}/2[\xi_4, \xi_5, \dots])$. We are now ready to give a full description of the $\mathcal{A}$-module structure of WH^*.

Theorem 5.29. *WH^* is a free $\mathcal{A}$-module with*

$$WH^*/\mathcal{A}^+ WH^* \cong \bigoplus_{k=-\infty}^{\infty} (\xi_2^k \cdot S_{\mathcal{A}})^\vee$$

where $(\xi_2^k \cdot S_{\mathcal{A}})^\vee \subseteq \mathrm{Hom}_{\mathcal{A}}^{\mathrm{top}}(R^\wedge, \mathbb{Z}/2)$ are the $\mathcal{A}$-homomorphisms that vanish on all $\xi_2^l \cdot p$ with $l \neq k$ and $p \in S_{\mathcal{A}}$.

Proof. By Theorem 5.25, $WH^* \cong \mathrm{Hom}^{\mathrm{top}}(R^\wedge, \mathbb{Z}/2)$ where $R^\wedge$ can be thought of as the space of power series in $\xi_2^{\pm 1}$ with coefficients in $S\{1, v_1\}$. In each degree, this is the same as the completion of $R^\wedge$ in the subspaces

$$I_l = \big\{\xi_2^{l-1} p_1 + \xi_2^{l-2} p_2 + \cdots \mid p_k \in S\{1, v_1\}\big\}.$$

Note that I_l is an $\mathcal{A}$-submodule of $R^\wedge$. The composition

$$\xi_2^l \cdot H_* \to R^\wedge \to R^\wedge / I_l$$

is an isomorphism of $\mathcal{A}$-modules where

$$\xi_2^l \cdot H_* \cong \mathbb{Z}/2\{\xi_2^k, k \geq l\} \otimes S_{\mathcal{A}} \otimes \mathcal{A}_*.$$

For each l, there is a split short exact sequence

$$0 \to \xi_2^{l+1} \cdot S\{1, v_1\} \to \xi_2^{l+1} \cdot H_* \to \xi_2^l \cdot H_* \to 0$$

of $\mathcal{A}$-modules. Thus

$$\xi_2^l \cdot H_* \cong \bigoplus_{k=l}^{\infty} \xi_2^k \cdot S\{1, v_1\}.$$

Furthermore, the diagram

$$\begin{tikzcd} \xi_2^{l+1} \cdot H_* \arrow[r] \arrow[d] & R^\wedge/I_{l+1} \arrow[d] \\ \xi_2^l \cdot H_* \arrow[r] & R^\wedge/I_l \end{tikzcd}$$

commutes. This means that as $\mathcal{A}$-modules

$$\begin{aligned} \operatorname{Hom}^{\mathrm{top}}(R^\wedge, \mathbb{Z}/2) &\cong \varinjlim_l (R^\wedge/I_l)^\vee \\ &\cong \varinjlim_l (\xi_2^l \cdot H_*)^\vee \\ &\cong \bigoplus_{k=-\infty}^{\infty} (\xi_2^k \cdot S\{1, v_1\})^\vee \\ &\cong \bigoplus_{k=-\infty}^{\infty} (\xi_2^k \cdot S_{\mathcal{A}})^\vee \otimes \mathcal{A}, \end{aligned}$$

proving the claim. □

Corollary 5.30.

$$\operatorname{Ext}_{\mathcal{A}}^{0,*}(WH^*, \mathbb{Z}/2) \cong \operatorname{Hom}_{\mathcal{A}}^*(WH^*, \mathbb{Z}/2) \cong \prod_{l=-\infty}^{\infty} \xi_2^l \cdot S_{\mathcal{A}}.$$

5.5 The Oriented Case

We conclude this chapter by considering the oriented situation. Hence, WH^* will now denote the oriented version of the cohomology group from Definition 5.2 and H^* will denote $H^*(MTSO)$. In the unoriented case, we have seen that WH^* is a free module, generalizing the situation for H^*. In the oriented case, the generalization from H^* is not so obviuos, but we do get something similar. So far, the results we have for the oriented case are not as complete as those obtained in the unoriented case.

Recall that the $\mathcal{A}$-module H^* is a direct sum of copies of $\mathcal{A}$ and $\mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1$, see e.g. [42]. In particular, the Thom class generates an $\mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1$ summand. This means that one can choose an $\mathcal{A}$-linear projection $H^* \to \mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1$ onto this summand.

There is map $WH^* \to H^* \hat{\otimes} WH^*$ constructed exactly as in the unoriented case. Combining this with the projection $H^* \to \mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1$, we get:

Proposition 5.31. *There is an $\mathcal{A}$-homomorphism*

$$WH^* \to \mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1 \hat{\otimes} WH^*.$$

In H^* the sequence $H^* \xrightarrow{\mathrm{Sq}^1} H^* \xrightarrow{\mathrm{Sq}^1} H^*$ is not exact; in fact, the cohomology is a polynomial algebra on one generator in each dimension divisible by 4. These generators correspond to the $\mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1$ summands of H^*.

In WH^* we do not have a similar phenomenon:

Proposition 5.32. *The sequence $WH^* \xrightarrow{\mathrm{Sq}^1} WH^* \xrightarrow{\mathrm{Sq}^1} WH^*$ is exact.*

Proof. Let $x = \sum_i p_{2i}\tilde{w}_{2i} + \sum_i p_{2i+1}\tilde{w}_{2i+1}$ be some element of WH^*. If $x \in \mathrm{Ker}(\mathrm{Sq}^1)$, then

$$0 = \mathrm{Sq}^1(x) = \sum_i (\mathrm{Sq}^1(p_{2i})\tilde{w}_{2i} + p_{2i}\tilde{w}_{2i+1}) + \sum_i \mathrm{Sq}^1(p_{2i+1})\tilde{w}_{2i+1}.$$

This happens precisely if $p_{2i} = \mathrm{Sq}^1(p_{2i+1})$ for all i. But then

$$x = \sum_i \mathrm{Sq}^1(p_{2i+1})\tilde{w}_{2i} + p_{2i+1}\tilde{w}_{2i+1} = \mathrm{Sq}^1(\sum_i p_{2i+1}\tilde{w}_{2i}),$$

proving the claim. □

However, there are relations. For instance, $\tilde{w}_0$ and $\tilde{w}_4$ are both indecomposable, but

$$\mathrm{Sq}^{2,1,2}(\tilde{w}_0) = \mathrm{Sq}^1(\tilde{w}_4).$$

In particular, there is no chance that WH^* splits as a sum of a free module and some $\mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1$ summands, but we shall see that the Ext-groups behave as if it did. As in the unoriented case, the proof goes by considering WH^* as a module over $\mathcal{A}(n)$.

Lemma 5.33. *Suppose that $x_i \in WH^k$ represent linearly independent elements of $WH^k/\mathcal{A}(n)WH^{<k}$ with $\mathrm{Sq}^1(x_i)$ linearly independent in $WH^{k+1}/\mathcal{A}(n)WH^{<k}$. Then $\sum_i a_i(x_i) \neq 0$ in $WH^*/\mathcal{A}(n)WH^{<k}$ for any $a_i \in \mathcal{A}(n)$.*

Proof. The map in Proposition 5.31 takes $\mathcal{A}(n)WH^{<k}$ to $\mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1 \hat{\otimes} \mathcal{A}(n)WH^{<k}$. Thus there is an $\mathcal{A}(n)$-linear map

$$WH^*/\mathcal{A}(n)WH^{<k} \to \mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1 \otimes WH^*/(\mathcal{A}(n)WH^{<k} + WH^{>k+1}).$$

Here x_i maps to $1 \otimes x_i$ and thus $\mathrm{Sq}^1(x_i)$ maps to $1 \otimes \mathrm{Sq}^1(x_i)$, which is non-zero.

If all $a_i \in \mathcal{A}(n)\,\mathrm{Sq}^1$, then $\sum_i a_i(x_i)$ maps to $\sum_i a_i' \otimes \mathrm{Sq}^1(x_i)$ where $a_i = a_i'\,\mathrm{Sq}^1$ and $a_i' \in \mathcal{A}(n)/\mathcal{A}(n)\,\mathrm{Sq}^1$ is non-zero (otherwise $a_i = 0$). It follows from the assumptions that $\sum_i a_i' \otimes \mathrm{Sq}^1(x_i)$ is non-zero.

If at least one $a_j \notin \mathcal{A}(n)\,\mathrm{Sq}^1$, then $\sum_i a_i(x_i)$ maps to $\sum_i (a_i \otimes x_i + a_i' \otimes \mathrm{Sq}^1(x_i))$ for suitable a_i'. This is non-zero because at least one a_j is non-zero in $\mathcal{A}(n)/\mathcal{A}(n)\,\mathrm{Sq}^1$ and the x_i's are linearly independent. □

Let $M_n^* = WH^*/\mathcal{A}(n)^+WH^*$ be the space of indecomposables. Define the subspace

$$K_n^k = \mathrm{Ker}(\mathrm{Sq}^1 : M_n^k \to WH^{k+1}/\mathcal{A}(n)WH^{<k}).$$

Let D_n^* be any complement so that $M_n^* = D_n^* \oplus K_n^*$ and choose a lift $M_n^* \to WH^*$ of the natural projection. This defines a map

$$\mathcal{A}(n) \otimes M_n^* \to WH^*.$$

This is surjective because $\mathcal{A}(n)$ is finite.

Lemma 5.34.

$$\mathrm{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)D_n^*) \cong K_n^{t-s}.$$

The projection $WH^*/\mathcal{A}(n)D_n^* \to K_n^{t-s}$ *induces an isomorphism*

$$\mathrm{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)D_n^*) \to \mathrm{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, K_n^{t-s}).$$

Proof. We claim that the composition

$$\mathcal{A}(n) \otimes K_n^l \to WH^* \to WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l})$$

induces an injective $\mathcal{A}(n)$-homomorphism

$$\mathcal{A}(n)/\mathcal{A}(n)\,\mathrm{Sq}^1 \otimes K_n^l \to WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l}).$$

Clearly, $\mathcal{A}(n)\,\mathrm{Sq}^1 \otimes K_n^l$ is in the kernel of the map by construction of K_n^l. On the other hand, assume $\sum a_i \otimes k_i$ maps to zero, i.e. there is a relation

$$\sum_i a_i(k_i) = \sum_j b_j(k_j) + \sum_m c_m(d_m) \tag{5.16}$$

for some $k_i \in K_n^l$, $k_j \in K_n^{<l}$, $d_m \in D_n^*$, and $a_i, b_j, c_m \in \mathcal{A}(n)$. Note that there can be no d_m of dimension greater than l by Lemma 5.33.

Consider the map induced by the one in Proposition 5.31

$$WH^* \to (\mathcal{A}/\mathcal{A}\,\mathrm{Sq}^1) \otimes M_n^l.$$

Then the first sum in (5.16) is mapped to $\sum_i a_i \otimes k_i$, the second sum is mapped to 0, and the third sum is mapped to

$$\sum_{\dim d_m = l} c_m \otimes d_m.$$

Thus we get the equality $\sum_i a_i \otimes k_i = \sum_{\dim d_m = l} c_m \otimes d_m$. But $K_n^* \cap D_n^* = 0$, so this can only happen if $a_i, c_l \in \mathcal{A}\,\mathrm{Sq}^1$. But then $a_i\,\mathrm{Sq}^1 = 0$ which implies that $a_i \in \mathcal{A}(n)\,\mathrm{Sq}^1$ since $\mathcal{A}(n)$ is free over $\mathcal{A}(0)$. This proves the claim.

It follows that we have a short exact sequence of $\mathcal{A}(n)$-modules

$$\mathcal{A}(n)/\mathcal{A}(n)\,\mathrm{Sq}^1 \otimes K_n^l \to WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l}) \to WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l+1}).$$

This induces a long exact sequence of Tor-groups

$$\begin{aligned} \ldots \to \mathrm{Tor}_{s,t}^{\mathcal{A}(n)}&(\mathbb{Z}/2^\vee, \mathcal{A}(n)/\mathcal{A}(n)\,\mathrm{Sq}^1 \otimes K_n^l) \\ &\to \mathrm{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l})) \\ &\quad \to \mathrm{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l+1})) \\ &\qquad \to \mathrm{Tor}_{s-1,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, \mathcal{A}(n)/\mathcal{A}(n)\,\mathrm{Sq}^1 \otimes K_n^l) \to \cdots . \end{aligned} \tag{5.17}$$

Note that

$$\mathrm{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, \mathcal{A}(n)/\mathcal{A}(n)\,\mathrm{Sq}^1 \otimes K_n^l) = \begin{cases} K_n^l & \text{for } s = t-l, \\ 0 & \text{otherwise,} \end{cases}$$

as one can see directly from a resolution using the fact that $\mathcal{A}(n)$ is free over $\mathcal{A}(0)$. A direct construction of resolutions also shows that the composition

$$\begin{aligned}\operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, \mathcal{A}(n)/\mathcal{A}(n)\operatorname{Sq}^1 \otimes K_n^l) \to \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l})) \\ \to \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, K_n^l)\end{aligned} \tag{5.18}$$

is injective for all s, so the long exact sequence (5.17) breaks up into short exact sequences. When $s = t - l$, this is the sequence

$$\begin{aligned}0 \to \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, \mathcal{A}(n)/\mathcal{A}(n)\operatorname{Sq}^1 \otimes K_n^l) \\ \to \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l})) \\ \to \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l+1})) \to 0,\end{aligned} \tag{5.19}$$

while for all other s, we get isomorphisms

$$\begin{aligned}\operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l})) \\ \cong \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l+1})).\end{aligned} \tag{5.20}$$

We now compute $\operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)D_n^*)$ for fixed s and t. For $l > t - s$,

$$\operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l})) = 0$$

because $WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l})$ is zero below degree l. Thus, (5.19) becomes an isomorphism

$$\operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, \mathcal{A}(n)/\mathcal{A}(n)\operatorname{Sq}^1 \otimes K_n^{t-s}) \cong \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<t-s})).$$

Note that

$$\operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)D_n^*) \to \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l})) \tag{5.21}$$

is an isomorphism for l sufficiently small. More precisely, the kernel of

$$WH^*/\mathcal{A}(n)D_n^* \to WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<l})$$

is zero above dimension $l + \dim \mathcal{A}(n)$ and thus one can choose a resolution such that the sth term is zero above dimension $l + (s+1)\dim \mathcal{A}(n)$. So if l is so small that $t > l + (s+1)\dim \mathcal{A}(n)$, the Tor-groups of the kernel vanish, and the long exact sequence of Tor-groups yields the isomorphism. In combination with (5.20), this shows that (5.21) is actually an isomorphism for all $l \leq t - s$. Hence the first map in

$$\begin{aligned}\operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)D_n^*) \to \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, WH^*/\mathcal{A}(n)(D_n^* \oplus K_n^{<t-s})) \\ \to \operatorname{Tor}_{s,t}^{\mathcal{A}(n)}(\mathbb{Z}/2^\vee, K_n^{t-s}) \cong K_n^{t-s}\end{aligned}$$

is an isomorphism, and the second is an isomorphism because (5.18) is a composition of isomorphisms when $l = t - s$. □

Lemma 5.35.

$$\operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*) \cong \begin{cases} M_n^{t-s} & \textit{for } s = 0, \\ K_n^{t-s} & \textit{for } s > 0. \end{cases}$$

For all $s > 0$, the projection $WH^* \to K_n^{t-s}$ induces an isomorphism

$$\operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*) \to \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, K_n^{t-s}).$$

Proof. By Lemma 5.33, the map

$$\mathcal{A}(n) \otimes D_n^* \to WH^*$$

is injective, so there is a long exact sequence

$$\begin{aligned} &\to \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, \mathcal{A}(n) \otimes D_n^*) \to \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*) \\ &\qquad \to \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*/(\mathcal{A}(n) \otimes D_n^*)) \to \operatorname{Tor}^{\mathcal{A}(n)}_{s-1,t}(\mathbb{Z}/2^\vee, \mathcal{A}(n) \otimes D_n^*) \to . \end{aligned}$$

The $s = 0$ part is the short exact sequence $0 \to D_n^* \to M_n^* \to K_n^* \to 0$, and for $s > 0$, we get isomorphisms

$$\operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*) \to \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*/(\mathcal{A}(n) \otimes D_n^*)).$$

The claim now follows from Lemma 5.34. □

Define

$$\begin{aligned} M^* &= WH^*/\mathcal{A}^+ WH^* \\ K^k &= \operatorname{Ker}(\mathrm{Sq}^1 : M^k \to WH^*/\mathcal{A}WH^{<k}). \end{aligned}$$

The projection $M_n^* \to M_{n+1}^*$ takes K_n^* to K_{n+1}^*. Clearly,

$$\begin{aligned} M^* &= \varinjlim_n M_n^* \\ K^* &= \varinjlim_n K_n. \end{aligned}$$

Let $h_0 \in \operatorname{Ext}^{1,1}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$ be the generator. Recall from Section 3.1 that multiplication by h_0 induces a map $\operatorname{Ext}^{s,t}_{\mathcal{A}}(WH^*, \mathbb{Z}/2) \to \operatorname{Ext}^{s+1,t+1}_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$.

Theorem 5.36.

$$\operatorname{Tor}_{\mathcal{A}}^{s,t}(\mathbb{Z}/2^\vee, WH^*) \cong \begin{cases} M^t & \textit{for } s = 0, \\ K^{t-s} & \textit{for } s > 0. \end{cases}$$

$$\operatorname{Ext}_{\mathcal{A}}^{s,t}(WH^*, \mathbb{Z}/2) \cong \begin{cases} (M^t)^\vee & \textit{for } s = 0, \\ (K^{t-s})^\vee & \textit{for } s > 0. \end{cases}$$

For $s > 0$,

$$h_0^s : \operatorname{Ext}^{0,t-s}_{\mathcal{A}}(WH^*, \mathbb{Z}/2) \to \operatorname{Ext}^{s,t}_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$$

is the projection $(M^{t-s})^\vee \to (K^{t-s})^\vee$.

Proof. We first determine $\operatorname{Tor}^{\mathcal{A}}_{s,t}(\mathbb{Z}/2^\vee, WH^*)$. Obviously, $\operatorname{Tor}^{\mathcal{A}}_{0,t}(\mathbb{Z}/2^\vee, WH^*) \cong M^t$ by definition. For higher s, we must compute $\varinjlim_n \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*)$, so we need to see what the maps in the direct system look like.

Suppose $M^*_n = D^*_n \oplus K^*_n$ is any splitting and that we have chosen $M^*_n \to WH^*$. Look at the composite map

$$WH^* \to M^{t-s}_n \to K^{t-s}_n.$$

This induces a diagram on Tor-groups

$$\begin{tikzcd}
\operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*) \arrow[r] & \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, M^{t-s}_n) \arrow[r] \arrow[d, "\cong"] & \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, K^{t-s}_n) \arrow[d, "\cong"] \\
 & M^{t-s}_n \arrow[r] & K^{t-s}_n.
\end{tikzcd}$$

The composition $\psi : \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*) \to K^{t-s}_n$ is an isomorphism by Corollary 5.35.

We claim that the image K'_n of $\operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*)$ in M^{t-s}_n is exactly the subspace K^{t-s}_n. Indeed, assume that $x \in K'_n$ is not contained in K^{t-s}_n. Hence we may choose D^{t-s}_n such that $x \in D^{t-s}_n$. Then $\psi(x) = 0$, which is a contradiction. Thus $K'_n \subseteq K^{t-s}_n$, and by surjectivity of ψ they must be equal.

The composition $WH^* \to M^{t-s}_n \to M^{t-s}_{n+1}$ yields a diagram of Tor-groups

$$\begin{tikzcd}
\operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, WH^*) \arrow[r] \arrow[d] & \operatorname{Tor}^{\mathcal{A}(n)}_{s,t}(\mathbb{Z}/2^\vee, M^{t-s}_n) \arrow[d] \\
\operatorname{Tor}^{\mathcal{A}(n+1)}_{s,t}(\mathbb{Z}/2^\vee, WH^*) \arrow[r] & \operatorname{Tor}^{\mathcal{A}(n+1)}_{s,t}(\mathbb{Z}/2^\vee, M^{t-s}_{n+1}).
\end{tikzcd}$$

Here the right vertical map is the projection $M^{t-s}_n \to M^{t-s}_{n+1}$ and the left vertical map is the map of subspaces $K_n \to K_{n+1}$.

Taking the direct limit over n shows that the injection

$$\operatorname{Tor}^{\mathcal{A}}_{s,t}(\mathbb{Z}/2^\vee, WH^*) \to \operatorname{Tor}^{\mathcal{A}}_{s,t}(\mathbb{Z}/2^\vee, M^{t-s})$$

has image exactly K^{t-s}.

Dualizing, we see that

$$\operatorname{Ext}^{s,t}_{\mathcal{A}}(M^{t-s}, \mathbb{Z}/2) \to \operatorname{Ext}^{s,t}_{\mathcal{A}}(WH^*, \mathbb{Z}/2) \tag{5.22}$$

is an isomorphism for $s = 0$ and is exactly the map $(M^{t-s})^\vee \to (K^{t-s})^\vee$ otherwise. But the map (5.22) commutes with multiplication by h_0, and

$$\operatorname{Ext}^{s,t}_{\mathcal{A}}(M^{t-s}, \mathbb{Z}/2) \cong h_0^s \operatorname{Ext}^{0,t-s}_{\mathcal{A}}(M^{t-s}, \mathbb{Z}/2) = h_0^s (M^{t-s})^\vee.$$

$\square$

The above computes the E_2-term of the Adams spectral sequence in Theorem 5.1. As opposed to the unoriented case, there may be differentials in this spectral sequence. A more explicit description of the spaces M^* and K^* would give a better picture of the spectral sequence. So far, we do not have any general results in this direction.

First look at $\mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)$. A direct computation shows that this contains the element $\xi_2^2 \in \mathrm{Hom}^4_{\mathcal{A}}(H^*(MTO), \mathbb{Z}/2)$ that takes the value 1 on both w_4 and w_2^2. The proof of Theorem 5.13 carries over to show that:

Proposition 5.37. *The homomorphism ξ_2^2 is invertible in $\mathrm{Hom}_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$ and this becomes a module over $\mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)[\xi_2^{-2}]$.*

We would like to show that $\mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)[\xi_2^{-2}]$ detect all of M^* as in the unoriented case, since this would imply:

Theorem 5.38. *Assume that $\mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)[\xi_2^{-2}]$ detects all elements of M^*. Then the space K^* is trivial except in dimensions divisible by 4. In particular, the Adams spectral sequence collapses.*

Proof. Let $[x] \in K^*$. Then there is a $\xi \cdot \xi_2^{-2n} \in \mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)[\xi_2^{-2}]$ such that $\xi \cdot \xi_2^{-2n}(x) = 1$. Let $\mathrm{Sq}^1(x) = \sum_i a_i(x_i)$ for suitable $a_i \in \mathcal{A}$. Then for a sufficiently large N

$$\begin{aligned} \xi \cdot \xi_2^{2^{N-1}-2n}(t^{2^N} x) &= 1 \\ \mathrm{Sq}^1(t^{2^N} x) &= \sum_i a_i(t^{2^N} x_i) \end{aligned}$$

where $\xi \cdot \xi_2^{2^{N-1}-2n} \in \mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)$ and multiplication by t is as in Definition 5.5. That is, the image of $t^{2^N} x$ in H^* is indecomposable and $\mathrm{Sq}^1(t^{2^N} x)$ is decomposable over $\mathcal{A}^{>1}$. Hence its dimension must be divisible by 4. Thus also x must have dimension divisible by 4.

This implies that there can only be higher Ext-groups in dimensions divisible by 4. Then it follows from the multiplicative structure that there can be no non-trivial differentials. □

We noted in the beginning of the section that K^* is non-trivial. We can say a bit more about K^* and its compliment. Let

$$B^k = \mathrm{Ker}(\mathrm{Hom}^k_{\mathcal{A}}(WH^*, \mathbb{Z}/2) \to \mathrm{Hom}^k_{\mathcal{A}}(K^*, \mathbb{Z}/2)).$$

If we knew that the spectral sequence collapsed, then B^* would be the torsion subgroup of $\pi_*(\widehat{MT}(d)^\wedge_2)$.

Proposition 5.39. *$\xi_2^2 : \mathrm{Hom}^k_{\mathcal{A}}(WH^*, \mathbb{Z}/2) \to \mathrm{Hom}^{k+4}_{\mathcal{A}}(WH^*, \mathbb{Z}/2)$ restricts to an isomorphism $B^k \to B^{k+4}$ with inverse ξ_2^{-2}.*

Proof. Let $\xi \in B^k$. We claim that $\xi_2^{2n} \cdot \xi \in B^{k+4n}$ for all $n \in \mathbb{Z}$. That is, $\xi_2^{2n} \cdot \xi(x) = 0$ for all $x \in K^{k+4n}$. Note that it is enough to show this for positive n because

$$\xi_2^{2n} \cdot \xi(x) = \xi_2^{2n+2^{N-1}} \cdot \xi(t^{2^N} x)$$

and $x \in K^{k+4n}$ implies $t^{2^N} x \in K^{k+4n+2^N}$ for N large enough.

But for positive n, $\xi_2^{2n} \cdot \xi$ factors as an $\mathcal{A}$-linear map

$$WH^* \to H^* \hat{\otimes} WH^* \xrightarrow{\xi_2^{2n} \otimes \mathrm{id}} (\mathbb{Z}/2)^{4n} \otimes WH^* \xrightarrow{1 \otimes \xi} \mathbb{Z}/2.$$

Assume that x maps to $1 \otimes x'$ in $(\mathbb{Z}/2)^{4n} \otimes WH^*$, and hence $\mathrm{Sq}^1(x)$ maps to $1 \otimes \mathrm{Sq}^1(x')$. But $\mathrm{Sq}^1(x)$ decomposes as $\sum_i a_i(x_i)$, so if x_i maps to $1 \otimes x_i'$, then

$$1 \otimes \mathrm{Sq}^1(x') = \sum_i 1 \otimes a_i(x_i').$$

Thus $\xi(x') = 0$, proving the claim. The proposition now follows because ξ_2^2 restricts to $B^k \to B^{k+4}$ and ξ_2^{-2} restricts to an inverse. □

Define

$$\begin{aligned} K(H)^k &= \mathrm{Ker}(\mathrm{Sq}^1 : H^k/\mathcal{A}H^{<k} \to H^k/\mathcal{A}H^{<k}) \\ B(H)^k &= \mathrm{Ker}(\mathrm{Hom}^k_{\mathcal{A}}(H^*, \mathbb{Z}/2) \to \mathrm{Hom}^k_{\mathcal{A}}(K(H)^*, \mathbb{Z}/2)). \end{aligned}$$

Lemma 5.40. *The map $WH^*/\mathcal{A}^+WH^* \to H^*/\mathcal{A}^+H^*$ maps K^* surjectively onto $K(H)^*$.*

Proof. As representatives for a basis of $K(H)^*$ we may take all products $\prod w_{2k}^{2n_k}$, c.f. [42], Chapter 20. Such a $w^2 = \prod w_{2k}^{2n_k}$ lifts to $w^2\tilde{w}_0 + \mathrm{Sq}^2(w^2)\tilde{w}_{-2}$, which represents an element of M^*. In fact, this lies in K^* because

$$\mathrm{Sq}^1(w^2\tilde{w}_0 + \mathrm{Sq}^2(w^2)\tilde{w}_{-2}) = w^2\tilde{w}_1 + \mathrm{Sq}^2(w^2)\tilde{w}_{-1} = \mathrm{Sq}^2(w^2\tilde{w}_{-1}).$$

Thus $K^* \to K(H)^*$ is surjective. □

Proposition 5.41.

$$\mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)[\xi_2^{-2}] \cap B^* = B(H)^*[\xi_2^{-2}].$$

Proof. The inclusion $\supseteq$ follows from Proposition 5.39.

Assume $\xi_2^{2n} \cdot \xi \in B^*$ for some $\xi \in \mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2)$. Then also $\xi_2^{2n+2N} \cdot \xi \in B^*$, and for N large,

$$\xi_2^{2n+2N} \cdot \xi \in \mathrm{Hom}_{\mathcal{A}}(H^*, \mathbb{Z}/2) \cap B^*.$$

But then $\xi_2^{2n+2N} \cdot \xi$ vanishes on K^*. Thus by Lemma 5.40, $\xi_2^{2n+2N} \cdot \xi \in B(H)^*$. □

Corollary 5.42. *M^*/K^* is infinite in all dimensions and K^* is infinite in all dimensions divisible by 4.*

Corollary 5.43. *The map $MT(d) \to \widehat{MT}(d)^\wedge_2$ induces an injection on the $E^2_{s,t}$-term of the Adams spectral sequences for $t - s \leq d$.*

Proof. This follows on $E_2^{0,t} = \mathrm{Ext}_{\mathcal{A}}^{0,t} = \mathrm{Hom}_{\mathcal{A}}^{0,t}$ because $WH^* \to H^*$ is surjective. For $s > 0$, the map $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*, \mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}}^{s,t}(WH^*, \mathbb{Z}/2)$ is $(K(H)^{t-s})^\vee \to (K^{t-s})^\vee$, which is injective by Lemma 5.40. □

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