Non-perturbative results in quantum field theory.

Selfadjointness, spectral theory and renormalisation.

PhD Thesis

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Supervised by Jacob Schach Møller.

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July 2018

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Submitted July 31, 2018

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Preface

This thesis presents the results I obtained during my PhD studies at the Department of Mathematics at Aarhus University. The project was supervised by Jacob Schach Møller but I also had many helpful discussions with Oliver Matte, Wojciech Dybalski, Fumio Hiroshima and Masao Hirokawa. My studies were mainly funded by the Independent Research Fund Denmark but I also received travel grants from the Augustinus Foundation, the Oticon Foundation and Aarhus University for which I am grateful.

The thesis contains an introduction to the framework, a presentation and discussion of the results obtained and four papers in which the results are proved. These papers are

- **Paper A: Spin-Boson type models analysed through symmetries:** submitted.
- **Paper B: Large interaction asymptotics of spin-boson type models:** ready for submission.
- **Paper C: Non-existence of ground states in the translation invariant Nelson model:** ready for submission.
- **Paper D: Rigorous Results on the Bose-Polaron:** small corrections needed (typos).

An old version of Paper A appeared in my Part A thesis. Since then, the conclusions have been vastly generalised and the final version is presented here. Both Paper A and B started from a note given to me by Jacob Schach Møller, who had tried to prove existence of exited states in the massive Spin-Boson model at sufficiently strong interactions. I managed to prove a convergence theorem which removed all constraints in one of the results by Jacob and even added an unexpected result on ultraviolet renormalisation.

The idea for Paper C came to me while working on Paper D. After reading a Paper by Ira Herbst and David Hasler, I realised that their difficulties could be avoided by rotational symmetry and non degeneracy of ground states. Paper D was finished last minute and may be a little rough around the edges. It started a project to investigate a model for polarons in Bose Einstein condensations. I wanted to prove theorems similar to those that hold for Nelson type Hamiltonians.

During my time as a PhD student I have spend quite a lot of time abroad. I spend 6 months at the TUM in Munich, 4 Months at Kyushu University and 2 weeks at Hiroshima University. Apart from this I have spend several months abroad during various summer schools and conferences. I have also presented my work at the following conferences/seminars:

1. Mathematical Physics seminar, Kyushu University. September 2016.
2. Qmath13, Institute of Technology. October 2016.
3. LQP39, University of Münster, January 2017.
4. Mathematical Physics seminar, Kyoto University. May 2017.
5. Probability Seminar, Kyushu University. July 2017.
6. Mathematical Physics seminar, Aalborg University. September 2017.
7. Thiele Seminar, Aarhus University. October 2017.
8. QMath/QUSCOPE Joint Quantum Theory Seminar, Aarhus University. November 2017.
9. Mathematical Challenges in Quantum Mechanics, La Sapienza, Rome. February 2018.
10. Quantum fields and related topics, RIMS Kyoto. July 2018.
11. Young Researchers symposium (in connection with ICMP), Mcgill University. July 2018.

This thesis marks the end of many happy years at the university. I had the chance to learn great mathematics, make friends and literally travel around the globe. First and foremost I would like to thank Jacob for his continued support and the hours he spend on my supervision. Secondly, I would like to thank Wojciech Dybalski, Fumio Hiroshima and Masao Hirokawa for letting me stay at their respective universities and helping me with my research. Thirdly, I would like to thank family and friends who have solved many practical problems for me while I was either physically or mentally absent from the real world.

Thomas Norman Dam
July 2018, Aarhus

Summary

In this thesis, several families of operators related to quantum field theory are investigated. It is emphasised that all results are obtained through non perturbative techniques so there is no restrictions on the size of interactions.

The first paper is concerned with a family of models describing a qubit coupled to a bosonic field. Let H_{SB} denote the corresponding Hamiltonian. It is shown, that there is a unitary map U such that $UH_{SB}U^* = F_+ \oplus F_-$, where $F_\pm$ are selfadjoint operators on Fock space. The operators $F_\pm$ are referred to as fiber operators. We calculate the essential spectrum of F_-, F_+ and H_{SB} under minimal conditions and identify which fiber operator corresponds to the ground state. Using this we find a simple criterion for the existence of an exited state.

Under the additional assumption of linear coupling we find the strong interaction limit of spin-boson Hamiltonians. The limit is independent of the qubit, which has two main applications. First of all, it is proven that exited states exists at sufficiently strong coupling. Secondly, one can conclude that ultraviolet renormalisation using Nelsons original method will never give physically interesting result.

The thesis also treats the massless translation invariant Nelson model. This model describes a spinless particle with no charge coupled to a photon field. After a unitary transformation we may write the Hamiltonian for this system as a direct integral of fiber operators $\{H(\xi)\}_{\xi\in\mathbb{R}^3}$. We prove that $H(\xi)$ does not have a ground state for any ξ. The physical significance of this result is that construction of scattering states cannot be done the usual way, as it requires existence of ground states for sufficiently many of the ξ.

We also treat polarons in Bose Einstein condenstates. Recently a new model has been proposed and we prove a collection of fundamental theorems for this Hamiltonian including selfadjointness. The main problem is, that it adds 4 terms to the old model and it is not clear if this destroys selfadjointness or other nice properties. However rewriting the operator it is possible to prove that many properties of the old model carries over to the new model.

Resumé

Denne afhandling omhandler matematisk stringent behandling af modeller fra kvantefeltteori. Det skal understreges, at alle resultater i denne afhandling er ikke-perturbative. Faktisk tages koblingens styrke til ∞ i et af resultaterne, så perturbative metoder ville slet ikke kunne virke i dette tilfælde.

Den første model som behandles, er en generalisering af spin-boson modellen. Denne model beskriver et to-niveau system koblet til et kvantiseret felt. Lad H_{SB} være den tilhørende Hamiltonoperator. Det vises, at der findes en unitær transformation U således $UH_{SB}U^* = F_+ \oplus F_-$, hvor $F_\pm$ er selvadjungerede operatorer på Fockrum. Operatorerne $F_\pm$ kaldes for fiberoperatorer. I afhandlingen bestemmes det essentielle spektrum af $F_\pm$ og H_{SB} under minimale antagelser. Desuden identificeres hvilken af fiberoperatorerne svarer til grundtilstanden. Ved brug af et variationelt argument finder man et simpelt kriterium for eksistensen af en exciteret tilstand.

Antages at koblingen til feltet er lineær, kan man bestemme en asymptotisk grænse for H_{SB} når koblingsstyrken går mod uendeligt. Denne grænse er uafhængig af to-niveau systemet, hvilket har to konsekvenser. Den første konsekvens er, at Hamiltonoperatoren har en exciteret tilstand, når koblingsstyrken er stor nok. Den anden konsekvens er, at "standardmetoden" til ultraviolet renormalisering ikke kan give et fysisk interessant resultat.

Afhandlingen indeholder også et kapitel dedikeret til den masseløse Nelson model. I dette tilfælde beskriver Hamiltonoperatoren en partikel uden spin eller ladning, der vekselvirker med et kvantiseret felt af fotoner. Efter en unitær transformation kan Hamiltonoperatoren skrives som et direkte integral af fiberoperatorer $\{H(\xi)\}_{\xi\in\mathbb{R}^3}$. Det vises, at $H(\xi)$ ikke har en grundtilstand for noget ξ. Dette resultat har den konsekvens, at spredningsteori bliver enormt besværligt, da disse grundtilstande normalt bruges til konstruktionen af spredningstilstande.

Den sidste model som analyseres i denne afhandling er en ny model for en urenhed der bevæger sig i et Bose-Einstein kondensat. Den minder meget om en ældre model for samme fysiske system, men Hamiltonoperatoren i den nye model indeholder 4 led mere en Hamiltonoperatoren i den gamle model. Det vises, at en lang række af egenskaber som den gamle Hamilton operator besidder også gælder for den nye Hamiltonoperator. Dette er dog langt fra oplagt og kræver mange omskrivninger af den nye Hamiltonoperator.

Chapter 1

A quick introduction to Fock spaces.

The aim of this section is to give the reader an up to date list of various identities and inequalities needed to analyse operators on Fock Spaces. The proofs of many of these claims are long and will not be given here. I have a PDF (available upon request) with most of the proofs which was given to during a course by Oliver Matte. Since then I added the remaining proofs along with an introduction to direct sums and abstract tensor products. I will assume the reader is familiar with tensor products of Hilbert-spaces and direct sums of Hilbert spaces and operators as this is covered in most graduate courses on functional analysis. I will start from tensor products of operators and work my way through the material. The reason for this is that not all people agree on what the tensor product of two operators is and this might create confusion later on.

1 Tensor products of operators

This subsection is taken ore or less directly from [6] (see also [38]). Throughout this section let $\mathcal{H}_1,\dots,\mathcal{H}_n$ be a finite collection of Hilbert spaces. Let $D_i \subset \mathcal{H}_i$ be a subset and define the algebraic tensor product

$$D_1 \widehat{\otimes} \cdots \widehat{\otimes} D_n = \mathrm{Span}\{x_1 \otimes \cdots \otimes x_n \mid x_i \in V_i\}.$$

The following theorem establishes notation and a few definitions:

Theorem 1.1. *Let T_i be and operator on $\mathcal{H}_i$ for $i \in \{1,\dots,n\}$. Then there is a unique linear map $T = T_1 \widehat{\otimes} \cdots \widehat{\otimes} T_n$ defined on $\mathcal{D}(T_1) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{D}(T_n)$ such that*

$$T_1 \widehat{\otimes} \cdots \widehat{\otimes} T_n (x_1 \otimes \cdots \otimes x_n) = T_1 x_1 \otimes \cdots \otimes T_n x_n, \tag{1.1}$$

for all $x_i \in \mathcal{D}(T_i)$ and $i \in \{1,\dots,n\}$. Furthermore, we have

(1) *If all T_i are densely defined then T is densely defined and $T_1^* \widehat{\otimes} \cdots \widehat{\otimes} T_n^* \subset T^*$.*

(2) *If all T_i are closable, then T is closable. We will then write $\overline{T} = T_1 \otimes \cdots \otimes T_n$. Furthermore, we have*

$$T_1 \otimes \cdots \otimes T_n = \overline{T}_1 \otimes \cdots \otimes \overline{T}_n$$
$$T_1^* \otimes \cdots \otimes T_n^* = (T_1 \otimes \cdots \otimes T_n)^*.$$

(3) *If all the T_i are symmetric (selfadjoint, unitary, a projection), then T is symmetric (selfadjoint, unitary, a projection).*

(4) *If $T_i \geq 0$ for all $i \in \{1,\dots,n\}$ then $T \geq 0$.*

(5) *If all the T_i are bounded then T is bounded and*

$$\|T\| = \|T_1\| \cdots \|T_n\| = \|T_1 \otimes \cdots \otimes T_n\|.$$

The following result is also important. Many of the operators that we will encounter are on the form given in this theorem

Theorem 1.2. *For each $j \in \{1,\dots,n\}$ let T_j be a selfadjoint operator on $\mathcal{H}_i$ and define*

$$\begin{aligned} H_i &= 1 \otimes \cdots \otimes T_i \otimes \cdots \otimes 1, \\ H &= H_1 + H_2 + \cdots + H_n. \end{aligned}$$

Then

(1) *$(H_1, \cdots, H_n)$ is a touple of strongly commuting selfadjoint operators with $\sigma(H_i) = \sigma(T_j)$. The joint spectrum is $\sigma(T_1) \times \cdots \times \sigma(T_n)$ and if $f : \mathbb{R} \to \mathbb{C}$ is Borel measurable then $f(H_j) = 1 \otimes \cdots \otimes f(T_j) \otimes \cdots \otimes 1$.*

(2) *H is essentially selfadjoint with*

$$e^{it\overline{H}} = e^{itT_1} \otimes \cdots \otimes e^{itT_n} \quad t \in \mathbb{R}.$$

(3) *If V_j is a core for T_j then $V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_n$ is a core for $\overline{H}$.*

(4) *Assume T_j is semibounded with $\inf(\sigma(T_j)) = \lambda_j$ for all j. Then H is selfadjoint and semibounded with $\inf(\sigma(H)) = \lambda := \lambda_1 + \cdots + \lambda_n$. Let P_B denote the spectral measure for an operator $B \in \{H, T_1, \dots, T_n\}$. Then*

$$\begin{aligned} e^{-tH} &= e^{-tT_1} \otimes \cdots \otimes e^{-tT_n} \quad t \geq 0 \\ P_H(\{\lambda\}) &= P_{T_1}(\{\lambda_1\}) \otimes \cdots \otimes P_{T_n}(\{\lambda_n\}). \end{aligned}$$

In particular, $\mathrm{Dim}(P_H(\{\lambda\})) = \mathrm{Dim}(T_1(\{\lambda\})) \cdots \mathrm{Dim}(T_n(\{\lambda\}))$. Let $\mu_j = \inf(\sigma_{\text{ess}}(T_j))$ which may be ∞. Then

$$\inf(\sigma_{\text{ess}}(\overline{H})) \geq \min_j \left\{ \mu_j + \sum_{l \neq i} \lambda_j \right\} := m.$$

(5) *Assume B_i is selfadjoint on $\mathcal{H}_i$. If $\mathcal{D}(T_i) \subset \mathcal{D}(B_i)$ for some $i \in \{1,\dots,n\}$ then $\mathcal{D}(H_i) \subset \mathcal{D}(1 \otimes \cdots \otimes B_i \otimes \cdots \otimes 1)$.*

(6) *Assume B_i is selfadjoint on $\mathcal{H}_i$ and $T_i + B_i$ is selfadjoint. Then*

$$H_i + 1 \otimes \cdots \otimes B_i \otimes \cdots \otimes 1 = 1 \otimes \cdots \otimes (T_i + B_i) \otimes \cdots \otimes 1 := S_i$$

1.1 Fock spaces

In this section we present some facts about symmetric Fock spaces. This section is based on (Oliver). No proofs are presented but most of the results are easy to prove and can be found in e.g. [31].

Let $\mathcal{H}$ be a separable Hilbert space. For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we will write $\mathcal{H}^{\otimes n}$ for the n-fold tensor product of $\mathcal{H}$. Here $\mathcal{H}^{\otimes 0} = \mathbb{C}$ by convention. The following lemma allows us to define symmetric tensors.

Lemma 1.3. *Let $n \in \mathbb{N}$ and $\mathcal{S}_n$ be the permutations of $\{1,\dots,n\}$. For each $\pi \in \mathcal{S}_n$ there is a unique unitary map $\widehat{\pi}$ on $\mathcal{H}^{\otimes n}$ such that*

$$\widehat{\pi}(f_1 \otimes \cdots \otimes f_n) = f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}$$

for all $f_1,\dots,f_n \in \mathcal{H}$. The following also holds :

1. *Let $\pi,\sigma \in \mathcal{S}_n$. Then $\widehat{\pi}\widehat{\sigma} = \widehat{\sigma \circ \pi}$ and $(\widehat{\pi})^{-1} = \widehat{\pi^{-1}}$.*
2. *Define*
$$S_n = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \widehat{\pi}.$$
Then S_n is a projection.

We also define $S_0 = 1$ on $\mathbb{C} = \mathcal{H}^{\otimes 0}$. The range $S_n(\mathcal{H}^{\otimes n})$ is called the symmetric tensor product and is sometimes written as $\mathcal{H}^{\otimes_s n}$. Furthermore, we will write

$$S_n(f_1 \otimes_s \cdots \otimes_s f_n) = f_1 \otimes_s \cdots \otimes_s f_n.$$

If $\mathcal{H} = L^2(\mathcal{M},\mathcal{F},\mu)$ where $(\mathcal{M},\mathcal{F},\mu)$ is σ-finite then $\mathcal{H}^{\otimes n} = L^2(\mathcal{M}^n,\mathcal{F}^{\otimes n},\mu^{\otimes n})$ and we have the formula

$$(\widehat{\pi} f)(k_1,\dots,k_n) = f(k_{\pi^{-1}(1)},\dots,k_{\pi^{-1}(n)})$$

Let $f \in \mathcal{H}^{\otimes n}$. Then $f \in \mathcal{H}^{\otimes_s n}$ of and only if $f(k_1,\dots,k_n) = f(k_{\pi(1)},\dots,k_{\pi(n)})$ for almost all $(k_1,\dots,k_n) \in \mathcal{M}_n$ and $\pi \in \mathcal{S}_n$. So $\mathcal{H}^{\otimes_s n}$ is simply the square integrable functions which are permutation symmetric in the variables $k_1,\dots,k_n$.

We now define the bosonic (or symmetric) Fock space as

$$\mathcal{F}_b(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s n}.$$

We will write an element $\psi \in \mathcal{F}_b(\mathcal{H})$ in terms of its coordinates $\psi = (\psi^{(n)})$ and define the vacuum $\Omega = (1,0,0,\dots)$. For any $g \in \mathcal{H}$ we define the corresponding exponential vector (or coherent state) as

$$\epsilon(g) = \sum_{n=0}^{\infty} \frac{g^{\otimes n}}{\sqrt{n!}}. \tag{1.2}$$

Let $\mathcal{D} \subset \mathcal{H}$. We define the following families of vectors

$$\begin{aligned}
\mathcal{N} &= \{(\psi^{(n)}) \in \mathcal{F}_b(\mathcal{H}) \mid \exists K \in \mathbb{N} \text{ s.t. } \psi^{(n)} = 0 \text{ for all } n \geq K\},\\
\mathcal{J}(\mathcal{D}) &= \{\Omega\} \cup \{f_1 \otimes_s \cdots \otimes_s f_n \mid f_i \in \mathcal{D} \text{ and } n \in \mathbb{N}\},\\
\mathcal{E}(\mathcal{D}) &= \{\epsilon(f) \mid f \in \mathcal{D}\}.
\end{aligned}$$

We have the following Lemma

Lemma 1.4. *$\mathcal{N}$ is a dense subspace of $\mathcal{F}_b(\mathcal{H})$. If $\mathcal{D} \subset \mathcal{H}$ is dense then $\mathcal{J}(D)$ and $\mathcal{E}(D)$ are total in $\mathcal{F}_b(\mathcal{H})$. Furthermore, $\mathcal{E}(\mathcal{D})$ is a collection of linearly independent vectors.*

We can now introduce the Wyel representation. Let $\mathcal{U}(\mathcal{H})$ be the unitaries from $\mathcal{H}$ into $\mathcal{H}$. Fix $U \in \mathcal{U}(\mathcal{H})$ and $h \in \mathcal{H}$. Then there is a unique unitary map $W(h,U)$ such that

$$W(h,U)\epsilon(g) = e^{-\|h\|^2/2-\langle f,Ug\rangle}\epsilon(h+Ug). \quad \forall g \in \mathcal{H}$$

It is easily seen that that $(h,U) \mapsto W(h,U)$ is strongly continuous and

$$W(h_1,U_1)W(h_2,U_2) = e^{-i\mathrm{Im}(\langle f,Ug\rangle)}W((h_1,U_1)(h_2,U_2)),$$

where $(h_1,U_1)(h_2,U_2) = (h_1+U_1h_2, U_1U_2)$. If A is selfadjoint on $\mathcal{H}$ and $f \in \mathcal{H}$ then $t \mapsto W(0,e^{itA})$ and $t \mapsto W(-itf,1)$ are strongly continuous unitary representations of $\mathbb{R}$ on $\mathcal{F}_b(\mathcal{H})$. By Stones theorem we may define selfadjoint operators $d\Gamma(A)$ and $\varphi(f)$ on $\mathcal{F}_b(\mathcal{H})$ by

$$\begin{aligned} e^{itd\Gamma(A)} &= W(0,e^{itA}) \\ e^{it\varphi(f)} &= W(-itf,1). \end{aligned}$$

for all $t \in \mathbb{R}$. One may then prove the following

Theorem 1.5. *Let A be selfadjoint on $\mathcal{H}$. Then $\mathcal{H}^{\otimes_s n}$ reduces $d\Gamma(A)$ and the restriction to $\mathcal{H}^{\otimes_s n} \cap \mathcal{D}(d\Gamma(A))$ is given by*

$$d\Gamma^{(n)}(A) := \overline{\sum_{i=1}^{n}(1\otimes)^{i-1}A(\otimes 1)^{n-i}} \,|_{\mathcal{H}^{\otimes_s n}} .$$

We may also calculate the spectrum

$$\begin{aligned} \sigma(d\Gamma^{(n)}(A)) &= \overline{\{\lambda_1+\cdots+\lambda_n \mid \lambda_i \in \sigma(A)\}} \;\; n \geq 1 \\ \sigma(d\Gamma(A)) &= \{0\} \cup \bigcup_{n=1}^{\infty} \overline{\{\lambda_1+\cdots+\lambda_n \mid \lambda_i \in \sigma(A)\}} \end{aligned}$$

Assume now $A \geq 0$ and injective. Then the following holds

(1) 0 *is an eigenvalue for $d\Gamma(\omega)$ with multiplicity 1. The eigenspace is spanned by Ω.*

(2) *Let $m = \inf(\sigma(A))$ and $m_{\mathrm{ess}} = \inf(\sigma_{\mathrm{ess}}(A))$. Then*

$$\inf(\sigma_{\mathrm{ess}}(d\Gamma^{(n)}(\omega))) \geq m_{\mathrm{ess}} + (n-1)m$$

(3) $d\Gamma(\omega)$ *will have compact resolvents if and only if ω has compact resolvents.*

If $\mathcal{H} = L^2(\mathcal{M},\mathcal{F},\mu)$ where $(\mathcal{M},\mathcal{F},\mu)$ is σ-finite and A is a multiplication operator on $\mathcal{H}$ then $d\Gamma^{(n)}(A)$ is multiplication by

$$A_n(k_1,\dots,k_n) = A(k_1)+\cdots+A(k_n).$$

Regarding the field operators we have the following result

Lemma 1.6. *Let $f \in \mathcal{H}$ and $\mathcal{D} \subset \mathcal{H}$ be dense. Then both $\mathcal{J}(\mathcal{D})$ and $\mathcal{E}(\mathcal{D})$ spans a core for $\varphi(f)$.*

In order to get a better understanding of field operators we introduce creation and annihilation operators.

Lemma 1.7. *Let $f \in \mathcal{H}$. There exists unique closed operators $a(f)$ and $a^\dagger(f)$ with the property that $a(g)\Omega = 0, a^\dagger(g)\Omega = g$ and*

$$a(g)(f_1 \otimes_s \cdots \otimes_s f_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \langle g, f_i \rangle f_1 \otimes_s \cdots \otimes_s \widehat{f_i} \otimes_s \cdots \otimes_s f_n$$
$$a^\dagger(g)(f_1 \otimes_s \cdots \otimes_s f_n) = \sqrt{n+1} g \otimes_s f_1 \otimes_s \cdots \otimes_s f_n$$

for all $f_1, \ldots, f_n \in \mathcal{H}$. Here $\widehat{f_i}$ means that f_i is omitted from the tensor product. $a(f)$ is called an annihilation operator while $a^\dagger(f)$ is called a creation operator. We also have:

(1) $\varphi(f) = \overline{a(f) + a^\dagger(f)}$

(2) *The canonical commutation relations hold*

$$\overline{[a(f), a(g)]} = 0 = \overline{[a^\dagger(f), a^\dagger(g)]} \text{ and } \overline{[a(f), a^\dagger(g)]} = \langle f, g \rangle.$$

(3) *The following holds*

$$\overline{[\varphi(f), \varphi(g)]} = 2i\mathrm{Im}(\langle f, g \rangle).$$

(4) *If $\mathcal{D} \subset \mathcal{H}$ is dense then $\mathcal{J}(\mathcal{D})$ and $\mathcal{E}(\mathcal{D})$ span cores for $a(f)$ and $a^\dagger(f)$.*

(5) *If A is selfadjoint on $\mathcal{H}$ and $f \in \mathcal{D}(A)$ then*

$$\overline{[d\Gamma(A), a^\dagger(f)]} = a^\dagger(Av)$$
$$\overline{[a(f), d\Gamma(A)]} = a(Av)$$
$$\overline{[d\Gamma(A), \varphi(f)]} = -i\varphi(iAv).$$

Furthermore $\mathcal{N} \cap \mathcal{D}(d\Gamma(A))$ is contained in the domains of $[d\Gamma(A), a^\dagger(f)]$, $[a(f), d\Gamma(A)]$ and $[d\Gamma(A), \varphi(f)]$.

If $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ where $(\mathcal{M}, \mathcal{F}, \mu)$ is σ-finite we have explicit formulas for creation and annihilation operators. For $\psi \in \mathcal{H}^{\otimes_s n}$ with $n \geq 1$ we have

$$(a(f)\psi)(k_1, k_2, \ldots, k_{n-1}) = \sqrt{n} \int_{\mathcal{M}} \overline{f(k)} \psi(k, k_1, \ldots, k_{n-1}) d\mu(k)$$
$$(a^\dagger(f)\psi)(k_1, k_2, \ldots, k_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n} f(k_i) \psi(k_1, \ldots, \widehat{k_i}, \ldots, k_{n+1})$$

where $\widehat{k_i}$ means that the variable k_i is omitted. If $\mathcal{K}$ is an other Hilbert space and $U : \mathcal{H} \to \mathcal{K}$ is a bounded operator with $\|U\| \leq 1$ then we define

$$\Gamma(U) = 1 \oplus \bigoplus_{n=1}^{\infty} U^{\otimes n} |_{\mathcal{H}^{\otimes_s n}} .$$

Note that $\Gamma(U)$ will be unitary if U is unitary. We have the following Lemma

Lemma 1.8. *Let $U:\mathcal{H}\to\mathcal{K}$ be unitary, A be selfadjoint on $\mathcal{H}$, $V\in\mathcal{U}(\mathcal{H})$ and $f\in\mathcal{H}$ then*

$$\Gamma(U)d\Gamma(A)\Gamma(U)^*=d\Gamma(UAU^*).$$
$$\Gamma(U)W(f,V)\Gamma(U)^*=W(Uf,UVU^*).$$
$$\Gamma(U)\varphi(f)\Gamma(U)^*=\varphi(Uf).$$
$$\Gamma(U)a(f)\Gamma(U)^*=a(Uf).$$
$$\Gamma(U)a^\dagger(f)\Gamma(U)^*=a^\dagger(Uf).$$

Furthermore, $\Gamma(U)(f_1\otimes_s\cdots\otimes_s f_n)=Uf_1\otimes_s\cdots\otimes_s Uf_n$ and $\Gamma(U)\Omega=\Omega$.

An important aspect of this Lemma is that we may assume $\mathcal{H}=L^2(\mathcal{M},\mathcal{F},\mu)$ when proving spectral properties and various inequalities. It is needed to prove the following relative bounds.

Theorem 1.9. *Let A be a non negative, selfadjoint and injective operator on $\mathcal{H}$. Let $f_1,\dots,f_n\in\mathcal{D}(\omega^{-1/2})$ and $a\geq 0$. Then*

(1) *$a(f_1)\cdots a(f_n)$ maps $\mathcal{D}(d\Gamma(A)^{a+n/2})$ continuously into $\mathcal{D}(d\Gamma(A)^a)$ with respect to the graph norm. We have the following specific bound for $\lambda\geq 0$ and $\psi\in\mathcal{D}(d\Gamma(A)^{a+n/2})$:*

$$\|(d\Gamma(A)+\lambda)^a a(f_1)\cdots a(f_n)\psi\|\leq\left(\prod_{i=1}^n\|\omega^{-1/2}f_i\|\right)\|(d\Gamma(A)+\lambda)^{a+n/2}\psi\|$$

(2) *$a^\dagger(f_1)\cdots a^\dagger(f_n)$ maps $\mathcal{D}(d\Gamma(A)^{n/2})$ continuously into $\mathcal{F}_b(\mathcal{H})$ with respect to the graph norm. We have the following specific bound for $\psi\in\mathcal{D}(d\Gamma(A)^{n/2})$:*

$$\|a^\dagger(f_1)\cdots a^\dagger(f_n)\psi\|^2\leq n!2^n\prod_{i=1}^n\|(1+\omega^{-1/2})f_i\|\sum_{\ell=0}^n\frac{1}{\ell!}\|d\Gamma(A)^{\ell/2}\psi\|^2$$

One can now apply these relative bounds to obtain relative bounds of the field operators. We sum up the important conclusions

Theorem 1.10. *Let A be a non negative, selfadjoint and injective operator on $\mathcal{H}$. Let $f_1,\dots,f_n\in\mathcal{D}(\omega^{-1/2})$. Then $\mathcal{D}(d\Gamma(A)^{n/2})\subset\mathcal{D}(\varphi(f_1)\cdots\varphi(f_n))$ and $\phi(f_1)$ is infinitesimally $d\Gamma(A)$ bounded.*

Furthermore, $\sigma(d\Gamma(A)+\varphi(f_1))=-\|A^{-1/2}f_1\|^2+\sigma(d\Gamma(A))$ and $d\Gamma(A)+\varphi(f_1)$ has a ground state if and only if $f_1\in\mathcal{D}(A^{-1})$.

The following transformation statements are also very good to know as they will be used frequently in the papers. This statement below taken from [6]. For a proof see [26].

Lemma 1.11. *Let $f,h\in\mathcal{H}$ and $U\in\mathcal{U}(\mathcal{H})$. Then*

$$W(h,U)\varphi(g)W(h,U)^*=\varphi(Ug)-2\mathrm{Re}(\langle Ug,h\rangle)$$
$$W(h,U)a(g)W(h,U)^*=a(Ug)-\langle Ug,h\rangle$$
$$W(h,U)a^\dagger(g)W(h,U)^*=a^\dagger(Ug)-\langle h,Ug\rangle$$

Furthermore, if ω is selfadjoint, non negative and injective on $\mathcal{H}$ and $h\in\mathcal{D}(\omega U^*)$ then

$$W(h,U)d\Gamma(\omega)W(h,U)^*=d\Gamma(U\omega U^*)-\varphi(U\omega U^*h)+\langle h,U\omega U^*h\rangle$$

on the domain $\mathcal{D}(d\Gamma(U\omega U^*))$.

The next result is commonly used. In the papers below it is used to split Hamiltonians into two parts. In one part one usually has compact resolvents and the other part can usually be analysed using Theorem 1.5. The statement here is taken from [6] where a proof can also be found.

Theorem 1.12. *There is a unique isomorphism $U : \mathcal{F}_b(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathcal{F}_b(\mathcal{H}_1) \otimes \mathcal{F}_b(\mathcal{H}_2)$ such that $U(\epsilon(f \oplus g)) = \epsilon(f) \otimes \epsilon(g)$. If $f_1, \dots, f_j \in \mathcal{H}_1$ and $g_1, \dots, g_\ell \in \mathcal{H}_2$ then*

$$\begin{aligned} &U((f_1, 0) \otimes_s \cdots \otimes_s (f_j, 0) \otimes_s (0, g_1) \otimes_s \cdots \otimes_s (0, g_\ell)) \\ &\qquad\qquad = \left(\frac{j!\ell!}{(j+\ell)!}\right)^{1/2} (f_1 \otimes_s \cdots \otimes_s f_j) \otimes (g_1 \otimes_s \cdots \otimes_s g_\ell). \end{aligned}$$

The map also has the following transformation properties. If A_i is selfadjoint on $\mathcal{H}_i$, V_i is unitary on $\mathcal{H}_i$ and $f \in \mathcal{H}_1, g \in \mathcal{H}_2$ then

$$\begin{aligned} UW(f \oplus g, V_1 \oplus V_2)U^* &= W(f, V_1) \otimes W(g, V_2) \\ Ud\Gamma(A_1 \oplus A_2)U^* &= \overline{d\Gamma(A_1) \otimes 1 + 1 \otimes d\Gamma(A_2)} \\ U\varphi(f, g)U^* &= \overline{\varphi(f) \otimes 1 + 1 \otimes \varphi(g)} \\ Ua(f, g)U^* &= \overline{a(f)\widehat{\otimes}1 + 1\widehat{\otimes}a(g)} \\ Ua^\dagger(f, g)U^* &= \overline{a^\dagger(f)\widehat{\otimes}1 + 1\widehat{\otimes}a^\dagger(g)}. \end{aligned}$$

One very essential observation is the relationship between quantum field theory and Malliavin calculus. In simple terms it can be shown that the symmetric Fock space is unitarily equivalent in a "nice way" to $L^2(\mathcal{X}, \mathcal{F}, \mathbb{P})$ where $(\mathcal{X}, \mathcal{F}, \mathbb{P})$ is probability space. Here "nice way" refers to the fact that many field operators are transformed to multiplication operators which simplifies many calculations. It also provides the Fock space with a very nice positive cone. The following statement is taken from [6]. See [5] for a proof.

Theorem 1.13. *Let $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$ be a real Hilbert space such that $\mathcal{H} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$. Then there exists a probability space $(\mathcal{X}, \mathcal{F}, \mathbb{P})$ such that $\mathcal{F}_b(\mathcal{H})$ is unitarily isomorphic to $L^2(\mathcal{X}, \mathcal{F}, \mathbb{P})$ via a map $\mathcal{I}$. Furthermore, the following properties hold*

(1) *If U is a bounded operator on $\mathcal{H}$ with $\|U\| \leq 1$ such that $U\mathcal{H}_\mathbb{R} \subset \mathcal{H}_\mathbb{R}$ then $\mathcal{I}\Gamma(U)\mathcal{I}^*$ is positivity preserving.*

(2) *Assume $\omega \geq 0$ is selfadjoint and injective. If $e^{-t\omega}$ maps $\mathcal{H}_\mathbb{R}$ into $\mathcal{H}_\mathbb{R}$ for all $t \geq 0$ then $\mathcal{I}e^{-td\Gamma(\omega)}\mathcal{I}^*$ is positivity improving. If $\inf(\sigma(\omega)) > 0$ then $\mathcal{I}e^{-td\Gamma(\omega)}\mathcal{I}^*$ is hypercontractive.*

(3) *If $v \in \mathcal{H}_\mathbb{R}$ then $\mathcal{I}\varphi(v)\mathcal{I}^*$ acts like multiplication by a normally distributed variable $\widetilde{\varphi}(v)$ with mean 0 and variance $\|v\|^2$. In fact, $\{\widetilde{\varphi}(v)\}_{v \in \mathcal{H}_\mathbb{R}}$ is a Gaussian process indexed by $\mathcal{H}_\mathbb{R}$ with mean 0 and covariance function given by $\mathrm{Cov}(\widetilde{\varphi}(v_1), \widetilde{\varphi}(v_2)) = \langle v_1, v_2 \rangle$.*

The duality between Fock spaces and Malliavin calculus can be pushed quite far and forms the foundations for white noise analysis. Even though white noise analysis is very exiting we shall not pursue it further here.

1.2 Direct Integrals

We willl now consider a special kind of operators defined on Hilbert space valued L^2 spaces. Let $\mathcal{Q} = (\mathcal{M}, \mathcal{F}, \mu)$ be a σ-finite measure space and $\mathcal{H}$ a separable Hilbert space. We have the identification

$$L^2(\mathcal{M}, \mathcal{F}, \mu) \otimes \mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu, \mathcal{H})$$

where $L^2(\mathcal{M}, \mathcal{F}, \mu, \mathcal{H})$ denotes the $\mathcal{H}$ valued L^2 space generated by $\mathcal{Q}$. Let $f : \mathcal{M} \to B(\mathcal{H})$ be strongly measurable (i.e. $x \mapsto f(x)\psi$ is measurable for all $\psi \in \mathcal{H}$) and bounded. Then we define the direct integral

$$I_\oplus(f) = \int_{\mathcal{M}}^{\oplus} f(x) d\mu(x)$$

as the bounded operator on $L^2(\mathcal{Q}, \mathcal{H})$ defined by $I_\oplus(U_x)\psi(x) = U_x\psi(x)$. One also has a direct integral for unbounded selfadjoint operators. Let $\{A_x\}_{x\in\mathcal{M}}$ be a collection on selfadjoint operators on $\mathcal{H}$. We say $\{A_x\}_{x\in\mathcal{M}}$ is strong resolvent measurable if $x \mapsto (A_x + i)^{-1}$ is strongly measurable. Then we define

$$\begin{aligned} &I_\oplus(A_x)\psi(x) = U_x\psi(x) \\ &\mathcal{D}(I_\oplus(A_x)) = \{\psi \in L^2(\mathcal{Q}, \mathcal{H}) \mid \psi(x) \in \mathcal{D}(A_x) \text{ and } x \mapsto \|A_x\psi(x)\| \in L^2(\mathcal{Q})\} \end{aligned}$$

The following Theorem sums up the results about direct integrals we shall need

Theorem 1.14. *Let $\{A_x\}_{x\in\mathcal{M}}$ be a collection on selfadjoint operators on $\mathcal{H}$. Then $x \mapsto (A_x + i)^{-1}$ is strongly measureable if and only if $x \mapsto e^{itA_x}$ is weakly measurable. In this case $I_\oplus(A_x)$ is selfadjoint and $x \mapsto (i + f(A_x))^{-1}$ is strongly measurable for all $f : \mathbb{R} \to \mathbb{R}$. Furthermore,*

$$f(I_\oplus(A_x)) = I_\oplus(f(A_x)).$$

If $A_x \geq \lambda$ for all x we find $I_\oplus(A_x) \geq \lambda$ (use $f = 1_{(-\infty,\lambda)}$). If A is selfadjoint or bounded on $\mathcal{H}$ we may identify $1 \otimes A = I_\oplus(A)$ and if V is a multiplication operator on $L^2(\mathcal{Q})$ then $V \otimes 1 = I_\oplus(V)$.

Proof. This is easily done using results from [26] and [35]. □

The following Lemma shows what happens in the special case where one considers operators on Fock space. This is used in [9]

Proposition 1.15. *Let $Q = (\mathcal{M}, \mathcal{F}, \mu)$ be a sigma-finite measure space. Let $x \mapsto f_x \in \mathcal{H}$ and $x \mapsto g_x \in \mathcal{H}$ be measurable, $\{A_x\}_{x\in\mathcal{M}}$ a be strong resolvent measurable family of selfadjoint operators on $\mathcal{H}$ and $x \mapsto U_x \in B(\mathcal{H})$ be strongly measurable with $\|U_x\| \leq 1$. Then*

(1) $\{\varphi(f_x)\}_{x\in\mathcal{M}}$, $\{a^\dagger(f_x)a(f_x)\}_{x\in\mathcal{M}}$ *and* $\{d\Gamma(A_x)\}_{x\in\mathcal{M}}$ *are strong resolvent measurable and $x \mapsto \Gamma(U_x)$ is strongly measurable. We will use the notation $\varphi_\oplus(f_x) = I_\oplus(\varphi(f_x))$, $a_\oplus^\dagger(f_x)a_\oplus(f_x) = I_\oplus(a^\dagger(f_x)a(f_x))$, $d\Gamma_\oplus(A_x) = I_\oplus(d\Gamma(A_x))$ and $\Gamma_\oplus(U_x) = I_\oplus(\Gamma(U_x))$.*

(2) If U_x is unitary for all x then $\Gamma_\oplus(U_x)$ is unitary and

$$\Gamma_\oplus(U_x)\varphi_\oplus(f_x)\Gamma_\oplus(U_x)^* = \varphi_\oplus(U_x f_x)$$

(3) *Assume $x \mapsto f_x$ and $x \mapsto g_x$ are bounded, $A_x \geq 0$ is injective for all $x \in \mathcal{M}$, $f_x, g_x \in \mathcal{D}(A_x^{-1/2})$ for all $x \in \mathcal{M}$ and the two maps $x \mapsto A_x^{-1/2} f_x$ and $x \mapsto A_x^{-1/2} g_x$ are bounded. Then $\varphi_\oplus(f_x)$ is $d\Gamma_\oplus(A)^{1/2}$ bounded, $\varphi_\oplus(g_x)\varphi_\oplus(f_x)$ is $d\Gamma_\oplus(A)$ bounded and $a_\oplus^\dagger(g_x)a_\oplus(f_x)$ is $d\Gamma_\oplus(A)$ bounded. In particular, $\varphi_\oplus(f_x)$ is infinitesimally $d\Gamma_\oplus(A)$ bounded.*

We will need the following definition

Definition 1.16. *Consider the ν-dimensional Lebesguespace $\mathcal{L}_\nu = (\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$. Let $x \mapsto f_x \in \mathcal{H}$ be bounded and measurable. We say it is weakly differentiable if for all $i \in \{1, \dots, \nu\}$ there is $x \mapsto g_x^{(i)} \in \mathcal{H}$ such that for all $\phi \in C_0^\infty(\mathbb{R}^\nu)$ and $\psi \in \mathcal{H}$ we have*

$$\int_{\mathbb{R}^\nu} \partial_{x_i}\phi(x)\langle \psi, f_x\rangle dx = -\int_{\mathbb{R}^\nu} \partial_{x_i}\phi(x)\langle \psi, g_x^{(i)}\rangle dx$$

in this case we write $\partial_{x_i} f = g_x^{(i)}$.

Shall need on last result about differential operators

Lemma 1.17. *Define*

$$\mathcal{K} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu, \mathcal{F}_b(\mathcal{H})) = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu) \otimes \mathcal{F}_b(\mathcal{H}),$$

$p_i = -i\partial_{x_i} \otimes 1$ *and* $|p| = (-\Delta_x)^{1/2} \otimes 1$.

(1) $\mathcal{D}(|p|) = \bigcap_{i=1}^\nu \mathcal{D}(p_i)$ *and for* $\psi \in \mathcal{D}(|p|)$ *we have* $\| |p|\psi\|^2 = \sum_{i=1}^\nu \|p_i\psi\|^2$.

(2) *If $x \mapsto f_x$ is weakly differentiable the $[\varphi_\oplus(f_x), p_i] = -i\varphi_\oplus(\partial_{x_i} f_x)$ holds on $C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D})$. In particular $\varphi_\oplus(f_x)\psi \in \mathcal{D}(|p|)$ for $\psi \in C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D})$.*

Chapter 2

The papers

In this chapter, I will comment on the results obtained in this thesis and compare them to results found in the literature. In general, the operators I investigated take the form $H = T + gV$ where T is the free (kinetic) energy and V models the interaction. The results obtained in this thesis are all non perturbative which means g is not assumed to be small. In fact we will send g to infinity in the second paper.

The interaction V always depends on elements $(f_1, \dots, f_n)$ from the bosonic state space $\mathcal{H}$ and T depends on a non negative, injective and selfadjoint operator on $\mathcal{H}$ called ω. The interaction V is called infrared regular if $f_i \in \mathcal{D}(\omega^{-1})$ for all i and infrared singular if this is not the case. As we shall see below, infrared regularity usually guarantees that H has a ground state, while infrared singularities can imply H does not have a ground state.

1 Paper A: Spin-Boson type models analysed through symmetries.

In this paper, we analyse so-called spin-boson type Hamiltonians with singular perturbations. Physically, such hamiltonians describe dynamics of a two level system interacting strongly with a boson field. The Hilbert space for the operator is $\mathbb{C}^2 \otimes \mathcal{F}_b(\mathcal{H})$ where $\mathcal{H}$ is the state space of a single Boson. Let σ_x, σ_y and σ_z denote the Pauli matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ \ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We also write $e_1 = (1,0)$ and $e_{-1} = (0,1)$. The operators investigated in this paper are of the form

$$H_\eta(\alpha, f, \omega) := \eta \sigma_z \otimes 1 + 1 \otimes d\Gamma(\omega) + \sum_{i=1}^{2n} \alpha_i (\sigma_x \otimes \varphi(f_i))^i,$$

which is here parametrised by $\alpha \in \mathbb{C}^{2n}, f \in \mathcal{H}^{2n}, \eta \in \mathbb{C}$ and ω selfadjoint and non negative on $\mathcal{H}$. One should note that the standard spin-boson model corresponds to the case where $\alpha_i =$ for $i \geq 2$. This operator possesses a special symmetry, called the spin-parity symmetry, which implies that there is a unitary map V such that

$$VH_\eta(\alpha, f, \omega)V^* = F_{-\eta}(\alpha, f, \omega) \oplus F_\eta(\alpha, f, \omega)$$

where the fiber operators $F_\eta(\alpha, f, \omega)$ are defined as

$$F_\eta(\alpha, f, \omega) = \eta\Gamma(-1) + d\Gamma(\omega) + \sum_{i=1}^{2n} \alpha_i \varphi(f_i)^i.$$

This project started as an attempt to prove that the standard spin-boson model has an exited state for strong couplings. The general strategy was to localize the essential spectrum of the of the $F_\eta(\alpha, f, \omega)$ and then establish the existence of a ground state in each fiber operator via a variational argument.

First problem on the agenda is proving that $F_\eta(\alpha, f, \omega)$ and $H_\eta(\alpha, f, \omega)$ makes sense. To do so we need to introduce some more notation and assumptions. For an element $f \in \mathcal{H}^{2n}$ we define the leading terms

$$\mathcal{L}(f) = \{i \in \{2, 3, \dots, 2n\} \mid f_i \neq f_j \ \forall j > i\}.$$

The expression $\mathcal{L}(f)^c$ is to be interpreted as the complement within $\{1, 2, \dots, 2n\}$. For ω selfadjoint on $\mathcal{H}$ we define the numbers

$$m = \inf\{\sigma(\omega)\} \quad \text{and} \quad m_{\text{ess}} = \inf\{\sigma_{\text{ess}}(\omega)\}.$$

We will need the following definition

Definition 1.1. *Let $(\mathcal{M}, \mathcal{F}, \mu)$ be a measure space.*

(1) We say that $(\mathcal{M}, \mathcal{F}, \mu)$ has strong topological properties if $\mathcal{M}$ is a locally compact, Hausdorff and second countable topological space, $\mathcal{F}$ is the Borel σ-algebra and μ is finite on compact sets.

(2) Let $\mathcal{M}$ be a metric space. We say that $\mathcal{M}$ can be **cut nicely** *if for each $n \in \mathbb{N}$ there is a sequence of disjoint sets $\{G_\alpha^n\}_{\alpha\in\mathbb{N}} \subset \mathcal{B}(\mathcal{M})$ that covers M such that $\sup_{\alpha\in\mathbb{N}} \operatorname{Diam}(G_\alpha^n)$ converges to 0 as n tends to infinity, $\overline{G_\alpha^n}$ is compact and for any $B \subset \mathcal{M}$ bounded the set*

$$\{\alpha \in \mathbb{N} \mid G_\alpha^n \cap B \neq \emptyset\}$$

is finite.

In order to get essential selfadjointness one needs to use either hyper contractive bounds (see [34]) or a result due to Arai (see [2]). For this reason the conditions we work under are sometimes split in two. The general, assumptions are

Hypothesis 1.1. *$\alpha \in \mathbb{C}^{2n}, f \in \mathcal{H}^{2n}$ and ω selfadjoint on $\mathcal{H}$ fulfil Hypothesis 1.1 if*

(1) *$\mathcal{L}(f)$ consists only of even numbers, $\alpha_i > 0$ for all $i \in \mathcal{L}(f)\backslash\{2\}$ and $\alpha_2 \geq 0$ if $2 \in \mathcal{L}(f)$.*

(2) *ω is injective and non negative.*

(3) *$f_i \in \mathcal{D}(\omega^{-\frac{1}{2}}) \cap \mathcal{D}(\omega^{\frac{1}{2}})$ for all $i \in \{2, \dots, 2n\}$ and $f_1 \in \mathcal{D}(\omega^{-\frac{1}{2}})$.*

Hypothesis 1.2. *$f \in \mathcal{H}^{2n}$ and ω selfadjoint on $\mathcal{H}$ fulfil Hypothesis 1.2 if and only if $\langle f_i, g(\omega) f_j\rangle \in \mathbb{R}$ for all $g : \mathbb{R} \to \mathbb{R}$ measurable and bounded on $\sigma(\omega)$.*

Hypothesis 1.3. *$f \in \mathcal{H}^{2n}$ and ω selfadjoint on $\mathcal{H}$ fulfil Hypothesis 1.3 if either $n \leq 2$ or $m > 0$ and Hypothesis 1.2 holds.*

Hypothesis 3 ensures that we may use hypercontrative bounds if $n > 2$.

Hypothesis 1.4. *$f \in \mathcal{H}^{2n}$ and ω selfadjoint on $\mathcal{H}$ fulfil Hypothesis 1.4 if*

(1) *$\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ where $(\mathcal{M}, \mathcal{F}, \mu)$ has strong topological properties and $\mathcal{M}$ can be cut nicely.*

(2) *ω is a multiplication operator on $\mathcal{H}$.*

(3) *There is a measurable function $h : \mathcal{M} \to \mathbb{C}$ with $|h| = 1$ such that hf is $\mathbb{R}^{2n}$ valued almost everywhere. A function h with these properties is called a **phase function** for f.*

Hypothesis 1.5. *We say $f \in \mathcal{H}^{2n}$ and ω selfadjoint on $\mathcal{H}$ fulfil Hypothesis 1.5 if $f_i \in \mathcal{D}(\omega^{-1})$ for all i.*

The following proposition gives precise conditions under which $F_\eta(\alpha, f, \omega)$ and $H_\eta(\alpha, f, \omega)$ are selfadjoint.

Proposition 1.2. *Assume $\eta \in \mathbb{C}$ and (α, f, ω) satisfies Hypotheses 1.1 and 1.3. Then the operators $F_\eta(\alpha, f, \omega)$ and $H_\eta(\alpha, f, \omega)$ are closed on the respective domains*

$$\mathcal{D}(F_\eta(\alpha, f, \omega)) = \mathcal{D}(d\Gamma(\omega)) \cap_{i \in \mathcal{L}(f) \setminus \{2\}} \mathcal{D}(\varphi(f_i)^i)$$
$$\mathcal{D}(H_\eta(\alpha, f, \omega)) = \mathcal{D}(1 \otimes d\Gamma(\omega)) \cap_{i \in \mathcal{L}(f) \setminus \{2\}} \mathcal{D}(1 \otimes \varphi(f_i)^i).$$

Given any core $\mathcal{D}$ of ω the linear span of the following sets

$$\mathcal{J}(\mathcal{D}) := \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{g_1 \otimes_s \cdots \otimes_s g_n \mid g_j \in \mathcal{D}\}$$
$$\widetilde{\mathcal{J}}(\mathcal{D}) := \{v_1 \otimes v_2 \mid v_1 \in \{e_1, e_{-1}\}, v_2 \in \mathcal{J}(\mathcal{D})\}$$

are cores for $F_\eta(\alpha, f, \omega)$ and $H_\eta(\alpha, f, \omega)$ respectively. Also both operators are selfadjoint and semibounded if $(\alpha, \eta) \in \mathbb{R}^{2n+1}$ and they have compact resolvents if ω has compact resolvents.

The overall strategy for proving this proposition is outlined in [21] and is a rather technical argument involving a lot of commutators. If one wishes to find the essential spectrum of the fiber operators (and thereby the full Hamiltonian) it is not really that important to know the full domain of the operators. However the estimates needed to prove proposition 1.2 were actually central in proving the following Theorem which we will call the HVZ Theorem. In the remaining part of this section we will be suppress α, f and ω from the notation as they are fixed in the initial part of each Theorem discussed below.

Theorem 1.3 (HVZ-theorem). *Let $\alpha \in \mathbb{R}^{2n}, \eta \in \mathbb{R}, f \in \mathcal{H}^{2n}$ and ω be selfadjoint on $\mathcal{H}$ and assume they satisfy Hypothesis 1.1,1.3 and either $n \leq 2$ or Hypothesis 1.4. Then the* *following holds*

$$\inf\{\sigma_{\text{ess}}(F_\eta)\} \geq \min\{\mathcal{E}_{-\eta} + m_{\text{ess}}, \mathcal{E}_\eta + m + m_{\text{ess}}\}$$
$$\bigcup_{q=1}^{\infty}\{\mathcal{E}_{(-1)^q\eta} + \lambda_1 + \cdots + \lambda_q \mid \lambda_i \in \sigma_{\text{ess}}(\omega)\} \subset \sigma_{\text{ess}}(F_\eta)$$
$$\inf(\sigma_{\text{ess}}(H_\eta)) = E_\eta + m_{\text{ess}}$$
$$\bigcup_{q=1}^{\infty}\{E_\eta + \lambda_1 + \cdots + \lambda_q \mid \lambda_i \in \sigma_{\text{ess}}(\omega)\} \subset \sigma_{\text{ess}}(H_\eta).$$

In particular, H_η has a ground state of finite multiplicity if $m_{\text{ess}} > 0$ and if $m = m_{\text{ess}}$ then $\inf(\sigma_{\text{ess}}(F_\eta)) = \mathcal{E}_{-\eta} + m_{\text{ess}}$. *Furthermore*

(1) *Assume $m = m_{\text{ess}}$, $[m_{\text{ess}}, 3m_{\text{ess}}] \subset \sigma_{\text{ess}}(\omega)$ and if $m_{\text{ess}} = 0$ then m_{ess} is not isolated in $\sigma_{\text{ess}}(\omega)$. Then $\sigma_{\text{ess}}(F_\eta) = [\mathcal{E}_{-\eta} + m_{\text{ess}}, \infty)$.*

(2) *Assume $[m_{\text{ess}}, 2m_{\text{ess}}] \subset \sigma_{\text{ess}}(\omega)$ and if $m_{\text{ess}} = 0$ then m_{ess} is not isolated in $\sigma_{\text{ess}}(\omega)$. Then $\sigma_{\text{ess}}(H_\eta) = [E_\eta + m_{\text{ess}}, \infty)$.*

(3) *If we assume Hypothesis 1.1,1.2,1.3 and either $n \leq 2$ or Hypothesis 1.4, then $\mathcal{E}_{-|\eta|} \leq \mathcal{E}_{|\eta|}$ with equality if and only if $\eta = 0$ or $m = 0$. In particular we find* $\inf(\sigma_{\text{ess}}(F_{|\eta|})) = \mathcal{E}_{-|\eta|} + m_{\text{ess}}$.

The HVZ-theorem for H_η does not really come as a surprise as similar results has been seen in e.g. [10] for the standard spin-boson model. The statement given here is much more general and the argument is also rather long and technical. In the paper [10] the authors really need a fourier transform along with many nice properties of ω and f. If $f_i = f_j$ for all $i, j \in \{1, \ldots, 2n\}$ then Theorem 1.3 applies as soon as the operator exists. Hence Theorem 1.3 is a vastly more general result. The proof is based on the method by Glimm and Jaffe introduced in the paper [16].

One really interesting thing is an equality such as $\inf(\sigma_{\text{ess}}(F_\eta)) = \mathcal{E}_{-\eta} + m_{\text{ess}}$. Statements of this sort can be combined with variational methods to extract ground states. In fact one can use a coherent trial state, and prove there is an exited state for large interactions in the standard spin boson model. We also have

Theorem 1.4. *Let $\alpha \in \mathbb{R}^{2n}, \eta \in \mathbb{R}, f \in \mathcal{H}^{2n}$ and ω be selfadjoint on $\mathcal{H}$ and assume they satisfy Hypothesis 1.1,1.2 and 1.3. Let U be the map such that*

$$UH_\eta(\alpha, f, \omega)U^* = F_{-\eta}(\alpha, f, \omega) \oplus F_\eta(\alpha, f, \omega).$$

Then

(1) *If $\eta \neq 0$ then ground states of H_η are non degenerate and if ψ is a ground state for H_η then $U\psi = e_{-\text{sign}(\eta)} \otimes \phi$ where ϕ is an eigenvector for $F_{-|\eta|}$ corresponding to the energy E_η.*

(2) *$F_{-|\eta|}$ has non degenerate ground states and any ground state eigenvector will have nonzero inner product with Ω. In particular H_0 will have a doubly degenerate ground states if they exists. If ψ is a ground state for H_0 then $U\psi = e_1 \otimes \phi_1 + e_{-1} \otimes \phi_{-1}$ where ϕ_i is either 0 an eigenvector for F_0 corresponding to the energy E_0.*

(3) *If we further assume Hypothesis 1.4 when $n > 2$ then $\mathcal{E}_{-|\eta|} = E_\eta$. Hence H_η has a ground state if and only if $F_{-|\eta|}$ has a ground state. Also, if $m = 0$ then $F_{|\eta|}$ has no ground state when $\eta \neq 0$.*

(4) *If we further assume Hypothesis 1.4 when $n > 2$ and $m, \eta \neq 0$ then H_η will have an exited state in $(E_\eta, E_\eta + m_{\mathrm{ess}}]$ if $F_{|\eta|}$ has a ground state. This is the case if $2|\eta| < m_{\mathrm{ess}}$.*

Parts of this result are similar to results found in [20] and [22] for the standard spin-boson model. In [20], non degeneracy of the ground state of H_η is investigated and in [22] the subspace containing the ground state is pinpointed. Our main contribution in these cases, is that these results works in much higher generality.

However, the observation that $F_{|\eta|}$ has no ground state if $m = 0$ and $\eta \neq 0$ is new and nearly impossible to prove directly. It is only the connection with H_η that allows us to give a simple proof. Furthermore the simple criteria $2|\eta| < m_{\mathrm{ess}}$ for the existence of an exited state is also new. Unfortunately this does not say anything about what happens when the interaction strength goes to infinity, but this is the main topic of paper B.

We now have the following result about ground states in the massless but infrared regular case.

Theorem 1.5. *Let $\alpha \in \mathbb{R}^{2n}, \eta \in \mathbb{R}, f \in \mathcal{H}^{2n}$ and ω be selfadjoint on $\mathcal{H}$ and assume they satisfy Hypothesis 1.1,1.2,1.3,1.5 and either $n \leq 2$ or Hypothesis 1.4.*

(1) *If $F_{-|\eta|}$ has a ground state ψ and H_η has a ground state ϕ then $\psi \in \mathcal{D}(N^a)$ and $\phi \in \mathcal{D}(1 \otimes N^a)$ for any $a > 0$.*

(2) *Assume $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu})$, ω is a multiplication operator and $n \leq 2$. Then E_η is an eigenvalue for $F_{-|\eta|}$ and H_η. Here $\lambda^{\otimes\nu}$ is the Lebesgue measure on $\mathcal{B}(\mathbb{R}^\nu)$.*

Part (2) was proven in [3] and [14] for the standard spin-boson model. Our result is proven along the same lines but contains some new ideas to improve generality. Usually it is assumed that ω has bounded derivatives away from 0 and that $\omega(x)$ goes to infinity as x goes to infinity. These criteria have been removed, so it is really only the infrared singularity that poses a problem for existence. In [20] and [4] it is proven, that the standard spin-boson model has a ground state in some infrared singular cases. However, that result is only perturbative and existence for all coupling strengths is an open problem.

Part (1) follows from completely novel ideas based on pull through formulas. Assume $\mathcal{H} = L^2(\mu, \mathcal{F}, \mu)$. Then one can define the pointwise annihilation operator of order n as a map $A_n : \mathcal{D}(N^{n/2}) \to L^2(\mathcal{M}^n, \mathcal{F}^{\otimes n}, \mu^{\otimes n}, \mathcal{F}_b(\mathcal{H}))$. This works well if regularity with respect to the number operator is already known. In paper A, we develop a new approach to the pointwise annihilation operators such that they can be applied to any state. In this framework we prove, that $A_n\psi \in L^2(\mathcal{M}^n, \mathcal{F}^{\otimes n}, \mu^{\otimes n}, \mathcal{F}_b(\mathcal{H}))$ if and only if $\psi \in \mathcal{D}(N^{n/2})$. Under the infrared regularity condition it is easy to see $A_n\psi \in L^2(\mathcal{M}^n, \mathcal{F}^{\otimes} n, \mu^{\otimes n}, \mathcal{F}_b(\mathcal{H}))$ for any ground state ψ and $n \in \mathbb{N}$. Thus the conclusion in Theorem 1.5 follows. In principle this could also be applied to almost every other model in non relativistic quantum field theory as long infrared singularities are absent.

2 Paper B: Large interaction asymptotics of spin-boson type models.

In this paper we consider the standard spin-boson model

$$H_\eta(v,\omega) := \eta\sigma_z \otimes 1 + 1 \otimes d\Gamma(\omega) + \sigma_x \otimes \varphi(v),$$

which is here parametrised by $v \in \mathcal{H}, \eta \in \mathbb{C}$ and ω selfadjoint and non negative on $\mathcal{H}$. As in Paper A, there is a unitary map V such that

$$VH_\eta(v,\omega)V^* = F_{-\eta}(v,\omega) \oplus F_\eta(v,\omega)$$

where the fiber operators $F_\eta(v,\omega)$ are defined as

$$F_\eta(v,\omega) = \eta\Gamma(-1) + d\Gamma(\omega) + \varphi(v).$$

For ω selfadjoint on $\mathcal{H}$ we define the numbers

$$m = \inf\{\sigma(\omega)\} \text{ and } m_{\text{ess}} = \inf\{\sigma_{\text{ess}}(\omega)\}.$$

Selfadjointness is now a walk in the park via Kato-Rellich theorem. Alternatively one can just use the Lemma 1.2.

Proposition 2.1. *Let $\omega \geq 0$ be selfadjoint and injective, $v \in \mathcal{D}(\omega^{-1/2})$ and $\eta \in \mathbb{C}$. Then the operators $F_\eta(v,\omega)$ and $H_\eta(v,\omega)$ are closed on the respective domains*

$$\mathcal{D}(F_\eta(v,\omega)) = \mathcal{D}(d\Gamma(\omega))$$
$$\mathcal{D}(H_\eta(v,\omega)) = \mathcal{D}(1 \otimes d\Gamma(\omega))$$

and given any core $\mathcal{D}$ of ω the linear span of the following sets

$$\mathcal{J}(\mathcal{D}) := \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{f_1 \vee \cdots \vee f_n \mid f_j \in \mathcal{D}\}$$
$$\widetilde{\mathcal{J}}(\mathcal{D}) := \{f_1 \otimes f_2 \mid f_1 \in \{e_1, e_{-1}\}, f_2 \in \mathcal{J}(\mathcal{D})\}$$

is a core for $F_\eta(v,\omega)$ and $H_\eta(v,\omega)$ respectively. Furthermore, both operators are selfadjoint and semibounded if $\eta \in \mathbb{R}$ and they have compact resolvents if ω has compact resolvents.

The physically relevant assumptions are:

Remark 2.2. *In the physical model we have $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$ with $\nu \leq 3$, $\omega(k) = \sqrt{m^2 + \|k\|^2}$ and $v_{g,\Lambda}(k) = g\omega(k)^{-1/2}\chi(\omega(k))$ where χ is a cutoff function (i.e. $0 \leq \chi \leq 1$ and ensures $v_{g,\Lambda} \in \mathcal{D}(\omega^{-1/2})$). The model is said to be massive if $m > 0$.*

The aim of this paper is to investigate limits of $F_\eta(v,\omega)$ and $H_\eta(v,\omega)$ as the interaction strength $|v|$ tends to infinity. As with paper A, the original motivation was to prove existence of exited states in the spin-boson model at large interaction strengths. In the special case where $\mathcal{H} = \mathbb{C}$ this was done in the paper [29] using a simple Weyl transformation and a compactness argument. The main technical result is the following theorem

Theorem 2.3. *Let $\{v_g\}_{g\in(0,\infty)} \subset \mathcal{D}(\omega^{-1/2})$ and P_ω denote the spectral measure corresponding to ω. Assume that there is $\widetilde{m} > 0$ such that:*

(1) *$\{P_\omega([0,\widetilde{m}])v_g\}_{g\in(0,\infty)}$ converges to $v \in \mathcal{D}(\omega^{-1/2})$ in the graph norm of $\omega^{-1/2}$.*

(2) *$\|\omega^{-1}P_\omega(\widetilde{m},\infty)v_g\|$ diverges to ∞ as g tends to infinity.*

Then the g-dependent family of operators given by

$$\begin{aligned} W(\omega^{-1}P_\omega(\widetilde{m},\infty)v_g,1)&F_\eta(v_g,\omega)W(\omega^{-1}P_\omega(\widetilde{m},\infty)v_g,1)^* + \|\omega^{-1/2}P_\omega(\widetilde{m},\infty)v_g\|^2 \\ &= \eta W(2\omega^{-1}P_\omega(\widetilde{m},\infty)v_g,-1) + d\Gamma(\omega) + \varphi(P_\omega(0,\widetilde{m}])v_g) \qquad (2.1)\\ &:= \widetilde{F}_{\eta,\widetilde{m}}(v_g,\omega) \end{aligned}$$

is uniformly bounded below by $-|\eta| - \sup_{g\in(0,\infty)}\|P_\omega(0,\widetilde{m}])v_g\|^2$. Furthermore, the family $\{\widetilde{F}_{\eta,\widetilde{m}}(v_g,\omega)\}_{g\in(0,\infty)}$ converges to $d\Gamma(\omega)+\varphi(v)$ in norm resolvent sense as g tends to ∞.

This result is interesting for a number of reasons. First of all the convergence is in norm resolvent sense and the transformation is unitary which means we can find limiting spectrum using standard theory. Secondly, there is almost no restriction on how $\omega^{-1}v_g$ goes to infinity. This means that it can be applied to both ultraviolet renormalisation analysis and to the case $v_g = gv$ for some scaling $g > 0$. Thirdly the limit found does not depend on η. From a physical point of view this means that the qubit becomes degenerate in the limit. Applying this theorem yields the following two corollaries regarding the strong interaction limit

Corollary 2.4. *Let $v \in \mathcal{H}$, $\eta \in \mathbb{R}$ and assume $m > 0$. Then there exists $g_0 > 0$ such that $\mathcal{E}_\eta(gv,\omega)$ is a non degenerate eigenvalue of $F_\eta(gv,\omega)$ when $g > g_0$. Furthermore, one may pick a family of normalised vectors $\{\psi_g\}_{g\in[g_0,\infty)}$ such that $g \mapsto \psi_g$ is smooth, $F_\eta(gv,\omega)\psi_g = \mathcal{E}_\eta(gv,\omega)\psi_g$ and*

$$\begin{aligned} \lim_{g\to\infty}\|\psi_g - e^{-g^2\|\omega^{-1}v\|^2}\epsilon(-g\omega^{-1}v)\| &= 0,\\ \lim_{g\to\infty}\frac{\langle\psi_g, N\psi_g\rangle - g^2\|\omega^{-1}v\|^2}{g} &= 0,\\ \lim_{g\to\infty}(\mathcal{E}_\eta(gv,\omega) + g^2\|\omega^{-1/2}v\|^2) &= 0. \end{aligned}$$

If $\eta < 0$ then $g \mapsto \mathcal{E}_\eta(gv,\omega) + g^2\|\omega^{-1/2}v\|$ is strictly increasing and the range is contained in $[-\eta, 0]$.

Corollary 2.5. *Assume ω is selfadjoint, non-negative and injective on $\mathcal{H}$. Let $v \in \mathcal{H}$ and $\eta \in \mathbb{R}$. If $m(\omega) > 0$ there is $g_0 > 0$ such that $H_\eta(gv,\omega)$ has an exited state with energy $\widetilde{E}_\eta(gv,\omega)$ for $g > g_0$. Furthermore*

$$\lim_{g\to\infty}(E_\eta(gv,\omega) - \widetilde{E}_\eta(gv,\omega)) = 0.$$

These two corollaries shows the claim that we set out to prove: For sufficiently strong coupling there is an exited state. However it also tells much more. I shows that as the interaction becomes large, the contribution from the qubit vanishes, so the eigenstates become approximately coherent and the energy approximates that of a free system. In particular, the energy difference between the ground state and the exited state will go to 0. Note that by Remark 2.2 the above conclusions applies to the massive spin-boson model. In the massless but infrared regular case one has the following result.

Theorem 2.6. *Assume also $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$ where λ_ν is the Lebesgue measure. Assume also ω is a selfadjoint, non-negative and injective multiplication operator. Let $v \in \mathcal{D}(\omega^{-1})$ and $\eta \leq 0$. Then there is a family $\{\psi_g\}_{g\in\mathbb{R}}$ of normalised ground states for $F_\eta(gv,\omega)$ and*

$$\lim_{g\to\infty}(\mathcal{E}_\eta(gv,\omega) + g^2\|\omega^{-1/2}v\|^2) = 0.$$

$$\lim_{g\to\infty}\frac{\langle\psi_g, N\psi_g\rangle - g^2\|\omega^{-1}v\|^2}{g^2} = 0.$$

We cannot conclude that an exited state exists in the full model, because one of the fiber operators will not have a ground state by Theorem 1.4 in Paper A. We now turn our attention to ultraviolet renormalisation. We have the following corollary to Theorem 2.3.

Corollary 2.7. *Assume $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ and ω is a multiplication operator on this space. Let $v : \mathcal{M} \to \mathbb{C}$ is measurable and that $\{\chi_g\}_{g\in(0,\infty)}$ is a collection of functions from $\mathbb{R}$ into $[0,1]$. Assume $g \mapsto \chi_g(x)$ is increasing and converges to 1 for all $x \in \mathbb{R}$. Assume furthermore that $k \mapsto \chi_g(\omega(k))v(k) \in \mathcal{D}(\omega^{-1/2})$ and that there is $\widetilde{m} > 0$ such that $\widetilde{v} := 1_{\{\omega\leq\widetilde{m}\}}v \in \mathcal{D}(\omega^{-1/2})$. If $k \mapsto \omega(k)^{-1}v(k)1_{\{\omega>1\}}(k) \notin \mathcal{H}$ there are unitary maps $\{V_g\}_{g\in(0,\infty)}$ and $\{U_g\}_{g\in(0,\infty)}$ independent of η such that:*

(1) *$\{V_g F_\eta(v_g,\omega)V_g^* + \|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2\}_{g\in(0,\infty)}$ is uniformly bounded below and converges in norm resolvent sense to the operator $d\Gamma(\omega) + \varphi(\widetilde{v})$ as g tends to infinity.*

(2) *$\{U_g H_\eta(v_g,\omega)U_g^* + \|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2\}_{g\in(0,\infty)}$ is uniformly bounded below and converges in norm resolvent sense to the operator*

$$\widetilde{H} := (d\Gamma(\omega) + \varphi(\widetilde{v})) \oplus (d\Gamma(\omega) + \varphi(\widetilde{v}))$$

as g tends to ∞. This implies

$$(H_\eta(v_g,\omega) + \|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2 + i)^{-1} - (H_0(v_g,\omega) + \|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2 + i)^{-1}$$

will converge to 0 in norm as g tends to ∞.

Let $\widetilde{H}_\eta(v_g,\omega) = H_\eta(v_g,\omega) + \|\omega^{-1/2}v_g\|^2$. By Remark 2.2 we see that the physical spin-boson model with $\nu = 3$ fulfils the criteria in Corollary 2.7. So if a limit of $\widetilde{H}_\eta(v_g,\omega)$ exists in strong or uniform resolvent sense then the limit will be independent of η. In particular the resulting model will not be physically interesting. In conclusion: One cannot hope to renormalise the spin-boson model in a physically interesting way using the scheme introduced by E. Nelson to renormalise the Nelson model (see [30]). For a longer discussion see paper B below.

Paper B contains two more results which are interesting even though they are not directly related to the main topic of the paper. The first one is

Theorem 2.8. *Let ω be selfadjoint, non-negative and injective on $\mathcal{H}$, $v \in \mathcal{D}(\omega^{-1/2})$, $g \in (0,\infty)$ and $\eta \leq 0$. Assume $\mathcal{H} = L^2(\mathcal{M},\mathcal{F},\mu)$ and ω is a multiplication operator on this space. If that $F_\eta(gv,\omega)$ has a ground state $\psi_{g,\eta} = (\psi_{g,\eta}^{(n)})$ then*

(1) *We may choose $\psi_{g,\eta}$ such that $\psi_{g,\eta}^{(0)} > 0$ and $(-1)^n \overline{v}^{\otimes n} \psi_{g,\eta}^{(n)} > 0$ almost everywhere on $\{v \neq 0\}^n$.*

(2) *Almost everywhere the following inequality holds*

$$|\psi_{g,\eta}^{(n)}(k_1,\dots,k_n)| \leq \frac{g^n}{\sqrt{n!}} \frac{|v(k_1)|\cdots|v(k_n)|}{\omega(k_1)\cdots\omega(k_n)}.$$

In particular $\psi_{g,\eta}^{(n)}$ is zero outside $\{v \neq 0\}^n$ almost everywhere and if $v \in \mathcal{D}(\omega^{-1})$ then $\|\psi_{g,\eta}^{(n)}\|$ goes to zero like g^n for g tending to 0.

(3) *Assume $v \in \mathcal{D}(\omega^{-1})$, $f : \mathbb{N}_0 \to [0,\infty)$ is a function and assume $F_\eta(gv,\omega)$ has a ground state for all $\eta \leq 0$. Then $H_a(gv,\omega)$ has a ground state $\phi_{g,a}$ for all $a \in \mathbb{R}$ and we have*

$$\begin{aligned}\alpha_{g,f,v,\omega} := \sum_{n=0}^{\infty} \frac{f(n)^2 g^{2n} \|\omega^{-1} v\|^{2n}}{n!} < \infty &\iff \psi_{g,\eta} \in \mathcal{D}(f(N)) \ \forall \eta \leq 0 \\ &\iff \phi_{a,\eta} \in \mathcal{D}(1 \otimes f(N)) \ \forall a \in \mathbb{R}\end{aligned}$$

In particular $\psi_{g,\eta} \in \mathcal{D}(\sqrt[p]{N!})$ and $\phi_{g,\eta} \in \mathcal{D}(1 \otimes \sqrt[p]{N!})$ for all $p > 2$.

A result similar to (2) was derived in Frölichs paper [13]. However the application (3) was never mentioned in that paper. One should note that part (3) vastly generalises the result in paper [22], were it is proven $\psi_g \in \mathcal{D}(e^{tN})$ for all $t > 0$. The only downside of part (3) is the infrared condition, which does not apply to the massless physical model. The last result is

Theorem 2.9. *Assume ω is a selfadjoint, non-negative and injective multiplcation operator on $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$. If $m(\omega) > 0$, $v \in \mathcal{H}\backslash\{0\}$ and*

$$\int_{\mathbb{R}^\nu} \frac{|v(k)|^2}{\omega(k) - m} dk = \infty. \tag{2.2}$$

then both $F_e(v,\omega)$ and $F_{-\eta}(v,\omega)$ have a ground state so $H_\eta(gv,\omega)$ will have an excited state. The condition is satisfied if $\omega \in C^2(\mathbb{R}^\nu,\mathbb{R})$, $\nu \leq 2$ and there is $x_0 \in \mathbb{R}^\nu$ such that $\omega(x_0) = m$ and $|v|$ is bounded from below by a positive number on a ball around x_0.

The technique for proving this result has been deployed for the translation invariant Nelson model in the papers [29] and [39]. Note that Theorem 2.9 applies to the massive physical spin-boson model in dimension 1 and 2 so an exited state always exist in this case.

3 Paper C: Non-existence of ground states in the translation invariant Nelson model.

In this paper we analyse operators arising from a spinless particle interacting with a boson field. The bosonic Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$ and the total Hilbertspace is $L^2(\mathbb{R}^n u, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu) \otimes \mathcal{F}_b(\mathcal{H}) = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu, \mathcal{F}_b(\mathcal{H}))$ and the total Hamiltonian takes the form

$$H = K(\Delta_x) \otimes 1 + 1 \otimes d\Gamma(\omega) + \mu\varphi_\oplus(U_x v)$$

where ω is a selfadjoint, non-negative and injective multiplication operator, K is the dispersion relation for the matter particle and $(U_x v)(k) = e^{ikx}v$ with $v \in \mathcal{D}(\omega^{1/2})$. Write k for the identity map from $\mathbb{R}^\nu$ to $\mathbb{R}^\nu$. H is called translation invariant because in commutes with the $p_x \otimes 1 + 1 \otimes d\Gamma(k)$ which is the total momentum of the field and the particle. From this fact it may be proven that there is a unitary transform U of $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu, \mathcal{F}_b(\mathcal{H}))$ such that

$$U^* H U = \int_{\mathbb{R}^\nu} H(\xi) d\lambda_\nu(\xi)$$

where

$$H_\mu(\xi) = K(\xi - d\Gamma(k)) + d\Gamma(\omega) + \mu\varphi(v).$$

The standard assumptions under which the above discussion is true are.

Hypothesis 3.1. *We assume*

(1) $K \in C^2(\mathbb{R}^\nu, \mathbb{R})$ *is non negative and there is* $C_K > 0$ *such that* $\|\nabla K\|^2 \leq C_K(1 + K)$ *and* $\|D^2 K\| \leq C_K$ *where* $D^2 K$ *is the Hessian of* K.

(2) $\omega : \mathbb{R}^\nu \to [0, \infty)$ *is continuous and* $\omega > 0$ λ_ν *almost everywhere.*

(3) $v \in \mathcal{D}(\omega^{-1/2})$.

As $H_0(\xi)$ acts on the n'th particle sector as multiplication by

$$G_n(k_1, \dots, k_n) = K(\xi - k_1 - \dots - k_n) + \omega(k_1) + \dots + \omega(k_n).$$

Thus $H_0(\xi)$ is selfadjoint on $\mathcal{D}(H_0(\xi)) = \mathcal{D}(d\Gamma(\omega)) \cap \mathcal{D}(K(\xi - d\Gamma(k)))$. Using Theorem 1.10 and that $d\Gamma(\omega)$ is $H_0(\xi)$-bounded one immediately gets $H_\mu(\xi)$ is selfadjoint. In fact

Lemma 3.1. *Assume Hypothesis 1. Then* $\mathcal{D}(H_\mu(\xi))$ *is independent of* ξ *and* μ. *Let* $\mathcal{D} \subset \{f \in \mathcal{H} \mid f$ *has compact suppot*$\}$ *be a dense subspace. Then* $\mathcal{E}(\mathcal{D})$ *and* $\mathcal{J}(\mathcal{D})$ *span cores for* $H_\mu(\xi, A)$.

Our main result can be stated under the conditions:

Hypothesis 3.2. *We assume*

(1) K, ω *and* v *are rotation invariant. Furthermore* $k \mapsto e^{-tK(k)}$ *is positive definite for all* t.

(2) *ω is sub-additive and $\omega(x_1) < \omega(x_2)$ if $|x_1| < |x_2|$. Furthermore, $C_\omega = \lim_{k\to 0}|k|^{-1}\omega(k)$ exists and is strictly positive.*

(3) $v \notin \mathcal{D}(\omega^{-1})$

For the 3-dimensional Nelson model we have $K \in \{k \mapsto |k|^2, k \mapsto \sqrt{|k|^2+m} - m\}$, $\omega(k) = |k|$ and $v = \omega^{-1/2}\chi$ where $\chi : \mathbb{R}^\nu \to \mathbb{R}$ is a spherically symmetric ultraviolet cutoff. It is well known that Hypothesis 1 and 2 are fulfilled in this case. We can now state the main theorem of this paper:

Theorem 3.2. *Assume Hypothesis 1 and 2 along with $\nu \geq 3$. Then $H_\mu(\xi)$ has no ground state for any ξ and $\mu \neq 0$.*

This theorem proves that infrared singularities in the interaction can imply non-existence of ground states. This is physically significant because ground states of the $H_\mu(\xi)$ are used to construct scattering states for the full system. Hence the result above implies that the construction of scattering states becomes a very hard problem. However a lot of work has actually already been done on this problem, since the conclusion of Theorem 3.2 has been widely anticipated.

The first indication of non-existence was provided by in the PhD-thesis of J. Frölich which was published in the two papers [12] and [13]. In the paper [32] it is proven that ground states exists in a non-equivalent Fock representation. This proves, that the method used in [3] and [14] cannot be used to prove existence of ground states, but not ground states are absent.

Absence of ground states was proven for the minimally coupled model in the paper [20]. The proof given in that paper requires that the map $\Sigma(\xi) = \inf(\sigma(H_\mu(\xi)))$ is differentiable and that the derivative is nonzero. However, proving Σ is differentiable is very hard and has only been done perturbatively (see [1]). Furthermore the differentiability criterion does not work at $\xi = 0$ where Σ has a global minimum.

In this paper we mimic the proof given in [20], but we rely on rotation-invariance, non degeneracy of ground states and the HVZ-theorem instead of the existence of a non zero a derivative. The techniques in this paper could potentially be extended to the renormalised model, as most of the Lemmas used here remains true for the renormalised model. But there are issues with domains in the last steps of the proof and proving the pullthrough formula is also a challenge.

4 Paper D: Rigorous Results on the Bose-Polaron

In this paper we look at a new model for an impurity moving in a Bose gas. In recent papers [25], [36] and [37] a more complicated model and rather successful model has been used in the physics literature. It considers the impurity as a spinless particle interacting with a bosonic field (the condensate). We define and analyse a generalised version of the model in [25], [36] and [37]. In this generalised model, the bosonic space is assumed to be an abstract, separable Hilbert space $\mathcal{H}$. The total hilbert space is $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu) \otimes \mathcal{F}_b(\mathcal{H}) = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu, \mathcal{F}_b(\mathcal{H}))$ and the full Hamiltonian takes the form

$$H^V_{g_1,g_2} = \left(\frac{1}{2M}\Delta_x + V\right) \otimes 1 + d\Gamma_\oplus(\omega) + g_1\varphi_\oplus(u_x - v_x) + g_2 H_I(v_x, u_x)$$

where V is an external potential and

$$H_I(u_x,v_x) = \int_{\mathbb{R}^\nu}^{\oplus} a^\dagger(u_x)a(u_x) + a^\dagger(v_x)a(v_x) - a^\dagger(u_x)a^\dagger(v_x) - a(u_x)a(v_x) d\lambda_\nu(x)$$

At this point it is not clear if $H_I(u_x,v_x)$ makes sense as an operator, as it is not clear that we are taking the direct integral of a selfadjoint operator. We will need the following lemma

Lemma 4.1. *Assume $u,v \in \mathcal{D}(\omega^{-1/2})$ where ω is selfadjoint, injective and non negative on $\mathcal{H}$. For $\psi \in \mathcal{D}(\omega)$ we have*

$$\begin{aligned} &a^\dagger(u)a(u) + a^\dagger(v)a(v) - a^\dagger(u)a^\dagger(v) - a(u)a(v) \\ &\qquad = \frac{1}{4}\varphi(u-v)^2 + \frac{1}{4}\varphi(i(u+v))^2 - C(v,u) \\ &\qquad = a^\dagger(u+v)a(u+v) + \varphi(v)\varphi(u) + D(v,u) \end{aligned}$$

where $C(v,u) = \frac{1}{2}(\|u\|^2 + \|v\|^2)$ and $D(v,u) = \frac{1}{2}(\langle u,v\rangle + Re(\langle u,v\rangle))$.

Thus if we define the function $h(x) = C(v_x,u_x)$ and $u_x, v_x \in \mathcal{D}(\omega^{-1/2})$ for all x then we may interpret

$$H_I(u_x,v_x) = \frac{1}{4}\varphi_\oplus(u_x - v_x)^2 + \frac{1}{4}\varphi_\oplus(i(u_x+v_x))^2 + h \otimes 1.$$

I will now describe the hypothesis we are working under

Hypothesis 4.1. *We assume the following minimal properties*

(1) *$V \in L^2_{loc}(\mathbb{R}^\nu)$ and $-\frac{1}{2M}\Delta_x + V$ is essentially selfadjoint on $C^\infty(\mathbb{R}^\nu)$. Defining $V_= \max\{0,-V\}$ we also assume $V_-^{1/2}$ is relatively $(-\frac{1}{2M}\Delta_x)^{1/2}$ bounded with bound smaller than 1.*

(2) *ω is selfadjoint, non negative and injective on $\mathcal{H}$.*

(3) *$x \mapsto v_x$ and $x \mapsto u_x$ are weakly differentiable maps. Both maps takes values in $\mathcal{D}(\omega^{-1/2}) \cap \mathcal{D}(\omega^{-1/2})$ and the partial derivatives takes values in $\mathcal{D}(\omega^{-1/2})$. Furthermore*

$$\sup_{x\in\mathbb{R}^\nu} \{\|(1+\omega^{-1/2}+\omega^{1/2})v_x\|, \|(1+\omega^{-1/2}+\omega^{1/2})u_x\|\} < \infty$$
$$\sup_{x\in\mathbb{R}^\nu, i\in\{1,\dots,\nu\}} \{\|(1+\omega^{-1/2})\partial_{x_i}v_x\|, \|(1+\omega^{-1/2})\partial_{x_i}u_x\|\} < \infty$$

One may now prove

Theorem 4.2. *Assume Hypothesis 1 holds, that $g_1 \in \mathbb{R}$ and $g_2 \geq 0$. Let S be the selfadjoint closure of $\frac{1}{2M}\Delta_x + V$. Then S is bounded below and $H_{g_1,g_2}(V)$ is selfadjoint on $\mathcal{D}(S\otimes 1) \cap \mathcal{D}(d\Gamma_\oplus(\omega))$, bounded below and essentially selfadjoint on any core for $S \otimes 1 + d\Gamma_\oplus(\omega)$. One example of a core is $C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D}(\omega))$.*

Assume in addition that $\langle a, e^{-t\omega}b\rangle \in \mathbb{R}$ for all $t \geq 0$ and $a,b \in \{v_x\}_{x\in\mathbb{R}^\nu} \cup \{u_x\}_{x\in\mathbb{R}^\nu}$. If $H^V_{g_1,g_2}$ has a ground state, then it is non degenerate and any eigenvector will have non zero inner product with any vector of the form $\phi \otimes \Omega$ with $\phi \neq 0$ and non negative.

The selfadjointness part of this statement more or less appears in [21]. However, the author does have stronger conditions and uses the result in [2]. Unfortunately there is a small mistake in [21, Lemma 3.3], so I decided to prove Theorem 4.2 from scratch in the paper. The proof is not that long (3 pages) and does not really rely on any specialised knowledge.

The hardest part to prove is the second part. One has to find a unitary map U into an L^2-space such that $U^* H^V_{g,g_2} U$ is positivity improving. The usual choice would be to take $U = 1 \otimes \widetilde{U}$ where $\widetilde{U}$ is a Q-space isomorphism. However, it is not clear that this will work as $H_I(u_x, v_x)$ is not a multiplication operator in Q-space. One needs to rewrite $H_I(u_x, v_x)$ using Lemma 4.1 and even then, standard theory does not quite apply. This is why this paper has a section devoted to positivity preserving and positivity improving semi groups.

As in the case of the Nelson model, we also have fiber Hamiltonians defined on $\mathcal{F}_b(\mathcal{H})$

$$\begin{aligned} H_{g_1,g_2}(\xi) =& \frac{1}{2M}(\xi - d\Gamma(m))^2 + d\Gamma(\omega) + g_1\varphi(u-v) \\ &+ g_2 a^\dagger(u)a(u) + g_2 a^\dagger(v)a(v) + g_2 a^\dagger(u)a^\dagger(v) + g_2 a(u)a(v) \end{aligned}$$

where $m = (m^{(1)}, \dots, m^{(\nu)})$ is a vector of operators and $\xi \in \mathbb{C}^\nu$. The operator $(\xi - d\Gamma(m))^2$ should be interpreted as $(\xi_1 - d\Gamma(m^{(1)}))^2 + \cdots + (\xi_\nu - d\Gamma(m^{(\nu)}))^2$. The basic assumptions for these operators are

Hypothesis 4.2. *Under hypothesis 4.2 we assume*

1. $\omega, m^{(1)}, \dots, m^{(\nu)}$ *are strongly commuting selfadjoint operators. Furthermore,* ω *is non negative and injective.*
2. $v, u \in \mathcal{D}(\omega^{-1/2}) \cap \mathcal{D}(\omega^{1/2}) \cap \bigcap_{j=1}^{\nu} \mathcal{D}(m^{(j)}) \cap \mathcal{D}(\omega^{-1/2} m^{(j)})$.

Hypothesis 4.3. $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$, ω *is multiplication by a continuous function and* $m^{(j)}$ *is multiplication by* $m^{(j)}(k) = k_i$.

Hypothesis 4.4. *Assume in addition* $\langle a, e^{-t\omega} e^{it_1 m^{(1)}} \dots e^{it_\nu m^{(\nu)}} b \rangle \in \mathbb{R}$ *for all* $t \geq 0$, $t_1, \dots, t_\nu \in \mathbb{R}$ *and* $a, b \in \{u, v\}$

We have the following Theorem

Theorem 4.3. *Assume Hypothesis 1 holds. If* $g_1 \in \mathbb{R}$, $g_2 \geq 0$ *and* $\xi \in \mathbb{R}^\nu$ *then* $H(\xi)$ *is selfadjoint on* $\mathcal{D}(d\Gamma(\omega)) \cap \mathcal{D}((d\Gamma(m))^2)$, *bounded below and essentially selfadjoint on* $\mathcal{D}(d\Gamma(\omega)) \cap \mathcal{D}((d\Gamma(m))^2) \cap \mathcal{N}$. *Furthermore, we also have:*

(1) $\xi \mapsto H(\xi)$ *is an analytic family of type A, so the map* $\xi \mapsto (H(\xi) + i)^{-1}$ *is smooth.*

(2) *The map* $\Sigma(\xi) = \inf(\sigma(H(\xi)))$ *is locally Lipshitz and almost everywhere twice differentiable.*

(3) *Assume Hypothesis 4.3 holds as well. Then*

$$\Sigma(\xi - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n) \in \sigma_{ess}(H(\xi))$$

for all $k_1,\dots,k_n \in \mathbb{R}^\nu$*. If in addition* $\inf_{k\in\mathbb{R}^\nu}\omega(k)>0$ *or* $\omega(0)=0$ *then*

$$\inf(\sigma_{ess}(H(\xi))) = \inf_{n\in\mathbb{N}_0}\inf_{\xi\in\mathbb{R}^\nu}\Sigma(\xi-k_1-\cdots-k_n)+\omega(k_1)+\cdots+\omega(k_1)(k_n).$$

If ω *is also unbounded we have* $\sigma_{ess}(H(\xi)) = [\inf(\sigma_{ess}(H(\xi))),\infty)$.

(4) *Assume Hypothesis 4.3 holds as well. Define the elements* $u_x = e^{ix_1 m^{(1)}}\dots e^{ix_\nu m^{(\nu)}}u$ *and* $v_x = e^{ix_1 m^{(1)}}\dots e^{ix_\nu m^{(\nu)}}v$*. Then there is a unitary map (the Lee Low Pines transformation) such that*

$$UH^0_{g_1,g_2}U^* = \int_{\mathbb{R}^\nu}^{\oplus} H_{g_1,g_2}(\xi)d\lambda_\nu(\xi)$$

If in addition we assume Hypothesis 4.4 then $H^0_{g_1,g_2}$ *has no ground state.*

(5) *Assume Hypothesis 4.4 holds as well. Then* Σ *has a global minimum at* $\xi=0$ *and if* $H_{g_1,g_2}(0)$ *has a ground state then it is non degenerate. If* $\inf(\sigma(\omega))>0$ *and we additionally assume Hypothesis 4.3 holds, then* 0 *is the unique minima.*

All of these results are known for the Nelson model (see [15], [18] and [23]). The main contribution is selfadjointness, statement (5) and the last part of statement (4). The proof of the remaining properties are almost identical to the proof given for the Nelson model. In part (5) we also follow the general strategy outlined in [15], [18] and [23] but the proofs become much longer due to technical problems with positive cones. Again we are saved by the general theory developed in this paper.

One interesting observation is the fact that $H^0_{g_1,g_2}$ has no ground state in the translation invariant case. From a physical perspective this should be a no-brainier: Translation invariant systems should not have bound states, as bound states should be localised. However, translation invariance is not with respect to impurity coordinates, so the "localisation argument" does not quite work.

Proving that there is no bound states has actually not been done so far and it is by no means trivial. We can only exclude the existence of a ground state due to the following simple argument: If a ground state eigenvector exists, then the eigenspace has infinite dimension by the direct integral decomposition. However this contradicts Theorem 4.2. A similar strategy would work for any bound state as long one can exclude infinite dimension of the eigenspace. Tools such as Mourre theory could be useful for this, but that would be future work.

We now turn our attention to the machinery that makes everything work. Let

$$\begin{aligned}\mathcal{H} =&L^2(\mathcal{M},\mathcal{F},\mu)\\ \mathcal{H}_+ :=&L^2(\mathcal{M},\mathcal{F},\mu) := \{f\in\mathcal{H} \mid f\geq 0 \text{ almost everywhere}\}\end{aligned}$$

be an L^2-space. We want to answer the following question: If B is a multiplcation operator and A is selfadjoint, bounded below and generates a positivity improving semigroup does it follow that $A+B$ generates a positivity improving semigroup? The general answer to this question is no (see [35, Theorem XIII.48]), but it is often true. To use the results in the literature (see [11], [27] and [35]) one needs to approximate B by bounded multiplication operators in such a way $A+B_n$ and $A+B-B_n$ remains uniformly bounded below. To find such lower bounds is often easy but not in the situations encountered in this paper.

Instead of working directly with operators we do instead work with forms.

Definition 4.4. *Let A be selfadjoint on a Hilbert space $\mathcal{H}$. The form of A is the sesquilinear map $q_A : \mathcal{D}(|A|^{1/2}) \times \mathcal{D}(|A|^{1/2}) \to \mathbb{C}$ given by*

$$q_A(\psi, \phi) = \langle \mathrm{Sign}(A)|A|^{1/2}\psi, |A|^{1/2}\phi\rangle$$

We now present the following theorem

Theorem 4.5. *Let A be selfadjoint and bounded below on $\mathcal{H}$. Assume B is a multiplication operator on $\mathcal{H}$ and define $B_+ = \max\{0, B\}$ and $B_- = \max\{0, -B\}$. Assume*

(1) *A generates a positivity improving semigroup.*

(2) *$\mathcal{D}(q_{B_+})$ contains a core for q_A and $\mathcal{D}(q_A) \cap \mathcal{D}(q_{B_+}) \subset \mathcal{D}(q_{B_-})$.*

(3) *The quadratic form $q = q_A + q_{B_+} - q_{B_-}$ is bounded below and closed.*

is bounded below and closed. If C is the operator corresponding to q then C will generate a positivity improving semi group.

One key example is Schrödinger operators on a connected, weighted Riemannian manifold (M, g, Υ) (see [17] for definitions). In this case A is the Laplace-Beltrami operator which is positivity improving because the heat kernel is positive everywhere. Furthermore, if B is a potential where q_{B_+} is locally integrable and q_{B_-} is $q_A + q_{B_+}$-bounded with bound strictly smaller than 1 then the form $q = q_A + q_B$ will be closed and bounded below. The operator corresponding q will generate a positivity improving semigroup. A similar result was found in [19] but we do not require potential to be Kato-class and we work on a weighted manifold.

The following result is a bit different but we use it a part of our proofs.

Theorem 4.6. *Let A, B, C be selfadjoint operators in $L^2(\mathcal{M}, \mathcal{F}, \mu)$. Assume*

(1) *A is bounded below and e^{-tA} is positivity improving for all $t \geq 0$.*

(2) *B is a multiplication operator which is bounded from below.*

(3) *$-C \geq 0$ and C is a multiplication operator.*

(4) *$\mathcal{D}(q_B)$ contains a form core for q_A and $\mathcal{D}(q_A) \cap \mathcal{D}(q_B) \subset \mathcal{D}(q_C)$.*

(5) *The form $q = q_A + q_B + q_C$ is closable and bounded below.*

Then the operator H corresponding to q is bounded below and e^{-tH} is positivity improving.

Let A be selfadjoint, bounded below and assume it generates a positivity improving semigroup. Assume also B is an A bounded multiplication operator and $H := A + B = A + B_+ - B_-$ is selfadjoint and bounded below. By standard theory (see e.g. [42]) we have $\mathcal{D}(q_A) \subset \mathcal{D}(q_B) = \mathcal{D}(q_{B_+}) \cap \mathcal{D}(q_{B_-})$ and it is not hard to see that $q_A + q_{B_+} + q_{-B_-}$ is closable and the operator corresponding to that closure is H. Hence we find H generates a positivity improving semi group. This is used often in Paper D.

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Paper A

Spin-Boson type models analysed using symmetries.

By T. N. Dam and J. S. Møller

SPIN-BOSON TYPE MODELS ANALYSED USING SYMMETRIES

THOMAS NORMAN DAM, JACOB SCHACH MØLLER

Abstract. In this paper we analyse a family of models for a qubit interacting with a bosonic field called spin-boson type models. The Hamiltonian has a special symmetry called spin-parity symmetry, which plays a central role in our analysis. Furthermore, higher order perturbations of field operators are added to the Hamiltonian. We find the domain of selfadjointness and decompose the Hamiltonian into two fiber operators each defined on Fock space. We then prove a HVZ theorem for the fiber operators and single out a particular fiber operator, which has a ground state if and only if the full Hamiltonian has a ground state. From these results we can deduce a simple criterion for the existence of an exited state.

2010 *Mathematics Subject Classification*: Primary 81Q10; Secondary 81T10.

Keywords. Spectral analysis, spin-boson model, Non-relativistic quantum field theory, higher order perturbations.

July 31, 2018

1. Introduction

This paper is devoted to the analysis of so called spin-boson type models, which is a family of models describing a qubit interacting with a bosonic field. The assumptions in our framework are very weak, which allows us to cover both the Rabi model and the standard spin-boson model simultaneously. Furthermore, higher order perturbations of field operators are also considered. QFT Models with higher order perturbations have lately become relevant in physics. They appear in cavity QED (see [11]) and in the theory of polarons (see [19]).

The analysis in this paper relies on the fact that spin-boson type Hamiltonians commute with the spin-parity operator. This fact was used in [3] and [8] to prove that ground states exist in the massless spin-boson model. The spin-parity operator has two invariant subspaces, which are both isomorphic to the Fock space. In our paper, we investigate the restriction of the full model to each of these subspaces. These two restrictions are referred to as the fiber operators. We shall see, that the two fiber operators differ only by the value of a scalar parameter, but they behave quite differently.

Models with higher order perturbations were treated in [9], [11], [13] and [21]. Spin-boson type models are treated in [11], [13] and [21], but the authors assume either that the field is massive or that the coupling is weak. The results in [9] does not assume weak coupling or a massive field, but the model treated in that paper is not the spin-boson model and rather strong infrared conditions are assumed. Furthermore, the author of [9] only proves selfadjointness of the Hamiltonian and existence of ground states, while we treat several other questions as well.

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We start by proving selfadjointness of all involved operators and move on to prove an HVZ theorem for the fiber operators. The method we use is related to the approach in [14], but is written up in a more general way, which allows one to handle massless fields and abstract Hilbert spaces. The HVZ theorem for the fiber operators also gives an HVZ theorem for the full Hamiltonian.

Using arguments similar to those presented in [8], we prove that if ground states exists for the full Hamiltonian, then the bottom of the spectrum is a non degenerate eigenvalue. Using this result, we single out a particular fiber which has a ground state if and only if the full Hamiltonian has a ground state. Ground states for the other fiber operator must therefore correspond to exited states. The HVZ theorem then gives a simple criterion for the existence of an exited state.

The reader is then encouraged to have a look at Appendix D, where a new framework for pointwise annihilation operators is developed. Most maps are continuous in this framework, so calculations are reduced to simple algebraic manipulations. This makes it very easy to rigorously prove higher order pull-through formulas. Using these pull-through formulas, we prove that ground states are in the domain of the number operator raised to any positive power (if infrared regularity is assumed).

Lastly, we follow the general strategy outlined in [7] to prove the existence of ground states in massless (but infrared regular) models. Our proofs are simpler than the ones presented in [7] and we are able to work under weaker assumptions on the bosonic dispersion relation. This is possible due to a novel approach to the last step in [7].

2. Notation and definitions

We start by fixing notation. If X is a topological space we will write $\mathcal{B}(X)$ for the Borel σ-algebra. Furthermore if $(\mathcal{M},\mathcal{F},\mu)$ is a measure space and X is a Banach space we will for $1\leq p\leq\infty$ write $L^p(\mathcal{M},\mathcal{F},\mu,X)$ for the vector valued L^p space. If $X=\mathbb{C}$ we will drop X from the notation. Also we will write $B(X)$ for the bounded linear operators from X to X.

Let $\mathcal{H}$ be the state space of a single boson which we will assume to be a separable Hilbert space. Write $\mathcal{H}^{\otimes n}$ for the n-fold tensor product of $\mathcal{H}$ and let $\mathcal{H}^{\otimes_s n}\subset\mathcal{H}^{\otimes n}$ be the subspace of symmetric tensors. The bosonic (or symmetric) Fock space is defined as

$$\mathcal{F}_b(\mathcal{H})=\bigoplus_{n=0}^{\infty}\mathcal{H}^{\otimes_s n}.$$

If $\mathcal{H}=L^2(\mathcal{M},\mathcal{F},\mu)$ where $(\mathcal{M},\mathcal{F},\mu)$ is σ-finite, then $\mathcal{H}^{\otimes_s n}=L^2_{sym}(\mathcal{M}^n,\mathcal{F}^{\otimes n},\mu^{\otimes n})$. We will write an element $\psi\in\mathcal{F}_b(\mathcal{H})$ in terms of its coordinates as $\psi=(\psi^{(n)})$ and define the vacuum $\Omega=(1,0,0,\dots)$. The finite particle vectors are defined by

$$\mathcal{N}=\{(\psi^{(n)})\in\mathcal{F}_b(\mathcal{H})\mid\ \exists K\in\mathbb{N}\text{ s.t. }\psi^{(n)}=0\text{ for all }n\geq K\}.$$

For $g\in\mathcal{H}$ one defines the annihilation operator $a(g)$ and creation operator $a^\dagger(g)$ on symmetric tensors in $\mathcal{F}_b(\mathcal{H})$ using $a(g)\Omega=0, a^\dagger(g)\Omega=g$ and

$$a(g)(f_1\otimes_s\cdots\otimes_s f_n)=\frac{1}{\sqrt{n}}\sum_{i=1}^n\langle g,f_i\rangle f_1\otimes_s\cdots\otimes_s\widehat{f_i}\otimes_s\cdots\otimes_s f_n$$
$$a^\dagger(g)(f_1\otimes_s\cdots\otimes_s f_n)=\sqrt{n+1}g\otimes_s f_1\otimes_s\cdots\otimes_s f_n$$

where $\widehat{f_i}$ means that f_i is omitted from the tensor product. One can show that these operators extends to closed operators on $\mathcal{F}_b(\mathcal{H})$ and that $(a(g))^*=a^\dagger(g)$. Furthermore, we have the canonical commutation relations which are:

$$\overline{[a(f),a(g)]}=0=\overline{[a^\dagger(f),a^\dagger(g)]}\text{ and }\overline{[a(f),a^\dagger(g)]}=\langle f,g\rangle.$$

We also define the field operators

$$\varphi(g) = \overline{a(g) + a^\dagger(g)}.$$

They are selfadjont and

$$\overline{[\varphi(f), \varphi(g)]} = 2i\mathrm{Im}(\langle f, g\rangle). \tag{2.1}$$

Let A be a selfadjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(A)$. Then we define the second quantisation of A to be the selfadjoint operator

$$d\Gamma(A) = 0 \oplus \bigoplus_{n=1}^{\infty} \overline{\sum_{k=1}^{n} (1\otimes)^{k-1} A (\otimes 1)^{n-k} \mid_{\mathcal{H}^{\otimes_s n}}}\ . \tag{2.2}$$

If ω is a multiplication operator then $d\Gamma(\omega)$ acts on elements in $\mathcal{H}^{\otimes_s n}$ as multiplication by $\omega_n(k_1, \dots, k_n) = \omega(k_1) + \cdots + \omega(k_n)$. The number operator is defined as $N = d\Gamma(1)$. If $\mathcal{K}$ is an other Hilbert space and $U : \mathcal{H} \to \mathcal{K}$ is a bounded operator with $\|U\| \leq 1$ then we define

$$\Gamma(U) = 1 \oplus \bigoplus_{n=1}^{\infty} U^{\otimes n} \mid_{\mathcal{H}^{\otimes_s n}}\ .$$

Note that $\Gamma(U)$ will be unitary if U is unitary. We will write $d\Gamma^{(n)}(A) = d\Gamma(A) \mid_{\mathcal{H}^{\otimes_s n}}$ and $\Gamma^{(n)}(U) = \Gamma(U) \mid_{\mathcal{H}^{\otimes_s n}}$ throughout the text. If $v \in \mathcal{D}(A)$ one has the commutation relation

$$\overline{[d\Gamma(A), \varphi(v)]} = -i\varphi(iAv) \tag{2.3}$$

where $\mathcal{N} \cap \mathcal{D}(d\Gamma(A)) \subset \mathcal{D}([d\Gamma(A), \varphi(v)])$. We now introduce the Weyl representation. For any $g \in \mathcal{H}$ we define the corresponding exponential vector

$$\epsilon(g) = \sum_{n=0}^{\infty} \frac{g^{\otimes n}}{\sqrt{n!}}. \tag{2.4}$$

One may prove that if $\mathcal{D} \subset \mathcal{H}$ is a dense subspace then $\{\epsilon(f) \mid f \in \mathcal{D}\}$ is a linearly independent and total subset of $\mathcal{F}_b(\mathcal{H})$. Let $\mathcal{U}(\mathcal{H})$ be the unitaries from $\mathcal{H}$ into $\mathcal{H}$. Fix $U \in \mathcal{U}(\mathcal{H})$ and $h \in \mathcal{H}$. Then there is a unique unitary map $W(h, U)$ such that

$$W(h, U)\epsilon(g) = e^{-\|h\|^2/2 - \langle f, Ug\rangle}\epsilon(h + Ug). \quad \forall g \in \mathcal{H}$$

One may easily check that $(h, U) \mapsto W(h, U)$ is strongly continuous. Furthermore one may check the relation

$$W(h_1, U_1)W(h_2, U_2) = e^{-i\mathrm{Im}(\langle f, Ug\rangle)}W((h_1, U_1)(h_2, U_2)),$$

where $(h_1, U_1)(h_2, U_2) = (h_1 + U_1 h_2, U_1 U_2)$. If A is selfadjoint and $f \in \mathcal{H}$ we have

$$\begin{aligned} e^{itd\Gamma(A)} &= \Gamma(e^{itA}) = W(0, e^{itA}) \\ e^{it\varphi(if)} &= W(tf, 1). \end{aligned}$$

The following lemma is important and well known (see e.g [4] or [12]):

Lemma 2.1. *Assume $\omega \geq 0$ is selfadjoint an injective on $\mathcal{H}$ and let $g_1, g_2, \dots, g_n \in \mathcal{D}(\omega^{-\frac{1}{2}})$. Then $\varphi(g_1)\cdots\varphi(g_n)$ is $d\Gamma(\omega)^{\frac{n}{2}}$ bounded. In particular $\varphi(g_1)\cdots\varphi(g_n)$ is $N^{n/2}$ bounded so $\mathcal{N} \subset \mathcal{D}(\varphi(g_1)\cdots\varphi(g_n))$. We have the following bounds*

$$\begin{aligned} \|\varphi(g_1)\psi\| &\leq 2\|(\omega^{-\frac{1}{2}} + 1)g_1\|\|(d\Gamma(\omega) + 1)^{\frac{1}{2}}\psi\| \\ \|\varphi(g_1)\varphi(g_2)\psi\| &\leq 15\|(\omega^{-\frac{1}{2}} + 1)g_1\|\|(\omega^{-\frac{1}{2}} + 1)g_2\|\|(d\Gamma(\omega) + 1)\psi\| \end{aligned}$$

which holds on respectively $\mathcal{D}(d\Gamma(\omega)^{\frac{1}{2}})$ and $\mathcal{D}(d\Gamma(\omega))$. In particular $\varphi(g_1)$ is infinitesimally $d\Gamma(\omega)$ bounded. Furthermore, $d\Gamma(\omega) + \varphi(g_1) \geq -\|\omega^{-\frac{1}{2}}g_1\|^2$.

Lemma 2.2. *Let $U : \mathcal{H} \to \mathcal{K}$ be unitary, A be selfadjoint on $\mathcal{H}$, $V \in \mathcal{U}(\mathcal{H})$ and $f \in \mathcal{H}$. Then*

$$\begin{aligned}\Gamma(U)d\Gamma(A)\Gamma(U)^* &= d\Gamma(UAU^*).\\ \Gamma(U)W(f,V)\Gamma(U)^* &= W(Uf,UVU^*).\\ \Gamma(U)\varphi(f)\Gamma(U)^* &= \varphi(Uf).\\ \Gamma(U)a(f)\Gamma(U)^* &= a(Uf).\\ \Gamma(U)a^\dagger(f)\Gamma(U)^* &= a^\dagger(Uf).\end{aligned}$$

Furthermore, $\Gamma(U)(f_1 \otimes_s \cdots \otimes_s f_n) = Uf_1 \otimes_s \cdots \otimes_s Uf_n$ *and* $\Gamma(U)\Omega = \Omega$.

3. The spin-boson model

Let $\sigma_x, \sigma_y, \sigma_z$ denote the Pauli matrices and define $e_1 = (1,0)$ and $e_{-1} = (0,1)$. Note that e_j is an eigenvector for σ_z with eigenvalue j. We consider a qubit coupled to a radiation field. The state space for the qubit is $\mathbb{C}^2$ and the energy of the qubit can be represented by $\eta\sigma_z$. Let $\mathcal{H}$ be the state space for a single boson and ω be the energy operator for a single boson. Then the state space for the field is $\mathcal{F}_b(\mathcal{H})$ and the energy operator of the field is $d\Gamma(\omega)$. This leads to the state space $\mathbb{C}^2 \otimes \mathcal{F}_b(\mathcal{H})$ for the total system and we have the Hamiltonian

$$H_\eta(\alpha, f, \omega) := \eta\sigma_z \otimes 1 + 1 \otimes d\Gamma(\omega) + \sum_{i=1}^{2n} \alpha_i(\sigma_x \otimes \varphi(f_i))^i,$$

which is here parametrised by $\alpha \in \mathbb{C}^{2n}, f \in \mathcal{H}^{2n}, \eta \in \mathbb{C}$ and ω selfadjoint on $\mathcal{H}$. We will also need the fiber operators:

$$F_\eta(\alpha, f, \omega) = \eta\Gamma(-1) + d\Gamma(\omega) + \sum_{i=1}^{2n} \alpha_i \varphi(f_i)^i.$$

If the spectra are real we define

$$\begin{aligned}E_\eta(\alpha, f, \omega) &:= \inf(\sigma(H_\eta(\alpha, f, \omega)))\\ \mathcal{E}_\eta(\alpha, f, \omega) &:= \inf(\sigma(F_\eta(\alpha, f, \omega))).\end{aligned}$$

For an element $f \in \mathcal{H}^{2n}$ we define the leading terms

$$\mathcal{L}(f) = \{i \in \{2,3,\dots,2n\} \mid f_i \neq f_j \ \forall j > i\}.$$

The expression $\mathcal{L}(f)^c$ is to be interpreted as the complement within $\{1,2,\dots,2n\}$. For ω selfadjoint on $\mathcal{H}$ we define the numbers

$$m(\omega) = \inf\{\sigma(\omega)\} \quad \text{and} \quad m_{\text{ess}}(\omega) = \inf\{\sigma_{\text{ess}}(\omega)\}.$$

The basic set of assumptions are:

Hypothesis 1. *$\alpha \in \mathbb{C}^{2n}, f \in \mathcal{H}^{2n}$ and ω selfadjoint on $\mathcal{H}$ fulfil Hypothesis 1 if*

1. *$\mathcal{L}(f)$ consists only of even numbers, $\alpha_i > 0$ for all $i \in \mathcal{L}(f)\backslash\{2\}$ and $\alpha_2 \geq 0$ if $2 \in \mathcal{L}(f)$.*
2. *ω is injective and nonnegative.*
3. *$f_i \in \mathcal{D}(\omega^{-\frac{1}{2}}) \cap \mathcal{D}(\omega^{\frac{1}{2}})$ for all $i \in \{2,\dots,2n\}$ and $f_1 \in \mathcal{D}(\omega^{-\frac{1}{2}})$.*

Hypothesis 2. *$f \in \mathcal{H}^{2n}$ and ω selfadjoint on $\mathcal{H}$ fulfil Hypothesis 2 if $\langle f_i, g(\omega) f_j \rangle \in \mathbb{R}$ for all $i,j \in \{1,\dots,2n\}$ and $g : \mathbb{R} \to \mathbb{R}$ measurable and bounded on $\sigma(\omega)$.*

Hypothesis 3. *$f \in \mathcal{H}^{2n}$ and ω selfadjoint on $\mathcal{H}$ fulfil Hypothesis 3 if either $n \leq 2$ or $m(\omega) > 0$ and Hypothesis 2 holds.*

Hypothesis 3 ensures that we may use hypercontrative bounds if $n > 2$.

Hypothesis 4. *$f \in \mathcal{H}^{2n}$ and ω selfadjoint on $\mathcal{H}$ fulfil Hypothesis 4 if*

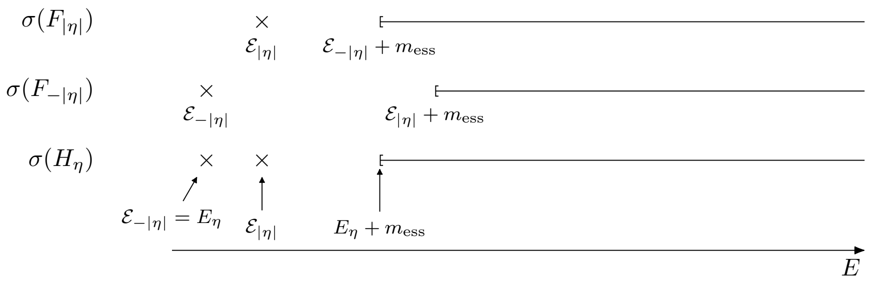

FIGURE 1. The picture established by Theorems 3.3 and 3.4 in the case $0 < 2|\eta| < m_{\text{ess}}$, $m = m_{\text{ess}}$ and $[m_{\text{ess}}, 3m_{\text{ess}}] \subset \sigma_{\text{ess}}(\omega)$.

(1) $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ *where* $(\mathcal{M}, \mathcal{F}, \mu)$ *satisfies the assumptions in Theorems A.5 and A.8.*
(2) ω *is a multiplication operator on* $\mathcal{H}$.
(3) *There is a measurable function* $h : \mathcal{M} \to \mathbb{C}$ *with* $|h| = 1$ *such that* hf *is* $\mathbb{R}^{2n}$ *valued almost everywhere. A function* h *with these properties is called a* ***phase function*** *for* f.

Hypothesis 5. *We say* $f \in \mathcal{H}^{2n}$ *and* ω *selfadjoint on* $\mathcal{H}$ *fulfil Hypothesis 5 if* $f_i \in \mathcal{D}(\omega^{-1})$ *for all* i.

This now brings us to our results.

Proposition 3.1. *Assume* $\eta \in \mathbb{C}$ *and* (α, f, ω) *satisfies Hypotheses 1 and 3. Then the operators* $F_\eta(\alpha, f, \omega)$ *and* $H_\eta(\alpha, f, \omega)$ *are closed on the respective domains*

$$\mathcal{D}(F_\eta(\alpha, f, \omega)) = \mathcal{D}(d\Gamma(\omega)) \cap_{i \in \mathcal{L}(f) \setminus \{2\}} \mathcal{D}(\varphi(f_i)^i)$$
$$\mathcal{D}(H_\eta(\alpha, f, \omega)) = \mathcal{D}(1 \otimes d\Gamma(\omega)) \cap_{i \in \mathcal{L}(f) \setminus \{2\}} \mathcal{D}(1 \otimes \varphi(f_i)^i).$$

Given any core $\mathcal{D}$ *of* ω *the linear span of the following sets*

$$\mathcal{J}(\mathcal{D}) := \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{g_1 \otimes_s \cdots \otimes_s g_n \mid g_j \in \mathcal{D}\}$$
$$\widetilde{\mathcal{J}}(\mathcal{D}) := \{v_1 \otimes v_2 \mid v_1 \in \{e_1, e_{-1}\}, v_2 \in \mathcal{J}(\mathcal{D})\}$$

are cores for $F_\eta(\alpha, f, \omega)$ *and* $H_\eta(\alpha, f, \omega)$ *respectively. Also both operators are selfadjoint and semibounded if* $(\alpha, \eta) \in \mathbb{R}^{2n+1}$ *and they have compact resolvents if* ω *has compact resolvents.*

Proposition 3.2. *Let* $\phi = (\phi_1, \phi_{-1}) = e_1 \otimes \phi_1 + e_{-1} \otimes \phi_{-1}$ *be an element in* $\mathcal{F}_b(\mathcal{H})^2 = \mathcal{F}_b(\mathcal{H}) \oplus \mathcal{F}_b(\mathcal{H}) \approx \mathbb{C}^2 \otimes \mathcal{F}_b(\mathcal{H})$. *Write* $\phi_j = (\phi_j^{(k)})$ *for* $j \in \{-1, 1\}$. *Let* $j \in \{-1, 1\}$. *Define* $\widetilde{\phi}_j = (\widetilde{\phi}_j^{(k)})$ *where*

$$\widetilde{\phi}_j^{(k)} = \begin{cases} \phi_j^{(k)} & k \text{ is even} \\ \phi_{-j}^{(k)} & k \text{ is odd} \end{cases}$$

Then $\widetilde{\phi}_j \in \mathcal{F}_b(\mathcal{H})$ *and the map* $U : \phi \mapsto (\widetilde{\phi}_1, \widetilde{\phi}_{-1})$ *is* U *is selfadjoint and unitary. Furthermore*

$$UH_\eta(\alpha, f, \omega)U^* = F_{-\eta}(\alpha, f, \omega) \oplus F_\eta(\alpha, f, \omega).$$

In the remaining part of this section we will be suppress α, f and ω from the notation as they are fixed in the initial part of each Theorem. The first result we present is about the location of the essential spectrum.

Theorem 3.3. *Let $\alpha \in \mathbb{R}^{2n}, \eta \in \mathbb{R}, f \in \mathcal{H}^{2n}$ and ω be selfadjoint on $\mathcal{H}$ and assume they satisfy Hypothesis 1,3 and either $n \leq 2$ or Hypothesis 4. Then the following holds*

$$\inf\{\sigma_{\text{ess}}(F_\eta)\} \geq \min\{\mathcal{E}_{-\eta} + m_{\text{ess}}, \mathcal{E}_\eta + m + m_{\text{ess}}\}$$

$$\bigcup_{q=1}^{\infty}\{\mathcal{E}_{(-1)^q\eta} + \lambda_1 + \cdots + \lambda_q \mid \lambda_i \in \sigma_{\text{ess}}(\omega)\} \subset \sigma_{\text{ess}}(F_\eta)$$

$$\inf(\sigma_{\text{ess}}(H_\eta)) = E_\eta + m_{\text{ess}}$$

$$\bigcup_{q=1}^{\infty}\{E_\eta + \lambda_1 + \cdots + \lambda_q \mid \lambda_i \in \sigma_{\text{ess}}(\omega)\} \subset \sigma_{\text{ess}}(H_\eta).$$

In particular, H_η has a ground state of finite multiplicity if $m_{\text{ess}} > 0$. We also have:

(1) *Assume $m = m_{\text{ess}}$, $[m_{\text{ess}}, 3m_{\text{ess}}] \subset \sigma_{\text{ess}}(\omega)$ and if $m_{\text{ess}} = 0$ then m_{ess} is not isolated in $\sigma_{\text{ess}}(\omega)$. Then $\sigma_{\text{ess}}(F_\eta) = [\mathcal{E}_{-\eta} + m_{\text{ess}}, \infty)$.*
(2) *Assume $[m_{\text{ess}}, 2m_{\text{ess}}] \subset \sigma_{\text{ess}}(\omega)$ and if $m_{\text{ess}} = 0$ then m_{ess} is not isolated in $\sigma_{\text{ess}}(\omega)$. Then $\sigma_{\text{ess}}(H_\eta) = [E_\eta + m_{\text{ess}}, \infty)$.*
(3) *If we assume Hypothesis 1,2,3 and either $n \leq 2$ or Hypothesis 4, then $\mathcal{E}_{-|\eta|} \leq \mathcal{E}_{|\eta|}$ with equality if and only if $\eta = 0$ or $m = 0$. In particular we find $\inf(\sigma_{\text{ess}}(F_{|\eta|})) = \mathcal{E}_{-|\eta|} + m_{\text{ess}}$.*

In the following result we single out which fiber operator is associated with the ground state and which fiber operator is associated with exited states.

Theorem 3.4. *Let $\alpha \in \mathbb{R}^{2n}, \eta \in \mathbb{R}, f \in \mathcal{H}^{2n}$ and ω be selfadjoint on $\mathcal{H}$ and assume they satisfy Hypothesis 1,2 and 3. Let U be the map from Proposition 3.2.*

(1) *If $\eta \neq 0$ and E_η is an eigenvalue for H_η then E_η is non degenerate. If ψ is a ground state for H_η then $U\psi = e_{-\text{sign}(\eta)} \otimes \phi$ where ϕ is an eigenvector for $F_{-|\eta|}$ corresponding to the energy E_η.*
(2) *If $\mathcal{E}_{-|\eta|}$ is an eigenvalue for $F_{-|\eta|}$ then $\mathcal{E}_{-|\eta|}$ is non degenerate. In particular, if E_0 is an eigenvalue for H_0 then E_0 will have multiplicity two. Furthermore, if ψ is a ground state for H_0 then $U\psi = e_1 \otimes \phi_1 + e_{-1} \otimes \phi_{-1}$ where ϕ_i is either 0 or an eigenvector for F_0 corresponding to the energy $E_0 = \mathcal{E}_0$.*
(3) *If we further assume Hypothesis 4 when $n > 2$ then $\mathcal{E}_{-|\eta|} = E_\eta$. Hence H_η has a ground state if and only if $F_{-|\eta|}$ has a ground state. Also, if $m = 0$ then $F_{|\eta|}$ has no ground state for $\eta \neq 0$.*
(4) *If we further assume Hypothesis 4 when $n > 2$ and $m, \eta \neq 0$ then H_η will have an exited state in $(E_\eta, E_\eta + m_{\text{ess}}]$ if $F_{|\eta|}$ has a ground state. This is the case if $2|\eta| < m_{\text{ess}}$.*

Assuming weak infrared regularity one can prove the following theorem. Note that the assumptions imposed on ω are much weaker than in e.g. [7].

Theorem 3.5. *Let $\alpha \in \mathbb{R}^{2n}, \eta \in \mathbb{R}, f \in \mathcal{H}^{2n}$ and ω be selfadjoint on $\mathcal{H}$ and assume they satisfy Hypothesis 1,2,3,5 and either $n \leq 2$ or Hypothesis 4.*

(1) *If $F_{-|\eta|}$ has a ground state ψ and H_η has a ground state ϕ then $\psi \in \mathcal{D}(N^a)$ and $\phi \in \mathcal{D}(1 \otimes N^a)$ for any $a > 0$.*
(2) *Assume $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu})$, ω is a multiplication operator and $n \leq 2$. Then E_η is an eigenvalue for $F_{-|\eta|}$ and H_η. Here $\lambda^{\otimes\nu}$ is the Lebesgue measure on $\mathcal{B}(\mathbb{R}^\nu)$.*

4. Important estimates

In this section we prove series of estimates which will become useful later. We start with the following lemma

Lemma 4.1. *Fix* $\alpha \in \mathbb{R}^{2n}$ *and define*

$$\mathcal{K} = \{f \in \mathcal{H}^{2n} \mid (\alpha, f) \text{ satisfies part (1) of Hypothesis 1 }\}.$$

There is a constant $C := C(\alpha)$*, such that for any collection* $\{A(v)\}_{v\in\mathcal{H}}$ *of selfadjoint operators and* $f \in \mathcal{K}$ *we have*

$$\sum_{j=2}^{2n} \alpha_j A(f_j)^j \geq C. \tag{4.1}$$

Proof. Let $K = \{i \in \{2, 4, \ldots, 2n\} \mid \alpha_i > 0\} = \{i_1, \ldots, i_k\}$. For each $b \leq k$ we consider polynomials of the form

$$\alpha_{i_b} X^{i_b} + \sum_{j=2}^{i_b-1} \widetilde{\alpha}_j X^j,$$

where $\widetilde{\alpha}_j$ is either 0 or α_j. Since there are only finitely many choices of b and $\widetilde{\alpha}_j$ we find a uniform lower bound $C_0 < 0$ of all these polynomials. Using the spectral calculus we find

$$\alpha_{i_b} A^{i_b} + \sum_{j=2}^{i_b-1} \widetilde{\alpha}_j A^j \geq C_0, \tag{4.2}$$

for all A selfadjoint on $\mathcal{F}_b(\mathcal{H})$, $b \in \{1, \ldots, k\}$ and choices of $\widetilde{\alpha}_j$ as either 0 or α_j. Since the sum of operators in equation (4.1) is a sum of at most n operators of the form in equation (4.2) we find nC_0 is a uniform lower bound. □

In the remaining part of the section we fix ω to be a selfadjoint, nonnegative and injective operator on $\mathcal{H}$ with domain $\mathcal{D}(\omega)$.

Lemma 4.2. *For any* $\varepsilon > 0$ *and* $r > 0$ *there is* $C := C(r, \varepsilon)$ *such that for all* $v_1, v_2 \in \mathcal{D}(\omega^{-\frac{1}{2}})$ *and* $a, b \geq 0$ *with* $\|(1 + \omega^{-\frac{1}{2}})v_1\| + \|(1 + \omega^{-\frac{1}{2}})v_2\| + a + b < r$ *we have*

$$2\mathrm{Re}(\langle a\varphi(v_1)^4\psi, b\varphi(v_2)^2\psi\rangle) \geq -\varepsilon\|d\Gamma(\omega)\psi\|^2 - C\|\psi\|^2$$

for all $\psi \in \mathcal{N} \cap \mathcal{D}(d\Gamma(\omega))$.

Proof. On elements in $\mathcal{N}$ we may calculate using equation (2.1)

$$\varphi(v_2)\varphi(v_1)^4 = \varphi(v_1)^4\varphi(v_2) + 4(2i\mathrm{Im}(\langle v_2, v_1\rangle))\varphi(v_1)^3.$$

This implies

$$\begin{aligned} 2\mathrm{Re}(\langle a\varphi(v_1)^4\psi, b\varphi(v_2)^2\psi\rangle) &= 2ab\|\varphi(v_1)^2\varphi(v_2)\psi\|^2 \\ &\quad - 16ab\mathrm{Im}(\langle v_2, v_1\rangle)\mathrm{Im}(\langle\varphi(v_1)^3\psi, \varphi(v_2)\psi\rangle). \end{aligned}$$

Now

$$\mathrm{Im}(\langle\varphi(v_1)^3\psi, \varphi(v_2)\psi\rangle) = \frac{1}{2i}\langle[\varphi(v_2), \varphi(v_1)^3]\psi, \psi\rangle = 3\mathrm{Im}(\langle v_2, v_1\rangle)\|\varphi(v_1)\psi\|^2.$$

Hence we find

$$2\mathrm{Re}(a\langle\varphi(v_1)^4\psi, b\varphi(v_2)^2\psi\rangle) \geq -48r^6\|\varphi(v_1)\psi\|^2.$$

Using Cauchy-Schwarz inequality and Lemma 2.1 we find

$$\|\varphi(v_1)\psi\|^2 \leq 4\|(\omega^{-\frac{1}{2}} + 1)v_1\|^2(\langle\psi, d\Gamma(\omega)\psi\rangle + \|\psi\|^2) \leq 4r^2(\langle\psi, d\Gamma(\omega)\psi\rangle + \|\psi\|^2),$$

and so

$$\begin{aligned} 2\mathrm{Re}(\langle a\varphi(v_1)^4\psi, b\varphi(v_2)^2\psi\rangle) &\geq -196r^8\|\psi\|\|d\Gamma(\omega)\psi\| - 196r^8\|\psi\|^2 \\ &\geq -\varepsilon\|d\Gamma(\omega)\psi\|^2 - 196r^8\|\psi\|^2 - \frac{(196r^8)^2}{4\varepsilon}\|\psi\|^2, \end{aligned}$$

which finishes the proof. □

Lemma 4.3. *For any $\varepsilon > 0, r > 0, n \in \mathbb{N}$ there is $C := C(r, \varepsilon, n)$ such that for all $v \in \mathcal{D}(\omega^{\frac{1}{2}})$ and $a \geq 0$ with $\|v\|+\|\omega^{\frac{1}{2}}v\|+a < r$ we have*

$$2\mathrm{Re}(\langle a\varphi(v)^{2n}\psi, d\Gamma(\omega)\psi\rangle) \geq -\varepsilon\|a\varphi(v)^{2n}\psi\|^2-C\|\psi\|^2,$$

for all $\psi \in \mathcal{N} \cap \mathcal{D}(d\Gamma(\omega))$.

Proof. Define $\omega_k = \max\{\omega, k\}$ via the spectral calculus. Using equation (2.3) we find the following operator identity holds on $\mathcal{N}$

$$\varphi(v)^n d\Gamma(\omega_k) = d\Gamma(\omega_k)\varphi(v)^n + i\sum_{j=0}^{n-1}\varphi(v)^{n-j-1}\varphi(i\omega_k v)\varphi(v)^j.$$

This yields

$$\begin{aligned}2\mathrm{Re}(\langle a\varphi(v)^{2n}\psi, d\Gamma(\omega_k)\psi\rangle) = 2a\|d\Gamma(\omega_k)^{\frac{1}{2}}\varphi(v)^n\psi\|^2\\ - 2a\sum_{j=0}^{n-1}\mathrm{Im}(\langle\varphi(v)^n\psi, \varphi(v)^{n-j-1}\varphi(i\omega_k v)\varphi(v)^j\psi\rangle).\end{aligned}$$

Now for each $j \leq n-1$ we have

$$\mathrm{Im}(\langle\varphi(v)^n\psi, \varphi(v)^{n-j-1}\varphi(i\omega_k v)\varphi(v)^j\psi\rangle) = \frac{1}{2i}\langle[\varphi(v)^{n-j-1}\varphi(i\omega_k v)\varphi(v)^j, \varphi(v)^n]\psi, \psi\rangle.$$

Using equation (2.1) we may calculate on $\mathcal{N}$

$$\begin{aligned}[\varphi(v)^{n-j-1}\varphi(i\omega_k v)\varphi(v)^j, \varphi(v)^n] &= \varphi(v)^{n-j-1}[\varphi(i\omega_k v), \varphi(v)^n]\varphi(v)^j\\ &= n2i\mathrm{Im}(-i\langle\omega_k v, v\rangle)\varphi(v)^{2(n-1)}.\end{aligned}$$

Using the above equalities we find

$$\begin{aligned}2\mathrm{Re}(\langle a\varphi(v)^{2n}\psi, d\Gamma(\omega)\psi\rangle) &\geq -2an^2\|\omega_k^{\frac{1}{2}}v\|^2\|\varphi(v)^{n-1}\psi\|^2\\ &= -2a^{1/n}n^2\|\omega_k^{\frac{1}{2}}v\|^2\|(a^{\frac{1}{2n}}\varphi(v))^{n-1}\psi\|^2.\end{aligned}$$

Now for any $\varepsilon' > 0$ there is a constant A depending only on ε', n such that $x^{2(n-1)} \leq \varepsilon' x^{4n} + A$. Pick such A for $\varepsilon' = 2^{-1}n^{-2}r^{-2-1/n}\varepsilon$. Then since $\|\omega_k^{\frac{1}{2}}v\|^2 \leq \|\omega^{\frac{1}{2}}v\|^2 \leq r^2$ for all k we find that

$$2\mathrm{Re}(\langle a\varphi(v)^{2n}\psi, d\Gamma(\omega_k)\psi\rangle) \geq -\varepsilon\|a\varphi(v)^{2n}\psi\|^2-2n^2Ar^{2+1/n}\|\psi\|^2.$$

Taking k to ∞ finishes the proof. ☐

Lemma 4.4. *Let $r > 0, 1 > \varepsilon > 0$ and $n \in \mathbb{N}$. Define*

$$\mathcal{K} = \{(\alpha, v) \in [0,\infty) \times (\mathcal{D}(\omega^{1/2}) \cap \mathcal{D}(\omega^{1/2})) \mid \alpha_j + \|(1+\omega^{-\frac{1}{2}}+\omega^{\frac{1}{2}})v\| < n^{-1}r\}$$

$$\mathcal{A} = \begin{cases}\mathcal{H}^n & n \leq 2\\ \{v \in \mathcal{H}^n \mid \langle v_i, v_j\rangle \in \mathbb{R}\ \forall i,j \in \{1,...,n\}\} & n > 2\end{cases}$$

There is a constant $C := C(\varepsilon, r, n)$ such that for all $(\alpha_1, v_1), \ldots, (\alpha_n, v_n) \in \mathcal{K}$ with $v = (v_1, \ldots, v_n) \in \mathcal{A}$ we have

$$\|d\Gamma(\omega)\psi\|^2+\sum_{j=1}^n\|\alpha_j\varphi(v_j)^{2j}\psi\|^2 \leq \frac{1}{1-\varepsilon}\left\|d\Gamma(\omega)\psi+\sum_{j=1}^n\alpha_j\varphi(v_j)^{2j}\psi\right\|^2+C\|\psi\|^2.$$

for all for $\psi \in \mathcal{N} \cap \mathcal{D}(d\Gamma(\omega))$.

Proof. First we note that

$$\|d\Gamma(\omega)\psi\|^2+\sum_{j=1}^n\|\alpha_j\varphi(v_j)^{2j}\psi\|^2=\left\|d\Gamma(\omega)\psi+\sum_{j=1}^n\alpha_j\varphi(v_j)^{2j}\psi\right\|^2$$
$$-\sum_{j=1}^n 2\mathrm{Re}(\langle\alpha_j\varphi(v_j)^{2j}\psi,d\Gamma(\omega)\psi\rangle)$$
$$-\sum_{j_1=1}^n\sum_{j_2=j_1+1}^n 2\alpha_{j_2}\alpha_{j_1}\mathrm{Re}(\langle\varphi(v_{j_1})^{2j_1}\psi,\varphi(v_{j_2})^{2j_2}\psi\rangle.$$

Let $\widetilde{C}(r,\varepsilon,n)$ be the constant from Lemma 4.3 and define $C_1=\widetilde{C}(r,\varepsilon,1)+\cdots+\widetilde{C}(r,\varepsilon,n)$ which depends only on n,r,ε. Then we find

$$-\sum_{j=1}^n 2\mathrm{Re}(\langle\alpha_j\varphi(v_j)^{2j}\psi,d\Gamma(\omega)\psi\rangle)\leq\sum_{j=1}^n\varepsilon\|\alpha_j\varphi(v_j)^{2j}\psi\|^2+C_1\|\psi\|^2.$$

We now turn to the double sum. If $n\leq 2$ we only have one term which can be estimated using Lemma 4.2. Therefore we find a constant $C_2>0$ such that

$$-\sum_{j_1=1}^n\sum_{j_2=j_1+1}^n 2\alpha_{j_2}\alpha_{j_1}\mathrm{Re}(\langle\varphi(v_{j_1})^{2j_1}\psi,\varphi(v_{j_2})^{2j_2}\psi\rangle)\leq\varepsilon\|d\Gamma(\omega)\psi\|^2+C_2\|\psi\|^2.$$

If $n>2$ then $\varphi(v_j)$ and $\varphi(v_i)$ commute on $\mathcal{N}$ for all i,j and so

$$-\sum_{j_1=1}^n\sum_{j_2=j_1+1}^n 2\alpha_{j_2}\alpha_{j_1}\mathrm{Re}(\langle\varphi(v_{j_1})^{2j_1}\psi,\varphi(v_{j_2})^{2j_2}\psi\rangle)$$
$$=-\sum_{j_1=1}^n\sum_{j_2=j_1+1}^n 2\alpha_{j_2}\alpha_{j_1}\|\varphi(v_{j_1})^{j_1}\varphi(v_{j_2})^{j_2}\psi\|^2\leq 0\leq\varepsilon\|d\Gamma(\omega)\psi\|^2+C_2\|\psi\|^2.$$

Using these inequalities we find the desired result with $C=\frac{C_1+C_2}{1-\varepsilon}$. □

Lemma 4.5. *Let $r>0, 1>\varepsilon>0$ and $n\in\mathbb{N}$. Then there is a constant C such that for all $f\in\mathcal{H}^{2n},\alpha\in\mathbb{C}^{2n},\eta\in\mathbb{C}$ and ω selfadjoint on $\mathcal{H}$ that fulfils Hypothesis 1, 3 and*

$$|\eta|+\|\alpha\|+\|(\omega^{-\frac{1}{2}}+1)f_1\|+\sum_{j=2}^{2n}\|(\omega^{-\frac{1}{2}}+1+\omega^{\frac{1}{2}})f_j\|<r,\qquad\max_{j\in\mathcal{L}(f)\backslash\{2\}}\{\alpha_j^{-1}\}<r,$$

we have

$$\|\eta\Gamma(-1)\psi\|+\sum_{j\in\mathcal{L}(f)^c}\|\alpha_j\varphi(f_j)^j\psi\|\leq\varepsilon\left\|d\Gamma(\omega)\psi+\sum_{j\in\mathcal{L}(f)}\alpha_j\varphi(f_j)^j\psi\right\|+C\|\psi\|$$
$$\|\alpha_i\varphi(f_i)^i\psi\|,\|d\Gamma(\omega)\psi\|\leq\frac{1}{(1-\varepsilon)^2}\|F_\eta(\alpha,f,\omega)\psi\|+C\|\psi\|.$$

for all $\psi\in\mathcal{N}\cap\mathcal{D}(d\Gamma(\omega))$ and $i\in\mathcal{L}(f)$.

Proof. For a fixed ε,r,n pick C_1 such that

$$r^2\sum_{j=1}^{2\ell-1}|x|^{2j}\leq\frac{\varepsilon^2}{16n^2r^2}|x|^{4l}+C_1,$$

for all $\ell\in\{1,\ldots,n\}$. For each $j\in\mathcal{L}(f)^c\backslash\{1\}$ we find $q\in\mathcal{L}(f)\backslash\{2\}$ such that $f_j=f_q$ and $j<q$. Noting that $\alpha_q^{-1}<r\iff r^{-1}\leq\alpha_q$ we find

$$\|\alpha_j\varphi(f_j)^j\|\leq\sqrt{r^2\|\varphi(f_j)^j\|^2}\leq\frac{\varepsilon}{4nr}\|\varphi(f_q)^q\|+\sqrt{C_1}\|\psi\|\leq\frac{\varepsilon}{4n}\|\alpha_q\varphi(f_q)^q\|+\sqrt{C_1}\|\psi\|.$$

We know from Lemma 2.1 that

$$\begin{aligned}\|\alpha_1\varphi(f_1)\psi\| &\leq 2r\|(\omega^{-\frac{1}{2}}+1)f_1\|(\langle\psi, d\Gamma(\omega)\psi\rangle+\|\psi\|^2)^{\frac{1}{2}}\\ &\leq \frac{\varepsilon}{2}\|d\Gamma(\omega)\psi\|+2r^2\|\psi\|+\frac{2r^4}{\varepsilon}\|\psi\|.\end{aligned}$$

Then it is clear that there is a constant C_2 depending only on r,ε and n such that

$$\sum_{j\in\mathcal{L}(f)^c}\|\alpha_j\varphi(f_j)^j\psi\|\leq\frac{\varepsilon}{2}\left(\|d\Gamma(\omega)\psi\|+\sum_{j\in\mathcal{L}(f)}\|\alpha_j\varphi(f_j)^j\psi\|\right)+C_2\|\psi\|.$$

Combining this with Lemma 4.4 (applied with $\varepsilon=\frac{1}{2}$) there is a constant C_3 again depending only on r,ε and n such that

$$\sum_{j\in\mathcal{L}(f)^c}\|\alpha_j\varphi(f_j)^j\psi\|\leq\varepsilon\left\|d\Gamma(\omega)\psi+\sum_{j\in\mathcal{L}(f)}\alpha_j\varphi(f_j)^j\psi\right\|+C_3\|\psi\|. \tag{4.3}$$

This proves the first relation. For the next we note that

$$\left\|d\Gamma(\omega)\psi+\sum_{j\in\mathcal{L}(f)}\alpha_j\varphi(f_j)^j\psi\right\|\leq\|F_\eta(\alpha,f,\omega)\psi\|+\sum_{j\in\mathcal{L}(f)^c}\|\alpha_j\varphi(f_j)^j\psi\|+|\eta|\|\psi\|.$$

Using equation (4.3) we obtain

$$\left\|d\Gamma(\omega)\psi+\sum_{j\in\mathcal{L}(f)}\alpha_j\varphi(f_j)^j\psi\right\|\leq\frac{1}{1-\varepsilon}\|F_e(\alpha,f,\omega)\psi\|+\frac{C_3+r}{1-\varepsilon}\|\psi\|.$$

Combining this and Lemma 4.4 we find a constant C_4 such that for all $q\in\mathcal{L}(f)$

$$\|\alpha_q\varphi(f_q)^q\|,\|d\Gamma(\omega)\psi\|\leq\frac{1}{(1-\varepsilon)^2}\|F_\eta(\alpha,f,\omega)\psi\|+C_4\|\psi\|.$$

This finishes the proof. □

5. Proof of Proposition 3.1 and Proposition 3.2

We start by proving a lemma regarding the map U in Proposition 3.2.

Lemma 5.1. *The map U defined in Proposition 3.2 is unitary with inverse $U^*=U$. Furthermore, for any $v\in\mathcal{H}$ and A selfadjoint on $\mathcal{H}$ we have*

$$U(\sigma_x\otimes\varphi(v))U^*=\varphi(v)\oplus\varphi(v)=1\otimes\varphi(v) \tag{5.1}$$

$$U(1\otimes d\Gamma(A))U^*=d\Gamma(A)\oplus d\Gamma(A)=1\otimes d\Gamma(A) \tag{5.2}$$

$$U(\sigma_z\otimes 1)U^*=(-\Gamma(-1))\oplus\Gamma(-1)=\sigma_z\otimes\Gamma(-1). \tag{5.3}$$

In particular we have for $\alpha\in\mathbb{C}^{2n}, f\in\mathcal{H}^{2n},\eta\in\mathbb{C}$ and ω selfadjoint on $\mathcal{H}$ that

$$UH_\eta(\alpha,f,\omega)U^*=F_{-\eta}(\alpha,f,\omega)\oplus F_\eta(\alpha,f,\omega).$$

Proof. First we note that

$$\begin{aligned}\sum_{k=0}^\infty\|\widetilde{\psi}_1^{(k)}\|^2+\sum_{k=0}^\infty\|\widetilde{\psi}_{-1}^{(k)}\|^2 &= \sum_{k\ even}\|\psi_1^{(k)}\|^2+\sum_{k\ odd}\|\psi_{-1}^{(k)}\|^2+\sum_{k\ even}\|\psi_{-1}^{(k)}\|^2+\sum_{k\ odd}\|\psi_1^{(k)}\|^2\\ &=\|\psi_1\|^2+\|\psi_{-1}\|^2=\|e_1\otimes\psi_1+e_{-1}\otimes\psi_{-1}\|^2\end{aligned}$$

which shows that the $\widetilde{\psi}_i$ are elements in Fock space and U gives rise to an isometric map from $\mathcal{F}_b(\mathcal{H})^2$ to $\mathcal{F}_b(\mathcal{H})^2$. To prove surjectivity we fix $(\psi_1,\psi_{-1})\in\mathcal{F}_b(\mathcal{H})^2$ and write $U^2(\psi_1,\psi_{-1})=U(\widetilde{\psi}_1,\widetilde{\psi}_{-1})=(\phi_1,\phi_{-1})$. Fixing $j\in\{1,-1\}$ we have

$$\phi_j^{(k)}=\begin{cases}\widetilde{\psi}_j^{(k)} & \text{k even}\\ \widetilde{\psi}_{-j}^{(k)} & \text{k odd}\end{cases}=\begin{cases}\psi_j^{(k)} & \text{k even}\\ \psi_j^{(k)} & \text{k odd}\end{cases}=\psi_j^{(k)},$$

and hence U is bijective with inverse $U^{-1} = U$. It is clear from the definition of $\widetilde{\psi}_j$ that the map $(\psi_1, \psi_{-1}) \mapsto \widetilde{\psi}_j$ is linear and hence U is also linear. We have thus proven that U is unitary with $U = U^{-1} = U^*$. It remains to prove equations (5.1),(5.2) and (5.3). Both sides of each equation is a selfadjoint map and the maps on the left hand side of each equation is essentially selfadjoint on the set spanned by $e_j \otimes \Omega$ and $e_j \otimes g_1 \otimes_s \cdots \otimes_s g_k$ with $j \in \{\pm 1\}$ and $g_\ell \in \mathcal{D}(A)$. Hence we just need to show equality on this set. Now

$$\begin{aligned} U^*(e_j \otimes \Omega) &= e_j \otimes \Omega \\ U^*(e_j \otimes (g_1 \otimes_s \cdots \otimes_s g_k)) &= e_{(-1)^k j} \otimes (g_1 \otimes_s \cdots \otimes_s g_k), \end{aligned}$$

which is in the domain of $\sigma_x \otimes \varphi(v), 1 \otimes d\Gamma(A)$ and $\sigma_z \otimes 1$. Using $\sigma_x e_j = e_{-j}$ and $\sigma_z e_j = j e_j$ we find

$$\begin{aligned} \sigma_x \otimes \varphi(v)(e_j \otimes \Omega) &= e_{-j} \otimes v = U^*(1 \otimes \varphi(v))(e_j \otimes \Omega) \\ \sigma_x \otimes \varphi(v)(e_{(-1)^k j} \otimes (g_1 \otimes_s \cdots \otimes_s g_k)) &= e_{(-1)^{k+1} j} \otimes a^\dagger(v) g_1 \otimes_s \cdots \otimes_s g_k \\ &\quad + e_{(-1)^{k-1} j} \otimes a(v) g_1 \otimes_s \cdots \otimes_s g_k \\ &= U^*(1 \otimes \varphi(v))(e_j \otimes (g_1 \otimes_s \cdots \otimes_s g_k)) \\ 1 \otimes d\Gamma(A)(e_j \otimes \Omega) &= 0 = U^*(1 \otimes d\Gamma(A))(e_j \otimes \Omega) \\ 1 \otimes d\Gamma(A)(e_{(-1)^k j} \otimes (g_1 \otimes_s \cdots \otimes_s g_k)) &= e_{(-1)^k j} \otimes d\Gamma(A) g_1 \otimes_s \cdots \otimes_s g_k \\ &= U^*(1 \otimes d\Gamma(A))(e_j \otimes (g_1 \otimes_s \cdots \otimes_s g_k)) \\ \sigma_z \otimes 1(e_j \otimes \Omega) &= j e_j \otimes \Omega = U^*(\sigma_z \otimes \Gamma(-1))(e_j \otimes \Omega) \\ \sigma_z \otimes 1(e_{(-1)^k j} \otimes (g_1 \otimes_s \cdots \otimes_s g_k)) &= (-1)^k j e_{(-1)^k j} \otimes (g_1 \otimes_s \cdots \otimes_s g_k) \\ &= j e_{(-1)^k j} \otimes \Gamma(-1) g_1 \otimes_s \cdots \otimes_s g_k \\ &= U^*(\sigma_z \otimes \Gamma(-1))(e_j \otimes (g_1 \otimes_s \cdots \otimes_s g_k)). \end{aligned}$$

This finishes the proof. □

Now Proposition 3.1 will follow as soon as we prove the statements for $F_\eta(\alpha, f, \omega)$. We start by proving the following lemma

Lemma 5.2. *The conclusions of Proposition 3.1 hold under Hypothesis 1, 3 and the assumption*

$$L := d\Gamma(\omega) + \sum_{j \in \mathcal{L}(f)} \alpha_j \varphi(f_j)^j$$

is essentially selfadjoint on $\mathcal{N} \cap \mathcal{D}(d\Gamma(\omega))$.

Proof. Combining the assumption with Lemma 4.4 we see that L is selfadjoint on

$$\mathcal{C} = \mathcal{D}(d\Gamma(\omega)) \cap_{j \in \mathcal{L}(f) \setminus \{2\}} \mathcal{D}(\varphi(f_j)^j).$$

Now simple perturbation theory along with Lemma 4.5 shows that $F_\eta(\alpha, f, \omega)$ is closed on $\mathcal{C}$ and any core for L is a core for $F_\eta(\alpha, f, \omega)$. If $\alpha \in \mathbb{R}^{2n}$ and $\eta \in \mathbb{R}$, the Kato-Rellich Theorem shows that $F_\eta(\alpha, f, \omega)$ is selfadjoint and bounded below.

We now prove that $\mathcal{J}(\mathcal{D})$ is a core for L. It is enough to approximate elements in $\mathcal{N} \cap \mathcal{D}(\omega)$. Any such element ψ can be approximated by a sequence $\{\psi_j\}_{j=1}^\infty$ from the span of $\mathcal{J}(\mathcal{D})$ with respect to $d\Gamma(\omega)$-norm. Pick c so large $1_{(-\infty,c)}(N)\psi = \psi$, and write $P = 1_{(-\infty,c)}(N)$. Then by the decomposition in equation (2.2) we see that $P\psi_j$ converges to ψ in $\mathcal{D}(d\Gamma(\omega))$-norm and in N^n norm. It follows from Lemma 2.1 that ψ_j converges to ψ in L-norm as desired.

If ω has compact resolvents then so does $d\Gamma(\omega)$ by lemma B.4. That $F_\eta(\alpha,f,\omega)$ has compact resolvents will now follow from the equality

$$(F_\eta(\alpha,f,\omega)+i)^{-1} = (d\Gamma(\omega)+i)^{-1} + (d\Gamma(\omega)+i)^{-1}(F_\eta(\alpha,f,\omega)-d\Gamma(\omega))(F_\eta(\alpha,f,\omega)+i)^{-1}.$$

This finishes the proof. □

Proof of Proposition 3.1. It remains to prove that

$$L := d\Gamma(\omega) + \sum_{j\in\mathcal{L}(f)} \alpha_j \varphi(f_j)^j$$

is essentially selfadjoint on $\mathcal{N}\cap\mathcal{D}(d\Gamma(\omega))$ under Hypothesis 1 and 3. The case $n\leq 2$ is simply done by appealing to [1]. If $n>2$ one appeals to the theory of hypercontractive semigroups (See Lemma E.1, Theorem E.2 and [17, Theorem X.58]) and obtains L is essentially selfadjoint on $\cap_{n\in\mathbb{N}}\mathcal{D}(d\Gamma(\omega)^n)$.

Using Lemma 2.1 we see that L is $d\Gamma(\omega)^n$ bounded. Recall that a vector $g\in\mathcal{H}$ is said to be bounded for ω if $g\in\cap_{k\in\mathbb{N}}\mathcal{D}(\omega^k)$ and there is $C>0$ such that $\|\omega^k g\|\leq C^k\|g\|$ for all $k\in\mathbb{N}$. The set of vectors which are bounded for ω is dense in $\mathcal{H}$ since

$$g=\lim_{\ell\to\infty} 1_{[-\ell,\ell]}(\omega)g$$

for any $g\in\mathcal{H}$. Let $g_1,\dots,g_q$ be bounded for ω. Then we have $g_1\otimes_s\cdots\otimes_s g_q\in\cap_{k\in\mathbb{N}}\mathcal{D}(d\Gamma(\omega)^k)$ and

$$\begin{aligned}\|d\Gamma(\omega)^k g_1\otimes_s\cdots\otimes_s g_q\| &= \left\|\sum_{\alpha\in\mathbb{N}_0^q,|\alpha|=k}\binom{k}{\alpha}\omega^{\alpha_1}g_1\otimes_s\cdots\otimes_s\omega^{\alpha_q}g_q\right\|\\ &\leq \sum_{\alpha\in\mathbb{N}_0^q,|\alpha|=k}\binom{k}{\alpha}C_1^{\alpha_1}\cdots C_q^{\alpha_n}\|g_1\|\cdots\|g_q\|\\ &\leq (C_1+\cdots+C_q)^k\|g_1\|\cdots\|g_q\|.\end{aligned}$$

Hence $g_1\otimes_s\cdots\otimes_s g_q$ is an analytic vector for $d\Gamma(\omega)^n$. In particular

$$\{\Omega\}\cup\{g_1\otimes_s\cdots\otimes_s g_q\mid g_i \text{ is bounded for } \omega, q\in\mathbb{N}\}\subset\mathcal{N}\cap\mathcal{D}(d\Gamma(\omega))$$

will span a core for $d\Gamma(\omega)^n$ by Nelsons analytic vector theorem. Since L is $d\Gamma(\omega)^n$ bounded, we find that elements from $\mathcal{N}\cap\mathcal{D}(d\Gamma(\omega))$ can approximate every element in $\mathcal{D}(d\Gamma(\omega)^n)$ with respect to the graph norm of L. Since L is essentially selfadjoint on $\mathcal{D}(d\Gamma(\omega)^n)$ we find $\mathcal{N}\cap\mathcal{D}(d\Gamma(\omega))$ is a core for $\overline{L}$. □

6. Lemmas for the HVZ theorem

In this chapter we discuss some of the technical machinery needed to prove the HVZ theorem.

Lemma 6.1. *Let $f\in\mathcal{H}^{2n},\alpha\in\mathbb{R}^{2n},\eta\in\mathbb{R}$, ω be selfadjoint on $\mathcal{H}$ and assume Hypothesis 1 and 3 are satisfied. If there is a unitary map $V:\mathcal{H}\to\mathcal{H}_1\oplus\mathcal{H}_2$ such that $Vf_i=(\widetilde{f}_i,0)$ for all $i\in\{1,\dots,2n\}$ and $V\omega V^*=\omega_1\oplus\omega_2$. Then $(\alpha,\widetilde{f},\omega_1)$ satisfies Hypothesis 1 and 3. Furthermore there is a unitary map*

$$U:\mathcal{F}_b(\mathcal{H})\to\mathcal{F}_b(\mathcal{H}_1)\oplus\bigoplus_{k=1}^{\infty}\left(\mathcal{F}_b(\mathcal{H}_1)\otimes\mathcal{H}_2^{\otimes_s k}\right)$$

such that

$$UF_\eta(\alpha,f,\omega)U^*=F_\eta(\alpha,\widetilde{f},\omega_1)\oplus\bigoplus_{k=1}^{\infty}\left(F_{(-1)^k\eta}(\alpha,\widetilde{f},\omega_1)\otimes 1+1\otimes d\Gamma^{(k)}(\omega_2)\right).$$

In fact $U = U_2U_1\Gamma(V)$, where U_1 is the unitary map from Theorem C.1 and U_2 is the unitary map from Theorem C.2.

Proof. It is easy to see that Hypothesis 1 and 3 are preserved under the isomorphism. Using Lemma 2.2 one calculates

$$\Gamma(V)F_\eta(\alpha,f,\omega)\Gamma(V)^* = \eta\Gamma(-1\oplus -1) + d\Gamma(\omega_1\oplus\omega_2) + \sum_{i=1}^{2n}\alpha_i\varphi(Vf_i)^i.$$

Let U_1 be the isomorphism from Theorem C.1. Using Theorems B.2 and C.1 we see

$$U_1\Gamma(V)F_\eta(\alpha,f,\omega)\Gamma(V)^*U_1^* = \eta\Gamma(-1)\otimes\Gamma(-1) + F_0(\alpha,\widetilde{f},\omega_1)\otimes 1 + 1\otimes d\Gamma(\omega_2).$$

Let U_2 be the unitary transform from Theorem C.2. Defining $U = U_2U_1\Gamma(V)$ we calculate

$$\begin{aligned} UF_\eta(\alpha,f,\omega)U^* &= \eta U_2\Gamma(-1)\otimes\Gamma(-1)U_2^* + U_2F_0(\alpha,\widetilde{f},\omega_1)\otimes 1U_2^* + U_21\otimes d\Gamma(\omega_2)U_2^* \\ &= \left(\eta\Gamma^{(0)}(-1)\Gamma(-1) + F_0(\alpha,\widetilde{f},\omega_1)\right) \\ &\oplus\bigoplus_{k=1}^{\infty}\left(\eta\Gamma(-1)\otimes\Gamma^{(k)}(-1) + F_0(\alpha,\widetilde{f},\omega_1)\otimes 1 + 1\otimes d\Gamma^{(k)}(\omega_2)\right).\end{aligned}$$

The fact that $\Gamma^{(k)}(-1) = (-1)^k$ finishes the proof. □

Lemma 6.2. *Let $f\in\mathcal{H}^{2n}, \alpha\in\mathbb{R}^{2n}, \eta\in\mathbb{R}$, ω be selfadjoint on $\mathcal{H}$ and assume Hypotheses 1 and 3 are satisfied. Let $\mathcal{H}_1,\mathcal{H}_2\subset\mathcal{H}$ be closed subspaces with $\mathcal{H}_1^\perp = \mathcal{H}_2$ and let P_i denote the orthogonal projection onto $\mathcal{H}_i$. If $f\in\mathcal{H}_1^{2n}$ and ω is reduced by $\mathcal{H}_1$, then we may take $\omega_i = \omega\mid_{\mathcal{H}_i}$ and $Vf = (P_1f, P_2f)$ in Lemma 6.1. Let U be the corresponding map. For $g_1,\dots,g_q\in\mathcal{H}_2$ we define*

$$B = \{\Omega\}\cup\bigcup_{b=1}^{\infty}\{h_1\otimes_s\cdots\otimes_s h_b \mid h_i\in\mathcal{H}_1\cap\mathcal{D}(\omega)\}$$

$$C = \{g_1\otimes_s\cdots\otimes_s g_q\}\cup\bigcup_{b=1}^{\infty}\{h_1\otimes_s\cdots\otimes_s h_b\otimes_s g_1\otimes_s\cdots\otimes_s g_q \mid h_i\in\mathcal{H}_1\cap\mathcal{D}(\omega)\}.$$

If $\psi\in\operatorname{Span}(B)$ then we may interpret ψ as an element in both $\mathcal{F}_b(\mathcal{H})$ and $\mathcal{F}_b(\mathcal{H}_1)$. Using this identification for ψ we find that

$$U^*(\psi\otimes(g_1\otimes_s\cdots\otimes_s g_q))\in\operatorname{Span}(C). \tag{6.1}$$

$$U^*(\psi) = \psi. \tag{6.2}$$

$$\|(F_\eta(\alpha,f,\omega)-\lambda)\psi\| = \|(F_\eta(\alpha,f,\omega_1)-\lambda)\psi\|. \tag{6.3}$$

where $\lambda\in\mathbb{C}$.

Proof. V is clearly unitary and satisfies the properties needed in Lemma 6.1. Let $j_i:\mathcal{H}_i\to\mathcal{H}_1\oplus\mathcal{H}_2$ be the embedding defined by either $j_1(f) = (f,0)$ or $j_2(g) = (0,g)$ and define $Q_i = V^*j_1$. Then Q_i is the inclusion map from $\mathcal{H}_i$ into $\mathcal{H}$. Lemma C.3 immediately yields equation 6.1 and

$$\Gamma(Q) = U^*\mid_{\mathcal{F}_b(\mathcal{H}_1)}.$$

This map acts as the identity on the set spanning B proving equation (6.2). To prove (6.3) we note $\psi = UU^*\psi = U\psi$ and so

$$\|(F_\eta(\alpha,f,\omega)-\lambda)\psi\| = \|U(F_\eta(\alpha,f,\omega)-\lambda)U^*U\psi\| = \|(F_\eta(\alpha,f,\omega_1)-\lambda)\psi\|.$$

This finishes the proof. □

Lemma 6.3. *Let $\{f^k\}_{k=1}^\infty \subset \mathcal{H}^{2n}, \alpha \in \mathbb{R}^{2n}, \eta \in \mathbb{R}$ and ω be selfadjoint on $\mathcal{H}$. Assume (α, f^k, ω) fulfils Hypothesis 1, 3 and $\mathcal{L}(f^k) = \mathcal{L}(f^1)$ for all $k \in \mathbb{N}$. Assume furthermore that*

$$C := \sup_{k\in\mathbb{N}, q\in\{2,\dots,2n\}} \{\|f_q^k\|, \|\omega^{\pm\frac{1}{2}} f_q^k\|, \|\omega^{-\frac{1}{2}} f_1^k\|, \|f_1^k\|\} < \infty.$$

Then for each $\lambda \in \mathbb{R}$ there is $K < \infty$ such that

$$\|\varphi(f_q^k)^j (F_\eta(\alpha, f^k, \omega) + \lambda \pm i)^{-1}\|, \|d\Gamma(\omega)(F_\eta(\alpha, f^k, \omega) + \lambda \pm i)^{-1}\| \leq K$$

for all $k \in \mathbb{N}, q \in \{1, \dots, 2n\}$ and $1 \leq j \leq q$.

Proof. Define

$$r = \max\left\{2n3C + \|\alpha\| + |\eta|, \left(\max_{q\in\mathcal{L}(f^1)\backslash\{2\}} \alpha_q^{-1}\right)\right\} + 1$$

and $\varepsilon = \frac{1}{2}$. Then by Lemma 4.5 we have for all $\psi \in \mathcal{D}(d\Gamma(\omega)) \cap \mathcal{N}$ and $q \in \mathcal{L}(f^1)\backslash\{2\}$ the inequalities

$$\|\alpha_q \varphi(f_q^k)^q \psi\|, \|d\Gamma(\omega)\psi\| \leq 4\|F_\eta(\alpha, f^k, \omega)\psi\| + \widetilde{C}\|\psi\|$$

where $\widetilde{C}$ depends only on n, r, ε and not on k. Now $\mathcal{D}(d\Gamma(\omega)) \cap \mathcal{N}$ is a core for $F_\eta(\alpha, f^k, \omega)$ and so the inequality extends to all $\psi \in \mathcal{D}(F_\eta(\alpha, f^k, \omega))$. Using

$$\|F_\eta(\alpha, f^k, \omega)(F_\eta(\alpha, f^k, \omega) \pm i + \lambda)^{-1}\| \leq 2 + |\lambda|$$

and $\alpha_q^{-1} \leq r$ for all $q \in \mathcal{L}(f^1)\backslash\{2\}$ we obtain the following uniform upper bounds

$$\|\varphi(f_q^k)^q (F_\eta(\alpha, f^k, \omega) \pm i + \lambda)^{-1}\psi\| \leq r(8 + 4|\lambda| + \widetilde{C})\|\psi\| \tag{6.4}$$

$$\|d\Gamma(\omega)(F_\eta(\alpha, f^k, \omega) \pm i + \lambda)^{-1}\psi\| \leq (8 + 4|\lambda| + \widetilde{C})\|\psi\| \tag{6.5}$$

for $q \in \mathcal{L}(f^1)\backslash\{2\}$. Assume now $q \in \{1, \dots, 2n\}$ and $1 \leq j \leq q$. Using Lemma 2.1 we find for $j \leq 2$ and $\psi \in \mathcal{D}(F_\eta(\alpha, f^k, \omega)) \subset \mathcal{D}(d\Gamma(\omega))$ that

$$\|\varphi(f_q^k)^j \psi\| \leq 15(2r)^j \|(d\Gamma(\omega) + 1)^{j/2}\psi\| \leq 60r^2 \|(d\Gamma(\omega) + 1)\psi\|$$

Using (6.5) we find $\|\varphi(f_q^k)^j (F_\eta(\alpha, f^k, \omega) \pm i + \lambda)^{-1}\| \leq 60r^2(8 + 4|\lambda| + \widetilde{C} + 1)$ which is a uniform upper bound. If $j \geq 3$ we may find $p \in \mathcal{L}(f^1)\backslash\{2\}$ such that $f_p = f_q$. For $\psi \in \mathcal{D}(F_\eta(\alpha, f^k, \omega)) \subset \mathcal{D}(\varphi(f_p)^p)$ we have

$$\|\varphi(f_q^k)^j \psi\| \leq \|\varphi(f_p^k)^p \psi\| + \|\psi\|.$$

Using equation (6.4) we find $\|\varphi(f_q^k)^j (F_\eta(\alpha, f^k, \omega) \pm i + \lambda)^{-1}\| \leq r(8 + 4|\lambda| + \widetilde{C}) + 1$ which is a uniform upper bound. ☐

Next is a crucial result regarding convergence of operators.

Lemma 6.4. *Assume $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ where $(\mathcal{M}, \mathcal{F}, \mu)$ is σ-finite, $\alpha \in \mathbb{R}^{2n}$ and $\eta \in \mathbb{R}$. Let $\omega, \omega_1, \omega_2, \dots$ be a collection of multiplication operators on $L^2(\mathcal{M}, \mathcal{F}, \mu)$ and $f, f^1, f^2, \dots$ be a collection of elements from $\mathcal{H}^{2n}$ such that $(\alpha, f, \omega), (\alpha, f^k, \omega_k)$ satisfy Hypothesis 1, 3 and $\mathcal{L}(f) = \mathcal{L}(f^k)$ for all k. Assume that*

$$\lim_{k\to\infty} \frac{\omega_k}{\omega} = 1 = \lim_{k\to\infty} \frac{\omega}{\omega_k}$$

in $L^\infty(\mathcal{M}, \mathcal{F}, \mu)$ and that

$$\lim_{k\to\infty} f_1^k = f_1 \qquad \lim_{k\to\infty} \omega_k^{-\frac{1}{2}} f_1^k = \omega^{-\frac{1}{2}} f_1 \tag{6.6}$$

$$\lim_{k\to\infty} f_j^k = f_j \qquad \lim_{k\to\infty} \omega_k^{\pm\frac{1}{2}} f_j^k = \omega^{\pm\frac{1}{2}} f_1 \tag{6.7}$$

in $\mathcal{H}$ for all $j \geq 2$. Furthermore we assume either $n \leq 2$ or there is a function $h : \mathcal{M} \to S^1 \subset \mathbb{C}$ such that hf and hf_k are almost surely $\mathbb{R}^{2n}$-valued for all k. Then

$F_\eta(\alpha, f^k, \omega_k) - \lambda_k$ converges to $F_\eta(\alpha, f, \omega) - \lambda$ in norm resolvent sense whenever $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{R}$ converges to λ.

Proof. We check convergence at the point i in the resolvent set. For convenience we will sometimes write $\omega = \omega_\infty$ or $f = f^\infty$. Since ω_k/ω and ω/ω_k are essentially bounded functions we see (α, f^k, ω) fulfils Hypothesis 1 and 3. Furthermore the limits in equations (6.6) and (6.7) also exists if we write ω instead of ω_k since the $\omega_k/\omega, \omega/\omega_k$ converges to 1 in $L^\infty(\mathcal{M}, \mathcal{F}, \mu)$. We now prove

$$(F_\eta(\alpha, f_k, \omega_k) + \lambda_n - i)^{-1} - (F_\eta(\alpha, f_k, \omega) + \lambda - i)^{-1} \tag{6.8}$$

converges to 0 since this will reduce the problem to the case $\omega_k = \omega$ and $\lambda_k = \lambda = 0$ for all k. For any $\psi \in \mathcal{F}_b(\mathcal{H})$ and $k, k' \in \mathbb{N} \cup \{\infty\}$ have

$$\begin{aligned}\sum_{\ell=1}^\infty \int_{\mathcal{M}^\ell} &(\omega_k(k_1) + \cdots + \omega_k(k_\ell))^2 |\psi^{(\ell)}(k_1, \dots, k_\ell)|^2 d\mu^{\otimes \ell}(k_1, \dots, k_\ell) \\ &\leq \left\| \frac{\omega_k}{\omega_{k'}} \right\|_\infty^2 \sum_{l=1}^\infty \int_{\mathcal{M}^\ell} (\omega_{k'}(k_1) + \cdots + \omega_{k'}(k_\ell))^2 |\psi^{(\ell)}(k_1, \dots, k_\ell)|^2 d\mu^{\otimes \ell}(k_1, \dots, k_\ell).\end{aligned}$$

so $\mathcal{D}(d\Gamma(\omega_k)) = \mathcal{D}(d\Gamma(\omega))$ for all $k \in \mathbb{N}$. On this set $\|(d\Gamma(\omega_k) - d\Gamma(\omega))\psi\|^2$ is now estimated by

$$\begin{aligned}\sum_{\ell=1}^\infty \int_{\mathcal{M}^\ell} &(\omega_k(k_1) - \omega(k_1) + \cdots + \omega_k(k_\ell) - \omega(k_\ell))^2 |\psi^{(\ell)}(k_1, \dots, k_\ell)|^2 d\mu^{\otimes \ell}(k_1, \dots, k_\ell) \\ &\leq \left\| \frac{\omega_k - \omega}{\omega} \right\|_\infty^2 \sum_{\ell=1}^\infty \int_{\mathcal{M}^\ell} (\omega(k_1) + \cdots + \omega(k_\ell))^2 |\psi^{(l)}(k_1, \dots, k_\ell)|^2 d\mu^{\otimes \ell}(k_1, \dots, k_l).\end{aligned}$$

Hence we find with $C_k = \|\frac{\omega_k - \omega}{\omega}\|_\infty$ that

$$\begin{aligned}\|(F_\eta(\alpha, f_k, \omega_k) + \lambda_n - i)^{-1} &- (F_\eta(\alpha, f_k, \omega) + \lambda - i)^{-1}\| \\ &\leq |\lambda_k - \lambda| + C_k \|d\Gamma(\omega)(F_\eta(\alpha, f_k, \omega) + \lambda - i)^{-1}\|.\end{aligned}$$

Now $\|d\Gamma(\omega)(F_\eta(\alpha, f_k, \omega) + \lambda - i)^{-1}\|$ is uniformly bounded by Lemma 6.3 and C_k converges to 0. Thus the operator in equation (6.8) converges to 0 as desired, and so we have reduced to the case $\omega_k = \omega$ and $\lambda_k = \lambda = 0$ for all k.

In case $n > 2$, we let $\mathcal{H}_\mathbb{R}$ be the real Hilbert space from Lemma E.1, corresponding to the elements f_i^k for $k \in \mathbb{N} \cup \{\infty\}$ and $i \in \{1, \dots, 2n\}$. Let $L^2(X, \mathcal{X}, \mathbb{Q})$ be a Q-space corresponding to $\mathcal{H}_\mathbb{R}$ and V is unitary map from Theorem E.2. Now define

$$I(f^k) = \alpha_1 \varphi(f_1^k) + \sum_{i=2}^{2n} \alpha_i \varphi(f_i^k), \tag{6.9}$$

for all $k \in \mathbb{N} \cup \{\infty\}$. By Theorem E.2 we know that $V e^{-t d\Gamma(\omega)} V^*$ is hypercontractive and the interaction terms $V I(f^k) V^*$ are a multiplication operators on the same Q-space for all k. Convergence in norm resolvent sense now follows if $\eta = 0$ from Theorem E.2 and [17, Theorem X.60]. For $\eta \neq 0$ we apply Lemma E.3.

Assume now $n \leq 2$ and define $I(f^k)$ as in equation (6.9) for all $k \in \mathbb{N} \cup \{\infty\}$ and write $F(f) := F_\eta(\alpha, f, \omega)$ and $F(f^k) := F_\eta(\alpha, f^k, \omega)$. Define

$$\begin{aligned}C_k &= \max_{0 \leq b \leq 1} \{\|\varphi(f - f^k)\varphi(f_k)^b (d\Gamma(\omega) + 1)^{-1}\|, \|\varphi(f - f^k)\varphi(f)^b (d\Gamma(\omega) + 1)^{-1}\|\} \\ D &= \sup_{0 \leq a \leq 3, k \in \mathbb{N} \cup \{\infty\}} \{\|\varphi(f_k)^a (F(f^k) \pm i)^{-1}\|, \|(d\Gamma(\omega) + 1)(F(f^k) \pm i)^{-1}\|\},\end{aligned}$$

where $D<\infty$ follows from Lemma 6.3. On $\mathcal{N}$ we may calculate

$$\begin{aligned} I(f^k)-I(f) &= \alpha_1\varphi(f_1^k-f)+\alpha_2(\varphi(f_2^k)\varphi(f_2^k-f_2)+\varphi(f_2^k-f_2)\varphi(f_2)) \\ &+\alpha_3(\varphi(f_3^k)^2\varphi(f_3^k-f_3)+\varphi(f_3^k)\varphi(f_3^k-f_3)\varphi(f_3)+\varphi(f_3^k-f_3)\varphi(f_3)^2) \\ &+\alpha_4\varphi(f_4^k)^3\varphi(f_4^k-f_4)+\alpha_4\varphi(f_4^k)^2\varphi(f_4^k-f_4)\varphi(f_4) \\ &+\alpha_4\varphi(f_4^k)\varphi(f_4^k-f_4)\varphi(f_4)^2+\alpha_4\varphi(f_4^k-f_4)\varphi(f_4)^3. \end{aligned}$$

Fix $k\in\mathbb{N}$. Let $\psi,\phi\in\mathcal{F}_b(\mathcal{H})$ with $(F(f^k)+i)^{-1}\phi$, $(F(f)-i)^{-1}\psi\in\mathcal{N}$. We note that this set is dense since $\mathcal{N}$ contains a core for $F(f^k)$ and $F(f)$. We calculate

$$\begin{aligned} &\langle\phi,((F(f^k)-i)^{-1}-(F(f)-i)^{-1})\psi\rangle \\ &\qquad=\langle(F(f^k)+i)^{-1}\phi,(I(f)-I(f^k))(F(f)-i)^{-1}\psi\rangle. \end{aligned}$$

This is a sum of 10 terms of the form

$$\begin{aligned} &-\alpha_j\langle\varphi(f_j^k)^a(F(f^k)+i)^{-1}\phi,\varphi(f_j^k-f_j)\varphi(f_j)^b(F(f)-i)^{-1}\psi\rangle \\ &-\alpha_j\langle\varphi(f_j-f_j^k)\varphi(f_j^k)^b(F(f^k)+i)^{-1}\phi,\varphi(f_j)^a(F(f)-i)^{-1}\psi\rangle. \end{aligned}$$

with $0\le a\le 3$ and $0\le b\le 1$. Hence we see that

$$|\langle\phi,((F(f^k)-i)^{-1}-(F(f)-i)^{-1})\psi\rangle|\le 10\max\{|\alpha_1|,|\alpha_2|,|\alpha_3|,|\alpha_4|\}D^2C_k\|\psi\|\|\phi\|.$$

C_k converges to 0 by Lemma 2.1 which finishes the proof. □

Lemma 6.5. *Let $\mathcal{H}=L^2(\mathcal{M},\mathcal{F},\mu)$ where $(\mathcal{M},\mathcal{F},\mu)$ is σ-finite, $\alpha\in\mathbb{R}^{2n},\eta\in\mathbb{R},f\in\mathcal{H}^{2n}$ and $\omega:\mathcal{M}\to\mathbb{R}$ be measurable. Assume that (α,f,ω) satisfies Hypothesis 1,3 and either $n\le 2$ or Hypothesis 4. Let $\{A_n\}_{n=1}^\infty$ be an increasing sequence of sets covering $\mathcal{M}$ up to a zeroset and define $f^k=1_{A_k}f$. Then (α,f^k,ω) satisfies Hypothesis 1, 3 and either $n\le 2$ or Hypothesis 4 for all k. Furthermore $F_\eta(\alpha,f^k,\omega)$ is uniformly bounded from below and converges to $F_\eta(\alpha,f,\omega)$ in norm resolvent sense. In particular, if $\lambda_k\in\sigma_{\mathrm{ess}}(F_\eta(\alpha,f^k,\omega))$ and $\{\lambda_k\}_{k=1}^\infty$ converges to λ, then $\lambda\in\sigma_{\mathrm{ess}}(F_\eta(\alpha,f,\omega))$ and*

$$\lim_{k\to\infty}\mathcal{E}_\eta(\alpha,f^k,\omega)=\mathcal{E}_\eta(\alpha,f,\omega).$$

Proof. (α,f^k,ω) satisfies Hypothesis 1 obviously. In case $n\le 2$ Hypothesis 3 is obtained directly and if $n>2$ the phase function for f will also be a phase function for f^k. Since Hypothesis 4 implies Hypothesis 2 we have proven the first part. Norm resolvent convergence follows directly from Lemma 6.4. Write

$$F_\eta(\alpha,f^k,\omega)=\eta\Gamma(-1)+d\Gamma(\omega)+\alpha_1\varphi(f_1^k)+\sum_{j=2}^{2n}\alpha_j\varphi(f_j^k)^j.$$

Using $\eta\Gamma(-1)\ge-|\eta|$ and $d\Gamma(\omega)+\alpha_1\varphi(f_1^k)\ge-\alpha_1^2\|\omega^{-\frac12}f_1^k\|^2\ge-\alpha_1^2\|\omega^{-\frac12}f_1\|^2$ by Lemma 2.1 we can find a uniform lower bound of the first three terms. The sum is also uniformly bounded from below by Lemma 4.1 since $\mathcal{L}(f^k)=\mathcal{L}(f)$ for all k. The functional calculus now implies the claims regarding convergence of the spectra. □

7. The HVZ theorem

In this section we prove Theorem 3.3 except from part 3. Fix $\eta\in\mathbb{R},\alpha\in\mathbb{R}^{2n},f\in\mathcal{H}^{2n}$, ω selfadjoint on $\mathcal{H}$ and assume they fulfil Hypothesis 1,3 and either $n\le 2$ or Hypothesis 4. We introduce the notation $F_{(-1)^k}:=F_{(-1)^k\eta}(\alpha,f,\omega)$, $\mathcal{E}_{(-1)^k}:=\mathcal{E}_{(-1)^k\eta}(\alpha,f,\omega)$, $m:=m(\omega)$ and $m_{\mathrm{ess}}:=m_{\mathrm{ess}}(\omega)$.

Since spectral properties are conserved under unitary transformations we may (using Lemmas A.10 and 2.2) assume that $\mathcal{H}=L^2(\mathcal{M},\mathcal{F},\mu)$ where $(\mathcal{M},\mathcal{F},\mu)$ satisfies the assumptions in Theorems A.5 and A.8 and ω is a multiplication operator on $\mathcal{H}$ with $\omega > 0$ almost everywhere. In particular $\mathcal{M}$ is a locally compact metric space. In case $n > 2$ and we assume Hypothesis 4 to hold, we may assume that the phase function h is 1 by using the unitary transformation $\Gamma(h)$. Hence we may assume the f_i to be almost everywhere real valued when $n > 2$.

Lemma 7.1. $\{\mathcal{E}_{(-1)^q} + \lambda_1 + \cdots + \lambda_q \mid \lambda_i \in \sigma_{\text{ess}}(\omega)\} \subset \sigma_{\text{ess}}(F_1)$ *for all* $q \in \mathbb{N}$.

Proof. Fix $q \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_q \in \sigma_{\text{ess}}(\omega)$. By Theorem A.5 we may for each $i \in \{1, \dots, q\}$ pick a collection of sets $\{A_k^i\}_{k=1}^{\infty}$ such that $0 < \mu(A_k^i) < \infty$, $|\omega - \lambda_i| \leq \frac{1}{k}$ on A_k^i, $A_k^i \cap A_\ell^j = \emptyset$ for $i \neq j$ or $k \neq \ell$ and

$$\sum_{k=1}^{\infty} \mu(A_k^i) < \infty$$

for all $i \in \{1, \dots, q\}$. Define for each and $k \in \mathbb{N}$ the set

$$B_k = \bigcup_{i=1}^{q} \bigcup_{j=k}^{\infty} A_j^i \ \Rightarrow\ \mu(B_k) = \sum_{i=1}^{q} \sum_{j=k}^{\infty} \mu(A_j^i) < \infty,$$

and note that $\mu(B_k) \downarrow 0$. Since the B_k is a decreasing collection of sets we find

$$B = \bigcap_{k=1}^{\infty} B_k$$

has measure 0. Define now for each $\ell \in \mathbb{N}$ the set

$$\mathcal{H}_\ell = \{f \in \mathcal{H} \mid 1_{B_\ell^c} f = f \ \mu - a.e\} = 1_{B_\ell^c} \mathcal{H}.$$

Assume first that $f \in \mathcal{H}_K^{2n}$ for some K and hence that $f \in \mathcal{H}_\ell^{2n}$ for all $\ell \geq K$. Define the following sets for $\ell \geq K$

$$\mathcal{A}_\ell = \bigcup_{k=K}^{\ell} \mathcal{H}_k \cap \mathcal{D}(\omega) = \mathcal{H}_\ell \cap \mathcal{D}(\omega) \qquad \mathcal{A}_\infty = \bigcup_{k=K}^{\infty} \mathcal{H}_k \cap \mathcal{D}(\omega)$$

We now claim that $\mathcal{A}_\infty$ is a core for ω. If $\phi \in \mathcal{D}(\omega)$ then $\phi_k = \phi 1_{B_k^c} \in \mathcal{A}_\infty$ for all $k \geq K$ and using dominated convergence we find

$$\lim_{k \to \infty} \|\phi - \phi_k\|^2 = 0 = \lim_{k \to \infty} \|\omega(\phi - \phi_k)\|^2$$

so $\mathcal{A}_\infty$ is a core for ω. Defining

$$\mathcal{J}(\mathcal{A}_\infty) = \{\Omega\} \cup \bigcup_{k=1}^{\infty} \{g_1 \otimes_s \cdots \otimes_s g_k \mid g_i \in \mathcal{A}_\infty\}$$

$$\mathcal{J}(\mathcal{A}_\ell) = \{\Omega\} \cup \bigcup_{k=1}^{\infty} \{g_1 \otimes_s \cdots \otimes_s g_k \mid g_i \in \mathcal{A}_\ell\}$$

we find that $\mathcal{J}(\mathcal{A}_\infty)$ spans a core for $F_{\pm 1}$ by Proposition 3.1. Note that any element in $g \in \text{Span}(\mathcal{J}(\mathcal{A}_\infty))$ is of the form

$$g = a\Omega + \sum_{i=1}^{b} \sum_{j=1}^{c} \alpha_{i,j} g_1^j \otimes_s \cdots \otimes_s g_i^j$$

for some $a, b, c, \alpha_{i,j}$ constants and $g_i^j \in \mathcal{A}_\infty$. Note that each g_i^j is in fact contained in some $\mathcal{A}_{\ell(i,j)}$ by definition so defining $u = \max_{i,j}\{\ell(i,j)\}$ we see that $g \in \text{Span}(\mathcal{J}(\mathcal{A}_\ell))$ for any $\ell \geq u$. Hence we have now proven the following statements

- For any $g \in \text{Span}(\mathcal{J}(\mathcal{A}_\infty))$ there is $u \in \mathbb{N}$ with $u \geq K$ such that $g \in \text{Span}(\mathcal{J}(\mathcal{A}_\ell))$ for any $\ell \geq u$.
- $\text{Span}(\mathcal{J}(\mathcal{A}_\infty))$ is a core for $F_{\pm 1}$.

For each $p \in \mathbb{N}$ we pick a $\nu_p \in \mathrm{Span}(\mathcal{J}(\mathcal{A}_\infty))$ such that $\|(F_{(-1)^q} - \mathcal{E}_{(-1)^q})\nu_p\| \leq \frac{1}{p}$ and $\|\nu_p\| = 1$. Pick for each $p \in \mathbb{N}$ an $u(p) \geq K$ such that $\nu_p \in \mathrm{Span}(\mathcal{J}(\mathcal{A}_\ell))$ for any $\ell \geq u(p)$ and $u(p+1) > u(p)$ for all $p \in \mathbb{N}$. For each $p \in \mathbb{N}$ and $i \in \{1, \dots, q\}$ we define

$$g_i^p = \mu(A^i_{u(p)})^{-\frac{1}{2}} 1_{A^i_{u(p)}}$$

and note $g_i^p \in \mathcal{D}(\omega)$ since ω is bounded by $\lambda_i + \frac{1}{u(p)}$ on $A^i_{u(p)}$. Note also that $g_i^p \in \mathcal{H}^\perp_{u(p)}$ since $A^i_{u(p)} \subset B_{u(p)}$ so g_i^p and elements in $\mathcal{H}_{u(p)}$ have disjoint support. Furthermore the collection $\{g_i^p\}_{p \in \mathbb{N}, i \in \{1,\dots,q\}}$ is orthogonal since the elements have disjoint support. Let U_p be the unitary map in Lemma 6.2 corresponding to $\mathcal{H}_{u(p)}$ which exists since $f \in \mathcal{H}_K^{2n} \subset \mathcal{H}^{2n}_{u(p)}$. Define

$$\phi_p = \sqrt{q!} U_p^* (\nu_p \otimes g_1^p \otimes_s \cdots \otimes_s g_q^p).$$

We are done in the case $f \in \mathcal{H}_K^{2n}$ for some K if we can prove that $\{\phi_p\}_{p=1}^\infty$ is a Weyl sequence for F_1 corresponding to the value $\mathcal{E}_{(-1)^q} + \lambda_1 + \cdots + \lambda_q$. We check:

(1) $\phi_p \in \mathcal{D}(F_1)$.
(2) $\|\phi_p\| = 1$ for all $p \in \mathbb{N}$.
(3) ϕ_p is orthogonal to ϕ_r for $p \neq r$.
(4) $\|(F_{(-1)^q} - (\mathcal{E}_{(-1)^q} + \lambda_1 + \cdots + \lambda_q))\phi_p\|$ converges to 0.

(1): Lemma 6.2 shows $\phi_p \in \mathcal{N} \cap \mathcal{D}(d\Gamma(\omega)) \subset \mathcal{D}(F_1)$ for all $p \in \mathbb{N}$.

(2): For each fixed $p \in \mathbb{N}$ we have that the g_i^p are orthogonal. Let $\mathcal{S}_q$ be the permutations of $\{1, \dots, q\}$. Then we find

$$\begin{aligned}
\|g_1^p \otimes_s \cdots \otimes_s g_q^p\|^2 &= \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} \langle g_1^p \otimes \cdots \otimes g_q^p, g^p_{\sigma(1)} \otimes \cdots \otimes g^p_{\sigma(q)} \rangle \\
&= \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} \langle g_1^p, g^p_{\sigma(1)} \rangle \cdots \langle g_q^p, g^p_{\sigma(q)} \rangle \\
&= \frac{1}{q!} \langle g_1^p, g_1^p \rangle \cdots \langle g_q^p, g_q^p \rangle = \frac{1}{q!}.
\end{aligned}$$

(3): Define for all $p \in \mathbb{N}$ the set

$$C_p = \{g_1^p \otimes_s \cdots \otimes_s g_q^p\} \cup \bigcup_{\ell=1}^\infty \{h_1 \otimes_s \cdots \otimes_s h_\ell \otimes_s g_1^p \otimes_s \cdots \otimes_s g_q^p \mid h_i \in \mathcal{H}_{u(p)} \cap \mathcal{D}(\omega)\}$$

and let $r < p$. Then $\phi_r \in \mathrm{Span}(C_r)$ and $\phi_p \in \mathrm{Span}(C_p)$ by Lemma 6.2, so we just need to see that every element in C_p and C_r are orthogonal. Let $\psi_1 \in C_p$ and $\psi_2 \in C_r$. Note every tensor in C_p has a factor g_1^p and that this factor is orthogonal to g_i^r for all $i \in \{1, \dots, q\}$ by construction. Furthermore for any $h \in \mathcal{H}_{u(r)}$ we see that h is supported in $B^c_{u(r)} \subset B^c_{u(p)} \subset (A^1_{u(p)})^c$ and hence $g_1^p h = 0$, so g_1^p is orthogonal to any element in $\mathcal{H}_{u(r)}$. This implies ψ_1 contains a factor orthogonal to all factors in ψ_2. This finishes the proof of (3).

(4): Using Lemma 6.2 we find

$$\begin{aligned}
&\|(F_1 - \mathcal{E}_{(-1)^q} - \lambda_1 - \cdots - \lambda_q)\phi_p\| \\
&\quad = \sqrt{q!}\|U_p(F_1 - \mathcal{E}_{(-1)^q} + \lambda_1 + \cdots + \lambda_q)U_p^* \nu_p \otimes g_1^p \otimes_s \cdots \otimes_s g_q^p\| \\
&\quad \leq \sqrt{q!}\|(F_{(-1)^q}(\alpha, f, \omega_1) - \mathcal{E}_{(-1)^q})\nu_p \otimes g_1^p \otimes_s \cdots \otimes_s g_q^p\| \\
&\quad + \sqrt{q!}\sum_{i=1}^{q}\|\nu_p \otimes g_1^p \otimes_s \cdots \otimes_s (\omega_2 g_i^p - \lambda_i g_i^p) \otimes_s \cdots \otimes_s g_q^p\| \\
&\quad \leq \|(F_{(-1)^q} - \mathcal{E}_{(-1)^q})\nu_p\| + \sqrt{q!}\sum_{i=1}^{q}\|(\omega - \lambda_i)g_i^p\| \\
&\quad \leq \frac{1}{p} + \sqrt{q!}\sum_{i=1}^{q}\frac{1}{u(p)}
\end{aligned}$$

which converges to 0. This finishes the case where $f \in \mathcal{H}_K^{2n}$ for some K. To prove the general case let $f^k = 1_{B_k^c} f$ and note that $\mathcal{E}_{(-1)^q\eta}(\alpha, f^k, \omega) + \lambda_1 + \cdots + \lambda_q \in \sigma_{\mathrm{ess}}(F_\eta(\alpha, f^k, \omega))$ for all k. Applying Lemma 6.5 finishes the proof. □

Lemma 7.2. *Define* $\widetilde{m} = \min\{m_{\mathrm{ess}} + \mathcal{E}_{-1}, \mathcal{E}_1 + m_{\mathrm{ess}} + m\}$. *Then* $(-\infty, \widetilde{m}) \cap \sigma_{\mathrm{ess}}(F_1) = \emptyset$.

Proof. First we note that if $m = 0$ then $m_{\mathrm{ess}} = 0$ by injectivity of ω and so the statement is trivial since $(-\infty, \widetilde{m})$ does not contain any spectral points of F_1. Hence we may assume $m > 0$, so $\omega \geq m > 0$ almost everywhere. If $m_{\mathrm{ess}} = \infty$ the conclusion will follow from Proposition 3.1. Hence we may assume $m_{\mathrm{ess}} < \infty$.

Define $B = \cup_{j \leq 2n}\{f_j \neq 0\}$ and assume that the lemma has been proven under the extra assumption that B is bounded and there is an $R > 0$ such that $R^{-1} < \omega < R$ on B. To prove the general case case we fix $x_0 \in \mathcal{M}$ and define

$$A_k = (B \cap \{k^{-1} < \omega < k\} \cap B_k(x_0)) \cup B^c$$

where $B_k(x_0)$ is the ball centred at x_0 of radius k. Note that the A_k are increasing and cover $\mathcal{M}$ up to a zeroset. Let $f^k = 1_{A_k} f$ and note that Lemma 6.5 implies (α, f^k, ω) satisfies Hypothesis 1,3 and either $n \leq 2$ or Hypothesis 4. Furthermore $\cup_{j \leq 2n}\{f_j^k \neq 0\}$ is now bounded and contained in $\{k^{-1} < \omega < k\}$. Defining $F_{\pm 1,k} := F_{\pm\eta}(\alpha, f^k, \omega)$ and $\mathcal{E}_{\pm 1,k} := \mathcal{E}_{\pm\eta}(\alpha, f^k, \omega)$ we therefore have

$$(-\infty, \widetilde{m}_k) \cap \sigma_{\mathrm{ess}}(F_{1,k}) = \emptyset, \tag{7.1}$$

where $\widetilde{m}_k = \min\{m_{\mathrm{ess}} + \mathcal{E}_{-1,k}, \mathcal{E}_{1,k} + m_{\mathrm{ess}} + m\}$. Note that $\widetilde{m}_k$ converges to $\widetilde{m}$ by Lemma 6.5. Equation (7.1) implies that $h(F_{k,1})$ is compact for all $h \in C_c^\infty((-\infty, \widetilde{m}_k))$. For any $h \in C_c^\infty((-\infty, \widetilde{m}))$ we have $h \in C_c^\infty((-\infty, \widetilde{m}_k))$ for k large enough and Lemma 6.5 implies

$$\lim_{k \to \infty} h(F_{1,k}) = h(F_1)$$

in norm. Therefore $h(F_1)$ is compact finishing the proof.

What remains is the special case where B is bounded and there is an $R > 0$ such that $R^{-1} < \omega < R$ on B. Pick now a sequence of maps $\{\omega_k\}_{k=1}^\infty$ as in Lemma A.1. Then $1_B\omega_k$ is a simple function

$$1_B\omega_k = \sum_{j=1}^{\widetilde{q}_k} \widetilde{\alpha}_{k,j} 1_{\widetilde{B}_j^k},$$

where $2R > \widetilde{\alpha}_{k,j} > (2R)^{-1}$, the $\widetilde{B}_1^k, \dots, \widetilde{B}_{q_k}^k$ are disjoint and

$$B = \bigcup_{j=1}^{\widetilde{q}_k} \widetilde{B}_j^k.$$

From Lemma A.1 one obtains ω/ω_k and ω_k/ω converges 1 in $L^\infty(\mathcal{M}, \mathcal{F}, \mu)$ and defining the masses $m_k := m(\omega_k)$ and $m_{\text{ess},k} := m_{\text{ess}}(\omega_k)$ we have m_k converges to m and $m_{\text{ess},k}$ converges to m_{ess}. We may thus assume $m_k > 0$ for all k. Define $A = B^c$ and use Theorem A.8 to pick disjoint subdivisions

$$A = H_k \cup \bigcup_{\ell=1}^{c_k} A_\ell^k \quad \text{and} \quad B = J_k \cup \bigcup_{j=1}^{q_k} B_j^k$$

which fulfils the criteria in Theorem A.8 and let P_k be the projection

$$P_k(f) = \sum_{j=1}^{q_k} \frac{1}{\mu(B_j^k)} \int_{B_j^n} f(x) d\mu(x) 1_{B_j^k} + \sum_{\ell=1}^{c_k} \frac{1}{\mu(A_\ell^k)} \int_{A_\ell^n} f(x) d\mu(x) 1_{A_l^k}.$$

Now $0 < \mu(B_j^k) < \infty$ by Theorem A.8 so the elements $1_{B_j^k}$ are nonzero, orthogonal elements in $\mathcal{H}$. We define

$$\mathcal{H}_k = \text{Span}\left(\left\{1_{B_j^k} \mid j \in \{1, \dots, q_k\}\right\}\right).$$

Furthermore, Theorem A.8 also gives that for all $k \in \mathbb{N}, j \leq q_k$ there is a j' such that $B_j^k \subset \widetilde{B}_{j'}^k$. Hence ω_k is constant on B_j^k. For each $\beta \in \mathbb{R}$ we thus find

$$\omega_k^\beta = \omega^\beta 1_A + \omega_k^\beta 1_{J_k} + \sum_{j=1}^{q_k} \alpha_{j,k}^\beta 1_{B_j^k},$$

with $2R > \alpha_{j,k} > (2R)^{-1}$. Hence ω_k^β is an operator defined on all of $\mathcal{H}_k$ and it acts on $1_{B_l^k}$ like multiplication with $\alpha_{j,k}^\beta$. In particular the projection Q_k onto $\mathcal{H}_k$ will map $\mathcal{H}$ into $\mathcal{D}(\omega_k^\beta)$ for all $\beta \in \mathbb{R}$. Note that $\{\mu(B_j^k)^{-\frac{1}{2}} 1_{B_j^k} \mid j \in \{1, \dots, q_k\}\}$ is an orthonormal basis for $\mathcal{H}_k$ and hence Q_k is given by

$$Q_k f = \sum_{j=1}^{q_k} \frac{1}{\mu(B_j^k)} \int_{B_j^k} f(x) d\mu 1_{B_j^k}.$$

Hence we find for $f \in \mathcal{D}(\omega_k^\beta)$ that

$$\begin{aligned} Q_k \omega_k^\beta f &= \sum_{j=1}^{q_k} \frac{1}{\mu(B_j^n)} \int_{B_j^k} \omega_k^\beta(x) f(x) d\mu 1_{B_j^n} = \sum_{j=1}^{q_k} \frac{\alpha_{k,j}^\beta}{\mu(B_j^k)} \int_{B_j^k} f(x) d\mu 1_{B_j^k} \\ &= \omega_k^\beta \sum_{j=1}^{k} \frac{1}{\mu(B_j^k)} \int_{B_j^k} f(x) d\mu 1_{B_j^k} = \omega_k^\beta Q_k f, \end{aligned}$$

so ω_k^β is reduced by $\mathcal{H}_k$. Note that $\omega^{-\frac{1}{2}} f_i$ is supported in B. Hence $P_k(\omega^{-\frac{1}{2}} f_i) = Q_k(\omega^{-\frac{1}{2}} f_i) \in \mathcal{H}_k \subset \mathcal{D}(\omega_k^{\frac{1}{2}})$ so we may define

$$f_i^k = \omega_k^{\frac{1}{2}} P_k(\omega^{-\frac{1}{2}} f_i) \in \mathcal{H}_k \subset \mathcal{D}(\omega_k^\beta) \ \forall \beta \in \mathbb{R}. \tag{7.2}$$

P_k converges strongly to 1 by Theorem A.8 which implies $\omega_k^{-\frac{1}{2}} f_i^k = P_k(\omega^{-\frac{1}{2}} f_i)$ converges to $\omega^{-\frac{1}{2}} f_i$ for all $i \in \{1, 2, \dots, 2n\}$. Since the f_i and f_i^k are 0 almost everywhere outside B and $(2R)^{-1} < \omega_k, \omega < 2R$ on B we find that

$$\|f_i^k - f_i\| \leq \left(\int_B |f_i^k - \omega_k^{\frac{1}{2}}\omega^{-\frac{1}{2}} f_i|^2 d\mu\right)^{\frac{1}{2}} + \left(\int_B |\omega_k^{\frac{1}{2}}\omega^{-\frac{1}{2}} f_i - f_i|^2 d\mu\right)^{\frac{1}{2}}$$
$$\leq \sqrt{2R}\|P_k(\omega^{-\frac{1}{2}} f_i) - \omega^{-\frac{1}{2}} f_i\|^2 + \sqrt{2R}\left(\int_B (\omega^{\frac{1}{2}} - \omega_k^{\frac{1}{2}})|f_i|^2 d\mu\right)^{\frac{1}{2}},$$

which converges to 0 by dominated convergence since ω_k converges to ω by Lemma A.1. Hence f_i^k converges to f_i for all $i \in \{1, 2, \ldots, 2n\}$. For $i \geq 2$ we calculate

$$\|\omega_k^{\frac{1}{2}} f_i^k - \omega^{\frac{1}{2}} f_i\| \leq \left(\int_B |\omega_k^{\frac{1}{2}} f_i^k - \omega_k^{\frac{1}{2}} f_i|^2 d\mu\right)^{\frac{1}{2}} + \left(\int_B |(\omega^{\frac{1}{2}} - \omega_k^{\frac{1}{2}}) f_i|^2 d\mu\right)^{\frac{1}{2}}$$
$$\leq 2R\|f_i^k - f_i\| + \left(\int_B |(\omega^{\frac{1}{2}} - \omega_k^{\frac{1}{2}}) f_i|^2 d\mu\right)^{\frac{1}{2}}$$

which converges to 0. Hence $\omega_k^{\frac{1}{2}} f_i^k$ converges to $\omega^{\frac{1}{2}} f_i$ for all $i \in \{2, 3, \ldots, 2n\}$.

Noting that $\mathcal{L}(f^k) = \mathcal{L}(f)$ for all k sufficiently large and the f_i^k are real if $n > 2$ we find that (α, f^k, ω_k) satisfies Hypothesis 1,3 and either $n \leq 2$ or Hypothesis 4. Hence we may now define

$$F_{\pm 1,k} := F_{\pm\eta}(\alpha, f^k, \omega_k).$$

Theorem 6.4 now applies so $F_{\pm 1,k}$ converges to $F_{\pm 1}$ in norm resolvent sense. Let $\mathcal{E}_{\pm 1,k} = \inf(\sigma(F_{\pm 1,k}))$ and define

$$\widetilde{m}_k = \min\{m_{\mathrm{ess},k} + \mathcal{E}_{-1,k}, \mathcal{E}_{-1,k} + m_{\mathrm{ess},k} + m_k\}.$$

Applying the bounds in Lemmas 4.1 and 2.1 along with the bound $\eta\Gamma(-1) \geq -|\eta|$ we see

$$F_{\pm 1,k} \geq -|\eta| - \|\omega_k^{-\frac{1}{2}} f_1^k\|^2 + C,$$

which is uniformly bounded below in k, since $\|\omega_k^{-\frac{1}{2}} f_1^k\|$ converges. This implies that $\mathcal{E}_{\pm 1,k}$ converges to $\mathcal{E}_{\pm 1}$ and so $\widetilde{m}_k$ converges to $\widetilde{m}$.

Assume now we have proven that $h(F_{1,k})$ is compact for any $h \in C_c^\infty((-\infty, \widetilde{m}_k))$. Then for any $h \in C_c^\infty((-\infty, \widetilde{m}))$ we would have $h \in C_c^\infty((-\infty, \widetilde{m}_k))$ for k large enough. This together with norm resolvent converges gives $h(F_1)$ is compact, which would finish the proof.

Let $h \in C_c^\infty((-\infty, \widetilde{m}_k))$. Now $f^k \in \mathcal{H}_k^{2n}$ by construction and ω_k is reduced by $\mathcal{H}_k$. Defining $g_1 = \omega_k \mid_{\mathcal{H}_k}$ and $g_2 = \omega_k \mid_{\mathcal{H}_k^\perp}$ we may apply Lemma 6.1 to obtain a unitary map

$$U_k : \mathcal{F}_b(\mathcal{H}) \to \mathcal{F}_b(\mathcal{H}_k) \oplus \bigoplus_{j=1}^{\infty} \left(\mathcal{F}_b(\mathcal{H}_k) \otimes (\mathcal{H}_k^\perp)^{\otimes_s j}\right),$$

such that

$$U_k F_{\pm 1,k} U_k^* = F_{\pm\eta}(\alpha, f^k, g_1) \oplus \bigoplus_{j=1}^{\infty} \left(F_{\pm(-1)^j\eta}(\alpha, f^k, g_1) \otimes 1 + 1 \otimes d\Gamma^{(j)}(g_2)\right).$$

Thus we find $\mathcal{E}_{\pm\eta}(\alpha, f^k, g_1) \geq \mathcal{E}_{\pm 1,k}$ and

$$U_k h(F_{1,k}) U_k^* = h(F_\eta(\alpha, f^k, g_1)) \oplus \bigoplus_{j=1}^{\infty} h\left(F_{(-1)^j\eta}(\alpha, f^k, g_1) \otimes 1 + 1 \otimes d\Gamma^{(j)}(g_2)\right).$$

Now $h(F_\eta(\alpha, f^k, g_1))$ is compact by Proposition 3.1 since $\mathcal{H}_k$ has finite dimension. For $j \geq 1$ let $C_j = F_{(-1)^j\eta}(\alpha, f^k, g_1) \otimes 1 + 1 \otimes d\Gamma^{(j)}(g_2)$. Using Theorem B.2, Proposition 3.1 and Lemma B.4 we find for $j \geq 1$

$$\begin{aligned}\inf(\sigma_{\rm ess}(C_j)) &\geq \mathcal{E}_{(-1)^j\eta}(\alpha, f^k, g_1) + (j-1)\inf(\sigma(g_2)) + \inf(\sigma_{\rm ess}(g_2)) \\ &\geq \mathcal{E}_{(-1)^j,k} + (j-1)m_k + m_{{\rm ess},k} \geq \widetilde{m}_k \\ \inf(\sigma(C_j)) &\geq \mathcal{E}_{(-1)^j\eta}(\alpha, f^k, g_1) + j\inf(\sigma(g_2)) \geq \mathcal{E}_{(-1)^j,k} + jm_k.\end{aligned}$$

Thus $h(C_j)$ is compact for all j and since $m_k > 0$ we find $h(C_j) = 0$ for j large enough. Hence $U_k h(F_{1,k})U_k^*$ is a direct sum of compact operators where only finitely many are nonzero. This shows $U_k h(F_{1,k})U_k^*$ is compact as desired. □

Combining the two previous lemmas with Proposition 3.2 proves the first part of Theorem 3.3. Statements (1) and (2) will follow from the corollaries below.

Corollary 7.3. *Assume $m = m_{\rm ess}$, that $[m, 3m] \subset \sigma_{\rm ess}(\omega)$ and if $m = 0$ then 0 is not isolated in $\sigma_{\rm ess}(\omega)$. Then $\sigma_{\rm ess}(F_1) = [\mathcal{E}_{-1} + m, \infty)$.*

Proof. If $m = m_{\rm ess}$ then $\mathcal{E}_1 \leq \mathcal{E}_{-1} + m$ by Lemma 7.1. Hence the minimum in Lemma 7.2 is $\mathcal{E}_{-1} + m_{\rm ess} = \mathcal{E}_{-1} + m$. Now fix $x \in [\mathcal{E}_{-1} + m, \infty)$. If $m \neq 0$ the result is direct from Lemma 7.1. If $m = 0$ then for any $\varepsilon > 0$ we may find $\lambda \in \sigma_{\rm ess}(\omega)$ with $\lambda \leq \varepsilon$. We may then pick $q \in \mathbb{N} \cup \{0\}$ such that

$$|x - \mathcal{E}_{(-1)^{2q+1}} - (2q+1)\lambda| \leq \varepsilon.$$

Now $\mathcal{E}_{(-1)^{2q+1}} + (2q+1)\lambda \in \sigma_{\rm ess}(F_1)$ so $x \in \overline{\sigma_{\rm ess}(F_1)} = \sigma_{\rm ess}(F_1)$. □

Corollary 7.4. *Assume $m = m_{\rm ess}$, that $[m, 2m] \subset \sigma_{\rm ess}(\omega)$ and if $m = 0$ then 0 is not isolated in $\sigma_{\rm ess}(\omega)$. Then $\sigma_{\rm ess}(H_\eta(\alpha, f, \omega)) = [m_{\rm ess} + E_\eta(\alpha, f, \omega), \infty)$.*

Proof. Combining Proposition 3.2 and Lemma 7.1 we see

$$\{E_\eta(\alpha, f, \omega) + \lambda_1 + \cdots + \lambda_q \mid \lambda_i \in \sigma_{\rm ess}(\omega)\} \subset \sigma_{\rm ess}(H_\eta(\alpha, f, \omega))$$

for all $q \in \mathbb{N}$. The proof is now the same as for Corollary 7.3. □

8. Uniqueness

In this chapter we fix $\eta \in \mathbb{R}, \alpha \in \mathbb{R}^{2n}, f \in \mathcal{H}^{2n}$ and ω be selfadjoint on $\mathcal{H}$ such that the Hypothesis 1,2 and 3 are satisfied. Let $F_\eta := F_\eta(\alpha, f, \omega)$, $\mathcal{E}_\eta := \mathcal{E}_\eta(\alpha, f, \omega)$, $H_\eta := H_\eta(\alpha, f, \omega)$, $m_{\rm ess} = m_{\rm ess}(\omega)$, $m = m(\omega)$ and $E_\eta := E_\eta(\alpha, f, \omega)$. We let $\mathcal{H}_\mathbb{R}$ be the real Hilbert space from Lemma E.1 and $L^2(Q, \mathcal{G}, \mathbb{P})$ be the corresponding Q-space.

Lemma 8.1. *Define the unitary matrix*

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Let V be the Q-space isomorphism and define $U = A \otimes V$. This defines a unitary map from $\mathbb{C}^2 \otimes \mathcal{F}_b(\mathcal{H})$ to

$$\mathbb{C}^2 \otimes L^2(Q, \mathcal{G}, \mathbb{P}) = L^2(\{\pm 1\} \times Q, \mathcal{B}(\{\pm 1\}) \otimes \mathcal{G}, \tau \otimes \mathbb{P}) := L^2(X, \mathcal{X}, \nu),$$

where τ is the counting measure. Here we use the tensor product

$$((v_1, v_{-1}) \otimes f)(a, x) = \delta_{1,a} v_1 f(x) + \delta_{-1,a} v_{-1} f(x).$$

where $\delta_{i,j}$ is the Kronecker delta. For $v \in \mathcal{H}_\mathbb{R}$ we have

$$U\sigma_x \otimes \varphi(v) U^* = \Phi(v) \tag{8.1}$$

$$U\sigma_z \otimes 1 U = -\sigma_x \otimes 1 \tag{8.2}$$

$$U 1 \otimes d\Gamma(\omega) U^* = 1 \otimes V d\Gamma(\omega) V^* \tag{8.3}$$

where $\Phi(v)$ is a multiplication operator. Furthermore $UH_\eta U^*$ generates a positivity improving semigroup for $\eta < 0$.

Proof. We begin by noting that $(e_i \otimes f)(a,x) = \delta_{i,a} f(x)$ and $V\varphi(v)V^* = \widetilde{\varphi}(v)$ is a multiplication operator. We now prove equations (8.1),(8.2) and (8.3). Equation (8.3) is trivial. To prove the other two one calculates

$$A\sigma_z A^* = \sigma_x \quad \text{and} \quad A\sigma_x A^* = -\sigma_z$$

and so $U\sigma_x \otimes \varphi(v)U = -\sigma_z \otimes \widetilde{\varphi}(v)$ and $U\sigma_z \otimes \varphi(v)U = \sigma_x \otimes 1$. Since $-\sigma_z \otimes \widetilde{\varphi}(v)$ obviously acts like multiplication by the map $(\Phi(v))(a,x) = -a(\widetilde{\varphi}(v))(x)$ we are done proving equations (8.1) and (8.2).

From Lemma B.3 we find that every element $\psi \in L^2(X, \mathcal{X}, \nu)$ is of the form

$$\psi = e_1 \otimes \psi_1 + e_{-1} \otimes \psi_{-1}.$$

Hence ψ is (strictly) positive if and only if ψ_1 and ψ_{-1} are (strictly) positive. Using Theorem E.2 we therefore find that $1 \otimes \exp(-tVd\Gamma(\omega)V^*)$ is positivity preserving for all $t \geq 0$. Furthermore, the map $\sigma_x \otimes 1$ is positivity preserving since it maps $e_1 \otimes \psi_1 + e_{-1} \otimes \psi_{-1}$ into $e_{-1} \otimes \psi_1 + e_1 \otimes \psi_{-1}$ and so $\exp(t1 \otimes \sigma_x)$ is positivity preserving for all $t \geq 0$. It follows that for all $\eta < 0$ and $t \geq 0$

$$\exp(-tUH_\eta(0,0,\omega)U^*) = \exp(-t\eta 1 \otimes \sigma_x)(1 \otimes \exp(-tVd\Gamma(\omega)V^*))$$

is positivity preserving. We now calculate

$$U(e_1 \otimes \Omega) = Ae_1 \otimes V\Omega = \frac{1}{\sqrt{2}}(e_1 \otimes 1 + e_2 \otimes 1) = \frac{1}{\sqrt{2}}. \tag{8.4}$$

Fix $\eta < 0$. Since $e_1 \otimes \Omega$ spans the non degenerate ground state eigenspace of $\eta\sigma_z \otimes 1 + 1 \otimes d\Gamma(\omega) = H_\eta(0,0,\omega)$ by Theorem B.2, the above calculation shows that $\frac{1}{\sqrt{2}}$ does the same for $-\eta\sigma_x \otimes 1 + 1 \otimes Vd\Gamma(\omega)V^* = UH_\eta(0,0,\omega)U^*$. So $UH_\eta(0,0,\omega)U^*$ generates a positivity preserving semi group and the ground state is spanned by a strictly positive vector. This implies that the semi group generated by $UH_\eta(0,0,\omega)U^*$ is ergodic by [18, Theorem XIII.43]. Note that

$$UH_\eta U^* = UH_\eta(0,0,\omega)U^* + \sum_{j=1}^{2n} \alpha_j \Phi(f_j)^j := UH_\eta(0,0,\omega)U^* + B$$

and define

$$B_k = \sum_{j=1}^{2n} \alpha_j \Phi(f_j)^j 1_{\{|\Phi(f_j)| \leq k\}},$$

which is a bounded multiplication operator. Assume now that we have proven the following statements

(1) If $u, v \geq 0$ and $\langle u, \exp(-tB_k)v\rangle = 0$ then $\langle u, v\rangle = 0$
(2) $UH_\eta(0,0,\omega)U^* + B_k, UH_\eta U^* - B_k$ are uniformly bounded below and converge in strong resolvent sense to $UH_\eta U^*$ and $UH_\eta(0,0,\omega)U^*$ respectively.

Then we may appeal to the proof of [6, Theorem 3] to see that $UH_\eta(\alpha, f, \omega)U^*$ generates an ergodic semigroup, which by [18, Theorem XIII.44] will be positivity improving.

The first statement is trivial since B_k is a multiplication operator. To prove the second statement, note that $\widetilde{\mathcal{J}}(\mathcal{D}(\omega))$ spans a core for $H_\eta(0,0,\omega)$ and H_η by Proposition 3.1. Thus for any element $\psi \in \mathrm{Span}(\widetilde{\mathcal{J}}(\mathcal{D}(\omega)))$ we have $U\psi \in \mathcal{D}(\Phi(f_j)^j)$ for all j so by dominated convergence we find

$$\lim_{k\to\infty} B_k U\psi = BU\psi.$$

Since $U\widetilde{\mathcal{J}}(\mathcal{D}(\omega))$ spans a core for $UH_\eta(0,0,\omega)U^*$ and $UH_\eta U^*$ we find $U\widetilde{\mathcal{J}}(\mathcal{D}(\omega))$ spans a core for $UH_\eta(0,0,\omega)U^* + B_k$ and $UH_\eta U^* - B_k$ for all k. Using standard theorems about strong resolvent convergence (e.g. [16, Theorem VIII.25]) we find $UH_\eta(0,0,\omega)U^*+B_k$ and $UH_\eta U^*-B_k$ converges to $UH_\eta U^*$ respectively $UH_\eta(0,0,\omega)U^*$ in strong resolvent sense.

What remains is the lower bound. We calculate

$$\begin{aligned}UH_\eta U^*-B_k &= -\eta\sigma_x\otimes 1+1\otimes Vd\Gamma(\omega)V^*+\alpha_1\Phi(f_1)1_{\{|\Phi(f_1)|>k\}}\\ &\quad+\sum_{j=2}^{2n}\alpha_j\Phi(f_j)^j 1_{\{|\Phi(f_j)|>k\}}.\\ UH_\eta(0,0,\omega)U^*+B_k &= -\eta\sigma_x\otimes 1+1\otimes Vd\Gamma(\omega)V^*+\alpha_1\Phi(f_1)1_{\{|\Phi(f_1)|\le k\}}\\ &\quad+\sum_{j=2}^{2n}\alpha_j\Phi(f_j)^j 1_{\{|\Phi(f_j)|\le k\}}.\end{aligned}$$

In both expressions, the first term on the right hand side is bounded below by $-|\eta|$, and the sum is bounded below uniformly in k by Lemma 4.1. Now $\alpha_1\Phi(f_1)$ is infinitesimally $1\otimes Vd\Gamma(\omega)V^*$ bounded by Lemmas 2.1 and B.3. Hence there are $0\le a<1, b\ge 1$ such that for all $\psi\in\mathcal{D}(1\otimes Vd\Gamma(\omega)V^*)$ we have

$$\|\alpha_1\Phi(f_1)\psi\|\le a\|(1\otimes Vd\Gamma(\omega)V^*)\psi\|+b\|\psi\|.$$

Now $1_C\alpha_1\Phi(f_1)$ will satisfy the same inequality for any $C\in\mathcal{X}$ and so [23, Theorem 9.1] provides a lower bound of $1_C\alpha_1\Phi(f_1)+1\otimes Vd\Gamma(\omega)V^*$ independent of C. □

Lemma 8.2. *If $\eta\neq 0$ and E_η is an eigenvalue for H_η then E_η is non degenerate. If ψ is any ground state for H_η then $U\psi=e_{-\mathrm{sign}(\eta)}\otimes\psi$ where ψ is an eigenvalue for the fiber $F_{-|\eta|}$ corresponding to the energy E_η. Also E_η is not an eigenvalue for $F_{|\eta|}$.*

Proof. If $\eta<0$ non degeneracy of the ground state follows from Lemma 8.1 along with [18, Theorem XIII.43]. If a ground state exist then it will have nonzero inner product with $e_1\otimes\Omega$, since this vector is mapped the positive element $\frac{1}{\sqrt{2}}$ under the map U from Lemma 8.1 (see equation (8.4)). If $\eta>0$ then $\sigma_x\otimes 1$ transforms H_η into $H_{-\eta}$, showing that non degeneracy holds in this case as well, but now the nonzero inner product will be with the vector $\sigma_x\otimes 1(e_1\otimes\Omega)=e_{-1}\otimes\Omega$. In conclusion if a ground state for H_η exists then it is non generate and $0\neq\langle\psi,e_{-\mathrm{sign}(\eta)}\otimes\Omega\rangle$.

Non degeneracy of the ground state for H_η implies that E_η must be a non degenerate eigenvalue for $F_{-|\eta|}$ or $F_{|\eta|}$, but never both. It remains to see that this always will be $F_{-|\eta|}$. Let U be the unitary map from Proposition 3.2, and let ψ be a ground state for H_η. Then $U\psi=e_{-1}\otimes\psi_{-1}+e_1\otimes\psi_1=(\psi_1,\psi_{-1})$ is a ground state for $F_{-\eta}\oplus F_\eta$ corresponding to the eigenvalue E_η. By non degeneracy of the ground state for H_η, we must have either $\psi_{-1}=0$ or $\psi_1=0$. Now

$$\begin{aligned}0\neq\langle\psi,e_{-\mathrm{sign}(\eta)}\otimes\Omega\rangle &= \langle(\psi_1,\psi_2),U^*e_{-\mathrm{sign}(\eta)}\otimes\Omega\rangle\\ &= \langle(\psi_1,\psi_2),e_{-\mathrm{sign}(\eta)}\otimes\Omega\rangle=\langle\psi_{-\mathrm{sign}(\eta)},\Omega\rangle.\end{aligned}$$

This implies $\psi_{-\mathrm{sign}(\eta)}\neq 0$. Hence $\psi_{-\mathrm{sign}(\eta)}$ is an eigenvector for $F_{-|\eta|}$ corresponding to the eigenvalue E_η. □

Lemma 8.3. *If $\mathcal{E}_{-|\eta|}$ is an eigenvalue for $F_{-|\eta|}$ then $\mathcal{E}_{-|\eta|}$ is non degenerate and every eigenvector will have nonzero inner product with Ω.*

Proof. We start with the case $\eta=0$. Let V be the Q-space isomorphism from Theorem E.2. From Thorem E.2 we know that $Vd\Gamma(\omega)V^*$ generates a positivity improving semigroup and $V\Omega=1$. We now prove that the semigroup of VF_0V^* is positivity improving. Note

$$VF_0V^*=VF_0(0,0,\omega)V^*+\sum_{j=1}^{2n}\alpha_j\widetilde{\varphi}(v_j)^j:=VH_\eta(0,0,\omega)V^*+A,$$

and define

$$A_k = \sum_{j=1}^{2n} \alpha_j \widetilde{\varphi}(v_j)^j 1_{\{|\widetilde{\varphi}(v_j)| \leq k\}},$$

which is now a bounded multiplication operator. With the exact same proof as in Lemma 8.1 we check

(1) If $u, v \geq 0$ and $\langle u, \exp(-tA_k)v\rangle = 0$ then $\langle u, v\rangle = 0$.
(2) $VF_0(0,0,\omega)V^* + A_k, VF_0V^* - A_k$ are uniformly bounded below and converge in strong resolvent sense to respectively $VF_0V^*, VF_0(0,0,\omega)V^*$.

An appeal to the proof of [6, Theorem 3] along with [16, Theorem XIII.43] finishes the proof when $\eta = 0$. For general $\eta \neq 0$ one may combine Theorem E.2 part 1 with [6, Theorem 2] to obtain the conclusion in this case. □

We can now prove some spectral properties of the fiber operators. In the remaining part of this section we will also assume Hypothesis 4 is satisfied if $n > 2$, so we may use Theorem 3.3 except from part (3), which is proven in the next lemma

Lemma 8.4. *In general we have $\mathcal{E}_{-|\eta|} = E$ and $\mathcal{E}_{-|\eta|} \leq \mathcal{E}_{|\eta|}$. Furthermore $\mathcal{E}_{-|\eta|} < \mathcal{E}_{|\eta|}$ if and only if $m > 0$ and $\eta \neq 0$. In particular if $\eta \neq 0$ and $m = 0$ then $F_{|\eta|}$ will have no ground state.*

Proof. If $m = 0$ then $m_{\text{ess}} = 0$ by injectivity of ω. Using Theorem 3.3 we obtain $\mathcal{E}_{\pm|\eta|} \leq \mathcal{E}_{\mp|\eta|}$ since $\mathcal{E}_{\mp|\eta|} \in \sigma(F_{\pm|\eta|})$. Hence equality follows in this case, and it is trivial if $\eta = 0$. The statement regarding no ground state of $F_{|\eta|}$ now follows from $\mathcal{E}_{-|\eta|} = \mathcal{E}_{|\eta|}$ and Lemma 8.2.

Assume that $m > 0$ and $\eta \neq 0$. Now $m > 0$ implies that E_η is an eigenvalue for H_η by Theorem 3.3, and so by Lemma 8.2 we have $E_\eta = \mathcal{E}_{-|\eta|}$ is an eigenvalue for $F_{-|\eta|}$. Now since $\mathcal{E}_{-|\eta|} = E_\eta \leq \mathcal{E}_{|\eta|}$ we just have to prove that $\mathcal{E}_{-|\eta|} = \mathcal{E}_{|\eta|}$ is impossible. Assume $\mathcal{E}_{-|\eta|} = \mathcal{E}_{|\eta|}$. Then Theorem 3.3 implies that

$$\inf(\sigma_{\text{ess}}(F_{|\eta|})) = \mathcal{E}_{-|\eta|} + m_{\text{ess}} > \mathcal{E}_{|\eta|},$$

and so $\mathcal{E}_{-|\eta|} = \mathcal{E}_{|\eta|} = E_\eta$ would be an eigenvalue for $F_{|\eta|}$, but this gives a contradiction with Lemma 8.2. □

Regarding exited states we now deduce the following

Lemma 8.5. *If $\eta \neq 0$ and $\mathcal{E}_{|\eta|}$ is an eigenvalue for $F_{|\eta|}$ then it is an eigenvalue for H_η contained in $(E, E + m_{\text{ess}}]$. This is the case if $2|\eta| < m_{\text{ess}}$.*

Proof. Assume $\mathcal{E}_{|\eta|}$ is an eigenvalue for $F_{|\eta|}$. Then $m_{\text{ess}} \geq m > 0$ by Lemma 8.4 and we calculate using Theorem 3.3 and Lemma 8.4

$$E_\eta = \mathcal{E}_{-|\eta|} < \mathcal{E}_{|\eta|} \leq \mathcal{E}_{-|\eta|} + m_{\text{ess}} = E_\eta + m_{\text{ess}}.$$

Assume now $0 < 2|\eta| < m_{\text{ess}}$. By Theorem 3.3 it is enough to prove the inequality $\mathcal{E}_{|\eta|} < \mathcal{E}_{-|\eta|} + m_{\text{ess}}$. For any $\varepsilon > 0$ we may pick normalised $\psi \in \mathcal{D}(F_{|\eta|}) = \mathcal{D}(F_{-|\eta|})$ such that

$$\varepsilon + \mathcal{E}_{|\eta|} - \mathcal{E}_{-|\eta|} \leq \langle \psi, F_{|\eta|} - F_{-|\eta|}\psi\rangle = 2|\eta|\langle \psi, \Gamma(-1)\psi\rangle \leq 2|\eta|.$$

This proves the desired inequality. □

Theorem 3.4 is now a combination of all lemmas in this chapter.

9. The massless case

In this chapter we prove the last half of Theorem 3.5. A proof of the first half can be found in Appendix D. First we shall need the following lemma.

Lemma 9.1. *Assume $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ with $(\mathcal{M}, \mathcal{F}, \mu)$ σ-finite. Let $\eta \leq 0, \alpha \in \mathbb{R}^{2n}, f \in \mathcal{H}^{2n}$, ω be a selfadjoint multiplication operator on $\mathcal{H}$ and assume (α, f, ω) satisfies Hypothesis 1,2,3 and either $n \leq 2$ or Hypothesis 4. Let $A = \bigcup_{i\leq 2n}\{f_i \neq 0\}$, $\mathcal{H}_1 = L^2(X, \mathcal{F}, 1_A\mu)$, $\mathcal{H}_2 = L^2(X, \mathcal{F}, 1_{A^c}\mu)$, ω_i be multiplication with ω on the space $\mathcal{H}_i$ and define $\widetilde{f}_i \in \mathcal{H}_1$ by $\widetilde{f}_i = f_i\ 1_A\mu$-almost everywhere. Then $(\alpha, \widetilde{f}, \omega_1)$ satisfies Hypothesis 1,2,3 and either $n \leq 2$ or Hypothesis 4. We also have*

1. *$\mathcal{E}_\eta(\alpha, f, \omega) = \mathcal{E}_\eta(\alpha, \widetilde{f}, \omega_1)$ and $\mathcal{E}_\eta(\alpha, f, \omega)$ is an eigenvalue for $F_\eta(\alpha, f, \omega)$ if and only if $\mathcal{E}_\eta(\alpha, f, \omega)$ is an eigenvalue for $F_\eta(\alpha, f, \omega_1)$. In particular if $\omega \geq q > 0$ almost everywhere on A then $\mathcal{E}_\eta(\alpha, f, \omega)$ is an eigenvalue for $F_\eta(\alpha, \widetilde{f}, \omega_1)$ and thus for $F_\eta(\alpha, f, \omega)$.*
2. *If $\psi = (\psi^{(k)})$ is a ground state for $F_\eta(\alpha, f, \omega_1)$, then $\psi = (1_{A^k}\psi^{(k)})$ is a ground state for $F_\eta(\alpha, f, \omega)$.*

Proof. Define $P_i : \mathcal{H} \to \mathcal{H}_i$ by $P_1(f) = f\ 1_A\mu$-almost everywhere and $P_2(f) = f\ 1_{A^c}\mu$-almost everywhere. Let $V : \mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_2$ be $V(f) = (P_1(f), P_2(f))$. Then we see V is unitary with $V^*(f, g) = 1_A f + 1_{A^c} g$ μ-almost everywhere. Clearly we have $Vf_i = (\widetilde{f}_i, 0)$ along with $V\omega V^* = (\omega_1, \omega_2)$. The properties in Hypothesis 1,2 and 3 are easily checked. If $n > 2$ one uses the same phase function for $\widetilde{f}$ as for f to verify Hypothesis 4. Using Lemma 6.1 we find a unitary map U from $\mathcal{F}_b(\mathcal{H})$ to

$$\mathcal{F}_b(\mathcal{H}_1) \oplus \bigoplus_{j=1}^{\infty} \Big(\mathcal{F}_b(\mathcal{H}_1) \otimes \mathcal{H}_2^{\otimes_s j}\Big)$$

such that

$$UF_\eta(\alpha, f, \omega)U^* = F_\eta(\alpha, \widetilde{f}, \omega_1) \oplus \bigoplus_{j=1}^{\infty} \Big(F_{(-1)^j\eta}(\alpha, \widetilde{f}, \omega_1) \otimes 1 + 1 \otimes d\Gamma^{(j)}(\omega_2)\Big).$$

Since $\eta \leq 0$, Theorem 3.3 implies $\mathcal{E}_\eta(\alpha, \widetilde{f}, \omega_1) \leq \mathcal{E}_{-\eta}(\alpha, \widetilde{f}, \omega_1)$ with equality if and only if $\eta = 0$ or $\inf \sigma_{\text{ess}}(\omega_1) = 0$. Hence we find using Theorem B.2 and Lemma B.4 that

$$\mathcal{E}_\eta(\alpha, f, \omega) = \min\{\mathcal{E}_\eta(\alpha, \widetilde{f}, \omega_1), \inf_{j\in\mathbb{N}}(\mathcal{E}_{(-1)^j\eta}(\alpha, \widetilde{f}, \omega_1) + j\inf(\sigma(\omega_2)))\} = \mathcal{E}_\eta(\alpha, \widetilde{f}, \omega_1)$$

Assume now that $\mathcal{E}_\eta(\alpha, f, \omega)$ is an eigenvalue for $F_\eta(\alpha, \tilde{f}, \omega_1)$. Then by the decomposition we find $\mathcal{E}_\eta(\alpha, f, \omega)$ is an eigenvalue for $F_\eta(\alpha, f, \omega)$. Assume now $\mathcal{E}_\eta(\alpha, f, \omega)$ is an eigenvalue for $F_\eta(\alpha, f, \omega)$, and that it is not an eigenvalue for $F_\eta(\alpha, \widetilde{f}, \omega_1)$. Then there is an $j \in \mathbb{N}$ such that

$$\begin{aligned}\mathcal{E}_\eta(\alpha, f, \omega) &= \inf(\sigma(F_{(-1)^j\eta}(\alpha, \widetilde{f}, \omega_1) \otimes 1 + 1 \otimes d\Gamma^{(j)}(\omega_2))) \\ &= \mathcal{E}_{(-1)^j\eta}(\alpha, \widetilde{f}, \omega_1) + j\inf(\sigma(\omega_2)).\end{aligned}$$

This can only hold if $\inf(\sigma(\omega_2)) = 0$ and therefore

$$\inf(\sigma(F_{(-1)^j\eta}(\alpha, \widetilde{f}, \omega_1) \otimes 1 + 1 \otimes d\Gamma^{(j)}(\omega_2)) = \mathcal{E}_{(-1)^j\eta}(\alpha, \widetilde{f}, \omega_1) + 0.$$

Injectivity of ω_2 and Lemma B.4 implies that 0 is no eigenvalue for $d\Gamma^{(j)}(\omega_2)$. By Theorem B.2 we find that $\mathcal{E}_{(-1)^j\eta}(\alpha, \widetilde{f}, \omega_1) + 0$ is not an eigenvalue for the operator $F_{(-1)^j\eta}(\alpha, f, \omega_1) \otimes 1 + 1 \otimes d\Gamma^{(j)}(\omega_2)$ which is a contradiction. Hence $\mathcal{E}_\eta(\alpha, f, \omega)$ is an eigenvalue for $F_\eta(\alpha, \widetilde{f}, \omega_1)$. The statement regarding the dimension of the eigenspaces is contained in Theorem 3.4. The last part of statement (1) follows from $m(\omega_1) \geq q > 0$ and Theorem 3.3.

To prove statement (2) we let $j_1 : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ be the embedding $j_1(f) = (f, 0)$ and define $Q = V^* j_1$. Now $U^*\psi = \Gamma(Q)\psi$ by Lemma C.3 and $U^*\psi$ is the desired eigenvector for $F_\eta(\alpha, f, \omega)$. Noting $Q(f) := V^* j_1(f) = 1_A f$ we see $\Gamma(Q)\psi = (1_{A^k}\psi^{(k)})$ as desired. □

From now on we assume $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu})$, ω is a multiplication operator, $\eta \in \mathbb{R}, \alpha \in \mathbb{R}^{2n}$, $n \leq 2$ and Hypothesis 1,2,3 and 5 are satisfied.

Define $B_\ell = \{\omega \geq \ell^{-1}\}$ and $f^\ell = 1_{B_\ell} f$. Then $F_{\pm 1,\ell} := F_{\pm|\eta|}(\alpha, f^\ell, \omega)$ converges in norm resolvent sense to $F_{\pm 1} := F_{\pm|\eta|}(\alpha, f, \omega)$ by Lemma 6.5 and $\mathcal{E}_\ell = \mathcal{E}_{-|\eta|}(\alpha, f^\ell, \omega)$ converges to $\mathcal{E} := \mathcal{E}_{-|\eta|}(\alpha, f, \omega)$. Furthermore $F_{-1,\ell}$ has a ground state ψ_ℓ for all ℓ by Lemma 9.1. Taking a subsequence we may assume that ψ_ℓ converges weakly to ψ. The last half of Theorem 3.5 will be proven by [2, Lemma 4.9] if we can prove that $\|\psi\| = 1$. First a few observations which we will summarise in Lemma 9.2a below. For a strict definition of the pointwise annihilation operator see the discussion after Lemma D.9 in Appendix D.

Lemma 9.2. *The following holds*

(1) *Let A_1 be the point wise annihilation operator of order 1. We have*

$$(A_1\psi_\ell)(k) = \sum_{j=1}^{2n} f_j^\ell(k)(F_{1,\ell} - \mathcal{E}_\ell + \omega(k))^{-1} j\alpha_j \varphi(f_j^\ell)^{j-1}\psi_\ell.$$

(2) *There is a constant C independent of ℓ and j such that $\|\alpha_j\varphi(f_j^\ell)^{j-1}\psi_\ell\| \leq C$*

(3) *$\psi_\ell \in \mathcal{D}(N)$ and $\langle\psi_\ell, N\psi_\ell\rangle$ is uniformly bounded in ℓ. In particular we find $A_1\psi_\ell \in L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, \mathcal{F}_b(\mathcal{H}))$ for all $\ell \in \mathbb{N}$ and the sequence $\{A_1\psi_\ell\}_{\ell=1}^\infty$ is bounded in this space.*

(4) *We have*

$$A_1\psi_\ell - \sum_{j=1}^{2n} f_j(k)(F_1 - \mathcal{E} + \omega(k))^{-1} j\alpha_j\varphi(f_j^\ell)^{j-1}\psi_\ell \tag{9.1}$$

converges to 0 in $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, \mathcal{F}_b(\mathcal{H}))$.

Proof. Statement (1) follows directly from Theorem D.16 in Appendix D. To prove statement (2), we note that $j\alpha_j\varphi(f_j^\ell)^{j-1}(F_{-1,\ell} - i)^{-1}$ is uniformly bounded for $\ell \in \mathbb{N}$ and $j \leq 2n$ by Lemma 6.3. Let C be the bounding constant. Then

$$\|j\alpha_j\varphi(f_j^\ell)^{j-1}\psi_\ell\| \leq C\|(\mathcal{E}_\ell - i)\psi_\ell\|.$$

Now $\mathcal{E}_\ell$ is convergent and $\|\psi_\ell\| = 1$ for all ℓ and so the conclusion follows.

To prove statement (3) we note that $\psi_\ell \in \mathcal{D}(N)$ by the first half of Theorem 3.5. Using (1) and $\mathcal{E}_\ell \leq \mathcal{E}_{|\eta|}(\alpha, f^\ell, \omega)$ by Theorem 3.3 we estimate

$$\|(A_1\psi_\ell)(k)\|^2 \leq C^2 \left(\sum_{j=1}^{2n} \frac{|f_j^\ell(k)|}{\omega(k)}\right)^2 \leq 2nC^2 \sum_{j=1}^{2n} \frac{|f_j(k)|^2}{\omega(k)^2}. \tag{9.2}$$

Integrating and appealing to Theorem D.15 yields the result.

To prove statement (4), note that $(F_{1,\ell} - \mathcal{E}_\ell + \omega(k))^{-1} - (F_1 - \mathcal{E} + \omega(k))^{-1}$ converges to 0 in norm by Lemma 6.4. Since $j\alpha_j\varphi(f_j^\ell)^{j-1}\psi_\ell$ is uniformly bounded, we see that the function in equation (9.1) converges to 0 pointwise. Using estimates as in equation (9.2), the conclusion follows by dominated convergence. □

We need one last lemma before we can put it all together. One should note that a similar statement also appears in the paper [7], but the proof presented here is faster and much more general.

Lemma 9.3. *Let $G \in C_0^\infty(\mathbb{R}^\nu)$ such that $G(0) = 1$ and $0 \leq G \leq 1$. Define $G_R = G(x/R)$ and let A be either $x = -i\nabla_k$ or k. For any $\varepsilon > 0$ there is $\ell', R' > 0$ such that $\|(1 - \Gamma(G_R(A)))\psi_\ell\| \leq \varepsilon$ for $R > R', \ell > \ell'$ and any choice of A.*

Proof. We start by noting

$$(1 - \Gamma(G_R(A)))^2 = 1 - \Gamma(G_R(A)) + \Gamma(G_R(A))(\Gamma(G_R(A)) - 1).$$

On j particle vectors we see that $\Gamma(G_R(A))(\Gamma(G_R(A)) - 1)$ acts like a negative multiplication operator in position/momentum space depending on the choice of A. Hence

$$(1 - \Gamma(G_R(A)))^2 \leq 1 - \Gamma(G_R(A)).$$

On j particle vectors in position/momentum space (depending on A) we find that $1 - \Gamma(G_R(A))$ acts like multiplication by

$$1 - G(k_1)G(k_2)\cdots G(k_j) = \sum_{i=1}^{j}(1 - G(k_i))G(k_{i+1})\cdots G(k_j) \leq \sum_{i=1}^{j}(1 - G(k_i)).$$

Hence $1 - \Gamma(G_R(A)) \leq d\Gamma(1 - G_R(A))$ so it is enough to prove that

$$\langle \psi_\ell, d\Gamma(1 - G_R(A))\psi_\ell\rangle$$

goes to 0 for R, ℓ tending to ∞. First we note that $\psi_\ell \in \mathcal{D}(N) \subset \mathcal{D}(d\Gamma(1 - G_R(A)))$ by Lemma 9.2, so the above quantity is well defined. Using Theorem D.15 we see that

$$\langle \psi_\ell, d\Gamma(1 - G_R(A))\psi_\ell\rangle = \langle A_1\psi_\ell, ((1 - G_R(A)) \otimes 1)A_1\psi_\ell\rangle.$$

Define the maps

$$q_\ell(t) = \sum_{j=1}^{2n} f_j(k)(F_1(f) - \mathcal{E} + \omega(k))^{-1} j\alpha_j\varphi(f_j^\ell)^{j-1}\psi_\ell.$$

By Lemma 9.2 we know $A_1\psi_\ell - q_\ell$ converges to 0 in $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, \mathcal{F}_b(\mathcal{H}))$ and the $A_1\psi_\ell$ are uniformly bounded in $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, \mathcal{F}_b(\mathcal{H}))$. In particular the q_ℓ are uniformly bounded in $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, \mathcal{F}_b(\mathcal{H}))$, and since $\|(1 - G_R(A)) \otimes 1\| = 1$ for all R, we find an ℓ' such that

$$\langle A_1\psi_\ell, (1 - G_R(A)) \otimes 1A_1\psi_\ell\rangle \leq \frac{\varepsilon}{3} + \langle q_\ell, ((1 - G_R(A)) \otimes 1)q_\ell\rangle$$

for all $R > 0, \ell > \ell'$. Write

$$\widetilde{q}_j(t) = f_j(k)(F_1 - \mathcal{E} + \omega(k))^{-1}$$

and note that $\widetilde{q}_j \in L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, B(\mathcal{F}_b(\mathcal{H})))$. Hence there is a sequence $\{\widetilde{q}_{j,p}\}_{p=1}^\infty$ of elements in $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, B(\mathcal{F}_b(\mathcal{H})))$ such that $\widetilde{q}_{j,p}$ converges to $\widetilde{q}_j$ in $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, B(\mathcal{F}_b(\mathcal{H})))$ and each $\widetilde{q}_{j,p}$ is a linear combination of functions of the form $g(k)B$ where $B \in B(\mathcal{F}_b(\mathcal{H}))$ and $g \in \mathcal{H}$. Since $j\alpha_j\varphi(f_j^l)^{j-1}\psi_\ell$ is uniformly bounded in ℓ, we see that

$$q_{\ell,p} := \sum_{j=1}^{2n} \widetilde{q}_{j,p} j\alpha_j\varphi(f_j^\ell)^{j-1}\psi_\ell$$

converges to $q_\ell \in L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, \mathcal{F}_b(\mathcal{H}))$ uniformly in ℓ, for p tending to ∞. In particular the $q_{\ell,p}$ are uniformly bounded in $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, \mathcal{F}_b(\mathcal{H}))$ since this holds for the q_ℓ. Picking p large enough we may thus estimate as above

$$\langle A_1\psi_\ell, (1 - G_R(A)) \otimes 1A_1\psi_\ell\rangle \leq \frac{2\varepsilon}{3} + \langle q_{\ell,p}, (1 - G_R(A)) \otimes 1q_{\ell,p}\rangle$$

for all $\ell > \ell', R > 0$. Now each of the terms in $q_{\ell,p}$ is of the form $g \otimes v_\ell$ where v_ℓ is uniformly bounded in ℓ and g is independent of ℓ. Furthermore the number of terms in $q_{\ell,p}$ is also independent of ℓ (it depends only on p by construction).

Since $1 - G_R(A)$ converges to 1 strongly by the functional calculus, we find that $((1 - G_R(A)) \otimes 1)q_{\ell,p}$ goes to 0 for R tending to ∞, uniformly in ℓ. Thus picking R larger than some R' we find for $\ell > \ell'$ that

$$\langle A_1\psi_\ell, ((1 - G_R(A)) \otimes 1)A_1\psi_\ell\rangle \leq \varepsilon,$$

since the $q_{\ell,p}$ are uniformly bounded in $L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^{\otimes\nu}, \mathcal{F}_b(\mathcal{H}))$. □

The following lemma finishes the proof of Theorem 3.5.

Lemma 9.4. $\|\psi\|= 1$.

Proof. Let $\varepsilon > 0$. Pick R', ℓ' so large that $\|(1-\Gamma(G_R(A)))\psi_\ell\|\leq \frac{\varepsilon}{3}$ when A is either $x = -i\nabla_k$ or k and $R > R', \ell > \ell'$. By Lemma 9.2, $\langle\psi_\ell, N\psi_\ell\rangle$ is uniformly bounded by a constant C, thus we find

$$\|(1 - 1_{[0,p]}(N))\psi_\ell\|= \|1_{(p,\infty)}(N)\psi_\ell\|\leq \frac{1}{\sqrt{p}}\|1_{(p,\infty)}(N)N^{\frac{1}{2}}\psi_\ell\|\leq \frac{\sqrt{C}}{\sqrt{p}}$$

Hence we may pick p so large that $\|(1 - 1_{[0,p]}(N))\psi_\ell\|\leq \frac{\varepsilon}{3}$ uniformly in ℓ. We now find

$$\begin{aligned}
1 &= \|\psi_\ell\| \\
&= \|(1 - \Gamma(G_R(k)))\psi_\ell\|+\|\Gamma(G_R(k))(1 - \Gamma(G_R(-i\nabla)))\psi_\ell\| \\
&+ \|\Gamma(G_R(k))\Gamma(G_R(-i\nabla))(1 - 1_{[0,p]}(N))\psi_\ell\|+\|\Gamma(G_R(k))\Gamma(G_R(-i\nabla))1_{[0,p]}(N)\psi_\ell\| \\
&\leq \varepsilon + \|\Gamma(G_R(k))\Gamma(G_R(-i\nabla))1_{[0,p]}(N)\psi_\ell\|.
\end{aligned}$$

Since $\Gamma(G_R(k))\Gamma(G_R(-i\nabla))1_{\{[0,p]\}}(N)$ is compact, we may take ℓ to ∞ and find

$$1 - \varepsilon \leq \|\Gamma(G_R(k))\Gamma(G_R(-i\nabla))1_{\{[0,p]\}}(N)\psi\|\leq \|\psi\|\leq \liminf_{\ell\to\infty}\|\psi_\ell\|= 1.$$

This finishes the proof. □

Appendix A. Measure Theory.

In this section we introduce the necessary measure theoretic tools to prove the HVZ theorem. We fix a σ-finite measurable space $(\mathcal{M}, \mathcal{F}, \mu)$. If $f\colon \mathcal{M} \to \mathbb{R}$ is a measurable map and M_f is the corresponding multiplication operator then it is easy to see

$$\begin{aligned}
\sigma(M_f) &= \{\lambda \in \mathbb{R} \mid \mu((\lambda - \varepsilon, \lambda + \varepsilon)) > 0 \text{ for all } \varepsilon > 0\} := \operatorname{essran}(f) \\
\sigma_{\mathrm{ess}}(M_f) &= \{\lambda \in \mathbb{R} \mid \operatorname{Dim}(1_{\{\lambda-\varepsilon<\omega<\lambda+\varepsilon\}}L^2(\mathcal{M}, \mathcal{F}, \mu)) = \infty \text{ for all } \varepsilon > 0\}.
\end{aligned}$$

Here $\operatorname{essran}(f)$ is called the essential range of f under μ.

Lemma A.1. *Let $f : \mathcal{M} \to \mathbb{R}$ be measurable such that $f > 0$ almost everywhere. Let $B \in \mathcal{F}$ and assume there is an $R > 0$ such that*

$$B \subset f^{-1}([R^{-1}, R]).$$

Then there is a sequence of real valued measurable functions $\{f_n\}_{n=1}^\infty$ such that f_n converges uniformly to f, f_n takes only finitely many values in B, $(2R)^{-1} < f_n < 2R$ on B, f_n/f and f/f_n converges to 1 in $L^\infty(\mathcal{M}, \mathcal{F}, \mu)$ and

$$\lim_{n\to\infty} \inf(\sigma(M_{f_n})) = \inf(\sigma(M_f)).$$

Furthermore, $\sigma_{\mathrm{ess}}(M_{f_n}) \neq \emptyset$ if $\sigma_{\mathrm{ess}}(M_f) \neq \emptyset$ and in this case

$$\lim_{n\to\infty} \inf(\sigma_{\mathrm{ess}}(M_{f_n})) = \inf(\sigma_{\mathrm{ess}}(M_f)).$$

Proof. Define

$$f_n = f 1_{B^c} + \sum_{k=1}^{\infty} \frac{k}{2^n R^{-1}} 1_{f^{-1}([\frac{k}{2^n R^{-1}}, \frac{k+1}{2^n R^{-1}}))\cap B}.$$

Note the above sum is pointwise finite, and so defines a measurable, nonnegative function. In fact $C = \{f \leq 0\} = \{f_n \leq 0\} \subset B^c$ for all n and

$$\sup_{x\in\mathcal{M}} |f(x) - f_n(x)| \leq \frac{1}{R2^n}, \tag{A.1}$$

so $\frac{1}{2R} < f_n < 2R$ on B for all n. Hence uniform convergence is implied and we may calculate

$$\left| \frac{f}{f_n} - 1 \right| = 1_B \left| \frac{f - f_n}{f_n} \right| \leq 2R \frac{1}{R2^n} = \frac{1}{2^{n-1}}.$$

Hence $f/f_n - 1 \in L^\infty(\mathcal{M}, \mathcal{F}, \mu)$ and converges to 0 in this topology. A similar argument works for $f_n/f - 1$.

Equation (A.1) shows $\mathcal{D}(M_f) = \mathcal{D}(M_{f_n})$ for all n and on this set we have the inequality $\|(M_f - M_{f_n})\psi\| \leq R^{-1}2^{-n}\|\psi\|$ which shows M_{f_n} converges to M_f in norm resolvent sense (see [16, Theorem VIII.25]). Since the operators M_{f_n} are uniformly bounded below by 0, we conclude

$$\lim_{n\to\infty} \inf(\sigma(M_{f_n})) = \inf(\sigma(M_f)).$$

Write $\lambda = \inf(\sigma_{\text{ess}}(M_f))$. For every $q \in \mathbb{N}$ we have

$$1_{f^{-1}((\lambda - q^{-1}, \lambda + q^{-1}))} = 1_{B^c \cap f^{-1}((\lambda - q^{-1}, \lambda + q^{-1}))} + 1_{f^{-1}((\lambda - q^{-1}, \lambda + q^{-1}))\cap B}$$

and the left hand side defines an infinite dimensional projection. Then one of the two projections on right hand side must have infinite dimension for infinitely many $q \in \mathbb{N}$. If it is the first projection, we find

$$1_{f_n^{-1}((\lambda - q^{-1}, \lambda + q^{-1}))} = 1_{B^c \cap f^{-1}((\lambda - q^{-1}, \lambda + q^{-1}))} + 1_{f_n^{-1}((\lambda - q^{-1}, \lambda + q^{-1}))\cap B}$$

has infinite dimension for for infinitely many q and so $\lambda \in \sigma_{\text{ess}}(M_{f_n})$. Defining $\lambda_n = \lambda$ we even found a sequence of elements converging to λ such that $\lambda_n \in \sigma_{\text{ess}}(M_{f_n})$.

Assume now $1_{f^{-1}(\lambda - q^{-1}, \lambda + q^{-1})\cap B}$ has infinite dimensional range for infinitely many q. Fix $n \in \mathbb{N}$ and pick k such that

$$\lambda \in [k(R2^n)^{-1}, (k+1)(R2^n)^{-1}),$$

and note that either $k(R2^n)^{-1}$ or $(k-1)(R2^n)^{-1}$ belongs to the essential spectrum of M_{f_n}, since it is an eigenvalue of infinite dimension. In particular $\sigma_{\text{ess}}(M_{f_n})$ contains a point λ_n such that $|\lambda - \lambda_n| \leq 2(R2^n)^{-1}$.

Hence we have now proven that for each $n \in \mathbb{N}$ there is a $\lambda_n \in \sigma_{\text{ess}}(M_{f_n})$ such that λ_n converges to λ. In particular $\mu_n = \inf(\sigma_{\text{ess}}(M_{f_n})) < \infty$ and bounded from above, since the sequence of λ_n is bounded.

Let μ be any limit point of the μ_n. Then $\mu \leq \lambda$ and by elementary properties of norm resolvent convergence, it is an element in the essential spectrum of M_f. This implies $\mu = \lambda$, and so λ is the only accumulation point of the bounded sequence μ_n, which implies the desired convergence. □

We have the following definition:

Definition A.2. *Write* $\mathbb{R}_+ = [0, \infty)$. *A continuous resolution for the measure space* $(\mathcal{M}, \mathcal{F}, \mu)$ *is a collection* $(A_x)_{x\in\mathbb{R}_+} \subset \mathcal{F}$ *such that* $\mu(A_0) = 0$, $A_x \subset A_y$ *when* $x \leq y$, $\mu(A_x) < \infty$ *for all* $x \in \mathbb{R}^+$, $1_{A_x} \to 1_{A_y}$ μ*-a.e for* $x \to y$ *and*

$$\bigcup_{x\geq 0} A_x = \mathcal{M}.$$

Lemma A.3. *Assume that $(\mathcal{M}, \mathcal{F}, \mu)$ allows a continuous resolution $(A_x)_{x\in\mathbb{R}_+}$. Let $A \in \mathcal{F}$ with $0 < \mu(A)$. Then for every $\lambda \in (0, \mu(A))$ there is $B \subset A$ with $B \in \mathcal{F}$ and $\mu(B) = \lambda$. Furthermore there is a partition of A into disjoint measurable sets $\{B_n\}_{n\in\mathbb{N}}$ such that $0 < \mu(B_n) < \infty$.*

Proof. We start by defining $f : [0, \infty) \to [0, \infty)$ by

$$f(x) = \int_{\mathcal{M}} 1_A(y) 1_{A_x}(y) d\mu(y).$$

Then f is increasing and continuous by the dominated convergence theorem. Furthermore $f(0) = 0$ and by monotone convergence we find that

$$\mu(A) = \lim_{x\to\infty} f(x).$$

Let $\lambda \in (0, \mu(A))$. The intermediate value theorem now gives $x_0 \in [0, \infty)$ such that $\lambda = f(x_0)$ implying $B = A_{x_0} \cap A$ has the properties claimed in the Lemma. We now prove that the subdivision of A exists. For each $n \in \mathbb{N}$ pick $x_n \in [0, \infty)$ such that

$$f(x_n) = \begin{cases} (1 - 2^{-n})\mu(A) & \mu(A) < \infty \\ n & \mu(A) = \infty \end{cases}$$

Since f is increasing and $f(x_n) < f(x_{n+1})$ we find that $x_n < x_{n+1}$. Define

$$E_n = A \cap A_{x_n}$$

and put $\widetilde{B}_1 = E_1$ and $\widetilde{B}_n = E_n \backslash E_{n-1}$ for $n \geq 2$. Note that $\mu(\widetilde{B}_1) = \mu(E_1) = f(x_1)$ so $0 < \mu(\widetilde{B}_1) < \infty$. Now $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ so we find for $n \geq 2$ that

$$\mu(\widetilde{B}_n) = \mu(E_n \backslash E_{n-1}) = f(x_n) - f(x_{n-1}),$$

so we conclude $0 < \mu(\widetilde{B}_n) < \infty$. Furthermore, $\widetilde{B}_n \cap \widetilde{B}_m = \emptyset$ for $n \neq m$ by construction. Define

$$x = \lim_{n\to\infty} x_n.$$

If $x = \infty$ we find

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} E_n = A \cap \bigcup_{n=1}^{\infty} A_{x_n} = A,$$

so we may pick $B_n = \widetilde{B}_n$. If $x < \infty$ we note that

$$\infty > \int_{\mathcal{M}} 1_A(y) 1_{A_x}(y) d\mu(x) = f(x) = \lim_{n\to\infty} f(x_n) = \mu(A),$$

so $\mu(A) < \infty$ and $f(x) = \mu(A)$. Furthermore,

$$\mu\left(\bigcup_{n=1}^{\infty} \widetilde{B}_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \mu(A).$$

Let $B = A \backslash \bigcup_{n=1}^{\infty} \widetilde{B}_n$ and note that $\mu(B) = 0$. Define $B_1 = \widetilde{B}_1 \cup B$ and $B_n = \widetilde{B}_n$ for $n \geq 2$. Then $\mu(B_n) = \mu(\widetilde{B}_n) \in (0, \infty)$ for all n, and $B_n \cap B_m = \emptyset$ for $n \neq m$. Furthermore,

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \widetilde{B}_n \cup \left(A \backslash \bigcup_{n=1}^{\infty} \widetilde{B}_n\right) = A,$$

which shows that a subdivision exists. $\square$

Using this we may prove the following preliminary result.

Lemma A.4. *Assume that $(\mathcal{M},\mathcal{F},\mu)$ allows a continuous resolution. Let $f:\mathcal{M}\to\mathbb{R}$ and $z\in\sigma(M_f)$. Then there is a collection of sets $\{A_n\}_{n=1}^\infty\subset\mathcal{F}$ such that $|f(x)-z_i|\leq\frac{1}{n}$ on A_n, $A_n\cap A_m=\emptyset$ if $m\neq n$, $\mu(A_n^i)>0$ and*

$$\mu\left(\bigcup_{n=1}^\infty A_n\right)=\sum_{n=1}^\infty\mu(A_n)<\infty.$$

If M_f denotes the corresponding multiplication operator we find $\sigma(M_f)=\sigma_{\text{ess}}(M_f)$.

Proof. Fix $z\in\sigma(M_f)$ and define

$$B_n=\{f\in(z-n^{-1},z+n^{-1})\}.$$

There are now several cases. First assume that $\mu(B_n)=\infty$ for all $n\in\mathbb{N}$. Then define A_n recursively as follows: By Lemma A.3 we may pick $A_1\subset B_1$ such that $\mu(A_1)=1$. Assume now we have constructed disjoint sets $A_1,\dots,A_{n-1}$ such that $A_j\subset B_j$ for each $j\in\{1,\dots,n-1\}$ and $\mu(A_j)=\frac{1}{j^2}$. Then

$$\infty=\mu(B_n)\leq\mu(B_n\backslash(A_1\cup\dots\cup A_{n-1}))+\sum_{j=1}^{n-1}\frac{1}{j^2}.$$

so $\mu(B_n\backslash(A_1\cup\dots\cup A_{n-1}))=\infty$. By Lemma A.3 there is $A_n\subset B_n\backslash(A_1\cup\dots\cup A_{n-1})$ such that $\mu(A_n)=\frac{1}{n^2}$. Hence we have now constructed a sequence of disjoint sets $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $A_n\subset B_n$ and $\mu(A_n)=\frac{1}{n^2}$. Since

$$\mu\left(\bigcup_{n=1}^\infty A_n\right)=\sum_{n=1}^\infty\mu(A_n)=\sum_{n=1}^\infty\frac{1}{n^2}<\infty$$

we are done. Assume now that there is a $N\in\mathbb{N}$ such that $\mu(B_N)<\infty$. Define

$$C_n=B_{N+n}.$$

Since the B_n are decreasing we find that $C_n\subset B_n$ and

$$\lim_{n\to\infty}\mu(C_n)=\mu(\{f=z\})<\infty.$$

If $\mu(\{f=z\})>0$ we apply Lemma A.3 and obtain a disjoint subdivision $\{A_n\}_{n=1}^\infty$ of $\{f=z\}$. Since $A_n\subset\{f=z\}\subset B_n$ for all n and

$$\mu\left(\bigcup_{n=1}^\infty A_n\right)=\sum_{n=1}^\infty\mu(A_n)=\mu(\{f=z\})<\infty,$$

we are finished.

What remains is the case $\mu(\{f=z\})=0$. We know that $\mu(C_n)>0$ for all n since $z\in\text{essran}(f)$. We now claim that there are natural numbers $n_1<n_2<n_3<\dots$ such that $\mu(C_{n_k}\backslash C_{n_{k+1}})>0$. Define $n_1=1$ and assume that $n_1<n_2<\dots<n_k$ has been constructed. Define

$$n_{k+1}=\min\{n\in\mathbb{N}\mid\mu(C_{n_k}\backslash C_n)>0,n>n_k\}.$$

The minimum exists because if the set is empty then

$$\mu(C_{n_k})=\mu(C_n)$$

for all $n>n_k$ implying that $\mu(\{f=z\})=\mu(C_{n_k})>0$ which is a contradiction. Define

$$A_k=C_{n_k}\backslash C_{n_{k+1}}.$$

Then $A_k\subset C_{n_k}\subset C_k\subset B_k$ so $|f(x)-z|\leq\frac{1}{k}$ holds on A_k. Furthermore, $0<\mu(A_k)\leq\mu(C_{n_k})<\infty$ for all k and the A_k are disjoint by construction. Note also

$$\sum_{n=1}^\infty\mu(A_n)=\mu\left(\bigcup_{n=1}^\infty A_n\right)\leq\mu(C_1)<\infty,$$

which proves the existence in the last case. The collection of maps $\mu(A_n)^{-\frac{1}{2}}1_{A_n}$ is a Weyl sequence for z and hence $z \in \sigma_{\text{ess}}(M_f)$. □

This leads to the following theorem which we shall need. The reader is reminded that singletons are sets of the form $\{x\}$ for some $x \in \mathcal{M}$.

Theorem A.5. *Assume that $(\mathcal{M}, \mathcal{F}, \mu)$ is σ-finite and that all singletons are measurable. Let A be the set of points in $(\mathcal{M}, \mathcal{F}, \mu)$ such that the corresponding singleton has positive measure. Then A is countable and in particular measurable. Let μ_{A^c} denote the measure $\mu_{A^c}(B) = \mu(A^c \cap B)$ and assume that $(\mathcal{M}, \mathcal{F}, \mu_{A^c})$ has a continous resolution. Let $f : \mathcal{M} \to \mathbb{R}$ be measurable, B denote the essential range of f with respect to μ_{A_c} and define*

$$C = \{\lambda \in \mathbb{R} \mid \exists \{\lambda_n\}_{n=1}^{\infty} \subset A, \lambda_n \neq \lambda_m \ \forall n \neq m \text{ and } |f(\lambda_n) - \lambda| \leq n^{-1}\}.$$

Then

$$\sigma_{\text{ess}}(M_f) = B \cup C.$$

If furthermore $\mu(A) < \infty$ then for $z_1, \ldots, z_k \in \sigma_{\text{ess}}(M_f)$ there are k collections of sets $\{A_n^i\}_{n=1}^{\infty} \subset \mathcal{F}$ for $i \in \{1, \ldots, k\}$ such that $|f(x) - z_i| \leq \frac{1}{n}$ on A_n^i, $A_n^i \cap A_m^j = \emptyset$ if $i \neq j$ or $m \neq n$, $\mu(A_n^i) > 0$ and

$$\mu\left(\bigcup_{n=1}^{\infty} A_n^i\right) = \sum_{n=1}^{\infty} \mu(A_n^i) < \infty.$$

Proof. By σ-finiteness of $(\mathcal{M}, \mathcal{F}, \mu)$ we know $\mathcal{M}$ can be divided into countably many disjoint sets of finite measure. Each of these sets can only contain countably many elements from A and these elements must all have finite measure. Hence A must be countable and all singletons must have finite measure.

Now if $\lambda \in B$ then we may by Lemma A.4 pick a sequence of disjoint elements $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{F}$ such that $0 < \mu_{A^c}(E_i) < \infty$ and $|f - \lambda| \leq \frac{1}{i}$ on E_i for all i. After removing the μ_{A^c} zero set A, we may assume $E_i \subset A^c$ for all i, and then $\mu(E_i)^{-\frac{1}{2}}1_{E_i}$ will be a Weyl sequence for λ. If $\lambda \in C$, we let $\{\lambda_n\}_{n=1}^{\infty}$ be the corresponding sequence. Then $\mu(\{\lambda_i\})^{-\frac{1}{2}}1_{\{\lambda_i\}}$ will be a Weyl sequence for λ.

To prove the other inclusion, let $\lambda \in (B \cup C)^c$. If $\mu(\{|f - \lambda| < n^{-1}\} \cap A^c) > 0$ or $A \cap \{0 < |f - \lambda| < n^{-1}\} \neq \emptyset$ for all $n \in \mathbb{N}$ then $\lambda \in B \cup C$ which would be a contradiction. Also $A \cap \{|f - \lambda| = 0\}$ must also be finite since we would otherwise have $\lambda \in C$.

Hence there is an $N \in \mathbb{N}$ such that $A^c \cap \{|f - \lambda| < N^{-1}\}$ is a zero set and $\{|f - \lambda| < N^{-1}\} \cap A = \{|f - \lambda| = 0\} \cap A$ is a finite set. In particular the spectral projection of f onto $(\lambda - N^{-1}, \lambda + N^{-1})$ is given by $1_{\{|f-\lambda|<N^{-1}\}} = 1_{\{|f-\lambda|=0\} \cap A}$ almost everywhere. Since $\{|f - \lambda| = 0\} \cap A$ is finite we see that $1_{\{|f-\lambda|=0\} \cap A}$ has finite-dimensional range and so $\lambda \in \sigma_{\text{ess}}(M_f)^c$.

The construction of the sets goes as follows. Assume first that $z_1, \ldots, z_k$ are different. For each $i \in \{1, \ldots, k\}$ we either have $z_i \in B$ or $z_i \in C$. If $z_i \in B$ then by Lemma A.4 pick a sequence of disjoint elements $\{\widetilde{A}_n^i\}_{n=1}^{\infty}$ in $\mathcal{F}$ such that $0 < \mu_{A^c}(\widetilde{A}_n^i)$ and $|f - \lambda| \leq \frac{1}{n}$ on $\widetilde{A}_n^i$ for all n. After removing the μ_{A^c} zero set A, we may assume $\widetilde{A}_n^i \subset A^c$ for all i and so we have

$$\mu\left(\bigcup_{n=1}^{\infty} \widetilde{A}_n^i\right) = \sum_{n=1}^{\infty} \mu(\widetilde{A}_n^i) = \sum_{n=1}^{\infty} \mu_{A^c}(\widetilde{A}_n^i) < \infty.$$

If $z_i \in C$ and $\{\lambda_n^i\}_{n=1}^{\infty}$ is the corresponding sequence we let $\widetilde{A}_n^i = \{\lambda_n^i\}$ which is a disjoint collection. Then $0 < \mu(\widetilde{A}_n^i)$ and $|f - \lambda| \leq \frac{1}{n}$ on $\widetilde{A}_n^i$ for all n. Furthermore, since $\mu(A) < \infty$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} \widetilde{A}_n^i\right) = \sum_{n=1}^{\infty} \mu(\widetilde{A}_n^i) \leq \mu(A) < \infty.$$

Now pick N so large that $2N^{-1} < \max_{i \neq j} |z_i - z_j|$ and define $A_n^i = \widetilde{A}_{N+n}^i$. Then

$$A_n^i \subset \left\{ |f - z_i| < \frac{1}{N+n} \right\} \subset \left\{ |f - z_i| < \frac{1}{n} \right\}$$
$$\mu\left(\bigcup_{n=1}^{\infty} A_n^i\right) = \sum_{n=1}^{\infty} \mu(A_n^i) \leq \sum_{n=1}^{\infty} \mu(\widetilde{A}_n^i) < \infty$$

and $0 < \mu(A_n^i)$ for all i, n. If $x \in A_n^i \cap A_m^j$ for $i \neq j$ we would have $|z_i - z_j| \leq |z_i - f(x)| + |f(x) - z_j| < \frac{2}{N}$ which is a contradiction. So $A_n^i \cap A_m^j = \emptyset$ if $i \neq j$. If $i = j$ and $n \neq m$ we find $A_n^i \cap A_m^j = \widetilde{A}_{N+n}^i \cap \widetilde{A}_{N+m}^i = \emptyset$. Thus we are now finished in the case where $z_1, \dots, z_k$ are different.

If $z_1, \dots, z_k$ are not all different, let $\lambda_1, \dots, \lambda_\ell$ be the different elements and $h(i)$ be such that $z_i = \lambda_{h(i)}$. We now pick sequences $\{\widetilde{A}_n^h\}_{n=1}^{\infty} \subset \mathcal{F}$ as in the theorem for the collection $\lambda_1, \dots, \lambda_\ell \in \sigma_{\text{ess}}(M_f)$ and define $A_n^i = \widetilde{A}_{i+kn}^{h(i)}$. Observe that

$$A_n^i \subset \left\{ |f - \lambda_{h(i)}| < \frac{1}{i+kn} \right\} \subset \left\{ |f - z_i| < \frac{1}{n} \right\}$$
$$\mu\left(\bigcup_{n=1}^{\infty} A_n^i\right) = \sum_{n=1}^{\infty} \mu(A_n^i) \leq \sum_{n=1}^{\infty} \mu(\widetilde{A}_n^i) < \infty$$

and $0 < \mu(A_n^i)$ for all i, n. Note that $A_n^i \cap A_m^j = \widetilde{A}_{i+kn}^{h(i)} \cap \widetilde{A}_{j+mk}^{h(j)}$. If $i \neq j$ or $n \neq m$ then $j + mk \neq i + nk$ since $1 \leq i, j \leq k$ and so the intersection is empty. This finishes the proof. □

We now have one more definition.

Definition A.6. *Let $(\mathcal{M}, \mathcal{F}, \mu)$ be a measure space.*

(1) We say that $(\mathcal{M}, \mathcal{F}, \mu)$ has strong topological properties if $\mathcal{M}$ is a locally compact, Hausdorff and second countable topological space, $\mathcal{F}$ is the Borel σ-algebra and μ is finite on compact sets.

(2) Let $\mathcal{M}$ be a metric space. We say that $\mathcal{M}$ can be **cut nicely** *if for each $n \in \mathbb{N}$ there is a sequence of disjoint sets $\{G_\alpha^n\}_{\alpha \in \mathbb{N}} \subset \mathcal{B}(\mathcal{M})$ that covers M such that $\sup_{\alpha \in \mathbb{N}} \text{Diam}(G_\alpha^n)$ converges to 0 as n tends to infinity, $\overline{G_\alpha^n}$ is compact and for any $B \subset \mathcal{M}$ bounded the set*

$$\{\alpha \in \mathbb{N} \mid G_\alpha^n \cap B \neq \emptyset\}$$

is finite.

The following result can be found in standard references on measure theory and topological spaces.

Lemma A.7. *We have the following statements:*

(1) *Let $\mathcal{M}$ be a locally compact metric space, and $A \subset \mathcal{M}$ a compact set. Then there is an $r > 0$ such that*

$$V_r = \{x \in \mathcal{M} \mid dist(x, A) \leq r\}$$

is compact. In particular for $\alpha \leq r$ the set

$$\{x \in \mathcal{M} \mid dist(x, A) < \alpha\}$$

is an open neighbourhood of A with compact closure.

(2) *If $(\mathcal{M},\mathcal{F},\mu)$ has strong topological properties then the subspace $C_c(\mathcal{M})$ is dense in $L^p(\mathcal{M},\mathcal{F},\mu)$ for all $1\leq p<\infty$. Furthermore $L^p(\mathcal{M},\mathcal{F},\mu)$ is separable for all $1\leq p<\infty$ and elements in $C_c(\mathcal{M})$ are uniformly continuous if $\mathcal{M}$ is metric.*

We have the following important theorem:

Theorem A.8. *Assume now that $(\mathcal{M},\mathcal{F},\mu)$ has strong topological properties and that $\mathcal{M}$ can be cut nicely. Assume that $\mathcal{M}=A\cup B$ with $A,B\in\mathcal{F}$ and $A\cap B=\emptyset$ such that B is bounded. Assume furthermore that for each $n\in\mathbb{N}$ we have a subdivision*

$$B=\bigcup_{\ell=1}^{\widetilde{k}_n}\widetilde{B}_\ell^n$$

such that $\widetilde{B}_\ell^n\cap\widetilde{B}_j^n=\emptyset$ when $j\neq\ell$. Then there are subdivisions

$$A=H_n\cup\bigcup_{\ell=1}^{K_n}A_\ell^n$$
$$B=J_n\cup\bigcup_{j=1}^{k_n}B_j^n$$

such that $K_n\in\mathbb{N}\cup\{\infty\}$ and $k_n\in\mathbb{N}$ for all n. Furthermore

(1) *For each j,n there is j' such that $B_j^n\subset\widetilde{B}_{j'}^n$.*
(2) *$\mu(H_n)=0=\mu(J_n)$ and $0<\mu(A_\ell^n),\mu(B_\ell^n)<\infty$ for all ℓ,n.*
(3) *$A_\ell^n\cap A_{\ell'}^n=\emptyset=B_j^n\cap B_{j'}^n$ when $\ell\neq\ell'$ or $j\neq j'$ and $H_n\cap A_\ell^n=\emptyset=J_n\cap B_j^n$ for all ℓ,j and n.*
(4) *Let P_n be the projection onto $\overline{\mathrm{Span}(\mathcal{B}_n)}$ where*

$$\mathcal{B}_n=\{1_{B_j^n}\mid j\in\{1,\dots,k_n\}\}\cup\{1_{A_\ell^n}\mid \ell\in\mathbb{N}\cap[0,K_n]\}.$$

Then P_n converges strongly to the identity and P_n is given by

$$P_n(f)=\sum_{j=1}^{k_n}\frac{1}{\mu(B_j^n)}\int_{B_j^n}f(x)d\mu(x)1_{B_j^n}+\sum_{\ell=1}^{K_n}\frac{1}{\mu(A_\ell^n)}\int_{A_\ell^n}f(x)d\mu(x)1_{A_\ell^n}.$$

Proof. For each $n\in\mathbb{N}$ let $\{G_\alpha^n\}_{\alpha\in\mathbb{N}}$ be the sets from Definition A.6. Fix $n\in\mathbb{N}$. Define for any set $C\in\mathcal{F}$

$$I_n(C)=\{\alpha\in\mathbb{N}\mid \mu(G_\alpha^n\cap C)>0\}.$$

If C is bounded then $I_n(C)$ is finite since there is only finitely many α such that $G_\alpha^n\cap C\neq\emptyset$. We have

$$\begin{aligned}B&=\bigcup_{j=1}^{\widetilde{k}_n}\Bigg(\bigcup_{\alpha\in I_n(\widetilde{B}_j^n)}\widetilde{B}_j^n\cap G_\alpha^n\Bigg)\cup\bigcup_{j=1}^{\widetilde{k}_n}\Bigg(\bigcup_{\alpha\in I_n(\widetilde{B}_j^n)^c}\widetilde{B}_j^n\cap G_\alpha^n\Bigg)\\&:=\bigcup_{j=1}^{\widetilde{k}_n}\Bigg(\bigcup_{\alpha\in I_n(\widetilde{B}_j^n)}\widetilde{B}_j^n\cap G_\alpha^n\Bigg)\cup J_n\\A&=\Bigg(\bigcup_{\alpha\in I_n(A)}A\cap G_\alpha^n\Bigg)\cup\Bigg(\bigcup_{\alpha\in I_n^c(A)}A\cap G_\alpha^n\Bigg):=\Bigg(\bigcup_{\alpha\in I_n(A)}A\cap G_\alpha^n\Bigg)\cup H_n\end{aligned}$$

Since $I_n(\widetilde{B}_j^n)^c$ and $I_n(A)^c$ are countable for all j we find that the sets J_n and H_n are a nullsets. The A_j^n are obtained by numbering the countable collection

$(A \cap G^n_\alpha)_{\alpha \in I_n(A)}$. Similarly the B^n_j are obtain by numbering the finite collection $\widetilde{B}^n_j \cap G^n_\alpha$ for $j \in \{1, \dots, \widetilde{k}_n\}$ and $\alpha \in I_n(\widetilde{B}^n_j)$.

The properties in statements (1) and (3) are clear. Statement (2) follows from the definition of the $I_n(\widetilde{B}^n_j)$ and $I_n(A)$ and the fact that elements of the form $C \cap G^n_\alpha$ have compact closure (and thus finite measure) since G^n_α has compact closure. Note now for later reference that

$$\sup_{\ell,j}\{\mathrm{Diam}(A^n_\ell), \mathrm{Diam}(B^n_j)\} \leq \sup_{\alpha \in \mathbb{N}} \mathrm{Diam}(G^n_\alpha) \tag{A.2}$$

which converges to 0 for n tending to ∞. What remains is statement (4). The projection onto $\overline{\mathrm{Span}(\mathcal{B})}$ is obviously given by

$$P_n f = \sum_{j=1}^{k_n} \frac{1}{\mu(B^n_j)} \int_{B^n_j} f(x) d\mu(x) 1_{B^n_j} + \sum_{l=1}^{K_n} \frac{1}{\mu(A^n_\ell)} \int_{A^n_\ell} f(x) d\mu(x) 1_{A^n_\ell}$$

where the limit is taken in $L^2(\mathcal{M}, \mathcal{F}, \mu)$ if $K_n = \infty$. Since the corresponding functions also converge pointwise, we can take the pointwise limit to represent the limit in $L^2(\mathcal{M}, \mathcal{F}, \mu)$. It remains to prove that P_n converges strongly to the identity. It is enough to check this on $C_c(\mathcal{M})$ since this space is dense in $L^2(\mathcal{M}, \mathcal{F}, \mu)$.

To prove the statement for $f \in C_c(\mathcal{M})$ we start by noting that if $\mu(\{f \neq 0\}) = 0$ then $P_n f = 0$ for all n and $f = 0$ in $L^2(\mathcal{M}, \mathcal{F}, \mu)$ so the convergence holds in this case. Assume now that $\{f \neq 0\}$ has positive measure and hence that $\mathrm{supp} f$ has positive but finite measure since it is compact. We start by proving that there is a compact set K and a number $N_1 \in \mathbb{N}$ such that K will contain $\mathrm{supp}(f)$ and $\mathrm{supp}(P_n f)$ for all $n \geq N_1$.

By Lemma A.7 there is an $r > 0$ such that $K = \{x \in \mathcal{M} \mid \mathrm{dist}(x, \mathrm{supp}(f)) \leq r\}$ is compact. Now pick $N_1 \in \mathbb{N}$ such that $\sup_{\ell,j}\{\mathrm{Diam}(A^n_\ell), \mathrm{Diam}(B^n_j)\} < r$ for all $n \geq N_1$ (this is possible by equation (A.2)). Let $n \geq N_1$. If $P_n f(x) \neq 0$ then there is a set C of diameter smaller than r such that $x \in C$ and the integral of f over C is not 0. In particular $\mathrm{supp}(f) \cap C \neq \emptyset$ so $x \in K$. This implies that $\{P_n f \neq 0\} \subset K$ and hence $\mathrm{supp}(P_n f) \subset K$ for all $n \geq N_1$, proving the claim.

We now prove convergence. Let $\varepsilon > 0$. Pick a compact set K containing $\mathrm{supp}(f)$ and $\mathrm{supp}(P_n(f))$ for all $n \geq N_1$. Since f is uniformly continuous on $\mathcal{M}$ there is a $\delta > 0$ such that when $x, y \in \mathcal{M}$, $d(x, y) \leq \delta$ then $|f(x) - f(y)| \leq \frac{\varepsilon}{\sqrt{\mu(K)}}$. Pick $N_2 \in \mathbb{N}$ such that $\sup_{\ell,j}\{\mathrm{Diam}(A^n_\ell), \mathrm{Diam}(B^n_j)\} < \delta$ for all $n \geq N_2$ and fix $n \geq N = \max\{N_1, N_2\}$. For $x \in \mathcal{M} \backslash (H_n \cup J_n)$ there is a set C of diameter less than δ such that $x \in C$ and

$$P_n f(x) = \frac{1}{\mu(C)} \int_C f(y) d\mu(y).$$

This implies

$$|P_n f(x) - f(x)| \leq \frac{1}{\mu(C)} \int_C |f(y) - f(x)| d\mu(y) \leq \frac{\varepsilon}{\mu(K)^{\frac{1}{2}}},$$

which gives

$$\|P_n f - f\| = \left(\int_K |P_n f(x) - f(x)|^2 d\mu(x) \right)^{\frac{1}{2}} \leq \frac{\mu(K)^{\frac{1}{2}} \varepsilon}{\mu(K)^{\frac{1}{2}}} = \varepsilon.$$

This finishes the proof. □

The following two lemmas show that Theorems A.5 and A.8 can be applied to a wide range of L^2-spaces.

Lemma A.9. *Let $A \subset \mathbb{Z}$ and let μ be some measure on $(A \times \mathbb{R}^\nu, \mathcal{B}(A \times \mathbb{R}^\nu))$ which is finite on compact sets. Then the assumptions of Theorems A.5 and A.8 are satisfied if $\mu(B) < \infty$ where $B = \{x \in A \times \mathbb{R}^\nu \mid \mu(\{x\}) > 0\}$ and μ_{B^c} is zero on sets of the form $\{i\} \times C$ with $C = \{x \in \mathbb{R}^\nu \mid \|x\| = c\}$.*

Proof. Since $A \times \mathbb{R}^\nu \subset \mathbb{R}^{\nu+1}$ is closed we see that $A \times \mathbb{R}^\nu$ is a locally compact metric space. It is second countable since both A and $\mathbb{R}^\nu$ are second countable. $A \times \mathbb{R}^\nu$ can be covered by compacts so $(A \times \mathbb{R}^\nu, \mathcal{B}(A \times \mathbb{R}^\nu), \mu)$ is σ-finite. In particular B is countable and therefore measurable.

Define $U_x = \{y \in \mathbb{R}^{\nu+1} \mid \|y\| \leq x\} \cap (A \times \mathbb{R}^\nu)$. Then 1_{U_x} converges to 1_{U_y} pointwise for $x \to y$ except at points in ∂U_y. Note that ∂U_y is a finite union of sets of the form $\{i\} \times \{x \in \mathbb{R}^\nu \mid \|x\| = c\}$ with $c \geq 0$ and $i \in \mathbb{Z}$. Hence ∂U_y is a μ_{B^c} zeroset, proving that $\{U_x\}_{x \in [0,\infty)}$ defines a continuous resolution for μ_{B^c}.

To show that $\mathcal{M}$ cuts nicely we define

$$G^n_{\alpha,i} = \{i\} \times \left(\bigtimes_{i=1}^{\nu} (n^{-1}\alpha_i, n^{-1}(\alpha_i + 1)] \right)$$

for $\alpha \in \mathbb{Z}^n$ and $i \in A$. For each n this is a disjoint cover and elements have diameter $\sqrt{\nu} n^{-1}$. Given any bounded set D, there is $A_1 \subset A$ finite and $R \in \mathbb{N}$ such that $D \subset A_1 \times [-R, R]^\nu$. It will only take finitely many of the $G^n_{(\alpha,i)}$ to cover $A_1 \times [-R, R]^\nu$ and hence D. □

The following lemma is central to the spectral analysis.

Lemma A.10. *Let $\mathcal{H}$ be a separable Hilbert space and let A be selfadjoint on $\mathcal{H}$. Then $\mathcal{H}$ is unitarily equivalent to $L^2(\mathcal{M}, \mathcal{F}, \mu)$ such that A is transformed into a multiplication operator ω. If $A \geq \gamma$ for some $\gamma \in \mathbb{R}$ then we may pick $\omega \geq \gamma$ everywhere. Furthermore if γ is not an eigenvalue then $\omega > \gamma$ almost everywhere. Also $(\mathcal{M}, \mathcal{F}, \mu)$ will fulfil the conditions in Theorems A.5 and A.8.*

Proof. We follow the construction found in [22]. Let $\{\psi_n\}_{n \in B}$ (where $B \subset \mathbb{N}$) be a complete collection of normed cyclic vectors, and let μ_n be the measure generated my ψ_n with respect to the spectral measure P_A of A. That is $\mu_n(C) = \langle \psi_n, P_A(C)\psi_n \rangle$. By [22] we see that $\mathcal{H}$ is unitarily equivalent to

$$\mathcal{K}_1 = \bigoplus_{n \in B} L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_n),$$

and A acts like multiplication by the identity map $f(x) = x$ on each of the component spaces. If $A \geq \gamma$ then μ_n will be supported on $[\gamma, \infty)$ showing we may pick $\widetilde{\omega} = \max\{f, \gamma\}$. By standard measure theory of kernels, there is measure $\tilde{\mu}$ on $\mathcal{B}(B \times \mathbb{R})$ defined by

$$\int_{B \times \mathbb{R}} f(n,k) d\tilde{\mu}(n,k) = \sum_{n \in B} \int f(n,x) d\mu_n(x)$$

for any non negative and measurable map $f : B \times \mathbb{R} \to \mathbb{R}$. Then the direct sum of L^2-spaces is clearly unitarily equivalent to

$$\mathcal{K}_2 = L^2\left(B \times \mathbb{R}, \mathcal{B}(B \times \mathbb{R}), \widetilde{\mu}\right)$$

and A is now multiplication by the map $\omega(n,x) = \widetilde{\omega}(x)$. Note that each $\{n\} \times \mathbb{R}$ has measure 1 by construction, so the measure space is σ-finite. Hence there is a strictly positive measurable map f on $B \times \mathbb{R}$ which integrates to 1. Defining $\mu = f\widetilde{\mu}$ we thus obtain a finite measure, and multiplication by $f^{-\frac{1}{2}}$ defines a unitary map from $\mathcal{K}_2$ to

$$L^2(B \times \mathbb{R}, \mathcal{B}(B \times \mathbb{R}), \mu).$$

Note that A still acts like multiplication by ω on this space. Since $\mu(B \times \mathbb{R}) = 1$, and sets of the form $\{i\} \times \{x \in \mathbb{R} \mid |x| = c\}$ are finite we see that the measure space satisfies the conditions in Lemma A.9. This finishes the proof. □

Appendix B. Spectral Theory of tensor products

I this section we list a few results regarding the tensor product of operators. A good reference for these results are [20]. Throughout this section let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be a finite collection of Hilbert spaces. For $V_i \subset \mathcal{H}_i$ subspaces, we define the algebraic tensor product

$$V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_n = \operatorname{Span}\{x_1 \otimes \cdots \otimes x_n \mid x_i \in V_i\}.$$

Most of the content in the following theorem can be found in [20]. The remaining items can easily be deduced.

Theorem B.1. *Let T_i be an operator on $\mathcal{H}_i$ for $i \in \{1, \dots, n\}$. Then there is a unique linear map $T = T_1 \widehat{\otimes} \cdots \widehat{\otimes} T_n$ defined on $\mathcal{D}(T_1) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{D}(T_n)$ such that*

$$T_1 \widehat{\otimes} \cdots \widehat{\otimes} T_n (x_1 \otimes \cdots \otimes x_n) = T_1 x_1 \otimes \cdots \otimes T_n x_n,$$

for all $x_i \in \mathcal{D}(T_i)$ and $i \in \{1, \dots, n\}$. We also have the following:

(1) *If all T_i are densely defined then T is densely defined and $T_1^* \widehat{\otimes} \cdots \widehat{\otimes} T_n^* \subset T^*$.*
(2) *If all T_i are closable, then T is closable. We will then write $\overline{T} = T_1 \otimes \cdots \otimes T_n$. Furthermore, the following identities hold*

$$T_1 \otimes \cdots \otimes T_n = \overline{T}_1 \otimes \cdots \otimes \overline{T}_n$$
$$T_1^* \otimes \cdots \otimes T_n^* = (T_1 \otimes \cdots \otimes T_n)^*.$$

(3) *If all the T_i are symmetric (selfadjoint, unitary, a projection), then T is symmetric (selfadjoint, unitary, a projection).*
(4) *If $T_j \geq 0$ for all $j \in \{1, \dots, n\}$ then $T \geq 0$.*
(5) *If all the T_i are bounded then T is bounded and*

$$\|T\| = \|T_1\| \cdots \|T_n\| = \|T_1 \otimes \cdots \otimes T_n\|.$$

The following result is also important.

Theorem B.2. *For each $j \in \{1, \dots, n\}$ let T_j be a selfadjoint operator on $\mathcal{H}_i$ and define*

$$H_i = 1 \otimes \cdots \otimes T_i \otimes \cdots \otimes 1,$$
$$H = H_1 + H_2 + \cdots + H_n.$$

Then

(1) *$(H_1, \cdots, H_n)$ is a touple of strongly commuting selfadjoint operators with $\sigma(H_i) = \sigma(T_j)$. The joint spectrum is $\sigma(T_1) \times \cdots \times \sigma(T_n)$ and if $f : \mathbb{R} \to \mathbb{C}$ is Borel measurable then $f(H_j) = 1 \otimes \cdots \otimes f(T_j) \otimes \cdots \otimes 1$.*
(2) *H is essentially selfadjoint with*

$$e^{it\overline{H}} = e^{itT_1} \otimes \cdots \otimes e^{itT_n} \quad t \in \mathbb{R}.$$

(3) *If V_j is a core for T_j then $V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_n$ is a core for $\overline{H}$.*
(4) *Assume T_j is semibounded with $\inf(\sigma(T_j)) = \lambda_j$ for all j. Then H is selfadjoint and semibounded with $\inf(\sigma(H)) = \lambda := \lambda_1 + \cdots + \lambda_n$. Let P_B denote the spectral measure for an operator $B \in \{H, T_1, \dots, T_n\}$. Then*

$$e^{-tH} = e^{-tT_1} \otimes \cdots \otimes e^{-tT_n} \quad t \geq 0$$
$$P_H(\{\lambda\}) = P_{T_1}(\{\lambda_1\}) \otimes \cdots \otimes P_{T_n}(\{\lambda_n\}).$$

In particular, $\mathrm{Dim}(P_H(\{\lambda\})) = \mathrm{Dim}(T_1(\{\lambda\})) \cdots \mathrm{Dim}(T_n(\{\lambda\}))$*. Let* $\mu_j = \inf(\sigma_{\mathrm{ess}}(T_j))$ *which may be* ∞*. Then*

$$\inf(\sigma_{\mathrm{ess}}(\overline{H})) \geq \min_j \left\{ \mu_j + \sum_{l \neq i} \lambda_j \right\} := m.$$

(5) *Assume* B_i *is selfadjoint on* $\mathcal{H}_i$*. If* $\mathcal{D}(T_i) \subset \mathcal{D}(B_i)$ *for some* $i \in \{1, ..., n\}$ *then* $\mathcal{D}(H_i) \subset \mathcal{D}(1 \otimes \cdots \otimes B_i \otimes \cdots \otimes 1)$*.*

(6) *Assume* B_i *is selfadjoint on* $\mathcal{H}_i$ *and* $T_i + B_i$ *is selfadjoint. Then*

$$H_i + 1 \otimes \cdots \otimes B_i \otimes \cdots \otimes 1 = 1 \otimes \cdots \otimes (T_i + B_i) \otimes \cdots \otimes 1 := S_i$$

Proof. Statements (1)-(3) can more or less be found in either [20] or [23]. It is proven in [20] that $f(H_j) = 1 \otimes \cdots \otimes f(T_j) \otimes \cdots \otimes 1$ holds for $f(x) = (x \pm i)^{-1}$. From there one may simply use standard approximation arguments. To prove (4), let P be a joint spectral measure of $A = (H_1, \ldots, H_n)$. Using that the joint spectrum is $\sigma(T_1) \times \cdots \times \sigma(T_n)$, one may show that $H = P(f)$ where $f(x_1, ..., x_n) = x_1 + \cdots + x_n$. So H is selfadjoint and bounded below by λ. The formula for e^{-tH} is now immediate from the spectral theorem. We also find

$$\begin{aligned} P_H(\lambda) &= P(\{x_1 + \cdots + x_n = \lambda\} \cap \sigma(T_1) \times \cdots \times \sigma(T_n)) \\ &= P(\{x_1 = \lambda_1\} \times \cdots \times \{x_n = \lambda_n\}) \\ &= P_{T_1}(\{\lambda_1\}) \otimes \cdots \otimes P_{T_n}(\{\lambda_n\}). \end{aligned}$$

Let $f \in C_c^\infty((-\infty, m))$. Then there is $\varepsilon > 0$ such that f is supported in $(-\infty, m - \varepsilon)$. We observe

$$P_H(f) = \int_{\sigma(T_1) \times \cdots \times \sigma(T_n)} f(x_1 + \cdots + x_n) dP(x_1, \ldots, x_n).$$

If $(x_1, \ldots, x_n) \in \sigma(T_1) \times \cdots \times \sigma(T_n)$ and $f(x_1 + \cdots + x_n) \neq 0$, then $x_i < \mu_i - \varepsilon$ for all i. Since only finitely many x_i have this property, we find $P_H(f)$ is a finite linear combination of terms of the form

$$P_{T_1}(\{x_1\}) \otimes \cdots \otimes P_{T_n}(\{x_n\})$$

with x_i in the discrete spectrum of T_i. The above projection has finite rank and is therefore compact, so $P_H(f)$ is compact.

To prove (5), note $B_j(T_j + i)^{-1}$ is bounded and

$$(1 \otimes \cdots \otimes B_j \otimes \cdots \otimes 1)(H_j + i)^{-1} = 1 \otimes \cdots \otimes B_j(T_j + i)^{-1} \otimes \cdots \otimes 1$$

holds on a dense set. Thus $(1 \otimes \cdots \otimes B_j \otimes \cdots \otimes 1)(H_j + i)^{-1}$ extends to a bounded operator implying the claim.

To prove (6), note that the equality holds on $\mathcal{H}_1 \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{D}(T_j + B_j) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{H}_n$ which is a core for S_j. Therefore

$$S_j = \overline{H_j + 1 \otimes \cdots \otimes B_j \otimes \cdots \otimes 1} \tag{B.1}$$

By part (5) we note $\mathcal{D}(S_j) \subset \mathcal{D}(H_j) \cap \mathcal{D}(1 \otimes \cdots \otimes B_j \otimes \cdots \otimes 1)$ so the closure on the right side of $(B.1)$ is not necessary. $\square$

The next two results are important to the theory developed in this paper.

Lemma B.3. *Let* A *and* B *be selfadjoint on* $\mathcal{H}_2$*. If* B *is* A*-bounded with bound* a*, and* $C \in B(\mathcal{H}_1)$ *then* $C \otimes B$ *is* $1 \otimes A$ *bounded with relative bound* $a\|C\|$*.*

Proof. On simple tensors we see that

$$C \otimes B(1 \otimes A - i\lambda)^{-1} = C \otimes B(A - i\lambda)^{-1}$$

which is bounded. Hence $\mathcal{D}(C \otimes B) \subset \mathcal{D}(1 \otimes A)$ and the above identity extends to the full tensor product. Calculating norms and taking λ to ∞ gives the $1 \otimes A$-bound by [22, Lemma 6.3]. Using Theorem B.1 to calculate the norm finishes the proof. □

We now wish to consider second quantised observables. Let ω be selfadjoint on the space $\mathcal{H}$. Let $d\Gamma(\omega)$ denote the second quantised observable. By standard theory of reducing subspaces we have

$$\sigma_p(d\Gamma^{(n)}(\omega)) \subset \sigma_p\left(\sum_{k=1}^{n}(1\otimes)^{k-1}\omega(\otimes 1)^{n-k}\right) \tag{B.2}$$

$$\sigma_{\text{ess}}(d\Gamma^{(n)}(\omega)) \subset \sigma_{\text{ess}}\left(\sum_{k=1}^{n}(1\otimes)^{k-1}\omega(\otimes 1)^{n-k}\right) \tag{B.3}$$

Lemma B.4. *Let $\omega \geq 0$ be a selfadjoint operator defined on the Hilbert space $\mathcal{H}$ and let $m = \inf(\sigma(\omega))$ and $m_{\text{ess}} = \inf(\sigma_{\text{ess}}(\omega))$. Let $d\Gamma(\omega)$ denote the second quantised observable. Then for $n \geq 1$ we have*

$$\begin{aligned}\sigma(d\Gamma^{(n)}(\omega)) &= \overline{\{\lambda_1 + \cdots + \lambda_n \mid \lambda_i \in \sigma(\omega)\}}\\ \inf(\sigma(d\Gamma^{(n)}(\omega))) &= nm\end{aligned}$$

Now let $\omega \geq 0$ be injective. Then

(1) 0 *is an eigenvalue for $d\Gamma(\omega)$ with multiplicity 1. The eigenspace is spanned by Ω.*
(2) $\inf(\sigma_{\text{ess}}(d\Gamma^{(n)}(\omega))) \geq m_{\text{ess}} + (n-1)m$
(3) *$d\Gamma(\omega)$ will have compact resolvents if and only if this is the case for ω.*

Proof. The statements regarding the spectrum is easy and can be found in e.g, [12]. We prove the statements (1), (2) and (3).

To prove statement (1), we note that Ω is an eigenvector as desired. Assume that there exists an eigenvector ψ orthogonal to Ω. We may then assume that there is $n \geq 1$ such that ψ is in the n'th particle sector and an eigenvector for $d\Gamma^{(n)}(\omega)$ with eigenvalue 0. Since $d\Gamma^{(n)}(\omega) \geq nm$ we find $m = 0$ and thus $m \in \sigma(\omega)$ but is not an eigenvalue. By Theorem B.2 and equation (B.2) we find that $d\Gamma^{(n)}(\omega)$ is injective, reaching a contradiction.

Statement (2) follows from Theorem B.2 and equation (B.3).

If $d\Gamma(\omega)$ has compact resolvents then projection onto the one particle subspace shows that ω has compact resolvents. If ω has compact resolvents, then $m_{\text{ess}} = \infty$ and so $m > 0$. Statement (2) now gives that $d\Gamma^{(n)}(\omega)$ has compact resolvents for all $n \in \mathbb{N}$. Now $\|(d\Gamma^{(n)}(\omega) + i)^{-1}\| \leq \frac{1}{nm}$ which converges to 0 as n tends to ∞. Hence we find

$$(d\Gamma(\omega) + i)^{-1} = \bigoplus_{n=0}^{\infty}(d\Gamma^{(n)}(\omega) + i)^{-1}$$

is compact. □

Appendix C. Isomorphism theorems

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Then

$$\mathcal{F}_b(\mathcal{H}_1 \oplus \mathcal{H}_2) \approx \mathcal{F}_b(\mathcal{H}_1) \otimes \mathcal{F}_b(\mathcal{H}_2) \approx \bigoplus_{n=0}^{\infty}\left(\mathcal{F}_b(\mathcal{H}_1) \otimes \mathcal{H}_2^{\otimes_s n}\right).$$

In this chapter we investigate these isomorphisms. See also [5] and [15].

Theorem C.1. *There is a unique isomorphism $U\colon \mathcal{F}_b(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathcal{F}_b(\mathcal{H}_1) \otimes \mathcal{F}_b(\mathcal{H}_2)$ such that $U(\epsilon(f \oplus g)) = \epsilon(f) \otimes \epsilon(g)$. If $f_1, \dots, f_j \in \mathcal{H}_1, g_1$ and $\dots, g_\ell \in \mathcal{H}_2$ then*

$$U((f_1,0) \otimes_s \cdots \otimes_s (f_j,0) \otimes_s (0,g_1) \otimes_s \cdots \otimes_s (0,g_\ell)) = \left(\frac{j!\ell!}{(j+\ell)!}\right)^{1/2} (f_1 \otimes_s \cdots \otimes_s f_j) \otimes (g_1 \otimes_s \cdots \otimes_s g_\ell)$$

The map also has the following transformation properties. If A_i is selfadjoint on $\mathcal{H}_i$, V_i is unitary on $\mathcal{H}_i$ and $f \in \mathcal{H}_1, g \in \mathcal{H}_2$ then

$$UW(f \oplus g, V_1 \oplus V_2)U^* = W(f,V_1) \otimes W(g,V_2) \tag{C.1}$$

$$Ud\Gamma(A_1 \oplus A_2)U^* = \overline{d\Gamma(A_1) \otimes 1 + 1 \otimes d\Gamma(A_2)} \tag{C.2}$$

$$U\varphi(f,g)U^* = \overline{\varphi(f) \otimes 1 + 1 \otimes \varphi(g)} \tag{C.3}$$

$$Ua(f,g)U^* = \overline{a(f)\widehat{\otimes}1 + 1\widehat{\otimes}a(g)} \tag{C.4}$$

$$Ua^\dagger(f,g)U^* = \overline{a^\dagger(f)\widehat{\otimes}1 + 1\widehat{\otimes}a^\dagger(g)}. \tag{C.5}$$

Proof. The set of exponential vectors are total and linearly independent. Hence at most map can satisfy $U(\epsilon(f \oplus g)) = \epsilon(f) \otimes \epsilon(g)$. By the linear independence we may define $U(\epsilon(f \oplus g)) = \epsilon(f) \otimes \epsilon(g)$ and extend by linearity. Note that the image of exponential vectors is total and the map conserves the inner product since

$$\langle \epsilon(h_1 \oplus h_2), \epsilon(f_1 \oplus f_2)\rangle = e^{\langle h_1 \oplus h_2, f_1 \oplus f_2\rangle} = \langle \epsilon(h_1), \epsilon(f_1)\rangle \langle \epsilon(h_2), \epsilon(f_2)\rangle.$$

Hence it extends by continuity to a unitary map. To prove equation (C.1), it is enough to check a total set. We calculate

$$\begin{aligned} UW(f \oplus g, V_1 \oplus V_2)U^*\epsilon(h_1) \otimes \epsilon(h_2) &= U\epsilon(V_1h_1 \oplus V_2h_2 + f \oplus g) \\ &= \epsilon(V_1h_1 + f) \otimes \epsilon(V_2h_2 + g) \\ &= W(f,V_1) \otimes W(g,V_2)(\epsilon(h_1) \otimes \epsilon(h_2)). \end{aligned}$$

This also proves equations (C.2) and (C.3) since both sides generate the same unitary group. To prove equations (C.4) and (C.5) define

$$\mathcal{C} = \{\epsilon(f_1) \otimes \epsilon(f_2) \mid f_i \in \mathcal{H}_i\} = U(\{\epsilon(f_1 \oplus f_2) \mid f_i \in \mathcal{H}_i\}).$$

for $\psi = \epsilon(f_1) \otimes \epsilon(f_2) \in \mathcal{C}$ we have

$$\begin{aligned} Ua(f \oplus g)U^*\epsilon(f_1) \otimes \epsilon(f_2) &= Ua(f \oplus g)\epsilon(f_1 \oplus f_2) \\ &= \langle f \oplus g, f_1 \oplus f_2\rangle \epsilon(f_1) \otimes \epsilon(f_2) \\ &= (a(f)\widehat{\otimes}1 + 1\widehat{\otimes}a(g))\epsilon(f_1) \otimes \epsilon(f_2) \end{aligned}$$

showing the first relation. For $\phi = \epsilon(g_1) \otimes \epsilon(g_2) \in \mathcal{C}$ we find

$$\langle \phi, Ua^\dagger(f \oplus g)U^*\psi\rangle = \langle Ua(f \oplus g)U^*\phi, \psi\rangle = \langle \phi, (a^\dagger(f)\widehat{\otimes}1 + 1\widehat{\otimes}a^\dagger(g))\psi\rangle$$

as $\mathcal{C}$ is total we can now conclude that equations (C.4) and (C.5) hold on $\mathcal{C}$. Let $\sharp$ denote either $\dagger$ or nothing. Exponential vectors span a core for both creation and annihilation operators (see [12]) so

$$Ua^\sharp(f \oplus g)U^* = \overline{a^\sharp(f)\widehat{\otimes}1 + 1\widehat{\otimes}a^\sharp(g) \mid_{\mathrm{Span}(\mathcal{C})}}$$

Using that exponential vectors span a core for both creation and annihilation operators it is not hard to see that $\overline{a^\sharp(f)\widehat{\otimes}1 + 1\widehat{\otimes}a^\sharp(g) \mid_{\mathrm{Span}(\mathcal{C})}}$ is an extension of

$a^\sharp(f)\widehat{\otimes}1 + 1\widehat{\otimes}a^\sharp(g)$. Thus we see $\overline{a^\sharp(f)\widehat{\otimes}1 + 1\widehat{\otimes}a^\sharp(g) \mid_{\mathrm{Span}(\mathcal{C})}} = \overline{a^\sharp(f)\widehat{\otimes}1 + 1\widehat{\otimes}a^\sharp(g)}$ proving equations (C.4) and (C.5). Lastly we note

$$\begin{aligned}
&U(f_1,0)\otimes_s\cdots\otimes_s(f_j,0)\otimes_s(0,g_1)\otimes_s\cdots\otimes_s(0,g_\ell)\\
&\qquad=\left(\frac{1}{(j+\ell)!}\right)^{1/2} Ua^\dagger(f_1,0)\cdots a^\dagger(f_j,0)a^\dagger(0,g_1)\cdots a^\dagger(0,g_\ell)\Omega\\
&\qquad=\left(\frac{\ell!j!}{(j+\ell)!}\right)^{1/2}(f_1\otimes_s\cdots\otimes_s f_j)\otimes(g_1\otimes_s\cdots\otimes_s g_\ell)
\end{aligned}$$

finishing the proof. □

The following result is obvious.

Theorem C.2. *There is a unique isomorphism*

$$U:\mathcal{F}_b(\mathcal{H}_1)\otimes\mathcal{F}_b(\mathcal{H}_2)\to\mathcal{F}_b(\mathcal{H}_1)\oplus\bigoplus_{n=1}^{\infty}\mathcal{F}_b(\mathcal{H}_1)\otimes\mathcal{H}_2^{\otimes_s n}$$

such that

$$U(w\otimes\{\psi_2^{(n)}\}_{n=0}^{\infty})=\psi^{(0)}w\oplus\bigoplus_{n=1}^{\infty}w\otimes\psi_2^{(n)}.$$

Let A be a selfadjoint operator on $\mathcal{F}_b(\mathcal{H}_1)$ and B be selfadjoint on $\mathcal{F}_b(\mathcal{H}_2)$ such that B is reduced by all of the subspaces $\mathcal{H}_2^{\otimes_s n}$. Write $B^{(n)} = B\mid_{\mathcal{H}_2^{\otimes_s n}}$. Then

$$\begin{aligned}
U(A\otimes 1+1\otimes B)U^* &= A+B^{(0)}\oplus\bigoplus_{n=1}^{\infty}(A\otimes 1+1\otimes B^{(n)})\\
U(A\otimes B)U^* &= B^{(0)}A\oplus\bigoplus_{n=1}^{\infty}(A\otimes B^{(n)}).
\end{aligned}$$

Lemma C.3. *Let $\mathcal{H}$ be a Hilbert space and assume there is a unitary map $V:\mathcal{H}\to\mathcal{H}_1\oplus\mathcal{H}_2$. Let U_1 be the map from Theorem C.1, U_2 be the map from Theorem C.2 and $j_i:\mathcal{H}_i\to\mathcal{H}_1\oplus\mathcal{H}_2$ be the embedding defined by $j_1(f)=(f,0)$ or $j_2(g)=(0,g)$. Define the maps $U=U_2U_1\Gamma(V)$ and $Q_i=V^*j_i$. Then*

$$\Gamma(Q_1)=U^*\mid_{\mathcal{F}_b(\mathcal{H}_1)} \tag{C.6}$$

Fix a subspace $\mathcal{K}\subset\mathcal{H}_1$ and $g_1,\dots,g_q\in\mathcal{H}_2$. Define

$$\begin{aligned}
B&=\bigcup_{b=0}^{\infty}\{h_1\otimes_s\cdots\otimes_s h_b\mid h_i\in\mathcal{K}\}\\
C&=\bigcup_{b=0}^{\infty}\{Q_1h_1\otimes_s\cdots\otimes_s Q_1h_b\otimes_s Q_2g_1\otimes_s\cdots\otimes_s Q_2g_q\mid h_i\in\mathcal{K}\}.
\end{aligned}$$

Let $\psi\in\mathrm{Span}(B)$. Then

$$U^*(\psi\otimes g_1\otimes_s\cdots\otimes_s g_q)\in\mathrm{Span}(C). \tag{C.7}$$

.

Proof. It is enough to prove equation (C.6) on elements of the form $\epsilon(f)$ for $f\in\mathcal{H}_1$. We calculate using Theorems C.1 and C.2:

$$U^*\epsilon(f)=\Gamma(V)^*U_1^*\epsilon(f)\otimes\Omega=\Gamma(V)^*\epsilon(f,0)=\epsilon(V^*j_1f)=\epsilon(Q_1f)=\Gamma(Q_1)\epsilon(f).$$

By linearity it is enough to prove equation (C.7) in the case $\psi=h_1\otimes_s\cdots\otimes_s h_b$ for $h_i\in\mathcal{K}$ and some $b\in\{0\}\cup\mathbb{N}$. Using Theorems C.1 and C.2 along with Lemma 2.2

we find

$$\begin{aligned}
U^*((h_1 \otimes_s \cdots \otimes_s h_b) \otimes (g_1 \otimes_s \cdots \otimes_s g_q))\\
&= \Gamma(V)^* U_1^*((h_1 \otimes_s \cdots \otimes_s h_b) \otimes (g_1 \otimes_s \cdots \otimes_s g_q))\\
&= \left(\frac{(b+q)!}{q!b!}\right)^{1/2} \Gamma(V)^*((h_1,0) \otimes_s \cdots \otimes_s (h_b,0) \otimes_s (0,g_1) \otimes_s \cdots \otimes_s (0,g_q))\\
&= \left(\frac{(b+q)!}{q!b!}\right)^{1/2} Q_1 h_1 \otimes_s \cdots \otimes_s Q_1 h_b \otimes_s Q_2 g_1 \otimes_s \cdots \otimes_s Q_2 g_q
\end{aligned}$$

finishing the proof. □

Appendix D. Pointwise annihilation operators

In this appendix we define pointwise annihilation operators and show associated pull through formulas. We will need this when discussing regularity of ground states. Let $\mathcal{H} = L^2(\mathcal{M}, \mathcal{E}, \mu)$, where $(\mathcal{M}, \mathcal{E}, \mu)$ is assumed to be σ-finite. We define the extended symmetric Fock space to be the product

$$\mathcal{F}_+(\mathcal{H}) = \bigtimes_{n=0}^{\infty} \mathcal{H}^{\otimes_s n}$$

with coordinate projections P_n. For elements $(\psi^{(n)}), (\phi^{(n)}) \in \mathcal{F}_+(\mathcal{H})$ we define

$$d((\psi^{(n)}), (\phi^{(n)})) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|\psi^{(n)} - \phi^{(n)}\|}{1 + \|\psi^{(n)} - \phi^{(n)}\|}$$

where $\|\cdot\|$ is the Fock space norm. This makes sense since $P_n(\mathcal{F}_+(\mathcal{H})) \subset \mathcal{F}_b(\mathcal{H})$. Standard theorems from measure theory and topology now gives the following lemma.

Lemma D.1. *The map d defines a metric on $\mathcal{F}_+(\mathcal{H})$ and turns this space into a complete separable metric space and a topological vector space. The topology and Borel σ-algebra is generated by the projections P_n. If a sequence $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{F}_b(\mathcal{H})$ is convergent/Cauchy then it is also convergent/Cauchy with respect to d. Also any total/dense set in $\mathcal{F}_b(\mathcal{H})$ will be total/dense in $\mathcal{F}_+(\mathcal{H})$ as well.*

Define

$$A = \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{g^{\otimes n} \mid g \in \mathcal{H}\}. \tag{D.1}$$

Then A is total in $\mathcal{F}_b(\mathcal{H})$ since the span of A can approximate any exponential vector. By Lemma D.1 we find A is total in $\mathcal{F}_+(\mathcal{H})$ as well. For each $a \in \mathbb{R}$ we define

$$\|\cdot\|_{a,+} = \lim_{n\to\infty} \left(\sum_{k=0}^{n} (k+1)^{2a} \|P_k(\cdot)\|^2\right)^{\frac{1}{2}}.$$

which is measurable from $\mathcal{F}_+(\mathcal{H})$ into $[0,\infty]$. Let

$$\mathcal{F}_{a,+}(\mathcal{H}) = \{\psi \in \mathcal{F}_+(\mathcal{H}) \mid \|\psi\|_{a,+} < \infty\}.$$

Note $\|\cdot\|_{a,+}$ restricts to a norm on $\mathcal{F}_{a,+}(\mathcal{H})$ that comes from an inner product. In particular $\mathcal{F}_{a,+}(\mathcal{H})$ is a Hilbert space and for $a \geq 0$ we have $\mathcal{F}_{a,+}(\mathcal{H}) = \mathcal{D}((N+1)^a)$. We summarise as follows

Lemma D.2. *$\|\cdot\|_{a,+}$ defines measurable map from $\mathcal{F}_+(\mathcal{H})$ to $[0,\infty]$ and restricts to a norm on the spaces $\mathcal{F}_{a,+}(\mathcal{H})$. This norm comes from an inner product so $\mathcal{F}_{a,+}(\mathcal{H})$ is a Hilbert space. Furthermore, the set A from equation (D.1) is total in $\mathcal{F}_+(\mathcal{H})$.*

The point of defining a metric on $\mathcal{F}_+(\mathcal{H})$ and finding a dense set is that most of the operations we will encounter in this chapter are continous on $\mathcal{F}_+(\mathcal{H})$. Therefore many operator identities can be proven by checking the identity on the set A from (D.1). Fix now $v \in \mathcal{H}$. We now define the following maps on $\mathcal{F}_+(\mathcal{H})$

$$\begin{aligned} a_+(v)(\psi^{(n)}) &= (a_n(v)\psi^{(n+1)}) \\ a_+^\dagger(v)(\psi^{(n)}) &= (0, a_0^\dagger(v)\psi^{(0)}, a_1^\dagger(v)\psi^{(1)}, \dots) \\ \varphi_+(v) &= a_+(v) + a_+^\dagger(v) \end{aligned}$$

Where $a_n(v)$ is annihilation from $\mathcal{H}^{\otimes_s(n+1)}$ to $\mathcal{H}^{\otimes_s n}$ and $a_n^\dagger(f)$ is creation from $\mathcal{H}^{\otimes_s n}$ to $\mathcal{H}^{\otimes(n+1)}$ which are both continuous. The following lemma is almost automatic.

Lemma D.3. *The maps $a_+(v)$,$a_+^\dagger(v)$ and $\varphi_+(v)$ are all continuous. For $B \in \{a, a^\dagger, \varphi\}$ we have*

$$B_+(v)\psi = B(v)\psi \ \ \textit{if} \ \psi \in \mathcal{D}(B(v)). \tag{D.2}$$

Proof. Equation (D.2) holds for $B \in \{a, a^\dagger\}$ simply by definition. The topology on $\mathcal{F}_+(\mathcal{H})$ is generated by the projections P_n. Therefore continuity of $a_+(v)$, $a_+^\dagger(v)$ and $\varphi_+(v)$ follows from continuity of

$$\begin{aligned} &P_n a_+(v) = a_n(v)P_{n+1} \\ &P_n a_+^\dagger(v) = a_{n-1}^\dagger(v)P_{n-1} \quad n \geq 1 \\ &P_0 a_+^\dagger(v) = 0. \end{aligned}$$

We now prove equation (D.2) for $B = \varphi$. The relation

$$\varphi(v)\psi = \varphi_+(v)\psi$$

easily holds on the span of A where A is the set from equation (D.1). For $\psi \in \mathcal{D}(\varphi(v))$ we may pick as sequence $\{\psi_n\}_{n=1}^\infty \subset \mathrm{Span}(A)$ that converges to ψ in $\varphi(f)$ norm (use e.g. [15, Corollary 20.5]). Continuity of $\varphi_+(v)$ together with Lemma D.1 now yields the desired result. □

We now move on to the second quantisation of unitaries and selfadjoint operators. Let U be unitary on $\mathcal{H}$ and $\omega = (\omega_1, \dots, \omega_p)$ be a tuple of strongly commuting selfadjoint operators on $\mathcal{H}$. We then define

$$\begin{aligned} d\Gamma(\omega) &= (d\Gamma(\omega_1), \dots, d\Gamma(\omega_p)) \\ d\Gamma^{(n)}(\omega) &= (d\Gamma^{(n)}(\omega_1), \dots, d\Gamma^{(n)}(\omega_p)) \end{aligned}$$

which are now tuples of strongly commuting selfadjoint operators (It is easily checked that the unitary groups commute). Let furthermore $f : \mathbb{R}^p \to \mathbb{C}$ be a map. We then define

$$\begin{aligned} f(d\Gamma_+(\omega)) &= \bigtimes_{n=0}^{\infty} f(d\Gamma^{(n)}(\omega)) \quad \mathcal{D}(f(d\Gamma_+(\omega))) = \bigtimes_{n=0}^{\infty} \mathcal{D}(f(d\Gamma^{(n)}(\omega))) \\ \Gamma_+(U) &= \bigtimes_{n=0}^{\infty} \Gamma^{(n)}(U). \end{aligned}$$

If $\omega : \mathcal{M} \to \mathbb{R}^p$ is measurable then we may identify ω as such a touple of commuting selfadjoint operators. In this case $f(d\Gamma^{(n)}(\omega))$ is multiplication by the map $f(\omega(k_1) + \cdots + \omega(k_n))$. The following lemma is obvious.

Lemma D.4. *The map $\Gamma_+(U)$ is an isometry on $\mathcal{F}_+(\mathcal{H})$ and is thus continuous. Furthermore we have*

$$\begin{aligned} f(d\Gamma_+(\omega))\psi &= f(d\Gamma(\omega))\psi, \quad \psi \in \mathcal{D}(f(d\Gamma(\omega))) \\ \Gamma_+(U)\psi &= \Gamma(U)\psi, \qquad\quad \psi \in \mathcal{F}_b(\mathcal{H}) \end{aligned}$$

We will now consider a class of linear functionals on $\mathcal{F}_+(\mathcal{H})$. For each $n \in \mathbb{N}$ we let $Q_n : \mathcal{F}_+(\mathcal{H}) \to \mathcal{N}$ denote the linear projection which preserves the first n entries of $(\psi^{(n)})$ and projects the rest of them to 0. For $\psi \in \mathcal{N}$ there is $K \in \mathbb{N}$ such that $Q_n\psi = \psi$ for $n \geq K$. For $\phi \in \mathcal{F}_+(\mathcal{H})$ we may thus define the pairing

$$\langle \psi, \phi \rangle_+ := \langle \psi, Q_n\phi \rangle = \sum_{i=0}^{K} \langle \psi^{(i)}, \phi^{(i)} \rangle, \tag{D.3}$$

where $n \geq K$.

Lemma D.5. *The map Q_n above is linear and continuous into $\mathcal{F}_b(\mathcal{H})$. The paring $\langle \cdot, \cdot \rangle_+$ is sesquilinear, and continuous in the second entry. If $\phi \in \mathcal{F}_{a,+}(\mathcal{H})$ then $\psi \mapsto \langle \psi, \phi \rangle_+$ is continous with respect to $\|\cdot\|_{-a,+}$. Furthermore, the collection of maps $\{\langle \psi, \cdot \rangle_+\}_{\psi \in \mathcal{N}}$ will separate points of $\mathcal{F}_+(\mathcal{H})$.*

Proof. The paring $\langle \cdot, \cdot \rangle_+$ is trivially sesquilinear. Let $\{\psi_k\}_{k=1}^\infty$ converge to ψ in $\mathcal{F}_+(\mathcal{H})$. Then $\psi_k^{(i)}$ will converge to $\psi^{(i)}$ for all i. Now $\|Q_n(\psi_k - \psi)\|^2$ is the sum

$$\|Q_n(\psi_k - \psi)\|^2 = \sum_{i=0}^{n} \|\psi_k^{(i)} - \psi^{(i)}\|^2$$

which converges to 0. Hence Q_n is continuous from $\mathcal{F}_+(\mathcal{H})$ into $\mathcal{F}_b(\mathcal{H})$. This also shows continuity in the second entry of $\langle \cdot, \cdot \rangle_+$. If $\phi \in \mathcal{F}_{a,+}(\mathcal{H})$ and $\psi \in \mathcal{N}$ we find some $K \in \mathbb{N}$ such that

$$|\langle \psi, \phi \rangle_+| \leq \sum_{i=0}^{K} (i+1)^a \|\phi^{(i)}\| (i+1)^{-a} \|\psi^{(i)}\| \leq \|\phi\|_{a,+} \|\psi\|_{-a,+}$$

showing the desired continuity. Fix now $\phi \in \mathcal{F}_+(\mathcal{H})$ and assume that $\langle \psi, \phi \rangle_+ = 0$ for all $\psi \in \mathcal{N}$. Then $\langle \psi, \phi^{(n)} \rangle = 0$ for all $\psi \in \mathcal{H}^{\otimes_s n}$ showing $\phi^{(n)} = 0$. ☐

Corollary D.6. *Let $\phi \in \mathcal{F}_{a,+}(\mathcal{H})$ for some $a \leq 0$, $\mathcal{D} \subset \mathcal{N}$ be dense in $\mathcal{F}_b(\mathcal{H})$ and assume $\langle \psi, \phi \rangle_+ = 0$ for all $\psi \in \mathcal{D}$. Then $\phi = 0$.*

Proof. Note $\mathcal{D}$ consists of elements which are analytic for $(N+1)^{-a}$ so $\mathcal{D}$ is a core for $(N+1)^{-a}$. Let ψ in $\mathcal{N}$ and pick $\{\psi_n\}_{n=1}^\infty \subset \mathcal{D}$ converging to ψ in $(N+1)^{-a}$-norm. Using Lemma D.5 we see $\langle \psi, \phi \rangle_+ = 0$ and thus $\phi = 0$ by Lemma D.5. ☐

Lemma D.7. *Let $\psi \in \mathcal{N}$, $\phi \in \mathcal{F}_+(\mathcal{H})$, $v \in \mathcal{H}$ and U be unitary on $\mathcal{H}$. Then we have*

$$\begin{aligned} \langle a^\dagger(v)\psi, \phi \rangle_+ &= \langle \psi, a_+(v)\phi \rangle_+, \qquad & \langle a(v)\psi, \phi \rangle_+ &= \langle \psi, a_+^\dagger(v)\phi \rangle_+, \\ \langle \varphi(v)\psi, \phi \rangle_+ &= \langle \psi, \varphi_+(v)\phi \rangle_+, \qquad & \langle \Gamma(U)\psi, \phi \rangle_+ &= \langle \psi, \Gamma_+(U^*)\phi \rangle_+. \end{aligned}$$

Let $\omega = (\omega_1, \ldots, \omega_p)$ be a tuple of commuting selfadjoint operators, $f : \mathbb{R}^p \to \mathbb{C}$, $\psi \in \mathcal{N} \cap \mathcal{D}(f(d\Gamma(\omega)))$ and $\phi \in \mathcal{D}(\overline{f}(d\Gamma_+(\omega)))$ we have

$$\langle f(d\Gamma(\omega))\psi, \phi \rangle_+ = \langle \psi, \overline{f}(d\Gamma_+(\omega))\phi \rangle_+.$$

Proof. Since $\psi \in \mathcal{N}$ we may pick K such that $\psi^{(n)} = 0$ for all $n \geq K$. Then we may calculate

$$\begin{aligned}
\langle a^\dagger(v)\psi, \phi\rangle_+ &= \langle \psi, a(v)Q_{K+1}\phi\rangle = \langle \psi, Q_K a_+(v)\phi\rangle = \langle \psi, a_+(v)\phi\rangle_+ \\
\langle a(v)\psi, \phi\rangle_+ &= \langle \psi, a^\dagger(v)Q_{K-1}\phi\rangle = \langle \psi, Q_K a_+^\dagger(v)\phi\rangle = \langle \psi, a_+^\dagger(v)\phi\rangle_+ \\
\langle \varphi(v)\psi, \phi\rangle_+ &= \langle \psi, a_+(v)\phi\rangle_+ + \langle \psi, a_+^\dagger(v)\phi\rangle_+ = \langle \psi, \varphi_+(v)\phi\rangle_+ \\
\langle \Gamma(U)\psi, \phi\rangle_+ &= \langle \psi, \Gamma(U^*)Q_K\phi\rangle = \langle \psi, Q_K\Gamma_+(U^*)\phi\rangle = \langle \psi, \Gamma_+(U^*)\phi\rangle_+
\end{aligned}$$

Assume now that $\psi \in \mathcal{N} \cap \mathcal{D}(f(d\Gamma(\omega)))$ and $\phi \in \mathcal{D}(\overline{f}(d\Gamma_+(\omega)))$. Then $Q_K\phi \in \mathcal{D}(\overline{f}(d\Gamma(\omega)))$ and

$$\langle f(d\Gamma(\omega))\psi, \phi\rangle_+ = \langle \psi, \overline{f}(d\Gamma(\omega))Q_K\phi\rangle = \langle \psi, Q_K\overline{f}(d\Gamma_+(\omega))\phi\rangle = \langle \psi, \overline{f}(d\Gamma_+(\omega))\phi\rangle_+$$

this finishes the proof. $\square$

We now consider functions with values in $\mathcal{F}_+(\mathcal{H})$. Let $(X, \mathcal{X}, \nu)$ be a σ-finite and countably generated measure space. Define the quotient

$$\mathcal{M}(X, \mathcal{X}, \nu) = \{f : X \to \mathcal{F}_+(\mathcal{H}) \mid f \text{ is } \mathcal{X} - \mathcal{B}(\mathcal{F}_+(\mathcal{H})) \text{ mesurable}\}/\sim,$$

where we define $f \sim g \iff f = g$ almost everywhere. We are interested in the subspace

$$\mathcal{C}(X, \mathcal{X}, \nu) = \{f \in \mathcal{M}(X, \mathcal{X}, \nu) \mid x \mapsto P_n f(x) \in L^2(X, \mathcal{X}, \nu, \mathcal{H}^{\otimes_s n})\ \forall n \in \mathbb{N}_0\}.$$

Lemma D.2 shows that $x \mapsto \|f(x)\|_{a,+}$ is measurable for functions $f \in \mathcal{C}(X, \mathcal{X}, \nu)$ and so the integral

$$\int_X \|f(x)\|_{a,+}^2 d\nu(x)$$

always makes sense. If $a = 0$ then it is finite if and only if $f \in L^2(X, \mathcal{X}, \nu, \mathcal{F}_b(\mathcal{H}))$. We write $f \in \mathcal{C}(X, \mathcal{X}, \nu)$ as $(f^{(n)})$ where $f^{(n)} = x \mapsto P_n f(x)$. For $f, g \in \mathcal{C}(X, \mathcal{X}, \nu)$ we define

$$d(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|f^{(n)} - g^{(n)}\|_{L^2(X,\mathcal{X},\nu,\mathcal{H}^{\otimes_s n})}}{1 + \|f^{(n)} - g^{(n)}\|_{L^2(X,\mathcal{X},\nu,\mathcal{H}^{\otimes_s n})}}.$$

We can now summarise.

Lemma D.8. *d is a complete metric on $\mathcal{C}(X, \mathcal{X}, \nu)$ such that $\mathcal{C}(X, \mathcal{X}, \nu)$ becomes separable topological vector space. The topology is generated by the maps $f \mapsto (x \mapsto P_n f(x))$. Furthermore $L^2(X, \mathcal{X}, \nu, \mathcal{F}_b(\mathcal{H})) \subset \mathcal{C}(X, \mathcal{X}, \nu)$ and convergence in $L^2(X, \mathcal{X}, \nu, \mathcal{F}_b(\mathcal{H}))$ implies convergence in $\mathcal{C}(X, \mathcal{X}, \nu)$. Also the map $x \mapsto \|f(x)\|_{a,+}$ is measurable for any f in $\mathcal{C}(X, \mathcal{X}, \nu)$ and $a \in \mathbb{R}$.*

We now move on to discuss some actions on this space. This is strongly related to the direct integral and readers should look up the results in [18]. Let $n \geq 1$, $v \in \mathcal{H}$, U be unitary on $\mathcal{H}$, $\omega = (\omega_1, \dots, \omega_p)$ a tuple of selfadjoint multiplication operators on $\mathcal{H}$, $m : \mathcal{M}^n \to \mathbb{R}^p$ measurable and $g : \mathbb{R}^p \to \mathbb{R}$ a measurable map. Then we wish to define operators on $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ for $\ell \geq 1$ by

$$\begin{aligned}
(a_{\oplus,\ell}^\dagger(v)f)(k) &= a_+^\dagger(v)f(k) \\
(a_{\oplus,\ell}(v)f)(k) &= a_+(v)f(k) \\
(\varphi_{\oplus,\ell}(v)f)(k) &= \varphi_+(v)f(k) \\
(\Gamma_{\oplus,\ell}(U)f)(k) &= \Gamma_+(U)f(k) \\
(g(d\Gamma_{\oplus,\ell}(\omega) + m)f)(k) &= g(d\Gamma_+(\omega) + m(k))f(k).
\end{aligned}$$

We further define $\mathcal{C}(\mathcal{M}^0, \mathcal{E}^{\otimes 0}, \mu^{\otimes 0}) = \mathcal{F}_+(\mathcal{H})$ along with $a_{\oplus,0}^\dagger(v) = a_+^\dagger(v)$, $a_{\oplus,0}(v) = a_+(v)$, $\varphi_{\oplus,0}(v) = \varphi_+(v)$ and $\Gamma_{\oplus,0} = \Gamma_+(U)$. We have the following lemma.

Lemma D.9. *The $a_{\oplus,\ell}^\dagger(v), a_{\oplus,\ell}(v), \varphi_{\oplus,\ell}(v)$ and $\Gamma_{\oplus,\ell}(U)$ are well defined and continuous for all $\ell \in \mathbb{N}_0$. Let $f \in \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$. If $f(k) \in \mathcal{D}(g(d\Gamma_+(\omega) + m(k)))$ for all $k \in \mathcal{M}^\ell$ then $k \mapsto P_n(g(d\Gamma_+(\omega)+m(k))f(k))$ is measurable. Thus as domain of $g(d\Gamma_{\oplus,\ell}(\omega) + m)$ we may choose*

$$\Big\{ f \in \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \Big| f(k) \in \mathcal{D}(g(d\Gamma_+(\omega) + m(k))) \text{ for almost every } k \in \mathcal{M}^\ell,$$
$$\int_{\mathcal{M}^\ell} \| P_n g(d\Gamma_+(\omega) + m(k)) f(k) \|^2 d\mu^{\otimes \ell}(k) < \infty \ \forall n \in \mathbb{N} \Big\}.$$

Proof. To deal with the first three maps, it is enough to handle the first two, since the last follows by addition of continuous maps. We have

$$\begin{aligned} k \mapsto P_n a_+(v) h(k) &= k \mapsto a_n(v) P_{n+1} h(k) \\ k \mapsto P_n a_+^\dagger(v) h(k) &= k \mapsto a_{n-1}^\dagger(v) P_{n-1} h(k) \quad n \geq 1 \\ k \mapsto P_0 a_+^\dagger(v) h(k) &= k \mapsto 0 \\ k \mapsto P_n \Gamma_+(U) h(k) &= k \mapsto \Gamma^{(n)}(U) P_n h(k). \end{aligned}$$

Continuity of $a_n(v), a_{n-1}^\dagger(v)$ and $\Gamma^{(n)}(U)$ and Lemma D.8 now implies that the first four maps are well defined and continous. For the next claim we note

$$P_n g(d\Gamma_+(\omega) + m(k)) f(k) = g(d\Gamma^{(n)}(\omega) + m(k)) P_n f(k)$$

since $d\Gamma^{(n)}(\omega_i) + m_i(k)$ is strongly resolvent measurable for each $i \in \{1, \dots, p\}$, we find that $g(d\Gamma^{(n)}(\omega) + m(k))$ is strongly resolvent measurable and so the conclusion follows from standard theorems (See e.g [18]). □

We will now introduce the pointwise annihilation operators. For $\psi = (\psi^{(n)}) \in \mathcal{F}_+(\mathcal{H})$ we define $A_\ell \psi \in \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ by

$$P_n(A_\ell \psi)(k_1, \dots, k_\ell) = \sqrt{(n+\ell)(n+\ell-1)\cdots(n+1)} \psi^{(n+\ell)}(k_1, \dots, k_\ell, \cdot, \dots, \cdot)$$

Since $\psi^{(n+\ell)}$ is symmetric and square integrable, we may pick a representative such that the above map is symmetric in $k_1, \dots, k_\ell$ and has values in $\mathcal{H}^{\otimes_s n}$. It is easy to see, that the choice of representative only changes $A_\ell \psi$ up to a zeroset. Hence A_ℓ is well defined. Clearly

$$\| (A_\ell \psi)^{(n)} - (A_\ell \phi)^{(n)} \|_n = \sqrt{(n+\ell)(n+\ell-1)\cdots(n+1)} \| \psi^{(n+\ell)} - \phi^{(n+\ell)} \|$$

so A_ℓ is continuous from $\mathcal{F}_+(\mathcal{H})$ into $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$. One immediately observes that $A_\ell \psi \in L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}_{a,+}(\mathcal{H}))$ if and only if

$$\begin{aligned} \infty &> \int_{\mathcal{M}^\ell} \| A_\ell \psi(k_1, \dots, k_\ell) \|_{a,+}^2 d\mu^{\otimes n}(k_1, \dots, k_\ell) \\ &= \sum_{n=0}^\infty (n+1)^{2a} (n+\ell)(n+\ell-1)\cdots(n+1) \| \psi^{(n+\ell)} \|^2 \\ &\iff \sum_{n=0}^\infty (n+\ell)^{2a+\ell} \| \psi^{(n+\ell)} \|^2 < \infty \end{aligned}$$

which is equivalent to $\psi \in \mathcal{D}(N^{\frac{\ell}{2}+a})$ if $\frac{\ell}{2} + a \geq 0$. In particular we see $A_\ell \psi$ is almost everywhere $\mathcal{F}_{-\frac{\ell}{2},+}(\mathcal{H})$ valued if $\psi \in \mathcal{F}_b(\mathcal{H})$. If $\psi, \phi \in \mathcal{D}(N^{\frac{\ell}{2}})$ we apply the above calculations with $a = 0$ to obtain

$$\text{(D.4)} \qquad \begin{aligned} \| A_\ell \psi - A_\ell \phi \|^2 &= \sum_{n=0}^\infty (n+\ell)(n+\ell-1)\cdots(n+1) \| \psi^{(n+\ell)} - \phi^{(n+\ell)} \|^2 \\ &\leq \| N^{\frac{\ell}{2}} (\psi - \phi) \|. \end{aligned}$$

We summarise:

Lemma D.10. *A_ℓ is a continuous linear map from $\mathcal{F}_+(\mathcal{H})$ to $\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})$ and from $\mathcal{D}(N^{\frac{\ell}{2}})$ into $L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{F}_b(\mathcal{H}))$. Furthermore $\psi\in\mathcal{D}(N^{\ell/2})$ if and only if $A_\ell\psi\in L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{F}_b(\mathcal{H}))$ and if $\psi\in\mathcal{F}_b(\mathcal{H})$ we have $A_\ell\psi$ is almost everywhere $\mathcal{F}_{-\frac{\ell}{2},+}(\mathcal{H})$ valued.*

Fix $v\in\mathcal{H}$ and $\ell\in\mathbb{N}_0$. We then define a map $z_\ell(v):\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})\to\mathcal{C}(\mathcal{M}^{\ell+1},\mathcal{E}^{\otimes((\ell+1))},\mu^{\otimes\ell+1})$ by

$$(z_0(v)\psi)(k)=v(k)\psi \text{ and } (z_\ell(v)\psi)(x,k)=v(x)\psi(k)$$

when $\ell\geq 1$. Note this defines a measurable map from $\mathcal{M}^{\ell+1}$ into $\mathcal{F}_+(\mathcal{H})$ and

$$\int_{\mathcal{M}^{\ell+1}}\|P_n(z_\ell(v)\psi)(k)\|d\mu^{\otimes\ell+1}(k)=\|v\|^2\|\psi^{(n)}\|^2_{L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{H}^{\otimes_s n})} \tag{D.5}$$

where we define $L^2(\mathcal{M}^0,\mathcal{E}^{\otimes 0},\mu^{\otimes 0},\mathcal{H}^{\otimes_s n})=\mathcal{H}^{\otimes_s n}$. This implies $z_\ell(v)$ is well defined and obviously linear. Equation (D.5) also shows $z_\ell(v)$ is continuous and maps $L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{F}_b(\mathcal{H}))$ continuously into $L^2(\mathcal{M}^{\ell+1},\mathcal{E}^{\otimes(\ell+1)},\mu^{\otimes(\ell+1)},\mathcal{F}_b(\mathcal{H}))$. We summarise

Lemma D.11. *The map $z_\ell(v)$ introduced above is linear and continuous. Both as a map from $\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})$ into the space $\mathcal{C}(\mathcal{M}^{\ell+1},\mathcal{E}^{\otimes(\ell+1)},\mu^{\otimes(\ell+1)})$ and from $L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{F}_b(\mathcal{H}))$ into $L^2(\mathcal{M}^{\ell+1},\mathcal{E}^{\otimes(\ell+1)},\mu^{\otimes(\ell+1)},\mathcal{F}_b(\mathcal{H}))$.*

Lastly, we look at permutation and symmetrisation operators. Let $\ell\geq 1$ and $\sigma\in\mathcal{S}_\ell$ where $\mathcal{S}_\ell$ is the set of permutations of $\{1,\dots,\ell\}$. Define $\widetilde{\sigma}:\mathcal{M}^\ell\to\mathcal{M}^\ell$ by $\widetilde{\sigma}(k_1,\dots,k_\ell)=(k_{\sigma(1)},\dots,k_{\sigma(\ell)})$ and observe that $\widetilde{\sigma}$ is $\mathcal{E}^{\otimes\ell}$-$\mathcal{E}^{\otimes\ell}$ measurable. Define $\widehat{\sigma}:\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})\to\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})$ by

$$(\widehat{\sigma}f)(k_1,\dots,k_\ell)=f(k_{\sigma(1)},\dots,k_{\sigma(\ell)})=(f\circ\widetilde{\sigma})(k_1,\dots,k_\ell).$$

$\widehat{\sigma}$ is a well defined isometry on $\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})$ since $\widetilde{\sigma}$ is measurable and $\mu^{\otimes\ell}=\mu^{\otimes\ell}\circ\widetilde{\sigma}^{-1}$ so

$$\int_{\mathcal{M}^\ell}\|f^{(n)}(k_1,\dots,k_\ell)\|^2d\mu^{\otimes\ell}(k)=\int_{\mathcal{M}^\ell}\|f^{(n)}(k_{\sigma(1)},\dots,k_{\sigma(\ell)})\|^2d\mu^{\otimes\ell}(k),$$

A similar calculation shows that $\widehat{\sigma}$ is also isometric on $L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{F}_b(\mathcal{H}))$. For $\pi\in\mathcal{S}_\ell$ we have

$$\widehat{\sigma}\widehat{\pi}f=f\circ\widetilde{\pi}\circ\widetilde{\sigma}=f\circ\widetilde{(\sigma\circ\pi)}=\widehat{\sigma\circ\pi}f$$

and hence the inverse map of $\widehat{\sigma}$ is $\widehat{\sigma^{-1}}$. Note also that $\widehat{\sigma}A_\ell\psi=A_\ell\psi$ since the $\psi^{(n)}$ are symmetric in all coordinates. Define now

$$S_\ell:=\frac{1}{(\ell-1)!}\sum_{\sigma\in\mathcal{S}_\ell}\widehat{\sigma}.$$

For $\pi\in\mathcal{S}_\ell$ we have

$$\widehat{\pi}S_\ell=\frac{1}{(\ell-1)!}\sum_{\sigma\in\mathcal{S}_\ell}\widehat{\pi}\widehat{\sigma}=\frac{1}{(\ell-1)!}\sum_{\sigma\in\mathcal{S}_\ell}\widehat{\pi\circ\sigma}=S_\ell.$$

Hence $S_\ell^2=\ell S_\ell$. We summarise:

Lemma D.12. *Let $\ell\in\mathbb{N}$. For $\sigma\in\mathcal{S}_\ell$ the map $\widehat{\sigma}$ defines a linear bijective isometry from $\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})$ to $\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})$ and from $L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{F}_b(\mathcal{H}))$ to $L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{F}_b(\mathcal{H}))$. Also $\widehat{\sigma}A_\ell\psi=A_\ell\psi$ and if $\pi\in\mathcal{S}_\ell$ then $\widehat{\pi}\widehat{\sigma}=\widehat{\pi\circ\sigma}$.*

Furthermore S_ℓ is continuous and linear from $\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})$ into the space $\mathcal{C}(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell})$ and it satisfies relation $S_\ell^2=\ell S_\ell$. Furthermore S_ℓ is also continuous from $L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{F}_b(\mathcal{H}))$ into $L^2(\mathcal{M}^\ell,\mathcal{E}^{\otimes\ell},\mu^{\otimes\ell},\mathcal{F}_b(\mathcal{H}))$.

We can now calculate commutators

Lemma D.13. *Let $\omega : \mathcal{M} \to \mathbb{R}^p$ be measurable, $v, g \in \mathcal{H}$ and let $f : \mathbb{R}^p \to \mathbb{R}$ be measurable. Define $A_0 = 1$, $A_{-1} = 0$, $z_0^\dagger(v) = 0$ and $z_\ell^\dagger(v) = S_\ell z_{\ell-1}(v)$ for $\ell \geq 1$. We have the following operator identities for $\ell, n \in \mathbb{N}_0$*

(D.6) $\quad a_{\oplus,\ell}(g)A_\ell\psi = A_\ell a_+(g) \qquad A_\ell a_+^\dagger(g) - a_{\oplus,\ell}^\dagger(g)A_\ell = z_\ell^\dagger(g)A_{\ell-1}$

(D.7) $\quad a_{\oplus,\ell+1}(g)z_\ell(v) = z_\ell(v)a_{\oplus,\ell}(g) \qquad a_{\oplus,\ell+1}^\dagger(g)z_\ell(v) = z_\ell(v)a_{\oplus,\ell}^\dagger(g)$

(D.8) $\quad a_{\oplus,\ell+1}(g)S_{\ell+1} = S_{\ell+1}a_{\oplus,\ell+1}(g) \qquad a_{\oplus,\ell+1}^\dagger(g)S_{\ell+1} = S_{\ell+1}a_{\oplus,\ell+1}^\dagger(g)$

(D.9) $\quad \varphi_{\oplus,\ell+1}(g)z_\ell(v) = z_\ell(v)\varphi_{\oplus,\ell}(g) \qquad \varphi_{\oplus,\ell+1}(g)S_{\ell+1} = S_{\ell+1}\varphi_{\oplus,\ell+1}(g)$

(D.10) $$A_\ell\varphi_+(v)^n = \sum_{q=0}^{\min\{\ell,n\}} \binom{n}{q} \varphi_{\oplus,\ell}(v)^{n-q} \left(\prod_{c=0}^{q-1} z_{\ell-c}^\dagger(v) \right) A_{\ell-q}$$

(D.11) $\quad \Gamma_{\oplus,\ell}(-1)A_\ell = (-1)^\ell A_\ell \Gamma_+(-1).$

Let $\ell \geq 1$. If $\psi \in \mathcal{D}(f(d\Gamma(\omega)))$ then $A_\ell\psi \in \mathcal{D}(f(d\Gamma_\oplus(\omega) + \omega_\ell))$ where we define $\omega_\ell(k_1, \dots, k_\ell) = \omega(k_1) + \dots + \omega(k_\ell)$ and

$$f(d\Gamma_\oplus(\omega) + \omega_\ell)A_\ell\psi = A_\ell f(d\Gamma_+(\omega))\psi.$$

Proof. First we note that equation (D.9) follow directly from (D.7) and (D.8). Secondly we note that by continuity and linearity it is enough to prove the relations (D.6)-(D.11) directly on the set A from equation (D.1). Thirdly we note that the identities involving A_ℓ are trivial for $\ell = 0$, so we only need to prove these when $\ell > 0$. We start by proving the identities in equation (D.6).

Let $h^{\otimes n} \in A$ (with $h^{\otimes 0} = \Omega$). If $n < \ell+1$ then $a_{\oplus,\ell}(v)A_\ell h^{\otimes n} = 0 = A_\ell a_+(v)h^{\otimes n}$. Otherwise we calculate

$$\begin{aligned}(a_{\oplus,\ell}(g)A_\ell h^{\otimes n})(k_1, \dots, k_\ell) &= \sqrt{n(n-1)\dots(n-\ell)}h(k_1)\dots h(k_\ell)\langle g, h\rangle h^{\otimes n-\ell-1} \\ &= (A_\ell a_+(g)h^{\otimes n})(k_1, \dots, k_\ell)\end{aligned}$$

If $n < \ell - 1$ we find

$$A_\ell a_+^\dagger(g)h^{\otimes n} = 0 = a_{\oplus,\ell}^\dagger(g)A_\ell h^{\otimes n} = 0 = z_{\ell-1}^\dagger(g)A_{\ell-1}h^{\otimes n}$$

If $n \geq \ell - 1$ we have (in the calculation we define $h^{\otimes -1} = 0$)

$$\begin{aligned}(A_\ell a_+^\dagger(v)h^{\otimes n})(k_1, \dots, k_\ell) &= A_\ell \left(\frac{1}{\sqrt{n+1}} \sum_{a=1}^{n+1} h^{\otimes a-1} \otimes v \otimes h^{\otimes n-a+1} \right)(k_1, \dots, k_\ell) \\ &= \sqrt{n(n-1)\dots(n-\ell+2)} \sum_{a=1}^{\ell} h(k_1)\dots v(k_a)\dots h(k_\ell)h^{\otimes n-\ell+1} \\ &+ \sqrt{n(n-1)\dots(n-\ell+2)} \sum_{a=\ell+1}^{n+1} h(k_1)\dots h(k_\ell)h^{\otimes a-1-\ell} \otimes v \otimes h^{\otimes n-a+1} \\ &= \sum_{a=1}^{\ell} v(k_a)(A_{\ell-1}h^{\otimes n})(k_1, \dots, \widehat{k}_a, \dots, k_\ell) \\ &+ \sqrt{n(n-1)\dots(n-\ell+1)}h(k_1)\dots h(k_\ell)a_+^\dagger(v)h^{\otimes n-\ell} \\ &= (S_\ell z_{\ell-1}(v)A_{\ell-1}h^{\otimes n})(k_1, \dots, k_\ell) + (a_{\oplus,\ell}^\dagger(v)A_\ell h^{\otimes n})(k_1, \dots, k_\ell)\end{aligned}$$

This finishes the proof of equation (D.6). If $n < \ell$ we have $(-1)^\ell \Gamma_{\oplus,\ell}(-1) A_\ell h^{\otimes n} = 0 = A_\ell \Gamma_+(-1) h^{\otimes n}$. Writing $k = (k_1, \dots, k_\ell)$ we obtain for $n \geq \ell$

$$\begin{aligned}(-1)^\ell \Gamma_{\oplus,\ell}(-1) A_\ell h^{\otimes n}(k) &= (-1)^\ell \sqrt{n(n-1)\dots(n-\ell+1)} h(k_1)\dots h(k_\ell)(-h)^{\otimes n-\ell} \\ &= A_\ell(-h)^{\otimes n}(k) \\ &= A_\ell \Gamma_+(-1) h^{\otimes n}(k).\end{aligned}$$

This proves equation (D.11). Next we let $\psi \in C(\mathcal{M}^{\ell+1}, \mathcal{E}^{\otimes(\ell+1)}, \mu^{\otimes(\ell+1)})$ and $\sigma \in \mathcal{S}_{\ell+1}$. We find

$$(a^\sharp_{\oplus,\ell}(v)\widehat{\sigma}\psi)(k) = a^\sharp_+(v)(\psi \circ \widetilde{\sigma})(k) = (\widehat{\sigma} a^\sharp_{\oplus,\ell}(v)\psi)(k) \tag{D.12}$$

$$(a^\sharp_{\oplus,\ell+2}(v) z_{\ell+1}(g)\psi)(x,k) = a^\sharp_+(g) v(x)\psi(k) = (z_{\ell+1}(g) a^\sharp_{\oplus,\ell+1}(v)\psi)(x,k) \tag{D.13}$$

where $\sharp$ is either nothing or $\dagger$. Equation (D.12) shows (D.8) and equation (D.13) shows (D.7) in the special case $\ell \geq 1$. The $\ell = 0$ case is similar. We will now prove (D.10). It clearly holds in the $\ell = 0$ case. We proceed by induction in ℓ. Adding the two equations in (D.6) we find the $n = 1$ case. Using the $n = 1$ case and the induction hypothesis we find

$$\begin{aligned}A_\ell \varphi_+(v)^{n+1} &= \varphi_{\oplus,\ell}(v)^{n+1} A_\ell + \sum_{a=0}^{n} \varphi_{\oplus,\ell}(v)^a (A_\ell \varphi_+(v) - \varphi_{\oplus,\ell}(v) A_\ell) \varphi_+(v)^{n-a} \\ &= \varphi_{\oplus,\ell}(v)^{n+1} A_\ell + \sum_{a=0}^{n} \varphi_{\oplus,\ell}(v)^a z^\dagger_{\ell-1}(v) A_{\ell-1} \varphi_+(v)^{n-a} \\ &= \varphi_{\oplus,\ell}(v)^{n+1} A_\ell + \sum_{a=0}^{n} \sum_{q=0}^{\min\{\ell-1,n-a\}} \binom{n-a}{q} \varphi_{\oplus,\ell}(v)^{n-q} \left(\prod_{c=-1}^{q-1} z^\dagger_{\ell-c-1}(v) \right) A_{\ell-q-1} \\ &= \varphi_{\oplus,\ell}(v)^{n+1} A_\ell + \sum_{q=1}^{\min\{\ell,n+1\}} \sum_{a=0}^{n+1-q} \binom{n-a}{q-1} \varphi_{\oplus,\ell}(v)^{n+1-q} \left(\prod_{c=0}^{q} z^\dagger_{\ell-c}(v) \right) A_{\ell-q} \\ &= \varphi_{\oplus,\ell}(v)^{n+1} A_\ell + \sum_{q=1}^{\min\{\ell,n+1\}} \binom{n+1}{q} \varphi_{\oplus,\ell}(v)^{n-q-1} \prod_{c=0}^{q} z^\dagger_{\ell-c}(v) A_{\ell-q}\end{aligned}$$

as desired. To prove the last statement we let $\psi \in \mathcal{D}(f(d\Gamma_+(\omega)))$. Note that

$$(f(d\Gamma_+(\omega))\psi)^{(n+\ell)}(k_1, \dots, k_{n+\ell}) = f(\omega(k_1) + \dots + \omega(k_{n+\ell}))\psi^{(n+\ell)}(k_1, \dots, k_{n+\ell})$$

is in $S_{n+\ell}(\mathcal{H}^{\otimes(n+\ell)})$. Standard integration theory yields $\psi^{(n+\ell)}(k_1, \dots, k_\ell, \cdot, \dots, \cdot) \in \mathcal{D}(f(d\Gamma^{(n)}(\omega) + \omega_\ell(k_1, \dots, k_\ell)))$ for almost all $(k_1, \dots, k_\ell) \in \mathcal{M}^\ell$. Furthermore we observe

$$\begin{aligned}&f(d\Gamma^{(n)}(\omega) + \omega_\ell(k_1, \dots, k_\ell))\psi^{(n+\ell)}(k_1, \dots, k_\ell, \cdot, \dots, \cdot) \\ &\qquad = (f(d\Gamma_+(\omega))\psi)^{(n+\ell)}(k_1, \dots, k_\ell, \cdot, \dots, \cdot) \\ &\int_{\mathcal{M}^\ell} \| f(d\Gamma^{(n)}(\omega) + \omega_\ell(k_1, \dots, k_\ell))\psi^{(n+\ell)}(k_1, \dots, k_\ell, \cdot, \dots, \cdot)\|^2 d\mu^{\otimes \ell}(k_1, \dots, k_\ell) \\ &\qquad = \|P_{n+\ell} f(d\Gamma_+(\omega))\psi\|^2 < \infty.\end{aligned}$$

Since $(P_n A_\ell \psi)(k_1,\dots,k_\ell) = \sqrt{n(n-1)\cdots(n-\ell+1)}\psi^{(n+\ell)}(k_1,\dots,k_\ell,\cdot,\dots,\cdot)$ we find that $A_\ell\psi \in \mathcal{D}(f(d\Gamma_{\oplus,\ell}(\omega)+\omega_\ell))$ by Lemma D.9. We calculate

$$\begin{aligned}
&(P_n f(d\Gamma_{\oplus,\ell}(\omega)+\omega_\ell)A_\ell\psi)(k_1,\dots,k_\ell)\\
&= \sqrt{n(n-1)\cdots(n-\ell+1)}f(d\Gamma^{(n)}(\omega)+\omega_\ell(k_1,\dots,k_\ell))\psi^{(n+\ell)}(k_1,\dots,k_\ell,\cdot,\dots,\cdot)\\
&= \sqrt{n(n-1)\cdots(n-\ell+1)}(f(d\Gamma_+(\omega))\psi)^{(n+\ell)}(k_1,\dots,k_\ell,\cdot,\ \dots,\cdot)\\
&= (P_n A_\ell f(d\Gamma_+(\omega))\psi)(k_1,\dots,k_\ell).
\end{aligned}$$

This finishes the proof. □

Commutation relations with Weyl operators can also be calculated but only on restricted domains. For future reference we prove

Lemma D.14. *Let $\psi \in \mathcal{D}(N^{\frac{1}{2}})$ and $g \in \mathcal{H}$. Then the following holds*

$$A_1 W(g,1)\psi = \int_{\mathcal{M}}^{\oplus} W(g,1)d\mu(k)A_1\psi + z_0(g)W(g,1)\psi \tag{D.14}$$

Proof. We calculate on an exponential vector $\epsilon(v)$

$$\begin{aligned}
(A_1 W(g,1)\epsilon(v))(k) &= e^{-\|v\|^2/2-\mathrm{Im}(\langle f,g\rangle)}A_1(\epsilon(v+g))(k)\\
&= (v(k)+g(k))W(g,1)\epsilon(v)\\
&= \left(\int_{\mathcal{M}}^{\oplus} W(g,1)d\mu(k)A_1\psi\right)(k) + z_0(g)W(g,1)\psi
\end{aligned}$$

Hence the result holds on the span of exponential vectors. The collection of exponential vectors span a core for the number operator N and thus for $N^{\frac{1}{2}}$. Hence a general element in $\psi \in \mathcal{D}(N^{\frac{1}{2}})$ may be approximated in $N^{\frac{1}{2}}$-norm by a sequence $\{\psi_n\}_{n=1}^\infty$ inside the span of exponential vectors. Now Lemmas D.10 and D.11 imply

$$\begin{aligned}
\lim_{n\to\infty} &\int_{\mathcal{M}}^{\oplus} W(g,1)d\mu(k)A_1\psi_n + z_0(g)W(g,1)\psi_n\\
&= \int_{\mathcal{M}}^{\oplus} W(g,1)d\mu(k)A_1\psi + z_0(g)W(g,1)\psi
\end{aligned}$$

in $L^2(\mathcal{M},\mathcal{E},\mu,\mathcal{F}_b(\mathcal{H}))$. Applying equation (D.4) (which hold with equality if $\ell=1$) we see that $W(g,1)\psi_n$ is Cauchy in $N^{\frac{1}{2}}$ norm and thus convergent. This implies that $W(g,1)\psi \in \mathcal{D}(N^{\frac{1}{2}})$ and $N^{\frac{1}{2}}W(g,1)\psi_n$ converges to $N^{\frac{1}{2}}W(g,1)\psi$. Appealing to Lemma D.10 we see that

$$\lim_{n\to\infty} A_1 W(g,1)\psi_n = A_1 W(g,1)\psi,$$

in $L^2(\mathcal{M},\mathcal{E},\mu,\mathcal{F}_b(\mathcal{H}))$ finishing the proof. □

The pointwise annihilation operators are useful for calculating expectation values. Before we start note that $L^2(\mathcal{M},\mathcal{F},\mu,\mathcal{F}_b(\mathcal{H}))$ is a tensor product $\mathcal{H}\otimes\mathcal{F}_b(\mathcal{H})$ under the identification $f\otimes\phi = k\mapsto f(k)\phi$. If ω is a multiplication operator on $\mathcal{H}$ then

$$\begin{aligned}
\omega\otimes 1 &= \int_{\mathcal{M}}^{\oplus}\omega(k)d\mu(k)\\
\mathcal{D}(\omega\otimes 1) &= \{f\in L^2(\mathcal{M},\mathcal{F},\mu,\mathcal{F}_b(\mathcal{H}))\mid \omega f\in L^2(\mathcal{M},\mathcal{F},\mu,\mathcal{F}_b(\mathcal{H}))\}.
\end{aligned}$$

In particular $\mathcal{D}(\omega\otimes 1) = \mathcal{D}(|\omega|\otimes 1)$. We now prove:

Theorem D.15. *Let $\psi,\phi \in \mathcal{F}_b(\mathcal{H})$ and B be a selfadjoint operator on $\mathcal{H}$. Let $B_+ = B1_{[0,\infty)}(B), B_- = B1_{(-\infty,0)}(B)$. Then:*

(1) *We have*

$$\mathcal{D}(d\Gamma(B_+)^{\frac{1}{2}})\cap\mathcal{D}(d\Gamma(B_-)^{\frac{1}{2}})=\mathcal{D}(d\Gamma(|B|)^{\frac{1}{2}}) \tag{D.15}$$

$$\mathcal{D}(d\Gamma(|B|))\subset\mathcal{D}(d\Gamma(B_+)),\mathcal{D}(d\Gamma(B_-)),\mathcal{D}(d\Gamma(B)) \tag{D.16}$$

and $d\Gamma(B_+)-d\Gamma(B_-)=d\Gamma(B)$ *on* $\mathcal{D}(d\Gamma(|B|))$.

(2) *Assume* B *is a multiplication operator. Then* $\psi\in\mathcal{D}(d\Gamma(|B|)^{\frac{1}{2}})\iff|B|^{\frac{1}{2}}A_1\psi\in L^2(\mathcal{M},\mathcal{E},\mu,\mathcal{F}_b(\mathcal{H}))$. *Furthermore, for* $\phi,\psi\in\mathcal{D}(d\Gamma(|B|)^{\frac{1}{2}})$ *we have*

$$\sum_{\sigma\in\pm}\sigma\langle d\Gamma(B_\sigma)^{\frac{1}{2}}\phi,d\Gamma(B_\sigma)^{\frac{1}{2}}\psi\rangle=\int_{\mathcal{M}}B(k)\langle A_1\phi(k),A_1\psi(k)\rangle d\mu(k), \tag{D.17}$$

along with $A_1\psi(k)\in\mathcal{F}_b(\mathcal{H})$ *almost everywhere on* $\{|B(k)|>0\}$.

(3) *For* $\psi,\phi\in\mathcal{D}(d\Gamma(|B|)^{\frac{1}{2}})\cap\mathcal{D}(N^{\frac{1}{2}})$ *we find* $A_1\psi,A_1\phi\in\mathcal{D}(|B|^{\frac{1}{2}}\otimes 1)$ *and*

$$\langle d\Gamma(|B|)^{\frac{1}{2}}\phi,d\Gamma(|B|)^{\frac{1}{2}}\psi\rangle=\langle(|B|^{\frac{1}{2}}\otimes 1)A_1\phi,(|B|^{\frac{1}{2}}\otimes 1)A_1\psi\rangle. \tag{D.18}$$

(4) *For* $\psi\in\mathcal{D}(d\Gamma(|B|))\cap\mathcal{D}(N^{\frac{1}{2}})$ *and* $\phi\in\mathcal{D}(N^{\frac{1}{2}})$ *we find* $A_1\psi\in\mathcal{D}(|B|\otimes 1)=\mathcal{D}(B\otimes 1)$ *and*

$$\langle\phi,d\Gamma(B)\psi\rangle=\langle A_1\phi,(B\otimes 1)A_1\psi\rangle. \tag{D.19}$$

(5) *Let* $v\in\mathcal{H}$ *and* $\psi\in\mathcal{F}_b(\mathcal{H})$. *If* $x\mapsto\overline{v}(k)(A_1\psi)(k)$ *is Fock space valued and integrable in the weak sense then* $\psi\in\mathcal{D}(a(v))$ *and*

$$a(v)\psi=\int_{\mathcal{M}}\overline{v}(k)(A_1\psi)(k)d\mu(k). \tag{D.20}$$

Proof. We start by proving the first four statements of the theorem when B is a multiplication operator. Let $A\in\{B,B_+,B_-\}$ and note $A\leq|B|$ everywhere. We prove equations (D.15) and (D.16) as follows

$$\begin{aligned}
&\mathcal{D}(d\Gamma(B_+)^{\frac{1}{2}})\cap\mathcal{D}(d\Gamma(B_-)^{\frac{1}{2}})\\
&=\left\{(\psi^{(n)})\middle|\sum_{n=1}^\infty\int_{\mathcal{M}^n}(B_\pm(k_1)+\cdots+B_\pm(k_n))|\psi^{(n)}|^2d\mu^{\otimes n}<\infty\right\}\\
&=\left\{(\psi^{(n)})\middle|\sum_{n=1}^\infty\int_{\mathcal{M}^n}(|B(k_1)|+\cdots+|B(k_n)|)|\psi^{(n)}|^2d\mu^{\otimes n}<\infty\right\}=\mathcal{D}(d\Gamma(|B|)^{\frac{1}{2}})\\
&\mathcal{D}(d\Gamma(|B|))=\left\{(\psi^{(n)})\middle|\sum_{n=1}^\infty\int_{\mathcal{M}^n}(|B(k_1)|+\cdots+|B(k_n)|)^2|\psi^{(n)}|^2d\mu^{\otimes n}<\infty\right\}\\
&\subset\left\{(\psi^{(n)})\middle|\sum_{n=1}^\infty\int_{\mathcal{M}^n}(A(k_1)+\cdots+A(k_n))^2|\psi^{(n)}|^2d\mu^{\otimes n}<\infty\right\}=\mathcal{D}(d\Gamma(A)).
\end{aligned}$$

The identity $d\Gamma(B_+)-d\Gamma(B_-)=d\Gamma(B)$ on $\mathcal{D}(d\Gamma(|B|))$ is now a simple computation. To prove statement (2), we calculate using symmetry

$$\begin{aligned}
&\sum_{n=1}^\infty\int_{\mathcal{M}^n}(|B(k_1)|+\cdots+|B(k_n)|)|\psi^{(n)}(k_1,\dots,k_n)|^2d\mu^{\otimes n}(k_1,\dots,k_n)\\
&=\int_{\mathcal{M}}|B(k_1)|\sum_{n=1}^\infty n\int_{\mathcal{M}^{n-1}}|\psi^{(n)}(k_1,\dots,k_n)|^2d\mu^{\otimes n-1}(k_2,\dots,k_n)d\mu(k_1) \qquad (D.21)\\
&=\int_{\mathcal{M}}|B(k)|\|A_1\psi(k)\|^2d\mu(k).
\end{aligned}$$

This shows statement (2) except equation (D.17). We have however proven equation (D.17) in the case $\phi=\psi$ and $B\geq 0$. Using linearity and statement (1), we find equation (D.17) holds for $\phi=\psi$. One may now apply the polarisation identity to finish the proof of statement (2). Statement (3) follows trivially from statement (2) when B is a multiplication operator.

We now prove statement (4). First we note that

$$B(k_1)^2 + \cdots + B(k_n)^2 \leq (|B(k_1)| + \cdots + |B(k_n)|)^2$$

so $\mathcal{D}(d\Gamma(|B|)) \subset \mathcal{D}(d\Gamma(B^2)^{\frac{1}{2}})$. This implies $A_1\psi \in \mathcal{D}(|B|\otimes 1) = \mathcal{D}(B\otimes 1)$ by statement (3). If $\phi,\psi \in \mathcal{D}(N^{\frac{1}{2}}) \cap \mathcal{D}(d\Gamma(|B|))$ the formula in statement (4) will follow from statements (1) and (2). To finish the proof, it is by Lemma D.10 enough to find a sequence $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{D}(N^{\frac{1}{2}}) \cap \mathcal{D}(d\Gamma(|B|))$ that converges to $\phi \in \mathcal{D}(N^{\frac{1}{2}})$ in the graph norm of $\mathcal{D}(N^{\frac{1}{2}})$.

Let $\phi \in \mathcal{D}(N^{\frac{1}{2}})$ and let $\phi_n = 1_{[-n,n]}(d\Gamma(|B|))\phi$. Since $d\Gamma(|B|)$ and $N^{\frac{1}{2}}$ commute strongly, we find $\phi_n \in \mathcal{D}(N^{\frac{1}{2}}) \cap \mathcal{D}(d\Gamma(|B|))$ and

$$\| N^{\frac{1}{2}}(\phi_n - \phi)\| = \|(1 - 1_{[-n,n]}(d\Gamma(|B|)))N^{\frac{1}{2}}\phi\|$$

which converges to 0. This finishes the proof when B is a multiplication operator.

For general B we may pick an L^2 space $\mathcal{K}$ and unitary $U : \mathcal{H} \to \mathcal{K}$ such that $UBU^* = \omega$ is a multiplication operator on $\mathcal{K}$. Note that $\Gamma(U)$ transforms $d\Gamma(f(B))$ into $d\Gamma(f(\omega))$ for any real, measurable f. This implies that statement (1) holds, since it holds with $\Gamma(U)$ applied on both sides.

Let $\widetilde{N}$ be the number operator on $\mathcal{F}_b(\mathcal{K})$ and $\widetilde{A}_1$ denote the pointwise annihilation operator with respect to $\mathcal{F}_b(\mathcal{K})$. First we prove that

$$U^* \otimes \Gamma(U)^* \widetilde{A}_1 \Gamma(U) = A_1$$

as maps on $\mathcal{D}(N^{\frac{1}{2}})$. Since $\Gamma(U)\mathcal{D}(N^{\frac{1}{2}}) = \mathcal{D}(\widetilde{N}^{\frac{1}{2}})$ so both operators are defined on the same sets. Now a sequence that converges in $N^{\frac{1}{2}}$-norm will be mapped by $\Gamma(U)$ to a sequence that converges in $\widetilde{N}^{\frac{1}{2}}$ norm. Therefore we just need to show the equality on a set that spans a core for $N^{\frac{1}{2}}$. The set $A = \{h^{\otimes n} \mid h \in \mathcal{H}, n \in \mathbb{N}_0\}$ satisfies this so we fix $h^{\otimes n} \in A$ and calculate

$$\begin{aligned} U^* \otimes \Gamma(U)^* \widetilde{A}_1 \Gamma(U) h^{\otimes n}(k) &= \sqrt{n}(U^* \otimes \Gamma(U)^*)(Uh)(k)(Uh)^{\otimes n-1} \\ &= \sqrt{n} h(k) h^{\otimes n-1} \\ &= A_1 h^{\otimes n} \end{aligned}$$

as desired.

We now prove statement (3). Under the assumptions in statement (3) we have $\Gamma(U)\psi, \Gamma(U)\phi \in \mathcal{D}(d\Gamma(|\omega|)^{\frac{1}{2}}) \cap \mathcal{D}(\widetilde{N}^{\frac{1}{2}})$ so $A_1\psi, A_1\phi \in U^* \otimes \Gamma(U)^*\mathcal{D}(|\omega|^{\frac{1}{2}}\otimes 1) = \mathcal{D}(|B|^{\frac{1}{2}}\otimes 1)$. We may then calculate

$$\begin{aligned} \langle d\Gamma(|B|)^{\frac{1}{2}}\phi, d\Gamma(|B|)^{\frac{1}{2}}\psi\rangle &= \langle d\Gamma(|\omega|)^{\frac{1}{2}}\Gamma(U)\phi, d\Gamma(|\omega|)^{\frac{1}{2}}\Gamma(U)\psi\rangle \\ &= \langle (|\omega|^{\frac{1}{2}}\otimes 1)\widetilde{A}_1\Gamma(U)\phi, (|\omega|^{\frac{1}{2}}\otimes 1)\widetilde{A}_1\Gamma(U)\psi\rangle \\ &= \langle (|B|^{\frac{1}{2}}\otimes 1)A_1\phi, (|B|^{\frac{1}{2}}\otimes 1)A_1\psi\rangle. \end{aligned}$$

We now prove statement (4). Under the given assumptions $\Gamma(U)\psi \in \mathcal{D}(d\Gamma(|\omega|)) \cap \mathcal{D}(\widetilde{N}^{\frac{1}{2}})$ and so $A_1\psi \in U^* \otimes \Gamma(U)^*\mathcal{D}(|\omega|\otimes 1) = \mathcal{D}(|B|\otimes 1)$. Hence

$$\begin{aligned} \langle \phi, d\Gamma(B)\psi\rangle &= \langle \Gamma(U)\phi, d\Gamma(\omega)\Gamma(U)\psi\rangle \\ &= \langle \widetilde{A}_1\Gamma(U)\phi, (\omega\otimes 1)\widetilde{A}_1\Gamma(U)\psi\rangle \\ &= \langle A_1\phi, (B\otimes 1)A_1\psi\rangle. \end{aligned}$$

To prove statement (5) fix $\phi \in \mathcal{H}^{\otimes_s n}$. Then

$$\left\langle \phi, P_n \int_{\mathcal{M}} \overline{v}(k)(A_1\psi)(k)d\mu(k) \right\rangle$$
$$= \sqrt{n+1} \int_{\mathcal{M}} \int_{\mathcal{M}^n} \overline{v(k)\phi(k_1,\dots,k_n)} \psi^{(n+1)}(k,k_1,\dots,k_n)d\mu^{\otimes n}(k_1,\dots,k_n)d\mu(k).$$

Using Fubinis Theorem we see

$$P_n \int_{\mathcal{M}} \overline{v}(k)(A_1\psi)(k)d\mu(k) = a_{n+1}(v)\psi^{(n+1)}.$$

Hence $(a_{n+1}(v)\psi^{(n+1)}) \in \mathcal{F}_b(\mathcal{H})$ so $\psi \in \mathcal{D}(a(v))$ and the desired equality holds. $\square$

We can now prove the pull-trough formula.

Theorem D.16. *Let $\alpha \in \mathbb{R}^{2n}, \eta \in \mathbb{R}, f \in \mathcal{H}^{2n}$ and ω be a selfadjoint multiplication operator on $\mathcal{H}$. Assume (α, f, ω) satisfies Hypothesis 1,2,3 and either $n \leq 2$ or Hypothesis 4. Define now $\mathcal{E}_\ell = \mathcal{E}_{(-1)^\ell\eta}(\alpha, f, \omega)$, $F_\ell := F_{(-1)^\ell\eta}(\alpha, f, \omega)$ and $\omega_\ell(k_1,\dots,k_n) = \omega(k_1) + \cdots + \omega(k_\ell)$. Let $\lambda \leq \mathcal{E}_\ell$ for all ℓ and let $R_\ell(a) = (F_\ell - \lambda + a)^{-1}$ for $a > 0$. If $\psi \in \mathcal{D}(F_0) = \mathcal{D}(F_1)$ and $A_q(F_0 - \lambda)\psi$ is Fock space valued for all $q \leq \ell$, then $(A_\ell\psi)(k) \in \mathcal{D}(F_0) = \mathcal{D}(F_1)$ for almost every $k \in \mathcal{M}^\ell$ and*

$$\begin{aligned} A_\ell\psi = &- R_\ell(\omega_\ell(\cdot)) \sum_{i=1}^{2n} \alpha_i \sum_{q=1}^{\min\{i,\ell\}} \binom{i}{q} \varphi_{\oplus,\ell}(f_i)^{i-q} \left(\prod_{c=0}^{q-1} P_{\ell-c} z_{\ell-c-1}(f_i) \right) A_{\ell-q}\psi \\ &+ R_\ell(\omega_\ell(\cdot))A_\ell(F_0 - \lambda)\psi. \end{aligned} \tag{D.22}$$

Assume furthermore, that Hypothesis 5 holds, $\eta \leq 0$ and ψ is a ground state for F_0. Then we may take $\lambda = \mathcal{E}_0$ and we have $A_\ell\psi \in L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes\ell}, \mu^{\otimes\ell}, \mathcal{F}_b(\mathcal{H}))$.

Proof. By definition $F_\ell - \lambda \geq 0$ for all ℓ and so the resolvent $R(\omega_\ell(k))$ exists almost everywhere since $\{\omega \leq 0\}$ is a μ-zero set. Define the lifted operators on $\mathcal{F}_+(\mathcal{H})$ and $C(\mathcal{M}^\ell, \mathcal{E}^{\otimes\ell}, \mu^{\otimes\ell})$ respectively

$$F_{+,\ell} = (-1)^\ell \eta \Gamma_+(-1) + d\Gamma_+(\omega) + \sum_{i=1}^{2n} \alpha_i \varphi_+(f_i)^i$$
$$F_{\oplus,\ell} = (-1)^\ell \eta \Gamma_{\oplus,\ell}(-1) + d\Gamma_{\oplus,\ell}(\omega) + \omega_\ell + \sum_{i=1}^{2n} \alpha_i \varphi_{\oplus,\ell}(f_i)^i$$

with domains $\mathcal{D}(F_+) = \mathcal{D}(d\Gamma_+(\omega))$ and $\mathcal{D}(F_\oplus) = \mathcal{D}(d\Gamma_{\oplus,\ell}(\omega) + \omega_\ell)$. Let $\psi \in \mathcal{D}(F_0) = \mathcal{D}(F_1)$ and assume $A_q(F_0 - \lambda)\psi$ is Fock space valued for all $q \leq \ell$. By Lemma D.13 we have $A_\ell\psi \in \mathcal{D}(F_\oplus)$. Using Lemmas D.3, D.4 and D.13 we also obtain

$$\begin{aligned} g_\ell := &- \sum_{i=1}^{2n} \alpha_i \sum_{q=1}^{\min\{i,\ell\}} \binom{i}{q} \varphi_{\oplus,\ell}(f_i)^{i-q} \left(\prod_{c=0}^{q-1} S_{\ell-c} z_{\ell-c-1}(f_i) \right) A_{\ell-q}\psi + A_\ell(F_0 - \lambda)\psi \\ &= (F_{\oplus,\ell} - \lambda)A_\ell\psi. \end{aligned}$$

Assume now we have proven that g_ℓ is almost everywhere Fock space valued. Let M be a zeroset such that:

- $A_\ell\psi$ is $\mathcal{F}_{-\ell/2,+}(\mathcal{H})$ valued on M^c (see Lemma D.10).
- $g_\ell(k) = (F_{+,\ell} + \omega_\ell(k) + \lambda)(A_\ell\psi)(k)$ and $g_\ell(k) \in \mathcal{F}_b(\mathcal{H})$ for $k \in M^c$.
- $R(\omega_\ell(k))$ exists on M^c.

Fix $k \in M^c$. For any vector ϕ such that both $R_\ell(\omega_\ell(k))\phi$ and ϕ is in $\mathcal{N}$ (this set is dense by Proposition 3.1) we find using Lemma D.7 that

$$\begin{aligned}\langle \phi, A_\ell\psi(k)\rangle_+ &= \langle (F_\ell + \omega_\ell(k) - \lambda)R_\ell(\omega_\ell(k))\phi, A_\ell\psi(k)\rangle_+ \\ &= \langle R_\ell(\omega_\ell(k))\phi, g_\ell(k)\rangle = \langle \phi, R_\ell(\omega_\ell(k))g_\ell(k)\rangle_+.\end{aligned}$$

Corollary D.6 shows that $A_\ell\psi(k) = R_\ell(\omega_\ell(k))g_\ell(k)$ for every $k \in M^c$ proving equation $(D.22)$ and that $A_\ell\psi$ is almost everywhere $\mathcal{D}(F_0) = \mathcal{D}(F_1)$-valued. We now prove g_ℓ is Fock space valued by induction.

If $\ell = 1$ then g_ℓ is a linear combination of $A_1(F_0 - \lambda)\psi$ and functions of the form $k \mapsto f^i(k)\varphi(f_i)^{i-1}\psi$ which all takes values in Fock space. Hence g_1 is almost everywhere Fock space valued and so equation (D.22) will hold for A_1. Assume now that $g_1, \dots, g_{\ell-1}$ are almost everywhere Fock space valued. Then equation (D.22) holds for $A_1\psi, \dots, A_{\ell-1}\psi$ and so $A_i\psi$ is almost everywhere $\mathcal{D}(F_0) = \mathcal{D}(F_1)$-valued for $i \leq \ell - 1$. Using Proposition 3.1 and Lemma D.3 we find for all $q \geq 1$ that

$$\varphi_{\oplus,\ell-q}(f_i)^{i-q}A_{\ell-q}\psi = k \mapsto \varphi_+(f_i)^q(A_{\ell-q}\psi)(k) = k \mapsto \varphi(f_i)^q(A_{\ell-q}\psi)(k).$$

In particular $\varphi_\oplus(f_i)^{i-q}A_{\ell-q}\psi$ is almost everywhere Fock space valued for $q \geq 1$. Since $z_q(f_i)$ and S_q map Fock space valued maps into Fock space valued maps, we see that g_ℓ is Fock space valued, finishing the proof of (D.22).

For the second part we note that $\lambda := cE_0 \leq \mathcal{E}_\ell$ for all ℓ by Theorem 3.3 and so we may apply the formula since $(F_0 - \lambda)\psi = 0$. We have already seen that $A_\ell\psi$ is $\mathcal{D}(F_0) = \mathcal{D}(F_1)$ valued almost everywhere. Hence the maps

$$k \mapsto \varphi(f_i)^q(A_\ell\psi)(k) = \varphi_{\oplus,\ell}(f_i)^qA_\ell\psi$$

will also be measurable into $\mathcal{F}_b(\mathcal{H})$ for all $q \leq i$. We will prove that they are square integrable. First we note that there is a constant $C_{q,i,l}$ such that

$$\|\varphi(f_i)^qR_\ell(\omega_\ell(k))\|^2 \leq C_{q,i,\ell}\left(1 + \frac{1}{\omega_\ell(k)}\right)^2.$$

Hence it is enough to prove that $\omega_\ell^{-2}\|g_\ell\|^2$ and $\|g_\ell\|^2$ are integrable which will now be done via induction. If $\ell = 1$ then g_ℓ is a linear combination of elements of the form $k \mapsto f_c(k)\varphi(f_c)^{c-1}\psi$ and since $f_c \in \mathcal{D}(\omega^{-1})$ the claim follows.

Inductively we now assume that $\omega_u^{-2}\|g_u\|^2, \|g_u\|^2$ are integrable for all $u < \ell$. Then $k \mapsto \varphi(f_i)^q(A_u\psi)(k)$ is square integrable for all $1 \leq i \leq 2n$, $q \leq i$ and $u < \ell$. Now g_ℓ is a linear combination of functions of the form

$$(k_1, \dots, k_\ell) \mapsto f_c(k_{\sigma(1)}) \cdots f_c(k_{\sigma(b)})\varphi(f_c)^{c-b}(A_{\ell-b}\psi)(k_{\sigma(b+1)}, \dots, k_{\sigma(\ell)})$$

where $\sigma \in \mathcal{S}_\ell, 1 \leq b \leq \ell$ and $c \in \{1, \dots, 2n\}$. Combining the observations that $\frac{1}{\omega_\ell(k)} \leq \frac{1}{\omega(k_{\sigma(1)})}$, $f_c \in \mathcal{D}(\omega^{-1})$ and $(\varphi(f_c)^{c-b}A_{\ell-b}\psi)(k_{\sigma(b+1)}, \dots, k_{\sigma(\ell)})$ is square integrable with respect to $(k_{\sigma(b+1)}, \dots, k_{\sigma(\ell)})$ we find the desired result. □

Proof of Theorem 3.5 part (1). Lemma 5.1 and Theorem 3.4 shows it is enough to prove the claim for the fiber operator. Lemmas 2.2 and A.10 show we may assume $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ with $(\mathcal{M}, \mathcal{F}, \mu)$ a σ-finite measure space. This case is dealt with in Lemma D.10 and Theorem D.16. □

Appendix E. Q-spaces and functional analysis

Following the approach in [8] we have

Lemma E.1. *Let $\{f_\alpha\}_{\alpha\in I} \subset \mathcal{H}$ and $\omega \geq 0$ be selfadjoint on $\mathcal{H}$. Assume that $\langle f_\alpha, g(\omega)f_\beta\rangle \in \mathbb{R}$ for all $\alpha, \beta \in I$ and $g \in \mathcal{M}_{b,+}(\mathbb{R}, \mathbb{R})$ where $\mathcal{M}_{b,+}(\mathbb{R}, \mathbb{R})$ is the set of real measurable maps from $\mathbb{R}$ to $\mathbb{R}$ which are bounded on $[0, \infty)$. Then there is a real Hilbert space $\mathcal{H}_\mathbb{R}$ such that $\mathcal{H} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$, $e^{-t\omega}$ maps $\mathcal{H}_\mathbb{R}$ to $\mathcal{H}_\mathbb{R}$ for all $t \geq 0$ and $f_\alpha \in \mathcal{H}_\mathbb{R}$ for all $\alpha \in I$.*

Proof. Let

$$\mathcal{H}' = \overline{\mathrm{Span}_{\mathbb{R}}\{g(\omega)f_\alpha \mid g \in \mathcal{M}_{b,+}(\mathbb{R},\mathbb{R}), \alpha \in I\}}.$$

Note that $\mathcal{H}'$ is a real Hilbert space. For every $f \in (\mathcal{H}')^\perp \backslash \{0\}$ we define

$$\mathcal{H}(f) = \overline{\mathrm{Span}_{\mathbb{R}}\{g(\omega)f \mid g \in \mathcal{M}_{b,+}(\mathbb{R},\mathbb{R})\}}.$$

It is clear that $e^{-t\omega}$ maps $\mathcal{H}'$ to $\mathcal{H}'$ and $\mathcal{H}(f)$ to $\mathcal{H}(f)$, since it maps the spanning set to the spanning set. Furthermore we define

$$\mathcal{A} = \{A \subset (\mathcal{H}')^\perp \mid \mathcal{H}(f) \perp \mathcal{H}(g) \ \forall\ f \neq g \in A\}.$$

We partially order $\mathcal{A}$ by inclusion and take a maximal totally ordered subset $\mathcal{B}$. Let B be the union of all elements in $\mathcal{B}$. If $f, g \in B$, then there is an element in $\mathcal{B}$ that contains both f and g (since $\mathcal{B}$ is totally ordered). This implies $\mathcal{H}(f) \perp \mathcal{H}(g)$ and so $B \in \mathcal{B}$ and is clearly the largest element. Define now

$$\mathcal{H}_{\mathbb{R}} := \mathcal{H}' \oplus \bigoplus_{a \in B} \mathcal{H}(a),$$

which is clearly a real Hilbert space containing $\{f_\alpha\}_{\alpha \in I}$ and it is left invariant by $e^{-t\omega}$ since each component is. Assume now towards contradiction that there is an element $f \in \mathcal{H}_{\mathbb{R}}^\perp \backslash \{0\}$. Then for every $g_1, g_2 \in \mathcal{M}_b(\mathbb{R},\mathbb{R}), h \in B$ we would have

$$\langle g_2(\omega)f, g_1(\omega)h\rangle = \langle f, g_2(\omega)g_1(\omega)h\rangle = 0$$

and so $\mathcal{H}(f)$ is orthogonal to $\mathcal{H}(h)$ for all $h \in B$. In particular $B \cup \{f\} \in \mathcal{A}$, and so $\mathcal{B} \cup \{B \cup \{f\}\}$ is larger than $\mathcal{B}$ and totally ordered which is not possible. Hence $\mathcal{H}_{\mathbb{R}}^\perp \backslash \{0\} = \emptyset$.

Let $\{e_n\}_{n=1}^N$ be an orthonormal basis for $\mathcal{H}_{\mathbb{R}}$ ($N \leq \infty$) which is then also an orthonormal basis for $\mathcal{H}$. Hence we may write any element in $\mathcal{H}$ as

$$f = \sum_{j=1}^N (a_j + ib_j)e_j = \sum_{j=1}^N a_j e_j + i \sum_{j=1}^N b_j e_j$$

as desired. This finishes the proof. □

Theorem E.2. *Let $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}$ be a real Hilbert space such that $\mathcal{H} = \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$. Then there exists a probability space $(X, \mathcal{X}, \mathbb{Q})$ such that $\mathcal{F}(\mathcal{H})$ is unitarily isomorphic to $L^2(X, \mathcal{X}, \mathbb{Q})$ via a map $\mathcal{I}$. Furthermore the following properties hold*

(1) *If U is a bounded operator on $\mathcal{H}$ with $\|U\| \leq 1$ such that $U\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{R}}$ then $\mathcal{I}\Gamma(U)\mathcal{I}^*$ is positivity preserving.*
(2) *Assume $\omega \geq 0$ is selfadjoint and injective. If $e^{-t\omega}$ maps $\mathcal{H}_{\mathbb{R}}$ into $\mathcal{H}_{\mathbb{R}}$ for all $t \geq 0$ then $\mathcal{I}e^{-td\Gamma(\omega)}\mathcal{I}^*$ is positivity improving. If $\inf(\sigma(\omega)) > 0$ then $\mathcal{I}e^{-td\Gamma(\omega)}\mathcal{I}^*$ is hypercontractive.*
(3) *If $v \in \mathcal{H}_{\mathbb{R}}$ then $\mathcal{I}\varphi(v)\mathcal{I}^*$ acts like multiplication by a normally distributed variable $\widetilde{\varphi}(v)$ with mean 0 and variance $\|v\|^2$.*
(4) *If $\{v_n\}_{n=1}^\infty \subset \mathcal{H}_{\mathbb{R}}$ converges to $v \in \mathcal{H}_{\mathbb{R}}$ then $\widetilde{\varphi}(v_n)^\ell$ converges to $\widetilde{\varphi}(v)^\ell$ in $L^q(X, \mathcal{X}, \mathbb{Q})$ for all $\ell \in \mathbb{N}$ and $q \geq 1$.*
(5) *Fix $\alpha \in \mathbb{R}^{2n}$ and $q, r > 0$ and define*

$$\mathcal{K} = \{f \in \mathcal{H}^{2n} \mid (\alpha, f) \text{ satisfies part (1) of Hypothesis 1 and } \|f_1\| < r\}$$

There is a constant $C := C(\alpha, r, q)$ such that for all $f \in \mathcal{K}$ we have

$$\|e^{\widetilde{H}_I(\alpha,f)}\|_q \leq C,$$

where $\widetilde{H}_I(\alpha, f) = \sum_{j=1}^{2n} \alpha_i \widetilde{\varphi}(f_i)$.

Proof. Everything in the first three points can be found in [4] and [17]. To prove the fourth part note that for any $N(0,\sigma^2)$ distributed variable X we have

$$\| |X|^a \|_q = \sigma^a E[|X/\sigma|^{aq}]^{1/q}$$

Since X/σ is $N(0,1)$ distributed we find that $E[|X/\sigma|^{qa}]^{1/q}$ depends only on q and a. Write $B(q,a)$ for this constant. Then we may calculate using Hölders inequality

$$\begin{aligned}\|\widetilde{\varphi}(v_n)^\ell - \widetilde{\varphi}(v)^\ell\|_q &\leq \sum_{j=0}^{\ell-1} \|\widetilde{\varphi}(v_n)^{\ell-j-1}\widetilde{\varphi}(v_n - v)\widetilde{\varphi}(v)^j\|_q \\ &\leq \sum_{j=0}^{\ell-1} \|\widetilde{\varphi}(v_n)^{(\ell-j-1)}\|_{3q}\|\widetilde{\varphi}(v_n - v)\|_{3q}\|\widetilde{\varphi}(v)^j\|_{3q} \\ &\leq \sum_{j=0}^{\ell-1} B(3q,\ell-j-1)B(3q,1)B(3q,j)\|v_n\|^{\ell-j-1}\|v_v - v\|\|v\|^j\end{aligned}$$

showing the desired result.

We now prove statement (5). The sum from $j = 2$ to $j = 2n$ is uniformly bounded below by a constant C_1 by Lemma 4.1. Thus we now find

$$\|e^{-\widetilde{H}_I(\alpha,f)}\|_q \leq e^{-C_1} E[e^{-q\alpha_1\widetilde{\varphi}(f_1)}]^{1/q} = e^{-C_1}(e^{-q^2\alpha_1^2\|f_1\|^2/2})^{1/q} \leq e^{-C_1}e^{-r^2\alpha_1^2 q/2}.$$

This finishes the proof. □

Lemma E.3. *Let $\{A_n\}_{n=1}^\infty$ be a sequence of selfadjoint operators on the Hilbert space $\mathcal{H}$ converging to A in norm resolvent sense. If B is a bounded selfadjoint operator on $\mathcal{H}$ then $\{A_n + B\}_{n=1}^\infty$ will converge in norm resolvent sense to $A + B$*

Proof. For $\lambda > \|B\|+1$ we have $\|B(A-i\lambda)^{-1}\|, \|B(A_n-i\lambda)^{-1}\| < \frac{\|B\|}{1+\|B\|}$ and so we may calculate

$$\begin{aligned}&(A + B - i\lambda)^{-1} - (A_n + B - i\lambda)^{-1} \\ &\qquad = \sum_{k=0}^\infty (A - i\lambda)^{-1}(B(A - i\lambda)^{-1})^k - (A_n - i\lambda)^{-1}(B(A_n - i\lambda)^{-1})^k\end{aligned}$$

now each term in the series converge and for fixed k we may estimate

$$\|(A - i\lambda)^{-1}(B(A - i\lambda)^{-1})^k - (A_n - i\lambda)^{-1}(B(A_n - i\lambda)^{-1})^k\| \leq \frac{2}{\lambda}\left(\frac{\|B\|}{1+\|B\|}\right)^k$$

which is summable. The conclusion now follows by dominated convergence. □

Acknowledgements. Thomas Norman Dam was supported by the Independent Research Fund Denmark through the project "Mathematics of Dressed Particles".

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Paper B

Large interaction asymptotics of spin-boson type models

By T. N. Dam and J. S. Møller

Asymptotics in Spin-Boson type models

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Abstract: In this paper we investigate a family of models for a qubit interacting with a bosonic field. More precisely, we find asymptotic limits of the Hamiltonian as the strength of the interaction tends to infinity. The main result has two applications. First of all, we show that self-energy renormalisation schemes similar to that of the Nelson model will never give a physically interesting result. This is because any limit obtained through such a scheme would be independent of the qubit. Secondly, we find that exited states exist in the massive Spin-Boson models for sufficiently large interaction strengths. We are also able to compute the asymptotic limit of many physical quantities.

1. Introduction

In this paper we consider a family of models for a qubit coupled to a bosonic field, which we will call spin-boson type models. These models has been investigated in many papers, so many properties are well known. Asymptotic completeness along with basic spectral properties were discussed in [4] and [22]. Existence and regularity of ground states were discussed in [1],[8],[11] and [13]. Furthermore, properties at positive temperature were discussed in [15] and [18].

One of the main ingredients in the papers [1] and [11] is the so-called spin-parity symmetry. In the paper [3], this symmetry is used to decompose the Hamiltonian into two so-called fiber Hamiltonians, which are both perturbations of Van Hove Hamiltonians. This symmetry is also essential to the analysis conducted in this paper, and we will need the results from [3].

To avoid a full technical description in the introduction, we will specialise to the 3-dimensional Spin-Boson model. In this case the bosons have dispersion relation $\omega(k) = \sqrt{m^2 + \|k\|^2}$ with $m \geq 0$ (here $k \in \mathbb{R}^3$). The interaction between the field and the qubit is parametrised by the functions

$$v_{g,\Lambda}(k) = g\frac{\chi_\Lambda(\omega(k))}{\sqrt{\omega(k)}}$$

where $\{\chi_\Lambda\}_{\Lambda\in(0,\infty)}$ is a family functions such that $v_{g,\Lambda} \in \mathcal{D}(\omega^{-1/2})$. We will assume that $\Lambda \mapsto \chi_\Lambda(k)$ increases to 1 as Λ tends to infinity for all $k \in \mathbb{R}^3$. Let $2\eta > 0$ be the size of the energy gap in the qubit and $H_{g,\Lambda,\eta}$ be the Hamiltonian of the full system. Then we show the following (See corollaries 4.4 and 4.3 below) two things:

1. First we consider self-energy renormalisation schemes. In such schemes one defines $f_{g,\eta}(\Lambda) = \inf(\sigma(H_{g,\Lambda,\eta}))$ and proves that $\{H_{g,\Lambda,\eta} - f_{g,\eta}(\Lambda)\}_{\Lambda\in(0,\infty)}$ converges in strong or norm resolvent sense to an operator $H^{\text{Ren}}_{g,\eta}$. Using Corollary 4.4 and Lemma 5.5 below we see:
   (1) $\Lambda \mapsto \inf(\sigma(H_{g,\Lambda,\eta})) + \|\omega^{-1/2}1_{\{\omega>1\}}v_{g,\Lambda}\|^2$ has a limit independent of η.
   (2) $(H_{g,\Lambda,\eta} - \|\omega^{-1/2}1_{\{\omega>1\}}v_{g,\Lambda}\|^2 + i)^{-1} - (H_{g,\Lambda,0} - \|\omega^{-1/2}1_{\{\omega>1\}}v_{g,\Lambda}\|^2 + i)^{-1}$ converges to 0 in norm as Λ tends to ∞ .
   From this we conclude that if a self-energy renormalisation scheme exists then $H^{\text{Ren}}_{g,\eta}$ must be independent of η, which is not physically interesting. In other words, the contribution from the qubit disappears, as the ultraviolet cutoff is removed. This result is similar to the result in [6], where it is shown, that the mass-shell in a certain model becomes "almost flat" as the ultraviolet cutoff is removed. Thus the contribution from the matter particle vanishes as the ultraviolet cutoff is removed.
2. If $m > 0$ we can take g to infinity instead. In this case the result yields that an exited state exists for g very large. Furthermore, the energy difference between the exited state and the ground state converges to 0. Taking g to infinity is not a purely mathematical exercise as experiments can go beyond the ultra deep coupling regime. This was achieved by Yoshihara, K. et al. and published in Nature Physics [25].

We will also prove two smaller results. The first result is about regularity of ground states with respect to the number operator. The result only applies to the infrared regular case, but is close to optimal and extends the results found in [13]. The second result is a condition under which the massive spin-boson model has an exited state in the mass gap.

2. Notation and preliminaries

We start by fixing notation. If X is a topological space we will write $\mathcal{B}(X)$ for the Borel σ-algebra. Furthermore if $(\mathcal{M}, \mathcal{F}, \mu)$ is a measure space we will for $1 \le p \le \infty$ write $L^p(\mathcal{M}, \mathcal{F}, \mu)$ for the corresponding L^p space.

Throughout this paper $\mathcal{H}$ will denote the state space of a single boson which we will assume to be a separable Hilbert space. Let S_n denote projection of $\mathcal{H}^{\otimes n}$ onto the subspace of symmetric tensors. The bosonic (or symmetric) Fock space is defined as

$$\mathcal{F}_b(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_n(\mathcal{H}^{\otimes n}).$$

If $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ where $(\mathcal{M}, \mathcal{F}, \mu)$ is a σ-finite measure space then $S_n(\mathcal{H}^{\otimes n}) = L^2_{sym}(\mathcal{M}^n, \mathcal{F}^{\otimes n}, \mu^{\otimes n})$. An element $\psi \in \mathcal{F}_b(\mathcal{H})$ is an infinite sequence of elements which is written as $\psi = (\psi^{(n)})$. We also define the vacuum $\Omega = (1, 0, 0, \dots)$. Furthermore, we will write

$$S_n(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes_s \cdots \otimes_s f_n.$$

For $g \in \mathcal{H}$ one defines the annihilation operator $a(g)$ and creation operator $a^\dagger(g)$ on symmetric tensors in $\mathcal{F}_b(\mathcal{H})$ by $a(g)\Omega = 0, a^\dagger(g)\Omega = g$ and

$$a(g)(f_1 \otimes_s \cdots \otimes_s f_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \langle g, f_i \rangle f_1 \otimes_s \cdots \otimes_s \hat{f}_i \otimes_s \cdots \otimes_s f_n$$
$$a^\dagger(g)(f_1 \otimes_s \cdots \otimes_s f_n) = \sqrt{n+1} g \otimes_s f_1 \otimes_s \cdots \otimes_s f_n$$

where $\widehat{f_i}$ means that f_i is omitted from the tensor product. One can show that these operators extends to closed operators on $\mathcal{F}_b(\mathcal{H})$ and that $(a(g))^* = a^\dagger(g)$. Furthermore we have the canonical commutation relations which states

$$\overline{[a(f), a(g)]} = 0 = \overline{[a^\dagger(f), a^\dagger(g)]} \text{ and } \overline{[a(f), a^\dagger(g)]} = \langle f, g \rangle.$$

One now introduces the selfadjoint field operators

$$\varphi(g) = \overline{a(g) + a^\dagger(g)}.$$

Let ω be a selfadjoint and non-negative operator on $\mathcal{H}$ with domain $\mathcal{D}(\omega)$. Write $(1\otimes)^{k-1}\omega(\otimes 1)^{n-k}$ for the operator $B_1 \otimes ... \otimes B_n$ where $B_k = \omega$ and $B_j = 1$ if $j \neq k$. We then define the second quantisation of ω to be the selfadjoint operator

$$d\Gamma(\omega) = 0 \oplus \bigoplus_{n=1}^{\infty} \left(\sum_{k=1}^{n} (1\otimes)^{k-1} \omega (\otimes 1)^{n-k} \right) |_{S_n(\mathcal{H}^{\otimes n})} . \tag{2.1}$$

If ω is a multiplication operator then $d\Gamma(\omega)$ acts on elements in $S_n(\mathcal{H}^{\otimes n})$ as multiplication by $\omega_n(k_1, \dots, k_n) = \omega(k_1) + \cdots + \omega(k_n)$. The number operator is defined as $N = d\Gamma(1)$. Let U be unitary from $\mathcal{H}$ to $\mathcal{K}$. Then we define the unitary from $\mathcal{F}_b(\mathcal{H})$ to $\mathcal{F}_b(\mathcal{K})$ by

$$\Gamma(U) = 1 \oplus \bigoplus_{n=1}^{\infty} U^{\otimes n} |_{S_n(\mathcal{H}^{\otimes n})},$$

For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we also define the operators $d\Gamma^{(n)}(\omega) = d\Gamma(\omega) |_{S_n(\mathcal{H}^{\otimes n})}$ and $\Gamma^{(n)}(U) = \Gamma(U) |_{S_n(\mathcal{H}^{\otimes n})}$. See [3] for a proof of the following lemma:

Lemma 2.1. *Let $\omega \geq 0$ be a selfadjoint operator defined on the Hilbert space $\mathcal{H}$ and let $m = \inf(\sigma(\omega))$. For $n \geq 1$ we have*

$$\sigma(d\Gamma^{(n)}(\omega)) = \overline{\{\lambda_1 + \cdots + \lambda_n \mid \lambda_i \in \sigma(\omega)\}},$$
$$\inf(\sigma(d\Gamma^{(n)}(\omega))) = nm.$$

Furthermore, $d\Gamma(\omega)$ will have compact resolvents if and only if ω has compact resolvents. Furthermore $d\Gamma^{(n)}(\omega)$ is injective for $n \geq 1$ if ω is injective.

We now introduce the Weyl representation. For any $g \in \mathcal{H}$ we define the corresponding exponential vector

$$\epsilon(g) = \sum_{n=0}^{\infty} \frac{g^{\otimes n}}{\sqrt{n!}}.$$

One may prove that if $\mathcal{D} \subset \mathcal{H}$ is dense then the set $\{\epsilon(f) \mid f \in \mathcal{D}\}$ is a linearly independent total subset of $\mathcal{F}_b(\mathcal{H})$. Let $\mathcal{U}(\mathcal{H})$ be the unitaries from $\mathcal{H}$ into $\mathcal{H}$. Fix now $U \in \mathcal{U}(\mathcal{H})$ and $h \in \mathcal{H}$. The corresponding Weyl transformation is the unique unitary map $W(h,U)$ satisfying

$$W(h,U)\epsilon(g) = e^{-\|h\|^2/2 - \langle f, Ug \rangle}\epsilon(h + Ug).$$

for all $g \in \mathcal{H}$. One may easily check that $(h,U) \mapsto W(h,U)$ is strongly continuous. Furthermore one may check the relation

$$W(h_1,U_1)W(h_2,U_2) = e^{-i\mathrm{Im}(\langle h_1, U_1 h_2 \rangle)}W((h_1,U_1)(h_2,U_2)), \tag{2.2}$$

where $(h_1,U_1)(h_2,U_2) = (h_1 + U_1 h_2, U_1 U_2)$. If ω is selfadjoint and $f \in \mathcal{H}$ then we have

$$e^{itd\Gamma(\omega)} = \Gamma(e^{it\omega}) = W(0, e^{it\omega}) \tag{2.3}$$

$$e^{it\varphi(if)} = W(tf, 1). \tag{2.4}$$

The following lemma is important and well known (see e.g [2] and [5]):

Lemma 2.2. *Let $\omega \geq 0$ be selfadjoint and injective. If $g \in \mathcal{D}(\omega^{-1/2})$ then $\varphi(g)$ is $d\Gamma(\omega)^{1/2}$ bounded. In particular $\varphi(g)$ is $N^{1/2}$ bounded. We have the following bound*

$$\|\varphi(g)\psi\| \leq 2\|(\omega^{-1/2} + 1)g\| \|(d\Gamma(\omega) + 1)^{1/2}\psi\|$$

which holds on $\mathcal{D}(d\Gamma(\omega)^{1/2})$. In particular $\varphi(g)$ is infinitesimally $d\Gamma(\omega)$ bounded. Furthermore $\sigma(d\Gamma(\omega) + \varphi(g)) = -\|\omega^{-1/2}g\|^2 + \sigma(d\Gamma(\omega))$.

3. The Spin-Boson model

Let $\sigma_x, \sigma_y, \sigma_z$ denote the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and define $e_1 = (1,0)$ and $e_{-1} = (0,1)$. The total system has the Hamiltonian

$$H_\eta(v,\omega) := \eta\sigma_z \otimes 1 + 1 \otimes d\Gamma(\omega) + \sigma_x \otimes \varphi(v),$$

which is here parametrised by $v \in \mathcal{H}, \eta \in \mathbb{C}$ and ω selfadjoint on $\mathcal{H}$. We will also need the fiber operators:

$$F_\eta(v,\omega) = \eta\Gamma(-1) + d\Gamma(\omega) + \varphi(v).$$

acting in $\mathcal{F}_b(\mathcal{H})$. If the spectra are real we define

$$\begin{aligned} E_\eta(v,\omega) &:= \inf(\sigma(H_\eta(v,\omega))) \\ \mathcal{E}_\eta(v,\omega) &:= \inf(\sigma(F_\eta(v,\omega))). \end{aligned}$$

For ω selfadjoint on $\mathcal{H}$ we define

$$m(\omega) = \inf\{\sigma(\omega)\} \quad \text{and} \quad m_{\text{ess}}(\omega) = \inf\{\sigma_{\text{ess}}(\omega)\}.$$

Standard perturbation theory and Lemma 2.2 yields:

Proposition 3.1. *Let $\omega \geq 0$ be selfadjoint and injective, $v \in \mathcal{D}(\omega^{-1/2})$ and $\eta \in \mathbb{C}$. Then the operators $F_\eta(v,\omega)$ and $H_\eta(v,\omega)$ are closed on the respective domains*

$$\begin{aligned} \mathcal{D}(F_\eta(v,\omega)) &= \mathcal{D}(d\Gamma(\omega)) \\ \mathcal{D}(H_\eta(v,\omega)) &= \mathcal{D}(1 \otimes d\Gamma(\omega)) \end{aligned}$$

and given any core $\mathcal{D}$ of ω the linear span of the following sets

$$\begin{aligned} \mathcal{J}(\mathcal{D}) &:= \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{f_1 \otimes_s \cdots \otimes_s f_n \mid f_j \in \mathcal{D}\} \\ \widetilde{\mathcal{J}}(\mathcal{D}) &:= \{f_1 \otimes f_2 \mid f_1 \in \{e_1, e_{-1}\}, f_2 \in \mathcal{J}(\mathcal{D})\} \end{aligned}$$

is a core for $F_\eta(v,\omega)$ and $H_\eta(v,\omega)$ respectively. Furthermore both operators are selfadjoint and semibounded if $\eta \in \mathbb{R}$ and they have compact resolvents if ω has compact resolvents.

From the paper [3] we find the following theorem:

Theorem 3.2. *Let $\phi = (\phi_1, \phi_{-1}) = e_1 \otimes \phi_1 + e_{-1} \otimes \phi_{-1}$ be an element in $\mathcal{F}_b(\mathcal{H})^2 = \mathcal{F}_b(\mathcal{H}) \oplus \mathcal{F}_b(\mathcal{H}) \approx \mathbb{C}^2 \otimes \mathcal{F}_b(\mathcal{H})$. Write $\phi_i = (\phi_i^{(k)})$ for $i \in \{-1, 1\}$. Let $i \in \{-1, 1\}$. Define $\widetilde{\phi}_i = (\widetilde{\phi}_i^{(k)})$ where*

$$\widetilde{\phi}_i^{(k)} = \begin{cases} \phi_i^{(k)} & k \text{ is even} \\ \phi_{-i}^{(k)} & k \text{ is odd} \end{cases}$$

and $V(\phi_1, \phi_{-1}) = (\widetilde{\phi}_1, \widetilde{\phi}_{-1})$. Then

(1) *V is unitary with $V^* = V$.*

(2) *If $\omega \geq 0$ is selfadjoint and injective then $V 1 \otimes d\Gamma(\omega) V^* = 1 \otimes d\Gamma(\omega)$. If furthermore $\eta \in \mathbb{R}$ and $v \in \mathcal{D}(\omega^{-1/2})$ then*

$$V H_\eta(v,\omega) V^* = F_{-\eta}(v,\omega) \oplus F_\eta(v,\omega).$$

(3) *Let $\omega \geq 0$ be selfadjoint and injective, $\eta \in \mathbb{R}$ and $v \in \mathcal{D}(\omega^{-1/2})$. Then $E_\eta(v,\omega) = \mathcal{E}_{-|\eta|}(v,\omega)$ and $H_\eta(v,\omega)$ has a ground state if and only if the operator $F_{-|\eta|}(v,\omega)$ has a ground state. This is the case if $m(\omega) > 0$, and it is non degenerate if $\eta \neq 0$. Also*

$$\begin{aligned} \inf(\sigma_{\text{ess}}(F_{|\eta|}(v,\omega))) &= \mathcal{E}_{-|\eta|}(v,\omega) + m_{\text{ess}}(\omega) \\ \inf(\sigma_{\text{ess}}(H_\eta(v,\omega))) &= E_\eta(v,\omega) + m_{\text{ess}}(\omega) \end{aligned}$$

and $\mathcal{E}_{|\eta|}(v,\omega) > \mathcal{E}_{-|\eta|}(v,\omega)$ if and only if both $\eta \neq 0$ and $m(\omega) \neq 0$.

(4) *Let $\omega \geq 0$ be selfadjoint and injective, $\eta \in \mathbb{R}$ and $v \in \mathcal{D}(\omega^{-1/2})$. If ϕ is a ground state for $H_\eta(v,\omega)$ then*

$$V\phi = \begin{cases} e_{-\mathrm{sign}(\eta)} \otimes \psi & \eta \neq 0 \\ e_{-1} \otimes \psi_{-1} + e_1 \otimes \psi_1 & \eta = 0 \end{cases}$$

where ψ is a ground state for $F_{-|\eta|}(v,\omega)$ and ψ_1, ψ_{-1} are either 0 or a ground state for $F_0(v,\omega)$.

4. Results

In this section we state the results which are proven in this paper. Throughout this section ω will always denote an injective, non negative and selfadjoint operator on $\mathcal{H}$. Furthermore, we will write $m = m(\omega)$ and $m_{\mathrm{ess}} = m_{\mathrm{ess}}(\omega)$. The main technical result is the following theorem:

Theorem 4.1. *Let $\{v_g\}_{g \in (0,\infty)} \subset \mathcal{D}(\omega^{-1/2})$ and P_ω denote the spectral measure corresponding to ω. Assume that there is $\widetilde{m} > 0$ such that:*

(1) *$\{P_\omega([0,\widetilde{m}])v_g\}_{g\in(0,\infty)}$ converges to $v \in \mathcal{D}(\omega^{-1/2})$ in the graph norm of $\omega^{-1/2}$.*
(2) *$\|\omega^{-1}P_\omega(\widetilde{m},\infty)v_g\|$ diverges to ∞ as g tends to infinity.*

Then the g-dependent family of operators given by

$$\begin{aligned} W(\omega^{-1}P_\omega(\widetilde{m},\infty)v_g,1)F_\eta(v_g,\omega)W(\omega^{-1}P_\omega(\widetilde{m},\infty)v_g,1)^* + \|\omega^{-1/2}P_\omega(\widetilde{m},\infty)v_g\|^2 \\ = \eta W(2\omega^{-1}P_\omega(\widetilde{m},\infty)v_g,-1) + d\Gamma(\omega) + \varphi(P_\omega(0,\widetilde{m}])v_g) \qquad (4.1) \\ := \widetilde{F}_{\eta,\widetilde{m}}(v_g,\omega) \end{aligned}$$

is uniformly bounded below by $-|\eta| - \sup_{g\in(0,\infty)}\|P_\omega(0,\widetilde{m}])v_g\|^2$. Furthermore, $\{\widetilde{F}_{\eta,\widetilde{m}}(v_g,\omega)\}_{g\in(0,\infty)}$ converges to $d\Gamma(\omega) + \varphi(v)$ in norm resolvent sense as g tends to ∞.

The assumption in part (1) is critical. Divergence where ω is small can lead to problems. This is proven in proposition 5.8 below.

In the strongly coupled Spin-Boson model one usually has $v_g = g\widetilde{v}$ where $\widetilde{v} \in \mathcal{D}(\omega^{-1/2})$ and $g \in (0,\infty)$ is the strength of the interaction. We can now answer what happens as g goes to ∞.

Corollary 4.2. *Let $v \in \mathcal{H}$, $\eta \in \mathbb{R}$ and assume $m > 0$. Then there exists $g_0 > 0$ such that $\mathcal{E}_\eta(gv,\omega)$ is a non degenerate eigenvalue of $F_\eta(gv,\omega)$ when $g > g_0$. Furthermore, one may pick a family of normalised vectors $\{\psi_g\}_{g\in[g_0,\infty)}$ such that $g \mapsto \psi_g$ is smooth, $F_\eta(gv,\omega)\psi_g = \mathcal{E}_\eta(gv,\omega)\psi_g$ and*

$$\lim_{g\to\infty} \|\psi_g - e^{-g^2\|\omega^{-1}v\|^2}\epsilon(-g\omega^{-1}v)\| = 0,$$

$$\lim_{g\to\infty} \frac{\langle \psi_g, N\psi_g\rangle - g^2\|\omega^{-1}v\|^2}{g} = 0,$$

$$\lim_{g\to\infty} (\mathcal{E}_\eta(gv,\omega) + g^2\|\omega^{-1/2}v\|^2) = 0.$$

If $\eta < 0$ then $g \mapsto \mathcal{E}_\eta(gv,\omega) + g^2\|\omega^{-1/2}v\|$ is strictly increasing.

Corollary 4.3. *Let $v \in \mathcal{H}$ and $\eta \in \mathbb{R}$. If $m > 0$ there is $g_0 > 0$ such that $H_\eta(gv,\omega)$ has an exited state with energy $\widetilde{E}_\eta(gv,\omega)$ for $g > g_0$. Furthermore*

$$\lim_{g\to\infty}(E_\eta(gv,\omega) - \widetilde{E}_\eta(gv,\omega)) = 0.$$

Corollary 4.4. *Assume $\mathcal{H} = L^2(\mathcal{M},\mathcal{F},\mu)$ and ω is a multiplication operator on this space. Let $v : \mathcal{M} \to \mathbb{C}$ is measurable and that $\{\chi_g\}_{g\in(0,\infty)}$ is a collection of functions from $\mathbb{R}$ into $[0,1]$. Assume $g \mapsto \chi_g(x)$ is increasing and converges to 1 for all $x \in \mathbb{R}$. Assume furthermore that $k \mapsto \chi_g(\omega(k))v(k) \in \mathcal{D}(\omega^{-1/2})$ and that there is $\widetilde{m} > 0$ such that $\widetilde{v} := 1_{\{\omega\leq\widetilde{m}\}}v \in \mathcal{D}(\omega^{-1/2})$. If $k \mapsto \omega(k)^{-1}v(k)1_{\{\omega>1\}}(k) \notin \mathcal{H}$ there are unitary maps $\{V_g\}_{g\in(0,\infty)}$ and $\{U_g\}_{g\in(0,\infty)}$ independent of η such that:*

(1) *$\{V_gF_\eta(v_g,\omega)V_g^*+\|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2\}_{g\in(0,\infty)}$ converges in norm resolvent sense to the operator $d\Gamma(\omega)+\varphi(\widetilde{v})$ as g tends to infinity.*

(2) *$\{U_gH_\eta(v_g,\omega)U_g^*+\|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2\}_{g\in(0,\infty)}$ is uniformly bounded below and converges in norm resolvent sense to the operator*

$$\widetilde{H} := (d\Gamma(\omega)+\varphi(\widetilde{v}))\oplus(d\Gamma(\omega)+\varphi(\widetilde{v}))$$

as g tends to ∞. This implies

$$(H_\eta(v_g,\omega)+\|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2+i)^{-1}-(H_0(v_g,\omega)+\|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2+i)^{-1}$$

will converge to 0 in norm as g tends to ∞.

To prove a result similar to Corollary 4.2 in the massless case one needs to work a bit harder. First we shall need

Theorem 4.5. *Assume $\mathcal{H} = L^2(\mathcal{M},\mathcal{K},\nu)$ and ω is multiplication by a measurable function. Let $v \in \mathcal{D}(\omega^{-1/2})$, $g \in (0,\infty)$ and $\eta \leq 0$. Assume that $F_\eta(gv,\omega)$ has a ground state $\psi_{g,\eta} = (\psi_{g,\eta}^{(n)})$. Then*

(1) *We may choose $\psi_{g,\eta}$ such that $\psi_{g,\eta}^{(0)} > 0$ and $(-1)^n\overline{v}^{\otimes n}\psi_{g,\eta}^{(n)} > 0$ almost everywhere on $\{v \neq 0\}^n$.*

(2) *Almost everywhere the following inequality holds*

$$|\psi_{g,\eta}^{(n)}(k_1,\dots,k_n)| \leq \frac{g^n}{\sqrt{n!}}\frac{|v(k_1)|\cdots|v(k_n)|}{\omega(k_1)\cdots\omega(k_n)}.$$

In particular $\psi_{g,\eta}^{(n)}$ is zero outside $\{v \neq 0\}^n$ almost everywhere and if $v \in \mathcal{D}(\omega^{-1})$ then $\|\psi_{g,\eta}^{(n)}\|$ goes to zero like g^n for g tending to 0.

(3) *Assume $v \in \mathcal{D}(\omega^{-1})$, $f : \mathbb{N}_0 \to [0,\infty)$ is a function and assume $F_\eta(gv,\omega)$ has a ground state for all $\eta \leq 0$. Then $H_a(gv,\omega)$ has a ground state $\phi_{g,a}$ for all $a \in \mathbb{R}$ and we have*

$$\alpha_{g,f,v,\omega} := \sum_{n=0}^{\infty}\frac{f(n)^2g^{2n}\|\omega^{-1}v\|^{2n}}{n!} < \infty \iff \psi_{g,\eta} \in \mathcal{D}(f(N)) \ \forall \eta \leq 0$$

$$\iff \phi_{g,a} \in \mathcal{D}(1\otimes f(N)) \ \forall a \in \mathbb{R}$$

In particular $\psi_{g,\eta} \in \mathcal{D}(\sqrt[p]{N!})$ and $\phi_{g,\eta} \in \mathcal{D}(1\otimes\sqrt[p]{N!})$ for all $p > 2$.

This extends the result which was proven using path measures in [13]. Similar point wise estimates can also be found in [7]. In the last two results we will assume $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$ where λ_ν is the Lebesgue measure. Furthermore we assume ω is a multiplication operator.

Theorem 4.6. *Let $v \in \mathcal{D}(\omega^{-1})$ and $\eta \leq 0$. Then there is a family $\{\psi_g\}_{g\in\mathbb{R}}$ of normalised ground states for $F_\eta(gv, \omega)$ and*

$$\lim_{g\to\infty} (\mathcal{E}_\eta(gv, \omega) + g^2 \|\omega^{-1/2} v\|^2) = 0.$$
$$\lim_{g\to\infty} \frac{\langle \psi_g, N\psi_g\rangle - g^2\|\omega^{-1}v\|^2}{g^2} = 0.$$

The following is a simple criterion for the existence of an exited state in the massive Spin-Boson model.

Theorem 4.7. *Assume $m > 0$ and*

$$\int_{\mathbb{R}^\nu} \frac{|v(k)|^2}{\omega(k) - m} dk = \infty. \tag{4.2}$$

Then both $F_\eta(v, \omega)$ and $F_{-\eta}(v, \omega)$ have a ground state and $H_\eta(gv, \omega)$ will have an excited state. The condition is satisfied if $\omega \in C^2(\mathbb{R}^\nu, \mathbb{R})$, $\nu \leq 2$ and there is $x_0 \in \mathbb{R}^\nu$ such that $\omega(x_0) = m$ and $|v|$ is bounded from below by a positive number on a ball around x_0. This holds for the physical model with $\nu \leq 2$.

5. Proof of the main technical result

In this section we shall investigate operators of the form

$$\widetilde{F}_\eta(v, \omega) := \Gamma(\omega) + \eta W(v, -1)$$

indexed by $\eta \in \mathbb{R}$, $v \in \mathcal{H}$ and ω selfadjoint and non negative on $\mathcal{H}$.

Proposition 5.1. *Assume $\eta \in \mathbb{R}$, $v \in \mathcal{H}$ and ω is selfadjoint, non negative and injective on $\mathcal{H}$. Then $\widetilde{F}_\eta(v, \omega)$ is selfadjoint on $\mathcal{D}(d\Gamma(\omega))$. Furthermore $\widetilde{F}_\eta(v, \omega)$ is bounded from below by $-|\eta|$ and $\widetilde{F}_\eta(v, \omega)$ has compact resolvents if ω has compact resolvents.*

Proof. Using equation (2.2) we see

$$W(v, -1)W(v, -1) = e^{-i\mathrm{Im}(\langle v, -v\rangle)} W(v - v, (-1)^2) = W(0, 1) = 1$$

so $W(v, -1) = W(v, -1)^{-1} = W(v, -1)^*$ since $W(v, -1)$ is unitary. Hence $\widetilde{F}_\eta(v, \omega) = d\Gamma(\omega) + \eta W(v, -1)$ is selfadjoint on $\mathcal{D}(d\Gamma(\omega))$. Furthermore the lower bound follows from $d\Gamma(\omega) \geq 0$ by Lemma 2.1 and $-1 \leq W(v, -1) \leq 1$. If ω has compact resolvents, then so does $d\Gamma(\omega)$ by Lemma 2.1 and hence

$$(F_\eta(v, \omega) + i)^{-1} = (d\Gamma(\omega) + i)^{-1} + \eta(d\Gamma(\omega) + i)^{-1} W(v, -1)(F_\eta(v, \omega) + i)^{-1}$$

will be compact. □

Lemma 5.2. *Assume that* $\{v_g\}_{g\in(0,\infty)}$ *is a collection of elements in* $\mathcal{H}$ *such that* $\|v_g\|$ *diverges to* ∞*. Then* $W(v_g,-1)$ *converges weakly to 0 as* g *goes to* ∞*.*

Proof. By [26, Theorem 4.26] it is enough to check a dense subset. By linearity it is enough to check a set that spans a dense set. Hence it is enough to check exponential vectors $\epsilon(g)$ for any $g\in\mathcal{H}$. We calculate

$$\begin{aligned}\langle\epsilon(g_1),W(v_g,-1)\epsilon(g_2)\rangle &= e^{-\|v_g\|^2/2+\langle v_g,g_2\rangle}\langle\epsilon(g_1),\epsilon(v_g-g_2)\rangle\\ &= e^{-\|v_g\|^2/2+\langle v_g,g_2\rangle+\langle g_1,v_g\rangle-\langle g_1,g_2\rangle},\end{aligned}$$

which converges to 0. □

The following Lemma contains all the technical constructions we need. The techniques goes back to Glimm and Jaffe (see [9]) but has also been used in [3].

Lemma 5.3. *Assume* ω *is selfadjoint, non negative and injective on* $\mathcal{H}$*. Let* P_ω *be the spectral measure of* ω *and let* $\widetilde{m}>0$*. Define the measurable function* $f_k:\mathbb{R}\to\mathbb{R}$

$$f_k(x)=x1_{(0,\widetilde{m}]}(x)+\sum_{n=0}^{\infty}(n+1)2^{-k}1_{(n2^{-k},(n+1)2^{-k}]\cap(\widetilde{m},\infty)}(x).$$

along with $\omega_k=\int_{\mathbb{R}}f_k(\lambda)dP_\omega(\lambda)$*. Then the following holds*

1. $\widetilde{F}_\eta(v,\omega_k)$ *converges to* $\widetilde{F}_\eta(v,\omega)$ *in norm resolvent sense uniformly in* v*.*
2. *Let* $\{v_g\}_{g\in(0,\infty)}$ *be a collection of elements in* $P_\omega((\widetilde{m},\infty))\mathcal{H}$*. For each* $k\in\mathbb{N}$*, there are Hilbert spaces* $\mathcal{H}_{1,k},\mathcal{H}_{2,k}$*, selfadjoint operators* $\omega_{1,k},\omega_{2,k}\geq 0$*, a collection of elements* $\{\widetilde{v}_{g,k}\}_{g\in(0,\infty)}\subset\mathcal{H}_{1,k}$ *and a collection of unitary maps* $\{\mathcal{U}_{g,k}\}_{g\in(0,\infty)}$ *such that*

$$\mathcal{U}_{g,k}:\mathcal{F}_b(\mathcal{H})\to\mathcal{F}_b(\mathcal{H}_{1,k})\oplus\left(\bigoplus_{n=1}^{\infty}\mathcal{F}_b(\mathcal{H}_{1,k})\otimes S_n((\mathcal{H}_{2,k})^{\otimes n})\right),$$

$\omega_{1,k}\geq 2^{-k}$ *has compact resolvents,* $\|v_g\|=\|\widetilde{v}_{k,g}\|$ *for all* $g>0$ *and*

$$\begin{aligned}\mathcal{U}_{g,k}\widetilde{F}_\eta(v_g,\omega_k)\mathcal{U}_{g,k}^* =& \widetilde{F}_\eta(\widetilde{v}_{g,k},\omega_{1,k})\\ &\oplus\bigoplus_{n=1}^{\infty}\left(\widetilde{F}_{(-1)^n\eta}(\widetilde{v}_{g,k},\omega_{1,k})\otimes 1+1\otimes d\Gamma^{(n)}(\omega_{2,k})\right)\end{aligned}$$

for all $\eta\in\mathbb{R}$*.*

Proof. (1): We may pick a σ-finite measure space $(\mathcal{M},\mathcal{F},\mu)$ and a unitary map $U:\mathcal{H}\to L^2(\mathcal{M},\mathcal{F},\mu)$ such that $\widetilde{\omega}=U\omega U^*$ is multiplication by a strictly positive and measurable map. Conjugation with the unitary map $\Gamma(U)$, Lemma A.1 and $U\omega_kU^*=f_k(U\omega_kU^*)$ gives us

$$\begin{aligned}&\|(\widetilde{F}_\eta(v,\omega_k)-\xi)^{-1}-(\widetilde{F}_\eta(v,\omega)-\xi)^{-1}\|\\ &\quad=\|(\widetilde{F}_\eta(Uv,f_k(U\omega U^*))-\xi)^{-1}-(\widetilde{F}_\eta(Uv,U\omega U^*)-\xi)^{-1}\|\end{aligned}$$

for all $\xi \in \mathbb{R}\backslash\mathbb{C}$. Hence we may assume ω is multiplication by a strictly positive map, which we shall also denote ω. Using standard theory for the spectral calculus (see [23]) we find ω_k is multiplication by $\omega_k(x) := f_k(\omega(x))$. Write $\omega = \omega_\infty$ and note that $\omega_k > 0$ for all $k \in \mathbb{N} \cup \{\infty\}$ by construction. Furthermore,

$$\sup_{x\in\mathcal{M}} \left| \frac{\omega_k(x) - \omega(x)}{\omega(x)} \right| \leq \frac{2^{-k}}{\widetilde{m}}. \tag{5.1}$$

Now $d\Gamma^{(n)}(\omega_k)$ acts on $L^2_{\mathrm{sym}}(\mathcal{M}^n, \mathcal{F}^{\otimes n}, \mu^{\otimes n})$ like multiplication with the map

$$\omega_k^{(n)}(x_1, \dots, x_n) = \omega_k(x_1) + \cdots + \omega_k(x_n)$$

for all $k \in \mathbb{N}\cup\{\infty\}$. Equation (5.1) gives that $|\omega^{(n)}(x) - \omega_k^{(n)}(x)| \leq 2^{-k}\widetilde{m}^{-1}\omega^{(n)}(x)$ for all $x \in \mathcal{M}^n$ so $\mathcal{D}(d\Gamma^{(n)}(\omega)) \subset \mathcal{D}(d\Gamma^{(n)}(\omega_k))$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Furthermore we find for $\psi \in \mathcal{D}(d\Gamma^{(n)}(\omega))$ that

$$\|(d\Gamma^{(n)}(\omega) - d\Gamma^{(n)}(\omega_k))\psi\| \leq \widetilde{m}^{-1}2^{-k}\|d\Gamma^{(n)}(\omega)\psi\|.$$

Hence for all $\psi \in \mathcal{D}(d\Gamma(\omega))$ we have $\psi \in \mathcal{D}(d\Gamma(\omega_k))$ and

$$\|(\widetilde{F}_\eta(v,\omega) - \widetilde{F}_\eta(v,\omega_k))\psi\| = \|(d\Gamma(\omega) - d\Gamma(\omega_k))\psi\| \leq \frac{2^{-k}}{\widetilde{m}}\|d\Gamma(\omega)\psi\|.$$

Let $\varepsilon > 0$, $\xi \in \mathbb{C}\backslash\mathbb{R}$. We now estimate

$$\begin{aligned}
&\|((\widetilde{F}_\eta(v,\omega) + \xi)^{-1} - (\widetilde{F}_\eta(v,\omega_k) + \xi)^{-1})\psi\| \\
&\qquad \leq \frac{1}{|\mathrm{Im}(\xi)|}\|(d\Gamma(\omega) - d\Gamma(\omega_k))(\widetilde{F}_\eta(v,\omega) + \xi)^{-1}\psi\| \\
&\qquad \leq \frac{1}{|\mathrm{Im}(\xi)|}\frac{2^{-k}}{\widetilde{m}}\|d\Gamma(\omega)(\widetilde{F}_\eta(v,\omega) + \xi)^{-1}\widetilde{\psi}\| \\
&\qquad \leq \frac{1}{|\mathrm{Im}(\xi)|}\frac{2^{-k}}{\widetilde{m}}\left(1 + \frac{1}{|\mathrm{Im}(\xi)|} + \frac{|\xi|}{|\mathrm{Im}(\xi)|}\right)\|\psi\|
\end{aligned}$$

which shows norm resolvent convergence uniformly in v.

(2): For each $k \in \mathbb{N}$ we define

$$C_k = \left\{ c \in \mathbb{N}_0 \,\middle|\, P_{c,k} := P_\omega((\widetilde{m},\infty) \cap (c2^{-k}, (c+1)2^{-k}]) \neq 0 \right\}$$

For each $c \in C_k$ let $\mathcal{K}_{c,k}$ be a Hilbert space with dimension $\dim(P_{c,k}) - 1$. In case this number is infinity we pick a Hilbert space we countably infinite dimension. Define $\mathcal{K} = P_\omega([0,\widetilde{m}])\mathcal{H}$ and note that $\mathcal{K}$ reduces ω. Define the spaces

$$\mathcal{H}_{1,k} = L^2(C_k, \mathcal{B}(C_k), \tau_{C_k}) = \ell^2(C_k) \quad \text{and} \quad \mathcal{H}_{2,k} = \mathcal{K} \oplus \bigoplus_{c\in C_k} \mathcal{K}_{c,k}$$

where τ_{C_k} is the counting measure on C_k. We now define $\omega_{1,k}$ to be multiplication by the map $f_k(c) = (c+1)2^{-k}$ in $\mathcal{H}_{1,k}$ and

$$\omega_{2,k} = \omega\mid_{\mathcal{K}} \oplus \bigoplus_{c\in C_k} (c+1)2^{-k}.$$

Note $\omega_{1,k} \geq 2^{-k}$ and $\omega_{2,k} \geq 0$ since $C_k \subset \mathbb{N}_0$. Write $C_k = \{n_{i,k}\}_{i=1}^K$ where $K \in \mathbb{N} \cup \{\infty\}$ and $n_{i,k} < n_{i+1,k}$. Then $\{1_{\{n_{i,k}\}}\}_{i=1}^K$ is an orthonormal basis of eigenvectors for $\omega_{1,k}$ corresponding to the eigenvalues $\{(n_{i,k}+1)2^{-k}\}_{i=1}^K$. This collection of eigenvalues is either finite or diverges to infinity so $\omega_{1,k}$ will have compact resolvents. For each $g \in (0,\infty)$ and $c \in C_k$ we define the vector

$$\psi_{c,g,k} = \begin{cases} \frac{P_{c,k}v_g}{\|P_{c,k}v_g\|}, & P_{c,k}v_g \neq 0 \\ \text{Some normalized element in } P_{c,k}\mathcal{H} & \text{otherwise} \end{cases}$$

and note $\{\psi_{c,g,k} \mid c \in C_k\}$ is an orthonormal collection of states. We also define

$$\widetilde{\mathcal{H}}_{c,g,k} = \left\{\psi \in P_{c,k}\mathcal{H} \,\middle|\, \psi \perp \psi_{c,g,k}\right\}$$

and note $\{\widetilde{\mathcal{H}}_{c,g,k} \mid c \in C_k\}$ consists of orthogonal subspaces. We then define

$$\mathcal{H}_{1,g,k} = \overline{\operatorname{Span}\{\psi_{c,g,k} \mid c \in C_k\}} \quad \text{and} \quad \mathcal{H}_{2,g,k} = \bigoplus_{c \in C_k} \widetilde{\mathcal{H}}_{c,g,k}.$$

Now $\omega \geq 0$ is injective and so

$$I = P_\omega((0,\widetilde{m}]) + \sum_{c=0}^{\infty} P_{c,k} = P_\omega([0,\widetilde{m}]) + \sum_{c \in C_k} P_{c,k},$$

which implies $\mathcal{H} = \mathcal{H}_{1,g,k} \oplus \mathcal{K} \oplus \mathcal{H}_{2,g,k}$. Note that $v_g \in \mathcal{H}_{1,g,k}$ by construction. Let $\mathcal{B}_{c,g,k}$ be an orthonormal basis for $\widetilde{\mathcal{H}}_{c,g,k}$ and let $\mathcal{B}_{g,k} = \cup_{c \in C_k} \mathcal{B}_{c,g,k}$ which is an orthonormal basis for $\mathcal{H}_{2,g,k}$. Let $B \subset \mathcal{K}$ be an orthonormal basis for $\mathcal{K}$ and define $B_{g,k} = \{\psi_{c,g,k} \mid c \in C_k\}$ which is an orthonormal basis for $\mathcal{H}_{1,g,k}$. Define $D = \mathcal{B}_{g,k} \cup B_{g,k} \cup B$ which is an orthonormal basis for $\mathcal{H}$.

Let $V_{c,g,k}$ be a unitary from $\widetilde{\mathcal{H}}_{c,g,k}$ to $\mathcal{K}_{c,k}$ which exists since the spaces have the same dimension. Define $Q_{g,k} : \mathcal{H}_{1,g,k} \to \mathcal{H}_{1,k}$ to be the unique unitary map which satisfies $Q_{g,k}\psi_{c,g,k} = 1_{\{c\}}$. Then we define

$$U_{g,k} = Q_{g,k} \oplus 1 \oplus \bigoplus_{c \in C_k} V_{c,g,k} : \mathcal{H} \to \mathcal{H}_{1,k} \oplus \mathcal{H}_{2,k}.$$

We now prove that

$$U_{g,k}^* \omega_{1,k} \oplus \omega_{2,k} U_{g,k} = \omega_k. \tag{5.2}$$

Let $\psi \in \mathcal{B}_{c,g,k} \cup \{\psi_{c,g,k}\}$ for some $c \in C_k$. Using the functional calculus we find $\psi = P_{c,k}\psi \in \mathcal{D}(\omega_k)$ and

$$\omega_k \psi = \omega_k P_{c,k} \psi = (c+1)2^{-k} P\psi_{c,g,k} = (c+1)2^{-k}\psi. \tag{5.3}$$

Furthermore for $\psi \in B \subset \mathcal{K}$ we find that $\psi \in \mathcal{D}(\omega_k^p)$ for all $p \in \mathbb{N}$ and we have the inequality $\|\omega_k^p \psi\| \leq \widetilde{m}^p \|\psi\|$. In particular D is an orthonormal basis for $\mathcal{H}$ consisting of analytic vectors for ω_k so D spans a core for ω_k. Hence it is enough to prove equation (5.2) on D.

Let $\psi \in \mathcal{B}_{g,k} \cup B_{g,k}$ and pick $c \in C_k$ such that $\psi \in \mathcal{B}_{c,g,k} \cup \{\psi_{c,g,k}\}$. If $\psi = \psi_{c,g,k}$ then $U_{g,k}\psi = (1_{\{c\}}, 0)$. Now $1_{\{c\}} \in \mathcal{D}(\omega_{1,k})$ with $\omega_{1,k}1_{\{c\}} = (c+1)2^{-k}1_{\{c\}}$ so $U_{g,k}\psi = (1_{\{c\}}, 0) \in \mathcal{D}(\omega_{1,k} \oplus \omega_{2,k})$ and

$$U_{g,k}^* \omega_{1,k} \oplus \omega_{2,k} U_{g,k}\psi = (c+1)2^{-k} U_{g,k}^*(1_{\{c\}}, 0) = (c+1)2^{-k}\psi = \omega_k \psi$$

by equation (5.3). If $\psi \in \mathcal{B}_{c,g,k}$ then $U_{g,k}\psi = (0, V_{c,g,k}\psi)$. By definition we have $V_{c,g,k}\psi \in \mathcal{K}_{c,k} \subset \mathcal{D}(\omega_2)$ with $\omega_{2,k}V_{c,g,k}\psi = (c+1)2^{-k}V_{c,g,k}\psi$. Hence $U_{g,k}\psi = (0, V_{c,g,k}\psi) \in \mathcal{D}(\omega_{1,k} \oplus \omega_{2,k})$ and

$$U_{g,k}^* \omega_{1,k} \oplus \omega_{2,k} U_{g,k}\psi = (c+1)2^{-k} U_{g,k}^*(1_{\{c\}}, 0) = (c+1)2^{-k}\psi = \omega_k \psi$$

by equation (5.3). If $\psi \in B \subset \mathcal{K}$ we have $\psi \in \mathcal{D}(\omega_k) \cap \mathcal{D}(\omega)$ and $\omega_k\psi = \omega\psi \in \mathcal{K}$. In particular $U_{g,k}\psi = (0, \psi) \in \mathcal{D}(\omega_{1,k} \oplus \omega_{2,k})$ and $U_{g,k}\omega_k\psi = (0, \omega_k\psi) = (0, \omega\psi)$. Thus we find

$$U_{g,k}^* \omega_{1,k} \oplus \omega_{2,k} U_{g,k}\psi = U_{g,k}^*(0, \omega\psi) = \omega_k\psi.$$

This proves equation (5.2). As earlier noted $v_g \in \mathcal{H}_{1,g,k}$ so $U_{g,k}v_g$ is of the form $(\widetilde{v}_{g,k}, 0)$ with $\|\widetilde{v}_{g,k}\| = \|v_g\|$. Using Lemma A.1 we find

$$\Gamma(U_{g,k})\widetilde{F}_\eta(v_g, \omega_k)\Gamma(U_{g,k})^* = \widetilde{F}_\eta((\widetilde{v}_{g,k}, 0), \omega_{k,1} \oplus \omega_{k,2}).$$

Letting L_1 be the isomorphism from Lemma A.3 we see that

$$\begin{aligned} L_1\widetilde{F}_\eta((\widetilde{v}_{g,k}, 0), \omega_{k,1} \oplus \omega_{k,2})L_1^* =& d\Gamma(\omega_1) \otimes 1 + 1 \otimes d\Gamma(\omega_{k_2}) \\ &+ \eta W(\widetilde{v}_{g,k}, -1) \otimes \Gamma(-1). \end{aligned}$$

Letting L_2 be the isomorphism from Lemma A.4 we see that

$$\begin{aligned} &L_2L_1\widetilde{F}_\eta((\widetilde{v}_{g,k}, 0), \omega_{k,1} \oplus \omega_{k,2})L_1^*L_2^* \\ &\quad= d\Gamma(\omega_{k,1}) + d\Gamma^{(0)}(\omega_{k,2}) + \Gamma^{(0)}(-1) \mid_{\mathbb{C}} \eta W(\widetilde{v}_{g,k}, -1) \oplus \\ &\quad\quad \bigoplus_{n=1}^{\infty} d\Gamma(\omega_{k,1}) \otimes 1 + 1 \otimes d\Gamma^{(n)}(\omega_{k,2}) + \eta W(\widetilde{v}_{g,k}, -1) \otimes \Gamma^{(n)}(-1) \\ &\quad= \widetilde{F}_\eta(\widetilde{v}_{g,k}, \omega_{k,1}) \oplus \bigoplus_{n=1}^{\infty} \widetilde{F}_{(-1)^n\eta}(\widetilde{v}_{g,k}, \omega_{k,1}) \otimes 1 + 1 \otimes d\Gamma^{(n)}(\omega_{k,2}) \end{aligned}$$

where we used $\Gamma^{(n)}(-1) = (-1)^n$. Hence $\mathcal{U}_{g,k} = L_2L_1\Gamma(U_{g,k})$ will work. □

Lemma 5.4. *Assume that ω is selfadjoint, injective and non negative operator on $\mathcal{H}$ which have compact resolvents. Let $\{v_g\}_{g\in(0,\infty)}$ be a collection of elements in $\mathcal{H}$ such that $\|v_g\|$ diverges to ∞. Then $\widetilde{F}_\eta(v_g, \omega)$ converges in norm resolvent sense to $\widetilde{F}_0(0, \omega) = d\Gamma(\omega)$ as g goes to ∞ for all $\eta \in \mathbb{R}$.*

Proof. We calculate

$$\begin{aligned} &(\widetilde{F}_\eta(v_g, \omega) - i)^{-1} - (d\Gamma(\omega) - i)^{-1} \\ &\quad= \eta(\widetilde{F}_\eta(v_g, \omega) - i)^{-1}W(v_g, -1)(d\Gamma(\omega) - i)^{-1} \\ &\quad= \eta(\widetilde{F}_\eta(v_g, \omega) - i)^{-1}W(v_g, -1)(d\Gamma(\omega) - i)^{-1}W(v_g, -1)(d\Gamma(\omega) - i)^{-1} \\ &\quad+ \eta^2(d\Gamma(\omega) - i)^{-1}W(v_g, -1)(d\Gamma(\omega) - i)^{-1}. \end{aligned}$$

This implies

$$\begin{aligned}&\|(\widetilde{F}_\eta(v_g,\omega)-i)^{-1}-(d\Gamma(\omega)-i)^{-1}\|\\&\qquad\leq(|\eta|+1)|\eta|\|(d\Gamma(\omega)-i)^{-1}W(v_g,-1)(d\Gamma(\omega)-i)^{-1}\|,\end{aligned}$$

which converges to 0 by Lemma 5.2 and compactness of $(d\Gamma(\omega)-i)^{-1}$. □

Lemma 5.5. *Let $\mathcal{H}$ be a Hilbert space. Let $\{A_n\}_{n=1}^\infty$ be a sequence of selfadjoint operators on $\mathcal{H}$ that are uniformly bounded below by γ. Let A be selfadjoint on $\mathcal{H}$ and bounded below. Then A_n converges to A in norm resolvent sense if and only if e^{-tA_n} converges to e^{-tA} in norm for all $t<0$. In this case $\inf(\sigma(A_n))$ converges to $\inf(\sigma(A))$.*

Proof. Norm resolvent convergence along with existence the uniform lower bound implies convergence of the semigroup (see [20, Theorem VIII.20]). To prove the converse we apply the formula

$$(A-\lambda)^{-1}\psi=\int_0^\infty e^{-t(A-\lambda)}\psi dt$$

for all $\lambda<\gamma$ along with dominated convergence. To prove the last part note that by the spectral theorem

$$\inf(\sigma(A))=-\log(\|\exp(-A)\|)=\lim_{n\to\infty}-\log(\|\exp(-A_n)\|)=\lim_{n\to\infty}\inf(\sigma(A_n))$$

finishing the proof. □

Lemma 5.6. *Assume that ω is a selfadjoint, injective and non negative operator on $\mathcal{H}$. Let $\{v_g\}_{g\in(0,\infty)}$ be a collection of elements in $\mathcal{H}$ such that $\|v_g\|$ diverges to ∞. Assume there is $\widetilde{m}>0$ such that $P_\omega((0,\widetilde{m}])v_g=0$ for all g where P_ω is the spectral measure corresponding to ω. Then $\widetilde{F}_\eta(v_g,\omega)$ converges in norm resolvent sense to $\widetilde{F}_0(0,\omega)=d\Gamma(\omega)$ as g tends to ∞ for all $\eta\in\mathbb{R}$.*

Proof. For each $k\in\mathbb{N}$ let $\mathcal{H}_{1,k},\mathcal{H}_{2,k},\omega_k,\omega_{1,k},\omega_{2,k}$ and $\widetilde{v}_{g,k}$ be the quantities from Lemma 5.3 corresponding to the family $\{v_g\}_{g\in(0,\infty)}$ and the number $\widetilde{m}>0$. For each $n\in\mathbb{N}_0$ we define

$$F_{\pm\eta,k,g,n}=\widetilde{F}_{\pm\eta}(\widetilde{v}_{g,k},\omega_{k,1})\otimes 1+1\otimes d\Gamma^{(n)}(\omega_{k,2}).$$

By Lemma 5.3 statement (1), it is enough to prove that $\widetilde{F}_\eta(v_g,\omega_k)$ converges to $d\Gamma(\omega_k)$ in norm resolvent sense as g tends to ∞. Noting that $\widetilde{F}_\eta(v_g,\omega_k)\geq-|\eta|$ for all g we may use Lemma 5.5. Using the unitary transformations in Lemma 5.3 we see

$$\begin{aligned}&\|\exp(-t\widetilde{F}_\eta(v_g,\omega_k))-\exp(-td\Gamma(\omega_k))\|\\&=\sup_{n\in\mathbb{N}_0}\{\|\exp(-tF_{(-1)^n\eta,k,g,n})-\exp(-tF_{0,k,g,n})\|\}\\&=\sup_{n\in\mathbb{N}_0}\{\|\exp(-t\widetilde{F}_{(-1)^n\eta}(\widetilde{v}_{g,k},\omega_{k,1}))-\exp(-td\Gamma(\omega_{1,k}))\|\|\exp(-td\Gamma^{(n)}(\omega_{2,k}))\|\}\\&\leq\sup_{n\in\mathbb{N}_0}\{\|\exp(-t\widetilde{F}_{(-1)^n\eta}(\widetilde{v}_{g,k},\omega_{k,1}))-\exp(-td\Gamma(\omega_{1,k}))\|\}\\&=\max_{n\in\{1,2\}}\{\|\exp(-t\widetilde{F}_{(-1)^n\eta}(\widetilde{v}_{g,k},\omega_{k,1}))-\exp(-td\Gamma(\omega_{1,k}))\|\}\end{aligned}$$

which converges to 0 by Lemma 5.4. This finishes the proof. □

Lemma 5.7. *Assume that ω is selfadjoint, injective and non negative operator on $\mathcal{H}$. Let $v \in \mathcal{D}(\omega^{-1/2})$ and $f \in \mathcal{D}(\omega)$. Then*

$$W(f,1)F_\eta(v,\omega)W(f,1)^* = \eta W(2f,-1) + d\Gamma(\omega) + \varphi(v-\omega f) + \|\omega^{1/2}f\|^2 - 2\mathrm{Re}(\langle v,f\rangle).$$

Proof. Use equation (2.2), (2.3) and Lemma A.2. □

We can now prove Theorem 4.1.

Proof (of Theorem 4.1). The formula in equation (4.1) is obtained via Lemma 5.7 and the lower bound is trivial from Lemma 2.2. For $c \in (0,\infty)$ we will write $P_c = P_\omega((c,\infty))$ and $\overline{P}_c = 1 - P_c = P_\omega((0,c])$. Note that $P_c v_g, P_c v, \overline{P}_c v_g, \overline{P}_c v \in \mathcal{D}(\omega^{-1/2})$ holds trivially by the spectral theorem. Define for $0 < c \le \widetilde{m}$ and $\eta \in \mathbb{R}$

$$\widetilde{F}_{\eta,c,g} := \eta W(2\omega^{-1}P_c v_g, -1) + d\Gamma(\omega) + \varphi(\overline{P}_c v_g) + \|\omega^{-1/2}P_\omega((c,\widetilde{m}])v_g\|^2$$
$$A_{\eta,c,g} := \eta W(2\omega^{-1}P_c v_g, -1) + d\Gamma(\omega) + \|\omega^{-1/2}P_\omega((c,\widetilde{m}])v_g\|^2$$

and note they are all selfadjoint on $\mathcal{D}(d\Gamma(\omega))$ by the Kato-Rellich theorem and Lemma 2.2. For $\psi \in \mathcal{D}(d\Gamma(\omega))$ we have $\|(1+d\Gamma(\omega))^{1/2}\psi\| \le \|(1+d\Gamma(\omega))\psi\|$ by the spectral theorem. Using this and Lemma 2.2 we find for all $c \in (0,\widetilde{m}]$

$$\begin{aligned}
\|(\widetilde{F}_{\eta,c,g}+i)^{-1} - (\widetilde{A}_{\eta,c,g}+i)^{-1}\| \\
&\le \|\varphi(\overline{P}_c v_g)(\widetilde{A}_{\eta,c,g}+i)^{-1}\| \\
&\le 2\|\overline{P}_c(1+\omega^{-1/2})v_g\|\|(1+d\Gamma(\omega))(\widetilde{A}_{\eta,c,g}+i)^{-1}\| \\
&\le 2\|\overline{P}_c(1+\omega^{-1/2})v_g\|(1+1+1+|\eta|+\|\omega^{-1/2}P_\omega((0,\widetilde{m}])v_g\|^2).
\end{aligned}$$

where we in the last step used $\|\omega^{-1/2}P_\omega((c,\widetilde{m}])v_g\|^2 \le \|\omega^{-1/2}P_\omega((0,\widetilde{m}])v_g\|^2$. We now define

$$C_1 := 3 + |\eta| + \sup_{g\in(0,\infty)} \|\omega^{-1/2}P_\omega((0,\widetilde{m}])v_g\|^2$$

which is finite since $\omega^{-1/2}P_\omega((0,\widetilde{m}])v_g$ is convergent. Let $B = d\Gamma(\omega) + \varphi(v)$ and $C_2 = \|(1+d\Gamma(\omega))(B+i)^{-1}\|$. We estimate using Lemma 2.2

$$\begin{aligned}
\|(\widetilde{F}_{0,\widetilde{m},g}+i)^{-1} - (B+i)^{-1}\| &\le \|\varphi(\overline{P}_{\widetilde{m}}v_g - v)(B+i)^{-1}\| \\
&\le 2\|(1+\omega^{-1/2})(\overline{P}_{\widetilde{m}}v_g - v)\|C_2
\end{aligned}$$

Let $U_c = W(P_\omega((c_1,\widetilde{m}])v, 1)$ for some $c \in (0,\widetilde{m}]$. Using equation (2.2) and Lemma A.2 we obtain $U_c\widetilde{F}_{\eta,\widetilde{m},g}U_c^* = \widetilde{F}_{\eta,c,g}$ for all $\eta \in \mathbb{R}$. Using this transformation and the previous estimates we find for all $c \in (0,\widetilde{m}]$ and $g > 0$ that

$$\begin{aligned}
\|(\widetilde{F}_{\eta,\widetilde{m},g}+i)^{-1} - (B+i)^{-1}\| &\le \|(\widetilde{F}_{\eta,c,g}+i)^{-1} - (\widetilde{F}_{0,c,g}+i)^{-1}\| \\
&\quad + \|(\widetilde{F}_{0,\widetilde{m},g}+i)^{-1} - (B+i)^{-1}\| \\
&\le 2(2C_1\|\overline{P}_c(1+\omega^{-1/2})v_g\|) \\
&\quad + \|(\widetilde{A}_{\eta,c,g}+i)^{-1} - (\widetilde{A}_{0,c,g}+i)^{-1}\| \\
&\quad + 2C_2\|(1+\omega^{-1/2})(\overline{P}_{\widetilde{m}}v_g - v)\|
\end{aligned}$$

Noting that

$$\|\overline{P}_c(1+\omega^{-1/2})v_g\| \leq \|(1+\omega^{-1/2})(\overline{P}_{\widetilde{m}}v_g - v)\| + \|\overline{P}_c(1+\omega^{-1/2})v\|$$

we see that

$$\limsup_{g\to\infty} \|(\widetilde{F}_{\eta,\widetilde{m},g}+i)^{-1} - (B+i)^{-1}\| \leq 4C_1\|\overline{P}_c(1+\omega^{-1/2})v\|$$

for all $c \in (0, \widetilde{m}]$ by Lemma 5.6. Taking c to 0 finishes the proof since $\widetilde{F}_{\eta,\widetilde{m},g} = \widetilde{F}_{\eta,\widetilde{m}}(v_g, \omega)$. □

Proposition 5.8. *Let* $\mathcal{H} = L(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3), \lambda^{\otimes 3}), \omega(k) = |k|$, $v_g = \omega^{-1/2}1_{\{g^{-1}\leq|k|\leq 2\}}$ *and* $\eta < 0$. *Then* $\|\omega^{-1}v_g\|$ *converges to* ∞ *for* g *going to* ∞, *but there is* $h \neq 0$ *such that* $\widetilde{F}_\eta(hv_g,\omega) = d\Gamma(\omega) + \eta W(h\omega^{-1}v_g, -1)$ *does not go to* $d\Gamma(\omega)$ *in norm resolvent sense.*

Proof. Define $v = \omega^{-1/2}1_{\{|k|\leq 2\}}$. It is easy to see that $\|\omega^{-1}v_g\|$ goes to ∞ as g tends to infinity. Assume that convergence in norm resolvent sense holds for all $h \neq 0$. Applying Lemma 5.7 with $f = h\omega^{-1}v_g$ we see

$$\inf\{\sigma(F_\eta(hv_g,\omega))\} + h^2\|\omega^{-1/2}v_g\|^2 = \inf\{\sigma(\widetilde{F}_\eta(hv_g,\omega))\}$$

converges to 0 for g going to ∞. In [3] it is proven that $F_\eta(hv_g,\omega)$ converges in norm resolvent sense to $F_\eta(hv,\omega)$, and the bottom of the spectrum also converges. This can also be done directly as an easy exercise left to the reader. Taking g to infinity yields

$$\inf\{\sigma(F_\eta(hv,\omega))\} = -h^2\|\omega^{-1/2}v\|^2$$

for any $h \neq 0$. Since $\langle \Omega, F_\eta(hv,\omega)\Omega\rangle = \eta$ we see $-h^2\|\omega^{-1/2}v\|^2 \leq \eta < 0$ for all $h \neq 0$. Taking h to 0 yields $0 \leq \eta < 0$. □

We now prove Corollaries 4.2, 4.3 and 4.4.

Proof (of Corollary 4.2). Define $U_g = W(g\omega^{-1}v, 1)$ and $m := m(\omega) > 0$. Note that

$$\widetilde{F}_\eta(2g\omega^{-1}v,\omega) := U_gF_\eta(gv,\omega)U_g^* + g^2\|\omega^{-1/2}v\|^2$$

converges in norm resolvent sense to $d\Gamma(\omega)$ as g tends to infinity by Theorem 4.1 (use $\widetilde{m} = m(\omega)$ and $v = 0$). Since $\widetilde{F}_\eta(2g\omega^{-1}v,\omega) \geq -|\eta|$ for all $g > 0$, Lemma 5.5 implies

$$\lim_{g\to\infty} \mathcal{E}_\eta(gv,\omega) + g^2\|\omega^{-1/2}v\|^2 = \lim_{g\to\infty} \inf(\sigma(\widetilde{F}_\eta(2g\omega^{-1}v,\omega))) = 0.$$

Let P_g be the spectral projection of $\widetilde{F}_\eta(gv,\omega)$ onto $[-\frac{m}{2}, \frac{m}{2}]$. Using [20, Theorem VIII.23] and Lemma 2.1 we find P_g converges in norm to $P = |\Omega\rangle\langle\Omega|$. Pick g_0 such that $\mathcal{E}_\eta(gv,\omega) + g^2\|\omega^{-1/2}v\|^2 \in (-\frac{m}{2}, \frac{m}{2})$ and $\|P_g - P\| < 1$ for all $g > g_0$. Then P_g has dimension 1 by [26, Theorem 4.35], so $\widetilde{F}_\eta(gv,\omega)$ and $F_\eta(gv,\omega)$ will have a non degenerate isolated ground state for all $g > g_0$. Let $\{\psi_g\}_{g\geq g_0}$ be a real analytic collection of normalized eigenstates for $F_\eta(gv,\omega)$ and write $\widetilde{\psi}_g = U_g\psi_g$ which is a ground state for $\widetilde{F}_\eta(gv,\omega)$. We calculate

$$|\langle e^{-g^2\|\omega^{-1}v\|}\varepsilon(-g\omega^{-1}v),\psi_g\rangle|=|\langle\Omega,\widetilde{\psi}_g\rangle|=\|P\widetilde{\psi}_g\|\geq\|\widetilde{\psi}_g\|-\|P-P_g\|\|\widetilde{\psi}_g\|>0.$$

Hence $h(g)=\langle e^{-g^2\|\omega^{-1}v\|}\varepsilon(-g\omega^{-1}v),\psi_g\rangle$ is nonzero and smooth. Multiplying with $\frac{1}{h(g)}$ and normalising, we may pick the family $\{\psi_g\}_{g\geq g_0}$ smooth such that

$$\langle e^{-g^2\|\omega^{-1}v\|}\varepsilon(-g\omega^{-1}v),\psi_g\rangle=\langle\Omega,U_g\psi_g\rangle>0.$$

This implies

$$1\geq|\langle\Omega,\widetilde{\psi}_g\rangle|=\|P\widetilde{\psi}_g\|\geq\|\widetilde{\psi}_g\|-\|P-P_g\|\|\widetilde{\psi}_g\|=1-\|P-P_g\|.$$

Therefore $|\langle\Omega,\widetilde{\psi}_g\rangle|=\langle\Omega,\widetilde{\psi}_g\rangle$ converges to 1, and hence $\widetilde{\psi}_g$ converges to Ω. This implies

$$0=\lim_{g\to\infty}\|\widetilde{\psi}_g-\Omega\|=\lim_{g\to\infty}\|U_g^*\widetilde{\psi}_g-U_g^*\Omega\|=\lim_{g\to\infty}\|\psi_g-e^{-g^2\|\omega^{-1}v\|^2}\epsilon(-g\omega^{-1}v)\|.$$

Using Lemma 5.2 we thus find

$$\langle\widetilde{\psi}_g,d\Gamma(\omega)\widetilde{\psi}_g\rangle=\mathcal{E}_\eta(gv,\omega)+g^2\|\omega^{-1/2}v\|^2-\eta\langle\widetilde{\psi}_g,W(2g\omega^{-1}v,-1)\widetilde{\psi}_g\rangle$$

converges to 0. Hence ψ_g converges to Ω in $d\Gamma(\omega)^{1/2}$ norm, and hence also in $N^{1/2}$ norm since $m>0$. Note that $\psi_g,\widetilde{\psi}_g\in\mathcal{D}(d\Gamma(\omega))\subset\mathcal{D}(N)$ since $m>0$. Using Theorem A.2 we see that

$$\langle\psi_g,N\psi_g\rangle=\langle\widetilde{\psi}_g,U_gNU_g^*\widetilde{\psi}_g\rangle=\langle\widetilde{\psi}_g,N\widetilde{\psi}_g\rangle+g\langle\widetilde{\psi}_g,\varphi(\omega^{-1}v)\widetilde{\psi}\rangle+g^2\|\omega^{-1}v\|.$$

Since $\widetilde{\psi}_g$ goes to Ω in $N^{1/2}$ norm and $\varphi(\omega^{-1}v)$ is $N^{1/2}$ bounded by Lemma 2.2 we find that $\varphi(\omega^{-1}v)\widetilde{\psi}_g$ converges to $\varphi(\omega^{-1}v)\Omega$ in norm. Hence $\langle\widetilde{\psi}_g,\varphi(\omega^{-1}v)\widetilde{\psi}_g\rangle$ and $\langle\widetilde{\psi}_g,N\widetilde{\psi}_g\rangle$ converges to 0 which implies

$$(g^{-1}\langle\psi_g,N\psi_g\rangle-g\|\omega^{-1}v\|)=g^{-1}\langle\widetilde{\psi}_g,N\widetilde{\psi}_g\rangle+\langle\widetilde{\psi}_g,\varphi(\omega^{-1}v)\widetilde{\psi}\rangle$$

converges to 0 as g tends to ∞. Define $f(g)=\mathcal{E}_\eta(gv,\omega)+g^2\|\omega^{-1/2}v\|^2$ and assume $\eta<0$. Since $f(0)=\eta$ and f converges to 0, we just need to see f is increasing. There is a unitary map $U:\mathcal{H}\to L^2(X,\mathcal{F},\mu)$ such that $U\omega U^*$ is a multiplication operator. Using Lemma A.1 we see $\Gamma(U)F_\eta(gv,\omega)\Gamma(U)^*=F_\eta(gUv,U\omega U^*)$ so

$$f(g)=\mathcal{E}_\eta(gUv,U\omega U^*)+g^2\|(U\omega U^*)^{-1/2}Uv\|^2$$

Hence we may assume $\mathcal{H}=L^2(\mathcal{M},\mathcal{F},\mu)$ and ω is multiplication by a strictly positive and measurable map, which we shall also denote ω. We note ψ_g exists for all $g\geq 0$ by Theorem 3.2 and we have the pull through formula (see [3])

$$a(k)\psi_g=-gv(k)(F_{-\eta}(gv,\omega)-\mathcal{E}_\eta(gv,\omega)+\omega(k))^{-1}\psi_g. \tag{5.4}$$

Note that $g \mapsto \mathcal{E}_\eta(gv,\omega)$ is real analytic since it is a an isolated non degenerate eigenvalue by Theorem 3.2. We may then calculate

$$\begin{aligned}\frac{d}{dg}\mathcal{E}_\eta(gv,\omega) &= \langle \psi_g, \varphi(v)\psi_g\rangle = 2\mathrm{Re}(\langle \psi_g, a(v)\psi_g\rangle) \\ &= -2g\int_X |v(k)|^2 \|(F_{-\eta}(gv,\omega)-\mathcal{E}_\eta(gv,\omega)+\omega(k))^{-1/2}\psi_g\|^2 dk \\ &> -2g\|\omega^{-1/2}v\|^2 = -\frac{d}{dg}g^2\|\omega^{-1/2}v\|\end{aligned}$$

because $\|(F_{-\eta}(gv,\omega)-\mathcal{E}_\eta(gv,\omega)+\omega(k))^{-1/2}\psi_g\|^2 < \omega(k)^{-1}$ almost everywhere by Theorem 3.2. This proves the claim. □

Proof (of Corollary 4.3). By Theorem 3.2 we may pick a a unitary map V such that $VH_\eta(gv,\omega)V^* = F_{-\eta}(gv,\omega)\oplus F_\eta(gv,\omega)$. Noting that

$$\lim_{g\to\infty} \mathcal{E}_\eta(gv,\omega) - \mathcal{E}_{-\eta}(gv,\omega) = 0$$

and that $\mathcal{E}_{\pm\eta}(gv,\omega)$ is an eigenvalue for $F_{\pm\eta}(gv,\omega)$ for sufficiently large g by Corollary 4.2, we see that for g large enough $H_\eta(gv,\omega)$ will have at least two eigenvalues in the mass gap $[E_\eta(gv,\omega), E_\eta(gv,\omega)+m_{\mathrm{ess}}]$, and the energy difference will converge to 0. □

Proof (of Corollary 4.4). Define $V_g = W(\omega^{-1}v_g, 1)$ which is independent of η. If $\omega^{-1}v1_{\{\omega\geq 1\}} \notin \mathcal{H}$, we see part (1) follows from Theorem 4.1 and Lemma 2.2.

We now prove part (2). By Theorem 3.2 there is a unitary map U with the property that $UH_\eta(v_g,\omega)U^* = F_{-\eta}(v_g,\omega)\oplus F_\eta(v_g,\omega)$. Let $U_g = U^*(V_g\oplus V_g)U$. Convergence to $\widetilde{H}$ and the uniform lower bound now follows from part (1) and Lemma 2.2. Let $C_g = \|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2$. Then

$$\begin{aligned}&\|(H_\eta(v_g,\omega)+C_g+i)^{-1} - (H_0(v_g,\omega)+C_g+i)^{-1}\| \\ &\quad = \|(U_gH_\eta(v_g,\omega)U_g^*+C_g+i)^{-1} - (U_gH_0(v_g,\omega)U_g^*+C_g+i)^{-1}\|\end{aligned}$$

which converges to 0. This finishes the proof. □

6. Proof of Theorem 4.5

In this chapter we prove Theorem 4.5. We will in this section assume that $\mathcal{H} = L^2(X,\mathcal{F},\mu)$ where $(X,\mathcal{F},\mu)$ is σ-finite and countably generated. We will also assume ω is a multiplication operator which satisfies $\omega > 0$ almost everywhere. We also fix $v \in \mathcal{D}(\omega^{-1/2})$ and define

$$h(x) = \begin{cases} -1 & v(x) = 0 \\ -\frac{v(x)}{|v(x)|} & v(x) \neq 0 \end{cases} \tag{6.1}$$

Note that h is measurable, $|h| = 1$ and $h^*v = -|v|$. Define also $h^{(n)}(k_1,\dots,k_n) = h(k_1)\dots h(k_n)$ and note

$$\Gamma(h) = \bigoplus_{n=0}^{\infty} h^{(n)} \qquad \Gamma(h)^* = \Gamma(h^*) = \bigoplus_{n=0}^{\infty} (h^{(n)})^*.$$

Define

$$\mathcal{C}_+ = \{\psi = (\psi^{(n)}) \in \mathcal{F}_b(\mathcal{H}) \mid \psi^{(0)} \geq 0, (h^{(n)})^* \psi^{(n)} \geq 0 \text{ a. e. for } n \geq 1\}.$$

We have

Lemma 6.1. *$\mathcal{C}_+$ is a selfddual cone inside $\mathcal{F}_b(\mathcal{H})$. The strictly positive elements are*

$$\mathcal{C}_{>0} = \{\psi = (\psi^{(n)}) \in \mathcal{F}_b(\mathcal{H}) \mid \psi^{(0)} > 0, (h^{(n)})^* \psi^{(n)} > 0 \text{ a. e. for } n \geq 1\}.$$

Proof. We note

$$\mathcal{C}_+ = \Gamma(h)\{\psi = (\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}_b(\mathcal{H}) \mid \psi^{(0)} \geq 0, \psi^{(n)} \geq 0 \text{ a. e. for } n \geq 1\}.$$

The result now follows by the theory developed in [16].

Lemma 6.2. *Let $g > 0$ and T be a selfadjoint operator on $\mathcal{F}_b(\mathcal{H})$ such that*

$$T = \bigoplus_{n=0}^{\infty} T^{(n)}$$

with $T^{(n)} \geq \gamma$ a multiplication operator for all $n \in \mathbb{N}_0$. Assume $g\varphi(v)$ infinitesimally T-bounded. Define $H = T + g\varphi(v)$. Then H is bounded below, selfadjoint and $(H-\lambda)^{-1}\mathcal{C}_+ \subset \mathcal{C}_+$ for all $\lambda < \inf(\sigma(H))$. If $v \neq 0$ almost everywhere then $(H-\lambda)^{-1}\mathcal{C}_+ \backslash \{0\} \subset \mathcal{C}_{>0}$ for all $\lambda < \inf(\sigma(H))$. So if $\inf(\sigma(H))$ is an eigenvalue then it is non degenerate and spanned by an element in $\mathcal{C}_{>0}$.

Proof. The Kato Rellich theorem implies H is selfadjoint and bounded below. For $\lambda < -\gamma$ we note that $(T-\lambda)^{-1}$ acts on each particle sector as multiplication with a positive bounded map. Hence it will map $\mathcal{C}_+$ into $\mathcal{C}_+$. Assume now that $\psi = (\psi^{(n)}) \in \mathcal{C}_+ \cap \mathcal{D}(d\Gamma(\omega))$. Then we have almost everywhere that

$$(h^{(n-1)})^*(-a(v)\psi^{(n)})(k_2, \dots, k_n) = \sqrt{n} \int_{\mathcal{M}} |v(k)|((h^{(n)})^* \psi^{(n)})(k, k_2, \dots, k_n) dk$$

$$\begin{aligned}
&(h^{(n+1)})^*(-a^\dagger(v)\psi^{(n)})(k_1, \dots, k_{n+1}) \\
&\quad = \frac{1}{\sqrt{n+1}} \sum_{l=1}^{n+1} -h^*(k_l)v(k_l)((h^{(n)})^*\psi^n)(k_1, \dots, \hat{k}_l, \dots, k_{n+1})
\end{aligned}$$

which implies $-g\varphi(v)\psi \in \mathcal{C}_+$. In particular we obtain

$$\begin{aligned}
&(-g\varphi(v)(T-\lambda)^{-1})^n \mathcal{C}_+ \subset \mathcal{C}_+ \\
&(-1)^n g^n \prod_{k=1}^{n} a^{\sharp_k}(v)(T-\lambda)^{-1}\mathcal{C}_+ \subset \mathcal{C}_+. \qquad (6.2)
\end{aligned}$$

where $\sharp_k$ can be either a $\dagger$ or nothing. For $\lambda \in \mathbb{R}$ sufficiently negative we may expand

$$(H-\lambda)^{-1} = \sum_{n=0}^{\infty} (T-\lambda)^{-1}(-g\varphi(v)(T-\lambda)^{-1})^n. \qquad (6.3)$$

Since each term preserves the closed set $\mathcal{C}_+$ we find $(H-\lambda)^{-1}\mathcal{C}_+ \subset \mathcal{C}_+$ for λ small enough. Assume now $v \neq 0$ almost everywhere. Let I_n denote the integral over $\mathcal{M}^n$ with respect to $\mu^{\otimes n}$. For $u \in S_n(\mathcal{H}^{\otimes n})\backslash\{0\}$ with $u \in \mathcal{C}_+$ we have

$$\begin{aligned}&(-1)^n(ga(v)(T-\mu)^{-1})^n u \qquad (6.4)\\ &= I_n\left((-1)^n g^n \overline{v(k_1)\dots v(k_n)}u(k_1,\dots,k_n)\prod_{\ell=1}^{n}(T^{(\ell)}(k_1,\dots,k_\ell)-\lambda)^{-1}\right)\end{aligned}$$

which is strictly positive. Let $u, w \in \mathcal{C}_+\backslash\{0\}$. Pick n_1 such that $u^{(n_1)} \neq 0$ and n_2 such that $w^{(n_2)} \neq 0$. Consider now the $n = n_1 + n_2$ term in equation (6.3). This term can again be written as a sum of terms of the form (6.2) multiplied to the left by $(T-\lambda)^{-1}$. Since all terms are positivity preserving we find

$$\begin{aligned}\langle u, (H-\lambda)^{-1}w\rangle &\geq \langle (T-\lambda)^{-1}u, (-ga^\dagger(v)(T-\lambda)^{-1})^{n_1}(-ga(v)(T-\lambda)^{-1})^{n_2}w\rangle\\ &= \langle (T-\lambda)^{-1}(-ga(v)(T-\lambda)^{-1})^{n_1}u, (-ga(v)(T-\lambda)^{-1})^{n_2}w\rangle\end{aligned}$$

Since $u - u^{(n_1)} \in \mathcal{C}_+$ and $w - w^{(n_2)} \in \mathcal{C}_+$ we find the following lower bound:

$$(T^{(0)}-\lambda)^{-1}(-ga(v)(T-\lambda)^{-1})^{n_1}u^{(n_1)}(-ga(v)(T-\lambda)^{-1})^{n_2}w^{(n_2)}$$

which is strictly positive by Equation (6.4). Hence we have proven the lemma for λ sufficiently negative. Now fix λ such that the lemma is true. For any $\mu \in (\lambda, \inf(\sigma(H)))$ we can use standard theory of resolvents to write

$$(H-\mu)^{-1} = \sum_{k=0}^{\infty}(\mu-\lambda)^n((H-\lambda)^{-1})^{n+1}$$

which is positivity preserving/improving since each term is. □

The following lemma can be found in [3].

Lemma 6.3. *Define $A = \{v \neq 0\}$, $\mu_A(B) = \mu(A\cap B)$ and $\mu_{A^c}(B) = \mu(A^c \cap B)$. Let $\mathcal{H}_1 = L^2(X, \mathcal{F}, \mu_A)$ and ω_1 be multiplication with ω but on the space $\mathcal{H}_1$. Assume that $\eta \leq 0$ and $g > 0$. Then*

1. *$\mathcal{E}_\eta(gv,\omega) = \mathcal{E}_\eta(gv,\omega_1)$ and $\mathcal{E}_\eta(gv,\omega)$ is an eigenvalue for $F_\eta(gv,\omega)$ if and only if $\mathcal{E}_\eta(gv,\omega)$ is an eigenvalue for $F_\eta(gv,\omega_1)$. In this case the dimension of the eigenspace is 1.*
2. *If $\psi = (\psi^{(n)})_{n=0}^\infty$ is a ground state for $F_\eta(gv,\omega_1)$, then $\psi = (1_{A^n}\psi^{(n)})_{n=0}^\infty$ is a ground state for $F_\eta(gv,\omega)$.*

We can now finally prove Theorem 4.5.

Proof (Proof of Theorem 4.5). Statement (1) follows from Lemmas 6.2 and 6.3 since $g\phi(v) = \phi(gv)$ and h defined in equation (6.1) does not depend on g as long as $g > 0$. To prove statement (2) we let ψ be a ground state for $F_\eta(gv,\omega)$. Define for $\lambda > 0$ and $\ell \in \mathbb{N}$ the operator

$$R_\ell(\lambda) = (F_{(-1)^\ell\eta}(gv,\omega) + \lambda - \mathcal{E}_\eta(gv,\omega))^{-1}.$$

This makes sense since $\mathcal{E}_\eta(gv,\omega) \leq \mathcal{E}_{-\eta}(gv,\omega)$ by Proposition 3.1. Using the pull through formula found in [3] we find

$$a(k_1,\dots,k_n)\psi_{g,\eta} = \sum_{i=1}^n gv(k_i)R_n(\omega(k_1)+\cdots+\omega(k_n))a(k_1,\dots,\hat{k}_i,\dots,k_n)\psi_{g,\eta},$$

where $\widehat{k}_i$ means that the variable k_i is omitted. We proceed by induction to show that $\|a(k_1,\dots,k_n)\psi_g\| \leq g^n \frac{|v(k_1)|\dots|v(k_n)|}{\omega(k_1)\dots\omega(k_n)}$. For $k=1$ this follows since $\mathcal{E}_\eta(gv,\omega) \leq \mathcal{E}_{-\eta}(gv,\omega)$ and so

$$\|a(k)\psi_{g,\eta}\| = \left\| \frac{gv(k)}{F_{-\eta}(gv,\omega)+\omega(k)+\mathcal{E}_\eta(gv,\omega)}\psi_g \right\| \leq g\frac{|v(k)|}{\omega(k)}.$$

Using the induction hypothesis we may now compute

$$\begin{aligned}\|a(k_1,\dots,k_n)\psi_{g,\eta}\| &\leq \sum_{i=1}^n \frac{\omega(k_i)}{\omega(k_1)+\cdots+\omega(k_n)} g^n \frac{|v(k_1)|\dots|v(k_n)|}{\omega(k_1)\dots\omega(k_n)} \\ &= g^n \frac{|v(k_1)|\dots|v(k_n)|}{\omega(k_1)\dots\omega(k_n)}.\end{aligned}$$

Now $\sqrt{n!}|\psi_{g,\eta}^{(n)}(k_1,\dots,k_n)| \leq \|a(k_1,\dots,k_n)\psi_{g,\eta}\|$ and so the desired inequality follows.

Statement (3): By Theorem 3.2 and $N = d\Gamma(1)$ we see that the conclusions about $\phi_{g,\eta}$ follows from those of $F_\eta(gv,\omega)$. It is easily seen that $\psi_{g,0} = e^{-2^{-1}g^2\|\omega^{-1}v\|}\varepsilon(g\omega^{-1}v)$ and $\varepsilon(g\omega^{-1}v) \in \mathcal{D}(f(N)) \iff \alpha_{g,f,v,\omega} < \infty$. This proves the " $\Leftarrow$ " part. If $\alpha_{g,f,v,\omega} < \infty$ then we may use the point wise bounds to obtain

$$\sum_{n=0}^\infty \frac{f(n)^2\|\psi_{g,\eta}^{(n)}\|^2}{n!} \leq \sum_{n=0}^\infty \frac{f(n)^2 g^{2n}\|\omega^{-1}v\|^{2n}}{n!} < \infty,$$

which proves the " $\Rightarrow$ ". □

7. Convergence in the massless case

In this section we will assume $\mathcal{H} = L^2(\mathbb{R}^\nu,\mathcal{B}(\mathbb{R}^\nu),\lambda^\nu)$ and that ω is a selfadjoint, non negative and injective multiplication operator on this space with $m(\omega)=0$. Fix an element $v \in \mathcal{D}(\omega^{-1})\backslash\{0\}$. In [3] it is proven that if $\eta \leq 0$ then $F_\eta(gv,\omega)$ has a normalised ground state ψ_g for any $g \in \mathbb{R}$ and $\mathcal{E}_\eta(gv,\omega) = \mathcal{E}_{-\eta}(gv,\omega)$. Furthermore we will for $\eta, g \in \mathbb{R}$ write $F_{\eta,g} := F_\eta(gv,\omega)$ and $\mathcal{E}_{\eta,g} := \mathcal{E}_\eta(gv,\omega)$.

Lemma 7.1. *Assume $\eta \leq 0$. Define $U_g = W(g\omega^{-1}v,1)$ and $\widetilde{\psi}_g = U_g\psi_g$. Then*

$$0 \leq \langle \widetilde{\psi}_g, d\Gamma(\omega)\widetilde{\psi}_g\rangle \leq |\eta|\langle \psi_g, \Gamma(-1)\psi_g\rangle = -\eta\langle \psi_g, \Gamma(-1)\psi_g\rangle,$$

and $\langle \psi_g, \Gamma(-1)\psi_g\rangle$ converges to 0 for g tending to ∞. Furthermore, given any sequence of elements $\{g_n\}_{n=1}^\infty \subset \mathbb{R}$ tending to ∞ there is a subsequence $\{g_{n_i}\}_{i=1}^\infty$ such that

$$\lim_{i\to\infty} |v(k)|^2 \|(F_{-\eta,g_{n_i}} - \mathcal{E}_{\eta,g_{n_i}} + \omega(k))^{-1}\psi_{g_{n_i}} - \omega(k)^{-1}\psi_{g_{n_i}}\|^2 = 0$$

almost everywhere.

Proof. We have

$$U_g F_{e,g} U_g^* + g^2\|\omega^{-1/2}v\|^2 = d\Gamma(\omega) + \eta W(2g\omega^{-1}v, -1) = \widetilde{F}_\eta(2g\omega^{-1}v, \omega).$$

Note that

$$\langle \Omega, \widetilde{F}_\eta(2g\omega^{-1}v, \omega)\Omega\rangle = \eta \exp(-2g^2\|\omega^{-1}v\|^2) \leq 0$$

so $\mathcal{E}_{e,g} + g^2\|\omega^{-1/2}v\| = \inf(\sigma(\widetilde{F}_\eta(2g\omega^{-1}v, \omega))) \leq 0$. This implies

$$0 \leq \langle \widetilde{\psi}_g, d\Gamma(\omega)\widetilde{\psi}_g\rangle \leq -\eta\langle \widetilde{\psi}_g, W(2g\omega^{-1}v, -1)\widetilde{\psi}_g\rangle = |\eta|\langle \psi_g, \Gamma(-1)\psi_g\rangle \leq |e|.$$

Since $\psi_g \in \mathcal{D}(N^{1/2})$ by Theorem 4.5 we find (see [3])

$$a(k)U_g\psi_g = U_g a(k)\psi_g + gv(k)\omega(k)^{-1}U_g\psi_g,$$

and so the pull through formula from equation (5.4) gives

$$a(k)\widetilde{\psi}_g = -gv(k)U_g(F_{-\eta,g} - \mathcal{E}_{e,g} + \omega(k))^{-1}\psi_g + gv(k)\omega(k)^{-1}U_g\psi_g.$$

Hence we find

$$\begin{aligned}&\langle \widetilde{\psi}_g, d\Gamma(\omega)\widetilde{\psi}_g\rangle \\ &= g^2\int_{\mathcal{M}} \omega(k)|v(k)|^2\|(F_{-\eta,g}(v,\omega) - \mathcal{E}_{e,g} + \omega(k))^{-1}\psi_g - \omega(k)^{-1}\psi_g\|^2 dk.\end{aligned}$$

Since this remains bounded by $|\eta|$ as g tends to infinity, we conclude that the integral converges to 0 as g tends to infinity. Thus existence of the desired subsequence follows from standard measure theory. Assume now that the conclusion about convergence of $\langle \psi_g, \Gamma(-1)\psi_g\rangle$ is false. We may then pick $\varepsilon > 0$ and sequence $\{g_n\}_{n=1}^\infty$ such that $-e\langle \psi_{g_n}, \Gamma(-1)\psi_{g_n}\rangle \geq \varepsilon$ for all n and

$$\lim_{g\to\infty} |v(k)|^2\|(F_{-\eta,g_n} - \mathcal{E}_{e,g_n} + \omega(k))^{-1}\psi_{g_n} - \omega(k)^{-1}\psi_{g_n}\|^2 = 0$$

for almost every $k \in \mathbb{R}^\nu$. Let P_g be the spectral measure of $F_{-\eta,g} - \mathcal{E}_{e,g} = F_{-\eta,g} - \mathcal{E}_{-\eta,g}$ and define the measure $\mu_g(A) = \langle \psi_g, P_g(A)\psi_g\rangle$. Since $v \neq 0$ we see

$$\begin{aligned}&\|(F_{-\eta,g_n} - \mathcal{E}_{e,g_n} + \omega(k))^{-1}\psi_{g_n} - \omega(k)^{-1}\psi_{g_n}\|^2 \\ &\qquad = \int_{[0,\infty)} \left|\frac{1}{\lambda + \omega(k)} - \frac{1}{\omega(k)}\right|^2 d\mu_{g_n}(\lambda)\end{aligned}$$

converges to 0 for some $k \in \mathbb{R}^\nu$ where $\omega(k) > 0$. Since the integrals above converges to 0, the numbers $\mu_{g_n}([\varepsilon/2, \infty))$ must converge to 0, as the integrand has a positive lower bound on $[\varepsilon/2, \infty)$. In particular $P_{g_n}([0, \varepsilon/2))\psi_{g_n} - \psi_{g_n}$ will converge to 0. Hence we find for n larger than some K that

$$-\eta\langle P_{g_n}([0,\varepsilon/2))\psi_{g_n}, \Gamma(-1)P_{g_n}([0,\varepsilon/2))\psi_{g_n}\rangle \geq \frac{3\varepsilon}{4}\|P_{g_n}([0,\varepsilon/2))\psi_{g_n}\|^2.$$

Let $x_n = P_{g_n}([0,\varepsilon/2))\psi_{g_n}$. By Lemma 2.2 we find $d\Gamma(\omega)+g\varphi(v)+g^2\|\omega^{-1/2}v\|^2 \geq 0$. Using this and $\mathcal{E}_{e,g} = \mathcal{E}_{-\eta,g} \leq -g^2\|\omega^{-1/2}v\|^2$ we may calculate for $n \geq K$

$$\begin{aligned}(\mathcal{E}_{e,g_n}(v,\omega)+\varepsilon/2)\|x_n\|^2 &\geq \langle x_n, F_{-\eta,g_n}x_n\rangle \\ &= -\eta\langle x_n, \Gamma(-1)x_n\rangle + \mathcal{E}_{-\eta,g_n}\|x_n\|^2 \\ &\quad + \langle x_n, (d\Gamma(\omega)+g_n\varphi(v)+g_n^2\|\omega^{-1/2}v\|^2)x_n\rangle \\ &\quad - (\mathcal{E}_{-\eta,g_n}+g_n^2\|\omega^{-1/2}v\|^2)\|x_n\|^2 \\ &\geq -\eta\langle x_n, \Gamma(-1)x_n\rangle + \mathcal{E}_{-\eta,g_n}\|x_n\|^2 \\ &\geq (3\varepsilon/4 + \mathcal{E}_{e,g_n}(v,\omega))\|x_n\|^2,\end{aligned}$$

which is the desired contradiction. □

Proof (Proof of Theorem 4.6). For each $g \geq 0$ we let ψ_g be a ground state eigenvector for $F_{e,g}$. Define $U_g = W(g\omega^{-1}v, 1)$ and $\widetilde{\psi}_g = U_g\psi_g$. We see that

$$\begin{aligned}|\mathcal{E}_{e,g} + g^2\|\omega^{-1/2}v\|| &= |\langle\widetilde{\psi}_g, \widetilde{F}_\eta(2g\omega^{-1}v,\omega)\widetilde{\psi}_g\rangle| \\ &= |\eta\langle\psi_g, \Gamma(-1)\psi_g\rangle + \langle\widetilde{\psi}_g, d\Gamma(\omega)\widetilde{\psi}_g\rangle| \\ &\leq 2|\eta|\langle\psi_g, \Gamma(-1)\psi_g\rangle,\end{aligned}$$

which converges to 0 for g tending to ∞ by Lemma 7.1. It only remains to prove the statement regarding the number operator. Let $\{g_n\}_{n=1}^\infty$ be any sequence converging to ∞. Pick a subsequence $\{g_{n_i}\}_{i=1}^\infty$ such that

$$\lim_{i\to\infty}|v(k)|^2\|(F_{-\eta,g_{n_i}} - \mathcal{E}_{e,g_{n_i}} + \omega(k))^{-1}\psi_{g_{n_i}} - \omega(k)^{-1}\psi_{g_{n_i}}\|^2 = 0$$

almost everywhere. Using equation (5.4) we see that

$$a(k)\psi_g = -gv(k)(F_{-\eta,g} - \mathcal{E}_{e,g} + \omega(k))^{-1}\psi_g$$

and so

$$\begin{aligned}&\frac{|\langle\psi_{g_{n_i}}, N\psi_{g_{n_i}}\rangle - g_{n_i}^2\|\omega^{-1}v\|^2|}{g_{n_i}^2} \\ &\leq \int_{\mathcal{M}}|v(k)|^2|\|(F_{-\eta,g_{n_i}} - \mathcal{E}_{e,g_{n_i}} + \omega(k))^{-1}\psi_{g_{n_i}}\|^2 - \|\omega(k)^{-1}\psi_{g_{n_i}}\|^2|dk,\end{aligned}$$

which goes to 0 as i tends to infinity by dominated convergence. □

8. Proof of Theorem 4.7

In this section we will assume $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda^\nu)$ and that ω is a selfadjoint, non-negative and injective multiplication operator on this space. Then $m_{\mathrm{ess}}(\omega) = m(\omega) := m$ since $\sigma(\omega) = \sigma_{\mathrm{ess}}(\omega)$ (See [3]). Furthermore we define $P = |\Omega\rangle\langle\Omega|$ and $\overline{P} = 1-P$. Then $\overline{P}$ clearly reduces $d\Gamma(\omega)$ and $\Gamma(-1)$. Let $\overline{d\Gamma}(\omega)$ and $\overline{\Gamma}(-1)$ denote the restrictions to $\overline{\mathcal{F}_b(\mathcal{H})} = \overline{P}\mathcal{F}_b(\mathcal{H})$. For $v \in \mathcal{H}$ we define $\overline{\varphi}(v)$ as the restriction of $\overline{P}\varphi(v)\overline{P}$ to $\overline{\mathcal{F}_b(\mathcal{H})}$. Note that it is symmetric and infinitesimally $\overline{d\Gamma}(\omega)$ bounded when $v \in \mathcal{D}(\omega^{-1/2})$. Hence we may define

$$\overline{F}_\eta(v,\omega) = \overline{d\Gamma}(\omega) + \eta\overline{\Gamma}(-1) + \overline{\varphi}(v),$$

which is selfadjoint on $\mathcal{D}(\overline{d\Gamma}(\omega))$ and bounded below when $v \in \mathcal{D}(\omega^{-1/2})$. Note by the min-max principle that $\inf(\sigma(\overline{F}_\eta(v,\omega))) \geq \mathcal{E}_\eta(v,\omega)$ and one may repeat the argument for Lemma 6.2 to show that for every $\lambda < \mathcal{E}_\eta(v,\omega)$ we have

$$(\overline{F}_\eta(v,\omega) - \lambda)^{-1}\overline{P}\mathcal{C}_+ \subset \overline{P}\mathcal{C}_+.$$

To summarise

Lemma 8.1. *If $v \in \mathcal{D}(\omega^{-1/2})$ then $\overline{F}_\eta(v,\omega)$ is selfadjoint selfadjoint and bounded below by $\mathcal{E}_\eta(v,\omega)$. Furthermore $(\overline{F}_\eta(v,\omega) - \lambda)^{-1}\overline{P}\mathcal{C}_+ \subset \overline{P}\mathcal{C}_+$ for every $\lambda < \mathcal{E}_\eta(v,\omega)$.*

We shall also need the following lemma.

Lemma 8.2. *For all $\lambda < \mathcal{E}_\eta(v,\omega)$ we have*

$$0 < \langle \Omega, (F_\eta(v,\omega) - \lambda)^{-1}\Omega\rangle = (e - \lambda + \langle v, (\overline{F}_\eta(v,\omega) - \lambda)^{-1}v\rangle)^{-1}.$$

Proof. Let $\lambda < \mathcal{E}_\eta(v,\omega)$. One easily checks that $(F_\eta(v,\omega) - \lambda, d\Gamma(\omega) + \eta\Gamma(-1) - \lambda)$ is a Feshbach pair for P. Write $T = d\Gamma(\omega) + \eta\Gamma(-1) - \lambda$, $H = F_\eta(v,\omega) - \lambda$ and $W = H - T = \varphi(v)$. The Feshbach map F is now given by

$$\begin{aligned} F &= PHP - PW\overline{P}(\overline{F}_\eta(v,\omega) - \lambda)^{-1}\overline{P}WP \\ &= (e - \lambda)P + \langle v, (\overline{F}_\eta(v,\omega) - \lambda)^{-1}v\rangle P. \end{aligned}$$

This is invertible from Span(Ω) to Span(Ω) since H is invertible. To calculate the inverse using we use the formula in [10] and find

$$F^{-1} = PH^{-1}P = \langle \Omega, (F_\eta(v,\omega) - \lambda)^{-1}\Omega\rangle P.$$

If one identifies the the linear maps from Span(Ω) to Span(Ω) with $\mathbb{C}$ we find the desired relation. Positivity follows since H^{-1} maps $\mathcal{C}_+$ into $\mathcal{C}_+$, and we know that the matrix element is not zero since the Feshbach map is invertible. □

We may now prove Theorem 4.7. The basic technique for proving this result comes from the paper [24] where it is used for the translation invariant Nelson model.

Proof (Proof of Theorem 4.7). Let $\eta > 0$ and assume the conclusion does not hold. Since $F_{-\eta}(v,\omega)$ has a ground state by Theorem 3.2 the only option is that $F_\eta(v,\omega)$ does not have a ground state. By Theorems 3.2 and 4.5 we note that $\mathcal{E}_\eta(v,\omega) = \inf(\sigma_{\text{ess}}(F_\eta(v,\omega))) = \mathcal{E}_{-\eta}(v,\omega) + m$ and that $F_{-\eta}(v,\omega)$ has a ground state ψ which has non-zero inner product with Ω. By Lemma 8.2 we find

$$\lambda - \eta > \langle v, (\overline{F}_\eta(v,\omega) - \lambda)^{-1}v\rangle$$

for all $\lambda < \mathcal{E}_\eta(v,\omega) = \mathcal{E}_{-\eta}(v,\omega) + m$, and so $\langle v, (\overline{F}_\eta(v,\omega) - \lambda)^{-1}v\rangle$ is uniformly bounded from above for all $\lambda < \mathcal{E}_{-\eta}(v,\omega) + m$. We shall now prove that this leads to a contradiction with the assumption in equation (4.2). The following pull through formula, holds for $x \in \mathcal{D}(d\Gamma(\omega))$ such that $(F_\eta(v,\omega)-\lambda)x \in \mathcal{D}(N^{1/2})$ (see [3])

$$\begin{aligned} a(k)x =&(F_{-\eta}(v,\omega)+\omega(k)-\lambda)^{-1}a(k)(F_\eta(v,\omega)-\lambda)x \\ &-v(k)(F_{-\eta}(v,\omega)+\omega(k)-\lambda)^{-1}. \end{aligned} \tag{8.1}$$

We note that

$$(F_\eta(v,\omega)-\lambda)(\overline{F}_\eta(v,\omega)-\lambda)^{-1}v = PF_\eta(v,\omega)(\overline{F}_\eta(v,\omega)-\lambda)^{-1}v + v \in \mathcal{D}(N^{1/2}).$$

Hence we may apply equation (8.1) with $x = (\overline{F}_\eta(v,\omega)-\lambda)^{-1}v$. Now $a(k)P = 0$ so $a(k)(F_\eta(v,\omega)-\lambda)x = v(k)\Omega$. This implies

$$\begin{aligned} \overline{v}(k)a(k)(\overline{F}_\eta(v,\omega)-\lambda)^{-1}v = &|v(k)|^2(F_{-\eta}(v,\omega)+\omega(k)+\lambda)^{-1}\Omega \\ &-|v(k)|^2(F_{-\eta}(v,\omega)+\omega(k)-\lambda)^{-1}(\overline{F}_\eta(v,\omega)-\lambda)^{-1}v. \end{aligned}$$

Taking the inner product with Ω for each k, we obtain two terms. Both are non-negative by Lemmas 6.2 and 8.1 so

$$\begin{aligned} \langle \Omega, v(k)a(k)(\overline{F}_\eta(v,\omega)-\lambda)^{-1}v\rangle &\geq |v(k)|^2\langle \Omega, (F_{-\eta}(v,\omega)+\omega(k)-\lambda)^{-1}\Omega\rangle \\ &\geq |\langle \Omega,\psi\rangle|^2|v(k)|^2(\omega(k)+\mathcal{E}_{-\eta}(v,\omega)-\lambda). \end{aligned}$$

Hence we find

$$\begin{aligned} \langle v, (\overline{F}_\eta(v,\omega)-\lambda)^{-1}v\rangle &= \int_{\mathcal{M}} \langle \Omega, v(k)a(k)(\overline{F}_\eta(v,\omega)-\lambda)^{-1}v\rangle dk \\ &\geq |\langle \Omega,\psi\rangle|^2 \int_{\mathcal{M}} |v(k)|^2(\omega(k)+\mathcal{E}_{-\eta}(v,\omega)-\lambda)dk, \end{aligned}$$

which goes to infinity for λ tending to $\mathcal{E}_{-\eta}(v,\omega)+m$ by the monotone convergence theorem, equation (4.2) and the fact $|\langle \Omega,\psi\rangle|^2 \neq 0$. This contradicts the boundedness of $\langle v, (\overline{F}_\eta(v,\omega)-\lambda)^{-1}v\rangle$.

In the special case mentioned, let $\omega(x_0) = m$ be the global minimum of ω. Using Taylor approximations there is $r > 0$ such that for $x \in B_r(x_0)$ we have $0 \leq \omega(k) - m \leq C|k-x_0|^2$. Switching to polar coordinates yields the result. □

A. Various transformation statements.

In this appendix various useful transformation theorems is stated. Sources are [19], [4] and [3].

Lemma A.1. *Let U be unitary from $\mathcal{H}$ into some Hilbert space $\mathcal{K}$. Then there is a unique unitary map $\Gamma(U) : \mathcal{F}_b(\mathcal{H}) \to \mathcal{F}_b(\mathcal{K})$ such that $\Gamma(U)\epsilon(g) = \epsilon(Ug)$. If ω is selfadjoint on $\mathcal{H}$, V is unitary and $f \in \mathcal{H}$ then*

$$\begin{aligned} \Gamma(U)d\Gamma(\omega)\Gamma(U)^* &= d\Gamma(U\omega U^*). \\ \Gamma(U)W(f,V)\Gamma(U)^* &= W(Uf, UVU^*). \\ \Gamma(U)\varphi(f)\Gamma(U)^* &= \varphi(Uf). \end{aligned}$$

Furthermore $\Gamma(U)(f_1 \otimes_s \cdots \otimes_s f_n) = Uf_1 \otimes_s \cdots \otimes_s Uf_n$ and $U\Omega = \Omega$.

One may transform the field operators and second quantised observables by the Weyl transformations. One then obtains the following important statements that we shall need. The proof is an easy calculation using exponential vectors

Lemma A.2. *Let $f, h \in \mathcal{H}$ and $U \in \mathcal{U}(\mathcal{H})$. Then*

$$\begin{aligned} W(h,U)\varphi(g)W(h,U)^* &= \varphi(Ug) - 2\mathrm{Re}(\langle Ug, h\rangle) \\ W(h,U)a(g)W(h,U)^* &= a(Ug) - \langle Ug, h\rangle \\ W(h,U)a^\dagger(g)W(h,U)^* &= a^\dagger(Ug) - \langle h, Ug\rangle \end{aligned}$$

Furthermore if ω is selfadjoint, non negative and injective on $\mathcal{H}$ and $h \in \mathcal{D}(\omega U^*)$ then

$$W(h,U)d\Gamma(\omega)W(h,U)^* = d\Gamma(U\omega U^*) - \varphi(U\omega U^* h) + \langle h, U\omega U^* h\rangle$$

on the domain $\mathcal{D}(d\Gamma(U\omega U^*))$.

In what follows we consider two fixed Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We will need the following two lemmas.

Lemma A.3. *There is a unique isomorphism $U : \mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2)$ such that $U(\epsilon(f \oplus g)) = \epsilon(f) \otimes \epsilon(g)$. The map has the following transformation properties. If ω_i is selfadjoint on $\mathcal{H}_i$, V_i is unitary on $\mathcal{H}_i$ and $f_i \in \mathcal{H}_i$ then*

$$\begin{aligned} UW(f_1 \oplus f_2, V_1 \oplus V_2)U^* &= W(f_1, V_1) \otimes W(f_2, V_2) \\ Ud\Gamma(\omega_1 \oplus \omega_2)U^* &= d\Gamma(\omega_1) \otimes 1 + 1 \otimes d\Gamma(\omega_2) \\ U\varphi(f_1, f_2)U^* &= \varphi(f_1) \otimes 1 + 1 \otimes \varphi(f_2) \\ Ua(f_1, f_2)U^* &= a(f_1) \otimes 1 + 1 \otimes a(f_2) \\ Ua^\dagger(f_1, f_2)U^* &= a^\dagger(f_1) \otimes 1 + 1 \otimes a^\dagger(f_2). \end{aligned}$$

Lemma A.4. *There is a unique isomorphism*

$$U : \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2) \to \mathcal{F}(\mathcal{H}_1) \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}(\mathcal{H}_1) \otimes S_n(\mathcal{H}_2^{\otimes n})$$

such that

$$U(w \otimes \{\psi_2^{(n)}\}_{n=0}^{\infty}) = \psi^{(0)} w \oplus \bigoplus_{n=1}^{\infty} w \otimes \psi_2^{(n)}.$$

Let A be a selfadjoint operator on $\mathcal{F}(\mathcal{H}_1)$ and B be selfadjoint on $\mathcal{F}(\mathcal{H}_2)$ such that B is reduced by all of the subspaces $S_n(\mathcal{H}_2^{\otimes n})$. Write $B^{(n)} = B \mid_{S_n(\mathcal{H}_2^{\otimes n})}$. Then

$$U(A \otimes 1 + 1 \otimes B)U^* = A + B^{(0)} \oplus \bigoplus_{n=1}^{\infty} (A \otimes 1 + 1 \otimes B^{(n)})$$

$$UA \otimes BU^* = A \otimes B = B^{(0)} A \oplus \bigoplus_{n=1}^{\infty} A \otimes B^{(n)}.$$

Acknowledgements. Thomas Norman Dam was supported by the Independent Research Fund Denmark with through the project "Mathematics of Dressed Particles".

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Communicated by

Paper C

Non-existence of ground states in the translation invariant Nelson model

By T. N. Dam

Non-existence of ground states in the translation invariant Nelson model

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Abstract: In this paper we consider the massless translation invariant Nelson model with ultraviolet cutoff. It is proven that the fiber operators have no ground state if there is no infrared cutoff.

1. Introduction

In this paper we study the translation invariant massless Nelson model. The model can (after a unitary transformation) be written as a direct integral of fiber operators $\{H(\xi)\}_{\xi\in\mathbb{R}^3}$. The spectral properties of these operators were first investigated by by J. Frölich in his Phd-thesis, which was published in the two papers [5] and [6]. Frölich showed, that if the field is massive or there is an infrared cut-off then $H(\xi)$ has a ground state for ξ in an open ball around 0. He also proved, that if the field is massless, no infrared conditions are imposed and a ground state exists for sufficiently many of the $H(\xi)$, then one can reach some physically unacceptable conclusions. The aim of this paper is to prove that $H(\xi)$ does not have a ground state if the field is massless and no infrared conditions are assumed. We shall briefly review central results about existence of ground states in the massless Nelson model.

In the paper [10], it is proven that ground states exists in a non-equivalent Fock representation. A consequence of this result is that the usual "taking the massgap to 0" strategy for proving existence of ground states does not work. This strongly indicates that there should be no ground state.

A proof of absence of ground states in a similar model was given by I. Herbst and D. Hasler in the paper [8]. They consider the fiber operators of the massless and translation invariant Pauli-Fierz model $\{\mathcal{H}(\xi)\}_{\xi\in\mathbb{R}^3}$. They prove that $\mathcal{H}(\xi_0)$ has no ground state if $\xi \mapsto \inf(\sigma(\mathcal{H}(\xi)))$ is differentiable at ξ_0 and has a non-zero derivative. One may easily work out the same problem for the Nelson model and obtain the same conclusions. However proving the existence of a non-zero derivative is an extremely hard problem and such a result has only been achieved for weak coupling and small ξ (see [1]). Furthermore, $\xi = 0$ is a global minimum for $\xi \mapsto \inf(\sigma(H(\xi)))$ and therefore the derivative is 0. However, $H(0)$ has no ground states shall prove below.

In fact we shall prove that $H(\xi)$ has no ground state for any non-zero coupling strength and $\xi \in \mathbb{R}^3$. Our proof is based on strategy used by I. Herbst and D. Hasler, but we remove the assumption regarding the existence of a non-zero derivative. Instead we use rotation invariance of the map $\xi \mapsto \inf(\sigma(H(\xi)))$, non degeneracy of ground states and the HVZ-theorem.

2. Notation and preliminaries

We start by fixing the measure theoretic notation. Let $(\mathcal{M}, \mathcal{F}, \mu)$ be a σ-finite measure space and X be a separable Hilbert space. We will write $L^p(\mathcal{M}, \mathcal{F}, \mu, X)$ for the Hilbert space valued L^p-space. If $X = \mathbb{C}$ it will be omitted from the notation. In case $\mathcal{M}$ is a topological space we will write $\mathcal{B}(\mathcal{M})$ for the Borel σ-algebra.

Let $\mathcal{H}$ denote a Hilbert space and $n \geq 1$. We write $\mathcal{H}^{\otimes n}$ for the n-fold tensor product. Write $\mathcal{S}_n$ for the set of permutations of $\{1, \dots, n\}$ and let $\mathcal{H}$ be a Hilbert Space. The symmetric projection is the unique bounded extension of the map

$$S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}$$

and S_0 is the identity on $\mathcal{H}^{\otimes n} = \mathbb{C}$. In certain cases we can realise tensor produces as concrete spaces:

$$\begin{aligned} L^2(\mathcal{M}, \mathcal{F}, \mu, X) &= L^2(\mathcal{M}, \mathcal{F}, \mu) \otimes X \\ (L^2(\mathcal{M}, \mathcal{F}, \mu))^{\otimes n} &= L^2(\mathcal{M}^{\otimes n}, \mathcal{F}^{\otimes n}, \mu^{\otimes n}). \end{aligned}$$

with the tensor products $f \otimes x = k \mapsto f(k)x$ and $f_1 \otimes \cdots \otimes f_n = (k_1, \dots, k_n) \mapsto f_1(k_1) \dots f_n(k_n)$. In the case $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ we have for $n \geq 1$

$$(S_n f)(k_1, \dots, k_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(k_{\sigma(1)}, \dots, k_{\sigma(n)}).$$

We note that $f \in S_n(L^2(\mathcal{M}, \mathcal{F}, \mu)^{\otimes n})$ if and only if $f \in L^2(\mathcal{M}^{\otimes n}, \mathcal{F}^{\otimes n}, \mu^{\otimes n})$ and $f(k_1, \dots, k_n) = f(k_{\sigma(1)}, \dots, k_{\sigma(1)})$ for any $\sigma \in \mathcal{S}_n$. Write $\mathcal{H}^{\otimes_s n} = S_n(\mathcal{H}^{\otimes n})$. The bosonic Fock space is defined by

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s n}.$$

where $S_0 = 1$. We will write an element $\psi \in \mathcal{F}(\mathcal{H})$ in terms of its coordinates as $\psi = (\psi^{(n)})$ and define the vacuum $\Omega = (1, 0, 0, \dots)$. Furthermore, for $\mathcal{D} \subset \mathcal{H}$

and $f_1, \dots, f_n \in \mathcal{H}$ we introduce the notation

$$\begin{aligned} S_n(f_1 \otimes \cdots \otimes f_n) &= f_1 \otimes_s \cdots \otimes_s f_n \\ \epsilon(f_i) &= \sum_{n=0}^{\infty} \frac{f_i^{\otimes n}}{\sqrt{n!}} \\ \mathcal{J}(\mathcal{D}) &= \{\Omega\} \cup \{f_1 \otimes_s \cdots \otimes_s f_n \mid f_i \in \mathcal{D}, n \in \mathbb{N}\} \\ \mathcal{L}(D) &= \{\epsilon(f) \mid f \in \mathcal{D}\} \end{aligned}$$

where $f_i^{\otimes 0} = \Omega$. One may prove that if $\mathcal{D} \subset \mathcal{H}$ is dense then $\mathcal{L}(D)$ is a linearly independent total subset of $\mathcal{F}(\mathcal{H})$. From this one easily concludes $\mathcal{J}(\mathcal{D})$ is total.

For $g \in \mathcal{H}$ one defines the annihilation operator $a(g)$ and creation operator $a^\dagger(g)$ on symmetric tensors in $\mathcal{F}(\mathcal{H})$ using $a(g)\Omega = 0, a^\dagger(g)\Omega = g$ and

$$\begin{aligned} a(g)(f_1 \otimes_s \cdots \otimes_s f_n) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \langle g, f_i \rangle f_1 \otimes_s \cdots \otimes_s \widehat{f_i} \otimes_s \cdots \otimes_s f_n \\ a^\dagger(g)(f_1 \otimes_s \cdots \otimes_s f_n) &= \sqrt{n+1} g \otimes_s f_1 \otimes_s \cdots \otimes_s f_n \end{aligned}$$

where $\widehat{f_i}$ means that this element is omitted. One can show that these operators extends to closed operators on $\mathcal{F}(\mathcal{H})$ and that $(a(g))^* = a^\dagger(g)$. Furthermore, we have the canonical commutation relations which states

$$\overline{[a(f), a(g)]} = 0 = \overline{[a^\dagger(f), a^\dagger(g)]} \text{ and } \overline{[a(f), a^\dagger(g)]} = \langle f, g \rangle.$$

One now introduces the selfadjoint field operators

$$\varphi(g) = \overline{a(g) + a^\dagger(g)}.$$

If ω is a selfadjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(\omega)$ then we define the second quantisation of ω to be the selfadjoint operator

$$d\Gamma(\omega) = 0 \oplus \bigoplus_{n=1}^{\infty} \overline{\sum_{k=1}^{n} (1\otimes)^{k-1} \omega (\otimes 1)^{n-k}} \mid_{\mathcal{H}^{\otimes_s n}} . \tag{2.1}$$

If ω is a multiplication operator on $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ we define $\omega_n : \mathcal{M}^n \to \mathbb{R}$ by $\omega_0 = 0$ and $\omega_n(k_1, \dots, k_n) = \omega(k_1) + \cdots + \omega(k_n)$. Then $d\Gamma(\omega)$ acts on elements in $\mathcal{H}^{\otimes_s n}$ as multiplication by $\omega_n(k_1, \dots, k_n) = \omega(k_1) + \cdots + \omega(k_n)$. The number operator is defined as $N = d\Gamma(1)$. Let U be a unitary map from $\mathcal{H}$ to $\mathcal{K}$. Then we define the unitary map

$$\Gamma(U) = 1 \oplus \bigoplus_{n=1}^{\infty} U \otimes \cdots \otimes U \mid_{\mathcal{H}^{\otimes_s n}} .$$

For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we define the operators $d\Gamma^{(n)}(\omega) = d\Gamma(\omega) \mid_{\mathcal{H}^{\otimes_s n}}$ and $\Gamma^{(n)}(U) = \Gamma(U) \mid_{\mathcal{H}^{\otimes_s n}}$. The following lemma is important and well known (see e.g [2]):

Lemma 2.1. *Let $\omega \geq 0$ be selfadjoint and injective. If $g \in \mathcal{D}(\omega^{-1/2})$ then $\varphi(g)$, $a^\dagger(g)$ and $a(g)$ are $d\Gamma(\omega)^{1/2}$ bounded. In particular $\varphi(g)$ is $N^{1/2}$ bounded. We have the following bound*

$$\|\varphi(g)\psi\| \leq 2\|(\omega^{-1/2}+1)g\|\|(d\Gamma(\omega)+1)^{1/2}\psi\|$$

which holds on $\mathcal{D}(d\Gamma(\omega)^{1/2})$. In particular, $\varphi(g)$ is infinitesimally $d\Gamma(\omega)$ bounded. Furthermore, $\sigma(d\Gamma(\omega)+\varphi(g)) = -\|\omega^{-1/2}g\|^2 + \sigma(d\Gamma(\omega))$.

We have the following obvious lemma which is useful for calculations

Lemma 2.2. *Let $f, g \in \mathcal{H}$. Then $\epsilon(g) \in \mathcal{D}(N^n)$ for all $n \geq 0$. Furthermore:*

(1) *$a(g)\epsilon(f) = \langle g, f\rangle \epsilon(f)$ and $\langle \varepsilon(g), \varepsilon(f)\rangle = e^{\langle g,f\rangle}$.*
(2) *If $f \in \mathcal{D}(\omega)$ then $\epsilon(f) \in \mathcal{D}(d\Gamma(\omega))$ and $d\Gamma(\omega)\epsilon(f) = a^\dagger(\omega f)\epsilon(f)$. In particular we find $\langle \epsilon(g), d\Gamma(\omega)\epsilon(f)\rangle = \langle g, \omega f\rangle e^{\langle g,f\rangle}$.*

Let $A \in \mathcal{B}(\mathbb{R}^\nu)$. In this paper we shall mainly encounter spaces of the form

$$\mathcal{H}_A = (\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), 1_A \lambda_\nu)$$

where λ_ν is the Lebesgue measure. Note $\mathcal{H}_A^{\otimes n} = L^2((\mathbb{R}^\nu)^n, \mathcal{B}(\mathbb{R}^\nu)^{\otimes n}, 1_{A^n}\lambda_\nu^{\otimes n})$. We also define

$$\mathcal{CS}_A = \{f \in \mathcal{H}_A \mid \exists R > 0 \text{ such that } 1_{B_R(0)} f = f \ 1_A\lambda_\nu \text{ almost everywhere}\}.$$

which is obviously a dense subspace inside $\mathcal{H}_A$. We will also need the contraction $P_A : \mathcal{H}_{\mathbb{R}^\nu} \to \mathcal{H}_A$ defined by

$$P_A(v) = v$$

$1_A\lambda_\nu$ almost everywhere. Let $\omega : \mathbb{R}^\nu \to \mathbb{R}$ be a measurable map. Then ω_A is defined to be multiplication by ω on the space $\mathcal{H}_A$. Define furthermore $d\Gamma(k_A) = (d\Gamma((k_1)_A), \dots, d\Gamma((k_\nu)_A))$ where $k_i : \mathbb{R}^\nu \to \mathbb{R}$ is projection to the i'th coordinate and let $g^{(n)} : (\mathbb{R}^\nu)^n \to \mathbb{R}^\nu$ be given by $g^{(0)}(0) = 0$ and $g^{(n)}(k) = k_1 + \cdots + k_n$ for $n \geq 1$. Then for $K : \mathbb{R}^\nu \to \mathbb{R}$ we have

$$K(\xi - d\Gamma(k_A)) = \bigoplus_{n=0}^{\infty} K_A(\xi - g^{(n)})$$

where $K_A(\xi - g^{(n)})$ is to be interpreted as the corresponding multiplication operator on $\mathcal{H}_A^{\otimes_s n}$. In case $A = \mathbb{R}^\nu$ we will omit A from the notation.

We shall also encounter vectors of operators. Let $B_1, \dots, B_n$ be operators on a Hilbert space $\mathcal{H}$ and define $B = (B_1, \dots, B_n)$ from $\cap_{i=1}^n \mathcal{D}(B_i)$ into $\mathcal{H}^\nu$ by $B\psi = (B_1\psi, \dots, B_n\psi)$. Note $\mathcal{H}^\nu = \bigoplus_{k=1}^{\nu} \mathcal{H}$ and is also a Hilbert space. For any $k \in \mathbb{R}^\nu$ we define

$$k \cdot B = \sum_{i=1}^{n} k_i B_i.$$

In particular we find for $\psi \in \mathcal{D}(B)$

$$\begin{aligned}\|k \cdot B\psi\|^2 &= \sum_{i,j=1}^{n} \langle k_i B_i \psi, k_j B_j \psi\rangle \leq \sum_{i,j=1}^{n} |k_i||k_j|\|B_i\psi\|\|B_j\psi\| \\ &\leq \sum_{i,j=1}^{n} \frac{1}{2}|k_i|^2\|B_i\psi\|\|B_j\psi\|^2 + \frac{1}{2}|k_j|\|B_i\psi\|^2 = \|k\|^2\|B\psi\|^2 \qquad (2.2)\end{aligned}$$

3. The operator - basic properties and the main result

Fix $K, \omega : \mathbb{R}^\nu \to [0, \infty)$ measurable and let $v \in \mathcal{H}$. Define for $A \in \mathcal{B}(\mathbb{R}^\nu)$ and $\xi \in \mathbb{R}^\nu$ the Hamiltonian

$$H_\mu(\xi, A) = K(\xi - d\Gamma(k_A)) + d\Gamma(\omega_A) + \mu\varphi(v_A)$$

where $v_A = P_A(v)$. We have

Lemma 3.1. *Assume $\omega > 0$ λ_ν almost everywhere, $v \in \mathcal{D}(\omega^{-1/2})$ and $A \in \mathcal{B}(\mathbb{R}^\nu)$. Then $\omega_A \geq 0$ is injective and $v_A \in \mathcal{D}(\omega_A^{-1/2})$. Furthermore, $H_\mu(\xi, A)$ is selfadjoint on $\mathcal{D}(H_0(\xi, A)) = \mathcal{D}(d\Gamma(\omega)) \cap \mathcal{D}(d\Gamma(K(\xi - d\Gamma(k_A)))$ and essentially selfadjoint on any core for $H_0(\xi, A)$. Also, $H_\mu(\xi, A) \geq -\mu^2 \|\omega^{-1/2} v\|$ independent of A and ξ.*

Proof. We know $\{\omega \leq 0\}$ is a λ_ν 0 set and therefore a $1_A \lambda_\nu$ 0 set. Hence $\omega_A \geq 0$ is injective. That $v_A \in \mathcal{D}(\omega_A^{-1/2})$ is obvious as $\omega^{-1/2} v$ is square integrable over $\mathbb{R}^\nu$. For each $n \in \mathbb{N}_0$ we define a map $G_\xi^{(n)} = K(\xi - g^{(n)}) + \omega_n$. and define the selfadjoint operator $B_\xi = \bigoplus_{n=0}^\infty G_\xi^{(n)}$ on $\mathcal{F}(\mathcal{H}_A)$. Using $\max\{K(\xi - g^{(n)}), \omega\} \leq G_\xi^{(n)} = K(\xi - g^{(n)}) + \omega_n$ we note

$$\mathcal{D}(B_\xi) = \mathcal{D}(K_A(\xi - d\Gamma(k))) \cap \mathcal{D}(d\Gamma(\omega)) \text{ and } H_0(\xi, A) = B_\xi.$$

In particular, $H_0(\xi, A)$ is selfadjoint. For $\psi \in \mathcal{D}(H_0(\xi, A))$ we have $\|d\Gamma(\omega)\psi\| \leq \|H_0(\xi, A)\psi\|$ and so we find via Lemma 2.1 and the Kato Rellich theorem that

$$H_\mu(\xi, A) := H_0(\xi, A) + \mu\varphi(v_A)$$

is selfadjoint on $\mathcal{D}(H_0(\xi, A))$ and any core for $H_0(\xi, A)$ is a core for H_μ. Using Lemma 2.1 again we find $H_\mu(\xi, A) \geq 0 - \mu^2 \|\omega^{-1/2} v\|^2 \geq -\mu^2 \|\omega^{-1/2} v\|$. □

Hypothesis 1: We assume

(1) $K \in C^2(\mathbb{R}^\nu, \mathbb{R})$ is non negative and there is $C_K > 0$ such that $\|\nabla K\|^2 \leq C_K(1 + K)$ and $\|D^2 K\| \leq C_K$ where $D^2 K$ is the Hessian of K.
(2) $\omega : \mathbb{R}^\nu \to [0, \infty)$ is continuous and $\omega > 0$ λ_ν almost everywhere.
(3) $v \in \mathcal{D}(\omega^{-1/2})$.

Under these hypothesis we define maps

$$\nabla K(\xi - d\Gamma(k_A)) = (\partial_1 K(\xi - d\Gamma(k_A)), \dots, \partial_\nu K(\xi - d\Gamma(k_A)))$$
$$\Sigma_A(\xi) = \inf(\sigma(H_\mu(\xi, A)))$$

We have the following lemma

Lemma 3.2. *Assume Hypothesis 1. The following holds*

(1) *$\mathcal{D}(K(\xi - d\Gamma(k_A))) \subset \mathcal{D}(\nabla K(\xi - d\Gamma(k_A)))$ and for $\psi \in \mathcal{D}(K(\xi - d\Gamma(k_A)))$ we have $\|\nabla K(\xi - d\Gamma(k_A))\psi\|^2 \leq C_K \|K(\xi - d\Gamma(k_A))\psi\|^2 + C_K \|\psi\|^2$*
(2) *$\mathcal{D}(K(\xi - d\Gamma(k_A)))$ is independent of ξ. On $\mathcal{D}(K(\xi - d\Gamma(k_A)))$ we have*

$$K(\xi + a - d\Gamma(k_A)) = K(\xi - d\Gamma(k_A)) + a \cdot \nabla K(\xi - d\Gamma(k_A)) + E_{\xi,A}(a) \tag{3.1}$$

where $\|E_{\xi,A}(a)\| \leq C_K \|a\|^2$. In particular, $\mathcal{D}(H_\mu(\xi, A))$ is independent of ξ.

(3) *Let* $\psi \in \mathcal{D}(K(\xi - d\Gamma(k_A)))$*. Then*

$$\begin{aligned}&\|K(\xi + a - d\Gamma(k_A)) - K(\xi - d\Gamma(k_A))\psi\|^2\\ &\quad \leq C_K^2\|a\|^2\|K(\xi - d\Gamma(k_A))\psi\|^2+(1 + \|a\|^2)C_K\|a\|^2\|\psi\|^2. \end{aligned}\tag{3.2}$$

Furthermore, $\xi \mapsto H_\mu(\xi, A)\psi$ *is continuous for any* $\psi \in cD(H_\mu(0, A))$ *and* $\xi \mapsto H_\mu(\xi, A)$ *is continuous in norm resolvent sense. In particular, the map* $\xi \mapsto \Sigma_A(\xi)$ *is continuous.*

(4) *Let* $\mathcal{D} \subset \mathcal{CS}_A$ *be a dense subspace. Then* $\mathcal{L}(\mathcal{D})$ *and* $\mathcal{J}(\mathcal{D})$ *span cores for* $H_\mu(\xi, A)$*.*

Proof. To prove (1) we calculate for $\psi \in \mathcal{D}(K(\xi - d\Gamma(k_A)))$

$$\begin{aligned}&\sum_{i=1}^{\nu}\sum_{n=0}^{\infty}\int_{A^n}|\psi^{(n)}(k)\partial_i K(\xi - g^{(n)}(k))|^2 d\lambda_\nu^{\otimes n}\\ &\quad \leq \sum_{n=0}^{\infty}\int_{A^n} C_k|\psi^{(n)}(k)K(\xi - g^{(n)}(k))|^2 d\lambda_\nu^{\otimes n} + C_k\|\psi^{(n)}\|\\ &\quad = C_K\|K(\xi - d\Gamma(k))\psi\|^2+C_K\end{aligned}$$

This proves (1). To prove (2) we use the fundamental theorem of calculus twice and arrive at

$$K(\xi + a - k) = K(\xi - k) + a\cdot\nabla K(\xi - k) + a\cdot\int_0^1\int_0^1 D^2K(\xi + sta - k)adsdt$$

Define $G_a(k) = a\cdot\int_0^1\int_0^1 D^2K(k + sta)adsdt$, and note $|G_{\xi,a}(k)|\leq C_K\|a\|^2$ uniformly in k and ξ. Thus if we define $E_{\xi,A}(a) = G_{\xi,a}(\xi - d\Gamma(k_A))$ we find that $E_{\xi,A}(a)$ is bounded with norm bound $C_K\|a\|^2$. Let $\psi \in \mathcal{D}(K(\xi - d\Gamma(k_A)))$. Then $\psi \in \mathcal{D}(K(\xi - d\Gamma(k_A)) + \langle a, \nabla K(\xi - d\Gamma(k_A))\rangle + E_{\xi,A}(a))$ by part 1. We have the point wise identity:

$$((K(\xi - d\Gamma(k_A)) + a\cdot\nabla K(\xi - d\Gamma(k_A)) + E_{\xi,A}(a))\psi)^{(n)} = K(\xi + a - g^{(n)})\psi^{(n)}$$

showing $K(\xi + a - g^{(n)})\psi^{(n)}$ is square integrable and the sum of squared norms is finite. Hence $\psi \in \mathcal{D}(K(\xi + a - d\Gamma(k_A)))$ and equation (3.1) holds. We have thus proven $\mathcal{D}(K(\xi + a - d\Gamma(k_A))) \subset \mathcal{D}(K(\xi - d\Gamma(k_A)))$ for all $\xi \in \mathbb{R}^\nu$ however using $\xi' = \xi - a$ we find the other inclusion. This proves (2).

To prove (3) we note that equation (3.2) is easily obatined from statements (1) and (2). Using

$$(H_\mu(\xi + a, A) - H_\mu(\xi, A))\psi = (K(\xi + a - d\Gamma(k_A)) - K(\xi - d\Gamma(k_A)))\psi$$

for any $\psi \in \mathcal{D}(H_\mu(\xi, A))$ and equation (3.2) we immediately obtain continuity for $\xi \mapsto H_\mu(\xi, A)\psi$. To prove the statement regarding norm resolvent convergence we calculate using equation (3.2)

$$\begin{aligned}&\|(H_\mu(\xi + a, A) + i)^{-1} - (H_\mu(\xi, A) + i)^{-1}\|^2\\ &\quad \leq C_K\|a\|^2\|K_A(\xi - d\Gamma(k))(H_\mu(\xi, A) + i)^{-1}\|+(1 + \|a\|^2)C_K\|a\|^2\end{aligned}$$

which goes to 0 for a tending to 0. Continuity of $\xi \mapsto \inf(\sigma(H_\mu(\xi, A)))$ now follows from continuity of the spectral calculus and the existence of a ξ-independent lower bound by Lemma 3.1.

It only remains to prove statement (4). By Lemma 3.1 it is enough to check that $\mathcal{J}(\mathcal{D})$ and $\mathcal{L}(\mathcal{D})$ span a core for $H_0(\xi, A)$. Let $f_1, \dots, f_n \in \mathcal{CS}_A$. Pick $R > 0$ such that $1_{B_R(0)} f_i = f_i \, 1_A \lambda_\nu$ almost everywhere for all $i \in \{1, \dots, n\}$ and note that $1_{B_R(0)^n} f_1 \otimes_s \cdots \otimes_s f_n = f_1 \otimes_s \cdots \otimes_s f_n \, 1_{A^n} \lambda_\nu^{\otimes n}$ almost everywhere. Let $C = \sup_{k \in B_R(0)} \omega(k)$. Using the fundamental theorem of calculus we find the following point wise inequality for $k \in B_R(0)^n$:

$$|K(\xi - g^n(k))| = K(\xi) + \|-g^{(n)}(k)\| \|\nabla K(\xi)\| + \|-g^{(n)}(k)\|^2 C_K \leq \widetilde{C}(1 + n^2 R^2)$$

Where $\widetilde{C} = \max\{K(\xi) + \frac{1}{2}\|\nabla K(\xi)\|, (1 + C_K)\}$ and we used that $\|-g^{(n)}(k)\| \leq nR$ for $k \in B_R(0)^n$. We therefore find the following point wise estimates on $B_R(0)^n$:

$$(K(\xi - g^n) + \omega_n)^{2p} |f_1 \otimes_s \cdots \otimes_s f_n| \leq (\widetilde{C}(1 + n^2 R^2) + nC)^{2p} |f_1 \otimes_s \cdots \otimes_s f_n|$$

Integrating yields $f_1 \otimes_s \cdots \otimes_s f_n \in \mathcal{D}(H_0(\xi, A)^p)$ and

$$\|H_0(\xi, A)^p f_1 \otimes_s \cdots \otimes_s f_n\| \leq (\widetilde{C}(1 + n^2 R^2) + nC)^p \|f_1 \otimes_s \cdots \otimes_s f_n\| \tag{3.3}$$

Multiplying by $\frac{1}{p!}$ and summing over p yields a finitie number so $f_1 \otimes_s \cdots \otimes_s f_n$ is analytic for $H_0(\xi)$. Now, Ω is an eigenvector for $H_0(\xi)$ and therefore analytic we see $\mathcal{J}(D)$ is a total set of analytic vectors for $H_0(\xi, A)$ and therefore it spans a core for $H_0(\xi, A)$ by Nelson analytic vector theorem.

By equation (3.3) we see $f^{\otimes n} \in \mathcal{D}(H_0(\xi, A)^p)$ and

$$\begin{aligned} \|H_0(\xi, A)^p f_1^{\otimes n}\|^2 &\leq \|(\widetilde{C}(1 + n^2 R^2) + nC)^{2p} \|f_1\|^{2n} \\ &\leq (\widetilde{C}^{1/2}(1 + nR) + \sqrt{nC})^{4p} \|f_1\|^{2n} \end{aligned}$$

This also holds for $n = 0$ as we in this case obtain $\|H_0(\xi, A)^p \Omega\|^2 = \sqrt{K(\xi)}^{4p} \leq (\widetilde{C}^{1/2})^{4p}$. Multiplying by $\frac{1}{n!}$ and summing over n yields a finitie number so $\epsilon(f_1) \in \mathcal{D}(H_0(\xi, A)^p)$ for all p. Now

$$\begin{aligned} \sum_{p=0}^{\infty} \frac{1}{(2p)!} \|H_0(\xi, A)^p \epsilon(f_1)\| &\leq \sum_{p=0}^{\infty} \frac{1}{(2p)!} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \|H_0(\xi, A)^p f_1^{\otimes n}\| \\ &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{p=0}^{\infty} \frac{1}{(2p)!} (\widetilde{C}^{1/2}(1 + nR) + \sqrt{nC})^{2p} \|f_1\|^n \\ &\leq \sum_{n=0}^{\infty} \frac{\|f\|^n e^{(\widetilde{C}^{1/2}(1+nR) + \sqrt{nC})}}{\sqrt{n!}} < \infty \end{aligned}$$

Thus $\varepsilon(f_1)$ is semi analytic for $H_0(\xi)$. This implies $\{\epsilon(f) \mid f \in \mathcal{D}\}$ spans a dense subspace of semi analytic vectors for $H_0(\xi, A)$, which is a core by the Masson-McClary theorem. □

Hypothesis 2: We assume

(1) K, ω and v are rotation invariant. Furthermore $k \mapsto e^{-tK(k)}$ is positive definite for all t.
(2) ω is sub-additive, $\omega(x_1) < \omega(x_2)$ of $|x_1| < |x_2|$. Also $C_\omega = \lim_{k\to 0}\|k\|^{-1}\omega(k)$ exists and is strictly positive.
(3) $v \notin \mathcal{D}(\omega_{\mathbb{R}^\nu}^{-1})$

The physical choices for the 3-dimensional Nelson model are $\omega(k) = |k|$, $K \in \{k \mapsto |k|^2, k \mapsto \sqrt{|k|^2 + m} - m\}$ and $v = \omega^{-1/2}\chi$ where $\chi : \mathbb{R}^\nu \to \mathbb{R}$ is a spherically symmetric ultra violet cutoff. It is well known that Hypothesis 1 and 2 are fulfilled in this case. We can now state the main theorem of this paper:

Theorem 3.3. *Assume Hypothesis 1 and 2 along with $\nu \geq 3$. Then $H_\mu(\xi)$ has no ground states for any ξ and $\mu \neq 0$.*

4. Proof of Theorem 3.3

We start with proving series of lemmas which we shall need. We work under Hypothesis 1 and 2. The first Lemma is known and we only sketch the proof.

Lemma 4.1. *The map $\xi \mapsto \Sigma(\xi)$ is rotation invariant.*

Proof L. et O denote any orthogonal matrix with dimensions ν. Define the unitary map $\widehat{O} : \mathcal{H} \to \mathcal{H}$ by $(\widehat{O}f)(k) = f(Ok)$ λ_ν almost everywhere. Let $f, g \in \mathcal{CS}$ and note $\widehat{O}f, \widehat{O}g \in \mathcal{CS}$. In particular, $\Gamma(\widehat{O})\epsilon(f) = \epsilon(\widehat{O}f) \in \mathcal{D}(H_\mu(\xi))$ for all ξ. One now easily calculates using Lemma 2.2

$$\langle \epsilon(g), \Gamma(\widehat{O})^* H_\mu(\xi)\Gamma(\widehat{O})\epsilon(f)\rangle = \langle \epsilon(g), H_\mu(O\xi)\epsilon(f)\rangle.$$

Now $\mathcal{L}(\mathcal{CS})$ is total so we find $H_\mu(O\xi) = \Gamma(\widehat{O})^* H_\mu(\xi)\Gamma(\widehat{O})$ on $\mathcal{L}(\mathcal{CS})$ which is spans a core for $H_\mu(O\xi)$ and so $\Gamma(\widehat{O})^* H_\mu(\xi)\Gamma(\widehat{O}) = H_\mu(O\xi)$. □

For any $x \in \mathbb{R}^n \backslash \{0\}$ we write $\widehat{x} = \|x\|^{-1}x$. The next small lemma is basically spherical coordinates.

Lemma 4.2. *Let $U \subset \mathbb{R}^\nu$ be invariant under multiplication by elements in $(0, \infty)$. Then for any positive, rotation invariant, measurable map f we have*

$$\int_U f(x) d\lambda_\nu(x) = n\lambda_\nu(U \cap B_1(0)) \int_0^\infty f(ke_1)k^{\nu-1} d\lambda_1(k)$$

where e_1 is the first standard basis vector. If U is open then $\lambda_\nu(U \cap B_1(0)) \neq 0$.

Proof. Consider the map $g : \mathbb{R}^\nu \to [0, \infty)$ given by $g(x) = |x|$. Define the transformed measure on $([0, \infty), \mathcal{B}([0, \infty)))$ by

$$\mu = (1_U \lambda_\nu) \circ g^{-1}$$

The transformation theorem implies

$$\mu([0, a]) = \lambda_\nu(a(U \cap B_1(0))) = \nu\lambda_\nu(\widetilde{U} \cap B_1(0)) \int_0^a k^{\nu-1} d\lambda_1(k)$$

By uniqueness of measures (see [13, chapter 5]) we find that μ has density $\nu\lambda_\nu(\widetilde{U}\cap B_1(0))r^{\nu-1}$ with respect to λ_1. Using that $f(g(x)e_1)=f(x)$ we find

$$\lambda_\nu(U\cap B_1(0))\nu\int_0^\infty f(ke_1)k^{\nu-1}d\lambda_1(k)=\int_0^\infty f(ke_1)d\mu(k)=\int_U f(x)d\lambda_\nu(x)$$

as desired. If U is not empty we can pick $k\in U$. If $\|k\|<1$ then $k\in U\cap B_1(0)$. If $\|k\|\geq 1$ then $\frac{1}{2\|k\|}k\in U\cap B_1(0)$ so $U\cap B_1(0)\neq\emptyset$. Hence if U is open and not empty we find $U\cap B_1(0)$ is open and non empty so $\lambda_\nu(U\cap B_1(0))\neq 0$. □

Lemma 4.3. *Σ has a global minimum at $\xi=0$.*

Proof. This result was proven in the paper [7] under the extra assumption that there is $m>0$ such that $\omega\geq m$. The proof used in [7] does however generalise to our setting. Another way to derive it to consider $\omega_n=1/n+\omega$ and let

$$H_n(\xi)=K(\xi-d\Gamma(k))+d\Gamma(\omega_n)+\mu\varphi(v)$$

Write $\Sigma_n(\xi)=\inf(\sigma(H_n(\xi)))$. Now $\operatorname{Span}(\mathcal{J}(\mathcal{CS}))$ is a common core for the $H_n(\xi)$ and $H(\xi)$ by Lemma 3.2 and for ψ in this set we see

$$\lim_{n\to\infty}(H_n(\xi)-H(\xi))\psi=\lim_{n\to\infty}\frac{1}{n}N\psi=0$$

implying $H_n(\xi)$ converges to $H(\xi)$ in strong resolvent sense by [11, Theorem VIII.25]. For any $\varepsilon>0$ we may pick $\psi\in\operatorname{Span}(\mathcal{J}(\mathcal{CS}))$ such that

$$\Sigma_n(\xi)+\varepsilon\geq\langle\psi,H_n(\xi)\psi\rangle\geq\langle\psi,H(\xi)\psi\rangle\geq\Sigma(\xi)$$

In particular, $\Sigma_n(\xi)\geq\Sigma(\xi)$ for all $n\in\mathbb{N}$. By [11, Theorem VIII.24] we find a sequence $\{\lambda_n\}_{n=1}^\infty$ converging to $\Sigma(\xi)$ with $\lambda_n\in\sigma(H_n(\xi))$.

Hence $0\leq\Sigma_n(\xi)-\Sigma(\xi)\leq\lambda_n-\Sigma(\xi)$ so $\Sigma_n(\xi)$ converges to $\Sigma(\xi)$. Now Σ_n has a global minimum at $\xi=0$ and so

$$\Sigma(0)=\lim_{n\to\infty}\Sigma_n(0)\leq\lim_{n\to\infty}\Sigma_n(\xi)=\Sigma(\xi)$$

finishing the proof. □

For every $\xi\in\mathbb{R}^n$ and $0<\varepsilon<1$ we define

$$S_\varepsilon(\xi)=\{k\in\mathbb{R}^\nu\backslash\{0\}\mid|\widehat{k}\cdot\xi|<(1-\varepsilon)\|\xi\|\}.$$

where $\widehat{k}=k/\|k\|$. The following Lemma is essential:

Lemma 4.4. *Let $\xi\in\mathbb{R}^\nu$. Then*

(1) $\Sigma(\xi-k)+\omega(k)>\Sigma(\xi)$ *if* $k\notin\mathbb{R}\xi$.
(2) *For any $1>\varepsilon>0$ there exists $D:=D(\varepsilon,\xi)<1$ and $r:=r(\varepsilon,\xi)>0$ such that for all $k\in B_r(0)\cap S_\varepsilon(\xi)$ we have*

$$\Sigma(\xi-k)-\Sigma(\xi)\geq-D\omega(k)$$

Proof. We start by proving (1). Assume $\xi = 0$ and $k \neq 0$. If $\omega(k) = 0$ then by Hypothesis 2 we have $\omega(k') < 0$ for all $k' \in B_{|k|}(0)$ which contradicts Hypothesis 1. So if $\xi = 0$ the result is trivial since $\Sigma(\xi - k) - \Sigma(\xi) > -\frac{1}{2}\omega(k) > -\omega(k)$ holds for all $k \neq 0$ by Lemma 4.3.

Assume now $\xi \neq 0$ and let $k \notin \mathbb{R}\xi$. By rotation invariance of Σ (Lemma 4.1) we may calculate

$$\Sigma(\xi - k) - \Sigma(\xi) = \Sigma(\xi - k) - \Sigma\left(\frac{\|\xi\|}{\|\xi - k\|}(\xi - k)\right) \tag{4.1}$$

By Lemma A.5 we have $\Sigma(\xi-k)+\omega(k) \in \sigma(H(\xi))$ and so $\Sigma(\xi) \leq \Sigma(\xi-k)+\omega(k)$ implying

$$\begin{aligned}\Sigma(\xi - k) - \Sigma\left(\frac{\|\xi\|}{\|\xi - k\|}(\xi - k)\right) &\geq -\omega\left(\frac{\|\xi\|}{\|\xi - k\|}(\xi - k) - \xi - k\right)\\ &= -\omega\left((\|\xi\| - \|\xi - k\|)\frac{\xi - k}{\|\xi - k\|}\right)\end{aligned} \tag{4.2}$$

Now $|\|\xi\| - \|\xi - k\|| \leq \|k\|$ by the reverse triangle inequality. If equality holds we have either $\|\xi\| = \|\xi - k\| + \|k\|$ or $\|\xi - k\| = \|\xi\| + \|-k\|$. By [14, Page 9] either k and $\xi - k$ are linearly dependent or k and ξ are linearly dependent. In any case ξ and k are linearly independent which (as $\xi \neq 0$) implies $k = a\xi$ for some $a \in \mathbb{R}$. So since $k \notin \mathbb{R}\xi$ we find $|\|\xi\| - \|\xi - k\|| < \|k\|$ and so

$$\omega\left((\|\xi\| - \|\xi - k\|)\frac{\xi - k}{\|\xi - k\|}\right) < \omega(k)$$

by Hypothesis 2. Combining this and equations (4.1) and (4.2) we find statement (1). To prove statement (2) we continue to calculate for $k \in S_\varepsilon(\xi)$ (which is disjoint from $\mathbb{R}\xi$)

$$|\|\xi - k\| - \|\xi\|| = \left|\frac{\|\xi - k\|^2 - \|\xi\|^2}{\|\xi - k\| + \|\xi\|}\right| = |k|\left|\frac{-\xi \cdot \widehat{k} + \|k\|}{\|\xi - k\| + \|\xi\|}\right| \leq |k|\left(1 - \varepsilon + \frac{\|k\|}{\|\xi\|}\right) \tag{4.3}$$

Pick n such that $D := (1 + 1/n)(1 - 1/n)^{-1}(1 - \varepsilon/2) < 1$ and $R > 0$ such that

$$C_\omega(1 - 1/n)\|k\| \leq \omega(k) \leq C_\omega\omega(1 + 1/n)\|k\| \tag{4.4}$$

for all $k \in B_R(0)$. Pick $r = \min\{\frac{\|\xi|\varepsilon}{2}, R\}$. Using equations (4.1), (4.2), (4.3) and (4.4) we find

$$\Sigma(\xi - k) - \Sigma(\xi) \geq -C(1 + 1/n)\left(1 - \varepsilon + \frac{|k|}{|\xi|}\right)|k| \geq -D\omega(k)$$

for $k \in B_r(0) \cap S_\varepsilon(\xi)$. □

The following lemma is well known see e.g. [4].

Lemma 4.5. *Define $A = \{v \neq 0\}$. Assume $H_\mu(\xi, A)$ has a ground state for some $\mu \neq 0$ and $\xi \in \mathbb{R}^\nu$. Then the corresponding eigenspace is non degenerate.*

We will now sharpen this result.

Lemma 4.6. *Assume $H_\mu(\xi)$ has a ground state for some $\mu \neq 0$ and $\xi \in \mathbb{R}^\nu$. Then the corresponding eigenspace is non degenerate if $\nu \geq 2$.*

Proof. Define $A = \{v \neq 0\}$. By Lemma A.3 there is a unitary map

$$U : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}_A) \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^c}^{\otimes_s n}$$

such that

$$UH_\mu(\xi)U^* = H_\mu(\xi, A) \oplus \bigoplus_{n=1}^{\infty} H_{n,\mu}(\xi, A) \mid_{\mathcal{F}(\mathcal{H}_A)\otimes\mathcal{H}_{A^c}^{\otimes_s n}} \tag{4.5}$$

for all $\xi \in \mathbb{R}^\nu$ where

$$H_{n,\mu}(\xi, A) = \int_{(A^c)^n}^{\oplus} H_\mu(\xi - k_1 - \cdots - k_n, A) + \omega(k_1) + \cdots + \omega(k_n) d\lambda_\nu^{\otimes n}(k)$$

Let ψ be any ground state for $H_{\mathbb{R}^\nu}(\xi)$. We prove $U\psi = (\widetilde{\psi}^{(0)}, 0, 0, \dots)$. Write $U\psi = (\widetilde{\psi}^{(n)})$ and assume towards contradiction that $\widetilde{\psi}^{(n)} \neq 0$ for some $n \geq 1$. Then $\widetilde{\psi}^{(n)}$ is an eigenvector for $H_{n,A}(\xi)$ corresponding to the eigenvalue $\Sigma(\xi)$. The spectral projection of $H_{n,A}(\xi)$ onto $\Sigma(\xi)$ is given by

$$\int_{(A^c)^n}^{\oplus} 1_{\{\Sigma(\xi)\}}(H(\xi - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n)) d\lambda_\nu^{\otimes n}(k) \neq 0.$$

Hence $\Sigma(\xi)$ is an eigenvalue for $H_\mu(\xi - k_1 - \cdots - k_n, A) + \omega(k_1) + \cdots + \omega(k_n)$ on a set of positive $\lambda_\nu^{\otimes n}$ measure. Sub-additivity of ω along with Lemmas A.3 and A.5 gives

$$\begin{aligned}\Sigma(\xi) &\geq \Sigma_A(\xi - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n)\\ &\geq \Sigma(\xi - k_1 - \cdots - k_n) + \omega(k_1 + \cdots + k_n) \geq \Sigma(\xi)\end{aligned}$$

most hold on a set of positive $\lambda_\nu^{\otimes n}$ measure. By Lemma 4.4 we se that this can only hold for $k \in (\mathbb{R}^\nu)^n$ with $k_1 + \cdots + k_n \in \mathrm{Span}(\xi)$. But the rank theorem implies that the set of k satisfying this is a subspace of $(\mathbb{R}^\nu)^n$ of dimension $\nu n - (\nu - 1) < \nu n$. However such a subspace must have λ_ν measure 0 which is a contradiction.

We now finish the proof as follows. Assume ψ_1, ψ_2 are orthogonal eigenvectors corresponding to the eigenvalue $\Sigma(\xi)$. Then $U\psi_i = (\widetilde{\psi_i}, 0, 0, \dots)$. Now U preserves the inner product so $\widetilde{\psi_1}$ and $\widetilde{\psi_2}$ are orthogonal eigenvectors for $H_\mu(\xi, A)$ corresponding to the eigenvalue $\Sigma(\xi)$ so in particular $\Sigma(\xi) \geq \Sigma_A(\xi)$. By equation (4.5) we conclude that $\Sigma(\xi) = \Sigma_A(\xi)$ and therefore $H_\mu(\xi, A)$ has two orthogonal ground states. This is a contradiction with Lemma 4.5. $\square$

The next two Lemmas are an adapted version of the corresponding ones found in [8]. For $\xi \in \mathbb{R}^\nu$ and $k \neq 0$ we define

$$\begin{aligned}Q_0(k, \xi) &= \omega(k)(H(\xi) - \Sigma(\xi) + \omega(k))^{-1}\\ P_0(\xi) &= 1_{\Sigma(\xi)}(H(\xi))\end{aligned}$$

Lemma 4.7. *Fix $\xi \in \mathbb{R}^\nu$ and $R > 0$. Then $\widehat{k} \cdot \nabla K(\xi - d\Gamma(k))Q_0(k,\xi)$ is uniformly bounded for k in $B(0,R)\backslash\{0\}$. We also have*

$$s - \lim_{k \to 0} \widehat{k} \cdot \nabla K(\xi - d\Gamma(k))Q_0(k,\xi)(1 - P_0(\xi)) = 0 \tag{4.6}$$

Proof. Note $\widehat{k} \cdot \nabla K(\xi - d\Gamma(k))Q_0(k)$ is bounded for $k \neq 0$ by the closed graph theorem and Lemma 3.2. For $\psi \in \mathcal{F}(\mathcal{H})$ we find by equation (2.2) that

$$\|\widehat{k} \cdot \nabla K(\xi - d\Gamma(k))Q_0(k,\xi)\psi\|^2 \leq \sum_{i=1}^{\nu} \|\partial_i K(\xi - d\Gamma(k))Q_0(k,\xi)\psi\|^2$$

so it is enough to see $\partial_i K(\xi - d\Gamma(k))Q_0(k,\xi)$ is uniformly bounded on $B(0,R)\backslash\{0\}$ for any $R > 0$ and converges strongly to $\partial_i K(\xi - d\Gamma(k))P_0(\xi)$. We have

$$\begin{aligned}\partial_i K(\xi - d\Gamma(k))Q_0(k,\xi) &= \partial_i K(\xi - d\Gamma(k)) \frac{\omega(k)}{H(\xi) - \Sigma(\xi) + \omega(k) + 1} \\ &+ \partial_i K(\xi - d\Gamma(k)) \frac{1}{H(\xi) - \Sigma(\xi) + \omega(k) + 1} Q_0(k,\xi)\end{aligned}$$

Now ω is continous and goes to 0 as k tends to 0 so $Q_0(k,\xi)$ goes strongly to $P_0(\xi)$. Hence it is enough to see $\partial_i K(\xi - d\Gamma(k))(H(\xi) - \Sigma(\xi) + \omega(k) + 1)^{-1}$ is uniformly bounded in k and converges to $\partial_i K(\xi - d\Gamma(k))(H(\xi) - \Sigma(\xi) + 1)^{-1}$ in norm. But this is obvious from the equality

$$\begin{aligned}\partial_i K(\xi - d\Gamma(k)) \frac{1}{H(\xi) - \Sigma(\xi) + \omega(k) + 1} &= \partial_i K(\xi - d\Gamma(k)) \frac{1}{H_\mu(\xi) - \Sigma(\xi) + 1} \\ + \partial_i K(\xi - d\Gamma(k)) &\frac{1}{H_\mu(\xi) - \Sigma(\xi) + 1} \frac{\omega(k)}{H_\mu(\xi) - \Sigma(\xi) + 1 + \omega(k)}\end{aligned}$$

because the first term is constant and the other term is uniformly bounded and goes to 0. $\square$

For $\xi \in \mathbb{R}^\nu$ and $k \notin \mathbb{R}\xi$ we may by Lemma 4.4 define

$$Q(k,\xi) = \omega(k)(H(\xi - k) - \Sigma(\xi) + \omega(k))^{-1}$$

Lemma 4.8. *Fix $\xi \in \mathbb{R}^\nu$. There is a vector $v(\xi) \in \mathbb{R}^\nu$ such that*

$$P_0(\xi)\widehat{k} \cdot \nabla K(\xi - d\Gamma(k))P_0(\xi) = \widehat{k} \cdot v(\xi)P_0(\xi)$$

for any $k \in \mathbb{R}^\nu\backslash\{0\}$. Pick $0 < \varepsilon < 1$ such that $\widehat{k} \cdot C_\omega v(\xi) < \frac{1}{2}$ for all $k \in S_\varepsilon(C_\omega v(\xi))$. Define

$$\widetilde{S}_\varepsilon(\xi) = S_\varepsilon(\xi) \cap S_\varepsilon(C_\omega v(\xi)).$$

If $\nu \geq 3$ then S_ε is open, non-empty and invariant under positive scalings. Furthermore,

$$w - \lim_{k \to 0, k \in \widetilde{S}_\varepsilon(\xi)} Q(k,\xi) - (1 - C_\omega \widehat{k} \cdot v(\xi))^{-1} P_0(\xi) = 0. \tag{4.7}$$

Proof. As ξ is fixed in this proof it will be omitted from the nation of Q, Q_0, B and P_0. If $P_0 = 0$ we can pick $v(\xi) = 0$. If $P_0(\xi) = 0$ then is has dimension 1 by Lemma 4.6 and is spanned by a vector $\psi \in \mathcal{D}(H(\xi))$. Using $P_0 = |\psi\rangle\langle\psi|$ we find that $v(\xi)_i = \langle\psi, \partial_i K(\xi - d\Gamma(k))\psi\rangle$ does the trick. Furthermore, S_ε is obviously open and invariant under positive scaling since this holds for $S_\varepsilon(\xi)$ and $S_\varepsilon(C_\omega v(\xi))$. Furthermore any non-zero vector which is orthogonal to ξ and $v(\xi)$ is in S_ε and such vector will always exist if $\nu \geq 3$.

It remains only to prove equation 4.7. By Lemma 4.4 we may pick $R(\xi, \varepsilon) > 0$ such that for $k \in \widetilde{S}_\varepsilon(\xi) \cap B_{R(\xi,\varepsilon)}(0)$ we have

$$\Sigma(k - \xi) - \Sigma(\xi) + \omega(k) \geq (1 - A(\xi, \varepsilon))\omega(k)$$

with $D(\xi, \varepsilon) < 1$. Hence we find

$$\|Q(k)\| \leq (1 - D(\xi, \varepsilon))^{-1} \quad \forall k \in \widetilde{S}_\varepsilon(\xi) \cap B_{R(\xi,\varepsilon)}(0) \tag{4.8}$$

Using Lemma 3.2 we may calculate for $k \in \widetilde{S}_\varepsilon(\xi)$:

$$\begin{aligned} Q(k) &= Q_0(k) + \omega(k)^{-1} Q_0(k)(H(\xi) - H(\xi - k))Q(k) && (4.9)\\ &= \frac{|k|}{\omega(k)} Q_0(k)(\widehat{k} \cdot \nabla K(\xi - d\Gamma(\omega)) - |k|^{-1} E_\xi(-k))Q(k) && (4.10)\\ &= Q_0(k) + \frac{|k|}{\omega(k)} Q_0(k)(\widehat{k} \cdot \nabla K(\xi - d\Gamma(\omega)))Q(k) + o_1(k) && (4.11)\end{aligned}$$

where $o_1(k) := -Q_0(k)\omega(k)^{-1} E_\xi(-k)Q(k)$. We also have

$$\begin{aligned} Q(k) &= Q_0(k) + \omega(k)^{-1} Q(k)(H(\xi) - H(\xi - k))Q_0(k) && (4.12)\\ &= Q_0(k) + \frac{|k|}{\omega(k)} Q(k)(\widehat{k} \cdot \nabla K(\xi - d\Gamma(\omega)))Q_0(k) + o_2(k) && (4.13)\end{aligned}$$

where $o_2(k) := -Q_0(k)\omega(k)^{-1} E_\xi(-k)Q(k)$. Note $o_i(k)$ goes to 0 in norm for k tending to 0 in $\widetilde{S}_\varepsilon(\xi)$ by equation (4.8), Lemma 3.2 and the uniform bound $\|Q_0(k)\| \leq 1$. Inserting equation (4.13) into equation (4.11) we find

$$\begin{aligned} Q(k) = Q_0(k) &+ \frac{|k|}{\omega(k)} Q_0(k)(\widehat{k} \cdot \nabla K(\xi - d\Gamma(\omega))Q_0(k) && (4.14)\\ &+ \frac{|k|^2}{\omega(k)^2} Q_0(k)(\widehat{k} \cdot \nabla K(\xi - d\Gamma(\omega)))Q(k)(\widehat{k} \cdot \nabla K(\xi - d\Gamma(\omega)))Q_0(k) + o(k) \end{aligned}$$

Where

$$\begin{aligned} o(k) &= Q_0(k)(\widehat{k} \cdot \nabla K(\xi - d\Gamma(\omega)))o_2(k) + o_1(k)\\ &= -\frac{|k|}{\omega(k)} Q_0(k)(\widehat{k} \cdot \nabla K(\xi - d\Gamma(\omega)))Q_0(k)|k|^{-1} E_\xi(-k)Q(k) + o_1(k) \end{aligned}$$

Note $o(k)$ goes to 0 in norm for k tending to 0 in $\widetilde{S}_\varepsilon(\xi)$ by equation (4.8), Lemmas 3.2 and 4.7, the uniform bound $\|Q_0(k)\| \leq 1$ and the fact that $|k|\omega(k)^{-1}$ has a limit for k tending to 0. Using equation (4.14) and appealing to the limit found in Lemma 4.7 along with the uniform bounds in Lemma 4.7 and equation (4.8) we now see $(1-P_0)Q(k)$ and $Q(k)(1-P_0)$ goes to 0 weakly for k tending to 0 inside $\widetilde{S}_\varepsilon(\xi)$. Hence we find

$$w - \lim_{k\to 0, k\in \widetilde{S}_\varepsilon(\xi)} Q(k) - P_0Q(k)P_0 = 0. \tag{4.15}$$

From equation (4.11) we find

$$\begin{aligned} P_0Q(k)P_0 &= P_0Q_0(k)P_0 + \frac{|k|}{\omega(k)} P_0Q_0(k)(\nabla K(\xi - d\Gamma(\omega))Q(k)P_0 + P_0 o_1(k)P_0 \\ &= P_0 + \frac{|k|}{\omega(k)} P_0(\widehat{k}\cdot \nabla K(\xi - d\Gamma(\omega)))(1-P_0)Q(k)P_0 \\ &+ \left(\frac{|k|}{\omega(k)} - C_\omega\right)\widehat{k}\cdot v(\xi)P_0Q(k)P_0 + C_\omega \widehat{k}\cdot v(\xi)P_0Q(k)P_0 + P_0o_1(k)P_0 \end{aligned}$$

Write $D_k = (1 - C_\omega \widehat{k}\cdot v(\xi))^{-1}$ and that for $k \in \widetilde{S}_\varepsilon(\xi)$ we have $|D_k| \leq 2$. A little algebra yields

$$\begin{aligned} P_0Q(k)P_0 - D_kP_0 = D_k \frac{|k|}{\omega(k)} P_0(\widehat{k}\cdot \nabla K(\xi - d\Gamma(\omega)))(1-P_0)Q(k)P_0 \\ + D_k\left(\frac{|k|}{\omega(k)} - C_\omega\right)\widehat{k}\cdot v(\xi)P_0Q(k)P_0 + D_kP_0o_1(k)P_0 \end{aligned}$$

The second and third term converges to 0 in norm since for k tending to 0 inside $\widetilde{S}_\varepsilon(\xi)$ since D_k and $\widehat{k}\cdot v(\xi)P_0Q(k)P_0$ are uniformly bounded by equation (4.8) and $o_1(k)$ converges to 0 in norm since for k tending to 0 inside $\widetilde{S}_\varepsilon(\xi)$. Sandwiching the first term with two vectors $\phi,\psi \in \mathcal{F}(\mathcal{H})$ we find

$$D_k\frac{|k|}{\omega(k)}\sum_{i=1}^n \widehat{k}_i \langle \partial_i K(\xi - d\Gamma(k))P_0\psi, (1-P_0)Q(k)P_0\phi\rangle$$

Now $\langle \partial_i K(\xi - d\Gamma(k))P_0\psi, (1-P_0)Q(k)P_0\phi\rangle$ converges to 0 for k going to 0 inside $\widetilde{S}_\varepsilon(\xi)$ by equation (4.15) and $\frac{|k|}{\omega(k)}\widehat{k}_i$ remains bounded as k goes to 0. Therefore first term goes weakly to 0 for k going to 0 inside $\widetilde{S}_\varepsilon(\xi)$. □

Proof (Theorem 3.3). Fix notation from Lemma 4.8. Assume that a ground state ψ exist and pick $\eta \in \mathcal{D}(N^{1/2})$ such that $\langle \psi, \eta\rangle > \frac{1}{2}$. Then by Lemma B.14 in Appendix B we have the pull through formula

$$\langle \eta, A_1\psi(k)\rangle = \mu \frac{v(k)}{\omega(k)}\langle \eta, Q(k)\psi\rangle.$$

Now

$$\lim_{k\to 0, k\in \widetilde{S}_\varepsilon(\xi)} \langle \eta, Q(k)\psi\rangle - (1 - C_\omega \widehat{k}\cdot v(\xi))^{-1}\langle \eta, \psi\rangle = 0$$

and since $(1 - C_\omega \widehat{k} \cdot v(\xi))^{-1} \langle \eta, \psi \rangle$ is uniformly bounded from below in $\widetilde{S}_\varepsilon(\xi)$ by $\frac{1}{2}$ we find that there is $R > 0$ such that

$$|\langle \eta, A_1 \psi(k) \rangle|^2 \geq \frac{\mu^2}{16} \frac{|v(k)|^2}{\omega(k)^2}$$

for all $k \in \widetilde{S}_\varepsilon(\xi) \cap B_R(0)$. Using Hypothesis 1 and 2 we see $\omega(Re_1)^2 > 0$ because if that was not true then $\omega \leq 0$ on $B_R(0)$ which is a contradiction. Hence we find

$$\infty = \int_{\mathbb{R}^\nu} \frac{|v(k)|^2}{\omega(k)^2} d\lambda_\nu \leq \frac{1}{\omega(Re_1)^2} \int_{B_R(0)^c} |v(k)|^2 d\lambda_\nu + \int_{B_R(0)} \frac{|v(k)|^2}{\omega(k)^2} d\lambda_\nu$$

as $v \in \mathcal{H}$ we find that the integral of $\omega(k)^{-2}|v(k)|^2$ over $B_R(0)$ must be infinite. Using Lemma 4.2 we find

$$\infty = \int_{B_R(0)} \frac{|v(k)|^2}{\omega(k)^2} d\lambda_\nu = \lambda_\nu(B_1(0)) \int_0^\infty 1_{B_R(0)}(xe_1) \frac{|v(ke_1)|^2}{\omega(ke_1)^2} k^{\nu-1} d\lambda_1(k)$$

as $\lambda_\nu(B_1(0))$ we see that the latter integral must be infinite. Furthermore since $\widetilde{S}_\varepsilon(\xi)$ is open and not empty we have

$$\begin{aligned}\int_{\widetilde{S}_\varepsilon(\xi) \cap B_R(0)} \frac{|v(k)|^2}{\omega(k)^2} d\lambda_\nu &= \nu \lambda_\nu(\widetilde{S}_\varepsilon(\xi) \cap B_1(0)) \int_0^\infty 1_{B_R(0)}(xe_1) \frac{|v(ke_1)|^2}{\omega(ke_1)^2} k^{\nu-1} d\lambda_1 \\ &= \infty\end{aligned}$$

by Lemma 4.2 so $|\langle \eta, A_1 \psi(k) \rangle|^2$ is not integrable. On the other hand we find

$$\begin{aligned}|\langle \eta, A_1 \psi(k) \rangle|^2 &\leq \|(N+1)^{1/2}\eta\|^2 \|(N+1)^{-1/2} A_1 \psi(k)\|^2 \\ &= \|(N+1)^{1/2}\eta\|^2 \sum_{i=1}^\infty \int_{\mathbb{R}^{(n-1)\nu}} |\psi^{(n)}(k, k_1, \dots, k_{n-1})|^2 d\lambda_\nu^{\otimes n-1}(k_1, \dots, k_{n-1})\end{aligned}$$

which is integrable with integral $\|(N+1)^{1/2}\eta\|^2 \|\psi\|^2$ by definition of the Fock space norm. This is the desired contradiction. □

A. Partitions of unity and the essential spectrum.

In this section we prove a few technical ingredients. Hypothesis 1 will be assumed throughout this section. Define $V_A : \mathcal{H} \to \mathcal{H}_A \oplus \mathcal{H}_{A^c}$ by

$$V_A(f) = (P_A f, P_{A^c} f).$$

Then V_A is unitary with $V_A^*(f, g) = f 1_A + g 1_{A^c}$ λ^ν almost everywhere. The following Lemma can be found in e.g. [9]:

Lemma A.1. *There is a unique isomorphism $U : \mathcal{F}(\mathcal{H}_A \oplus \mathcal{H}_{A^c}) \to \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{F}(\mathcal{H}_{A^c})$ with the property that $U(\epsilon(f_1 \oplus f_2)) = \epsilon(f_1) \otimes \epsilon(f_2)$.*

The following Lemma is obvious

Lemma A.2. *There is a unique isomorphism*

$$U : \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{F}(\mathcal{H}_{A^c}) \to \mathcal{F}(\mathcal{H}_A) \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^c}^{\otimes_s n}$$

such that

$$U(w \otimes \{\psi^{(n)}\}_{n=0}^{\infty}) = \psi^{(0)} w \oplus \bigoplus_{n=1}^{\infty} w \otimes \psi^{(n)}.$$

Note that we may identify

$$\mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^c}^{\otimes_s n} = (1 \otimes S_n) L^2(\mathbb{R}^{n\nu}, \mathcal{B}(\mathbb{R}^{n\nu}), 1_{(A^c)^n} \lambda_{n\nu}, \mathcal{F}(\mathcal{H}_A))$$

where $1 \otimes S_n$ acts on $L^2(\mathbb{R}^{n\nu}, \mathcal{B}(\mathbb{R}^{n\nu}), \lambda_{n\nu}, \mathcal{F}(\mathcal{H}_A))$ like

$$(S_n f)(k_1, \dots, k_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(k_{\sigma(1)}, \dots, k_{\sigma(n)}).$$

Now we define

$$H_A^{(n)}(\xi, k_1, \dots, k_n) = H_\mu(\xi - k_1 - \cdots - k_n, A) + \omega(k_1) + \cdots + \omega(k_n)$$

which is strongly resolvent measurable in $(k_1, \dots, k_n) \in (A^c)^n$ since $\xi \mapsto H(\xi)$ is strong resolvent measurable by Lemma 3.2. In particular

$$H_{n,A}(\xi) = \oint_{(A^c)^n} H^{(n)}(\xi, k_1, \dots, k_n) d\lambda^{n\nu}(k_1, \dots, k_n)$$

defines a selfadjoint operator on $L^2(\mathbb{R}^{n\nu}, \mathcal{B}(\mathbb{R}^{n\nu}), \lambda^{n\nu}, \mathcal{F}(\mathcal{H}_A))$ and it is reduced by the projection $1 \otimes S_n$. To see this we note that $1 \otimes S_n$ commutes with the unitary group of $H_{n,A}(\xi)$ since $H_A^{(n)}(\xi, k_1, \dots, k_n)$ symmetric in the variables $k_1, \dots, k_n$. Combining the above observations one arrives at the following lemma.

Lemma A.3. *Let $A \in \mathcal{B}(\mathbb{R}^\nu)$ and assume $1_A v = v$ λ_ν almost everywhere. Define $j_i : \mathcal{H}_i \to \mathcal{H}_A \oplus \mathcal{H}_{A^c}$ by $j_A(f) = (f, 0)$ and $j_{A^c}(f) = (0, f)$ and define $Q_i = V_A^* j_i$. There is a unitary map*

$$U : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}_A) \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^c}^{\otimes_s n}$$

such that

$$U H_\mu(\xi) U^* = H_\mu(\xi, A) \oplus \bigoplus_{n=1}^{\infty} H_{n,A}(\xi) \mid_{\mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^c}^{\otimes_s n}} := G_A(\xi) \tag{A.1}$$

for all $\xi \in \mathbb{R}^\nu$. In particular $\Sigma_A(\xi) \geq \Sigma(\xi)$ for all $\xi \in \mathbb{R}^\nu$. Furthermore

$$U \mid_{\mathcal{F}(\mathcal{H}_A)} = \Gamma(Q_A).$$

Let $g_1, \dots, g_n \in \mathcal{H}_{A^c}$ *and let* $\mathcal{K} \subset \mathcal{CS}_A$ *be a subspace. Define*

$$\begin{aligned} D =& \{Q_{A^c} g_1 \otimes_s \cdots \otimes_s Q_{A^c} g_n\} \\ & \cup \bigcup_{b=1}^{\infty} \{h_1 \otimes_s \cdots \otimes_s h_b \otimes_s Q_{A^c} g_1 \otimes_s \cdots \otimes_s Q_{A^c} g_n \mid h_i \in \mathcal{K}\}. \end{aligned}$$

If $\psi \in \mathrm{Span}(\mathcal{J}(\mathcal{K}))$ *we have*

$$U^*(\psi \otimes g) \in \mathrm{Span}(D). \tag{A.2}$$

$$\|(H_\mu(\xi - k) - H_\mu(\xi))\Gamma(Q_A)\psi\| = \|(H_\mu(\xi - k, A) - H_\mu(\xi, A))\psi\|. \tag{A.3}$$

$$\|(H_\mu(\xi) - \lambda)\Gamma(Q_A)\psi\| = \|(H_\mu(\xi, A) - \lambda)\psi\|. \tag{A.4}$$

where $\lambda \in \mathbb{C}$.

Proof. Define $U = U_2 U_1 \Gamma(V_A)$. Let $f, h \in \mathcal{CS}$ and write for $C \in \{A, A^c\}$ $f_C = P_C(f), h_C = P_C(h) \in \mathcal{CS}_C$. Then

$$U\epsilon(f) = U_2 U_1 \epsilon(f_A, f_{A^c}) = U_2 \epsilon(f_A) \otimes \epsilon(f_{A^c}) = \epsilon(f_A) \oplus \bigoplus_{n=1}^{\infty} \epsilon(f_A) \otimes \frac{1}{\sqrt{n!}} f_{A^c}^{\otimes n}$$

which one may check is in $\mathcal{D}(G(\xi))$. A long but easy calculation using Lemma 2.2 yields

$$\langle \epsilon(h), U^* G(\xi) U \epsilon(f) \rangle = \langle U\epsilon(h), G(\xi) U \epsilon(f) \rangle = \langle \epsilon(h), H(\xi) \epsilon(f) \rangle$$

As $\mathcal{L}(\mathcal{CS})$ is total we find $H_\mu(\xi)$ and $U^* G(\xi) U = H(\xi)$ on $\mathcal{L}(\mathcal{CS})$ which is spans a core for $H_\mu(\xi)$. Hence $U^* G(\xi) U = H_\mu(\xi)$ as both operators are selfadjoint. This proves the claim regarding the transformations. The remaining statements except equations (A.3) and (A.4) can be found in [3]. However equations (A.3) and (A.4) follows from $U \mid_{\mathcal{F}(\mathcal{H}_A)} = \Gamma(Q_A)$ and equation (A.1). □

We have the following Lemma

Lemma A.4. *Let* $k_1, \dots, k_\ell \in \mathbb{R}^\nu$ *be different. If there is* $\varepsilon > 0$ *such that* $(B_\epsilon(k_1) \cup \cdots \cup B_\epsilon(k_\ell)) \cap \{v \neq 0\}$ *is a* λ_ν *0-set then* $\Sigma(\xi - k_1 - \cdots - k_\ell) + \omega_n(k_1, \dots, k_\ell) \in \sigma_{ess}(H_\mu(\xi))$.

Proof. Pick $\varepsilon > 0$ such that the balls $B_\epsilon(k_1), \dots, B_\epsilon(k_\ell)$ are pairwise disjoint and we have $(B_\epsilon(k_1) \cup \cdots \cup B_\epsilon(k_\ell)) \cap \{v \neq 0\}$ is a λ_ν 0-set. Let $\varepsilon_n = \frac{\varepsilon}{n}$, $B_n^{(i)} = B_{\varepsilon_n}(k_i)$, $B_n = B_n^{(1)} \cup \cdots \cup B_n^{(\ell)}$, $k_0 = k_1 + \cdots + k_\ell$, $A_n = B_n^{(1)} \times \cdots \times B_n^{(\ell)}$ and let

$$\begin{aligned} & g_n^{(i)} = \lambda_\nu (B_n^{(i)} \backslash B_{n+1}^{(i)})^{-1/2} 1_{B_n^{(i)} \backslash B_{n+1}^{(i)}} \\ & \mathcal{A}_n = \{f \in \mathcal{CS} \mid f 1_{(B_n^{(i)})^c} = f \ \lambda_\nu \text{ almost everywhere for all } i \in \{1, \dots, n\}\} \\ & \mathcal{A}_\infty = \bigcup_{n=1}^{\infty} \mathcal{A}_n. \end{aligned}$$

Note that $\mathcal{CS} \subset \overline{\mathcal{A}_\infty}$ so $\mathcal{A}_\infty$ is a dense subspace of $\mathcal{H}$. In particular, $\mathcal{J}(\mathcal{A}_\infty)$ spans a core for $H_\mu(\xi - k_0)$ by Lemma 3.2. For each $p \in \mathbb{N}$ we may thus pick

$\psi_p \in \mathcal{J}(\mathcal{A}_\infty)$ such that $\|(H_\mu(\xi-k_0)-\Sigma(\xi-k_0))\psi_p\| \leq 1/p$. By Lemma 3.2 there is $u_1(p)$ such that

$$\sup_{x=(x_1,\dots,x_\ell)\in A_n} \|(H(\xi-x_1-\cdots-x_\ell)-H(\xi-k_0))\psi_p\| \leq \frac{1}{p}.$$

for all $n \geq u_1(p)$. Note now that ψ_p may we written as

$$\psi_p = a(p)\Omega + \sum_{i=1}^{b}\sum_{j=1}^{c(p)} \alpha_{i,j}(p) f_1^j(p)\otimes_s \cdots \otimes_s f_i^j(p)$$

for some $a(p), b(p), c(p), \alpha_{i,j}(p)$ constants and $f_i^j(p) \in \mathcal{A}_\infty$. Note that each $f_i^j(p)$ is in fact contained in some $\mathcal{A}_{l(i,j,p)}$ by definition so defining $u_2(p) = \max_{i,j}\{l(i,j,p)\}$ we see that $\psi_p \in \mathrm{Span}(\mathcal{J}(\mathcal{A}_l))$ for any $l \geq u_2(p)$. Define now u_p inductively by $u_1 = \max\{u_1(p), u_2(p)\}$ and $u_{p+1} = \max\{u_1(p), u_2(p), u_{p-1}\}+1$.

To summarise we have found vectors $\psi_p \in \mathcal{D}(H(\xi))$ and a strictly increasing sequence of numbers $\{u_p\}_{p=1}^\infty \subset \mathbb{N}$ such that

(1) $\|(H(\xi-k_0)-\Sigma(\xi-k_0))\psi_p\| \leq 1/p$.
(2) $\sup_{k\in B_{\delta_p}(k_1+\cdots+k_\ell)}\|(H(\xi-k)-H(\xi-k_0))\psi_p\| \leq \frac{1}{p}$ and $\ell\varepsilon_{u_p} \leq \delta$
(3) $\psi_p \in \mathrm{Span}(\mathcal{J}(\mathcal{A}_{u_p}))$.

For each $n \in \mathbb{N}$ and $A \in \{B_n^c, B_n\}$ define $V_n = V_{B_{\varepsilon_n}(k_0)^c}$ and $j_{n,A} : \mathcal{H}_i \to \mathcal{H}_{B_n^c} \oplus \mathcal{H}_{B_n}$ by $j_{n,B_n^c}(f) = (f,0)$ and $j_{n,B_n} f = (0,f)$. Furthermore we set $Q_{n,A} = V_n^* j_{n,A}$ and let U_n be the unitary map from Lemma A.4 corresponding to B_n^c. Fix $f \in \mathcal{H}$. Then the following equalities holds λ_ν almost everywhere:

$$Q_{n,B_n}P_{B_n}(f) = V_n^*(0, P_{B_n}(f)) = 1_{B_n}P_{B_n}(f) = 1_{B_n}f \tag{A.5}$$
$$Q_{n,B_n^c}P_{B_n^c}(f) = V_n^*(P_{B_n^c}(f),0) = 1_{B_n^c}P_{B_n^c}(f) = 1_{B_n^c}f \tag{A.6}$$

since $P_{B_n}(f) = f\ 1_{B_n}\lambda_\nu$-almost everywhere and $P_{B_n^c}(f) = f\ 1_{B_n^c}\lambda_\nu$-almost everywhere. For $f \in \mathcal{A}_n$ we have $1_{B_n^c}f = f$ and so we obtain the two equalities

$$\Gamma(Q_{n,B_n^c})\Gamma(P_{B_n^c})\psi = \Gamma(1_{B_n^c})\psi = \psi \ \ \forall\, \psi \in \mathrm{Span}(\mathcal{J}(\mathcal{A}_n)) \tag{A.7}$$
$$Q_{n,B_n}P_{B_n}g_n^{(i)} = 1_{B_n}g_n^{(i)} = g_n^{(i)} \tag{A.8}$$

for all $i \in \{1,\dots,\ell\}$. We now define the Weyl sequence as follows:

$$\phi_p = \sqrt{\ell!}U_{u_p}^*(\Gamma(P_{B_{u_p}^c})\psi_p \otimes P_{B_{u_p}}g_{u_p}^{(1)} \otimes_s \cdots \otimes_s P_{B_{u_p}}g_{u_p}^{(\ell)})$$

We will now prove

(1) $\phi_p \in \mathcal{D}(F_1)$.
(2) ϕ_p is orthogonal to ϕ_r for $p \neq r$.
(3) $\|\phi_p\| = 1$ for all $p \in \mathbb{N}$.
(4) $\|(H(\xi)-\Sigma(\xi-k_0)-\omega_n(k_1,\dots,k_n))\phi_p\|$ converges to 0.

(1): Define for all $p \in \mathbb{N}$ the set

$$\begin{aligned} C_p =& \{g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)}\} \\ &\cup \bigcup_{q=1}^{\infty} \{h_1 \otimes_s \cdots \otimes_s h_q \otimes_s g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)} \mid h_i \in \mathcal{A}_{u_p}\} \subset \mathcal{J}(\mathcal{CS}) \end{aligned}$$

and let $\mathcal{K}_p = P_{B_{u_p}^c} \mathcal{A}_{u_p} \subset \mathcal{CS}_{B_{u_p}^c}$ since $P_{B_{u_p}^c}$ maps $\mathcal{CS}$ into $\mathcal{CS}_{B_{u_p}^c}$. Using equation (A.5) we find $Q_{u_p, B_{u_p}^c} \mathcal{K}_p = 1_{B_{u_p}} \mathcal{A}_{u_p} = \mathcal{A}_{u_p}$ and Lemma A.3 implies $\psi_p \in C_p \subset \mathcal{J}(\mathcal{CS}) \subset \mathcal{D}(H_\mu(\xi))$ as required.

(2): Let $r < p$. Then $\phi_r \in \mathrm{Span}(C_r)$ and $\phi_p \in \mathrm{Span}(C_p)$, so we just need to see that every element in C_p and C_r are orthogonal. Let $\psi_1 \in C_p$ and $\psi_2 \in C_r$. Note every tensor in C_p has a factor $g_{u_p}^{(1)}$ and that this factor is orthogonal to $g_{u_r}^{(i)}$ for all i by construction. Furthermore for any $h \in \mathcal{A}_{u_r}$ we see that h is supported in $B_{u_r}^c \subset B_{u_p}^c$ and hence $g_{u_p}^{(1)} h = 0$, so $g_{u_p}^{(1)}$ is orthogonal to any element in $\mathcal{A}_{u_r}$. This implies ψ_1 contains a factor orthogonal to all factors in ψ_2 and thus ψ_1 is orthogonal to ψ_2.

(3): $Q_{u_p, B_{u_p}^c}$ and $Q_{u_p, B_{n_p}}$ are isometrics and which implies $\Gamma(Q_{n_p, B_{u_p}^c})$ and $\Gamma(Q_{n_p, B_{u_p}})$ are isometrics. Using equations (A.7) and (A.8) we calculate

$$\begin{aligned} \|\phi_p\| &= \sqrt{\ell!} \|\Gamma(P_{B_{n_p}^c}) \psi_p\| \|P_{B_{u_p}} g_{u_p}^{(1)} \otimes_s \cdots \otimes_s P_{B_{u_p}} g_{u_p}^{(\ell)}\| \\ &= \sqrt{\ell!} \|\Gamma(Q_{n_p, B_{u_p}^c}) \Gamma(P_{B_{u_p}^c}) \psi_p\| \|\Gamma(Q_{u_p, B_{u_p}^c}) P_{B_{u_p}} g_{u_p}^{(1)} \otimes_s \cdots \otimes_s P_{B_{u_p}} g_{u_p}^{(\ell)}\| \\ &= \sqrt{\ell!} \|\psi_p\| \|g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)}\| = 1 \end{aligned}$$

where we used $g_{u_p}^{(i)}$ and $g_{u_p}^{(j)}$ are normalised and orthogonal if $i \neq j$ and

$$\|g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)}\|^2 = \frac{1}{\ell!} \sum_{\sigma \in \mathcal{S}_\ell} \langle g_{u_p}^{(1)} \otimes \cdots \otimes g_{u_p}^{(\ell)}, g_{u_p}^{(\sigma(1))} \otimes_s \cdots \otimes_s g_{u_p}^{(\sigma(\ell))} \rangle = \frac{1}{\ell!}.$$

(4): Define the function $g_{u_p} = g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)}$. Using Lemma A.3 we see that $\|(H(\xi) - \Sigma(\xi - k_0) - \omega_n(k_1, \dots, k_n))\phi_p\|$ is given by

$$\begin{aligned} \sqrt{\ell!} \Bigg(\int_{B_{u_p}^\ell} & \|(H_{B_{u_p}^c}(\xi - x_1 - \cdots - x_\ell) + \omega_\ell(x_1, \dots, x_\ell) \\ & - \Sigma(\xi - k_0) - \omega_\ell(k_1, \dots, k_\ell)) \Gamma(P_{B_{u_p}^c}) \psi_p\|^2 |g_{u_p}(x)|^2 d\lambda_\nu(x) \Bigg)^{1/2} := \sqrt{\ell!} \gamma \end{aligned}$$

Using the triangle inequality, $\|\Gamma(P_{B^c_{n_p}})\psi_p\|= 1$, $\Gamma(Q_{n_p,B^c_{u_p}})\Gamma(P_{B^c_{n_p}})\psi_p = \psi_p$ and Lemma A.3 we find $\gamma \leq C_1 + C_2 + C_3$ where

$$C_1 = \left(\int_{B^\ell_{u_p}} \|(H(\xi - x_1 - \cdots - x_n) - H(\xi - k_0))\psi_p\|^2 |g_{u_p}(x)|^2 d\lambda_\nu(x)\right)^{1/2}$$
$$C_2 = \left(\int_{B^\ell_{u_p}} |(\omega_n(x_1,\dots,x_\ell) - \omega_n(k_1,\dots,k_\ell))|^2 |g_{u_p}(x)|^2 d\lambda_\nu(x)\right)^{1/2}$$
$$C_3 = \|(H(\xi - k_0) - \Sigma(\xi - k_0))\psi_p\| \left(\int_{B^\ell_{u_p}} |g_{u_p}(x)|^2 d\lambda_\nu(x)\right)^{1/2}$$

Let $f : (\mathbb{R}^\nu)^n \to \mathbb{R}_+$ be non negative and symmetric. Using that the $g^{(i)}_{u_p}$ have disjoint support one finds

$$|g_{u_p}(x_1,\dots,x_\ell)|^2 = \frac{1}{\ell!^2} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{\nu} \overline{g^{(\pi(i))}_{u_p}(x_i)} g^{(\sigma(i))}_{u_p}(x_i) = \frac{1}{\ell!^2} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{\nu} |g^{(\sigma(i))}_{u_p}(x_i)|^2$$

Thus using permutation invariance of f we find

$$\int_{B^\ell_{u_p}} f(x)|g_{u_p}(x)|^2 d\lambda_\nu(x) = \frac{1}{\ell!} \int_{A_{u_p}} f(x) \prod_{i=1}^{\nu} |g^{(i)}_{u_p}(x_i)|^2 d\lambda_\nu(x)$$

Thus $\sqrt{\ell!}C_3 = \|(H(\xi - k_0) - \Sigma(\xi - k_0))\psi_p\| \leq p^{-1}$. Furthermore

$$\sqrt{\ell!}C_1 \leq \sup_{(x_1,\dots,x_n)\in A_{u_p}} \|(H(\xi - x_1 - \cdots - x_\ell) - H(\xi - k_0))\psi_p\| \leq p^{-1}$$
$$\sqrt{\ell!}C_2 \leq \sup_{(x_1,\dots,x_n)\in A_{u_p}} |\omega_\ell(x_1,\dots,x_\ell) - \omega_\ell(k_1,\dots,k_\ell)|$$

By continuity of ω we now see $\sqrt{\ell!}\gamma$ goes to 0 for p tending to ∞. □

Lemma A.5. *Let $k_1,\dots,k_\ell \in \mathbb{R}^\nu$. Then $\Sigma(\xi - k_1 - \cdots - k_\ell) + \omega_n(k_1,\dots,k_\ell) \in \sigma_{ess}(H_\mu(\xi))$.*

Proof. Assume first $k_1,\dots,k_\ell \in \mathbb{R}^\nu$ are different elements and define $A_n = B_{1/n}(k_1) \cup \cdots \cup B_{1/n}(k_\ell)$. Let $v_n = 1_{A^c_n} v$ and note that $v_n \in \mathcal{D}(\omega^{-1/2})$ and

$$\lim_{n\to\infty} \|(v_n - v)(\omega^{-1/2} + 1)\| = 0$$

by dominated convergence. Define

$$H^{(n)}(\xi) = \Omega(\xi - d\Gamma(k)) + d\Gamma(\omega) + \mu\varphi(v_n) \geq -\mu^2\|\omega^{-1/2}v_n\|^2 \geq -\mu^2\|\omega^{-1/2}v\|^2$$
$$\Sigma_n(\xi) = \inf(\sigma(H(\xi)))$$

Using Lemma 2.1 we find

$$\begin{aligned}\|(H_\mu(\xi) + i)^{-1} - (H^{(n)}(\xi) + i)^{-1}\| &\leq |\lambda| \|\varphi(v - v_n)(H_\mu(\xi) + i)^{-1})^{-1}\| \\ &\leq |\lambda| \|(v_n - v)(\omega^{-1/2} + 1)\| \|(d\Gamma(\omega) + 1)^{1/2}(H_\mu(\xi) + i)^{-1}\|\end{aligned}$$

so $H^{(n)}(\xi)$ converges to $H_\mu(\xi)$ in norm resolvent sense for all $\xi \in \mathbb{R}^\nu$. The uniform lower bound of $\Sigma_n(\xi)$ and norm resolvent convergence now implies $\Sigma_n(\xi)$ converges to $\Sigma(\xi)$ for all ξ.

By Lemma A.4 we have $\Sigma_n(\xi-k_1-\cdots-k_\ell)+\omega_n(k_1,\dots,k_\ell) \in \sigma_{ess}(H^{(n)}(\xi))$. Now $\Sigma_n(\xi-k_1-\cdots-k_\ell)+\omega_n(k_1,\dots,k_\ell)$ converges to $\Sigma(\xi-k_1-\cdots-k_\ell)+\omega_n(k_1,\dots,k_\ell)$ and $H^{(n)}(\xi)$ converges to $H(\xi)$ in norm resolvent sense so we are done in the case where $k_1,\dots,k_\ell$ are different. The conclusion now follows since Σ and ω_ℓ are continous, $\{(k_1,\dots,k_\ell) \mid k_i \neq k_j \;\; \forall i,j\}$ is dense and $\sigma_{ess}(H(\xi))$ is closed. □

B. Proof of pull though formula

This appendix is devoted to proving the pull through formula. The in case $K(k)=|k|^2$ one could compute everything directly using tools as in [8]. However the other possible choices of K require a more sophisticated approach ao we use the formalised developed in [3] and the reader should consult this paper for the proofs. Let $\mathcal{H}=L^2(\mathcal{M},\mathcal{E},\mu)$, where $(\mathcal{M},\mathcal{E},\mu)$ is assumed to be σ-finite. We start by defining

$$\mathcal{F}_+(\mathcal{H})=\bigtimes_{n=0}^{\infty}\mathcal{H}^{\otimes_s n}$$

with coordinate projections P_n and $\mathcal{H}=L^2(\mathbb{R}^\nu,\mathcal{B}(\mathbb{R}^\nu),\lambda_\nu)$. For $(\psi^{(n)}),(\phi^{(n)}) \in \mathcal{F}_+(\mathcal{H})$ we define

$$d((\psi^{(n)}),(\phi^{(n)}))=\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{\|\psi^{(n)}-\phi^{(n)}\|}{1+\|\psi^{(n)}-\phi^{(n)}\|}$$

where $\|\cdot\|$ is the Fock space norm. This makes sense since $P_n(\mathcal{F}_+(\mathcal{H}))\subset\mathcal{F}(\mathcal{H})$. We now have

Lemma B.1. *The map d defines a metric on $\mathcal{F}_+(\mathcal{H})$ and turns this space into a complete separable metric space and a topological vector space. The topology and Borel σ-algebra is generated by the projections P_n. If a sequence $\{\psi_n\}_{n=1}^\infty \subset \mathcal{F}(\mathcal{H})$ is convergent/Cauchy then it is also convergent/Cauchy with respect to d. Also any total/dense set in $\mathcal{F}_b(\mathcal{H})$ will be total/dense in $\mathcal{F}_+(\mathcal{H})$ as well.*

For each $a\in\mathbb{R}$ we define

$$\|\cdot\|_{a,+}=\lim_{n\to\infty}\left(\sum_{k=0}^{n}(k+1)^{2a}\|P_k(\cdot)\|^2\right)^{\frac{1}{2}}.$$

which is measurable from $\mathcal{F}_+(\mathcal{H})$ into $[0,\infty]$. Let

$$\mathcal{F}_{a,+}(\mathcal{H})=\{\psi\in\mathcal{F}_+(\mathcal{H}) \mid \|\psi\|_{a,+}<\infty\}.$$

Note $\|\cdot\|_{a,+}$ restricts to a norm on $\mathcal{F}_{a,+}(\mathcal{H})$ that comes from an inner product. In particular $\mathcal{F}_{a,+}(\mathcal{H})$ is a Hilbert space and for $a\geq 0$ we have $\mathcal{F}_{a,+}(\mathcal{H})=\mathcal{D}((N+1)^a)$. We summarise as follows

Lemma B.2. $\|\cdot\|_{a,+}$ *defines measurable map from* $\mathcal{F}_+(\mathcal{H})$ *to* $[0,\infty]$*, and restricts to a norm on the spaces* $\mathcal{F}_{a,+}(\mathcal{H})$ *that comes from an inner product turning* $\mathcal{F}_{a,+}(\mathcal{H})$ *into a Hilbert space.*

The point of defining a metric on $\mathcal{F}_+(\mathcal{H})$ and finding a dense set is that most of the operations we will encounter in this chapter are continous on $\mathcal{F}_+(\mathcal{H})$. Therefore many operator identities only needs to be proven on well behaved vectors. Fix now $v \in \mathcal{H}$. We now define the following maps on $\mathcal{F}_+(\mathcal{H})$

$$\begin{aligned} a_+(v)(\psi^{(n)}) &= (a_n(v)\psi^{(n+1)}) \\ a_+^\dagger(v)(\psi^{(n)}) &= (0, a_0^\dagger(v)\psi^{(0)}, a_1^\dagger(v)\psi^{(1)}, \dots) \\ \varphi_+(v) &= a_+(v) + a_+^\dagger(v) \end{aligned}$$

Where $a_n(v)$ is annihilation from $\mathcal{H}^{\otimes_s (n+1)}$ to $\mathcal{H}^{\otimes_s n}$ and $a_n^\dagger(f)$ is creation from $\mathcal{H}^{\otimes_s n}$ to $\mathcal{H}^{\otimes (n+1)}$.

Lemma B.3. *The maps* $a_+(v)$,$a_+^\dagger(v)$ *and* $\varphi_+(v)$ *are all continuous. For* $B \in \{a, a^\dagger, \varphi\}$ *we have*

$$B_+(v)\psi = B(v)\psi \ \text{ if } \psi \in \mathcal{D}(B(v)). \tag{B.1}$$

Furthermore we have the commutation relations

$$\begin{aligned} [a_+(v), a_+^\dagger(g)] &= \langle v, g\rangle \\ [\varphi_+(v), \varphi_+(g)] &= 2i\mathrm{Im}(\langle v, g\rangle) \end{aligned}$$

We now move on to the second quantisation of unitaries and selfadjoint operators. Let U be unitary on $\mathcal{H}$ and $\omega = (\omega_1, \dots, \omega_p)$ be a tuple of strongly commuting selfadjoint operators on $\mathcal{H}$. We then define

$$\begin{aligned} d\Gamma(\omega) &= (d\Gamma(\omega_1), \dots, d\Gamma(\omega_p)) \\ d\Gamma^{(n)}(\omega) &= (d\Gamma^{(n)}(\omega_1), \dots, d\Gamma^{(n)}(\omega_p)) \end{aligned}$$

which are now tuples of strongly commuting selfadjoint operators (this is easily checked using the unitary group). Let furthermore $f : \mathbb{R}^p \to \mathbb{C}$ be a map. We then define

$$\begin{aligned} f(d\Gamma_+(\omega)) &= \bigtimes_{n=0}^{\infty} f(d\Gamma^{(n)}(\omega)) \quad \mathcal{D}(f(d\Gamma_+(\omega))) = \bigtimes_{n=0}^{\infty} \mathcal{D}(f(d\Gamma^{(n)}(\omega))) \\ \Gamma_+(U) &= \bigtimes_{n=0}^{\infty} \Gamma^{(n)}(U). \end{aligned}$$

If $\omega : \mathcal{M} \to \mathbb{R}^p$ is measurable then we may identify ω as such a touple of commuting selfadjoint operators. In this case $f(d\Gamma^{(n)}(\omega))$ is multiplication by the map $f(\omega(k_1) + \cdots + \omega(k_n))$. The following lemma is now obvious.

Lemma B.4. *The map* $\Gamma_+(U)$ *is an isometry on* $\mathcal{F}_+(\mathcal{H})$ *and is thus continuous. Furthermore we have*

$$\begin{aligned} f(d\Gamma_+(\omega))\psi &= f(d\Gamma(\omega))\psi, \quad && \psi \in \mathcal{D}(f(d\Gamma(\omega))) \\ \Gamma_+(U)\psi &= \Gamma(U)\psi, && \psi \in \mathcal{F}_b(\mathcal{H}) \end{aligned}$$

We will now consider a class of linear functionals on $\mathcal{F}_+(\mathcal{H})$. For each $n \in \mathbb{N}$ we let $Q_n : \mathcal{F}_+(\mathcal{H}) \to \mathcal{N}$ denote the linear projection which preserves the first n entries of $(\psi^{(n)})$ and projects the rest of them to 0. For $\psi \in \mathcal{N}$ there is $K \in \mathbb{N}$ such that for $n \geq K$ we have $Q_n\psi = \psi$. For $\phi \in \mathcal{F}_+(\mathcal{H})$ we may thus define the pairing

$$\langle \psi, \phi \rangle_+ := \langle \psi, Q_n\phi \rangle = \sum_{i=0}^{K} \langle \psi^{(i)}, \phi^{(i)} \rangle, \tag{B.2}$$

where $n \geq K$.

Lemma B.5. *The map Q_n above is linear and continuous into $\mathcal{F}(\mathcal{H})$. The paring $\langle \cdot, \cdot \rangle_+$ is sesquilinear, and continuous in the second entry. If $\phi \in \mathcal{F}_{a,+}(\mathcal{H})$ then $\psi \mapsto \langle \psi, \phi \rangle_+$ is continous with respect to $\|\cdot\|_{-a,+}$. Furthermore, the collection of maps of the form $\langle \psi, \cdot \rangle_+$ will separate points of $\mathcal{F}_+(\mathcal{H})$.*

Corollary B.6. *Let $\phi \in \mathcal{F}_{a,+}(\mathcal{H})$ for some $a \leq 0$, $\mathcal{D} \subset \mathcal{N}$ be dense in $\mathcal{F}(\mathcal{H})$ and assume $\langle \psi, \phi \rangle_+ = 0$ for all $\psi \in \mathcal{D}$. Then $\phi = 0$.*

We also have the following formal adjoint relations

Lemma B.7. *Let $\psi \in \mathcal{N}$, $\phi \in \mathcal{F}_+(\mathcal{H})$, $v \in \mathcal{H}$ and U be unitary on $\mathcal{H}$. Then we have*

$$\begin{aligned} \langle a^\dagger(v)\psi, \phi \rangle_+ &= \langle \psi, a_+(v)\phi \rangle_+, & \langle a(v)\psi, \phi \rangle_+ &= \langle \psi, a_+^\dagger(v)\phi \rangle_+, \\ \langle \varphi(v)\psi, \phi \rangle_+ &= \langle \psi, \varphi_+(v)\phi \rangle_+, & \langle \Gamma(U)\psi, \phi \rangle_+ &= \langle \psi, \Gamma_+(U^*)\phi \rangle_+. \end{aligned}$$

Let $\omega = (\omega_1, \dots, \omega_p)$ be a tuple of commuting selfadjoint operators, $f : \mathbb{R}^p \to \mathbb{C}$, $\psi \in \mathcal{N} \cap \mathcal{D}(f(d\Gamma(\omega)))$ and $\phi \in \mathcal{D}(\overline{f}(d\Gamma_+(\omega)))$ we have

$$\langle f(d\Gamma(\omega))\psi, \phi \rangle_+ = \langle \psi, \overline{f}(d\Gamma_+(\omega))\phi \rangle_+.$$

We now consider functions with values in $\mathcal{F}_+(\mathcal{H})$. Let $(X, \mathcal{X}, \nu)$ be a σ-finite and countably generated measure space. Define the quotient

$$\mathcal{M}(X, \mathcal{X}, \nu) = \{f : X \to \mathcal{F}_+(\mathcal{H}) \mid f \text{ is } \mathcal{X} - \mathcal{B}(\mathcal{F}_+(\mathcal{H})) \text{ mesurable}\}/\sim,$$

where we define $f \sim g \iff f = g$ almost everywhere. We are interested in the subspace

$$\mathcal{C}(X, \mathcal{X}, \nu) = \{f \in \mathcal{M}(X, \mathcal{X}, \nu) \mid x \mapsto P_n f(x) \in L^2(X, \mathcal{X}, \nu, \mathcal{H}^{\otimes_s n}) \ \forall\, n \in \mathbb{N}_0\}.$$

Lemma B.2 shows that $x \mapsto \|f(x)\|_{a,+}$ is measurable for functions $f \in \mathcal{C}(X, \mathcal{X}, \nu)$ and so the integral

$$\int_X \|f(x)\|_{a,+}^2 d\nu(x)$$

always makes sense. If $a = 0$ then it is finite if and only if $f \in L^2(X, \mathcal{X}, \nu, \mathcal{F}_b(\mathcal{H}))$. We write $f \in \mathcal{C}(X, \mathcal{X}, \nu)$ as $(f^{(n)})$ where $f^{(n)} = x \mapsto P_n f(x)$. For $f, g \in \mathcal{C}(X, \mathcal{X}, \nu)$ we define

$$d(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|f^{(n)} - g^{(n)}\|_{L^2(X,\mathcal{X},\nu,\mathcal{H}^{\otimes_s n})}}{1 + \|f^{(n)} - g^{(n)}\|_{L^2(X,\mathcal{X},\nu,\mathcal{H}^{\otimes_s n})}}.$$

We can now summarise.

Lemma B.8. *d is a complete metric on $\mathcal{C}(X,\mathcal{X},\nu)$ such that $\mathcal{C}(X,\mathcal{X},\nu)$ becomes separable topological vector space. The topology is generated by the maps $f \mapsto (x \mapsto P_n f(x))$. Furthermore $L^2(X,\mathcal{X},\nu,\mathcal{F}_b(\mathcal{H})) \subset \mathcal{C}(X,\mathcal{X},\nu)$ and convergence in $L^2(X,\mathcal{X},\nu,\mathcal{F}_b(\mathcal{H}))$ implies convergence in $\mathcal{C}(X,\mathcal{X},\nu)$. Also the map $x \mapsto \|f(x)\|_{a,+}$ is measurable for any f in $\mathcal{C}(X,\mathcal{X},\nu)$ and $a \in \mathbb{R}$.*

We now move on to discuss some actions on this space. This is strongly related to the direct integral and readers should look up the results in [12]. Let $n \geq 1$, $v \in \mathcal{H}$, U be unitary on $\mathcal{H}$, $\omega = (\omega_1, \dots, \omega_p)$ a tuple of selfadjoint multiplication operators on $\mathcal{H}$, $m : \mathcal{M}^n \to \mathbb{R}^p$ measurable and $g : \mathbb{R}^p \to \mathbb{R}$ a measurable map. Then we wish to define operators on $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ for $\ell \geq 1$ by

$$\begin{aligned}
(a^\dagger_{\oplus,\ell}(v) f)(k) &= a^\dagger_+(v) f(k) \\
(a_{\oplus,\ell}(v) f)(k) &= a_+(v) f(k) \\
(\varphi_{\oplus,\ell}(v) f)(k) &= \varphi_+(v) f(k) \\
(\Gamma_{\oplus,\ell}(U) f)(k) &= \Gamma_+(U) f(k) \\
(g(d\Gamma_{\oplus,\ell}(\omega) + m) f)(k) &= g(d\Gamma_+(\omega) + m(k)) f(k).
\end{aligned}$$

We further define $\mathcal{C}(\mathcal{M}^0, \mathcal{E}^{\otimes 0}, \mu^{\otimes 0}) = \mathcal{F}_+(\mathcal{H})$ along with $a^\dagger_{\oplus,0}(v) = a^\dagger_+(v)$, $a_{\oplus,0}(v) = a_+(v)$, $\varphi_{\oplus,0}(v) = \varphi_+(v)$ and $\Gamma_{\oplus,0} = \Gamma_+(U)$. We have the following lemma.

Lemma B.9. *The $a^\dagger_{\oplus,\ell}(v), a_{\oplus,\ell}(v), \varphi_{\oplus,\ell}(v)$ and $\Gamma_{\oplus,\ell}(U)$ are well defined and continuous for all $\ell \in \mathbb{N}_0$. Let $f \in \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$. If $f(k) \in \mathcal{D}(g(d\Gamma_+(\omega) + m(k)))$ for all k then $k \mapsto P_n(g(d\Gamma_+(\omega) + m(k)) f(k))$ is measurable. Thus as domain of $g(d\Gamma_{\oplus,\ell}(\omega) + m)$ we may choose*

$$\bigcap_{\ell=0}^{\infty} \Big\{ f \in \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \Big| f(k) \in \mathcal{D}(g(d\Gamma_+(\omega) + m(k))) \text{ for a.e. } k \in \mathcal{M}^\ell, \int_{\mathcal{M}^\ell} \| P_n g(d\Gamma_+(\omega) + m(k)) f(k) \|^2 d\mu^{\otimes \ell}(k) < \infty \Big\}.$$

We will now introduce the pointwise annihilation operators. For $\psi = (\psi^{(n)}) \in \mathcal{F}_+(\mathcal{H})$ we define $A_\ell \psi \in \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ by

$$P_n(A_\ell \psi)(k_1, \dots, k_\ell) = \sqrt{(n+\ell)(n+\ell-1)\cdots(n+1)}\, \psi^{(n+\ell)}(k_1, \dots, k_\ell, \cdot, \dots, \cdot)$$

which is easily seen to be well defined and take values in $\mathcal{H}^{\otimes_s n}$. We can prove

Lemma B.10. *A_ℓ is a continuous linear map from $\mathcal{F}_+(\mathcal{H})$ to $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ and from $\mathcal{D}(N^{\frac{\ell}{2}})$ into $L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}(\mathcal{H}))$. Furthermore $\psi \in \mathcal{D}(N^{\ell/2}) \iff A_\ell \psi \in L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}(\mathcal{H}))$ and if $\psi \in \mathcal{F}(\mathcal{H})$ we have $A_\ell \psi$ is almost everywhere $\mathcal{F}_{-\frac{\ell}{2},+}(\mathcal{H})$ valued.*

Fix $v \in \mathcal{H}$ and $\ell \in \mathbb{N}_0$. We then define a map $z_\ell(v) : \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \to \mathcal{C}(\mathcal{M}^{\ell+1}, \mathcal{E}^{\otimes (\ell+1)}, \mu^{\otimes (\ell+1)})$ by

$$(z_0(v)\psi)(k) = v(k)\psi \text{ and } (z_\ell(v)\psi)(x,k) = v(x)\psi(k)$$

when $\ell \geq 1$. One may prove

Lemma B.11. *The map $z_\ell(v)$ introduced above is linear and continuous. Both as a map from $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ into the space $\mathcal{C}(\mathcal{M}^{\ell+1}, \mathcal{E}^{\otimes(\ell+1)}, \mu^{\otimes(\ell+1)})$ and from $L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}(\mathcal{H}))$ into the space $L^2(\mathcal{M}^{\ell+1}, \mathcal{E}^{\otimes(\ell+1)}, \mu^{\otimes(\ell+1)}, \mathcal{F}(\mathcal{H}))$.*

Lastly we look at permutation and symmetrisation operators. Let $\ell \geq 1$ and $\sigma \in \mathcal{S}_\ell$ where $\mathcal{S}_\ell$ is the set of permutations of $\{1, \ldots, \ell\}$. Defining $\widetilde{\sigma} : \mathcal{M}^\ell \to \mathcal{M}^\ell$ by $\widetilde{\sigma}(k_1, \ldots, k_\ell) = (k_{\sigma(1)}, \ldots, k_{\sigma(\ell)})$. Define $\widehat{\sigma} : \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \to \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ by

$$(\widehat{\sigma} f)(k_1, \ldots, k_\ell) = f(k_{\sigma(1)}, \ldots, k_{\sigma(\ell)}) = (f \circ \widetilde{\sigma})(k_1, \ldots, k_\ell).$$

Define now

$$S_\ell := \frac{1}{(\ell-1)!} \sum_{\sigma \in \mathcal{S}_\ell} \widehat{\sigma}.$$

One may prove:

Lemma B.12. *Let $\ell \in \mathbb{N}$. For $\sigma \in \mathcal{S}_\ell$ the map $\widehat{\sigma}$ defines a linear bijective isometry from $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ to $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ and from $L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}(\mathcal{H}))$ to $L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}(\mathcal{H}))$. Also $\widehat{\sigma} A_\ell \psi = A_\ell \psi$ and if $\pi \in \mathcal{S}_\ell$ then $\widehat{\pi}\widehat{\sigma} = \widehat{\pi \circ \sigma}$.*

Furthermore S_ℓ is continuous and linear from $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ into the space $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ and it satisfies relation $S_\ell^2 = \ell S_\ell$. Furthermore S_ℓ is also continuous from $L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}(\mathcal{H}))$ into $L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}(\mathcal{H}))$.

We can now calculate commutators (more commutation relations can be found in [3] but we will only cite those used here)

Lemma B.13. *Let $\omega : \mathcal{M} \to \mathbb{R}^p$ be measurable, $v \in \mathcal{H}$ and let $f : \mathbb{R}^p \to \mathbb{R}$ be measurable. Then*

$$\varphi_\oplus(v) A_1 = A_1 \varphi_+(v) - z_0(v) \tag{B.3}$$

Let $\ell \geq 1$. If $\psi \in \mathcal{D}(f(d\Gamma(\omega)))$ then $A_\ell \psi \in \mathcal{D}(f(d\Gamma_\oplus(\omega) + \omega_\ell))$ where we define $\omega_\ell(k_1, \ldots, k_\ell) = \omega(k_1) + \cdots + \omega(k_\ell)$ and

$$f(d\Gamma_\oplus(\omega) + \omega_\ell) A_\ell \psi = A_\ell f(d\Gamma_+(\omega))\psi.$$

We can now prove the pull-trough formula.

Lemma B.14. *Let $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$ and ω, v, K satisfy Hypothesis 1 and 2 and let $\mu \in \mathbb{R}, \xi \in \mathbb{R}^\nu, \nu \geq 2$. Assume ψ is a ground state for $H_\mu(\xi)$. Then we have*

$$(A_1\psi)(k) = -\mu v(k)(H_\mu(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}\psi$$

almost everywhere.

Proof. First we note $(H_\mu(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}$ exists as a bounded operator away from the zero set $\mathbb{R}\xi$ by Lemma 4.4. Define the lifted operators on $\mathcal{F}_+(\mathcal{H})$ and $C(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell})$ respectively

$$\begin{aligned}
H_+(\xi) &= K(\xi - d\Gamma_+(k)) + d\Gamma_+(\omega) + \mu\varphi_+(v) \\
H_\oplus(\xi) &= K(\xi - g - d\Gamma_{\oplus,1}(k)) + d\Gamma_{\oplus(v),1}(\omega) + \omega + \mu\varphi_{\oplus(v),1}
\end{aligned}$$

where $g:\mathbb{R}^\nu \to \mathbb{R}^\nu$ is given by $g(k)=k$. The domains are

$$\mathcal{D}(H_+(\xi)) = \mathcal{D}(d\Gamma_+(\omega)) \cap \mathcal{D}(K(\xi - d\Gamma_+(g)))$$
$$\mathcal{D}(H_\oplus(\xi)) = \mathcal{D}(d\Gamma_{\oplus,1}(\omega) + \omega_\ell) \cap \mathcal{D}(K(\xi - g - d\Gamma_{\oplus,1}(g)))$$

By Lemma B.13 we have $A_1\psi \in \mathcal{D}(H_\oplus(\xi))$ since $\psi \in \mathcal{D}(H(\xi)) \subset \mathcal{D}(H_+(\xi))$. Using Lemmas B.3, B.4 and B.13 we also obtain

$$h := (H_\oplus(\xi) - \Sigma(\xi))A_1\psi = -\mu z_0^\dagger(v)\psi + A_1(H_+(\xi) - \Sigma(\xi))\psi = -\mu z_0(v)\psi$$

which is Fock space valued. Let M be a zeroset such that:

1. $A_1\psi$ is $\mathcal{F}_{-1/2,+}(\mathcal{H})$ valued on M^c (see Lemma B.10).
2. $h(k) = (H_+(\xi - k) + \omega(k))(A_1\psi)(k)$ and $h(k) \in \mathcal{F}(\mathcal{H})$ for $k \in M^c$.
3. $(H_\mu(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}$ exists on M^c.

Fix $k \in M^c$. For any vector ϕ such that both $(H_\mu(\xi-k)+\omega(k)-\Sigma(\xi))^{-1}\phi$ and ϕ is in $\mathcal{N}$ (this set is dense by Proposition 3.2) we find using Lemma B.7 that

$$\begin{aligned}
&\langle \phi, A_1\psi(k)\rangle_+ \\
&= \langle (H_\mu(\xi - k) + \omega(k) - \Sigma(\xi))(H_\mu(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}\phi, A_1\psi(k)\rangle_+ \\
&= \langle (H_\mu(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}\phi, h(k)\rangle \\
&= \langle \phi, (H_\mu(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}h(k)\rangle_+ .
\end{aligned}$$

Corollary B.6 finishes the proof.

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Paper D

Rigorous Results on the Bose-Polaron

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Rigorous Results on the Bose-Polaron

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Abstract: We consider a new model for an impurity in a Bose-Einstein condensate. The Hamiltonian is translation invariant and so it can be represented as a direct integral of fiber Hamiltonians $\{H(\xi)\}_{\xi\in\mathbb{R}^3}$ each corresponding to a fixed value of total momentum. We prove selfadjointness of the Hamiltonian and fiber operators and find the essential spectrum of the fiber Hamiltonians. We then extend (and correct) certain results on operators generating a positivity improving semigroup and apply them to the model. From these results we obtain directly that $\xi \mapsto \inf(\sigma(H(\xi)))$ has a global minimum at $\xi = 0$ and that H does not have a ground state.

1. Introduction

The model we investigate in this paper is an extension of the usual polaron model and has been used in the papers [5], [7], [10] and [19]. Informally, the Hamiltonian appears using a Bogoliubov approximation but one still keeps some of the second order terms that would normally be ignored. However these terms are important to model Effimov physics and the new model does give rather good results when compared with data (see [7]).

The point of departure of this paper is the model encountered immediately after the formal manipulations have been finished. We then rewrite it in an convenient form which is practical for proving the theorems we are after. The reader is warned that the Hamiltonians in [19] contains a small misprint which is eventually corrected in [10].

The first step is proving selfadjointness of the Hamiltonian and the fiber Hamiltonians arising from the Lee-Low-Pines transformation. This is non trivial because the "perturbation" is a rather large expression. So it is not at all clear that one ends up be a selfadjoint Hamiltonian.

After proving selfadjointness we try to generalise the results from [8], [9] and [12] to our setting. Especially the theory from [9] is used in [19] (without proof that it is correct). The main problem is that the "perturbation" is not a multiplication operator. In fact, to generalise the results in [9] and [8] we actually have to develop new abstract theorems on positivity improving semi groups. We improve and correct results found in [4], [11] and [16] on the subject and apply them along the lines found in [9] and [8].

One new point being made in this paper is the absence of ground states for translation invariant Hamiltonians. From a physical perspective this sounds trivial and actually there should be no bound states at all in translation invariant systems. However our proof only works to exclude a ground state and the general problem is future work.

2. Notation and preliminaries

We start by fixing notation. If X is a topological space we will write $\mathcal{B}(X)$ for the Borel σ-algebra. Furthermore, if $(\mathcal{M}, \mathcal{F}, \mu)$ is a measure space and X is a Banach space we will for $1 \leq p \leq \infty$ write $L^p(\mathcal{M}, \mathcal{F}, \mu, X)$ for the vector valued L^p space. If $X = \mathbb{C}$ we will drop X from the notation. Also we will write $B(X)$ for the bounded linear operators from X to X.

Now let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be Hilbert spaces and let $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ denote the tensor product. If $\mathcal{D}_i \subset \mathcal{H}_i$ we will write

$$\mathcal{D}_1 \widehat{\otimes} \dots \widehat{\otimes} \mathcal{D}_n = \operatorname{Span}\{f_1 \otimes \cdots \otimes f_n \mid f_i \in \mathcal{D}_i\}$$

We will be using the results from the following Theorem

Theorem 2.1. *Let A_i be a closable operator on $\mathcal{H}_i$ for all i. Then*

(1) *There is a unique, closable map $A_1 \widehat{\otimes} \dots \widehat{\otimes} A_n$ on $\mathcal{D}_1 \widehat{\otimes} \dots \widehat{\otimes} \mathcal{D}_n$ such that if $f_i \in \mathcal{D}_i$ for all i then $(A_1 \widehat{\otimes} \dots \widehat{\otimes} A_n) f_1 \otimes \cdots \otimes f_n = A_1 f_1 \otimes \cdots \otimes A_n f_n$. We define*

$$A_1 \otimes \cdots \otimes A_n = \overline{A_1 \widehat{\otimes} \dots \widehat{\otimes} A_n}.$$

Then $A_1 \otimes \cdots \otimes A_n = \overline{A_1} \otimes \cdots \otimes \overline{A_n}$ and $A_1 \otimes \cdots \otimes A_n$ is unitary if $A_1, \dots, A_n$ are unitary.

(2) *Define $T_i := (1 \otimes)^{i-1} A_i (\otimes 1)^{n-i} = B_1 \otimes \cdots \otimes B_n$ with $B_j = 1$ if $i \neq j$ and $B_i = A_i$. If A_i is essentially selfadjoint then T_i is selfadjoint and T_i will commute strongly with T_j. Furthermore, for any $f : \mathbb{R} \to C$ we have*

$$f(T_i) = (1 \otimes)^{i-1} f(A_i) (\otimes 1)^{n-i}.$$

In particular, $\sigma(T_i) = \sigma(A_i)$.

(3) *Define $T = T_1 + \cdots + T_n$ and assume A_i is essentially selfadjoint for all $1 \leq i \leq n$. Then T is essentially selfadjoint on $\mathcal{D}_1 \widehat{\otimes} \dots \widehat{\otimes} \mathcal{D}_n$ and the unitary group is given by $e^{it\overline{T}} = e^{itA_1} \otimes \cdots \otimes e^{itA_n}$. If $-\infty < \lambda_i := \inf(\sigma(A_i))$ for all $1 \leq i \leq n$ then $\inf(\sigma(T)) = \lambda := \lambda_1 + \cdots + \lambda_n$, T is selfadjoint and $e^{-tT} = e^{-tA_1} \otimes \cdots \otimes e^{-tA_n}$. If λ_i is a non degenerate eigenvalue of A_i for all $1 \leq i \leq n$, then λ is a non degenerate eigenvalue for T.*

(4) *Let $Q_i = (\mathcal{M}_i, \mathcal{F}_i, \mu_i)$ be a σ-finite measure spaces for $i \in \{1, 2, \dots, n\}$ and $\mathcal{H}$ be a separable Hilbert space. Then we may identify*

$$L^2(Q_1) \otimes \cdots \otimes L^2(Q_n) = L^2(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n, \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n, \mu_1 \otimes \cdots \otimes \mu_n)$$
$$L^2(Q_1) \otimes \mathcal{H} = L^2(Q_1, \mathcal{H})$$

where $(f_1 \otimes \cdots \otimes f_n)(x_1, \dots, x_n) = f_1(x_1) \cdot \cdots \cdot f(x_n)$ and $f \otimes \psi(x) = f(x)\psi$. In particular, one may identify $L^2(\mathcal{M}_1 \times \mathcal{M}_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2) = L^2(Q_1, L^2(Q_2))$ where $\psi \in L^2(\mathcal{M}_1 \times \mathcal{M}_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ is identified with the element $x \mapsto \psi(x, \cdot)$.

Proof. See [2], [18] and [21].

Of special interest to this paper is vector valued L^2 spaces. Let $\mathcal{Q} = (\mathcal{M}, \mathcal{F}, \mu)$ be a σ-finite measure space and $\mathcal{H}$ a separable Hilbert space. Let $f : \mathcal{M} \to B(\mathcal{H})$ be strongly measureable (i.e. $x \mapsto f(x)\psi$ is measurable for all $\psi \in \mathcal{H}$) and bounded. Then we define the direct integral

$$I_\oplus(f) = \int_{\mathcal{M}}^{\oplus} f(x) d\mu(x)$$

as the bounded operator on $L^2(Q, \mathcal{H})$ defined by $I_\oplus(f)\psi(x) = f(x)\psi(x)$. One also has a direct integral for unbounded selfadjoint operators. Let $\{A_x\}_{x \in \mathcal{M}}$ be a collection on selfadjoint operators on $\mathcal{H}$. We say $\{A_x\}_{x \in \mathcal{M}}$ is strong resolvent measurable if $x \mapsto (A_x + i)^{-1}$ is strongly measurable. Then we define

$$I_\oplus(A_x)\psi(x) = A_x\psi(x)$$
$$\mathcal{D}(I_\oplus(A_x)) = \{\psi \in L^2(Q, \mathcal{H}) \mid \psi(x) \in \mathcal{D}(A_x) \text{ and } x \mapsto \|A_x\psi(x)\| \in L^2(Q)\}$$

The following Theorem sums up the results about direct integrals we shall need

Theorem 2.2. *Let $\{A_x\}_{x \in \mathcal{M}}$ be a collection on selfadjoint operators on $\mathcal{H}$. Then $x \mapsto (A_x + i)^{-1}$ is strongly measureable if and only if $x \mapsto e^{itA_x}$ is weakly measurable. In this case $I_\oplus(A_x)$ is selfadjoint and $x \mapsto (i + f(A_x))^{-1}$ is strongly measurable for all $f : \mathbb{R} \to \mathbb{R}$. Furthermore*

$$f(I_\oplus(A_x)) = I_\oplus(f(A_x)).$$

If $A_x \geq \lambda$ for all x we find $I_\oplus(A_x) \geq \lambda$ (use $f = 1_{(-\infty, \lambda)}$). If A is selfadjoint or bounded on $\mathcal{H}$ we may identify $1 \otimes A = I_\oplus(A)$ and if V is a multiplication operator on $L^2(Q)$ then $V \otimes 1 = I_\oplus(V)$.

Proof. See [16] and some easy calculations.

Throughout this paper we will write $\mathcal{H}$ for the state space of a single boson which is always assumed to be a separable Hilbert space. Let $n \in \mathbb{N}$ and $\mathcal{H}^{\otimes n}$ be the n-fold tensor product. If $B \in B(\mathcal{H})$ we also write $B^{\otimes n}$ for the n-fold tensor product of B. Write $\mathcal{S}_n$ for the set of permutations of $\{1, \dots, n\}$. The symmetric projection S_n is the unique bounded map satisfying

$$S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} := f_1 \otimes_s \cdots \otimes_s f_n$$

We further define $S_0 = 1 = B^{\otimes 0}$ on $\mathcal{H}^{\otimes 0} = \mathbb{C}$ and write $\mathcal{H}^{\otimes_s n} = S_n(\mathcal{H}^{\otimes n})$. Note $B^{\otimes n}$ maps symmetric tensors to symmetric tensors so $B^{\otimes n}\mid_{\mathcal{H}^{\otimes_s n}}$ is well defined as an operator on $\mathcal{H}^{\otimes_s 0}$. This implies that if ω is selfadjoint on $\mathcal{H}$ then $(e^{it\omega})^{\otimes n}$ leaves $\mathcal{H}^{\otimes_s n}$ invariant so by Theorem 2.1 we may define

$$d\Gamma(\omega) = 0 \oplus \bigoplus_{n=1}^{\infty} \overline{\sum_{k=1}^{n} (1\otimes)^{k-1}\omega(\otimes 1)^{n-k}\mid_{\mathcal{H}^{\otimes_s n}}} \text{ and } \Gamma(B) = \bigoplus_{n=0}^{\infty} B^{\otimes n}\mid_{\mathcal{H}^{\otimes_s n}}.$$

as operators on the bosonic (or symmetric) Fock space

$$\mathcal{F}_b(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s n}.$$

We will write an element $\psi \in \mathcal{F}_b(\mathcal{H})$ in terms of its coordinates $\psi = (\psi^{(n)})$ and define the vacuum $\Omega = (1, 0, 0, \dots)$. For $g \in \mathcal{H}$ we define a coherent state

$$\epsilon(g) = \sum_{n=0}^{\infty} \frac{g^{\otimes n}}{\sqrt{n!}}. \tag{2.1}$$

Let $\mathcal{D} \subset \mathcal{H}$ be a dense subspace. We also define

$$\mathcal{N} = \{(\psi^{(n)}) \in \mathcal{F}_b(\mathcal{H}) \mid \ \exists K \in \mathbb{N} \text{ s.t. } \psi^{(n)} = 0 \text{ for all } n \geq K\} \tag{2.2}$$
$$\mathcal{J}(\mathcal{D}) = \{\Omega\} \cup \{f_1 \otimes_s \cdots \otimes_s f_n \mid n \in \mathbb{N}\ f_i \in \mathcal{D}\}. \tag{2.3}$$
$$\mathcal{E}(\mathcal{D}) = \{\epsilon(f) \mid f \in \mathcal{D}\} \tag{2.4}$$

One may prove $\mathcal{N}, \mathcal{J}(\mathcal{D})$ and $\mathcal{E}(\mathcal{D})$ are dense. For $g \in \mathcal{H}$ one defines the annihilation operator $a(g)$ and creation operator $a^\dagger(g)$ on symmetric tensors in $\mathcal{F}_b(\mathcal{H})$ using $a(g)\Omega = 0, a^\dagger(g)\Omega = g$ and

$$a(g)(f_1 \otimes_s \cdots \otimes_s f_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \langle g, f_i\rangle f_1 \otimes_s \cdots \otimes_s \widehat{f_i} \otimes_s \cdots \otimes_s f_n$$
$$a^\dagger(g)(f_1 \otimes_s \cdots \otimes_s f_n) = \sqrt{n+1} g \otimes_s f_1 \otimes_s \cdots \otimes_s f_n$$

where $\widehat{f_i}$ means that f_i is omitted from the tensor product. One can show that these operators extends to closed operators on $\mathcal{F}_b(\mathcal{H})$ and that $(a(g))^* = a^\dagger(g)$. One may thus define the symmetric operator

$$\varphi(g) = \overline{a(g) + a^\dagger(g)}.$$

Let $\mathcal{U}(\mathcal{H})$ be the unitaries from $\mathcal{H}$ into $\mathcal{H}$. Fix now $U \in \mathcal{U}(\mathcal{H})$ and $h \in \mathcal{H}$. Then there is a unique unitary map $W(h, U)$, called a Weyl operator, such that

$$W(h, U)\epsilon(g) = e^{-\|h\|^2/2 - \langle f, Ug\rangle}\epsilon(h + Ug).$$

for all $g \in \mathcal{H}$. The properties of the above operators are

Proposition 2.3. *Let ω and C be selfadjoint on $\mathcal{H}$, U be unitary on $\mathcal{H}$ and $v, g \in \mathcal{H}$. Then*

(1) *$d\Gamma(\omega)$ is selfadjoint, essentially selfadjoint on the span of $\mathcal{J}(\mathcal{D}(\omega))$ and $e^{itd\Gamma(\omega)} = \Gamma(e^{it\omega})$. If $\omega \geq 0$ then $d\Gamma(\omega) \geq 0$ and $e^{-td\Gamma(\omega)} = \Gamma(e^{-t\omega})$. If $\omega \geq 0$ and injective then 0 is a non degenerate eigenvalue. Ω spans the eigenspace.*
(2) *If ω and B strongly commute then so does $d\Gamma(\omega)$ and $d\Gamma(B)$.*
(3) *$\varphi(v)$ is selfadjoint and $e^{it\varphi(v)} = W(-itv, 0)$.*
(4) *The following commutation relations holds*

$$\overline{[a(v), a(g)]} = 0 = \overline{[a^\dagger(v), a^\dagger(g)]} \text{ and } \overline{[a(v), a^\dagger(g)]} = \langle v, g\rangle. \tag{2.5}$$

$$\overline{[\varphi(v), \varphi(g)]} = 2i\mathrm{Im}(\langle v, g\rangle). \tag{2.6}$$

Furthermore, $a^\dagger(v)a(v) = d\Gamma(|v\rangle\langle v|)$.
(5) *If $v \in \mathcal{D}(\omega)$ then $\mathcal{N} \cap \mathcal{D}(\omega) \subset \mathcal{D}([d\Gamma(\omega), \varphi(v)])$ and*

$$\overline{[d\Gamma(\omega), \varphi(v)]} = -i\varphi(i\omega v) \tag{2.7}$$

(6) *$\Gamma(U)$ is unitary and $\Gamma(U)\varphi(v)\Gamma(U)^* = \varphi(Uv)$.*
(7) *Assume $\omega \geq 0$ is selfadjoint an injective on $\mathcal{H}$, $D, E \in \{a, a^\dagger, \varphi\}$ and let $g_1, g_2 \in \mathcal{D}(\omega^{-\frac{1}{2}})$. Then $D(g_1)E(g_2)$ is $d\Gamma(\omega)$ bounded and $D(g_1)$ is $d\Gamma(\omega)^{1/2}$ bounded. In particular $D(g_1)E(g_2)$ is N bounded so $\mathcal{N} \subset \mathcal{D}(D(g_1)E(g_2))$. We have the following bounds*

$$\|D(g_1)\psi\| \leq 2\|(\omega^{-\frac{1}{2}} + 1)g_1\|\|(d\Gamma(\omega) + 1)^{\frac{1}{2}}\psi\|$$

$$\|D(g_1)E(g_2)\psi\| \leq 15\|(\omega^{-\frac{1}{2}} + 1)g_1\|\|(\omega^{-\frac{1}{2}} + 1)g_2\|\|(d\Gamma(\omega) + 1)\psi\|$$

which holds on respectively $\mathcal{D}(d\Gamma(\omega)^{\frac{1}{2}})$ and $\mathcal{D}(d\Gamma(\omega))$. In particular $\varphi(g_1)$ is infinitesimally $d\Gamma(\omega)$ bounded. Furthermore $d\Gamma(\omega) + \varphi(g_1) \geq -\|\omega^{-\frac{1}{2}}g_1\|^2$.

Proof. See [1], [13] and some easy calculations.

Using the above results one can now easily conclude the following:

Proposition 2.4. *Let $Q = (\mathcal{M}, \mathcal{F}, \mu)$ be a sigma-finite measure space. Let $x \mapsto f_x \in \mathcal{H}$ and $x \mapsto g_x \in \mathcal{H}$ be measurable, $\{\omega_x\}_{x\in\mathcal{M}}$ a be strong resolvent measurable family of selfadjoint operators on $\mathcal{H}$ and $x \mapsto U_x \in B(\mathcal{H})$ be strongly measurable with $\|U_x\| \leq 1$. Then*

(1) *$\{\varphi(f_x)\}_{x\in\mathcal{M}}$, $\{a^\dagger(f_x)a(f_x)\}_{x\in\mathcal{M}}$ and $\{d\Gamma(\omega_x)\}_{x\in\mathcal{M}}$ are strong resolvent measurable and $x \mapsto \Gamma(U_x)$ is strongly measurable. We will write $\varphi_\oplus(f_x) = I_\oplus(\varphi(f_x))$, $a^\dagger_\oplus(f_x)a_\oplus(f_x) = I_\oplus(a^\dagger(f_x)a(f_x))$, $d\Gamma_\oplus(\omega_x) = I_\oplus(d\Gamma(\omega_x))$ and $\Gamma_\oplus(U_x) = I_\oplus(\Gamma(U_x))$.*
(2) *If U_x is unitary for all x then $\Gamma_\oplus(U_x)$ is unitary and $\Gamma_\oplus(U_x)\varphi_\oplus(f_x)\Gamma_\oplus(U_x)^* = \varphi_\oplus(U_xf_x)$.*
(3) *Assume $x \mapsto f_x$ and $x \mapsto g_x$ are bounded, $\omega_x \geq 0$ is injective for all $x \in \mathcal{M}$, $f_x, g_x \in \mathcal{D}(\omega_x^{-1/2})$ for all $x \in \mathcal{M}$ and the two maps $x \mapsto \omega_x^{-1/2}f_x$ and $x \mapsto \omega_x^{-1/2}g_x$ are bounded. Then $\varphi_\oplus(f_x)$ is $d\Gamma_\oplus(\omega)^{1/2}$ bounded, $\varphi_\oplus(g_x)\varphi_\oplus(f_x)$ is $d\Gamma_\oplus(\omega)$ bounded and $a^\dagger_\oplus(g_x)a_\oplus(f_x)$ is $d\Gamma_\oplus(\omega)$ bounded. We have the bounds*

$$\|\varphi_\oplus(f_x)\psi\| \leq 2 \sup_{x\in\mathcal{M}} (\|(\omega_x^{-\frac{1}{2}} + 1)f_x\|)\|(d\Gamma_\oplus(\omega_x) + 1)^{\frac{1}{2}}\psi\|$$

$$\|\varphi_\oplus(g_x)\varphi_\oplus(f_x)\psi\| \leq 15 \sup_{x\in\mathcal{M}} (\|(\omega_x^{-\frac{1}{2}} + 1)g_1\|\|(\omega_x^{-\frac{1}{2}} + 1)g_2\|)\|(d\Gamma_\oplus(\omega_x) + 1)\psi\|$$

In particular, $\varphi_\oplus(f_x)$ is infinitesimally $d\Gamma_\oplus(\omega)$ bounded.

We will need the following definition

Definition 2.5. *Let $\mathcal{L}_\nu = (\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$ be the ν-dimensional Lebesgue measure space. Let $x \mapsto f_x \in \mathcal{H}$ be bounded and measurable. We say it is weakly differentiable if for all $i \in \{1, \dots, \nu\}$ there is $x \mapsto g_x^{(i)} \in \mathcal{H}$ such that for all $\phi \in C_0^\infty(\mathbb{R}^\nu)$ and $\psi \in \mathcal{H}$ we have*

$$\int_{\mathbb{R}^\nu} \partial_{x_i}\phi(x)\langle \psi, f_x\rangle d\lambda_\nu(x) = -\int_{\mathbb{R}^\nu} \phi(x)\langle \psi, g_x^{(i)}\rangle d\lambda_\nu(x).$$

In this case we write $\partial_{x_i} f = g_x^{(i)}$.

We shall need the following result about differential operators

Lemma 2.6. *Define*

$$\mathcal{K} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu, \mathcal{F}_b(\mathcal{H})) = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu) \otimes \mathcal{F}_b(\mathcal{H}),$$

$p_i = -i\partial_{x_i} \otimes 1$ and $|p|= (-\Delta)^{1/2} \otimes 1$. Here Δ is the Laplace operator. Then

(1) *$\mathcal{D}(|p|) = \bigcap_{i=1}^\nu \mathcal{D}(p_i)$ and for $\psi \in \mathcal{D}(|p|)$ we have $\||p|\psi\|^2 = \sum_{i=1}^\nu \|p_i\psi\|^2$.*
(2) *If $x \mapsto f_x$ is weakly differentiable the $[\varphi_\oplus(f_x), p_i] = -i\varphi_\oplus(\partial_{x_i} f_x)$ holds on $C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D})$. In particular, $\varphi_\oplus(f_x)\psi \in \mathcal{D}(|p|)$ for $\psi \in C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D})$.*

Proof. To prove statement (1) we let F denote the Fourier transform and define the functions $f(k) = |k|$ and $f_i(k) = k_i$ form $\mathbb{R}^\nu$ to $\mathbb{R}$. By Theorem 2.2 we see $(F \otimes 1)|p|(F \otimes 1)^* = I_\oplus(f)$ and $(F \otimes 1)p_i(F \otimes 1)^* = I_\oplus(f_i)$ and so

$$\begin{aligned}(F \otimes 1)\mathcal{D}(|p|) &= \left\{\psi \in \mathcal{K} \mid |k|^2\|\psi(k)\|^2 \in L^1(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)\right\} \\ &= \left\{\psi \in \mathcal{K} \mid |k_i|^2\|\psi(k)\|^2 \in L^1(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu) \ \forall i \in \{1, \dots, \nu\}\right\} \\ &= (F \otimes 1)\bigcap_{i=1}^\nu \mathcal{D}(p_i)\end{aligned}$$

showing the domain identity. Now let $\psi \in C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{H}$. Then

$$\||p|\psi\|^2 = \langle \psi, -\Delta \otimes 1\psi\rangle = \sum_{i=1}^\nu \langle \psi, p_i^2\psi\rangle = \sum_{i=1}^\nu \|p_i\psi\|^2 \tag{2.8}$$

For general $\psi \in \mathcal{D}(|p|)$ we may pick a sequence $\{\psi_n\}_{n=1} \subset C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{H}$ approximating $\psi \in |p|$ norm. Using equation (2.8) we see $\{\psi_n\}_{n=1}$ is Cauchy in p_i norm for all $i \in \{1, \dots, \nu\}$, so as the p_i are closed we find $\{\psi_n\}_{n=1}$ converges to ψ in p_i norm for all $i \in \{1, \dots, \nu\}$. The result now follows taking limits in equation (2.8).

Statement (2) is proven in [6] where the author concludes that $\varphi_\oplus(f_x)\psi \in \mathcal{D}(|p|^2)$ for $\psi \in C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D})$. This is not true, but his proof works well enough to prove statement (2).

3. The Hamiltonian(s) - definition and results

The full Hamiltonian is defined on $L^2(\mathcal{L}_\nu, \mathcal{F}_b(\mathcal{H}))$ where $\mathcal{H}$ is for now taken to be an abstract Hilbert space. Let $\{u_x\}_{x\in\mathbb{R}^\nu}, \{v_x\}_{x\in\mathbb{R}^\nu} \subset \mathcal{H}$ be strongly measurable families and ω be selfadjoint on $\mathcal{H}$. The full Hamiltonian takes the form

$$H^V_{g_1,g_2} = \left(-\frac{1}{2M}\Delta + V\right) \otimes 1 + 1 \otimes d\Gamma(\omega) + g_1\varphi_\oplus(u_x - v_x) + g_2 a^\dagger_\oplus(v_x)a_\oplus(v_x)$$
$$+ g_2 a^\dagger_\oplus(u_x)a_\oplus(u_x) - g_2 \int^\oplus_{\mathbb{R}^\nu} a(U_x v)a(U_x u) + a^\dagger(U_x v)a^\dagger(U_x u)dx$$

where V is a multiplication operator. A priori it is not clear that this operator makes sense on any domain since the last direct integral is not of a selfadjoint operator. We shall show later that under certain assumptions it will make sense however. Here $V = 0$ corresponds to the translation invariant case.

We shall also need the fiber Hamiltonians. Let $u, v \in \mathcal{H}$ and ω be selfadjoint on $\mathcal{H}$. Let $m = (m^{(1)}, \dots, m^{(\nu)})$ be a touple of commuting selfadjoint operators on $\mathcal{H}$. Then we define

$$H_{g_1,g_2}(\xi) = \frac{1}{2M}(\xi - d\Gamma(m))^2 + d\Gamma(\omega) + g_1\varphi(u - v) + g_2 a^\dagger(u)a(u)$$
$$+ g_2 a^\dagger(v)a(v) - g_2 a(v)a(u) - g_2 a^\dagger(v)a^\dagger(u)$$

where $(\xi - d\Gamma(k))^2 = \sum_{i=1}^\nu (\xi_i - d\Gamma(m^{(i)}))^2$ by definition.

Remark 3.1. For the reader comparing this to the papers [10] and [19] please note that in this case $W := (u - v)(k) = \sqrt{\frac{\frac{1}{2M}\|k\|^2}{\omega}}$ and $(u+v)(u-v) = 1$ which fixes the functions u and v. One then introduces a ultraviolet cutoff in u and v to make sense of the operator.

Furthermore, defining $U_x\psi(k) = e^{ikx}\psi(k)$ we have $u_x = U_x u, v_x = U_x v$. The operator ω is multiplication by $\omega(k) = \sqrt{a|k|^2 + |k|^4}$ and $m^{(i)}$ is multiplication by the projection $m^{(i)}(k) = k_i \in \mathbb{R}$. All results except uniqueness of the minimum obtained in Theorem 3.2 part (5) below applies to this situation. In particular the Hamiltonians above are selfadjoint.

Hypothesis 1: Under Hypothesis 1 we assume

1. $\omega, m^{(1)}, \dots, m^{(\nu)}$ are strongly commuting selfadjoint operators. Furthermore ω is non negative and injective.
2. $v, u \in \mathcal{D}(\omega^{-1/2}) \cap \mathcal{D}(\omega^{1/2}) \cap \bigcap_{j=1}^\nu \mathcal{D}(m^{(j)}) \cap \mathcal{D}(\omega^{-1/2}m^{(j)})$.

Hypothesis 2: $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$, ω is multiplication by a continuous function and $m^{(j)}$ is multiplication by $m^{(j)}(k) = k_j$.

Hypothesis 3: Assume in addition $\langle a, e^{-t\omega}e^{it_1 m^{(1)}} \dots e^{it_\nu m^{(\nu)}} b\rangle \in \mathbb{R}$ for all $t \geq 0$, $t_1, \dots, t_\nu \in \mathbb{R}$ and $a, b \in \{u, v\}$

Theorem 3.2. *Assume Hypothesis 1 holds. If $g_1 \in \mathbb{R}$, $g_2 \geq 0$ and $\xi \in \mathbb{R}^\nu$ then $H_{g_1,g_2}(\xi)$ is selfadjoint on $\mathcal{D}(d\Gamma(\omega)) \cap \mathcal{D}((d\Gamma(m))^2)$, bounded below and essentially selfadjoint on $\mathcal{D}(d\Gamma(\omega)) \cap \mathcal{D}((d\Gamma(m))^2) \cap \mathcal{N}$. Furthermore, we also have:*

(1) *$\xi \mapsto H_{g_1,g_2}(\xi)$ is an analytic family of type A, so the map $\xi \mapsto (H_{g_1,g_2}(\xi)+i)^{-1}$ is smooth.*
(2) *The map $\Sigma(\xi) = \inf(\sigma(H_{g_1,g_2}(\xi)))$ is locally Lipshitz and almost everywhere twice differentiable.*
(3) *Assume Hypothesis 2 holds as well. Then*

$$\Sigma(\xi - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n) \in \sigma_{ess}(H_{g_1,g_2}(\xi))$$

for all $k_1, \ldots, k_n \in \mathbb{R}^\nu$. If in addition $\inf_{k\in\mathbb{R}^\nu} \omega(k) > 0$ or $\omega(0) = 0$ then

$$\inf(\sigma_{ess}(H_{g_1,g_2}(\xi))) = \inf_{n\in\mathbb{N}_0} \inf_{\xi\in\mathbb{R}^\nu} \Sigma(\xi - k_1 - \cdots - k_n) + \omega_n(k_1, \ldots, k_n).$$

If ω is also unbounded we have $\sigma_{ess}(H_{g_1,g_2}(\xi)) = [\inf(\sigma_{ess}(H_{g_1,g_2}(\xi))), \infty)$.
(4) *Assume Hypothesis 2 holds as well. Define $u_x = e^{ix_1 m^{(1)}} \ldots e^{ix_\nu m^{(\nu)}} u$ and $v_x = e^{ix_1 m^{(1)}} \ldots e^{ix_\nu m^{(\nu)}} v$. Then there is a unitary map (called the Lee Low Pines transformation) such that*

$$UH^0_{g_1,g_2}U^* = \int_{\mathbb{R}^\nu}^{\oplus} H_{g_1,g_2}(\xi) d\lambda_\nu(\xi)$$

If in addition we assume Hypothesis 3 then $H^0_{g_1,g_2}$ has no ground state.
(5) *Assume Hypothesis 3 holds as well. Then Σ has a global minimum at $\xi = 0$ and if $H_{g_1,g_2}(0)$ has a ground state then it is non degenerate. If $\inf(\sigma(\omega)) > 0$ and we additionally assume Hypothesis 2 holds, then 0 is the unique minima.*

Hypothesis 4: We assume the following minimal properties

1. $V \in L^2_{loc}(\mathbb{R}^\nu)$ and $-\frac{1}{2M}\Delta + V$ is essentially selfadjoint on $C^\infty(\mathbb{R}^\nu)$. Defining $V_- = \max\{0, -V\}$ we also assume $V_-^{1/2}$ is relatively $(-\frac{1}{2M}\Delta)^{1/2}$ bounded with bound smaller than 1.
2. ω is selfadjoint, non negative and injective on $\mathcal{H}$.
3. $x \mapsto v_x$ and $x \mapsto u_x$ are weakly differentiable maps. Both maps takes values in $\mathcal{D}(\omega^{-1/2}) \cap \mathcal{D}(\omega^{-1/2})$ and the partial derivatives takes values in $\mathcal{D}(\omega^{-1/2})$. Furthermore

$$\sup_{x\in\mathbb{R}^\nu} \{\|(1 + \omega^{-1/2} + \omega^{1/2})v_x\|, \|(1 + \omega^{-1/2} + \omega^{1/2})u_x\|\} < \infty$$
$$\sup_{x\in\mathbb{R}^\nu, i\in\{1,\ldots,\nu\}} \{\|(1 + \omega^{-1/2})\partial_{x_i} v_x\|, \|(1 + \omega^{-1/2})\partial_{x_i} u_x\|\} < \infty$$

Theorem 3.3. *Assume Hypothesis 1 holds, that $g_1 \in \mathbb{R}$ and $g_2 \geq 0$. Let S be the selfadjoint closure of $-\frac{1}{2M}\Delta + V$. Then S is bounded below and $H^V_{g_1,g_2}$ is selfadjoint on $\mathcal{D}(S \otimes 1) \cap \mathcal{D}(d\Gamma_\oplus(\omega))$, bounded below and essentially selfadjoint on any core for $S \otimes 1 + d\Gamma_\oplus(\omega)$. One example of a core is $C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D}(\omega))$.*

Assume in addition that $\langle a, e^{-t\omega} b\rangle \in \mathbb{R}$ for all $t \geq 0$ and $a, b \in \{v_x\}_{x\in\mathbb{R}^\nu} \cup \{u_x\}_{x\in\mathbb{R}^\nu}$. If $H^V_{g_1,g_2}$ has a ground state, then it is non degenerate and any eigenvector will have non zero inner product with any vector of the form $\phi \otimes \Omega$ with $\phi \neq 0$ and non negative.

The reason for bringing a potential is the the particles are really confined by an external potential so it is really of some interest to bring it along. The reason for looking at compatible interactions is that it makes the notation somewhat smoother and the argument is really the same.

The proofs of both Theorem 3.2 and Theorem 3.3 relies heavily on properties of positivity improving semigroups which usually on gives information about the bottom of the spectrum. Actually one should be able to prove that $H_0 = H$ has no bound states, however this is not an easy task even though it seems obvious that translation invariant systems cannot have bound states. To prove that certain operators generate a positivity improving semigroup we shall prove a perturbation theorem which will be presented below. Similar theorems can be found in [4], [11] and [16]. However both [4] and [11] are missing a key assumption about uniform lower bounds of approximating Hamiltonians in their proof. This assumption is not appropriate for the setting in this paper which will become apparent later. Let A be selfadjoint operator. In the following we will let q_A denote the quadratic form

$$q_A(\psi, \phi) = \langle \mathrm{Sign}(A)|A|^{1/2}\psi, |A|^{1/2}\phi\rangle \quad \mathcal{D}(q_A) = \mathcal{D}(|A|^{1/2})$$

Instead of working directly with operators it is easier to work directly with quadratic forms. This results in the following two results which are each of independent interest.

Theorem 3.4. *Let A be bounded below and selfadjoint on $L^2(\mathcal{M}, \mathcal{F}, \mu)$. Let B_+ and B_- multiplication operators such that B_+ is bounded below and assume*

1. *e^{-tA} is positivity improving for all $t \geq 0$*
2. *$\mathcal{D}(q_{B_+})$ contains a form core for q_A and $\mathcal{D}(q_A) \cap \mathcal{D}(q_{B_+}) \subset \mathcal{D}(q_{B_-})$.*
3. *The form $q = q_A + q_{B_+} + q_{B_-}$ is closed and bounded below.*

Then the operator H corresponding to q is bounded below and e^{-tH} is positivity improving.

Theorem 3.5. *Let A, B, C be selfadjoint operators in $L^2(\mathcal{M}, \mathcal{F}, \mu)$. Assume*

1. *A is bounded below and e^{-tA} is positivity improving for all $t \geq 0$.*
2. *B is a multiplication operator which is bounded from below.*
3. *$-C \geq 0$ and C is a multiplication operator.*
4. *$\mathcal{D}(q_B)$ contains a form core for q_A and $\mathcal{D}(q_A) \cap \mathcal{D}(q_B) \subset \mathcal{D}(q_C) \subset \mathcal{D}(q_C)$.*
5. *The form $q = q_A + q_B + q_C$ is closable and bounded below.*

Then the operator H corresponding to q is bounded below and e^{-tH} is positivity improving.

4. Proof of selfadjointness

We shall need the following Lemma.

Lemma 4.1. *Assume $v, u \in \mathcal{D}(\omega^{-1/2})$ where ω is selfadjoint on $\mathcal{H}$, injective and non negative. The following identities hold on $\mathcal{D}(d\Gamma(\omega))$*

$$\begin{aligned}\varphi(u-v)^2 + \varphi(i(u+v))^2 =& 4a^\dagger(u)a(u) + 4a^\dagger(v)a(v) - 4(a^\dagger(u)a^\dagger(v) + a(u)a(v)) \\ &+ 2\|u\|^2 + 2\|v\|^2 \\ =& 4a^\dagger(u+v)a(u+v) + 2\varphi(v)\varphi(u) + 2\|u\|^2 + 2\|v\|^2 \\ &+ 4\mathrm{Re}(\langle u, v\rangle) + 2i\mathrm{Im}(\langle u, v\rangle)\end{aligned}$$

Proof. The following calculations holds on $\mathcal{D}(d\Gamma(A))$

$$\begin{aligned}\varphi(u-v)^2+\varphi(i(u+v))^2 &= \varphi(u)^2+\varphi(v)^2-\varphi(v)\varphi(u)-\varphi(u)\varphi(v)\\ &\quad+\varphi(iu)^2+\varphi(iv)^2+\varphi(iv)\varphi(iu)+\varphi(iu)\varphi(iv).\end{aligned}$$

Using Proposition 2.3 we find

$$\begin{aligned}\varphi(u)^2+\varphi(iu)^2 =& a^\dagger(u)a(u)+a(u)a^\dagger(u)+a^\dagger(u)^2+a(u)^2\\ &+a^\dagger(iu)a(iu)+a(iu)a^\dagger(iu)+a^\dagger(iu)^2+a(iu)^2\\ =& a^\dagger(u)a(u)+a(u)a^\dagger(u)+a^\dagger(u)^2+a(u)^2\\ &+a^\dagger(u)a(u)+a(u)a^\dagger(u)-a^\dagger(u)^2-a(u)^2\\ =& 4a^\dagger(u)a(u)+2\|u\|^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\varphi(iv)\varphi(iu)-\varphi(v)\varphi(u) =& a^\dagger(iv)a(iu)+a^\dagger(iu)a(iv)+a^\dagger(iu)a^\dagger(iv)+a(iu)a(iv)\\ &-a^\dagger(v)a(u)-a^\dagger(u)a(v)-a^\dagger(u)a^\dagger(v)-a(u)a(v)\\ =& a^\dagger(v)a(u)+a^\dagger(u)a(v)-a^\dagger(u)a^\dagger(v)-a(u)a(v)\\ &-a^\dagger(v)a(u)-a^\dagger(u)a(v)-a^\dagger(u)a^\dagger(v)-a(u)a(v)\\ =& -2(a^\dagger(u)a^\dagger(v)+a(u)a(v))\end{aligned}$$

Thus we finally arrive at

$$\begin{aligned}\varphi(u-v)^2+\varphi(i(u+v))^2 &= 4a^\dagger(u)a(u)+4a^\dagger(v)a(v)-4(a^\dagger(u)a^\dagger(v)+a(u)a(v))\\ &\quad+2\|u\|^2+2\|v\|^2.\end{aligned}$$

We may also calculate

$$\begin{aligned}&\varphi(v-u)^2+\varphi(i(v+u))^2 = \varphi(v-u)^2+\varphi(i(v+u))^2+\varphi(v+u)^2-\varphi(v+u)^2\\ &= 4a^\dagger(u+v)a(u+v)+2\|u+v\|^2+\varphi(v-u)^2-\varphi(v+u)^2\\ &= 4a^\dagger(u+v)a(u+v)+2\|u+v\|^2+\varphi(u)\varphi(v)+\varphi(v)\varphi(u)\\ &= 4a^\dagger(u+v)a(u+v)+2\varphi(v)\varphi(u)+2\|u\|^2+2\|v\|^2+4\mathrm{Re}(\langle u,v\rangle)+2i\mathrm{Im}(\langle u,v\rangle)\end{aligned}$$

finishing the proof. □

Before we proceed we will make the following definitions for $u,v\in\mathcal{H}$

$$\begin{aligned}C(u,v,g_2) &:= \frac{1}{2}(\|u\|^2+\|v\|^2)\\ D(u,v,g_2) &:= \frac{1}{2}(\langle u,v\rangle+\mathrm{Re}(\langle u,v\rangle))\end{aligned}$$

We will start with proving $H_{g_1,g_2}(\xi)$ is selfadjoint under the given assumptions. The main calculations are contained in the following Lemma.

Lemma 4.2. *Let ω, m be a selfadjoint and strongly commuting operators on $\mathcal{H}$. Assume $\omega \geq 0$ is injective and $v \in \mathcal{D}(\omega^{-1/2}) \cap \mathcal{D}(\omega^{1/2}) \cap \mathcal{D}(m) \cap \mathcal{D}(\omega^{-1/2}m)$. Then*

$$2Re(\langle d\Gamma(\omega)\psi, \varphi(v)^2\psi\rangle) \geq -2\|\omega^{1/2}v\|\|\psi\|^2 \qquad \psi \in \mathcal{D}(d\Gamma(\omega)) \cap \mathcal{N}$$
$$2Re(\langle (c - d\Gamma(m))^2\psi, \varphi(v)^2\psi\rangle) \geq \varepsilon(\|(c - d\Gamma(m))^2\psi\|^2 + \|d\Gamma(\omega)\psi\|^2) - K\|\psi\|^2$$

for all $\varepsilon > 0$, $c \in \mathbb{R}$ and $\psi \in \mathcal{D}(d\Gamma(\omega)) \cap \mathcal{D}(d\Gamma(m^2)) \cap \mathcal{N}$. Here we may choose

$$K = \varepsilon + 16\varepsilon^{-1}R^4 + 64R^2(1 + 8\varepsilon^{-1}R^2)$$

as long $\|v\|, \|mv\|, \|\omega^{-1/2}mv\|$ are smaller than some $R > 0$.

Proof. Let $\psi \in \mathrm{Span}(\mathcal{J}(\mathcal{D}(\omega)))$ and define $\omega_\ell = \omega 1_{[0,\ell]}(\omega)$ via the spectral calculus. Using Proposition 2.3 we calculate on $\mathcal{N}$

$$d\Gamma(\omega_\ell)\varphi(v)^2 = \varphi(v)d\Gamma(\omega_\ell)\varphi(v) - i\varphi(i\omega_\ell v)\varphi(v)$$

So

$$2\mathrm{Re}(\langle d\Gamma(\omega_\ell)\psi, \varphi(v)^2\psi\rangle) = 2\|d\Gamma(\omega_\ell)^{1/2}\varphi(v)\psi\|^2 + 2\mathrm{Im}(\langle \psi, \varphi(i\omega_\ell v)\varphi(v)\psi\rangle)$$

Now

$$\begin{aligned} 2\mathrm{Im}(\langle \psi, \varphi(i\omega_\ell v)\varphi(v)\psi\rangle) &= \frac{1}{i}\langle \psi, [\varphi(i\omega_\ell v), \varphi(v)]\psi\rangle \\ &= 2\mathrm{Im}(\langle i\omega_\ell v, v\rangle)\|\psi\|^2 \\ &= -2\|\omega_\ell^{1/2}v\|\|\psi\|^2 \end{aligned}$$

So we end up with

$$2\mathrm{Re}(\langle d\Gamma(\omega_\ell)\psi, \varphi(v)^2\psi\rangle) \geq -2\|\omega_\ell^{1/2}v\|\|\psi\|^2.$$

As ℓ goes to infinity we see $\|\omega_\ell^{1/2}v\|$ converges to $\|\omega^{1/2}v\|$ by the functional calculus. For $\phi \in \mathcal{J}(\mathcal{D}(\omega))$ one easily sees $d\Gamma(\omega_\ell)\phi$ converges to $d\Gamma(\omega)\phi$ applying the functional calculus to each factor in the tensor product. Thus taking ℓ to infinity yields the result for $\psi \in \mathrm{Span}(\mathcal{J}(\mathcal{D}(\omega)))$. For general $\psi \in \mathcal{D}(d\Gamma(\omega))$ we may find a sequence of elements $\{\psi_n\}_{n=1}^\infty \subset \mathrm{Span}(\mathcal{J}(\mathcal{D}(\omega)))$ converging to ψ in $\mathcal{D}(d\Gamma(\omega))$-norm. As $\varphi(v)^2$ is $\mathcal{D}(d\Gamma(\omega))$ bounded we see convergence holds in $\varphi(v)^2$-norm as well. Taking n to infinity in

$$2\mathrm{Re}(\langle d\Gamma(\omega)\psi_n, \varphi(v)^2\psi_n\rangle) \geq -2\|\omega^{-1/2}v\|\|\psi_n\|^2$$

yields the result.

Let $\psi \in \mathcal{D}(d\Gamma(m)^2) \cap \mathcal{D}(d\Gamma(\omega)) \cap \mathcal{N}$. Define $d\Gamma_c(m) = c - d\Gamma_c(m)$. Using Proposition 2.3 and $d\Gamma_c(m)\psi \in \mathcal{D}(d\Gamma_c(m)) \cap \mathcal{N}$ we find

$$\begin{aligned} \langle d\Gamma_c(m)^2\psi, \varphi(v)^2\psi\rangle =& \langle \varphi(v)d\Gamma_c(m)^2\psi, \varphi(v)\psi\rangle \\ =& \langle d\Gamma_c(m)\varphi(v)d\Gamma_c(m)\psi, \varphi(v)\psi\rangle - i\langle \varphi(imv)d\Gamma_c(m)\psi, \varphi(v)\psi\rangle \\ =& \|d\Gamma_c(m)\varphi(v)\psi\|^2 - i\langle \varphi(imv)d\Gamma_c(m)\psi, \varphi(v)\psi\rangle \\ &- i\langle \varphi(imv)\psi, d\Gamma_c(m)\varphi(v)\psi\rangle \end{aligned}$$

Now

$$\langle \varphi(imv)d\Gamma_c(m)\psi, \varphi(v)\psi\rangle = 2i\mathrm{Im}(\langle imv, v\rangle)\langle d\Gamma_c(m)\psi,\psi\rangle + \langle d\Gamma_c(m)\psi, \varphi(v)\varphi(imv)\psi\rangle$$
$$\langle \varphi(imv)\psi, d\Gamma_c(m)\varphi(v)\psi\rangle = -i\|\varphi(imv)\psi\|^2 + \langle \varphi(v)\varphi(imv)\psi, d\Gamma_c(m)\psi\rangle$$

Defining $a = \langle d\Gamma_c(m)\psi, \varphi(v)\varphi(imv)\psi\rangle$ we finally arrive at

$$\begin{aligned}2\mathrm{Re}(\langle d\Gamma_c(m)^2\psi, \varphi(v)^2\psi\rangle) =& 2\|d\Gamma_c(m)\varphi(v)\psi\|^2 \\ &+ 2\mathrm{Re}(2\mathrm{Im}(\langle imv, v\rangle)\langle \psi, d\Gamma_c(m)\psi\rangle - ia) \\ &+ 2\mathrm{Re}(-\|\varphi(imv)\psi\|^2 - i\overline{a}) \\ \geq& -\langle mv, v\rangle\langle \psi, d\Gamma_c(m)\psi\rangle + 2\mathrm{Im}(a+\overline{a}) - 2\|\varphi(imv)\psi\|^2 \\ =& -4\langle mv, v\rangle\langle \psi, d\Gamma_c(m)\psi\rangle - 2\|\varphi(imv)\psi\|^2\end{aligned}$$

Using Proposition 2.3 we estimate

$$\begin{aligned}&\|\varphi(imv)\psi\|^2 \leq 4\|(1+\omega^{-1/2})mv\|^2\|(d\Gamma(\omega)+1)^{1/2}\psi\|^2 \\ &\leq 4\|(1+\omega^{-1/2})mv\|^2(\|\psi\|^2 + \|\psi\|\|d\Gamma(\omega)\psi\|) \\ &\leq 2^{-1}\varepsilon\|d\Gamma(\omega)\psi\|^2 + 4\|(1+\omega^{-1/2})mv\|^2(1+2\varepsilon^{-1}\|(1+\omega^{-1/2})mv\|^2)\|\psi\|^2\end{aligned}$$

We also estimate

$$\begin{aligned}|\langle v, mv\rangle\langle \psi, d\Gamma_c(m)\psi\rangle| &\leq \|v\|\|mv\|\|\psi\|\|(d\Gamma_c(m)^2+1)^{1/2}\psi\| \\ &= (4\varepsilon^{-1}\|v\|^2\|mv\|^2 + 4^{-1}\varepsilon)\|\psi\|^2 + 4^{-1}\varepsilon\|d\Gamma_c(m)^2\psi\|^2\end{aligned}$$

This implies

$$2\mathrm{Re}(\langle d\Gamma_c(m)^2\psi, \varphi(v)^2\psi\rangle) \geq \varepsilon(\|d\Gamma_c(m)^2\psi\|^2 + \|d\Gamma(\omega)\psi\|^2) - K\|\psi\|^2$$

where

$$K = \varepsilon + 16\varepsilon^{-1}\|v\|^2\|mv\|^2 + 16\|(1+\omega^{-1/2})mv\|^2(1+2\varepsilon^{-1}\|(1+\omega^{-1/2})mv\|^2)$$

We find the desired inequality when $\|(1+\omega^{-1/2})mv\|^2$ is estimated by $4R^2$, $\|mv\|$ is estimated by R and $\|v\|$ is estimated by R.

Lemma 4.3. *Assume Hypothesis 1. For $\psi \in \mathcal{D}(H_{0,0}(0))$ and $\xi \in \mathbb{R}^\nu$ we have*

$$\max_{1\leq i\leq \nu}\{\|d\Gamma(\omega)\psi\|, \|(\xi_i - d\Gamma(m^{(i)}))^2\psi\|\} \leq \|H_{0,0}(\xi)\| \tag{4.1}$$

There is a constant γ independent of ξ such that

$$\frac{1}{2}\|H_{0,0}(\xi)\psi\|^2 \leq \gamma\|\psi\|^2 + \|H_{0,g_2}(\xi)\psi\|$$

for all $\psi \in \mathcal{D}(H_{0,0}(0)) \cap \mathcal{N}$. Given $R > 0$ one may choose the same γ for all choices of $u, v, \omega, m^{(i)}$ where

$$\|u\|, \|m^{(i)}u\|, \|\omega^{-1/2}m^{(i)}u\|, \|\omega^{1/2}u\|, \|v\|, \|m^{(i)}v\|, \|\omega^{-1/2}m^{(i)}v\|, \|\omega^{1/2}v\| \leq R$$

Proof. As $d\Gamma(m^{(i)})$ and $d\Gamma(\omega)$ commute strongly and $d\Gamma(\omega) \geq 0$ we find from the spectral theorem that

$$\langle (\xi_i - d\Gamma(m^{(i)}))^2\psi, d\Gamma(\omega)\psi\rangle \geq 0.$$

Similarly $\langle (\xi_i - d\Gamma(m^{(i)}))^2\psi, (\xi_j - d\Gamma(m^{(j)}))^2\psi\rangle \geq 0$ which implies

$$\begin{aligned}\|H_{0,0}(\xi)\psi\|^2 = &\sum_{i=1}^{\nu}\|(\xi_i - d\Gamma(m^{(i)}))^2\psi\|^2 + 2\mathrm{Re}(\langle (\xi_i - d\Gamma(m^{(i)}))^2\psi, d\Gamma(\omega)\psi\rangle)\\ &+ \|d\Gamma(\omega)\psi\|^2 + \sum_{i>j=1}^{\nu} 2\mathrm{Re}(\langle (\xi_i - d\Gamma(m^{(i)}))^2\psi, (\xi_j - d\Gamma(m^{(j)}))^2\psi\rangle)\\ &\geq \max_{1\leq i\leq\nu}\{\|d\Gamma(\omega)\psi\|^2, \|(\xi_i - d\Gamma(m^{(i)}))^2\psi\|^2\}\end{aligned}$$

Define now $a = u - v$, $b = i(u + v)$ and $g = 4^{-1}g_4$. For $\psi \in \mathcal{N} \cap \mathcal{D}(H_{0,0}(\xi))$ we calculate using Lemma 4.1

$$\begin{aligned}\|(H_{0,g_2}(\xi) - C(g_2, u, v))\psi\|^2 = &\|H_{0,0}(\xi)\psi\|^2 + g^2\|(\varphi(a)^2 + \varphi(b)^2)\psi\|^2\\ &+ g2\mathrm{Re}(d\Gamma(\omega)\psi, (\varphi(a)^2 + \varphi(b)^2)\psi\rangle)\\ &+ \sum_{i=1}^{\nu} g2\mathrm{Re}(d\Gamma(k_i)^2\psi, (\varphi(a)^2 + \varphi(b)^2)\psi\rangle)\end{aligned}$$

Using Lemma 4.2 with $\varepsilon' = \frac{\varepsilon}{4\nu g(\nu+1)}$ we now arrive at

$$\begin{aligned}\|(H_{0,g_2}(\xi) - C(g_2, u, v))\psi\|^2 \geq &\|H_{0,0}(\xi)\psi\|^2\\ &- \frac{1}{2(\nu+1)}\left(\|d\Gamma(\omega)\psi\|^2 + \sum_{i=1}^{\nu}\|(\xi_i - d\Gamma(k_i))^2\psi\|^2\right) - \widetilde{K}\|\psi\|\end{aligned}$$

where $\widetilde{K}$ is independent of ξ and $\widetilde{K}$ may chosen to have the same value for all choices of $u, v, \omega, m^{(i)}$ where

$$\|u\|, \|m^{(i)}u\|, \|\omega^{-1/2}m^{(i)}u\|, \|\omega^{1/2}u\|, \|v\|, \|m^{(i)}v\|, \|\omega^{-1/2}m^{(i)}v\|, \|\omega^{1/2}v\| \leq R$$

Define $\gamma = \sqrt{\widetilde{K}} + C(g_2, u, v)$ and use equation (4.1) along with the triangle inequality to find the desired result.

Lemma 4.4. *Assume Hypothesis 1. Then $H_{g_1,g_2}(\xi)$ is closed on $\mathcal{D}(H_{g_1,g_2}(0)) = \mathcal{D}(d\Gamma(\omega)) \cap \bigcap_{i=1}^{\nu}\mathcal{D}(d\Gamma(m^{(i)})^2)$ for $g_2 \in [0,\infty)$, $\xi \in \mathbb{C}^\nu$ and $g_1 \in \mathbb{C}$. It is selfadjoint and bounded below if $g_1 \in \mathbb{R}$ and $\xi \in \mathbb{R}^\nu$. Furthermore, $\mathcal{D}(H_{g_1,g_2}(0)) \cap \mathcal{N}$ is a core.*

Proof. We start by noting $H_{0,0}(0)$ is symmetric is a sum of non negative and commuting operators. Thus it is selfadjoint (see [14]). Both $d\Gamma(m^{(i)})$ $d\Gamma(\omega)$ strongly commute with the number operator N by Proposition 2.3. Using [18, 5.26 and 5.27] we see $1_{\{N\leq k\}}\psi \in \mathcal{D}(H_{0,0}(0))$ and $H_{0,0}(0)1_{\{N\leq k\}}\psi = 1_{\{N\leq k\}}H_{0,0}(0)\psi$ for any $\psi \in \mathcal{D}(H_{0,0}(0))$. Taking k to infinity shows $1_{\{N\leq k\}}\psi$ converges to ψ in $H_{0,0}(0)$ norm so $\mathcal{N} \cap \mathcal{D}(H_{0,0}(0))$ is a core for $H_{0,0}(0)$. By Lemmas 4.1, 4.3 and

A.3 we see $H_{0,g_2}(0)$ is selfadjoint, bounded below and has $\mathcal{D}(H_{g_1,g_2}(0)) \cap \mathcal{N}$ as a core. To finish the proof we note that

$$H_{g_1,g_2}(\xi) = H_{0,g_2}(0) + g_1\varphi(u-v) + \sum_{i=1}^{n} u\xi_i^2 + \xi_i d\Gamma(m^{(i)}).$$

Now $d\Gamma(m^{(i)})$ is infinitesimally $d\Gamma(m^{(i)})^2$ bounded and $\varphi(u-v)$ is infinitesimally $d\Gamma(\omega)$ bounded. As both $d\Gamma(m^{(i)})^2$ and $d\Gamma(\omega)$ are $H_{0,g_2}(0)$ bounded by [21, Theorem 5.9] we see that $H_{g_1,g_2}(\xi)$ arises from $H_{0,g_2}(0)$ by an infinitesimally bounded perturbation. Thus $H_{g_1,g_2}(\xi)$ is closed on $\mathcal{D}(H_{0,0}(0))$ and has $\mathcal{D}(H_{0,0}(0)) \cap \mathcal{N}$ as a core. Selfadjointness in the case $g_1 \in \mathbb{R}$, $\xi \in \mathbb{R}^\nu$ and existence of a lower bound follows directly from the Kato-Rellich theorem.

Lemma 4.5. *Assume Hypothesis 1 holds. There is a constant γ independent of ξ such that*

$$\|H_{0,0}(\xi)\psi\|^2 \leq \gamma(\|H_{g_1,g_2}(\xi)\psi\|^2 + \|\psi\|^2)$$

for all $\psi \in \mathcal{D}(H_{0,0}(0))$. Given $R > 0$ one may choose the same γ for all $u, v, \omega, m^{(i)}$ where

$$\|u\|, \|m^{(i)}u\|, \|\omega^{-1/2}m^{(i)}u\|, \|\omega^{1/2}u\|, \|v\|, \|m^{(i)}v\|, \|\omega^{-1/2}m^{(i)}v\|, \|\omega^{1/2}v\| \leq R$$

and $\|\omega^{-1/2}u\|, \|\omega^{-1/2}v\| \leq R$.

Proof. By Lemma 4.4 it is enough to prove the statement for $\psi \in \mathcal{D}(H_{0,0}(0)) \cap \mathcal{N}$ due to this being a core for $H_{g_1,g_2}(\xi)$. By Lemma 4.3 there is $\gamma' \geq 0$ such that

$$\frac{1}{2}\|H_{0,0}(\xi)\psi\|^2 \leq \|H_{0,g_2}(\xi)\psi\| + \gamma'\|\psi\|$$

By Lemma 4.3 there is $\alpha > 0$ depending only on $g_1\|(1+\omega^{-1/2})(u-v)\|$ such that $\|g_1\varphi(u-v)\psi\| \leq \frac{1}{4}\|d\Gamma(\omega)\| + \alpha\|\psi\|$. Using this we find

$$\frac{1}{2}\|H_{0,0}(\xi)\psi\|^2 \leq \|H_{g_1,g_2}(\xi)\psi\| + (\gamma' + \alpha)\|\psi\| + \frac{1}{4}\|H_{0,0}(\xi)\|$$

rearranging yields the inequality with $\gamma = \max\{4, \gamma' + \alpha\}$.

We can now prove most of Theorem 1.

Proof (Proof of Theorem 1 parts (1)-(3)). The selfadjointness and existence on a lower bound is clear from Lemma 4.4. Obviously $\xi \mapsto H_{g_1,g_2}(\xi)\psi$ is analytic for any $\psi \in \mathcal{D}_0$ and since $H_{g_1,g_2}(\xi)$ is closed on this set for all ξ statement (1) follows. Statement (2) follows since

$$\Sigma(\xi) - \frac{1}{2M}\xi^2 = \inf_{\psi \in \mathcal{D}_0, \|\psi\|=1} \sum_{i=1}^{\nu} \xi_i \langle \psi, d\Gamma(k_i)\psi \rangle + \langle \psi, H_{g_1,g_2}(0)\psi \rangle$$

we see $\Sigma(\xi) - \frac{1}{2M}\xi^2$ is an infimum of concave functions and thus concave. In particular is almost everywhere twice differentiable and locally Lipschitz. Thus the conclusion in (2) follows. To prove part (3) one should follow the calculations in [3] to find that

$$\Sigma(\xi - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n) \in \sigma_{ess}(H_{g_1,g_2}(\xi)) \tag{4.2}$$

To prove the second part of (3) if $\inf(\sigma(\omega)) > 0$ is easily done in the same way as [12]. The only thing one needs is Lemma 4.5 at some point during the calculations. If $\inf(\sigma(\omega)) = 0$ and $\omega(0) = 0$ then $\Sigma_e ss(\xi) = \Sigma(\xi)$ by equation (4.2) so the statement is obvious in this case.

We now consider the full Hamiltonian First we shall need a technical Lemma

Lemma 4.6. *Assume Hypothesis 4 holds. Then the following holds*

(1) $\overline{S}$ *is bounded below. Write* $E = \inf(\sigma(\overline{S}))$.
(2) $|p|$ *is* $S \otimes 1$ *bounded.*
(3) *For any* $\varepsilon > 0$ *there is* $\gamma \in (0, \infty)$ *such that*

$$\|(S \otimes 1)\psi\|^2 + \|d\Gamma_\oplus \psi\|^2 \leq \|H^V_{0,0}\psi\|^2 + |E|\|\psi\|^2 \tag{4.3}$$

$$2Re(\langle S \otimes 1\psi, \varphi_\oplus(f_x)^2\psi\rangle) \geq -\varepsilon\|H^V_{0,0}\psi\|^2 - \gamma\|\psi\|^2 \tag{4.4}$$

$$2Re(\langle d\Gamma_\oplus(\omega)\psi, \varphi_\oplus(f_x)^2\psi\rangle) \geq -2 \sup_{x\in\mathbb{R}^\nu} (\|\omega^{1/2} f_x\|)\|\psi\|^2 \tag{4.5}$$

for $\psi \in C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D}(\omega))$.

Proof. Define the form

$$q(\psi, \phi) = \frac{1}{2M}\langle\sqrt{-\Delta}\psi, \sqrt{-\Delta}\phi\rangle + \langle V_+\psi, V_+\phi\rangle - \langle V_-\psi, V_-\phi\rangle$$

which is closed on $\mathcal{D}((-\Delta)^{1/2}) \cap \mathcal{D}(V_+^{1/2})$ by the KLMN theorem and the fact that adding two non negative, closed forms gives a closed form if the intersection domain is dense (see [15]). Let A be the selfadjoint and lower bounded operator corresponding to q and note that $A = S$ on $C_0^\infty(\mathbb{R}^\nu)$. As S is selfadjoint we see $A = \overline{S}$. Now A is bounded below and $\mathcal{D}(A) \subset \mathcal{D}(q) \subset (-\Delta)^{1/2}$ so $(-\Delta)^{1/2}$ is $\overline{S}$ bounded. From Lemma A.1 we see $|p| = (-\Delta)^{1/2} \otimes 1$ is $S \otimes 1 = \overline{S} \otimes 1$ bounded. This proves statements (1) and (2).

Fix $\psi \in C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D}(\omega))$. We calculate

$$\|H^V_{0,0}\psi\|^2 = \|(S \otimes 1)\psi\|^2 + \|d\Gamma_\oplus\psi\|^2 + 2\mathrm{Re}(\langle(S \otimes 1)\psi, d\Gamma_\oplus\psi\rangle)$$

Using that

$$\langle(S \otimes 1)\psi, d\Gamma_\oplus\psi\rangle = \langle(S \otimes 1)d\Gamma_\oplus(\omega)^{1/2}\psi, d\Gamma_\oplus(\omega)^{1/2}\psi\rangle \geq E\|\phi\|^2$$

we have proven equation (4.3). Note that equation (4.5) follows from the point wise estimate in 4.2. Thus we only need to prove equation (4.4). We calculate

$$\begin{aligned}
2\mathrm{Re}(\langle S \otimes 1\psi, \varphi_\oplus(f_x)^2\psi\rangle) =& 2\mathrm{Re}(\langle V_+ \otimes 1\psi, \varphi_\oplus(f_x)^2\psi\rangle) && (4.6)\\
&- 2\mathrm{Re}(\langle V_- \otimes 1\psi, \varphi_\oplus(f_x)^2\psi\rangle) && (4.7)\\
&+ \sum_{j=1}^{\nu} 2\mathrm{Re}(\langle p_j^2\psi, \varphi_\oplus(f_x)^2\psi\rangle). && (4.8)
\end{aligned}$$

Let $a \in \{\pm\}$. ψ takes values in $\mathcal{N}$ so

$$\langle V_a \otimes 1\psi, \varphi_\oplus(f_x)^2\psi\rangle = \int_{\mathbb{R}^\nu} V_a \|\phi(f_x)\Psi(x)\|^2 d\lambda_\nu(x) = \|V_a^{1/2} \otimes 1\varphi_\oplus(f_x)\psi\|^2$$

Here the first equality ensures $\varphi_\oplus(f_x)\psi \in \mathcal{D}(V_a^{1/2} \otimes 1)$. Furthermore, since $p_j^2\psi, p_j\psi \in C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D}(\omega))$ we may use Lemma 2.6 to calculate

$$\begin{aligned}\langle p_j^2\psi, \varphi_\oplus(f_x)^2\psi\rangle =& \langle \varphi_\oplus(f_x)p_j^2\psi, \varphi_\oplus(f_x)\psi\rangle \\ =& \langle \varphi_\oplus(f_x)p_j\psi, p_j\varphi_\oplus(f_x)\psi\rangle - \langle -i\varphi_\oplus(\partial_{x_j}f_x)p_j\psi, \varphi_\oplus(f_x)\psi\rangle \\ =& \|p_i\varphi_\oplus(f_x)\psi\|^2 \\ &- \langle -i\varphi_\oplus(\partial_{x_j}f_x)p_j\psi, \varphi_\oplus(f_x)\psi\rangle \\ &- \langle -i\varphi_\oplus(\partial_{x_j}f_x)\psi, p_j\varphi_\oplus(f_x)\psi\rangle\end{aligned}$$

Write $U_j(x) = \mathrm{Im}(\langle \partial_{x_j}f_x, f_x\rangle)$ and note that by Theorem 2.2 and Proposition 2.3 we have

$$\langle -i\varphi_\oplus(\partial_{x_j}f_x)p_j\psi, \varphi_\oplus(f_x)\psi\rangle = \langle p_j\psi, i\varphi_\oplus(f_x)\varphi_\oplus(\partial_{x_j}f_x)\psi\rangle + \langle p_j\psi, i2i(U \otimes 1)\psi\rangle$$
$$\langle -i\varphi_\oplus(\partial_{x_j}f_x)\psi, p_j\varphi_\oplus(f_x)\psi\rangle = -\langle i\varphi_\oplus(f_x)\varphi_\oplus(\partial_{x_j}f_x)\psi, p_j\psi\rangle - \|\varphi_\oplus(\partial_{x_j}f_x)\psi\|^2$$

Noting $\langle i\varphi_\oplus(f_x)\varphi_\oplus(\partial_{x_j}f_x)\psi, p_j\psi\rangle$ and $\langle p_j\psi, i\varphi_\oplus(f_x)\varphi_\oplus(\partial_{x_j}f_x)\psi\rangle$ are complex conjugates we find that

$$2\mathrm{Re}(\langle p_j^2\psi, \varphi_\oplus(f_x)^2\psi\rangle) \geq -4\mathrm{Re}(\langle p_j\psi, (U_j \otimes 1)\psi\rangle) + \|\varphi_\oplus(\partial_{x_j}f_x)\psi\|^2$$

Using Lemma 2.6 we new arrive at

$$\begin{aligned}\mathrm{Re}(\langle S \otimes 1\psi, \varphi_\oplus(f_x)^2\psi\rangle) =& \|V_+^{1/2} \otimes 1\varphi_\oplus(f_x)\psi\|^2 - \|V_-^{1/2} \otimes 1\varphi_\oplus(f_x)\psi\|^2 && (4.9)\\ &+ \frac{1}{2M}\||p|\varphi_\oplus(f_x)\psi\|^2 - \sum_{j=1}^{\nu}\|\varphi_\oplus(\partial_{x_j}f_x)\psi\|^2 && (4.10)\\ &+ \sum_{j=1}^{\nu} -2\mathrm{Re}(\langle p_j\psi, (U_j \otimes 1)\psi\rangle) && (4.11)\end{aligned}$$

Using Lemma A.1 we find constants $b > 0$ and $a < 1$ such that

$$\|V_-^{1/2} \otimes 1\varphi_\oplus(f_x)\psi\|^2 \leq \frac{a}{2M}\||p|\varphi_\oplus(f_x)\psi\|^2 + b\|\varphi_\oplus(f_x)\psi\|^2$$

which implies

$$\begin{aligned}\mathrm{Re}(\langle S \otimes 1\psi, \varphi_\oplus(f_x)^2\psi\rangle) \geq& \sum_{j=1}^{\nu} -\|\varphi_\oplus(\partial_{x_j}f_x)\psi\|^2 - 2\mathrm{Re}(\langle p_j\psi, (U_j \otimes 1)\psi\rangle) \\ &- b\|\varphi_\oplus(f_x)\psi\|^2\end{aligned}$$

Write $\widetilde{b} = \max\{b, 1\}$

$$R = \max\left\{ \sup_{x\in\mathbb{R}^\nu} \|(1+\omega^{-1/2}+\omega^{1/2})f_x\|, \sup_{x\in\mathbb{R}^\nu, j\in\{1,..,\nu\}} \|(1+\omega^{-1/2})\partial_{x_j} f_x\| \right\}$$

Using Proposition 2.3, $2\alpha\beta \leq \varepsilon^{-1}\alpha^2 + \varepsilon\beta^2$ for all $\varepsilon, \alpha, \beta > 0$ and equation (4.3) we find

$$\begin{aligned}
&\max\{\widetilde{b}\|\varphi_\oplus(\partial_{x_j} f_x)\psi\|^2, \widetilde{b}\|\varphi_\oplus(f_x)\psi\|^2\} \\
&\quad \leq 4\widetilde{b}R^2\|(d\Gamma_\oplus(\omega)+1)^{1/2}\psi\|^2 \\
&\quad \leq \widetilde{b}4R^2(\|\psi\|^2 + \|\psi\|\|d\Gamma_\oplus(\omega)\psi\|) \\
&\quad \leq (\widetilde{b}4R^2 + \varepsilon|E|)\|\psi\|^2 + \frac{16\widetilde{b}^2R^4(\nu+1)}{\varepsilon}\|\psi\|^2 + \frac{\varepsilon}{4(\nu+1)}\|H_{0,0}^V\psi\|^2
\end{aligned}$$

By statement 2 we find $c, d \in [0, \infty)$ such that

$$\||p|\psi\|^2 \leq c\|(S\otimes 1)\psi\|^2 + d\|\psi\|^2 \tag{4.12}$$

Using this, Cauchy-Schwatz, $\|U_j \otimes 1\| \leq R^2$, $2\alpha\beta \leq \varepsilon^{-1}\alpha^2 + \varepsilon\beta^2$ for all $\varepsilon, \alpha, \beta > 0$ equation (4.3) we find

$$\begin{aligned}
\sum_{j=1}^{\nu} |\langle p_j\psi, (U_j\otimes 1)\psi\rangle| &\leq \frac{2c\nu^2R^4}{\varepsilon}\|\psi\|^2 + \frac{\varepsilon}{c8}\||p|\psi\|^2 \\
&\leq \frac{2c\nu R^4 + d\varepsilon}{\varepsilon}\|\psi\|^2 + \frac{\varepsilon}{8}\|H_{0,0}^V\psi\|^2
\end{aligned}$$

Defining

$$\frac{1}{2}\gamma = \frac{2(2c\nu^2R^4 + d\varepsilon)}{\varepsilon} + 4\widetilde{b}R^2(\nu+1) + (\nu+1)\varepsilon|E| + \frac{16\widetilde{b}^2R^4(\nu+1)^2}{\varepsilon}$$

we have the desired inequality.

We can now prove selfadjointness of H_{g_1,g_2}^V and the decomposition.

Lemma 4.7. *H_{g_1,g_2}^V is selfadjoint on $\mathcal{D}(S\otimes 1)\cap\mathcal{D}(d\Gamma_\oplus(\omega))$ and essentially selfadjoint on any core for $H_{0,0}^V$.*

Assume now $v_x(k) = e^{ix\cdot k}v(k), u_x(k) = e^{ix\cdot k}u(k)$ where $u, v \in \mathcal{D}(\omega^{\pm 1/2}) \cap \bigcap_{i=1}^{\nu}\mathcal{D}(k_i) \cap \mathcal{D}(\omega^{-1/2}k_i)$. Define F to be the Fourier transform and let $U = (F\otimes 1)\Gamma_\oplus(e^{-ikx})$ be the Lee-Low-Pines transformation. Then

$$UH_{g_1,g_2}(0)U^* = \int_{\mathbb{R}^\nu}^{\oplus} H_{g_1,g_2}(\xi)d\xi$$

Proof. We start by noting $H^V_{0,0}$ is selfadjoint on $\mathcal{D}_0 = \mathcal{D}(S \otimes 1) \cap \mathcal{D}(d\Gamma_\oplus(\omega))$ as it is the sum of two semibounded, selfadjoint and strongly commuting operators (see [14] and Theorem 2.1). By Proposition 2.3 and Theorem 2.1 we see $C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D}(\omega))$ is a core for $H^V_{0,0}$. Using Lemma 4.1 we realise that

$$\begin{aligned} H^V_{g_1,g_2} = &\left(-\frac{1}{2M}\Delta + V\right) \otimes 1 + 1 \otimes d\Gamma(\omega) + g_1\varphi_\oplus(u_x - v_x) \\ &+ \frac{g_2}{4}\varphi_\oplus(u_x - v_x) + \frac{g_2}{4}\varphi_\oplus(i(u_x + v_x)) + C \otimes 1 \end{aligned}$$

where C is multiplication by $x \mapsto C(u_x, v_x, g_2)$. Selfadjointness now follows from Lemmas 4.6 and A.3.

To prove the next part let U_x denote multiplication by $e^{ik\cdot x}$ and note that U_x is a unitary and strongly continous representation of $\mathbb{R}^\nu$. Furthermore, since $v_x = U_{-x}v, u_x = U_{-x}u$ we see v_x, u_x are continuously differentiable by the domain properties of u and v with derivatives $\partial_{x_i}u_x = U_x(ix_iu), \partial_{x_i}v_x = U_x(ix_iv)$. The $\omega^{\pm1/2}$ norms of u_x, v_x and the $\omega^{-1/2}$ norm of the derivatives does not depend on x, so Hypothesis 4 is fulfilled. Let $x \mapsto a_x \in \{x \mapsto u_x - v_x, x \mapsto i(u_x + v_x)\}$. Note that $U_{-x}a_x = a$ independent of x so by Proposition 2.4 we see

$$U\varphi_\oplus(a_x)U^* = (F \otimes 1)\varphi_\oplus(a)(F \otimes 1)^* = 1 \otimes \varphi(a) = \varphi_\oplus(a)$$

Using this equality, that $C(g_2, u_x, v_x)$ is a constant and that the statement is true for $H_{g_1,0}(0)$ we find $U^*\psi(\xi) \in \mathcal{D}(H_{g_1,0}(\xi)) = \mathcal{D}(H_{g_1,g_2}(\xi))$ for all ξ and

$$\begin{aligned} (UH^0_{g_1,g_2}U^*)\psi(\xi) =& H_{g_1,0}(\xi)\psi(\xi) - C(g_2, u, v)\psi(\xi) \\ &+ \frac{g_2}{4}\varphi(u - v)^2\psi(\xi) + \frac{g_2}{4}\varphi(i(u + v))^2\psi(\xi) \\ =& H_{g_1,g_2}(\xi)\psi(\xi). \end{aligned}$$

this finishes the proof.

5. Abstract results on positivity

In this section we will fix a σ-finite measure space $(\mathcal{M}, \mathcal{F}, \mu)$ and write $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$. Define

$$\begin{aligned} L^2_\mathbb{R}(\mathcal{M}, \mathcal{F}, \mu) &:= \{f \in L^2(\mathcal{M}, \mathcal{F}, \mu) \mid f(k) \in \mathbb{R} \text{ almost everywhere}\} = \mathcal{H}_\mathbb{R} \\ L^2_+(\mathcal{M}, \mathcal{F}, \mu) &:= \{f \in L^2(\mathcal{M}, \mathcal{F}, \mu) \mid f(k) \geq 0 \text{ almost everywhere}\} = \mathcal{H}_+ \\ L^2_{>0}(\mathcal{M}, \mathcal{F}, \mu) &:= \{f \in L^2(\mathcal{M}, \mathcal{F}, \mu) \mid f(k) > 0 \text{ almost everywhere}\} = \mathcal{H}_{>0} \end{aligned}$$

the following definition is central and con be found in [15]:

Definition 5.1. *Let $f, g \in \mathcal{H}$ we say $f \geq g$ if $f - g \in \mathcal{H}_+$. $A \in B(\mathcal{H})$ is called positivity preserving if $A\mathcal{H}_+ \subset \mathcal{H}_+$, positivity improving if $A\mathcal{H}_+\backslash\{0\} \subset \mathcal{H}_{>0}$ and ergodic if for all $\psi, \phi \in \mathcal{H}_+\backslash\{0\}$ we have $\langle\psi, A^n\phi\rangle > 0$ for some $n \in \mathbb{N}$.*

We now define two maps abs : $\mathcal{H} \to \mathcal{H}_+$ and $\Re, \Im : \mathcal{H} \to \mathcal{H}_\mathbb{R}$ by $\text{abs}(f)(x) = |f(x)|$, $\Re(f)(x) = \text{Re}(f(x))$ and $\Im(f)(x) = \text{Im}(f(x))$ for almost all $x \in \mathcal{M}$. The following lemma is essential

Lemma 5.2. *Let $(X, \mathcal{X}, \tau)$ be a σ-finite measurable space and $A \in B(\mathcal{H})$ be positivity preserving. The following hold*

(1) $f \in \mathcal{H}_+ \iff \langle f, g\rangle \geq 0$ *for all* $g \in \mathcal{H}_+$. *In particular* $\mathcal{H}_+$ *is closed.*
(2) abs, $\Im$ *and* $\Re$ *are well defined and continous. If* $f, g \in \mathcal{H}$ *and* $\text{abs}(f) \leq g$ *then* $g \in \mathcal{H}_+$ *and* $\|f\| \leq \|g\|$
(3) *For all* $f \in \mathcal{H}$ *there is a multiplication operator* ϕ_f *with* $|\phi_v(x)| = 1$ *for all* $x \in M$, $\phi_f \text{abs}(f) = f$ *and* $\text{abs}(f) = \phi_f^* f$. *If* $f \in \mathcal{H}_+$ *then* $\phi_f = 1$ *so* $\text{abs}(f) = f$. *If* $a \in \mathbb{C}$ *then* $\text{abs}(af) = |a|\text{abs}(f)$
(4) *If* $f, g \in \mathcal{H}$ *and* $f \leq g$ *then* $Af \leq Ag$.
(5) $\Re$ *and* $\Im$ *are real linear and* $1 = \Re + i\Im$. *If* $f \in \mathcal{H}_\mathbb{R}$ *then* $Af \in \mathcal{H}_\mathbb{R}, \Re(f) = f$ *and* $\Re(if) = 0$.
(6) $\Re A = A\Re$ *and for all* $f \in \mathcal{H}$ *we have* $\text{abs}(Af) \leq A(\text{abs}(f))$.
(7) *if* $x \mapsto f(x) \in L^1(X, \mathcal{X}, \tau, \mathcal{H})$ *then* $x \mapsto \text{abs}(f(x)) \in L^1(X, \mathcal{X}, \tau, \mathcal{H})$ *and*

$$\text{abs}\left(\int_X f(x) d\tau(x)\right) \leq \int_X \text{abs}(f(x)) d\tau(x)$$

(8) *If* $x \mapsto f(x), x \mapsto g(x) \in L^1(X, \mathcal{X}, \tau, \mathcal{H})$ *and* $f(x) \leq g(x)$ *for all* $x \in X$ *then*

$$\int_X f(x) d\tau(x) \leq \int_X g(x) d\tau(x)$$

Proof. (1): The implication " $\Rightarrow$ " in is clear. To prove the other implication we assume that it is false. Then $\mu(\{f < 0\}) \neq 0$. Now μ is σ-finite so there is $B \subset \{f < 0\}$ such that $0 < \mu(B) < \infty$. Thus we now have $0 \leq \langle 1_B, f\rangle < 0$ (see [17]) which is a contradiction. Now $\mathcal{H}_+ = \bigcap_{\psi \in \mathcal{H}_+} \{\phi \in \mathcal{H} \mid \langle \psi, \phi\rangle \geq 0\}$ which is closed.

(2): Let $C \in \{\text{abs}, \Re, \Im\}$. We see

$$\int_\mathcal{M} |Cf(m) - Cg(m)|^2 d\mu(m) \leq \int_\mathcal{M} |f(m) - g(m)|^2 d\mu(m) = \|f - g\|^2.$$

Thus C is well defined and continous. Let $\widetilde{g}$ be a representative for g and $\widetilde{f}$ a representative for f. Then by assumption $\widetilde{g}(k) - |\widetilde{f}(k)| \geq 0$ almost everywhere, so $g(k) = \widetilde{g}(k) - |\widetilde{f}(k)| + |\widetilde{f}(k)| \geq 0$ almost everywhere. Hence

$$\|g\|^2 - \|f\|^2 = \int_\mathcal{M} \widetilde{g}(k)^2 - |\widetilde{f}(k)|^2 d\mu(k) = \int_\mathcal{M} (\widetilde{g}(k) + |\widetilde{f}(k)|)(\widetilde{g}(k) - |\widetilde{f}(k)|) d\mu(k)$$

which is non negative.

(3): Fix some representative $\widetilde{f}$ of f. Define $\phi_f(k) = 1$ if $\widetilde{f}(k) = 0$ and $\phi_f(k) = \frac{\widetilde{f}(k)}{|\widetilde{f}|}$ otherwise. Note $\phi_f = 1$ almost everywhere if $f \in \mathcal{H}_+$. As $\widetilde{f}$ is measurable we see that ϕ_f is measurable. Furthermore, the following calculations hold almost everywhere

$$\phi_f(k)\text{abs}(f)(k) = \phi_f(k)|\widetilde{f}(k)| = \widetilde{f}(k) = f(k)$$
$$\phi_f(k)^* f(k) = \phi_f(k)^* \widetilde{f}(k) = |\widetilde{f}(k)| = |f(k)|$$

showing $\phi_f \mathrm{abs}(f) = f$ and $\phi_f^* f = \mathrm{abs}(f)$. Now, $a\widetilde{f}$ is a representative for af so $\mathrm{abs}(af)(k) = |a\widetilde{f}(k)| = |a|\mathrm{abs}(f)(k)$ almost everywhere. Hence $|a|\mathrm{abs}(f) = \mathrm{abs}(af)$ proving (3).

(4): We see $g - f \in \mathcal{H}_+$, so $A(g-f) \in \mathcal{H}_+$.

(5): Let $f, g \in \mathcal{H}$ and $a, b \in \mathbb{R}$. Let $\widetilde{f}$ and $\widetilde{g}$ be representatives of f and g respectively. Then $a\widetilde{f} + b\widetilde{g}$ is a representative for $af + bg$ and

$$\mathrm{Re}(a\widetilde{f} + b\widetilde{g}) = a\mathrm{Re}(\widetilde{f}) + b\mathrm{Re}(\widetilde{g}) \qquad \mathrm{Im}(a\widetilde{f} + b\widetilde{g}) = a\mathrm{Im}(\widetilde{f}) + b\mathrm{Im}(\widetilde{g})$$

so for $C \in \{\Re, \Im\}$ we have almost everywhere $C(af + bg)(k) = (aC(f) + bC(g))(k)$. Furthermore, $\widetilde{f} = \mathrm{Re}(\widetilde{f}) + i\mathrm{Im}(\widetilde{f})$ showing $f(k) = \Re(f)(k) + i\Im(f)(k)$ almost everywhere so $1 = \Re + i\Im$.

Let $f \in \mathcal{H}_\mathbb{R}$. Then $f = f_+ - f_-$ with $f_\pm \in \mathcal{H}_+ \subset \mathcal{H}_\mathbb{R}$ so $Af = Af_+ - Af_- \in \mathcal{H}_+ - \mathcal{H}_+ = \mathcal{H}_\mathbb{R}$. Furthermore $\widetilde{f}(k) = \mathrm{Re}(\widetilde{f})(k)$ and $0 = \mathrm{Re}(i\widetilde{f})(k)$ for almost all k implying $f = \Re(f)$ and $0 = \Re(if)$.

(6). Using (5) we find

$$\Re(Af) = \Re(A\Re(f)) + \Re(iA\Im(f)) = A\Re(f) + 0 = A\Re(f)$$

Let $\Theta \in \mathbb{Q}$ and $f \in \mathcal{H}$. Let $\widetilde{f}$ be a representative for f and note $\Re(e^{i\Theta}\widetilde{f}(k)) \leq |\widetilde{f}(k)|$ for all k. As $\Re(e^{i\Theta}\widetilde{f}(k))$ is a representative for $\Re(e^{i\Theta} f)$ we see from (4) that $\Re(e^{i\Theta} Af) = A\Re(e^{i\Theta} f) \leq A(\mathrm{abs}(f))$.

Let g be a representative for Af and h be a representative for $A(\mathrm{abs}(f))$. As $\Re(e^{i\Theta} g)$ is a representative of $\Re(e^{i\Theta} Af)$ we see $\Re(e^{i\Theta} Af) \leq h$ except on a nullset C_Θ. Let $C = \bigcup_{\Theta \in \mathbb{Q}} C_\Theta$ which is still a nullset. For $k \notin C$ we have

$$|g(k)| = \sup_{\Theta \in \mathbb{Q}} \Re(e^{i\Theta} g(k)) \leq h(k)$$

As C is a null-set we have $\mathrm{abs}(Af) \leq A(\mathrm{abs}(f))$.

(7): $x \mapsto \mathrm{abs}(f(x))$ is measurable since abs is continous. We also see

$$\int_X \|\mathrm{abs}(f(x))\| d\tau(x) = \int_X \|f(x)\| d\tau(x) < \infty$$

Let $g = \int_X f(x) d\tau(x), v = \int_X \mathrm{abs}(f(x)) d\tau(x)$ and $h \in \mathcal{H}_+$. Then

$$\langle h, \mathrm{abs}(g) \rangle = \int_X \langle h, \phi_g^* f(x) \rangle d\tau(x) \leq \int_X \langle h, \mathrm{abs}(f(x)) \rangle d\tau(x)$$

where we used $|\langle h, \phi_g^* f(x) \rangle| \leq \langle h, \mathrm{abs}(f(x)) \rangle$. By (1) we now see $v - g \in \mathcal{H}_+$.

(8): Write $u = \int_X f(x) d\tau(x), v = \int_X g(x) d\tau(x)$ and let $h \in \mathcal{H}_+$. Then we see $\langle h, v - u \rangle = \int_X \langle h, g(x) - f(x) \rangle d\tau(x) > 0$ proving (8).

We now have a new definition

Definition 5.3. *Let A be selfadjoint and bounded below. We say A generates a positivity preserving (respectively ergodic or positivity improving) semigroup if e^{-tA} is positivity preserving, (respectively ergodic or positivity improving) for all $t > 0$.*

The following Lemma is essential. The proof can be found in [16]

Theorem 5.4. *Let A be selfadjoint, bounded below and generates a positivity preserving semi group. Write $\lambda = \inf(\sigma(A))$*

1. *If λ is a non degenerate eigenvalue and a ground state $\psi \in \mathcal{H}_{>0}$ exists then A generates a positivity improving semi group.*
2. *If A generates an ergodic semigroup then it generates a positivity improving semigroup.*
3. *If A generates a positivity improving semigroup and λ is an eigenvalue then it is non degenerate and the eigenspace is spanned by an element $\psi \in \mathcal{H}_{>0}$.*

The following result is found combing the results in [16] with in [4, Theorem 3].

Theorem 5.5. *Let A, C be bounded below and selfadjoint. Assume*

1. *e^{-tA} is positivity improving for all $t \geq 0$*
2. *There is bounded multiplication operators $\{B_n\}_{n=1}^{\infty}$ such that $A + B_n$ and $C - B_n$ are uniformly bounded below and converges to respectively C and A in strong resolvent sense.*

Then C generates a positivity improving semi group.

We will now prove Theorems 3.4 and 3.5.

Proof (Theorem 3.4). **Step 1**. We start by assuming $B_- = 0$. Then

$$q = q_A + q_{B_+} \tag{5.1}$$

is closed on $\mathcal{D}(q_A) \cap \mathcal{D}(q_{B_+})$ since $\mathcal{D}(q_A) \cap \mathcal{D}(q_{B_+})$ is dense (see [15]). Let H be the corresponding selfadjoint and lower bounded operator. Define $B_n = 1_{(-\infty,n)}(B_+)B_+$. Then B_n is a bounded multiplication operator and $q_n = q_A + q_{B_n} = q_{A+B_n}$ is increasing. On $\mathcal{D}(q)$ we see q_n converges to q so $A + B_n$ has a uniform lower bound and converges strongly to C by [15, Theorem S.14].

Furthermore, $q - q_{B_n}$ will be decreasing and converge to q_A restricted to $\mathcal{D}(q_A) \cap \mathcal{D}(q_{B_+})$. $\mathcal{D}(q_A) \cap \mathcal{D}(q_{B_+})$ is a form core for q_A so by [15, Theorem S.16] we see $C - B_n$ converges to A in strong resolvent sense. We also note, that $q - q_{B_n} \geq q_A$ so $H - B_n$ is uniformly bounded below and so we may now apply 5.5 to finish the proof in the case $B_- = 0$.

Step 2. Letting H be the operator associated with $q_A + q_{B_+}$ we see that $H_1, \widetilde{B}^+ = 0, \widetilde{B}^- := B_-$ satisfies the assumptions in the Theorem. Hence we may assume $B_+ = 0$ and write $B := B_-$. We now define $B_n = 1_{(-n,\infty)}(B_+)B_+$. Then $q_A + q_{B_n}$ is decreasing with limit $q = q_A + q_B$. This shows $A + B_n$ is uniformly bounded below and converges to C in strong resolvent sense. Furthermore, $q_C - q_{B_n}$ is increasing with limit $q = q_A$. This shows $C - B_n$ is uniformly bounded below and converges to A in strong resolvent sense. So we may apply Theorem 5.5 to finish the proof.

Proof (Theorem 3.5). Pick γ so large that $q \geq -\gamma$ and set $\widetilde{q} = q_A + q_B$. Then we note that

$$|q_C(\psi)| = -q_C(\psi) \leq q_A(\psi) + q_B(\psi) + \gamma\langle\psi, \phi\rangle$$

For any $0 \leq a < 1$ we note that $q_{aC} = a^2 q_C$ is relativity $\widetilde{q}$ bounded with bound smaller than $a^2 < 1$. Hence $q_a = q_A + q_B + aq_C$ is closed and uniformly bounded below by $q \geq -\gamma$. Let H_a be the operator corresponding to q_A. By Theorem 3.4 we know that H_a generates a positivity improving semigroup and thus $(H_a + \gamma + \ell)^{-1}$ is positivity improving for all $0 \leq a < 1, \ell > 0$ (see [15]).

Now the collection of closed forms q_a is uniformly bounded below and converges monotonously to q, which is closable. By [15, Theorem S.14] we find that $(H_a + \gamma + \ell)^{-1}$ converges strongly to $(H + \gamma + 1)^{-1}$ for all $\ell > 0$ and so this map is positivity preserving. Using standard theory of forms (see [20]) we have

$$\begin{aligned}(H_a + \gamma + 1)^{-1} - &(H_0 + \gamma + 1)^{-1} \\ &= a^2(|C|^{1/2}(H_0 + \gamma + 1)^{-1})^* |C|^{1/2}(H_a + \gamma + 1)^{-1}\end{aligned}$$

Now since $|C|^{-1}(H_0 + \gamma + 1)^{-1}, |B|^{-1/2}(H_a + \gamma + 1)^{-1}$ are positivity preserving we find for fixed element in $\psi \in \mathcal{H}_+ \backslash \{0\}$ that

$$(H_a + \gamma + 1)^{-1}\psi - (H_0 + \gamma + 1)^{-1}\psi \in \mathcal{H}_+$$

taking the limit a tending to 1, we find

$$(H + \gamma + 1)^{-1}\psi - (H_0 + \gamma + 1)^{-1}\psi \in \mathcal{H}_+.$$

Since $(H_0 + \gamma + 1)^{-1}\psi \in \mathcal{H}_{>0}$ we are finished.

The following Corollary is useful

Corollary 5.6. *Let A, B and C be selfadjoint. Assume A and C are bounded below and*

1. *e^{-tA} is positivity improving for all $t \geq 0$*
2. *B is a multiplication operator with $\mathcal{D}(B) \subset \mathcal{D}(A)$ and $C = A + B$.*

Then C generates a positivity improving semi group.

Proof. Let $B_- = 1_{(-\infty,0)}(B)B$ and $B_+ = 1_{[0,\infty)}(B)B$. Note that $\mathcal{D}(B_\pm) \subset \mathcal{D}(A)$ and so $\mathcal{D}(q_{B_\pm}) \subset \mathcal{D}(q_A)$ (see [21, Theorem 9.4]). It only remains to see that

$$q_C = q_A + q_{B_+} + q_{B_-}.$$

However this holds on $\mathcal{D}(C) = \mathcal{D}(A)$ obviously and using $\mathcal{D}(q_A) = \mathcal{D}(q_C) \subset \mathcal{D}(q_{B_\pm})$ (see [21, Theorem 9.4]) we can extend the equality by continuity (use [21, Theorem 5.9]).

We will use the following two lemmas

Lemma 5.7. *Let $A_1, \ldots, A_n$ be strongly commuting selfadjoint operators on $\mathcal{H}$ which are bounded below. Assume also C is selfadjoint operators on $\mathcal{H}$ an that $B = A_1 + \cdots + A_n + C$ is essentially selfadjoint. If $A_1, .., A_n, C$ all generate a positivity presering semigroup we find $\overline{B}$ generates one as well.*

Proof. $A = A_1 + \cdots + A_n$ is selfadjoint (see [14]) and using the joint functional calculus for $B = (A_1, \ldots, A_n)$ we see $e^{-tA}\mathcal{H}_+ = e^{-tA_1} \ldots e^{-tA_n}\mathcal{H}_+ \subset \mathcal{H}_+$. Using trotters product formula we have

$$e^{-t\overline{B}}\psi = \lim_{n\to\infty} (e^{n^{-1}tA}e^{n^{-1}C})^n\psi$$

As $\mathcal{H}_+$ is closed and $(e^{n^{-1}tA}e^{n^{-1}C})^n\mathcal{H}_+ \subset \mathcal{H}_+$ we find $e^{-t\overline{B}}$ preserves $\mathcal{H}_+$.

Lemma 5.8. *Assume $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2, \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ and $\mu_1 \otimes \mu_2$. Let $i, j \in \{1, 2\}$ with $i \neq j$. Then*

1. *If $B : \mathcal{M}_1 \to B(L^2(\mathcal{M}_2, \mathcal{F}_2, \mu_2))$ is strongly measurable and B_x is positivity preserving for all x then so is $I_\oplus(B_x)$.*
2. *I $B_i \in B(L^2(\mathcal{M}_i, \mathcal{F}_i, \mu_i))$ is positivity preserving so is $B_1 \otimes 1, B_1 \otimes B_2$ and $1 \otimes B_2$*

Proof. (1): Let $\psi \in \mathcal{H}_+$. ψ is identified with the element $x \mapsto \psi(x, \cdot)$ under the identification $\mathcal{H} = L^2(\mathcal{M}_1, \mathcal{F}_1, \mu_1, L^2(\mathcal{M}_2, \mathcal{F}_2, \mu_2))$. Noting that

$$0 = \mu\{|\psi| \neq \psi\} = \int_{\mathcal{M}_1} \mu_2(\{|\psi(x, \cdot)| \neq \psi(x, \cdot)\}) d\mu_1(x)$$

we see $\psi(x, \cdot) \in L^2(\mathcal{M}_2, \mathcal{F}_2, \mu_2)_+$ for almost all x. Let $\phi \in \mathcal{H}_+$ as well. Then we see

$$\langle \phi, I_\oplus(B_x)\psi \rangle = \int_{\mathcal{M}_1} \langle \phi(x, \cdot), B_x \psi(x, \cdot) \rangle d\mu_1(x) \geq 0$$

we are now finished by Lemma 5.2.

(2): By Theorem 2.2 we have dealt with the case $1 \otimes B_2$ in (1). Let U be the unitary map from $\mathcal{H}$ to $L^2(\mathcal{M}_2 \times \mathcal{M}_1, \mathcal{F} = \mathcal{F}_2 \otimes \mathcal{F}_1, \mu_2 \otimes \mu_1)$ defined by $U\psi(x, y) = \psi(y, x)$. U obviously have the inverse $U^{-1}\psi(y, x) = \psi(x, y)$ and is isometric by fubinis theorem. As $U(\psi \otimes \phi) = \phi \otimes \psi$ we see $U^* 1 \otimes B_1 U = B_1 \otimes 1$ since this will hold on simple tensors and simple tensors are total in $\mathcal{H}$.

Now U maps $\mathcal{H}_+$ to $L^2_+(\mathcal{M}_2 \times \mathcal{M}_1, \mathcal{F} = \mathcal{F}_2 \otimes \mathcal{F}_1, \mu_2 \otimes \mu_1)$ and U^* maps $L^2_+(\mathcal{M}_2 \times \mathcal{M}_1, \mathcal{F} = \mathcal{F}_2 \otimes \mathcal{F}_1, \mu_2 \otimes \mu_1)$ to $\mathcal{H}_+$. As $1 \otimes B_2$ is positivity preserving by statement 1 we find $B_1 \otimes 1$ is positivity preserving. Now $B_1 \otimes B_2 \mathcal{H}_+ = (B_1 \otimes 1)(1 \otimes B_2)\mathcal{H}_+ \subset \mathcal{H}_+$ shows $B_1 \otimes B_2$ is positivity preserving.

6. Application of positivity results

In order to apply the above theorems one needs to find an L^2 space to work in. The following Theorem is well known and can be found in [1]

Theorem 6.1. *There is a probability space $\mathcal{Q} = (\mathcal{Y}, \mathfrak{Y}, \mathbb{P})$ and a unitary map U from $\mathcal{F}_b(\mathcal{H})$ to $L^2(\mathcal{Q})$ with the following properties*

1. *$U\Omega = 1$.*
2. *If B is a bounded operator on $\mathcal{H}$ with $\|B\| \leq 1$ which maps $\mathcal{H}_\mathbb{R}$ to $\mathcal{H}_\mathbb{R}$ then $U\Gamma(B)U^*$ is positivity preserving.*
3. *For all $v \in \mathcal{H}_\mathbb{R}$ the operator $U\varphi(v)U^*$ acts like multiplication by a random variable $\Phi(v)$. Furthermore, the collection $\{\Phi(v)\}_{v \in \mathcal{H}_\mathbb{R}}$ is normally distributed with covariance function $\mathbb{E}[\Phi(v)\Phi(u)] = \langle v, u \rangle$.*

Lemma A.4 in the appendix gives conditions for specific spaces to exist. We may now prove

Lemma 6.2. *Let $(\mathcal{M}, \mathcal{F}, \mu)$ be a σ-finite measurable space and A be a multiplication operator on this space. Assume $x \mapsto v_x \in L^\infty(\mathcal{M}, \mathcal{F}, \mu, \mathcal{H}_\mathbb{R})$ and let U be the isomorphism from Theorem 6.1. Then $(1 \otimes U)\varphi_\oplus(v_x)(1 \otimes U)^*$ and $A \otimes 1$ is a multiplication operator on $L^2(\mathcal{M} \times \mathcal{Y}, \mathcal{F} \otimes \mathfrak{Y}, \mu \otimes \mathcal{P})$.*

Proof. Let f be a strictly positive element in $L^2(\mathcal{M},\mathcal{F},\mu)$ which exists since $(\mathcal{M},\mathcal{F},\mu)$ is sigma finite. Define $\psi(x)=f(x)\Phi(v_x)\in L^2(\mathbb{R}^\nu,\mathcal{B}(\mathbb{R}^\nu),\lambda_\nu,L^2(\mathcal{Q}))$ since $\|\psi(x)\|^2=|f(x)|^2\|v_x\|^2$ which is integrable. Under the identification

$$L^2(\mathbb{R}^\nu,\mathcal{B}(\mathbb{R}^\nu),\lambda_\nu,L^2(\mathcal{Q}))=L^2(\mathcal{M}\times\mathcal{Y},\mathcal{F}\otimes\mathfrak{Y},\mu\otimes\mathbb{P})$$

there is a jointly measurable map $p(x,y)$ such that $p(x,\cdot)=f(x)\Phi(v_x)$ for almost all $x\in\mathbb{R}^\nu$. Define $V(x,y)=\frac{p(x,y)}{f(x)}$ and note that $V(x,\cdot)=\Phi(v_x)$ in $L^2(\mathcal{Q})$. Obviously

$$(1\otimes U)\varphi_\oplus(v_x)(1\otimes U)^*=\int_{\mathbb{R}^\nu}^\oplus \Phi(v_x)d\lambda_\nu(x)$$

Using standard properties of direct integrals we see

$$\begin{aligned}\psi\in\mathcal{D}(V)&\iff\int_{\mathbb{R}^\nu}\int_{\mathcal{Y}}|V(x,y)\psi(x,y)|^2d\mathbb{P}(y)d\lambda_\nu(x)<\infty\\&\iff\int_{\mathbb{R}^\nu}\|\Phi(v_x)\psi(x,\cdot)\|^2d\lambda_\nu(x)<\infty\iff\psi\in\mathcal{D}\left(\int_{\mathbb{R}^\nu}^\oplus\Phi(v_x)d\lambda_\nu(x)\right)\end{aligned}$$

Furthermore, for almost all x we have $V(x,\cdot)\psi(x,\cdot)=\Phi(v_x)\psi(x,\cdot)$ when $\psi\in\mathcal{D}(V)$. This proves $(1\otimes U)\varphi_\oplus(v_x)(1\otimes U)^*$ is multiplication by V.

We now check $A\otimes 1$ is multiplication by $V(x,y)=A(x)$. For $\psi\in\mathcal{D}(A),\phi\in L^2(Q)$ we see $(A\otimes 1)\psi\otimes\phi=A\psi\otimes\phi=(x,y)\mapsto A(x)\psi(x)\phi(y)$. This implies $(x,y)\mapsto A(x)\psi(x)\phi(y)=V(x,y)\psi(x)\phi(y)$ is square integrable so $\psi\otimes\phi\in\mathcal{D}(V)$ and $V\psi\otimes\phi=(A\otimes 1)\psi\otimes\phi$. As $A\otimes 1$ and V are selfadjoint and $A\otimes 1$ essentially selfadjoint on $\mathcal{D}(A)\widehat{\otimes}L^2(Q)$ we see $A\otimes 1=V$.

Lemma 6.3. *There a bounded function $V_\nu:\mathbb{R}^\nu\to\mathbb{R}$ such that $-(2M)^{-1}\Delta+V_\nu$ has a ground state.*

Proof. By [15, Theorem XIII.11] the Lemma holds if $\nu=1$ and in this case the ground state eigenvalue is a negative number a. Let ψ_1 denote a ground state eigenvector. Define

$$V_\nu(x_1,\dots,x_\nu)=V_1(x_1)+\cdots+V_1(x_\nu)$$

Then $\psi_\nu(x_1,\dots,x_\nu)=\psi_1(x_1)\cdot\dots\cdot\psi_1(x_\nu)$ is an eigenvector for $-(2M)^{-1}\Delta+V$ corresponding to the eigenvalue $\nu a<0$. As the essential spectrum of $-(2M)^{-1}\Delta+V$ is $[0,\infty)$ (see [20]) the conclusion is obvious.

Lemma 6.4. *Let $v\in\mathcal{H}$. If $v\in\mathcal{H}_\mathbb{R}$ then $e^{-t|v\rangle\langle v|}$ maps $\mathcal{H}_\mathbb{R}$ to $\mathcal{H}_\mathbb{R}$.*

Proof. We may calculate

$$e^{-t|v\rangle\langle v|}=\sum_{n=0}^\infty\frac{(-t)^n(|v\rangle\langle v|)^n}{n!}=1+C|v\rangle\langle v|$$

where $C\in\mathbb{R}$ depends on t and $\|v\|$. This clearly maps $\mathcal{H}_\mathbb{R}$ to $\mathcal{H}_\mathbb{R}$.

The remaining part of Theorem 3.3 will now be proven in the following Lemma.

Lemma 6.5. *The remaining conclusion of Theorem 3.3 is true.*

Proof. Define $\omega_x = |u_x + v_x\rangle\langle u_x + v_x|$. Applying Lemma 4.7 with $g_1 = 0$, $u'_x = v_x + u_x$ and $v'_x = 0$ we find

$$\widetilde{H}^V = S \otimes 1 + d\Gamma_\oplus(\omega) + g_2 d\Gamma_\oplus(\omega_x)$$

is selfadjoint on $\mathcal{D}(S\otimes 1)\cap\mathcal{D}(d\Gamma_\oplus(\omega))$ and $C_0^\infty(\mathbb{R}^\nu)\widehat{\otimes}\mathcal{J}(\mathcal{D}(\omega))$ is a core. Let V_ν be the potential from Lemma 6.3 and note that

$$\widetilde{H}^0 + V_\nu \otimes 1 = \widetilde{H}^{V_\nu} \tag{6.1}$$

Pick $\mathcal{H}_\mathbb{R}$ such that $v_x, u_x \in \mathcal{H}_\mathbb{R}$ for all $x \in \mathbb{R}^\nu$ and $e^{-t\omega}\mathcal{H}_\mathbb{R} \subset \mathcal{H}_\mathbb{R}$. Let $\mathcal{Q}$ be the corresponding probability space from Theorem 6.1. Write U for the unitary map from $\mathcal{F}_b(\mathcal{H})$ to $L^2(\mathcal{Q})$ and note that

$$\begin{aligned} A = (1\otimes U)(\widetilde{H}^0{+}V_\nu\otimes 1)(1\otimes U)^* & \\ = (-(2M)^{-1}\Delta + V_\nu)\otimes 1 + I_\oplus(Ud\Gamma(\omega)U^*) &+ g_2 I_\oplus(Ud\Gamma(\omega_x)U^*) \end{aligned}$$

By corollary 5.6 we see $-(2M)^{-1}\Delta + V_\nu$ generates a positivity improving semi group. Combining this with theorem 6.1 and Lemmas 5.7 and 6.4 we see that A generates a positivity preserving semigroup. So A will generate a positivity improving semigroup if we can show that $\widetilde{H}^0 + V_\nu \otimes 1$ has a ground state ϕ such that $(1\otimes U)\phi$ is strictly positive and any other ground state is a multiple of ϕ.

Let $\lambda = \inf(\sigma(-(2M)^{-1}\Delta + V_\nu)) < 0$ and let ψ be a strictly positive eigenvector (such a vector exists because $-(2M)^{-1}\Delta + V_\nu$ generates a positivity improving semi group). By Theorems 2.1 and 6.1 we see $\widetilde{H} \geq \lambda$ and $\psi\otimes\Omega$ is a ground state satisfying $(1\otimes U)\psi\otimes\Omega$ is positive.

Let now Ψ be a ground state for $\widetilde{H}^0 + V_\nu \otimes 1$. Then

$$\lambda = \langle\Psi, (-(2M)^{-1}\Delta + V_\nu)\otimes 1\Psi\rangle + \langle\Psi, d\Gamma_\oplus(\omega)\Psi\rangle + \langle\Psi, d\Gamma_\oplus(\omega_x)\Psi\rangle$$

Since $(-(2M)^{-1}\Delta{+}V_\nu)\otimes 1 \geq \lambda$, $d\Gamma_\oplus(\omega)\geq 0$ and $d\Gamma_\oplus(\omega_x)\geq 0$ we must have the equalities $\langle\Psi, (-(2M)^{-1}\Delta{+}V_\nu)\otimes 1\Psi\rangle = \lambda$ and $\langle\Psi, d\Gamma_\oplus(\omega)\Psi\rangle = 0$. In particular,

$$\lambda = \langle\Psi, (-(2M)^{-1}\Delta + V_\nu)\otimes 1\Psi\rangle + \langle\Psi, d\Gamma_\oplus(\omega)\Psi\rangle = \langle\Psi, H_{0,0}^{V_\nu}\Psi\rangle.$$

By Theorem 2.1 we see $\lambda = \inf(\sigma(H_{0,0}(V_\nu)))$ is a non degenerate eigenvalue for $H_{0,0}(V_\nu)$. Now Ψ must be a ground state for $H_{0,0}(V_\nu)$ since it minimizes the quadratic form and $\psi\otimes\Omega$ is also a ground state for $H_{0,0}(V_\nu)$. In particular Ψ must be proportional to $\psi\otimes\Omega$. Thus we have now established that A generates a positivity improving semigroup. Noting that $V_\nu\otimes 1$ acts like bounded multiplication operator we find $(1\otimes U)(\widetilde{H}^0)(1\otimes U)^*$ generates a positivity improving semigroup (use Corollary 5.6).

Let $\widetilde{V} \in \{0, V\}$ and define a form on $\mathcal{D}_0 = \mathcal{D}(|p|)\cap\mathcal{D}(I_\oplus(Ud\Gamma(\omega)U^*)^{1/2})\cap\mathcal{D}(\widetilde{V}_+^{1/2}\otimes 1)$

$$q^{\widetilde{V}} = q_{I_\oplus(Ud\Gamma(\omega)U^*)} + q_{g_2 I_\oplus(Ud\Gamma(\omega_x)U^*)} + q_{-(2M)^{-1}\Delta\otimes 1} + q_{\widetilde{V}_+\otimes 1} - q_{\widetilde{V}_-\otimes 1}$$

where we used $\mathcal{D}(d\Gamma_{\oplus}(\omega_x)^{1/2}) \subset \mathcal{D}(d\Gamma_{\oplus}(\omega)^{1/2})$ by [21, Theorem 9.4]. The first four terms are non negative and thus defines a closed form. Using $V_- \otimes 1$ is $-(2M)^{-1}\Delta \otimes 1$ form bounded with bound $a < 1$ strictly smaller than 1 we see (using Lemma A.1) that

$$q_{\widetilde{V}_-\otimes 1}(\psi,\psi) \leq aq_{-(2M)^{-1}\Delta\otimes 1}(\psi,\psi) + b\|\psi\|^2 \leq a(q(\psi,\psi) + q_{\widetilde{V}_-\otimes 1}(\psi,\psi)) + b\|\psi\|^2$$

so $q^{\widetilde{V}}$ is becomes closed by the KLMN theorem. Write $A^{\widetilde{V}}$ for the corresponding operator. For $\psi \in C_0^\infty(\mathbb{R}) \otimes \mathcal{J}(\mathcal{D}(\omega))$ and $\phi \in \mathcal{D}_0$ we see that

$$q^{\widetilde{V}}(\psi,\phi) = \langle (1\otimes U)\widetilde{H}^{\widetilde{V}}(1\otimes U)^*\psi, \phi\rangle$$

Hence $(1 \otimes U)\widetilde{H}^{\widetilde{V}}(1 \otimes U)^* = A^{\widetilde{V}}$ on $C_0^\infty(\mathbb{R}) \otimes (1 \otimes U)\mathcal{J}(\mathcal{D}(\omega))$ which is a core for $(1 \otimes U)\widetilde{H}^{\widetilde{V}}(1 \otimes U)^*$, implying $(1 \otimes U)\widetilde{H}^{\widetilde{V}}(1 \otimes U)^* = A^{\widetilde{V}}$. Noting that $C_0^\infty(\mathbb{R}) \otimes U\mathcal{J}(\mathcal{D}(\omega)) \subset \mathcal{D}(q_{V_+\otimes 1})$ is a core for $q_{\widetilde{H}^0} = q^0$ by Lemma A.2 we find via Theorem 3.4 that $(1 \otimes U)\widetilde{H}^V(1 \otimes U)^*$ generates a positivity improving semigroup.

To finish the proof note that $(1\otimes U)H^V_{g_1,g_2}(1\otimes U)^* - (1\otimes U)\widetilde{H}^V(1\otimes U)^*$ is a multiplication operator that is relatively $(1 \otimes U)d\Gamma_{\oplus}(\omega)(1 \otimes U)^*$ bounded (use Lemmas 4.1 and 6.2). In particular, it is $(1 \otimes U)\widetilde{H}^V(1 \otimes U)^*$ bounded by [21, Theorem 5.9] and so we may apply Corollary 5.6 to see $(1\otimes U)\widetilde{H}^V(1\otimes U)^*$ generates a positivity improving semi group. Note that for all $\psi \in L^2_+(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$ we have $(1 \otimes U)\psi \otimes \Omega$ is non negative.

Lemma 6.6. *Let $A = (A_1, \dots, A_\nu)$ be a touple of commuting and selfadjoint operators on $\mathcal{H}$ and assume $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$ is a real Hilbert space satisfying $\mathcal{H} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$. Let Q be the probability space from Theorem 6.1 corresponding to $\mathcal{H}_\mathbb{R}$ and $U : \mathcal{F}_b(\mathcal{H}) \to L^2(Q)$ the isomorphism. Assume lastly that $\mathcal{H}_\mathbb{R}$ is invariant under the action of $e^{itB_1}, \dots, e^{itB_n}$.*

(1) *Define $K(\xi) = U^*\frac{1}{2}(\xi - d\Gamma(B))^2U$. Then $K(\xi)$ is selfadjoint, bounded below and $e^{-tK(0)}$ is positivity preserving for all t. Furthermore, for any $\psi \in L^2(Q)$ $\mathrm{abs}(e^{-tK(\xi)}\psi) \leq e^{-tK(0)}\mathrm{abs}(\psi)$.*
(2) *Define the measure spaces $X_1 = (\mathbb{Z}^\nu, \mathcal{B}(\mathbb{Z}^\nu), \tau)$ and $X_2 = (T^\nu, \mathcal{B}(T^\nu), h)$ where τ is the counting measure and T^ν is the ν dimensional torus with normalised Haar measure h. Let $F : L^2(X_1) \to L^2(X_2)$ be the furrier transform and A be a diagonal matrix with diagonal entries $a_1, \dots, a_\nu > 0$. Then*

$$K = (F\otimes U)\int_{\mathbb{Z}^d}^{\oplus} (A\alpha - d\Gamma(B))^2 d\tau(\alpha)(F\otimes U)^* \tag{6.2}$$

is positivity preserving on $L^2(X_2) \otimes L^2(Q)$.

Proof. (1): Let N_t be the density of a ν-dimensional normally distributed vector with variance $\frac{t}{m}I$ and mean 0. By Lemma A.7 we find

$$\begin{aligned} e^{-tK(\xi)}\psi &= U^* e^{\frac{1}{2M}(\xi - d\Gamma(B))^2}U\psi \\ &= \int_{\mathbb{R}^\nu} N_t(x)e^{i\xi\cdot x}U^*e^{ix_1 d\Gamma(A_1)}\cdots e^{ix_\nu d\Gamma(A_\nu)}U\psi d\lambda_\nu(x) \end{aligned}$$

Using Proposition 2.3, Lemma 5.2 and Theorem 6.1 we see

$$\begin{aligned}\mathrm{abs}(e^{-tK(\xi)}\psi) &\leq \int_{\mathbb{R}^\nu} \mathrm{abs}(N_t(x)e^{i\xi\cdot x}U^*\Gamma(e^{ix_1A_1}\cdots e^{ix_\nu A_\nu})U\psi)d\lambda_\nu(x)\\ &\leq \int_{\mathbb{R}^\nu} N_t(x)U^*\Gamma(e^{ix_1A_1}\cdots e^{ix_\nu A_\nu})U\mathrm{abs}(\psi)d\lambda_\nu(x)\\ &= e^{-tK(0)}\mathrm{abs}(\psi)\end{aligned}$$

If $\psi \in L^2_+(Q)$ we see $\mathrm{abs}(\psi)=\psi$ by Lemma 5.2 so $e^{-tK(0)}\psi \in L^2_+(Q)$ by Lemma 5.2.

(2): For $(\phi_1,\dots,\phi_\nu)\in T^\nu$ we define the unitary map $V(\psi_1,\dots,\phi_\nu)$ on $L^2(X_2)$ given by $(V(\phi_1,\dots,\phi_\nu)\psi)(b_1,\dots,b_\nu)=\psi(\phi_1 b_1,\dots,\phi_\nu b_\nu)$. It is well known (see folland) that this is a strongly continous, unitary representation. The observation $(t_1,\dots,t_n)\mapsto(e^{it_1},\dots,e^{it_n})\in T^n$ is continuous a continous homomorphism shows $(t_1,\dots,t_n)\mapsto V(e^{it_1},\dots,e^{it})$ is a strongly continous homomorphism. Let $V(x)=\widetilde{V}(e^{ix_1},\dots,e^{ix_n})$. Define now the map

$$\begin{aligned}\pi(x) &= V(Ax)\otimes U^*\Gamma(e^{ix_1A_1}\cdots e^{ix_\nu A_\nu})\\ &= V(Ax)\psi_1\otimes U^*e^{-ix_1 d\Gamma(A_1)}\cdots e^{-ix_\nu d\Gamma(A_\nu)}U\end{aligned}$$

which is strongly continous by Lemmas A.5 and A.6. We can now define an operator bounded B_t on $L^2(X_2)\otimes L^2(Q)$ by

$$B_t\psi = \int_{\mathbb{R}^\nu} N_t(x)\pi(x)\psi d\lambda_\nu(x)$$

since $\|N_t(x)(V(Ax)\otimes U^*\Gamma(e^{ix_1A_1}\cdots e^{ix_\nu A_\nu})U)\psi\| = N_t(x)\|\psi\|$ which is integrable and $x\mapsto N_t(x)\pi(x)\psi$ is continous.

We claim $e^{-tK}=B_t$. Clearly $\{1_{\{\alpha\}}\}_{\alpha\in\mathbb{Z}^d}$ is an orthonormal basis for $L^2(X_1)$. Now $F1_{\{\alpha\}}=f_\alpha$ where

$$f_\alpha(\phi_1,\dots,\phi_\nu)=\phi_1^{\alpha_1}\dots\phi_\nu^{\alpha_\nu}.$$

This implies $\mathcal{A}=\{f_\alpha\otimes\psi \mid \alpha\in\mathbb{Z}^\nu, \psi\in L^2(Q)\}$ spans a dense subspace of $L^2(X_2)\otimes L^2(Q)$, so it is enough to check

$$\langle f_{\{\alpha_1\}}\otimes\psi_1, e^{-tK}f_{\{\alpha_2\}}\otimes\psi_2\rangle = \langle f_{\{\alpha_1\}}\otimes\psi_1, B_t f_{\{\alpha_2\}}\otimes\psi_2\rangle$$

for all $\alpha_1,\alpha_2\in\mathbb{Z}^\nu$ and $\psi_1,\psi_2\in L^2(Q)$. We calculate

$$\begin{aligned}&\langle f_{\{\alpha_1\}}\otimes\psi_1, e^{-tK}f_{\{\alpha_2\}}\otimes\psi_2\rangle\\ &= \left\langle 1_{\{\alpha_1\}}\otimes U^*\psi_1, \int_{\mathbb{Z}^d}^{\oplus} e^{-t\frac{1}{2M}(A\alpha-d\Gamma(A)^2)}d\tau(\alpha)1_{\{\alpha_2\}}\otimes U^*\psi_2\right\rangle\\ &= \left\langle 1_{\{\alpha_1\}}\otimes U^*\psi_1, 1_{\{\alpha_2\}}\otimes e^{-t\frac{1}{2M}(A\alpha_1-d\Gamma(A))^2}U^*\psi_2\right\rangle\\ &= \delta_{\alpha_1,\alpha_2}\int_{\mathbb{R}^\nu} N_t(x)e^{iAx\cdot\alpha_1}\langle\psi_1, U\Gamma(e^{ix_1A_1}\cdots e^{ix_\nu A_\nu})U^*\psi_2\rangle d\lambda_\nu(x)\\ &= \int_{\mathbb{R}^\nu} N_t(x)e^{iAx\cdot\alpha_1}\langle f_{\{\alpha_1\}}\otimes\psi_1, f_{\{\alpha_2\}}\otimes U\Gamma(e^{ix_1A_1}\cdots e^{ix_\nu A_\nu})U^*\psi_2\rangle d\lambda_\nu(x)\end{aligned}$$

Now $e^{iA\beta\cdot x} = e^{i\beta\cdot Ax}$ since A is symmetric. Furthermore

$$e^{i\alpha\cdot Ax} f_\alpha(\phi_1,\dots,\phi_\nu) = \phi_1^{\alpha_1}\dots\phi_\nu^{\alpha_\nu} e^{ia_1x_1\alpha_1}\dots e^{ia_\nu x_\nu \alpha_\nu} = V(Ax) f_\alpha(\phi_1,\dots,\phi_\nu)$$

So

$$\begin{aligned}&\langle f_{\{\alpha_1\}}\otimes\psi_1, e^{-tK} f_{\{\alpha_2\}}\otimes\psi_2\rangle\\ &= \int_{\mathbb{R}^\nu} N_t(x)\langle f_{\{\alpha_1\}}\otimes\psi_1, V(Ax) f_{\{\alpha_2\}}\otimes U\Gamma(e^{ix_1A_1}\cdots e^{ix_\nu A_\nu})U^*\psi_2\rangle d\lambda_\nu(x)\\ &= \langle f_{\{\alpha_1\}}\otimes\psi_1, B_t f_{\{\alpha_2\}}\otimes\psi_2\rangle\end{aligned}$$

Now $V(Ax)$ is positivity preserving and so is $U\Gamma(e^{ix_1A_1}\cdots e^{ix_\nu A_\nu})U^*$. It follows (by Lemma 5.8) that $\pi(x)$ is positivity preserving for all x. Thus we may use Lemma 5.2 to see that for $\psi\in L^2_+(T^\nu\times\mathcal{Y},\mathcal{B}(T^\nu)\otimes\mathfrak{Y}, h\otimes\mathbb{P})$ we have

$$\begin{aligned}\mathrm{abs}(e^{-tK}\psi) &\leq \int_{\mathbb{R}^\nu}\mathrm{abs}(N_t(x)\pi(x)\psi)d\lambda_\nu(x)\\ &\leq \int_{\mathbb{R}^\nu} N_t(x)\pi(x)\mathrm{abs}(\psi)d\lambda_\nu(x) = e^{-tK}\mathrm{abs}(\psi) = e^{-tK}\psi.\end{aligned}$$

Hence $e^{-tK}\psi\in L^2_+(T^\nu\times\mathcal{Y},\mathcal{B}(T^\nu)\otimes\mathfrak{Y}, h\otimes\mathbb{P})$.

Lemma 6.7. *Let $a_1,\dots,a_\nu\in(0,\infty)$ and A be the diagonal matrix with $A_{i,i} = a_i$. Let $\omega\geq 0$ be injective and selfadjoint on $\mathcal{H}$ and let $m^{(1)},\dots,m^{(\nu)}$ be selfadjoint operators on $\mathcal{H}$ such that $\omega, m^{(1)},\dots,m^{(\nu)}$ are strongly commuting. Assume $v,u\in\mathcal{D}(\omega^{-1/2})\cap\mathcal{D}(\omega^{1/2})\cap\bigcap_{i=1}^\nu\mathcal{D}(m^{(i)})\cap\mathcal{D}(m^{(i)}\omega^{-1/2})$ and that*

$$\langle u, e^{-it_1m^{(1)}},\dots,e^{-it_\nu m^{(\nu)}}e^{-t_{\nu+1}\omega}v\rangle\in\mathbb{R}$$

for all $t_1,\dots,t_\nu\in\mathbb{R}$ and $t_{\nu+1}\geq 0$. Let $M = (\mathbb{Z}^\nu,\mathcal{B}(\mathbb{Z}^\nu),\tau)$ where τ is the counting measure on $\mathbb{Z}^\nu$ and define

$$H_{g_1,g_2} = \int_{\mathbb{Z}^\nu}^\oplus H_{g_1,g_2}(A\alpha)d\tau(\alpha) = I_\oplus(H_{g_1,g_2}(A\alpha)).$$

Then $H_{g_1,g_2} = \widetilde{H}_{g_1,g_2}$ where

$$\begin{aligned}\widetilde{H}_{g_1,g_2} =& I_\oplus((A\alpha - d\Gamma(m))^2) + d\Gamma_\oplus(\omega) + g_2 a_\oplus^\dagger(u+v)a_\oplus(u+v)\\ &+ g_2\varphi_\oplus(v)\varphi_\oplus(u) + g_1\varphi_\oplus(u-v) + D(g_2,u,v)\end{aligned}$$

and H_{g_1,g_2} can have at most one ground state counted with multiplicity.

Proof. Clearly $\widetilde{H}_{g_1,g_2}$ is well defined on $\mathcal{D}_0 = \mathcal{D}(I_\oplus((A\alpha - d\Gamma(m))^2))\cap\mathcal{D}(d\Gamma_\oplus(\omega))$. Let $\psi\in\mathcal{D}_0$. For each $\alpha\in\mathbb{Z}^\nu$ we see $\psi(\alpha)\in\mathcal{D}(d\Gamma(\omega))\cap\mathcal{D}((A\alpha - d\Gamma(m))^2) = \mathcal{D}(H_{g_1,g_2}(A\alpha))$ and (using Lemma 4.1) we see

$$(\widetilde{H}_{g_1,g_2}\psi)(\alpha) = H_{g_1,g_2}(A\alpha)\psi(\alpha)$$

so $\alpha\mapsto\|H_{g_1,g_2}(A\alpha)\psi(\alpha)\|^2 = \alpha\mapsto\|(\widetilde{H}_{g_1,g_2}\psi)(\alpha)\|^2$ is integrable. Thus $\widetilde{H}_{g_1,g_2}\subset H_{g_1,g_2}$. To prove equallity we fix $\psi\in\mathcal{D}(H_{g_1,g_2})$ and note that $\psi(\alpha)\in\mathcal{D}(d\Gamma(\omega))\cap$

$\mathcal{D}((A\alpha - d\Gamma(m))^2) = \mathcal{D}(H_{g_1,g_2}(A\alpha))$ for each $\alpha \in \mathbb{Z}^\nu$. Using Lemmas 4.5 and 4.3 we find γ independent of α such that

$$\max\{\|d\Gamma(\omega)\psi(\alpha)\|^2, \|(A\alpha - d\Gamma(m))^2\psi(\alpha)\|^2\} \leq \gamma\|H_{g_1,g_2}(A\alpha)\psi(\alpha)\|^2 + \gamma\|\psi(\alpha)\|^2$$

which is integrable by assumption. Hence $\psi \in \mathcal{D}_0$ proving $H_{g_1,g_2} = \widetilde{H}_{g_1,g_2}$.

Taking $g_1 = 0$, $v' = u + v$ and $u' = 0$ we find

$$\begin{aligned} K &= I_\oplus((A\alpha - d\Gamma(m))^2) + d\Gamma_\oplus(\omega) + g_2 a_\oplus^\dagger(u+v)a_\oplus(u+v) \\ &= I_\oplus((A\alpha - d\Gamma(m))^2) + d\Gamma_\oplus(\omega) + g_2 d\Gamma_\oplus(\widetilde{\omega}) \end{aligned}$$

is selfadjoint on $\mathcal{D}_0$ where $\widetilde{\omega} = |v+u\rangle\langle u+v|$. Let $F : L^2(M) \to L^2(T^\nu, \mathcal{B}(T^\nu), h)$ be the furrier transform. Here T^ν is the ν-dimensional torus and h is the normalised Haar-measure so for $\alpha \in \mathbb{Z}^\nu$ we see $F(1_\alpha) = (\phi_1, \dots, \phi_n) \mapsto \phi_1^{\alpha_1}, \dots, \phi_\nu^{\alpha_\nu}$. Pick a real Hilbert space $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$ as in Lemma A.4, let Q be the probability space corresponding to $\mathcal{H}_\mathbb{R}$ from Theorem 6.1 and let U be the isomorphism. Using Theorem 6.1 and Lemmas 5.7, 6.4 and 6.6 we see $(F \otimes U)K(F \otimes U)^*$ generates a positivity preserving semi group. Define $\psi = 1_{\{0\}} \otimes \Omega$. To prove that $(F \otimes U)K(F \otimes U)^*$ generates a positivity improving semi group it is enough to see $(F \otimes U)\psi$ is strictly positive, that $K\psi = 0$ and any eigenvector for K corresponding to the eigenvalue 0 is a multiple of ψ.

Now $(F \otimes U)\psi = 1 \otimes 1 = 1$. Furthermore, since $d\Gamma(C)\Omega = 0$ for any selfadjoint C we see

$$(K\psi)(\alpha) = 1_{\{0\}}(\alpha)(A\alpha - d\Gamma(m))^2\Omega = 0$$

for all $\alpha \in \mathbb{Z}^\nu$ proving $K\psi = 0$. Now let ϕ be a ground state for K. Then

$$0 = \langle \phi, K\phi \rangle = \langle \phi, I_\oplus((A\alpha - d\Gamma(m))^2)\phi \rangle + \langle \phi, d\Gamma_\oplus(\omega)\phi \rangle + g_2\langle \phi, d\Gamma_\oplus(\widetilde{\omega})\phi \rangle$$

as all terms are non negative we see that each of the must be 0. In particular

$$0 = \sum_{\alpha \in \mathbb{Z}^\nu} \|d\Gamma(\omega)^{1/2}\phi(\alpha)\|^2.$$

By Proposition 2.3 we see that for all $\alpha \in \mathbb{Z}^d$ we must have $\phi(\alpha) = f(\alpha)\Omega$ for some $f(\alpha) \in \mathbb{C}$. Thus

$$0 = \langle \phi, I_\oplus((A\alpha - d\Gamma(m))^2)\phi \rangle = \sum_{\alpha \in \mathbb{Z}^\nu} |f(\alpha)|^2 (A\alpha)^2$$

and hence $f(\alpha) = 0$ for all $\alpha \neq 0$. This implies $\phi = f(0)1_{\{0\}} \otimes \Omega = f(0)\psi$ proving $(F \otimes U)K(F \otimes U)^*$ generates a positivity improving semi group. By Lemma 6.2 and Theorem 6.1 we see

$$(F \otimes U)H_{g_1,g_2}(F \otimes U)^* - (F \otimes U)K(F \otimes U)^*$$

is a multiplication operator which is relatively bounded to $d\Gamma_\oplus(\omega)$. In particular $(F \otimes U)H_{g_1,g_2}(F \otimes U)^*$ generates a positivity improving semi group by Corollary 5.6 finishing the proof.

We can now prove part (5) of Theorem 3.2

Proof (Theorem 3.2 part (5)). Define $\widetilde{\omega} = |u+v\rangle\langle u+v|$, let $\widetilde{m} \in \{0, m\}$ and define the operators

$$K(\widetilde{m}) = d\Gamma(\widetilde{m})^2 + d\Gamma(\omega) + g_2 d\Gamma(\widetilde{\omega})$$
$$P(\widetilde{m}) = d\Gamma(\widetilde{m})^2 + d\Gamma(\omega) + g_2 d\Gamma(\widetilde{\omega}) - g_2\varphi(v)\varphi(u) + g_1\varphi(y-v) + D(g_2, u, v)$$

which are both selfadjoint using Lemma 4.4. Pick a real Hilbert space $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}$ as in Lemma A.4, let Q be the probability space corresponding to $\mathcal{H}_{\mathbb{R}}$ from Theorem 6.1 and let U be the isomorphism. Using Theorem 6.1 and Lemmas 5.7 and 6.6 we see $UK(\widetilde{m})U^*$ generates a positivity preserving semigroup. Furthermore, $K(\widetilde{m})\Omega = 0$ and $K(\widetilde{m}) \geq 0$ showing Ω is a ground state an eigenvector. If ψ is a ground state we have

$$0 = \langle \psi, K(\widetilde{m})\psi\rangle \geq \langle \psi, d\Gamma(\omega)\psi\rangle = \|d\Gamma(\omega)1/2\psi\|^2 \geq 0$$

Thus $d\Gamma(\omega)^{1/2}\psi = 0$ and which implies ψ is proportional to Ω by Proposition 2.3. As $U\Omega = 1 > 0$ we see $UK(\widetilde{m})U^*$ generates a positivity improving semigroup. Now $U(P(\widetilde{m}) - K(\widetilde{m}))U^*$ is a multiplication operator which is $Ud\Gamma(\omega)U^*$-bounded. Since $U^*d\Gamma(\omega)U^*$ is $U^*K(\widetilde{m})U$ bounded by [21, Theorem 5.9] we see $U(P(\widetilde{m}) - K(\widetilde{m}))U^*$ is $U^*K(\widetilde{m})U$ bounded. Corollary 5.6 now gives $UP(\widetilde{m})U^*$ generates a positivity improving semi group. This yields that the ground state of $P(m) = H_{g_1,g_2}(0)$ must have dimension 0 or 1. Furthermore, Ω is mapped to a strictly positive element ad advertised.

Let abs denote the action from Lemma 5.2. Using Lemmas 5.2 and 6.6 we see that

$$\begin{aligned}\mathrm{abs}(e^{-tH_{g_1,g_2}(\xi)}\psi) &= \lim_{n\to\infty} \mathrm{abs}(e^{-tn^{-1}P(0)}e^{-tn^{-1}(\xi - d\Gamma(m))^2}\psi) \\ &\leq \lim_{n\to\infty} e^{-tn^{-1}P(0)}e^{-tn^{-1}(d\Gamma(m))^2}\mathrm{abs}(\psi) = e^{-tH_{g_1,g_2}(0)}\mathrm{abs}(\psi)\end{aligned}$$

which implies

$$\|e^{-H_{g_1,g_2}(\xi)}\| = \sup_{\|\psi\|=1}\|e^{-H_{g_1,g_2}(\xi)}\psi\| \leq \sup_{\|\psi\|=1}\|e^{-H_{g_1,g_2}(0)}\mathrm{abs}(\psi)\| \leq \|e^{-H_{g_1,g_2}(0)}\|$$

So $e^{-\Sigma(\xi)} = \|e^{-H_{g_1,g_2}(\xi)}\| \leq \|e^{-H_{g_1,g_2}(0)}\| = e^{-\Sigma(0)}$ proving 0 is a global minimum.

It remains to prove that 0 is a unique minimum in case ω is a multiplication operator with $\inf(\omega(k)) > 0$ and $m^{(i)}(k) = k_i$. Assume that $\xi_0 \neq 0$ is an other minimum for Σ. Then there is $a_1, \dots, a_\nu > 0$ and an element $\alpha \in \mathbb{Z}^d$ such that $A\alpha_0 = \xi_0$ where A is the diagonal matrix with $A_{i,i} = a_i$. Using part (3) of Theorem 3.2 we that if Σ is minimal at a point x then $\Sigma_{ess}(x) - \Sigma(x) \geq \inf(\omega(k)) > 0$ so $H_{g_1,g_2}(x)$ has a ground state. In particular $H_{g_1,g_2}(0)$ and $H_{g_1,g_2}(\xi)$ has a ground state ψ_0 and ψ_ξ. Let $M = (\mathbb{Z}^\nu, \mathcal{B}(\mathbb{Z}^\nu), \tau)$ where τ is the counting measure on $\mathbb{Z}^\nu$ and define

$$H_{g_1,g_2} = \int_{\mathbb{Z}^\nu}^{\oplus} H_{g_1,g_2}(A\alpha)d\tau(\alpha) = I_\oplus(H_{g_1,g_2}(A\alpha)).$$

Now $H_{g_1,g_2} \geq \Sigma(0) = \Sigma(A\alpha_0)$ and $\Sigma(0) = \Sigma(A\alpha_0)$ is an eigenvalue for H_{g_1,g_2} with two orthogonal eigenvectors $1_{\{0\}} \otimes \psi_0$ and $1_{\{\alpha_0\}} \otimes \psi_0$. This is a contradiction with Lemma 6.7 finishing the proof.

A. Collection of facts

In this chapter we collect some small results which are used throughout the paper. We begin by the following lemma

Lemma A.1. *Let A be selfadjoint on $\mathcal{H}_1$ with domain $\mathcal{D}(A)$ and B be closed on $\mathcal{H}_1$ with domain $\mathcal{D}(B)$. Assume B is A-bounded with bound b. If $\mathcal{H}_2$ is an other Hilbert space then $B\otimes 1$ is $A\otimes 1$ bounded with bound b.*

Proof. See Appendix B of [2].

We will be working quite extensively with forms in the last part of the paper. For this reason we will need a few general results on the square roots of operators.

Lemma A.2. *Let A and B be closed operators with domains $\mathcal{D}(A),\mathcal{D}(B)\subset\mathcal{H}$.*

1. *If $\mathcal{D}(A)=\mathcal{D}(B)$ then any core for A is a core for B.*
2. *If A is selfadjoint and bounded below then any core $\mathcal{D}$ for A is a core for q_A.*

Proof. (1): We note A and B have equivalent graph norms by [21, Theorem 5.9]. Thus $\mathcal{D}_0$ is dense in $(\|\cdot\|_A,\mathcal{D}(A)) \iff \mathcal{D}_0$ is dense in $(\|\cdot\|_B,\mathcal{D}(B))$. (1) now follows by definition of a core.

(2): Let γ be a lower bound of A. By [18, Proposition 10.5] we see $\mathcal{D}_0$ is a core for q_A of and only if it is a core for $(A-\gamma)^{1/2}$. Now $\mathcal{D}((A-\gamma)^{1/2})=\mathcal{D}(q_A)=\mathcal{D}(|A|^{1/2})$ by [18, Proposition 10.5] so by (1) we see $\mathcal{D}_0$ is a core for q_A of and only if it is a core for $|A|^{1/2}$.

Let $\mathcal{D}_0$ be a core for $\mathcal{D}(A)$. Noting that $\mathcal{D}(|A|^{1/2})\subset\mathcal{D}(|A|)=\mathcal{D}(A)$ we see $|A|^{1/2}$ is A bounded by [21, Theorem 5.9], so since $\mathcal{D}_0$ is dense in $(\|\cdot\|_A,\mathcal{D}(A))$ we see that it is also dense in $(\|\cdot\|_{|A|^{1/2}},\mathcal{D}(A))$. By [18, Proposition 3.18] we see $\mathcal{D}(|A|)=\mathcal{D}(A)$ is dense in $(\|\cdot\|_{|A|^{1/2}},\mathcal{D}(|A|^{1/2}))$ so in total $\mathcal{D}_0$ is a core for $|A|^{1/2}$.

Lemma A.3. *Let B be selfadjoint on $\mathcal{H}$. Assume C,D are symmetric operators on $\mathcal{H}$ such that $\mathcal{D}(B)\subset\mathcal{D}(C),\mathcal{D}(D)$ and D is infinitesismally B-bounded. Assume also that there is a core $\mathcal{D}$ for B such that for all $\varepsilon>0$ there is b_ε such that*

$$2\mathrm{Re}(\langle B\psi,C\psi\rangle)\geq-\varepsilon\|B\psi\|^2-b_\varepsilon\psi$$

for all $\psi\in\mathcal{D}_0$. Then $H_{g,h}=B+gC+hD$ is selfadjoint on $\mathcal{D}(B)$ for all $g\geq 0$ and $h\in\mathbb{R}$. Furthermore $H_{g,h}$ is essentially selfadjoint on any core for B.

Proof. We start by proving $H_{g,0}$ is closed for all $g\geq 0$. It is clearly true for $g=0$ so we fix $g>0$. We note $H_{g,0}$ is symmetric and therefore closable. To see $H_{g,0}=\overline{H_{g,0}}$ it is enough to see $\mathcal{D}(\overline{H_{g,0}})=\mathcal{D}(H_{g,0})=\mathcal{D}(B)$ where $\mathcal{D}(B)\subset\mathcal{D}(\overline{H_{g,0}})$ is obvious. For $\phi\in\mathcal{D}(\overline{H_{g,0}})$ there is a sequence $\{\phi_n\}_{n=1}^\infty$ converging to ψ such that $\{H_{g,0}\phi_n\}_{n=1}^\infty$ is Cauchy. Note that for any $\varepsilon>0$ we have

$$2\mathrm{Re}(\langle B\psi,gC\psi\rangle)\geq-\varepsilon\|B\psi\|^2-b_{\varepsilon/g}\|\psi\|^2$$

for all $\psi\in\mathcal{D}_0$. Using this with $\varepsilon=1/2$ we find

$$\|H_{g,0}\psi\|^2=\|B\psi\|^2+g^2\|C\psi\|^2+2\mathrm{Re}(\langle B\psi,gC\psi\rangle)\geq\frac{1}{2}\|B\psi\|^2-b_{1/(2g)}\|\psi\|^2 \quad (A.1)$$

for all $\psi \in \mathcal{D}_0$. Let $\psi \in \mathcal{D}(B)$. As $\mathcal{D}_0$ is a core for B we may for is a sequence $\{\psi_n\}_{n=1}^{\infty}$ converging to ψ in the graph norm of B. As gC is B-bounded we see $H_{g,0}\psi_n$ converges to $H_{g,0}\psi$. Taking limits we see

$$b_{1/(2g)}\|\psi\|^2 + \|H_{g,0}\psi\|^2 \geq \frac{1}{2}\|B\psi\|^2 \tag{A.2}$$

on for all $\psi \in \mathcal{D}(B)$. In particular we now see $\{B\phi_n\}_{n=1}^{\infty}$ is cauchy so $\phi \in \mathcal{D}(B)$ as we wanted to prove. By Wüst theorem we see $H_{g,0}$ is selfadjoint on $\mathcal{D}(B)$ for all $g \geq 0$.

By [21, Theorem 5.9] we see A is $H_{g,0}$ bounded, so we get hD is infinitesimally $H_{g,0}$ bounded. Thus $H_{g,h}$ is selfadjoint on $\mathcal{D}(B)$. By Lemma A.2 we see any core for B is a core for $H_{g,h}$ finishing the proof.

Lemma A.4. *Let $\{x_\alpha\}_{\alpha\in I} \subset \mathcal{H}$ and $\mathcal{A} \subset B(\mathcal{H})$. Assume $\mathcal{A}$ is closed under multiplication, $\mathcal{A}$ is closed under taking adjoins, and for all $\alpha, \beta \in I$ and $A \in \mathcal{A}$ we have $\langle x_\alpha, Ax_\beta\rangle \in \mathbb{R}$. Then there is a real hilbertspace $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$ such that $\{x_\alpha\}_{\alpha\in I} \subset \mathcal{H}_\mathbb{R}$, $\mathcal{H}_\mathbb{R}$ is invariant under $\mathcal{A}$ and $\mathcal{H} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$.*

Proof. Let

$$\mathcal{H}' = \overline{\mathrm{Span}_\mathbb{R}\{Ax_\alpha \mid A \in \mathcal{A}, \alpha \in I\}}.$$

Note that $\mathcal{H}'$ is a real Hilbert space since $\mathcal{A}$ is closed under multiplication and taking adjoints. For every $f \in (\mathcal{H}')^\perp \backslash \{0\}$ we define

$$\mathcal{H}(f) = \overline{\mathrm{Span}_\mathbb{R}\{Af \mid A \in \mathcal{A}}.$$

It is clear that the elements of $\mathcal{A}$ maps $\mathcal{H}'$ to $\mathcal{H}'$ and $\mathcal{H}(f)$ to $\mathcal{H}(f)$, since it maps the spanning set to the spanning set. Furthermore we define

$$\mathcal{B} = \{B \subset (\mathcal{H}')^\perp \mid \mathcal{H}(f) \perp \mathcal{H}(g) \ \forall \ f \neq g \in A\}.$$

We partially order $\mathcal{B}$ by inclusion and take a maximal totally ordered subset $\mathcal{C}$. Let C be the union of all elements in $\mathcal{C}$. If $f, g \in C$, then there is an element in $\mathcal{C}$ that contains both f and g (since $\mathcal{C}$ is totally ordered). This implies $\mathcal{H}(f) \perp \mathcal{H}(g)$ and so $B \in \mathcal{C}$ and is clearly the largest element. Define now

$$\mathcal{H}_\mathbb{R} := \mathcal{H}' \oplus \bigoplus_{a\in B} \mathcal{H}(a),$$

which is clearly a real Hilbert space containing $\{x_\alpha\}_{\alpha\in I}$ and it is left invariant by $\mathcal{A}$ since each component is. Assume now towards contradiction that there is an element $f \in \mathcal{H}_\mathbb{R}^\perp \backslash \{0\}$. Then for every $A_1, A_2 \in \mathcal{A}, h \in C$ we would have

$$\langle A_2 f, A_1 h\rangle = \langle f, A_2^* A_1 h\rangle = 0$$

and so $\mathcal{H}(f)$ is orthogonal to $\mathcal{H}(h)$ for all $h \in C$. In particular $C \cup \{f\} \in \mathcal{B}$, and so $\mathcal{C} \cup \{C \cup \{f\}\}$ is larger than $\mathcal{C}$ and totally ordered which is not possible. Hence $\mathcal{H}_\mathbb{R}^\perp \backslash \{0\} = \emptyset$.

Let $\{e_n\}_{n=1}^N$ be an orthonormal basis for $\mathcal{H}_\mathbb{R}$ $(N \leq \infty)$ which is then also an orthonormal basis for $\mathcal{H}$. Hence we may write any element in $\mathcal{H}$ as

$$f = \sum_{j=1}^N (a_j + ib_j)e_j = \sum_{j=1}^N a_j e_j + i\sum_{j=1}^N b_j e_j$$

as desired. This finishes the proof.

Lemma A.5. *Let $\mathcal{H}$ be a Hilbert space and (X,d) a metric space. Let $\mathcal{U}(\mathcal{H})$ be the set of unitary operators on $\mathcal{H}$ and assume $\pi_i : X \to \mathcal{U}(\mathcal{H})$ is strongly continous for $i \in \{1,\dots,n\}$. Then $\pi(x) = \pi_1(x)\dots\pi_n(x)$ is strongly continous*

Proof. Let $\{x_n\}_{n=1}^\infty \subset X$ converge to $x \in X$. For $\psi \in \mathcal{H}$ we have

$$\begin{aligned}\|\pi(x_n)\psi - \pi(x)\psi\| &= \left\|\sum_{i=1}^n \pi_1(x_n)\dots\pi_{i-1}(x_n)(\pi_i(x_n) - \pi_i(x))\pi_{i+1}(x)\dots\pi_n(x)\psi\right\| \\ &\leq \sum_{i=1}^n \|(\pi_i(x_n) - \pi_i(x))\pi_{i+1}(x)\dots\pi_n(x)\psi\|\end{aligned}$$

which goes to 0.

Lemma A.6. *Let $\mathcal{H}_1, \mathcal{H}_2$ be a Hilbert space and (X,d) a metric space. Let $\mathcal{U}(\mathcal{H}_i)$ be the set of unitary operators on $\mathcal{H}_i$ and assume $\pi_i : X \to \mathcal{U}(\mathcal{H}_i)$ is strongly continous for $i \in \{1,2\}$. Then $\pi(x) = \pi_1(x) \otimes \pi_2(x)$ is strongly continous and takes values in the unitary operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

Proof. $\pi(x)$ is unitary for all $x_0 \in X$ by Theorem 2.1. Let $\{x_n\}_{n=1}^\infty \subset X$ converge to $x \in X$. For a simple tensor $\psi_1 \otimes \psi_2$ we see

$$\begin{aligned}\|\pi(x_n)\psi_1 \otimes \psi_2 - \pi(x_0)\psi_1 \otimes \psi_2\| \leq{}& \|\pi_1(x_n)\psi_1\|\|(\pi_2(x_n) - \pi_2(x_0))\psi_2\| \\ &+ \|(\pi_1(x_n) - \pi_1(x_0))\psi_1\|\|\pi_2(x_0)\psi_1\|\end{aligned}$$

as $\|\pi_1(x_n)\psi_1\| = \|\psi\|$ we find the above converges to 0. Thus one sees $\pi(x)\psi$ is continous when ψ is a simple tensor. If ψ is a linear combination of simple tensor then $\pi(x)\psi$ is a line combination of continous maps and so continous. Hence $\pi(x)\psi$ is continous for $\psi \in \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$. Let $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and pick $\{\psi_n\}_{n=1}^\infty$ converging to ψ. For any $\varepsilon > 0$ pick N_1 such that $\|\psi - \psi_{N_1}\| < \varepsilon/3$. Pick now N such that $\|(\pi(x_n) - \pi(x_0))\psi_{N_1}\| < \varepsilon/3$. Then

$$\begin{aligned}\|(\pi(x_n) - \pi(x_0))\psi\| \leq{}& \|\pi(x_n)(\psi - \psi_{N_1})\| + \|(\pi(x_n) - \pi(x_0))\psi_{N_1}\| \\ &+ \|\pi(x_0)(\psi - \psi_{N_1})\| < \varepsilon.\end{aligned}$$

finishing the proof.

Lemma A.7. *Assume $B = (A_1,\dots,A_n)$ be selfadjoint and strongly commuting on $\mathcal{H}$, $\xi \in \mathbb{R}^n$ and $M > 0$. Then*

$$A = \frac{1}{2M}(\xi_1 - A_1)^2 + \dots + \frac{1}{2M}(\xi_1 - A_n)^2$$

is selfadjoint and bounded below. Let $N_t : \mathbb{R}^\nu \to \mathbb{R}$ be the density of a Gaussian random variable with mean 0 and variance $\frac{t}{m}I$. Then for all $\psi \in \mathcal{H}$

$$e^{-tA} = \int_{\mathbb{R}^\nu} N_t(x) e^{i\xi\cdot x} e^{-ix_1A_1}\dots e^{-ix_nA_n}\psi \, d\lambda(x) \tag{A.3}$$

Proof. A is a sum of non negative and strongly commuting selfadjoint operators so it is non negative and selfadjoint (see [14]). Let P be the spectral measure of B. Then

$$\begin{aligned} A &= \int_{\mathbb{R}^\nu} \frac{1}{2M}(\xi_1-\lambda_1)^2 dP(\lambda) + \cdots + \int_{\mathbb{R}^\nu} \frac{1}{2M}(\xi_n-\lambda_n)^2 dP(\lambda) \\ &\subset \int_{\mathbb{R}^\nu} \frac{1}{2M}(\xi_1-\lambda_1)^2 + \cdots + \frac{1}{2M}(\xi_n-\lambda_n)^2 dP(\lambda). \end{aligned}$$

By selfadjointness of A we see equality must hold. In particular, we must have

$$e^{-tA} = \int_{\mathbb{R}^\nu} e^{-\frac{t}{2M}(\xi_1-\lambda_1)^2-\cdots-\frac{t}{2M}(\xi_n-\lambda_n)^2} dP(\lambda).$$

Let $\psi \in \mathcal{H}$ and μ_ψ be the measure $\mu_\psi(C) = \langle \psi, P(C)\psi \rangle$. Using that

$$e^{-\frac{t}{2M}(\xi_1-\lambda_1)^2-\cdots-\frac{t}{2M}(\xi_n-\lambda_n)^2} = \int_{\mathbb{R}^\nu} N_t(x) e^{i\xi\cdot x} e^{-ix_1\lambda_1} \dots e^{-ix_n\lambda} d\lambda(x)$$

we see

$$\langle \psi, e^{-tA}\psi \rangle = \int_{\mathbb{R}^\nu}\int_{\mathbb{R}^\nu} N_t(x) e^{i\xi\cdot x} e^{-ix_1\lambda_1} \dots e^{-ix_n\lambda_n} d\lambda(x) d\mu_\psi(\lambda)$$

The aboslute value of the integrand is $N_t(x)$ which integrates to $\mu_\psi(\mathbb{R}^\nu) = \|\psi\|^2$. Thus we may use fubini to obtain

$$\langle \psi, e^{-tA}\psi \rangle = \int_{\mathbb{R}^\nu} N_t(x) e^{i\xi\cdot x} \langle \psi, e^{-ix_1A_1} \dots e^{-ix_nA_n}\psi \rangle d\lambda(x)$$

For any $\psi \in \mathcal{H}$ we see $x \mapsto e^{-ix_1A_1} \dots e^{-ix_nA_n}\psi$ is continous by Lemma A.5 and the fact $x \mapsto x_i$ is continous. Furthermore, $\|N_t(x)e^{-ix_1A_1} \dots e^{-ix_nA_n}\psi\| = N_t(x)\|\psi\|$ which is integrable with integral $\|\psi\|$. Hence vector valued integral in equation (A.3) exists for each ψ and defines a bounded linear operator C_t with norm smaller than 1. What we have proven so far is that $\langle \psi, e^{-tA}\psi \rangle = \langle \psi, C_t\psi \rangle$ for all $\psi \in \mathcal{H}$ so $C_t = e^{-tA}$.

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