Characteristic Classes of Surface Bundles

PhD Thesis

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July 1, 2004

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Table of Contents

Preface and Acknowledgements

This is my thesis. It reflects the scientific research I have done in my four years as a PhD student at the University of Aarhus.

I would like to thank my thesis advisor *Ib Madsen* for being such an excellent advisor. You have taught me so much, Ib, about mathematics and about how to do scientific research, and I hope we can continue to collaborate in the years to come.

I would also like to thank all the wonderful people who made my eight years in Aarhus so interesting. In particular Frank Nasser for enduring me as an office mate for six years.

INTRODUCTION

SØREN GALATIUS

My scientific work can all be placed under the heading "Characteristic classes of surface bundles", and I will begin this introduction be explaining what that means, and by what methods it has been studied in the last years. Along the way I shall try to explain my own contributions. This introduction is supposed to be understandable for non-experts. It is not supposed to be precise in any way.

The subject can be seen from at least two quite different points of view: the topological point of view and the algebraic geometrical point of view. Historically it began in algebraic geometry; the "Mumford conjecture" was formulated completely in algebraic geometrical terms. The latest major breakthrough in the subject, the Madsen-Weiss theorem, is topological in nature (both its formulation and proof), but has implications in algebraic geometry. In particular it proves the Mumford conjecture. I will try to give an idea what the Mumford conjecture and the Madsen-Weiss theorem say.

1. Bundles

A bundle over B consists, loosely speaking, of an object E_x for each $x \in B$ such that E_x "depends continuously on x". These are put together into a "total space" E which maps to B via $\pi : E \to B$ such that $E_x = \pi^{-1}(x)$. The bundle is a *vector bundle* if each E_x is a vectorspace, and it is a *surface bundle* if each E_x is a surface, etc. Two properties of being a bundle are very important. The first is that two bundles over B can be isomorphic. Usually $\pi : E \to B$ and $\pi' : E' \to B$ are called isomorphic if there is a map $\varphi : E \to E'$ which is over X (i.e. $\pi' \circ \varphi = \pi$) with an inverse that is also over X. The second is that bundles can be pulled back: If $E \to B$ is a bundle and $f : X \to B$ is a map, then there is a bundle $f^*E \to X$ and a diagram

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B. \end{array}$$

Pullback is usually constructed by letting f^*E be the bundle over X whose fibre over x is $E_{f(x)}$. Pullback is associative up to isomorphism: $(f \circ g)^*(E)$ is isomorphic to $g^*(f^*E)$. This means that we get a contravariant *functor*

$$B \mapsto \{\text{bundles } E \to B\}/\text{isomorphism}.$$

The above discussion was deliberately vague, for two reasons. Firstly because I didn't specify what kind of bundles I am considering, i.e. what do the fibres look like. Secondly because I didn't specify to what category B belongs. Depending on point of view, B may be a variety (over some ground field k), it may be a smooth manifold, or it may be any topological space.

As a well understood example, let us consider first vector bundles. In algebraic geometry a vector bundle over B is a locally free sheaf on X. Isomorphism and pullback makes sense for these. In the topological situation it is a fibre bundle with fibre $\mathbb{R}^n$ and structure group $\mathrm{Gl}_n(\mathbb{C})$ (or $\mathrm{Gl}_n(\mathbb{R})$).

For surface bundles we would in algebraic geometry (notice that an algebraic geometer calls a curve what a topologist calls a surface), mean a proper smooth morphism $\pi : E \to X$ such that all the fibres are curves. In the manifold situation we would consider proper submersions $\pi : E \to B$ such that all the fibres are connected oriented surfaces (and thus determined up to diffeomorphism by their *genus* g). Let us, for the moment, write V and S for the associated contravariant functors:

$$V(B) = \{\text{vector bundles } E \to B\}/\text{isomorphism}$$
$$S(B) = \{\text{surface bundles } E \to B\}/\text{isomorphism}$$

Wishfully, one would like to calculate $V(B)$ and $S(B)$ for "reasonable" B. Then we could say that two bundles are isomorphic if and only if they define the same class in $V(B)$ or $S(B)$. In practice that appears to be impossible for all but the most trivial cases (e.g. when B is a point), unfortunately. Instead, one looks for characteristic classes.

2. Characteristic classes

Very generally, a characteristic class with values in some contravariant functor h is a natural transformation $c : V \to h$ or $c : S \to h$. Usually, h has the property of being computable: $h(B)$ can be explicitly determined for a reasonable class of B's. Then the characteristic class combines the nice property of V or S that they are related to bundles with the nice property of h that it is computable. In algebraic geometry, $h(B)$ would typically be the Chow ring $A(B)$ or some version of cohomology. In topology h would typically be some kind of cohomology, e.g. de Rham cohomology or more generally cohomology with coefficients in some ring. Or maybe K-theory.

An example is the Chern characters ch_i. They associate to a vectorbundle $E \to B$ a class $\mathrm{ch}_i(E)$ in $H^{2i}(B, \mathbb{Q})$ (or de Rham cohomology), or in $A(B) \otimes \mathbb{Q}$. They can be constructed in several ways, cf [MS] or [RH]. Some of their properties are listed in the following theorem.

Theorem 2.1.

(i) $\mathrm{ch}_i(E \oplus E') = \mathrm{ch}_i(E) + ch_i(E')$.

(ii) Any characteristic class c of vectorbundles satisfying $c(E \oplus E') = c(E) + c(E')$ *is a unique linear combination of the* ch_i.
(iii) Any characteristic class of vectorbundles is a unique polynomial in the ch_i.
(iv) Characteristic classes for n-dimensional vector bundles are exactly the polynomials in $\mathrm{ch}_1, \ldots, \mathrm{ch}_n$.

Thus we have a complete description of the ring of characteristic classes of vectorbundles with values in $H^*(-;\mathbb{Q})$. The properties *(i)* and *(ii)* uses that the functor V takes values in monoids: Two vector bundles can be added by taking fibrewise direct sum.

Now let us ask if there is a theorem similar to Theorem 2.1, but for surface bundles instead of vector bundles. The answer to that question has two sides: Firstly we should *define* some characteristic classes. Secondly we could try to prove a uniqueness property similar to that of the Chern characters in the theorem above. Historically, the relevant definition was first given by Mumford in algebraic geometry. He also conjectured a uniqueness property similar to theorem 2.1 above. The definition in the manifold case was given by Miller and Morita. I will give a rough sketch of the definition of the κ-classes (also known as the MMM-classes, named after Mumford, Miller and Morita).

3. Miller-Morita-Mumford classes

At the heart of the definition is a construction that to a surface bundle $\pi : E^{n+2} \to B^n$ associates a map

$$\pi_! : H^{k+2}(E) \to H^k(B)$$

called "integration along the fibre", "transfer", "shriek" or something similar. It can be defined in several ways. For instance if E and B are oriented manifolds, then they satisfy Poincaré duality: $H^{k+2}(E) \cong H_{n-k}(E)$ and $H^k(B) \cong H_{n-k}(B)$. Then $\pi_!$ can be defined as $\pi_* : H_{n-k}(E) \to H_{n-k}(B)$ using that H_{n-k} is covariant. This definition has an analogue in algebraic geometry. Alternatively if we use de Rham cohomology we can map the $(k+2)$-form $\omega \in \Omega^{k+2}(E)$ to the k-form $\int_F \omega \in \Omega^k(B)$ given by

$$\left(\int_F \omega\right)_x (\xi_1, \ldots, \xi_k) = \int_{E_x} (\omega(\xi_1, \ldots, \xi_k, -, -))$$

where $\int_{E_x} : \Omega^2(E_x) \to \mathbb{R}$ is the usual integration.

Then we can associate to the surface bundle $\pi : E \to B$ a characteristic class $\kappa_i(E) \in H^{2i}(B)$ by

$$\kappa_i(E) = \pi_!(e^{i+2})$$

where $e \in H^2(E)$ is the Euler class (= first Chern character) of the "tangent bundle along the fibre" $T^\pi E$. This is a vector bundle over E whose restriction to $E_x \subseteq E$ is the tangentbundle $T(E_x)$.

This defines a series of characteristic classes κ_i for surface bundles, i.e. natural transformations

$$\kappa_i : S(B) \to H^{2i}(B).$$

With the proper interpretation, the above definition makes sense also in algebraic geometry.

The classes κ_i have many of the formal properties that ch_i have for vector bundles. It is natural to ask to what extend one can formulate and prove an analogue of Theorem 2.1 for the κ-classes. Recall that properties *(i)* and *(ii)* used that V took values in monoids because vectorbundles can be added. We shall make S monoid-valued in the following way (which does *not* work in algebraic geometry). Let us consider bundles $\pi : E \to B$ whose fibres are surfaces with boundary. The boundary should consist of one "incoming" circle and one "outgoing". Thus $\partial E = B \times (S^1 \amalg S^1)$. Then S is monoid-valued: Given bundles $E \to B$ and $E' \to B$ we let $E \cup_\partial E'$ be the manifold obtained by gluing the outgoing boundary of E to the incoming boundary of E'. This is again a bundle over B, and this glueing is associative up to isomorphism. In particular we can always increase the genus of the fibre by glueing the trivial bundle $B \times F_{1,g}$ whose fibre $F_{1,2}$ is a torus with two boundary components. Then we have the following theorem about characteristic classes with values in $H^*(-;\mathbb{Q})$ (or $H^*(-;\mathbb{R})$ or de Rham cohomology).

Theorem 3.1.

(i) $\kappa_i(E \cup_\partial E') = \kappa_i(E) + \kappa_i(E')$. *In particular if* $E' = B \times F_{1,2}$ *is the trivial bundle whose fibre is a torus with two boundary circles, then* $\kappa_i(E) = \kappa_i(E \cup_\partial (B \times F_{1,2}))$.

(ii) Any characteristic class κ *satisfying* $\kappa(E \cup_\partial E') = \kappa(E) + \kappa(E')$ *is a unique linear combination of the* κ_i.

(iii) Any characteristic class κ *satisfying* $\kappa(E \cup_\partial E') = \kappa(E) + \kappa(E')$ *when* $E' = B \times F_{1,2}$ *is a unique polynomial in the* κ_i.

(iv) Characteristic classes of surface bundles of genus g *are, for* $* < (g-1)/2$, *exactly the polynomials in the* κ_i.

This theorem is an analogue for surface bundles of Theorem 2.1. Direct sum is replaced by glueing, and dimension of vector bundles is replaced by genus of surfaces. Property *(i)* is a rather direct consequence of the definition. Given *(i)*, *(ii)* and *(iii)* are equivalent. They are not, however easy to prove. In fact they are equivalent to the Mumford conjecture (which is now a consequence of the Madsen-Weiss theorem). Property *(iv)* is noticeable weaker than its analogue for Chern characters: We do not claim to have a complete description of the characteristic classes of bundles of a given genus. Only of the classes in the "stable range". This is related to the Harer stability theorem.

4. Madsen's conjecture and the Madsen-Weiss theorem

Next let me say something about the proof of Theorem 3.1. A key observation is that $\pi : E \to B$ really does not quite need to be a bundle for the definition of $\kappa_i(E)$ to make sense. Slightly less will do. In fact we consider instead of S the contravariant functor that takes B to the set of equivalence classes of proper smooth maps $\pi : E \to B$ together with an epimorphism of vectorbundles $\Pi : TE \times \mathbb{R}^j \to TB \times \mathbb{R}^j$, $j \geq 0$. The kernel of Π (which is a vector bundle over E) is required to be oriented. The equivalence relation is firstly that we are allowed to increase j by crossing Π with $\mathbb{R}$. Secondly, we can change (π, Π) by a *concordance*. This means that if $\tilde{\pi} : \tilde{E} \to \mathbb{R} \times B$ is a proper smooth map and $\tilde{\Pi} : T\tilde{E} \times \mathbb{R}^j \to T(B \times \mathbb{R}) \times \mathbb{R}^j$ is an epimorphism of vectorbundles with oriented kernel, then if $\tilde{\pi}$ is transverse to $\{0,1\} \times B$ then we identify $E_0 = \tilde{\pi}^{-1}(0)$ with $E_1 = \tilde{\pi}^{-1}(1)$.

It is clear that there is a natural transformation $S \to \tilde{S}$, namely we can set $j = 0$ and let Π be the differential of π (this is an epimorphism of vectorbundles $TE \to TB$ because $\pi : E \to B$ is a surface bundle). While at first $\tilde{S}$ may look more complicated than S, it really is much easier to understand. This is because it is a *cohomology theory*. This means roughly that if $B = B_1 \cup B_2$ where $B_i \subseteq B$ are open submanifolds, then $\tilde{S}(B)$ is related to $\tilde{S}(B_1)$, $\tilde{S}(B_2)$ and $\tilde{S}(B_1 \cap B_2)$ by a Mayer-Vietoris long exact sequence. Furthermore it is a cohomology theory of a kind which is rather well understood. It is namely a *cobordism theory*. This kind of cohomology theories were invented by Thom in his celebrated work on cobordism in the 1950's. It would lead too far away from the subject to try to describe this in any generality, so I will just mention that given this, by now classical, theory (in particular the "Thom isomorphism theorem"), it is easy to prove theorem 3.1 *(ii)* for S replaced by the functor $\tilde{S}$.

Now the situation is as follows. We have defined the characteristic classes κ_i as natural transformations $S \to H^{2i}(-)$. We have noted that they extend to natural transformations $\tilde{S} \to H^{2i}(-)$, and we have proved Theorem 3.1 for $\tilde{S}$ in place of S. Now in [MT], Madsen proposed the following conjecture (slightly reformulated):

If h^* is any cohomology theory, then natural transformations $\kappa : S \to h^*$ satisfying $\kappa(E \cup_\partial E') = \kappa(E) + \kappa(E')$ when $E' = B \times F_{1,2}$ are in bijection with natural transformations $\tilde{S} \to h^*$.

Given what we have said, this clearly implies the Mumford conjecture (by setting $h^* = H^*(-;\mathbb{Q})$). Madsen's conjecture was proved by Madsen and Weiss in [MW].

All my scientific work has been closely related to the Madsen-Weiss theorem. The first obvious question is to let h^* be a cohomology theory other than

$H^*(-;\mathbb{Q})$. An obvious cohomology theory is $H^*(-;\mathbb{F}_p)$ for p a prime. I gave the answer in [G1], calculating "all" characteristic classes with values in $H^*(-;\mathbb{F}_p)$. The answer has the advantage of being complete, i.e. determining all characteristic classes, and the disadvantage of being abstract, in the sense that the answer is rather large and it is difficult from [G1] to pick a single characteristic class and evaluate it on a given bundle. [G2] is an attempt to compensate for that by defining explicit characteristic classes $\lambda_i : S \to H^*(-;\mathbb{F}_p)$. They associate to a bundle $\pi : E \to B$ a characteristic class $\lambda_i E \in H^{2i(p-1)-2}(B;\mathbb{F}_p)$. These classes have the interesting property that under the map

$$H^*(B;\mathbb{Z}/p) \xrightarrow{p} H^*(B;\mathbb{Z}/p^2)$$

they map to the reduction of $\kappa_{i(p-1)-1}$ mod p^2.

The paper [GMT] goes in another direction. Inside the $\mathbb{Q}$-vectorspace $H^*(B;\mathbb{Q})$ there is an integral lattice, namely the image of $H^*(B;\mathbb{Z})$. If $\pi : E \to B$ is a bundle, then $\kappa_i(E)$ will lie in that lattice (in fact it is a well defined class in $H^*(B;\mathbb{Z})$). It is natural to ask how divisible $\kappa_i(E)$ is, i.e. what is the largest natural number D_i such that $\kappa_i(E)/D_i \in H^*(B;\mathbb{Q})$ is always in the integral lattice. The answer is given in [GMT] and is somewhat surprising. We recall it here.

The expression $\log(\frac{e^z-1}{z})$ defines a holomorphic function on the disk of radius 2π and is given by a power series

$$\log\left(\frac{e^z-1}{z}\right) = \sum_{i=1}^{\infty} \alpha_i \frac{z^i}{i!}$$

It is easy to see that the coefficients α_i are rational numbers (related to the socalled Bernoulli numbers), so they may be written uniquely as a fraction in lowest terms

$$\alpha_i = \frac{N_i}{D_i}$$

Then the main theorem of [GMT] is that this D_i is the largest number such that $\kappa_i(E)/D_i \in H^{2i}(B;\mathbb{Q})$ is always in the integral lattice.

Finally in my last paper [G3] I consider a variation of the Madsen-Weiss theorem. Instead of considering bundles $\pi : E \to B$ with oriented fibres, one may consider bundles of surfaces with some other structure. The structure that I consider in [G3] is that of a *spin structure*. This may be relevant to physics. A spin structure is an extra structure on a manifold, like an orientation. In [G3] I prove the analogue of the Madsen-Weiss theorem for such surfaces. I also carry out the analogue of [G1], namely determine "all" characteristic classes (in an abstract way) with values in $H^*(-;\mathbb{F}_p)$ for p a prime. For p odd there is no difference between the answer for oriented surfaces and the answer for spin surfaces.

5. Historical notes

This final section of my introduction is meant to be slightly more precise and "historically correct", but also slightly more technical. I will try to highlight the important theorems of the subject.

All the functors in the preceding sections are representable, and usually theorems are expressed in terms of the representing spaces instead of the functors themselves. Let me briefly introduce the representing spaces.

In topology the functor S_F sending a space B to the set of isomorphism classes of bundles $E \to B$ with fibre F is represented by a space $B\mathrm{Diff}(F)$. This means that there is a natural bijection

$$S_F(B) \cong [B, B\mathrm{Diff}(F)]$$

where $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$. Here $\mathrm{Diff}(F)$ is the topological group of (orientation preserving) diffeomorphisms of F which fixes ∂F pointwise. $B(-)$ is a certain functor from (topological) groups to spaces. By the "Yoneda lemma" there is a bijections between $h(B\mathrm{Diff}(F))$ and the set of natural transformations $S_F \to h$, where h is any contravariant functor from spaces to sets. In particular $H^*(B\mathrm{Diff}(F))$ is the set of characteristic classes of fibre bundles with fibre F.

Now we can let $F = F_{g,b}$ be a surface of genus g and with b boundary components. Then we define

$$\Gamma_{g,b} = \pi_0 \mathrm{Diff}(F_{g,b})$$

to be the group of components of $\mathrm{Diff}(F_{g,b})$. Then a theorem of Earle and Eells [EE] states that the natural map

$$B\mathrm{Diff}(F_{g,b}) \to B\Gamma_{g,b}$$

is a homotopy equivalence when $g \geq 2$.

A very important theorem in the subject is the Harer stability theorem. To explain it, notice that if $b > 0$ and $F_{g,b} \to F_{g+g',b'}$ is an embedding, then there is an induced map $\mathrm{Diff}(F_{g,b}) \to \mathrm{Diff}(F_{g+g',b'})$ and in turn a map

$$B\Gamma_{g,b} \to B\Gamma_{g+g',b'}.$$

The Harer stability theorem [JH] states that the induced map

$$H^*(B\Gamma_{g+g',b'}) \to H^*(B\Gamma_{g,b})$$

is an isomorphism for $* < (g-1)/2$. In fact Harer only proved this for $* < (g-1)/3$; the bound was later improved by Ivanov [I]. Combined with the Earle-Eells theorem this implies that the map $B\mathrm{Diff}(F_{g,b}) \to B\mathrm{Diff}(F_{g+g',b'})$ induces an isomorphism in H^* when $* < (g-1)/2$, and thus that the set of characteristic classes for surface bundles is independent of the genus in a "stable range".

The current approach to the subject, which has now lead to a proof of the Mumford conjecture, began with Tillmann's paper [T]. To explain the main result, consider the monoid-value functor S from section 3. It is represented by the topological monoid

$$M = \coprod_{g \geq 0} B\mathrm{Diff}(F_{g,2}).$$

The operation $- \cup_\partial (B \times F_{1,2})$ of glueing a torus to all the fibres is represented by a map $M \to M$ and we can form the direct limit

$$\mathbb{Z} \times B\Gamma_\infty := \mathrm{colim} \left(\coprod_{g \geq 0} B\mathrm{Diff}(F_{g,2}) \right)$$

where the colimit is over the map $M \to M$ glueing tori. Then $H^*(\mathbb{Z} \times B\Gamma_\infty)$ will be the set of characteristic classes κ of surface bundles satisfying

$$\kappa(E \cup_\partial E') = \kappa(E) + \kappa(E') \quad \text{when } E' = B \times F_{1,2}.$$

Tillmann's theorem says that there is an "infinite loop space" $\Omega^\infty E$ and a map

$$\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty E$$

which is an isomorphism in H^*. Being an infinite loop space means roughly that the functor $[-, \Omega^\infty E]$ is a cohomology theory.

Thus the Mumford conjecture becomes equivalent to a statement about the cohomology of $\Omega^\infty E$. Tillmann's insight lead to the more ambitious hope of determining completely the space $\Omega^\infty E$, not just its rational cohomology. In fact Madsen in [MT] conjectured that $\Omega^\infty E = \Omega^\infty \mathbb{C}P^\infty_{-1}$, where $\Omega^\infty \mathbb{C}P^\infty_{-1}$ is a certain (well understood) infinite loop space. The space $\Omega^\infty \mathbb{C}P^\infty_{-1}$ is the infinite loop space representing the functor $\tilde{S}$ from section 4 above:

$$\tilde{S}(B) \cong [B, \Omega^\infty \mathbb{C}P^\infty_{-1}]$$

and what became known as Madsen's conjecture says that

$$\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1}$$

is an isomorphism in H^*. This is what Madsen and Weiss proved in [MW]. My own paper [G1] calculates $H_*(\Omega^\infty \mathbb{C}P^\infty_{-1}; \mathbb{F}_p)$ for any prime p.

Finally a few words about algebraic geometry. Here the functor S, whose value at a variety B is the set of isomorphism classes of smooth proper morphisms $E \to B$ whose fibres are curves of genus g, is represented by the moduli space $\mathcal{M}_g$. It is an irreducible variety of dimension $3g - 3$. Strictly speaking it is not really a representing object, it is only a "coarse moduli space". For $k = \mathbb{C}$ there is a direct construction of $\mathcal{M}_g$ as the quotient

$$\mathcal{M}_g = \mathcal{T}_g / \Gamma_g.$$

Here $\mathcal{T}_g$ is the socalled Teichmüller space and is homeomorphic to $\mathbb{R}^{6g-6}$. The mapping class group $\Gamma_g = \pi_0 \mathrm{Diff}(F)$ acts on $\mathcal{T}_g$. It is known that the action has finite isotropy groups, i.e. for all $x \in \mathcal{T}_g$, the subgroup $\{\varphi \in \Gamma_g \mid \varphi x = x\}$ is finite. This implies that the canonical map

$$B\Gamma_g \to \mathcal{T}_g/\Gamma_g = \mathcal{M}_g$$

is an isomorphism in $H^*(-;\mathbb{Q})$. Hence by the Earle-Eells theorem we have

$$H^*(B\mathrm{Diff}(F);\mathbb{Q}) \cong H^*(\mathcal{M}_g;\mathbb{Q})$$

which provides a direct connection between characteristic classes in topology and characteristic classes in algebraic geometry, at least rationally. Mumford in [M] originally defined the classes κ_i in $H^*(\mathcal{M}_g;\mathbb{Q})$ (in fact he defined them in a compactification $\bar{\mathcal{M}}_g$ of $\mathcal{M}_g$) and conjectured that the induced map

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \to H^*(\mathcal{M}_g;\mathbb{Q})$$

were an isomorphism when g is large compared to $*$.

References

[EE] C. J. Earle, C. Eells: *A fibre bundle description of Teichmüller theory*, J. Differential Geometry **3** 1969 19–43.

[G1] S. Galatius: *Mod p homology of the stable mapping class group*, to appear in Topology.

[G2] S. Galatius: *Secondary characteristic classes of surface bundles*, preprint, math.AT:0402226.

[G3] S. Galatius: *Mod 2 homology of the stable spin mapping class group*, preprint.

[GMT] S. Galatius, I. Madsen, U. Tillmann: *Divisibility of the stable Miller-Morita-Mumford classes*, preprint.

[I] N. V. Ivanov: *Stabilization of the homology of the Teichmüller modular groups*, Algebra i Analiz **1** (1989) 120–126; translation in: Leningrad Math. J. **1** (1990) 675–691.

[JH] J. L. Harer: *Stability of the homology of the mapping class groups of oriented surfaces*, Ann. of Math. **121** (1985) 215–249.

[RH] R. Hartshorne: *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[M] D. Mumford: *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and geometry, Vol. II, 271–328, Progr. Math. 36, Birkhäuser Boston, Boston, MA, 1983.

[MS] J. Milnor, J. Stasheff: *Characteristic Classes*, Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J., 1974.

[MT] I. Madsen, U. Tillmann: *The stabe mapping class group and* $Q(\mathbb{C}P^\infty_+)$, Invent. Math. **145** (2001), 509–544.

[MW] I. Madsen, M. Weiss: *The stable moduli space of Riemann Surfaces: Mumford's conjecture*, preprint.

[T] U. Tillmann: *On the homotopy of the stable mapping class group*. Invent. Math. **130** (1997), no. 2, 257–275.

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MOD p HOMOLOGY OF THE STABLE MAPPING CLASS GROUP

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Abstract. We calculate the homology $H_*(\Gamma_{g,n};\mathbb{F}_p)$ of the mapping class group $\Gamma_{g,n}$ in the stable range. The calculation is based on Madsen and Weiss' proof of the "Generalised Mumford Conjecture": $\Gamma_{g,n}$ has the same homology as a component of the space $\Omega^\infty \mathbb{C}P^\infty_{-1}$ in the stable range.

1. Introduction

Let $F_{g,n}$ be an oriented surface of genus g and with n boundary components. Let $\mathrm{Diff}(F_{g,n},\partial)$ be the topological group of orientation preserving diffeomorphisms of $F_{g,n}$ fixing pointwise a neighbourhood of the boundary. The mapping class group is the group $\Gamma_{g,n} = \pi_0 \mathrm{Diff}(F_{g,n},\partial)$ of components. There are group maps

$$\Gamma_{g,n} \to \Gamma_{g,n-1} \quad \text{and} \quad \Gamma_{g,n} \to \Gamma_{g+1,n}$$

induced by gluing a disk, resp. a torus with two boundary components, to one of the boundary components of $F_{g,n}$. By a theorem of Harer and Ivanov, these maps induce isomorphisms in $H_*(-;\mathbb{Z})$ for $* \leq (g-1)/2$, and thus there is a stable range in which the group homology $H_*(\Gamma_{g,n};\mathbb{Z})$ is independent of g and n. In this range it agrees with $H_*(\Gamma_\infty;\mathbb{Z})$ where $\Gamma_\infty = \mathrm{colim}_g \Gamma_{g,1}$ is the stable mapping class group.

1.1. Madsen-Weiss' theorem.

The Mumford conjecture predicts that

$$H^*(\Gamma_\infty;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\dots]$$

for certain classes $\kappa_i \in H^{2i}(\Gamma_\infty)$. This was recently proved by Madsen and Weiss, but their result gives more. To state the full result, consider the classifying space $B\Gamma_\infty$. Its homology is the group homology of Γ_∞. By the Quillen plus-construction we get a simply connected space $B\Gamma_\infty^+$ and a map $B\Gamma_\infty \to B\Gamma_\infty^+$ inducing an isomorphism in homology. The Madsen-Weiss theorem determines the homotopy type of $\mathbb{Z} \times B\Gamma_\infty^+$ to be that of $\Omega^\infty \mathbb{C}P^\infty_{-1}$. The space $\Omega^\infty \mathbb{C}P^\infty_{-1}$ (to be defined below), can be examined by methods from stable homotopy theory. In particular it is easy to calculate $H^*(\Omega^\infty \mathbb{C}P^\infty_{-1};\mathbb{Q})$. This implies the Mumford conjecture.

Key words and phrases. mapping class groups, moduli spaces, Thom spectra, homology of infinite loop spaces.

In this paper we calculate $H_*(\Omega^\infty \mathbb{C}P^\infty_{-1}; \mathbb{F}_p)$ for any prime p and hence by the above $H_*(\Gamma_{g,n}; \mathbb{F}_p)$ for $* \leq (g-1)/2$.

1.2. **Outline and statement of results.** Let p be a prime number, and let $H_* = H_*(-; \mathbb{F}_p)$.

Let L be the canonical complex line bundle over $\mathbb{C}P^\infty$ and let $\mathbb{C}P^\infty_{-1} = \mathbb{T}\mathrm{h}(-L)$ be the Thom spectrum of the -2-dimensional virtual bundle $-L$. Inclusion of a point in $\mathbb{C}P^\infty$ induces a map $S^{-2} \to \mathbb{C}P^\infty_{-1}$ and the zero section of the line bundle L induces a map

$$\mathbb{C}P^\infty_{-1} = \mathbb{T}\mathrm{h}(-L) \to \mathbb{T}\mathrm{h}(-L+L) = \Sigma^\infty \mathbb{C}P^\infty_+$$

These fit together into a cofibration sequence

$$S^{-2} \to \mathbb{C}P^\infty_{-1} \to \Sigma^\infty \mathbb{C}P^\infty_+$$

and there is an induced fibration sequence of infinite loop spaces

$$\Omega^\infty \Sigma \mathbb{C}P^\infty_{-1} \xrightarrow{\omega} Q(\Sigma \mathbb{C}P^\infty_+) \xrightarrow{\partial} QS^0 \tag{1.1}$$

where we write $Q = \Omega^\infty \Sigma^\infty$. This fibration sequence is the starting point for the calculation of the mod p homology of $\Omega^\infty \Sigma \mathbb{C}P^\infty_{-1}$ and $\Omega^\infty \mathbb{C}P^\infty_{-1}$.

The mod p homology of $Q\Sigma\mathbb{C}P^\infty_+$ and QS^0 is completely known ([1], [3], [2]), as is the induced map ∂_* in homology ([6]). The first main result of this paper is a calculation of the Hopf algebra $H_*(\Omega^\infty \Sigma \mathbb{C}P^\infty_{-1}; \mathbb{F}_p)$. We need to introduce the following notation to state the results (see Section 2 for further details). All Hopf algebras will be commutative and cocommutative. The Hopf algebra cokernel and kernel of a map $f : A \to B$ of Hopf algebras will be denoted $B /\!\!/ f$ and $A \backslash\!\!\backslash f$, respectively. PA is the vectorspace of primitive elements, and QA is the vectorspace of indecomposable elements. For a graded vectorspace V, $s^{-1}V$ will denote the desuspension of V: $(s^{-1}V)_{\nu-1} = V_\nu$. We also need to introduce the following functors from vectorspaces to algebras. Let V be a non-negatively graded vectorspace and let $B \subseteq V_0$ be a basis for the degree zero part of V. Let $V = V^+ \oplus V^-$ be the splitting of V into even and odd dimensional parts. Then $E[V^-]$ is the exterior algebra generated by V^- and $\mathbb{F}_p[V^+]$ is the polynomial algebra generated by V^+. Furthermore $\mathbb{F}_p[B, B^{-1}]$ is the algebra of Laurent polynomials in the elements of B and $\mathbb{F}_p[V^+, B^{-1}] = \mathbb{F}_p[V^+] \otimes_{\mathbb{F}_p[B]} \mathbb{F}_p[B, B^{-1}]$ is the polynomial algebra generated by V^+, with the elements of B inverted. The *free commutative algebra* generated by V is $S[V] = E[V^-] \otimes \mathbb{F}_p[V^+, B^{-1}]$.

The calculations use the theory of *homology operations*. These are defined on the mod p homology of infinite loop spaces, and are natural with respect to infinite loop maps, cf. [1], [3], [2]. The basic operations are

$$\beta^\varepsilon Q^s : H_n(X) \to H_{n+2s(p-1)-\varepsilon}(X), \qquad (p > 2)$$
$$Q^s : H_n(X) \to H_{n+s}(X), \qquad (p = 2)$$

where $\varepsilon \in \{0,1\}$ and $s \in \mathbb{Z}_{\geq\varepsilon}$. Given a sequence $I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$ (with all $\varepsilon_i = 0$ if $p = 2$) there is an iterated operation $Q^I = \beta^{\varepsilon_1} Q^{s_1} \dots \beta^{\varepsilon_k} Q^{s_k}$. The mod p homology of QS^0 then has the following form: Let $\iota \in H_0(QS^0)$ be the image of the non-basepoint in S^0. Then $H_*(QS^0)$ is the free commutative algebra on the set

$$\mathbf{T} = \{Q^I\iota | I \text{ admissible, } e(I) + b(I) > 0\} \tag{1.2}$$

(see section 3 for the definition of $e(I)$, $b(I)$ and the notion of admissibility).

As a step towards calculating $H_*(\Omega^\infty \Sigma \mathbb{C}P^\infty_{-1})$ we determine the Hopf algebra cokernel of $\partial_* : H_*(Q\Sigma\mathbb{C}P^\infty_+) \to H_*(QS^0)$. In the following theorem, $Q_0S^0 \subseteq QS^0$ is the basepoint component. Then $H_*(QS^0) = H_0(QS^0) \otimes H_*(Q_0S^0)$.

Theorem 1.1. *Let* $\mathbf{T}$ *be as in* (1.2). *Let* $H_*(QS^0)^{(0)}$ *denote the subalgebra of* $H_*(QS^0)$ *generated by the set*

$$\{Q^I\iota \in \mathbf{T} | \text{all } \varepsilon_i = 0\} \qquad (p > 2)$$
$$\{Q^I\iota \in \mathbf{T} | \text{all } s_i \text{ even}\} \qquad (p = 2)$$

Then the composite

$$H_*(QS^0)^{(0)} \to H_*(QS^0) \to H_*(QS^0)/\!\!/\partial_*$$

is an isomorphism. In particular the Hopf algebra $H_*(QS^0)/\!\!/\partial_*$ *is concentrated in degrees* $\equiv 0 \pmod{2(p-1)}$. *Similarly for* $H_*(Q_0S^0)/\!\!/\partial_*$. *Furthermore the dual algebra* $H^*(Q_0S^0)\backslash\!\!\backslash\partial^*$ *is a polynomial algebra.*

Theorem 1.2. *The sequence*

$$\mathbb{F}_p \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_* \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \xrightarrow{\omega_*} H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \longrightarrow \mathbb{F}_p$$

is an exact sequence of Hopf algebras. It is split but not canonically. Furthermore there is a canonical isomorphism

$$H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_* \cong E[s^{-1}P(H_*(QS^0)/\!\!/\partial_*)]$$

In particular, $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ *is primitively generated and for* $p > 2$ *it is free commutative.*

Theorem 1.1 is an algebraic consequence of the known structure of $H_*(Q\Sigma\mathbb{C}P^\infty_+)$ and $H_*(QS^0)$ and the induced map ∂_* in homology. The proof of Theorem 1.2 uses the Eilenberg-Moore spectral sequence of the fibration sequence (1.1).

Next we calculate $H_*(\Omega^\infty\mathbb{C}P^\infty_{-1})$.

Theorem 1.3. *For* $p = 2$, *the sequence*

$$\mathbb{F}_2 \longrightarrow H_*(\Omega^\infty\mathbb{C}P^\infty_{-1}) \xrightarrow{\Omega\omega_*} H_*(Q\mathbb{C}P^\infty_+) \xrightarrow{\Omega\partial_*} H_*(\Omega QS^0) \longrightarrow \mathbb{F}_2$$

is exact. In particular $\Omega\omega_*$ *induces an isomorphism*

$$H_*(\Omega^\infty\mathbb{C}P^\infty_{-1}) \xrightarrow{\cong} H_*(Q\mathbb{C}P^\infty_+)\backslash\!\!\backslash\Omega\partial_*$$

The short exact sequence in Theorem 1.3 determines $H_*(\Omega^\infty \mathbb{C}P^\infty_{-1};\mathbb{F}_2)$ completely. Indeed, as part of the proof of Theorem 1.3 we determine $H_*(\Omega QS^0)$ and the induced map $\Omega\partial_*$ in homology. The result can be summarised by the diagram

$$\begin{tikzcd} QH_*(Q\mathbb{C}P^\infty_+) \arrow[r, "Q(\Omega\partial_*)"] \arrow[d, "\cong"] & QH_*(\Omega QS^0) \arrow[d, "\cong"] \\ PH_*(Q\Sigma\mathbb{C}P^\infty_+) \arrow[r, "P(\partial_*)"] & PH_*(QS^0) \end{tikzcd} \tag{1.3}$$

where the vertical isomorphisms in (1.3) are the homology suspensions. The homology $H_*(\Omega_0 QS^0)$ of the basepoint component of ΩQS^0 is a *divided power algebra*, i.e. its dual is a primitively generated polynomial algebra.

For odd primes p our results are less precise in that $H_*(\Omega^\infty \mathbb{C}P^\infty_{-1};\mathbb{F}_p)$ is only determined up to algebra isomorphism. The main technical theorem is the following

Theorem 1.4. *For odd primes p, the homology suspension*

$$\sigma_* : QH_*(\Omega^\infty \mathbb{C}P^\infty_{-1}) \to PH_*(\Omega^\infty \Sigma \mathbb{C}P^\infty_{-1})$$

is surjective.

Proving Theorem 1.4 is the most difficult part of the paper. It uses that σ_* commutes with the homology operations $\beta^\varepsilon Q^s$.

Corollary 1.5. *Let $Y \subseteq H_*(\Omega^\infty \mathbb{C}P^\infty_{-1})$ be a subset such that $\sigma_*(Y)$ is a basis of $PH_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$. Then $H_*(\Omega^\infty \mathbb{C}P^\infty_{-1})$ is the free commutative algebra on the set*

$$Y \cup \{\beta Q^s y | y \in Y^-,\ \deg(y) = 2s-1\}$$

Corollary 1.5 is a formal consequence of Theorem 1.4 and the fact that the Hopf algebra $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ is primitively generated. The proof uses the "Kudo transgression theorem", cf. [2], Theorem 1.1(7): If $\deg(y) = 2s-1$, then in the Leray-Serre spectral sequence we have that $\sigma_*(y)$ transgresses to y and that $(\sigma_* y)^{p-1} \otimes y$ transgresses to $-\beta Q^s y$.

1.3. **Acknowledgements.** This calculation is part of my phd-project at the University of Aarhus. It is a great pleasure to thank my thesis advisor Ib Madsen for his help and encouragement during my years as a graduate student.

2. Recollections

In this introductory section we collect the results we need later in the paper. We start by recalling some important results on the structure of Hopf algebras from [5] and proceed to review the functor Cotor and the closely related Eilenberg-Moore spectral sequence, cf. [4], [8].

2.1. **Hopf algebras.** Here and elsewhere, the field $\mathbb{F}_p$ with p elements is the ground field, and $\otimes = \otimes_{\mathbb{F}_p}$. Until further notice, p is assumed odd. Algebras and coalgebras are as in [5] and in particular they always have units resp. counits.

Definition 2.1. When A is a coalgebra and M_A, ${}_AN$ are A-comodules with structure maps $\Delta_M : M \to M \otimes A$ and $\Delta_N : N \to A \otimes N$, the cotensor product is defined by the exact sequence

$$0 \longrightarrow M \square_A N \longrightarrow M \otimes N \longrightarrow M \otimes A \otimes N$$

where the right-hand morphism is $\Delta_M \otimes N - M \otimes \Delta_N$. The functors $M\square_A -$ and $-\square_A N$ are left exact functors from A-comodules to $\mathbb{F}_p$-vectorspaces in general, and to A-comodules when A is cocommutative.

Definition 2.2. For a morphism $f : A \to B$ of Hopf algebras, define the *kernel* and *cokernel*

$$A\backslash\!\!\backslash f = A \square_B k, \quad B /\!\!/ f = B \otimes_A k$$

A priori, the kernel and cokernel are vectorspaces, but when A and B are commutative and cocommutative, they become Hopf algebras and are the kernel and cokernel in the categorical sense. Hopf algebras that are both commutative and cocommutative are called *abelian*, and the category of those is an abelian category (this essentially follows from [5, Prop. 4.9]).

The Hopf algebras appearing in this paper will (except for R and $\mathscr{R}$ defined below) be of the form $A = H_*(X;\mathbb{F}_p)$ for X an infinite loop space. Such Hopf algebras will always be abelian. We will often have that $H_*(X,\mathbb{F}_p)$ is of finite type, and in this case $H^*(X;\mathbb{F}_p)$ will also be a Hopf algebra. However, if $\pi_0(X)$ is infinite, $H^*(X;\mathbb{F}_p)$ will not be a Hopf algebra (e.g. $X = QS^0$ with $\pi_0 X = \mathbb{Z}$). Usually it will then be the case that the basepoint component $X_0 \subseteq X$ is of finite type, and thus we can consider $H^*(X_0;\mathbb{F}_p)$. Hopf algebras A with $A_i = 0$ for $i < 0$ and $A_0 = \mathbb{F}_p$ are called *connected*. In general we will have a natural splitting of Hopf algebras $H_*(X) = H_*(X_0) \otimes \mathbb{F}_p\{\pi_0 X\}$ where $\mathbb{F}_p\{\pi_0 X\} = H_0(X)$ is the group algebra.

Definition 2.3. For an algebra A with augmentation ε, let $IA = \mathrm{Ker}(\varepsilon : A \to k)$ and dually for a coalgebra A with augmentation η, let $JA = \mathrm{Cok}(\eta : k \to A)$. Let Q and P be the functors defined by the exact sequences

$$IA \otimes IA \xrightarrow{\varphi} IA \longrightarrow QA \longrightarrow 0$$

and

$$0 \longrightarrow PA \longrightarrow JA \xrightarrow{\Delta} JA \otimes JA$$

P and Q satisfies $P(A \otimes B) = PA \oplus PB$ and $Q(A \otimes B) = QA \otimes QB$, and as functors from abelian Hopf algebras to vectorspaces, Q is right exact and P is left exact ([5, Prop 4.10]). When A is connected, $PA \subseteq A$ is the subset consisting of elements x satisfying $\Delta x = x \otimes 1 + 1 \otimes x$. If $A = \mathbb{F}_p\{\pi_0 X\}$ is a group algebra, then $PA = 0$.

The functors P and Q are related by the short exact sequence of [5, Thm. 4.23]:

Theorem 2.4. *For an abelian Hopf algebra A, let $\xi : A \to A$ be the Frobenius map $x \mapsto x^p$ and let $\lambda : A \to A$ be the dual of $\xi : A^* \to A^*$. Let $\xi A \subseteq A$ be the image of ξ and let $A \to \lambda A$ be the coimage of λ. Then there is the following natural exact sequence*

$$0 \longrightarrow P\xi A \longrightarrow PA \longrightarrow QA \longrightarrow Q\lambda A \longrightarrow 0 \tag{2.1}$$

In particular $PA \to QA$ is an isomorphism except possibly in degrees $\equiv 0 \pmod{2p}$ if $p > 2$. For $p = 2$ it is an isomorphism in odd degrees.

Finally, we recall Borel's structure theorem ([5, Theorem 7.11])

Theorem 2.5. *Any connected abelian Hopf algebra A is isomorphic as an algebra to a tensor product of algebras of the form $E[x]$, $\mathbb{F}_p[x]$ and $\mathbb{F}_p[x]/(x^{p^n})$, $n \geq 1$.*

Corollary 2.6. *A connected abelian Hopf algebra A is isomorphic as an algebra to a polynomial algebra if and only if $\xi : A \to A$ is injective. Dually if A is of finite type, A^* is polynomial if and only if $\lambda : A \to A$ is surjective.* $\square$

2.2. **The functor** Cotor**.** When A is a coalgebra and B and C are left resp. right A-comodules, the functor

$$\mathrm{Cotor}^A(B, C)$$

is defined as the right derived functor of the cotensor product $\square_A$. To be explicit (and to fix grading conventions), choose an injective resolution $0 \to B \to I_0 \to I_{-1} \to \dots$ of B in the category of right A-comodules and set

$$\mathrm{Cotor}^A_n(B, C) = H_n(I_* \square_A C)$$

When A, B and C are in the graded category, Cotor gets an inner grading and is thus bigraded with $\mathrm{Cotor}^A_{n,m}(B, C) = (\mathrm{Cotor}^A_n(B, C))_m$. When A, B, C are all positively graded, Cotor is concentrated in the second quadrant.

When A, B and C are of finite type over a field, this functor is dual to the more common Tor:

$$\mathrm{Cotor}^A(B, C) = \left(\mathrm{Tor}^{A^*}(B^*, C^*)\right)^*$$

This follows immediately from the duality between $\square_A$ and $\otimes_{A^*}$.

We shall consider Cotor as a functor from diagrams of cocommutative coalgebras

$$\mathscr{S} = \left\{ \begin{array}{ccc} & & B \\ & & \downarrow \\ C & \longrightarrow & A \end{array} \right\}$$

to coalgebras. The external product is an isomorphism (see [10, Theorem 3.1, p. 209])

$$\mathrm{Cotor}^A(B,C)\otimes \mathrm{Cotor}^{A'}(B',C') \to \mathrm{Cotor}^{A\otimes A'}(B\otimes B', C\otimes C')$$

and under this isomorphism the comultiplication in $\mathrm{Cotor}^A(B,C)$ is given by the comultiplication $\Delta : \mathscr{S} \to \mathscr{S}\otimes\mathscr{S}$ in the diagram $\mathscr{S}$.

Dually, when $\mathscr{S}$ is a diagram of Hopf algebras, $\mathrm{Cotor}^A(B,C)$ is a Hopf algebra with multiplication induced by the multiplication $\varphi : \mathscr{S}\otimes\mathscr{S} \to \mathscr{S}$ of the diagram $\mathscr{S}$.

Later we will need the structure of $\mathrm{Cotor}^A(B,\mathbb{F}_p)$ where $\mathbb{F}_p$ denotes the trivial Hopf algebra and $f : B \to A$ is a morphism of Hopf algebras. From the change of rings spectral sequence and [5, Theorem 4.9] we get

Proposition 2.7. *For a map $f : B \to A$ of Hopf algebras, there is a natural isomorphism of Hopf algebras*

$$\mathrm{Cotor}^A(B,\mathbb{F}_p) \xrightarrow{\cong} B\backslash\!\backslash f \otimes \mathrm{Cotor}^{A/\!/f}(\mathbb{F}_p,\mathbb{F}_p)$$

□

To complete the description of $\mathrm{Cotor}^A(B,\mathbb{F}_p)$ we need to calculate the Hopf algebra $\mathrm{Cotor}^A(\mathbb{F}_p,\mathbb{F}_p)$. This is easily done by applying Borel's structure theorem to the dual algebra A^* and using Lemma 2.8 below. The Hopf algebra $\Gamma[x]$ is dual to a polynomial algebra: $\Gamma[x] = (k[x^*])^*$ and $s^{-\nu}$ denotes bigraded desuspension: $(s^{-\nu}V)_{-\nu,n} = V_n$ for a singly graded object V.

Lemma 2.8. *The following isomorphisms hold as Hopf algebras*

$$\begin{aligned}
\mathrm{Tor}^{E[x]}(\mathbb{F}_p,\mathbb{F}_p) &= \Gamma[s^{-1}x]\\
\mathrm{Tor}^{\mathbb{F}_p[x]}(\mathbb{F}_p,\mathbb{F}_p) &= E[s^{-1}x]\\
\mathrm{Tor}^{\mathbb{F}_p[x]/(x^{p^n})}(\mathbb{F}_p,\mathbb{F}_p) &= E[s^{-1}x]\otimes\Gamma[s^{-2}x^{p^n}]
\end{aligned}$$

□

By the duality between Tor and Cotor we obtain the Hopf algebra structure of $\mathrm{Cotor}^A(\mathbb{F}_p,\mathbb{F}_p)$ in terms of a set of generators of the dual algebra A^*.

Corollary 2.9. *For any connected Hopf algebra A of finite type,* $\mathrm{Cotor}^A(\mathbb{F}_p,\mathbb{F}_p)$ *is freely generated by the primitive elements in* $\mathrm{Cotor}^A_{-1,*}(\mathbb{F}_p,\mathbb{F}_p)$ *and* $\mathrm{Cotor}^A_{-2,*}$. *Choosing generators of A^* (according to Borel's structure theorem), the generators of* $\mathrm{Cotor}^A(\mathbb{F}_p,\mathbb{F}_p)$ *are in bidegrees*

$$\begin{aligned}
(-1,k) &\quad \text{for } x\in A^*_k \text{ an odd generator}\\
(-1,k) &\quad \text{for } x\in A^*_k \text{ an even generator}\\
(-2,p^m k) &\quad \text{for } x\in A^*_k \text{ an even generator of height } p^m
\end{aligned}$$

The primitive elements of $\mathrm{Cotor}^A(k,k)$ *are in bidegrees*

$$\begin{array}{rl} p^n(-1,k) & \text{for } x \in A_k^* \text{ an odd generator} \\ (-1,k) & \text{for } x \in A_k^* \text{ an even generator} \\ p^n(-2,p^m k) & \text{for } x \in A_k^* \text{ an even generator of height } p^m \end{array}$$

∎

More functorially, one defines for $p > 2$ the functor $\hat{P}A = P\mathrm{Cotor}^A_{-2,*}(\mathbb{F}_p, \mathbb{F}_p)$. Then the result in Corollary 2.9 is that $\mathrm{Cotor}^A(\mathbb{F}_p, \mathbb{F}_p) \cong S[s^{-1}PA] \otimes S[s^{-2}\hat{P}A]$ combined with the facts that $\hat{Q}(\mathbb{F}_p[x]) = \hat{Q}(E[x]) = 0$ and that $\hat{Q}(\mathbb{F}_p[x]/(x^{p^n})) = \mathbb{F}_p.\{x^{p^n}\}$, where $\hat{Q}A = (\hat{P}A^*)^*$.

In particular, the only primitive elements of *odd* total degree are in bidegrees $(-1,k)$ for even generators $x \in A_k^*$.

Finally, we shall need a criterion for left exactness of the functor Q, namely

Proposition 2.10. *Let*

$$k \to A \to B \to C \to k$$

be a short exact sequence of abelian Hopf algebras. If C is a free commutative algebra, then the sequence

$$0 \to QA \to QB \to QC \to 0$$

is short exact.

Proof. Since C is free, we may split $B \to C$ with a map of algebras. Thus $B \cong A \otimes C$ as an algebra, and $Q(B)$ depends only on the algebra structure of B. ∎

A peculiar consequence of Corollary 2.6 is that if A is a Hopf algebra that is free as an algebra, then any Hopf subalgebra of A is also free as an algebra.

2.3. **The spectral sequence.**

In this section, we recall the spectral sequence of [4] and some of its properties.

We consider *homotopy cartesian squares*

$$\mathscr{C} = \left\{ \begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array} \right\}$$

of connected spaces, and with B simply connected (homotopy cartesian means that $F \simeq \mathrm{holim}(X \to B \leftarrow E)$. One can always find a model that is a *fibre square*, i.e. where $E \to B$ is a fibration, and $F \to X$ is the pullback fibration). In the following, H_* denotes $H_*(-;\mathbb{F}_p)$.

Definition 2.11. The Eilenberg-Moore spectral sequence E^r is a functor from fibre squares $\mathscr{C}$ as above to spectral sequences of coalgebras. It has

$$E^2 = \mathrm{Cotor}^{H_*(B)}(H_*(E), H_*(X))$$

and converges as coalgebra to H_*F.

Theorem 2.12 ([4, Proposition 16.4])**.** *The external product induces an isomorphism*

$$E^r(\mathscr{C}) \otimes E^r(\mathscr{C}') \to E^r(\mathscr{C} \times \mathscr{C}')$$

Under this isomorphism, the coalgebra structure is induced by the diagonal $\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$. □

Dually, when $\mathscr{C}$ is a diagram of H-spaces and H-maps (here meaning maps commuting *strictly* with the multiplication such as loop spaces and loop maps), there is a multiplication $m : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ inducing a multiplication $\varphi = m_* : E^r(\mathscr{C}) \otimes E^r(\mathscr{C}) \to E^r(\mathscr{C})$. In this case, the spectral sequence is one of Hopf algebras. Furthermore it is clear that on the E^2-term, the Hopf algebra structure is the same as the one on Cotor described above.

2.4. **The loop suspension.** We shall use the spectral sequence only in the case when X is a point. This corresponds to a fibration sequence

$$F \to E \to B$$

and the spectral sequence computes homology of the fibre. When E is also a point, we have the path-loop fibration sequence

$$\Omega X \to * \to X$$

In this case, the line

$$E^2_{0,*} = \mathrm{Cotor}^{H_*(X)}_{0,*}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p \square_{H_*(X)} \mathbb{F}_p = \mathbb{F}_p$$

is concentrated in degree 0 and hence there is a "secondary edge homomorphism"

$$H_*(\Omega X) \to E^\infty_{-1,*} \hookrightarrow E^2_{-1,*} \cong PH_*X \tag{2.2}$$

Proposition 2.13 ([11, Proposition 4.5])**.** *The morphism in* (2.2) *is the loop suspension*

$$\sigma_* : QH_*(\Omega X) \to PH_*X$$

□

We shall also need

Lemma 2.14. *Let C_* be a connected differential graded Hopf algebra. If x is an element of minimal degree with $dx \neq 0$, then x is indecomposable and dx is primitive.*

Proof. Immediate from the Leibniz rules for product and coproduct. □

Corollary 2.15. *Minimal differentials in the spectral sequence of a path-loop fibration correspond to minimal elements in the cokernel of σ_*.*

Proof. Since dx is primitive and not in $E^2_{-1,*}$ it is of even total degree and x is of odd total degree. By Corollary 2.9, the only odd dimensional indecomposable elements are in $E^2_{-1,*}$ and the result follows. □

3. Unstable R-modules

As sketched in the introduction, homology of an infinite loop space has homology operations $\beta^\varepsilon Q^s$. In this section we recall the precise definitions and explain how to express homology of QX as a free algebra on certain iterated operations on the homology of X. We follow the notation from [2]. In this section we consider only $p > 2$. Small changes, which we recall later, are needed for $p = 2$.

We define several categories of graded vectorspaces with a set of linear transformations $\{\beta^\varepsilon Q^s \mid \varepsilon \in \{0,1\}, s \in \mathbb{Z}_{\geq\varepsilon}\}$ of degree $2s(p-1) - \varepsilon$.

$$\begin{tikzcd} Q\text{-unstable } R\text{-modules} \arrow[r] \arrow[d] & Q\text{-unstable } \mathscr{R}\text{-modules} \arrow[d] \\ \text{unstable } R\text{-modules} \arrow[r] & \text{unstable } \mathscr{R}\text{-modules} \arrow[d] \\ & \mathscr{R}\text{-modules} \arrow[d] \\ & \text{graded vectorspaces} \end{tikzcd} \tag{3.1}$$

Here, $\mathscr{R}$ is the free non-commutative algebra on the set $\{\beta^\varepsilon Q^s \mid \varepsilon \in \{0,1\}, s \in \mathbb{Z}_{\geq\varepsilon}\}$, and the various entries in (3.1) differ in what relations the action of the operations $\beta^\varepsilon Q^s$ are assumed to satisfy. It is the left part of the diagram that is geometrically relevant, since the homology of an infinite loop space X is naturally an unstable R-modules, and so is the vectorspace of primitive elements $PH_*(X)$. The space of indecomposable elements $QH_*(X)$ is naturally a Q-unstable R-module.

All of the above forgetful functors to graded vectorspaces have left adjoint "free" functors. From $\mathscr{R}$-modules it is the functor $V \mapsto \mathscr{R} \otimes V$, and the other four are quotients thereof.

In 3.2, we define the algebras $\mathscr{R}$ and R and the four categories of unstable modules. In 3.3 we construct the four adjoint functors $\mathscr{D}$, $\mathscr{D}'$, D and D'. Finally, in 3.4 we recall the computation of $H_*(QX)$ in terms of $H_*(X)$. It should be noted that the algebra $\mathscr{R}$ and the related categories are needed only in the proof of Theorem 4.4. It is R that is geometrically relevant but $\mathscr{R}$ has the property that a submodule of a free $\mathscr{R}$-module is again free and similarly for submodules of free (Q-)unstable $\mathscr{R}$-modules. This makes $\mathscr{R}$ simpler from the viewpoint of homological algebra.

3.1. **Araki-Kudo-Dyer-Lashof operations.** Recall that an *infinite loop space* is a sequence $E_0, E_1, \dots$ of spaces and homotopy equivalences $\Omega E_{i+1} \to E_i$. One thinks of E_0 as the "underlying space" of the infinite loop space. In particular, $E_0 = \Omega^2 E_2$ is a homotopy commutative H-space. Thus, as mentioned in the introduction, $H_*(E_0)$ is a commutative algebra under the Pontrjagin product. Furthermore $H_*(E_0)$ naturally carries a set of linear transformations $\beta^\varepsilon Q^s$, $\varepsilon \in \{0,1\}, s \in \mathbb{Z}_{\geq \varepsilon}$. These linear transformations are commonly called *Dyer-Lashof operations* (or *Araki-Kudo* operations) and are operations

$$\beta^\varepsilon Q^s : H_n(E_0) \to H_{n+2s(p-1)-\varepsilon}(E_0)$$

natural with respect to infinite loop maps. They measure the failure of chain level commutativity of the Pontrjagin product.

They satisfy a number of relations that makes $H_*(E_0)$ an *unstable R-module*, the notion of which is defined below.

3.2. **The algebras $\mathscr{R}$ and R and categories of unstable modules.**

Definition 3.1. Let $\mathscr{R}$ be the free (non-commutative) algebra generated by symbols

$$\beta^\varepsilon Q^s, \quad \varepsilon \in \{0,1\}, s \in \mathbb{Z}_{\geq \varepsilon}.$$

and write $\beta Q^s = \beta^1 Q^s$ and $Q^s = \beta^0 Q^s$. $\mathscr{R}$ is a graded algebra with

$$\deg(\beta^\varepsilon Q^s) = 2s(p-1) - \varepsilon$$

It will occasionally be convenient to consider $\mathscr{R}$ as a *bigraded* algebra with gradings

$$\deg_Q(\beta^\varepsilon Q^s) = 2s(p-1), \quad \deg_\beta(\beta^\varepsilon Q^s) = -\varepsilon$$

$\mathscr{R}$ is a cocommutative Hopf algebra with comultiplication

$$\Delta(\beta^\varepsilon Q^s) = \sum_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ s_1+s_2=s}} \beta^{\varepsilon_1} Q^{s_1} \otimes \beta^{\varepsilon_2} Q^{s_2}$$

Remark 3.2. $\mathscr{R}$ is a Hopf algebra in the sense of [5], i.e. a monoid object in the category of cocommutative coalgebras. Notice however that $\mathscr{R}$ is not a group object, since Q^0 is not invertible.

Definition 3.3. An $\mathscr{R}$-module is called *unstable*, if

$$\beta^\varepsilon Q^s x = 0 \quad \text{whenever } 2s - \varepsilon < \deg(x) \tag{3.2}$$

It is called *Q-unstable* if furthermore

$$Q^s x = 0 \quad \text{whenever } 2s = \deg(x) \tag{3.3}$$

On homology of an infinite loop space we also have the relation

$$Q^s x = x^p \quad \text{whenever } 2s = \deg(x) \tag{3.4}$$

For an infinite loop space X, $H_*(X)$ is naturally an unstable $\mathscr{R}$-module and $QH_*(X)$ is Q-unstable because of (3.4). However, the ideal in $\mathscr{R}$ of elements with universally trivial action is nonzero, and hence the action of $\mathscr{R}$ on H_*X factors through a quotient of $\mathscr{R}$. This quotient is the *Dyer-Lashof algebra R*.

Definition 3.4. For each $r, s \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$ with $r > ps$, define elements in $\mathscr{R}$

$$\mathscr{A}^{(\varepsilon,r,0,s)} = \beta^\varepsilon Q^r Q^s - \left(\sum_{i=0}^{r+s} (-1)^{r+i} (pi - r, r - (p-1)s - i - 1) \beta^\varepsilon Q^{r+s-i} Q^i \right)$$

For $r \geq ps$ define elements

$$\begin{aligned}\mathscr{A}^{(0,r,1,s)} = Q^r \beta Q^s - \Bigg(&\sum_{i=0}^{r+s} (-1)^{r+i} (pi - r, r - (p-1)s - i) \beta Q^{r+s-i} Q^i \\ &- \sum_{i=0}^{r+s} (-1)^{r+i} (pi - r - 1, r - (p-1)s - i) Q^{r+s-i} \beta Q^i \Bigg)\end{aligned}$$

and

$$\mathscr{A}^{(1,r,1,s)} = \beta Q^r \beta Q^s - \left(- \sum_{i=0}^{r+s} (-1)^{r+i} (pi - r - 1, r - (p-1)s - i) \beta Q^{r+s-i} \beta Q^i \right)$$

where $(i, j) = (i + j)!/(i!j!)$. These elements are the socalled *Adem relations*.

Let $\mathscr{A} \subseteq \mathscr{R}$ be the $\mathbb{F}_p$-span of all Adem elements. This is a bigraded subspace of $\mathscr{R}$. Let $\langle \mathscr{A} \rangle \subseteq \mathscr{R}$ be the two-sided ideal generated by $\mathscr{A}$. Let $\mathscr{J} \subseteq \mathscr{R}$ be the two-sided ideal (or equivalently the left ideal) generated by the relations (3.2) (for $x \in \mathscr{R}$). $\mathscr{J}$ is the smallest ideal such that $\mathscr{R}/\mathscr{J}$ is unstable as a left $\mathscr{R}$-module.

Definition 3.5. The *Dyer-Lashof algebra* is the quotient

$$R = \mathscr{R}/(\langle \mathscr{A} \rangle + \mathscr{J})$$

The action of $\mathscr{A}$ and hence $\langle \mathscr{A} \rangle$ on homology of infinite loop spaces is trivial by results from [2], dual to Adem's result for the Steenrod algebra. So is the action of $\mathscr{J}$. Hence $H_*(X)$ is an R-module when X is an infinite loop space. Conversely ([1], [3]) the map $R \to H_*(QS^0)$ induced by acting on the zero-dimensional class ι, corresponding to the non-basepoint of S^0, is an injection, so there are no further relations.

The set of all products of generators form a vector space basis of $\mathscr{R}$. To have an explicit basis for R, we recall the notion of admissible monomials, [2, p. 16].

A sequence

$$I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$$

of integers $\varepsilon_i \in \{0,1\}$ and $s_i \in \mathbb{Z}_{\geq \varepsilon_i}$ determines the iterated homology operation

$$Q^I = \beta^{\varepsilon_1} Q^{s_1} \dots \beta^{\varepsilon_k} Q^{s_k} \in \mathscr{R}$$

This sequence is called *admissible* if for all $i = 2, \dots, k$,

$$s_i \leq p s_{i-1} - \varepsilon_{i-1} \tag{3.5}$$

The corresponding iterated homology operations $Q^I \in \mathscr{R}$ are called *admissible monomials*. The *length* and *excess* of I are

$$\ell(I) = k, \quad e(I) = 2s_1 - \varepsilon_1 - \sum_{j=2}^{k} [2s_j(p-1) - \varepsilon_j]$$

Furthermore, define

$$b(I) = \varepsilon_1$$

Using the Adem relations one may rewrite an arbitrary element of R as a linear combination of admissible monomials. Applying Adem relations does not raise the excess.

There is a natural quotient map $\mathscr{R} \to R$. Thus R-modules are also $\mathscr{R}$-modules.

Definition 3.6. An R-module is called *unstable*, respectively Q*-unstable*, if it is so as an $\mathscr{R}$-module.

3.3. **Free functors.**

Definition 3.7. For a graded vectorspace V we define $\mathscr{D}V$ to be the quotient of $\mathscr{R} \otimes V$ by the relations (3.2) and $\mathscr{D}'V$ to be the quotient of $\mathscr{D}V$ by the relations (3.3). Define also

$$DV = R \otimes_{\mathscr{R}} \mathscr{D}V, \quad D'V = R \otimes_{\mathscr{R}} \mathscr{D}'V$$

The functor $\mathscr{D}$ is left adjoint to the forgetful functor from unstable $\mathscr{R}$-modules to vectorspaces. Thus $\mathscr{D}V$ is the "free unstable $\mathscr{R}$-module" generated by V. Similarly, D is left adjoint to the forgetful functor from unstable R-modules to graded vectorspaces. Analogous remarks apply to $\mathscr{D}'$ and D'. The functors appear in the following exact sequences, natural in V

$$\langle \mathscr{A} \rangle \otimes_{\mathscr{R}} \mathscr{D}V \to \mathscr{D}V \to DV \to 0 \tag{3.6}$$

$$\langle \mathscr{A} \rangle \otimes_{\mathscr{R}} \mathscr{D}'V \to \mathscr{D}'V \to D'V \to 0 \tag{3.7}$$

When $V = \mathbb{F}_p \iota$ for a homogeneous element ι, DV has basis

$$\{Q^I \iota \mid I \text{ admissible}, e(I) \geq \deg(\iota)\}$$

Together with additivity of D, this describes DV as a $\mathbb{F}_p$-vectorspace. Since $R \cong D\mathbb{F}_p$ as a left R-module, we also have a basis of R over $\mathbb{F}_p$.

3.4. **Homology of QX.** Here we recall the computation of $H_*(QX)$. It can be expressed as a functor of $H_*(X)$ which is left adjoint to a suitable forgetful functor, forgetting the Pontrjagin product and the R-action, see [2]. We shall give a non-functorial description in terms of a basis of $JH_*(X)$.

Theorem 3.8. *Let $B \subseteq JH_*(X)$ be a basis consisting of homogeneous elements. Then $H_*(QX)$ is the free commutative algebra on the set*

$$\mathbf{T} = \{Q^I x | x \in B,\ I \text{ admissible},\ e(I) + b(I) > \deg(x)\}$$

□

Corollary 3.9. *The natural map*

$$\varphi_Q : D' JH_*(X) \to QH_*(QX)$$

is an isomorphism of Q-unstable R-modules.

If X is connected and $H_*(X)$ has trivial comultiplication (e.g. if X is a suspension), then the natural map

$$\varphi_P : DJH_*(X) \to PH_*(QX)$$

is an isomorphism of unstable R-modules. □

Remark 3.10. QX is connected if and only if X is connected. More generally the group of components of QX is determined by the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[\pi_0 X] \to \pi_0(QX) \to 0$$

where the first arrow is induced by the inclusion of the basepoint in X. When X is nonconnected we are sometimes only interested in homology of the component Q_0X of the basepoint in QX. This can be described as follows. Let $\tau : QX \to Q_0X$ be the map that on a component Q_iX, $i \in \pi_0(QX)$ multiplies by an element of $Q_{-i}X$. This defines a welldefined homotopy class of maps $QX \to Q_0X$ which is left inverse to the inclusion. Then we have that $H_*(Q_0X)$ is the free commutative algebra on the set

$$\tilde{\mathbf{T}} = \{\tau_* Q^I x | x \in B,\ I \text{ admissible},\ e(I) + b(I) > \deg(x),\ \deg(Q^I x) > 0\}$$

4. Homological algebra of unstable modules

The map

$$Q(\partial_*) : QH_*(Q\Sigma\mathbb{C}P^\infty_+) \to QH_*(Q_0S^0)$$

was computed in [6, Theorem 4.5]. The left hand side is $D'JH_*(\Sigma\mathbb{C}P^\infty_+)$ and the right hand side is $D'\mathbb{F}_p$. The starting point of our theorems is

Theorem 4.1 ([6])**.** *Let $a_s \in H_s(\Sigma\mathbb{C}P^\infty_+)$ be the generator, s odd. Let $\iota \in JH_0(S^0)$ be the generator. Then*

$$Q(\partial_*)(a_s) = \begin{cases} (-1)^r \beta Q^r \iota & s = 2r(p-1) - 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. The map $\partial : \Sigma\mathbb{C}P^\infty_+ \to QS^0$ coincides with the universal S^1-transfer denoted t_0 in [6]. The formula for $Q(\partial_*)(a_s)$ in the theorem now follows from ignoring all decomposable terms in [6, Theorem 4.5]. □

4.1. **Main technical theorems.** To state the theorems, recall from subsection 3.2 that $\mathscr{R}$ may be bigraded by $\deg = \deg_Q + \deg_\beta$. Since the Adem relations are homogeneous with respect to $\deg_Q$ and $\deg_\beta$, there is an induced bigrading of R. If V is bigraded, $\mathscr{R} \otimes V$ is a bigraded left $\mathscr{R}$-module. Since the unstability relations (3.2) are homogeneous, there is an induced bigrading of $\mathscr{D}V$. Similarly for $\mathscr{D}'V$, DV and $D'V$. Thus by Corollary 3.9 a bigrading of $JH_*(X)$ will induce a bigrading of $QH_*(QX)$ and, for X a suspension, a bigrading of $PH_*(QX)$. However, $H_*(QX)$ will only have $\deg_\beta$ welldefined up to multiplication with p because of the unstability relation (3.4).

For bigraded modules V with $\deg = \deg_Q + \deg_\beta$ as above, we shall write $V^{i,j} = \{x \in V \mid \deg_Q(x) = i, \deg_\beta(x) = j\}$ and $V^n = \oplus_{i+j=n} V^{i,j}$ and $V^{(n)} = \oplus_i V^{i,n}$. We will only consider gradings in the fourth quadrant, i.e. $V^{i,j} = 0$ unless $i \geq 0$ and $j \leq 0$. Write $V^{(-)} = \oplus_{n<0} V^{(n)}$.

Theorem 4.2. *Bigrade $JH_*(S^0)$ by setting $\deg_\beta(\iota) = 0$ and give QH_*QS^0 the induced bigrading. Then we have*

$$\operatorname{Im}(Q(\partial_*)) = QH_*(QS^0)^{(-)}$$

Proof. The inclusion $\operatorname{Im}(Q\partial_*) \subseteq QH_*(QS^0)^{(-)}$ is immediate from Theorem 4.1. The other inclusion follows from Lemma 4.3 below. Indeed, the two-sided ideal in R generated by the set $\{\beta Q^s \mid s \geq 1\}$ is spanned by operations Q^I with at least one β. By Lemma 4.3 below, any such operation is also in the left ideal with the same generators, i.e. is a linear combination of elements of the form $Q^J \beta Q^s$. In particular, any element in $QH_*(QS^0)^{(-)}$ is also in $\operatorname{Im}(Q\partial_*)$ because $Q\partial_*$ is R-linear. □

Lemma 4.3. *The left ideal in R generated by the set $\{\beta Q^s \mid s \geq 1\}$ is also a right ideal.*

Proof. Write $I \subseteq R$ for the left ideal generated by $\{\beta Q^s \mid s \geq 1\}$.

For $r \leq ps$, consider the Adem relation $\mathscr{A}^{(0,ps,1,r-(p-1)s)}$:

$$\begin{aligned} Q^{ps}\beta Q^{r-(p-1)s} &= \beta Q^r Q^s \\ &\quad + \sum_{i>s} \lambda_i \beta Q^{r+s-i} Q^i \\ &\quad + \text{terms of form } Q^{r+s-i}\beta Q^i \end{aligned}$$

where we have singled out the term in the Adem relation corresponding to $i = s$, and where the $\lambda_i \in k$ are certain binomial coefficients. This shows that in the left R-module R/I we can write $\beta Q^r Q^s$ as a linear combination of $\beta Q^a Q^b$ with $a < r$. In particular, $\beta Q^1 Q^s = 0 \in R/I$ and by induction $\beta Q^r Q^s = 0 \in R/I$.

Thus we have $\beta Q^r Q^s \in I$ whenever $\beta Q^r Q^s$ is admissible. Since a nonadmissible $\beta Q^r Q^s$ is a linear combination of admissible ones, we have $\beta Q^r Q^s \in I$ for any r, s. This shows that I is invariant under right multiplication with Q^s. Since it is obviously invariant under right multiplication with βQ^s it follows that I is a right ideal. □

The kernel of $Q\partial_*$ is harder to determine explicitly. The partial information contained in Theorem 4.4 below suffices for the calculation.

Notice that for any $\mathscr{R}$-module V, the augmentation of $\mathscr{R}$ gives a natural quotient map $V \to \mathbb{F}_p \otimes_{\mathscr{R}} V$ identifying $\mathbb{F}_p \otimes_{\mathscr{R}} V$ with the quotient of V by the relations $\beta^\varepsilon Q^s x = 0$ for $x \in V, \varepsilon \in \{0,1\}, s \geq \varepsilon$. The functor $\mathbb{F}_p \otimes_{\mathscr{R}} -$ agrees with the functor $\mathbb{F}_p \otimes_R -$ on R-modules. Thus the vectorspace $\mathbb{F}_p \otimes_{\mathscr{R}} V = \mathbb{F}_p \otimes_R V$ measures the dimensions of a minimal set of R-module generators of an unstable R-module V.

In the next theorem, $a_s \in JH_s(\Sigma\mathbb{C}P^\infty_+)$ denotes the generator for s odd.

Theorem 4.4. *Bigrade $JH_*(\Sigma\mathbb{C}P^\infty_+)$ by concentrating it in $\deg_\beta = -1$ and give $QH_*(Q\Sigma\mathbb{C}P^\infty_+)$ the induced bigrading. Then the bigraded vectorspace*

$$\mathbb{F}_p \otimes_R \operatorname{Ker}(Q\partial_*) = \mathbb{F}_p \otimes_R Q(H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash \partial_*)$$

is concentrated in bidegrees $\deg_\beta = -1$ and $\deg_\beta = -2$. In particular $\operatorname{Ker}(Q\partial_*)$ is generated as an R-module by the elements $a_s \in \operatorname{Ker}(Q\partial_*)$ with $s \not\equiv -1 \pmod{2(p-1)}$ together with elements of degree $\equiv -1$ and $\equiv -2 \pmod{2(p-1)}$.

Proof. The equality $\operatorname{Ker}(Q\partial_*) = Q(H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash \partial_*)$ in the theorem follows from Proposition 2.10 because $H_*(QS^0)$ is a free commutative algebra.

The last statement of the theorem follows from the first. Indeed the elements $Q^I a_s$ are all in the kernel of $Q(\partial_*)$ when $s \not\equiv -1 \pmod{2(p-1)}$ because a_s is in the kernel. These elements give rise to one "tautological" element $a_s \in F_p \otimes_R \operatorname{Ker}(Q\partial_*)$. On the span of the $Q^I a_s$ with $s \equiv -1 \pmod{2(p-1)}$ the claim about degrees of generators follows since on these elements $\deg \equiv \deg_\beta \pmod{2(p-1)}$. Thus we need only prove the first statement of the theorem.

We have the short exact sequence of Q-unstable R-modules

$$0 \longrightarrow \operatorname{Ker}(Q\partial_*) \longrightarrow QH_*(Q\Sigma\mathbb{C}P^\infty_+) \xrightarrow{Q\partial_*} QH_*(QS^0)^{(-)} \longrightarrow 0 \qquad (4.1)$$

If were to apply the functor $\mathbb{F}_p \otimes_R -$ from R-modules to vectorspaces, we would get a long exact sequence involving $\operatorname{Tor}^R_*(\mathbb{F}_p, -)$, and a determination of the map induced by $Q\partial_*$ in Tor_1 would give the result. This is more or less what we do, except that it is technically more convenient to replace the functor $\mathbb{F}_p \otimes_R -$ by $\mathbb{F}_p \otimes_{\mathscr{R}} -$ and to replace Tor by a suitable functor taking unstability into account. We proceed to make these ideas precise.

The category of Q-unstable $\mathscr{R}$-modules is abelian and has enough projectives. The functor $\mathbb{F}_p \otimes_{\mathscr{R}} -$ from Q-unstable $\mathscr{R}$-modules is right exact, hence the left derived functors $L_r(\mathbb{F}_p \otimes_{\mathscr{R}} -)$ are defined. These are unstable versions of $\mathrm{Tor}_r^{\mathscr{R}}(\mathbb{F}_p, -)$. For brevity, let us write $T_1^{\mathscr{R}}(\mathbb{F}_p, -) = L_1(\mathbb{F}_p \otimes_{\mathscr{R}} -)$.

With these definitions, applying the functor $\mathbb{F}_p \otimes_{\mathscr{R}} -$ to the sequence (4.1) induces the exact sequence

$$0 \longrightarrow \mathrm{Cok}(T_1^{\mathscr{R}}(\mathbb{F}_p, Q\partial_*)) \longrightarrow \mathbb{F}_p \otimes_R \mathrm{Ker}(Q\partial_*) \longrightarrow \mathrm{Ker}(\mathbb{F}_p \otimes_R Q\partial_*) \longrightarrow 0 \tag{4.2}$$

Claim 1. The elements $a_s \in \mathrm{Ker}(Q\partial_*)$ with $s \not\equiv -1 \pmod{2(p-1)}$ maps in (4.2) to a generating set in $\mathrm{Ker}(\mathbb{F}_p \otimes_R Q\partial_*)$.

Proof of Claim 1. This is the kernel of the map

$$\mathbb{F}_p \otimes_R Q\partial_* : \mathbb{F}_p \otimes_R QH_*(Q\Sigma\mathbb{C}P_+^\infty) \to \mathbb{F}_p \otimes_R QH_*(Q_0S^0)^{(-)}$$

Clearly, the natural map $JH_*(\Sigma\mathbb{C}P_+^\infty) \to k \otimes_R QH_*(Q\Sigma\mathbb{C}P_+^\infty)$ is an isomorphism, and by Lemma 4.3 we get that $\mathbb{F}_p \otimes_R QH_*(QS^0)^{(-)}$ is spanned by $\{\beta Q^s \iota \mid s \geq 1\}$. Thus Claim 1 follows from Theorem 4.1. $\square$

Claim 2: $\mathrm{Cok}(T_1^{\mathscr{R}}(\mathbb{F}_p, Q\partial_*))$ is concentrated in $\deg_\beta = -1$ and $\deg_\beta = -2$.

Proof of Claim 2. We will compute $T_1^{\mathscr{R}}(\mathbb{F}_p, Q\partial_*)$ using suitable free resolutions. For brevity, write $V = JH_*(\Sigma\mathbb{C}P_+^\infty)$. By Corollary 3.9 we may consider $Q\partial_*$ as a map from $D'V$ onto $D'\mathbb{F}_p^{(-)}$. Let $W \subseteq (\mathscr{D}'\mathbb{F}_p)^{(-)}$ denote the subspace with basis $\{\beta Q^{s_1} Q^{s_2} \ldots Q^{s_k} \mid s_1 \geq 1, s_2, \ldots, s_k \geq 0\}$. In the diagram

$$\begin{tikzcd} & & W \arrow[d] & D'V \arrow[d, "Q\partial_*"] & \\ 0 \arrow[r] & (\langle \mathscr{A} \rangle \cdot \mathscr{D}'\mathbb{F}_p)^{(-)} \arrow[r] & \mathscr{D}'\mathbb{F}_p^{(-)} \arrow[r] & D'\mathbb{F}_p^{(-)} \arrow[r] & 0 \end{tikzcd} \tag{4.3}$$

in which the lower exact sequence is an instance of (3.7), we may choose a lifting $\rho : W \to D'V$ since $Q\partial_*$ is surjective. Writing $V = V_0 \oplus V_1$ where $V_0 = \mathrm{span}\{a_s \mid s \equiv -1 \pmod{2(p-1)}\}$ and $V_1 = \mathrm{span}\{a_s \mid s \not\equiv -1 \pmod{2(p-1)}\}$, we may choose the lifting ρ to have $\rho(W) \subseteq D'V_0$ since $D'V = D'V_0 \oplus D'V_1$ and since $Q\partial_*$ vanishes on $D'V_1$. We may also choose the lifting to have $\rho(\beta Q^s) = a_{2s(p-1)-1}$ and extend (4.3) to the following exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\rho) \arrow[r, "j"] \arrow[d, "\sigma"] & \mathscr{D}'W \arrow[r, "\rho"] \arrow[d] & D'V_0 \arrow[r] \arrow[d, "Q\partial_*"] & 0 \\ 0 \arrow[r] & (\langle \mathscr{A} \rangle \cdot \mathscr{D}'\mathbb{F}_p)^{(-)} \arrow[r, "i"] & \mathscr{D}'\mathbb{F}_p^{(-)} \arrow[r] & D'\mathbb{F}_p^{(-)} \arrow[r] & 0 \end{tikzcd} \tag{4.4}$$

Note that the middle map in (4.4) is an isomorphism.

Next we apply the functor $\mathbb{F}_p \otimes_{\mathscr{R}} -$ to (4.4). This gives a diagram involving the left derived functor $T_1^{\mathscr{R}}(\mathbb{F}_p, -) = L_1(\mathbb{F}_p \otimes_{\mathscr{R}} -)$. This functor vanishes on the middle part of (4.4) since these (isomorphic) objects are free. Thus, a part of the induced diagram looks like this

$$\begin{tikzcd}
0 \arrow[r] & T_1^{\mathscr{R}}(\mathbb{F}_p, D'V_0) \arrow[r] \arrow[d, "T_1^{\mathscr{R}}(\mathbb{F}_p, Q\partial_*)"] & \mathbb{F}_p \otimes_{\mathscr{R}} \mathrm{Ker}\rho \arrow[r, "j_*"] \arrow[d, "\sigma_*"] & \mathbb{F}_p \otimes_{\mathscr{R}} \mathscr{D}'W \arrow[d, "\cong"] \\
0 \arrow[r] & T_1^{\mathscr{R}}(\mathbb{F}_p, D'\mathbb{F}_p^{(-)}) \arrow[r] & \mathbb{F}_p \otimes_{\mathscr{R}} (\langle \mathscr{A} \rangle \cdot \mathscr{D}'\mathbb{F}_p)^{(-)} \arrow[r, "i_*"] & \mathbb{F}_p \otimes_{\mathscr{R}} (\mathscr{D}'\mathbb{F}_p)^{(-)}
\end{tikzcd} \tag{4.5}$$

where a star in subscript is shorthand for $\mathbb{F}_p \otimes_{\mathscr{R}} -$ on morphisms. Thus we have represented $T_1^{\mathscr{R}}(\mathbb{F}_p, D'V_0)$ and $T_1^{\mathscr{R}}(\mathbb{F}_p, \mathscr{D}'\mathbb{F}_p^{(-)})$ as the kernels of j_* and i_*, and the map $T_1^{\mathscr{R}}(\mathbb{F}_p, Q\partial_*)$ as the restriction of σ_*.

To calculate the cokernel of $T_1^{\mathscr{R}}(\mathbb{F}_p, Q\partial_*)$ and to prove Claim 2, note that

$$(\langle \mathscr{A} \rangle \cdot \mathscr{D}'\mathbb{F}_p)^{(-)} = \mathscr{R}^{(-)} \cdot \mathscr{A}^{(0)} \cdot \mathscr{D}'\mathbb{F}_p^{(0)} + \mathscr{R} \cdot \mathscr{A}^{(-)} \cdot \mathscr{D}'\mathbb{F}_p^{(0)} + \mathscr{R} \cdot \mathscr{A} \cdot \mathscr{D}'\mathbb{F}_p^{(-)}$$

This is generated over $\mathscr{R}$ by the subspace

$$\mathscr{R}^{(-1)} \cdot \mathscr{A}^{(0)} \cdot \mathscr{D}'\mathbb{F}_p^{(0)} + \mathscr{A}^{(-)} \cdot \mathscr{D}'\mathbb{F}_p^{(0)} + \mathscr{A} \cdot \mathscr{D}'\mathbb{F}_p^{(-)} \tag{4.6}$$

The corresponding $\mathscr{R}$-indecomposable classes will span $\mathbb{F}_p \otimes_{\mathscr{R}} (\langle \mathscr{A} \rangle \cdot \mathscr{D}'\mathbb{F}_p)^{(-)}$ as a vectorspace, and since the first and the second term in (4.6) has $\deg_\beta \in \{-1, -2\}$, it suffices to prove that the last term $\mathscr{A} \cdot \mathscr{D}'\mathbb{F}_p^{(-)}$ does not contribute to the cokernel of $T_1^{\mathscr{R}}(\mathbb{F}_p, Q\partial_*)$.

To this end, notice that $\mathscr{A} \cdot \mathscr{D}'\mathbb{F}_p^{(-)}$ corresponds to $\mathscr{A} \cdot \mathscr{D}'W$ under the middle isomorphism in (4.4), and that $\mathscr{A} \cdot \mathscr{D}'W$ is in the kernel of ρ since the action of $\mathscr{A}$ is trivial in $D'V$. Notice also that $\mathscr{A} \cdot \mathscr{D}'W$ vanishes under the projection $\mathscr{D}'W \to \mathbb{F}_p \otimes_{\mathscr{R}} \mathscr{D}'W$ and thus by exactness of (4.4) and (4.5) the classes corresponding to $\mathscr{A} \cdot \mathscr{D}'\mathbb{F}_p^{(-)}$ in $\mathbb{F}_p \otimes_{\mathscr{R}} (\langle \mathscr{A} \rangle \cdot \mathscr{D}'\mathbb{F}_p)^{(-)}$ lifts all the way to $T_1^{\mathscr{R}}(\mathbb{F}_p, D'V_0)$ and therefore does not contribute to the cokernel of $T_1^{\mathscr{R}}(\mathbb{F}_p, Q\partial_*)$. □

Now Theorem 4.4 follows from the exact sequence (4.2) and the Claims above. □

5. Homology of $\Omega^\infty \Sigma \mathbb{C}P^\infty_{-1}$

The spectral sequence associated to the fibration (1.1) has

$$E^2 = \mathrm{Cotor}^{H_*(Q_0 S^0)}(H_*(Q\Sigma\mathbb{C}P^\infty_+), \mathbb{F}_p) \Rightarrow H_*(\Omega^\infty \Sigma \mathbb{C}P^\infty_{-1}) \tag{5.1}$$

By Proposition 2.7 the E^2-term splits as

$$E^2 \cong \mathrm{Cotor}^{H_*(Q_0 S^0) /\!\!/ \partial_*}(\mathbb{F}_p, \mathbb{F}_p) \otimes H_*(Q\Sigma\mathbb{C}P^\infty_+) \backslash\!\!\backslash \partial_* \tag{5.2}$$

In this section, p is odd so after localising the fibration (1.1), the base-space is simply connected and the spectral sequence converges.

As explained in the introduction, we will first prove Theorem 1.1 about the coalgebra structure on $H_*(Q_0 S^0) /\!\!/ \partial_*$, or, equivalently the algebra structure of

$H^*(Q_0S^0)\backslash\!\backslash\partial^*$, and then use this to prove that the spectral sequence (5.1) collapses. Then a close examination of the $E^\infty = E^2$ term will prove Theorem 1.2.

5.1. **The Hopf algebra cokernel of ∂_*.** To state the results, let us introduce a bigrading of $H_*(QS^0)$. Recall that $H_*(QS^0)$ is the free commutative algebra on the set

$$\{Q^I\iota \mid I \text{ admissible, } e(I)+b(I)>0\}$$

Make it a bigraded algebra by setting $\deg_\beta(Q^I\iota) = \deg_\beta(Q^I)$. By the Cartan formula for the coproduct we get that the subalgebra $H_*(QS^0)^{(0)}$ is a Hopf subalgebra, but notice that $H_*(QS^0)$ is not a bigraded R-module because of the relation (3.4). We are now ready to prove Theorem 1.1 in the case $p>2$.

Proof of Theorem 1.1 for $p>2$. We first prove that the composition

$$H_*(QS^0)^{(0)} \to H_*(QS^0) \to H_*(QS^0)/\!\!/\partial_* \tag{5.3}$$

is an isomorphism of Hopf algebras.

With the bigrading introduced above, we have $H_*(QS^0) = H_*(QS^0)^{(0)} \oplus H_*(QS^0)^{(-)}$ where the first summand is a subalgebra and the second is an ideal. Since $\mathrm{Im}(\partial_*) \subseteq \mathbb{F}_p \oplus H_*(QS^0)^{(-)}$, the composition (5.3) is injective.

To see surjectivity, note that $Q(H_*(QS^0)/\!\!/\partial_*) = \mathrm{Cok}(Q\partial_*)$ since Q is right exact. By Theorem 4.2 we have $\mathrm{Im}(Q\partial_*) = QH_*(QS^0)^{(-)}$ and hence $\mathrm{Cok}(Q\partial_*) = (QH_*(QS^0))^{(0)} = Q(H_*(QS^0)^{(0)})$.

To prove that $H_*(Q_0S^0)/\!\!/\partial_*$ is dual to a polynomial, notice that we have $H_*(Q_0S^0)/\!\!/\partial_* \cong H_*(Q_0S^0)^{(0)}$ and that it suffices to prove that $\lambda : H_*(Q_0S^0)^{(0)} \to H_*(Q_0S^0)^{(0)}$ is surjective. λ is given by the dual Steenrod operations: If $\deg(x) = 2ps$, $\lambda x = \mathcal{P}^s_* x$. By the Nishida relations ([2, Theorem 1.1 (9)]), one gets $\lambda Q^{ps} = Q^s\lambda$ and thus

$$\lambda(Q^{ps_1}Q^{ps_2}\dots Q^{ps_k}[1]*[-p^k]) = Q^{s_1}Q^{s_2}\dots Q^{s_k}[1]*[-p^k]$$

Thus λ hits the generators of $H_*(Q_0S^0)^{(0)}$ and since it is a map of algebras, it is surjective. □

5.2. **The spectral sequence.** We are now ready to compute the E^2-term of the spectral sequence (5.1) and to prove that it collapses at the E^2-term.

Theorem 5.1. *The spectral sequence collapses at the E^2-term. The E^2-term is given by*

$$E^2 = H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_* \otimes E[s^{-1}P(H_*(QS^0)/\!\!/\partial_*)]$$

as a Hopf algebra.

Proof. We need to identify the factor $\mathrm{Cotor}^{H_*(Q_0S^0)/\!\!/\partial_*}(\mathbb{F}_p,\mathbb{F}_p)$ in the splitting (5.2) of the E^2-term. By Theorem 1.1, the dual algebra $H^*(Q_0S^0)\backslash\!\backslash\partial^*$ is polynomial and hence by Corollary 2.9 we get

$$\mathrm{Cotor}^{H_*(Q_0S^0)/\!\!/\partial_*}(\mathbb{F}_p,\mathbb{F}_p) \cong E[s^{-1}P(H_*(QS^0)/\!\!/\partial_*)]$$

as claimed.

In this E^2-term, primitives and generators are concentrated in bidegrees $(0,*)$ and $(-1,*)$ and hence by Lemma 2.14 there can be no non-zero differentials in the spectral sequence. □

Proof of Theorem 1.2. By Theorem 5.1 we get that the sequence

$$\mathbb{F}_p \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_* \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \xrightarrow{\omega_*} H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_* \longrightarrow \mathbb{F}_p$$

is exact (i.e. ω_* is onto).

To identify $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*$ recall that the spectral sequence defines a filtration $F_0 \supseteq F_{-1} \supseteq \dots$ on $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ and hence on $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\partial_*$ and an isomorphism of graded vectorspaces

$$s^{-1}P(H_*QS^0/\!/\partial_*) \to F_{-1}(H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\partial_*)/F_{-2}$$

Choosing any lifting

$$s^{-1}P(H_*QS^0/\!/\partial_*) \xrightarrow{l} H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\partial_*$$

we will get an isomorphism of algebras

$$E[s^{-1}P(H_*QS^0/\!/\partial_*)] \to H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\partial_*$$

and since $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*$ is a Hopf algebra, Theorem 2.4 defines a unique choice of lifting l into $P(H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*)$.

The splitting follows from Lemma 5.2 below. □

Lemma 5.2. *Let*

$$\mathbb{F}_p \longrightarrow A \longrightarrow B \xrightarrow{\pi} C \longrightarrow \mathbb{F}_p$$

be a short exact sequence of Hopf algebras. If either A or C is exterior, the sequence is split exact in the category of Hopf algebras.

Proof. Assume C is exterior. Then by Theorem 2.4 we have that $PC \cong QC$ and the diagram

$$\begin{tikzcd} PB \arrow[r, "P\pi"] \arrow[d] & PC \arrow[r] \arrow[d, "\cong"] & 0 \\ QB \arrow[r] & QC \arrow[r] & 0 \end{tikzcd}$$

is exact since $Q(-)$ is right exact. Thus $PB \to PC$ is surjective and a choice of splitting $PC \to PB$ of $P\pi$ induces a splitting $C \cong E[PC] \to B$ of π.

The case where A is exterior follows by duality. □

Corollary 5.3. *The vectorspace*

$$\mathrm{Ker}(P\omega_*) = P(H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\partial_*) = Q(H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*)$$

is concentrated in degrees $\equiv -1 \pmod{2(p-1)}$

Proof. This follows from Theorem 1.1 and Theorem 1.2. □

6. Homology of $\Omega^\infty \mathbb{C}P^\infty_{-1}$

The goal of this section is to prove Theorem 1.4 and Corollary 1.5.

As mentioned in the introduction, we will consider the Eilenberg-Moore spectral sequence of the path-loop fibration over $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$. From the fibration (1.1) one easily gets that $\pi_1(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) = \mathbb{Z}$ and therefore we have a homotopy equivalence

$$\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1} \simeq S^1 \times \tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1}$$

where $\tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1} \to \Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$ is the universal covering map. Furthermore we have $\Omega(\tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1}) = \Omega^\infty_0\mathbb{C}P^\infty_{-1}$, the basepoint component of $\Omega^\infty\mathbb{C}P^\infty_{-1}$. Similarly $Q\Sigma\mathbb{C}P^\infty_+ \simeq S^1 \times \tilde{Q}\Sigma\mathbb{C}P^\infty_+$ and under these splittings the map ω in the fibration (1.1) restricts to a map $S^1 \to S^1$ of degree 2. Since p is odd, the effect of replacing $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$ and $Q\Sigma\mathbb{C}P^\infty_+$ by their universal convering spaces is to remove a factor of $H_*(S^1) = E[\sigma]$, $\sigma = [S^1] \in H_1(S^1)$, from each of the terms $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ and $H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*$ in Theorem 1.2.

The Eilenberg-Moore spectral sequence associated to the path-loop fibration over $\tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1}$ is

$$E^2 = \operatorname{Cotor}^{H_*(\tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1})}(\mathbb{F}_p,\mathbb{F}_p) \Rightarrow H_*(\Omega^\infty_0\mathbb{C}P^\infty_{-1}) \tag{6.1}$$

and by Theorem 1.2, the E^2-term splits (non-canonically) as

$$E^2 \cong \operatorname{Cotor}^{H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*}(\mathbb{F}_p,\mathbb{F}_p) \otimes \operatorname{Cotor}^{H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*}(\mathbb{F}_p,\mathbb{F}_p) \tag{6.2}$$

More canonically there is a short exact sequence of Cotor's, but for the following arguments we will assume that a splitting has been chosen.

I claim it must collapse. As before, we consider a possibly nonzero differential $dx = y \neq 0$ with $\deg(x)$ minimal. We will reach a contradiction in a number of steps. The argument is based on Theorem 4.4 and a careful analysis of degrees modulo $2(p-1)$ in the spectral sequence.

By Theorem 1.2 and Lemma 2.9, the first factor in (6.2) is a polynomial algebra on generators of total degree $\equiv -2 \pmod{2(p-1)}$. To gain information about the second factor, we map the spectral sequence (6.1) into the spectral sequence of the path-loop fibration over $\tilde{Q}\Sigma\mathbb{C}P^\infty_+$ via the map $\omega : \tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1} \to \tilde{Q}\Sigma\mathbb{C}P^\infty_+$. This is a map $E^r(\omega)$ of spectral sequences whose restriction to the first factor in the splitting (6.2) is zero, and whose restriction to the second factor in (6.2) is induced by the inclusion $H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_* \to H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+)$. The next lemma says that this second factor in (6.2) injects under $E^2(\omega)$.

Lemma 6.1. *Let $f : A \to B$ be an injection of primitively generated Hopf algebras. Then* $\operatorname{Cotor}^f(\mathbb{F}_p,\mathbb{F}_p) : \operatorname{Cotor}^A(\mathbb{F}_p,\mathbb{F}_p) \to \operatorname{Cotor}^B(\mathbb{F}_p,\mathbb{F}_p)$ *is also injective.*

Proof. By Theorem 2.4, A^* and B^* are tensor products of exterior algebras and polynomial algebras truncated at height p. Thus we can split $f^* : B^* \to A^*$ in the category of algebras (since a splitting can be chosen on the generators of A^*). Dually, $f : A \to B$ is split injective as a map of coalgebras and thus $\mathrm{Cotor}^f(\mathbb{F}_p, \mathbb{F}_p)$ is injective. $\square$

Corollary 6.2. *Relative to the splitting* (6.2)*, a differential $dx = y \neq 0$ with x of minimal degree will have x in the right factor and y in the left.*

Proof. Recall that P and Q are additive: $P(A \otimes B) = PA \oplus PB$ and $Q(A \otimes B) = QA \oplus QB$. Thus x and y does not contain products between the two factors in (6.2).

Since y is primitive and in bidegree $(\leq -3, *)$, it must be of even total degree by Corollary 2.9, and thus x is of odd total degree. By Theorem 1.2 this is only possible if x is in the right factor.

By Lemma 6.1, the right factor injects into the spectral sequence of $Q\Sigma\mathbb{C}P^\infty_+$, and since all differentials vanish in this spectral sequence, y must map to 0 there, and hence y is in the left factor. $\square$

The remaining part of the collapse proof is to eliminate the possibility of differentials from the right factor to the left. This is the hardest part of the proof, the main ingredient of which is Theorem 4.4.

Theorem 6.3. *The spectral sequence* (6.1) *collapses.*

Proof. Assume there is a differential $dx = y \neq 0$ with $\deg(x)$ minimal. Then y is a primitive element in $\mathrm{Cotor}^{H_*(\tilde\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})}(\mathbb{F}_p, \mathbb{F}_p)$. By Corollary 6.2 and Corollary 2.9 it is of the form

$$y = (s^{-1}z)^{p^k}$$

for a $z \in P(H_*(\tilde\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*)$. By Theorem 1.2 we must have $\deg(z) \equiv -1 \pmod{2(p-1)}$. Write

$$\deg(z) = 2n(p-1) - 1$$

Then

$$\deg y = p^k(2n(p-1) - 2) = 2p^k(n(p-1) - 1) \equiv -2 \pmod{2(p-1)}$$

and thus $\deg x \equiv -1 \pmod{2(p-1)}$ because the differential has degree -1. By Proposition 2.14 we get that x corresponds to a minimal element in the cokernel of $\sigma_* : QH_*(\Omega^\infty_0\mathbb{C}P^\infty_{-1}) \to PH_*(\tilde\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$, of degree $\equiv 0 \pmod{2(p-1)}$. By Corollary 6.2, x is also a minimal element in the cokernel of the composition

$$QH_*(\Omega^\infty_0\mathbb{C}P^\infty_{-1}) \xrightarrow{\sigma_*} PH_*(\tilde\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \xrightarrow{P\omega_*} P(H_*(\tilde Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*)$$

By minimality this element is not a pth power and hence maps to a non-zero element of $Q(H_*(\tilde Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*)$. Again by minimality, and because the loop suspension σ_* is R-linear, this element is R-indecomposable and hence since σ_* has degree 1, x will map to a nonzero element of degree $\equiv 0 \pmod{2(p-1)}$ in

$$\mathbb{F}_p \otimes_R Q(H_*(\tilde Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*)$$

in contradiction with Theorem 4.3. $\square$

Theorem 1.4 no follows from Theorem 6.3 and Proposition 2.15.

Proof of Corollary 1.5. This is completely analogous to the inductive step in the classical calculations of homology of QX or of cohomology of $K(\mathbb{F}_p, n)$. We sketch the details.

Consider the Leray-Serre spectral sequence

$$E^2 = H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \otimes H_*(\Omega^\infty\mathbb{C}P^\infty_{-1}) \Rightarrow H_*(point) \tag{6.3}$$

Since σ_* is onto, we can pick a basis $B \subseteq PH_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ and for each $x \in B$ pick an element $\tau x \in H_*(\Omega^\infty\mathbb{C}P^\infty_{-1})$ with $\sigma_*(\tau x) = x$. We can now form a model spectral sequence

$$\tilde{E}^2 = \bigotimes_{x\in B} E^r(x)$$

where, if $x \in B$ has odd degree,

$$E^2(x) = E[x] \otimes \mathbb{F}_p[\tau x]$$

with the differential determined by requiring that x transgresses to τx. If x has even degree $\deg(x) = 2s$, we set

$$E^2(x) = \mathbb{F}_p\{1, x, \dots, x^{p-1}\} \otimes E[\tau x] \otimes \mathbb{F}_p[\beta Q^s(\tau x)]$$

with the differential determined by requiring that x transgresses to τx and that $x^{p-1} \otimes \tau x$ transgresses to $\beta Q^s(\tau x)$.

The choices of $\tau x \in H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ determines a map of spectral sequences $\tilde{E}^r \to E^r$ and the comparison theorem implies that it is an isomorphism and then Corollary 1.5 follows. □

7. The case $p = 2$

At the prime 2, the calculation of $H_*(\Omega^\infty\mathbb{C}P^\infty_{-1})$ and $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ can also be made. Some details are quite different however. In particular, we will use the looped fibration

$$\Omega^\infty\mathbb{C}P^\infty_{-1} \to Q(\mathbb{C}P^\infty_+) \to \Omega QS^0 \tag{7.1}$$

to compute $H_*(\Omega^\infty\mathbb{C}P^\infty_{-1})$, instead of the path-loop fibration over $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$. At $p = 2$ our base spaces in the fibrations are no longer simply connected. The following lemma deals with this

Lemma 7.1. *As spaces we have*

$$QS^0 \simeq \mathbb{Z} \times \mathbb{R}P^\infty \times \tilde{Q}_0S^0$$

$$\Omega QS^0 \simeq \mathbb{Z}/2 \times \mathbb{R}P^\infty \times \tilde{\Omega}_0QS^0$$

where $\tilde{X} \to X$ denotes the universal covering.

Proof. Let X be an $(n-1)$-connected H-space with $\pi_n(X) = G$. There is an H-map $X \to K(G,n)$ inducing an isomorphism in π_n and with fibre the n-connected cover $X\langle n\rangle$. If one can find a map $K(G,n) \to X$ inducing an isomorphism in π_n, this map will give a splitting $X \simeq X\langle n\rangle \times K(G,n)$.

For $n = 0$ this is automatic.

For $X = Q_2S^0 \simeq Q_0S^0$, $\pi_1(X) = \mathbb{Z}/2$ and the definition of the Dyer-Lashof operation $Q^1\iota \in H_1(Q_2S^0;\mathbb{F}_2)$ gives a map

$$\mathbb{R}P^\infty = B\mathbb{Z}/2 \to Q_2S^0 \simeq Q_0S^0$$

inducing an isomorphism in H_1 and thus by the Hurewicz theorem an isomorphism in π_1 and the splitting of QS^0 follows.

For $X = \Omega_0Q_0S^0$, $\pi_1(X) = \mathbb{Z}/2$. The Hopf map gives an infinite loop map $\eta : Q(S^1) \to Q_0S^0$. I claim it is nonzero in π_2. To see this it suffices to show that $(\eta\langle 1\rangle)_*$ is nonzero in H_2 which can be seen as follows. Let $\sigma \in H_1(QS^1)$ be the fundamental class. Since $QS^1 \simeq S^1 \times QS^1\langle 1\rangle$, the element $Q^1\sigma \in H_2(QS^1)$ must be in the image from $H_*(QS^1\langle 1\rangle)$. Since $\eta_*(Q^1\sigma) = Q^1(Q^1[1] * [-2]) \neq 0$, $\eta\langle 1\rangle_*$ is indeed nonzero in H_2.

Hence, $\Omega_0\eta : Q_0S^0 \to \Omega_0Q_0S^0$ is nonzero in π_1 and thus the composition

$$\mathbb{R}P^\infty \to Q_0S^0 \to \Omega_0Q^0S^0$$

is nonzero in π_1 and the splitting of ΩQS^0 follows. □

Lemma 7.1 ensures that our spectral sequences has trivial local coefficients and hence that the spectral sequences converges.

7.1. **Recollections.** The Dyer-Lashof algebra is slightly different at $p = 2$. Let $\mathscr{R}$ be the free non-commutative algebra on the set $\{Q^s \mid s \geq 0\}$ with $\deg(Q^s) = s$. The Adem relation $\mathscr{A}^{(0,r,0,s)}$ in Definition 3.4 still makes sense, and we let $\mathscr{A} \subseteq \mathscr{R}$ be the span of the $\mathscr{A}^{(0,r,0,s)}$. The unstability relations at $p = 2$ are

$$Q^s x = \begin{cases} x^2 & \text{if } \deg x = s \\ 0 & \text{if } \deg x > s \end{cases}$$

and the algebra R is defined from these data as before. Corresponding to $I = (s_1, s_2, \dots, s_k)$ there is an iterated operation $Q^I = Q^{s_1} \dots Q^{s_k}$, and this operation is called admissible if $s_i \leq 2s_i$ for all i. The definition of excess at $p = 2$ is

$$e(I) = s_1 - \sum_{j=2}^{k} s_j$$

Given a basis $B \subseteq JH_*(X)$, then $H_*(QX)$ is the polynomial algebra on the set

$$\mathbf{T} = \{Q^I x \mid x \in B,\ I \text{ admissible},\ e(I) > \deg(x)\}$$

and similarly for $H_*(Q_0X)$.

One pleasant feature of $p = 2$ is the following

Lemma 7.2. *The cohomology algebra $H^*(Q_0X)$ is polynomial if the Frobenius $\xi : H^*(X) \to H^*(X)$ is injective.*

Proof. This is because the Nishida relation $\lambda Q^{2s} = Q^s\lambda$ makes $\lambda : H_*(Q_0X) \to H_*(Q_0X)$ surjective if $\lambda : H_*(X) \to H_*(X)$ is surjective. □

In particular, $H^*(Q_0S^0)$ and $H^*(Q_0\mathbb{C}P^\infty_+)$ are both polynomial.

The calculation in Theorem 2.8 is valid with the remark that $\mathbb{F}_2[x]/(x^2)$ must be interpreted as $E[x]$ and thus it does not produce generators of Cotor in bidegree $(-2, *)$. Only truncations at height $p^n, n \geq 2$ does that.

An important difference is that for odd primes, $\mathrm{Cotor}^A(\mathbb{F}_p, \mathbb{F}_p)$ is automatically a free algebra. This is no longer true for $p = 2$, since $\mathrm{Tor}^{\mathbb{F}_2[x]}(\mathbb{F}_2, \mathbb{F}_2) = E[s^{-1}x]$, and exterior algebras are not free in characteristic 2.

One consequence of the above remarks is the following

Proposition 7.3. *Let X be a simply connected space with $H^*(X)$ polynomial. Then $H_*(\Omega X)$ is an exterior algebra and the suspension*

$$\sigma_* : QH_*(\Omega X) \to PH_*(X)$$

is an isomorphism. The spectral sequence

$$\mathrm{Cotor}^{H_*(X)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(\Omega X)$$

collapses.

Proof. This is because

$$\mathrm{Cotor}^{H_*(X)}(\mathbb{F}_2, \mathbb{F}_2) \cong E[s^{-1}PH_*(X)]$$

has generators and primitives in bidegrees $(-1, *)$. Together with Lemma 2.14, this proves the collapse claim and that σ_* is an isomorphism. Dually we have that

$$\sigma^* : QH^*(X) \to PH^*(\Omega X)$$

is an isomorphism and since the image generates $H^*(\Omega X)$ as an algebra, $H^*(\Omega X)$ is primitively generated. By Theorem 2.4 we get that $H_*(\Omega X)$ is exterior. □

In particular this applies to $X = \tilde{Q}_0S^0$.

Similarly, we have

Proposition 7.4. *For any space X, the spectral sequence*

$$\mathrm{Cotor}^{H_*(\tilde{Q}\Sigma X)}(k, k) \Rightarrow H_*(Q_0X)$$

collapses and the suspension

$$\sigma_* : QH_*(QX) \to PH_*(Q\Sigma X)$$

is an isomorphism.

Proof. σ_* is surjective since it hits $JH_*(\Sigma X)$ and since it is R-linear. Thus by Corollary 2.15, the spectral sequence must collapse. Now $H_*(Q\Sigma X)$ is primitively generated, so by Theorem 2.4 we get that $H^*(Q\Sigma X)$ is exterior and hence the spectral sequence has

$$E^2 = \mathrm{Cotor}^{H_*(\tilde{Q}\Sigma X)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[s^{-1}PH_*(\tilde{Q}\Sigma X)]$$

Since this is free as an algebra, there are no extension problems in homology, and since $QH_*(Q_0X)$ is in linear bijection with $E^{\infty}_{-1,*}$, we get that σ_* is injective. □

7.2. **Homology of $\Omega^\infty \mathbb{C}P^\infty_{-1}$.** The lemmas in subsection 7.1 imply the following diagram

$$\begin{array}{ccc} QH_*(Q_0\mathbb{C}P^\infty_+) & \xrightarrow{Q(\Omega_0\partial)_*} & QH_*(\Omega_0 QS^0) \\ \downarrow{\cong} & & \downarrow{\cong} \\ PH_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+) & \xrightarrow{P\partial_*} & PH_*(\tilde{Q}_0 S^0) \end{array} \tag{7.2}$$

in which the vertical isomorphisms are the suspensions and in which $H_*(\Omega_0 QS^0)$ is an exterior algebra, dual to a polynomial algebra.

The formula for ∂_* has an extra term because of the Hopf map η. We quote the result from [6, Theorem 4.4]:

Theorem 7.5 ([6])**.** *Let $a_s \in H_*(\mathbb{C}P^\infty_+)$ be the generator, s odd. Then*

$$Q(\partial_*)(a_s) = Q^{2s+1}\iota + Q^{s+1}Q^s\iota = Q^{2s+1}\iota + Q^{2s}Q^1\iota$$

□

We shall need a lemma analogous to Lemma 4.3.

Lemma 7.6. *The left ideal in R generated by $\{Q^{2s+1} \mid s \geq 0\}$ is also a right ideal.*

Proof. This is completely analogous to the proof of Lemma 4.3. One uses the Adem relation

$$Q^{2s}Q^{r-s} = Q^rQ^s + \sum_{i>s} \lambda_i Q^{r+s-i}Q^i$$

valid for $r \leq 2s$, for r odd and s even. □

Lemma 7.7. *Let $b_{2s+1} \in PH_*(QS^0) = PH_*(Q_0S^0)$ be the unique primitive element with $b_{2s+1} - Q^{2s+1}\iota$ decomposable. Then $PH_*(QS^0)$ is generated over R by the set $\{b_{2s+1} \mid s \geq 0\}$.*

Proof. Let $\lambda : QH_*(Q_0S^0) \to QH_*(Q_0S^0)$ be the dual of the squaring. By the Nishida relation $\lambda Q^{2s} = Q^s\lambda$, the coimage of λ has basis

$$\{Q^I\iota | I \text{ admissible, } e(I) > 0,\, 2|I\}$$

where $2|I$ means that all entries of I are even. Thus Theorem 2.4 implies that the image of $PH_*(QS^0) \to QH_*(QS^0)$ has basis

$$\{Q^I\iota \mid I \text{ admissible}, e(I) > 0, 2 \not| I\}$$

and by Lemma 7.6, this is generated over R by the subset

$$\{Q^{2s+1}\iota \mid s \geq 0\}$$

Thus the subspace of $PH_*(QS^0)$ generated over R by $\{b_{2s+1} \mid s \geq 0\}$ contains all indecomposable primitives. But this generated subspace is clearly preserved by the Frobenius map $\xi : x \mapsto x^2$, so it contains all primitives and the claim follows from Theorem 2.4. □

We are now ready to prove the mod 2 analogue of Theorem 1.1. The result is much simpler, and the extra term in Theorem 7.5 does not give much trouble.

Theorem 7.8. *The map*

$$P\partial_* : PH_*(Q\Sigma\mathbb{C}P^\infty_+) \to PH_*(QS^0)$$

is surjective.

Proof. By the previous lemma, it suffices to prove that $Q\partial_*$ hits the classes $Q^{2s+1}\iota$. Indeed, any indecomposable class mapping to $Q^{2s+1}\iota$ is odd-dimensional and thus by Theorem 2.4 has a unique primitive representative that will map to b_{2s+1}.

For $s = 0$, this is immediate, since $\partial_*(a_1) = (Q^1\iota) * \iota^{-2}$. For general s we use the Adem relation $Q^{2s}Q^1 = Q^{s+1}Q^s$ to get

$$\begin{aligned} Q(\partial_*)(a_{2s+1}) &= Q^{2s+1}\iota + Q^{s+1}Q^s\iota \\ &= Q^{2s+1}\iota + Q^{2s}Q^1\iota \end{aligned}$$

Thus we have

$$Q(\partial_*)(a_s - Q^{2s}a_1) = Q^{2s+1}\iota$$

□

Remark 7.9. The claim of [6, Cor. 7.5] that ∂_* and thus $P(\partial_*)$ is *injective* is incorrect. The Q^IQ^{2r+1} of [6, Cor. 7.4] is not necessarily admissible, and in fact an application of the Adem relations shows that

$$\partial_*(Q^3a_1 - Q^2Q^1a_1) = 0$$

Together with the diagram (7.2), Theorem 7.8 makes the spectral sequence

$$\mathrm{Cotor}^{H_*(\tilde{\Omega}QS^0)}(H_*(\tilde{Q}\mathbb{C}P^\infty_+), k) \Rightarrow H_*(\Omega^\infty_0\mathbb{C}P^\infty_{-1}) \tag{7.3}$$

very simple. We can now prove Theorem 1.3.

Proof of Theorem 1.3. It follows from diagram (7.2) and Theorem 7.8 that the map $Q(\Omega_0\partial_*)$ is surjective. Therefore the E^2-term of the Eilenberg-Moore spectral sequence is

$$E^2 = \mathrm{Cotor}^{H_*(\Omega_0 QS^0)}(H_*(Q_0\mathbb{C}P^\infty_+), \mathbb{F}_2) \cong H_*(Q_0\mathbb{C}P^\infty_+)\backslash\!\!\backslash \Omega\omega_*$$

and is concentrated on the line $E^2_{0,*}$. Therefore it collapses and we get the short exact sequence

$$\mathbb{F}_2 \longrightarrow H_*(\Omega^\infty\mathbb{C}P^\infty_{-1}) \xrightarrow{\Omega\omega_*} H_*(Q\mathbb{C}P^\infty_+) \xrightarrow{\Omega\partial_*} H_*(\Omega QS^0) \longrightarrow \mathbb{F}_2$$

□

7.3. **Homology of $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$.** This part of the calculation is similar to the odd primary case. We consider again the spectral sequence (5.1) with the splitting (5.2). Notice that the fibration (1.1) splits off the fibration $S^1 \to S^1 \to \mathbb{R}P^\infty$ and hence it has trivial local coefficients. As for odd primes, we need to determine the coalgebra structure on $H_*(QS^0)/\!\!/\partial_*$,so we first prove Theorem 1.1 in the case $p = 2$.

Proof of Theorem 1.1, $p = 2$. Since Q is right exact we have $Q(H_*(QS^0)/\!\!/\partial_*) = \mathrm{Cok}(Q\partial_*)$, and from the calculation in the proof of Theorem 7.8 it follows that the image of $Q\partial_*$ contains all $Q^I\iota$ where I has at least one *odd* entry, and therefore that the composition in Theorem 1.1 is surjective.

To prove injectivity, consider again the dual squaring $\lambda : H_*(Q_0S^0) \to H_*(Q_0S^0)$. It is a map of Hopf algebras, and since $\lambda Q^{2s} = Q^s\lambda$ and $\lambda Q^{2s+1} = 0$ we get that

$$\lambda : H_*(QS^0)^{(0)} \to H_*(QS^0)$$

is an isomorphism. Hence

$$H_*(QS^0) = H_*(QS^0)^{(0)} \oplus \mathrm{Ker}(\lambda)$$

where the first summand is a subalgebra and the second is an ideal. Now the injectivity of the map in the theorem follows from the fact that $\mathrm{Ker}(\lambda)$ is an ideal and that $\mathrm{Im}(\partial_*) \subseteq \mathbb{F}_2 \oplus \mathrm{Ker}(\lambda)$. □

Theorem 7.10. *$H_*(Q_0S^0)/\!\!/\partial_*$ is dual to a polynomial algebra.*

Proof. This follows since $\lambda : H_*(Q_0S^0) \to H_*(Q_0S^0)$ is surjective. □

Notice that $H^*(Q_0S^0)$ itself is polynomial. This is in contrast to the odd primary case, where only the subalgebra $H^*(Q_0S^0)\backslash\!\!\backslash\partial^* \subseteq H^*(Q_0S^0)$ is polynomial.

Proof of Theorem 1.3, $p = 2$. Completely as for odd primes, Theorem 7.10 makes the spectral sequence collapse, and the collapse gives a short exact sequence of Hopf algebras

$$\mathbb{F}_2 \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_* \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \xrightarrow{\omega_*} H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \longrightarrow \mathbb{F}_2 \tag{7.4}$$

Since $H_*(Q\Sigma\mathbb{C}P^\infty_+)$ is primitively generated, so is $H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*$. Hence the sequence is split if and only if $P(\omega_*)$ is surjective. We have the diagram

$$\begin{tikzcd}
QH_*(\Omega^\infty\mathbb{C}P^\infty_{-1}) \arrow[r] \arrow[d, "\sigma_*"] & QH_*(Q\mathbb{C}P^\infty_+) \arrow[r] \arrow[d, "\cong"] & QH_*(\Omega QS^0) \arrow[r] \arrow[d, "\cong"] & 0 \\
PH_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \arrow[r] & PH_*(Q\Sigma\mathbb{C}P^\infty_+) \arrow[r, "P\partial_*"] & PH_*(QS^0) \arrow[r] & 0
\end{tikzcd}$$

which we know is exact except possibly at $PH_*(Q\Sigma\mathbb{C}P^\infty_+)$. But it follows from the rest of the diagram that it is also exact at $PH_*(Q\Sigma\mathbb{C}P^\infty_+)$. Since $\mathrm{Ker}(P\partial_*) = P(H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*)$ we get that the sequence (7.4) splits. $\square$

References

[1] S. Araki, T. Kudo: *Topology of H_n-spaces and H_n-squaring Operations*, Mem. Fac. Sci. Kyushu Univ. Ser. A, **10** (1956), 85–120.

[2] F. R. Cohen, T. J. Lada, J. P. May: *The Homology of Iterated Loop Spaces*, Lecture Notes in Mathematics **533**, Springer-Verlag, 1976.

[3] E. Dyer, R. Lashof: *Homology of Iterated Loop Spaces*, Amer. J. Math. **84** (1962), 35–88.

[4] S. Eilenberg, J. C. Moore: *Homology and Fibrations I Coalgebras, cotensor products and its derived functors*, Comm. Math. Helv. **40** (1965), 199–236.

[5] J. Milnor, J. C. Moore: *On the Structure of Hopf algebras*, Ann. Math. **81** (1965), 211–264.

[6] B. M. Mann, E. Y. Miller, H. R. Miller: *S^1-equivariant function spaces*, Trans. Amer. Math. Soc. **295** (1989), 233–256.

[7] I. Madsen, U. Tillmann: *The Stable Mapping Class Group and $Q(\mathbb{C}P^\infty_+)$*, Invent. Math. **145** (2001), 509–544.

[8] J. C. Moore, L. Smith: *Hopf Algebras and Multiplicative Fibrations II*, Amer. J. Math. **90** (1968), 1113–1150.

[9] I. Madsen, M. Weiss: *Cohomology of the Stable Mapping Class Group*, in preparation.

[10] H. Cartan, S. Eilenberg: *Homological algebra*, Princeton University Press, 1956.

[11] L. Smith: *Homological algebra and the Eilenberg-Moore spectral sequence*, Trans. Amer. Math. Soc. **129** (1967), 58–93.

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SECONDARY CHARACTERISTIC CLASSES OF SURFACE BUNDLES

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Abstract. The Miller-Morita-Mumford classes associate to an oriented surface bundle $E \to B$ a class $\kappa_i(E) \in H^{2i}(B;\mathbb{Z})$. In this note we define for each prime p and each integer $i \geq 1$ a secondary characteristic class $\lambda_i(E) \in H^{2i(p-1)-2}(B;\mathbb{Z})/\mathbb{Z}\kappa_{i(p-1)-1}(E)$. The mod p reduction $\lambda_i(E) \in H^*(B;\mathbb{F}_p)$ has zero indeterminacy and satisfies $p\lambda_i(E) = \kappa_{i(p-1)-1}(E) \in H^*(B;\mathbb{Z}/p^2)$.

1. Introduction and statement of results

Recall that any bundle $\pi\colon E \to B$ of oriented surfaces with finite dimensional base B has an embedding $j\colon E \to B \times \mathbb{R}^{N+2}$ over B. For N large, j is unique up to isotopy. A choice of embedding j induces a transfer ("collapse") map

$$B_+ \wedge S^{N+2} \xrightarrow{\pi_!} \mathrm{Th}(\nu j).$$

The embedding $j\colon E \to B \times \mathbb{R}^{N+2}$ also induces classifying maps

$$\begin{tikzcd} T^\pi E \arrow[r, "\mathrm{cl}(T^\pi E)"] \arrow[d] & \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^2 \arrow[d] \\ E \arrow[r] & \mathrm{SO}(N+2)/(\mathrm{SO}(N)\times\mathrm{SO}(2)) \end{tikzcd}$$

and

$$\begin{tikzcd} \nu j \arrow[r, "\mathrm{cl}(\nu j)"] \arrow[d] & \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^N \arrow[d] \\ E \arrow[r] & \mathrm{SO}(N+2)/(\mathrm{SO}(N)\times\mathrm{SO}(2)). \end{tikzcd}$$

For brevity, write $U = U_N = \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^2$ and $U^\perp = U_N^\perp = \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^N$. We get the composition

$$\alpha = \mathrm{Th}(\mathrm{cl}(\nu j)) \circ \pi_!\colon B_+ \wedge S^{N+2} \to \mathrm{Th}(U_N^\perp).$$

Recall that there is a Thom class $u_{U^\perp} \in H^N(\mathrm{Th}(U^\perp), \star; \mathbb{Z})$ and that we have $H^{N+*}(\mathrm{Th}(U^\perp), \star; \mathbb{Z}) = \mathbb{Z}[e(U)].u_{U^\perp}$ for $* < N$. The definition of the κ-classes is

$$\kappa_i(E) = \alpha^*(e(U)^{i+1}.u_{U^\perp}) = \pi_!^*(e(T^\pi E)^{i+1}.u_{\nu j}) \in H^{2i}(B;\mathbb{Z}).$$

In this paper we define secondary characteristic classes of surface bundles. The definition involves Toda brackets. In section 2 we recall some generalities about Toda brackets. By a surface bundle we shall mean a smooth fibre bundle with closed oriented two-dimensional fibres.

Lemma 1.1. *Let p be a prime, and let $\mathcal{P}^i$ denote the Steenrod power operation. When $p = 2$, write $\mathcal{P}^i = \mathrm{Sq}^{2i}$ and $\beta\mathcal{P}^i = \mathrm{Sq}^{2i+1}$. Given a surface bundle $\pi\colon E \to B$, let $\alpha\colon B_+ \wedge S^{N+2} \to \mathrm{Th}(U_N^\perp)$ be as before and let $u\colon \mathrm{Th}(U_N^\perp) \to K(\mathbb{Z}, N)$ be the Thom class. Then the Toda bracket*

$$\{\beta\mathcal{P}^i, u, \alpha\} \subseteq H^{2i(p-1)-2+N}(B_+ \wedge S^{N+2}; \mathbb{Z}) = H^{2i(p-1)-2}(B; \mathbb{Z})$$

is defined with indeterminacy $\mathbb{Z}\kappa_{i(p-1)-1}(E)$.

Definition 1.2. With notation as in Lemma 1.1 define

$$\lambda_i(E) = (-1)^i\{\beta\mathcal{P}^i, u, \alpha\} \in H^{2i(p-1)-2}(B; \mathbb{Z})/\mathbb{Z}\kappa_{i(p-1)-1}(E).$$

Theorem 1.3. *The mod p reduction $\lambda_i(E) \in H^*(B; \mathbb{F}_p)$ has zero indeterminacy and satisfies*

$$p\lambda_i(E) = \kappa_{i(p-1)-1}(E) \in H^*(B; \mathbb{Z}/p^2).$$

More generally we have in integral cohomology that

$$p\lambda_i(E) = (1 + p\mathbb{Z})\kappa_{i(p-1)-1}(E).$$

Theorem 1.4.

(i) If $\pi\colon E \to B$ and $\pi'\colon E' \to B$ are surface bundles, then

$$\lambda_i(E \amalg E') = \lambda_i(E) + \lambda_i(E').$$

(ii) Let E_1, E_2 and E_2' be bundles of compact surfaces with boundary and assume that the oriented boundaries satisfy $\partial E_1 = \partial E_2 = \partial E_2'$. Then we can form the surface bundles

$$E = E_1 \cup_\partial \bar{E}_2,$$
$$E = E_1 \cup_\partial \bar{E}_2',$$

and

$$D = E_2' \cup_\partial \bar{E}_2,$$

where the bars denote orientation reversal. In this case we have

$$\lambda_i(E') = \lambda_i(E) + \lambda_i(D).$$

(iii) If $\pi\colon E \to B$ is a bundle of compact surfaces with boundary satisfying $\partial E = S^0 \times S^1 \times B$, then

$$\lambda_i(E \cup_\partial (D^1 \times S^1 \times B)) = \lambda_i(E \cup_\partial (S^0 \times D^2 \times B)).$$

The additivity properties *(ii)* and *(iii)* above are formally the same as those of the κ-classes.

As an application of secondary classes we prove the following strengthening of a theorem of [GMT].

Theorem 1.5. *Let p be a prime and $s \geq 1$. Then the reduction of $\kappa_{ps(p-1)-1}(E)$ mod p^2 vanishes:*

$$\kappa_{ps(p-1)-1}(E) = 0 \in H^*(B; \mathbb{Z}/p^2).$$

Theorem 1.5 proves part of the following conjecture.

Conjecture 1.6. *Let $s \geq 1$ and $v \geq 0$. Then*

$$\kappa_{p^v s(p-1)-1}(E) = 0 \in H^*(B; \mathbb{Z}/p^{v+1}).$$

If the conjecture is true, then $\kappa_{p^v s(p-1)-1}(E)$ can be divided by p^{v+1}. In [GMT] we prove that this holds modulo torsion. It is also proved in [GMT] that the statement of Conjecture 1.6 is best possible in the sense that if $s \not\equiv 0 \pmod p$, then $\kappa_{p^v s(p-1)-1}(E) \neq 0 \in H^*(B; \mathbb{Z}/p^{v+2})$. I hope to return to Conjecture 1.6 at a later time.

Acknowledgements. The author acknowledges John Rognes for very inspiring conversations during the author's visit to the University of Oslo.

2. Secondary composition

We recall the definition of secondary compositions (Toda brackets). For further details see [Toda].

All spaces and all maps in this section are pointed. The reduced suspension SX is regarded as the pushout of $X \wedge [-1,0] \longleftarrow X \longrightarrow X \wedge [0,1]$ where $-1 \in [-1,0]$ and $1 \in [0,1]$ are the basepoints. Thus, two nullhomotopies $F\colon X \wedge [-1,0] \to Y$ and $G\colon X \wedge [0,1] \to Y$ induce a map $G - F\colon SX \to Y$.

For a sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

with $g \circ f \simeq 0$ and $h \circ g \simeq 0$, a choice of null-homotopies $F\colon g \circ f \simeq 0$ and $G\colon h \circ g \simeq 0$ determines a map

$$h \circ F - G \circ (f \wedge [-1,0])\colon SX \to W.$$

We define the *secondary composition* to be the subset $\{h, g, f\} \subseteq [SX, W]$ of homotopy classes of maps obtained in this fashion, as F, G ranges over all null-homotopies.

Recall that $[SX, W] = [X, \Omega W]$ is a group.

Lemma 2.1. *$\{h, g, f\}$ depends only on the homotopy classes of h, g, and f. If $\{h, g, f\}$ is defined, then it gives a unique element in the double coset,*

$$\{h, g, f\} \in h \circ [SX, Z] \setminus [SX, W]/[SY, W] \circ Sf.$$

If $[SX, W]$ is abelian, then

$$\{h, g, f\} \in [SX, W]/\big(h \circ [SX, Z] + [SY, W] \circ Sf\big).$$

Proof. See [Toda, Lemma 1.1]. □

Proposition 2.2. *For a sequence of maps*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V$$

we have

(i) $\{k, h, g\} \circ f \subseteq \{k, h, g \circ f\}$
(ii) $\{k, h, g \circ f\} \subseteq \{k, h \circ g, f\}$
(iii) $\{k \circ h, g, f\} \subseteq \{k, h \circ g, f\}$
(iv) $k \circ \{h, g, f\} \subseteq \{k \circ h, g, f\}$

Proof. See [Toda, Proposition 1.2]. □

Proposition 2.3. *Let*

$$K(\mathbb{Z}, n) \xrightarrow{p} K(\mathbb{Z}, n) \xrightarrow{\rho} K(\mathbb{F}_p, n) \xrightarrow{\beta} K(\mathbb{Z}, n+1)$$

represent multiplication by p, reduction mod p, and the mod p Bockstein, respectively. Then

$$\mathrm{id} \in \{\beta, \rho, p\} \subseteq [SK(\mathbb{Z}, n), K(\mathbb{Z}, n+1)] = [K(\mathbb{Z}, n), K(\mathbb{Z}, n)]$$

□

Corollary 2.4. *Let* $c\colon X \to K(\mathbb{Z}, n)$ *represent a cohomology class. Let* ρ *and* β *be as in Proposition 2.3. Then*

$$\{\beta, \rho, c\} = \tfrac{1}{p}c + \mathbb{Z}c \subseteq H^n(X) = [SX, K(\mathbb{Z}, n+1)],$$

where

$$\tfrac{1}{p}c = \{c' \mid pc' = c\}$$

Proof. Clearly the two sides have the same indeterminacy $\mathbb{Z}c + \beta H^{n-1}(X; \mathbb{F}_p)$, so all we need to check is that if $pc' = c$, then $c' \in \{\beta, \rho, c\}$. But this follows from Propositions 2.2 and 2.3:

$$\{\beta, \rho, p \circ c'\} \supseteq \{\beta, \rho, p\} \circ c' \ni c'$$ □

3. Elementary properties of the secondary classes

Consider the oriented Grassmannian $\mathrm{SO}(N+2)/(\mathrm{SO}(N) \times \mathrm{SO}(2))$. Let $U = U_N = \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^2$ be the canonical oriented 2-dimensional vectorbundle and let $U^\perp = U_N^\perp = \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^N$ be its orthogonal complement.

Lemma 3.1 ([GMT])**.** *In* $H^*(\mathrm{Th}(U^\perp), \star; \mathbb{F}_p)$ *we have that*

$$\mathcal{P}^i u_{U^\perp} = (-1)^i e^{i(p-1)} u_{U^\perp}.$$

Proof. Let $\mathcal{P} = \sum_i \mathcal{P}^i$. Then $\mathcal{P}(u_U) = (1+e(U)^{p-1})u_U$. Since $u_{U\oplus U^\perp} = u_U \smile u_{U^\perp}$ we get

$$u_U \smile u_{U^\perp} = u_{U\oplus U^\perp} = \mathcal{P}(u_{U\oplus U^\perp}) = \mathcal{P}(u_U) \smile \mathcal{P}(u_{U^\perp}) = (1+e(U)^{p-1})u_U \smile \mathcal{P}(u_{U^\perp})$$

and hence

$$\mathcal{P}(u_{U^\perp}) = (1 + e(U)^{p-1})^{-1} u_{U^\perp} = \left(\sum_i (-1)^i e(U)^{i(p-1)}\right) u_{U^\perp}. \qquad \square$$

Proof of Lemma 1.1. Clearly $u \circ \alpha \simeq 0$. It follows from Lemma 3.1 that $\mathcal{P}^i u$ is the reduction of an integral class, so $\beta\mathcal{P}^i \circ u \simeq 0$. Therefore $\{\beta\mathcal{P}^i, u, \alpha\}$ is defined.

The indeterminacy can be computed from Lemma 2.1. Indeed we have

$$\beta\mathcal{P}^i[B_+ \wedge S^{N+3}, K(\mathbb{Z}, N)] = 0$$

and

$$\begin{aligned}[S\mathrm{Th}(U^\perp), K(\mathbb{Z}, N + 2i(p-1) + 1)] \circ \alpha &= \alpha^* H^{N+2i(p-1)}(\mathrm{Th}(U^\perp); \mathbb{Z}) \\ &= \alpha^*(\mathbb{Z}e^{i(p-1)} u_{U^\perp}) = \mathbb{Z}\kappa_{i(p-1)-1}(E).\end{aligned}$$

$\square$

Proof of Theorem 1.3. This follows from Proposition 2.2 and Corollary 2.4 and the diagram

$$\begin{tikzcd}
B_+ \wedge S^{N+2} \arrow[r, "\alpha"] \arrow[dr, "\kappa_{i(p-1)-1}"'] & \mathrm{Th}(U_N^\perp) \arrow[r, "u"] \arrow[d, "e^{i(p-1)}u"] & K(\mathbb{Z}, N) \arrow[d, "\mathcal{P}^i"] \\
& K(\mathbb{Z}, N + 2i(p-1)) \arrow[r, "\rho"] & K(\mathbb{F}_p, N + 2i(p-1)) \arrow[d, "\beta"] \\
& & K(\mathbb{Z}, N + 2i(p-1) + 1).
\end{tikzcd}$$

Indeed, Proposition 2.2 gives the inclusions

$$\begin{aligned}\{\beta, \rho, \kappa_{i(p-1)-1}(E)\} = \{\beta, \rho, (e^{i(p-1)}u) \circ \alpha\} &\subseteq \{\beta, \rho \circ (e^{i(p-1)}u), \alpha\} \\ &= (-1)^i\{\beta, \mathcal{P}^i u, \alpha\} \supseteq (-1)^i\{\beta\mathcal{P}^i, u, \alpha\} = \lambda_i(E).\end{aligned}$$

Then Lemma 2.1 proves that the first inclusion is an equality since the two sides have the same indeterminacy $\mathrm{Im}(\beta) + \mathbb{Z}\kappa_{i(p-1)-1}$. Therefore by Corollary 2.4

$$\lambda_i(E) \subseteq \{\beta, \rho, \kappa_{i(p-1)-1}(E)\} = \tfrac{1}{p}\kappa_{i(p-1)-1}(E) + \mathbb{Z}\kappa_{i(p-1)-1}(E),$$

and hence

$$p\lambda_i(E) \subseteq (1 + p\mathbb{Z})\kappa_{i(p-1)-1}(E).$$

Since they have the same indeterminacy, they are equal. $\square$

Proof of Theorem 1.4. *(i)* follows from the additivity of α under disjoint union, i.e. the property that

$$\alpha(E \amalg E') = \alpha(E) + \alpha(E') \in [B_+ \wedge S^{N+2}, \mathrm{Th}(U_N^\perp)].$$

Similarly *(ii)* follows from the "additivity" of α under glueing. Explicitly, a choice of embedding $j_\partial \colon \partial E_1 \to B \times \mathbb{R}^{N+1}$ over B will induce a map

$$\alpha_\partial \colon B_+ \wedge S^{N+1} \to \mathrm{Th}(U^\perp).$$

A choice of embedding $j_{E_1} : E_1 \to B \times [0,\infty) \times \mathbb{R}^{N+1}$ extending j_∂ will induce a nullhomotopy α_{E_1} of α_∂. Then, in the notation of paragraph 2 we have

$$\begin{aligned} \alpha_E &= \alpha_{E_1} - \alpha_{E_2}, \\ \alpha_{E'} &= \alpha_{E_1} - \alpha_{E_2'}, \\ \alpha_D &= \alpha_{E_2'} - \alpha_{E_2}. \end{aligned}$$

Thus we get

$$\alpha_{E'} = \alpha_E + \alpha_D \in [B_+ \wedge S^{N+2}, \mathrm{Th}(U^\perp)].$$

Finally *(iii)* follows from *(ii)* because

$$D^1 \times S^1 \times B \cup_{S^0 \times S^1 \times B} S^0 \times D^2 \times B = S^2 \times B$$

and $\lambda_i(S^2 \times B) = 0$. □

4. A variant of λ_{ps}

The goal of this section is to prove Theorem 1.5. The definition and properties of λ_i proves that $\kappa_{i(p-1)}$ is divisible by p. When $i = ps$, a variant of λ_{ps} can be used to prove that $\kappa_{ps(p-1)-1}$ is divisible by p^2.

Definition 4.1. Let $s \geq 0$ and let $\mathscr{A}_p$ be the Steenrod algebra. When $p = 2$ we write $\mathcal{P}^i = \mathrm{Sq}^{2i}$ and $\beta\mathcal{P}^i = \mathrm{Sq}^{2i+1}$ as before. Define $\theta_s \in \mathscr{A}_p$ by

$$\theta_s = \sum_{j=0}^{s} (-1)^j \binom{(p-1)(s-j)}{j} \mathcal{P}^{ps-j}\mathcal{P}^j = \mathcal{P}^{ps} + \text{terms of length 2}.$$

Define vectors $v_s, w_s \in \mathscr{A}_p^{s+1}$ by

$$w_s = (\mathcal{P}^0, \dots \mathcal{P}^s), \quad v_s = (\mathcal{P}^{ps}, \dots, (-1)^j \binom{(p-1)(s-j)-1}{j} \mathcal{P}^{ps-j}, \dots, \mathcal{P}^{(p-1)s}).$$

Lemma 4.2.

(i) In $H^*(\mathrm{Th}(U^\perp), \star; \mathbb{F}_p)$ we have that $\theta_s u_{U^\perp} = e^{ps(p-1)} u_{U^\perp}$.
(ii) $v_s^T \beta w_s = \beta\theta_s$.

Proof. *(i)* This is similar to Lemma 3.1, using the fact that the admissible terms of length 2 act trivially on $u_{U^\perp}$. Formula *(ii)* is the Adem relation for $\mathcal{P}^{(p-1)s}\beta\mathcal{P}^s$. □

Definition 4.3. Let α, u, θ_s be as above. Define the secondary characteristic class

$$\tilde{\lambda}_{ps}(E) = (-1)^s\{\beta\theta_s, u, \alpha\} \in H^{2ps(p-1)-2}(B,\mathbb{Z})/\mathbb{Z}\kappa_{ps(p-1)-1}(E).$$

Notice that $\tilde{\lambda}_{ps}$ satisfies the same formal properties as λ_{ps}. In particular $p\tilde{\lambda}_{ps} = (1+p\mathbb{Z})\kappa_{ps(p-1)-1}$. In general $\tilde{\lambda}_{ps} \neq \lambda_{ps}$.

Proof of Theorem 1.5. We have

$$\begin{aligned}(-1)^s\rho\circ\{\beta\theta_s, u, \alpha\} \subseteq (-1)^s\{\rho\circ\beta\theta_s, u, \alpha\} &= (-1)^s\{v_s^T\beta w_s, u, \alpha\} \\ &\supseteq (-1)^s v_s^T\{\beta w_s, u, \alpha\}\end{aligned}$$

and it is seen that all the inclusions are equalities since the indeterminacy vanishes. Since

$$(-1)^s\{\beta w_s, u, \alpha\} \in \prod_{i=0}^{s} H^{N+2i(p-1)}(B_+\wedge S^{N+2};\mathbb{F}_p) = \prod_{i=0}^{s} H^{2i(p-1)-2}(B;\mathbb{F}_p),$$

v^T will vanish because $H^*(B;\mathbb{F}_p)$ is an unstable $\mathscr{A}_p$-module.

Hence the mod p reduction of $\tilde{\lambda}_{ps}(E)$ vanishes, so $\kappa_{ps(p-1)-1}(E) = p\tilde{\lambda}_{ps}(E) = 0 \in H^*(B;\mathbb{Z}/p^2)$. $\square$

References

[GMT] S. Galatius, I. Madsen, U. Tillmann: *Divisibility of the stable Miller-Morita-Mumford classes*, in preparation.

[Toda] H. Toda: *Composition methods in homotopy groups of spheres*, Ann. Math. Studies, No. 49. Princeton University Press, 1962.

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MOD 2 HOMOLOGY OF THE STABLE SPIN MAPPING CLASS GROUP

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Abstract. The aim of this paper is twofold. Firstly we adapt the proof in [MW] to cover *spin mapping class groups*, that is, mapping class groups of surfaces with spin structures. The result is that in a stable range, the spin mapping class groups has the same homology as $\Omega^\infty\mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)})$. Here $U_{\mathrm{Spin}(2)} = E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} \mathbb{R}^2$ is the canonical $\mathrm{Spin}(2)$ vectorbundle over $B\mathrm{Spin}(2)$, and $-U_{\mathrm{Spin}(2)}$ is the -2-dimensional virtual inverse. Secondly we calculate the mod 2 homology of the space $\Omega^\infty\mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)})$ in analogy with [G].

1. Introduction and statement of results

The main result of this paper is a calculation of the mod 2 homology of the "spin mapping class groups" in a stable range, in the spirit of [G], which rests heavily on [MW]. The paper consists of two parts. In the second part we adapt the proof in [MW] to the case of surfaces with a spin structure[1]. The result is that these groups have the same homology, in a stable range, as the infinite loop space $\Omega^\infty\mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)})$ where $U_{\mathrm{Spin}(2)} = E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} \mathbb{R}^2$ is the canonical $\mathrm{Spin}(2)$-vectorbundle over $B\mathrm{Spin}(2)$ and $-U_{\mathrm{Spin}(2)}$ is the -2-dimensional virtual inverse. $\mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)})$ is the Thom spectrum with Thom class in dimension -2 (see section 2 for a more precise definition). In the first part of the paper we calculate the mod 2 homology of the infinite loop space $\Omega^\infty\mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)})$. Let us introduce some notation before giving a more precise description of our results.

Let $\theta\colon P_3 \to B_3$ be a principal $\mathrm{Gl}_3(\mathbb{R})$-bundle. For the moment it can be arbitrary, but we shall later specialise to the case $\theta = \theta_{\mathrm{Spin}}\colon E\mathrm{Spin}(3) \to B\mathrm{Spin}(3)$. Let $U_3 \to B_3$ be the associated vectorbundle. Let $B_2 = P_3/\mathrm{Gl}_2(\mathbb{R})$ and $U_2 = P_3 \times_{\mathrm{Gl}_2(\mathbb{R})} \mathbb{R}_2$. Then B_2 fibres over B_3 with fibre $\mathrm{Gl}_3(\mathbb{R})/\mathrm{Gl}_2(\mathbb{R}) \simeq S^2$. In fact B_2 is fibre homotopy equivalent to the sphere bundle of U_3, but it has the advantage that we have a canonical homeomorphism

$$\mathrm{Bun}(V, U_2) = \mathrm{Bun}(V \times \mathbb{R}, U_3)$$

where Bun denotes the space of bundle maps.

[1]The version of [MW] on which this manuscript is based, only treats *oriented* surfaces. A new version of [MW] that handles surfaces with a more general tangential structure, equivalent to the "θ-structures" considered in the second part of this paper, is soon to be available.

Definition 1.1. Let F be a surface, possibly with boundary, and let $\theta\colon U_3 \to B_3$ be as above. Then the space of θ-structures on F is the space $\mathrm{Bun}(TF, U_2)$ of bundle maps (these are suppose to be standard near the boundary if F has boundary). This has a left action of $\mathrm{Diff}(F)$. The space of (F,θ)-surfaces is the space

$$\mathscr{M}^\theta(F) := E\mathrm{Diff}(F) \times_{\mathrm{Diff}(F)} \mathrm{Bun}(TF, U_2).$$

The space $\mathscr{M}^\theta(F)$ is the classifying space for pairs (π,ξ) of a fibre bundle $\pi\colon E \to X$ with fibre F and a bundle map $\xi\colon T^\pi E \to U_2$, where $T^\pi E$ denotes the fibrewise tangentbundle of E. Notice that $\mathrm{Bun}(TF, U_2)$ may be empty. This will be the case e.g. if U_2 is orientable but F is not. Notice also that $\mathscr{M}^\theta(F)$ may be non-connected. We describe its components.

The action of $\mathrm{Diff}(F)$ on $\mathrm{Bun}(TF, U_2)$ induces an action of $\mathrm{Diff}(F)$ on $\pi_0\mathrm{Bun}(TF, U_2)$. For $\gamma \in \pi_0\mathrm{Bun}(TF, U_2)$ we write $\mathrm{Diff}(F,\gamma) \subseteq \mathrm{Diff}(F)$ for the subgroup that fixes γ. Define

$$\mathscr{M}^\theta(F,\gamma) = E\mathrm{Diff}(F,\gamma) \times_{\mathrm{Diff}(F,\gamma)} \mathrm{Bun}_\gamma(TF, U_2)$$

This is a connected space, and in general we have

$$\mathscr{M}^\theta(F) \simeq \coprod_\gamma \mathscr{M}^\theta(F,\gamma),$$

where the disjoint union is over one $\gamma \in \pi_0\mathrm{Bun}(TF, U_2)$ in each $\mathrm{Diff}(F)$-orbit.

There is a fibration sequence

$$\mathrm{Bun}_\gamma(TF, U_2) \to \mathscr{M}^\theta(F,\gamma) \to B\mathrm{Diff}(F,\gamma). \tag{1.1}$$

In particular (for genus ≥ 2), $\mathscr{M}^\theta(F,\gamma)$ is a $K(\pi,1)$ if and only if $\mathrm{Bun}_\gamma(TF, U_2)$ is. In this case we have $\mathscr{M}^\theta(F,\gamma) = B\Gamma^\theta(F,\gamma)$ where $\Gamma^\theta(F,\gamma) = \pi_1\mathscr{M}^\theta(F,\gamma)$ is what we could call the mapping class group of (F,γ).

The parametrised Pontryagin-Thom construction defines a map

$$\alpha\colon \mathscr{M}^\theta(F,\gamma) \to \Omega^\infty\mathbb{T}\mathrm{h}(-U_2)$$

and in favorable cases this will be an isomorphism in $H_n(-;\mathbb{Z})$ when n is small compared to the genus of F.

The case $\theta = \theta_{\mathrm{SO}}\colon E\mathrm{SO}(3) \times_{\mathrm{SO}(3)} \mathbb{R}^3 \to B\mathrm{SO}(3)$ is equivalent to the case considered in [MW]: An element $\gamma \in \pi_0\mathrm{Bun}(TF, U_2)$ is an orientation of F, and $\mathscr{M}^\theta(F,\gamma) \simeq B\mathrm{Diff}(F,\gamma)$ is the classifying space of the group of orientation preserving diffeomorphisms. Furthermore $\Omega^\infty\mathbb{T}\mathrm{h}(-U_2) = \Omega^\infty\mathbb{C}P^\infty_{-1}$. The homology of this space is calculated in [G].

For the rest of this introduction we specialise to the case $\theta = \theta_{\mathrm{Spin}}\colon E\mathrm{Spin}(3) \times_{\mathrm{Spin}(3)} \mathbb{R}^3 \to B\mathrm{Spin}(3)$. Then an element $\gamma \in \pi_0\mathrm{Bun}(TF, U_2)$ is a “spin structure” on F, given equivalently by a “quadratic refinement of the intersection form” on $H_1(F,\mathbb{F}_2)$ cf [J]. Any two spin structures on F differ by an element in $H^1(F,\mathbb{F}_2)$ so there are 4^g spin structures. There are only two $\mathrm{Diff}(F)$-orbits, however. They are distinguished by the Arf invariant of the quadratic form. Therefore

$$\mathscr{M}^\theta(F) = \mathscr{M}^\theta(F,\gamma_0) \amalg \mathscr{M}^\theta(F,\gamma_1)$$

where γ_0 is an Arf invariant 0 spin structure and γ_1 is an Arf invariant 1 spin structure.

The fibration sequence (1.1) specialises to

$$\mathbb{R}P^\infty \to \mathscr{M}^\theta(F,\gamma) \to B\mathrm{Diff}(F,\gamma)$$

and in case F has genus ≥ 2 these are all $K(\pi,1)$-spaces. The fundamental groups of $\mathscr{M}^\theta(F,\gamma)$ and $B\mathrm{Diff}(F,\gamma)$ could both be called "spin mapping class groups". Both are studied in [B] who uses the notation $G_\gamma(F) = \pi_1 B\mathrm{Diff}(F,\gamma) = \pi_0\mathrm{Diff}(F,\gamma)$ and $\tilde{G}_\gamma(F) = \pi_1\mathscr{M}^\theta(F,\gamma)$ and attributes the latter to Gregor Masbaum. [H] and [B] proves homological stability of these groups: If F' is obtained from F by glueing along boundaries of F, then the natural maps $G_\gamma(F) \to G_\gamma(F')$ and $\tilde{G}_\gamma(F) \to \tilde{G}_\gamma(F')$ are both isomorphisms in $H_k(-;\mathbb{Z})$ when $g \geq 2k^2+6k-2$, where g is the genus of F. [H] proves this in the case where F' has boundary, and [B] extends Harer's proof to the case where $\partial F' = \emptyset$. He also proves homological stability for the groups $\tilde{G}_\gamma(F)$.

The parametrised Pontryagin-Thom construction defines a map

$$\alpha\colon \mathscr{M}^\theta(F,\gamma) \to \Omega^\infty \mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)}) \tag{1.2}$$

and we prove that (on components) it is an isomorphism in $H_k(-;\mathbb{Z})$ whenever $g \geq 2k^2+6k-2$. The number $2k^2+6k-2$ is the stability range of $\tilde{G}_\gamma(F)$ in [B], and an improvement of the stability range would give an improvement of the isomorphism range of the map (1.2).

It is easily seen (using e.g. the fibration sequence (1.3) below) that $\pi_0\Omega^\infty\mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)}) \cong \mathbb{Z}\times\mathbb{Z}/2$. One may verify that the image of the map (1.2) is in the component given by the genus of F and the Arf invariant of γ. We conclude this introduction by stating the theorems about the homology of $\Omega^\infty\mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)})$, and thus the homology in a stable range, of $\mathscr{M}^\theta(F,\gamma)$.

The starting point of the calculation is a fibration sequence of infinite loop spaces

$$\Omega^\infty\mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)}) \xrightarrow{\Omega\omega} Q(B\mathrm{Spin}(2)_+) \xrightarrow{\Omega\partial} \Omega Q(S(U_{\mathrm{Spin}(2)})_+). \tag{1.3}$$

Here $S(U_{\mathrm{Spin}(2)})$ is the sphere bundle of $U_{\mathrm{Spin}(2)}$ and Q denotes the functor $\Omega^\infty\Sigma^\infty$. If we identify $B\mathrm{Spin}(2)$ with $\mathbb{C}P^\infty$ then $U_{\mathrm{Spin}(2)} = L\otimes_{\mathbb{C}} L$, where L is the canonical complex line bundle, and $S(U_{\mathrm{Spin}(2)}) = \mathbb{R}P^\infty$. We give a concrete description of (1.3) in Section 2.

For brevity we shall write U for $U_{\mathrm{Spin}(2)}$. In the following, all Hopf algebras are commutative and cocommutative. Recall that any map $f: A \to B$ of such Hopf algebras have a kernel denoted $A\backslash\!\backslash f$ and a cokernel $f/\!\!/f$ in the category of Hopf algebras. Homology and cohomology is always with coefficients in $\mathbb{F}_2$.

Theorem 1.2. *The fibration sequence* (1.3) *induces a short exact sequence of Hopf algebras*

$$H_*(\Omega^\infty \mathbb{T}\mathrm{h}(-U))\backslash\!\backslash \Omega\omega_* \hookrightarrow H_*(\Omega^\infty \mathbb{T}\mathrm{h}(-U)) \xrightarrow{\Omega\omega_*} H_*(Q(B\mathrm{Spin}(2)_+))\backslash\!\backslash \Omega\partial_* \tag{1.4}$$

and dually

$$H^*(Q_0(B\mathrm{Spin}(2)_+))/\!\!/\Omega\partial_* \xhookrightarrow{\Omega\omega^*} H^*(\Omega_0^\infty \mathbb{T}\mathrm{h}(-U)) \twoheadrightarrow H^*(\Omega_0^\infty \mathbb{T}\mathrm{h}(-U))/\!\!/\Omega\omega^*. \tag{1.5}$$

It remains to determine the Hopf algebras $H_*(Q(B\mathrm{Spin}(2)_+))\backslash\!\backslash\Omega\partial_*$ and $H_*(\Omega^\infty\mathbb{T}\mathrm{h}(-U))\backslash\!\backslash\Omega\omega_*$. The next theorem determines the Hopf algebra $H_*(Q(B\mathrm{Spin}(2)_+))\backslash\!\backslash\Omega\partial_*$. We also produce an explicit splitting of the sequence (1.4), although the splitting is only as algebras, not as Hopf algebras.

The action of $\mathrm{Spin}(3) = SU(2)$ on S^2 gives an S^2-bundle $E\mathrm{Spin}(3)\times_{\mathrm{Spin}(3)} S^2 \to B\mathrm{Spin}(3)$. The vertical tangent bundle $E\mathrm{Spin}(3)\times_{\mathrm{Spin}(3)} TS^2$ has a canonical spin-structure, and the classifying map $E\mathrm{Spin}(3)\times_{\mathrm{Spin}(3)} S^2 \to B\mathrm{Spin}(2)$ is a homotopy equivalence. Consequently we get a map $B\mathrm{Spin}(3) \to \mathscr{M}^\theta(S^2)$. The composition

$$B\mathrm{Spin}(3) \to \mathscr{M}^\theta(S^2) \xrightarrow{\alpha} \Omega^\infty\mathbb{T}\mathrm{h}(-U) \to Q(B\mathrm{Spin}(2)_+)$$

is the Becker-Gottlieb transfer for the fibration sequence $S^2 \to B\mathrm{Spin}(2) \to B\mathrm{Spin}(3)$.

Before stating the next theorem, let us recall that for a Hopf algebra A over $\mathbb{F}_2$ there is a Frobenius map $\xi\colon A \to A$ given by $\xi x = x^2$ which is a morphism of Hopf algebras. Write $a_i \in H_{2i}(B\mathrm{Spin}(2))$ and $b_i \in H_{4i}(B\mathrm{Spin}(3))$ for the generators, $i \geq 0$. Recall that $H_*(Q(B\mathrm{Spin}(2)_+))$ is the free commutative algebra on the set $\mathbf{T}_2$ of generators given by

$$\mathbf{T}_2 = \{Q^I a_i \mid i \geq 0,\ I \text{ admissible},\ e(I) > 2i\},$$

where Q^I are the iterated Dyer-Lashof operations (see [CLM] for definitions and proofs). Similarly $H_*(Q(B\mathrm{Spin}(3)_+))$ is the free commutative algebra on the set of generators given by

$$\mathbf{T}_3 = \{Q^I b_i \mid i \geq 0,\ I \text{ admissible},\ e(I) > 4i\}.$$

Theorem 1.3.

(i) We have $H_*(Q(B\mathrm{Spin}(2)_+))\backslash\!\backslash\partial_* = \xi H_*(Q(B\mathrm{Spin}(2)_+))$. Both the algebra $H_*(Q(B\mathrm{Spin}(2)_+))\backslash\!\backslash\partial_*$ and the dual algebra $H^*(Q_0(B\mathrm{Spin}(2)_+))/\!\!/\partial^*$ are free commutative.

(ii) The composition

$$H_*(Q(B\mathrm{Spin}(3)_+)) \to H_*(\Omega^\infty\mathbb{T}\mathrm{h}(-U)) \to H_*(Q(B\mathrm{Spin}(2)_+))\backslash\!\backslash\partial_*$$

is surjective. It maps b_i to a_i^2 and more generally it maps $Q^{2I}b_i$ to $(Q^I a_i)^2$.

It remains to describe the Hopf algebra $H_*(\Omega^\infty \mathbb{T}\mathrm{h}(-U))\backslash\!\!\backslash \Omega\omega_*$ in Theorem 1.2. This is done by first describing the (co-)homology of $\Omega Q(\mathbb{R}P^\infty_+)$ and $\Omega^2 Q(\mathbb{R}P^\infty_+)$.

To state the results about $\Omega Q(\mathbb{R}P^\infty_+)$ and $\Omega^2 Q(\mathbb{R}P^\infty_+)$, let us recall a certain functor from [MM]. It is called V in [MM, definition 6.2], but we shall call it A. We shall as usual let PA denote the vectorspace of primitive elements in A and QA denote the vectorspace of indecomposable elements in A.

Definition 1.4 ([MM])**.** Let V be a graded vectorspace and $\xi\colon V \to V$ a linear map such that $\xi V_n \subseteq V_{2n}$. Let SV denote the free commutative (i.e. polynomial) algebra generated by V, and let $I \subseteq SV$ be the ideal generated by the elements $x^2 - \xi x$, $x \in V$. Let $AV = A(V, \xi) = SV/I$.

The functor A satisfies $A(V \oplus V') = AV \otimes AV'$ and therefore the diagonal $V \to V \oplus V$ induces a comultiplication on AV making it a Hopf algebra. The vectorspace of primitive elements is V itself, $PAV = V$.

Theorem 1.5.

(i) The suspension

$$\sigma^*\colon QH^*(Q_0\mathbb{R}P^\infty_+) \to PH^*(\Omega Q\mathbb{R}P^\infty_+)$$

is an isomorphism (of degree -1).

(ii) The suspension σ^* above induces an isomorphism

$$A(s^{-1}QH^*(Q_0\mathbb{R}P^\infty_+), s^{-1}\mathrm{Sq}_1) \cong H^*(\Omega Q\mathbb{R}P^\infty_+).$$

Here s^{-1} denotes desuspension of graded vector spaces and Sq_1 is the Steenrod operation given by $\mathrm{Sq}_1(x) = \mathrm{Sq}^{k-1}(x)$ if $\deg(x) = k$.

(iii) The Hopf algebra $H^*(\Omega_0 Q(\mathbb{R}P^\infty_+))$ is primitively generated and polynomial.

(iv) The suspension induces an isomorphism

$$\sigma^*\colon \mathrm{Coker}(\mathrm{Sq}_1) \to QH^*(\Omega Q\mathbb{R}P^\infty_+).$$

Theorem 1.6.

(i) The suspension

$$\sigma^*\colon QH^*(\Omega_0 Q\mathbb{R}P^\infty_+) \to PH^*(\Omega^2 Q\mathbb{R}P^\infty_+)$$

is an isomorphism (of degree -1).

(ii) The suspension σ^* above induces an isomorphism

$$A(s^{-2}\mathrm{Coker}(\mathrm{Sq}_1), s^{-2}\mathrm{Sq}_2) \cong H^*(\Omega^2 Q\mathbb{R}P^\infty_+).$$

Here $\mathrm{Sq}_2\colon \mathrm{Coker}(\mathrm{Sq}_1) \to \mathrm{Coker}(\mathrm{Sq}_1)$ is the Steenrod operation given by $\mathrm{Sq}_2(x) = \mathrm{Sq}^{k-2}(x)$ if $\deg(x) = k$.

(iii) The Hopf algebra $H^*(\Omega_0 Q(\mathbb{R}P^\infty_+))$ is primitively generated but not polynomial.

(iv) The suspension induces an isomorphism

$$\sigma^* \circ \sigma^*\colon \mathrm{Coker}(\mathrm{Sq}_2) \to QH^*(\Omega^2 Q\mathbb{R}P^\infty_+)$$

of degree -2.

Using this description of $H^*(\Omega^2 Q\mathbb{R}P^\infty_+)$ we describe the Hopf algebra $H_*(\Omega^\infty \mathbb{T}\mathrm{h}(-U))\backslash\!\!\backslash \omega_*$ and its dual $H^*(\Omega^\infty \mathbb{T}\mathrm{h}(-U)) /\!\!/ \omega^*$.

Theorem 1.7.

(i) The Hopf algebra $H_*(\Omega^\infty \mathbb{T}\mathrm{h}(-U))\backslash\!\!\backslash \Omega\omega_*$ *is precisely the image of* $H_*(\Omega^2 Q\mathbb{R}P^\infty) \to H_*(\Omega^\infty \mathbb{T}\mathrm{h}(-U))$.

(ii) $H^*(\Omega^\infty_0 \mathbb{T}\mathrm{h}(-U)) /\!\!/ \Omega\omega^*$ *injects into* $H^*(\Omega^2_0 Q(\mathbb{R}P^\infty_+))$ *and is primitively generated.*

(iii) Under the isomorphism in Theorem 1.6.(ii), $H^*(\Omega^\infty_0 \mathbb{T}\mathrm{h}(-U)) /\!\!/ \Omega\omega^*$ *is precisely the subalgebra generated by the double suspension of the sub vectorspace*

$$\mathrm{Ker}\left(Q\partial^* \colon QH^*(Q\mathbb{R}P^\infty_+) \to QH^*(Q\Sigma(B\mathrm{Spin}(2)_+))\right)$$

of $QH^*(Q\mathbb{R}P^\infty_+)$.

Finally we can combine the above to conclude the following corollary.

Corollary 1.8. *The infinite loop map*

$$\Omega^2 Q(\mathbb{R}P^\infty_+) \times Q(B\mathrm{Spin}(3)_+) \to \Omega^\infty \mathbb{T}\mathrm{h}(-U),$$

which on the first factor is the map $\Omega^2 Q(\mathbb{R}P^\infty_+) \to \Omega^\infty \mathbb{T}\mathrm{h}(-U)$ *induced by* (1.3) *and which on the second factor is the map* $Q(B\mathrm{Spin}(3)_+) \to \Omega^\infty \mathbb{T}\mathrm{h}(-U)$ *from Theorem 1.3, induces an injection*

$$H^*(\Omega^\infty \mathbb{T}\mathrm{h}(-U)) \to H^*(\Omega^2 Q(\mathbb{R}P^\infty_+)) \otimes H^*(Q(B\mathrm{Spin}(3)_+)).$$

2. A cofibration sequence

Let us first describe a concrete model for the maps of spectra underlying the fibration sequence (1.3).

Let $q\colon \mathbb{C}P^n \to \mathbb{C}P^n$ denote the map $q([z_0 : \cdots : z_n]) = [z_0^2 : \cdots : z_n^2]$. Let L_n denote the canonical complex line bundle over $\mathbb{C}P^n$ and $L_n^\perp$ its orthogonal complement. There is a bundle map

$$\begin{array}{ccc} L_n \otimes L_n & \xrightarrow{\hat{q}} & L_n \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & \xrightarrow{q} & \mathbb{C}P^n \end{array}$$

where $\hat{q}\colon (z_0,\dots,z_n)\otimes(w_0,\dots,w_n) \mapsto (z_0w_0,\dots,z_nw_n)$. Thus $\hat{q}$ identifies $L_n \otimes L_n$ with q^*L_n. We shall write $L_n^2 = q^*L_n$ and $L_n^{2\perp} = q^*L_n^\perp$.

There is an obvious bundle map

$$\begin{array}{ccc} L_{n-1}^\perp \times \mathbb{C} & \longrightarrow & L_n^\perp \\ \downarrow & & \downarrow \\ \mathbb{C}P^{n-1} & \longrightarrow & \mathbb{C}P^n \end{array}$$

and an induced bundle map

$$\begin{tikzcd} {L^2_{n-1}}^{\perp} \times \mathbb{C} \arrow[r] \arrow[d] & {L^2_n}^{\perp} \arrow[d] \\ \mathbb{C}P^{n-1} \arrow[r] & \mathbb{C}P^n \end{tikzcd}$$

These gives maps of Thom spaces $\mathrm{Th}(L_{n-1}^{\perp}) \wedge S^2 \to \mathrm{Th}(L_n^{\perp})$ and $\mathrm{Th}({L^2_{n-1}}^{\perp}) \wedge S^2 \to \mathrm{Th}({L^2_n}^{\perp})$. Therefore we get spectra $\mathbb{T}\mathrm{h}(-L)$ and $\mathbb{T}\mathrm{h}(-L^2)$ with $(2n+2)$-nd space $\mathrm{Th}(L_n^{\perp})$ and $\mathrm{Th}({L^2_n}^{\perp})$, respectively. The associated infinite loop spaces are

$$\begin{aligned} \Omega^\infty \mathbb{T}\mathrm{h}(-L) &= \operatorname{colim} \Omega^{2n+2} \mathbb{T}\mathrm{h}(L_n^{\perp}) \\ \text{and} \quad \Omega^\infty \mathbb{T}\mathrm{h}(-L^2) &= \operatorname{colim} \Omega^{2n+2} \mathbb{T}\mathrm{h}({L^2_n}^{\perp}) \end{aligned} \tag{2.1}$$

The bundle $L \to \mathbb{C}P^\infty$ above is isomorphic to $U_{\mathrm{SO}(2)} = E\mathrm{SO}(2) \times_{\mathrm{SO}(2)} \mathbb{R}^2 \to B\mathrm{SO}(2)$ and $L^2 \to \mathbb{C}P^\infty$ is isomorphic to $U_{\mathrm{Spin}(2)} = E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} \mathbb{R}^2 \to B\mathrm{Spin}(2)$. The map q above is induced from the double cover $\mathrm{Spin}(2) \to \mathrm{SO}(2)$. Therefore we shall write $\Omega^\infty \mathbb{T}\mathrm{h}(-U_{\mathrm{SO}(2)})$ and $\Omega^\infty \mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)})$ for the spaces of (2.1).

For a vector bundle $\xi \to X$, let $\mathrm{Th}(\xi) = \xi \cup \{\infty\}$ be the one-point compactification of the total space.

Lemma 2.1. *Let ξ and η be vector bundles over X. Then there is a cofibration sequence*

$$\mathrm{Th}(\xi) \xrightarrow{z} \mathrm{Th}(\xi \oplus \eta) \xrightarrow{\partial} \mathrm{Th}(\mathbb{R} \oplus \xi_{|S(\eta)})$$

where z is induced from the zero section of η and $\xi_{|S(\eta)}$ denotes pullback of ξ to the sphere bundle of η. If $\xi \oplus \eta = \mathbb{R}^n \times X$, then ∂ is the parametrised Pontryagin-Thom construction of the sphere bundle $S(\eta) \to X$.

Proof. The normal bundle of the embedding $S(\eta) \to \eta$ is $\mathbb{R} \times S(\eta)$. This embeds via "polar coordinates" onto $\eta - X$. Therefore the normal bundle of the composition $S(\eta) \to \eta \to \eta \oplus \xi$ is $\mathbb{R} \oplus \xi_{|S(\eta)}$ and this embeds onto $\xi \oplus \eta - \xi \subseteq \xi \oplus \eta$. This defines a homeomorphism

$$\mathrm{Th}(\xi \oplus \eta)/\mathrm{Th}(\eta) = (\xi \oplus \eta - \eta) \cup \{\infty\} \cong \mathrm{Th}(\mathbb{R} \oplus \xi_{|S(\eta)})$$

If $\xi \oplus \eta = X \times \mathbb{R}^n$, then ∂ is exactly the Thom-Pontryagin construction applied to the embedding $S(\eta) \subseteq X \times \mathbb{R}^n$ over X. □

Lemma 2.2. *The map $\mathbb{R}P^{2n+1} \to \mathbb{C}P^n \times \mathbb{C}^{n+1}$ given by*

$$[x_0 : y_0 : \cdots : x_n : y_n] \mapsto ([z_0 : \cdots : z_n], (z_0^2, \ldots, z_n^2)),$$

where $z_j = x_j + iy_j$, is a homeomorphism onto $S(L_n^2) \subseteq \mathbb{C}P^n \times \mathbb{C}^{n+1}$. Thus $S(L_n^2) \to \mathbb{C}P^n$ is identified with the quotient map

$$\mathbb{R}P^{2n+1} = S^{2n+1}/\{\pm 1\} \to S^{2n+1}/S^1 = \mathbb{C}P^n$$

□

Corollary 2.3. *There is a cofibration sequence*

$$\mathrm{Th}(\mathbb{R} \oplus L_n^{2\perp}) \xrightarrow{z} \Sigma^{2n+3}\mathbb{C}P_+^n \xrightarrow{\partial} \Sigma^{2n+2}\mathbb{R}P_+^{2n+1} \tag{2.2}$$

Proof. Let $\xi = \mathbb{R} \oplus L_n^{2\perp}$ and $\eta = L_n^2$ in Lemma 2.1. Then $\xi \oplus \eta = \mathbb{C}P^n \times \mathbb{C}^{n+1} \times \mathbb{R}$ and $\mathbb{R} \oplus \xi_{|S(\eta)} = \mathbb{C} \oplus L^{2\perp}_{n|S(L_n^2)} = L_n^2 \oplus L^{2\perp}_{n|S(L_n^2)} = S(L_n^2) \times \mathbb{C}^{n+1}$, using the canonical trivialisation of $L^2_{n|S(L_n^2)}$. Now lemmas 2.1 and 2.2 gives the desired result. □

Corollary 2.4. *There is a cofibration sequence of spectra*

$$\Sigma\mathbb{T}\mathrm{h}(-L^2) \longrightarrow \Sigma^{\infty+1}(\mathbb{C}P_+^\infty) \longrightarrow \Sigma^\infty \mathbb{R}P_+^\infty$$

and associated fibration sequences

$$\Omega^\infty \Sigma\mathbb{T}\mathrm{h}(-L^2) \xrightarrow{\omega} Q\Sigma(\mathbb{C}P_+^\infty) \xrightarrow{\partial} Q\mathbb{R}P_+^\infty \tag{2.3}$$

and

$$\Omega^\infty \mathbb{T}\mathrm{h}(-L^2) \xrightarrow{\Omega\omega} Q(\mathbb{C}P_+^\infty) \xrightarrow{\Omega\partial} \Omega Q\mathbb{R}P_+^\infty \tag{2.4}$$

Proposition 2.5. *The map*

$$\partial \colon Q\Sigma\mathbb{C}P_+^\infty \to Q\mathbb{R}P_+^\infty$$

is the "S^1-transfer" denoted t_1 in [MMM]

Proof. The map t_1 in [MMM] is exactly the pretransfer of the S^1-bundle $ES^1 \times_{S^1} (S^1/\{\pm 1\}) \to BS^1$, and this is ∂. □

Theorem 2.6 ([MMM])**.** *Let $\bar{a}_r \in H_{2r+1}(\Sigma\mathbb{C}P_+^\infty)$ and $e_r \in H_r(\mathbb{R}P^\infty)$ be the generator. Then*

$$\partial_*(\bar{a}_r) \equiv e_{2r+1} + Q^{r+1}e_r$$

modulo decomposable elements.

Proof. This follows from [MMM, Theorem 4.4] by ignoring the decomposable terms. □

Corollary 2.7. *The map*

$$\partial_* \colon H_*(Q\Sigma\mathbb{C}P_+^\infty) \to H_*(Q\mathbb{R}P_+^\infty)$$

is injective.

Proof. This follows from Theorem 2.6 and the known structure of $H_*(Q\Sigma\mathbb{C}P_+^\infty)$ and $H_*(Q\mathbb{R}P_+^\infty)$, cf [CLM]. □

3. Cohomology of $\Omega Q\mathbb{R}P^\infty_+$ and $\Omega^2 Q\mathbb{R}P^\infty_+$

The goal of this section is to prove Theorems 1.5 and 1.6. This is done via the following proposition.

Proposition 3.1. *Let X be a simply connected, homotopy commutative, homotopy associative H-space. Assume that $H_*(X)$ and $H_*(\Omega X)$ are of finite type. Then $H^*(X)$ is a polynomial algebra is and only if $\xi\colon PH^*(X) \to PH^*(X)$ is injective. In this case we have*

(i) The suspension

$$\sigma^*\colon QH^*(X) \to PH^*(\Omega X)$$

is an isomorphism (of degree -1).

(ii) The suspension σ^* above induces an isomorphism

$$A[s^{-1}QH^*(X), s^{-1}\mathrm{Sq}_1] \cong H^*(\Omega X).$$

Here s^{-1} denotes desuspension of graded vectorspaces and $Sq_1\colon QH^*(X) \to QH^*(X)$ is the Steenrod operation given by $\mathrm{Sq}_1(x) = \mathrm{Sq}^{k-1}(x)$ if $\deg(x) = k$.

(iii) The Hopf algebra $H^*(\Omega X)$ is primitively generated. It is polynomial if and only if $\mathrm{Sq}_1\colon QH^*(X) \to QH^*(X)$ is injective.

Proof. It follows from Borel's structure theorem that $H^*(X)$ is polynomial if and only if $\xi\colon H^*(X) \to X^*(X)$ is injective. And this happens if and only if $\xi\colon PH^*(X) \to PH^*(X)$ is injective. The proof of the proposition is based on the Eilenberg-Moore spectral sequence, see [EM] or the review in [G]. The E_2-term is $\mathrm{Tor}_{H^*(X)}(\mathbb{F}_2, \mathbb{F}_2)$ and it converges to $H^*(\Omega X)$.

When $H^*(X)$ is a polynomial algebra, the E_2-term of the spectral sequence is

$$\mathrm{Tor}_{H^*(X)}(\mathbb{F}_2, \mathbb{F}_2) = E[s^{-1}QH^*(X)]$$

which has generators and primitives concentrated on the line $E_2^{-1,*}$. Therefore it must collapse, because it is a spectral sequence of Hopf algebras. The suspension can be identified with the map

$$QH^*(X) \cong \mathrm{Tor}^{-1,*}_{H^*(X)}(\mathbb{F}_2, \mathbb{F}_2) = E_2^{-1,*} \to E_2^{-1,*} \subseteq \tilde{H}^*(\Omega X)$$

and the image is within the vectorspace of primitive elements. Therefore σ^* is injective because $E^2 = E^\infty$.

The image of σ^* generated the algebra $H^*(\Omega X)$ since it even does so after a filtration. In particular we have proved that $H^*(\Omega X)$ is primitively generated.

The suspension σ^* commutes with Steenrod operations. In particular we have

$$(\sigma^*(x))^2 = \sigma^*(\mathrm{Sq}_1 x)$$

so the image of σ^* is closed under the Frobenius map $\xi\colon x \mapsto x^2$. That σ^* is surjective now follows from the Milnor-Moore exact sequence,

$$0 \to P\xi H^*(\Omega X) \to PH^*(\Omega X) \to QH^*(\Omega X) \to 0.$$

Namely, if σ^* were not surjective, there would be an element of minimal degree in $PH^*(\Omega X)$ not in the image of σ^*. This element would have to map to zero in $QH^*(\Omega X)$ because the image of σ^* generates. Hence, by the exact sequence, it would have to be a square of some other element. But this contradicts minimality because the image of σ^* is closed under ξ.

We have proved (i) and the first part of (iii). Now (ii) follows from the fact that $H^*(\Omega X)$ is primitively generated and that $\xi\colon PH^*(\Omega X) \to PH^*(\Omega X)$ corresponds under σ^* to Sq_1. Finally, by (ii) we have that $\xi\colon H^*(\Omega X) \to H^*(\Omega X)$ is inective if and only if $\mathrm{Sq}_1\colon QH^*(X) \to QH^*(X)$ is injective. □

Remark 3.2. Without the assumption on simple connectivity the above proposition is generally false. It does hold in the following very special case, however. Namely, if $\pi_1 X$ is an $\mathbb{F}_2$-vectorspace and X splits as $X \simeq \tilde{X} \times B\pi_1 X$. In this case we have $PH_1(X) = H_1(X) = \pi_1(X)$ and $QH_0(\Omega X) = \pi_0(\Omega X) = \pi_1(X)$, and for $k \geq 2$ we have $PH_k(X) = PH_k(\tilde{X})$ and $QH_{k-1}(\Omega X) = QH_{k-1}(\Omega\tilde{X})$.

Let $e_r \in H_r(\mathbb{R}P^\infty)$ be the generator. Recall from [CLM] that $H_*(Q\mathbb{R}P^\infty_+)$ is the free commutative algebra on the set

$$\mathbf{T} = \{Q^I e_r \mid r \geq 0,\, I \text{ admissible},\, e(I) > r\}$$

We shall also need a basis for $PH_*(Q\mathbb{R}P^\infty_+)$

Definition 3.3. Let $p_{2r+1} \in PH_*(Q\mathbb{R}P^\infty_+)$ be the unique primitive class with $p_{2r+1} - e_{2r+1}$ decomposable. For an admissible sequence of the form $I = (2s+1, 2I')$ with $e(I) \geq 2i$, let $p_{(I,2i)}$ be the unique primitive class with $p_{(I,2i)} - Q^I e_{2i}$ decomposable. For an admissible sequence $I = (I', 2s+1, 2I'')$ with $e(I) \geq 2i$, let $p_{(I,2i)} = Q^{I'} p_{(2s+1,2I'',2i)}$.

Thus $p_{(I,i)} \in PH_*(Q\mathbb{R}P^\infty_+)$ is defined for alle (I,i) with $2 \nmid (I,i)$.

Lemma 3.4. *The set*

$$\{p_{(I,i)} \mid\ i \geq 0,\, I \text{ admissible},\, e(I) \geq i,\, 2 \nmid (I,i)\}$$

is a basis of $PH_*(Q\mathbb{R}P^\infty_+) = PH_*(Q_0\mathbb{R}P^\infty_+)$.

Proof. This is well known. That $p_{(I,i)}$ spans all of $PH_*(Q\mathbb{R}P^\infty)$ follows from the Milnor-Moore exact sequence. See [G] for more details □

Definition 3.5. Define operations $H_*(Q\mathbb{R}P^\infty) \to H_*(Q\mathbb{R}P^\infty)$ by

$$\lambda x = \mathrm{Sq}^k_* x, \quad \deg(x) = 2k,$$
$$\lambda' x = \mathrm{Sq}^k_* x, \quad \deg(x) = 2k+1$$
$$\lambda'' x = \mathrm{Sq}^k_* x, \quad \deg(x) = 2k+2$$

We write λ, λ' and λ'' for the induced operations on $PH_*(Q\mathbb{R}P^\infty_+)$ and $QH_*(Q\mathbb{R}P^\infty_+)$ also. These are dual to $\xi = \mathrm{Sq}_0$, Sq_1, and Sq_2 on cohomology, respectively.

Lemma 3.6. *In $H_*(\mathbb{R}P^\infty)$ we have*

$$\lambda e_{2r} = e_r$$
$$\lambda' e_{2r-1} = r e_r$$
$$\lambda'' e_{2r-2} = \binom{r}{2} e_r$$

Proof. This is dual to the formula $\mathrm{Sq}^k w_1^n = \binom{n}{k} w_1^{n+k} \in H^*(\mathbb{R}P^\infty)$. $\square$

Lemma 3.7. *The operations λ, λ' and λ'' satisfiy the relations*

$$\lambda Q^{2s} x = Q^s \lambda x \tag{3.1}$$
$$\lambda' Q^{2s} x = Q^s \lambda' x \tag{3.2}$$
$$\lambda' Q^{2s-1} x = (\deg Q^s \lambda x) Q^s \lambda x \tag{3.3}$$
$$\lambda'' Q^{2s} x = Q^s \lambda'' x, \qquad \text{if } \lambda x = 0 \tag{3.4}$$
$$\lambda'' Q^{2s-1} x = (1 + \deg Q^s \lambda' x) Q^s \lambda' x \tag{3.5}$$

Proof. This follows from the Nishida relations (cf [CLM]). $\square$

Proposition 3.8. $\lambda\colon QH_*(Q\mathbb{R}P^\infty_+) \to QH_*(Q\mathbb{R}P^\infty_+)$ *is surjective.*

Proof. This is because $\lambda\colon H_*(\mathbb{R}P^\infty) \to H_*(\mathbb{R}P^\infty)$ is surjective. Explicitly, (3.1) and Lemma 3.6 implies that

$$\lambda Q^{2I} e_{2r} = Q^I e_r$$

so the basis $\mathbf{T}$ of $QH_*(Q\mathbb{R}P^\infty_+)$ is hit. $\square$

Proposition 3.9. $\lambda'\colon PH_*(Q\mathbb{R}P^\infty_+) \to PH_*(Q\mathbb{R}P^\infty_+)$ *is surjective.*

Proof. Lemma 3.6 and Lemma 3.7 imply that

$$\lambda' e_{4r+1} = e_{2r+1}$$

and that

$$\lambda'(Q^{4s+1} Q^{4I'} e_{2i}) = Q^{2s+1} Q^{2I'} e_i$$

Hence, since λ' preserves decomposables

$$\lambda' p_{4r+1} = p_{2r+1}$$

and

$$\lambda' p_{(4s+1, 4I', 4i)} = p_{(2s+1, 2I', 2i)}$$

and hence

$$\lambda' p_{(2I', 4s+1, 4I'', 4i)} = p_{(I', 2s+1, 2I'', 2i)}$$

Therefore, by Lemma 3.4, λ' is surjective. $\square$

Proposition 3.10. $\lambda''\colon PH_*(Q\mathbb{R}P^\infty_+) \to \mathrm{Ker}(\lambda')$ *is not surjective.*

Proof. The element $p_3 = e_3 + e_1e_2 + e_1^3$ satisfies $\lambda'(p_3) = Q^2e_1 = p_{(1,1)}$ and the element $p_{(2,1)}$ satisfies $\lambda' p_{(2,1)} = p_{(1,1)}$. So $p_{(2,1)} + p_3 \in \operatorname{Ker}(\lambda')$. But $PH_4(Q\mathbb{R}P^\infty)$ has basis $\{Q^3e_1, Q^2Q^1e_1\}$ and $\lambda''(Q^3e_1) = \lambda''(Q^2Q^1e_1) = 0$, so $p_{(2,1)} + p_3$ is not hit by λ''. $\square$

Proof of Theorem 1.5. This follows from Proposition 3.1, using Propositions 3.8 and 3.9. $\square$

Proof of Theorem 1.6. This follows from Proposition 3.1, using Theorem 1.5.

That $H^*(\Omega_0^2 Q(\mathbb{R}P_+^\infty))$ is not polynomial follows from proposition 3.10. Indeed there must be an a generator of degree two with square zero. $\square$

4. The spectral sequence

The aim of this section is to prove theorems 1.2 and 1.7. The starting point is the fibration (1.3). None of the spaces in the fibration are connected. In fact we have

$$\pi_0(\Omega Q\mathbb{R}P_+^\infty) = \mathbb{Z}/2 \times \mathbb{Z}/2, \quad \pi_0 Q(B\mathrm{Spin}(2)_+) = \mathbb{Z}, \quad \pi_0\Omega^\infty\mathbb{T}\mathrm{h}(-U) = \mathbb{Z} \times \mathbb{Z}/2$$

and

$$\pi_1 Q(B\mathrm{Spin}(2)_+) = \mathbb{Z}/2, \quad \pi_1(\Omega Q(\mathbb{R}P_+^\infty)) = \mathbb{Z}/2 \times \mathbb{Z}/2$$

The claim in theorem (1.3) is clearly equivalent to the claim that the sequence

$$H_*(\Omega^\infty\mathbb{T}\mathrm{h}(-U)) \xrightarrow{\Omega\omega_*} H_*(Q(B\mathrm{Spin}(2)_+)) \xrightarrow{\Omega\partial_*} H_*(\Omega Q(\mathbb{R}P_+^\infty)) \qquad (4.1)$$

is exact (both means that $\Omega\omega_*$ maps onto the kernel of $\Omega\partial_*$). This is equivalent to proving that the sequence

$$H_*(\Omega_0^\infty\mathbb{T}\mathrm{h}(-U)) \xrightarrow{\Omega\omega_*} H_*(Q_0(B\mathrm{Spin}(2)_+)) \xrightarrow{\Omega\partial_*} H_*(\hat{\Omega}_0 Q(\mathbb{R}P_+^\infty)) \qquad (4.2)$$

is exact. Here $\hat{\Omega}_0 Q(\mathbb{R}P_+^\infty)$ is the double cover of $\Omega_0 Q(\mathbb{R}P_+^\infty)$ corresponding to the image of $\Omega\partial$ in π_1. This is equivalent because there is a map from (4.2) to (4.1), the kernel of which is the sequence

$$H_0(\Omega^\infty\mathbb{T}\mathrm{h}(-U)) \xrightarrow{\Omega\omega_*} H_0(Q(B\mathrm{Spin}(2)_+)) \xrightarrow{\Omega\partial_*} H_0(\Omega Q(\mathbb{R}P_+^\infty)) \otimes H_*(\mathbb{R}P^\infty)$$

which is exact.

Now (4.2) corresponds to the following modified version of (1.3)

$$\Omega_0^\infty\mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)}) \xrightarrow{\Omega\omega} Q_0(B\mathrm{Spin}(2)_+) \xrightarrow{\Omega_0\partial} \hat{\Omega} Q(\mathbb{R}P_+^\infty).$$

To this fibration there is an associated Eilenberg-Moore spectral sequence

$$\begin{aligned} E^2 &= \mathrm{Cotor}^{H_*(\hat{\Omega}_0 Q\mathbb{R}P_+^\infty)}(H_*(Q_0\mathbb{C}P_+^\infty), \mathbb{F}_2) \\ &\cong \mathrm{Cotor}^{H_*(\hat{\Omega}_0 Q\mathbb{R}P_+^\infty)/\!\!/\Omega\partial_*}(\mathbb{F}_2, \mathbb{F}_2) \otimes H_*(Q(B\mathrm{Spin}(2)_+))\backslash\!\!\backslash\Omega\partial_* \\ &\Rightarrow H_*\Omega_0^\infty\mathbb{T}\mathrm{h}(-U) \end{aligned} \qquad (4.3)$$

Lemma 4.1. *The dual algebra* $H^*(\hat{\Omega}_0 Q\mathbb{R}P^\infty_+)\backslash\!\backslash\Omega\partial^*$ *is polynomial.*

Proof. It is a subalgebra of $H^*(\hat{\Omega}_0 Q(\mathbb{R}P^\infty_+))$ which again is a subalgebra of $H^*(\Omega_0 Q(\mathbb{R}P^\infty_+))$ because $\Omega_0 Q(\mathbb{R}P^\infty_+) \simeq \mathbb{R}P^\infty \times \hat{\Omega}_0 Q(\mathbb{R}P^\infty_+)$. Therefore the lemma follows from Theorem 1.5. $\square$

Proof of Theorem 1.2. From Lemma 4.1 we get that

$$\mathrm{Cotor}^{H_*(\hat{\Omega}_0 Q\mathbb{R}P^\infty_+)/\!/\Omega\partial_*}(\mathbb{F}_2,\mathbb{F}_2) = E[s^{-1}P(H_*(\hat{\Omega}_0 Q\mathbb{R}P^\infty)/\!/\Omega\partial_*)]$$

Therefore the spectral sequence (4.3) has primitives and generators concentrated in $E^2_{0,*}$ and $E^2_{-1,*}$. Since it is a spectral sequence of Hopf algebras, it must collapse. Therefore the map

$$\Omega\omega_*\colon H_*(\Omega_0\mathbb{T}\mathrm{h}(-U)) \to H_*(Q_0(B\mathrm{Spin}(2)_+))\backslash\!\backslash\Omega\partial_*$$

is surjective. $\square$

We next prove Theorem 1.7. We need a lemma.

Lemma 4.2. *The map*

$$PH_*(\tilde{\Omega}_0 Q\mathbb{R}P^\infty_+) \to P(H_*(\hat{\Omega}_0 Q\mathbb{R}P^\infty_+)/\!/\Omega\partial_*)$$

is surjective.

Proof. The fuctor P is left exact and has a right derived functor $\hat{P}$. See [G] for a survey and references. The important property is that it vanishes when the dual algebra is polynomial. There is an exact sequence of Hopf algebras

$$\mathbb{F}_2 \longrightarrow \mathrm{Im}(\Omega\partial_*) \longrightarrow H_*(\hat{\Omega}_0 Q\mathbb{R}P^\infty_+) \longrightarrow H_*(\hat{\Omega}_0 Q\mathbb{R}P^\infty_+)/\!/\Omega\partial_* \longrightarrow \mathbb{F}_2$$

Now $(\mathrm{Im}(\Omega\partial_*))^* = \mathrm{Im}(\Omega\partial^*)$ is a subalgebra of $H^*(Q_0(B\mathrm{Spin}(2)_+))$ and hence is polynomial. Therefore $\hat{P}(\mathrm{Im}(\Omega\partial_*)) = 0$ and the lemma follows. $\square$

Corollary 4.3. *The map*

$$\mathrm{Cotor}^{H_*(\tilde{\Omega}_0 Q\mathbb{R}P^\infty_+)}(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Cotor}^{H_*(\hat{\Omega}_0 Q\mathbb{R}P^\infty_+)/\!/\Omega\partial_*}(\mathbb{F}_2,\mathbb{F}_2)$$

is surjective.

Proof. This is because $\mathrm{Cotor}^A(\mathbb{F}_2,\mathbb{F}_2) = E[s^{-1}PA]$ when A^* is polynomial. $\square$

Proof of Theorem 1.7. The spectral sequence gives a filtration $F^0 \supseteq F^{-1} \supset \dots$ of $H_*(\Omega^\infty_0\mathbb{T}\mathrm{h}(-U))$ which restricts to a filtration of $H_*(\Omega^\infty_0\mathbb{T}\mathrm{h}(-U))\backslash\!\backslash\Omega\omega_*$. With respect to this filtration we have

$$E^0(H_*(\Omega^\infty_0\mathbb{T}\mathrm{h}(-U))\backslash\!\backslash\Omega\omega_*) \cong \mathrm{Cotor}^{H_*(\hat{\Omega}_0 Q\mathbb{R}P^\infty_+)/\!/\Omega\partial_*}(\mathbb{F}_2,\mathbb{F}_2)$$

There is a map of fibrations

$$\begin{tikzcd}
\Omega^2_0 Q\mathbb{R}P^\infty_+ \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \tilde{\Omega}_0 Q\mathbb{R}P^\infty_+ \arrow[d] \\
\Omega^\infty_0 \mathbb{T}\mathrm{h}(-U) \arrow[r, "\Omega\omega"] & Q_0(B\mathrm{Spin}(2)_+) \arrow[r, "\Omega\partial"] & \hat{\Omega}_0 Q\mathbb{R}P^\infty_+
\end{tikzcd}$$

and an associated map of spectral sequences which on the E^2-term is

$$\mathrm{Cotor}^{H_*(\tilde{\Omega}_0 Q\mathbb{R}P^\infty_+)}(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Cotor}^{H_*(\hat{\Omega}_0 Q\mathbb{R}P^\infty_+)/\!/\Omega\partial_*}(\mathbb{F}_2,\mathbb{F}_2)\otimes H_*(Q(B\mathrm{Spin}(2)_+))\backslash\!\backslash\partial_*$$

Since both spectral sequences collapse, we get that the map

$$H_*(\Omega^2_0 Q\mathbb{R}P^\infty_+) \to H_*(\Omega^\infty_0\mathbb{T}\mathrm{h}(-U))\backslash\!\backslash\Omega\omega_* \tag{4.4}$$

is filtered and on filtration quotients the map is identified with

$$\mathrm{Cotor}^{H_*(\tilde{\Omega}_0 Q\mathbb{R}P^\infty_+)}(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Cotor}^{H_*(\hat{\Omega}_0 Q\mathbb{R}P^\infty_+)/\!/\Omega\partial_*}(\mathbb{F}_2,\mathbb{F}_2)$$

Since this is surjective by Lemma 4.2, then also the map (4.4) is surjective.

This proves (i). (ii) is just the dual statement of (i). To prove (iii) we see that the quotient

$$QH_*(\Omega^2 Q(\mathbb{R}P^\infty_+)) \to Q(H_*(\Omega^\infty\mathbb{T}\mathrm{h}(-U))\backslash\!\backslash\Omega\omega_*)$$

is identified under suspension with

$$PH_*(\Omega Q(\mathbb{R}P^\infty_+)) \to P(H_*(\Omega Q(\mathbb{R}P^\infty_+))/\!/\Omega\partial_*)$$

which again by suspension is mapped to

$$PH_*(Q(\mathbb{R}P^\infty_+)) \to P(H_*(Q(\mathbb{R}P^\infty_+))/\!/\partial_*) = \mathrm{Coker}(P\partial_*).$$

By dualising we get $\mathrm{Ker}(Q\partial_*)$ as claimed. □

5. Proof of Theorem 1.3

We know from theorem 1.5 that $H_*(\Omega Q(\mathbb{R}P^\infty_+))$ is an exterior algebra. We also know that $H_*(QB\mathrm{Spin}(2)_+)$ is a polynomial algebra. We have the following commutative diagram

$$\begin{tikzcd}
QH_*(Q(B\mathrm{Spin}(2)_+)) \arrow[r, "Q(\Omega\partial_*)"] \arrow[d, "\cong"] & QH_*(\Omega Q(\mathbb{R}P^\infty_+)) \arrow[d, "\cong"] \\
PH_*(Q\Sigma(B\mathrm{Spin}(2)_+)) \arrow[r, "P(\partial_*)"] & PH_*(Q(\mathbb{R}P^\infty_+))
\end{tikzcd}$$

It follows that $Q(\Omega\partial_*)$ is injective. These three facts prove that $H_*(Q\mathbb{C}P^\infty_+)\backslash\!\backslash\Omega\partial_* = \xi H_*(Q\mathbb{C}P^\infty_+)$. We calculate the Becker-Gottlieb transfer of the bundle

$$E\mathrm{Spin}(2)\times_{\mathrm{Spin}(2)} S^2 \to B\mathrm{Spin}(2).$$

Lemma 5.1. *Let $N, S\colon B\mathrm{Spin}(2) \to E\mathrm{Spin}(2) \times_{Spin(2)} S^2$ denote the sections at the north and south pole, respectively. Then the Becker-Gottlieb transfer is*

$$\tau = N + S \in [B\mathrm{Spin}(2), Q(E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} S^2_+)]$$

Proof. This is similar to the Becker-Gottlieb calculations in [GMT]: S^2 is the $\mathrm{Spin}(2)$-equivariant pushout of $D^2 \leftarrow S^1 \to D^2$ and therefore the bundle $E\mathrm{Spin}(2)\times_{\mathrm{Spin}(2)} S^2$ is the fibrewise pushout of $E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} D^2 \leftarrow E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} S^1 \to E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} D^2$. Then properties (A1)–(A3) in [GMT, p. 15] proves the proposition. Indeed the transfer of $E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} S^1$ vanishes by (A3) and the transfer of $E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} D^2$ is the section at the center of D^2 by (A1). Then the additivity (A2) proves that the transfer of the whole bundle is $N + S$. □

Corollary 5.2. *Let $E\mathrm{Spin}(2) \times_{\mathrm{Spin}(2)} S^2 \to B\mathrm{Spin}(2)$ classify the vertical tangentbundle. Then*

$$\alpha = \iota + c \in [B\mathrm{Spin}(2), Q(B\mathrm{Spin}(2))]$$

where ι is the usual inclusion of $B\mathrm{Spin}(2)$ and c is the orientation reversal map.

Proof of theorem 1.3. We have $\iota_* a_i = a_i$ and $c_* a_i = (-1)^i a_i$. Therefore

$$(\iota + c)_* a_i = \sum_{r+s=i} (-1)^s a_r a_s$$

Reducing mod 2 we get

$$(i + c)_*(a_{2i}) = a_i^2 \quad \text{and} \quad (i + c)_* a_{2i+1} = 0$$

Since $B\mathrm{Spin}(2) \to B\mathrm{Spin}(3)$ maps $a_{2i} \mapsto b_i$, we have proved that the composition in theorem 1.3 maps b_i to a_i^2 as claimed.

Then it will also map $Q^{2I} b_i$ to $(Q^I a_i)^2$ and hence the composition is surjective.

Both $H_*(Q(B\mathrm{Spin}(2)_+))$ and $H^*(Q_0(B\mathrm{Spin}(2)_+))$ are free commutative. This follows from the fact that $\lambda\colon H_*(B\mathrm{Spin}(2)) \to H_*(B\mathrm{Spin}(2))$ is surjective, similarly to the case of $Q(\mathbb{R}P^\infty_+)$. But then

$$\xi\colon H_*(Q(B\mathrm{Spin}(2)_+)) \to \xi H_*(Q(B\mathrm{Spin}(2)_+))$$

is an isomorphism so the same holds for $\xi H_*(Q(B\mathrm{Spin}(2)_+))$. □

Proof of Corollary 1.8. By the exact sequence in Theorem 1.2 and by Theorem 1.7, the kernel of

$$H^*(\Omega_0^\infty \mathbb{T}\mathrm{h}(-U)) \to H^*(\Omega_0^2 Q(\mathbb{R}P^\infty_+))$$

is exactly $H^*(Q_0(B\mathrm{Spin}(2)_+))/\!\!/\Omega\partial^*$. By theorem 1.3 (ii), this injects into $H^*(Q(B\mathrm{Spin}(3)_+))$. □

6. Adapting [MW]

This is the second part of the paper, and the aim is to adapt the proof in [MW]. As explained in the introduction we can let $\theta = \theta_{\mathrm{SO}}$ and then for genus ≥ 2 we have $\mathscr{M}^\theta(F,\gamma) = B\Gamma(F,\gamma)$, where γ is an orientation of F and $\Gamma(F,\gamma) = \pi_0 \mathrm{Diff}(F,\gamma)$ is the oriented mapping class group of F. Then we can let $F = F_{g,2}$ and let $\Gamma_{\infty,2} = \operatorname{colim} \Gamma_{g,2}$ where the colimit is over glueing an oriented torus. Then [MW] proves that there is a homology equivalence

$$\mathbb{Z} \times B\Gamma_{\infty,2} \to \Omega^\infty \mathbb{T}\mathrm{h}(-U_{SO})$$

For $\theta = \theta_{\mathrm{Spin}}$ we can again let $F = F_{g,2}$ and let

$$\mathscr{M}^\theta(F_{\infty,2}) := \operatorname{hocolim} \mathscr{M}^\theta(F_{g,2})$$

where the hocolim is over glueing a torus. There are two essentially different ways of doing this because we can choose either an Arf invariant 0 torus or an Arf invariant 1 torus. Which one we use is not important however, because the composition of two tori will be a surface of genus 2 and with an Arf invariant 0 spin structure anyhow.

Then we adapt the proof to showing that there is a homology equivalence

$$\mathbb{Z} \times \mathscr{M}^\theta(F_{\infty,2}) \to \Omega^\infty \mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)})$$

Since the $\mathscr{M}^\theta(-)$ satisfies Harer stability, we also get that $\mathscr{M}^\theta(F,\gamma)$ has the same homology as $\Omega^\infty \mathbb{T}\mathrm{h}(-U_{\mathrm{Spin}(2)})$ in a stable range.

Most of the modifications are straightforward and the proofs are valid for any vectorbundle $\theta \colon U_3 \to B_3$. Only at the very end shall we specialise to the case $\theta = \theta_{\mathrm{Spin}}$. The idea is roughly as follows. All the sheaves in [MW] are made out of either submersions $\pi \colon E \to X$ with oriented three-dimensional fibres, or surface bundles $q \colon M \to X$ with oriented two-dimensional fibres, with some extra structure. Then we can modify the definition by removing the word "oriented" anbd instead include a bundle map $T^\pi E \to U_3$ or $T^q M \to U_2$. The original case in [MW] can the be recovered by setting $\theta = \theta_{\mathrm{SO}}$ (the sheaves will be slightly fattened versions of those in [MW]).

This procedure works very well, and for most of the chapters we shal just give the modified definitions and claim that the proofs work in our more general situation as well. There is one point that needs attention, however. Namely the definition of E^{rg} and the sheaf map $\mathcal{L}_T \to \mathcal{W}_T$ in [MW, Chapter 5]. To do this properly in the added generality we shall need to give a new definition of fibrewise surgery. Also [MW, Chapter 6] about the "connectivity problem" need attention.

7. The sheaves

This section defines the appropriate generalisations of the sheaves on $\mathscr{X}$ defined in [MW, Section 2]. Let $\theta \colon U_3 \to B_3$ be a 3-dimensional real vectorbundle with inner product. Let $q_0 \colon T(S^1 \times [0,1] \times \mathbb{R}) \to U_3$ be a fixed bundle map, constant in the $[0,1] \times \mathbb{R}$-directions.

Definition 7.1. Let $\mathcal{V}^\theta$ be the sheaf on $\mathscr{X}$ defined such that $\mathcal{V}(X)$ is the set of (π, f, q) such that $(\pi, f)\colon E^{k+3} \to X^k \times \mathbb{R}$ is a proper smooth map, $\pi\colon E \to X$ is a graphic submersion, f is fibrewise regular, and $q\colon T_\pi E \to U_3$ is a bundle map. We assume that near the boundary of E, (π, f) agrees over $X \times \mathbb{R}$ with $S^1 \times [0,1] \times \mathbb{R}$ and q agrees with q_0.

Define $h\mathcal{V}^\theta$, $\mathcal{W}^\theta$, $h\mathcal{W}^\theta$, $\mathcal{W}^\theta_{\mathrm{loc}}$ and $h\mathcal{W}^\theta_{\mathrm{loc}}$ similarly.

Remark 7.2. For $\theta = E\mathrm{SO}(3) \times_{\mathrm{SO}(3)} \mathbb{R}^3 \to B\mathrm{SO}(3)$, the map $q\colon T_\pi E \to U$ induces an orientation on the fibres of $\pi\colon E \to X$. Thus for this θ there is a sheaf map

$$\mathcal{V}^\theta \to \mathcal{V}$$

and this is a weak equivalence. Thus $\mathcal{V}^\theta$ is a fat version of the sheaf $\mathcal{V}$ in [MW].

Following [MW] we get a diagram of classifying spaces

$$\begin{array}{ccccc}
|\mathcal{V}^\theta_c| & \longrightarrow & |\mathcal{W}^\theta| & \longrightarrow & |\mathcal{W}^\theta_{\mathrm{loc}}| \\
\downarrow & & \downarrow & & \downarrow \\
|h\mathcal{V}^\theta| & \longrightarrow & |h\mathcal{W}^\theta| & \longrightarrow & |h\mathcal{W}^\theta_{\mathrm{loc}}|
\end{array}$$

where the vertical maps are induced by taking the 2-jet prolongation of f.

We aim at generalising [MW] to the statement $\Omega B|\mathcal{V}^\theta_c| \simeq |h\mathcal{V}^\theta|$.

Lemma 7.3. *Let $\mathbb{T}\mathrm{h}(-U_2)$ denote the Thom spectrum of the vitual bundle $-U_2$ over $B(2)$. Then*

$$|h\mathcal{V}^\theta| \simeq \Omega^\infty \mathbb{T}\mathrm{h}(-U_2)$$

Lemma 7.4. *We have*

$$|\mathcal{V}^\theta_c| \simeq \coprod_F \mathscr{M}^\theta(F).$$

where the disjoint union is taken over surfaces with two boundary components, one in each diffeomorphism class.

If $U \to B$ is orientable, this means that the disjoint union is over the surfaces $F_{g,2}$, $g \geq 0$.

Proof. This is proved similarly to the case considered in [MW]. □

8. Adjusting the proof

Most of the proof given in [MW] goes through with little or no change also in this more general situation. We describe the necessary changes chapter for chapter.

8.1. **Chapter 3.** [MW] determines the homotopy types of $|h\mathcal{V}|$, $|h\mathcal{W}|$ and $|h\mathcal{W}_{\mathrm{loc}}|$ and proves that

$$|h\mathcal{V}| \to |h\mathcal{W}| \to |h\mathcal{W}_{\mathrm{loc}}|$$

is a homotopy fibre sequence.

Let $\mathrm{Bun}(\mathbb{R}^3, U_3)$ denote the space of bundle maps from $\mathbb{R}^3$, considered as a bundle over a point, to the bundle U_3. As in [MW] we let $S(\mathbb{R}^3)$ be the vectorspace of quadratic forms on $\mathbb{R}^3$ and $\Delta \subseteq S(\mathbb{R}^3)$ be the subset of degenerate quadratic forms.

Define an $\mathrm{O}(3)$-space $A^\theta(\mathbb{R}^3)$ by

$$A^\theta = ((\mathbb{R}^3)^* \times S(\mathbb{R}^3) - \{0\} \times \Delta) \times \mathrm{Bun}(\mathbb{R}^3, U_3)$$

and define

$$\mathrm{G}\mathcal{W}^\theta(3,n)) = \mathrm{O}(n+3) \times_{\mathrm{O}(n)\times\mathrm{O}(3)} A^\theta(\mathbb{R}^3).$$

Thus a point in $\mathrm{G}\mathcal{W}(3,n)$ is a quadruple (V, l, q, ξ) where $V \subseteq \mathbb{R}^{3+n}$ is a three-dimensional subspace, $l\colon V \to \mathbb{R}$ is a linear map, $q\colon V \to \mathbb{R}$ is a quadratic map, and $\xi\colon V \to U_3$ is a bundle map, subject to the condition that q is non-degenerate if $l = 0$.

Example 8.1. For $U_3 = E\mathrm{O}(3) \times_{\mathrm{O}(3)} \mathbb{R}^3$, the space $A^\theta(\mathbb{R}^3)$ has the same (equivariant) homotopy type as the $A(\mathbb{R}^3)$ of [MW]. For $U_3 = E\mathrm{SO}(3) \times_{\mathrm{SO}(3)} \mathbb{R}^3$, the space $\mathrm{G}\mathcal{W}^\theta(3,n)$ has the same homotopy type as $\mathrm{G}\mathcal{W}(3,n)$ of [MW].

Let $\Sigma^\theta(3,n) \subseteq \mathrm{G}\mathcal{W}^\theta(3,n)$ be the subspace corresponding to $\{0\} \times (S(\mathbb{R}^3) - \Delta) \times \mathrm{Bun}(\mathbb{R}^3, U_3) \subseteq A^\theta(\mathbb{R}^3)$, and let

$$\mathrm{G}\mathcal{V}^\theta(3,n) = \mathrm{G}\mathcal{W}^\theta(3,n) - \Sigma^\theta(3,n)$$

Let $\mathcal{U}_n^\theta \to \mathrm{G}\mathcal{W}^\theta(3,n)$ be the universal bundle. We get a cofibration sequence

$$\mathrm{Th}(\mathcal{U}_n^{\theta\perp}|\mathrm{G}\mathcal{V}^\theta(3,n)) \to \mathrm{Th}(\mathcal{U}_n^{\theta\perp}) \to \mathrm{Th}(\mathcal{U}_n^{\theta\perp} \oplus \mathcal{U}_n^{\theta*}|\Sigma^\theta(3,n))$$

and an associated fibration sequence of infinite loop spaces

$$\Omega^\infty \mathbf{h}\mathcal{V}^\theta \to \Omega^\infty \mathbf{h}\mathcal{W}^\theta \to \Omega^\infty \mathbf{h}{\mathcal{W}_{\mathrm{loc}}}^\theta$$

as in [MW, Paragraph 3.1].

We have the following generalisations of [MW]:

Theorem 8.2.

(i) $|h\mathcal{W}^\theta| \simeq \Omega^\infty \mathbf{h}\mathcal{W}^\theta$
(ii) $|h\mathcal{V}^\theta| \simeq \Omega^\infty \mathbf{h}\mathcal{V}^\theta$
(iii) $|h\mathcal{W}^\theta_{\mathrm{loc}}| \simeq \Omega^\infty \mathbf{h}{\mathcal{W}_{\mathrm{loc}}}^\theta$
(iv) $|\mathcal{W}^\theta_{\mathrm{loc}}| \simeq \Omega^\infty \mathbf{h}{\mathcal{W}_{\mathrm{loc}}}^\theta$

Proof. Similar to [MW]. □

8.2. **Chapter 4.** In 4.2, we define $\mathcal{W}_\theta^{\mathscr{A}}$ and $h\mathcal{W}_\theta^{\mathscr{A}}$ in the obvious way. These are sheaves of posets.

In 4.3, we define a sheaf $\mathcal{T}_\theta^{\mathscr{A}}$ as in [MW, Definition 4.3.1], but with the added data of a bundle map $q\colon T^\pi E \to U_3$. Notice that this is a small errata to [MW]: Their $\mathcal{T}^{\mathscr{A}}$ should consist of $(\pi,\psi)\colon E \to X \times \mathbb{R}$ such that $\pi\colon E \to X$ is a submersion with *oriented* fibres.

With these modifications, the proof in [MW, Section 4.3] goes through without further difficulties. Thus we get

Theorem 8.3. $|\mathcal{W}^\theta| \simeq |h\mathcal{W}^\theta|$

8.3. **Chapter 5: Surgery.** [MW, Chapter 5.2] is about fibrewise surgery. The idea is roughly as follows. Given a bundle $q\colon M \to X$ of manifolds, a finite set T, a Riemannian vectorbundle $\omega\colon V \to T \times X$ with isometric involution $\rho\colon V \to V$, and an embedding $e\colon D(V^\rho)\times_{T\times X} S(V^{-\rho}) \to M - \partial M$, then one performs surgery by removing the interior of the embedded $D(V^\rho)\times_{T\times X} S(V^{-\rho})$ and replacing it with $S(V^\rho)\times_{T\times X} D(V^{-\rho})$.

In our generalised setting, M will be equipped with a bundle map $\xi\colon T^q M \to U_2$. We would like to perform surgery in a way that we end up with a bundle $\bar q\colon \bar M \to X$, equipped with a bundle map $\bar\xi\colon T^{\bar q}\bar M \to U_2$. We describe how to do this.

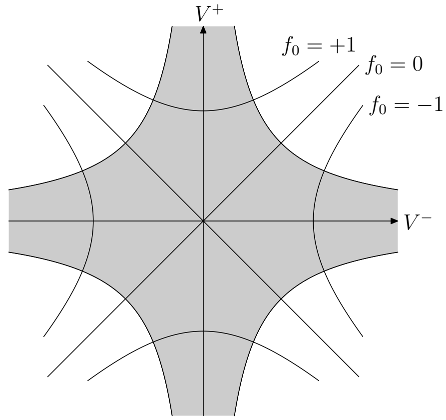
8.3.1. *Saddles.* Choose once and for all a smooth function $\tau\colon [0,1] \to [0,1]$ which is 0 near 0 and 1 near 1. Let Y be a manifold and $\omega\colon V \to Y$ a Riemannian vectorbundle with isometric involution $\rho\colon V \to V$. Let $g\colon Y \to \mathbb{R}$ be smooth. As in [MW] we define the saddle of V to be the subset

$$\mathrm{Sad}(V) = \{v \in V \mid |v_+||v_-| \le 1\}$$

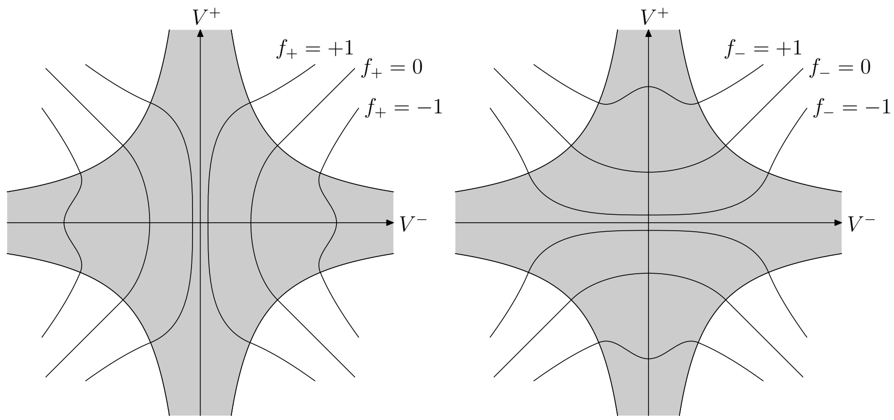
Define three smooth functions by

$$\begin{aligned}
f_0(v) &= g\omega(v) + |v_+|^2 - |v_-|^2\\
f_+(v) &= g\omega(v) + \frac{1}{|v_-|^2}\big(|v_+|^2|v_-|^2\tau(|v_+||v_-|) + (1-\tau(|v_+||v_-|))\big) - |v_-|^2\\
f_-(v) &= g\omega(v) + |v_+|^2 - \frac{1}{|v_+|^2}\big(|v_+|^2|v_-|^2\tau(|v_+||v_-|) + (1-\tau(|v_+||v_-|))\big)
\end{aligned}$$

The map f_0 is defined on all of $\mathrm{Sad}(V)$ and is fibrewise regular except at the zero section of V, where it has a Morse singularity with critical value given by $g\omega$. The maps $f_\pm$ is defined on $\mathrm{Sad}(V) - V^{\pm\rho}$, is fibrewise regular and proper, and agrees with f_0 near $\partial\mathrm{Sad}(V)$. The following picture shows the level curves of f_0 in V. $\mathrm{Sad}(V) \subseteq V$ is the shaded area.



This should be compared with the level curves of f_+ and f_-, shown in the following pictures.



Moreover, f_+ defines a diffeomorphism

$$\begin{aligned} \mathrm{Sad}(V) - V^\rho &\to D(V^\rho) \times_Y S(V^{-\rho}) \times \mathbb{R} \\ v &\mapsto (|v_-|v_+, |v_-|^{-1}v_-, f_+(v)) \end{aligned} \tag{8.1}$$

Similarly, f_- defines a diffeomorphism

$$\begin{aligned} \mathrm{Sad}(V) - V^{-\rho} &\to S(V^\rho) \times_Y D(V^{-\rho}) \times \mathbb{R} \\ v &\mapsto (|v_+|v_+, |v_+|^{-1}v_-, f_-(v)) \end{aligned} \tag{8.2}$$

Remark 8.4. Comparing (8.1) to equation [MW, equation (5.3)] we see that, up to diffeomorphism, the process of removing V^ρ and replacing f with f_+ is equivalent to glueing $D(V^\rho) \times_Y S(V^{-\rho}) \times \mathbb{R}$ to $\mathrm{Sad}(V) - V^\rho$ along [MW, equation (5.3)]. Similarly for (8.2) and [MW, equation (5.4)].

Definition 8.5. Given a vectorbundle $\omega\colon V \to Y$ and a smooth $g\colon Y \to \mathbb{R}$ as above, we let

$$M_+(V,g) = f_+^{-1}(0) \subseteq \mathrm{Sad}(V)$$
$$M_-(V,g) = f_-^{-1}(0) \subseteq \mathrm{Sad}(V)$$

By our earlier remarks we see that both $M_+(V,g)$ and $M_-(V,g)$ agrees near $\partial\mathrm{Sad}(V)$ with $f_0^{-1}(0)$. By restriction of (8.1) we get a diffeomorphism over Y

$$M_+(V,g) \to D(V^\rho) \times_Y S(V^{-\rho})$$

and the fibrewise differential induces an isomorphism

$$T^\omega M_+(V,g) \times \mathbb{R} \to T^\omega V_{|M_+(V,g)}$$

Similarly for $M_-(V,g)$.

This gives an alternative description of surgery. Namely, given a surface bundle $q\colon M \to X$, a finite set T, a Riemannian vectorbundle $V \to T \times X$ with isometric involution $\rho\colon V \to V$, a smooth function $g\colon T \times X \to \mathbb{R}$, and an embedding over X $\lambda\colon M_+(V,g) \to M - \partial M$, then one performs surgery by replacing the embedded $M_+(V,g)$ by $M_-(V,g)$. Since $M_+(V,g)$ and $M_-(V,g)$ agree near their boundary, this gives a welldefined smooth bundle $\bar{q}\colon \bar{M} \to X$. Moreover the following is true. If M is equipped with a bundle map $\xi\colon T^q M \to U_2$ and V is equipped with $\xi\colon T^\omega V_{|\mathrm{Sad}(V)} \to U_3$, and the fibrewise differential of λ is over U_2, then $\bar{M}$ gets a canonical map $T^{\bar{q}}\bar{M} \to U_2$.

8.3.2. *The sheaves.* Keeping these remarks in mind, we make the following definitions. $\mathcal{W}^\theta_{\mathrm{loc},T}(X)$ is the set of

(i) $\omega\colon V \to T \times X$ a Riemannian vector bundle with isometric involution ρ, as in [MW].
(ii) $g\colon T \times X \to \mathbb{R}$ a smooth function.
(iii) $\xi\colon T^\omega V_{|\mathrm{Sad}(V)} \to U_3$ a vectorbundle map

and $\mathcal{W}^\theta_T(X)$ is the set of

(1) $(V, g, \xi) \in \mathcal{W}_{\mathrm{loc},T}(X)$
(2) $q\colon M \to X$ a bundle of surfaces
(3) $\xi\colon T^q M \to U_2$ a bundle map
(4) $e\colon M_+(V,g) \to M - \partial M$ an embedding over X such that the fibrewise differential De is over U_2.

8.3.3. *The proofs.* We go through the definitions and proofs in [MW, Chapter 5] and describe what modifications are needed in this more general situation. Again this is summarised in the diagram

$$\begin{tikzcd}
\mathcal{W}^\theta \arrow[r] & \mathcal{W}^\theta_{\mathrm{loc}} \\
\mathcal{L}^\theta \arrow[r] \arrow[u] & \mathcal{L}^\theta_{\mathrm{loc}} \arrow[u] \\
\operatorname{hocolim} \mathcal{L}^\theta_T \arrow[r] \arrow[u] \arrow[d] & \operatorname{hocolim} \mathcal{L}^\theta_{\mathrm{loc},T} \arrow[u] \arrow[d] \\
\operatorname{hocolim} \mathcal{W}^\theta_T \arrow[r] & \operatorname{hocolim} \mathcal{W}^\theta_{\mathrm{loc},T}
\end{tikzcd} \tag{8.3}$$

8.3.4. *Second row.* Define $\mathcal{L}^\theta_{\mathrm{loc}}(X)$ to be the set of

(i) $(p, g)\colon Y \to X \times \mathbb{R}$ proper smooth maps such that p is etale and graphic and such that g is smooth.
(ii) $\omega\colon V \to Y$ is a Riemannian vectorbundle with isometric involution $\rho\colon V \to V$.
(iii) $\xi\colon T^\omega V_{|\mathrm{Sad}(V)} \to U_3$ a bundle map

and let $\mathcal{L}^\theta(X)$ be the set of

(i) $(\pi, f, \xi) \in \mathcal{W}^\theta(X)$ with $(\pi, f)\colon E \to X \times \mathbb{R}$, $\xi\colon T^\pi E \to U_3$.
(ii) $(p, g, V, \xi) \in \mathcal{L}^\theta_{\mathrm{loc}}(X)$
(iii) $\lambda\colon \mathrm{Sad}(V) \to E - \partial E$ an embedding over $X \times \mathbb{R}$ such that the fibrewise differential $D\lambda$ is over U_3.

The proofs given in [MW] of [MW, Proposition 5.3.3] and [MW, Proposition 5.3.7] goes through with the obvious changes and proves that the sheaf maps $\mathcal{L}^\theta_{\mathrm{loc}} \to \mathcal{W}^\theta_{\mathrm{loc}}$ and $\mathcal{L}^\theta \to \mathcal{W}^\theta$ are weak equivalences.

8.3.5. *Third row.* Let $\mathcal{L}^\theta_{\mathrm{loc},T}(X)$ be the set of

(i) $(p, g, V, \xi) \in \mathcal{L}^\theta_{\mathrm{loc}}(X)$
(ii) $h\colon S \times X \to Y$ an embedding over $3 \times X$
(iii) $\delta\colon Y - \mathrm{Im}(h) \to \{\pm 1\}$ continuous

and let $\mathcal{L}^\theta_T(X)$ be the set of

(i) $(p, g, V, \xi, h, \delta) \in \mathcal{L}^\theta_{\mathrm{loc},T}(X)$
(ii) $(\pi, f, \xi) \in \mathcal{W}^\theta(X)$
(iii) $\lambda\colon \mathrm{Sad}(V) \to E - \partial E$ embedding over $X \times \mathbb{R}$ such that the fibrewise differential $D\lambda$ is over U_3.

The proofs given in [MW] of [MW, Proposition 5.4.2] and [MW, Proposition 5.4.4] goes through with the obvious changes and proves that the sheaf maps $\operatorname{hocolim} \mathcal{L}^\theta_{\mathrm{loc},T} \to \mathcal{L}^\theta_{\mathrm{loc}}$ and $\operatorname{hocolim} \mathcal{L}^\theta_T \to \mathcal{L}^\theta$ are weak equivalences.

8.3.6. *Fourth row, right hand column.* [MW, Lemma 5.5.2] and [MW, Corollary 5.5.3] goes through as in [MW].

8.3.7. *Fourth row left hand column.* This is more technical, and more changes are needed to adapt the proof in [MW]. The modified definitions of $\mathcal{W}^\theta_T$ and $\mathcal{W}^\theta_{\mathrm{loc},T}$ were made with this in mind. The problem is to give a definition of E^{rg} and to define a sheaf map $\mathcal{L}^\theta_T \to \mathcal{W}^\theta_T$, natural in $T \in \mathscr{K}$.

Take an element of $\mathcal{L}^\theta_T(X)$. This consists of $(p, g, V, \xi, h, \delta) \in \mathcal{L}^\theta_{\mathrm{loc},T}(X)$, $(\pi, f, \xi) \in \mathcal{W}^\theta(X)$, and $\lambda\colon \mathrm{Sad}(V) \to E - \partial E$. Define $Y_0, Y_+, Y_- \subseteq Y$ and $V_+, V_-, V_0 \subseteq V$ as in [MW]. Define E^{rg}, f^{rg} in the following way

- On the embedded $\mathrm{Sad}(V_+)$, remove V_+^ρ and replace f by f_+.
- On the embedded $\mathrm{Sad}(V_-)$, remove $V_-^{-\rho}$ and replace f by f_-.
- On the embedded $\mathrm{Sad}(V_0)$, remove V_+^ρ and replace f by f_+.

This defines a bundle $(\pi^{\mathrm{rg}}, f^{\mathrm{rg}})\colon E^{\mathrm{rg}} \to X \times \mathbb{R}$ of smooth compact surfaces. Now let $M = (f^{\mathrm{rg}})^{-1}(0)$. This is a bundle of smooth compact surfaces over X, and is equipped with the following extra structure

(i) A bundle map $\xi\colon T^\pi M \to U_2$
(ii) A Riemannian vectorbundle $\omega\colon h^*V_0 \to T \times X$ with isometric involution ρ.
(iii) A bundle map $\xi\colon T^\omega(h^*V_0)_{|\mathrm{Sad}(V)} \to U_3$.
(iv) A smooth function $g\colon T \times X \to Y \to \mathbb{R}$.
(v) An embedding (over X) $e\colon M_+(V, g) \to M$ such that the fibrewise differential is over U_2.

That is, we have an element of $\mathcal{W}^\theta_T(X)$. This defines a sheaf map $\mathcal{L}^\theta_T \to \mathcal{W}^\theta_T$ which is natural in $T \in \mathscr{K}$. Just as in [MW] one proves that $\mathcal{L}^\theta_T \to \mathcal{W}^\theta_T$ is an equivalence.

8.3.8. *Using the concordance lifting property.* To prove that the sheaf maps $\mathcal{W}^\theta_T \to \mathcal{W}^\theta_{\mathrm{loc},T}$ has the concordance lifting property we need the following lemma

Lemma 8.6. *Let $A \subseteq X$ be a cofibration and let $V \to [0,1]\times X$ be a vectorbundle. Let $U \to B$ be another vectorbundle. Then any bundle map $\xi\colon V_{|\{0\}\times X \cup [0,1]\times X} \to U$ extends to a bundle map $V \to U$*

Proof. Choose a retraction $r\colon [0,1] \times X \to \{0\} \times X \cup [0,1] \times A$. Now the fibre bundle $\mathrm{Iso}(V, r^*V) \to [0,1] \times A$ has a canonical section over $\{0\} \times X \cup [0,1] \times A$. This section extends over all of $[0,1] \times X$ because $\{0\} \times X \cup [0,1] \times A \to [0,1] \times X$ is a trivial cofibration. This section defines a bundle map

$$\begin{array}{ccc} V & \xrightarrow{\hat{r}} & V_{|\{0\}\times X \cup [0,1]\times A} \\ \downarrow & & \downarrow \\ [0,1] \times X & \xrightarrow{r} & \{0\} \times X \cup [0,1] \times A \end{array}$$

and we can compose ξ with $\hat{r}$. $\square$

Proposition 8.7. *The map $\mathcal{W}^\theta_T \to \mathcal{W}^\theta_{\mathrm{loc},T}$ has the concordance lifting property.*

Proof. Let $\chi \in \mathcal{W}^\theta_T$ be an element given by

- $(V,\xi) \in \mathcal{W}^\theta_{\mathrm{loc},T}(X)$ with $\omega\colon V \to T \times X$ and $\xi\colon T^\omega V_{|\mathrm{Sad}(V)} \to U_3$
- $q\colon M \to X$ a surface bundle (with certain boundary conditions).
- $\xi\colon T^q M \to U_2$ a bundle map
- $e\colon M_+(V,g) \to M$ an embedding over X such that the fibrewise differential De is over U_2.

Suppose given a concordance of (V,ξ). This will be given by a vectorbundle $\tilde{\omega}\colon \tilde{V} \to (0,1) \times T \times X$ and $\tilde{\xi}\colon T^\omega \tilde{V}_{|\mathrm{Sad}(\tilde{V})} \to U$. We can choose an isomorphism $\tilde{V} \cong (0,1) \times V$ over $(0,1) \times T \times X$. Put $\tilde{M} = (0,1) \times M$ and $\tilde{q} = (0,1) \times q$. Let $\tilde{g} = g \circ \mathrm{pr}_{T\times X}\colon (0,1) \times T \times X \to \mathbb{R}$. Then we have the isomorphism $M_+(\tilde{V},\tilde{g}) \cong (0,1) \times M_+(V,g)$ and we can set $\tilde{e} = (0,1) \times e\colon M_+(\tilde{V},\tilde{g}) \cong (0,1) \times M_+(V,g) \to (0,1) \times M$.

It remains to define a bundle map $T^{\tilde{q}}\tilde{M} \to U_2$ which is specified on $T^{\tilde{q}}\tilde{M}_{|\{0\}\times M \cup [0,1]\times M_+(V,g)}$. This can be done by the previous lemma, using that $M_+(V,g) \to M$ is a cofibration. □

8.4. **Chapter 6: The connectivity problem.** We describe how to adapt the definition of the sheaf $\mathcal{C}_M$ and prove that it is contractible. Let $\mathbb{R}^2 \times \mathbb{R}$ have the standard euclidean metric and involution $\rho = \mathrm{diag}(1,1,-1)$. For any finite set T and a manifold X we have the trivial vectorbundle $V = \mathbb{R}^2 \times \mathbb{R} \times T \times X$ over $T \times X$ and we have canonical identifications

- $\mathrm{Sad}(V) = \mathrm{Sad}(\mathbb{R}^2 \times \mathbb{R}) \times T \times X$.
- $T^\omega V_{|\mathrm{Sad}(V)} = \mathbb{R}^2 \times \mathbb{R} \times \mathrm{Sad}(V)$
- $D^2 \times S^0 \times T \times X \cong M_+(0) \subseteq \mathrm{Sad}(V)$

Thus to promote V to an element of $\mathcal{W}^\theta_{\mathrm{loc},T}$ with $T \to \{1\}$ we must specify a bundle map $T^\omega V_{|\mathrm{Sad}(V)} \to U_3$, or equivalently a map $\mathrm{Sad}(V) \to \mathrm{Bun}(\mathbb{R}^2 \times \mathbb{R}, U_3)$.

Definition 8.8. Let M be a surface and $TM \to U_2$ a bundle map. Let $\mathcal{C}_{\theta,M}$ be the sheaf whose value at a connected manifold X is the set of

- A finite set T
- A map $\mathrm{Sad}(V) \to \mathrm{Bun}(\mathbb{R}^2 \times \mathbb{R}, U_3)$, where $V = \mathbb{R}^2 \times \mathbb{R} \times T \times X$ as above
- An embedding $e_T\colon M_+(0) \to (M - \partial M) \times X$ over X such that the fibrewise differential De_T is over U_2 and such that surgery along e_T results in a connected surface bundle over X.

We want to prove that $B|\mathcal{C}_{\theta,M}| \simeq |\beta\mathcal{C}^{\mathrm{op}}_{\theta,M}|$ is contractible. We proceed as in [MW]: Given a closed set $A \subseteq X$ and a germ $s_0 \in \mathrm{colim}_U\, \beta\mathcal{C}^{\mathrm{op}}_{\theta,M}(U)$ we extend this germ to an element of $\beta\mathcal{C}^{\mathrm{op}}_{\theta,M}(X)$. The germ s_0 consists of a locally finite open cover $(U_j)_{j\in J}$ of U and objects $\varphi_{RR} \in \mathcal{C}_{\theta,M}(U_R)$ for each finite non-empty $R \subseteq J$, and for each $R \subseteq S$ a morphism $\varphi_{RS}\colon \varphi_{SS} \to \varphi_{RR|U_S}$ satisfying the cocycle condition. Each of the φ_{RR} defines an embedding

$$D^2 \times S^0 \times T_R \times U_R \cong M_+(0) \to (M - \partial M) \times U_R$$

(really there should be one finite set T_R for each component of U_R, but we will suppress this from the notation).

[MW] shows how to extend this to an element of their $\beta\mathcal{C}_M^{\mathrm{op}}(X)$ by choosing contractible open sets $V_j \subseteq X$ and embeddings

$$D^2 \times S^0 \times Q_j \times V_j \to (M - \partial M) \times V_j \qquad (*)$$

and by taking coproducts they get an element of their $\beta\mathcal{C}_M^{\mathrm{op}}(X)$ which restricts to the given germ. To finish the proof that our $\beta\mathcal{C}_{\theta,M}^{\mathrm{op}}(X)$ is contractible we have to promote (*) to an object of our $\beta\mathcal{C}_{\theta,M}^{\mathrm{op}}(V_j)$. This can be done by the next lemma.

Lemma 8.9. *Let M be a surface and $TM \to U_2$ a bundle map. Let X be contractible and let $V = \mathbb{R}^2 \times \mathbb{R} \times T \times X$ be the trivial vectorbundle over $T \times X$. Then for any embedding*

$$e\colon D^2 \times S^0 \times T \times X \to (M - \partial M) \times X$$

over X there exists a bundle map $T^\omega V_{|\mathrm{Sad}(V)} \to U_3$ and a diffeomorphism $h\colon M_+(0) \to D^2 \times S^0 \times T \times X$ such that the fibrewise differential of $e \circ h$ is over U_2.

Proof. First let h be the inverse of the standard diffeomorphism given by (8.1). The requirement that $D(e \circ h)$ is over U_2 defines a unique bundle map $T^\omega V_{|M_+(0)} \to U$, or equivalently a map $M_+(0) \to \mathrm{Bun}(\mathbb{R}^2 \times \mathbb{R}, U_3)$. After possibly composing h with an orientation preserving diffeomorphism of D^2 we can extend this to $M_+(0) \cup (\{0\} \times D^1 \times T \times X)$. Now the inclusion

$$M_+(0) \cup (\{0\} \times D^1 \times T \times X) \to \mathrm{Sad}(V)$$

is a trivial cofibration so we can extend to all of $\mathrm{Sad}(V)$. □

8.5. **Chapter 7: Stabilisation.** This is almost as in [MW]. Start by choosing an element $z \in \mathcal{W}_\emptyset(*)$ of genus 2. This is a torus with two boundary components and with a spin structure. As already explained in paragraph 6, there are two essentially different choices of such tori, but which one we pick is not important for stabilisation.

As in [MW] we get a fibration sequence

$$|z^{-1}h\mathcal{V}| \to \operatorname{hocolim} z^{-1}|\mathcal{W}_T| \to \operatorname{hocolim} z^{-1}|\mathcal{W}_{\mathrm{loc},T}|$$

where $|z^{-1}h\mathcal{V}| \simeq \Omega^\infty \mathbb{T}\mathrm{h}(-U)$ and where

$$\mathrm{hofib}(z^{-1}|\mathcal{W}_T| \to z^{-1}|\mathcal{W}_{\mathrm{loc},T}|) \simeq \mathbb{Z} \times \mathscr{M}^\theta(F_{\infty,2+2|T|}).$$

When the spaces $\mathscr{M}^\theta(F_{\infty,2+2|T|})$ satisfies Harer stability, i.e. if any morphism $S \to T$ in $\mathscr{K}$ induces homology equivalences $\mathscr{M}^\theta(F_{\infty,2+2|T|}) \to \mathscr{M}^\theta(F_{\infty,2+2|S|})$, then the proof in [MW] goes through and proves that

$$\mathbb{Z} \times \mathscr{M}^\infty(F_{\infty,2}) \to \Omega^\infty \mathbb{T}\mathrm{h}(-U)$$

is a homology equivalence.

And we know from [H] and [B] that for $\theta = \theta_{\mathrm{Spin}}$ this Harer stability indeed does hold.

References

[B] T. Bauer: *An infinite loop space structure on the nerve of spin bordism categories*, Quart. J. Math. **55** (2004), 117–133.

[CLM] F. R. Cohen, T. J. Lada, J. P. May: *The Homology of Iterated Loop Spaces*, Lecture Notes in Mathematics **533**, Springer-Verlag, 1976.

[EM] S. Eilenberg, J. C. Moore: *Homology and Fibrations I Coalgebras, cotensor products and its derived functors*, Comm. Math. Helv. **40** (1965), 199–236.

[G] S. Galatius: *Mod p homology of the stable mapping class group*, to appear in Topology.

[GMT] S. Galatius, I. Madsen, U. Tillmann: *Divisibility of the stable Miller-Morita-Mumford classes*, preprint.

[H] J. Harer: *Stability of the homology of moduli spaces of Riemann surfaces with spin structure*, Math. Ann. **287** (1990), 323–334.

[J] Johnson: *Spin structures and quadratic forms on surfaces*, J. London Math. Soc. (2) **22** (1980), 365–373.

[MM] J. Milnor, J. C. Moore: *On the Structure of Hopf algebras*, Ann. Math. **81** (1965), 211–264.

[MMM] B. M. Mann, E. Y. Miller, H. R. Miller: *S^1-equivariant function spaces*, Trans. Amer. Math. Soc. **295** (1989), 233–256.

[MW] I. Madsen, M. Weiss: *The stable moduli space of Riemann surfaces: Mumford's conjecture*, arXiv:math.AT/0212321, Version 2, October 2003.

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DIVISIBILITY OF THE STABLE MILLER-MORITA-MUMFORD CLASSES

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Abstract. We determine the sublattice generated by the Miller-Morita-Mumford classes κ_i in the torsion free quotient of the integral cohomology ring of the stable mapping class group. We further decide when the mod p reductions $\kappa_i \in H^*(B\Gamma_\infty; \mathbb{F}_p)$ vanish.

1. Introduction and results.

Let $\Gamma^s_{g,b}$ denote the mapping class group of a surface of genus g with b ordered boundary components and s marked points. We will supress s or b when their value is zero. Gluing a disk or a torus with two boundary components to one of the boundary components induces homomorphisms

$$\Gamma^s_{g,b-1} \longleftarrow \Gamma^s_{g,b} \longrightarrow \Gamma^s_{g+1,b}. \tag{1.1}$$

Recall that by Harer-Ivanov's stability theory both homomorphisms induce a homology isomorphism in dimensions $*$ with $2*+1 < g$, cf. [H2], [I]. Let $\Gamma_\infty := \lim_{g\to\infty}\Gamma_{g,2}$ be the stable mapping class group.

Mumford in [Mu] introduced certain tautological classes in the cohomology of moduli spaces of Riemann surfaces. Miller [Mi] and Morita [Mo] studied topological analogues. Let $e \in H^2(B\Gamma^1_{g,b}; \mathbb{Z})$ be the Euler class of the central extension

$$\mathbb{Z} \longrightarrow \Gamma_{g,b+1} \longrightarrow \Gamma^1_{g,b} \tag{1.2}$$

which is induced by gluing a disk with a marked point to one of the boundary components. Define

$$\kappa_i := \pi_!(e^{i+1}) \quad \in H^{2i}(B\Gamma_{g,b}; \mathbb{Z})$$

where $\pi_!$ is the Umkehr (or integration along the fibre) map associated to the forgetful map $\Gamma^1_{g,b} \to \Gamma_{g,b}$. These correspond under the maps of (1.1) and hence define classes in $H^*(B\Gamma_\infty; \mathbb{Z})$. We will only be concerned with these stable classes in this paper.

By the solution of the Mumford conjecture [MW],

$$H^*(B\Gamma_\infty; \mathbb{Q}) \simeq \mathbb{Q}[\kappa_1, \kappa_2, \dots].$$

In contrast, little is known about κ_i in integral cohomology. It follows from [H1] that κ_1 is precisely divisible by 12 (cf. [MT, p. 537]). The following theorem

*The third author was supported by an Advanced Fellowship of the EPSRC.

determines the maximal divisor of κ_i in the torsion free cohomology for all $i \geq 1$. We write

$$H^*_{free}(B\Gamma_\infty) := H^*(B\Gamma_\infty; \mathbb{Z})/\text{Torsion}$$

for the integral lattice in $H^*(B\Gamma_\infty; \mathbb{Q})$. This is a Hopf algebra. The graded module of primitive elements $P(H^*_{free}(B\Gamma_\infty))$ is a copy of $\mathbb{Z}$ in each even degree, and κ_i is a primitive element of $H^*_{free}(B\Gamma_\infty)$. However, it is not a generator of $P(H^{2i}_{free}(B\Gamma_\infty))$.

Let D_i be the positive integer defined in terms of its p-adic valuation by the formula

$$\nu_p(D_i) = \begin{cases} 1 + \nu_p(i+1) & \text{if } i+1 \equiv 0 \mod (p-1) \\ 0 & \text{if } i+1 \not\equiv 0 \mod (p-1). \end{cases} \tag{1.3}$$

Theorem 1.1. *The class κ_i is D_i times an additive generator of $P(H^{2i}_{free}(B\Gamma_\infty))$.*

$D_{2i} = 2$ for all i, while the numbers D_{2i-1} are related to Bernoulli numbers as follows. Let

$$\log(\frac{e^x - 1}{x}) = \sum_{i=1}^{\infty} \alpha_i \frac{x^i}{i!}.$$

Then $\alpha_1 = 1/2$, $\alpha_{2i+1} = 0$ and $\alpha_{2i} = (-1)^{i-1} B_i/2i$ where B_i is the i-th Bernoulli number. Then D_{2i-1} is the denominator of α_{2i} when expressed as a fraction in its lowest terms, see e.g. [A2]. So $D_1 = 2^2 \cdot 3$, $D_3 = 2^3 \cdot 3 \cdot 5$, $D_5 = 2^2 \cdot 3^2 \cdot 7, \dots$.

Our Theorem 1.1 is inspired by a conjecture of T. Akita [Ak] which we also prove:

Theorem 1.2. *κ_i as an element in $H^{2i}(B\Gamma_\infty; \mathbb{F}_p)$ vanishes if and only if $i+1 \equiv 0$ mod $(p-1)$.*

The structure of the Hopf algebra $H^*_{free}(B\Gamma_\infty)$ is not completely understood at present, but we have the following partial result. There is an isomorphism of Hopf algebras over the p-adic numbers

$$H^*_{free}(B\Gamma_\infty) \otimes \mathbb{Z}_p \simeq H^*(BU; \mathbb{Z}_p) \tag{1.4}$$

for each odd prime p. Indeed this follows from the main result of [MW], listed in Theorem 2.4 below, and from Theorem 7.8 of [MS] combined with Bott periodicity. The precise structure of $H^*_{free}(B\Gamma_\infty) \otimes \mathbb{Z}_2$ is unknown. Low dimensional calculations reveal that (1.4) fails for $p = 2$: the algebra is not polynomial.

In outline, the proofs of the above theorems depend on previous results as follows. For Theorem 1.2, the proof of the "if" part in Section 2.5 is a calculation using characteristic classes which relies on the fact that there is a map of infinite loop spaces $\alpha : \mathbb{Z} \times B\Gamma^+_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1}$ (compare [T] and [MT] or Theorems 2.1 and 2.2 below). The "only if" part is implied by Theorem 1.1: if p divides κ_i then in particular it must divide its reduction to the free part.

For Theorem 1.1, the main theorem of [MT] (here Theorem 2.2), implies that the maximal divisor of κ_i as an element in $H^{2i}_{free}(B\Gamma_\infty)$ is less or equal D_i. For i even we have $D_i = 2$, and so the reverse inequality is implied by the "if" part of Theorem 1.2. For i odd, the reverse inequality follows readily by a well-known relation between the κ_i classes and the symplectic characteristic classes for surface bundles, (stated as Theorem 3.2), albeit up to a factor of 2. To eliminate this indeterminacy the main theorem of [MW] (here Theorem 2.4), as well as calculations from [G1] (see (3.3)) and a stronger version of the main result of [MT] (given in Theorem 3.5) are used.

Given the interest in the mapping class groups also outside the topology community we have strived to make this paper as self contained as possible. In particular we have spelled out some of the more obscure parts of [MT].

2. The kappa classes and spectrum cohomology.

2.1. Universal surface bundles. The methods used in this and the surrounding papers do not use the mapping class groups directly but rather the topological groups of orientation preserving diffeomorphisms of surfaces. We briefly review the correspondence.

Let $F_{g,b}$ be a connected surface of genus g with b boundary circles. We write $\mathrm{Diff}(F_{g,b};\partial)$ for the topological group of orientation preserving diffeomorphisms that keep (a neighborhood of) the boundary pointwise fixed. For $g \geq 2$, results from [EE] and [ES] yield

$$B\Gamma_{g,b} \simeq B\mathrm{Diff}(F_{g,b};\partial)$$

so that $B\Gamma_{g,b}$ classifies diffeomorphism classes of smooth fibre bundles $\pi : E \to X$ with fibre $F_{g,b}$ and standard boundary behavior:

$$\partial E = X \times \sqcup_1^b S^1, \quad \pi|\partial E = \mathrm{proj}_X.$$

Similarly,

$$B\Gamma^s_{g,b} \simeq B\mathrm{Diff}(F_{g,b};\partial \sqcup \{x_1,\dots,x_s\})$$

where $x_1,\dots,x_s$ are distinct interior points of $F_{g,b}$. Take $s=1$. Since $\mathrm{Diff}(F_{g,b};\partial)$ acts transitively on the interior of $F_{g,b}$,

$$E(F_{g,b}) := E\mathrm{Diff}(F_{g,b};\partial) \times_{\mathrm{Diff}(F_{g,b};\partial)} F_{g,b} \simeq B\mathrm{Diff}(F_{g,b};\partial \sqcup \{x\}) \simeq B\Gamma^1_{g,b}.$$

The forgetful map $\pi : B\Gamma^1_{g,b} \to B\Gamma_{g,b}$ corresponds to the universal smooth $F_{g,b}$ bundle

$$F_{g,b} \longrightarrow E(F_{g,b}) \longrightarrow B\mathrm{Diff}(F_{g,b};\partial). \tag{2.1}$$

The central extension (1.2) is classified by "the differential at x",

$$\mathrm{Diff}(F_{g,b};\partial \sqcup \{x\}) \longrightarrow \mathrm{GL}^+(T_xF_{g,b}) \simeq SO(2).$$

Hence the circle bundle induced from (1.2) by applying the classifying space functor corresponds to the circle bundle of the vertical tangent bundle associated with (2.1).

2.2. Spectra and spectrum cohomology. Let $E = \{E_n, \epsilon_n\}$ be a CW-spectrum[1] in the sense of [A1]: E_n is a sequence of pointed CW-complexes and $\epsilon_n : SE_n \to E_{n+1}$ a (pointed) isomorphism onto a subcomplex, where $S(-)$ denotes suspension. The associated infinite loop space is the direct limit

$$\Omega^\infty E = \operatorname{colim} \Omega^n E_n$$

of the n-th loop space of E_n; the limit is taken over the adjoint maps $\epsilon'_n : E_n \to \Omega E_{n+1}$.

The k-th homotopy group of E is defined to be the direct limit of $\pi_{n+k}(E_n)$. It is equal to the k-th homotopy group of the space $\Omega^\infty E$. In particular, the group of components of $\Omega^\infty E$ is the direct limit of $\pi_n(E_n)$. For $\alpha \in \pi_0(\Omega^\infty E)$ we let $\Omega^\infty_\alpha E$ be the component determined by α. In particular we write $\Omega^\infty_0 E$ for the component of the zero element.

The homology and cohomology groups of E are

$$H^k(E) = \lim \tilde{H}^{k+n}(E_n), \; H_k(E) = \lim \tilde{H}_{k+n}(E_n)$$

where the limits are induced from the maps ϵ_n together with the suspension isomorphisms. In contrast to homotopy groups the cohomology groups of a spectrum are usually much simpler than the cohomology groups of $\Omega^\infty E$.

The evident evaluation map from $S^n \Omega^n E_n$ to E_n induces maps

$$\sigma^* : H^*(E) \longrightarrow \tilde{H}^*(\Omega^\infty_0 E), \;\; \sigma_* : \tilde{H}_*(\Omega^\infty_0 E) \longrightarrow H_*(E). \tag{2.3}$$

If we use field coefficients in the cohomology groups then $H^*(\Omega^\infty_0 E)$ is a connected Hopf algebra and the image of σ^* is contained in the graded vector space $PH^*(\Omega^\infty_0 E)$ of primitive elements. We shall be particularly concerned with the torsion free integral homology and cohomology groups

$$H^*_{free}(\Omega^\infty_0 E) = H^*(\Omega^\infty_0 E; \mathbb{Z})/\text{Torsion}, \quad H^{free}_*(\Omega^\infty_0 E) = H_*(\Omega^\infty_0 E; \mathbb{Z})/\text{Torsion}.$$

They are lattices in $H^*(\Omega^\infty_0 E; \mathbb{Q})$ and $H_*(\Omega^\infty_0 E; \mathbb{Q})$ and are dual Hopf algebras. Moreover, the image of σ^* is contained in the module of primitive elements

$$\sigma^* : H^*_{free}(E) \longrightarrow P(H^*_{free}(\Omega^\infty_0 E)),$$

and dually σ_* factors over the indecomposable elements of $H^{free}_*(\Omega^\infty_0 E)$.

Given a pointed space X we have the associated suspension spectrum $S^\infty X$ whose n-th term is $S^n X$ with infinite loop space $\Omega^\infty S^\infty X$. There is an obvious inclusion $i : X \to \Omega^\infty S^\infty X$ inducing a splitting of σ^* (and σ_*):

$$H^*(S^\infty X) \xrightarrow{\sigma^*} \tilde{H}^*(\Omega^\infty_0 S^\infty X) \xrightarrow{i^*} \tilde{H}^*(X) \tag{2.4}$$

is the suspension isomorphism.

The spectra of most relevance to us are $\mathbb{C}P^\infty_{-1}$ and the suspension spectrum $S^\infty \mathbb{C}P^\infty_+$ of $\mathbb{C}P^\infty \sqcup \{+\}$. We recall the definition of the former. There are two

[1] If one does not assume the spaces to be CW-complexes then one should assume that ϵ_n is a closed cofibration.

complex vector bundles over the complex projective n-space $\mathbb{C}P^n$, namely the tautological line bundle L_n and its n-dimensional complement $L_n^\perp$ in $\mathbb{C}P^n \times \mathbb{C}^{n+1}$. Its Thom space (or one point compactification) is denoted by $\mathrm{Th}(L_n^\perp)$. Since the restriction of $L_n^\perp$ to $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ is equal to $L_{n-1}^\perp \oplus \mathbb{C}$ where $\mathbb{C}$ denotes the trivial line bundle over $\mathbb{C}P^{n-1}$ we get a map

$$\epsilon : S^2\mathrm{Th}(L_{n-1}^\perp) \longrightarrow \mathrm{Th}(L_n^\perp).$$

The spectrum $\mathbb{C}P_{-1}^\infty$ has

$$(\mathbb{C}P_{-1}^\infty)_{2n} = \mathrm{Th}(L_{n-1}^\perp), \qquad (\mathbb{C}P_{-1}^\infty)_{2n+1} = S\mathrm{Th}(L_{n-1}^\perp)$$

and the structure map ϵ_{2n+1} is given by the above ϵ. The associated infinite loop space is

$$\Omega^\infty \mathbb{C}P_{-1}^\infty = \operatorname{colim}_\epsilon \Omega^{2n}\mathrm{Th}(L_{n-1}^\perp).$$

The inclusion of $L_{n-1}^\perp$ into $L_{n-1}^\perp \oplus L_{n-1} = \mathbb{C}P^{n-1} \times \mathbb{C}^n$ via the zero section of L_{n-1} induces a map from $\mathrm{Th}(L_{n-1}^\perp)$ into $S^{2n}(\mathbb{C}P_+^{n-1})$ and hence a map

$$\omega : \Omega^\infty \mathbb{C}P_{-1}^\infty \longrightarrow \Omega^\infty S^\infty(\mathbb{C}P_+^\infty).$$

This map fits into a fibration sequence

$$(2.5) \qquad \Omega^\infty \mathbb{C}P_{-1}^\infty \xrightarrow{\omega} \Omega^\infty S^\infty(\mathbb{C}P_+^\infty) \xrightarrow{\partial} \Omega^\infty S^{\infty-1}$$

where the right-hand term is the direct limit of $\Omega^n S^{n-1}$. Indeed the inclusion of a fibre $\mathbb{C}^n \to L_n^\perp$ induces a map $S^{2n} \to \mathrm{Th}(L_n^\perp)$, and a map of its cofibre $\mathrm{Th}(L_n^\perp)/S^{2n} \to \mathrm{Th}(L_n^\perp \oplus L_n)$. One checks on cohomology that this map is $(4n+1)$-connected and gets a cofibre sequence of spectra $S^\infty(S^{-2}) \to \mathbb{C}P_{-1}^\infty \to S^\infty \mathbb{C}P_+^\infty \to S^\infty(S^{-1})$. (2.5) is the associated fibration sequence of infinite loop spaces.

The component groups of (2.5) are

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\pi_0(\omega)} \mathbb{Z} \xrightarrow{\pi_0(\partial)} \mathbb{Z}/2 \longrightarrow 0$$

so $\pi_0(\omega)$ is multiplication by ± 2, depending on the choice of generators. There is a canonical splitting of infinite loop spaces

$$(2.6) \qquad \Omega^\infty S^\infty(\mathbb{C}P_+^\infty) = \Omega^\infty S^\infty(\mathbb{C}P^\infty) \times \Omega^\infty S^\infty.$$

We fix the generator of $\pi_0 \Omega^\infty S^\infty(\mathbb{C}P_+^\infty)$ to be the element that maps to $+1$ under the isomorphisms

$$\pi_0(\Omega^\infty S^\infty(\mathbb{C}P_+^\infty)) \xrightarrow{\pi_0(c)} \pi_0(\Omega^\infty S^\infty) \xrightarrow{\text{degree}} \mathbb{Z},$$

where c collapses $\mathbb{C}P^\infty$ to the non-base point of S^0. We fix the generator of $\pi_0(\Omega^\infty \mathbb{C}P_{-1}^\infty)$ so that $\pi_0(\omega)$ is multiplication by -2.

2.3. Review of results used. Our divisibility result of theorem 1.1 is based upon the following three theorems.

Theorem 2.1. **[T].** *The spaces* $\mathbb{Z} \times B\Gamma_\infty^+$ *and* $B\Gamma_\infty^+ (= \{0\} \times B\Gamma_\infty^+)$ *are infinite loop spaces.*

Here the superscript $(+)$ denotes Quillen's plus construction, cf. [B]. For each prime p we pick a positive integer $k = k(p)$ so that $-k$ reduces to a generator of the units $(\mathbb{Z}/p^2)^\times$ when p is odd. We pick $k = 3$ when $p = 2$. Write ψ^{-k} for the self map of $\mathbb{C}P^\infty$ that multiplies by $-k$ on the second cohomology group. Composing with the inclusion into $\Omega^\infty S^\infty(\mathbb{C}P^\infty)$ and using the loop sum we have a map

$$1 + k\psi^{-k} : \mathbb{C}P^\infty \longrightarrow \Omega^\infty S^\infty(\mathbb{C}P^\infty),$$

and, since the target is an infinite loop space, an extension to a self map of $\Omega^\infty S^\infty(\mathbb{C}^\infty P^\infty)$, again denoted $1 + k\psi^{-k}$.

Theorem 2.2. **[MT].** *There are infinite loop maps*

$$\alpha : \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow \Omega^\infty \mathbb{C}P_{-1}^\infty, \quad \mu_p : \Omega^\infty S^\infty(\mathbb{C}P_+^\infty) \longrightarrow (\mathbb{Z} \times B\Gamma_\infty^+)_p^\wedge$$

such that the composition $\omega \circ \alpha \circ \mu_p$ *and the self map*

$$\begin{pmatrix} 1 + k\psi^{-k} & * \\ 0 & -2 \end{pmatrix} : \mathbb{C}P^\infty \times \Omega^\infty S^\infty \longrightarrow \mathbb{C}P^\infty \times \Omega^\infty S^\infty$$

become homotopic after p*-adic completion.*

The reader is referred to [BK] for the notion of p-adic completion (also called $\mathbb{F}_p$-completion). For infinite loop spaces $\Omega^\infty E$ of finite type one has

$$[X, (\Omega^\infty E)_p^\wedge] = [X, \Omega^\infty E] \otimes \mathbb{Z}_p, \quad H^*((\Omega^\infty E)_p^\wedge; \mathbb{Z}) = H^*(\Omega^\infty E; \mathbb{Z}_p).$$

Remark 2.3. The homotopy class of the map α in Theorem 2.2 is uniquely determined by its composition with $\mathbb{Z} \times B\Gamma_\infty \to \mathbb{Z} \times B\Gamma_\infty^+$. Indeed since $\Omega^\infty \mathbb{C}P_{-1}^\infty$ is an infinite loop space the induced map

$$[\mathbb{Z} \times B\Gamma_\infty^+, \Omega^\infty \mathbb{C}P_{-1}^\infty] \longrightarrow [\mathbb{Z} \times B\Gamma_\infty, \Omega^\infty \mathbb{C}P_{-1}^\infty]$$

is an isomorphism. This is a standard property of the plus construction.

Theorem 2.4. **([MW]).** *The map* α *is a homotopy equivalence.*

We may view $\Gamma_{g,2}$ as the mapping class group of surfaces with one incoming and one outgoing boundary component. Gluing the incoming boundary component of one surface to the outgoing component of the other defines a map

$$\Gamma_{g,2} \times \Gamma_{h,2} \longrightarrow \Gamma_{g+h,2}$$

and a corresponding map of classifying spaces that makes the disjoint union $\bigsqcup B\Gamma_{g,2}$ over all $g \geq 0$ into a topological monoid. Consider the map

$$\bigsqcup_{g\geq 0} B\Gamma_{g,2} \longrightarrow \mathbb{Z} \times B\Gamma_\infty^+$$

that sends $B\Gamma_{g,2}$ into the component $\{g\} \times B\Gamma_\infty^+$ by the stabilization map (1.1) followed by the map into the plus construction. The infinite loop space structure on $\mathbb{Z} \times B\Gamma_\infty^+$ is compatible with the monoidal structure on $\bigsqcup B\Gamma_{g,2}$, and the induced map

$$\Omega B(\bigsqcup_{g\geq 0} B\Gamma_{g,2}) \longrightarrow \mathbb{Z} \times B\Gamma_\infty^+$$

is a homotopy equivalence. (Indeed, the monoid is a subcategory of the symmetric monoidal surface category $\mathcal{S}$ defined in [T], and the proof the homotopy equivalence $\Omega B\mathcal{S} \simeq \mathbb{Z} \times B\Gamma_\infty^+$ in [T] easily adopts to this subcategory; cf. also proposition 4.1 of [T].)

2.4. The map α and the kappa classes. The map $\alpha : \mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1}$ constructed in Section 2 of [MT] restricts to a map $\alpha_{g,2} : B\Gamma_{g,2} \to \Omega^\infty_g \mathbb{C}P^\infty_{-1}$ that is homotopic to the composition

$$\alpha_{g,2} : B\Gamma_{g,2} \longrightarrow B\Gamma_{g+1} \xrightarrow{\alpha_{g+1}} \Omega^\infty_g \mathbb{C}P^\infty_{-1}. \tag{2.7}$$

The left hand map is induced from gluing the two parametrized boundary circles together.

We next recall a description of α_{g+1} which is well-suited for identifying the kappa classes. Let $\pi : E \to X$ be a smooth surface bundle with closed fiber F. Thus $E = P \times_{\mathrm{Diff}(F)} F$ where P is a principal $\mathrm{Diff}(F)$ bundle over X. We do not assume that X is smooth or finite dimensional, only that X is paracompact (or a CW-complex).

We denote by $\mathrm{Emb}(F, \mathbb{R}^n)$ the space of smooth embeddings in the C^∞-topology, and let $\mathbb{R}^\infty$ and $\mathrm{Emb}(F, \mathbb{R}^\infty)$ be the colimits of $\mathbb{R}^n$ and $\mathrm{Emb}(F, \mathbb{R}^n)$, respectively. We shall consider fiberwise embeddings $e : E \to X \times \mathbb{R}^\infty$, that is, fiberwise maps such that each $e_x : E_x \to \{x\} \times \mathbb{R}^\infty$ is an embedding and such that the adjoint map $P \to \mathrm{Emb}(F, \mathbb{R}^\infty)$ is continuous. Such an e is equivalent to a section of $P \times_{\mathrm{Diff}(F)} \mathrm{Emb}(F, \mathbb{R}^\infty)$.

An embedding $e_x : F \hookrightarrow \mathbb{R}^{n+2}$ extends to a map from the normal bundle $N^n e_x = \{(p, v) | v \perp T_p F\}$ into $\mathbb{R}^{n+2}$ by sending (p, v) to $e_x(p) + v$. We call the embedding e_x *fat* if this map restricts to an embedding of the unit disk bundle $D(N^n e_x)$. The subspace of fat embeddings $\mathrm{Emb}^f(F, \mathbb{R}^\infty) \subset \mathrm{Emb}(F, \mathbb{R}^\infty)$ is contractible, since the inclusion is a homotopy equivalence by the tubular neighborhood theorem and since $\mathrm{Emb}(F, \mathbb{R}^\infty)$ is contractible by Whitney's embedding theorem. A fibrewise fat embedding $e : E \to X \times \mathbb{R}^\infty$ is then a section of the fibre bundle $P \times_{\mathrm{Diff}(F)} \mathrm{Emb}^f(F, \mathbb{R}^\infty)$.

Suppose first that $e : E \to X \times \mathbb{R}^{n+2}$ is a fibrewise fat embedding of codimension n. The Pontryagin-Thom construction associates a "collapse " map onto the Thom space of the fibrewise normal bundle,

$$c_{\pi,e} : X_+ \wedge S^{n+2} \longrightarrow D(N^n_\pi e)/S(N^n_\pi e) = \mathrm{Th}(N^n_\pi e).$$

We are particularly interested in its adjoint map $X \to \Omega^{n+2}\mathrm{Th}(N_\pi^n e)$.

Let $G(2,n)$ be the Grassmann manifold of oriented 2-dimensional subspaces of $\mathbb{R}^{n+2}$, and let U_n and $U_n^\perp$ be the two complementary bundles over it of dimension 2 and n, respectively. The fat embedding induces bundle maps

$$T_\pi E \longrightarrow U_n, \quad N_\pi^n E \longrightarrow U_n^\perp$$

and a commutative diagram

$$\begin{array}{ccccc} X_+ \wedge S^{n+2} & \xrightarrow{c_{\pi,e}} & \mathrm{Th}(N_\pi^n E) & \xrightarrow{s} & \mathrm{Th}(T_\pi E \oplus N_\pi^n E) \\ \| & & \downarrow & & \downarrow \\ X_+ \wedge S^{n+2} & \xrightarrow{c_n} & \mathrm{Th}(U_n^\perp) & \longrightarrow & \mathrm{Th}(U_n \oplus U_n^\perp). \end{array} \tag{2.8}$$

In the general case of a fiberwise fat embedding $e : E \to X \times \mathbb{R}^\infty$, the base space X is the colimit of the subspaces

$$X_n := \{x \in X | e_x(E_x) \subset \{x\} \times \mathbb{R}^{n+2}\},$$

and the diagram

$$\begin{array}{ccc} (X_n)_+ \wedge S^{n+2} & \longrightarrow & \mathrm{Th}(U_n^\perp) \\ \downarrow & & \downarrow \\ (X_{n+1})_+ \wedge S^{n+3} & \longrightarrow & \mathrm{Th}(U_{n+1}^\perp) \end{array}$$

is commutative since $U_{n+1}^\perp|_{G(2,n)} = U_n^\perp$. Taking adjoins we get

$$\alpha_{\pi,e} : X \longrightarrow \operatorname{colim} \Omega^{n+2}\mathrm{Th}(U_n^\perp).$$

Since $\mathrm{Emb}^f(F, \mathbb{R}^\infty)$ is contractible, all sections of $P \times_{\mathrm{Diff}(F)} \mathrm{Emb}^f(F, \mathbb{R}^\infty)$ are homotopic, and consequently the homotopy class $[\alpha_{\pi,e}]$ is independent of the choice of e.

Realification gives a $(2n-1)$ connected map from $\mathbb{C}P^n$ into the oriented Grassmannian $G(2,2n)$ covered by a bundle map $L_n^\perp \to U_{2n}^\perp$. Thus $G(2,\infty) \simeq \mathbb{C}P^\infty$ and

$$\Omega^\infty \mathbb{C}P^\infty_{-1} = \operatorname{colim} \Omega^{2n+2}\mathrm{Th}(L_n^\perp) \xrightarrow{\simeq} \operatorname{colim} \Omega^{2n+2}\mathrm{Th}(U_{2n}^\perp)$$

is a homotopy equivalence. Altogether we have a well-defined homotopy class

$$\alpha_\pi : X \longrightarrow \Omega^\infty \mathbb{C}P^\infty_{-1}.$$

For $x = B\mathrm{Diff}(F_{g+1}) \simeq B\Gamma_{g+1}$ this is the map α_{g+1} of (2.7).

Let us check that the image of α_{g+1}, and hence the image of $\alpha_{g,2}$, lie in the g-component of $\Omega^\infty \mathbb{C}P^\infty_{-1}$, or equivalently that the composition

$$\mathrm{proj} \circ \omega \circ \alpha_{g+1} : B\Gamma_{g+1} \longrightarrow \Omega^\infty S^\infty(\mathbb{C}P^\infty_+) \longrightarrow \Omega^\infty S^\infty$$

lands in the $-2g$ component (with identification of components chosen at the end of Section 2.2). Consider (2.8) with X a single point and $E = F_{g+1}$. The component is given by

$$\begin{aligned} < c_\pi^* s^*(U_T \cdot U_N), [S^{2n+2}] > &=< c_\pi^*(e(TF_{g+1}) \cdot U_N, [S^{2n+2}] > \\ &=< e(TF_{g+1}), [F_{g+1}] >= -2g, \end{aligned}$$

as claimed.

Next we compute the maps $\alpha_{g,2}$ and $\alpha_{g+1,2}$ under the map $B\Gamma_{g,2} \to B\Gamma_{g+1,2}$ induced from gluing a torus with two boundary circles $F_{1,2}$ to $F_{g,2}$. Considering $F_{1,2}$ as a fibre bundle over a point the construction above gives an element $[1] \in \Omega_1^\infty \mathbb{C}P_{-1}^\infty$. Loop sum with $[1]$ in $\Omega^\infty \mathbb{C}P_{-1}^\infty$ translates the g component into the $(g+1)$ component and

$$\begin{array}{ccc} B\Gamma_{g,2} & \xrightarrow{\alpha_{g,2}} & \Omega_g^\infty \mathbb{C}P_{-1}^\infty \\ \downarrow & & \downarrow{\scriptstyle *[1]} \\ B\Gamma_{g+1,2} & \xrightarrow{\alpha_{g+1,2}} & \Omega_{g+1}^\infty \mathbb{C}P_{-1}^\infty \end{array} \tag{2.9}$$

is homotopy commutative. To see this observe that the left hand map in (2.7) is homotopic to the composition

$$B\mathrm{Diff}(F_{g,2};\partial) \longrightarrow B\mathrm{Diff}(F_{g+1}; S^1 \times I) \longrightarrow B\mathrm{Diff}(F_{g+1})$$

where the middle space classifies F_{g+1}-bundles that are trivialized on a band $S^1 \times I \subset F_{g+1}$. The diagram (2.9) then factors as

$$\begin{array}{ccccc} B\mathrm{Diff}(F_{g,2};\partial) & \longrightarrow & B\mathrm{Diff}(F_{g+1}; S^1 \times I) & \longrightarrow & \Omega_g^\infty \mathbb{C}P_{-1}^\infty \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle *[1]} \\ B\mathrm{Diff}(F_{g+1,2};\partial) & \longrightarrow & B\mathrm{Diff}(F_{g+2}) & \longrightarrow & \Omega_{g+1}^\infty \mathbb{C}P_{-1}^\infty, \end{array}$$

where the middle hand vertical map replaces the band $S^1 \times I$ by the torus $F_{1,2}$ with two boundary circles. It is evident from the construction of α_π that the right hand square homotopy commutes, and hence that (2.9) is homotopy commutative.

The restriction of α in Theorem 2.2 to the zero component is then

$$[\tilde{\alpha}] = \lim\,([\tilde{\alpha}_{g,2}]) \in [B\Gamma_\infty, \Omega_0^\infty \mathbb{C}P_{-1}^\infty],$$

where $\tilde{\alpha}_{g,2} = (*[-g]) \circ \alpha_{g,2}$.

We can now relate the kappa classes to spectrum cohomology. Consider

$$B\Gamma_{g,2} \xrightarrow{\tilde{\alpha}_{g,2}} \Omega_0^\infty \mathbb{C}P_{-1}^\infty \xrightarrow{\omega} \Omega_0^\infty S^\infty(\mathbb{C}P_+^\infty)$$

and recall the cohomology suspension from Section 2.2

$$\sigma^* : \tilde{H}^{2i}(\mathbb{C}P^\infty;\mathbb{Z}) \simeq H^{2i}(S^\infty \mathbb{C}P_+^\infty) \longrightarrow H^{2i}(\Omega_0^\infty S^\infty(\mathbb{C}P_+^\infty)).$$

Theorem 2.5. *The Miller-Morita-Mumford class κ_i is equal to $(\sigma \circ \omega \circ \tilde{\alpha}_{g,2})^*(e^i)$ where $e \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ is the Euler class of the canonical line bundle.*

Proof: Let $\pi : E \to X$ be a smooth fibre bundle with fibre F_{g+1}, classified by $f_\pi : X \to B\Gamma_{g+1}$. By definition

$$f_\pi^*(\kappa_i) = \pi_!(e(T_\pi E)^{i+1}) \in H^{2i}(X;\mathbb{Z})$$

where $\pi_!$ is the composition

$$\tilde{H}^{2i+2}(E;\mathbb{Z}) \xrightarrow{\simeq} \tilde{H}^{2i+2n+2}(\mathrm{Th}(N_\pi E);\mathbb{Z}) \xrightarrow{c^*_{\pi,e}} H^{2i+2n+2}(S^{2n+2}(X_+);\mathbb{Z})$$

followed by the $(2n+2)$-nd desuspension, where $c_{\pi,e}$ is as in (2.8). Moreover, for $x \in H^*(\mathbb{C}P^n;\mathbb{Z}) = H^*(G(2,2n);\mathbb{Z})$ (n large)

$$s^*(\Sigma^{2n+2}(x)) = s^*(\lambda_{L_n^\perp} \cdot \lambda_{L_n} \cdot x) = \lambda_{L_n^\perp} \cdot e(L_n) \cdot x.$$

Here the λ's denote the cohomology Thom classes. Apply (2.8) and take $x = e^i$ to complete the proof. □

2.5. Characteristic classes and one part of Akita's conjecture. Let $H(\mathbb{Z},k)$ denote the Eilenberg-MacLane space with non-trivial homotopy $\mathbb{Z}$ in dimension k. The Thom class $\lambda_n = \lambda_{L_n^\perp}$ is represented by a map from $\mathrm{Th}(L_n^\perp)$ to $H(\mathbb{Z},2n)$. We let Y_{2n+2} be its homotopy fibre, so that there is a fibration sequence

$$Y_{2n+2} \xrightarrow{j_{2n+2}} \mathrm{Th}(L_n^\perp) \xrightarrow{\lambda_n} H(\mathbb{Z},2n).$$

The spaces Y_{2n+2} are the $(2n+2)$-nd terms of a spectrum Y and the j_{2n+2} define a map $j : Y \to \mathbb{C}P^\infty_{-1}$ of spectra. Since $\Omega^{2n+2}H(\mathbb{Z},2n)$ has vanishing homotopy groups

$$\Omega^\infty Y \xrightarrow{\Omega^\infty j} \Omega^\infty \mathbb{C}P^\infty_{-1} \tag{2.10}$$

is a (weak) homotopy equivalence.

For our next theorem we need the relation between Steenrod operations and Stiefel-Whitney classes and their mod p analogues. Recall the i-th Steenrod operation:

$$P^i : H^k(X;\mathbb{F}_p) \longrightarrow H^{k+2i(p-1)}(X;\mathbb{F}_p), \quad p \text{ odd},$$
$$Sq^i : H^k(X,\mathbb{F}_2) \longrightarrow H^{k+i}(X,\mathbb{F}_2).$$

Let E be an oriented vector bundle over X and λ_E its cohomology Thom class. One defines $v_i(E) \in H^{2i(p-1)}(X;\mathbb{F}_2)$, respectively $v_i(E) \in H^i(X;\mathbb{F}_2)$ by

$$P^i(\lambda_E) = v_i(E)\lambda_E, \quad Sq^i(\lambda_E) = v_i(E)\lambda_E.$$

For an oriented 2-plane bundle (or complex line bundle) L,

$$\begin{aligned} v_1(L) = e(L)^{p-1} \qquad\qquad & \text{for } p \text{ odd}, \\ v_2(L) = e(L) \quad \text{and} \quad v_1(L) = 0 \quad & \text{for } p = 2. \end{aligned}$$

Moreover, the total class

$$v(E) = 1 + v_1(e) + v_2(E) + \cdots \in H^*(X;\mathbb{F}_p)$$

takes direct sums of oriented vector bundles into (graded) products.

Theorem 2.6. *The modulo p reduction of* $\kappa_i \in H^{2i}(B\Gamma_\infty;\mathbb{F}_p)$ *is zero when* $i+1 \equiv 0\,(mod\ p-1)$.

Proof:[2] By (2.10) the bottom vertical map in the commutative diagram

$$\begin{array}{ccc} H^{2i}(\mathbb{C}P^\infty_{-1};\mathbb{F}_p) & \xrightarrow{j^*} & H^{2i}(Y,\mathbb{F}_p) \\ \sigma^*\big\downarrow & & \sigma^*\big\downarrow \\ H^{2i}(\Omega^\infty_0\mathbb{C}P^\infty_{-1};\mathbb{F}_p) & \xrightarrow{(\Omega^\infty j)^*} & H^{2i}(\Omega^\infty Y;\mathbb{F}_p) \end{array}$$

is an isomorphism. Theorem 2.5 shows that

$$\sigma^*(e^{i+1}\lambda_{L^\perp}) = \kappa_i,$$

so it suffices to prove that j^* vanishes when $i+1 \equiv 0\,(\mathrm{mod}\,p-1)$. Equivalently, we must show that $e^{i+1}\lambda_n$ is in the image of

$$\lambda_n^* : H^*(H(\mathbb{Z},2n);\mathbb{F}_p) \longrightarrow H^*(Th(L_n^\perp);\mathbb{F}_p)$$

in the stated dimensions. This is implied by

$$v(L_n^\perp) = v(L_n)^{-1} = \begin{cases} (1+e^{p-1})^{-1}, & p>2 \\ (1+e)^{-1}, & p=2 \end{cases}$$

Since $P^i(\lambda_n) \in$ image (λ_n^*) the result follows. □

Remark 2.7. The relation $P^i(\lambda_{L^\perp}) = \kappa_{i(p-1)-1}\lambda_{L^\perp}$ used above is further exploited in [G2] to define secondary classes μ_i with $p\mu_i = \kappa_{i(p-1)-1}$ in cohomology with $\mathbb{Z}/p^2\mathbb{Z}$ coefficients.

Theorem 2.6 proves half of Akita's conjecture mentioned in the introduction. The other half is implied by Theorem 1.1 which is proved below.

3. Proof of the main theorem.

3.1. Segal's splitting. Given an infinite loop space $B = \Omega^\infty E$ the inclusion $i : B \to \Omega^\infty S^\infty B$ admits a canonical retraction

$$\theta : \Omega^\infty S^\infty B \longrightarrow B.$$

The inclusion of $\mathbb{C}P^\infty$ in BU that represents the reduced canonical line bundle (of virtual dimension zero) extends to a map

$$l : \Omega^\infty S^\infty(\mathbb{C}P^\infty) \longrightarrow \Omega^\infty S^\infty(BU) \xrightarrow{\theta} BU$$

where we have used that BU is an infinite loop space via Bott periodicity. We shall need

[2] We thank John Rognes for this proof; it replaces a more cumbersome earlier argument.

Theorem 3.1 ([S]). *The map l has a left inverse up to homotopy. In the resulting decomposition*

$$\Omega^\infty S^\infty(\mathbb{C}P^\infty) \simeq BU \times Fib(l)$$

the homotopy fiber $Fib(l)$ has vanishing rational cohomology.

In particular, this gives an identification of Hopf algebras

$$H^*_{free}(\Omega^\infty S^\infty(\mathbb{C}P^\infty)) \simeq H^*(BU;\mathbb{Z}) \tag{3.1}$$

induced by l^*. The graded module of primitive elements of the right hand side is a copy of $\mathbb{Z}$ in each even degree generated by the integral Chern character class $s_i = i!ch_i$. Since $l \circ i$ represents the (reduced) line bundle and $ch_i(L) = \frac{1}{i!}e^i$, (3.1) implies that

$$i^* : H^*_{free}(\Omega^\infty S^\infty(\mathbb{C}P^\infty)) \longrightarrow H^*(\mathbb{C}P^\infty;\mathbb{Z})$$

sends the primitive generator s_i to e^i. Note from (2.4) that we also have that

$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) = H^*(S^\infty\mathbb{C}P^\infty;\mathbb{Z}) \xrightarrow{\sigma^*} P(H^*_{free}(\Omega^\infty S^\infty(\mathbb{C}P^\infty)))$$

maps e^i to s_i.

3.2. The odd primary case. The action of Γ_{g+1} on $H^1(F_{g+1};\mathbb{Z})$ induces the standard symplectic representation, and then a representation of $\Gamma_{g,2}$ via the map $\Gamma_{g,2} \to \Gamma_{g+1}$ used above. We may let $g \to \infty$ and obtain

$$B\Gamma_\infty \longrightarrow B\mathrm{SP}(\mathbb{Z})$$

which can be composed with the map into $B\mathrm{SP}(\mathbb{R})$. Now $B\mathrm{SP}(\mathbb{R}) \simeq BU$ so that we have a map

$$\eta : B\Gamma_\infty \longrightarrow BU$$

Theorem 3.2 ([Mo], [Mu]). *In $H^*(B\Gamma_\infty;\mathbb{Q})$ one has the relation*

$$\eta^*(s_{2i-1}) = (-1)^i(\frac{B_i}{2i})\,\kappa_{2i-1}.$$

The isomorphisms

$$H^*(BU;\mathbb{Q}) \xrightarrow{l^*} H^*(\Omega^\infty_0 S^\infty(\mathbb{C}P^\infty_+);\mathbb{Q}) \xrightarrow{\omega^*} H^*(\Omega^\infty \mathbb{C}P^\infty_{-1};\mathbb{Q})$$

together with Theorem 2.4 gives an isomorphism

$$H^*(B\Gamma_\infty;\mathbb{Q}) \simeq H^*(BU;\mathbb{Q})$$

of Hopf algebras. Hence $P(H^*_{free}(B\Gamma_\infty))$ is a copy of $\mathbb{Z}$ in each even degree. We choose a generator

$$\tau_i \in P(H^{2i}_{free}(B\Gamma_\infty)).$$

As in Section 2.3, let k be a positive integer such that $-k$ generates $(\mathbb{Z}/p^2)^\times$ for odd p and let $k=3$ when $p=2$. We have the following well-known table of p-adic valuation (see e.g. [A2]):

$$
\begin{aligned}
&\nu_p(B_i/2i) = -(1+\nu_p(2i)) && \text{if } 2i = 0\,(\mathrm{mod}\ p-1)\\
&\nu_p(B_i/2i) \geq 0 && \text{if } 2i \neq 0\,(\mathrm{mod}\ p-1)\\
&\nu_p(1-(-k)^i) = 1+\nu_p(i) && \text{if } i = 0\,(\mathrm{mod}\ p-1),\ p \text{ odd}\\
&\qquad\qquad\qquad = 2+\nu_2(i) && \text{if } p=2\\
&\nu_p(1-(-k)^i) = 0 && \text{if } i \neq 0\,(\mathrm{mod}\ p-1).
\end{aligned}
\tag{3.2}
$$

We are now ready to prove the p-primary part of Theorem 1.1 when p is an odd prime.

Proposition 3.3. *Let $\kappa_i = D_i\tau_i$ in $P(H^{2i}_{free}(B\Gamma_\infty))$. For odd primes p,*

$$\nu_p(D_i) = 1+\nu_p(i+1) \quad \text{if } i+1 \equiv 0\,(\mathrm{mod}\ p-1)$$

and $\nu_p(D_i)=0$ otherwise.

Proof: If $i+1 \equiv 0\,(\mathrm{mod}\ p-1)$ then $i = 2j-1$ and Theorm 3.2 together with (3.2) gives $\nu_p(D_{2j-1}) \geq 1+\nu_p(2j)$. The converse inequality and the rest of the proposition is a consequence of Theorem 2.2: there is a factorization

$$1+k\psi^{-k} : \mathbb{C}P^\infty \xrightarrow{\mu_p} (B\Gamma_\infty^+)^\wedge_p \xrightarrow{\omega\circ\alpha} \Omega^\infty_0 S^\infty(\mathbb{C}P^\infty_+)^\wedge_p \xrightarrow{\mathrm{proj}} \Omega^\infty S^\infty(\mathbb{C}P^\infty)^\wedge_p.$$

Since $\Omega^\infty S^\infty(\mathbb{C}P^\infty)$ is of finite type

$$H^*_{free}(\Omega^\infty S^\infty(\mathbb{C}P^\infty)^\wedge_p) = H^*_{free}(\Omega^\infty S^\infty(\mathbb{C}P^\infty))\otimes\mathbb{Z}_p.$$

It follows from Theorem 2.5 and the discussion at the end of Section 3.1 that $\mu_p^*(\kappa_i)$ is the image of the generator s_i under

$$(1+k\psi^{-k})^* : H^{2i}_{free}(\Omega^\infty S^\infty(\mathbb{C}P^\infty))\otimes\mathbb{Z}_p \longrightarrow H^{2i}(\mathbb{C}P^\infty;\mathbb{Z}_p).$$

Both groups are $\mathbb{Z}_p$ and $(1+k\psi^{-k})^*$ in dimension $2i$ is multiplication by $1+k(-k)^i = 1-(-k)^{i+1}$. If $i+1\equiv 0\,(\mathrm{mod}\ p-1)$ then $\nu_p(1-(-k)^{i+1}) = 1+\nu_p(i+1)$. This proves the reverse inequality: $\nu_p(D_i)\leq 1+\nu_p(i+1)$. If $i+1\neq 0\,(\mathrm{mod}\ p-1)$ then $\nu_p(1-(-k)^{i+1})=0$. □

3.3. Diagonalizing the splitting map μ_p. It remains to determine the divisibility of κ_i at 2 for the argument of Theorem 3.3 gives only

$$1+\nu_2(2j)\leq \nu_2(D_{2j-1}) \leq 2+\nu_2(2j).$$

This requires two extra results. Firstly that

$$H_*(\Omega^\infty_0\mathbb{C}P^\infty_{-1};\mathbb{F}_2) \xrightarrow{\omega_*} H_*(\Omega^\infty_0 S^\infty(\mathbb{C}P^\infty_+);\mathbb{F}_2) \tag{3.3}$$

is injective, and secondly that μ_2 can be chosen so that

$$(\omega \circ \alpha \circ \mu_2)_* : H_*(\mathbb{C}P^\infty; \mathbb{F}_2) \longrightarrow H_*(\Omega_0^\infty S^\infty(\mathbb{C}P_+^\infty); \mathbb{F}_2) \tag{3.4}$$

is zero in degrees $* \equiv 2 \,(\mathrm{mod}\ 4)$.

The first result (3.3) is contained in theorem 7.10 of [G1]. The second result (3.4) requires a strengthening of Theorem 2.2: We need to calculate the undetermined map

$$\Omega^\infty S^\infty \longrightarrow \Omega^\infty S^\infty(\mathbb{C}P^\infty)$$

denoted by $(*)$ in Theorem 2.2.

Theorem 3.4. *For every prime p, $\omega \circ \alpha \circ \mu_p$ in Theorem 2.2 is homotopic to the diagonal self-map $Diag(1 + k\psi^k, -2)$ after p-adic completion.*

The rest of this subsection contains a proof of this theorem. The proof we give is a variation of the proof of Theorem 2.2 from sections 3.1-3.3 in [MT].

We first describe Riemann surfaces Σ with holomorphic actions of the q-th roots of unity $\mu_q \subset \mathbb{C}^\times$. This gives maps $B\mu_q \to B\mathrm{Diff}(\Sigma)$.

Consider a divisor $D = \Sigma n_i p_i$ of $\mathbb{C}P^1$ with support $A = \{p_0, p_1, \dots, p_k\}, n_i \in \mathbb{Z}$ and $n_0 + n_1 + \cdots + n_k = 0$. Let q be an integer and assume for simplicity that $\gcd(q, n_i) = 1$ for $i = 0, 1, \dots, k$. Let Σ_D be the branched cover associated with the Galois extension

$$\mathbb{C}(z) \hookrightarrow \mathbb{C}(z)[T]/(F(T)), \qquad F(T) = T^q - \Pi(z - p_i)^{n_i}$$

(see e.g. [F], chap. 1.8). The Galois group is the group μ_q. The surface Σ_D has a holomorphic action of μ_q with orbit space $\mathbb{C}P^1$. The induced map $\pi : \Sigma_D \to \mathbb{C}P^1$ is holomorphic, branched over A, and $\pi^{-1}(p_i)$ is a single point (since we assumed $\gcd(n_i, q) = 1$). Thus the μ_q action of Σ_D is free outside A, and A is fixed pointwise by all elements in μ_q.

Let γ_i be a small loop in $\mathbb{C}P^1$ around p_i. The fundamental group $\mathbb{C}P^1 \setminus A$ is the free group of rank k generated by $\gamma_0, \dots, \gamma_k$ with the single relation $\Pi\gamma_i = 1$. The covering $\Sigma_d \setminus A \to \mathbb{C}P^1 \setminus A$ is classified by the map from $\pi_1(\mathbb{C}P^1 \setminus A)$ to μ_q that sends γ_i to $e^{2\pi i n_i/q}$. The complex tangent line $T_{p_i}\Sigma_D$ at p_i is a μ_q representation; $u \in \mu_q$ multiplies by $\bar{u} = u^{\bar{n}_i}$ where $\bar{n}_i \in \mathbb{Z}/q$ is the multiplicative inverse of n_i.

If D and D' are two divisors and q divides their difference $D - D'$ then there is a biholomorphic map between Σ_D and $\Sigma_{D'}$ that is equivariant w.r.t. the μ_q action. Thus it is only the class of D in $\tilde{H}_0(A; \mathbb{Z}/q)$ that matters.

Recall from Section 2.4 that given a surface bundle $\pi : E \to X$ there is a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & \Omega^\infty(Th(T_\pi E^\perp)) \arrow[r] \arrow[d] & \Omega^\infty S^\infty(E_+) \arrow[d, "T_\pi"'] \\
X \arrow[r, "\alpha_{p_i}"] & \Omega^\infty \mathbb{C}P_{-1}^\infty \arrow[r, "\omega"] & \Omega^\infty S^\infty(\mathbb{C}P_+^\infty)
\end{tikzcd} \tag{3.5}$$

where $\Omega^\infty(Th(T_\pi E^\perp)) = \operatorname{colim} \Omega^{n+2}(N_\pi^n E)$. The upper horizontal composition is the Becker-Gottlieb transfer map $t_E = t_\pi : X \to \Omega^\infty S^\infty(E_+)$.

We need the following properties of the Becker-Gottlieb transfer for smooth manifold bundles with compact fiber and compact Lie structure group:

(A1) Let $f : E \to E'$ be a fiberwise homotopy equivalence. Then

$$t_{E'} = \Omega^\infty S^\infty(f_+) \circ t_E \in [X, \Omega^\infty S^\infty(E'_+)].$$

(A2) Suppose

$$\begin{array}{ccc} E_{12} & \xrightarrow{j_1} & E_1 \\ {\scriptstyle j_2}\big\downarrow & & \big\downarrow{\scriptstyle i_1} \\ E_2 & \xrightarrow{i_2} & E \end{array}$$

is fiberwise homotopy coCartesian. Then

$$t_E = \Omega^\infty S^\infty(i_{1+}) \circ t_{E_1} + \Omega^\infty S^\infty(i_{2+}) \circ t_{E_2} - \Omega^\infty S^\infty(i_{12+}) \circ t_{E_{12}}$$

in $[X, \Omega^\infty S^\infty(E_+)]$ where $i_{12} = j_1 \circ i_1 = j_2 \circ i_2$.

(A3) If the tangent bundle along fibers $T_\pi E$ admits a non-zero section, then $t_E \in [X, \Omega_0^\infty S^\infty(E_+)]$ is constant.

The proof of (A1) and (A2) can be found in [LMS], p. 189-190 or in [BS]. Property (A3) is much simpler. It follows because

$$Th(N_\pi E) \longrightarrow Th(N_\pi E \oplus T_\pi E)$$

is homotopic to the constant map at ∞ whenever $T_\pi E$ has an everywhere non-zero section.

Let $\Sigma = \Sigma_D$ be the μ_q-surface constructed aboce. We shall study the transfer of the associated smooth surface bundle

$$\pi : E\mu_q \times_{\mu_q} \Sigma \longrightarrow B\mu_q.$$

To shorten notation we write

$$t_\Sigma : B\mu_q \longrightarrow \Omega^\infty(E_+), \quad E = E\mu_q \times_{\mu_q} \Sigma$$

for the associated transfer.

Let $D = \Sigma n_i p_i$ and $q \in \mathbb{N}$ satisfy $A =$ supp$(D) = \{p_0, \dots, p_k\}, n_0 + \cdots + n_k = 0$, $\gcd(q, n_i) = 1$ for $i = 0, \dots, k$. For each i, the inclusion of $E\mu_q \times_{\mu_q} \{p_i\} \subset E$ induces a map

$$\hat{p}_i : B\mu_q \longrightarrow E \longrightarrow \Omega^\infty S^\infty(E_+).$$

The principal μ_q bundle $E\mu_q \to B\mu_q$ induces a transfer from $B\mu_q$ to $\Omega^\infty S^\infty(E\mu_{q+})$ $\simeq \Omega^\infty S^\infty$, and hence

$$\hat{t}_q : B\mu_q \longrightarrow \Omega^\infty S^\infty \longrightarrow \Omega^\infty S^\infty(E_+)$$

upon choosing a point of E.

Lemma 3.5. *The transfer* t_Σ *is equal to* $\Sigma \hat{p}_i + (1-k)\hat{t}_q$ *in* $[B\mu_q, \Omega^\infty S^\infty(E_+)]$.

Proof. We make a cell decomposition of $S^2 = \mathbb{C}P^1$ with two 0-cells $\{0,\infty\}$, $k+1$ 1-cells I_i and $k+1$ 2-cells D_i such that $p_i \in \text{ int } D_i$.

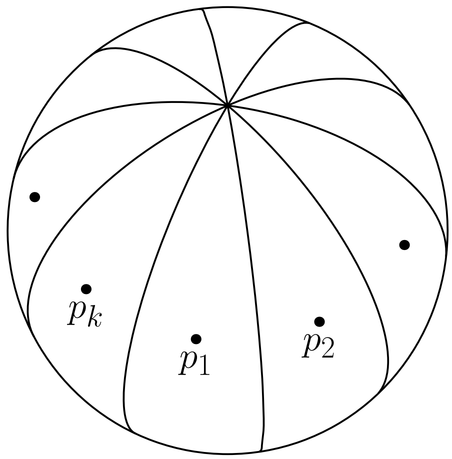

Figure 3.6

There are obvious coCartesian diagrams

$$\begin{array}{ccccccc} \sqcup \partial I_i & \longrightarrow & \sqcup I_i & \qquad & \sqcup \partial D_i & \longrightarrow & \sqcup D_i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{0,\infty\} & \longrightarrow & G & & G & \longrightarrow & S^2 \end{array}$$

where G denotes the 1-skeleton. This cell structure lifts to a cell structure of Σ:

$$\begin{array}{ccccccc} \sqcup \partial \tilde{I}_i & \longrightarrow & \sqcup \tilde{I}_i & \qquad & \sqcup \partial \tilde{D}_i & \longrightarrow & \sqcup \tilde{D}_i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{0,\infty\}^\sim & \longrightarrow & \tilde{G} & & \tilde{G} & \longrightarrow & \Sigma \end{array} \tag{3.7}$$

with $\tilde{G} = \pi^{-1}(G)$ etc. We can apply the functor $E\mu_q \times_{\mu_q} (-)$ to (3.7) and use (A1-3) to evaluate $t_{\tilde{G}}$ and t_Σ. First by (A1) and (A2),

$$t_{\tilde{G}} = (1+k)\hat{t}_q + 2\hat{t}_q - 2(k+1)\hat{t}_q = (1-k)\hat{t}_q,$$

since $\tilde{I}_i = \mu_q \times I_i, \partial\tilde{I}_i = \mu_q \times \partial I_i$ and $\{0,\infty\}^\sim = \mu_q \times \{0,\infty\}$. Second, the inclusion of p_i in $\tilde{D}_i$ is a homotopy equivalence, so $t_{\tilde{D}_i} = \hat{p}_i$. Moreover, $E\mu_q \times_{\mu_q} \partial\tilde{D}_i = E\mu_q \times S^1$ has trivial vertical tangent bundle, so $t_{\partial\tilde{D}_i}$ is homotopically constant. One more application of (A1-2) completes the proof. $\square$

The tangent representation $T_{p_i}\Sigma$ is given by multiplication with $e^{2\pi i \bar{n}_i/q}$, so an application of (3.5) gives

Corollary 3.6. *The homotopy class of*

$$B\mu_q \longrightarrow \Omega^\infty \mathbb{C}P^\infty_{-1} \longrightarrow \Omega^\infty S^\infty(\mathbb{C}P^\infty_+)$$

is $\Sigma\psi^{\bar{n}_i} + (1-k)t_{\mu_q}$. *Here* $\psi^{\bar{n}_i}$ *is the composition*

$$\psi^{\bar{n}_i} : B\mu_q \longrightarrow \mathbb{C}P^\infty \longrightarrow \Omega^\infty S^\infty(\mathbb{C}P^\infty_+)$$

with the left hand map induced from the group homomorphism $\mu_q \to S^1$ *that sends* u *to* $u^{\bar{n}_i}$. □

We are now ready to complete the proof of Theorem 3.4. As in section 3.3 of [MT] we let $q = p^n$ be a prime power and consider the divisor

$$D = p_0 + mp_1 + \cdots + mp_k, \qquad m \equiv -1/k (\text{mod } p^n).$$

We use the notation

$$F(n) = \Sigma_D, \quad C_{p^n} = \mu_{p^n}, \quad \tau_n \hat{t}_{p^n}$$

and consider diagram (3.5) with

$$X = B\text{Diff}(F(n)), \qquad E = E\text{Diff}(F(n)) \times_{\text{Diff}(F(n))} F(n).$$

Composing with the map $BC_{p^n} \to B\text{Diff}(F(n))$ induced by the C_{p^n} action on $F(n)$, and using that $B\Gamma_{g(n)} \simeq B\text{Diff}(F(n))$ with $g(n) = 1/2(p^n-1)(k-1)$ we get the diagram

$$\begin{array}{ccccc}
B\Gamma_g(n) & \xrightarrow{\alpha_n} & \Omega^\infty_{g(n)-1}\mathbb{C}P^\infty_{-1} & \xrightarrow{T} & \Omega^\infty_0\mathbb{C}P^\infty_{-1} \\
\mu_n\big\uparrow & & \omega\big\downarrow & & \omega\big\downarrow \\
BC_{p^n} & \longrightarrow & \Omega^\infty_{2-2g(n)}S^\infty(\mathbb{C}P^\infty_+) & \xrightarrow{T} & \Omega^\infty_0 S^\infty(\mathbb{C}P^\infty_+).
\end{array}$$

The right-hand horizontal maps are translations of the indicated component into the zero component. The lower horizontal composition is by Corollary 3.6 equal to

$$(1 + k\psi^{-k}, \tilde{\tau}_n) \in [BC_{p^n}, \Omega^\infty_0 S^\infty(\mathbb{C}P^\infty_+)]$$

with $\tilde{\tau}_n = T \circ \tau_n$.

Lemma 3.7. *Let* $i_{n-1} : BC_{p^n-1} \to BC_{p^n}$ *be the map associated with* $C_{p^{n-1}} \subset C_{p^n}$. *Then*

$$[\tau_n \circ i_{n-1}] = p[\tau_{n-1}] \in [BC_{p^{n-1}}, \Omega^\infty S^\infty].$$

Proof: Let E be a contractible space with a free action of C_{p^n}, e.g. the union of odd dimensional spheres $E = \bigcup_{m\geq 1} S^{2m-1}$. Consider the diagram

$$\begin{array}{ccccc}
E & \xrightarrow{\pi_{n-1}} & BC_{p^{n-1}} & \xrightarrow{i_{n-1}} & BC_{p^n} \\
\hat{i}\big\uparrow & & \hat{i}_{n-1}\big\uparrow & & i_{n-1}\big\uparrow \\
\bigsqcup_1^p E & \longrightarrow & \bigsqcup_1^p BC_{p^{n-1}} & \longrightarrow & BC_{p^{n-1}}
\end{array}$$

where $BC_{p^{n-1}}$ and BC_{p^n} are the orbit spaces $E/C_{p^{n-1}}$ and E/C_{p^n} and i_{n-1} is represented by the obvious quotient map. The lower sequence in the above diagram is the pull-back of the upper sequence, and $i_{n-1} \circ \pi_{n-1} = \pi_n$. The transfer of a composition is the composition of transfers, and transfers are natural for pull-backs. Thus $[\tau_n]$ is the transfer of the lower sequence composed with $\hat{i}$. This comes out to be $p[\tau_{n-1}]$. □

The Ivanov-Harer stability theorems imply that the map of plus constructions,

$$B\Gamma^+_{g(n)-1,2} \longrightarrow B\Gamma^+_{g(n)}$$

is b_n-connected with $b_n = [\frac{g(n)-1}{2}]$. Thus

$$[BC^{(b_n)}_{p^n}, B\Gamma^+_{g(n)-1,2}] \simeq [BC^{(b_n)}_{p^n}, B\Gamma^+_{g(n)}]$$

where the superscript indicates the b_n-skeleton. Let $\mu_{n,2} : BC^{(b_n)}_{p^n} \to B\Gamma^+_{g(n)-1,2}$ correspond to μ_n so that

$$T \circ \omega \circ \alpha_{n,2} \circ \mu_{n,2} = (1 + k\psi^{-k}, \tilde{\tau}_n) \tag{3.9}$$

in $[BC^{(b_n)}_{p^n}, \Omega^\infty_0 S^\infty(\mathbb{C}P^\infty_+)]$.

Consider the subset G_n of $[BC^{(b_n)}_{p^n}, (B\Gamma^+_{\infty,2})^\wedge_p]$ of elements that satisfy (3.9). It is a non-empty and finite (or at least compact), so $\varprojlim G_n \neq \emptyset$. We pick $\tilde{\mu}_p \in \varprojlim G_n$. Since $\operatorname{colim} BC^{(b_n)}_{p^n} = BC_{p^\infty}$ has p-adic completion homotopy equivalent to $(\mathbb{C}P^\infty)^\wedge_p$, the map $\tilde{\mu}_p$ extends to a map

$$\tilde{\mu}_p : \Omega^\infty S^\infty(\mathbb{C}P^\infty) \longrightarrow (B\Gamma^+_{\infty,2})^\wedge_p$$

and

$$T \circ \omega \circ \alpha \circ \mu_p \simeq (1 + k\psi^k, 0).$$

This proves that the first row in the matrix described in Theorem 2.2 is $(1+k\psi^{-k}, 0)$. The second row corresponds to the homotopy class of

$$\Omega^\infty S^\infty \xrightarrow{\mu} \mathbb{Z} \times B\Gamma^+_{\infty,2} \xrightarrow{\alpha} \Omega^\infty \mathbb{C}P^\infty_{-1} \xrightarrow{\omega} \Omega^\infty S^\infty(\mathbb{C}P^\infty_+). \tag{3.10}$$

An infinite loop map with source $\Omega^\infty S^\infty$ is determined by its restriction to $S^0 = \{-1, +1\} \hookrightarrow \Omega^\infty S^\infty$. The μ above maps the base point $+1$ of S^0 into $(0, *)$ and the non-base point into $(1, *)$. The composition (3.10) maps the base point into the base point of $\Omega^\infty_0 S^\infty(\mathbb{C}P^\infty)$ and the non-base point into the base point of $\Omega^\infty_{-2} S^\infty(\mathbb{C}P^\infty_+)$. This shows that the second row of the matrix in Theorem 2.2 is $(0, -2)$ as claimed. □

3.4. The case p=2. The difficulty with the divisibility of κ_i at 2 is that the 2-adic valuation of $B_i/2i$ is one less than the 2-adic valuation of $(3^{2i} - 1)$. The argument used in Proposition 3.3 thus contains an indeterminacy of one factor 2.

Proposition 3.6. *With the notation of Proposition 3.3, $\nu_2(D_i) = 1 + \nu_2(i+1)$.*

Proof: The inequality $\nu_2(D_i) \geq 1+\nu_2(i+1)$ follows from Theorem 2.6 and Theorem 3.2. We prove the reverse inequality. As in the proof of Proposition 3.3 we have

$$1-(-3)^{i+1} : P(H^{2i}_{free}(\Omega^\infty S^\infty \mathbb{C}P^\infty_+))) \otimes \mathbb{Z}_2 \xrightarrow{(\omega\circ\alpha)^*} P(H^{2i}(B\Gamma^+_\infty)) \otimes \mathbb{Z}_2$$
$$\xrightarrow{\mu_2^*} H^{2i}(\mathbb{C}P^\infty; \mathbb{Z}_2).$$

All groups are copies of $\mathbb{Z}_2$, $\kappa_i = D_i\tau_i$ with τ_i a generator and $\kappa_i = (\omega\circ\alpha)^*(s_i)$ where s_i is the generator of the left term.

For even i, $\nu_2(1-(-3)^{i+1}) = 1$ and by Theorem 2.6 $\nu_2(D_i) = 1$ as claimed. For odd $i = 2j-1$ suppose that $\nu_2(D_i) = 2+\nu_2(i+1)$. Then $\mu_2^*(\tau_i)$ would be a generator and dually

$$\mu_{2*} : H_{2i}(\mathbb{C}P^\infty; \mathbb{F}_2) \longrightarrow H^{free}_{2i}(B\Gamma^+_\infty) \otimes \mathbb{F}_2$$

would be non-zero. Then

$$\mu_{2*} : H_{2i}(\mathbb{C}P^\infty; \mathbb{F}_2) \longrightarrow H_{2i}(B\Gamma^+_\infty) \otimes \mathbb{F}_2$$

would also be non-zero. Now apply Theorem 2.4 together with the injectivity of (3.3) to conclude that

$$H_{4j-2}(\mathbb{C}P^\infty; \mathbb{F}_2) \xrightarrow{(\omega\circ\alpha\circ\mu_2)_*} H_{4j-2}(\Omega^\infty_0 S^\infty(\mathbb{C}P^\infty_+); \mathbb{F}_2)$$

would be non-zero under the assumption that $\nu_2(D_{2j-1}) = 2+\nu_2(2j)$. This contradicts Theorem 3.4. Indeed the self map ψ^{-3} of $\mathbb{C}P^\infty$ induces the identity on $H_*(\mathbb{C}P^\infty; \mathbb{F}_2)$. Hence $1+3\psi^{-3}$ induces the same map on $\mathbb{F}_2$ homology as

$$4i : \mathbb{C}P^\infty \longrightarrow \Omega^\infty S^\infty(\mathbb{C}P^\infty).$$

But $(4i)_* = 0$ on $\mathbb{F}_2$-homology in dimension $4j-2$. □

References.

[A1] J.F. Adams, Lectures on generalized cohomology, category theory, homology theory and their applications III, LNM 99, Springer (1969).

[A2] J.F. Adams, *On the groups $J(X)$ II*, Topology **3** (1965), 137–171.

[Ak] T. Akita, *Nilpotency and triviality of mod p Morita-Mumford classes of mapping class groups of surfaces*, Nagoya Math. J. **165** (2002), 1–22.

[B] J. Berrick, An approach to algebraic K-theory, Research Notes in Mathematics, 56, Pitman (Advanced Publishing Program), Boston, 1982.

[BK] A.K. Bousfield, D.M. Kan, Homotopy Limits, Completions and Localizations, Springer Verlag LNM **304**, 1972.

[BS] J.C. Becker, R.E. Schultz, *Axioms for transfers and traces*, Math. Z. **227** (1988), 583-605.

[EE] C.J. Earle, J. Eells, *A fibre bundle description of Teichmüller theory*, J. Diff. Geom. **3** (1969), 19–43.

[ES] C.J. Earle, A. Schatz, *Teichmüller theory for surfaces with boundary*, J. Diff. Geom. **4** (1970), 169–185.

[F] O. Forster: Lectures on Riemann surfaces, Graduate Texts in Mathematics, 81. Springer-Verlag, New York-Berlin, 1991.

[G1] S. Galatius, *Homology of* $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$ *and* $\Omega^\infty\mathbb{C}P^\infty_{-1}$, to appear in Topology.

[G2] S. Galatius, *Secondary characteristic classes of surface bundles*, preprint, math.AT/0402226.

[H1] J.L. Harer, *The second homology group of the mapping class group of an orientable surface* , Invent. Math. **72** (1983), 221–239.

[H2] J.L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Annals Math. **121** (1985), 215–249.

[I] N.V. Ivanov, *Stabilization of the homology of Teichmüller modular groups*, Original: Algebra i Analiz **1** (1989), 110–126; Translated: Leningrad Math. J. **1** (1990), 675–691.

[LMS] G. Lewis, J.P. May, M. Steinberger,*Equivariant stable homotopy theory*, Springer LNM **1213**, 1986.

[MS] I. Madsen, C. Schlichtkrull, *The circle transfer and K-theory*, AMS Contemporary Math. **258** (2000), 307–328.

[MT] I. Madsen, U. Tillmann, *The stable mapping class group and* $Q(\mathbb{C}P^\infty)$, Invent. Math. **145** (2001), 509–544.

[MW] I. Madsen, M. Weiss, *Cohomolgy of the stable mapping class group*, preprint 2003.

[Mi] E.Y. Miller, *The homology of the mapping class group*, J. Diff. Geom. **24** (1986), 1–14.

[Mo] S. Morita, *Characteristic classes of surface bundles*, Invent. Math. **90** (1987), 551-577.

[Mu] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and Geometry, M. Artin and J. Tate, editors, Progr. Math. **36** Birkhauser (1983), 271–328.

[S] G. Segal, *The stable homotopy of complex projective space*, Quart. J. Math. Oxford **24** (1973), 1–5.

[T] U. Tillmann, *On the homotopy of the stable mapping class group*, Invent. Math. **130** (1997), 257–275.

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