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Abstract

Local digital algorithms based on $n\times\cdots\times n$ configuration counts are commonly used within science for estimating intrinsic volumes from binary images. This paper investigates multigrid convergence of such algorithms. It is shown that local algorithms for intrinsic volumes other than volume are not multigrid convergent on the class of convex polytopes. In fact, counter examples are plenty. Also on the class of r-regular sets, counter examples to multigrid convergence are constructed for the surface area and the integrated mean curvature. Finally, a multigrid convergent local algorithm in 2D for the Euler characteristic of convex particles with a lower bound on the interior angles is suggested.

*Keywords:*Image analysis, Local algorithm, Multigrid convergence, Intrinsic volumes, Binary morphology

1 Introduction and main results

The purpose of this paper is to assess a certain class of algorithms that are widely used for analysing digital output data from e.g. microscopes and scanners. These algorithms yield a fast way of estimating the so-called intrinsic volumes of a given object. The intrinsic volumes V_q, $q = 0,\dots,d$, include many of the quantities, scientists are most frequently interested in, see e.g. [11], such as the volume V_d, the surface area $2V_{d-1}$, the integrated mean curvature $2\pi(d-1)^{-1}V_{d-2}$, and the Euler characteristic V_0.

The algorithms considered rely only on what the image looks like locally, thus we refer to them as local algorithms. The use of local algorithms goes back to [4], see also [8, 10] for an overview of the algorithms suggested in the literature. The popularity of local algorithms is due to the fact that they allow simple linear time implementations [12], as opposed to the more complex algorithms of [2, 9]. However, as we shall see below, this efficiency is often paid for by a lack of accuracy.

We model a digital image of an object $X \subseteq \mathbb{R}^d$ by a binary image, i.e. as the set $X \cap \mathbb{L}$ where $\mathbb{L}$ is some lattice in $\mathbb{R}^d$. In applications, such a binary image is usually obtained from an observed grey-scale image by thresholding. Each point in $\mathbb{L}$ may belong to either X or the background. For every $n \times \cdots \times n$ cell in the observation lattice, this yields 2^{n^d} possible configurations of foreground and background points. The idea of local algorithms is to estimate V_q as a weighted sum of configuration counts, see Definition 2.11.

Local algorithms are suggested many places in the literature [8, 10, 17, 18] and various partial definitions are given [5, 6, 8, 22]. In Section 2 we attempt to set up a unified, rigorous definition of local algorithms and, in particular, to justify the use of local algorithms for the estimation of intrinsic volumes.

The next question is, when a local algorithm yields a good approximation of V_q. A natural criterion for an algorithm is multigrid convergence, i.e. that the estimator converges to the true value when the resolution goes to infinity. This is a very strong and in applications often unnatural requirement. In practice, observations are often made in a design based setting where the lattice has been randomly translated before making the observation. The natural, and usually weaker, requirement in this situation is that the estimator should be unbiased, at least asymptotically when the resolution tends to infinity. The various convergence criteria are discussed in Section 2.2 in more detail.

In order for the digital image to contain enough information about X to enable us to estimate $V_q(X)$, some niceness assumptions on the underlying set X are needed. In this paper, we shall investigate which intrinsic volumes V_q allow asymptotically unbiased local estimators when X is assumed to belong to the class of compact convex polytopes with non-empty interior or the class of r-regular sets (see Definition 4.1).

1.1 Known results

Various results have already been obtained in this direction. It is well-known, see e.g. [12], that there is a local estimator for the volume V_d which is unbiased even in finite resolution given by counting lattice points in X and weighting them by the volume of the unit lattice cell.

In contrast, Jürgen Kampf has proved [5] that on the class of finite unions of polytopes, local algorithms for V_q based on $2 \times \cdots \times 2$ configurations in orthogonal lattices are always asymptotically biased for $0 \leq q \leq d-2$. In fact, he has shown that the worst case asymptotic bias is always $100\,\%$.

For $q = d-1$, Ziegel and Kiderlen showed in [24] that there exists no asymptotically unbiased local algorithm for the surface area in 3D based on $2 \times 2 \times 2$ configurations in an orthogonal lattice, but the asymptotic worst case bias is finite in this case.

It has been conjectured in [8] and [6] that no local algorithm for estimation of surface area is multigrid convergent in dimension $d = 2$ and $d = 3$, respectively. This was proved by Tajine and Daurat [23] in dimension $d = 2$ in the special case of length estimation for straight line segments. In fact, they show that any algorithm will be (asymptotically) biased for almost all slopes of the line segment. In [7, Theorem 5], Kiderlen and Rataj prove a formula for the asymptotic mean of a surface area estimator, on which a proof in arbitrary dimension d could be based.

On the other hand it is known that with suitable smoothness conditions (r-regularity) on the boundary ∂X there exists a multigrid convergent local algorithm for estimating the Euler characteristic V_0 in 2D [14] and in 3D [20]. In fact, this algorithm yields the correct value in sufficiently high finite resolution. It is still a partially open question whether the existence of such an algorithm is due to the smoothness conditions on the boundary or to the fact that V_0 is a topological invariant. However, it is shown in [22] that there is no asymptotically unbiased estimator for the integrated mean curvature V_{d-2} in dimensions $d > 2$ based on $2 \times \cdots \times 2$ configurations. This suggests that it is the topology invariance that makes V_0 special.

1.2 Main results of the paper

We first consider the estimation of V_q on the class $\mathcal{P}^d$ of compact convex polytopes with non-empty interior. Any $P \in \mathcal{P}^d$ can be written in the form

$$P = \bigcap_{i=1}^{N} H^-_{u_i,t_i}$$

where $H^-_{u,t}$ denotes the halfspace $\{x \in \mathbb{R}^d \mid \langle x, u\rangle \leq t\}$ for $u \in S^{d-1}$ and $t \in \mathbb{R}$. The parameters u_i, t_i can be used to define a measure ν on $\mathcal{P}^d$. This is made precise in Section 3.1.

When $1 \leq q \leq d-1$, we shall prove the following theorem, generalizing the results of [23]:

Theorem 1.1. *For $1 \leq q \leq d-1$, any local algorithm for V_q in the sense of Definition 2.11 is asymptotically biased (and hence not multigrid convergent) for ν-almost all $P \in \mathcal{P}^d$ if $d-q$ is odd and for a subset of $\mathcal{P}^d$ of positive ν-measure if $d-q$ is even.*

As simple examples of sets for which an asymptotic bias occur, one may take almost all rotations of almost all orthogonal boxes $\bigoplus_{i=d}^{d}[0, t_i e_i]$ where $t_1, \ldots, t_d \in \mathbb{R}$ and $e_1, \ldots, e_d \in \mathbb{R}^d$ is the standard basis.

If an algorithm were only asymptotically biased for a very small class of sets, for instance orthogonal boxes, this could well be acceptable in practice where objects are often randomly shaped with a probability of zero for hitting this class. Hence we state the theorem for all polytopes in a set of positive ν-measure. The reasonableness in choosing the measure ν on $\mathcal{P}^d$ may be disputed, see the discussion in Section 3.1, since it depends on how the studied particles arise. However, it would apply to many situations where the particles under study arise from random sections of some material.

In the case $q = 0$, we can obtain a similar theorem

Theorem 1.2. *Any local algorithm for V_0 in the sense of Definition 2.11 is asymptotically biased (and hence not multigrid convergent) on $\mathcal{P}^d$ if $d > 1$.*

However, constructing the counter examples is now harder. In fact, in $\mathbb{R}^2$ there is a sequence of local algorithms $\hat{V}_0^n$ for $n \in \mathbb{N}$ based on $n \times \cdots \times n$ configurations such that $\hat{V}_0^n$ is multigrid convergent for all $P \in \mathcal{P}^d$ (or even all relatively open compact convex sets) having no interior angles less than $\psi_n \in \mathbb{R}$ where $\lim_{n \to 0} \psi_n = 0$. In particular, for any $P \in \mathcal{P}^d$ there is an $N \in \mathbb{N}$ such that $\hat{V}_0^n(P) = V_0(P)$ whenever $n \geq N$ and the resolution is sufficiently high. Thus, if one studies convex particles with a lower bound on the interior angles, there exists a multigrid convergent local algorithm for V_0. The explicit construction of these algorithms and the precise conditions on the weights are given in Section 3.5.

As in [5], the proof goes by first constructing a counter example $P \subseteq \mathbb{R}^2$. This is then generalized to higher dimensions by the prism $P \times \bigoplus_{i=3}^d [0, e_i]$. This approach also provides the following generalization of Kampf's results:

Theorem 1.3. *For $0 \leq q \leq d-2$, any local algorithm for V_q as in Definition 2.11 has an asymptotic worst case bias of 100 % on $\mathcal{P}^d$.*

We finally move on to the case of r-regular sets. Using the main results of [7] and [22], we shall see:

Theorem 1.4. *For $q = d-1, d-2$ and $q > 0$, any local algorithm for V_q as in Definition 2.11 with homogeneous weights is asymptotically biased (and hence not multigrid convergent) on the class of r-regular sets.*

The definition of homogeneous weights is given in Definition 2.9 below. For $0 < q < d-2$, the asymptotic behavior of local estimators for V_q is not well enough understood to determine whether asymptotically unbiased estimators exist. However, Theorem 1.4 suggests that the Euler characteristic is the only V_q with $q < d$ that allows an asymptotically unbiased local estimator on the class of r-regular sets.

2 Local digital algorithms

2.1 Digital estimators

We first set up some notation and terminology and introduce digital estimators in general.

Let $\xi = \{\xi_1, \dots, \xi_d\}$ be a positively oriented basis of $\mathbb{R}^d$ and let $\mathbb{L}$ denote the lattice spanned by ξ. Let $C_\xi = \bigoplus_{i=1}^d [0, \xi_i]$ be the unit cell of the lattice with volume $\det(\mathbb{L})$. For $c \in \mathbb{R}^d$, we let $\mathbb{L}_c = \mathbb{L} + c$ denote the translated lattice.

Now suppose $X \subseteq \mathbb{R}^d$ is some subset of $\mathbb{R}^d$. We use the binary digitization model for a digital image, see e.g. [12]. That is, we think of a digital image as the set $X \cap a\mathbb{L}_c \subseteq a\mathbb{L}_c$ where $a > 0$ is the lattice distance. This set contains the same information about X as the Gauss digitization [8, Definition 2.7], which is the union of all translations of C_ξ having midpoint in $X \cap a\mathbb{L}_c$.

Let $V : \mathcal{S} \to \mathbb{R}$ be a function defined on some class $\mathcal{S}$ of subsets of $\mathbb{R}^d$. We want to estimate this function based on digital images of elements of $\mathcal{S}$.

Definition 2.1. *By a digital algorithm $\hat{V}$ for V, we mean a collection of functions $\hat{V}^{a\mathbb{L}_c} : \mathcal{P}(a\mathbb{L}_c) \to \mathbb{R}$ for every $a > 0$ and $c \in C_\xi$ where $\mathcal{P}(a\mathbb{L}_c)$ is the power set of $a\mathbb{L}_c$. For $X \in \mathcal{S}$ we use $\hat{V}^{a\mathbb{L}_c}(X) := \hat{V}^{a\mathbb{L}_c}(X \cap a\mathbb{L}_c)$ as a digital estimator for $V(X)$.*

A digital algorithm $\hat{V}$ is said to be

- ***translation invariant*** *if $\hat{V}^{a\mathbb{L}_0}(S) = \hat{V}^{a\mathbb{L}_c}(S + ac + az)$ for all $S \in \mathcal{P}(a\mathbb{L})$, $c \in C_\xi$, $z \in \mathbb{L}$, and $a > 0$.*

- ***rotation (reflection) invariant*** *if* $\hat{V}^{a\mathbb{L}_c}(S) = \hat{V}^{a\mathbb{L}_{Rc}}(RS)$ *for all* $S \in \mathcal{P}(a\mathbb{L})$, $c \in C_\xi$, $a > 0$, *and all rotations (reflections)* $R \in \mathrm{SO}(d)$ *preserving* $a\mathbb{L}$.
- ***motion invariant*** *if it is both translation and rotation invariant.*

Remark 2.2. Sometimes, e.g. in [19], $\hat{V}^{a\mathbb{L}}$ is only defined for a belonging to some sequence $a_k \to 0$ (typically, $a_k = 2^{-k}$). Though a weaker requirement, this will not affect the non-existence theorems of this paper, so we consider only the case of the definition.

Similarly, the algorithm is sometimes only defined for a subset of $\mathcal{P}(a\mathbb{L}_c)$, e.g. finite sets, or only for $c = 0$, but of course, such a definition can easily be extended.

2.2 Various convergence criteria

Having defined a digital algorithm, the next question is how it should relate to $V(X)$. Obviously, many different sets may have the same digital image, so $\hat{V}^{a\mathbb{L}_c}(X)$ will typically not give the correct value. However, $X \cap a\mathbb{L}_c$ will contain more and more information about X as a decreases. Thus it is reasonable to require that $\hat{V}^{a\mathbb{L}_c}(X)$ converges to the correct value when the lattice distance goes to zero. In [8], this is called multigrid convergence and the formal definition here is as follows:

Definition 2.3. *A digital algorithm* $\hat{V}$ *for* $V : \mathcal{S} \to \mathbb{R}$ *is called multigrid convergent if for all* $X \in \mathcal{S}$,

$$\lim_{a \to 0} \hat{V}^{a\mathbb{L}_0}(X) = V(X).$$

Note that the definition only involves the non-translated lattice $\mathbb{L}_0$. This definition does cause some problems. It depends on the choice of origo with respect to which the lattice is scaled. For instance, it could be that $\hat{V}^{a\mathbb{L}_c}(X)$ does not converge to $V(X)$, even if the algorithm is translation invariant. One could of course repair this by requiring $\lim_{a\to 0} \hat{V}^{a\mathbb{L}_c}(X) = V(X)$ for all $c \in C_\xi$. In practical applications, however, the lattice may not be scaled with respect to a fixed origo. Thus the following stronger condition would be natural:

Definition 2.4. *A digital algorithm* $\hat{V}$ *is called uniformly multigrid convergent if for all* $X \in \mathcal{S}$ *and* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$$|\hat{V}^{a\mathbb{L}_c}(X) - V(X)| \leq \varepsilon$$

for all $c \in C_\xi$ *and* $a < \delta$.

In other words, the convergence $\hat{V}^{a\mathbb{L}_c}(X) \to V(X)$ is uniform with respect to translations of $\mathbb{L}$. An equivalent formulation is that for every pair of sequences $a_k \to 0^+$ and $c_k \in \mathbb{R}^d$,

$$\lim_{k \to \infty} \hat{V}^{a_k\mathbb{L}_{c_k}}(X) = V(X).$$

Multigrid convergence is in many situations a much too strong requirement. Of the examples mentioned in the introduction, only the estimator for the Euler characteristic of r-regular sets is multigrid convergent.

In practice, a design based approach is often taken. Here the observation is made on a uniform random translation of the lattice. That is, the observed image is the random set $X \cap a\mathbb{L}_c$ where $c \in C_\xi$ is a uniform random translation vector. A digital algorithm is called integrable if $c \mapsto \hat{V}^{a\mathbb{L}_c}(X)$ is integrable over C_ξ for all $a > 0$ and $X \in \mathcal{S}$, i.e. the mean $E\hat{V}^{a\mathbb{L}_c}(X)$ is finite for all $X \in \mathcal{S}$. The natural requirement for an integrable digital algorithm is that $\hat{V}^{a\mathbb{L}_c}(X)$ is unbiased, at least when a tends to zero. More formally:

Definition 2.5. *Let $\hat{V}$ be an integrable digital algorithm for V defined on a class $\mathcal{S}$ of subsets of $\mathbb{R}^d$. Then $\hat{V}$ is called asymptotically unbiased if for all $X \in \mathcal{S}$,*

$$\lim_{a \to 0} E\hat{V}^{a\mathbb{L}_c}(X) = V(X).$$

It is clear that uniform multigrid convergence implies asymptotic unbiasedness. So does multigrid convergence in most nice situations, as the next proposition shows. Let $\mathcal{C}_\partial$ denote the collection of compact subsets of $\mathbb{R}^d$ whose boundary has $\mathcal{H}^d$-measure zero where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure.

Proposition 2.6. *Suppose $V : \mathcal{S} \to \mathbb{R}$ is a translation invariant function defined on some $\mathcal{S} \subseteq \mathcal{C}_\partial$ and that $\hat{V}^{a\mathbb{L}_c}$ is a translation invariant digital estimator for V. Then multigrid convergence implies asymptotic unbiasedness.*

Proof. Suppose $X \in \mathcal{S}$ and that $\hat{V}$ is multigrid convergent. It will be enough to show that for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $a < \delta$,

$$|\hat{V}^{a\mathbb{L}_0}((X - ac) \cap a\mathbb{L}) - V(X)| < \varepsilon$$

holds for almost all $c \in C_\xi$.

Assume this were not true. Then there would be an $\varepsilon > 0$, a sequence $a_m \to 0$, and $W_m \subseteq C_\xi$ with $\mathcal{H}^d(W_m) > 0$ such that

$$|\hat{V}^{a_m\mathbb{L}_0}((X - a_m c) \cap a_m\mathbb{L}) - V(X)| \geq \varepsilon$$

for all $c \in W_m$.

First assume that a is fixed. By compactness of X, $(X - ac) \cap a\mathbb{L}$ can take only finitely many values in $\mathcal{P}(a\mathbb{L})$ when $c \in C_\xi$. Thus also $\hat{V}^{a\mathbb{L}_0}((X - ac) \cap a\mathbb{L})$ takes only finitely many different values for $c \in C_\xi$.

Define

$$S_z = \{c \in C_\xi \mid az \in X - ac\} = C_\xi \cap (a^{-1}X - z)$$

for $z \in \mathbb{L}$ and note that only finitely many S_z are non-empty. Thus for $S \subseteq a\mathbb{L}$

$$\{c \in C_\xi \mid (X - ac) \cap a\mathbb{L} = S\} = \bigcap_{z \in S} S_z \cap \bigcap_{z \notin S} S_z^c \tag{2.1}$$

Observe that $S_z^c \cap \operatorname{int} C_\xi$ is open and equals $\operatorname{int} C_\xi$ for all but finitely many z. The boundary of S_z is contained in $\partial C_\xi \cup \partial(a^{-1}X - z)$ and therefore it has $\mathcal{H}^d$-measure zero. A point in (2.1) will either lie in the interior of all S_z, $z \in S$, or in the boundary of one of them. Thus (2.1) will either have non-empty interior or $\mathcal{H}^d$-measure zero.

Since W_m is the finite union of sets of the form (2.1) and $\mathcal{H}^d(W_m) > 0$, it must have non-empty interior U_m. Now choose a_{m_i} inductively. First let $a_{m_1} = a_1$ and let $K_{m_1} \subseteq U_1$ be a compact set with non-empty interior. For a_{m_2} sufficiently small,

$a_{m_2}(C_\xi + z) \subseteq K_{m_1}$ for some z. Therefore we may choose a compact set with non-empty interior $K_{m_2} \subseteq K_{m_1} \cap a_{m_2}(U_{m_2} + z)$. Continuing this way yields a decreasing sequence of compact sets K_{m_i}. In particular, $\bigcap K_{m_i}$ is non-empty, so we may choose $y \in \bigcap K_{m_i}$. By the translation invariance of $\hat{V}^{a\mathbb{L}_0}$ and V,

$$|\hat{V}^{a_{m_i}\mathbb{L}_0}((X-y) \cap a_{m_i}\mathbb{L}) - V(X)| \geq \varepsilon$$

for all i, so $\hat{V}$ is not multigrid convergent for $X - y$, which is a contradiction. □

2.3 Local digital algorithms

In this section we introduce the notion of local algorithms. The name 'local algorithm' is adopted from [6, Definition 4.1] and [8, Definition 8.3]. In these definitions, a local algorithm is really an algorithm for reconstructing the boundary of a solid in 2D or 3D as a union of line segments or polygons, respectively. The idea is that each of these building blocks should only depend on what the digital image looks like locally. From the reconstructed set, the length or surface area can be estimated as a sum of lengths or areas of the building blocks, respectively. The authors also refer to algorithms for estimating length and surface area arising in this way as local algorithms.

We choose the following definition for general digital algorithms:

Definition 2.7. *A digital algorithm $\hat{V}$ is called local if there is a finite collection of pairs (B_k, W_k) for $k \in K$ such that $B_k, W_k \subseteq \mathbb{L}$ are two finite disjoint sets and*

$$\hat{V}^{a\mathbb{L}_c}(S) = \sum_{k\in K}\sum_{z\in\mathbb{L}} w_k(a, a(z+c))\mathbb{1}_{\{a(B_k+z+c)\subseteq S, a(W_k+z+c)\subseteq a\mathbb{L}_c\setminus S\}} \tag{2.2}$$

for all finite $S \subseteq a\mathbb{L}_c$. Here $\mathbb{1}_A$ denotes the indicator function for the set A. The pair (B_k, W_k) is called a configuration and the elements of B_k are referred to as the 'foreground' or 'black' pixels, while W_k is referred to as the set of 'background' or 'white' pixels. The functions $w_k : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ are called the weights.

Thus each occurrence of a translation of the configuration (B_k, W_k) contributes to the estimate with a weight $w_k(a, z)$ depending only on the translation vector z and the lattice distance a.

The definitions of [6] and [8] correspond to the collection

$$\{(B_k, W_k) \mid B_k \cup W_k = B(R) \cap \mathbb{L}, B_k \cap W_k = \emptyset\}$$

for some $R > 0$ where $B(R)$ denotes the ball of radius R. Strictly speaking, their definition is not quite contained in Definition 2.7. However, all the examples of local algorithms for computing length and surface area mentioned in these references are of this form.

We introduce a bit more notation: An $n \times \cdots \times n$ cell is a set of the form $C_z^n = (z + \bigoplus_{i=1}^d [0, n\xi_i))$ for $z \in \mathbb{L}$. The set of lattice points lying in such a cell is denoted by $C_{z,0}^n = C_z^n \cap \mathbb{L}$. A lattice point in $C_{0,0}^n$ has the form $x = \sum_{i=1}^d \lambda_i \xi_i$ for some $\lambda_i \in \{0, \ldots, n-1\}$ and we write $x = x_j$ where the index is given by

$$j = \sum_{i=1}^d n^{i-1}\lambda_i.$$

An $n\times\cdots\times n$ configuration is a pair (B^n, W^n) where $B^n, W^n \subseteq C^n_{0,0}$ are disjoint with $B^n \cup W^n = C^n_{0,0}$. We index these by (B^n_l, W^n_l), $l = 0,\ldots,2^{n^d}-1$, where a configuration (B^n, W^n) is assigned the index

$$l = \sum_{i=0}^{n^d-1} 2^i \mathbb{1}_{\{x_i\in B^n\}}.$$

Proposition 2.8. *For every local algorithm $\hat{V}$ there is an $n\in\mathbb{N}$ such that for all finite $S\subseteq a\mathbb{L}_c$,*

$$\hat{V}^{a\mathbb{L}_c}(S) = \sum_{l=0}^{2^{n^d}-1}\sum_{z\in\mathbb{L}} \tilde{w}_l(a, a(z+c))\mathbb{1}_{\{a(B^n_l+z+c)\subseteq S, a(W^n_l+z+c)\subseteq\mathbb{R}^d\setminus S\}} \tag{2.3}$$

for suitable weights $\tilde{w}_l(a,z)$.

Proof. By finiteness of K, there is an $n\in\mathbb{N}$ and a $y\in\mathbb{L}$ with $B_k, W_k \subseteq C^n_{y,0}$ for all $k\in K$. Thus, (2.2) becomes an estimator of the form (2.3) with weights

$$w_l(a,z) = \sum_{k\in K} w_k(a,z)\mathbb{1}_{\{B_k-y\subseteq B^n_l, W_k-y\subseteq W^n_l\}}\left(\sum_{m=0}^{2^{n^d}-1}\mathbb{1}_{\{B_k-y\subseteq B^n_m, W_k-y\subseteq W^n_m\}}\right)^{-1}.$$

□

Thus, for the remainder of this paper we shall only consider local algorithms of the form (2.3). We usually skip the n from the notation and write (B_l, W_l) for the $n\times\cdots\times n$ configurations.

Clearly, the larger n is, the better accuracy of the algorithm can be expected, as more information is taken into account. For most algorithms used in practice [8, 12], $n = 2$. However, algorithms with $n = 3$ have been suggested, see [13]. Also, most theoretical studies of local algorithms only involve $n = 2$, see Section 1.1. One exception is [23].

Definition 2.9. *The weights are said to be*

- ***translation invariant*** *if $w_l(a,z)$ is independent of $z\in\mathbb{R}^d$.*
- ***rotation (reflection) invariant*** *if $w_{l_1}(a,z_1) = w_{l_2}(a,z_2)$ whenever there is a rotation (reflection) R preserving $\mathbb{L}$ such that $R(B_{l_1}+z_1) = B_{l_2}+z_2$.*
- ***motion invariant*** *if the weights are both translation and rotation invariant.*
- ***homogeneous (of degree q)*** *if $w_l(a,z) = a^q w_l(1,z)$ for all $a>0$ and $z\in\mathbb{R}^d$.*

The estimators for Minkowski tensors in e.g. [17, 18] are examples of local digital estimators where the weights are not translation invariant. If V is rotation (reflection) invariant, the following proposition justifies the choice of rotation (reflection) invariant weights, see also [22]:

Proposition 2.10. *Assume V is rotation (reflection) invariant. For every local algorithm $\hat{V}$, there is a local algorithm $\hat{W}$ with rotation (reflection) invariant weights such that for all compact $X \in \mathcal{S}$,*

$$\sup_{R\in\mathcal{R}} |\hat{W}^{a\mathbb{L}_{Rc}}(RX) - V(RX)| \leq \sup_{R\in\mathcal{R}} |\hat{V}^{a\mathbb{L}_{Rc}}(RX) - V(RX)| \tag{2.4}$$

where $\mathcal{R}$ denotes the group of rotations (reflections) preserving $\mathbb{L}$.

Proof. If $|\mathcal{R}|$ is the cardinality of $\mathcal{R}$, define for $S \subseteq \mathbb{L}$

$$\hat{W}^{a\mathbb{L}_c}(a(S+c)) = \frac{1}{|\mathcal{R}|}\sum_{R\in\mathcal{R}} \hat{V}^{a\mathbb{L}_{Rc}}(aR(S+c)).$$

This is a local estimator with rotation invariant weights and it clearly satisfies (2.4) since $V(RX) = V(X)$. □

Finally, we introduce a bit more notation: For $A, B \subseteq \mathbb{R}^d$, let $\check{B} = \{-b \mid b \in B\}$ and $A \ominus B = \{x \in \mathbb{R}^d \mid x + \check{B} \subseteq A\}$. The hit-or-miss transform of X with structure elements B and W is defined to be the set

$$X \ominus \check{B} \backslash X \oplus \check{W} = \{y \in \mathbb{R}^d \mid y + B \subseteq X, y + W \subseteq \mathbb{R}^d\backslash X\}.$$

A local estimator then takes the form

$$\hat{V}^{a\mathbb{L}_c}(X) = \sum_{l=0}^{2^{n^d}-1}\sum_{z\in\mathbb{L}} w_l(a, a(z+c))\mathbb{1}_{X\ominus a\check{B}_l\backslash X\oplus a\check{W}_l}(a(z+c)).$$

If $z \mapsto w_0(a,z)$ is integrable and $z \mapsto w_l(a,z)$ are locally integrable for $l > 0$, $\hat{V}^{a\mathbb{L}_c}(X)$ is always integrable for X compact since

$$\begin{aligned} E\Big(\sum_{z\in\mathbb{L}} w_l(a, a(z+c))\mathbb{1}_{X\ominus a\check{B}_l\backslash X\oplus a\check{W}_l}(a(z+c))\Big) \\ = a^{-d}\det(\mathbb{L})^{-1}\int_{X\ominus\check{B}_l\backslash X\oplus\check{W}_l} w_l(a,z)dz \end{aligned} \tag{2.5}$$

and hence

$$E\hat{V}(X \cap a\mathbb{L}_c) = a^{-d}\det(\mathbb{L})^{-1}\sum_{l=0}^{2^{n^d}-1}\int_{X\ominus\check{B}_l\backslash X\oplus\check{W}_l} w_l(a,z)dz.$$

2.4 Local digital estimators for intrinsic volumes

We finally specialize the definition of local digital estimators to intrinsic volumes. This resulting definition coincides with the ones used in [5, 22].

Suppose $X \subseteq \mathbb{R}^d$ is a compact convex set. The intrinsic volumes $V_q(X)$ are defined for $q = 0, \ldots, d$ to be the coefficients in the well-known Steiner formula

$$\mathcal{H}^d(X \oplus B(r)) = \sum_{q=0}^{d} r^{d-q}\kappa_{d-q}V_q(X)$$

for the volume of the Minkowski sum $X \oplus B(r)$ of X and the ball $B(r) \subseteq \mathbb{R}^d$ of radius r. Here κ_q is the volume of the unit ball in $\mathbb{R}^q$. The intrinsic volumes can be generalized to the class of sets of positive reach, see [3].

Each V_q is the total measure of the q'th curvature measure $\Phi_q(X; \cdot)$ on $\mathbb{R}^d$, see [16]. Thus

$$V_q(X) = n^{-d} \sum_{z \in \mathbb{L}} \Phi_q(X; aC_z^n).$$

This justifies the use of a local algorithm $\hat{V}_q$ for estimating $V_q(X)$, i.e. an algorithm of the form

$$\hat{V}_q^{a\mathbb{L}_c}(X) = \sum_{z \in \mathbb{L}} \sum_{l=0}^{2^{n^d}-1} w_l^{(q)}(a, a(z+c)) \mathbb{1}_{X \ominus a\check{B}_l \setminus X \oplus a\check{W}_l}(a(z+c))$$

where $w_l^{(q)}(a, a(z+c))$ can be thought of as an estimate of $n^{-d^2} \Phi_q(X; a(C_z^n + c))$. As $\Phi_q(X; \cdot)$ is rotation and reflection invariant, Proposition 2.10 justifies choosing the weights to be rotation and reflection invariant as well. Moreover, $\Phi_q(X; \cdot)$ is translation invariant so it is natural to require the weights to be so too, i.e. $w_l^{(q)}(a, z) = w_l^{(q)}(a)$. In order to get finite estimators for compact sets, we always assume that $w_0^{(q)}(a) = 0$.

We thus arrive at the following definition of a local digital estimator for V_q:

Definition 2.11. *For $0 \leq q \leq d$, a local digital estimator for V_q is an estimator of the form*

$$\hat{V}_q^{a\mathbb{L}_c}(X) = \sum_{l=1}^{2^{n^d}-1} w_l(a) N_l(X \cap a\mathbb{L}_c) \tag{2.6}$$

where

$$N_l(X \cap a\mathbb{L}_c) = \sum_{z \in \mathbb{L}} \mathbb{1}_{X \ominus a\check{B}_l \setminus X \oplus a\check{W}_l}(a(z+c))$$

is the total number of occurrences of the configuration (B_l, W_l) in the image $X \cap a\mathbb{L}_c$. The weights are assumed to be motion and reflection invariant.

Throughout this paper, a local digital estimator of for V_q will mean an estimator of the form (2.6). We often skip the superscripts $a\mathbb{L}_c$ and (q) in the notation for the estimator and the weights and write $\hat{V}_q(X)$ and $w_l(a)$, respectively.

In applications, the weights are usually chosen to be homogeneous of degree q: $w_l^{(q)}(a) = a^q w_l^{(q)}$ for some constants $w_l^{(q)} \in \mathbb{R}$, motivated by the homogeneity property:

$$\Phi_q(aX; aA) = a^q \Phi_q(X; A).$$

However, in [5], also the case of general functions is considered. In this paper, we will not assume homogeneity unless explicitly specified.

If an algorithm is not asymptotically unbiased, the worst case relative asymptotic bias measures the bias:

Definition 2.12. *The worst case relative asymptotic bias of an estimator $\hat{V}_q$ for V_q on a class of compact convex sets $\mathcal{S}$ is given by*

$$\sup_{X\in\mathcal{S}} \frac{|\lim_{a\to 0} E\hat{V}_q^{a\mathbb{L}_c}(X) - V_q(X)|}{V_q(X)}.$$

As long as we restrict ourselves to convex sets, this agrees with the definition in [5]. Note that by Proposition 2.10, the worst case relative asymptotic bias is minimized by an algorithm with rotation and reflection invariant weights.

3 Local estimators for the intrinsic volumes of polytopes

We first consider local digital estimators for intrinsic volumes on the class $\mathcal{P}^d$ of compact convex polytopes in $\mathbb{R}^d$ with non-empty interior.

We will use the following notation: for a set $A \subseteq \mathbb{R}^d$, we denote by $\mathrm{aff}(A) \subseteq \mathbb{R}^d$ the smallest affine linear subspace containing A and by $\mathrm{lin}(A) \subseteq \mathbb{R}^d$ the smallest linear subspace parallel to $\mathrm{aff}(A)$. For a set of vectors $u_1, \ldots, u_N$, we denote by $\mathrm{pos}(u_1, \ldots, u_N)$ the set of linear combinations of $u_1, \ldots, u_N$ with non-negative coefficients.

3.1 The space of polytopes

The set $\mathcal{P}^d$ is usually given the topology induced by the Hausdorff metric, see [16, Section 1.8]. As our main Theorem 1.1 is stated for almost all polytopes, we need an appropriate measure on the induced Borel σ-algebra in order to make sense of the statement. However, the choice of such a measure is not unambiguous. The most natural way of describing a polytope is either as the convex hull of its vertex set or as an intersection of halfspaces. The parameters describing the vertices and halfspaces, respectively, can be used to parametrize $\mathcal{P}^d$, but this leads to two very different measures. In the first case, almost all polytopes will be simplicial while non-simple polytopes constitute a set of positive measure. In the second case, it is the other way around. A polytope is called simple if every vertex is the intersection of exactly d facets and it is called simplicial if every facet is a simplex, see e.g. [25].

As we shall be viewing polytopes as intersections of halfspaces, we take the second approach. There may still be different ways of defining a measure, and the best choice depends on the application one has in mind. The main purpose here is to convince the reader that counter examples to multigrid convergence are plenty on $\mathcal{P}^d$. As Theorem 1.1 only claims something to be a zero-set, the theorem will also hold for any measure absolutely continuous with respect the one introduced below.

A convex polytope can always be written in the form

$$P = \bigcap_{i=1}^{N} H^-_{u_i,t_i} \tag{3.1}$$

where $t_i \in \mathbb{R}$ and $u_i \in S^{d-1}$. The idea is to use the parameters t_i, u_i to parametrize polytopes by. We denote by $S^{d,N} \subseteq (S^{d-1})^N$ the open subset consisting of N-tuples of pairwise different vectors in S^{d-1}. A point will be written either as a vector $(u_1, \dots, u_N)$ or as an $N \times d$-matrix U. Then (3.1) is the solution set to the matrix inequality $Ux \leq t$.

First note that (3.1) is unbounded if and only if the inequality $Ux \leq 0$ has a non-trivial solution x and (3.1) is non-empty. The set where $Ux \leq 0$ has a non-trivial solution is closed in $S^{d,N}$. Let $S_c^{d,N} \subseteq (S^{d-1})^N$ denote the complement. Then $S^{d,N} \cap S_c^{d,N}$ is open in $(S^{d-1})^N$.

Next observe that (3.1) has non-empty interior exactly if there exists a solution x to $Ux < t$. This happens for (U,t) in an open subset

$$\mathcal{U}^{d,\leq N} \subseteq (S^{d,N} \cap S_c^{d,N}) \times \mathbb{R}^N.$$

A point $(U,t) \in \mathcal{U}^{d,\leq N}$ defines a polytope with exactly N facets if and only if for every $i = 1, \dots, N$ there is a solution to $\tilde{U}^i x < \tilde{t}^i$ where $\tilde{U}^i$ and $\tilde{t}^i$ are U and t except the ith row and the ith coordinate have changed sign, respectively. This is again an open subset $\mathcal{U}^{d,N} \subseteq \mathcal{U}^{d,\leq N}$.

Let $\mathcal{P}^{d,N} \subseteq \mathcal{P}^d$ be the subset consisting of polytopes with exactly N facets. Then $\mathcal{P}^d$ is the disjoint union of the subsets $\mathcal{P}^{d,N}$.

There is a surjective map

$$P : \mathcal{U}^{d,N} \to \mathcal{P}^{d,N}$$

given by (3.1). This is continuous with respect to the Hausdorff metric on $\mathcal{P}^{d,N}$, as one can see e.g. by using [16, Theorem 1.8.7]. If Σ_N is the N'th symmetric group acting on $\mathcal{U}^{d,N}$ by permutation of the pairs (u_i, t_i), then P is the quotient map.

Definition 3.1. *The measure on $\mathcal{P}^d$ whose restriction to $\mathcal{P}^{d,N}$ is $\mathcal{H}^{dN} \circ P^{-1}$ is denoted by ν.*

We introduce the following notation for $P \in \mathcal{P}^d$: $\mathcal{F}_k(P)$ denotes the set of k-faces of P. The facet with normal vector u_i is denoted by F_i. If P is simple, every $F \in \mathcal{F}_k(P)$ is the intersection of exactly $d-k$ facets. See e.g. [25] for details on the combinatorics of simple polytopes. We index the facets containing F by

$$I_1(F) = \{i_1^F, \dots, i_{d-k}^F\} \subseteq \{1, \dots, N\},$$

i.e. $F = \bigcap_{i \in I_1(F)} F_i$. The ordering is not important here. Let

$$I_2(F) = \{i \in \{1, \dots, N\} \backslash I_1(F) \mid F_i \cap F \neq \emptyset\}$$

index the facets intersecting F in a lower dimensional faces. If P is simple, this lower dimensional face must have dimension $k-1$.

Let $\mathcal{US}^{d,N}$ denote the set

$$\mathcal{US}^{d,N} = \{(U,t) \in \mathcal{U}^{d,N} \mid P(U,t) \text{ is simple}\}$$

and let $\mathcal{US}_\mu^{d,N}$, $\mu \in M$, denote the connected components of $\mathcal{US}^{d,N}$.

Proposition 3.2.

(i) For $I \subseteq \{1, \dots, N\}$, the set

$$G_I = \{(U, t) \in \mathcal{U}^{d,N} \mid \exists x \in \mathbb{R}^d : \forall i \in I : \langle x, u_i \rangle = t_i,\, Ux \leq t\}$$

is relatively closed in $\mathcal{U}^{d,N}$.

(ii) $\mathcal{U}^{d,N} \backslash \mathcal{US}^{d,N}$ is relatively closed in $\mathcal{U}^{d,N}$ and has $\mathcal{H}^{dN}$-measure 0.

(iii) For any $I \subseteq \{1, \dots, N\}$ of cardinality $|I| = d - k$, $P(U, t)$ has a k-face F with $I_1(F) = I$ for either no or all $(U, t) \in \mathcal{US}^{d,N}_\mu$.

Proof. (i) To see this, take a sequence $(U^k, t^k) \in G_{i_1,\dots,i_s}$ such that $(U^k, t^k) \to (U, t)$ inside $\mathcal{U}^{d,N}$. Then there is a sequence x^k with $\langle u^k_{i_j}, x^k \rangle = t^k_{i_j}$ and $U^k x^k \leq t^k$. If the x^k are bounded, there is a convergent subsequence $x^{k_n} \to x$ and it follows by continuity that $\langle u_{i_j}, x \rangle = t_{i_j}$ and $Ux \leq t$. If x^k is unbounded, choose a subsequence such that $|x^{k_n}| \to \infty$ and $\frac{x^{k_n}}{|x^{k_n}|}$ converges to $x \in S^{d-1}$. Then $U^{k_n} \frac{x^{k_n}}{|x^{k_n}|} \leq \frac{t^{k_n}}{|x^{k_n}|}$ and thus in the limit $Ux \leq 0$, contradicting $U \in S^{d,N}_c$.

(ii) If $P(U, t)$ is not simple, it has a vertex v solving $d + 1$ of the equations $\langle u_{i_j}, v \rangle = t_{i_j}$, $j = 1, \dots, d + 1$. The claim now follows from (i) and the fact that

$$G_{\{i_1,\dots,i_{d+1}\}} \subseteq \{(U, t) \in \mathcal{U}^{d,N} \mid \exists x \in \mathbb{R}^d : \forall j = 1, \dots, d + 1 : \langle u_{i_j}, x \rangle = t_{i_j}\},$$

since the latter has $\mathcal{H}^{dN}$-measure 0.

(iii) By the definition of simple polytopes, the set of $(U, t) \in \mathcal{US}^{d,N}_\mu$ having a vertex v with $I_1(v) = I$ is $G_I \cap \mathcal{US}^{d,N}_\mu$. This is closed by (i). On the other hand,

$$\langle u_i, v \rangle = t_i \text{ for } i \in I \text{ and } \langle u_i, v \rangle < t_i \text{ for } i \notin I. \tag{3.2}$$

Uniqueness of v shows that the system $\langle u_i, v \rangle = t_i$ for $i \in I$ may be inverted in a neighborhood of (U, t), yielding a solution to (3.2) and thus showing that $G_I \cap \mathcal{US}^{d,N}_\mu$ is also open. Hence $G_I \cap \mathcal{US}^{d,N}_\mu \in \{\mathcal{US}^{d,N}_\mu, \emptyset\}$, proving the $k = 0$ case.

Given I with $|I| = d - k$,

$$F = \bigcap_{i \in I} F_i \in \mathcal{F}_k(P(U, t)) \cup \{\emptyset\}$$

whenever $(U, t) \in \mathcal{US}^{d,N}_\mu$. If there is a $(U, t) \in \mathcal{US}^{d,N}_\mu$ and a $v \in \mathcal{F}_0(P(U, t))$ with $I \subseteq I_1(v)$, the $k = 0$ case shows that $\bigcap_{i \in I_1(v)} F_i \in \mathcal{F}_0(P(U, t))$ must hold for all $(U, t) \in \mathcal{US}^{d,N}_\mu$ and hence, in particular, $\bigcap_{i \in I} F_i \neq \emptyset$ for all $(U, t) \in \mathcal{US}^{d,N}_\mu$. If there is no $v \in \mathcal{F}_0(P(U, t))$ with $I \subseteq I_1(v)$, F can have no vertices and is hence empty. □

The proposition shows that all $P \in P(\mathcal{US}^{d,N}_\mu)$ have the same combinatorial structure. A path $(U(s), t(s))$ in $\mathcal{US}^{d,N}_\mu$ defines a path of vertex sets $\mathcal{F}_0(P(U(s), t(s)))$ and hence an isotopy of $P(U(s), t(s))$ preserving the combinatorial structure. We therefore speak of the images $P(\mathcal{US}^{d,N}_\mu) = \mathcal{P}^{d,N}_\mu \subseteq \mathcal{P}_d$ as the combinatorial isotopy classes.

3.2 Hit-or-miss transforms of polytopes

In order to study the asymptotic bias of a local digital estimator $\hat{V}_q$ applied to $P \in \mathcal{P}^d$, we must consider

$$E\hat{V}_q(P) = \sum_{l=1}^{2^{n^d}-1} w_l(a) EN_l(P \cap a\mathbb{L}_c).$$

By (2.5),

$$EN_l(P \cap a\mathbb{L}_c) = a^{-d} \det(\mathbb{L})^{-1} \mathcal{H}^d(P \ominus a\check{B}_l \backslash P \oplus a\check{W}_l).$$

Thus, we need to describe the volume of hit-or-miss transforms of polytopes.

Suppose $P \in \mathcal{P}^{d,N}$ is given by

$$P(U,t) = \bigcap_{i=1}^{N} H^-_{u_i,t_i}.$$

Let $X_{i,l}$ denote the set

$$\begin{aligned} X_{i,l} &= (H^-_{u_i,t_i} \ominus a\check{B}_l) \backslash (H^-_{u_i,t_i} \oplus a\check{W}_l) \\ &= H^-_{u_i,t_i - ah(B_l,u_i)} \backslash H^-_{u_i,t_i + ah(\check{W}_l,u_i)} \end{aligned}$$

for $l = 1, \dots, 2^{n^d} - 2$ and

$$\begin{aligned} X_{i,0} &= \mathbb{R}^d \backslash H^-_{u_i,t_i + ah(\check{C}^n_{0,0},u_i)}, \\ X_{i,2^{n^d}-1} &= H^-_{u_i,t_i - ah(C^n_{0,0},u_i)}. \end{aligned}$$

Then $\mathbb{R}^d$ is the disjoint union of the sets $X_{i,l}$ for $l = 0, \dots, 2^{n^d} - 1$. Hence it is also the disjoint union of the sets

$$X_{l_1,\dots,l_N} = \bigcap_{i=1}^{N} X_{i,l_i}$$

for $l_1, \dots, l_N \in \{0, \dots, 2^{n^d} - 1\}$.

We also use the multi index notation $X_L = X_{l_1,\dots,l_N}$ for $L \in \mathcal{L} = \{1, \dots, 2^{n^d} - 1\}^N$. We associate to an index $L \in \mathcal{L}$ the index sets $I^L = \{i \mid l_i \neq 2^{n^d} - 1\}$ and $J^L = \{i \mid l_i = 2^{n^d} - 1\}$. Moreover, we associate the face of P given by $F_L = \bigcap_{i \in I^L} F_i$. If P is simple, this is either $|I^L|$-dimensional or the empty face.

Lemma 3.3. *For $(U,t) \in \mathcal{US}^{d,N}_\mu$, the volume of $P(U,t)$ is given by a polynomial in $t_1, \dots, t_N$ with coefficients depending only on U:*

$$\mathcal{H}^d(P(U,t)) = \frac{1}{d!} \sum_{j_1,\dots,j_d=1}^{N} a_{j_1,\dots,j_d}(U) \prod_{s=1}^{d} t_{j_s}.$$

In fact, there are functions $a_j(u_1,\dots,u_s)$ for all $j,s=1,\dots,d$, defined whenever $(u_1,\dots,u_s)\in(S^{d-1})^s$ are linearly independent, such that for $F\in\mathcal{F}_q(P(U,t))$,

$$\begin{aligned}\mathcal{H}^q(F)=\frac{1}{q!}\sum_{v\in\mathcal{F}_0(F)}\sum_{\sigma\in\Sigma_q}\sum_{j_{d-q+1},\dots,j_d=1}^{d}\prod_{s=d-q+1}^{d}&a_{j_s}(u_{i_1^F},\dots,u_{i_{d-q}^F},u_{i^v_{\sigma(d-q+1)}},\dots,u_{i^v_{\sigma(s)}})\\&\times t_{i_1^F,\dots,i_{d-q}^F,i^v_{\sigma(d-q+1)},\dots,i^v_{\sigma(s)}}(j_s)\end{aligned}\tag{3.3}$$

where $t_{i_1,\dots,i_s}(j)=t_{i_j}$ and indices are chosen so that $I_1(v)=I_1(F)\cup\{i^v_{d-q+1},\dots,i^v_s\}$.

In particular, one may take

$$a_{j_1,\dots,j_d}(U)=\sum_{v\in\mathcal{F}_0(\bigcap_{k=1}^d F_{j_k})}\sum_{\sigma\in\Sigma_d}\prod_{s=1}^{d}a_{\sigma^{-1}(j_s)}(u_{i^v_{\sigma(1)}},\dots,u_{i^v_{\sigma(s)}}).\tag{3.4}$$

Each a_j is rotation invariant and depends analytically on $u_i,\dots,u_s$. For $s<j$, $a_j(u_1,\dots,u_s)=0$ and $a_s(u_1,\dots,u_s)>0$. If u_s is orthogonal to all u_i with $i<s$,

$$a_j(u_1,\dots,u_s)=\begin{cases}1 & \textit{for } j{=}s,\\ 0 & \textit{otherwise.}\end{cases}$$

We sometimes write $a_j(u_{i_1},\dots,u_{i_s})=a_{i_1,\dots,i_s}(j)$ to keep notation short.

Proof. The first equation is [16, Lemma 5.1.2]. The remaining claims follow by writing out the details of the proof of that lemma.

Define $a_j(u_1,\dots,u_s)\in\mathbb{R}$ such that the normalized projection of u_s onto the subspace $\mathrm{lin}(u_1,\dots,u_{s-1})^\perp$ is given by

$$\sum_{j=1}^{s}a_j(u_1,\dots,u_s)u_j.\tag{3.5}$$

We set $a_j(u_1,\dots,u_s)=0$ for $j>s$. The listed properties of a_j then follows immediately.

The idea is to use of the identity

$$\mathcal{H}^d(P)=\frac{1}{d}\sum_{i=1}^{N}h(P,u_i)\mathcal{H}^{d-1}(F_i)$$

inductively on the faces of P. The identity (3.3) clearly holds for $q=0$, the empty product being equal to 1.

Let $F\in\mathcal{F}_q(P)$ be given and let $F'\in\mathcal{F}_{q-1}(P)$ be a face of F. We may assume that $I_1(F')=I_1(F)\cup\{i^{F'}_{d-q+1}\}$. The normal vector $u(F,F')$ of F at F' is exactly the normalized projection of u_s onto $\mathrm{lin}(u_{i_1^F},\dots,u_{i_{s-1}^F})^\perp$ given by (3.5). It follows that

$$h(F,u(F,F'))=\sum_{j=1}^{d-q+1}a_j(u_{i_1^F},\dots,u_{i_{d-q}^F},u_{i^{F'}_{d-q+1}})t_{i_1^F,\dots,i_{d-q}^F,i^{F'}_{d-q+1}}(j).$$

Thus by induction,

$$\begin{aligned}
\mathcal{H}^q(F) &= \frac{1}{q} \sum_{F' \in \mathcal{F}_{q-1}(F)} \sum_{j=1}^{d-q+1} a_{i_1^F,\dots,i_{d-q}^F,i_{d-q+1}^{F'}}(j) t_{i_1^F,\dots,i_{d-q}^F,i_{d-q+1}^{F'}}(j) \\
&\quad \times \frac{1}{(q-1)!} \sum_{v \in \mathcal{F}_0(F')} \sum_{\sigma \in \Sigma_{q-1}} \sum_{j_{d-q+2},\dots,j_d=1}^{d} \prod_{s=d-q+2}^{d} a_{i_1^F,\dots,i_{d-q}^F,i_{d-q+1}^{F'},i^v_{\sigma(d-q+2)},\dots,i^v_{\sigma(s)}}(j_s) \\
&\quad \times t_{i_1^F,\dots,i_{d-q}^F,i_{d-q+1}^{F'},i^v_{\sigma(d-q+2)},\dots,i^v_{\sigma(s)}}(j_s) \\
&= \frac{1}{q!} \sum_{F' \in \mathcal{F}_{q-1}(F)} \sum_{v \in \mathcal{F}_0(F')} \sum_{\sigma \in \Sigma_{q-1}} \sum_{j_{d-q+1},\dots,j_d=1}^{d} a_{i_1^F,\dots,i_{d-q}^F,i_{d-q+1}^{F'}}(j_{d-q+1}) \\
&\quad \times t_{i_1^F,\dots,i_{d-q}^F,i_{d-q+1}^{F'}}(j_{d-q+1}) \prod_{s=d-q+2}^{d} a_{i_1^F,\dots,i_{d-q}^F,i_{d-q+1}^{F'},i^v_{\sigma(d-q+2)},\dots,i^v_{\sigma(s)}}(j_s) \\
&\quad \times t_{i_1^F,\dots,i_{d-q}^F,i_{d-q+1}^{F'},i^v_{\sigma(d-q+2)},\dots,i^v_{\sigma(s)}}(j_s) \\
&= \frac{1}{q!} \sum_{v \in \mathcal{F}_0(F)} \sum_{\sigma \in \Sigma_q} \sum_{j_{d-q+1},\dots j_d=1}^{d} \prod_{s=d-q+1}^{d} a_{i_1^F,\dots,i_{d-q}^F,i^v_{\sigma(d-q+1)},\dots,i^v_{\sigma(s)}}(j_s) \\
&\quad \times t_{i_1^F,\dots,i_{d-q}^F,i^v_{\sigma(d-q+1)},\dots,i^v_{\sigma(s)}}(j_s).
\end{aligned}$$

□

Given a multi index $L \in \mathcal{L}$, we use the notation for $i \in I^L$:

$$\begin{aligned}
\beta_i &= -h(B_{l_i}, u_i), \\
\omega_i &= h(\check{W}_{l_i}, u_i), \\
\zeta_i &= -h(C^n_{0,0}, u_i), \\
\delta_L(U) &= \prod_{i \in I^L} \mathbb{1}_{\{\beta_i > \omega_i\}}.
\end{aligned}$$

Lemma 3.4. *Let $(U,t) \in \mathcal{US}^{d,N}_\mu$ and $L \in \mathcal{L}$ be given. Then $\mathcal{H}^d(X_L)$ is a homogeneous polynomial of degree d in the numbers $(t_i + a\beta_i)$ and $(t_i + a\omega_i)$ for $i \in I_1(F_L)$ and $(t_i + a\zeta_i)$ for $i \in I_2(F_L)$ with coefficients depending only on U. In particular, it is a homogeneous polynomial of degree d in $a, t_1, \dots, t_N$ given by*

$$\begin{aligned}
\mathcal{H}^d(X_L) = \delta_L(U) \frac{1}{d!} &\sum_{j_1,\dots,j_d \in I_1(F_L) \cup I_2(F_L)} a_{j_1,\dots,j_d}(U) \\
&\times \prod_{i \in I_1(F_L)} \sum_{s_i=1}^{n(i)} \binom{n(i)}{s_i} a^{s_i} t_i^{n(i)-s_i} (\beta_i^{s_i} - \omega_i^{s_i}) \prod_{j \in I_2(F_L)} (t_j + a\zeta_j)^{n(j)}.
\end{aligned} \tag{3.6}$$

In particular, $\mathcal{H}^d(X_L) = 0$ if $F_L = \emptyset$.

As a polynomial in a, the lowest order term is

$$a^{|I_1(F_L)|} \delta_L(U) \frac{1}{d!} \sum_{j_1,\dots,j_d \in I_1(F_L) \cup I_2(F_L)} a_{j_1,\dots,j_d}(U) \prod_{i \in I_1(F_L)} n(i) t_i^{n(i)-1} (\beta_i - \omega_i) \prod_{j \in I_2(F_L)} t_j^{n(j)}.$$

Proof. We must compute the volume of

$$X_L = \bigcap_{i\in I^L} (H^-_{u_i,t_i+a\beta_i} \backslash H^-_{u_i,t_i+a\omega_i}) \cap \bigcap_{j\in J^L} H^-_{u_j,t_j+a\zeta_j}.$$

Clearly, if $\delta_L(U) = 0$, this is empty. For $I \subseteq I^L$ let

$$X_I = \bigcap_{i\in I} H^-_{u_i,t_i+a\omega_i} \cap \bigcap_{j\in I^L\backslash I} H^-_{u_j,t_j+a\beta_j} \cap \bigcap_{k\in J^L} H^-_{u_k,t_k+a\zeta_k}.$$

Then $X_I \cap X_J = X_{I\cup J}$ and

$$X_L = X_\emptyset \backslash \bigcup_{i\in I^L} X_{\{i\}}.$$

For a sufficiently small, all $X_I \in \mathcal{P}^{d,N}_\mu$ by openness of $\mathcal{US}^{d,N}_\mu$. Let $Q(t) = \mathcal{H}^d(P(U,t))$ be the polynomial in (3.6), write

$$\xi_i(I) = \omega_i 1_{i\in I} + \beta_i 1_{i\in I^L\backslash I} + \zeta_i 1_{i\in J^L},$$

and for a given index set $j_1, \dots, j_d$, let

$$n(i) = |\{k \in \{1, \dots, d\} : j_k = i\}|.$$

Then the inclusion-exclusion principle yields:

$$\begin{aligned}
\mathcal{H}^d(X_L) = &\sum_{I\subseteq I^L} (-1)^{|I|}\mathcal{H}^d(X_I)\\
= &\sum_{I\subseteq I^L} (-1)^{|I|}Q(t_1 + a\xi_1(I), \dots, t_N + a\xi_N(I))\\
= &\frac{1}{d!}\sum_{j_1,\dots,j_d} a_{j_1,\dots,j_d} \sum_{I\subseteq I^L}(-1)^{|I|}\prod_{i\in I}(t_i + a\omega_i)^{n(i)}\\
&\times \prod_{j\in I^L\backslash I}(t_j + a\beta_j)^{n(j)} \prod_{k\in J^L}(t_k + a\zeta_k)^{n(k)}\\
= &\frac{1}{d!}\sum_{j_1,\dots,j_d} a_{j_1,\dots,j_d} \prod_{i\in I^L}((t_i + a\beta_i)^{n(i)} - (t_i + a\omega_i)^{n(i)}) \prod_{j\in J^L}(t_j + a\zeta_j)^{n(j)}\\
= &\frac{1}{d!}\sum_{j_1,\dots,j_d\in I_1(F_L)\cup I_2(F_L)} a_{j_1,\dots,j_d} \prod_{i\in I_1(F_L)} \sum_{s_i=1}^{n(i)} \binom{n(i)}{s_i} a^{s_i} t_i^{n(i)}(\beta_i^{s_i} - \omega_i^{s_i})\\
&\times \prod_{j\in I_2(F_L)} (t_j + a\zeta_j)^{n(j)}.
\end{aligned}$$

The last equality follows from the fact that $I^L = I_1(F_L)$ and since only terms with $I_1(F_L) \subseteq \{j_1, \dots, j_d\}$ contribute, the description of $a_{j_1,\dots,j_d}$ in Lemma 3.3 shows that $a_{j_1,\dots,j_d} = 0$ unless $\{j_1, \dots, j_d\} \subseteq I_1(F_L) \cup I_2(F_L)$. □

3.3 Asymptotic behavior of the estimators

For $x \in X_{l_1,\dots,l_N}$,

$$(x + aC^n_{0,0}) \cap P = x + a\bigcap_{i=1}^{N} B_{l_i}.$$

We denote the configuration $\bigcap_{i=1}^N B_{l_i}$ by $B_{l_1,\dots,l_N}$ and the corresponding weight is denoted by $w_{l_1,\dots,l_N}(a)$ or $w(\bigcap_{i=1}^N B_{l_i}, a)$. Note that if one of the l_i equals 0, then $B_{l_1,\dots,l_N} = B_0 = \emptyset$. For $L \in \mathcal{L}$, we also use the notation B_L and $w_L(a)$. The preceding section yields the following formula:

Corollary 3.5. *Let $P \in \mathcal{P}^{d,N}_\mu$ be a polytope. Then for $l \neq 0$,*

$$EN_l(P \cap a\mathbb{L}_c) = a^{-d}\det(\mathbb{L})^{-1}\sum_{L\in\mathcal{L}} \mathcal{H}^d(X_L)\mathbb{1}_{\{B_L = B_l\}}.$$

It follows that

$$E\hat{V}_q(P) = a^{-d}\det(\mathbb{L})^{-1}\sum_{L\in\mathcal{L}} w_L(a)\mathcal{H}^d(X_L).$$

where $\mathcal{H}^d(X_L)$ is given by Lemma 3.4.

For a local estimator $\hat{V}_q$, we introduce the following notation:

$$\mathcal{E}^N = \{P \in \mathcal{P}^{d,N} \mid \lim_{a\to 0} E\hat{V}_q(P) \text{ exists }\},$$
$$\mathcal{V}^N = \{P \in \mathcal{E}^N \mid \lim_{a\to 0} E\hat{V}_q(P) = V_q(P)\}.$$

Similarly, for a combinatorial isotopy class $\mathcal{P}^{d,N}_\mu$ of simple polytopes, $\mathcal{E}^N_\mu = \mathcal{P}^{d,N}_\mu \cap \mathcal{E}^N$ and $\mathcal{V}^{d,N}_\mu = \mathcal{P}^{d,N}_\mu \cap \mathcal{V}^N$.

Lemma 3.6. *There exist measurable subsets V^N_μ, E^N_μ of $(S^{d-1})^N$ satisfying*

$$\tilde{\mathcal{E}}^N_\mu := (E^N_\mu \times \mathbb{R}^N) \cap \mathcal{US}^N_\mu \subseteq \mathcal{E}^N_\mu,$$
$$\tilde{\mathcal{V}}^N_\mu := (V^N_\mu \times \mathbb{R}^N) \cap \mathcal{US}^N_\mu \subseteq \mathcal{V}^N_\mu,$$
$$\mathcal{H}^{dN}(\mathcal{E}^N_\mu \backslash \tilde{\mathcal{E}}^N_\mu) = \mathcal{H}^{dN}(\mathcal{V}^N_\mu \backslash \tilde{\mathcal{V}}^N_\mu) = 0,$$

such that on $\tilde{\mathcal{E}}^N_\mu$, $\lim_{a\to 0} E\hat{V}_q(P(U,t))$ is a polynomial in $t_1,\dots,t_N$ with coefficients depending only on U and on $\tilde{\mathcal{V}}^N_\mu \subseteq \tilde{\mathcal{E}}^N_\mu$, this is homogeneous of degree q.

Proof. Let

$$E^N_\mu = \{U \in (S^{d-1})^N \mid \mathcal{H}^N(\mathcal{E}^N_\mu \cap (\{U\}\times\mathbb{R}^N)) > 0\}.$$

Then

$$\mathcal{H}^{dN}(\mathcal{E}^N_\mu \backslash \tilde{\mathcal{E}}^N_\mu) = \int_{(S^{d-1})^N\backslash E^N_\mu}\int_{\mathbb{R}^N} 1_{\mathcal{E}^N_\mu} d\mathcal{H}^N d\mathcal{H}^{(d-1)N} = 0.$$

By Lemma 3.4 and Corollary 3.5, $E\hat{V}_q(P)$ has the form

$$\sum_{\substack{n_1,\dots,n_N=0,\\ \sum n_i \le d}}^{d-1} H_{n_1,\dots,n_N}(a)\prod_{i=1}^N t_i^{n_i}.$$

For a fixed $U \in E_\mu^N$, the function $H_{n_1,\dots,n_N}(a)$ depends only on a and the limit when $a \to 0$ exists for all $t_1, \dots, t_N$ in a set of non-zero $\mathcal{H}^N$-measure. It follows from linear independence of the monomials $\prod_{i=1}^N t_i^{n_i}$ that each limit $\lim_{a\to 0} H_{n_1,\dots,n_N}(a)$ must exist. Denote this limit by $H_{n_1,\dots,n_N}$. Then

$$\lim_{a\to 0} \sum_{\substack{n_1,\dots,n_N=0,\\ \sum n_i \le d}}^{d-1} H_{n_1,\dots,n_N}(a) \prod_{i=1}^N t_i^{n_i} \quad \sum_{\substack{n_1,\dots,n_N=0,\\ \sum n_i \le d}}^{d-1} H_{n_1,\dots,n_N} \prod_{i=1}^N t_i^{n_i} \tag{3.7}$$

and in particular, $\tilde{\mathcal{E}}_\mu^N \subseteq \mathcal{E}_\mu^N$.

Similarly, define

$$V_\mu^N = \{U \in (S^{d-1})^N \mid \mathcal{H}^N(\mathcal{V}_\mu^N \cap (\{U\} \times \mathbb{R}^N)) > 0\}.$$

Recall that

$$V_q(P) = \sum_{F \in \mathcal{F}_q(P)} \gamma(F, P)\mathcal{H}^q(F) \tag{3.8}$$

where

$$\gamma(F, P) = \frac{\mathcal{H}^{d-q-1}(\mathrm{pos}(u_{i_1^F}, \dots, u_{i_{d-q}^F}) \cap S^{d-1})}{\mathcal{H}^{d-q-1}(S^{d-q-1})}$$

is the external angle of P at F and clearly depends only on U. By Lemma 3.3, each $\mathcal{H}^q(F)$ is a homogeneous polynomial of degree q in $t_1, \dots, t_N$. Thus, for $U \in E_\mu^N$, either $\mathcal{H}^N(\mathcal{V}_\mu^N \cap (\{U\} \times \mathbb{R}^N)) = 0$ or the coefficients of (3.7) and (3.8) must agree. In particular, $H_{n_1,\dots,n_N} = 0$ unless $\sum n_i = q$. $\square$

Let $\tilde{\mathcal{E}}^N = \bigcup_{\mu \in M} \tilde{\mathcal{E}}_\mu^N$ and $\tilde{\mathcal{V}}^N = \bigcup_{\mu \in M} \tilde{\mathcal{V}}_\mu^N$.

Corollary 3.7. *Given a local estimator $\hat{V}_q$, there is a local estimator $\hat{V}_q'$ with polynomial weights such that on $\tilde{\mathcal{E}}^N$, $\lim_{a\to 0} E\hat{V}_q(P) = \lim_{a\to 0} E\hat{V}_q'(P)$. Moreover, there is an estimator $\hat{V}_q''$ with homogeneous weights of degree q and $\lim_{a\to 0} E\hat{V}_q''(P) = V_q(P)$ on $\tilde{\mathcal{V}}^N$.*

Proof. By Lemma 3.4 and Corollary 3.5, $E\hat{V}_q(P)$ takes the form

$$E\hat{V}_q(P) = \sum_{l=1}^{2^{n^d}-1} w_l(a) \sum_{k=0}^d a^{k-d} c_{l,k}(P)$$

where $c_{l,k}(P) \in \mathbb{R}$ are coefficients depending only on $P \in P(\mathcal{US}^{d,N})$.

For each $k = 0, \dots, d$, choose $M_k \subseteq \{1, \dots, 2^{n^d} - 1\}$ maximal with no linear relation between the coefficients $c_{l,k}(P)$ with $l \in M_k$ that holds for all $P \in P(\tilde{\mathcal{E}}^N)$. In particular, for $l \in M_k$ there are functions

$$w_{l,k}(a) = w_l(a) + \sum_{s \notin M_k} \alpha_{l,k}^s w_s(a)$$

for suitable $\alpha_{l,k}^s \in \mathbb{R}$ such that

$$\lim_{a\to 0} E\hat{V}_q(P) = \lim_{a\to 0} \sum_{k=0}^d \sum_{l \in M_k} w_{l,k}(a) a^{k-d} c_{l,k}(P) \tag{3.9}$$

for all $P \in P(\tilde{\mathcal{E}}^N)$. By the proof of Lemma 3.6, the limit exists in each degree on $P(\tilde{\mathcal{E}}^N)$.

Choose $P_m \in P(\tilde{\mathcal{E}}^N)$ for $m \in M_k$ such that the vectors $(c_{l,k}(P_m))_{l\in M_k}$ are linearly independent. The existence of the limit (3.9) for all P_m yields an invertible linear system, and solving this shows that also

$$w_{l,k} := \lim_{a\to 0} w_{l,k}(a)a^{k-d}$$

exists for all l.

Let W be the formal vector space spanned by the functions $w_l(a)$ and let W_q denote the subspace spanned by $\{w_{l,q}(a) \mid l \in M_q\}$.

We show by induction that it is possible to choose polynomials $\tilde{w}_{l,k}(a)$ of degree at least $d-k$ such that $\lim_{a\to 0} \tilde{w}_{l,k}(a)a^{k-d} = w_{l,k}$ consistently in the sense that it defines a linear map $\operatorname{span}\{W^q, q = 0,\dots,d\} \to \mathrm{Pol}_d$ where Pol_d is the set of polynomials with $\mathbb{R}$-coefficients of degree at most d.

For $k = 0$, choose $\tilde{w}_{l,0}(a) = w_{l,0}a^d$. Suppose now that we have chosen $\tilde{w}_{l,k}(a)$ for all $k < q$ defining a map $\operatorname{span}\{W^k, k < q\} \to \mathrm{Pol}_d$.

We know $\lim_{a\to 0} w(a)a^{q-d}$ exists for all $w(a) \in W^q$. Choose a maximal set of independent $w_{l_i,k_i}(a) \in W^q$, $i \in I$, with $k_i < q$ and extend this by $w_1^q,\dots,w_m^q$ to a basis of W^q. Then $\lim_{a\to 0} a^{q-d}w_{l_i,k_i}(a) = 0$ and $\lim_{a\to 0} a^{q-d}w_j^q(a) = w_j^q$. For

$$w_{l,q} = \sum_{i\in I} \alpha_i w_{l_i,k_i}(a) + \sum_{j=1}^m \beta_j w_j^q(a)$$

define

$$\tilde{w}_{l,q}(a) = \sum_{i\in I} \alpha_{l,q}^i \tilde{w}_{l_i,k_i}(a) + \sum_{j=1}^m \beta_{l,q}^j w_j^q a^{d-q}.$$

This clearly extends the map $\operatorname{span}\{W^k, k < q\} \to \mathrm{Pol}_d$ to $\operatorname{span}\{W^k, k \le q\} \to \mathrm{Pol}_d$, completing the induction step.

Extending the map $\operatorname{span}\{W^q, q = 0,\dots,d\} \to \mathrm{Pol}_d$ trivially to a map $W \to \mathrm{Pol}_d$ yields a way of choosing the $w_l(a)$ as polynomials.

The proof of the second claim is similar, except we now choose P_m only from the set $\tilde{\mathcal{V}}^N$. By Lemma 3.6, $w_{l,k} = 0$ for $k \neq q$. Thus the inductive construction yields an estimator with homogeneous weights. □

3.4 Intrinsic volumes of positive degree

We are finally ready to prove Theorem 1.1 which we restate as follows:

Theorem 3.8. *Let $\hat{V}_q$ be any local algorithm for V_q for $1 \le q \le d-1$*

If $d-q$ is odd, $\hat{V}_q$ is asymptotically biased ν-almost everywhere on $\mathcal{E}^{d,N}$.

For $d-q$ even, $\hat{V}_q$ is asymptotically biased ν-almost everywhere on $\mathcal{E}_\mu^N$ for all combinatorial isotopy classes $\mu \in M$ corresponding to polytopes having a $(d-q)$-face which is combinatorially isotopic to $\bigoplus_{i=1}^{d-q}[0, e_i]$. In particular, $\hat{V}_q$ is asymptotically biased on a set of positive ν-measure.

Proof. Suppose we are given an estimator $\hat{V}_q$. Fix a combinatorial isotopy class $\mathcal{US}^{d,N}_{\mu}$. We want to show that $\mathcal{H}^{dN}(\mathcal{V}^N_{\mu}) = 0$. It is enough to show $\mathcal{H}^{(d-1)N}(V^N_{\mu}) = 0$.

By Corollary 3.7, we may assume that the weights are homogeneous of degree q. Then

$$\lim_{a \to 0} E\hat{V}_q(P) = \det(\mathbb{L})^{-1} \sum_{L \in \mathcal{L}} w_L A^q_L(U,t) \tag{3.10}$$

where A^q_L is coefficient in front of a^{d-q} in the formula (3.6) for $\mathcal{H}^{d-1}(X_L)$. We write $w'_L = \det(\mathbb{L})^{-1} w_L$ to shorten notation. In particular, (3.10) is a homogeneous polynomial in $t_1, \dots, t_N$ of degree q. On $\tilde{\mathcal{V}}^N_{\mu}$, this must equal

$$V_q(P) = \frac{1}{q!} \sum_{F \in \mathcal{F}_q(P)} \gamma(F,P)\mathcal{H}^q(F). \tag{3.11}$$

Choose a $(d-q)$-face $F_I = \bigcap_{i \in I} F_i$ with $|I| = q$. We want to compare the coefficients in front of $\prod_{i \in I} t_i$. Denote the coefficient in (3.10) by H_I and the one in (3.11) by G_I. Then H_I must equal G_I on $\tilde{\mathcal{V}}^N_{\mu}$. Both H_I and G_I depend only on $U \in (S^{d-1})^N$. In order to show that $\mathcal{H}^{(d-1)N}(V^N_{\mu}) = 0$, it is enough to show that almost all points in V^N_{μ} have a small neighborhood $W \subseteq (S^{d-1})^N$ with

$$\mathcal{H}^{(d-1)N}(W \cap \{H_I = G_I\}) = 0.$$

For $c_1 \neq c_2 \in C^n_{0,0}$, let H_{c_1,c_2} denote the hyperplane $\{x \in \mathbb{R}^d \mid \langle x, c_1 \rangle = \langle x, c_2 \rangle\}$. Let

$$D = \bigcup_{c_1 \neq c_2 \in C^n_{0,0}} H_{c_1,c_2}.$$

Observe that for a set $S \subseteq C^n_{0,0}$ and a connected component E in $S^{d-1} \backslash D$, there is a unique $s \in S$ such that $h(S,u) = \langle s, u \rangle$ for all $u \in E$. Moreover, all the indicator functions δ_l are constant on E.

Since $\mathcal{H}^{d-1}(D) = 0$, almost all $(u_1, \dots, u_N) \in V^N_{\mu}$ belong to $(S^{d-1} \backslash D)^N$. Let such $U \in V^N_{\mu} \cap (S^{d-1} \backslash D)^N$ be given. Choose a small connected neighborhood W contained in $\mathcal{US}^{d,N}_{\mu} \cap ((S^{d-1} \backslash D)^N \times \mathbb{R}^N)$. Then there are vectors $b^i_l \in B_l \cup \{0\}$, $w^i_l \in \check{W}_l \cup \{0\}$, and $c_i \in C^n_{0,0}$ such that

$$\begin{aligned} h(B_l, u_i)\delta_l(u_i) &= \langle b^i_l, u_i \rangle, \\ h(\check{W}_l, u_i)\delta_l(u_i) &= \langle w^i_l, u_i \rangle, \\ h(C^n_{0,0}, u_i) &= \langle c_i, u_i \rangle, \end{aligned}$$

whenever $(u_1, \dots, u_N) \in W$. Thus H_I has the form

$$\begin{aligned} H_I(U) = \sum_{L \in \mathcal{L}} w'_L \mathbb{1}_{I \subseteq I_1(F_L) \cup I_2(F_L)} &\sum_{j_1, \dots, j_d \in I_1(F_L) \cup I_2(F_L)} d_{j_1,\dots,j_d} a_{j_1,\dots,j_d}(U) \\ \times \prod_{i \in I_1(F_L)} (\langle b^i_{l_i}, u_i \rangle^{e(i)} - \langle w_{l_i}, u_i \rangle^{e(i)}) &\prod_{j \in I_2(F_L)} \langle c_j, u_j \rangle^{e(j)} \end{aligned}$$

on W. Here $d_{j_1,\dots,j_d}$ are certain constants and $e(i)$ are certain exponents with

$$\sum_{i \in I_1(F_L) \cup I_2(F_L)} e(i) = d - q.$$

In particular, H_I is an analytic function, depending only on the u_i with $i \in I \cup I_2(F_I)$.

Similarly, by (3.11) and Lemma 3.3

$$G_I(U) = \frac{1}{q!} \sum_{\substack{F \in \mathcal{F}_q(P) \\ F \cap F_I \neq \emptyset}} \gamma(F,P) \sum_{v \in F} \sum_{\sigma \in \Sigma_q} \sum_{(j_{d-q+1},\dots,j_d) \in J_{v,\sigma}} \prod_{s=d-q+1}^{d} a_{i_1^F,\dots,i_{d-q}^F, i^v_{\sigma(d-q+1)},\dots,i^v_{\sigma(s)}}(j_s)$$

where $J_{v,\sigma}$ are certain index sets. Each $\gamma(F,P)$ is an analytic function of $u_{i_1^F},\dots,u_{i_{d-q}^F}$ which is defined whenever $u_{i_1^F},\dots,u_{i_{d-q}^F}$ are linearly independent. This follows from Schläfli's formula [15], see also [1], according to which $\gamma(F,P)$ is analytic as a function of the angles between the faces in $\operatorname{pos}(u_{i_1^F},\dots,u_{i_{d-q}^F})$, and these angles can again be expressed analytically as functions of $u_{i_1^F},\dots,u_{i_{d-q}^F}$. It follows that G_I is analytic on W.

The formulas for H_I and G_I, initially defined on W, naturally extend to analytic functions $\bar{H}_I, \bar{G}_I : W' \to \mathbb{R}$ where $W' \subseteq (S^{d-1})^{|I \cup I_2(F_I)|}$ is the largest connected subset containing W and such that $u_{i_1^v},\dots,u_{i_d^v}$ are linearly independent for every $v \in \mathcal{F}_0(F_I)$.

Choose a path through independent unit vectors inside $\operatorname{lin}(u_i, i \in I)^{|I|}$ from $(u_i)_{i \in I}$ to an orthonormal frame $(u_i')_{i \in I}$. Next, for each u_j with $j \in I_2(F)$, choose a path inside $\operatorname{lin}(u_j, u_i, i \in I) \setminus \operatorname{lin}(u_i, i \in I)$ from u_j to its normalized projection onto $\operatorname{lin}(u_i, i \in I)^\perp$ denoted by u_j'. Together, this defines a path inside W' from $(u_i)_{i \in I \cup I_2(F_I)}$ to $(u_i')_{i \in I \cup I_2(F_I)}$ such that the u_i' with $i \in I$ are orthogonal and each u_j' with $j \in I_2(F_I)$ is orthogonal to all u_i' with $i \in I$.

By Lemma 3.4, a term with index $j_1,\dots,j_d$ in the summation formula for $\mathcal{H}^d(X_L)$ can only contribute a $\prod_{i \in I} t_i$ term if $I \subseteq \{j_1,\dots,j_d\}$, and by Lemma 3.3, if $i_s^v \in I$, then

$$a_{j_s}(u'_{i_1^v},\dots,u'_{i_s^v}) = \begin{cases} 1, & \text{for } j_s = s, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, if $i_s^v \notin I$, and $j_s \in I$,

$$a_{j_s}(u'_{i_1^v},\dots,u'_{i_s^v}) = 0.$$

Thus, $a_{j_1,\dots,j_d}$ can only be non-zero if every element of I appears exactly once in $j_1,\dots,j_d$.

It follows from the formula for $\mathcal{H}^d(X_L)$ given in Lemma 3.4 that the term $\prod_{i \in I} t_i$ can only appear if $I \subseteq I_2(F_L)$. Define the index set

$$J_L = \{j_1,\dots,j_d \mid \exists v \in \mathcal{F}_0(F_I \cap F_L) : j_1,\dots,j_d \in I_1(v), \forall j \in I : n(j) = 1\}.$$

Then the coefficient in front of $\prod_{i \in I} t_i$ in the formula for $\mathcal{H}^d(X_L)$ extended to the point $(u_i')_{i \in I \cup I_2(F_I)}$ has the form

$$\frac{1}{q!} \sum_{j_1,\dots,j_d \in J_L} a_{j_1,\dots,j_d} \prod_{i \in I_1(F_L)} (\langle b^i_{l_i}, u_i' \rangle^{n(i)} - \langle w^i_{l_i}, u_i' \rangle^{n(i)}) \prod_{j \in I_2(F_L) \setminus I} \langle c_j, u_j' \rangle^{n(j)}.$$

and thus

$$\bar{H}_I((u_i')_{i\in I\cup I_2(F_I)}) = \frac{1}{q!}\sum_{L:I\subseteq I_2(F_L)} w_L' \sum_{j_1,\dots,j_d\in J_L} a_{j_1,\dots,j_d} \times \prod_{i\in I_1(F_L)} (\langle b_{l_i}^i, u_i'\rangle^{n(i)} - \langle w_{l_i}^i, u_i'\rangle^{n(i)}) \prod_{j\in I_2(F_L)\setminus I} \langle c_j, u_j'\rangle^{n(j)}.$$

On the other hand, $\bar{G}_I$ is given by

$$\bar{G}_I((u_i')_{i\in I\cup I_2(F_I)}) = \sum_{F\in\mathcal{F}_q(P):F\cap F_I\in\mathcal{F}_0(P)} \gamma(F,P) = V_0(F_I) = 1.$$

This follows from (3.3) because if $t_{i_1^F,\dots,i_{d-q}^F,i_{\sigma(d-q+1)}^v,\dots,i_{\sigma(s)}^v}(j_s) = t_i$ for some $i\in I$, then i must be the j_sth coordinate in $(i_1^F,\dots,i_{d-q}^F,i_{\sigma(d-q+1)}^v,\dots,i_{\sigma(s)}^v)$. But then

$$a_{j_s}(u'_{i_1^F},\dots,u'_{i_{d-q}^F},u'_{i_{\sigma(d-q+1)}^v},\dots,u'_{i_{\sigma(s)}^v}) = 0$$

unless $j_s = s$. Hence $\{i_{d-q+1}^v,\dots,i_s^v\} = I$.

Suppose $d-q$ is odd and $q>0$. Choose a rotation $R\in \mathrm{SO}(d)$ changing all signs in $\mathrm{lin}(u_i', i\in I)^\perp$. This is possible because $\dim(\mathrm{lin}(u_i', i\in I)^\perp) = d-q < d$. This clearly preserves $\bar{G}_I = 1$ since the orthogonality properties among the u_i' are not changed. Since the $a_{j_1,\dots,j_d}$ are rotation invariant and $d-q$ is odd, $\bar{H}_I$ changes sign. As $\mathrm{SO}(d)$ is connected, there is a path from $(u_i')_{i\in I\cup I_2(F_I)}$ to $(Ru_i')_{i\in I\cup I_2(F_I)}$ inside W'. It follows that $\bar{H}_I$ and $\bar{G}_I$ cannot agree everywhere on W'. As they are both analytic and W' is connected, $\mathcal{H}^{(d-1)N}(W'\cap\{\bar{H}_I = \bar{G}_I\}) = 0$. This proves the claim in the case where $d-q$ is odd.

If $d-q$ is even, we assume that $\mathcal{US}_\mu^N$ is chosen such that the elements have a $(d-q)$-face which is combinatorially isotopic to $[0,1]^{d-q}$. Assume that F_I is this face. Let $\bar{G}_{I'} = 0$ for $I\subsetneq I'$. It is enough to show that

$$\mathcal{H}^{(d-1)N}(W'\cap\{\bar{H}_{I'} = \bar{G}_{I'}\}) = 0$$

for some $I\subseteq I'$ since $V_\mu^N \subseteq A := \bigcap_{I\subseteq I'}\{\bar{H}_{I'} = \bar{G}_{I'}\}$.

Since $(u_i')_{i\in I_2(F_I)}$ is exactly the set of normal vectors of $F_I\subseteq \mathrm{aff}(F_I)$, it is possible to choose a path from $(u_i')_{i\in I_2(F_I)}$ inside $\mathrm{lin}(F_I)$ to $(u_i'')_{i\in I_2(F_I)}$ such that these are the normal vectors $\{\pm v_1,\dots,\pm v_{d-q}\}$ of an orthogonal box of the form $\bigoplus_{i=1}^{d-q}[0,v_i]$. This ensures that for all $v\in\mathcal{F}_0(F_I)$, the u_i'' with $i\in I_1(v)$ are orthogonal. It follows from the above reasoning that $a_{j_1,\dots,j_d} = 0$ unless $j_s = s$ for all s. By (3.4), $a_{1,\dots,d} = q!$ and hence

$$\bar{H}_I((u_i'')_{i\in I\cup I_2(F_I)}) = \sum_{L:I\subseteq I_2(F_L)} w_L' \prod_{i\in I_1(F_L)} (\langle b_{l_i}^i, u_i''\rangle - \langle w_{l_i}^i, u_i''\rangle) \prod_{j\in I_2(F_L)\setminus I} \langle c_j, u_j''\rangle.$$

A similar argument for I' with $I\subseteq I'\subseteq I\cup I_2(F_I)$ shows that the coefficient $\bar{H}_{I'}$ in front of $\prod_{i\in I'} t_i$ is

$$\bar{H}_{I'}((u_i'')_{i\in I\cup I_2(F_I)}) = \sum_{L:I'\subseteq I_2(F_L)} w_L' \prod_{i\in I_1(F_L)} (\langle b_{l_i}^i, u_i''\rangle - \langle w_{l_j}^i, u_i''\rangle) \prod_{j\in I_2(F_L)\setminus I'} \langle c_j, u_j''\rangle. \tag{3.12}$$

Suppose $(u_i'')_{i\in I\cup I_2(F_I)} \in A$. Then (3.12) vanishes. If $I \subsetneq I'$, multiplication by $\prod_{j\in I'\setminus I}\langle c_j, u_j''\rangle$ shows that also

$$\bar{K}_{I'}((u_i'')_{i\in I\cup I_2(F_I)}) := \sum_{L:I'\subseteq I_2(F_L)} w_L' \prod_{i\in I_1(F_L)} (\langle b_{l_i}^i, u_i''\rangle - \langle w_{l_i}^i, u_i''\rangle) \prod_{j\in I_2(F_L)\setminus I} \langle c_j, u_j''\rangle = 0.$$

Hence, on A

$$\begin{aligned}
\bar{H}_I((u_i'')_{i\in I\cup I_2(F_I)}) &= \sum_{I\subsetneq I'}(-1)^{|I'|-|I|+1}\bar{K}_{I'} + \sum_{L:I=I_2(F_L)} w_L' \prod_{i\in I_1(F_L)} (\langle b_{l_i}^i, u_i''\rangle - \langle w_{l_i}^i, u_i''\rangle) \\
&= \sum_{L:F_I\cap F_L\in\mathcal{F}_0(F_I)} w_L' \prod_{i\in I_1(F_L)} \langle b_{l_i}^i - w_{l_i}^i, u_i''\rangle \\
&= \sum_{l_1,\ldots,l_{d-q}=1}^{2^{n^d}-1} w_{l_1,\ldots,l_{d-q}}' \prod_{j=1}^{d-q} \sum_{\varepsilon_j\in\{\pm1\}} \langle b_{l_j}(\varepsilon_j v_j) - w_{l_j}(\varepsilon_j v_j), \varepsilon_j v_j\rangle
\end{aligned}$$

where $b_{l_j}(\varepsilon_j v_j) = b_{l_j}^i$ and $w_{l_j}(\varepsilon_j v_j) = w_{l_j}^i$ if $\varepsilon_j v_j = u_i''$.

For $l \in \{1, \ldots 2^{n^d} - 1\}$, let

$$\alpha(l) = \sum_{l_2,\ldots,l_{d-q}=1}^{2^{n^d}-1} w_{l,l_2,\ldots,l_{d-q}}' \sum_{\varepsilon_2,\ldots,\varepsilon_{d-q}\in\{\pm1\}} \prod_{j=2}^{d-q} \langle b_{l_j}(\varepsilon_j v_j) - w_{l_j}(\varepsilon_j v_j), \varepsilon_j v_j\rangle.$$

This depends only on l and v_j for $j = 2, \ldots, d-q$. Then

$$\bar{H}_I((u_i'')_{i\in I\cup I_2(F_I)}) = \sum_{l=1}^{2^{n^d}-1} \alpha(l)(\langle b_l^{j_1} - w_l^{j_1}, v_1\rangle + \langle b_l^{j_2} - w_l^{j_2}, -v_1\rangle) = \langle x, v_1\rangle.$$

where $v_1 = u_{j_1}''$ and $-v_1 = u_{j_2}''$ and $x \in \mathbb{R}^d$ is some vector depending only on $v_2, \ldots, v_{d-q}$. It follows that

$$\bar{H}_I((Ru_i'')_{i\in I\cup I_2(F_I)}) = \langle x, Rv_1\rangle \tag{3.13}$$

for any rotation $R \in \mathrm{SO}(K^\perp)$, where $\mathrm{SO}(K^\perp)$ is the subgroup of $\mathrm{SO}(d)$ that fixes $K = \mathrm{lin}(v_2, \ldots, v_{d-q})$. But v_1 is orthogonal to K and $\dim K^\perp = q+1 > 1$, so (3.13) cannot equal $\bar{G}_I((Ru_i'')_{i\in I\cup I_2(F_I)}) = 1$ for all rotations $R \in \mathrm{SO}(K^\perp)$. Thus, there must be an $R \in \mathrm{SO}(K^\perp)$ such that $(Ru_i'')_{i\in I\cup I_2(F_I)} \notin A$. But then

$$H_{I'}((Ru_i'')_{i\in I\cup I_2(F_I)}) \neq G_{I'}((Ru_i'')_{i\in I\cup I_2(F_I)})$$

for at least one $I \subseteq I'$. Since $\mathrm{SO}(K^\perp)$ is path connected, $(Ru_i'')_{i\in I\cup I_2(F_I)} \in W'$ and it follows that

$$\mathcal{H}^{(d-1)N}(W' \cap \{H_{I'} = G_{I'}\}) = 0$$

as in the odd case. □

The theorem does not explicitly construct the polytopes for which $\hat{V}_q$ is biased. However, consider the space of orthogonal boxes

$$B(U,t) = \bigoplus_{i=1}^{d} [0, t_i u_i]$$

parametrized by $U \in \mathrm{SO}(d)$ and $t \in (0,\infty)^d$.

Corollary 3.9. *Let $\hat{V}_q$ be a local algorithm for V_q where $1 \le q \le d-1$. Then $\hat{V}_q(B(U,t))$ is asymptotically biased for almost all $(U,t) \in \mathrm{SO}(d) \times (0,\infty)^d$.*

Proof. This follows from the proof of Theorem 1.1 in the case $d-q$ even since the proof does not use the fact that $d-q$ is even, only that $q \neq 0, d$. $\square$

Remark 3.10. It seems likely that Theorem 3.8 should hold for all combinatorial isotopy classes of simple polytopes in the case $d-q$ even as well, but a proof would require a different argument.

3.5 The Euler characteristic in 2D

In this section we investigate the estimation of the Euler characteristic V_0 on $\mathcal{P}^2$ and prove Theorem 1.2 in the case $d=2$.

From Section 3.2 we have:

Corollary 3.11. *Let $P \in \mathcal{P}^{2,N}$ be given and let θ_{ij} denote the interior angle between F_i and F_j, i.e. $\pi - \theta_{ij}$ is the angle between u_i and u_j.*

$$\begin{aligned} E\hat{V}_0(P) = &\sum_{l=1}^{2^{n^d}-1} w_l'(a) \sum_{i=1}^{N} \Bigg(\frac{1}{2}\delta_l(u_i)(h(\check{W}_l,u_i)^2 - h(B_l,u_i)^2) \sum_{j\in I_2(F_i)} \cot(\theta_{ij}) \Bigg) \\ &+ (-h(B_l \oplus \check{W}_l, u_i))^+ \Bigg(a^{-1}\mathcal{H}^1(F_i) + \sum_{j\in I_2(F_i)} h(\check{C}^n_{0,0}, u_j)\csc(\theta_{ij}) \Bigg) \qquad (3.14) \\ &+ \sum_{l,k=1}^{2^{n^d}-1} w_{lk}'(a) \sum_{v\in\mathcal{F}_0(P)} \csc(\theta_{i_1^v i_2^v})(-h(B_l\oplus\check{W}_l, u_{i_1^v}))^+(-h(B_k\oplus\check{W}_k, u_{i_2^v}))^+ \end{aligned}$$

where $\delta_l(u) = \mathbb{1}_{\{h(B_l\oplus\check{W}_l,u)<0\}}$ and w_l' is as in the proof of Theorem 3.8.

Proof. It follows from Lemma 3.4 or directly from plane geometric considerations that for a sufficiently small,

$$\begin{aligned} \mathcal{H}^2(X_{i,l} \cap \bigcap_{j\neq i} X_{j,2^{n^d}-1}) = &\frac{a^2}{2}\delta_l(u_i)(h(\check{W}_l,u_i)^2 - h(B_l,u_i)^2) \sum_{j\in I_2(F_i)} \cot(\theta_{ij}) \\ &+ (-h(B_l\oplus\check{W}_l,u_i))^+ \Bigg(a\mathcal{H}^1(F_i) + a^2 \sum_{j\in I_2(F_i)} h(\check{C}^n_{0,0},u_j)\csc(\theta_{ij}) \Bigg), \end{aligned}$$

$$\begin{aligned} &\mathcal{H}^2(X_{i_1^v,l} \cap X_{i_2^v,k} \cap \bigcap_{m\neq i_1^v,i_2^v} X_{m,2^{n^d}-1}) \\ &= a^2\csc(\theta_{i_1^v i_2^v})(-h(B_l\oplus\check{W}_l,u_{i_1^v}))^+(-h(B_k\oplus\check{W}_k,u_{i_2^v}))^+. \end{aligned}$$

$\square$

We introduce the following notation:

Definition 3.12. *Let $\mathbb{L} \subseteq \mathbb{R}^2$ be the lattice spanned by $\xi = \{\xi_1, \xi_2\}$. Define $D_{\mathbb{L}} \subseteq S^1$ by*

$$D_{\mathbb{L}} = \left\{ \tfrac{z}{|z|} \mid z \in C^{2n}_{-n(\xi_1+\xi_2),0} \backslash \{0\} \right\}.$$

We say that a vertex v of a polygon P is n-critical if $(P - v) \cap aC^{2n}_{-n(\xi_1+\xi_2),0} = \{0\}$ for all a small enough or equivalently if $a^{-1}(P - v) \cap S^1$ is contained in a connected component of $S^1 \backslash D_{\mathbb{L}}$.

Theorem 3.13. *Theorem 1.2 holds for $d = 2$.*

Proof. Suppose the weights $w_l(a)$ of an asymptotically unbiased estimator $\hat{V}_0$ are given. We just need to show the existence of one element in $\mathcal{P}^{2,N} \backslash \mathcal{V}^N$ for some N, so assume for contradiction that $\mathcal{V}^N = \mathcal{P}^{2,N}$. Since all polygons are simple, Corollary 3.7 allows us to assume that the weights are homogeneous, i.e. $w_l(a) = w_l$.

Let

$$\begin{aligned} v_1 &= (\cos\varphi, \sin\varphi), \\ v_2 &= (\cos(\varphi+\psi), \sin(\varphi+\psi)), \end{aligned}$$

where $(\varphi, \psi) \in U$ for some small open subset $U \subseteq \mathbb{R}^2$ such that v_1 and v_2 lie in the same connected component $E \subseteq S^1 \backslash D_{\mathbb{L}}$.

Consider a parallelogram

$$P(\varphi, \psi, s_1, s_2) = [0, s_1 v_1] \oplus [0, s_2 v_2] \tag{3.15}$$

for $s_1, s_2 > 0$. Then P has two n-critical vertices at 0 and $s_1 v_1 + s_2 v_2$. The normal vectors of P are

$$\begin{aligned} u_1 &= -u_3 = (-\sin\varphi, \cos\varphi), \\ u_2 &= -u_4 = (-\sin(\varphi+\psi), \cos(\varphi+\psi)). \end{aligned}$$

Observe that $\csc(\theta_{i_1^v i_2^v}) = \csc\psi$ for all $v \in \mathcal{F}_0(P)$, and if $I_2(F_i) = \{j_1, j_2\}$, then $\cot(\theta_{ij_1}) = -\cot(\theta_{ij_2})$.

Since $\lim_{a\to 0} E\hat{V}_0(P)$ exists, the coefficient in front of a^{-1} in (3.14)

$$\sum_{l=1}^{2^{n^d}-1} w'_l \sum_{i=1}^{2} s_i \sum_{\varepsilon=\pm 1} (-h(B_l \oplus \check{W}_l, \varepsilon u_i))^+$$

must vanish. This holds for all $s_1, s_2 > 0$, so for each $i = 1, 2$, also

$$\sum_{l=1}^{2^{n^d}-1} w'_l((-h(B_l \oplus \check{W}_l, u_i))^+ + (-h(B_l \oplus \check{W}_l, -u_i))^+) = 0$$

and Corollary 3.11 reduces to

$$E\hat{V}_0(P) = \csc\psi \sum_{l,k=1}^{2^{n^d}-1} w'_{lk} \sum_{v \in \mathcal{F}_0(P)} (-h(B_l \oplus \check{W}_l, u_{i_1^v}))^+ (-h(B_k \oplus \check{W}_k, u_{i_2^v}))^+$$

for all a sufficiently small.

Let R denote the reflection of $C^n_{0,0}$ in the point $(\frac{n}{2}\xi_1, \frac{n}{2}\xi_2)$ and observe that

$$h(B_l \oplus \check{W}_l, u) = h(RB_l \oplus R\check{W}_l, -u).$$

Thus, since the weights are reflection invariant,

$$\begin{aligned} EV_0(P) = 2\csc\psi \sum_{l,k=1}^{2^{n^d}-1} &(w'(B_l \cap B_k) + w'(RB_l \cap B_k)) \\ &\times (-h(B_l \oplus \check{W}_l, u_1))^+ (-h(B_k \oplus \check{W}_k, u_2))^+. \end{aligned} \tag{3.16}$$

for all sufficiently small a.

Let $\beta_l^+, \omega_l^- : S^1 \to C^n_{0,0}$ denote the functions given by $\beta_l^+(u) = h(B_l, u)$ and $\omega_l^-(u) = -h(\check{W}_l, u)$. In particular, $h(B_l \oplus \check{W}_l, u) = \langle \beta_l^+(u) - \omega_l^-(u), u\rangle$. Note that β_l^+ and ω_l^- are constant on the set $R_{-\frac{\pi}{2}}E \subseteq S^1$ where $R_{-\frac{\pi}{2}}$ is the rotation by $-\frac{\pi}{2}$. Thus, whenever $\varphi, \varphi+\psi \in E$,

$$\begin{aligned} \delta_l(u_1) &= \delta_l(u_2), \\ \beta_l = \beta_l^+(u_1) &= \beta_l^+(u_2), \\ \omega_l = \omega_l^-(u_1) &= \omega_l^-(u_2), \end{aligned}$$

for some fixed vectors $\beta_l, \omega_l \in \mathbb{R}^2$.

Write $\omega_l - \beta_l = (x_l, y_l)$. Then for $\varphi, \varphi+\psi \in E$,

$$\begin{aligned} &(-h(B_l \oplus \check{W}_l, u_1))^+(-h(B_k \oplus \check{W}_k, u_2))^+ + (-h(B_k \oplus \check{W}_k, u_1))^+(-h(B_l \oplus \check{W}_l, u_2))^+ \\ &\quad = \delta_l(u_1)\delta_k(u_1)(\langle \omega_l - \beta_l, u_1\rangle\langle \omega_k - \beta_k, u_2\rangle + \langle \omega_k - \beta_k, u_1\rangle\langle \omega_l - \beta_l, u_2\rangle) \\ &\quad = \delta_l(u_1)\delta_k(u_1)((-x_l\sin\varphi + y_l\cos\varphi)(-x_k\sin(\varphi+\psi) + y_k\cos(\varphi+\psi)) \\ &\qquad + (-x_k\sin\varphi + y_k\cos\varphi)(-x_l\sin(\varphi+\psi) + y_l\cos(\varphi+\psi))) \\ &\quad = \delta_l(u_1)\delta_k(u_1)((2x_lx_k\sin\varphi\sin(\varphi+\psi) + 2y_ly_k\cos\varphi\cos(\varphi+\psi)) \\ &\qquad - (x_ky_l + x_ly_k)(\sin\varphi\cos(\varphi+\psi) + \cos\varphi\sin(\varphi+\psi))). \end{aligned}$$

Since $w(B_l \cap B_k) = w(B_k \cap B_l)$ and

$$w(RB_l \cap B_k) = w(R(RB_l \cap B_k)) = w(RB_k \cap B_l),$$

the terms in (3.16) pair up, showing that $E\hat{V}_0(P(\varphi,\psi))$ is a linear combination of the functions

$$\begin{aligned} \cos\varphi\cos(\varphi+\psi)\csc\psi &= \cos^2\varphi\cot\psi - \sin\varphi\cos\varphi, \\ \sin\varphi\sin(\varphi+\psi)\csc\psi &= \sin^2\varphi\cot\psi + \sin\varphi\cos\varphi, \\ (\cos\varphi\sin(\varphi+\psi) + \sin\varphi\cos(\varphi+\psi))\csc\psi &= \sin\varphi\cos\varphi\cot\psi + \cos^2\varphi - \sin^2\varphi. \end{aligned} \tag{3.17}$$

On the other hand, (3.16) equals $V_0(P(\varphi,\psi)) = 1$ for all $(\varphi,\psi) \in U$. But the functions in (3.17) are clearly linearly independent of the constant function 1, yielding the contradiction. $\square$

Corollary 3.14. *Any local estimator for V_0 has a worst case asymptotic relative bias on $\mathcal{P}^2$ of either 0 or ∞.*

Proof. Let $P(\varphi,\psi)$ be as in the proof of Theorem 1.2 for $d=2$. The proof shows that $\lim_{a\to 0} E\hat{V_0}(P(\varphi,\psi))$ has the form

$$\begin{aligned}\alpha_1(\cos^2\varphi\cot\psi-\sin\varphi\cos\varphi)+\alpha_2(\sin^2\varphi\cot\psi+\sin\varphi\cos\varphi)\\ +\alpha_3(\sin\varphi\cos\varphi\cot\psi+\cos^2\varphi-\sin^2\varphi)\end{aligned}\tag{3.18}$$

for some $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$ and all $(\varphi,\psi)\in I\times(0,\varepsilon)\subseteq U$ for some small open interval I and some $\varepsilon>0$.

The functions $\cos^2\varphi$, $\sin^2\varphi$, and $\sin\varphi\cos\varphi$ are linearly independent, so if (3.18) is non-trivial, there must be a $\varphi\in I$ such that

$$\lim_{\psi\to 0}\lim_{a\to 0} E\hat{V_0}(P(\varphi,\psi))=\pm\infty. \qquad \square$$

Note how the fact that $P(\varphi,\psi)$ had an n-critical vertex was essential in the proof. The next proposition shows that the polygons with n-critical vertices are the only sets in $\mathcal{P}^2$ where the estimation of V_0 fails. To get a slightly more general result, we first extend the definition of an n-critical vertex to the class $\mathcal{K}^2$ of compact convex sets with non-empty interior.

Definition 3.15. *Let $K\in\mathcal{K}^2$. We say that $x\in\partial K$ is an n-critical boundary point if for all $a>0$,*

$$(K-x)\cap aC^{2n}_{-n(\xi_1+\xi_2),0}=\{0\}.$$

Note that K can have at most finitely many n-critical boundary points

Lemma 3.16. *Let $K\in\mathcal{K}^2$ have no n-critical boundary points. Then there exists a $\delta>0$ such that whenever $a<\delta$,*

$$(K-x)\cap aC^{2n}_{-n(\xi_1+\xi_2),0}\neq\{0\}.\tag{3.19}$$

Proof. Let $x\in\partial K$. Then $[x,x+a(x)c]\subseteq K$ for some $c\in C^{2n}_{-n(\xi_1+\xi_2),0}$. There is an open neighborhood U_x of x in ∂K such that $y+\frac{1}{2}a(x)c\in K$ for all $y\in U_x$. Cover ∂K by finitely many such U_x and choose a to be the smallest of the corresponding $\frac{1}{2}a(x)$. $\square$

Let (B^n_l,W^n_l) be a configuration. Define the corresponding weight

$$w_l=\sum_{k=1}^{n^2}(-1)^k\frac{1}{k}n_l^{k-1}\tag{3.20}$$

where n_l^k is the number

$$n_l^k=\left|\left\{S\subseteq\mathbb{L}\backslash\{0\}\;\middle|\;|S|=k,B_l\cap\bigcap_{z\in S}C^n_{z,0}\neq\emptyset\right\}\right|.$$

Proposition 3.17. *Let $\hat{V}_0^n$ be the local algorithm based on $n\times\cdots\times n$ configurations with weights given by* (3.20)*. For all $K\in\mathcal{K}^2$ with no n-critical boundary points, $\hat{V}_0^n(K)=1$ whenever a is sufficiently small.*

The idea is to approximate K by a polyconvex set. Let $P_z=\operatorname{conv}(C^n_{z,0}\cap K)$ be the convex hull of $C^n_{z,0}\cap K$ and define the approximation

$$\hat{K}=\bigcup_{z\in\mathbb{Z}^2}P_z.$$

Then the proof will show that $V_0(K)=V_0(\hat{K})$ and that $\hat{V}_0^n(K)=V_0(\hat{K})$.

Proof. Let $K\in\mathcal{K}^2$ with no n-critical boundary points be given. For simplicity, assume $\mathbb{L}=\mathbb{Z}^2$. The general case follows by considering a linear map $L:\mathbb{R}^2\to\mathbb{R}^2$ with $L(\mathbb{L})=\mathbb{Z}^2$. Then K has an n-critical vertex for $\mathbb{L}$ if and only if $L(K)$ has an n-critical vertex with respect to $\mathbb{Z}^2$. Moreover, $N_l(K\cap\mathbb{L})=N_l(L(K)\cap\mathbb{Z}^2)$ and thus $E\hat{V}_0^{a\mathbb{L}}(K)=E\hat{V}_0^{a\mathbb{Z}^2}(L(K))$. Choose a so small that (3.19) is satisfied and such that K contains a ball of radius $\sqrt{2}(n+1)a$. By possibly considering $a^{-1}K$ instead of K, we may assume that $a=1$ to keep notation simple.

We first claim that

$$V_0(\hat{K})=V_0(K)=1.$$

For this, it is enough to show that $\hat{K}$ and $\mathbb{R}^2\backslash\hat{K}$ are both connected.

In order to show that $\hat{K}$ is connected, we show that every $x=(x_1,x_2)\in\hat{K}\cap\mathbb{Z}^2$ is connected by a path in $\hat{K}$ to a fixed reference point $y=(y_1,y_2)\in\hat{K}\cap\mathbb{Z}^2$ with $y+B(\sqrt{2}n)\subseteq K$. We may assume that $x_1\le y_1$ and $x_2\le y_2$ such that $C_0^n\cap(K-x)\neq\{0\}$. Then $C^n_{0,0}\cap(K-x)$ must contain a point $z\neq x$. To see this, choose $p\in\partial K$ with $x\in[p,y]$. Then $C^n_{0,0}\cap(K-p)$ contains a point $z\neq 0$ since p is not n-critical. Since $p+z,y+z\in K$, also $x+z\in K$ by convexity. Thus x is connected to $x+z$ in $\hat{K}$, and the claim follows by induction on $|x_1-y_1|+|x_2-y_2|$.

In order to show that $\mathbb{R}^2\backslash\hat{K}$ is connected, assume for contradiction that $x\in K\backslash\hat{K}$ is contained in a compact component. Let l be the vertical line through x and let $b_1,b_2\in\hat{K}\cap l$ be such that

$$[b_1,b_2]\cap\hat{K}=\{b_1,b_2\},\tag{3.21}$$

$x\in[b_1,b_2]$, and the vector b_2-b_1 points upwards. Then for $i=1,2$ there are line segments $[x_i,y_i]\subseteq\partial P_{z_i}$ with $x_i,y_i,z_i\in\mathbb{Z}^2$ such that $b_i\in[x_i,y_i]$. After possibly reflecting the picture in the coordinate axes, we may assume:

$$\begin{aligned}x_1,x_2&\in H^-_{e_1,\langle x,e_1\rangle},\\ y_1,y_2&\in H^-_{-e_1,-\langle x,e_1\rangle},\\ \langle x_1,e_1\rangle&\le\langle x_2,e_1\rangle,\\ \operatorname{aff}[x_1,y_1]\cap\operatorname{aff}[x_2,y_2]&\subseteq H^-_{-e_1,-\langle x,e_1\rangle}.\end{aligned}$$

First observe that the vertical distance from x_2 to $[x_1,y_1]$ is at most 1. Assume this were not true. If $[x_1,y_1]$ has positive slope, either $x_2=y_1+(0,m)\in l$ for some $m\in\mathbb{N}$, implying $x\in[x_2,y_1]\subseteq\hat{K}$, or there is an $m\in\mathbb{N}$ such that $x_2-(0,m)$

lies above $[x_1, y_1]$ and $\operatorname{conv}(x_2 - (0, m), x_1, y_1) \subseteq P_z$ for some $z \in \mathbb{Z}^2$. If $[x_1, y_1]$ has non-positive slope, so must $[x_2, y_2]$ and hence $\operatorname{conv}(x_2, x_2 - (0, 1), y_2) \subseteq P_z$ for some $z \in \mathbb{Z}^2$. All three cases contradict the assumption (3.21).

Thus, either $\operatorname{conv}(x_1, y_1, x_2) \subseteq P_z$ for some $z \in \mathbb{L}$, or $x_2 = y_1 + (0, 1)$, or $[x_1, y_1]$ has negative slope and $x_2 = x_1 + (0, 1)$. The second case implies $[x_2, y_1] \subseteq \hat{K} \cap l$. In the third case, $[x_2, y_2]$ must also have negative slope. Hence $\operatorname{conv}(x_2 - (0, 1), x_2, y_2) \subseteq P_z$ for some $z \in \mathbb{L}$. Again, all three cases contradict the assumption (3.21).

The proof is now complete if we can show that

$$\hat{V}_0^n(K \cap \mathbb{Z}^2) = V_0(\hat{K}).$$

By the inclusion-exclusion principle,

$$\begin{aligned}
V_0(\hat{K}) &= \sum_{k=1}^{n^d} \sum_{\substack{S \subseteq \mathbb{L}, \\ |S|=k}} (-1)^k V_0\Big(\bigcap_{z \in S} P_z \Big) \\
&= \sum_{k=1}^{n^d} \sum_{z_0 \in \mathbb{L}} \sum_{\substack{S \subseteq \mathbb{L}, \\ |S|=k-1}} (-1)^k \frac{1}{k} V_0\Big(P_{z_0} \cap \bigcap_{z \in S} P_z \Big) \\
&= \sum_{k=1}^{n^d} (-1)^k \frac{1}{k} \sum_{z_0 \in \mathbb{L}} \sum_{\substack{S \subseteq \mathbb{L}, \\ |S|=k-1}} \mathbb{1}_{\{P_{z_0} \cap \bigcap_{z \in S} P_z \neq \emptyset\}}.
\end{aligned}$$

By construction of the weights, it remains to show that if $P_{z_1} \cap \dots \cap P_{z_k} \neq \emptyset$, then $C^n_{z_1,0} \cap \dots \cap C^n_{z_k,0} \cap K \neq \emptyset$.

For $k = 1$ this is trivial. Assume $P_{z_1} \cap P_{z_2} \neq \emptyset$. If $P_{z_1} \subseteq P_{z_2}$, then the claim is clearly true. Otherwise, $\partial P_{z_1} \cap \partial P_{z_2} \neq \emptyset$. Hence there are $x_i, y_i \in C^n_{z_i,0} \cap K$ such that the line segments $[x_1, y_1]$ and $[x_2, y_2]$ intersect in $C^n_{z_1} \cap C^n_{z_2}$. Assume that $x_1, y_1, x_2, y_2 \notin C^n_{z_1} \cap C^n_{z_2}$. Then $[x_1, y_1]$ divides $C^n_{z_2}$ into two components C^1 and C^2 with $C^1 \subseteq C^n_{z_1} \cap C^n_{z_2}$. As $[x_2, y_2]$ intersects $[x_1, y_1] \cap C^n_{z_2}$, either x_2 or y_2 must belong to $C^1 \cup [x_1, y_1] \subseteq C^n_{z_1}$, which is a contradiction.

For $k \geq 3$, assume that z_1 and z_2 have the smallest and largest 1st coordinate among the z_i, respectively. By the above, there is a $y_1 \in C^n_{z_1,0} \cap C^n_{z_2,0} \cap K$. If y_1 lies in $C^n_{z_1,0} \cap \dots \cap C^n_{z_k,0}$, we are done. Otherwise, suppose that the 2nd coordinate is too large for y_1 to belong to $C^n_{z_1,0} \cap \dots \cap C^n_{z_k,0}$. Let z_3 have the smallest 2nd coordinate among the z_i. There are points

$$\begin{aligned}
y_2 &\in C^n_{z_1,0} \cap C^n_{z_3,0} \cap \dots \cap C^n_{z_k,0} \cap K, \\
y_3 &\in C^n_{z_2,0} \cap C^n_{z_3,0} \cap \dots \cap C^n_{z_k,0} \cap K,
\end{aligned}$$

by induction. If $y_2, y_3 \notin C^n_{z_1} \cap \dots \cap C^n_{z_k}$, write $y_i = (r_i, s_i)$ with $r_2 < r_1 < r_3$. We may assume $s_1 > s_2 \geq s_3$. Then $(r_1, s_2) \in \operatorname{conv}(y_1, y_2, y_3) \subseteq K$ and thus

$$(r_1, s_2) \in C^n_{z_1,0} \cap \dots \cap C^n_{z_k,0} \cap K. \qquad \square$$

Example 3.18. For $n = 2$ and $\mathbb{L} = \mathbb{Z}^2$, $\hat{V}_0^2$ is the algorithm suggested by Pavlidis in [14], which is multigrid convergent on the class of r-regular sets. Theorem 3.17 shows that this algorithm is also multigrid convergent on the class of compact convex polygons with no interior angles of less than 45 degrees.

3.6 The Euler characteristic in higher dimensions

The results of the previous sections allow us to generalize Theorem 3.13 and Corollary 3.14 to higher dimensions.

Theorem 3.19. *For $d \geq 2$ and $q \leq d-2$, any local algorithm $\hat{V}_q$ for which $\lim_{a\to 0} E\hat{V}_q(P)$ exists for all $P \in \mathcal{P}^d$ has a worst case asymptotic relative bias of 100 % on $\mathcal{P}^d$. In particular, Theorem 1.2 holds.*

The proof uses the fact that if $P = \bigoplus_{i=1}^d [0, v_i]$ with $v_1, \ldots, v_d \in \mathbb{R}^d$ linearly independent, then

$$V_q(P) = \sum_{1\le i_1<\cdots<i_q\le d} \mathcal{H}^q\bigg(\bigoplus_{s=1}^q [0, v_{i_s}]\bigg). \tag{3.22}$$

This follows because

$$\sum_{S\subseteq\{1,\ldots,d\}\setminus\{i_1,\ldots,i_q\}} \gamma\bigg(\sum_{i\in S} v_i + \bigoplus_{s=1}^q [0, v_{i_s}], P\bigg) = 1.$$

Proof. First consider the case of the standard lattice $\mathbb{Z}^d$. Take $Q = P \oplus \bigoplus_{j=3}^d [0, e_j]$ where $e_1, \ldots, e_d \in \mathbb{R}^d$ is the standard basis and $P \subseteq \text{lin}\{e_1, e_2\} \cong \mathbb{R}^2$ is as in (3.15). Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map taking P to $[0, s_1e_1] \oplus [0, s_2e_2]$ and fixing $e_3, \ldots, e_d$. Then

$$\lim_{a\to 0} E\hat{V}_q^{a\mathbb{Z}^d}(Q) = \lim_{a\to 0} E\hat{V}_q^{aL(\mathbb{Z}^d)}(L(Q)). \tag{3.23}$$

The left hand side is a polynomial in $s_1, s_2, t_3, \ldots, t_d$.

On the other hand, (3.22) yields

$$V_q(Q) = \sum_{\substack{S\subseteq\{3,\ldots,d\},\\ |S|=q}} \prod_{i\in S} t_i + \sum_{\substack{S\subseteq\{3,\ldots,d\},\\ |S|=q-1}} (s_1+s_2) \prod_{i\in S} t_i + \sum_{\substack{S\subseteq\{3,\ldots,d\},\\ |S|=q-2}} s_1s_2 \sin\psi \prod_{i\in S} t_i.$$

If (3.23) contains monomials in $s_1, s_2, t_3, \ldots, t_d$ of degree larger than q, we can make the relative bias arbitrarily large and if it contains monomials of degree less than q, it can be made arbitrarily close to 0 just by scaling Q.

Otherwise, (3.23) is homogeneous in $s_1, s_2, t_3, \ldots, t_d$ of degree q, and the argument in the proof of Corollary 3.7 shows that the weights may be assumed to be homogeneous of degree q. Observing that

$$(-h(L(B_{l_i} \oplus \check{W}_{l_i}), \pm e_i))^+ \in \{0, 1\}$$

for $i = 3, \ldots, d$, the proof of Theorem 3.8 shows that

$$\begin{aligned}
\lim_{a\to 0} E\hat{V}_0^{aL(\mathbb{Z}^d)}(L(Q)) &= \sum_{l_1,\ldots,l_{d-q}=1}^{2^{n^d}-1} w_{l_1,\ldots,l_{d-q}} \prod_{i\in I_1^L} \sum_{\varepsilon_i\in\pm 1} (-h(L(B_{l_i} \oplus \check{W}_{l_i}), \varepsilon_i e_i))^+ \prod_{i\notin I_1^L} t_i \\
&= \lim_{a\to 0} \hat{V}_0'^{a\mathbb{Z}^d}(P) \sum_{\substack{S\subseteq\{3,\ldots,d\},\\ |S|=q}} \prod_{i\in S} t_i + \sum_{\substack{S\subseteq\{3,\ldots,d\},\\ |S|=q-1}} (\beta_S^1(Q)s_1 + \beta_S^2(Q)s_2) \prod_{i\in S} t_i \\
&\quad + \sum_{\substack{S\subseteq\{3,\ldots,d\},\\ |S|=q-2}} \beta_S^{12}(Q)s_1s_2 \prod_{i\in S} t_i
\end{aligned}$$

where $\beta_S^1(Q), \beta_S^2(Q), \beta_S^{12}(Q)$ are certain numbers depending only on (φ, ψ) and V_0' is the estimator for V_0 in $\mathbb{R}^2$ with weights

$$w_l' = \sum_{3 \le i_1 < \cdots < i_{d-q-2} \le d} \sum_{c_3, \ldots, c_{d-q} = -n+1}^{n-2} w(\pi^{-1}(B_l) \cap B_{i_1, c_1} \cap \cdots \cap B_{i_{d-q-2}, c_{d-q-2}}).$$

where $\pi : C_{0,0}^n(\mathbb{R}^d) \to C_{0,0}^n(\mathbb{R}^2)$ is the projection induced by $\mathbb{R}^d \to \operatorname{lin}\{e_1, e_2\}$ and $B_{j,c} = C_{0,0}^n \cap H_{e_j,c}^-$ for $c = 0, \ldots, n-2$, while $B_{j,c} = C_{0,0}^n \cap H_{-e_j,c}^-$ for $c = -n+1, \ldots, -1$.

By Corollary 3.14, $\lim_{a \to 0} \hat{V}_0'(P)$ is either zero or can be made arbitrarily large by properly choosing P. Thus, the asymptotic worst case error can be made arbitrarily close to zero or arbitrarily large, respectively, by choosing P first and then choosing s_1 and s_2 small compared to $t_3, \ldots, t_d$.

Now consider a general lattice $\mathbb{L}$. Choose a linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ such that $L(\mathbb{Z}^d) = \mathbb{L}$. Then

$$E\hat{V}_q^{a\mathbb{Z}^d}(Q) = E\hat{V}_q^{a\mathbb{L}}(L(Q)),$$

while

$$V_q(L(Q)) = \sum_{\substack{S \subseteq \{3,\ldots,d\},\\ |S|=q}} \alpha_S \prod_{i \in S} t_i + \sum_{\substack{S \subseteq \{3,\ldots,d\},\\ |S|=q-1}} (\alpha_S^1 s_1 + \alpha_S^2 s_2) \prod_{i \in S} t_i + \sum_{\substack{S \subseteq \{3,\ldots,d\},\\ |S|=q-2}} \alpha_S^{12} s_1 s_2 t_i$$

where α_S depends only on L while $\alpha_S^1, \alpha_S^2, \alpha_S^{12}$ may also depend on (φ, ψ). Thus the general case follows as before by first choosing P and then choosing s_1, s_2. $\square$

4 Local estimators on the class of r-regular sets

We now move on to local digital algorithms applied to r-regular sets. The formal definition of r-regular sets is as follows:

Definition 4.1. *$X \subseteq \mathbb{R}^d$ is called r-regular if for every $x \in \partial X$ there exist two balls $B_1, B_2 \subseteq \mathbb{R}^d$ both of radius r and containing x with $B_1 \subseteq X$ and $\operatorname{int}(B_2) \subseteq \mathbb{R}^d \backslash X$. For $x \in \partial X$, $n(x)$ denotes the unique outward pointing normal vector.*

The purpose of this section is to prove Theorem 1.4. As we only consider local estimators with homogeneous weights, we may assume $w_{2^{n^d}-1} = 0$, see [22, Section 3]. The case of the surface area $2V_{d-1}$ is an easy consequence of the corresponding theorem for polytopes and the following formula due to Kiderlen and Rataj [7, Theorem 5]:

Theorem 4.2 (Kiderlen, Rataj). *For any local estimator $\hat{V}_{d-1}$ with homogeneous weights and $w_{2^{n^d}-1} = 0$ and for any compact r-regular set $X \subseteq \mathbb{R}^d$:*

$$\lim_{a \to 0} E\hat{V}_{d-1}(X) = \det(\mathbb{L})^{-1} \sum_{l=1}^{2^{n^d}-2} w_l \int_{\partial X} (-h(B_l \oplus \check{W}_l, n))^+ d\mathcal{H}^{d-1}.$$

Proof of Theorem 1.4 for V_{d-1}. Suppose $\hat{V}_{d-1}$ is given. By Corollary 3.9, we may choose $v_1, \dots, v_d \in \mathbb{R}^d$ orthogonal such that

$$\lim_{a\to 0} E\hat{V}_{d-1}\Big(\bigoplus_{i=1}^{d}[0, v_i]\Big) \neq V_{d-1}\Big(\bigoplus_{i=1}^{d}[0, v_i]\Big).$$

Consider the r-regular set

$$X(r) = B(r) \oplus \bigoplus_{i=1}^{d}[0, v_i].$$

Observe that

$$\lim_{r\to 0} V_{d-1}(X(r)) = V_{d-1}\Big(\bigoplus_{i=1}^{d}[0, t_i u_i]\Big).$$

On the other hand, Theorem 4.2 yields

$$\begin{aligned}\lim_{a\to 0} E\hat{V}_{d-1}(X(r)) =\ & \det(\mathbb{L})^{-1}\sum_{l=1}^{2^{n^d}-2} w_l \int_{\partial X(r)} (-h(B_l \oplus \check{W}_l, n))^+ d\mathcal{H}^{d-1}\\ =\ & \lim_{a\to 0} E\hat{V}_{d-1}\Big(\bigoplus_{i=1}^{d}[0, v_i]\Big)\\ & + \det(\mathbb{L})^{-1}\sum_{l=1}^{2^{n^d}-2} w_l \int_{Y} (-h(B_l \oplus \check{W}_l, n))^+ d\mathcal{H}^{d-1}\end{aligned}$$

where

$$Y = X \backslash \bigcup_{F\in\mathcal{F}_{d-1}(\bigoplus_{i=1}^{d}[0,v_i])} (F + r u_{i_1^F}).$$

Since each $h(B_l \oplus \check{W}_l, n)$ is bounded for $n \in S^{d-1}$ and $\lim_{r\to 0}\mathcal{H}^{d-1}(Y) = 0$, it follows that

$$\lim_{r\to 0}\lim_{a\to 0} E\hat{V}_{d-1}(X(r)) = \lim_{a\to 0} E\hat{V}_{d-1}\Big(\bigoplus_{i=1}^{d}[0, v_i]\Big).$$

In particular,

$$\lim_{a\to 0} E\hat{V}_{d-1}(X(r)) \neq V_{d-1}(X(r))$$

when r is sufficiently small. □

It follows from the definition of r-regularity that the boundary of an r-regular set X is a C^1 manifold. The normal vector field n is almost everywhere differentiable on ∂X, see [3]. In particular, the second fundamental form II_x is defined on $T_x\partial X$ if n is differentiable at x. Define Q_x to be the quadratic form on $T_x\partial X \oplus \mathrm{lin}\{n(x)\} = \mathbb{R}^d$ given by

$$Q_x(\alpha, tn(x)) = -II_x(\alpha) + \mathrm{Tr}(II_x)t^2.$$

For a finite set $S \subseteq \mathbb{R}^d$, define

$$II_x^+(S) = \max\{II_x(s) \mid s \in S,\, h(S,n) = \langle s, n\rangle\},$$
$$II_x^-(S) = \min\{II_x(s) \mid s \in S,\, h(S,-n) = \langle s, -n\rangle\},$$

and if $s^\pm \in S$ are such that $II_x^\pm(S) = II_x^\pm(s^\pm)$, define

$$Q_x^\pm(S) = Q_x(s^\pm).$$

The following formula is shown in [22]:

Theorem 4.3. *For a local estimator $\hat{V}_{d-2}$ with homogeneous weights and $w_{2^{n^d}-1} = 0$ and an r-regular set X,*

$$\lim_{a\to 0} E\hat{V}_{d-2}(X) = \det(\mathbb{L})^{-1}\frac{1}{2}\sum_{l=1}^{2^d-2} w_l \int_{\partial X} (Q^+(B_l) - Q^-(W_l))\delta_l(n)d\mathcal{H}^{d-1}.$$

The proof of Theorem 1.4 for V_{d-2} follows from this:

Theorem 1.4 for V_{d-2}. We first introduce the sets that will serve as counter examples. For $0 < r < R$ and $\theta \in (0,\pi)$, let

$$T(R,r) = B(r) \oplus B^{d-1}((R-r)\sin\theta)$$

where $B^{d-1}(s)$ is the ball of radius s in $\mathrm{lin}(e_1,\dots,e_{d-1})$. We then consider r-regular sets of the form

$$X(R,r) = (B(R) \cap H^-_{R\cos\theta, e_d}) \cup (T(R,r) + (R-r)\cos\theta e_d).$$

Choose a rotation $\rho \in \mathrm{SO}(d)$ taking e_d to $S^{d-1}\backslash D$ where D is as in the proof of Theorem 3.8 and consider $\rho(X(R,r))$. Then

$$\hat{V}_{d-2}^{a\mathbb{L}_c}(\rho X(R,r)) = \hat{V}_{d-2}^{a\rho^{-1}\mathbb{L}_c}(X(R,r)),$$

so by possibly changing the lattice, we may assume that $\rho = I$ and $e_d \in S^{d-1}\backslash D$. Let $U \subseteq S^{d-1}\backslash D$ be the connected component containing e_d. This is open in S^{d-1}.

If $n(x) \notin D$, there exist unique vectors $b_l(n) \in B_l$ and $w_l(n) \in W_l$ depending only on $n \in S^{d-1}$ such that $Q_x^+(B_l) = Q_x(b_l(n))$ and $Q_x^-(W_l) = Q_x(w_l(n))$. This defines functions

$$\beta_l, \omega_l : S^{d-1}\backslash D \to C^n_{0,0}.$$

Note that these are locally constant and so is the indicator function δ_l on $S^{d-1}\backslash D$.

Let $\varepsilon_1,\dots,\varepsilon_{d-1} \in T\partial X(R,r)$ denote the principal directions corresponding to the principal curvatures $k_1,\dots,k_{d-1}$. Since $e_d \notin D$, $n(x) \in S^{d-1}\backslash D$ for almost all $x \in \partial X(R,r)$ and for such x

$$Q_x^+(B_l) - Q_x^-(W_l) = \sum_{j=1}^{d-1} k_j(-\langle\beta_l(n),\varepsilon_j\rangle^2 + \langle\beta_l(n),n\rangle^2 + \langle\omega_l(n),\varepsilon_1\rangle^2 - \langle\omega_l(n),n\rangle^2).$$

Observe that $\partial X(R,r)$ is the disjoint union of three sets S_1, S_2, and S_3 where

$$\begin{aligned}
S_1 &= (\partial B(R)) \cap H^-_{R\cos\theta, e_d},\\
S_2 &= (\partial T(R,r) + (R-r)\cos\theta e_d) \backslash (H^-_{R\cos\theta,e_d} \cup S_3),\\
S_3 &= B^{d-1}((R-r)\sin\theta) + ((R-r)\cos\theta + r)e_d.
\end{aligned}$$

On S_3, $k_1 = \cdots = k_{d-1} = 0$ and thus Q vanishes on S_3.

Parametrize S_1 by $g_1 : S^{d-2} \times (\theta, \pi) \to S_1$. Identifying S^{d-2} with the unit sphere in $\operatorname{lin}(e_1, \ldots, e_{d-1}) \subseteq \mathbb{R}^d$,

$$g_1(u,\varphi) = R(\sin\varphi u + \cos\varphi e_d).$$

Similarly, parametrize S_2 by $g_2 : S^{d-2} \times (0,\theta) \to S_2$ where

$$g_2(u,\varphi) = (R-r)\sin\theta u + r\cos\varphi e_d.$$

Note that on both S_1 and S_2,

$$\begin{aligned}
n(u,\varphi) &= \sin\varphi u + \cos\varphi e_d,\\
\varepsilon_{d-1}(u,\varphi) &= -\cos\varphi u + \sin\varphi e_d,\\
\varepsilon_j(u,\varphi) &= \varepsilon'_j(u),
\end{aligned}$$

for $j = 1, \ldots, d-2$, where $\varepsilon'_j(u)$ are the principal directions on S^{d-2}.

On S_1,

$$\begin{aligned}
k_j &= \frac{1}{R},\\
\int_{S_1} f &= \int_\theta^\pi \int_{S^{d-2}} f(u,\varphi) R^{d-1} \sin^{d-2}\varphi \mathcal{H}^{d-2}(du)d\varphi,
\end{aligned}$$

for all $j = 1, \ldots, d-1$ and any integrable function f. On S_2,

$$\begin{aligned}
k_{d-1} &= \frac{1}{r},\\
k_j(\varphi) &= \frac{\sin\varphi}{(R-r)\sin\theta + r\sin\varphi},\\
\int_{S_2} f &= \int_0^\theta \int_{S^{d-2}} f(u,\varphi) r((R-r)\sin\theta + r\sin\varphi)^{d-2} \mathcal{H}^{d-2}(du)d\varphi,
\end{aligned}$$

for $j = 1, \ldots, d-2$ and any integrable function f.

Define $F_1, F_2 : (0,\pi) \to \mathbb{R}$ by

$$\begin{aligned}
F_1 = {} & \det(\mathbb{L})^{-1} \frac{1}{2} \sum_{l=1}^{2^{n^d}-2} w_l \int_{S^{d-2}} (-\langle \beta_l, \varepsilon_{d-1}\rangle^2 + \langle \beta_l, n\rangle^2\\
& + \langle \omega_l, \varepsilon_{d-1}\rangle^2 - \langle \omega_l, n\rangle^2) \delta_l(n) d\mathcal{H}^{d-2},\\
F_2 = {} & \det(\mathbb{L})^{-1} \frac{1}{2} \sum_{l=1}^{2^{n^d}-2} w_l \sum_{j=1}^{d-2} \int_{S^{d-2}} (-\langle \beta_l, \varepsilon_j\rangle^2 + \langle \beta_l, n\rangle^2\\
& + \langle \omega_l, \varepsilon_j\rangle^2 - \langle \omega_l, n\rangle^2) \delta_l(n) d\mathcal{H}^{d-2}.
\end{aligned}$$

Then Theorem 4.3 yields

$$\lim_{a\to 0} E\hat{V}_{d-2}(X) = \int_{S_1\cup S_2} \left(k_{d-1}F_1 + \sum_{j=1}^{d-2} k_j F_2\right) d\mathcal{H}^{d-1} = I_2 + I_4, \tag{4.1}$$

while

$$V_{d-2}(X) = \frac{1}{2\pi}\int_{S_1\cup S_2} \left(k_{d-1} + \sum_{j=1}^{d-2} k_j\right) d\mathcal{H}^{d-1} = \frac{1}{2\pi}(I_1 + I_3). \tag{4.2}$$

Here

$$\begin{aligned}
I_1 &= \int_{S_1} \left(k_{d-1} + \sum_{j=1}^{d-2} k_j\right) d\mathcal{H}^{d-1}\\
&= (d-1)R^{d-2}\int_\theta^{2\pi} \sin^{d-2}\varphi d\varphi,\\
I_2 &= \int_{S_1} \left(k_{d-1}F_1 + \sum_{j=1}^{d-2} k_j F_2\right) d\mathcal{H}^{d-1}\\
&= R^{d-2}\int_\theta^{2\pi} (F_1(\varphi) + (d-2)F_2(\varphi))\sin^{d-2}\varphi d\varphi,\\
I_3 &= \int_{S_2} \left(k_{d-1} + \sum_{j=1}^{d-2} k_j\right) d\mathcal{H}^{d-1}\\
&= \int_0^\theta \left(\frac{1}{r} + (d-2)\left(\frac{\sin\varphi}{(R-r)\sin\theta + r\sin\varphi}\right)\right) r((R-r)\sin\theta + r\sin\varphi)^{d-2} d\varphi\\
&= \int_0^\theta (((R-r)\sin\theta + r\sin\varphi)^{d-2} + (d-2)r\sin\varphi((R-r)\sin\theta + r\sin\varphi)^{d-3}) d\varphi\\
&= R^{d-2}\sin^{d-2}\theta \int_0^\theta 1 d\varphi + rp(r,\theta),\\
I_4 &= \int_{S_2} \left(k_{d-1}F_1 + \sum_{j=1}^{d-2} k_j F_2\right) d\mathcal{H}^{d-1}\\
&= \int_0^\theta \left(\frac{1}{r}F_1(\varphi) + \left(\frac{(d-2)\sin\varphi}{(R-r)\sin\theta + r\sin\varphi}\right) F_2(\varphi)\right)\\
&\quad \times r((R-r)\sin\theta + r\sin\varphi)^{d-2} d\varphi\\
&= \int_0^\theta (((R-r)\sin\theta + r\sin\varphi)^{d-2} F_1(\varphi)\\
&\quad + (d-2)r\sin\varphi((R-r)\sin\theta + r\sin\varphi)^{d-3} F_2(\varphi)) d\varphi\\
&= R^{d-2}\sin^{d-2}\theta \int_0^\theta F_1(\varphi) d\varphi + r\tilde{p}(r,\theta).
\end{aligned}$$

Here p and $\tilde{p}$ are both polynomials in r with coefficients depending only on θ and R.

Since $\hat{V}_{d-2}$ is asymptotically unbiased, (4.1) must equal (4.2), i.e.

$$I_2 + I_4 = \tfrac{1}{2\pi}(I_1 + I_3).$$

This must hold for all $0 < r < R$, so letting $r \to 0$ shows that

$$\begin{aligned}\int_\theta^\pi (F_1(\varphi) + (d-2)F_2(\varphi)) \sin^{d-2}\varphi d\varphi + \sin^{d-2}\theta \int_0^\theta F_1(\varphi) d\varphi \\ = (d-1)\frac{1}{2\pi}\int_\theta^{2\pi} \sin^{d-2}\varphi d\varphi + \sin^{d-2}\theta \frac{1}{2\pi}\int_0^\theta 1 d\varphi\end{aligned} \tag{4.3}$$

holds for all $\theta \in (0, \pi)$.

The assumption $e_d \in U$ ensures that for small values of φ, $n(u, \varphi) \in U$ for all $u \in S^{d-1}$, and hence all b_l, w_l, and δ_l are constants. This shows that F_1 and F_2 are continuous for small φ. In fact, a direct computation shows that for such small φ,

$$\begin{aligned}F_1(\varphi) &= K_1(\sin^2\varphi - \cos^2\varphi) + K_2 \sin\varphi\cos\varphi,\\ F_2(\varphi) &= K_3 + K_4 \sin^2\varphi + K_5\cos^2\varphi + K_6 \sin\varphi\cos\varphi,\end{aligned}$$

where $K_1, \dots, K_6 \in \mathbb{R}$ are certain constants. In particular, (4.3) may be differentiated with respect to θ for small values of θ. This yields

$$\begin{aligned}(d-2)\Big(- F_2(\theta)\sin^{d-2}\theta + \cos\theta \sin^{d-3}\theta \int_0^\theta F_1(\varphi)d\varphi\Big) \\ = \frac{(d-2)}{2\pi}(-\sin^{d-2}\theta + \theta\cos\theta\sin^{d-3}\theta)\end{aligned} \tag{4.4}$$

for θ small.

Since $d - 2 \neq 0$, (4.4) shows that $\theta\cos\theta\sin^{d-3}\theta$ must be a polynomial in $\cos\theta$ and $\sin\theta$, which is a contradiction. □

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