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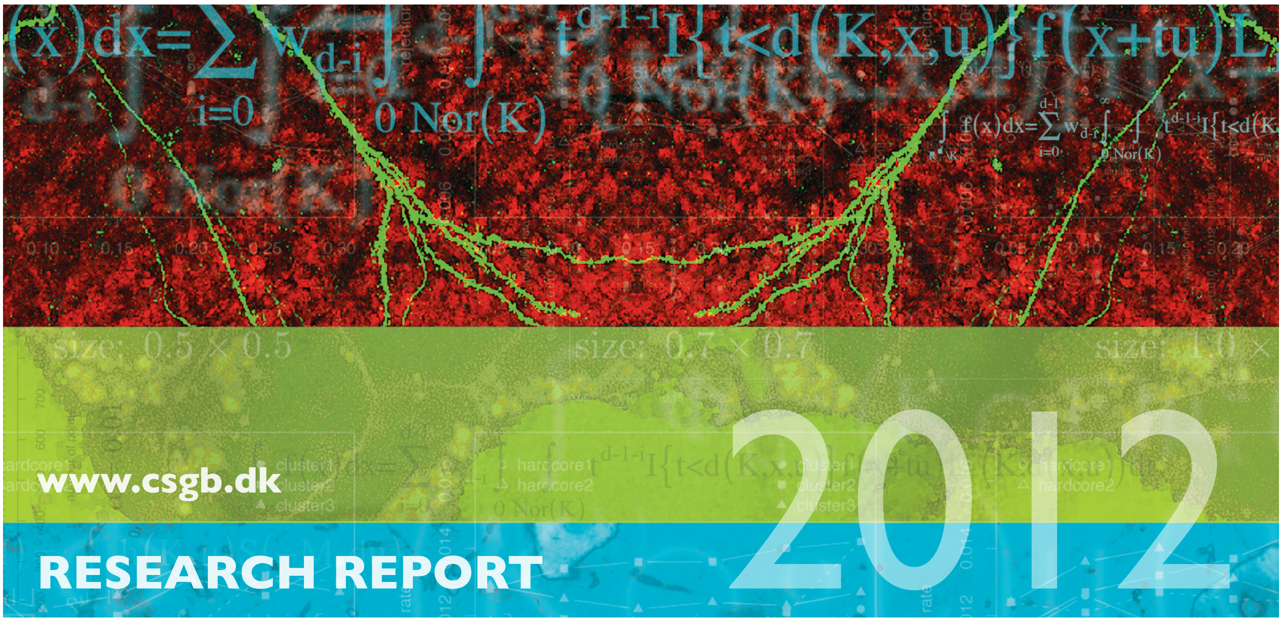

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Local digital estimators of intrinsic volumes for Boolean models and in the design based setting

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Abstract

In order to estimate the specific intrinsic volumes of a planar Boolean model from a binary image, we consider local digital algorithms based on weigted sums of 2×2 configuration counts. For Boolean models with balls as grains, explicit formulas for the bias of such algorithms are derived, resulting in a set of linear equations that the weights must satisfy in order to minimize the bias in high resolution. These results generalize to larger classes of random sets, as well as to the design based situation, where a fixed set is observed on a stationary isotropic lattice. Finally, the formulas for the bias obtained for Boolean models are applied to existing algorithms in order to compare their accuracy.

*Keywords:*Digitization in 2D; intrinsic volumes; local estimators; configurations; Boolean models; design based digitization.

1 Introduction

Let $X \subseteq \mathbb{R}^2$ be a compact subset of the plane. Suppose we are given a digital image of X, i.e. the only information about X available to us is the set $X \cap \mathbb{L}$ where $\mathbb{L} \subseteq \mathbb{R}^2$ is a square lattice. In the language of signal processing, we are thus using an *ideal sampler* to obtain a sample of the characteristic function of X at all the points of $\mathbb{L}$. In image analysis terms, $\mathbb{L}$ can be interpreted as the set of all pixel midpoints and the digitization $X \cap \mathbb{L}$ contains the same information about X as the commonly used Gauss digitization [10, p. 56]. From this binary representation of X, we would like to recover certain geometric properties of X. The quantities we are interested in are the so-called intrinsic volumes V_i. In the plane, these are simply the volume $V_2(X)$, the boundary length $2V_1(X)$, and the Euler characteristic $V_0(X)$. See [14, Chapter 4] for the definition when X is polyconvex.

In this paper, we exclusively consider local digital estimators based on 2×2 configuration counts in a square lattice. Using the additivity of intrinsic volumes, these may be described as follows: The plane is divided into a disjoint union of square cells with vertices in $\mathbb{L}$. For each 2×2 cell in the lattice, each vertex may belong to either X or $\mathbb{R}^2 \backslash X$, yielding $2^4 = 16$ different possible configurations. Each cell contributes to the estimator for $V_i(X)$ with a certain weight depending only on the configuration. Thus the estimator becomes a weighted sum of the configuration counts. The weights can in principle be chosen freely. Algorithms of this type are desirable as they are simple and efficiently implementable based on linearly filtering the image.

One way of testing the quality of local algorithms is by simulations on a fixed test set for various high resolutions, see e.g. [10, Section 10.3.4]. In contrast, we shall follow Ohser, Nagel, and Schladitz in [13], where the algorithms are applied to a standard model from stochastic geometry, namely the Boolean model. But rather than testing a known algorithm, we let the weights be arbitrary and derive conditions on the weights such that the bias of the estimator is minimal for high resolutions.

If the grains are almost surely balls, a Steiner-type result for finite sets shown by Kampf and Kiderlen in [7] yields a general formula for the estimator from which the asymptotic behaviour can be derived. The main result is that a local estimator is asymptotically unbiased if and only if the weights satisfy certain linear equations. Moreover, we obtain formulas for the approximate bias in high resolution. These results are stated in Theorem 4.2 and 4.4 below.

Local estimators are introduced in Section 2. This is specialized to Boolean models in Section 3 and the computations are performed in Section 4.

In Section 5, the main theorems are generalized to a larger class of Boolean models. This relies on a generalization, proved by Kampf in [5], of the formulas obtained in [7] to the case where grains are compact convex sets inside which an ε-ball slides freely. A formula by Kiderlen and Jensen presented in [8] also yields an immediate generalization of the first-order results to general standard random sets, see Section 6.

We then turn to the design based situation where a deterministic set X is observed on a randomly translated and rotated lattice. Under certain conditions on X, we obtain a generalization of the main theorems for Boolean models. This is done for the boundary length in Section 7, using a result of Kiderlen and Rataj from [9], and for the Euler characteristic in Section 8 by a refinement of their approach.

In the litterature, various algorithms for computing intrinsic volumes are suggested. The obtained formulas allow for a computation of the bias in high resolution and hence a comparison of the commonly used algorithms. This is the content of the last section of the paper, Section 9.

2 Local digital estimators

Let $\mathbb{Z}^2$ be the standard lattice in $\mathbb{R}^2$. Let C denote the unit square $[0,1] \times [0,1]$ in $\mathbb{R}^2$ and let C_0 be the set of vertices in C. We enumerate the elements of C_0 as follows: $x_0 = (0,0)$, $x_1 = (1,0)$, $x_2 = (0,1)$, and $x_3 = (1,1)$. A configuration is a subset $\xi \subseteq C_0$. We denote the 16 possible configurations by ξ_l, $l = 0, \ldots, 15$, where the configuration ξ is assigned the index

$$l = \sum_{i=0}^{3} 2^i 1_{x_i \in \xi}.$$

Here $1_{x_i \in \xi}$ is the indicator function.

More generally, we shall consider an orthogonal lattice $a\mathbb{L} = aR_v(\mathbb{Z}^2 + c)$ where $c \in C$ is a translation vector, R_v is the rotation by the angle $v \in [0, 2\pi]$, and $a > 0$ is the lattice distance. The configuration ξ_l is then understood to be the corresponding transformation $aR_v(\xi_l + c)$ of the configuration $\xi_l \subseteq \mathbb{Z}^2$.

The elements of ξ_l are referred to as the 'foreground' or 'black' pixels and will also sometimes be denoted by B_l, while the points in the complement $W_l = C_0 \backslash \xi_l = \xi_{15-l}$ are referred to as the 'background' or 'white' pixels.

The 16 possible configurations are divided into six equivalence classes under rigid motions. These are denoted by η_j for $j = 1, \dots, 6$. These are defined in Table 1.

Table 1: Configuration classes

j	η_j	d_j	Description	Example
1	$\{\xi_0\}$	1	4 white vertices	$\begin{pmatrix} \circ & \circ \\ \circ & \circ \end{pmatrix}$
2	$\{\xi_1, \xi_2, \xi_4, \xi_8\}$	4	3 white and 1 black vertices	$\begin{pmatrix} \circ & \circ \\ \bullet & \circ \end{pmatrix}$
3	$\{\xi_3, \xi_5, \xi_{10}, \xi_{12}\}$	4	2 adjacent white and 2 black vertices	$\begin{pmatrix} \circ & \circ \\ \bullet & \bullet \end{pmatrix}$
4	$\{\xi_6, \xi_9\}$	2	2 opposite white and 2 black vertices	$\begin{pmatrix} \circ & \bullet \\ \bullet & \circ \end{pmatrix}$
5	$\{\xi_7, \xi_{11}, \xi_{13}, \xi_{14}\}$	4	1 white 3 black vertices	$\begin{pmatrix} \bullet & \circ \\ \bullet & \bullet \end{pmatrix}$
6	$\{\xi_{15}\}$	1	4 black vertices	$\begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$

The number d_j is the number of elements in the equivalence class η_j.

Now let $X \subseteq \mathbb{R}^2$ be a compact set. Suppose we observe X on the lattice $a\mathbb{L}$. Based on the set $X \cap a\mathbb{L}$ we want to estimate the intrinsic volumes V_i introduced in Section 1.

In order for the V_i to be well-defined and for the digitization $X \cap a\mathbb{L}$ to carry enough information about X, we require that X is sufficiently 'nice'. The notion of a gentle set is introduced in Section 7 when dealing with V_1. This includes all topologically regular polyconvex sets. When we work with V_0, X will be assumed to be either a compact topologically regular polyconvex set or a compact full-dimensional manifold. A set is called topologically regular if it coincides with the closure of its interior.

Our approach is to consider a local algorithm based on the observations of X on the 2×2 cells of $a\mathbb{L}$. By additivity of the intrinsic volumes, $V_i(X)$ is a sum of contributions from each lattice cell $z + aR_v(C)$ for $z \in a\mathbb{L}$. We estimate this by a certain weight $w^{(i)}(a,z)$, depending only on the information we have about the cell, i.e. the configuration

$$X \cap (z + aR_v(C_0)) - (z-c) = (X - (z-c)) \cap \xi_{15}.$$

Recall here that $\xi_{15} = aR_v(C_0 + c)$ is the set of vertices in the unit cell of $a\mathbb{L}$.

Since V_i is invariant under rigid motions, we would like the estimator to satisfy

$$\hat{V}_i(X) = \hat{V}_i(MX)$$

for any rigid motion M preserving $a\mathbb{L}$. Thus $w^{(i)}(a,z)$ should only depend on the equivalence class η_j of $(X-(z-c)) \cap \xi_{15}$ under rigid motions.

As V_i is homogeneous of degree i, i.e. $V_i(aX) = a^i V_i(X)$, the estimator should also satisfy

$$\hat{V}_i(aX \cap a\mathbb{L}) = a^i \hat{V}_i(X \cap \mathbb{L}).$$

This corresponds to weights of the form $w^{(i)}(a,z) = a^i w_j^{(i)}$ where $w_j^{(i)} \in \mathbb{R}$ are some constants.

We are thus led to consider estimators of the form

$$\hat{V}_i(X) = a^i \sum_{j=1}^{6} w_j^{(i)} N_j$$

where N_j is the number of occurences of the configuration class η_j

$$N_j = \sum_{z \in a\mathbb{L}} 1_{(X-(z-c))\cap\xi_{15}\in\eta_j}.$$

It is also natural to require the estimators to be compatible with interchanging background and foreground as follows:

$$\hat{V}_1(X) = \hat{V}_1(\mathbb{R}^2 \backslash X), \tag{2.1}$$

$$\hat{V}_0(X) = -\hat{V}_0(\mathbb{R}^2 \backslash X). \tag{2.2}$$

The reason for the first condition is that interchanging foreground and background does not change the boundary. The second condition is natural because the Euler characteristic satisfies

$$V_0(X) = -V_0(\overline{\mathbb{R}^2 \backslash X})$$

for both topologically regular compact polyconvex sets, see [12], and compact 2-manifolds with boundary.

3 The 2D Boolean model

Let Ξ be a stationary Boolean model in the plane with compact convex grains and intensity γ. That is,

$$\Xi = \bigcup_i (x_i + K_i)$$

where $\{x_1, x_2, \dots\}$ is a stationary Poisson process in $\mathbb{R}^2$ with intensity γ and the K_i are i.i.d. random compact convex sets in $\mathbb{R}^2$ with distribution $\mathbb{Q}$. We assume throughout that the grain distribution $\mathbb{Q}$ is rotation invariant and hence Ξ is isotropic. See e.g. [15] for more details.

Since the Boolean model is a standard random set in the sense of [15, Definition 9.2.1.], one can define the specific intrinsic volumes. These may be thought of as the mean intrinsic volumes per unit volume. They are defined by

$$\overline{V}_i(\Xi) = \lim_{r\to\infty} \frac{EV_i(\Xi \cap rW)}{V_2(rW)} \tag{3.1}$$

where W is any compact convex set with non-empty interior, see [15].

Now assume that we observe Ξ on a lattice $a\mathbb{L}$ in a compact convex window W with non-empty interior. By the isotropy assumption, we may as well assume the lattice to be the standard lattice $a\mathbb{Z}^2$. Thus we are given the set $\Xi \cap a\mathbb{Z}^2 \cap W$. Based on this, we want to define local estimators for the specific intrinsic volumes.

The limit in (3.1) is introduced to correct for edge effects. However, we only observe in a bounded window. We can get rid of the limit as follows: Let $C_z = z + aC$ be a lattice cell with $z \in a\mathbb{Z}^2$ and let $\partial^+ C_z = z + a([0,1]\times\{1\} \cup \{1\}\times[0,1])$ be the upper right boundary. Write $C_{z,0} = C_z \backslash \partial^+ C_z$ and define

$$V_i(C_{z,0} \cap \Xi) = V_i(C_z \cap \Xi) - V_i(\partial^+ C_z \cap \Xi).$$

Then Theorem 9.2.1. in [15] implies that

$$EV_i(C_{z,0} \cap \Xi) = V_2(C_z)\overline{V}_i(\Xi) = a^2 \overline{V}_i(\Xi).$$

A summation over all lattice cells contained in W yields

$$\overline{V}_i(\Xi) = \sum_{z\in a\mathbb{Z}^2 \cap (W\ominus a\check{C})} \frac{EV_i(C_{z,0}\cap\Xi)}{V_2(C_z)N_0} = \sum_{z\in a\mathbb{Z}^2 \cap (W\ominus a\check{C})} \frac{EV_i(C_{z,0}\cap\Xi)}{a^2 N_0}. \tag{3.2}$$

whenever $N_0 \neq 0$ where N_0 is the total number of points in $a\mathbb{Z}^2 \cap (W \ominus a\check{C})$. Here $\check{C} = \{-x \mid x \in C\}$ and $W \ominus a\check{C} = \{x \in \mathbb{R}^2 \mid x + aC \subseteq W\}$. Thus $a\mathbb{Z}^2 \cap (W \ominus a\check{C})$ contains exactly those z such that C_z is contained in W.

As in Section 2, we estimate each contribution $EV_i(C_{z,0} \cap \Xi)$ by a weight of the form $a^i w_j^{(i)}$ depending on the configuration type η_j. Then (3.2) yields an estimator of the form

$$\hat{V}_i(\Xi) = a^{i-2} \sum_{j=1}^{6} w_j^{(i)} \frac{N_j}{N_0} \tag{3.3}$$

where $w_j^{(i)} \in \mathbb{R}$ are arbitrary weights and the number of configurations N_j are given by

$$N_j = \sum_{z\in a\mathbb{Z}^2 \cap (W\ominus a\check{C})} 1_{(\Xi - z)\cap \xi_{15} \in \eta_j}. \tag{3.4}$$

As opposed to the approach in [13], we make no a priori assumptions on the weights but leave them arbitrary and investigate the behavior of the estimator.

Ideally, $\hat{V}_i$ would define an unbiased estimator, i.e. $E\hat{V}_i(\Xi) = \overline{V}_i(\Xi)$. Generally, this is not possible with finite resolution, i.e. when $a > 0$. Instead, we shall obtain conditions for this to hold asymptotically when the lattice distance tends to zero, that is,

$$\lim_{a \to 0} E\hat{V}_i(\Xi) = \overline{V}_i(\Xi).$$

The mean value of $\hat{V}_i(\Xi)$ is

$$E\hat{V}_i(\Xi) = a^{i-2} \sum_{j=1}^{6} w_j^{(i)} E\left(\frac{N_j}{N_0}\right) = a^{i-2} \sum_{j=1}^{6} w_j^{(i)} P(\Xi \cap aC_0 \in \eta_j) \tag{3.5}$$

by (3.4) and stationarity of Ξ.

For each ξ_l, there are formulas of the form

$$P(\Xi \cap aC_0 = \xi_l) = \sum_{k=0}^{15} b_{lk} P(\xi_k \subseteq \mathbb{R}^2 \backslash \Xi) \tag{3.6}$$

for suitable integers b_{lk}, see also [13]. As Ξ is stationary and isotropic, $P(\Xi \cap aC_0 = \xi_l)$ and $P(\xi_k \subseteq \mathbb{R}^2 \backslash \Xi)$ depend only on the equivalence class of ξ_l and ξ_k under rigid motions. Let ξ_{k_i} and ξ_{l_j} be representatives for η_i and η_j, respectively. Then (3.6) reduces to

$$P(\Xi \cap aC_0 = \xi_{l_j}) = \sum_{i=1}^{6} b'_{ij} P(\xi_{k_i} \subseteq \mathbb{R}^2 \backslash \Xi) \tag{3.7}$$

with the integer b'_{ij} given as the ijth entry in the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & -2 & -2 & 3 & -4 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

The right hand side of (3.7) is now well-known, since

$$P(\xi_k \subseteq \mathbb{R}^2 \backslash \Xi) = e^{-\gamma EV_2(\xi_k \oplus K)} \tag{3.8}$$

where K is a random compact convex set of distribution $\mathbb{Q}$ and $\oplus$ denotes Minkowski addition, see [15]. Thus we only need to describe $EV_2(\xi_k \oplus K)$.

If $F_k = \operatorname{conv}(\xi_k)$ denotes the convex hull of ξ_k, an application of the rotational mean value formula, see [15, Theorem 6.1.1], shows that

$$EV_2(F_k \oplus K) = EV_2(K) + \frac{2}{\pi} V_1(F_k) EV_1(K) + V_2(F_k), \tag{3.9}$$

since the grain distribution is isotropic. In order to apply this, it remains to compute the error

$$EV_2(F_k \oplus K) - EV_2(\xi_k \oplus K). \tag{3.10}$$

4 Boolean models with random balls as grains

We first restrict ourselves to Boolean models where the grains are a. s. balls $B(r)$ of random radius r. For technical reasons we will assume throughout this section that there is an $\varepsilon > 0$ such that $r \geq \varepsilon$ a. s.

In [7, Proposition 1], Kampf and Kiderlen give an expression for the error (3.10). Applied to our situation, this becomes a power series in $\frac{a}{r}$:

$$\begin{aligned} V_2(F_k \oplus B(r)) - V_2(\xi_k \oplus B(r)) &= a^2 V_2\left(a^{-1}(F_k \oplus B(r))\right) - a^2 V_2\left(a^{-1}(\xi_k \oplus B(r))\right) \\ &= 2a^2 \sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n)!!} V_1^{(2n+1)}(a^{-1}\xi_k)\left(\frac{r}{a}\right)^{2n-1} \end{aligned} \tag{4.1}$$

whenever $\frac{a}{r}$ is sufficiently small. The coefficients $V_1^{(m)}(\xi_k)$ are called intrinsic power volumes in [7]. These are defined by

$$V_1^{(m)}(\xi_k) = \frac{1}{m2^{m-1}} \sum_{F \in \mathcal{F}_1(F_k)} \gamma(F_k, F) V_1(F)^m$$

where $\mathcal{F}_1(F_k)$ is the set of 1-dimensional faces of F_k and $\gamma(F_k, F)$ is the outer angle. In the plane, this equals 1 if $\dim F_k = 1$ and $\frac{1}{2}$ if $\dim F_k = 2$. See [7] for the definition of the double factorial. Note that the set $a^{-1}\xi_k$ is independent of a, so the coefficients $V_1^{(2n+1)}(a^{-1}\xi_k)$ are positive constants.

The condition $r \geq \varepsilon$ a. s. ensures that whenever a is sufficiently small, (4.1) holds a. s. Combining this with (3.9), we obtain a power series expansion

$$\begin{aligned} EV_2(\xi_k \oplus B(r)) = EV_2(B(r)) &+ a\frac{2}{\pi}V_1(a^{-1}F_k)EV_1(B(r)) + a^2 V_2(a^{-1}F_k) \\ &- a^3 V_1^{(3)}(a^{-1}\xi_k)E(r^{-1}) + O(a^5). \end{aligned}$$

The constants $V_i(a^{-1}F_k)$ and $V_1^{(3)}(a^{-1}\xi_k)$ can be computed directly for each k. Inserting this in the Taylor expansion for the exponential function in (3.8), yields a power series expansion

$$\begin{aligned} P(\xi_k \subseteq \mathbb{R}^2 \backslash \Xi) = c_1 + \Bigg(c_2 + ac_3\frac{\gamma}{\pi}EV_1(B(r)) + a^2\left(c_4\gamma + c_5\left(\frac{\gamma}{\pi}EV_1(B(r))\right)^2\right) \\ + a^3\left(c_6\gamma E(r^{-1}) + c_7\frac{\gamma^2}{\pi}EV_1(B(r)) + c_8\left(\frac{\gamma}{\pi}EV_1(B(r))\right)^3\right)\Bigg)e^{-\gamma EV_2(B(r))} + O(a^4) \end{aligned} \tag{4.2}$$

for a sufficiently small and constants $c_1, \ldots, c_8$ depending on k. If ξ_{k_j} is a representative for η_j, define A to be the matrix with entry a_{mj} the constant c_m occuring in the formula for $P(\xi_{k_j} \subseteq \mathbb{R}^2 \backslash \Xi)$ for $j = 1, \dots, 6$. A direct computation shows that

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2\sqrt{2} & -(2+\sqrt{2}) & -4 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & 2 & 4 & 3+2\sqrt{2} & 8 \\ 0 & 0 & \frac{1}{12} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}+1}{12} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{2+\sqrt{2}}{2} & 4 \\ 0 & 0 & -\frac{4}{3} & -\frac{8\sqrt{2}}{3} & -\frac{10+7\sqrt{2}}{3} & -\frac{32}{3} \end{pmatrix}.$$

Inserting this in (3.7), we obtain expressions for $P(\Xi \cap aC_0 = \xi_{l_j})$ of the form (4.2) with constants c_m given by the jth column in AB. Then by (3.5), $a^{2-i}E\hat{V}_i(\Xi)$ is also of this form with vector of constants $c^{(i)} = (c_1^{(i)}, \dots, c_8^{(i)})$ given by

$$(c^{(i)})^T = ABD(w^{(i)})^T$$

where $w^{(i)} = (w_1^{(i)}, \dots, w_6^{(i)})$ is the vector of weights and D is the diagonal matrix with jth diagonal entry the number d_j of elements in η_j. Writing this out, we get

$$\begin{aligned} E\hat{V}_i(\Xi) = {} & a^{i-2}\big(c_1^{(i)} + c_2^{(i)} e^{-\gamma EV_2(B(r))}\big) \\ & + a^{i-1} c_3^{(i)} \frac{\gamma}{\pi} EV_1(B(r)) e^{-\gamma EV_2(B(r))} \\ & + a^i \left(c_4^{(i)} \gamma + c_5^{(i)} \left(\frac{\gamma}{\pi} EV_1(B(r))\right)^2 \right) e^{-\gamma EV_2(B(r))} \\ & + a^{i+1} \left(c_6^{(i)} \gamma E(r^{-1}) + c_7^{(i)} \frac{\gamma^2}{\pi} EV_1(B(r)) + c_8^{(i)} \left(\frac{\gamma}{\pi} EV_1(B(r))\right)^3 \right) e^{-\gamma EV_2(B(r))} \\ & + O(a^{i+2}) \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} c_1^{(i)} &= w_6^{(i)} \\ c_2^{(i)} &= w_1^{(i)} - w_6^{(i)} \\ c_3^{(i)} &= 4(-w_1^{(i)} + (2-\sqrt{2})w_2^{(i)} + (-2+2\sqrt{2})w_3^{(i)} + (2-\sqrt{2})w_5^{(i)} - w_6^{(i)}) \\ c_4^{(i)} &= -w_1^{(i)} + 2w_2^{(i)} - 2w_5^{(i)} + w_6^{(i)} \\ c_5^{(i)} &= 4(2w_1^{(i)} + (-5+2\sqrt{2})w_2^{(i)} + (4-4\sqrt{2})w_3^{(i)} + (3-2\sqrt{2})w_4^{(i)} \\ &\qquad + (-7+6\sqrt{2})w_5^{(i)} + (3-2\sqrt{2})w_6^{(i)}) \\ c_6^{(i)} &= \tfrac{1}{6}(w_1^{(i)} + (2\sqrt{2}-2)w_2^{(i)} + (2-4\sqrt{2})w_3^{(i)} + (2\sqrt{2}-2)w_5^{(i)} + w_6^{(i)}) \\ c_7^{(i)} &= 2(2w_1^{(i)} + (-6+\sqrt{2})w_2^{(i)} + (4-2\sqrt{2})w_3^{(i)} + (2-\sqrt{2})w_4^{(i)} \\ &\qquad + (-2+3\sqrt{2})w_5^{(i)} - \sqrt{2}w_6^{(i)}) \\ c_8^{(i)} &= \tfrac{4}{3}(-8w_1^{(i)} + (22-7\sqrt{2})w_2^{(i)} + (-16+14\sqrt{2})w_3^{(i)} + (-6+3\sqrt{2})w_4^{(i)} \\ &\qquad + (10-13\sqrt{2})w_5^{(i)} + (-2+3\sqrt{2})w_6^{(i)}). \end{aligned} \tag{4.4}$$

Note that $c_8^{(i)} = -16c_6^{(i)} - 2c_7^{(i)}$.

We now look for weights $w_j^{(i)}$ such that $\lim_{a\to 0} E\hat{V}_i(\Xi) = \overline{V}_i(\Xi)$. In [15, Theorem 9.1.4], the following formulas for the specific intrinsic volumes are shown:

$$\overline{V}_2(Z) = 1 - e^{-\gamma EV_2(B(r))}, \tag{4.5}$$

$$\overline{V}_1(Z) = \gamma EV_1(B(r))e^{-\gamma EV_2(B(r))}, \tag{4.6}$$

$$\overline{V}_0(Z) = \left(\gamma - \frac{1}{\pi}(\gamma EV_1(B(r)))^2\right) e^{-\gamma EV_2(B(r))}. \tag{4.7}$$

These are truncated expressions of the form (4.3) with fixed constants $c_m^{(i)}$, so the bias of $E\hat{V}_i(\Xi)$ can be found by comparing coefficients.

First consider $\overline{V}_2(\Xi)$. From (4.3) we see that

$$\lim_{a\to 0} E\hat{V}_2(\Xi) = c_1^{(2)} + c_2^{(2)} e^{-\gamma EV_2(B(r))},$$

so by (4.5), we get an asymptotically unbiased estimator for $\overline{V}_2(\Xi)$ exactly if $c_1^{(2)} = 1$ and $c_2^{(2)} = -1$. By Equation (4.4), this means:

Proposition 4.1. *$\hat{V}_2(\Xi)$ is asymptotically unbiased if and only if the weights satisfy $w_1^{(2)} = 0$ and $w_6^{(2)} = 1$.*

It is well known that $\hat{V}_2(\Xi)$ is unbiased, not only asymptotically, with the choice $w^{(2)} = \left(0, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, 1\right)$, since this yields the estimator that computes the area of the approximation of X by a union of squares of sidelength a centered at the foreground points, see e.g. [11].

Next we compare $E\hat{V}_1(\Xi)$, given by (4.3), with (4.6) and obtain:

Theorem 4.2. *The limit $\lim_{a\to 0} E\hat{V}_1(\Xi)$ exists if and only if $c_1^{(1)} = c_2^{(1)} = 0$, or equivalently*

$$w_1^{(1)} = w_6^{(1)} = 0. \tag{4.8}$$

In this case,

$$\lim_{a\to 0} E\hat{V}_1(\Xi) = \frac{1}{\pi} c_3^{(1)} \overline{V}_1(\Xi).$$

In particular, $E\hat{V}_1(\Xi)$ is asymptotically unbiased if and only if the weights satisfy

$$c_3^{(1)} = 4((2-\sqrt{2})w_2^{(1)} + (-2+2\sqrt{2})w_3^{(1)} + (2-\sqrt{2})w_5^{(1)}) = \pi. \tag{4.9}$$

The bias is

$$a\left(c_4^{(1)}\gamma + c_5^{(1)}\left(\frac{\gamma}{\pi}EV_1(B(r))\right)^2\right)e^{-\gamma EV_2(B(r))} + O(a^2),$$

so the estimator converges as $O(a^2)$ exactly if $c_4^{(1)} = c_5^{(1)} = 0$ or equivalently if the weights satisfy:

$$w_2^{(1)} - w_5^{(1)} = 0, \tag{4.10}$$

$$(-5+2\sqrt{2})w_2^{(1)} + (4-4\sqrt{2})w_3^{(1)} + (3-2\sqrt{2})w_4^{(1)} + (-7+6\sqrt{2})w_5^{(1)} = 0. \tag{4.11}$$

If these equations are satisfied, the bias is

$$a^2\left(c_6^{(1)}\gamma E(r^{-1}) + c_7^{(1)}\frac{\gamma^2}{\pi}EV_1(B(r)) + c_8^{(1)}\left(\frac{\gamma}{\pi}EV_1(B(r))\right)^3\right) + O(a^3). \tag{4.12}$$

The first condition (4.8) is intuitive, since lattice cells of type η_1 and η_6 will typically not contain any boundary points. Equation (4.10) is also natural since it is exactly the condition (2.2), saying that interchanging foreground and background in the digital image should not change the value of the estimator. Equation (4.9) is not so obvious. The coefficient in front of $w_j^{(1)}$ in $\frac{1}{8}c_3^{(1)}$ is the asymptotic probability that a lattice square containing a piece of the boundary is of type η_j. Equation (4.11) does not seem to have a simple geometric interpretation. While (4.9) and (4.10) generalize to the design based setting, as we shall see in Section 7 and 8, (4.11) seems to be special for the Boolean model and the underlying distribution.

The equations (4.8), (4.9), (4.10), and (4.11) do not determine the weights uniquely. There is still one degree of freedom in the choice. However, this is not enough to remove the a^2-term in (4.12), since the system of linear equations the weights must satisfy becomes overdetermined. The following proposition gives the best possible choice of weights:

Proposition 4.3. *The complete solution to the system of linear equations* (4.8)*,* (4.9)*,* (4.10)*, and* (4.11) *is*

$$w^{(1)} = \frac{\pi}{16}(0, 1+\sqrt{2}, \sqrt{2}, 12+8\sqrt{2}, 1+\sqrt{2}, 0) + w(0, 1, -\sqrt{2}, -4-4\sqrt{2}, 1, 0)$$

where $w \in \mathbb{R}$ is arbitrary.

In general, the best choice of w depends on the intensity γ and the grain distribution $\mathbb{Q}$. Note that negative weights are allowed, even though this does not have an intuitive geometric interpretation.

Finally for the Euler characteristic, comparing (4.3) with (4.7) yields

Theorem 4.4. *The limit $\lim_{a\to 0} E\hat{V}_0(\Xi)$ exists if and only if $c_1^{(0)} = c_2^{(0)} = c_3^{(0)} = 0$, i.e.*

$$w_1^{(0)} = w_6^{(0)} = 0, \tag{4.13}$$

$$(2-\sqrt{2})w_2^{(0)} + (-2+2\sqrt{2})w_3^{(0)} + (2-\sqrt{2})w_5^{(0)} = 0. \tag{4.14}$$

In this case,

$$\lim_{a\to 0} E\hat{V}_0(\Xi) = \left(c_4^{(0)}\gamma + c_5^{(0)}\left(\frac{\gamma}{\pi}EV_1(B(r))\right)^2\right)e^{-\gamma EV_2(B(r))}$$

so $\hat{V}_0$ is asymptotically unbiased if and only if the following two equations are satisfied

$$c_4^{(0)} = 2w_2^{(0)} - 2w_5^{(0)} = 1, \tag{4.15}$$

$$\begin{aligned} c_5^{(0)} = 4\big(&(2\sqrt{2}-5)w_2^{(0)} + (4-4\sqrt{2})w_3^{(0)} \\ &+ (3-2\sqrt{2})w_4^{(0)} + (6\sqrt{2}-7)w_5^{(0)}\big) = -\pi. \end{aligned} \tag{4.16}$$

If these equations are satisfied, the bias is

$$a\left(c_6^{(0)}\gamma E(r^{-1}) + c_7^{(0)}\frac{\gamma^2}{\pi}EV_1(B(r)) + c_8^{(0)}\left(\frac{\gamma}{\pi}EV_1(B(r))\right)^3\right) + O(a^2). \tag{4.17}$$

Thus the best possible weights are given by:

Proposition 4.5. *The general solution to the linear equations* (4.13)*,* (4.14)*,* (4.15)*, and* (4.16) *is*

$$w^{(0)} = \left(0, \frac{1}{2}, -\frac{1}{2\sqrt{2}}, \left(\frac{3}{4} + \frac{1}{\sqrt{2}}\right)(2-\pi), 0, 0\right) + w\left(0, 1, -\sqrt{2}, -4-4\sqrt{2}, 1, 0\right)$$

with $w \in \mathbb{R}$ *arbitrary.*

Again there is one degree of freedom in the choice of weights, which is not enough to annihilate the leading term of (4.17).

Again the equations (4.13), (4.14), and (4.15) are geometric in the sense that they also show up in the design based setting, while (4.16) seems to be special for the Boolean model.

Also note that $\hat{V}_0$ does not satisfy the condition (2.2), not even asymptotically. For a choice of weights satisfying (4.13),

$$E\hat{V}_0(\Xi) = w_2^{(0)}N_2(\Xi) + w_3^{(0)}N_3(\Xi) + w_4^{(0)}N_4(\Xi) + w_5^{(0)}N_5(\Xi).$$

Since $N_j(\Xi) = N_{7-j}(\mathbb{R}^2\backslash\Xi)$ for $j = 2, 5$, while $N_j(\Xi) = N_j(\mathbb{R}^2\backslash\Xi)$ for $j = 3, 4$,

$$E\hat{V}_0(\mathbb{R}^2\backslash\Xi) = w_2^{(0)}N_5(\Xi) + w_3^{(0)}N_3(\Xi) + w_4^{(0)}N_4(\Xi) + w_5^{(0)}N_2(\Xi).$$

Under the condition (2.2), we would thus have

$$\begin{aligned} 2\overline{V}_0(\Xi) &= \lim_{a\to 0}(E\hat{V}_0(\Xi) - E\hat{V}_0(\mathbb{R}^2\backslash\Xi)) \\ &= \lim_{a\to 0} a^{-2}(w_2^{(0)} - w_5^{(0)})E(N_2 - N_5) \\ &= (w_2^{(0)} - w_5^{(0)})\left(4\gamma + 4(2-4\sqrt{2})\left(\frac{\gamma}{\pi}EV_1(B(r))\right)^2\right)e^{-\gamma EV_2(B(r))} \end{aligned}$$

which no choice of weights can satisfy by (4.7).

The two equations (4.11) and (4.16) become more important compared to (4.10) and (4.15) when r and γ are large. These are the only equations involving the configuration η_4, which can only occur where two different balls are close.

5 General Boolean models

The case where the grains are random balls generalizes to stationary Boolean models where the grain distribution is isotropic and satisfies the following extra condition: there is an $\varepsilon > 0$ such that for almost all grains K, $B(\varepsilon)$ slides freely inside K.

This means that for every $x \in \partial K$ there is a ball of radius ε contained in K and containing x. More formally, the condition is that for almost all K:

$$\forall x \in \partial K : x - \varepsilon n(x) + B(\varepsilon) \subseteq K. \tag{5.1}$$

Here $n(x)$ denotes the (necessarily unique) outward pointing unit normal vector at x. The condition (5.1) is equivalent to $B(\varepsilon)$ being a summand of K, i.e. there is a convex set L s.t. $K = L \oplus B(\varepsilon)$, see [14, Theorem 3.2.2]

Condition (5.1) is a generalization of the assumption $r \geq \varepsilon$ a. s. in the case where the grains are random balls.

Letting $\operatorname{diam}(X)$ denote the diameter of a compact set, we have:

Lemma 5.1. *For any finite set S and convex set K containing $B(\varepsilon)$ as a summand, there is a constant c depending only on* $\operatorname{diam}(K)$, $\operatorname{diam}(S)$, *and ε such that*

$$V_2(\operatorname{conv}(aS) \oplus K) - V_2((aS) \oplus K) \leq ca^3$$

for all $a < 1$.

Proof. It is shown in [5, Lemma 17] that if K has twice differentiable support function, then for all $\lambda \geq 1$,

$$V_2(\operatorname{conv}(S) \oplus \lambda K) - V_2(S \oplus \lambda K) \leq c' \max\{1, \operatorname{diam}(K)\}^2 \lambda^{-1}$$

where c' is a constant depending only on $\operatorname{diam}(S)$ and ε. Taking $\lambda = a^{-1}$ yields the claim in this situation.

For a general $K = L \oplus B(\varepsilon)$, we may approximate L by a sequence L_n of convex bodies with smooth support functions, see [14, Theorem 3.3.1]. Then $L_n \oplus B(\varepsilon)$ converges to $L \oplus B(\varepsilon)$ and $L_n \oplus B(\varepsilon)$ has smooth support function.

According to [5, Lemma 10], the map $B \mapsto V_2(M \oplus B)$ is continuous for M compact and B compact convex with interior points. Hence

$$V_2(\operatorname{conv}(aS) \oplus L_n \oplus B(\varepsilon)) - V_2((aS) \oplus L_n \oplus B(\varepsilon)) \leq c' \max\{1, \operatorname{diam}(L_n \oplus B(\varepsilon))\}^2 a^3$$

for all n implies

$$V_2(\operatorname{conv}(aS) \oplus L \oplus B(\varepsilon)) - V_2((aS) \oplus L \oplus B(\varepsilon)) \leq c' \max\{1, \operatorname{diam}(L \oplus B(\varepsilon))\}^2 a^3$$

by continuity of the diameter function. $\square$

Now let ξ_l be a configuration and write $F_l = \operatorname{conv}(\xi_l)$. Then Lemma 5.1 implies:

Corollary 5.2. *Let Ξ be a Boolean model such that the grains satisfy* (5.1) *almost surely. For $\sqrt{2}a < \varepsilon$ and $l = 0, \ldots, 15$,*

$$EV_2(F_l \oplus C) - EV_2(\xi_l \oplus C) \in O(a^3).$$

This allows us to compute $P(\xi_l \subseteq \mathbb{R}^2\backslash\Xi)$ as in Section 4, but only up to second order:

$$\begin{aligned}
P(\xi_l \subseteq \mathbb{R}^2\backslash\Xi) &= e^{-\gamma EV_2(\xi_l \oplus C)} \\
&= e^{-\gamma EV_2(F_l \oplus C) + O(a^3)} \\
&= e^{-\gamma(EV_2(C) + \frac{2}{\pi}V_1(F_l)EV_1(C) + V_2(F_l) + O(a^3))} \\
&= c_1 + e^{-\gamma EV_2(C)}\Bigg(c_2 + ac_3\frac{\gamma}{\pi}EV_1(C) \\
&\quad + a^2\left(c_4\gamma + c_5\left(\frac{\gamma}{\pi}EV_1(C)\right)^2\right)\Bigg) + O(a^3)
\end{aligned}$$

with exactly the same constants c_m as those in (4.2), since these depend only on $V_i(a^{-1}F_l)$.

Furthermore, for isotropic Boolean models, the specific intrinsic volumes are again given by

$$\begin{aligned}
\overline{V}_2(\Xi) &= 1 - e^{-\gamma EV_2(C)}, \\
\overline{V}_1(\Xi) &= \gamma EV_1(C)e^{-\gamma EV_2(C)}, \\
\overline{V}_0(\Xi) &= \left(\gamma - \frac{1}{\pi}(\gamma EV_1(C))^2\right)e^{-\gamma EV_2(C)},
\end{aligned}$$

so by exactly the same arguments as in Section 4, we find:

Theorem 5.3. *Theorem 4.2 and 4.4, except for Equation* (4.12) *and* (4.17)*, also hold for an isotropic Boolean model with grains satisfying* (5.1) *almost surely.*

6 Generalization to standard random sets

The first-order results for Boolean models generalize further to isotropic standard random sets. This is an easy consequence of well-known results obtained in [8]. A standard random set Z is a stationary random closed set, such that the realizations $Z(\omega)$ are locally polyconvex for a. a. ω. This means that for every compact convex set K, $Z(\omega) \cap K$ is a finite union of convex sets. Furthermore, Z should satisfy the integrability condition

$$E2^{N(Z \cap B(1))} < \infty$$

where $N(Z(\omega) \cap B(1))$ is the minimal number n such that $Z(\omega) \cap B(1)$ is a union of n convex sets. A stationary Boolean model with compact convex grains is an example of a standard random set.

The specific intrinsic volumes of a standard random set are again defined by (3.1) and we estimate $\overline{V}_1$ by

$$\hat{V}_1(Z) = a^{-1}\sum_{j=1}^{6} w_j^{(1)}\frac{N_j}{N_0}$$

as in (3.3) where N_j are defined as in (3.4) with Ξ replaced by Z. Since lower dimensional parts of Z are usually invisible in the digitization, we assume that Z is a. s. topologically regular.

Theorem 6.1. *Let Z be an isotropic standard random set in the plane which is a. s. topologically regular. Then $\lim_{a\to 0} E\hat{V}_1(Z)$ exists if and only if $w_1^{(1)} = w_6^{(1)}$. In this case,*

$$\lim_{a\to 0} E\hat{V}_1(Z) = \frac{1}{\pi} c_3^{(1)} \overline{V}_1(Z)$$

with $c_3^{(1)}$ as in (4.4). In particular, $\hat{V}_1(Z)$ is asymptotically unbiased exactly if (4.9) holds.

Proof. As in the case of the Boolean model,

$$E\hat{V}_1(Z) = a^{-1} \sum_{j=1}^{6} w_j^{(1)} P(Z \cap aC_0 \in \eta_j).$$

First let ξ_l, $l \neq 0, 15$, be a configuration with black and white points $B_l \neq \emptyset$ and $W_l \neq \emptyset$, respectively. Define the support function of a set A to be

$$h(A, n) = \sup\{\langle x, n\rangle \mid x \in \operatorname{conv}(A)\}$$

for $n \in S^1$ and $\langle\cdot,\cdot\rangle$ the standard Euclidean inner product. In [8], the following formula is shown as Theorem 4:

$$\lim_{a\to 0} a^{-1} P(B_l \subseteq Z, W_l \subseteq Z^c) = \int_{S^1} (-h(B_l \oplus \check{W}_l), n)^+ \bar{L}(dn).$$

Here $x^+ = \max\{x, 0\}$ and $\bar{L}$ is the mean normal measure on S^1:

$$\bar{L}(A) = \lim_{r\to\infty} \frac{ES_1(Z \cap B(r); A)}{V_2(B(r))}, \quad A \in \mathcal{B}(S^1),$$

where $S_1(K;\cdot)$ is the length measure of a polyconvex set K, see [14, Chapter 4]. In particular, the total measure $\bar{L}(S^1)$ is $2\overline{V}_1(Z)$.

By the isotropy of Z, $\bar{L}$ is rotation invariant, so Tonelli's theorem yields

$$\begin{aligned}\lim_{a\to 0} a^{-1} P(B_l \subseteq Z, W_l \subseteq Z^c) &= \int_{S^1} (-h(B_l \oplus \check{W}_l, n))^+ \bar{L}(dn)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{S^1} (-h(B_l \oplus \check{W}_l, R_{-v}n))^+ \bar{L}(dn)dv\\ &= \frac{1}{2\pi}\int_{S^1}\int_0^{2\pi} (-h(B_l \oplus \check{W}_l, u_v))^+ dv d\bar{L}\end{aligned}$$

where $u_v = (\cos v, \sin v)$. The inner integral depends only on the equivalence class η_j containing ξ_l. Thus we only need to compute it for one representative ξ_{l_j} of each η_j.

$$(-h(B_1 \oplus \check{W}_1, u_v))^+ = (-h(B_7 \oplus \check{W}_7, v))^+ = \begin{cases} \cos v, & v \in [0, \frac{\pi}{4}],\\ \sin v, & v \in [\frac{\pi}{4}, \frac{\pi}{2}],\\ 0, & \text{otherwise.}\end{cases}$$

$$(-h(B_3 \oplus \check{W}_3, u_v))^+ = \begin{cases} \sin v - \cos v, & v \in [\frac{\pi}{4}, \frac{\pi}{2}],\\ \cos v + \sin v, & v \in [\frac{\pi}{2}, \frac{3\pi}{4}],\\ 0, & \text{otherwise.}\end{cases}$$

$$(-h(B_6 \oplus \check{W}_6, u_v))^+ = 0.$$

A direct computation now shows that

$$\begin{aligned}\lim_{a\to 0} a\sum_{j=2}^{5} w_j^{(1)} EN_j &= \sum_{j=2}^{5} w_j^{(1)} d_j \frac{1}{2\pi}\int_{S^1}\int_0^{2\pi}(-h(B_{l_j}\oplus \check{W}_{l_j}, u_v))^+ dv d\bar{L}\\ &= \frac{1}{2\pi}\int_{S^1}(w_2^{(1)}4(2-\sqrt{2}) + w_3^{(1)}4(-2+2\sqrt{2}) + w_5^{(1)}4(2-\sqrt{2}))d\bar{L}\\ &= \frac{1}{2\pi}c_3^{(1)}2\overline{V}_1(Z).\end{aligned}$$

Finally, it is well-known that

$$\begin{aligned}&\lim_{a\to 0} P(Z\cap aC_0 \in \eta_6) = \overline{V}_2(Z),\\ &\lim_{a\to 0} P(Z\cap aC_0 \in \eta_1) = 1-\overline{V}_2(Z),\end{aligned}$$

so we must choose $w_1^{(1)} = w_6^{(1)} = 0$ in order for $\lim_{a\to 0} E\hat{V}_1(Z)$ to exist for all Z. □

7 Boundary length in the design based setting

Instead of considering random sets observed on a fixed lattice, we now turn to the design based setting. In this situation, we sample a deterministic compact set $X \subseteq \mathbb{R}^2$ with a lattice that has been randomly translated and rotated before making the observation. More formally, we let $\mathbb{L}$ be the random set $\mathbb{L}(c,v) = R_v(\mathbb{Z}^2 + c)$ where $v \in [0, 2\pi]$ and $c \in C$ are mutually independent uniform random variables and R_v denotes the rotation by the angle v.

We first consider estimators for the boundary length $2V_1$, as this is a fairly easy consequence of known results. Based on the digital image $X \cap a\mathbb{L}$, we consider an estimator of the form

$$\hat{V}_1(X) = a\sum_{j=1}^{6} w_j^{(1)} N_j(X\cap a\mathbb{L}), \tag{7.1}$$

as decribed in Section 2. Again we study the asymptotic behavior of $E\hat{V}_1(X)$.

We first need some conditions on X. A compact set $X \subseteq \mathbb{R}^2$ is called gentle if the following two conditions hold:

(i) $\mathcal{H}^1(\mathcal{N}(\partial X)) < \infty$,

(ii) For $\mathcal{H}^1$-almost all $x \in \partial X$, there exist two balls B_i and B_o with non-empty interior, both containing x, and such that $B_i \subseteq X$ and $\mathrm{int}(B_o) \subseteq \mathbb{R}^2\backslash X$.

Here and in the following $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure, and $\mathcal{N}(\partial X)$ is the reduced normal bundle

$$\mathcal{N}(\partial X) = \{(x,n) \in \partial X \times S^1 \mid \exists t > 0 : \forall y \in \partial X : |tn| < |tn + x - y|\}.$$

The last condition means that $(x,n) \in \mathcal{N}(\partial X)$ if there is a $t > 0$ such that x is the point in ∂X closest to $x + tn$.

The following is now a consequence of [9, Theorem 5]:

Theorem 7.1. *Let $X \subseteq \mathbb{R}^2$ be a compact gentle set. Assume that X is observed on a stationary and isotropic random lattice $a\mathbb{L}$ of grid distance a. Then $\lim_{a\to 0} E\hat{V}_1(X)$ exists iff*

$$w_6^{(1)} = w_1^{(1)} = 0.$$

In this case,

$$\lim_{a\to 0} E\hat{V}_1(X) = \frac{1}{\pi} c_3^{(1)} V_1(X)$$

with $c_3^{(1)}$ as in (4.4). In particular, $\hat{V}_1(X)$ is asymptotically unbiased if and only the weights satisfy Equation (4.10).

In Section 8 we shall see that under stronger conditions on X, the convergence is actually $O(a)$ and the weights can be chosen so that it is even $O(a^2)$.

Theorem 5 of [9] is only shown for a uniformly translated lattice, wheras we assume isotropy as well. Thus we need the following lemma.

Lemma 7.2. *For any compact gentle set X, there is a constant $M > 0$ such that for any square lattice $\mathbb{L}$ with unit grid distance,*

$$N_j(X \cap a\mathbb{L}) \le M + 4\sqrt{2} a^{-1} V_1(X)$$

for all $a > 0$ and $j = 2, \ldots, 5$.

Proof. Let $N_\partial(X \cap a\mathbb{L})$ be the number of $z \in a\mathbb{L}$ such that $(z + aR_vC) \cap \partial X \neq \emptyset$. Then

$$\begin{aligned} N_j(X \cap a\mathbb{L}) &\le N_\partial(X \cap a\mathbb{L}) \\ &\le a^{-2} V_2\big(\partial X \oplus B(\sqrt{2}a)\big). \end{aligned}$$

The second inequality holds because $(z + aR_vC) \cap \partial X \neq \emptyset$ implies that

$$z + aR_vC \subseteq \partial X \oplus B\big(\sqrt{2}a\big).$$

It is shown in [4] that for a bounded measurable function f with compact support,

$$\int_{\mathbb{R}^2} f d\mathcal{H}^2 = \sum_{i=1}^{2} i\kappa_i \int_{\mathcal{N}(\partial X)} \int_0^{\delta(\partial X;x,n)} t^{i-1} f(x+tn) dt \mu_{2-i}(\partial X; d(x,n)). \tag{7.2}$$

Here κ_i denotes the volume of the unit ball in $\mathbb{R}^i$ and the μ_i are signed measures on $\mathcal{N}(\partial X)$ with total variation $|\mu_i|$. For $(x,n) \in \mathcal{N}(\partial X)$, the reach is defined by

$$\delta(\partial X; x, n) = \sup\{t > 0 \mid \forall y \in \partial X : |tn| < |tn + x - y|\}.$$

Applying this to the indicator function $1_{\partial X \oplus B(\sqrt{2}a)}$ yields:

$$\begin{aligned} N_j(X \cap a\mathbb{L}) &\le a^{-2} \sum_{i=1}^{2} i\kappa_i \int_{\mathcal{N}(\partial X)} \int_0^{\delta(\partial X;x,n)} t^{i-1} 1_{\partial X \oplus B(\sqrt{2}a)} dt \mu_{2-i}(\partial X; d(x,n)) \\ &\le a^{-2} \sum_{i=1}^{2} i\kappa_i \int_{\mathcal{N}(\partial X)} \int_0^{\delta(\partial X;x,n)} t^{i-1} 1_{\partial X \oplus B(\sqrt{2}a)} dt |\mu_{2-i}|(\partial X; d(x,n)) \\ &\le a^{-2} \sum_{i=1}^{2} \kappa_i \int_{\mathcal{N}(\partial X)} \big(\sqrt{2}a\big)^i |\mu_{2-i}|(\partial X; d(x,n)) \\ &\le 2\pi |\mu_0|(\partial X; \mathcal{N}(\partial X)) + 2\sqrt{2} a^{-1} \mathcal{H}^1(\partial X). \end{aligned}$$

In the last step we used the identity (8) of [9]. It follows from [4, Corollary 2.5] that the total variation $|\mu_0|(\mathcal{N}(\partial X))$ is finite when $\mathcal{H}^1(\mathcal{N}(\partial X)) < \infty$. Hence, if we define $M = 2\pi|\mu_0|(\partial X; \mathcal{N}(\partial X))$, the lemma is proved. $\square$

Proof of Theorem 7.1. Since X is compact, N_1 is always infinite, so $w_1^{(1)}$ must equal zero in order for the estimator to be well-defined. Moreover, $\lim_{a\to 0} a^2 N_6 = V_2(X)$. Thus aN_6 diverges when $a \to 0$, while all other aN_j remain bounded according to Lemma 7.2. Hence the condition $w_6^{(1)} = 0$ is necessary in order for $\lim_{a\to 0} E\hat{V}_1(X)$ to exist.

Let ξ_l be a configuration with $l \neq 0, 15$. Theorem 5 of [9] then reads:

$$\lim_{a\to 0} a \int_C N_l(X \cap a\mathbb{L}(v,c))dc = \int_{S^1} (-h(R_v(B_l) \oplus R_v(\check{W}_l), n))^+ S_1(X; dn)$$

where $S_1(X;\cdot)$ again denotes the first length measure on S^1.

We must compute

$$\lim_{a\to 0} aEN_l(X \cap a\mathbb{L}) = \lim_{a\to 0} a\frac{1}{2\pi}\int_0^{2\pi}\int_C N_l(X \cap a\mathbb{L}(v,c))dcdv.$$

By Lemma 7.2,

$$aN_l(X \cap a\mathbb{L}(v,c)) \leq M'$$

for some constant M' depending only on ∂X. Thus the Lebesgue theorem of dominated convergence applies, and together with Tonelli's theorem it yields:

$$\begin{aligned}
\lim_{a\to 0} aEN_l(X \cap a\mathbb{L}(v,c)) &= \frac{1}{2\pi}\int_0^{2\pi} \lim_{a\to 0} a \int_C N_l(X \cap a\mathbb{L}(v,c))dcdv \\
&= \frac{1}{2\pi}\int_{S^1}\int_0^{2\pi} (-h(R_v(B_l) \oplus R_v(\check{W}_l), n))^+ dv S_1(X; dn) \\
&= \frac{1}{2\pi}\int_{S^1}\int_0^{2\pi} (-h(B_l \oplus \check{W}_l, R_{-v}n))^+ dv S_1(X; dn).
\end{aligned}$$

The claim now follows as in the proof of Theorem 6.1, since $S_1(X; S^1) = 2V_1(X)$. $\square$

Note how the isotropy of the lattice was crucial in the proof. This corresponds to the isotropy requirement for the Boolean model.

8 Euler characteristic in the design based setting

We remain in the design based setting of Section 7 where a deterministic set X is sampled on a stationary and isotropic lattice. We now turn the attention to the Euler characteristic and the higher order behavior of the boundary length estimator. For this, we need to put some stronger boundary conditions on X. For instance, Kampf shows in [5] that if we leave out the random rotation of the lattice, it is impossible to find local estimators for the Euler characteristic that are asymptotically unbiased for all polygons. On the other hand, it is well-known that there exists a local algorithm for the Euler characteristic which is asymptotically unbiased on the class of so-called

r-regular sets, see e.g. the discussion in [16]. We will assume throughout this section that X is a full-dimensional C^2 manifold, which is a special case of an r-regular set.

The estimator for the Euler characteristic was defined in Section 2 as

$$\hat{V}_0(X) = \sum_{j=1}^{6} w_j^{(0)} N_j(X \cap a\mathbb{L}).$$

Note that $a^{-1}\hat{V}_1(X)$, as defined in (7.1), is also of this form with weights $w_j^{(1)}$. To treat both cases, we sometimes just write $w_j^{(i)}$ for the weights.

The main result we shall obtain is the following:

Theorem 8.1. *Assume $X \subseteq \mathbb{R}^2$ is a compact 2-dimensional C^2 submanifold with boundary.*

The limit $\lim_{a\to 0} E\hat{V}_0(X)$ exists if and only if the weights satisfy (4.13) *and* (4.14). *The limit is then given by*

$$\lim_{a\to 0} E\hat{V}_0(X) = c_4^{(0)} V_0(X)$$

with $c_4^{(0)}$ as in (4.4). *Thus $\hat{V}_0(X)$ is asymptotically unbiased if and only if* (4.15) *holds. In this case, $E\hat{V}_0(X)$ satisfies condition* (2.2) *asymptotically.*

Under the condition (4.8), *$E\hat{V}_1(X)$ satisfies*

$$\lim_{a\to 0} a^{-1}(E\hat{V}_1(X) - \lim_{a\to 0} E\hat{V}_1(X)) = c_4^{(1)} V_0(X),$$

so $E\hat{V}_1(X)$ converges as $O(a)$, and if (4.10) *is satisfied, even as $o(a)$. In this case, $\hat{V}_1(X)$ satisfies* (2.1).

Theorem 8.1 generalizes the equations (4.10) and (4.15) to the design based setting. However, the equations (4.11) and (4.16) do not appear. These involve the configuration η_4, which cannot occur when the boundary is C^2 and a is sufficiently small.

It was noted already in Section 7 that we must choose $w_1^{(i)} = 0$ in order for $\hat{V}_i$ to be well-defined and $w_6^{(i)} = 0$ to make $a^{1-i} E\hat{V}_i(X)$ asymptotically convergent. Hence we assume $w_1^{(i)} = w_6^{(i)} = 0$ for the remainder of this section.

For the proof, we must thus compute

$$\sum_{j=2}^{5} w_j^{(i)} EN_j = \sum_{j=2}^{5} w_j^{(i)} \frac{1}{2\pi} \int_0^{2\pi} \int_C N_j(X \cap a\mathbb{L}(c,v)) dc dv.$$

We follow the same approach as in [9]. The idea is that

$$N_j(X \cap a\mathbb{L}(c,v)) = \sum_{l:\xi_l \in \eta_j} \sum_{z \in a\mathbb{L}(c,v)} 1_{\{z + aR_v(B_l) \subseteq X\}} 1_{\{z + aR_v(W_l) \subseteq \mathbb{R}^2 \backslash X\}}.$$

Integrating over all $c \in C$,

$$\begin{aligned}
&\int_C N_j(X \cap a\mathbb{L}(c,v)) dc \\
&\quad = a^{-2} \sum_{l:\xi_l \in \eta_j} \mathcal{H}^2(z \in \mathbb{R}^2 \mid z + aR_v(B_l) \subseteq X, z + aR_v(W_l) \subseteq \mathbb{R}^2 \backslash X) \\
&\quad = a^{-2} \sum_{l:\xi_l \in \eta_j} \int_{\mathbb{R}^2} f_l(z,v) \mathcal{H}^2(dz). \qquad (8.1)
\end{aligned}$$

where f_l denotes the indicator function

$$f_l(z,v) = 1_{\{z + aR_v(B_l) \subseteq X\}} 1_{\{z + aR_v(W_l) \subseteq \mathbb{R}^2 \backslash X\}}. \qquad (8.2)$$

As in the proof of Lemma 7.2, we apply [4, Theorem 2.1] to compute (8.1). By the assumptions on X, there is a unique outward pointing normal vector $n(x)$ at x. Since ∂X is an embedded C^2 submanifold, the tubular neighborhood theorem ensures that there is an $\varepsilon > 0$ such that all points in $\partial X \oplus B(\varepsilon)$ have a unique closest point in ∂X, that is, $\delta(\partial X; x, n(x)) \geq \varepsilon$ for all $x \in \partial X$. For $\sqrt{2}a < \varepsilon$, the support of f_l is contained in $\partial X \oplus B(\varepsilon)$.

Using [4, Corollary 2.5] to describe μ_i, Formula (7.2) applied to f_l simplifies to

$$\begin{aligned}\int_{\mathbb{R}^2} f_l(z,v)\mathcal{H}^2(dz) = &\int_{\partial X}\int_{-\varepsilon}^{\varepsilon} t f_l(x+tn,v)k(x)dt\mathcal{H}^1(dx)\\ &+\int_{\partial X}\int_{-\varepsilon}^{\varepsilon} f_l(x+tn,v)dt\mathcal{H}^1(dx)\end{aligned} \tag{8.3}$$

where $k(x)$ is the signed curvature at x.

The main part of the proof of Theorem 8.1 is now contained in Lemma 8.4 and 8.5, handling each of the two integrals in (8.3). Before proving these, we show two technical lemmas that will become useful later. The first auxiliary lemma describes the boundary structure of X.

Let $\tau(x)$ denote the unit tangent vector at x chosen such that $\{\tau(x), n(x)\}$ are positively oriented.

Lemma 8.2. *Let $X \subseteq \mathbb{R}^2$ be a C^2 submanifold with boundary. For some $\delta < 0$, there is a well-defined C^1 function $l : [-2\delta, 2\delta] \times \partial X \to \mathbb{R}$ such that $l(r,x)$ is the signed length of the line segment parallel to $n(x)$ from $x + r\tau(x)$ to ∂X. The sign is chosen such that $x + r\tau(x) + l(r,x)n(x) \in \partial X$. Moreover, the functions*

$$\frac{l(br,x)}{r}, \frac{l(br,x)}{r^2}$$

are bounded and continuous for $(b,r,x) \in [-2,2] \times [-\delta,\delta]\backslash\{0\} \times \partial X$ and

$$\begin{aligned}\lim_{r\to 0}\frac{l(br,x)}{r} &= 0,\\ \lim_{r\to 0}\frac{l(br,x)}{r^2} &= -\tfrac{1}{2}b^2k(x).\end{aligned}$$

Proof. By the assumptions on X, there are finitely many isometric C^2 parametrizations of the form $\alpha : (a-2\mu, b+2\mu) \to \partial X$ such that the sets $\alpha([a,b])$ cover ∂X. For any $t \in (a-2\mu, b+2\mu)$, let $n(t)$ denote the outward pointing unit normal vector at $\alpha(t)$. There are unique functions $l, r : (-\mu,\mu) \times (a-\mu, b+\mu) \to \mathbb{R}$ such that for any $(s,t) \in (-\mu,\mu) \times (a-\mu, b+\mu)$,

$$\alpha(s+t) - \alpha(t) = r(s,t)\alpha'(t) + l(s,t)n(t)$$

where

$$\begin{aligned}r(s,t) &= \langle \alpha(s+t) - \alpha(t), \alpha'(t)\rangle,\\ l(s,t) &= \langle \alpha(s+t) - \alpha(t), n(t)\rangle.\end{aligned}$$

In particular, note that both functions are C^1, and as functions of s they are even C^2. In an open neighborhood of $[a,b] \times 0$, $\frac{\partial}{\partial s}r(s,t) > 0$. By the inverse function theorem applied to $(r(s,t),t)$, there is a δ such that the inverse $s(r,t)$ is defined and is C^1 on $(-3\delta,3\delta)\times[a,b]$. In fact, $r\mapsto s(r,t)$ is C^2 as it is the inverse of $s\mapsto r(s,t)$. Then $l(s(r,t),t)$ is the distance from $\alpha(t)+r\alpha'(t)$ to $\alpha(s(r,t)+t)$. If $3\delta<\varepsilon$, this is the boundary point on the line parallel to $n(t)$ closest to $\alpha(t)+r\alpha'(t)$.

By the mean value theorem,

$$\begin{aligned}
\frac{l(s(br,t),t)}{r} &= b\frac{\partial}{\partial s}l(s,t)\Big|_{s=s(br_0,t)}\frac{\partial}{\partial r}s(r,t)\Big|_{r=br_0},\\
\frac{l(s(br,t),t)}{r^2} &= b^2\frac{r_0}{r}\frac{\partial^2}{\partial s^2}l(s,t)\Big|_{s=s(br_1,t)}\frac{\partial}{\partial r}s(r,t)\Big|_{r=br_0}\frac{\partial}{\partial r}s(r,t)\Big|_{r=br_1},
\end{aligned}\tag{8.4}$$

for some $0\le|r_1|\le|r_0|\le|r|$. The continuity of $\frac{\partial}{\partial s}l$, $\frac{\partial^2}{\partial s^2}l$ and $\frac{\partial}{\partial r}s$ on $[-2\delta,2\delta]\times[a,b]$ implies that (8.4) is bounded on $[-2,2]\times[-\delta,\delta]\backslash\{0\}\times[a,b]$.

Finally, since $l(s(0,t),t)=0$ and $\frac{\partial}{\partial s}l(s,t)\mid_{s=0}=0$, we obtain

$$\begin{aligned}
\lim_{r\to0}\frac{l(s(br,t),t)}{r} &= \frac{\partial}{\partial r}l(s(br,x))\Big|_{r=0}=0\\
\lim_{r\to0}\frac{l(s(br,t),t)}{r^2} &= \frac{1}{2}\frac{\partial^2}{\partial r^2}l(s(br,x))\Big|_{r=0}\\
&= \frac{1}{2}\bigg(\frac{\partial^2}{\partial s^2}l(s,t)\Big|_{s=0}\Big(\frac{\partial}{\partial r}s(br,x)\Big|_{r=0}\Big)^2\\
&\qquad+\frac{\partial}{\partial s}l(s,t)\Big|_{s=0}\Big(\frac{\partial^2}{\partial r^2}s(br,x)\Big|_{r=0}\Big)\bigg)\\
&= \tfrac{1}{2}b^2\langle\alpha''(t),n(t)\rangle\\
&= -\tfrac{1}{2}b^2k(\alpha(t)),
\end{aligned}$$

proving the last claim. □

Before proving the main lemmas, we set up some notation. Let $v\in[0,2\pi]$ and $x\in\partial X$. Let $v_0,\ldots,v_3$ be the elements of $R_v(C_0)$ ordered such that $s_i\ge s_{i+1}$ where $s_i=\langle v_i,n(x)\rangle$. Let $b_i=\langle v_i,\tau(x)\rangle$ and define

$$t_i=-as_i+l(b_ia,x).$$

The t_i are constructed such that for $t\in[-\varepsilon,\varepsilon]$,

$$x+tn(x)+av_i\in X \text{ if and only if } t\le t_i.\tag{8.5}$$

The values of s_i and b_i are given in Table 2 for values of $u=\theta(n(x))-v\in(-\frac{5\pi}{4},-\frac{\pi}{4})$ where $\theta(n(x))$ is the angle between $n(x)$ and the vector $(1,0)$. For u and $-u-\frac{\pi}{4}$ the value of s_i is the same, while b_i changes sign. This yields the values of s_i and b_i for $u\in(-\frac{\pi}{4},\frac{3\pi}{4})$. In the table, w is chosen such that $w\in(0,\frac{\pi}{4})$.

Let t_i' be a reordering of the t_i such that $t_i'\le t_{i+1}'$ and let v_i' be the corresponding ordering of the v_i. This ordering depends on both x, v and a. Since t_i may not equal t_i', we need the following lemma, ensuring that this does not happen too often:

Lemma 8.3. *There is a constant M such that for all $x\in\partial X$ and a sufficiently small,*

$$a^{-1}\mathcal{H}^1(v\in[0,2\pi]\mid\exists i:v_i\ne v_i')\le M.$$

Table 2: Values of s_i and b_i for $u \in (-\frac{5\pi}{4}, -\frac{\pi}{4})$.

	$u = \theta(n) - v$	i	s_i	b_i
$u \in (-\frac{\pi}{2}, -\frac{\pi}{4})$	$w = u + \frac{\pi}{2}$	0	$\cos w + \sin w$	$\cos w - \sin w$
		1	$\cos w$	$-\sin w$
		2	$\sin w$	$\cos w$
		3	0	0
$u \in (-\frac{3\pi}{4}, -\frac{\pi}{2})$	$w = -u - \frac{\pi}{2}$	0	$\cos w$	$\sin w$
		1	$\cos w - \sin w$	$\cos w + \sin w$
		2	0	0
		3	$-\sin w$	$\cos w$
$u \in (-\pi, -\frac{3\pi}{4})$	$w = u + \pi$	0	$\sin w$	$\cos w$
		1	0	0
		2	$-(\cos w - \sin w)$	$\cos w + \sin w$
		3	$-\cos w$	$\sin w$
$u \in (-\frac{5\pi}{4}, -\pi)$	$w = -u - \pi$	0	0	0
		1	$-\sin w$	$\cos w$
		2	$-\cos w$	$-\sin w$
		3	$-(\cos w + \sin w)$	$\cos w - \sin w$

Furthermore, there is a constant M' such that

$$|t_i - t_i'| \le 4 \sup\{|l(ba, x)| \mid (b, x) \in [-\sqrt{2}, \sqrt{2}] \times \partial X\} \le M'a^2.$$

Proof. If $v_i \neq v_i'$, then in particular there is a $j_1 < j_2$ with $t_{j_1} > t_{j_2}$. But then

$$0 \le t_{j_1} - t_{j_2} = a(s_{j_2} - s_{j_1}) + l(b_{j_1}a, x) - l(b_{j_2}a, x) \tag{8.6}$$

and hence

$$0 \le a(s_{j_1} - s_{j_2}) \le l(b_{j_1}a, x) - l(b_{j_2}a, x) \le Ca^2$$

for some uniform constant C, according to Lemma 8.2.

But then

$$0 \le \cos(\theta(x, v)) \le \langle (v_{j_1} - v_{j_2}), n(x) \rangle \le Ca$$

where $\theta(x, v)$ is the angle from $n(x)$ to $v_{j_1} - v_{j_2}$. Thus, $\theta(x, v) = \theta(x, 0) + v$ must lie in $\cos^{-1}([0, Ca])$. But

$$\mathcal{H}^1(v \in [0, 2\pi] \mid \theta(x, v) \in \cos^{-1}([0, Ca])) = \mathcal{H}^1(\cos^{-1}([0, Ca]) \cap [0, 2\pi]) \le C'a$$

and there are only 6 possible combinations of j_1 and j_2, so

$$a^{-1}\mathcal{H}^1(v \in [0, 2\pi] \mid \exists i : v_i \neq v_i') \le a^{-1} 6 \mathcal{H}^1(\cos^{-1}([0, Ca]) \cap [0, 2\pi]) \le 6C'.$$

Suppose $t_i < t_i' = t_j$. If $j < i$, the last claim of the lemma follows from Lemma 8.2 and (8.6) as $a(s_{j_2} - s_{j_1})$ is negative. If $i < j$, there must be a $k < i$ with $t_j < t_k$. Then

$$|t_i - t_i'| \le |t_i - t_k| + |t_k - t_j| \le 4 \sup\{|l(ba, x)| \mid (b, x) \in [-\sqrt{2}, \sqrt{2}] \times \partial X\}$$

by a double application of (8.6). The case $t_i > t_i'$ can be treated in a similar way. □

In order to prove Theorem 8.1, we need to describe the asymptotic behavior of EN_j. This is computed by integrating over all rotations in (8.3) and letting a tend to 0. The two terms on the right hand side of (8.3) are treated separately in the two next lemmas.

Lemma 8.4. *With f_l as in* (8.2),

$$\lim_{a\to 0} a^{-2} \sum_{l:\xi_l\in\eta_j} \frac{1}{2\pi}\int_0^{2\pi}\int_{\partial X}\int_{-\varepsilon}^{\varepsilon} tf_l(x+tn,v)k(x)dt\mathcal{H}^1(dx)dv = \begin{cases} V_0(X), & j=2,\\ 0, & j=3,4,\\ -V_0(X), & j=5.\end{cases}$$

Proof. By Fubini's theorem

$$\begin{aligned}
&\int_0^{2\pi}\int_{\partial X}\int_{-\varepsilon}^{\varepsilon} tf_l(x+tn,v)k(x)dt\mathcal{H}^1(dx)dv\\
&\quad = \int_{\partial X}\int_0^{2\pi}\int_{-\varepsilon}^{\varepsilon} t1_{\{x+tn+aR_v(B_l)\subseteq X\}}1_{\{x+tn+aR_v(W_l)\subseteq \mathbb{R}^2\backslash X\}}dtdvk(x)\mathcal{H}^1(dx)\\
&\quad = \int_{\partial X}\int_0^{2\pi}\int_{-\varepsilon}^{\varepsilon} t1_{\{x+tn+aR_{v-\theta(n)}(B_l)\subseteq X\}}1_{\{x+tn+aR_{v-\theta(n)}(W_l)\subseteq \mathbb{R}^2\backslash X\}}dtdvk(x)\mathcal{H}^1(dx).
\end{aligned}$$

For $x\in\partial X$ fixed, write $u=v-\theta(n(x))$ as in Table 2 and let

$$I_j(x,u)=\sum_{l:\xi_l\in\eta_j}\int_{-\varepsilon}^{\varepsilon} tf_l(x+tn,u)dt.$$

For $\sqrt{2}a<\varepsilon$, configurations of type η_4 can never occur, so $(x+tn+aR_u(C_0))\cap X$ corresponds to a configuration of type

$$\begin{array}{ll}
\eta_1 & \text{for } t>t_3',\\
\eta_2 & \text{for } t\in(t_2',t_3'],\\
\eta_3 & \text{for } t\in(t_1',t_2'],\\
\eta_5 & \text{for } t\in(t_0',t_1'],\\
\eta_6 & \text{for } t\le t_0',
\end{array}$$

according to (8.5).

As an example, consider the configuration type η_5. Then we get

$$I_5=\int_{t_0'}^{t_1'} tdt = \tfrac{1}{2}(t_1'^2-t_0'^2).$$

Thus we must compute

$$\lim_{a\to 0} a^{-2}\int_{\partial X}\int_0^{2\pi} I_5 dukd\mathcal{H}^1 = \lim_{a\to 0} a^{-2}\int_{\partial X}\int_0^{2\pi}\frac{1}{2}(t_1'^2-t_0'^2)dukd\mathcal{H}^1.$$

By Lemma 8.3, $\lim_{a\to 0}\mathcal{H}^1(u\in[0,2\pi]\mid t_i\neq t_i')=0$. Moreover, it follows from Lemma 8.2 that

$$a^{-2}t_i^2 = s_i^2-2s_i\frac{l(b_ia,x)}{a}+\frac{l(b_ia,x)^2}{a^2}$$

is uniformly bounded. Hence we may replace $t_i'^2$ by t_i^2 in the integral. Furthermore, the Lebesgue theorem of dominated convergence applies. This yields

$$\begin{aligned}\lim_{a\to 0} a^{-2}\int_{\partial X}\int_0^{2\pi} I_5 dukd\mathcal{H}^1 &= \int_{\partial X}\int_0^{2\pi} \lim_{a\to 0} a^{-2}\cdot \tfrac{1}{2}(t_1^2-t_0^2)duk(x)\mathcal{H}^1(dx)\\ &= \int_{\partial X}\int_0^{2\pi} \tfrac{1}{2}(s_1^2-s_0^2)duk(x)\mathcal{H}^1(dx)\end{aligned}$$

The last step used Lemma 8.2.

Using the values of s_i given in Table 2, we get

$$\begin{aligned}&\lim_{a\to 0} a^{-2}\int_{\partial X}\int_0^{2\pi} I_5 dukd\mathcal{H}^1 = \int_{\partial X}\int_0^{2\pi} \tfrac{1}{2}(s_1^2-s_0^2)dukd\mathcal{H}^1\\ &= 2\pi V_0(X)\cdot 2\int_0^{\frac{\pi}{4}} \frac{1}{2}\Big((\cos^2 w-(\cos w+\sin w)^2)\\ &\quad +((\cos w-\sin w)^2-\cos^2 w)+(-\sin^2 w)+\sin^2 w\Big)dw\\ &= 2\pi V_0(X)\int_0^{\frac{\pi}{4}}(-4\cos w\sin w)dw\\ &= -2\pi V_0(X).\end{aligned}$$

Similarly for the remaining configuration types:

$$\begin{aligned}\int_0^{2\pi}\lim_{a\to 0}\frac{1}{a^2}I_3 du &= \int_0^{2\pi}\tfrac{1}{2}(s_2^2-s_1^2)du = 2\int_0^{\frac{\pi}{4}} 0dw = 0,\\ \int_0^{2\pi}\lim_{a\to 0}\frac{1}{a^2}I_2 du &= \int_0^{2\pi}\tfrac{1}{2}(s_3^2-s_2^2)du = 2\int_0^{\frac{\pi}{4}} \tfrac{1}{2}\cdot 4\cos w\sin w dw = 1,\end{aligned}$$

and the claim follows. □

Lemma 8.5. *For $w_j^{(i)}\in\mathbb{R}$ and $c_3^{(i)}$ as in (4.4), the limit*

$$\lim_{a\to 0} a^{-2}\cdot\frac{1}{2\pi}\left(\sum_{j=2}^{5} w_j^{(i)}\int_{\partial X}\int_0^{2\pi}\int_{-\varepsilon}^{\varepsilon}\sum_{l:\xi_l\in\eta_j} f_l(x+tn,v)dtdv\mathcal{H}^1(dx)-2ac_3^{(i)}V_1(X)\right)$$

exists and equals

$$(w_2^{(i)}-w_5^{(i)})V_0(X).$$

Proof. Let $x\in\partial X$ be given and define

$$I_j(x,v)=\sum_{l:\xi_l\in\eta_j}\int_{-\varepsilon}^{\varepsilon} f_l(x+tn,v)dt.$$

By the same reasoning as in the proof of Lemma 8.4,

$$I_2 = t_3'-t_2',\ I_3 = t_2'-t_1' \text{ and } I_5 = t_1'-t_0'.$$

As an example, consider η_5. We will compute

$$\begin{aligned}\lim_{a\to 0} a^{-2}\int_{\partial X}\int_0^{2\pi}&(I_5+a(s_1-s_0))dvd\mathcal{H}^1\\ &=\lim_{a\to 0}\int_{\partial X}\int_0^{2\pi}(a^{-2}(t_1'-t_0')+a^{-1}(s_1-s_0))dvd\mathcal{H}^1.\end{aligned}\tag{8.7}$$

Since $a^{-2}|t_i-t_i'|\le M'$ and $\mathcal{H}^1(t_i\neq t_i')<Ma$ by Lemma 8.3 for some uniform constants M and M', we may replace t_i by t_i' in (8.7).

By another application of Lemma 8.2,

$$a^{-2}t_i+a^{-1}s_i=l(b_ia,x)$$

is uniformly bounded. This allows us to apply Lebesgue's theorem to (8.7). In the case of η_5, this yields

$$\begin{aligned}\lim_{a\to 0}\int_{\partial X}\int_0^{2\pi}&(a^{-2}I_5+a^{-1}(s_1-s_0))dvd\mathcal{H}^1\\ &=\int_{\partial X}\int_0^{2\pi}\lim_{a\to 0}(a^{-2}(t_1'-t_0')+a^{-1}(s_1-s_0))dvd\mathcal{H}^1\\ &=\int_{\partial X}\int_0^{2\pi}\lim_{a\to 0}a^{-2}(l(ab_1,x)-l(ab_0,x))dv\mathcal{H}^1(dx)\\ &=\int_{\partial X}\int_0^{2\pi}\frac{-k}{2}(b_1^2-b_0^2)dvd\mathcal{H}^1,\end{aligned}$$

where the last step also follows from Lemma 8.2.

Doing the same for the remaining configurations shows that

$$\begin{aligned}&\int_{\partial X}\int_0^{2\pi}\frac{-k}{2}(w_2^{(i)}(b_3^2-b_2^2)+w_3^{(i)}(b_2^2-b_1^2)+w_5^{(i)}(b_1^2-b_0^2))dvd\mathcal{H}^1 \qquad (8.8)\\ &\quad=\lim_{a\to 0}a^{-2}\int_{\partial X}\int_0^{2\pi}\Bigg(\sum_{j=2}^{5}w_j^{(i)}I_j\\ &\qquad\qquad -a(w_2^{(i)}(s_2-s_3)+w_3^{(i)}(s_1-s_2)+w_5^{(i)}(s_0-s_1))\Bigg)dvd\mathcal{H}^1\\ &\quad=\lim_{a\to 0}a^{-2}\Bigg(\sum_{j=2}^{5}w_j^{(i)}\int_{\partial X}\int_0^{2\pi}I_jdvd\mathcal{H}^1\\ &\qquad\qquad -a\mathcal{H}^1(\partial X)8\Bigg(\int_0^{\frac{\pi}{4}}(w_2^{(i)}\sin w+w_3^{(i)}(\cos w-\sin w)+w_5^{(i)}\sin w)dw\Bigg)\Bigg)\\ &\quad=\lim_{a\to 0}a^{-2}\Bigg(\sum_{j=2}^{5}w_j^{(i)}\int_{\partial X}\int_0^{2\pi}I_jdvd\mathcal{H}^1\\ &\qquad\qquad -2aV_1(X)\big((8-4\sqrt{2})w_2^{(i)}+(8\sqrt{2}-8)w_3^{(i)}+(8-4\sqrt{2})w_5^{(i)}\big)\Bigg).\end{aligned}$$

On the other hand, inserting the b_i from Table 2, a direct computation shows:

$$\int_0^{2\pi}(b_1^2-b_0^2)dv = 2\int_0^{\frac{\pi}{4}} 4\sin w\cos w dw = 2,$$
$$\int_0^{2\pi}(b_2^2-b_1^2)dv = 2\int_0^{\frac{\pi}{4}} 0 dw = 0,$$
$$\int_0^{2\pi}(b_3^2-b_2^2)dv = 2\int_0^{\frac{\pi}{4}} (-4\sin w\cos w) dw = -2.$$

Thus (8.8) equals

$$\int_{\partial X}\frac{-k(x)}{2}(-2w_2^{(i)}+2w_5^{(i)})\mathcal{H}^1(dx) = 2\pi V_0(X)(w_2^{(i)}-w_5^{(i)}),$$

from which the claim follows. □

Proof of Theorem 8.1. From Lemma 8.4 and 8.5, it follows that the limit

$$\begin{aligned}
&\lim_{a\to 0}\left(a^{-i}E\hat{V}_i(X) - a^{-1}\frac{1}{\pi}c_3^{(i)}V_1(X)\right) \qquad (8.9)\\
&= \lim_{a\to 0}\left(\sum_{j=2}^{5} w_j^{(i)}EN_j - a^{-1}\frac{1}{\pi}c_3^{(i)}V_1(X)\right)\\
&= \lim_{a\to 0} a^{-2}\Bigg(\sum_{j=2}^{5} w_j^{(i)}\sum_{l:\xi_l\in\eta_j}\frac{1}{2\pi}\int_0^{2\pi}\Bigg(\int_{\partial X}\int_{-\varepsilon}^{\varepsilon} tf_l(x+tn,v)k(x)dt\mathcal{H}^1(dx)\\
&\quad + \int_{\partial X}\int_{-\varepsilon}^{\varepsilon} f_l(x+tn,v)dt\mathcal{H}^1(dx)\Bigg)dv - a\frac{1}{\pi}c_3^{(i)}V_1(X)\Bigg)
\end{aligned}$$

exists and equals $(2w_2^{(i)}-2w_5^{(i)})V_0(X) = c_4^{(i)}V_0(X)$. Hence $\lim_{a\to 0}E\hat{V}_0(X)$ exists if and only if $c_3^{(0)}=0$, and in this case the limit equals $c_4^{(0)}V_0(X)$.

In the limit, the condition (2.2) is

$$\lim_{a\to 0}E\hat{V}_0(X) = \lim_{a\to 0}(w_2^{(0)}EN_2(X)+w_3^{(0)}EN_3(X)+w_5^{(0)}EN_5(X)) = V_0(X),$$
$$\lim_{a\to 0}E\hat{V}_0(\mathbb{R}^2\backslash X) = \lim_{a\to 0}(w_2^{(0)}EN_5(X)+w_3^{(0)}EN_3(X)+w_5^{(0)}EN_2(X)) = -V_0(X).$$

This is equivalent to

$$\lim_{a\to 0}(w_2^{(0)}EN_2+w_3^{(0)}EN_3+w_5^{(0)}EN_5) = V_0(X),$$
$$\lim_{a\to 0}(w_2^{(0)}-w_5^{(0)})(EN_2-EN_5) = 2V_0(X).$$

From (8.9) with $w_2^{(0)}=1$, $w_3^{(0)}=w_4^{(0)}=0$, and $w_5^{(0)}=-1$, it follows that

$$\lim_{a\to 0}(EN_2-EN_5) = 4V_0(X).$$

Thus Equation (4.15) ensures that (2.2) holds asymptotically.

The statement about $\hat{V}_1(X)$ follows from (8.9) in a similar way. □

When ∂X is actually a C^3 manifold, we can get slightly better asymptotic results:

Theorem 8.6. *Let $X \subseteq \mathbb{R}^2$ be a C^3 full-dimensional submanifold. Assume that the weights defining $\hat{V}_1(X)$ satisfy Equations (4.9) and (4.10) and the weights defining $\hat{V}_0(X)$ satisfy Equations (4.14) and (4.15). Then $E\hat{V}_1(X)$ and $E\hat{V}_0(X)$ converge as $O(a^2)$ and $O(a)$, respectively.*

Proof. It is enough to check that $a^{-i-1}(E\hat{V}_i(X) - \lim_{a\to 0} E\hat{V}_i(X))$ is bounded. Going through the proofs of Lemma 8.4 and 8.5, we see that it is enough to show that

$$a^{-3}(t'^2_{i+1} - t'^2_i) - a^{-1}(s^2_{i+1} - s^2_i) \tag{8.10}$$

and

$$a^{-1}\int_0^{2\pi} \left(a^{-2}(t'_{i+1} - t'_i) - a^{-1}(s_i - s_{i+1}) + \frac{k}{2}(b^2_{i+1} - b^2_i)\right) dv \tag{8.11}$$

are uniformly bounded.

The triangle inequality yields

$$\begin{aligned}|a^{-3}(t'^2_{i+1} - t'^2_i) - a^{-1}(s^2_{i+1} - s^2_i)| \le{}& |a^{-3}t^2_i - a^{-1}s^2_i| + |a^{-3}t^2_{i+1} - a^{-1}s^2_{i+1}| \\ &+ a^{-3}|t'^2_i - t^2_i| + a^{-3}|t^2_{i+1} - t'^2_{i+1}|\end{aligned}$$

The terms

$$|a^{-3}t^2_i - a^{-1}s^2_i| = \left| -2s_i\frac{l(b_i a, x)}{a^2} + \frac{l(b_i a, x)^2}{a^2}\right|$$

are uniformly bounded by Lemma 8.2. Furthermore,

$$\frac{|t'^2_i - t^2_i|}{a^3} = \frac{|t'_i + t_i|}{a}\frac{|t'_i - t_i|}{a^2}$$

is bounded by Lemma 8.3. This takes care of (8.10).

Similarly,

$$\begin{aligned}&\left|a^{-3}(t'_{i+1} - t'_i) - a^{-3}(s_i - s_{i+1}) + a^{-1}\frac{k}{2}(b^2_{i+1} - b^2_i)\right| \\ &\le \left|a^{-3}(t_{i+1} - t_i) - a^{-2}(s_i - s_{i+1}) + a^{-1}\frac{k}{2}(b^2_{i+1} - b^2_i)\right| \\ &\quad + a^{-3}|t_i - t'_i| + a^{-3}|t_{i+1} - t'_{i+1}|.\end{aligned}$$

Again by Lemma 8.3, $a^{-2}|t_i - t'_i|$ is uniformly bounded by some C and hence

$$\int_0^{2\pi} a^{-3}|t_i - t'_i| dv \le \int_0^{2\pi} a^{-1}C1_{\{t_i \neq t'_i\}} dv$$

is also bounded by Lemma 8.3. Finally,

$$\begin{aligned}&a^{-3}(t_{i+1} - t_i) - a^{-2}(s_i - s_{i+1}) + a^{-1}\frac{k}{2}(b^2_{i+1} - b^2_i) \\ &= a^{-3}(l(b_{i+1}a, x) - l(b_i a, x)) + a^{-1}\frac{k}{2}(b^2_{i+1} - b^2_i).\end{aligned}$$

But by a refinement of Lemma 8.2, $r \mapsto l(r, x)$ is C^3 when ∂X is a C^3 manifold and

$$\frac{l(br, x)}{r^3} + \frac{b^2 k(x)}{2r}$$

is bounded for $(b, r, x) \in [-\sqrt{2}, \sqrt{2}] \times [-\delta, \delta] \backslash \{0\} \times \partial X$. This takes care of (8.11). □

Remark 8.7. The proof of Theorem 8.1 easily generalizes to describe the asymptotic behavior of local estimators for V_{d-2} in higher dimensions d.

9 Classical choices of weights

For a stationary isotropic Boolean model Ξ with typical grain C having a ball $B(\varepsilon)$ as summand a. s., we found in Theorem 4.2 that the estimator for half the specific boundary length satisfies

$$\lim_{a \to 0} E\hat{V}_1(\Xi) = \frac{1}{\pi} c_3^{(1)} \overline{V}_1(\Xi).$$

In particular, $\hat{V}_1(\Xi)$ is asymptotically unbiased if and only if $c_3^{(1)} = \pi$. In this case, the bias for small values of a is approximately

$$E\hat{V}_1(\Xi) - \overline{V}_1(\Xi) \approx a \left(c_4^{(1)} \gamma + c_5^{(1)} \left(\frac{\gamma}{\pi} EV_1(C) \right)^2 e^{-\gamma EV_2(C)} \right)$$

with $c_m^{(1)}$ as in (4.4).

In the literature, various local algorithms are used for estimating the boundary length of a planar set. With the formulas above we can compute their asymptotic bias and thus compare their accuracy.

Ohser and Mücklich, [11], describe an estimator for the specific boundary length based on a discretized version the Cauchy projection formula. In the rotation invariant setting, the estimator corresponds to (3.3) with weights:

$$w^{(1)} = \left(0, \frac{\pi}{16} \left(1 + \frac{\sqrt{2}}{2} \right), \frac{\pi}{16} (1 + \sqrt{2}), \frac{\pi}{8}, \frac{\pi}{16} \left(1 + \frac{\sqrt{2}}{2} \right), 0 \right).$$

Inserting these weights in the equations shows that this estimator satisfies (4.9). Hence it defines an asymptotically unbiased estimator. The weights also satisfy (4.10) but not (4.11). For small values of a, the error is approximately

$$-a \frac{1 + \sqrt{2}}{2} \frac{\gamma^2}{\pi} EV_1(C)^2 e^{-\gamma EV_2(C)} \approx -1,207 a \frac{\gamma^2}{\pi} EV_1(C)^2 e^{-\gamma EV_2(C)}.$$

One of the oldest algorithms for estimating the boundary length is suggested by Bieri in [1]. The idea is to reconstructing the underlying object as a union of squares of sidelength a centered at the foreground pixels. The boundary length is then estimated by the boundary length of the reconstructed object. This corresponds to a local estimator with weights

$$w^{(1)} = \left(0, \tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{1}{2}, 0\right).$$

However, it is well-known that for a compact object X this is the boundary length of the smallest box containing X and hence is a very coarse estimate. The asymptotic mean is

$$\frac{4}{\pi} \overline{V}_0(X).$$

Of course, one can correct for the factor $\frac{4}{\pi}$ and consider the weights

$$w^{(1)} = \left(0, \tfrac{\pi}{8}, \tfrac{\pi}{8}, \tfrac{\pi}{4}, \tfrac{\pi}{8}, 0\right) \tag{9.1}$$

instead. These weights can be justified by the Cauchy formula in [11] using $\theta_1 = \frac{\pi}{2}$. It is also the unique unbiased estimator where all weights are equal, except that configurations of type η_4 are counted with double weight. These weigths satisfy Equations (4.9) and (4.10) but not (4.11). The bias for small a is approximately

$$-a\frac{\gamma^2}{\pi}EV_1(C)^2 e^{-\gamma EV_2(C)}.$$

The approach of Dorst and Smeulders in [2] is to reconstruct the underlying set by an 8-adjacency system and compute the length of the boundary of the reconstructed set, letting vertical and horizontal segments contribute with one weight and diagonal segments with another weight. The resulting estimators are of the forms

$$\begin{aligned} w^{(1)} &= \left(0, 0, \frac{\theta}{2}, \sqrt{2}\theta, \frac{\sqrt{2}\theta}{2}, 0\right), \\ w^{(1)} &= (0, 0, a, 2b, b, 0). \end{aligned} \tag{9.2}$$

These algorithms are only tested on straight lines in [2] and therefore it was not necessary to assign a value $w_4^{(1)}$. The weights chosen here are such that a diagonal segment coming from a configuration of type η_4 is counted double. In particular, the first line with $\theta = 1$ computes V_1 of the approximating polyconvex set.

The authors list some of the constants frequently used in the literature. The case $\theta = 1$ goes back to Freeman in [3]. This yields a biased estimator. But even if the constants are chosen such that the estimator is asymptotically unbiased, all weights of this form have the disadvantage of not satisfying Equation (4.10), which is the most desirable af the two equations (4.10) and (4.11), as it also appears in the design based setting.

The boundary is also sometimes approximated using a 4- or 6-adjacency graph. However, the same problem with Equation (4.10) arises.

Another classical approach is the marching squares algorithm. This is based on a reconstruction of both foreground and background. The boundary is then approximated by a digital curve lying between these, see e.g. [10], Figure 4.29. The corresponding weights are

$$w^{(1)} = \left(0, \frac{\sqrt{2}}{4}, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}, 0\right).$$

This estimator is not asymptotically unbiased either. In fact, the asymptotic mean is

$$\lim_{a\to\infty} E\hat{V}_1(\Xi) = (2\sqrt{2} - 2)\frac{4}{\pi}\overline{V}_1(\Xi) \approx 1,0548\overline{V}_1(\Xi).$$

Correcting for this factor, we obtain an asymptotically unbiased estimator satisfying (4.14) with approximate bias for small values of a

$$a\frac{\sqrt{2}-6}{4}\frac{\gamma^2}{\pi}EV_1(C)^2 e^{-\gamma EV_2(C)} \approx -1,146a\frac{\gamma^2}{\pi}EV_1(C)^2 e^{-\gamma EV_2(C)}.$$

One can compare the classical estimators for the Euler characteristic in a similar way. Ohser and Mücklich, [11], suggest an estimator based on the approximation of Ξ by a 6-neighborhood graph. This results in weights

$$w^{(0)} = \left(0, \tfrac{1}{4}, 0, 0, -\tfrac{1}{4}, 0\right). \tag{9.3}$$

These satisfy (4.14) and (4.15), but not (4.16). Hence it does not define an asymptotically unbiased estimator for Boolean models, but it does in the design based setting of Section 8. For Boolean models, the asymptotic bias is

$$\lim_{a\to 0} E\hat{V}_0 - \overline{V}_0 = \left(\frac{2-4\sqrt{2}}{\pi}+1\right)\frac{\gamma^2}{\pi}EV_1(C)^2e^{-\gamma EV_2(C)} \approx -0,164\frac{\gamma^2}{\pi}EV_1(C)^2e^{-\gamma EV_2(C)}.$$

The estimator for the Euler characteristic suggested in [1] corresponds to the weights

$$w^{(0)} = \left(0, \tfrac{1}{4}, 0, -\tfrac{1}{2}, -\tfrac{1}{4}, 0\right).$$

The bias of this estimator is

$$\lim_{a\to 0} E\hat{V}_0 - \overline{V}_0 = \left(\frac{-4}{\pi}+1\right)\frac{\gamma^2}{\pi}EV_1(C)^2e^{-\gamma EV_2(C)} \approx -0,273\frac{\gamma^2}{\pi}EV_1(C)^2e^{-\gamma EV_2(C)},$$

which is slightly worse.

The conclusion is that for Boolean models, the best of the estimators for $\overline{V}_1$ and $\overline{V}_0$ listed here are (9.1) and (9.3), respectively. However, the weights in Proposition (4.3) and (4.5), respectively, give better etimators.

In the design based setting, all of the classical algorithms listed here except (9.2) are equally good when assessed by means of the results of the present paper.

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