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Abstract

Intrinsic volumes and Minkowski tensors have been used to describe the geometry of real world objects. This paper presents an estimator that allows to approximate these quantities from digital images. It is based on a generalized Steiner formula for Minkowski tensors of sets of positive reach. When the resolution goes to infinity, the estimator converges to the true value if the underlying object is a set of positive reach. The underlying algorithm is based on a simple expression in terms of the cells of a Voronoi decomposition associated with the image.

Keywords: Minkowski tensor, digital algorithm, set of positive reach, digitization

1 Introduction

Minkowski tensors are tensor valued generalizations of the Minkowski functionals, associating with every sufficiently regular compact set in $\mathbb{R}^d$ a symmetric tensor, rather than a scalar. They carry information about geometric features of the set such as position, orientation and eccentricity. For instance, the volume tensor – defined formally in Section 2 – of rank 0 is just the volume of the set, while the volume tensors of rank 1 and 2 are closely related to the center of gravity and the tensor of inertia, respectively. For this reason, Minkowski tensors can be used as shape descriptors in materials science [23, 25], physics [12] and biology [2, 30].

The main purpose of this paper is to present estimators that approximate all the Minkowski tensors of a set K when only very weak information on K is available. More precisely, we assume that a finite set K_0 which is close to K in the Hausdorff metric is known. The estimators are based on the Voronoi decomposition of $\mathbb{R}^d$ associated with the finite set K_0, following an idea of Mérigot et al. [17]. What makes these estimators so interesting is that they are consistent; that is, they converge to the respective Minkowski tensors of K when applied to a sequence of finite approximations converging to K. We emphasize that the notion of 'estimator' is used here in the sense of digital geometry [14] meaning 'approximation of the true value based on discrete input' and should not be confused with the statistical concept related to the inference from data with random noise. The main application we have in mind is the case where K_0 is a digitization of K. This is detailed in the following.

As data is often only available in digital form, there is a need for estimators that allow us to approximate the Minkowski tensors from digital images. In a black-and-white image of a compact geometric object $K \subseteq \mathbb{R}^d$, each pixel (or voxel) is colored black if its midpoint belongs to K and white otherwise. Thus, the information about K contained in the image is the set of black pixel (voxel) midpoints $K_0 = K \cap a\mathbb{L}$, where $\mathbb{L}$ is the lattice formed by all pixel (voxel) midpoints and a^{-1} is the resolution. A natural criterion for the reliability of a digital estimator is that it yields the correct tensor when $a \to 0_+$. This property is called *multigrid convergence*. The estimators suggested in [22, 24] are all of local type; that is, they are obtained by scanning the digital image with a linear filter of predetermined fixed size and thus they avoid an explicit reconstruction of the object's boundary. The advantage of these estimators is that they are intuitive, easy to implement, and the computation time is linear in the number of pixels or voxels. However, it appears that none of these estimators is multigrid convergent. In fact, it is shown in [26] that there is no multigrid convergent estimator for a Minkowski tensor of rank 0 other than the volume. This also excludes multigrid convergent estimators for certain Minkowski tensors of higher rank; see the discussion in Section 7. In this paper, we therefore suggest an estimator that is not local, but which yields a multigrid convergent approximation for all Minkowski tensors. The present work is inspired by [17], and we therefore start by recalling some basic notions from this paper.

For a nonempty compact set K, the authors of [17] define a tensor valued measure, which they call the *Voronoi covariance measure*, defined on a Borel set $A \subseteq \mathbb{R}^d$ by

$$\mathcal{V}_R(K; A) = \int_{K^R} \mathbb{1}_A(p_K(y))(y - p_K(y))(y - p_K(y))^\top \, dy.$$

Here, K^R is the set of points at distance at most $R > 0$ from K and p_K is the *metric projection* on K: the point $p_K(x)$ is the point in K closest to x, provided that this closest point is unique. The metric projection of K is well-defined on $\mathbb{R}^d$ with the possible exception of a set of Lebesgue-measure zero; see, e.g., [6].

The paper [17] is mainly concerned with local features of surfaces and it is proved there that when $K \subseteq \mathbb{R}^3$ is a smooth surface, then

$$\mathcal{V}_R(K; B(x, r)) \approx \frac{2\pi}{3} R^3 r^2 \Big(u(x)u(x)^\top + \frac{r^2}{4} \sum_{i=1,2} \kappa_i(x)^2 P_i(x) P_i(x)^\top \Big), \tag{1.1}$$

where $B(x, r)$ is the Euclidean ball with midpoint $x \in K$ and radius r, $u(x)$ is the surface normal at $x \in K$, $P_1(x), P_2(x)$ are the principal directions and $\kappa_1(x), \kappa_2(x)$ the corresponding principal curvatures. Hence, the eigenvalues and -directions of the Voronoi covariance measure carry information about local curvatures and normal directions.

Assuming that a compact set K_0 approximates K, [17] suggests to estimate $\mathcal{V}_R(K; \cdot\,)$ by $\mathcal{V}_R(K_0; \cdot\,)$. It is shown in that paper, that $\mathcal{V}_R(K_0; \cdot\,)$ converges to

$\mathcal{V}_R(K;\,\cdot\,)$ in the bounded Lipschitz metric when $K_0 \to K$ in the Hausdorff metric. Moreover, if K_0 is a finite set, then the Voronoi covariance measure can be expressed in the form

$$\mathcal{V}_R(K_0; A) = \sum_{x \in K_0 \cap A} \int_{B(x,R) \cap V_x(K_0)} (y-x)(y-x)^\top \, dy.$$

Here, $V_x(K_0)$ is the Voronoi cell of x in the Voronoi decomposition of $\mathbb{R}^d$ associated with K_0. Thus, the estimator which is used to approximate $\mathcal{V}_R(K; A)$ is easily computed. Given the Voronoi cells of K_0, each Voronoi cell contributes with a simple integral. Figure 1 (a) shows the Voronoi cells of a finite set of points on an ellipse. The Voronoi cells are elongated in the normal direction. This is the intuitive reason why they can be used to approximate (1.1).

The Voronoi covariance measure $\mathcal{V}_R(K; A)$ can be identified with a symmetric 2-tensor. In the present work, we explore how natural extensions of the Voronoi covariance measure can be used for estimating general Minkowski tensors. We will work with full-dimensional sets K, and the finite point sample K_0 may be obtained from the representation $K_0 = K \cap a\mathbb{L}$ of a digital image of K. The Voronoi cells associated with $K_0 = K \cap a\mathbb{L}$ are sketched in Figure 1 (b). The generalizations of the Voronoi covariance measure, which we will introduce, will be called *Voronoi tensor measures*. We will then show how the Minkowski tensors can be recovered from these. Taking point samples from K with increasing resolution, convergence results will follow from an easy generalization of the convergence proof in [17].

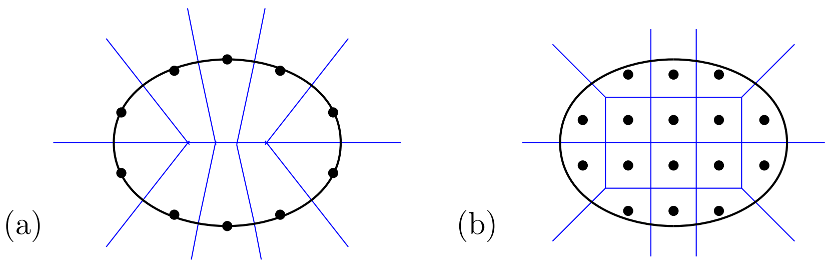

Figure 1: (a). The Voronoi cells of a finite set of points on a surface. (b). A digital image and the associated Voronoi cells.

The paper is structured as follows: Minkowski tensors are defined in Section 2. In Section 3, we define the Voronoi tensor measures, discuss how they can be estimated from finite point samples, and explain the Steiner formula that connects them with the Minkowski tensors. Section 4 is concerned with the convergence of the estimator. The results are specialized to digital images in Section 5. Section 6 discusses a generalization to the estimation of local Minkowski tensors on the normal bundle. Finally, the estimator is compared with existing approaches in Section 7.

2 Minkowski tensors

We work in Euclidean space $\mathbb{R}^d$ with scalar product $\langle \cdot\,, \cdot \rangle$ and norm $|\cdot|$. The Euclidean ball with center $x \in \mathbb{R}^d$ and radius $r \geq 0$ is denoted by $B(x, r)$, and we write S^{d-1} for the unit sphere in $\mathbb{R}^d$. The k-dimensional Hausdorff-measure in $\mathbb{R}^d$ is denoted by $\mathcal{H}^k$, $0 \le k \le d$. Let $\mathbb{T}^p$ denote the space of symmetric p-tensors (tensors of rank p) over $\mathbb{R}^d$. We denote by x^r the r-fold tensor product of $x \in \mathbb{R}^d$ and for two symmetric tensors a, b we denote their symmetric tensor product by ab. Identifying $\mathbb{R}^d$ with its dual (via the scalar product), a symmetric p-tensor defines a symmetric multilinear map $(\mathbb{R}^d)^p \to \mathbb{R}$. Letting $e_1, \dots, e_d$ be the standard basis in $\mathbb{R}^d$, a tensor $T \in \mathbb{T}^p$ is determined by its coordinates

$$T_{i_1 \dots i_p} = T(e_{i_1}, \dots, e_{i_p})$$

with respect to the standard basis, for all choices of $i_1, \dots, i_p \in \{1, \dots, d\}$. We use the norm on $\mathbb{T}^p$ given by

$$|T| = \sup\{|T(v_1, \dots, v_p)| : |v_1| = \dots = |v_p| = 1\}$$

for $T \in \mathbb{T}^p$. The same definition is used for arbitrary tensors of rank p.

For any compact $K \subseteq \mathbb{R}^d$, we can define an element of $\mathbb{T}^r$ called the *rth volume tensor*

$$\Phi_d^{r,0}(K) = \frac{1}{r!} \int_K x^r \, dx.$$

For $s \ge 1$ we define $\Phi_d^{r,s}(K) = 0$. We will now define Minkowski surface tensors, which require stronger regularity assumptions on K. Usually, like in [20, Section 5.4.2], the set K is assumed to be convex. However, as Minkowski tensors are tensor-valued integrals with respect to the generalized curvature measures (also called support measures) of K, they can be defined whenever the latter are available. We use this to define Minkowski tensors for sets of positive reach.

For a compact set $K \subseteq \mathbb{R}^d$, we let $d_K(x)$ denote the distance from $x \in \mathbb{R}^d$ to K. Then $K^R = \{x \in \mathbb{R}^d \mid d_K(x) \le R\}$ is the R-parallel set of K. The reach $\mathrm{Reach}(K)$ of K is defined as the supremum over all $R \ge 0$ such that for all $x \in \mathbb{R}^d$ with $d_K(x) < R$ there is a unique closest point $p_K(x)$ in K. We say that K has positive reach if $\mathrm{Reach}(K) > 0$. By definition, the map p_K is defined everywhere on K^R if $R < \mathrm{Reach}(K)$.

Let $K \subseteq \mathbb{R}^d$ be a set of positive reach. Zähle [29] introduced the *generalized curvature measures* $\Lambda_k(K; \cdot)$ of K, for $k = 0, \dots, d-1$. An extension to general closed sets is considered in [10]. The generalized curvature measures (also called support measures) are measures on $\Sigma = \mathbb{R}^d \times S^{d-1}$. Based on these measures, for every $k \in \{0, \dots, d-1\}$ and $r, s \ge 0$ we define the Minkowski tensor in $\mathbb{T}^{r+s}$ by

$$\Phi_k^{r,s}(K) = \frac{1}{r!s!} \frac{\omega_{d-k}}{\omega_{d-k+s}} \int_\Sigma x^r u^s \Lambda_k(K; d(x,u)).$$

Here ω_j is the surface area of the unit sphere S^{j-1} in $\mathbb{R}^j$. More information on Minkowski tensors can for instance be found in [11, 16, 21].

One can define local Minkowski tensors in a similar way (see [9]). For a Borel set $A \subseteq \mathbb{R}^d$ we put

$$\Phi_k^{r,s}(K; A) = \frac{1}{r!s!} \frac{\omega_{d-k}}{\omega_{d-k+s}} \int_{A \times S^{d-1}} x^r u^s \, \Lambda_k(K; d(x,u))$$

and

$$\Phi_d^{r,0}(K;A) = \frac{1}{r!}\int_{K\cap A} x^r\,dx.$$

Moreover, we define $\Phi_d^{r,s}(K;\,\cdot\,) = 0$ if $s \geq 1$. The local Minkowski tensors can be used to describe local boundary properties. For instance, local 1- and 2-tensors are used for the detection of sharp edges and corners on surfaces in [5]. They also carry information about normal directions and principal curvatures.

We conclude this section with a general remark on continuity properties of the Minkowski tensors. Let $\mathcal{C}^d$ be the family of nonempty compact subsets of $\mathbb{R}^d$ and $\mathcal{K}^d \subseteq \mathcal{C}^d$ the subset of nonemtpy compact convex sets. For two compact sets $K, M \in \mathcal{C}^d$, we define their *Hausdorff distance* by

$$d_H(K,M) = \inf\{\varepsilon > 0 \mid K \subseteq M^\varepsilon, M \subseteq K^\varepsilon\}.$$

Although the functions $K \mapsto \Phi_k^{r,s}(K)$ are continuous when considered in the metric space $(\mathcal{K}^d, d_H)$, they are not continuous on $\mathcal{C}^d$. (For instance, the volume tensors of a finite set are always vanishing, but finite sets can be used to approximate any compact set in the Hausdorff metric.) This is the reason why our approach requires an approximation argument with parallel sets as outlined below. The consistency of our estimator is mainly based on a continuity result for the metric projection map. We quote this result [3, Theorem 3.2] in a slightly different formulation which avoids dependence on the diameter $\operatorname{diam}(K)$ of K and is symmetric in the two bodies involved. Let $\|f\|_{L^1(E)}$ be the usual L^1-norm of the restriction of f to a Borel set $E \subseteq \mathbb{R}^d$.

Proposition 2.1. *Let $\rho > 0$ and let $E \subseteq \mathbb{R}^d$ be a bounded measurable set. Then there is a constant $C_1 = C_1\left(d, \operatorname{diam}(E \cup \{0\}), \rho\right) > 0$ such that*

$$\|p_K - p_{K_0}\|_{L^1(E)} \leq C_1 d_H(K, K_0)^{\frac{1}{2}}$$

for all $K, K_0 \in \mathcal{C}^d$ with $K, K_0 \subseteq B(0,\rho)$.

Proof. Let E' be the convex hull of E and observe that

$$\|p_K - p_{K_0}\|_{L^1(E)} \leq \|p_K - p_{K_0}\|_{L^1(E')}.$$

It is shown in [3, Lemma 3.3] (see also [7, Theorem 4.8]) that the map $v_K : \mathbb{R}^d \to \mathbb{R}$ given by $v_K(x) = |x|^2 - d_K^2(x)$ is convex and that its gradient coincides almost everywhere with $2p_K$. Since E' has rectifiable boundary, [3, Theorem 3.5] implies that

$$\begin{aligned}\|p_K - p_{K_0}\|_{L^1(E')} \leq{}& c_1(d)(\mathcal{H}^d(E') + (c_2 + \|d_K^2 - d_{K_0}^2\|_{\infty,E'}^{\frac{1}{2}})\mathcal{H}^{d-1}(\partial E')) \\ &\times \|d_K^2 - d_{K_0}^2\|_{\infty,E'}^{\frac{1}{2}}.\end{aligned}$$

Here $c_2 = \operatorname{diam}(2p_K(E') \cup 2p_{K_0}(E')) \leq 2\operatorname{diam}(K \cup K_0) \leq 4\rho$ and the supremum-norm $\|\cdot\|_{\infty,E'}$ on E' can be estimated by

$$\begin{aligned}\|d_K^2 - d_{K_0}^2\|_{\infty,E'} &\leq 2\operatorname{diam}(E' \cup K \cup K_0)\|d_K - d_{K_0}\|_{\infty,E'} \\ &\leq 2\left[\operatorname{diam}(E' \cup \{0\}) + \rho\right] d_H(K, K_0).\end{aligned}$$

Moreover, intrinsic volumes are increasing on the class of convex sets, so

$$\mathcal{H}^d(E') \le \mathcal{H}^d\big(B(0,\mathrm{diam}(E'\cup\{0\}))\big)$$
$$\mathcal{H}^{d-1}(\partial E') \le \mathcal{H}^{d-1}\big(\partial B(0,\mathrm{diam}(E'\cup\{0\}))\big).$$

Together with the trivial estimate $d_H(K, K_0) \le 2\rho$ and the equality $\mathrm{diam}(E\cup\{0\}) = \mathrm{diam}(E'\cup\{0\})$, this yields the claim. □

The authors of [3] argue that the exponent 1/2 in Proposition 2.1 is best possible.

3 Construction of the estimator

In Section 3.1 below, we define the Voronoi tensor measures and show how the Minkowski tensors can be obtained from these. We then explain in Section 3.2 how the Voronoi tensor measures can be estimated from finite point samples. As a special case, we obtain estimators for all intrinsic volumes. This is detailed in Section 3.3.

3.1 The Voronoi tensor measures

Let K be a compact set. Define the $\mathbb{T}^{r+s}$-valued measures $\mathcal{V}_R^{r,s}(K;\,\cdot\,)$ given on a Borel set $A \subseteq \mathbb{R}^d$ by

$$\mathcal{V}_R^{r,s}(K;A) = \int_{K^R} \mathbb{1}_A(p_K(x))\, p_K(x)^r (x - p_K(x))^s \, dx. \tag{3.1}$$

When K is a smooth surface, $\mathcal{V}_R^{0,2}(K;\,\cdot\,)$ corresponds to the Voronoi covariance measure of [17]. We will refer to the measures defined in (3.1) as the *Voronoi tensor measures*.

Note that if $f : \mathbb{R}^d \to \mathbb{R}$ is a bounded Borel function, then

$$\int_{\mathbb{R}^d} f(x)\, \mathcal{V}_R^{r,s}(K;dx) = \int_{K^R} f(p_K(x))\, p_K(x)^r (x - p_K(x))^s \, dx \in \mathbb{T}^{r+s}. \tag{3.2}$$

Suppose now that K has positive reach $\mathrm{Reach}(K) > R$. Then a special case of the generalized Steiner formula derived in [10] yields that

$$\begin{aligned}\mathcal{V}_R^{r,s}(K;A) &= \sum_{k=1}^{d} \omega_k \int_\Sigma \int_0^R \mathbb{1}_A(x) t^{s+k-1} x^r u^s \, dt \, \Lambda_{d-k}(K; d(x,u)) \\ &\qquad + \mathbb{1}_{\{s=0\}} \int_{K\cap A} x^r \, dx \\ &= r!s! \sum_{k=0}^{d} \kappa_{k+s} R^{s+k} \Phi_{d-k}^{r,s}(K;A),\end{aligned} \tag{3.3}$$

where κ_k is the volume of the unit ball in $\mathbb{R}^k$. In particular, the total measure is

$$\mathcal{V}_R^{r,s}(K) = \mathcal{V}_R^{r,s}(K;\mathbb{R}^d) = r!s! \sum_{k=0}^{d} \kappa_{k+s} R^{s+k} \Phi_{d-k}^{r,s}(K).$$

Equation (3.3), used for different parallel distances R, can be solved for the Minkowski tensors. More precisely, choosing $d+1$ different values $0 < R_0 < \ldots < R_d < \text{Reach}(K)$ for R, we obtain a system of $d+1$ linear equations:

$$\begin{pmatrix} \mathcal{V}_{R_0}^{r,s}(K;A) \\ \vdots \\ \mathcal{V}_{R_d}^{r,s}(K;A) \end{pmatrix} = r!s! \begin{pmatrix} \kappa_s R_0^s & \ldots & \kappa_{s+d} R_0^{s+d} \\ \vdots & & \vdots \\ \kappa_s R_d^s & \ldots & \kappa_{s+d} R_d^{s+d} \end{pmatrix} \begin{pmatrix} \Phi_d^{r,s}(K;A) \\ \vdots \\ \Phi_0^{r,s}(K;A) \end{pmatrix}. \tag{3.4}$$

Since the Vandermonde matrix

$$A_{R_0,\ldots,R_d}^{r,s} = r!s! \begin{pmatrix} \kappa_s R_0^s & \ldots & \kappa_{s+d} R_0^{s+d} \\ \vdots & & \vdots \\ \kappa_s R_d^s & \ldots & \kappa_{s+d} R_d^{s+d} \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)} \tag{3.5}$$

in (3.4) is invertible, the system can be solved for the tensors, and thus we get

$$\begin{pmatrix} \Phi_d^{r,s}(K;A) \\ \vdots \\ \Phi_0^{r,s}(K;A) \end{pmatrix} = \left(A_{R_0,\ldots,R_d}^{r,s}\right)^{-1} \begin{pmatrix} \mathcal{V}_{R_0}^{r,s}(K;A) \\ \vdots \\ \mathcal{V}_{R_d}^{r,s}(K;A) \end{pmatrix}. \tag{3.6}$$

If $s > 0$, then $\Phi_d^{r,s}(K;A) = 0$ by definition, so we may omit one of the equations in the system (3.4).

3.2 Estimation of Minkowski tensors from finite point samples

Let K be a compact set of positive reach. Suppose that we are given a finite set K_0 that is close to K in the Hausdorff metric. In the applications we have in mind, we have in addition that $K_0 \subseteq K$, but this is not necessary for the algorithm to work. Based on K_0, we want to estimate the local Minkowski tensors of K. We do this by approximating $\mathcal{V}_{R_k}^{r,s}(K;A)$ in Formula (3.6) by $\mathcal{V}_{R_k}^{r,s}(K_0;A)$, for $k = 0,\ldots,d$. This leads to the following set of estimators:

$$\begin{pmatrix} \hat{\Phi}_d^{r,s}(K_0;A) \\ \vdots \\ \hat{\Phi}_0^{r,s}(K_0;A) \end{pmatrix} = \left(A_{R_0,\ldots,R_d}^{r,s}\right)^{-1} \begin{pmatrix} \mathcal{V}_{R_0}^{r,s}(K_0;A) \\ \vdots \\ \mathcal{V}_{R_d}^{r,s}(K_0;A) \end{pmatrix} \tag{3.7}$$

with $A_{R_0,\ldots,R_d}^{r,s}$ given by (3.5). Setting $A = \mathbb{R}^d$ in (3.7), we obtain estimators

$$\hat{\Phi}_k^{r,s}(K_0) = \hat{\Phi}_k^{r,s}(K_0;\mathbb{R}^d)$$

of the (global) Minkowski tensors. Note that this approach requires an estimate for the reach of K because we need to choose $0 < R_0,\ldots,R_d < \text{Reach}(K)$.

Let

$$V_x(K_0) = \{y \in \mathbb{R}^d \mid p_{K_0}(y) = x\}$$

denote the Voronoi cell of $x \in K_0$ with respect to the set K_0. Since $\mathbb{R}^d$ is the union of the finitely many Voronoi cells of K_0, it follows that K_0^R is the union of the R-bounded parts of the Voronoi cells $B(x,R) \cap V_x(K_0)$, $x \in K_0$, which have pairwise disjoint interiors. Thus (3.1) simplifies to

$$\mathcal{V}_R^{r,s}(K_0; A) = \sum_{x \in K_0 \cap A} x^r \int_{B(x,R) \cap V_x(K_0)} (y-x)^s \, dy. \tag{3.8}$$

Like the Voronoi covariance measure, the Voronoi tensor measure $\mathcal{V}_R^{r,s}(K_0; A)$ is a sum of simple contributions from the individual Voronoi cells.

An example of a Voronoi cell decomposition associated with a digital image is sketched in Figure 2. The original set K is the disk bounded by the inner black circle, and the disk bounded by the outer black circle is its R-parallel set K^R. The finite point sample is $K_0 = K \cap \mathbb{Z}^2$, which is shown as the set of red dots in the picture and the red curve is the boundary of its R-parallel set. The Voronoi cells of K_0 are indicated by blue lines. The R-bounded part of one of the Voronoi cells is the part that is cut off by the red arc.

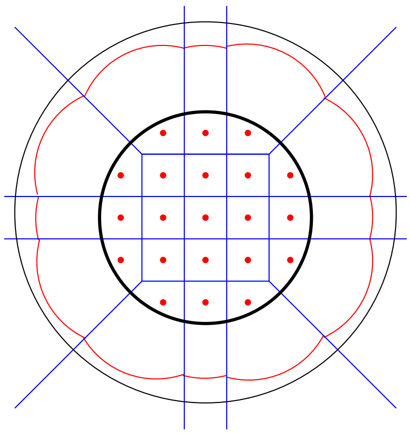

Figure 2: The Voronoi decomposition (blue lines) and R-parallel set (red curve) associated with a digital image.

3.3 The case of intrinsic volumes

Recall that $\Phi_k^{0,0}(K) = \Lambda_k(K; \mathbb{R}^d)$ is just the usual kth intrinsic volume. Thus, Section 3.2 provides an estimator for all intrinsic volumes as a special case. This case is particularly simple. The measure $\mathcal{V}_R^{0,0}(K; A)$ is simply the volume of a local parallel set

$$\begin{aligned} \mathcal{V}_R^{0,0}(K; A) &= \mathcal{H}^d(\{x \in K^R \mid p_K(x) \in A\}), \\ \mathcal{V}_R^{0,0}(K) &= \mathcal{H}^d(K^R). \end{aligned}$$

In particular, if $K \subseteq \mathbb{R}^d$ is a compact set with $\mathrm{Reach}(K) > R$, then Equation (3.3) reduces to the usual local Steiner formula

$$\mathcal{H}^d(\{x \in K^R \mid p_K(x) \in A\}) = \sum_{k=0}^{d} \kappa_k R^k \Lambda_{d-k}(K; A \times S^{d-1}),$$

and

$$\mathcal{H}^d(K^R) = \sum_{k=0}^{d} \kappa_k R^k \Phi^{0,0}_{d-k}(K).$$

In this case, our algorithm approximates the parallel volume $\mathcal{H}^d(K^R)$ by $\mathcal{H}^d(K_0^R)$. In the example in Figure 2, this corresponds to approximating the volume of the larger black disk by the volume of the region bounded by the red curve. This volume is again the sum of the volumes of the regions bounded by the red and blue curves. In other words, it is the sum of volumes of the R-bounded Voronoi cells on the right-hand side of the equation

$$\mathcal{V}^{0,0}_R(K_0; A) = \sum_{x \in K_0 \cap A} \mathcal{H}^d(B(x, R) \cap V_x(K_0)).$$

4 Convergence properties

In this section we prove the main convergence result. This is an immediate generalization of [17, Theorem 5.1].

4.1 The convergence theorem

For a bounded Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, we let $|f|_\infty$ denote the usual supremum norm,

$$|f|_L = \sup\left\{ \frac{|f(x) - f(y)|}{|x - y|} \;\middle|\; x \neq y \right\}$$

the Lipschitz semi-norm, and

$$|f|_{bL} = |f|_L + |f|_\infty$$

the bounded Lipschitz norm. Let d_{bL} be the bounded Lipschitz metric on the space of bounded $\mathbb{T}^p$-valued Borel measures on $\mathbb{R}^d$, given on two such measures μ and ν on $\mathbb{R}^d$ by

$$d_{bL}(\mu, \nu) = \sup\left\{ \left| \int f\, d\mu - \int f\, d\nu \right| \;\middle|\; |f|_{bL} \leq 1 \right\}.$$

The following theorem shows that the map

$$K \mapsto \mathcal{V}^{r,s}_R(K; \,\cdot\,)$$

is Hölder continuous with exponent $\frac{1}{2}$ with respect to the Hausdorff metric on $\mathcal{C}^d$ and the bounded Lipschitz metric. In the proof, we use the symmetric difference $A \Delta B = (A \setminus B) \cup (B \setminus A)$ of sets $A, B \subseteq \mathbb{R}^d$.

Theorem 4.1. *Let $R, \rho > 0$ and $r, s \in \mathbb{N}_0$ be given. Then there is a positive constant $C_2 = C_2(d, R, \rho, r, s)$ such that*

$$d_{bL}(\mathcal{V}^{r,s}_R(K; \,\cdot\,), \mathcal{V}^{r,s}_R(K_0; \,\cdot\,)) \leq C_2 d_H(K, K_0)^{\frac{1}{2}}$$

for all compact sets $K, K_0 \subseteq B(0, \rho)$.

Proof. Let f with $|f|_{bL} \leq 1$ be given. Then (3.2) yields

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} f(x)\, \mathcal{V}_R^{r,s}(K; dx) - \int_{\mathbb{R}^d} f(x)\, \mathcal{V}_R^{r,s}(K_0; dx)\Big| \\
&\quad = \Big|\int_{K^R} f(p_K(x))\, p_K(x)^r (x - p_K(x))^s\, dx \\
&\qquad\qquad - \int_{K_0^R} f(p_{K_0}(x)) p_{K_0}(x)^r (x - p_{K_0}(x))^s\, dx\Big| \\
&\quad \leq I + II, \qquad (4.1)
\end{aligned}$$

where I is the integral

$$\int_{K^R \cap K_0^R} \big|f(p_K(x)) p_K(x)^r (x - p_K(x))^s - f(p_{K_0}(x))\, p_{K_0}(x)^r (x - p_{K_0}(x))^s\big|\, dx$$

and

$$II = \rho^r R^s \mathcal{H}^d(K^R \Delta K_0^R).$$

By [3, Corollary 4.4], there is a constant $c_1 = c_1(d, R, \rho) > 0$ such that

$$\mathcal{H}^d(K^R \Delta K_0^R) \leq c_1\, d_H(K, K_0) \qquad (4.2)$$

when $d_H(K, K_0) \leq R/2$. Replacing c_1 by a possibly even bigger constant, we can ensure that (4.2) also holds when $R/2 \leq d_H(K, K_0) \leq 2\rho$. Hence,

$$II \leq c_2\, d_H(K, K_0)^{\frac{1}{2}} \qquad (4.3)$$

with some constant $c_2 = c_2(d, R, \rho, r, s) > 0$.

Using the inequality

$$\Big|\bigotimes_{i=1}^m y_i - \bigotimes_{i=1}^m z_i\Big| \leq \sum_{j=1}^m |y_j - z_j| \prod_{i=1}^{j-1} |y_i| \prod_{i=j+1}^m |z_i|, \qquad (4.4)$$

and the assumption $|f|_{bL} \leq 1$, we get

$$\begin{aligned}
I &\leq (r + s + 1)\rho^r R^s \int_{K^R \cap K_0^R} |p_K(x) - p_{K_0}(x)|\, dx \\
&\leq c_3\, d_H(K, K_0)^{\frac{1}{2}}. \qquad (4.5)
\end{aligned}$$

The existence of the constant $c_3 = c_3(d, R, \rho, r, s)$ in the last inequality is guaranteed by Proposition 2.1 with $K^R \cap K_0^R$ as the set E, because this choice of E satisfies $\operatorname{diam}(E \cup \{0\}) \leq 2(\rho + R)$. □

When $r = s = 0$ and $f = 1$, the above proof simplifies to Inequality (4.2) as I vanishes. Hence we obtain the following strengthening of the theorem, which is relevant for the estimation of intrinsic volumes.

Theorem 4.2. *Let $R, \rho > 0$. Then there is a constant $C_3 = C_3(d, R, \rho) > 0$ such that*

$$\left|\mathcal{V}_R^{0,0}(K) - \mathcal{V}_R^{0,0}(K_0)\right| \leq C_3\, d_H(K, K_0)$$

for all compact sets $K, K_0 \subseteq B(0, \rho)$.

For local tensors, the proof of Theorem 4.1 can also be adapted to show a convergence result.

Theorem 4.3. *If $K_i \to K$ with respect to the Hausdorff metric on $\mathcal{C}^d$, as $i \to \infty$, then $\mathcal{V}_R^{r,s}(K_i; A) \to \mathcal{V}_R^{r,s}(K; A)$ in the tensor norm, for every Borel set A which satisfies*

$$\mathcal{H}^d(p_K^{-1}(\partial A) \cap K^R) = 0.$$

Proof. Convergence of tensors is equivalent to coordinate-wise convergence. Hence, it is enough to show that the coordinates satisfy

$$\mathcal{V}_R^{r,s}(K_i; A)_{i_1 \ldots i_{r+s}} \to \mathcal{V}_R^{r,s}(K; A)_{i_1 \ldots i_{r+s}} \qquad \text{as } i \to \infty,$$

for all choices of indices $i_1 \ldots i_{r+s}$; see the notation at the beginning of Section 2.

We write $T_K(x) = p_K(x)^r (x - p_K(x))^s$. Then

$$\mathcal{V}_R^{r,s}(K; A)_{i_1 \ldots i_{r+s}} = \int_{K^R} \mathbb{1}_A(p_K(x)) T_K(x)_{i_1 \ldots i_{r+s}}\, dx$$

is a signed measure. Let $T_K(x)^+_{i_1 \ldots i_{r+s}}$ and $T_K(x)^-_{i_1 \ldots i_{r+s}}$ denote the positive and negative part of $T_K(x)_{i_1 \ldots i_{r+s}}$, respectively. Then

$$\mathcal{V}_R^{r,s}(K; A)^{\pm}_{i_1 \ldots i_{r+s}} = \int_{K^R} \mathbb{1}_A(p_K(x)) T_K(x)^{\pm}_{i_1 \ldots i_{r+s}}\, dx$$

are non-negative measures such that

$$\mathcal{V}_R^{r,s}(K;\, \cdot\,)_{i_1 \ldots i_{r+s}} = \mathcal{V}_R^{r,s}(K;\, \cdot\,)^+_{i_1 \ldots i_{r+s}} - \mathcal{V}_R^{r,s}(K;\, \cdot\,)^-_{i_1 \ldots i_{r+s}}.$$

The proof of Theorem 4.1 can immediately be generalized to show that the measure $\mathcal{V}_R^{r,s}(K_i;\, \cdot\,)^{\pm}_{i_1 \ldots i_{r+s}}$ converges to $\mathcal{V}_R^{r,s}(K;\, \cdot\,)^{\pm}_{i_1 \ldots i_{r+s}}$ in the bounded Lipschitz norm (as $i \to \infty$), and hence the measures converge weakly. In particular, they converge on every continuity set of $\mathcal{V}_R^{r,s}(K;\, \cdot\,)^{\pm}_{i_1 \ldots i_{r+s}}$. If $\mathcal{H}^d(p_K^{-1}(\partial A) \cap K^R) = 0$, then A is such a continuity set. $\square$

Though relatively mild, the condition $\mathcal{H}^d(p_K^{-1}(\partial A) \cap K^R) = 0$ can be hard to control if K is unknown. For instance, it is not satisfied if $A = K$ or if K is a polytope intersecting ∂A at a vertex.

As the matrix $A^{r,s}_{R_0, \ldots, R_d}$ in definition (3.7) of $\hat{\Phi}_k^{r,s}(K_0; A)$ does not depend on the set K_0, the above results immediately yield a consistency result for the estimation of the Minkowski tensors. We formulate this only for $A = \mathbb{R}^d$.

Corollary 4.4. *Let $\rho > 0$ and let K be a compact subset of $B(0,\rho)$ such that* $\mathrm{Reach}(K) > R_d > \ldots > R_0 > 0$. *There is a constant $C_4 = C_4(d, R_0, \ldots, R_d, \rho)$ such that*

$$\left|\hat{\Phi}_k^{0,0}(K_0) - \Phi_k^{0,0}(K)\right| \leq C_4\, d_H(K_0, K),$$

for all $k \in \{0, \ldots, d\}$, whenever the compact set K_0 is contained in $B(0,\rho)$.

For $r, s \in \mathbb{N}_0$ there is a constant $C_5 = C_5(d, R_0, \ldots, R_d, \rho, r, s)$ such that

$$\left|\hat{\Phi}_k^{r,s}(K_0) - \Phi_k^{r,s}(K)\right| \leq C_5\, d_H(K_0, K)^{\frac{1}{2}},$$

for all $k \in \{0, \ldots, d-1\}$, whenever the compact set K_0 is contained in $B(0,\rho)$.

5 Application to digital images

Our main motivation for this paper is the estimation of Minkowski tensors from digital images. Recall that we model a black-and-white digital image of $K \subseteq \mathbb{R}^d$ as the set $K \cap a\mathbb{L}$, where $\mathbb{L} \subseteq \mathbb{R}^d$ is a fixed lattice and $a > 0$. We refer to [1] for basic information about lattices.

The lower dimensional parts of K are generally invisible in the digital image. When dealing with digital images, we will therefore always assume that the underlying set is topologically regular, which means that it is the closure of its own interior.

In digital stereology, the underlying object K is often assumed to belong to one of the following two set classes:

- K is called *δ-regular* if it is topologically regular and the reach of its closed complement $\mathrm{cl}(\mathbb{R}^d \backslash K)$ and the reach of K itself are both at least $\delta > 0$. This is a kind of smoothness condition on the boundary, ensuring in particular that ∂K is a C^1 manifold (see the discussion after Definition 1 in [28]).
- K is called *polyconvex* if it is a finite union of compact convex sets. While convex sets have infinite reach, note that polyconvex sets do generally not have positive reach. Also note that for a compact convex set $K \subseteq \mathbb{R}^d$, the set $\mathrm{cl}(\mathbb{R}^d \backslash K)$ need not have positive reach.

It should be observed that for a compact set $K \subseteq \mathbb{R}^d$ both assumptions imply that the boundary of K is a $(d-1)$-rectifiable set (i.e., the image of a bounded subset of $\mathbb{R}^{d-1}$ under a Lipschitz map), which is a much weaker property that will still be sufficient for the analysis in Section 5.1.

5.1 The volume tensors

Simple and efficient estimators for the volume tensors $\Phi_d^{r,0}(K)$ of a (topologically regular) compact set K are already known and are usually based on the approximation of K by the union of all pixels (voxels) with midpoint in K. This leads to the estimator

$$\tilde{\Phi}_d^{r,0}(K \cap a\mathbb{L}) = \frac{1}{r!} \sum_{z \in K \cap a\mathbb{L}} \int_{z + aV_0(\mathbb{L})} x^r\, dx,$$

where $V_0(\mathbb{L})$ is the Voronoi cell of 0 in the Voronoi decomposition generated by $\mathbb{L}$. This, in turn, can be approximated by

$$\tilde{\tilde{\Phi}}_d^{r,0}(K \cap a\mathbb{L}) = \frac{a^d}{r!}\mathcal{H}^d\left(V_0(\mathbb{L})\right) \sum_{z \in K \cap a\mathbb{L}} z^r.$$

When $r \in \{0,1\}$, we even have $\tilde{\Phi}_d^{r,0}(K \cap a\mathbb{L}) = \tilde{\tilde{\Phi}}_d^{r,0}(K \cap a\mathbb{L})$.

Choose $C > 0$ such that $V_0(\mathbb{L}) \subseteq B(0, C)$. Then

$$K \Delta \bigcup_{z \in K \cap a\mathbb{L}} (z + aV_0(\mathbb{L})) \subseteq (\partial K)^{aC}.$$

In fact, if $x \in \left[\bigcup_{z \in K \cap a\mathbb{L}}(z + aV_0(\mathbb{L}))\right] \setminus K$, then there is some $z \in K \cap a\mathbb{L}$ such that $x \in z + aV_0(\mathbb{L})$ and $x \notin K$. Since $z \in K$ and $x \notin K$, we have $[x, z] \cap \partial K \neq \emptyset$. Moreover, $x - z \in aV_0(\mathbb{L}) \subset B(0, aC)$, and hence $|x - z| \leq aC$. This shows that $x \in (\partial K)^{aC}$. Now assume that $x \in K$ and $x \notin (\partial K)^{aC}$. Then $B(x, \rho) \subseteq K$ for some $\rho > aC$. Since $\bigcup_{z \in a\mathbb{L}}(z+aV_0(\mathbb{L})) = \mathbb{R}^d$, there is some $z \in a\mathbb{L}$ such that $x \in z+aV_0(\mathbb{L})$. Hence $x - z \in aV_0(\mathbb{L}) \subset B(0, aC)$. We conclude that $z \in B(x, aC) \subseteq K$, therefore $z \in K \cap a\mathbb{L}$ and thus $x \in \bigcup_{z \in K \cap a\mathbb{L}}(z + aV_0(\mathbb{L}))$.

Hence

$$\left|\tilde{\Phi}_d^{r,0}(K \cap a\mathbb{L}) - \Phi_d^{r,0}(K)\right| \leq \frac{1}{r!}\int_{(\partial K)^{aC}} |x|^r \, dx.$$

If $\mathcal{H}^d(\partial K) = 0$, then the integral on the right-hand side goes to zero by monotone convergence, so

$$\lim_{a \to 0_+} \tilde{\Phi}_d^{r,0}(K \cap a\mathbb{L}) = \Phi_d^{r,0}(K). \tag{5.1}$$

If ∂K is $(d-1)$-rectifiable, then $\mathcal{H}^d(\partial K) = 0$ and [8, Theorem 3.2.39] implies that $\mathcal{H}^d((\partial K)^{aC})$ is of order $O(a)$. Hence, the speed of convergence in (5.1) is $O(a)$ as $a \to 0_+$.

Inequality (4.4) yields that $|x^r - z^r| \leq aCr(|x|+aC)^{r-1}$ whenever $x \in z+aV_0(\mathbb{L})$ and $r \geq 1$. Therefore,

$$\begin{aligned}\left|\tilde{\tilde{\Phi}}_d^{r,0}(K \cap a\mathbb{L}) - \tilde{\Phi}_d^{r,0}(K \cap a\mathbb{L})\right| &\leq \frac{aC}{(r-1)!}\sum_{z \in K \cap a\mathbb{L}} \int_{z+aV_0(\mathbb{L})} (|x| + aC)^{r-1} \, dx \\ &\leq \frac{aC}{(r-1)!}\int_{K^{aC}} (|x| + aC)^{r-1} \, dx,\end{aligned}$$

which shows that

$$\lim_{a \to 0_+} \tilde{\tilde{\Phi}}_d^{r,0}(K \cap a\mathbb{L}) = \Phi_d^{r,0}(K),$$

provided that $\mathcal{H}^d(\partial K) = 0$. If ∂K is $(d-1)$-rectifiable, then the speed of convergence is of the order $O(a)$.

Hence, we suggest to simply use the estimators $\tilde{\tilde{\Phi}}_d^{r,0}(K \cap a\mathbb{L})$ for the volume tensors. This estimator can be computed faster and more directly than $\hat{\Phi}_d^{r,0}(K \cap a\mathbb{L})$. Moreover, it does not require an estimate for the reach of K, and it converges for a much larger class of sets than those of positive reach.

5.2 Convergence for digital images

For the estimation of the remaining tensors we suggest to use the Voronoi tensor measures. Choosing $K_0 = K \cap a\mathbb{L}$ in (3.8), we obtain

$$\mathcal{V}_R^{r,s}(K \cap a\mathbb{L}; A) = \sum_{x \in K \cap a\mathbb{L} \cap A} x^r \int_{B(x,R) \cap V_x(K \cap a\mathbb{L})} (y-x)^s \, dy. \tag{5.2}$$

To show some convergence results in Corollary 5.2, we first note that the digital image converges to the original set in the Hausdorff metric.

Lemma 5.1. *If K is compact and topologically regular, then*

$$\lim_{a \to 0_+} d_H(K, K \cap a\mathbb{L}) = 0.$$

If K is δ-regular, then $d_H(K, K \cap a\mathbb{L})$ is of order $O(a)$. The same holds if K is topologically regular and polyconvex.

Proof. Recall from [1, p. 311] that $\mu(\mathbb{L}) = \max_{x \in \mathbb{R}^d} \operatorname{dist}(x, \mathbb{L})$ is well defined and denotes the covering radius of $\mathbb{L}$.

Let $\varepsilon > 0$ be given. Since K is compact, there are points $x_1, \ldots, x_m \in K$ such that

$$K \subseteq \bigcup_{i=1}^{m} B(x_i, \varepsilon).$$

Using the fact that K is topologically regular, we conclude that there are points $y_i \in \operatorname{int}(K) \cap \operatorname{int}(B(x_i, 2\varepsilon))$ for $i = 1, \ldots, m$. Hence, there are $\varepsilon_i \in (0, 2\varepsilon)$ such that $B(y_i, \varepsilon_i) \subseteq K \cap B(x_i, 2\varepsilon)$ for $i = 1, \ldots, m$. Let $0 < a < \min\{\varepsilon_i/\mu(\mathbb{L}) \mid i = 1, \ldots, m\}$. Since $\varepsilon_i/a > \mu(\mathbb{L})$ it follows that $a\mathbb{L} \cap B(y_i, \varepsilon_i) \neq \emptyset$, for $i = 1, \ldots, m$. Thus we can choose $z_i \in a\mathbb{L} \cap B(y_i, \varepsilon_i) \subseteq a\mathbb{L} \cap K$ for $i = 1, \ldots, m$. By the triangle inequality, we have $|z_i - x_i| \leq \varepsilon_i + 2\varepsilon \leq 4\varepsilon$, and hence $x_i \in (K \cap a\mathbb{L}) + B(0, 4\varepsilon)$, for $i = 1, \ldots, m$. Therefore, $K \subseteq (K \cap a\mathbb{L}) + B(0, 5\varepsilon)$ if $a > 0$ is sufficiently small.

Assume that K is δ-regular, for some $\delta > 0$. We choose $0 < a < \delta/(2\mu(\mathbb{L}))$. Since $a\mu(\mathbb{L}) < \delta/2$, for any $x \in K$ there is a ball $B(y, a\mu(\mathbb{L}))$ of radius $a\mu(\mathbb{L})$ such that $x \in B(y, a\mu(\mathbb{L})) \subseteq K$. From $a\mathbb{L} \cap B(y, a\mu(\mathbb{L})) \neq \emptyset$ we conclude that there is a point $z \in K \cap a\mathbb{L}$ with $|x - z| \leq 2a\mu(\mathbb{L})$. Hence $x \in (K \cap a\mathbb{L}) + B(0, 2a\mu(\mathbb{L}))$, and therefore $d_H(K, K \cap a\mathbb{L}) \leq 2a\mu(\mathbb{L})$.

Finally, we assume that K is topologically regular and polyconvex. Then K is the union of finitely many compact convex sets with interior points. Hence, for the proof we may assume that K is convex with $B(0, \rho) \subseteq K$ for a fixed $\rho > 0$. Choose $0 < a < \rho/(2\mu(\mathbb{L}))$ and put $r = 2a\mu(\mathbb{L}) < \rho$. If $x \in K$, then $B((1 - r/\rho)x, r) \subseteq K$ and $B((1 - r/\rho)x, r)$ contains a point $z \in a\mathbb{L}$. Since

$$|x - z| \leq r + (r/\rho)|x| \leq 2a\mu(\mathbb{L})\,(1 + \operatorname{diam}(K)/\rho),$$

we get

$$K \subset (K \cap a\mathbb{L}) + B\big(0, 2a\mu(\mathbb{L})\,(1 + \operatorname{diam}(K)/\rho)\big),$$

which completes the argument. $\square$

Thus Theorems 4.1 and 4.2 and Corollary 4.4 together with Lemma 5.1 yield the following result.

Corollary 5.2. *If K is compact and topologically regular, then*

$$\lim_{a\to 0_+} d_{bL}\big(\mathcal{V}_R^{r,s}(K;\,\cdot\,), \mathcal{V}_R^{r,s}(K\cap a\mathbb{L};\,\cdot\,)\big) = 0,$$
$$\lim_{a\to 0_+} \mathcal{V}_R^{r,s}(K\cap a\mathbb{L}) = \mathcal{V}_R^{r,s}(K).$$

If, in addition, K has positive reach, then

$$\lim_{a\to 0_+} \hat{\Phi}_k^{r,s}(K\cap a\mathbb{L}) = \Phi_k^{r,s}(K). \tag{5.3}$$

If K is δ-regular or a topologically regular convex set, then the speed of convergence is $O(a)$ when $r = s = 0$ and $O(\sqrt{a})$ otherwise.

The property (5.3) expresses the fact that $\hat{\Phi}_k^{r,s}(K\cap a\mathbb{L})$ is multigrid convergent as $a \to 0_+$. A similar statement about local tensors, but without the speed of convergence, can be made. We omit this here.

5.3 Possible refinements of the algorithm for digital images

We first describe how the number of necessary radii $R_0 < R_1 < \ldots < R_d$ in (3.7) can be reduced by one if $s = 0$ and $A = \mathbb{R}^d$. Setting $s = 0$ and $A = \mathbb{R}^d$ and subtracting $(r!)\Phi_d^{r,0}(K)$ on both sides of Equation (3.3) yields

$$\int_{K^R\setminus K} p_K(x)^r\,dx = \mathcal{V}_R^{r,0}(K) - (r!)\Phi_d^{r,0}(K) = (r!)\sum_{k=1}^{d} \kappa_k R^k \Phi_{d-k}^{r,0}(K). \tag{5.4}$$

As noted in Section 5.1, the volume tensor $\Phi_d^{r,0}(K)$ can be estimated by $\tilde{\tilde{\Phi}}_d^{r,0}(K\cap a\mathbb{L})$. We may take $\mathcal{V}_R^{r,0}(K\cap a\mathbb{L}) - (r!)\tilde{\tilde{\Phi}}_d^{r,0}(K\cap a\mathbb{L})$ as an improved estimator for (5.4). This corresponds to replacing the integration domains $B(x,R)\cap V_x(K\cap a\mathbb{L})$ in (5.2) by

$$(B(x,R)\cap V_x(K\cap a\mathbb{L}))\setminus V_x(a\mathbb{L}).$$

This makes sense since $V_x(a\mathbb{L})$ is likely to be contained in K while the left-hand side of (5.4) is an integral over $K^R\setminus K$. The Minkowski tensors can now be isolated from only d equations of the form (5.4) with d different values of R.

We now suggest a slightly modified estimator for the Minkowski tensors satisfying the same convergence results as $\hat{\Phi}_k^{r,s}(K\cap a\mathbb{L})$ but where the number of summands in (5.2) is considerably reduced. As the volume tensors can easily be estimated with the estimators in Section 5.1, we focus on the tensors with $k < d$.

Let K be a compact set. We define the *Voronoi neighborhood* $N_{\mathbb{L}}(0)$ of 0 to be the set of points $y \in \mathbb{L}$ such that the Voronoi cells $V_0(\mathbb{L})$ and $V_y(\mathbb{L})$ of 0 and y, respectively, have exactly one common $(d-1)$-dimensional face. Similarly, for $z \in \mathbb{L}$ the Voronoi neighborhood $N_{\mathbb{L}}(z)$ of z is defined, and thus clearly $N_{\mathbb{L}}(z) = z + N_{\mathbb{L}}(0)$. When $\mathbb{L} \subset \mathbb{R}^2$ is the standard lattice, $N_{\mathbb{L}}(z)$ consists of the four points in $\mathbb{L}$ that are neighbors of z in the usual 4-neighborhood. Define $I(K\cap a\mathbb{L})$ to be the set of points

$z \in K \cap a\mathbb{L}$ such that $N_{a\mathbb{L}}(z) \subseteq K \cap a\mathbb{L}$. The relative complement $B(K \cap a\mathbb{L}) = (K \cap a\mathbb{L}) \setminus I(K \cap a\mathbb{L})$ of $I(K \cap a\mathbb{L})$ can be considered as the set of lattice points in $K \cap a\mathbb{L}$ that are close to the boundary of the given set K.

We modify (5.2) by removing contributions from $I(K \cap a\mathbb{L})$ and define

$$\check{\mathcal{V}}_R^{r,s}(K \cap a\mathbb{L}; A) = \sum_{x \in B(K\cap a\mathbb{L})\cap A} x^r \int_{B(x,R)\cap V_x(K\cap a\mathbb{L})} (y-x)^s \, dy. \tag{5.5}$$

Assuming that K has positive reach, let $0 < R_0 < R_1 < \ldots < R_d < \mathrm{Reach}(K)$. We write again K_0 for $K \cap a\mathbb{L}$. Then we obtain the estimators

$$\begin{pmatrix} \check{\Phi}_d^{r,s}(K_0; A) \\ \vdots \\ \check{\Phi}_0^{r,s}(K_0; A) \end{pmatrix} = \left(A_{R_0,\ldots,R_d}^{r,s}\right)^{-1} \begin{pmatrix} \check{\mathcal{V}}_{R_0}^{r,s}(K_0; A) \\ \vdots \\ \check{\mathcal{V}}_{R_d}^{r,s}(K_0; A) \end{pmatrix} \tag{5.6}$$

with $A_{R_0,\ldots,R_d}^{r,s}$ given by (3.5).

Working with $\check{\mathcal{V}}_R^{r,s}(K \cap a\mathbb{L}; A)$ reduces the workload considerably. For instance, when K is δ-regular or polyconvex and topologically regular, the number of elements in $I(K \cap a\mathbb{L})$ increases with a^{-d}, whereas the number of elements in $B(K \cap a\mathbb{L})$ only increases with $a^{-(d-1)}$ as $a \to 0_+$. The set $I(K \cap a\mathbb{L})$ can be obtained from the digital image of K in linear time using a linear filter. Moreover, we have the following convergence result.

Proposition 5.3. *Let K be a topologically regular compact set with positive reach and let C be such that $V_0(\mathbb{L}) \subseteq B(0, C)$. If A is a Borel set in $\mathbb{R}^d$ and $aC < R_0 < R_1 < \ldots < R_d < \mathrm{Reach}(K)$ and $K_0 = K \cap a\mathbb{L}$, then*

$$\check{\Phi}_k^{r,s}(K_0; A) = \hat{\Phi}_k^{r,s}(K_0; A)$$

for all $k \in \{0, \ldots, d-1\}$, whenever $s = 0$ or s is odd. If s is even and $k \in \{0, \ldots, d-1\}$, then

$$\lim_{a\to 0_+} \check{\Phi}_k^{r,s}(K_0; A) = \hat{\Phi}_k^{r,s}(K_0; A).$$

Proof. Let $aC < R < \mathrm{Reach}(K)$. For $x \in I(K \cap a\mathbb{L})$, we have

$$B(x, R) \cap V_x(K \cap a\mathbb{L}) = V_x(a\mathbb{L}),$$

so the contribution of x to the sum in (5.2) is $(s!)x^r \Phi_d^{s,0}(V_0(a\mathbb{L}))$. It follows that

$$\mathcal{V}_R^{r,s}(K \cap a\mathbb{L}; A) - \check{\mathcal{V}}_R^{r,s}(K \cap a\mathbb{L}; A) = (s!)\Phi_d^{s,0}(V_0(a\mathbb{L})) \sum_{x\in I(K\cap a\mathbb{L})\cap A} x^r. \tag{5.7}$$

For odd s we have $\Phi_d^{s,0}(V_0(a\mathbb{L})) = 0$, so the claim follows. For $s = 0$ the right-hand side of (5.7) does not vanish, but it is independent of R. A combination of

$$\left(A_{R_0,\ldots,R_d}^{r,0}\right)^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} (r!)^{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

with (5.7), (3.7) and (5.6) gives the claim.

For even $s > 0$, we have that $\Phi_d^{s,0}(V_0(a\mathbb{L})) = a^{d+s}\Phi_d^{s,0}(V_0(\mathbb{L}))$, while

$$\begin{aligned}\Big|\sum_{x\in I(K\cap a\mathbb{L})\cap A} x^r\Big| &\le \sum_{x\in I(K\cap a\mathbb{L})} |x|^r \\ &\le \sup_{x\in K}|x|^r \sum_{x\in I(K\cap a\mathbb{L})} \left(a^d\mathcal{H}^d(V_0(\mathbb{L}))\right)^{-1}\mathcal{H}^d(V_x(a\mathbb{L})) \\ &\le \sup_{x\in K}|x|^r\cdot a^{-d}\cdot\mathcal{H}^d(V_0(\mathbb{L}))^{-1}\cdot\mathcal{H}^d(K^{aC}).\end{aligned}$$

Therefore, the expression on the righ-hand side of (5.7) converges to 0. □

6 Generalization for local tensors

In this section, we consider an obvious generalization for local tensors, that is, for certain tensor valued measures. Namely, for $k < d$ it is natural to define $\Phi_k^{r,s}(K;\,\cdot\,)$ as a $\mathbb{T}^{r+s}$-valued measure on all of Σ. For a set K of positive reach and a measurable set $B \subseteq \Sigma$, we put

$$\overline{\Phi}_k^{r,s}(K;B) = \int_\Sigma \mathbb{1}_B(x,u)x^r u^s\,\Lambda_k(K,d(x,u)), \qquad k\in\{0,\dots,d-1\}.$$

Note that we use the 'bar'-notation in this section for all quantities that are related to generalized local tensors. Let $u_K(x) = (x-p_K(x))/|x-p_K(x)|$, whenever this is defined. The problem with estimating $\overline{\Phi}_k^{r,s}(K;B)$ from integrals

$$\int_{K^R\setminus K}\mathbb{1}_B(p_K(x),u_K(x))p_K(x)^r(x-p_K(x))^s\,dx$$

by solving an appropriate linear system is that the function $x\mapsto x/|x|$ on $\mathbb{R}^d\setminus\{0\}$ is not Lipschitz, meaning that the convergence proof will not work. Instead, we suggest to use the following modification of the Voronoi tensor measure:

$$\overline{\mathcal{V}}_R^{r,s}(K;B) = \int_{K^R\setminus K^{R/2}}\mathbb{1}_B(p_K(x),u_K(x))p_K(x)^r(x-p_K(x))^s\,dx. \tag{6.1}$$

If $0 < R < \mathrm{reach}(K)$, then the generalized Steiner formula yields in this case

$$\overline{\mathcal{V}}_R^{r,s}(K;B) = r!s!\sum_{k=1}^d \kappa_{s+k}R^{s+k}(1-2^{-(s+k)})\overline{\Phi}_{d-k}^{r,s}(K;B).$$

Again, we can recover the Minkowski tensors as follows:

$$\begin{pmatrix}\overline{\Phi}_{d-1}^{r,s}(K;B)\\ \vdots \\ \overline{\Phi}_0^{r,s}(K;B)\end{pmatrix} = \left(\overline{A}_{R_1,\dots,R_d}^{r,s}\right)^{-1}\begin{pmatrix}\overline{\mathcal{V}}_{R_1}^{r,s}(K;B)\\ \vdots \\ \overline{\mathcal{V}}_{R_d}^{r,s}(K;B)\end{pmatrix}$$

where

$$\overline{A}_{R_1,\dots,R_d}^{r,s} = \frac{1}{r!s!}\begin{pmatrix}\kappa_{s+1}(1-2^{-(s+1)})R_1^{s+1} & \dots & \kappa_{s+d}(1-2^{-(s+d)})R_1^{s+d}\\ \vdots & & \vdots\\ \kappa_{s+1}(1-2^{-(s+1)})R_d^{s+1} & \dots & \kappa_{s+d}(1-2^{-(s+d)})R_d^{s+d}\end{pmatrix}.$$

The map $x \mapsto x/|x|$ is Lipschitz on $\mathbb{R}^d \setminus \mathrm{int}(B(0,R/2))$ with Lipschitz constant $4/R$, and therefore

$$|u_K(x) - u_{K_0}(x)| \leq \tfrac{4}{R}|p_K(x) - p_{K_0}(x)|$$

on $K^R \setminus K^{R/2} \cap K_0^R \setminus K_0^{R/2}$. Moreover,

$$(K^R \setminus K^{R/2}) \Delta (K_0^R \setminus K_0^{R/2}) \subseteq (K^R \Delta K_0^R) \cup (K^{R/2} \Delta K_0^{R/2}).$$

Using this, it is straightforward to generalize the proof of Theorem 4.3 to show the next result.

Theorem 6.1. *If $K, K_i \in \mathcal{C}^d$ are compact sets such that $K_i \to K$ in the Hausdorff metric as $i \to \infty$, then $\overline{\mathcal{V}}_R^{r,s}(K_i; B)$ converges to $\overline{\mathcal{V}}_R^{r,s}(K; B)$ in the tensor norm for any B satisfying*

$$\mathcal{H}^d(\{x \in K^R \mid (p_K(x), u_K(x)) \in \partial B\}) = 0.$$

For digital images, the use of the modified Voronoi tensor measure (6.1) also has the advantage that when a is sufficiently small, Voronoi cells with $V_x(K \cap a\mathbb{L}) = V_x(a\mathbb{L})$ are ignored. These are likely to come from the interior of K, and should therefore not contribute.

The drawback of this algorithm is that we need twice as many radii R to estimate the Minkowski tensors. In particular, the set $K^R \setminus K^{R/2}$ may be quite small, which increases the risk of errors in practical computations.

7 Comparison to known estimators

Most existing estimators of intrinsic volumes [14, 15, 19] and Minkowski tensors [22, 24] are of local type. The idea is to look at all $n \times \cdots \times n$ pixel blocks in the image and count how many times each of the 2^{n^d} possible configurations of black and white points occur. Each configuration is weighted by an element of $\mathbb{T}^{r+s}$ and $\Phi_k^{r,s}(K)$ is estimated as a weighted sum of the configuration counts. It is known that estimators of this type for intrinsic volumes other than ordinary volume are not multigrid convergent, even when K is known to be a convex polytope; see [26]. It is not difficult to see that there cannot be a multigrid convergent local estimator for the (even rank) tensors $\Phi_k^{0,2s}(K)$ with $k = 0, \ldots, d-1$, $s \in \mathbb{N}$, for polytopes K, either. In fact, repeatedly taking the trace of such an estimator would lead to a multigrid convergent local estimator of the kth intrinsic volume, in contradiction to [26].

The algorithm presented in this paper is not local: it is required in the convergence proof that the parallel radius R is fixed while the resolution a^{-1} goes to infinity. The non-local operation in the definition of our estimator is the calculation of the Voronoi diagram. The computation time for Voronoi diagrams of k points is $O(k \log k + k^{\lfloor d/2 \rfloor})$, see [4], which is somewhat slower than local algorithms for which the computation time for k data points is $O(k)$.

The idea to base digital estimators for intrinsic volumes on an inversion of the Steiner formula as in (3.6) has occurred before in [13, 18]. In both references, the authors define estimators for polyconvex sets which are not necessarily of positive reach. This more ambitious aim leads to problems with the convergence.

In [13], the authors use a version of the Steiner formula for polyconvex sets given in terms of the Schneider index, see [20]. Since its definition is, however, local in nature, the authors choose a local algorithm to estimate it. As already mentioned, such algorithms are not multigrid convergent.

In [18], it is used that the intrinsic volumes of a polyconvex set can, on the one hand, be approximated by those of a parallel set with small parallel radius, and on the other hand, the closed complement of this parallel set has positive reach, so that its intrinsic volumes can be computed via the Steiner formula. The authors employ a discretization of the parallel volumes of digital images, but without showing that the convergence is preserved.

It is likely that the ideas of the present paper combined with the ones of [18] could be used to construct multigrid convergent digital algorithms for polyconvex sets. The price for this is that the notion of convergence in [18] is slightly artificial for practical purposes, requiring very small parallel radii in order to get good approximations and at the same time large radii compared to resolution.

In [27], local algorithms based on grey-valued images are suggested. They are shown to converge to the true value when the resolution tends to infinity. However, they only apply to surface and certain mean curvature tensors. Moreover, they are hard to apply in practice, since they require detailed information about the underlying point spread function which specifies the representation of the object as grey-value image. If grey-value images are given, the algorithm of the present paper could be applied to thresholded images, but there may be more efficient ways to exploit the additional information of the grey-values.

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