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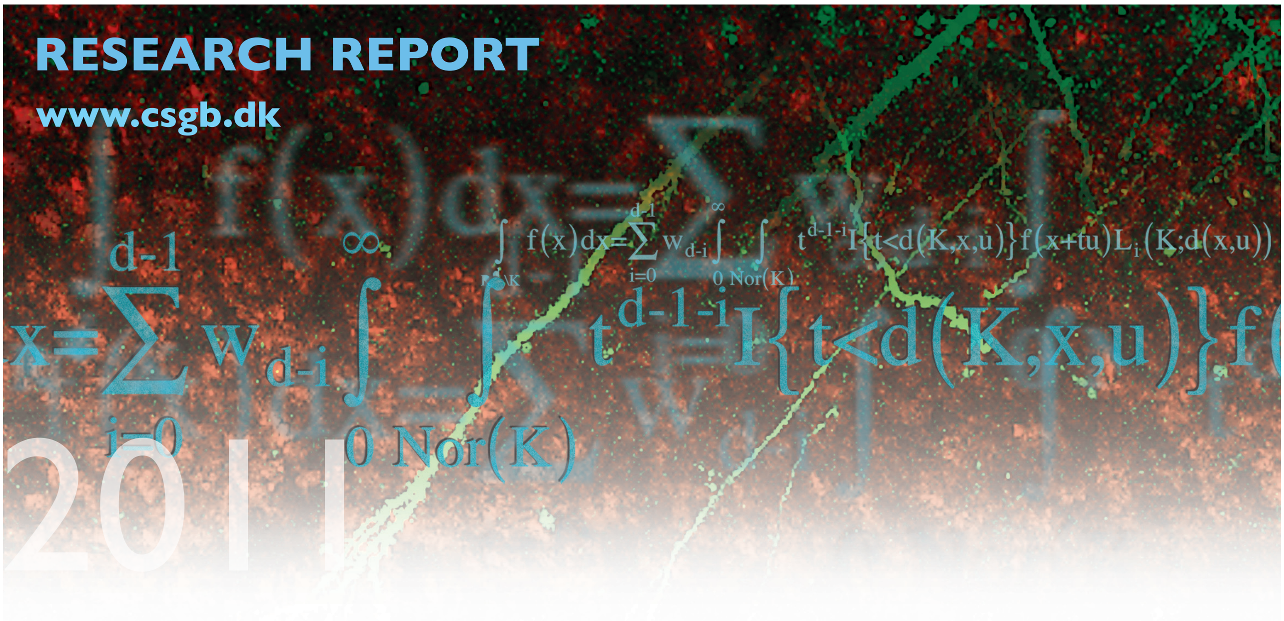

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Abstract

The r-parallel volume $V(C_r)$ of a compact subset C in d-dimensional Euclidean space is the volume of the set C_r of all points of Euclidean distance at most $r > 0$ from C. According to Steiner's formula, $V(C_r)$ is a polynomial in r when C is convex. For finite sets C satisfying a certain geometric condition, a Laurent expansion of $V(C_r)$ for large r is obtained. The dependence of the coefficients on the geometry of C is explicitly given by so-called intrinsic power volumes of C. In the planar case such an expansion holds for all finite sets C. Finally, when C is a compact set in arbitrary dimension, it is shown that the difference of large r-parallel volumes of C and of its convex hull behaves like cr^{d-3}, where c is an intrinsic power volume of C.

Keywords: Large parallel sets, Laurent expansion of parallel volume, Steiner formula, intrinsic power volume.

1 Introduction

In 1840, Jakob Steiner [10] showed that the r-parallel volume of certain compact convex sets in $\mathbb{R}^d$ are polynomials in $r \geq 0$ when $d = 2$ or $d = 3$. The generalization to arbitrary dimensions is now known as the *Steiner formula*

$$V_d(C_r) = \sum_{i=0}^{d} \kappa_{d-i} V_{d-i}(C) r^i, \qquad r \geq 0. \tag{1.1}$$

Here C is any compact convex subset of $\mathbb{R}^d$, C_r is the *(outer) r-parallel set* of C consisting of all points of distance at most r from C, V_d is the Lebesgue measure and $V_i(C)$ are the *intrinsic volumes* of C, which are defined by this relation. A possible proof of (1.1) uses the fact that the intrinsic volumes can be given explicitly in the case where the set C is a convex polytope P:

$$V_i(P) = \sum_{F \in \mathcal{F}_i(P)} \gamma(P, F) V_i(F), \tag{1.2}$$

$i = 0, \dots, d$. Here $\mathcal{F}_i(P)$ is the family of i-dimensional faces of P and $\gamma(P, F)$ is the outer angle of P at F defined in Equation (3.2) below. There are numerous generalizations of the Steiner formula, for instance local versions [8, Chapter 4] and even Steiner-type results for arbitrary closed sets C; see [3]. Often, Steiner-type formulas are only valid for sufficiently small r, e.g. in the case of sets of positive reach [1]. In the present work we focus on r-parallel volumes for large r and determine a Laurent series expansion or at least its leading coefficients. We discuss mainly the parallel volume of *finite* sets, which turns out to be already nontrivial, but Theorem 2, below, deals with compact sets $C \subseteq \mathbb{R}^d$. A first result for large parallel volumes has been obtained in [4] for compact sets $C \subseteq \mathbb{R}^d$, where it is shown that the volume of C_r is close to the volume of its convex hull $\operatorname{conv} C_r = (\operatorname{conv} C)_r$. In fact, there is a constant $c = c(C)$ such that

$$0 \leq V_d(\operatorname{conv} C_r) - V_d(C_r) \leq c r^{d-3} \tag{1.3}$$

for all sufficiently large r. In [4] an example set C was given where this volume difference behaves like r^{d-3}, so the exponent here is best possible. Independently, a weaker version of (1.3) was shown in [7, Lemma 2], where C was assumed to be an at most two-dimensional subset of the set $\{0, 1\}^d$, $d \geq 3$, and the volume difference was shown to converge to zero faster than r^{d-2}. In [7] this was used to obtain an (incomplete) collection of asymptotic Miles-type formulas for the specific intrinsic volumes of stationary digitized Boolean models of balls. It was the original motivation for the present research to complete and generalize these results. One might even ask for similar asymptotic formulas for the specific intrinsic volumes of digitized standard random sets. (Standard random sets are stationary, a.s. locally polyconvex random sets, satisfying a certain integrability condition; see [9, Definition 9.2.1]). While the Boolean model case only requires (truncated) Laurent expansions for the volume of the set C_r, being the Minkowski sum of C with the r-scaled Euclidean unit ball B^d, results for general standard random sets would require such expansions for the volume of the Minkowski sum of C with an arbitrary r-scaled polyconvex set. In [6, Corollary 2.2] the first two leading coefficients of such a Laurent expansion are determined and [6, § 5] discusses its application to the theory of random sets, including the calculation of the one-sided derivative of the contact distribution function at zero. However, higher order expansions in this general setting appear not to be known.

Other applications of our expansions appear to be in reach. For example, in [4] formula (1.3) was used to examine the expected value of the parallel volume of Brownian paths, when the time is small, and to relate analytical properties of $r \mapsto W(rK)$ to geometric properties of K, where W denotes the Wills functional and K is a fixed compact set.

In the next section, the main results will be stated starting with the definition of functionals of finite subsets of $\mathbb{R}^d$ capturing all geometric properties that are relevant when considering large parallel volumes. An optimal lower bound for the constant c in (1.3) will be given in Theorem 2. For finite sets C, we obtain an explicit Laurent expansion in the case $d = 2$ in Proposition 1 and, under an additional condition on C, in Theorem 5 for all $d \geq 2$. Section 3 provides proofs of the main results and discusses in particular the properties of the coefficients in the Laurent expansion.

2 Main results

The r-parallel volumes of one-dimensional compact sets are trivially affine functions for sufficiently large r. We therefore assume $d \geq 2$ throughout the following. In the spirit of (1.2) define

$$V_i^{(m)}(C) = \sum_{F \in \mathcal{F}_i(\operatorname{conv} C)} \gamma(\operatorname{conv} C, F) \int_F d(C \cap F, x)^{m-i}\, dx, \tag{2.1}$$

where $i = 0, \ldots, d$, $m \geq i$, and C is a finite subset of $\mathbb{R}^d$. We call the functionals $V_i^{(m)}$ the *intrinsic power volumes*. Here, integration is understood with respect to i-dimensional Lebesgue measure in the affine hull of F, and $d(C \cap F, x)$ is the smallest Euclidean distance between x and a point in $C \cap F$. Due to (1.2), we have

$$V_i^{(i)}(C) = V_i(\operatorname{conv} C)$$

for all $i = 0, \ldots, d$. The functionals $V_i^{(m)}$, defined on the family of finite subsets of $\mathbb{R}^d$, share many properties with usual intrinsic volumes (see Lemma 9) and are in particular independent of the dimension of the embedding space. The functional $V_1^{(m)}(C)$, where the sum in (2.1) extends over all edges of $\operatorname{conv} C$, is given more explicitly in (3.6). In particular, if C is the set of vertices of a convex polytope,

$$V_1^{(m)}(C) = \frac{1}{m 2^{m-1}} \sum_{F \in \mathcal{F}_1(\operatorname{conv} C)} \gamma(\operatorname{conv} C, F) V_1^m(F) \tag{2.2}$$

$m \geq 1$. In the case of planar finite sets, these (and the classical intrinsic volumes) are the only intrinsic power volumes that occur in a Laurent expansion of large parallel volumes.

Proposition 1. *Let $C \subseteq \mathbb{R}^2$ be a finite set and put $K = \operatorname{conv} C$. Then*

$$V_2(K_r) - V_2(C_r) = 2 \sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n)!!} V_1^{(2n+1)}(C) r^{-(2n-1)}$$

for all sufficiently large r. (The definition of the double factorial $n!!$ is recalled in Equation (3.1) *below.)*

Proposition 1 and Corollary 6 follow from Theorem 5 below. All other results in this section will be shown in Section 3. The leading coefficient of the Laurent expansion in Proposition 1 depends on C through the intrinsic power volume $V_1^{(3)}(C)$. An analogue statement holds when C is only assumed to be compact, and in all dimensions $d \geq 2$. To define $V_1^{(m)}(C)$, $m \geq 1$, also for compact $C \subseteq \mathbb{R}^d$, we choose for any $x \in \mathbb{R}^d$ two points p_x^C and q_x^C in C with x in the line-segment $[p_x^C, q_x^C]$ and $C \cap [p_x^C, q_x^C] = \{p_x^C, q_x^C\}$, whenever such points exist and are unique up to permutation. Otherwise we put $p_x^C = q_x^C = x$. We then define

$$V_1^{(m)}(C) = \frac{1}{\kappa_{d-1}} \int_{\mathbb{R}^d} d(\{p_x^C, q_x^C\}, x)^{m-1}\, C_1(\operatorname{conv} C, dx), \tag{2.3}$$

where $C_1(\operatorname{conv} C, \cdot)$ is the first curvature measure of $\operatorname{conv} C$. If C is finite, then the definitions (2.1) and (2.3) coincide as will be shown in Remark 7.

Theorem 2. *If K is the convex hull of a compact set $C \subseteq \mathbb{R}^d$, then*

$$\lim_{r\to\infty} \frac{V_d(K_r) - V_d(C_r)}{r^{d-3}} = \tfrac{\omega_{d-1}}{2} V_1^{(3)}(C). \tag{2.4}$$

Here ω_i is the surface area of the ($(i-1)$-dimensional) unit sphere in $\mathbb{R}^i$.

A natural question is whether the speed of convergence in Theorem 2 can be determined: is there an $\alpha > 0$ such that, for any compact $C \subseteq \mathbb{R}^d$, there is a constant $c = c(C)$ with

$$\left| \frac{V_d(K_r) - V_d(C_r)}{r^{d-3}} - \tfrac{\omega_{d-1}}{2} V_1^{(3)}(C) \right| \le c r^{-\alpha}$$

for all sufficiently large r? The following proposition (with $f(r) = r^{-\alpha/2}$) shows that already in $\mathbb{R}^2$, such a stability result cannot hold.

Proposition 3. *Let $f : (0,\infty) \to (0,\infty)$ be a continuous bijective map with $\lim_{r\to\infty} f(r) = 0$. Then there is a compact set $C \subseteq \mathbb{R}^2$ and a number $c > 0$ such that*

$$c f(r) \le \left| \frac{V_2(K_r) - V_2(C_r)}{r^{-1}} - V_1^{(3)}(C) \right|$$

for all sufficiently large r. As usual, we have put $K = \operatorname{conv} C$.

However, if the class of compact sets is replaced by the smaller class of all finite sets, a stability result with $\alpha = 1$ can be obtained. (That this speed of convergence is optimal for this class when $d \ge 3$ follows from Theorem 5 below.)

Theorem 4. *Let $C \subseteq \mathbb{R}^d$ be finite and put $K = \operatorname{conv} C$. Then there is a constant $c = c(C)$ such that*

$$\left| \frac{V_d(K_r) - V_d(C_r)}{r^{d-3}} - \tfrac{\omega_{d-1}}{2} V_1^{(3)}(C) \right| \le \frac{c}{r}, \tag{2.5}$$

for sufficiently large r.

In the remainder of this section, C will always be a finite subset of $\mathbb{R}^d$. Under the following condition on the facets (these are the $(d-1)$-dimensional faces) of $\operatorname{conv} C$, we can also obtain a Laurent series expansion of infinite order in higher dimensions, generalizing Proposition 1.

Condition (A). *For all facets G of the polytope $K = \operatorname{conv} C$ and all faces F of G we have*

$$d(C \cap F, x) = d(C \cap G, x) \quad \text{for all } x \in F. \tag{2.6}$$

Clearly (2.6) holds whenever F is a singleton or $F = G$. Hence Condition (A) must only be checked for faces F of dimension between 1 and $d-2$, and is in particular satisfied for all finite sets C in $\mathbb{R}^2$. In $\mathbb{R}^3$, only edges $F \in \mathcal{F}_1(K)$ must be considered. In particular, if C is the vertex set of a simplicial polytope in $\mathbb{R}^3$, Condition (A) is violated if and only if at least one facet of $\operatorname{conv} C$ is a triangle with a strictly obtuse angle.

For instance, Condition (A) is fulfilled if $C \subseteq \mathbb{R}^d$ is the set of vertices of a rectangular cuboid. If C is the set of the vertices of the standard simplex

$$S_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_{i=1}^{d} x_i \leq 1, x_1, \dots, x_d \geq 0\}$$

in $\mathbb{R}^d$, Condition (A) is satisfied if and only if $d \in \{2, 3\}$. This can be shown as follows. We have already seen that Condition (A) is trivially fulfilled when $d = 2$, and it also holds for $d = 3$ as the triangles forming the facets of S_3 do not contain a strictly obtuse angle. For $d \geq 4$ the point $x = (1/3, 1/3, 1/3, 0, \dots, 0) \in \mathbb{R}^d$ is contained in the face

$$F = \{(y_1, y_2, y_3, 0, \dots, 0) \in \mathbb{R}^d : \sum_{i=1}^{3} y_i = 1, y_1, y_2, y_3 \geq 0\}$$

of the facet $G = S_{d-1} \times \{0\}$. Hence (2.6) is violated, as

$$d(C \cap F, x) = \sqrt{\frac{2}{3}} > \sqrt{\frac{1}{3}} = \|x - o\| = d(C \cap G, x).$$

If $C \subseteq \mathbb{R}^d$ satisfies Condition (A), a Laurent expansion for large parallel volumes with explicit coefficients can be shown.

Theorem 5. *Let K be the convex hull of a finite set $C \subseteq \mathbb{R}^d$ that satisfies Condition (A). Then*

$$V_d(K_r) - V_d(C_r) = \sum_{n=3-d}^{\infty} a_n(C) r^{-n}$$

for all sufficiently large r. Here, the coefficients

$$a_n(C) = \sum_{\substack{i=1, \\ 2|(n+d-i)}}^{\min\{d-1, n+d-2\}} (-1)^{(n+d-i+2)/2} \binom{(d-i)/2}{(n+d-i)/2} \kappa_{d-i} V_i^{(n+d)}(C)$$

vanish for all even positive n.

As Condition (A) is satisfied for all *planar* finite sets, Proposition 1 is a direct consequence of Theorem 5. For $d = 3$ the representation in Theorem 5 simplifies considerably.

Corollary 6. *Let K be the convex hull of a finite set $C \subseteq \mathbb{R}^3$ that satisfies Condition (A). Then*

$$V_3(K_r) - V_3(C_r) = \pi V_1^{(3)}(C) + 2 \sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n)!!} V_2^{(2n+2)}(C)\, r^{-(2n-1)}. \tag{2.7}$$

This section is concluded by an example where Corollary 6 is applied to the vertex set

$$C^d = \mathrm{vert}([0,1]^d) = \bigcup_{F\in\mathcal{F}_0([0,1]^d)} F$$

of the unit cube $[0,1]^d \subseteq \mathbb{R}^d$ for $d=3$. It will be shown in Subsection 3.6 below, that (2.7) then becomes

$$V_3([0,1]^3_r) - V_3(C^3_r) = \frac{\pi}{4} + \sum_{n=1}^{\infty} b_n\, r^{-(2n-1)} \tag{2.8}$$

with

$$b_n = \frac{6}{(2n+1)(2n-1)(n+1)4^n}\sum_{i=0}^{n}\frac{(2i-1)!!}{i!}.$$

3 Proofs and auxiliary results

3.1 Notations and the intrinsic power volumes

The following subsection summarizes the notation and discusses basic properties of the intrinsic power volumes. For the reader's convenience we repeat the most important concepts already named in the previous sections.

The Euclidean norm in $\mathbb{R}^d$ is denoted by $\|\cdot\|$, the closed Euclidean unit ball by B^d and its boundary, the $(d-1)$-dimensional unit sphere by S^{d-1}. We write A^{C}, $\mathrm{diam}\,A$, $\mathrm{bd}\,A$, $\mathrm{int}A$, $\mathrm{relint}\,A$ for the set complement, the diameter, the topological boundary, the interior and the relative interior of a set $A \subseteq \mathbb{R}^d$, respectively. Let $d(A,x) = \inf_{y\in A}\|x-y\|$ be the distance of $x\in\mathbb{R}^d$ to $A\subseteq\mathbb{R}^d$. Let $[x,y]$ be the line-segment with endpoints $x,y\in\mathbb{R}^d$. We denote the m-dimensional Hausdorff measure by $\mathcal{H}^m$ and use the volume of the Euclidean unit ball $\kappa_d = \mathcal{H}^d(B^d)$ and the surface area of the $(d-1)$-dimensional unit sphere $\omega_d = \mathcal{H}^{d-1}(S^{d-1})$. Often, we will write $V_d = \mathcal{H}^d$ for the usual Lebesgue measure. The double factorial is defined to be

$$n!! = \begin{cases} n\cdot(n-2)\cdot\dots\cdot 2, & \text{if } n \text{ is even},\\ n\cdot(n-2)\cdot\dots\cdot 1, & \text{if } n \text{ is odd},\end{cases} \tag{3.1}$$

with the usual convention $(-1)!! = 0!! = 1$.

We recall some basic notions from convex geometry; see [8] for details. A *convex body* is a nonempty compact convex set in $\mathbb{R}^d$. For a convex body K let $p(K,x)$ be the *metric projection* of $x\in\mathbb{R}^d$, that is the point in K closest to x. Then $d(K,x) = \|x-p(K,x)\|$ is the distance between x and K. If $x\notin K$,

$$u(K,x) = \frac{x-p(K,x)}{d(K,x)} \in S^{d-1}$$

is the negative projection direction. Let $\mathcal{F}_i(K)$ be the family of all i-dimensional faces of K, $i=1,\dots,d$. We denote the normal cone of K at $F\in\mathcal{F}_i(K)$ by $N(K,F)$ and put $n(K,F) = N(K,F) \cap S^{d-1}$. Then $n(K,F) = \{u(K,x) \in S^{d-1} : x \notin K, p(K,x) \in \operatorname{relint} F\}$ and

$$\gamma(K,F) = \frac{\mathcal{H}^{d-i-1}(n(K,F))}{\omega_{d-i}} \tag{3.2}$$

is the *exterior angle of* K *at* $F \in \mathcal{F}_i(K)$, $i = 0, \dots, d-1$. For completeness we set $\gamma(K,K) = 1$ if K is full-dimensional. We recall that the *support measures* $\Theta_0(K,\cdot), \dots, \Theta_{d-1}(K,\cdot)$ of K are the measures on $\mathbb{R}^d \times S^{d-1}$ satisfying

$$\begin{aligned}\mathcal{H}^d(\{x \in \mathbb{R}^d \setminus K : d(K,x) \le r,\ (p(K,x), u(K,x)) \in \eta\})\\ = \sum_{m=0}^{d-1} \frac{\binom{d-1}{m}}{d-m} r^{d-m} \Theta_m(K,\eta)\end{aligned}$$

for all $r \ge 0$ and all Borel-sets $\eta \subseteq \mathbb{R}^d \times S^{d-1}$. More generally, for any measurable function $f \ge 0$ on $\mathbb{R}^d$ we have

$$\begin{aligned}\int_{\mathbb{R}^d \setminus K} f(x)\,dx = \sum_{m=0}^{d-1} \binom{d-1}{m} \int_{\mathbb{R}^d \times S^{d-1}} \int_0^\infty f(x+su)\\ \times s^{d-m-1}\,ds\,\Theta_m(K, d(x,u)).\end{aligned} \tag{3.3}$$

The support measures are concentrated on the *(generalized) normal bundle*

$$\mathcal{N}(K) = \{(p(K,y), u(K,y)) : y \notin K\} \subseteq (\operatorname{bd} K) \times S^{d-1}.$$

Their total mass is

$$\Theta_m(K, \mathbb{R}^d \times S^{d-1}) = \omega_{d-m} \binom{d-1}{m}^{-1} V_m(K). \tag{3.4}$$

The *area measures* of K are the projections of the support measures to their second component:

$$S_m(K,\omega) = \Theta_m(K, \mathbb{R}^d \times \omega)$$

where ω is a Borel-set in S^{d-1}. The *curvature measures* are their projections on the first component given by

$$C_m(K,\beta) = \Theta_m(K, \beta \times S^{d-1})$$

for Borel-sets $\beta \subseteq \mathbb{R}^d$. If K is a polytope, then

$$\binom{d-1}{m} \Theta_m(K,\eta) = \sum_{F \in \mathcal{F}_m(K)} \int_F \int_{n(K,F)} \mathbf{1}_\eta(x,u)\,d\mathcal{H}^{d-m-1}(u) d\mathcal{H}^m(x) \tag{3.5}$$

for any Borel set $\eta \subseteq \mathbb{R}^d \times S^{d-1}$; see [8, (4.2.2)].

Remark 7. If C is finite, then (3.5) implies

$$C_1(\operatorname{conv} C, \cdot) = \kappa_{d-1} \sum_{F \in \mathcal{F}_1(\operatorname{conv} C)} \gamma(\operatorname{conv} C, F) \mathcal{H}^1(F \cap \cdot),$$

and thus, (2.3) coincides with the definition (2.1) when $i = 1$.

For a convex body $K \subseteq \mathbb{R}^d$ we call

$$F_K(u) = \{x \in K : \langle x, u \rangle = h_K(u)\}$$

the *support set* of K in direction $u \in S^{d-1}$.

Lemma 8. *Let $K \subseteq \mathbb{R}^d$ be a convex body and $m \in \{0, \dots, d-2\}$. Set*

$$A_m = \{u \in S^{d-1} : \dim F_K(u) \geq m+1\}.$$

Then $S_m(K, A_m) = 0$.

Proof. As a straight-forward generalization of [8, Theorem 2.2.9], we obtain

$$\mathcal{H}^{d-m-1}(A_m) = 0.$$

By [8, Theorem 4.6.5] there is a constant a such that

$$S_m(K, \omega) \leq a \mathcal{H}^{d-m-1}(\omega)$$

for each $(\mathcal{H}^{d-m-1}, d-m-1)$-rectifiable set $\omega \subseteq S^{d-1}$, where we just want to mention that zero sets are always rectifiable and refer to [2, 3.2.14] for a complete definition. So, $S_m(K, A_m) = 0$. □

We now summarize properties of the intrinsic power volumes as functionals on the family $\mathcal{E}$ of finite subsets of $\mathbb{R}^d$. In order to get a simplified expression for $V_1(C)$, we define the set

$$\begin{aligned} \mathcal{F}_1^*(C) = \{[x, y] : x \neq y,\ [x, y] \cap C = \{x, y\}, \\ \text{there is } e \in \mathcal{F}_1(\operatorname{conv} C) \text{ with } [x, y] \subseteq e\} \end{aligned}$$

of all refined edges of $\operatorname{conv} C$, where every edge in $\mathcal{F}_1(\operatorname{conv} C)$ is partitioned into line segments such that exactly their endpoints are in C. Note that if $C = \mathcal{F}_0(K)$ for a convex polytope K, then $\mathcal{F}_1^*(C) = \mathcal{F}_1(\operatorname{conv} C)$, but in general this does not hold. For $e \in \mathcal{F}_1^*(C)$ we put $N(\operatorname{conv} C, e) = N(\operatorname{conv} C, \tilde{e})$ and $\gamma(\operatorname{conv} C, e) = \gamma(\operatorname{conv} C, \tilde{e})$, where $\tilde{e}$ is the unique edge of $\operatorname{conv} C$ with $e \subseteq \tilde{e}$.

Lemma 9 (Properties of $V_i^{(m)}$). *Let $i = 1, \dots, d$, and $m \geq i$ be given.*

(a) $V_i^{(i)}(C) = V_i(\operatorname{conv} C)$ *for any* $C \in \mathcal{E}$.

(b) $V_i^{(m)}$ *is homogeneous of degree* m*:*

$$V_i^{(m)}(\alpha C) = \alpha^m V_i^{(m)}(C)$$

for all $\alpha \geq 0$ *and all* $C \in \mathcal{E}$.

(c) $V_i^{(m)}$ *is motion invariant.*

(d) $V_i^{(m)}$ *is independent of the embedding space: Let C be a finite set in $\mathbb{R}^{d'} \subseteq \mathbb{R}^d$ for some $d' < d$. Then $V_i^{(m)}(C)$ for C considered as a subset of $\mathbb{R}^{d'}$ coincides with $V_i^{(m)}(C)$ for the (lower-dimensional) subset C of $\mathbb{R}^d$.*

(e) Simplified expression for $V_1^{(m)}$*: For finite* $C \subseteq \mathbb{R}^d$ *we have*

$$V_1^{(m)}(C) = \frac{1}{2^{m-1}m} \sum_{F \in \mathcal{F}_1^*(\operatorname{conv} C)} \gamma(\operatorname{conv} C, F) V_1(F)^m. \tag{3.6}$$

Thus, $V_i^{(m)}$, as a functional on $\mathcal{E}$, has similar properties as the intrinsic volume on the family of convex bodies. However, in contrast to the latter, $V_i^{(m)}$ is in general not a valuation and not continuous with respect to the Hausdorff metric.

3.2 Proof of Theorem 2

The proof of Theorem 2 is divided in a sequence of lemmas. Several times, the following analytical lemma will be needed. It can be shown using a Taylor expansion of order two.

Lemma 10. *Let* $n \in \mathbb{N}$*,* $0 \le a \le r$*, and* $f_n(r) = r^n - \sqrt{r^2 - a^2}^n$*. Then*

$$\begin{aligned}
\tfrac{1}{2}a^2r^{-1} &\le f_n(r) \le \tfrac{1}{2}a^2r^{-1} + \tfrac{\sqrt{2}}{4}a^4r^{-3}, && \text{if } n = 1, \text{ and } a \le \frac{r}{2},\\
\tfrac{3}{2}a^2r - \tfrac{3\sqrt{2}}{8}a^4r^{-1} &\le f_n(r) \le \tfrac{3}{2}a^2r, && \text{if } n = 3, \text{ and } a \le \frac{r}{2},\\
\tfrac{n}{2}a^2r^{n-2} - \tfrac{n(n-2)}{8}a^4r^{n-4} &\le f_n(r) \le \tfrac{n}{2}a^2r^{n-2}, && \text{if } n \ge 4.
\end{aligned}$$

In short,

$$\left| f_n(r) - \tfrac{n}{2}a^2r^{n-2} \right| \le c_n a^4 r^{n-4} \tag{3.7}$$

where $c_2 = 0$ *and* $c_n = n(n-2)/8$ *for* $n \ge 4$*. Inequality* (3.7) *also holds for* $n = 1$ *with* $c_1 = \sqrt{2}/4$ *and for* $n = 3$ *with* $c_3 = 3\sqrt{2}/8$ *if* $0 \le a \le r/2$*. In particular, the inequality* $f_n(r) \le na^2r^{n-2}$*, which is elementary for* $n = 1$*, holds for all* n*.*

For compact $C \subseteq \mathbb{R}^d$ we now show that the difference between the parallel volume of $K = \operatorname{conv} C$ and the parallel volume of C is approximately

$$I_C(r) = (d-1) \int_{\mathbb{R}^d \times S^{d-1}} \int_0^\infty \mathbf{1}_{K_r \setminus C_r}(x + su) s^{d-2} \, ds \, \Theta_1(K, d(x, u)).$$

Lemma 11. *Let* $C \subseteq \mathbb{R}^d$ *be a compact set and put* $K = \operatorname{conv} C$*. Then there is a constant* $c = c(C)$ *with*

$$0 \le V_d(K_r) - V_d(C_r) - I_C(r) \le c \cdot r^{d-4} \tag{3.8}$$

for all $r \ge \operatorname{diam} C$*.*

Proof. If $x \in K_r \setminus C_r$, then $\operatorname{ext} K \subseteq C$, where $\operatorname{ext} K := \bigcup_{F \in \mathcal{F}_0(K)} F$, implies

$$p(K, x) \notin \operatorname{ext} K, \tag{3.9}$$

and an application of the Pythagorean theorem implies

$$d(K, x) > \sqrt{r^2 - (\operatorname{diam} C)^2}, \tag{3.10}$$

whenever $r \geq \operatorname{diam} C$; see [5, Example 3.2 and Lemma 3.4]. Since $r \geq \operatorname{diam} C$ implies $(K_r \setminus C_r) \cap K = \emptyset$ and (3.9) together with Lemma 8 imply $\Theta_0(K, \{(p(K,x), u(K,x)) \mid x \in K_r \setminus C_r\}) = 0$, we get from (3.3) with $f = \mathbf{1}_{K_r \setminus C_r}$ that

$$\begin{aligned} V_d(K_r) - V_d(C_r) &= V_d(K_r \setminus C_r) \\ &= \sum_{m=1}^{d-1} \binom{d-1}{m} \int_{\mathbb{R}^d \times S^{d-1}} \int_0^\infty \mathbf{1}_{K_r \setminus C_r}(x+su) \\ &\qquad \times s^{d-m-1}\, ds\, \Theta_m(K, d(x,u)). \end{aligned} \tag{3.11}$$

The term on the right hand side of (3.11) corresponding to $m = 1$ is $I_C(r)$. We now consider the summands of the right hand side of (3.11) for which $m \geq 2$. For $d = 2$ no such summands exist. Since all these summands are non-negative for $d \geq 3$, the left inequality of the assertion is shown. The right inequality follows from the fact that – due to equations (3.10) and (3.4) and Lemma 10 – the right hand side of (3.11), without the summand for $m = 1$, is bounded from above by

$$\begin{aligned} &\sum_{m=2}^{d-1} \binom{d-1}{m} \int_{\mathbb{R}^d \times S^{d-1}} \int_{\sqrt{r^2 - (\operatorname{diam} C)^2}}^r s^{d-m-1}\, ds\, \Theta_m(K, d(x,u)) \\ &= \sum_{m=2}^{d-1} \omega_{d-m} \frac{1}{d-m} \left(r^{d-m} - (r^2 - (\operatorname{diam} C)^2)^{(d-m)/2} \right) V_m(K) \\ &\leq \sum_{m=2}^{d-1} \omega_{d-m} (\operatorname{diam} C)^2 r^{d-m-2} V_m(K) \\ &\leq c r^{d-4}, \end{aligned}$$

where $c = \sum_{m=2}^{d-1} \omega_{d-m} (\operatorname{diam} C)^{4-m} V_m(K)$. □

An upper bound for $I_C(r)$ is obtained easily.

Lemma 12. *Let $C \subseteq \mathbb{R}^d$ be a compact set and put $K = \operatorname{conv} C$. If $r \geq 2(\operatorname{diam} C)$, then*

$$I_C(r) \leq \tfrac{\omega_{d-1}}{2} V_1^{(3)}(C) r^{d-3}$$

for $d \geq 3$ and

$$I_C(r) \leq V_1^{(3)}(C) r^{-1} + \tfrac{\sqrt{2}}{4} V_1^{(5)}(C) r^{-3}$$

for $d = 2$.

Proof. Recall $A_1 := \{u \in S^{d-1} \mid \dim F_K(u) \geq 2\}$. For any $(x,u) \in ((\operatorname{bd} K) \times A_1^{\complement}) \cap \mathcal{N}(K)$ the set $F_K(u)$ is a (possibly degenerate) line segment, and thus there are points p and q with $x \in [p,q]$ and $[p,q] \cap C = \{p,q\}$. If $x \notin C$, then these points are (up to permutation) unique and then they are denoted by p_x^C and q_x^C. Otherwise we put $p_x^C = q_x^C = x$. Now

$$\mathbf{1}_{K_r \setminus C_r}(x+su) = \mathbf{1}_{[p_x^C, q_x^C]_r \setminus C_r}(x+su) \leq \mathbf{1}_{[p_x^C, q_x^C]_r \setminus \{p_x^C, q_x^C\}_r}(x+su)$$

holds for all $(x,u) \in ((\mathrm{bd}\, K) \times A_1^{\complement}) \cap \mathcal{N}(K)$ and hence for Θ_1-a.a. $(x,u) \in \mathcal{N}(K)$ by Lemma 8. Setting $a_x = d(\{p_x^C, q_x^C\}, x)$, we get

$$\begin{aligned} I_C(r) &\le (d-1) \int_{\mathbb{R}^d \times S^{d-1}} \int_0^\infty \mathbf{1}_{[p_x^C, q_x^C]_r \setminus \{p_x^C, q_x^C\}_r}(x+su)\, s^{d-2}\, ds\, \Theta_1(K, d(x,u)) \\ &= (d-1) \int_{\mathbb{R}^d \times S^{d-1}} \int_{\sqrt{r^2-a_x^2}}^r s^{d-2}\, ds\, \Theta_1(K, d(x,u)) \\ &= \int_{\mathbb{R}^d} \left(r^{d-1} - \sqrt{r^2 - a_x^2}^{\,d-1} \right) C_1(K, dx). \end{aligned}$$

Lemma 10 with $n = d-1$ and $a = a_x$ yields

$$I_C(r) \le \tfrac{d-1}{2} \int_{\mathbb{R}^d} a_x^2\, C_1(K, dx) r^{d-3}$$

for $d \ge 3$, and

$$I_C(r) \le \tfrac{1}{2} \int_{\mathbb{R}^d} a_x^2\, C_1(K, dx) r^{-1} + \tfrac{\sqrt{2}}{4} \int_{\mathbb{R}^d} a_x^4\, C_1(K, dx) r^{-3}$$

for $d = 2$. In view of (2.3) this shows the assertion. □

We now derive the corresponding asymptotic lower bound for $I_C(r) r^{-(d-3)}$.

Lemma 13. *For any compact $C \subseteq \mathbb{R}^d$ we have*

$$\liminf_{r\to\infty} \frac{I_C(r)}{r^{d-3}} \ge \tfrac{\omega_{d-1}}{2} V_1^{(3)}(C). \tag{3.12}$$

Proof. Let K be the convex hull of C. For $x \in \mathbb{R}^d$ set $\tau_x = (q_x^C - p_x^C)/\|q_x^C - p_x^C\|$ if $p_x^C \neq q_x^C$, and set $\tau_x = o$, otherwise. The following arguments do not depend on the orientation of τ_x. For $(x,u) \in \mathcal{N}(K)$, $\epsilon \ge 0$ and $\delta > 0$ we denote the indicator of the following event by $\zeta(x,u,\epsilon,\delta)$: for any $y \in C$ the implication

$$\langle x-y, u\rangle \le \epsilon \quad \Rightarrow \quad \langle y, \tau_x\rangle < \langle p_x^C, \tau_x\rangle + \delta \;\text{ or }\; \langle y, \tau_x\rangle > \langle q_x^C, \tau_x\rangle - \delta$$

holds. Note that if $p_x^C = q_x^C$ (and in particular if $x \in C$), we have $\zeta(x,u,\epsilon,\delta) = 1$.

Fix $(x,u) \in \mathcal{N}(K)$ with $x \notin C$, and numbers $\delta > 0, \epsilon, s \ge 0$. Put

$$a_x = d(\{p_x^C, q_x^C\}, x) \le \frac{\mathrm{diam}\, C}{2}$$

and $r_\epsilon = \max\{(\mathrm{diam}\, C)^2/(4\epsilon), (\mathrm{diam}\, C)/2\}$. In order to find a lower bound for $I_C(r)$, we will first show the inequality

$$\zeta(x,u,\epsilon,\delta) \mathbf{1}_{\{\sqrt{r^2 - ((a_x-\delta)^+)^2} \le s\}} \le \mathbf{1}_{\{d(C, x+su) \ge r\}}, \tag{3.13}$$

for all $r \ge r_\epsilon$.

So assume $\zeta(x,u,\epsilon,\delta) = 1$ and $\sqrt{r^2 - ((a_x-\delta)^+)^2} \le s$. Next let $y \in C$. If $\langle x-y, u\rangle \le \epsilon$, then $\zeta(x,u,\epsilon,\delta) = 1$ implies

$$d(y, x+su)^2 \ge s^2 + \langle x-y, \tau_x\rangle^2 \ge s^2 + ((a_x-\delta)^+)^2 \ge r^2.$$

If $\langle x-y,u\rangle > \epsilon$, then

$$d(y,x+su)^2 \geq (\langle x-y,u\rangle + s)^2 \geq \left(\epsilon + \sqrt{r^2-((a_x-\delta)^+)^2}\,\right)^2 \geq r^2$$

for all $r \geq r_\epsilon$ by Lemma 10. So $d(C,x+su) \geq r$ for all $r \geq r_\epsilon$, which completes the proof of (3.13).

Now let $\epsilon, \delta > 0$. For all $r \geq r_\epsilon$ Lemma 8 and inequality (3.13) imply

$$\begin{aligned} I_C(r) &= (d-1)\int_{(\operatorname{bd} K)\times A_1^{\complement}} \int_0^r \mathbf{1}_{\{d(C,x+su)\geq r\}} s^{d-2}\, ds\, \Theta_1(K, d(x,u)) \\ &\geq (d-1)\int_{(\operatorname{bd} K)\times A_1^{\complement}} \int_{\sqrt{r^2-((a_x-\delta)^+)^2}}^r \zeta(x,u,\epsilon,\delta) s^{d-2}\, ds\, \Theta_1(K, d(x,u)) \\ &= \int_{(\operatorname{bd} K)\times A_1^{\complement}} \zeta(x,u,\epsilon,\delta)\Big(r^{d-1} - \sqrt{r^2-((a_x-\delta)^+)^2}^{\,d-1}\Big)\Theta_1(K,d(x,u)). \end{aligned}$$

Since

$$\lim_{r\to\infty} \frac{r^{d-1} - \sqrt{r^2-((a_x-\delta)^+)^2}^{\,d-1}}{r^{d-3}} = \tfrac{d-1}{2}((a_x-\delta)^+)^2$$

due to Lemma 10, Fatou's lemma gives

$$\liminf_{r\to\infty} \frac{I_C(r)}{r^{d-3}} \geq \tfrac{d-1}{2}\int_{(\operatorname{bd} K)\times A_1^{\complement}} \zeta(x,u,\epsilon,\delta)\left((a_x-\delta)^+\right)^2 \Theta_1(K,d(x,u)).$$

Now we first let $\epsilon \to 0$ using $\lim_{\epsilon\to 0}\zeta(x,u,\epsilon,\delta) = \zeta(x,u,0,\delta)$, and the monotone convergence theorem. Since $\zeta(x,u,0,\delta)((a_x-\delta)^+)^2 \leq (\operatorname{diam} C)^2$, we can use the dominated convergence theorem to let $\delta \to 0$, and get

$$\liminf_{r\to\infty} \frac{I_C(r)}{r^{d-3}} \geq \tfrac{d-1}{2}\int_{(\operatorname{bd} K)\times A_1^{\complement}} \lim_{\delta\to 0}\zeta(x,u,0,\delta) a_x^2\, \Theta_1(K,d(x,u)).$$

For all $(x,u) \in (\operatorname{bd} K)\times A_1^{\complement}$, we have $\lim_{\delta\to 0}\zeta(x,u,0,\delta) = 1$, and (3.12) follows using (2.3) and Lemma 8. $\square$

Theorem 2 now follows directly from Lemmas 11, 12, and 13.

3.3 Proof of Proposition 3

Without loss of generality, may assume that $f(r) \geq \frac{1}{r}$ holds for all $r \in (0,\infty)$. Put $g : [0,1] \to (-\infty, 0]$, $x \mapsto -x/(6f^{-1}(x))$, and let $S = g(1) - 1$,

$$C = \{(x,g(x)) : x \in [0,1]\} \cup \{(x, g(2-x)) : x \in [1,2]\} \cup \{(0,S),(2,S)\}$$

and

$$C^0 = \{(0,0),(2,0),(0,S),(2,S)\}.$$

For any $r \in \mathbb{R}$, large enough that $f(r) \le 1/3$, we have $r \ge \frac{1}{f(r)} \ge 3$ and

$$\begin{aligned} V_2(C_r) - V_2(C_r^0) &\ge \int_0^1 \max\{y \in \mathbb{R} : (x,y) \in C_r\} \\ &\qquad\qquad - \max\{y \in \mathbb{R} : (x,y) \in C_r^0\}\, dx \\ &\ge \int_0^1 \max\{y \in \mathbb{R} : \|(x,y) - (f(r), g(f(r)))\| \le r\} \\ &\qquad\qquad - \max\{y \in \mathbb{R} : (x,y) \in C_r^0\}\, dx \\ &= \int_0^1 \sqrt{r^2 - (x - f(r))^2} + g(f(r)) - \sqrt{r^2 - x^2}\, dx \\ &\ge \int_0^1 \frac{2xf(r) - f(r)^2}{2\sqrt{r^2 - (x - f(r))^2}} + g(f(r))\, dx \end{aligned}$$

by using Lemma 10 with r, a, and n replaced by $\sqrt{r^2 - (x - f(r))^2}$, $\sqrt{2xf(r) - f(r)^2}$, and 1, respectively. Since $g(f(r)) = -\frac{f(r)}{6r}$, this integral can be estimated from below by

$$\int_0^1 \frac{2xf(r) - f(r)^2}{2r} - \frac{f(r)}{6r}\, dx = \frac{f(r)}{2r} - \frac{f(r)^2}{2r} - \frac{f(r)}{6r} \ge \frac{f(r)}{6r}.$$

Observing that $K = \operatorname{conv} C = \operatorname{conv} C^0$ and $V_1^{(3)}(C) = V_1^{(3)}(C^0)$ we conclude from Proposition 1 that there is a constant $c_1 \ge 0$ with

$$\begin{aligned} &V_1^{(3)}(C)\tfrac{1}{r} - \big(V_2(K_r) - V_2(C_r)\big) \\ &\quad = -\big(V_2((\operatorname{conv} C^0)_r) - V_2(C_r^0) - V_1^{(3)}(C^0)\tfrac{1}{r}\big) + \big(V_2(C_r) - V_2(C_r^0)\big) \\ &\quad \ge -\frac{c_1}{r^3} + \frac{f(r)}{6r} \\ &\quad \ge \frac{f(r)}{12r} \end{aligned}$$

for all sufficiently large r. Hence

$$\frac{V_2(K_r) - V_2(C_r)}{r^{-1}} - V_1^{(3)}(C) \le -\tfrac{1}{12} f(r)$$

for all sufficiently large r and Proposition 3 is shown.

Remark 14. In order to show an statement analogue to Proposition 3 in higher dimensions, one can consider bodies of revolution.

3.4 Proof of Theorem 4

Theorem 4 is a consequence of Lemmas 11, 12 and the following result.

Lemma 15. *For any finite set $C \subseteq \mathbb{R}^d$ there is a constant $c = c(C) > 0$ such that*

$$I_C(r) \ge \tfrac{\omega_{d-1}}{2} V_1^{(3)}(C) r^{d-3} - c r^{d-4} \tag{3.14}$$

for sufficiently large r.

Proof. We have

$$I_C(r) = (d-1)\int_{(\operatorname{bd} K)\times S^{d-1}} \int_0^r \mathbf{1}_{(C_r)^\complement}(x+su)s^{d-2}\,ds\,\Theta_1(K,d(x,u)).$$

As

$$\begin{aligned}\mathbf{1}_{(C_r)^\complement}(x+su) &= \mathbf{1}_{(\{p_x^C,q_x^C\}_r)^\complement}(x+su)\big(1-\mathbf{1}_{(C\setminus\{p_x^C,q_x^C\})_r}(x+su)\big)\\ &\geq \mathbf{1}_{(\{p_x^C,q_x^C\}_r)^\complement}(x+su)\big(1-\sum_{y\in C}\mathbf{1}_{[0,r]}(\|x+su-y\|)\big),\end{aligned}$$

and

$$\mathbf{1}_{(\{p_x^C,q_x^C\}_r)^\complement}(x+su) = \mathbf{1}_{\left(\sqrt{r^2-a_x^2},\infty\right)}(s)$$

(with $a_x = d(\{p_x^C, q_x^C\}, x)$), we get

$$I_C(r) \geq \int_{(\operatorname{bd} K)\times S^{d-1}} \left(r^{d-1} - \sqrt{r^2-a_x^2}^{\,d-1}\right)\Theta_1(K,d(x,u)) - \sum_{y\in C} J_C(r,y). \tag{3.15}$$

Here,

$$\begin{aligned}J_C(r,y) = (d-1)\int_{(\operatorname{bd} K)\times S^{d-1}} \int_0^r &\mathbf{1}_{(\{p_x^C,q_x^C\}_r)^\complement}(x+su)\\ &\times \mathbf{1}_{[0,r]}(\|(x+su)-y\|)s^{d-2}\,ds\,\Theta_1(K,d(x,u)).\end{aligned} \tag{3.16}$$

By Lemma 16 below, $J_C(r,y) = O(r^{d-4})$ for all $y \in C$, as $r\to\infty$, and thus the second term on the right hand side of (3.15) is $O(r^{d-4})$. Lemma 10 with $n = d-1$ and $a = a_x$ together with (2.3) shows that the first term on the right hand side of (3.15) is bounded from below by

$$\tfrac{\omega_{d-1}}{2}V_1^{(3)}(C)r^{d-3} - c_{d-1}\kappa_{d-1}V_1^{(5)}(C)r^{d-5}$$

for all sufficiently large r. Hence (3.14) follows from (3.15). □

Lemma 16. *Let $C \subseteq \mathbb{R}^d$ be finite and $y \in C$. Then there is a constant $c = c(C,y) > 0$ such that*

$$J_C(r,y) \leq cr^{d-4}$$

for all sufficiently large r, where $J_C(r,y)$ is defined by (3.16).

Proof. From (3.16) and (3.5) we get

$$J_C(r,y) = \sum_{e\in\mathcal{F}_1^*(K)} \int_e \int_{n(K,e)} \int_0^r \zeta(x+su)s^{d-2}\,ds\,d\mathcal{H}^{d-2}(u)\,d\mathcal{H}^1(x).$$

where $\zeta(z) = \mathbf{1}_{((\operatorname{relbd} e)_r)^\complement}(z)\mathbf{1}_{[0,r]}(\|z-y\|)$. Note that the relative boundary $\operatorname{relbd} e$ of e consists just of the two endpoints of e. Spherical coordinates in hyperplanes orthogonal to e and Fubini's theorem give

$$J_C(r,y) = \sum_{e\in\mathcal{F}_1^*(K)} \int_{\mathbb{R}^d} \mathbf{1}_e(p(K,z))\mathbf{1}_{n(K,e)}(u(K,z))\zeta(z)\,dz. \tag{3.17}$$

Fix $e \in \mathcal{F}_1^*(K)$ and let x_1 and x_2 be its endpoints. We assume without loss of generality that e contains the origin. Let g be the affine hull of e and let L be the affine hull of e and y. We may assume $y \notin g$, since we cannot have $y \in \operatorname{relint} e$ and we have $\mathbf{1}_e(p(K,z))\mathbf{1}_{n(K,e)}(u(K,z))\zeta(z) = 0$ for all $z \in \mathbb{R}^d$ if $y \in g \setminus \operatorname{relint} e$. Let H^+ be the closed half space containing e in its boundary with normal vector $y - p(g,y)$, such that $y \notin H^+$. Finally let V_y be the Voronoi cell of y with respect to the set $\{x_1, x_2, y\}$. The planar set

$$T = L \cap V_y \cap H^+ \cap (e + g^\perp)$$

is either empty or a bounded triangle. For nonempty T let $\delta > 0$ be the maximal distance from a point of T to $\{x_1, x_2\}$. If $T = \emptyset$ put $\delta = 0$. If we can show that

$$\mathbf{1}_e(p(K,z))\mathbf{1}_{n(K,e)}(u(K,z))\zeta(z) \leq \mathbf{1}_T(z|L)\mathbf{1}_{(\sqrt{r^2-\delta^2},r]}(\|z|L^\perp\|) \tag{3.18}$$

holds for all sufficiently large r, then

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbf{1}_e(p(K,z))\mathbf{1}_{n(K,e)}(u(K,z))\zeta(z)\,dz \\
&\quad\leq \int_{L^\perp}\int_L \mathbf{1}_T(y_1)\mathbf{1}_{(\sqrt{r^2-\delta^2},r]}(\|y_2\|)\,dy_1\,dy_2 \\
&\quad= V_2(T)\cdot\kappa_{d-2}\big(r^{d-2} - \sqrt{r^2-\delta^2}^{\,d-2}\big) \\
&\quad\leq (d-2)\kappa_{d-2}V_2(T)\delta^2 r^{d-4},
\end{aligned}$$

for all sufficiently large r. The last inequality is evident in the case $d = 2$, and follows for $d \geq 3$ from Lemma 10 with $n = d-2$ and $a = \delta$. Bounding all summands in (3.17) in such a way shows the assertion.

It remains to prove (3.18). Assume that the left hand side of (3.18) is one. Then $p(K,z) \in e$, $u(K,z) \in n(K,e)$, $\|z - x_1\| > r$, $\|z - x_2\| > r$, and $\|z - y\| \leq r$. The last three inequalities imply $z \in V_y$ and $z|L \in V_y|L = V_y \cap L$. The convexity of K and $y \in K$ imply $\langle z - p(K,z), y - p(K,z)\rangle \leq 0$. Since both $z - p(K,z)$ and $y - p(g,y)$ are perpendicular to g, this gives $z \in H^+$. Finally,

$$z \in e + N(K,e) \subseteq e + g^\perp$$

gives $z|L \in T$. As $\|z|L^\perp\| \leq \|z - y\| \leq r$ and

$$r^2 < d(\{x_1,x_2\},z)^2 = d(\{x_1,x_2\},z|L)^2 + d(z|L,z)^2 \leq \delta^2 + \|z|L^\perp\|^2,$$

we have $\sqrt{r^2-\delta^2} < \|z|L^\perp\| \leq r$, and (3.18) is shown. □

3.5 Proof of Theorem 5

We first show a key observation: If Condition (A) holds, then the part of the difference set $K_r \setminus C_r$ that is projected on a face F is independent of the points of C outside F.

Lemma 17. *Let $C \subseteq \mathbb{R}^d$ be a finite set satisfying Condition (A). If $K = \operatorname{conv} C$, $m \in \{0, \dots, d\}$, and $F \in \mathcal{F}_m(K)$ then*

$$(K_r \setminus C_r) \cap (F + N(K, F)) = (F_r \setminus (C \cap F)_r) \cap (F + N(K, F)) \tag{3.19}$$

for all sufficiently large r.

Proof. Since

$$K_r \cap (F + N(K, F)) = F_r \cap (F + N(K, F))$$

and $(C \cap F)_r \subseteq C_r$, the set on the left-hand side is contained in the set on the right-hand side. To show the opposite inclusion, let V_y be the Voronoi cell of $y \in C$ with respect to C, let $S_y = V_y \cap (F + N(K, F))$ be the set of all the points in V_y with metric projection in F, and define

$$r_0 = \min\{r \geq 0 : S_y \subseteq (C \cap F)_r \text{ for all } y \in C \text{ with bounded } S_y\}.$$

Let $r > r_0$ and assume

$$x \in (F_r \setminus (C \cap F)_r) \cap (F + N(K, F)).$$

Then, clearly, $x \in K_r$. Moreover, we have $\|x - y\| > r$ for all $y \in C \cap F$. As $\{V_y : y \in C\}$ covers $\mathbb{R}^d$, there is a $y \in C$ with $x \in V_y \cap (F + N(K, F)) = S_y$. The definition of r_0 and $x \notin (C \cap F)_r$ imply that the closed convex set S_y is unbounded. Hence, there is a ray with direction v, say, completely contained in S_y. It follows that $v \in N(K, F)$ and, as the ray is contained in V_y, that $\langle y, v\rangle \geq h_K(v)$. Hence F and y are contained in a supporting hyperplane of K (with normal v), and thus they are contained in some facet G of K. As $x' = p(K, x) \in F$, Condition (A) implies that there is a point $y' \in C \cap F$ with $\|y - x'\| \geq \|y' - x'\|$ and thus

$$\begin{aligned} d(C, x)^2 = \|y - x\|^2 &\geq \|y - x'\|^2 + \|x' - x\|^2 \\ &\geq \|y' - x'\|^2 + \|x' - x\|^2 = \|y' - x\|^2 > r^2, \end{aligned}$$

where the first equality is due to $x \in V_y$, and the last inequality is due to $x \notin (C \cap F)_r$. Hence $x \notin C_r$, which completes the proof of (3.19). □

We now prove Theorem 5. Let $C \subseteq \mathbb{R}^d$ be a finite set that satisfies Condition (A), and set $K = \operatorname{conv} C$. Assume $r > \operatorname{diam} C$. Due to (3.3) with $f = \mathbf{1}_{K_r \setminus C_r}$, (3.9), and (3.5) we have

$$V_d(K_r) - V_d(C_r) = \sum_{m=1}^{d-1} \sum_{F \in \mathcal{F}_m(K)} I_{F,m} \tag{3.20}$$

with

$$I_{F,m} = \int_F \int_{n(K,F)} \int_0^\infty \mathbf{1}_{K_r \setminus C_r}(x + su) s^{d-m-1} \, ds \, d\mathcal{H}^{d-m-1}(u) \, d\mathcal{H}^m(x),$$

$m \in \{1, \dots, d-1\}, F \in \mathcal{F}_m(K)$. Lemma 17 implies

$$\begin{aligned}
I_{F,m} &= \int_F \int_{n(K,F)} \int_0^\infty \mathbf{1}_{F_r \setminus (C \cap F)_r}(x + su) s^{d-m-1} \, ds \, d\mathcal{H}^{d-m-1}(u) \, d\mathcal{H}^m(x) \\
&= \int_F \int_{n(K,F)} \int_{\sqrt{r^2 - d(C \cap F, x)^2}}^r s^{d-m-1} \, ds \, d\mathcal{H}^{d-m-1}(u) \, d\mathcal{H}^m(x) \\
&= \frac{\omega_{d-m}}{d-m} \gamma(K,F) \int_F \left(r^{d-m} - \sqrt{r^2 - d(C \cap F, x)^2}^{\,d-m} \right) \, d\mathcal{H}^m(x).
\end{aligned}$$

Put $a_{F,x} = d(C \cap F, x)$. The binomial series

$$r^{d-m} - \sqrt{r^2 - a_{F,x}^2}^{\,d-m} = r^{d-m} \sum_{k=1}^\infty (-1)^{k+1} \binom{(d-m)/2}{k} \left(\frac{a_{F,x}}{r} \right)^{2k}$$

converges absolutely as $r > \operatorname{diam} C \geq a_{F,x}$. Hence

$$\begin{aligned}
I_{F,m} &= \sum_{k=1}^\infty (-1)^{k+1} \binom{(d-m)/2}{k} \kappa_{d-m} \gamma(K,F) \int_F a_{F,x}^{2k} \, d\mathcal{H}^m(x) r^{d-m-2k} \\
&= \sum_{\substack{n=2-(d-m), \\ 2|(n+d-m)}}^\infty (-1)^{(n+d-m+2)/2} \binom{(d-m)/2}{(n+d-m)/2} \\
&\qquad\qquad \times \kappa_{d-m} \gamma(K,F) \int_F a_{F,x}^{n+d-m} \, d\mathcal{H}^m(x) \, r^{-n}.
\end{aligned}$$

Substitution into (3.20) and definition (2.1) gives

$$\begin{aligned}
V_d(K_r) - V_d(C_r) &= \sum_{m=1}^{d-1} \sum_{\substack{n=2-(d-m), \\ 2|(n+d-m)}}^\infty (-1)^{(n+d-m+2)/2} \binom{(d-m)/2}{(n+d-m)/2} \\
&\qquad\qquad \times \kappa_{d-m} V_m^{(n+d)}(C) \, r^{-n} \\
&= \sum_{n=3-d}^\infty \sum_{\substack{m=1, \\ 2|(n+d-m)}}^{\min\{d-1, n+d-2\}} (-1)^{(n+d-m+2)/2} \binom{(d-m)/2}{(n+d-m)/2} \\
&\qquad\qquad \times \kappa_{d-m} V_m^{(n+d)}(C) \, r^{-n}.
\end{aligned}$$

This concludes the proof of Theorem 5.

3.6 The example of the unit cube.

We show (2.8). In arbitrary dimension d we have

$$\#\mathcal{F}_i([0,1]^d) = 2^{d-i} \binom{d}{d-i}$$

and, using orthogonal projections and symmetry,

$$\gamma([0,1]^d, F) = \gamma([0,1]^{d-i}, \{o\}) = (\#\mathcal{F}_0([0,1]^{d-i}))^{-1} = 2^{-(d-i)},$$

for any $F \in \mathcal{F}_i([0,1]^d)$, $i = 0, \ldots, d-1$. Thus definition (2.1) and a symmetry argument give

$$\begin{aligned} V_i^{(m+i)}(C^d) &= \binom{d}{d-i} \int_{[0,1]^i} d(C_i, x)^m dx \\ &= 2^i \binom{d}{d-i} \int_{[0,1/2]^i} \|x\|^m dx \\ &= 2^{-m} \binom{d}{d-i} \int_{[0,1]^i} \|x\|^m dx. \end{aligned}$$

This implies

$$V_1^{(m+1)}(C^d) = \frac{d}{(m+1)2^m} \tag{3.21}$$

for $m \geq 0$, and, introducing polar coordinates,

$$\begin{aligned} V_2^{(m+2)}(C^d) &= 2^{-m} \binom{d}{d-2} 2 \int_0^{\pi/4} \int_0^{1/\cos\varphi} \|(r \cdot (\cos\varphi, \sin\varphi)\|^m r \, dr \, d\varphi \\ &= \frac{d(d-1)}{(m+2)2^m} \int_0^{\pi/4} \cos^{-(m+2)}(\varphi) \, d\varphi. \end{aligned}$$

We put $d_n := \int_0^{\pi/4} \cos^{-n}(\varphi) \, d\varphi$. Integrating $\int_0^{\pi/4} \cos^{-(n+1)}(\varphi) \cos(\varphi) \, d\varphi$ by parts, we obtain the recurrence

$$(n+1)d_{n+2} = 2^{n/2} + n d_n, \qquad n \geq 0,$$

with starting value $d_0 = \frac{\pi}{4}$. Induction gives

$$d_{2m} = 2^{m-1} \frac{(m-1)!}{(2m-1)!!} \sum_{i=0}^{m-1} \frac{(2i-1)!!}{i!},$$

and we arrive at

$$\begin{aligned} V_2^{(2m+2)}(C^3) &= \frac{3}{(m+1)4^m} d_{2m+2} \\ &= \frac{3(m!)}{(2m+1)!! \, (m+1)2^m} \sum_{i=0}^{m} \frac{(2i-1)!!}{i!}. \end{aligned} \tag{3.22}$$

As C^3 satisfies Condition (A), (2.8) follows by substituting (3.21) with $d = 3$, $m = 2$, and (3.22) into (2.7).

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