Classification of smooth Fano polytopes

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Introduction

This thesis is about the classification of smooth Fano polytopes, which are simplicial lattice polytopes, containing the origin in the interior, and the vertex set of any facet is a basis of the lattice of points having integer coordinates.

Smooth Fano polytopes have inherited their name from the toric varieties they correspond to. A toric variety is a normal algebraic variety containing an algebraic torus as a dense open subset, such that the natural action of the torus on itself extends to an action on the whole variety. Toric varieties have a combinatorial description in terms of a fan of cones, and many geometric properties of the varieties translate to combinatorial properties of the fan. Due to the fan description, toric varieties provide algebraic geometers a wealth of concrete examples to work with and have been a valuable testing ground for general conjectures and theories.

Smooth Fano varieties are non-singular projective varieties over the complex numbers, such that the anti-canonical divisor is an ample Cartier divisor. Isomorphism classes of smooth Fano toric varieties are in one-to-one correspondence with isomorphism classes of smooth Fano polytopes. Hence smooth Fano toric varieties can be studied and classified by purely combinatorial methods used on the corresponding polytopes. This has proved to be quite a fruitful approach and many papers have been concerned about this topic.

Most importantly, it is known that there are finitely many isomorphism classes of smooth Fano d-polytopes in any given dimension $d \geq 1$. Before the work presented in this thesis complete classification of these isomorphism classes existed only for $d \leq 5$.

It is also known that smooth Fano d-polytopes can have at most $3d$ vertices. It turns out that smooth Fano d-polytopes with almost $3d$ vertices are closely related to the more general simplicial reflexive d-polytopes: A lattice polytope containing the origin in the interior, is called reflexive, if the dual polytope is also a lattice polytope. A simplicial reflexive d-polytope can also have at most $3d$ vertices, and in case of equality it is known to be a smooth

Fano d-polytope.
The new results obtained and presented in this thesis are the following:

- Classification of terminal simplicial reflexive d-polytopes with $3d-1$ vertices (theorem 2.17), where terminality means that the origin and the vertices are the only lattice points in the polytopes. These polytopes turn out to be smooth Fano d-polytopes. The result is published in the paper [29].

- Classification of smooth Fano polytopes whose dual polytopes have a fixed number of lattice points on every edge (corollary 2.4 and theorem 2.7). This corresponds to a classification of smooth Fano toric varieties, where all the torus invariant curves have the same anti-canonical degree.

- A counter example to a conjecture regarding the classification of higher dimensional smooth Fano polytopes (section 3.1.3). The counter example is presented in the preprint [28].

- An algorithm that classifies smooth Fano d-polytopes up to isomorhism for any given d (section 3.2). The SFP-algorithm, as we call it (SFP for <u>S</u>mooth <u>F</u>ano <u>P</u>olytopes), has been implemented and used to classify smooth Fano d-polytopes for $d \leq 8$. The algorithm is described in the preprint [30].

- Terminality of the polytope $\text{conv}(P \cup -P)$ for every smooth Fano polytope P (theorem 4.2).

All these results are obtained by using elementary combinatorial methods on the investigated polytopes.
One of the key ideas is the concept of a special facet of a smooth Fano polytope, or more generally, a simplicial reflexive polytope. This concept is new and due to the author of the thesis. The concept yields a short proof of the upper bound on the number of vertices, and allows the explicit determination of a finite subset $\mathcal{W}_d$ of the lattice with the property, that any smooth Fano d-polytope is isomorphic to one whose vertices lie in $\mathcal{W}_d$.
Furthermore, using the idea of a special facet we can divide the case of $3d-1$ vertices of a terminal simplicial reflexive d-polytope into three different distributions of vertices, which can be considered one by one.
Another key idea in the thesis is the following observation: Given a subset of the vertex set of a smooth Fano polytope, we can actually deduce a lot of information on the face lattice of the polytope. This is due to the strong assumptions on these polytopes. For some concrete subsets of lattice points

we can even show that this is a subset of the vertex set of only one smooth Fano polytope. This is how we show that a particular polytope is a counter example to a proposed conjecture. Conversely, for many concrete subsets of lattice points we can show that NO smooth Fano polytope have this set as a subset of its vertices.
Once we have identified the subset $\mathcal{W}_d$, we can use our observation to construct an efficient algorithm that runs through every subset of $\mathcal{W}_d$, hereby constructing at least one representative for each isomorpism class of smooth Fano d-polytopes.
A lot of time has been spent on the development and implementation of the SFP-algorithm. It is the author's hope that the obtained classification data will be a valuable database of examples for researchers working with toric geometry or lattice polytopes.
The structure of the thesis is as follows:

Chapter 1. The basic facts of polytopes are recalled, and we define our objects of study: Smooth Fano polytopes and the more general simplicial reflexive polytopes. We show some simple combinatorial identities and inequalities. The important concept of a special facet is introduced and used to prove some finiteness results.

Chapter 2. In this chapter several classification results on smooth Fano d-polytopes valid for any d are considered. New proofs are given for some well-known results. Some new results are proved. The two main results of this chapter are the classification of smooth Fano polytopes whose dual polytopes have a fixed number of lattice points on the edges and the classification of terminal simplicial reflexive d-polytopes with $3d-1$ vertices.

Chapter 3. In the first part of this chapter we show by means of an explicit counter example that a conjecture concerning the classification of higher dimensional smooth Fano polytopes is not true. In the second part of the chapter we present the SFP-algorithm that can classify smooth Fano d-polytopes (up to isomorphism) for any d.

Chapter 4. In the final chapter we discuss some observations and ideas for further research.

Two appendices are included: One containing a table of the number of isomorphism classes of smooth Fano d-polytopes for given number of vertices. The other containing the C++ code of the implementation of the SFP-algorithm.

To increase readability of the thesis, an index has been included after the bibliography. The three manuscripts by the author are also included.

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Contents

Chapter 1

Definitions, facts and special facets

The thesis is about a special class of lattice polytopes, namely the so-called smooth Fano polytopes. Apart from being an interesting class of polytopes in their own right, the study of them is motivated by their connection to algebraic geometry. More precisely, isomorphism classes of smooth Fano d-polytopes are in one-to-one correspondence with isomorphism classes of smooth projective toric d-folds with ample anti-canonical divisor.

In this chapter we define the objects we wish to study, namely smooth Fano polytopes together with their relatives, simplicial reflexive polytopes. We will also prove some basic properties and set up notation, that will be used throughout the thesis.

The concept of a special facet (definition 1.11) is new and due to the author of the thesis. The existence of special facets is essential in many of our arguments in chapters 2 and 3. Furthermore, the concept allows us to give a short proof of the upper bound on the number of vertices of a simplicial reflexive polytope (theorem 1.15) and of the finiteness of the number of isomorphism classes of smooth Fano d-polytopes (theorem 1.16).

The structure of the chapter is as follows: In section 1.1 we recall some basic concepts and facts about polytopes and fix the notation we use throughout the thesis. In section 1.2 we define the lattice polytopes we wish to study, that is smooth Fano polytopes and the more generel simplicial reflexive polytopes. In section 1.3 we define the notion of special facets and use this to prove some finiteness results.

1.1 Polytopes

We begin by recalling some basic concepts and facts about polytopes, together with fixing a notation. Proofs and details can be found in any textbook on convex polytopes, for example [14], [27] or the first half of [12].
Throughout the whole thesis d will be a positive integer. $\mathbb{R}^d$ will denote a d-dimensional vector space and $\{e_1, \ldots, e_d\}$ a fixed basis of this space. By $\mathbb{Z}^d$ we denote the lattice

$$\{a_1 e_1 + \ldots + a_d e_d \mid a_i \in \mathbb{Z}\} \subset \mathbb{R}^d$$

of points having integral coordinates. By $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ we denote the usual scalar product, i.e.

$$\langle \sum_i a_i e_i, \sum_i b_i e_i \rangle = \sum_i a_i b_i.$$

A set S of $\mathbb{R}^d$ is called *convex* if the line segment

$$[x, y] := \{\lambda x + (1 - \lambda) y \mid 0 \leq \lambda \leq 1\}$$

is contained in S for every pair $x, y \in S$.
Let K be a set of points in $\mathbb{R}^d$. The *convex hull* of K is defined as the set

$$\operatorname{conv}(K) := \bigcap \{S \subseteq \mathbb{R}^d \mid K \subseteq S, S \text{ convex}\}.$$

It is readily seen that

$$\operatorname{conv}(K) = \{a_1 x_1 + \ldots + a_k x_k \mid a_i \geq 0 \ , \ \sum_{i=1}^{k} a_i = 1 \ , \ x_1, \ldots, x_k \in K\}.$$

A set $S \subseteq \mathbb{R}^d$ is called *affine* if the line $\{\lambda x + (1 - \lambda) y | \lambda \in \mathbb{R}\}$ is contained in S for every pair $x, y \in S$.
When $K \subseteq \mathbb{R}^d$, the *affine hull* of K is the set

$$\operatorname{aff}(K) := \bigcap \{S \subseteq \mathbb{R}^d \mid K \subseteq S, S \text{ affine}\}.$$

Once again it is easy to see that

$$\operatorname{aff}(K) = \{a_1 x_1 + \ldots + a_k x_k \mid a_i \in \mathbb{R} \ , \ \sum_{i=1}^{k} a_i = 1 \ , \ x_1, \ldots, x_k \in K\}.$$

Non-empty affine hulls are translates of linear subspaces of $\mathbb{R}^d$, and have a well-defined dimension. The *dimension* $\dim(\operatorname{conv}(K))$ of $\operatorname{conv}(K)$ is defined to be the dimension of $\operatorname{aff}(K)$.

The *relative interior* relint(S) of a convex set S is the set of interior points of S relative to aff(S). When aff$(S) = \mathbb{R}^d$, the relative interior relint$S$ of S equals the interior int$S$ of S.
A *polytope* P is the convex hull conv(K) of a finite set $K \subset \mathbb{R}^d$. The *dimension* dim(P) of P is the dimension of conv(K). If dim$(P) = k$, we say that P is a k-polytope.
A set of the form $\{v \in \mathbb{R}^d | \langle u, v\rangle = a\}$ for some $u \in \mathbb{R}^d$ and $a \in \mathbb{R}$, is called a *hyperplane*. Sets of the form $\{v \in \mathbb{R}^d | \langle u, v\rangle \leq a\}$ for some $u \in \mathbb{R}^d$ and $a \in \mathbb{R}$ are called *closed halfspaces*.
The main theorem on the representation of polytopes is the following ([27] theorem 1.1).

Theorem 1.1. *A subset $P \subseteq \mathbb{R}^d$ is a polytope if and only if P is a bounded intersection of finitely many closed halfspaces.*

1.1.1 Faces

A *supporting hyperplane* of a convex set S is a hyperplane $\{v \in \mathbb{R}^d | \langle u, v\rangle = a\}$, such that $\sup\{\langle u, v\rangle | v \in S\} = a$.
A *face* of a convex set S is the intersection of S with a supporting hyperplane. The convex set S is a face of itself and is called an *improper face*. Other faces are called *proper*. The *dimension* of a face F of S is defined to be the dimension of aff(F).
The faces of a polytope P of dimension 0, 1, $\dim(P) - 2$ and $\dim(P) - 1$ are called *vertices*, *edges*, *ridges* and *facets*, respectively. The empty set $\emptyset$ is considered to be an improper face of P. A *k-face* of P is a face of dimension k. The dimension of the face $\emptyset$ is set to -1. The set of faces of a polytope P is partially ordered by inclusion, and is known as the *face lattice*. By $\mathcal{V}(P)$ we will denote the set of vertices of a polytope P.
Here is a fundamental property of the vertex set $\mathcal{V}(P)$ ([27] proposition 2.2):

Proposition 1.2. *Let $P \subseteq \mathbb{R}^d$ be a polytope.*

1. *Every polytope is the convex hull of its vertices: $P = \mathrm{conv}(\mathcal{V}(P))$.*

2. *If a polytope can be written as the convex hull of a finite point set, then the set contains all the vertices of the polytope: $P = \mathrm{conv}(V)$ implies that $\mathcal{V}(P) \subseteq V$.*

Some other fundamental properties of faces of polytopes are these ([27] proposition 2.3):

Proposition 1.3. *Let $P \subseteq \mathbb{R}^d$ be a polytope and F a face of P.*

1. *The face F is a polytope with $\mathcal{V}(F) = F \cap \mathcal{V}(P)$.*
2. *Every intersection of faces of P is a face of P.*
3. *The faces of F are exactly the faces of P that are contained in F.*
4. $F = P \cap \operatorname{aff}(F)$

As a consequence of proposition 1.3.(1), there are finitely many faces of a polytope.
The set of facets of a polytope P gives us a minimal set of closed halfspaces, such that the polytope is the intersection of these halfspaces ([14] p.31).

Proposition 1.4. *Each d-polytope $P \subseteq \mathbb{R}^d$ is the intersection of a finite family of closed halfspaces; the smallest such family consists of those closed halfspaces containing P whose boundaries are the affine hulls of the facets of P.*

1.1.2 Dual polytope

When P is a d-polytope in $\mathbb{R}^d$ containing the origin 0 in the interior int$P$, a convex set P^* in $\mathbb{R}^d$ is defined by

$$P^* := \{u \in \mathbb{R}^d \mid \langle u, x \rangle \leq 1 \ \forall x \in P\}.$$

This is a polytope ([14] p.47) and is called the *dual of* P. Obviously, $P^{**} = P$. There is an inclusion reversing one-to-one correspondence between k-faces of P and $(d-k-1)$-faces of P^* ([14] p.47). In particular, facets of P corresponds to vertices of P^*. The correspondence is given by

$$F \text{ face of } P \quad \longleftrightarrow \quad \{u \in P^* | \langle u, F \rangle = \{1\}\} \text{ face of } P^*$$

When F is a $(d-1)$-polytope in $\mathbb{R}^d$, $0 \notin \operatorname{aff}(F)$, we define u_F to be the unique element in $\mathbb{R}^d$, such that $\langle u_F, F \rangle = \{1\}$. In particular, if F is a facet of a d-polytope P with $0 \in \operatorname{int}P$, then u_F is the vertex of P^* corresponding to the facet F of P and $\langle u_F, v \rangle \leq 1$ for all $v \in P$ with equality if and only if $v \in F$. When F is a facet of a polytope P, $0 \in \operatorname{int}P$, u_F is also known as the *outer normal* of F.

1.1.3 Simplicial polytopes

If $K \subset \mathbb{R}^d$ consists of $k+1$ points and $\dim(\operatorname{conv}K) = k$ for some $0 \leq k \leq d$, then the polytope $\operatorname{conv}(K)$ is called a *k-simplex*. Any l-face F of a k-simplex

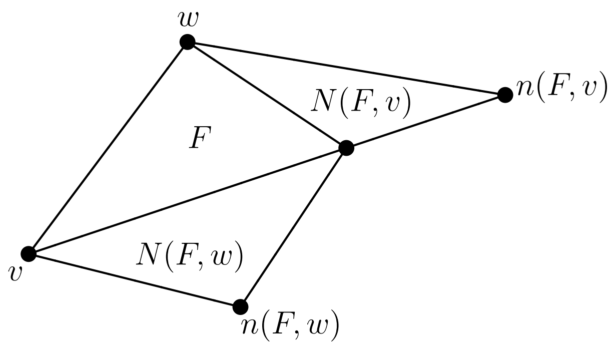


Figure 1.1: This illustrates the concepts of neighboring facets and neighboring vertices.

P, $0 \leq l \leq k$, is the convex hull of $l+1$ vertices of P, and vice versa ([14] p.53).
A polytope P is called *simplicial* if every proper face $F \in \mathcal{F}(P)$ is a simplex. Equivalently, any facet of a d-polytope P is a $(d-1)$-simplex ([27] proposition 2.16).
Let F be a facet of a simplicial d-polytope P and v a vertex of F. The set $R = \operatorname{conv}(\mathcal{V}(F) \setminus \{v\})$ is a facet of F, hence a $(d-2)$-face of P, i.e. a ridge. As every ridge of a polytope is the intersection of precisely two facets ([12] theorem II.1.10), there is a unique facet of P that intersects F in R. We call this facet a *neighboring facet* of F, and denote it by $N(F, v)$. The facet $N(F, v)$ is a $(d-1)$-simplex, so there is a unique vertex in $\mathcal{V}(N(F, v))$, that is not in R. Call this a *neighboring vertex* of F and denote it by $n(F, v)$. See figure 1.1 for an illustration.
Obviously, any facet of a simplicial d-polytope has exactly d different neighboring facets and at most d different neighboring vertices.
When F is a $(d-1)$-simplex in $\mathbb{R}^d$, where $0 \notin \operatorname{aff}(F)$, $\mathcal{V}(F)$ will be a basis of $\mathbb{R}^d$. The basis of $\mathbb{R}^d$ dual to this is a set of elements $\{u_F^v \mid v \in \mathcal{V}(F)\}$ in $\mathbb{R}^d$, such that

$$\langle u_F^v, w \rangle = \begin{cases} 1 & w = v \\ 0 & w \neq v \end{cases}$$

for every $w \in \mathcal{V}(F)$. In other words: For any $x \in \mathbb{R}^d$ the number $\langle u_F^v, x \rangle$ is the v-coordinate of x with respect to the basis $\mathcal{V}(F)$.
We will now see how u_F and $u_{N(F,v)}$ are related for any facet F of a simplicial polytope P, $0 \in \operatorname{int} P$, and any $v \in \mathcal{V}(F)$.

Lemma 1.5. *Let P be a simplicial polytope containing the origin in the* *interior. Let F be a facet of P and $v \in \mathcal{V}(F)$. Let F' be the neighboring facet $N(F,v)$ and v' the neighboring vertex $n(F,v)$.*

1. *For any point $x \in \mathbb{R}^d$,*

$$\langle u_{F'}, x\rangle = \langle u_F, x\rangle + (\langle u_{F'}, v\rangle - 1)\langle u_F^v, x\rangle \tag{1.1}$$

   In particular,

   - $\langle u_F^v, x\rangle < 0$ *if and only if* $\langle u_{F'}, x\rangle > \langle u_F, x\rangle$.
   - $\langle u_F^v, x\rangle > 0$ *if and only if* $\langle u_{F'}, x\rangle < \langle u_F, x\rangle$.
   - $\langle u_F^v, x\rangle = 0$ *if and only if* $\langle u_{F'}, x\rangle = \langle u_F, x\rangle$.

2. $\langle u_F^v, v'\rangle < 0$.

3. $\langle u_F^v, v'\rangle \cdot \langle u_{F'}^{v'}, v\rangle = 1$.

Proof. The vertices of F is a basis of $\mathbb{R}^d$, and equation (1.1) holds for every $w \in \mathcal{V}(F)$, hence for every $x \in \mathbb{R}^d$.
The vertex v is not in F', and then the term $\langle u_{F'}, v\rangle - 1$ is negative. From this the remaining statements in 1 follows.
Statement 2 is just a corollary to statement 1.
Statement 3 follows from

$$v' - \langle u_F^v, v'\rangle v = \sum_{w \in F \cap F'} \langle u_F^w, v'\rangle w.$$

□

1.2 Smooth Fano polytopes

In this section we define our object of study: Smooth Fano polytopes. We shall gradually zoom in on them. First we take a look at reflexive polytopes, then simplicial reflexive ones and finally smooth Fano polytopes.

1.2.1 Reflexive polytopes

A polytope P in $\mathbb{R}^d$ is called a *lattice* polytope if $\mathcal{V}(P) \subset \mathbb{Z}^d$. A lattice polytope P in $\mathbb{R}^d$ is called *reflexive*, if $0 \in \mathrm{int}P$ and P^* is also a lattice polytope. In other words, a lattice polytope P, $0 \in \mathrm{int}P$, is called reflexive if u_F is a lattice point for every facet F of P.
The notion of a reflexive polytope was introduced by Batyrev ([3]).

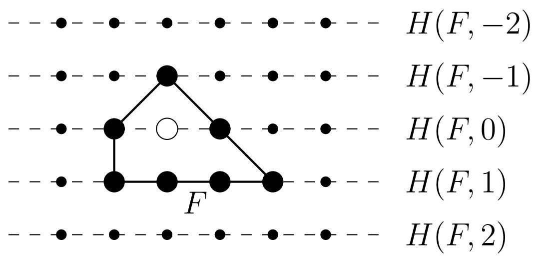


Figure 1.2: When P is a reflexive polytope, a facet F of P will "slice" the lattice into disjoint lattice hyperplanes.

Two lattice polytopes P_1, P_2, $0 \in \text{relint} P_1$ and $0 \in \text{relint} P_2$, are said to be *isomorphic*, denoted $P_1 \cong P_2$, if there exists a *unimodular transformation* $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, i.e. a bijective linear map φ with $\varphi(\mathbb{Z}^d) = \mathbb{Z}^d$, such that $\varphi(P_1) = P_2$.
For given $d \geq 1$ there are only finitely many reflexive d-polytopes up to isomorphism (see [5] section 2 for a survey).
Reflexive polytopes (up to isomorphism) have been completely classified up to dimension 4 by computer ([17],[18]), and their number grows fast: In dimension 2 there are 16 isomorphism classes (see [20] proposition 2.1 for a nice picture of representatives). In dimension 3 there are 4319 isomorphism classes, while in dimension 4 the number of isomorphism classes has grown to 473800776.
When F is a facet of a reflexive polytope $P \subset \mathbb{R}^d$, we define these lattice hyperplanes

$$H(F, i) := \{x \in \mathbb{Z}^d \mid \langle u_F, x \rangle = i\} \quad , \ i \in \mathbb{Z}.$$

As $u_F \in \mathbb{Z}^d$, every lattice point x is in exactly one of the lattice hyperplanes, namely $H(F, \langle u_F, x \rangle)$. Certainly, $\mathcal{V}(F) \subset H(F, 1)$. The remaining vertices of P are situated in the hyperplanes $H(F, i)$ for $i \in \{0, -1, -2, \ldots\}$. See figure 1.2 for an illustration. The integer $1 - \langle u_F, v \rangle$ is sometimes called the *integral distance between* F *and* $v \in \mathcal{V}(P)$.

1.2.2 Simplicial reflexive polytopes

Let us now turn to simplicial reflexive d-polytopes. Due to the complete classification of reflexive d-polytopes, $d \leq 4$, the simplicial ones have been classified as well. In dimension 2 there are (of course) 16 isomorphism classes of simplicial reflexive polytopes, while in dimension 3 and 4 there are 194 and 5450 isomorphism classes respectively ([20] remarks preceeding theorem 5.8).
Reflexivity guarentees that $u_F \in \mathbb{Z}^d$ for every facet F of a simplicial reflexive polytope P. But in general, the points $u_F^w \notin \mathbb{Z}^d$ for arbitrary facets F of P

and $w \in \mathcal{V}(F)$. However,

$$\{u_F^w \mid w \in \mathcal{V}(F)\} \subset \mathbb{Z}^d \quad \iff \quad \mathcal{V}(F) \text{ is a basis of } \mathbb{Z}^d. \tag{1.2}$$

The fact that for any $x \in \mathcal{V}(P)$,

$$\langle u_F, x\rangle \leq 0 \text{ if and only if } x \notin F,$$

can sharpen lemma 1.5 and put some restrictions on the points of P.

Lemma 1.6. *Let P be a simplicial reflexive polytope. For every facet F of P and every vertex $v \in \mathcal{V}(F)$ we have*

$$\langle u_F, x\rangle - 1 \leq \langle u_F^v, x\rangle$$

for any $x \in P$. In case of equality, x is on the facet $N(F, v)$.

Proof. The inequality is obvious, when $\langle u_F^v, x\rangle \geq 0$. So assume $\langle u_F^v, x\rangle < 0$. Since $x \in P$, $\langle u_{F'}, x\rangle \leq 1$ with equality if and only if $x \in F$. From lemma 1.5 we then have

$$\langle u_F, x\rangle - 1 \leq (1 - \langle u_{F'}, v\rangle)\langle u_F^v, x\rangle \leq \langle u_F^v, x\rangle$$

as $\langle u_{F'}, v\rangle \leq 0$. □

A vertex x of a simplicial reflexive polytope P will "jump" between hyperplanes $H(F, i)$ and $H(N(F, v), j)$, $v \in \mathcal{V}(F)$, according to its v-coordinate (lemma 1.5.(1)). In particular, if $x \in \mathcal{V}(P) \cap H(F, 0)$ for some facet F, then x is actually a neighboring vertex of F.

Lemma 1.7 ([10] section 2.3 remarks 5(2), [20] lemma 5.5)**.** *Let F be a facet and $x \in H(F, 0)$ be vertex of a simplicial reflexive d-polytope P. Then x is a neighboring vertex of F.*
More precisely, for every $w \in \mathcal{V}(F)$ where $\langle u_F^w, x\rangle < 0$, x is equal to $n(F, w)$.
In particular, for every $w \in \mathcal{V}(F)$ there is at most one vertex $x \in H(F, 0) \cap \mathcal{V}(P)$, with $\langle u_F^w, x\rangle < 0$.
As a consequence, there are at most d vertices of P in $H(F, 0)$.

Proof. Since $\langle u_F, x\rangle = \sum_{w\in\mathcal{V}(F)}\langle u_F^w, x\rangle = 0$ and $x \neq 0$, there is at least one w for which $\langle u_F^w, x\rangle < 0$. Choose such a w and consider the neighboring facet $F' = N(F, w)$. By lemma 1.5.(1) we get that $0 < \langle u_{F'}, x\rangle \leq 1$. As P is reflexive, $\langle u_{F'}, x\rangle = 1$ and then $x = n(F, w)$. A neighboring vertex is unique, and the third statement follows.
There are at most d different neighboring vertices of any facet F. And any vertex in $H(F, 0)$ is a neighboring vertex of F. Hence the result. □

We will use the lemmas 1.5–1.7 again and again in our arguments in chapter 2 and 3, so a good intuition of these basic facts is important.

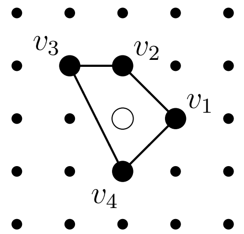


Figure 1.3: This is a smooth Fano 2-polytope with 4 vertices. Let $F = \text{conv}(\{v_1, v_2\})$. Then the vertex v_3 is in the lattice hyperplane $H(F, 0)$ and v_4 is in $H(F, -1)$. The neighboring vertex $n(F, v_1)$ of F is $v_3 = -v_1 + v_2$, and $\langle u_F^{v_1}, v_3 \rangle = -1$ as lemma 1.9 states.

1.2.3 Smooth Fano polytopes

Smooth Fano d-polytopes are our main objects of study, and now is the time to define them.

Definition 1.8. *A simplicial reflexive polytope P in $\mathbb{R}^d$ is called a* smooth Fano d-polytope *if the vertices of every facet F of P is a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^d$.*

Smooth Fano d-polytopes inherit their strange name from the toric varieties they correspond to: From a smooth Fano d-polytope one can construct a complete regular fan of cones in $\mathbb{R}^d$ (see [13] chapter 1). This fan defines a smooth projective d-dimensional toric variety with ample anti-canonical divisor (see [12] chapter VII.8). Such varieties are called smooth Fano toric d-folds. Isomorphism classes of smooth Fano d-polytopes are in one-to-one correspondence with isomorphism classes of smooth Fano toric d-folds (see [4] theorem 2.2.4), hence the classification of these varieties can be carried out by classifying the relevant polytopes. For more about the exciting connection between toric varieties and polytopes we refer to the books [12] and [13].
There are of course finitely many smooth Fano d-polytopes (up to isomorphism) for any d. Later we shall give a short proof of this well-known fact (theorem 1.16).
In the next two chapters we shall consider some results on the classification of smooth Fano d-polytopes.
In appendix A one can find a table of the number of isomorphism classes of smooth Fano d-polytopes for given $d \leq 8$ and given number of vertices.
Now we shall prove some simple identities and inequalities.

Lemma 1.9. *Let F be a facet of a smooth Fano polytope P and $v \in \mathcal{V}(F)$. For simplicity, set $F' = N(F, v)$ and $v' = n(F, v)$.*

Then

1. $\langle u_F^v, v' \rangle = -1.$

2. $\langle u_F, v' \rangle = \langle u_{F'}, v \rangle.$

3. *For every* $x \in \mathbb{R}^d$

$$\langle u_{F'}, x \rangle = \langle u_F, x \rangle + \langle u_F^v, x \rangle \big(\langle u_F, v' \rangle - 1 \big).$$

4. *For any* $w \in \mathcal{V}(P)$

$$\langle u_F^v, w \rangle \geq \begin{cases} 0 & \langle u_F, w \rangle = 1 \\ -1 & \langle u_F, w \rangle = 0 \\ \langle u_F, w \rangle & \langle u_F, w \rangle < 0 \end{cases}$$

5. *Let* $w \in \mathcal{V}(P)$. *If* $\langle u_F^v, w \rangle < 0$, *then* $\langle u_F, v' \rangle \geq \langle u_F, w \rangle$ *with equality if and only if* $w = v'$.

   In other words: Among the vertices of P *with negative* v*-coordinate (with respect to the basis* $\mathcal{V}(F)$*) the neighboring vertex* $n(F, v)$ *is the one with shortest integral distance to* F.

Proof. 1. By lemma 1.5 we know that $\langle u_F^v, v' \rangle < 0$. As both $\mathcal{V}(F)$ and $\mathcal{V}(F')$ are bases of the lattice $\mathbb{Z}^d$, $\langle u_F^v, v' \rangle$ must be equal to -1.

2. The second assertion follows from the $\mathbb{Z}$-linear relation

$$v + v' = \sum_{w \in \mathcal{V}(F \cap F')} \langle u_F^w, v' \rangle w = \sum_{w \in \mathcal{V}(F \cap F')} \langle u_{F'}^w, v \rangle w.$$

3. The equality follows from lemma 1.5 and statement 2.

4. If $\langle u_F, w \rangle = 1$, then $w \in \mathcal{V}(F)$ and $\langle u_F^v, w \rangle$ is either 0 or 1.

   If $\langle u_F, w \rangle = 0$ and $\langle u_F^v, w \rangle < 0$ then $w = v'$ (by lemma 1.7) and $\langle u_F^v, w \rangle = -1$.

   Suppose $\langle u_F, w \rangle < 0$ and suppose $\langle u_F^v, w \rangle < \langle u_F, w \rangle$. Then $\langle u_F^v, w \rangle = \langle u_F, w \rangle - 1$ and $w = v'$ by lemma 1.6. But $\langle u_F^v, w \rangle < -1$, which contradicts statement 1.

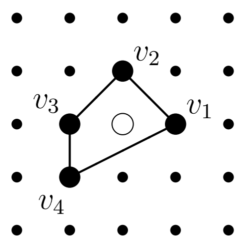


Figure 1.4: This illustrates lemma 1.7 and lemma 1.9. v_3 is a neighboring vertex of the facet $F = \operatorname{conv}(\{v_1, v_2\})$ and $v_3 \in H(F, -1)$. Let F' be the neighboring facet $N(F, v_1)$ of F. Since $v_4 = -v_1 - v_2$ and $v_4 \in H(F, -2)$ we get by lemma 1.9 that $\langle u_{F'}, v_4 \rangle = -2 + (-1)(-1 - 1) = 0$. As v_4 is in the hyperplane $H(F', 0)$, it must be a neighboring vertex of F' by lemma 1.7. More precisely, $v_4 = v_3 - v_2$ and is therefore equal to $n(F', v_2)$. Notice that both $\langle u_F^{v_1}, v_3 \rangle$ and $\langle u_F^{v_1}, v_4 \rangle$ are negative, but v_3 is the neighboring vertex $n(F, v_1)$ by lemma 1.9.(5), since $\langle u_F, v_3 \rangle > \langle u_F, v_4 \rangle$.

5. Assume $\langle u_F^v, w \rangle < 0$. Consider the equality in statement 3.

$$\langle u_{F'}, w \rangle = \langle u_F, w \rangle + \langle u_F^v, w \rangle \big(\langle u_F, v' \rangle - 1 \big).$$

The left hand side is smaller than or equal to 1.

$$1 \geq \langle u_F, w \rangle + \langle u_F^v, w \rangle \big(\langle u_F, v' \rangle - 1 \big).$$

The right hand side is greater than or equal to $\langle u_F, w \rangle - (\langle u_F, v' \rangle - 1)$. So

$$1 \geq \langle u_F, w \rangle - (\langle u_F, v' \rangle - 1),$$

and the last statement follows.

□

1.2.4 The dual of a smooth Fano polytope

Let us examine the dual $P^* \subset \mathbb{R}^d$ of a smooth Fano d-polytope $P \subset \mathbb{R}^d$. As P is a simplicial d-polytope, P^* is a simple d-polytope. Recall that a d-polytope is called *simple*, if each vertex of the polytope has exactly d outgoing edges. Lemma 1.10 below states a relation we shall later refer to.

Lemma 1.10. *Let F be a facet and $v \in \mathcal{V}(F)$ a vertex of a smooth Fano polytope $P \subset \mathbb{R}^d$.*

Then $-\langle u_F, n(F,v)\rangle$ is the number of lattice points in the relative interior of the edge E of P^* corresponding to the ridge $F \cap N(F,v)$ of P, i.e $|E \cap \mathbb{Z}^d| = 2 - \langle u_F, n(F,v)\rangle$.

Proof. By lemma 1.9.(3) the lattice points u_F and $u_{N(F,v)}$ satisfy

$$u_{N(F,v)} = u_F + (\langle u_F, n(F,v)\rangle - 1)u_F^v.$$

Since $\{u_F^v | v \in \mathcal{V}(F)\}$ is a basis of $\mathbb{Z}^d$, the lattice points on the line segment connecting u_F and $u_{N(F,v)}$ are

$$u_F \ , \ u_F - u_F^v \ , \ u_F - 2u_F^v \ , \ \ldots \ , \ u_F - (\langle u_F, n(F,v)\rangle - 1)u_F^v.$$

So there are $-\langle u_F, n(F,v)\rangle + 2$ lattice points on the edge E of P corresponding to the ridge $F \cap N(F,v)$ of P and $-\langle u_F, n(F,v)\rangle$ lattice points in the relative interior of E. □

1.3 Special facets

This section is devoted to a simple, but quite useful concept: The notion of special facets of a polytope containing the origin in the interior. The existence of special facets is essential for our classification results and for the SFP-algorithm
The concept of special facets is due to the author of this thesis.

1.3.1 Definition and basic properties

When P is any polytope, we define ν_P to be the sum of the vertices of P,

$$\nu_P := \sum_{v \in \mathcal{V}(P)} v.$$

The following definition is due to the author.

Definition 1.11. *Let P be a polytope containing the origin in the interior. A facet F of P called* special[1]*, if ν_P is a non-negative linear combination of $\mathcal{V}(F)$.*

Here are some obvious properties of special facets of polytopes having the origin in the interior.

[1]The author would have liked a better name for this concept, but could not think of any. The name *heavy facet* has been suggested, as the center of gravity of the vertex set is contained in the simplex $\operatorname{conv}(F \cup \{0\})$.

Lemma 1.12. *Let P be a d-polytope with $0 \in \text{int}P$.*
Then

1. *P has at least one special facet.*
2. *Every facet of P is special if and only if $\nu_P = 0$.*
3. *If P is simplicial, then a facet F is special if and only if $\langle u_F^v, \nu_P \rangle \geq 0$ for all $v \in \mathcal{V}(F)$.*
4. *If P is simplicial and reflexive, then $0 \leq \langle u_F, \nu_P \rangle \leq d - 1$ for any special facet F of P.*

Proof. 1. If ν_P is the origin, any facet of P is a special facet. Suppose $\nu_P \neq 0$. Consider the halfline $L = \mathbb{R}_{\geq 0}\nu_P$. L intersects the boundary of P in a unique point $q \neq 0$, which is on some facet F of P. Then

$$r\nu_P = q = \sum_{v \in \mathcal{V}(F)} a_v v, \quad \sum_{v \in \mathcal{V}(F)} a_v = 1\ ,\ 0 \leq a_v \leq 1\ \forall v \in \mathcal{V}(F)$$

for some $r > 0$. Divide by r.

2. If $\nu_P = 0$, then every facet is special. Conversely, suppose $\nu_P \neq 0$. By the arguments above we can find a facet F such that $-\nu_P$ is a positive linear combination of a subset V of $\mathcal{V}(F)$.

$$-\nu_P = \sum_{v \in V \subseteq \mathcal{V}(F)} b_v v,\ \ b_v > 0\ \forall v \in V \subseteq \mathcal{V}(F).$$

So $\langle u_F, -\nu_P \rangle > 0$.

Suppose that F is special. Then there is a subset W of $\mathcal{V}(F)$, so that

$$\nu_P = \sum_{v \in W \subseteq \mathcal{V}(F)} a_v v,\ \ a_v > 0\ \forall v \in W \subseteq \mathcal{V}(F).$$

Then $\langle u_F, \nu_P \rangle > 0$. But this is a contradiction. Hence F is not a special facet.

3. The vertex set $\mathcal{V}(F)$ of any facet is a basis of $\mathbb{R}^d$. So ν_P has a unique representation as a linear combination of $\mathcal{V}(F)$.

4. Since $\langle u_F, v \rangle \leq 1$ for every vertex v with equality if and only if $v \in \mathcal{V}(F)$, we have $0 \leq \langle u_F, \nu_P \rangle \leq d$. We only need to prove that $\langle u_F, \nu_P \rangle = d$ cannot happen. So assume this to be the case. Then there are no vertices of P in the hyperplanes $H(F, i)$ for $i < 0$, and then 0 is not an interior point. A contradiction.

□

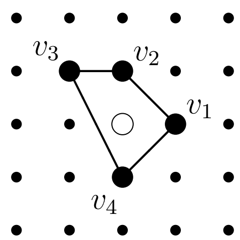


Figure 1.5: The sum of the vertices is the lattice point v_2. The facets $F = \text{conv}(\{v_1, v_2\})$ and $F' = \text{conv}(\{v_2, v_3\})$ are special, while the other two facets are not.

Corollary 1.13. *Any smooth Fano d-polytope has at least two special facets.*

Proof. Suppose not. Let F be the unique special facet of a smooth Fano d-polytope P. Then $\langle u_F^w, \nu_P \rangle \geq 1$ for every $w \in \mathcal{V}(F)$. And then $\langle u_F, \nu_P \rangle \geq d$, which contradicts lemma 1.12. □

Corollary 1.13 does not hold for simplicial reflexive polytopes, as the polytope in figure 1.2 has only one special facet.
Notice that $\langle u_F, \nu_P \rangle$ can take any value between 0 and $d-1$, as the convex hull of the points

$$e_1 \;,\; \ldots \;,\; e_d \;,\; -e_1 - \ldots - e_{d-1} + ke_d \;,\; -e_d \quad ,\; 0 \leq k \leq d-1.$$

is a smooth Fano d-polytope.
For arbitrary facets F and arbitrary vertices v of any smooth Fano d-polytope P there are lower bounds on $\langle u_F, v \rangle$ (see [10] lemma 11):

$$\langle u_F, v \rangle \geq \frac{d-1}{d-2}(1 - (d-1)^{|\mathcal{V}(P)|-d}).$$

If we restrict to special facets we can greatly improve this bound.

Lemma 1.14. *Let F be a special facet of a simplicial reflexive d-polytope P. Then $\langle u_F, v \rangle \geq -d$ for every vertex v of P. In other words, every vertex of P is in one of the hyperplanes $H(F, 1), \ldots, H(F, -d)$.*

Proof. Obvious, as

$$0 \leq \langle u_F, \nu_P \rangle = d + \sum_{i \leq -1} i \cdot |\mathcal{V}(P) \cap H(F, i)|.$$

□

1.3.2 Bounds on the number of vertices

When P is an arbitrary reflexive d-polytope the following bound on the number of vertices has been shown ([20] theorem 5.4): $|\mathcal{V}(P)| \leq 2d\alpha$, where $\alpha = \max\{|\mathcal{V}(F)| \mid F \text{ facet of } P\}$. It is conjectured that $|\mathcal{V}(P)| \leq 6^{\frac{d}{2}}$ for any reflexive d-polytope with equality if and only if d is even and P^* is isomorphic to the convex hull of the points ([20] conjecture 5.2)

$$\pm e_1, \ldots, \pm e_d, \pm(e_1 - e_2), \ldots, \pm(e_{d-1} - e_d).$$

If one restricts to simplicial reflexive polytopes, there exists a much better bound. It was conjectured by Batyrev in the case of smooth Fano polytopes (see [12] p.337) and by Nill in the simplicial reflexive case ([20] corollary 5.6). It was first proved by Casagrande.
The proof given here is simpler than the original proof in [9].

Theorem 1.15 ([9] theorem 1)**.** *Any simplicial reflexive d-polytope P has at most $3d$ vertices.*

Proof. Consider a special facet F. Now

$$0 \leq \langle u_F, \nu_P \rangle = \sum_{i \in \mathbb{Z}, i \leq 1} i \cdot |H(F,i) \cap \mathcal{V}(P)|.$$

There are exactly d vertices in $H(F,1)$, namely $\mathcal{V}(F)$. So we get

$$0 \leq \langle u_F, \nu_P \rangle = d + \sum_{i \in \mathbb{Z}, i \leq -1} i \cdot |H(F,i) \cap \mathcal{V}(P)|.$$

So there can be at most d vertices in the set

$$\mathcal{V}(P) \cap \bigcup_{i \leq -1} H(F,i).$$

There are at most d vertices in $H(F,0)$ (lemma 1.7), and we finally arrive at

$$|\mathcal{V}(P)| \leq 3d.$$

$\square$

In the next chapter we shall see that simplicial reflexive d-polytopes are very close to smooth Fano d-polytopes, when the number of vertices is close to $3d$.

1.3.3 Finitely many isomorphism classes

With the notion of special facets we can also make a short proof of the well-known fact, that there are finitely many isomorphism classes of smooth Fano d-polytopes for each given d. There are other short proofs of this fact ([10] corollary 7, [4] theorem 2.1.13).

Theorem 1.16. *For each $d \geq 1$ there are only finitely many isomorphism classes of smooth Fano d-polytopes.*

Proof. Let P be any smooth Fano d-polytope. We can apply a unimodular transformation, so that $\text{conv}\{e_1, \dots, e_d\}$ is a special facet of P. Then by lemma 1.14 $-d \leq \langle u_F, v\rangle \leq 1$ for every vertex v of P. As a consequence of the lower bounds on $\langle u_F^{e_i}, v\rangle$ (lemma 1.9.(4)), $\mathcal{V}(P)$ is contained in a finite set of $\mathbb{Z}^d$. Thus there are finitely many possibilities for $\mathcal{V}(P)$. $\square$

In the proof of theorem 1.16 we use the fact, that for any given d there exists a certain finite subset of $\mathbb{Z}^d$, such that $\mathcal{V}(P)$ is contained in this subset for every smooth Fano d-polytope P having $\text{conv}\{e_1, \dots, e_d\}$ as a special facet. We will exploit this fact again in section 3.2, where we describe an algorithm that systematically goes through certain finite subsets of $\mathbb{Z}^d$ in order to classify smooth Fano d-polytopes (up to isomorphism), by constructing at least one representative for each isomorphism class.

Chapter 2

Classifications under additional assumptions

In this chapter we consider some classification results on smooth Fano d-polytopes that hold in every dimension d. To obtain these results one has to assume something extra about the polytopes, some kind of central-symmetry or that the vertices of the polytope are few or close to the upper bound. Some of these results have been generalized to simplicial reflexive polytopes.
The following new material is obtained and presented in this chapter

- Classification of smooth Fano d-polytopes P having a fixed number of lattice points on the edges of the dual polytopes P^* (corollary 2.4 and theorem 2.7).

- Classification of terminal simplicial reflexive d-polytopes with $3d-1$ vertices (theorem 2.17), which is presented in the paper [29] by the author.

- New proofs of the classification of simplicial reflexive d-polytopes with $3d$ vertices (theorem 2.11) and of the classification of simplicial centrally symmetric reflexive d-polytopes with $3d-1$ vertices (theorem 2.16).

The structure of the chapter is as follows: In section 2.1 we recall the known classification of pseudo-symmetric smooth Fano polytopes. In section 2.2 we prove a new classification result on smooth Fano polytopes, whose dual polytopes have a fixed number of lattice points on the edges. In section 2.3 we recall some known classifications in case of few vertices, and in section 2.4 we reprove the classification of simplicial reflexive d-polytopes with $3d$ vertices and prove the new classification result in case of $3d-1$ vertices.

2.1 Central and pseudo symmetry

A polytope is called *centrally symmetric* if $v \in P$ implies $-v \in P$. Centrally symmetric smooth Fano polytopes have been classified ([24]). More generally, a polytope P is called *pseudo-symmetric*, if there exists a facet F of P, such that $-F$ is also a facet. The notion of pseudo-symmetry is due to Ewald, and pseudo-symmetric smooth Fano polytopes have also been classified ([11]).
We state these classification results below, as the smooth Fano polytopes we shall consider in the coming subsections often turn out to be centrally or pseudo-symmetric.
First, we need to define some particular pseudo-symmetric smooth Fano polytopes: Let k be an positive even integer. The convex hull (or any unimodular copy of it)

$$\operatorname{conv}\{\pm e_1, \ldots, \pm e_k, \pm(e_1 + \ldots + e_k)\}$$

is a smooth Fano k-polytope (when we embed it in its affine hull) and is called a *del Pezzo k-polytope* and denoted by V_k.
The convex hull (or any unimodular copy of it)

$$\operatorname{conv}\{\pm e_1, \ldots, \pm e_k, e_1 + \ldots + e_k\}$$

is a smooth Fano k-polytope (in its affine hull) and is called a *pseudo del Pezzo k-polytope* and denoted by $\tilde{V}_k$. These polytopes were introduced by Ewald in [11].
And now some terminology: Suppose K and L are convex sets in $\mathbb{R}^d$ containing 0 in their relative interior. If $K \cap L = \{0\}$ then we define $K \circ L := \operatorname{conv}(K \cup L)$ and we say that the a set equal to $K \circ L$ *splits* into K and L. In particular, if P_i, $i = 1, 2$, are smooth Fano d_i-polytopes in $\mathbb{R}^{d_i}$, the set $P_1 \circ P_2 \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is a smooth Fano $(d_1 + d_2)$-polytope which splits into P_1 and P_2.

Theorem 2.1 ([24], [11]). *Any pseudo-symmetric smooth Fano d-polytope P splits into line segments, del Pezzo polytopes and pseudo del Pezzo polytopes. That is, P is isomorphic to a polytope*

$$\underbrace{L \circ \ldots \circ L}_{i \text{ times}} \circ V_{a_1} \circ \ldots V_{a_j} \circ \tilde{V}_{b_1} \circ \ldots \circ \tilde{V}_{b_k},$$

where each L is a lattice polytope isomorphic to the line segment $[e_1, -e_1]$ and $i + a_1 + \ldots + a_j + b_1 + \ldots + b_k = d$.

Theorem 2.1 has been generalized to pseudo-symmetric simplicial reflexive polytopes by Nill ([21] theorem 2.5). It turns out that any pseudo-symmetric

simplicial reflexive polytope, whose vertices span the lattice $\mathbb{Z}^d$, is a smooth Fano polytope ([21] corollary 2.6).
Finally, it should be mentioned that Casagrande has proved some classification results in case of "many pairs of centrally vertices" ([8] theorem 5 and proposition 7). One of these is a direct generalization of theorem 2.1: For any smooth Fano d-polytope P, consider the linear subspace H spanned by centrally symmetric pairs of vertices of P. If $\dim H = d$, then P is pseudo-symmetric.

2.2 Fixed number of lattice points on dual edges

The main new result in this section is the classification of smooth Fano polytopes whose dual polytopes have a fixed number of lattice points on every edge.
For any facet F of a smooth Fano polytope P and any $v \in \mathcal{V}(F)$, recall the observation in lemma 1.10:

$$-\langle u_F, n(F,v)\rangle = |\mathbb{Z}^d \cap \mathrm{relint} E|,$$

where E is the edge of P^* corresponding to the ridge $F \cap N(F,v)$ of P.
We begin by studying the case where there are no lattice points in the relative interior of any edge of the dual (subsection 2.2.1). This is equivalent to $\langle u_F, n(F,v)\rangle = 0$ for every facet F and every vertex $v \in \mathcal{V}(F)$.
Then we move on to study the case where there exists a positive integer i, such that $\langle u_F, n(F,v)\rangle = -i$ for every facet F and every $v \in \mathcal{V}(F)$ (subsection 2.2.2).

2.2.1 Close neighbors

We find it convenient to say that a facet F of a simplicial reflexive polytope has *close neighbors*, if $n(F,w) \in H(F,0)$ for every $w \in \mathcal{V}(F)$.
The topic of this subsection is the study of simplicial reflexive polytopes where every facet has close neighbors. It turns out that these polytopes are indeed centrally symmetric smooth Fano polytopes (theorem 2.3).
We begin with a lemma.

Lemma 2.2 ([20] lemma 5.5). *Let F be a facet of a simplicial reflexive d-polytope P. Suppose there are at least $d-1$ vertices $v_1,\dots,v_{d-1}$ in $\mathcal{V}(F)$, such that $n(F,v_i) \in H(F,0)$ and $\langle u_F^{v_i}, n(F,v_i)\rangle = -1$ for every $1 \le i \le d-1$. Then $\mathcal{V}(F)$ is a basis of the lattice $\mathbb{Z}^d$.*

Proof. According to lemma 1.5 we have

$$\langle u_{N(F,v_i)}, v_j \rangle = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}$$

for every $1 \leq i \leq d-1$, $1 \leq j \leq d$. Consider the following set of lattice points in $\mathbb{Z}^d$

$$\{u_F - u_{N(F,v_1)}, \ldots, u_F - u_{N(F,v_{d-1})}, \sum_{i=1}^{d-1} u_{N(F,v_i)} - (d-2)u_F\}.$$

This set is obviously a basis of $\mathbb{R}^d$, dual to the basis $\mathcal{V}(F)$. Since both bases consists of lattice points, they are lattice bases of $\mathbb{Z}^d$. □

Now we can prove the main result of the subsection.

Theorem 2.3. *If every facet of a simplicial reflexive d-polytope P has close neighbors, then d is even and P is a centrally symmetric smooth Fano polytope that splits into del Pezzo polytopes,*

$$P \cong V_{c_1} \circ \ldots \circ V_{c_n}, \quad c_1 + \ldots + c_n = d.$$

Proof. Let F be any facet of P and let $v \in \mathcal{V}(F)$ be a vertex of F. Then the neighboring vertex $w = n(F, v)$ satisfies $\langle u_F, n(F, v)\rangle = 0$ by the assumptions. Consider the neighboring facet $G = N(F, v)$. By the assumptions we have $\langle u_G, v\rangle = 0$. Now use lemma 1.5.(1) to calculate $\langle u_F^v, w\rangle$.

$$1 = \langle u_G, w\rangle = \langle u_F, w\rangle + (\langle u_G, v\rangle - 1)\langle u_F^v, w\rangle = 0 + (0-1)\langle u_F^v, w\rangle.$$

So $\langle u_F^v, w\rangle = -1$. This holds for all the neighboring vertices of F. By lemma 2.2, $\mathcal{V}(F)$ is a basis of $\mathbb{Z}^d$. As F was arbitrary P is a smooth Fano polytope. We now wish to show that P is centrally symmetric. For this let v be any vertex of P, and F a facet such that v is a neighboring vertex of F. So $v \in H(F, 0)$. Without loss of generality we can assume that the vertices of F is the standard basis of $\mathbb{Z}^d$, and that v is of the form

$$v = -e_1 - \ldots - e_k + a_{k+1}e_{k+1} + \ldots + a_d e_d,$$

where $a_i \geq 0$ for every $k+1 \geq i \geq d$. At least one of these a_i is positive, say a_{k+1}. As F has close neighbors, there is a vertex $v \neq w \in H(F, 0)$, so that $w = n(F, e_{k+1})$. Without loss of generality we can assume that w has the form

$$w = b_1 e_1 + \ldots + b_k e_k - e_{k+1} - \ldots - e_l + b_{l+1}e_{l+1} + \ldots + b_d e_d,$$

where $b_i \geq 0$ for $i > l$. We also have that $b_i \geq 0$ for every $1 \leq i \leq k$.
Our aim is to show that $v = -w$. So, suppose $b_i = 0$ for some i, $1 \leq i \leq k$. Wlog $i = 1$. Consider the facet $F_1 = N(F, e_1)$.

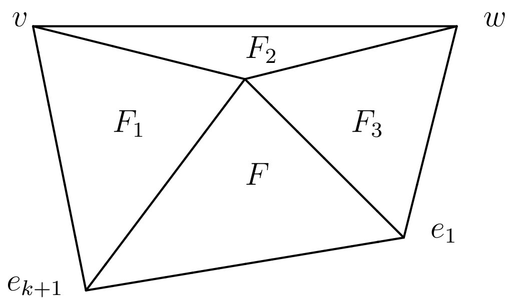
$$\mathcal{V}(F_1) = \{v, e_2, \ldots, e_d\}.$$

As $\langle u_{F_1}, w\rangle = 0$ (lemma 1.5), w is a neighboring vertex of F_1 (lemma 1.7). More precisely, $w = n(F_1, e_{k+1})$. Let $F_2 = N(F_1, e_{k+1})$. Then

$$\mathcal{V}(F_2) = \{v, e_2, \ldots, e_k, w, e_{k+2}, \ldots, e_d\}.$$

Consider the facet $F_3 = N(F, e_{k+1})$. It vertices are

$$\mathcal{V}(F_3) = \{e_1, e_2, \ldots, e_k, w, e_{k+2}, \ldots, e_d\}.$$



The facets F_2 and F_3 obviously share a common ridge, so $n(F_3, e_1) = v$. But (by lemma 1.9)

$$\langle u_{F_3}, v\rangle = \langle u_F, v\rangle + (\langle u_F, w\rangle - 1)\langle u_F^{e_{k+1}}, v\rangle = -a_{k+1} < 0.$$

And then F_3 does not have close neighbors. A contradiction.
We conclude that $b_i \geq 1$ for all $1 \leq i \leq k$.
Using the same argument we can conclude that $a_i \geq 1$ for all $k+1 \leq i \leq l$.
As $v \in H(F, 0)$ the sum of the coefficients $\langle u_F^{e_i}, v\rangle$ is 0, that is $-k + a_{k+1} + \ldots + a_d = 0$. Since $a_i \geq 1$ for $k+1 \leq i \leq l$, we have that $k \geq l - k$.
Similarly, $w \in H(F, 0)$, so $l - k \geq k$.
We conclude that $2k = l$ and that

$$v = -e_1 - \ldots - e_k + e_{k+1} + \ldots + e_l$$

and that

$$w = e_1 + \ldots + e_k - e_{k+1} - \ldots - e_l.$$

So v, w is a pair of centrally symmetric vertices. As v was arbitrary, we have shown that P is centrally symmetric.
Hence P splits into line segments and del Pezzo polytopes by theorem 2.1. As every facet has close neighbors there can be no line segments in this split. Then d must be even and P must split into del Pezzo polytopes. □

As an immediate consequence of lemma 1.10 and theorem 2.3 we get:

Corollary 2.4. *If P is a smooth Fano d-polytope and there are no lattice points in the relative interior of any edge of P^*, then d is even and P splits into del Pezzo polytopes.*

Finally we shall show that a certain condition on the f-vector of a smooth Fano polytope P, implies that P splits into del Pezzo polytopes. Recall, how the f-vector $(f_{-1}(P), f_0(P), f_1(P), \ldots, f_d(P))$ of a d-polytope P is defined: $f_k(P)$ is the number of k-faces of P. In particular, $f_{-1}(P) = f_d(P) = 1$ and $f_0(P) = |\mathcal{V}(P)|$.
There is a well-known inequality regarding $f_{d-3}(P)$ and $f_{d-2}(P)$ for any smooth Fano d-polytope P. Using the notion of close neighbors, the result can be formulated like this:

Theorem 2.5 ([4] theorem 2.3.7)**.** *Let P be a smooth Fano d-polytope. Then*

$$12 f_{d-3}(P) \geq (3d-4) f_{d-2}(P)$$

with equality if and only if every facet of P has close neighbors.

Combining this equivalence with theorem 2.3 we get

Corollary 2.6. *Let P be a smooth Fano d-polytope. If*

$$12 f_{d-3}(P) = (3d-4) f_{d-2}(P)$$

then P splits into del Pezzo polytopes.

2.2.2 A fixed positive number of lattice points in the relative interior of dual edges

In this subsection we deal with the other case: Smooth Fano polytopes whose duals have a fixed positive number of lattice points in the relative interior of any edge.
For every $k \in \mathbb{Z}_{>0}$ let T_k denote the simplex $\operatorname{conv}\{e_1, \ldots, e_k, -e_1 - \ldots - e_k\}$.

Theorem 2.7. *Let P be a smooth Fano d-polytope. Suppose there exists a positive integer h, such that $n(F,v) \in H(F,-h)$ for every facet F of P and any vertex v of F. Equivalently, suppose every edge on the dual polytope P contains exactly h lattice points in the relative interior.*
Then d is divisible by h and P is isomorphic to the convex hull of the points

$$\begin{gathered} e_1, \ldots, e_d \\ -e_1 - \ldots - e_h, \ldots, -e_{d-h+1} - \ldots - e_d \end{gathered}$$

In other words, P splits into $\frac{d}{h}$ copies of the simplex T_h.

Proof. The proof goes much like the proof of theorem 2.3.
Let v be a vertex of P, and F a facet, such that v is a neighboring vertex of F. Without loss of generality we can assume that $F = \operatorname{conv}\{e_1, \ldots, e_d\}$ and $v = n(F, e_1)$.

$$v = -e_1 + \langle u_F^{e_2}, v\rangle e_2 + \ldots.$$

Suppose there is a j, say $j = 2$, such that $\langle u_F^{e_j}, v\rangle > 0$. Then there exists a vertex $w \neq v$ of P, such that $w = n(F, e_2) \in H(F, -h)$. Consider the vertex w in the basis the facet $F_1 = N(F, e_1)$ provides:

$$\begin{aligned} w &= \langle u_F^{e_1}, w\rangle e_1 - e_2 + \ldots \\ &= -\langle u_F^{e_1}, w\rangle v + (\langle u_F^{e_1}, w\rangle\langle u_F^{e_2}, v\rangle - 1)e_2 + \ldots \end{aligned}$$

There are two cases: $\langle u_F^{e_1}, w\rangle = 0$ and $\langle u_F^{e_1}, w\rangle \geq 1$.
Suppose $\langle u_F^{e_1}, w\rangle = 0$. Then $w \in H(F_1, -h)$ and is a neighboring vertex of F_1. In fact, $w = n(F_1, e_2)$. The vertex set of the facet $N(F_1, e_2)$ is $\{v, w, e_3, \ldots, e_d\}$. Now, consider the facet $F_2 = N(F, e_2)$, whose vertex set is $\{e_1, w, e_3, \ldots, e_d\}$. Hence the neighboring vertex $n(F_2, e_1)$ is v. But $\langle u_{F_2}, v\rangle < -h$, which contradicts the assumptions on the polytope P. So, $\langle u_F^{e_1}, w\rangle \geq 1$. Then for any index k we have the implication:

$$\langle u_F^{e_k}, v\rangle = -1 \quad \Rightarrow \quad \langle u_F^{e_k}, w\rangle \geq 1.$$

As $\langle u_F, v\rangle = -h < 0$, we have

$$|\{k | \langle u_F^{e_k}, w\rangle > 0\}| > |\{k | \langle u_F^{e_k}, v\rangle = -1\}| > |\{k | \langle u_F^{e_k}, v\rangle > 0\}|.$$

Interchange the role of v and w, and get the implication

$$\langle u_F^{e_k}, w\rangle = -1 \quad \Rightarrow \quad \langle u_F^{e_k}, v\rangle \geq 1.$$

for any index k. Similarly,

$$|\{k | \langle u_F^{e_k}, v\rangle > 0\}| > |\{k | \langle u_F^{e_k}, w\rangle = -1\}| > |\{k | \langle u_F^{e_k}, w\rangle > 0\}|.$$

This gives us a contradiction. And we conclude: There does not exist an index j, such that $\langle u_F^{e_j}, v\rangle > 0$.
From this we see that any neighboring vertex v of F is on the form: $v = -e_1 - \ldots - e_h$ (up to a permutation of the basis vectors). Therefore h must divide d, and any vertex of P not on F is on the claimed form. $\square$

Theorem 2.7 is related to this result by Casagrande ([9] theorem 3.(iv)): Let P be a smooth Fano d-polytope, and define

$$\partial_P := \min\{-\langle u_F, v\rangle \mid u_F \in \mathcal{V}(P^*), v \in \mathcal{V}(P), v \notin F\}.$$

Suppose $\partial_P > 0$. Then $|\mathcal{V}(P)| \leq d + \frac{d}{\partial_P}$ with equality if and only if P splits into $\frac{d}{\partial_P}$ copies of T_{∂_P}.
Hence the conclusions are the same, but under different assumptions.

2.3 Few vertices

In this section we take a look at the available classifications of smooth Fano d-polytopes having only a few vertices. Everything here is well-known. We shall later refer to these classifications, so we have decided to collect them here.

2.3.1 $d+1$ vertices

The classification of smooth Fano simplices is easy.

Theorem 2.8. *Any smooth Fano d-simplex is isomorphic to T_d.*

Proof. Let F be any facet of P. By applying a suitable unimodular transformation, we can assume that $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$. The remaining vertex v is the neighboring vertex $n(F, e_i)$ for every $1 \leq i \leq d$. Hence $v = -e_1 - \ldots - e_d$ by lemma 1.9. $\square$

2.3.2 $d+2$ vertices

Regular complete fans in $\mathbb{R}^d$ with $d+2$ rays have been classified ([15]). The classification of smooth Fano d-polytopes with $d+2$ vertices follows immediately from this.
We can give an easy proof of this classification result, and since we will need the result in subsection 3.1.3, we have decided to give a proof for the sake of completeness. The proof is different from the one given in [15] (which concerns a far more general case).

Theorem 2.9 ([15] theorem 1). *Let P be a smooth Fano d-polytope with $d+2$ vertices.*
Then P is isomorphic to the convex hull of the points

$$\begin{gathered} e_1 \ , \ \dots \ , \ e_d \\ v_1 = -e_1 - \dots - e_k \\ v_2 = a_1 e_1 + \dots + a_k e_k - e_{k+1} - \dots - e_d, \end{gathered}$$

where $1 \leq k \leq d-1$ and $a_i \in \mathbb{Z}_{\geq 0}$ for every $1 \leq i \leq k$, such that $\sum_{i=1}^{k} a_i \leq d-k-1$.

Proof. Let F be a special facet of P. Without loss of generality we can assume $\mathcal{V}(F) = \{e_1, \dots, e_d\}$. We denote the remaining two vertices of P by v_1 and v_2. If one of these two vertices is equal to $n(F, e_i)$ for every $1 \leq i \leq d$, then P is the simplex T_d, which is not the case. We can then safely assume, that there exists a k, $1 \leq k \leq d-1$, such that

$$n(F, e_i) = \begin{cases} v_1 & 1 \leq i \leq k \\ v_2 & k+1 \leq i \leq d \end{cases}$$

Then by lemma 1.9 we have

$$v_1 = -e_1 - \dots - e_k + b_{k+1} e_{k+1} + \dots + b_d e_d$$

and

$$v_2 = a_1 e_1 + \dots + a_k e_k - e_{k+1} - \dots - e_d,$$

where $a_i, b_j \in \mathbb{Z}$ for every possible i and j. As the sum of the vertices of P is a non-negative linear combination of the vertices of F, we have $a_i, b_j \in \mathbb{Z}_{\geq 0}$. Suppose there exists a positive a_i and a positive b_j, say $a_1, b_d > 0$. Consider the neighboring facet $F' = N(F, e_1)$. The vertices on F' are

$$\mathcal{V}(F') = \{e_2, \dots, e_d, v_1\}.$$

Write the remaining two vertices of P in the basis F' provides

$$\begin{aligned} e_1 &= -v_1 - \dots - e_k + b_{k+1} e_{k+1} + \dots + b_d e_d \\ v_2 &= -a_1 v_1 + (a_2 - a_1) e_2 + \dots + (a_k - a_1) e_k \\ &\quad + (a_1 b_{k+1} - 1) e_{k+1} + \dots + (a_1 b_d - 1) e_d \end{aligned}$$

But none of these can be the neighboring vertex $n(F', e_d)$, as both have non-negative e_d-coordinate. A contradiction, and the polytope P is on the claimed form. $\square$

2.3.3 $d+3$ vertices

Batyrev has introduced the concepts of primitive collections and relations of a fan defining a smooth projective toric variety ([2]). These concepts are excellent tools for representation and classification of smooth projective toric varieties, and have been widely used to classify smooth Fano polytopes (see [4],[6],[8],[22],[23]).
Complete regular fans in $\mathbb{R}^d$ with $d+3$ rays define projective toric varieties ([16] theorem 1). Using the language of primitive collections and relations Batyrev has classified these fans ([2]). Hence a classification of smooth Fano d-polytopes with $d+3$ vertices is available to us.
We will now define the concepts of primitive collections and relations in the context of smooth Fano polytopes, but the generalization to regular complete fans defining projective varieties is straightforward.
Let $C = \{v_1, \dots, v_k\}$ be a subset of $\mathcal{V}(P)$, where P is a smooth Fano polytope. The set C is called a *primitive collection* (of P), if $\operatorname{conv}(C)$ is not a face of P, but $\operatorname{conv}(C \setminus \{v_i\})$ is a face of P for every $1 \leq i \leq k$. Consider the lattice point $x = v_1 + \dots + v_k$. There exists a unique face $\sigma(C) \neq P$ of P, called the *focus* of C, such that x is a positive $\mathbb{Z}$-linear combination of vertices of $\sigma(C)$, that is

$$x = a_1 w_1 + \dots + a_m w_m, \quad a_i \in \mathbb{Z}_{>0}$$

where $\{w_1, \dots, w_m\} = \mathcal{V}(\sigma(C))$. The linear relation

$$v_1 + \dots + v_k = a_1 w_1 + \dots + a_m w_m \tag{2.1}$$

is called a *primitive relation*. The intersection $C \cap \sigma(C)$ is empty ([2] proposition 3.1).
The integer $k - a_1 - \dots - a_m$ is called the *degree* of the primitive relation (2.1). It is easy to show that the degree of any primitive relation of P is strictly positive ([4] proposition 2.1.10).
The formulation of theorem 2.10 below is taken from [22] theorem 8.2.

Theorem 2.10 (Batyrev [2]). *Let P be a smooth Fano d-polytope with $d+3$ vertices. Then P is defined either by 3 or 5 primitive relations.*

- *If there are 3 primitive collections of P, then they are pairwise disjoint.*
- *If there are 5 primitive relations there exists $(p_0, p_1, p_2, p_3, p_4) \in \mathbb{Z}^5$, such that the primitive relations of P are*

$$v_1 + \dots + v_{p_0} + y_1 + \dots + y_{p_1} = c_2 z_2 + \dots + c_{p_2} z_{p_2} + (b_1+1) t_1 + \dots + (b_{p_3}+1) t_{p_3}$$
$$y_1 + \dots + y_{p_1} + z_1 + \dots + z_{p_2} = u_1 + \dots + u_{p_4}$$

$$
\begin{gathered}
z_1 + \ldots + z_{p_2} + t_1 + \ldots + t_{p_3} = 0 \\
t_1 + \ldots + t_{p_3} + u_1 + \ldots + u_{p_4} = y_1 + \ldots + y_{p_1} \\
u_1 + \ldots + u_{p_4} + v_1 + \ldots + v_{p_0} = c_2 z_2 + \ldots + c_{p_2} z_{p_2} + b_1 t_1 + \ldots + b_{p_3} t_{p_3}
\end{gathered}
$$

where

$$\mathcal{V}(P) = \{v_1, \ldots, v_{p_0}, y_1, \ldots, y_{p_1}, z_1, \ldots, z_{p_2}, t_1, \ldots, t_{p_3}, u_1, \ldots, u_{p_4}\}$$

and $c_2, \ldots, c_{p_2}, b_1, \ldots, b_{p_3}$ *are positive integers.*

2.4 Many vertices

As shown in the previous chapter, any simplicial reflexive d-polytope has at most $3d$ vertices (theorem 1.15). This was first proved in a paper by Casagrande ([9]), in which she also classified simplicial reflexive d-polytopes with $3d$ vertices using a theorem by Nill ([20] theorem 5.9).
In this section we will give another proof of the known classification of simplicial reflexive d-polytopes with $3d$ vertices, together with a proof of a new classification result concerning terminal simplicial reflexive d-polytopes with $3d-1$ vertices. In both cases the investigated polytopes turn out to be smooth Fano d-polytopes.

2.4.1 $3d$ vertices

Simplicial reflexive d-polytopes have been classified in the case of $3d$ vertices. We can give another proof of this using the classification theorem 2.3.

Theorem 2.11 ([9] theorem 1, [20] theorem 5.9)**.** *Let P be a simplicial reflexive d-polytope with $3d$ vertices.*
Then d is even and P is isomorphic to the convex hull of the points

$$
\begin{gathered}
\pm e_1 \ , \ \ldots \ , \ \pm e_d \\
\pm(e_1 - e_2) \ , \ \ldots \ , \ \pm(e_{d-1} - e_d),
\end{gathered}
$$

that is, P splits into $\frac{d}{2}$ copies of del Pezzo 2-polytopes.
In particular, P is a smooth Fano polytope.

Proof. Let F be a special facet of P. Then there are exactly d vertices in each of the hyperplanes $H(F,i)$, $i = 1, 0, -1$. So the sum ν_P of the vertices of P is the origin and every facet is special. In particular, every facet of P has close neighbors. By theorem 2.3 P is a centrally symmetric smooth Fano polytope. Using the classification of these (theorem 2.1), we can see d must be even and P isomorphic to the convex hull of the claimed points. □

2.4.2 $3d-1$ vertices - terminal case

As we saw above, any simplicial reflexive d-polytope with $3d$ vertices is a smooth Fano d-polytope. Now, we deal with the next case: Simplicial reflexive d-polytopes with $3d-1$ vertices. It turns out, that if we in addition assume terminality, the investigated polytopes are indeed smooth Fano (theorem 2.17). Recall that a lattice polytope P with $0 \in \mathrm{int}P$ is called *terminal*, if $P \cap \mathbb{Z}^d = \mathcal{V}(P) \cup \{0\}$.

Suppose P is a simplicial reflexive d-polytope and F is a special facet of P. Then

$$0 \leq \sum_{v \in \mathcal{V}(P)} \langle u_F, v\rangle = \sum_{i \leq 1} i|H(F,i) \cap \mathcal{V}(P)| = d + \sum_{i \leq -1} i|H(F,i) \cap \mathcal{V}(P)|.$$

When $|\mathcal{V}(P)|$ is close to $3d$, the vertices of P tend to be packed in the hyperplanes $H(F,i)$ for $i \in \{1,0,-1\}$. So in order to classify simplicial reflexive polytopes with many vertices, it seems reasonable to investigate cases of many vertices in $H(F,0)$. We will do so in the following. These results will be ingredients in the proof of theorem 2.17.
The first result concerns cases where we somehow know that each facet F of a simplicial reflexive d-polytope has at least $d-1$ of its neighboring vertices $n(F,v)$, $v \in \mathcal{V}(F)$, in $H(F,0)$.

Proposition 2.12. *Let P be a simplicial reflexive d-polytope, such that*

$$|\{v \in \mathcal{V}(F) \mid n(F,v) \in H(F,0)\}| \geq d-1$$

for every facet F of P.
Then there exists a facet G of P, such that $\mathcal{V}(G)$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^d$.

Proof. By lemma 2.2 we are done if there exists a facet G, such that the set

$$\{v \in \mathcal{V}(G) \mid n(G,v) \in H(G,0) \text{ and } \langle u_G^v, n(G,v)\rangle = -1\}$$

is of size at least $d-1$. So we suppose that no such facet exists.
Write every vertex of P in the basis $\{e_1, \ldots, e_d\}$. For every facet F of P, we let $\det A_F$ denote the determinant of the matrix

$$A_F := \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix},$$

where $\mathcal{V}(F) = \{v_1, \ldots, v_d\}$. As $\det A_F$ is determined up to a sign, the number $r_F := |\det A_F|$ is well-defined.

Now, let F_0 be an arbitrary facet of P. There must be at least one vertex v of F_0, such that $v' = n(F_0, v) \in H(F_0, 0)$ but $\langle u^v_{F_0}, v' \rangle \neq -1$. Then $\langle u^v_F, v' \rangle < -1$ by lemma 1.6. Let F_1 denote the neighboring facet $N(F_0, v)$. Then $r_{F_0} > r_{F_1}$.
We can proceed in this way to produce an infinite sequence of facets

$$F_0, F_1, F_2, \ldots \quad \text{where} \quad r_{F_0} > r_{F_1} > r_{F_2} > \ldots .$$

But there are only finitely many facets of P. A contradiction. $\square$

And now a technical lemma.

Lemma 2.13. *Let F be a facet of a simplicial reflexive polytope P. Let $v_1, v_2 \in \mathcal{V}(F)$, $v_1 \neq v_2$, and set $y_1 = n(F, v_1)$ and $y_2 = n(F, v_2)$.*
Suppose $y_1 \neq y_2$, $y_1, y_2 \in H(F, 0)$ and $\langle u^{v_1}_F, y_1 \rangle = \langle u^{v_2}_F, y_2 \rangle = -1$.
Then there are no vertex $x \in \mathcal{V}(P)$ in $H(F, -1)$ with $\langle u^{v_1}_F, x \rangle = \langle u^{v_2}_F, x \rangle = -1$.

Proof. Suppose the statement is not true.
For simplicity, let $G = \operatorname{conv}(\mathcal{V}(F) \setminus \{v_1, v_2\})$. The vertex x written as a linear combination of $\mathcal{V}(F)$ is then

$$x = -v_1 - v_2 + \sum_{w \in \mathcal{V}(G)} \langle u^w_F, x \rangle w.$$

The vertices of the facet $F_1 = N(F, v_1)$ are $\{y_1\} \cup (\mathcal{V}(F) \setminus \{v_1\})$, where

$$y_1 = -v_1 + \langle u^{v_2}_F, y_1 \rangle v_2 + \sum_{w \in \mathcal{V}(G)} \langle u^w_F, y_1 \rangle w.$$

In the basis (of $\mathbb{R}^d$) F_1 provides we have

$$x = y_1 + (-1 - \langle u^{v_2}_F, y_1 \rangle) v_2 + \sum_{w \in \mathcal{V}(G)} \langle u^w_F, x - y_1 \rangle w$$

The vertex x is in $H(F_1, 0)$ by lemma 1.5. Certainly, $\langle u^{v_2}_F, y_1 \rangle \leq 0$, otherwise we would have a contradiction to lemma 1.6. On the other hand, $\langle u^{v_2}_F, y_1 \rangle \geq 0$, as $n(F, v_2) \neq y_1$. So $\langle u^{v_2}_F, y_1 \rangle = 0$ and $x = n(F_1, v_2)$.
Similarly, $\langle u^{v_1}_F, y_2 \rangle = 0$.

$$y_2 = -v_2 + \sum_{w \in \mathcal{V}(G)} \langle u^w_F, y_2 \rangle w.$$

But then y_2 and x are both in $H(F_1, 0)$ and both have negative v_2-coordinate. This is a contradiction to lemma 1.7. $\square$

Lemma 2.13 tells us that a smooth Fano d-polytope P tends to have pairs of centrally symmetric vertices, when $\mathcal{V}(P)$ is close to $3d$.
Now, we restrict our attention to terminal simplicial reflexive d-polytopes, and show a lemma concerning the case of d vertices in $H(F,0)$ for some facet F.

Lemma 2.14. *Let P be a terminal simplicial reflexive d-polytope. If there are d vertices of P in $H(F,0)$ for some facet F of P, then*

$$\mathcal{V}(P) \cap H(F,0) = \{-y + z_y \mid y \in \mathcal{V}(F)\}$$

where $z_y \in \mathcal{V}(F)$ for every y.
In particular, $\mathcal{V}(F)$ is a basis of the lattice $\mathbb{Z}^d$.

Proof. Let $y \in \mathcal{V}(F)$. By lemma 1.7 there exists exactly one vertex $x \in H(F,0)$, such that $x = n(F,y)$. Conversely, there are no vertex $y' \neq y$ of F, such that $x = n(F,y')$. So x is on the form

$$x = -by + a_1 w_1 + \ldots + a_k w_k \quad , \; 0 < b \leq 1 \; , \; 0 < a_i \text{ and } w_i \in \mathcal{V}(F) \setminus \{y\} \; \forall i,$$

where $b = \sum_{i=1}^{k} a_i$.
Suppose there exists a facet G containing both x and y. Then

$$1 + b = \langle u_G, x + by \rangle = \langle u_G, a_1 w_1 + \ldots + a_k w_k \rangle \leq \sum_{i=1}^{k} a_i = b.$$

Which is a contradiction. So there are no such facets.
Consider the lattice point $z_y = x + y$. For any facet G of P, $\langle u_G, z_y \rangle \leq 1$ as both $\langle u_G, x \rangle, \langle u_G, y \rangle \leq 1$ and both cannot be equal to 1. So z_y is a lattice point in P. Since P is terminal, z_y is either a vertex of P or the origin.
As $1 = \langle u_F, x + y \rangle = \langle u_F, z_y \rangle$, z_y must be a vertex of F and $y \neq z_y$. And then we're done.
The vertex set $\mathcal{V}(F)$ is a basis of $\mathbb{Z}^d$ by lemma 2.2. □

The proof of lemma 2.14 is inspired by proposition 4.1 in [20].
The next lemma concerns vertices of P in $H(F,-1)$ when there are d vertices in $H(F,0)$ for some facet F of a terminal simplicial reflexive d-polytope P.

Lemma 2.15. *Let F be a facet of a terminal simplicial reflexive d-polytope $P \subset \mathbb{R}^d$, such that $|H(F,0) \cap \mathcal{V}(P)| = d$. If $x \in H(F,-1) \cap P$, then $-x \in \mathcal{V}(F)$.*

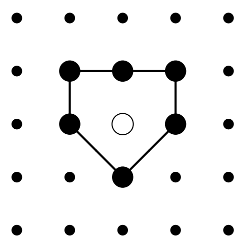


Figure 2.1: Terminality is important in lemma 2.14: This is a simplicial reflexive (self-dual) 2-polytope with 5 vertices. Consider the facet F containing 3 lattice points. The two vertices in $H(F, 0)$ are not on the form $-y+z_y$ for vertices $y, z_y \in \mathcal{V}(F)$.

Proof. The vertex set $\mathcal{V}(F)$ is a basis of the lattice $\mathbb{Z}^d$, and every vertex in $H(F, 0)$ is of the form $-y + z$ for some $y, z \in \mathcal{V}(F)$ (lemma 2.14).
Let x be vertex of P in $H(F, -1)$.

$$x = \sum_{w \in \mathcal{V}(F)} \langle u_F^w, x \rangle w,$$

where $\langle u_F^w, x \rangle \in \mathbb{Z}$ for every $w \in \mathcal{V}(F)$. If $\langle u_F^w, x \rangle \leq -2$ for some $w \in \mathcal{V}(F)$, then $x = n(F, w)$ (lemma 1.6), which is not the case. So $\langle u_F^w, x \rangle \geq -1$ for every $w \in \mathcal{V}(F)$. Furthermore, by lemma 2.13 x is only allowed one negative coordinate with respect to the basis $\mathcal{V}(F)$. The only possibility is then $x = -w$, where $w \in \mathcal{V}(F)$. □

The last thing we need before our main theorem is a result on centrally symmetric simplicial reflexive d-polytopes with $3d - 1$ vertices. The result is due to Nill. The proof given here is different from the one given in [20] and in the more general classification of pseudo-symmetric simplicial reflexive polytopes ([21] corollary 4.2).

Theorem 2.16 ([20] theorem 5.9)**.** *Let $P \subset \mathbb{R}^d$ be a centrally symmetric simplicial reflexive d-polytope with $3d - 1$ vertices.*
Then d is uneven and P is a smooth Fano polytope isomorphic to the convex hull of the points

$$\begin{array}{c} \pm e_1 \ , \ \ldots \ , \ \pm e_d \\ \pm(e_2 - e_3) \ , \ \ldots \ , \ \pm(e_{d-1} - e_d). \end{array}$$

Proof. As P is centrally symmetric, any facet of P is special. Furthermore, for any facet G of P, there must be d vertices of P in each of the hyperplanes $H(G, 1)$ and $H(G, -1)$, and then $d - 1$ vertices in $H(G, 0)$. Hence we may

apply proposition 2.12 to find a facet F of P, such that $\mathcal{V}(F)$ is a basis of $\mathbb{Z}^d$, and we may assume $\mathcal{V}(F) = \{e_1, \dots, e_d\}$. Then $-F = \operatorname{conv}\{-e_1, \dots, -e_d\}$ is also a facet of P.
Let v be a vertex in $H(F, 0)$. Then by lemma 1.6 $\langle u_F^{e_i}, v\rangle \in \{-1, 0, 1\}$ for every $1 \leq i \leq d$. Suppose v has at least two positive coordinates, say $\langle u_F^{e_i}, v\rangle = \langle u_F^{e_j}, v\rangle = 1$. The point $-v$ is then a vertex of P with at least two negative coordinates. As both v and $-v$ have at least two negative coordinates, there can be at most $d-2$ vertices in $H(F, 0)$. This is not the case. So v must be equal to $-e_i + e_j$ for suitable $1 \leq i, j \leq d$.
P must then be isomorphic to the claimed convex hull and then a smooth Fano polytope. □

Finally, we are ready to prove our main result. Notice, that an arbitrary simplicial reflexive d-polytope with $3d-1$ vertices is not necessarily smooth Fano (figure 2.1)

Theorem 2.17. *Let $P \subset \mathbb{R}^d$ be a terminal simplicial reflexive d-polytope with $3d-1$ vertices.*
If d is even, then P is isomorphic to the convex hull of the points

$$\begin{array}{c} e_1 \;,\; \pm e_2 \;,\; \dots \;,\; \pm e_d \\ \pm(e_1 - e_2) \;,\; \dots \;,\; \pm(e_{d-1} - e_d). \end{array} \tag{2.2}$$

If d is uneven, then P is isomorphic to either the convex hull of the points

$$\begin{array}{c} \pm e_1 \;,\; \dots \;,\; \pm e_{d-1} \;,\; e_d \\ \pm(e_1 - e_2) \;,\; \dots \;,\; \pm(e_{d-2} - e_{d-1}) \;,\; e_1 - e_d. \end{array} \tag{2.3}$$

or the convex hull of the points

$$\begin{array}{c} \pm e_1 \;,\; \dots \;,\; \pm e_d \\ \pm(e_2 - e_3) \;,\; \dots \;,\; \pm(e_{d-1} - e_d). \end{array} \tag{2.4}$$

In every case, P is a smooth Fano d-polytope.

Proof. By the existing classification we can check that theorem 2.17 holds for $d \leq 2$ ([20] proposition 2.1). So we may assume that $d \geq 3$.
Let ν_P be the sum of the vertices of P, that is

$$\nu_P = \sum_{v \in \mathcal{V}(P)} v.$$

Let F be a special facet of P, i.e. $\langle u_F^w, \nu_P\rangle \geq 0$ for every $w \in \mathcal{V}(F)$. Of course, there are d vertices of P in $H(F, 1)$. The remaining $2d-1$ vertices are in the hyperplanes $H(F, i)$ for $i \in \{0, -1, -2, \dots, -d\}$, such that

$$0 \leq \langle u_F, \nu_P\rangle = d + \sum_{i \leq -1} i \cdot |\mathcal{V}(P) \cap H(F, i)|.$$

So there are three cases to consider.

	Case 1	Case 2	Case 3
$\lvert\mathcal{V}(P) \cap H(F,1)\rvert$	d	d	d
$\lvert\mathcal{V}(P) \cap H(F,0)\rvert$	d	d	$d-1$
$\lvert\mathcal{V}(P) \cap H(F,-1)\rvert$	$d-1$	$d-2$	d
$\lvert\mathcal{V}(P) \cap H(F,-2)\rvert$	0	1	0
$\lvert\mathcal{V}(P)\rvert$	$3d-1$	$3d-1$	$3d-1$

We will consider these cases separately.

Case 1. There are d vertices in $H(F,0)$, so by lemma 2.14 $\mathcal{V}(F)$ is a basis of $\mathbb{Z}^d$. We may then assume that $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$.

The sum of the vertices is a lattice point on F, since $\langle u_F, \nu_P \rangle = 1$. As P is terminal, this must be a vertex e_i of F, say $\nu_P = e_1$. Then a facet F' of P is a special facet if and only if $e_1 \in \mathcal{V}(F')$.

There are $d-1$ vertices in $H(F,-1)$, so by lemma 2.15 we get

$$\mathcal{V}(P) \cap H(F,-1) = \{-e_1, \ldots, -e_{j-1}, -e_{j+1}, \ldots, -e_d\},$$

for some $1 \leq j \leq d$. Now, there are two possibilities: $j = 1$ or $j \neq 1$, that is $-e_1 \notin \mathcal{V}(P)$ or $-e_1 \in \mathcal{V}(P)$.

$-e_1 \notin \mathcal{V}(P)$**.** Then $-e_i \in \mathcal{V}(P)$ for every $2 \leq i \leq d$. There are d vertices in $H(F,0)$, so by lemma 2.14 there is a vertex $-e_1 + e_{a_1}$, which we can assume to be $-e_1 + e_2$.

Consider the facet $F' = N(F, e_2)$. This is a special facet, so we can show that

$$\mathcal{V}(P) \cap H(F',-1) = \mathcal{V}(-F') \setminus \{-e_1\}.$$

The vertex $-e_1 + e_2$ is in the hyperplane $H(F',-1)$. So $e_1 - e_2$ is a vertex of F' (lemma 2.15), and then of P.

For every $3 \leq i \leq d$ we use the same procedure to show that $-e_i + e_{a_i}$ and $-e_{a_i} + e_i$ are vertices of P. This shows that d is even and that P is isomorphic to the convex hull of the points in (2.2).

$-e_1 \in \mathcal{V}(P)$**.** We may assume $-e_d \notin \mathcal{V}(P)$. The sum of the vertices $\mathcal{V}(P)$ is e_1, so there are exactly two vertices in $H(F,0)$ of the form $-e_k + e_1$ and $-e_l + e_1$, $k \neq l$. We wish to show that $k = d$ or $l = d$. This is obvious for $d = 3$. So suppose $d \geq 4$ and $k, l \neq d$, that is $-e_k, -e_l \in \mathcal{V}(P)$.

Consider the facet $F' = N(F, e_k)$, which is a special facet. So by the arguments above we get that

$$\mathcal{V}(P) \cap H(F', -1) = \mathcal{V}(-F') \setminus \{-e_d\}.$$

As $\mathcal{V}(F') = \{e_1, \ldots, e_{k-1}, e_{k+1}, e_d, -e_k + e_1\}$, we have that $-e_1 + e_k$ must be a vertex of P.

In a similar way we get that $-e_1 + e_l$ is a vertex of P. But this is a contradiction. So k or l is equal to d, and without loss of generality, we can assume that $k = 2$ and $l = d$.

For $3 \leq i \leq d - 1$ we proceed in a similar way to get that both $-e_i + e_{a_i}$ and $-e_{a_i} + e_i$ are vertices of P, and that $a_i \neq d$.

And so we have showed that d must be uneven and that P is isomorphic to the convex hull of the points in (2.3).

Case 2. Since $\langle u_F, \nu_P \rangle = 0$, the sum of the vertices is the origin, so every facet of P is special. There are d vertices in $H(F, 0)$, so $\mathcal{V}(F)$ is a basis of $\mathbb{Z}^d$ (lemma 2.14). Without loss of generality, we can assume $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$. By lemma 2.15

$$x \in \mathcal{V}(P) \cap H(F, -1) \implies x = -e_i \text{ for some } 1 \leq i \leq d.$$

Consider the single vertex v in the hyperplane $H(F, -2)$. If $\langle u_F^{e_j}, v \rangle > 0$ for some j then $\langle u_{F'}, v \rangle < -2$ for the facet $F' = N(F, e_j)$ (lemma 1.5), which is not the case as F' is special. So $\langle u_F^{e_j}, v \rangle \leq 0$ for every $1 \leq j \leq d$. As v is a primitive lattice point we can without loss of generality assume $v = -e_1 - e_2$.

There are d vertices in $H(F, 0)$, so there is a vertex of the form $-e_1 + e_j$ for some $j \neq 1$. If $j = 2$, then $-e_1 \in \operatorname{conv}\{-e_1 + e_2, -e_1 - e_2\}$ which is not the case as P is terminal. So we may assume $j = 3$. In $H(F, 0)$ we also find the vertex $-e_2 + e_i$ for some i. A similar argument yields $i \neq 1$.

Let $G = N(F, e_1)$. Then $\mathcal{V}(G)$ is a basis of the lattice $\mathbb{Z}^d$. Write v in this basis.

$$v = (-e_1 + e_3) - e_3 - e_2.$$

As $i \neq 1$, $-e_2 + e_i$ is in $H(G, 0)$ and is equal to $n(G, e_2)$ (lemma 1.7).

Suppose $v \neq n(G, e_3)$. As there are no vertices of P in $H(G, -2)$, there are only three possibilities for $n(G, e_3)$.

1. $n(G, e_3) \in H(G, 0)$ and $\langle u_G^{e_3}, n(G, e_3) \rangle = -1$

2. $n(G, e_3) \in H(G, -1)$ and $\langle u_G^{e_3}, n(G, e_3) \rangle = -1$
3. $n(G, e_3) \in H(G, -1)$ and $\langle u_G^{e_3}, n(G, e_3) \rangle = -2$

The first possibility cannot occur by lemma 2.13. As v is not on the facet $N(G, e_3)$ we can rule out the second possibility. Vertices in $P \cap H(G, -1)$ are of the form: $-e_k$, $-e_1 - e_2$ or $-e_l + e_1 = -(-e_1 + e_3) + e_3 - e_l$ for some k, l. None of these have -2 as e_3-coordinate with respect to the basis $\mathcal{V}(G)$. Hence the third possibility does not occur.

Therefore $v = n(G, e_3)$, and $\mathrm{conv}\{v, -e_1 + e_3, e_2\}$ is a face of P.

As e_3 and $-e_1 + e_3$ are vertices of P, there are at least two vertices of P with positive e_3-coordinate (with respect to the basis F provides). There is exactly one vertex in $H(F, 0)$ with negative e_3-coordinate, namely $-e_3 + e_k$ for some k. Any other has to be in $H(F, -1)$. The vertices of P add to 0, so the point $-e_3$ must be a vertex of P.

But $-e_3 = -(-e_1 + e_3) + v + e_2$, which cannot be the case as P is simplicial.

We conclude that case 2 is not possible.

Case 3. In this case we also have $\langle u_F, \nu_P \rangle = 0$, so every facet is special. Case 2 was not possible, so $-1 \leq \langle u_G, v \rangle \leq 1$ for any facet G and any vertex v of P. Then P is centrally symmetric and d must be uneven. By theorem 2.16 P is isomorphic to the convex hull of the points in (2.4).

This ends the proof of theorem 2.17. $\square$

Chapter 3

Classifications in fixed dimension

The topic of this chapter is the complete classifications of isomorphism classes of smooth Fano d-polytopes for fixed d.
Smooth Fano 3-polytopes have been classified independently by Batyrev ([1]) and by Watanabe and Watanabe ([25]). In dimension 4 the classification is due to Batyrev ([4]) and Sato ([22]). Recently, the classification of smooth Fano 5-polytopes has been announced by Kreuzer and Nill ([19]).
In the first part of this chapter we will briefly describe the approach used by Sato ([22]) to classify smooth Fano 4-polytopes. We will show by means of an explicit counter example (subsection 3.1.3), that this approach does not work in higher dimensions, and neither does a certain generalization of it. The counter example is presented in the preprint [28] by the author.
In the second part we present an algorithm that has been developed and implemented by the author, and used to obtain the classification of smooth Fano d-polytopes for $d \leq 8$. The algorithm is presented in the preprint [30] by the author.

3.1 Inductive construction

In this section we examine one approach to classify smooth Fano d-polytopes for arbitrary d. The approach is due to Sato ([22]) and he used it to classify smooth Fano 4-polytopes.
The overall idea is this (precise definitions will be given below): We say that two smooth Fano d-polytopes P and Q are equivalent, if there is a sequence of smooth Fano d-polytopes $P_1, \ldots, P_k$, such that $P \cong P_1$ and $Q \cong P_k$ and such that each P_i is obtained from P_{i-1} by adding or removing a vertex according

to some rule. If we can somehow find a representative for each equivalence class, then (depending on the rule of addition of vertices) we can construct an algorithm that produces the list of smooth Fano d-polytopes for every d. Here we use a rule for vertex removal or addition that has a certain geometric meaning: The addition (resp. removal) of a vertex should correspond to an equivariant blow-up (resp. blow-down) of the corresponding smooth Fano toric varieties. Using this rule one can show that for $d \leq 4$ each smooth Fano d-polytope is either pseudo-symmetric or equivalent to the simplex T_d, corresponding to the toric variety $\mathbb{P}^d$. This is due to Sato in [22], and he conjectured that this would hold for every d.
In subsection 3.1.3 we shall see that Sato's conjecture does not hold for $d = 5$, as there exists a smooth Fano 5-polytope which is neither pseudo-symmetric nor equivalent to the simplex T_d. In fact, the examined 5-polytope is not equivalent to any other smooth Fano 5-polytope no matter what rule one uses for the addition and removal of vertices. This counter example is due to the author and presented in the preprint [28].

3.1.1 Sato's approach

Here we explain the notion of F-equivalence, which is due to Sato in [22].
We begin by stating the combinatorial conditions on the polytopes that correspond to an equivariant blow-up of the relevant smooth Fano toric varieties. Let P be a smooth Fano d-polytope and F a k-face of P, $1 \leq k \leq d-1$. Consider the sum ν_F of vertices of $\mathcal{V}(F)$, $\nu_F = \sum_{w \in \mathcal{V}(F)} w$. Consider the polytope $Q = \operatorname{conv}(\mathcal{V}(P) \cup \{\nu_F\})$. Suppose Q is a smooth Fano d-polytope. We say that Q is an *equivariant blow-up* of P and that P is an *equivariant blow-down* of Q, if the facets of Q is exactly the set

$$\begin{gathered} \{\ G \mid G \text{ facet of } P\ ,\ F \nsubseteq G\ \} \\ \cup \\ \{\ \operatorname{conv}(\{\nu_F\} \cup \mathcal{V}(G) \setminus \{w\}) \mid G \text{ facet of } P\ ,\ F \subseteq G\ ,\ w \in \mathcal{V}(F)\ \}. \end{gathered}$$

By the standard theory of toric varieties (see [12] chapter VI.7) a smooth Fano d-polytope Q is an equivariant blow-up of a smooth Fano d-polytope P if and only if X_Q is an equivariant blow-up of X_P, where X_P and X_Q are the toric varieties corresponding to the polytopes P and Q respectively.
Now we are ready to define the notion of F-equivalence. We define it in the context of smooth Fano d-polytopes, but the original definition regards smooth Fano toric d-folds. The definitions are of course equivalent.

Definition 3.1 ([22] definition 1.1). *Two smooth Fano d-polytopes P and Q are called* F-equivalent*, denoted $P \overset{F}{\sim} Q$, if there is a sequence of smooth Fano d-polytopes $P_0, \ldots, P_k$, $k \geq 0$, such that*

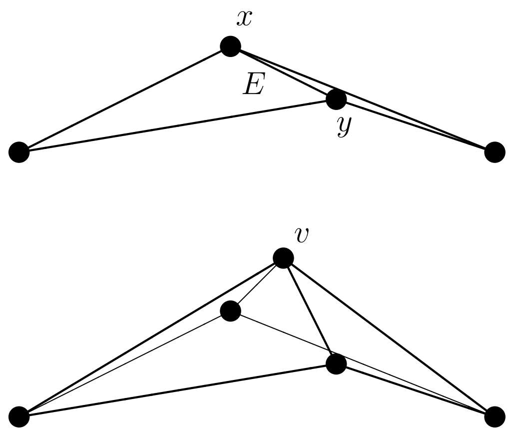


Figure 3.1: A vertex v is added so it corresponds to an equivariant blow-up of the corresponding toric variety: v is the sum of the two vertices x,y defining the edge $E = [x, y]$. Every face F containing E is replaced by two new faces, $F_x = \text{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{x\})$ and $F_y = \text{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{y\})$.

1. *$P \cong P_0$ and $Q \cong P_k$.*
2. *P_i is either an equivariant blow-up or blow-down of P_{i-1} for each $1 \leq i \leq k$.*

F-equivalence is clearly an equivalence relation on the set of smooth Fano d-polytopes, and F-equivalence respects isomorphism, i.e. if $P \cong Q$ then $P \overset{F}{\sim} Q$.
We are now facing the problem: Find representatives for each F-equivalence class.
For this problem, Sato proposes the following conjecture.

Conjecture 3.2 ([22] conjecture 1.3 and 6.3)**.** *Every smooth Fano d-polytope is either pseudo-symmetric or F-equivalent to the simplex T_d.*

The projective space $\mathbb{P}^d$, which is the toric variety corresponding to the simplex T_d ([12] example VI.3.5), can be transformed into any smooth Fano toric d-fold by a series of blow-ups and blow-downs ([26] theorem A), so conjecture 3.2 may very well be true.
In sections 7 and 8 of [22] Sato proves that conjecture 3.2 holds for $d = 3$ and $d = 4$. This is done by using the language of primitive relations and examining how these relations change under an equivariant blow-up. In fact Sato proves, that every smooth Fano 3-polytope is F-equivalent to the

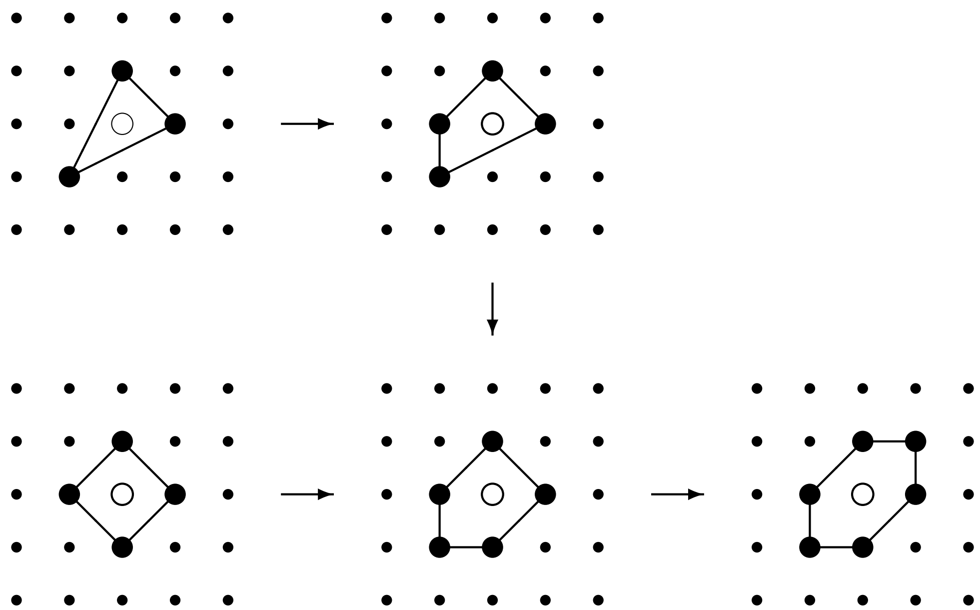


Figure 3.2: Any pair of smooth Fano 2-polytopes are F-equivalent. The arrows indicate that a smooth Fano 2-polytope is equivariantly blown-up to a new smooth Fano 2-polytope.

simplex T_3, and there are only 2 smooth Fano 4-polytopes not F-equivalent to the simplex T_4: These are the del Pezzo 4-polytope V_4 and the pseudo del Pezzo 4-polytope $\tilde{V}_4$. Both V_4 and $\tilde{V}_4$ are alone in their F-equivalence class. In arbitrary dimension $d \geq 1$, Sato proves that a smooth Fano d-polytope P is F-equivalent to T_d if one of the following conditions hold ([22] corollary 6.13 and proposition 8.3):

- $|\mathcal{V}(P)| = d + 2$.
- $|\mathcal{V}(P)| = d+3$ and P has three primitive collections (see theorem 2.10).
- $|\mathcal{V}(P)| = d + 3$ and P has five primitive collections and either $p_1 = 1$ or $p_4 = 1$ (if we write the primitive relations as in theorem 2.10).

So conjecture 3.2 holds in low dimensions and for smooth Fano polytopes with few vertices. But does it hold in general in higher dimensions?
Before we answer this question (subsection 3.1.3) we discuss a possible generalization of the notion of F-equivalence.

3.1.2 I-equivalence

What if there were no restrictions on the addition and removal of vertices, when we construct our smooth Fano d-polytopes inductively? What if we defined another equivalence relation, say I-equivalence, in the following way:

Definition 3.3. *Two smooth Fano d-polytopes P and Q are called* I-equivalent*, denoted $P \overset{I}{\sim} Q$, if there exists a sequence of smooth Fano d-polytopes $P_0, \ldots, P_k$, $k \geq 0$, such that*

1. *$P \cong P_0$ and $Q \cong P_k$.*
2. *For every $0 \leq i < k$, $\mathcal{V}(P_i) = \mathcal{V}(P_{i+1}) \cup \{v\}$ or $\mathcal{V}(P_{i+1}) = \mathcal{V}(P_i) \cup \{v\}$ for some lattice point v.*

I-euivalence is clearly an equivalence relation on the set of smooth Fano d-polytopes, and it respects isomorphism. Obviously, F-equivalence implies I-equivalence, which is (in a certain sense) the coarsest possible equivalence relation.
Below we show that any pseudo-symmetric smooth Fano d-polytope is I-equivalent to the simplex T_d. To do this we need the following result by Sato.

Theorem 3.4 ([22] theorem 6.7)**.** *Let $d, r, a_1, \ldots, a_r$ be positive integers such that $a_1 + \ldots + a_r = d$.
Then*

$$T_{a_1} \circ \ldots \circ T_{a_r} \overset{F}{\sim} T_d,$$

where T_{a_i} is a unimodular copy of the simplex $\operatorname{conv}\{e_1, \ldots, e_{a_i}, -e_1 - \ldots - e_{a_i}\}$.

Using this we can prove the promised property.

Proposition 3.5. *Any pseudo-symmetric smooth Fano d-polytope is I-equivalent to T_d.*

Proof. First notice that (for every positive even integer k):

$$V_k = \operatorname{conv}\left(\mathcal{V}(\tilde{V}_k) \cup \{-e_1 - \ldots - e_k\}\right)$$

and

$$\tilde{V}_k = \operatorname{conv}\left(\{\pm e_1, \ldots, \pm e_k\} \cup \{e_1 + \ldots + e_k\}\right).$$

So

$$V_k \overset{I}{\sim} \tilde{V}_k \overset{I}{\sim} \underbrace{T_1 \circ \ldots \circ T_1}_{k \text{ times}}.$$

As F-equivalence implies I-equivalence, we get by theorem 3.4 that $V_k \overset{I}{\sim} \tilde{V}_k \overset{I}{\sim} T_k$.
Let P be a pseudo-symmetric smooth Fano d-polytope. Then P splits into line segments, del Pezzo polytopes and pseudo del Pezzo polytopes (theorem 2.1). By the above remarks we see that $P \overset{I}{\sim} T_d$. □

Inspired by Satos conjecture one might then suspect:

Conjecture 3.6. *Every smooth Fano d-polytope is I-equivalent to T_d.*

This would indeed hold for $d \leq 4$.

3.1.3 A counter example to Sato's conjecture

The main result of this subsection is that conjecture 3.2 is not true. We show this by means of an explicit counter example, which is due to the author and presented in the preprint [28].
More precisely, we examine a smooth Fano 5-polytope P with 8 vertices with the following properties.

1. P is not pseudo-symmetric.
2. There does not exist a smooth Fano 5-polytope Q with 7 vertices, such that $Q \subset P$ (theorem 3.8).
3. There does not exist a smooth Fano 5-polytope R with 9 vertices, such that $P \subset R$ (theorem 3.9).

Furthermore, the example shows the existence of 'isolated' smooth Fano d-polytopes: It is not possible to obtain P from another smooth Fano 5-polytope by adding or removing a vertex, no matter what rule one uses for the inductive construction. As a consequence conjecture 3.6 does not hold.
In fact, by letting a computer examine the list of smooth Fano 5-polytopes which is available now, the author has obtained the following results: Of the 866 isomorphism classes of smooth Fano 5-polytopes only 3 of these are not I-equivalent to the simplex T_5. Regarding F-equivalence, 828 smooth Fano 5-polytopes are F-equivalent to T_5 and several of the remaining 38 5-polytopes are not pseudo-symmetric.
We begin by showing a lemma, which is a variant of lemma 1.7 and a special case of corollary 4.4 in [7].

Lemma 3.7. *Let*

$$v_1 + \ldots + v_k = a_1 w_1 + \ldots + a_m w_m \tag{3.1}$$

be a linear relation of vertices of P, such that $a_i \in \mathbb{Z}_{>0}$ and $\{v_1, \ldots, v_k\} \cap \{w_1, \ldots, w_m\} = \emptyset$. Suppose $k - a_1 - \ldots - a_m = 1$ and that $\operatorname{conv}\{w_1, \ldots, w_m\}$ *is a face of P.*
Then (3.1) is a primitive relation, and whenever $\{w_1, \ldots, w_m\}$ is contained in a face F, then $(F \cup \{v_1, \ldots, v_k\}) \setminus \{v_i\}$ is a face of P for every $1 \leq i \leq k$.

Proof. Let F be a facet of P containing the vertices $\{w_1, \ldots, w_m\}$. As

$$\langle u_F, a_1 w_1 + \ldots + a_m w_m \rangle = a_1 + \ldots + a_m = k - 1,$$

we must have that $\{v_1, \ldots v_k\} \setminus \{v_i\} \subset F$ for some $1 \leq i \leq k$. We therefore denote the facet F by F_i. There is a unique facet F_j for any $1 \leq j \leq k$, $j \neq i$, such that $F_j \cap F_i$ is a ridge with vertices $\mathcal{V}(F_i) \setminus \{v_j\}$. Then $\langle u_{F_j}, v_j \rangle \leq 0$, and $(\{v_1, \ldots v_k\} \setminus \{v_j\}) \subset F_j$.
The only thing left to show, is that $\operatorname{conv}\{v_1, \ldots, v_k\}$ is not a face of P. Suppose there exists a facet F' containing $\{v_1, \ldots, v_k\}$. Then

$$k = \langle u_{F'}, v_1 + \ldots + v_k \rangle = a_1 \langle u_{F'}, w_1 \rangle + \ldots + a_m \langle u_{F'}, w_m \rangle \leq k - 1,$$

which is a contradiction. $\square$

Now we are ready to state the counter example: Let $e_1, \ldots, e_5$ be the standard basis of the integral lattice $\mathbb{Z}^5 \subset \mathbb{R}^5$. Consider the smooth Fano 5-polytope P with 8 vertices, $\mathcal{V}(P) = \{v_1, \ldots, v_8\}$.

$$\begin{gathered} v_1 = e_1 \ , \ v_2 = e_2 \ , \ v_3 = e_3 \ , \ v_6 = e_4 \ , \ v_7 = e_5 \\ v_4 = -e_1 - e_2 - e_3 - 3e_4 \ , \ v_5 = -e_4 \ , \ v_8 = -e_1 - e_2 - 2e_4 - e_5. \end{gathered}$$

The primitive relations of P are

$$\begin{aligned} v_1 + v_2 + v_3 + v_4 &= 3v_5 && (3.2) \\ v_5 + v_7 + v_8 &= v_3 + v_4 && (3.3) \\ v_3 + v_4 + v_6 &= v_7 + v_8 && (3.4) \\ v_5 + v_6 &= 0 && (3.5) \\ v_1 + v_2 + v_7 + v_8 &= 2v_5. && (3.6) \end{aligned}$$

When F is a face of P, $\mathcal{V}(F)$ is a subset of $\mathcal{V}(P) = \{v_1, \ldots, v_8\}$. For simplicity we write $\{i_1, \ldots, i_k\}$ to denote the face $\operatorname{conv}\{v_{i_1}, \ldots, v_{i_k}\}$. In this notation the facets of P are

$$\begin{array}{ccc} \{1,2,3,5,7\} & \{1,2,3,5,8\} & \{1,2,4,5,7\} \\ \{1,2,4,5,8\} & \{1,3,4,5,7\} & \{1,3,4,5,8\} \\ \{1,3,4,7,8\} & \{1,3,6,7,8\} & \{1,4,6,7,8\} \\ \{2,3,4,5,7\} & \{2,3,4,5,8\} & \{2,3,4,7,8\} \\ \{2,3,6,7,8\} & \{2,4,6,7,8\} & \{1,2,3,6,7\} \\ \{1,2,3,6,8\} & \{1,2,4,6,7\} & \{1,2,4,6,8\} \end{array}$$

We will now show, that it is not possible to add or remove a lattice point from the vertex set $\mathcal{V}(P)$ and obtain another smooth Fano 5-polytope. As P is not pseudo-symmetric, our main result immediately follows.

Theorem 3.8. *There does not exist a smooth Fano* 5*-polytope* Q *with* 7 *vertices, such that* $Q \subset P$*.*

Proof. Suppose there does exist a smooth Fano 5-polytope Q, such that $\mathcal{V}(Q) = \mathcal{V}(P) \setminus \{v_m\}$ for some $1 \leq m \leq 8$. By the existing classification (theorem 2.9) we know that Q has exactly two primitive relations of positive degree

$$v_{i_1} + \ldots + v_{i_k} = 0 \quad , \quad v_{j_1} + \ldots + v_{j_{d-k}} = c_1 v_{i_1} + \ldots + c_k v_{i_k} .$$

There are two possibilities: Either $v_m = \pm v_5$ or $v_m \neq \pm v_5$.

$v_m = \pm v_5$ That is, $m = 5$ or $m = 6$. There must be a primitive collection of vertices of Q with empty focus. But for both possible v_m, no non-empty subset of $\mathcal{V}(P) \setminus \{v_m\}$ add to 0. So v_m cannot be equal to $\pm v_5$.

$v_m \neq \pm v_5$ Then $v_5 + v_6 = 0$ is a primitive relation of Q, and the other primitive collection is $C = \{v_1, v_2, v_3, v_4, v_7, v_8\} \setminus \{v_m\}$. The vertices in C must add up to cv_5, where $|c| \leq 4$. It is now easy to check for every possible v_m, that this is not the case.

And we're done. □

Theorem 3.9. *There does not exist a smooth Fano* 5*-polytope* R *with* 9 *vertices, such that* $P \subset R$*.*

Proof. Suppose there does exist a smooth Fano 5-polytope R, such that $\mathcal{V}(P) = \mathcal{V}(R) \setminus \{v_9\}$ for some $v_9 \in \mathcal{V}(R)$.
As v_5 is a vertex of R, the relation (3.2) is a primitive relation of R (lemma 3.7). Then $\{3, 4\}$ is a face of R. Relation (3.3) ensures that $\{7, 8\}$ is also a face of R. This means that the relations (3.2)-(3.4) are primitive relations of R.
As the relations (3.2)-(3.4) all have degree one, we can deduce a lot of the combinatorial structure of R: The set $\{3, 4\}$ is a face of R, thus

$$\{3, 4, 5, 7\}, \{3, 4, 5, 8\}, \{3, 4, 7, 8\}$$

are faces of R (relation (3.3)). Relation (3.2) implies that

$$\{1, 2, 3, 5, 7\}, \{1, 2, 4, 5, 7\}, \{1, 3, 4, 5, 7\}, \{2, 3, 4, 5, 7\},$$
$$\{1, 2, 3, 5, 8\}, \{1, 2, 4, 5, 8\}, \{1, 3, 4, 5, 8\}, \{2, 3, 4, 5, 8\}.$$

are facets of R. By using relation (3.3) we get 2 facets of R:

$$\{1,3,4,7,8\},\{2,3,4,7,8\}.$$

Relation (3.4) gives us 4 more facets of R.

$$\{1,3,6,7,8\},\{1,4,6,7,8\},\{2,3,6,7,8\},\{2,4,6,7,8\}.$$

Among the original 18 facets of P, 14 are also facets of R. The remaining 4 facets are:

$$\{1,2,3,6,7\},\{1,2,3,6,8\},\{1,2,4,6,7\},\{1,2,4,6,8\}.$$

So v_9 is in a cone over one of these four facets of P, i.e. v_9 is a non-negative $\mathbb{Z}$-linear combination of vertices of one of the four facets. Without loss of generality we can assume that

$$v_9 = a_1v_1 + a_2v_2 + a_3v_3 + a_6v_6 + a_7v_7 \quad , \ a_i \geq 0 \ \forall i \in \{1,2,3,6,7\}.$$

If this is not the case, apply an appropriate renumbering of the vertices of P, which fixes the primitive relations.
Then $\{1,2,3,6,7\}$ is not a facet of R. But $F = \{1,2,3,5,7\}$ is a facet of R, so on the other side of the ridge $\{1,2,3,7\}$, there must be the facet $F' = \{1,2,3,7,9\}$. Write v_9 is the basis $\mathcal{V}(F)$,

$$v_9 = a_1v_1 + a_2v_2 + a_3v_3 - a_6v_5 + a_7v_7.$$

By lemma 1.9, $a_6 = 1$ and $1 > \langle u_F, v_9\rangle > \langle u_F, v_6\rangle = -1$. So $0 = \langle u_F, v_9\rangle = a_1 + a_2 + a_3 - 1 + a_7$.
As $\{1,3,6,7,8\}$ and $\{2,3,6,7,8\}$ are facets of R, we must have $\{1,3,6,7,9\}$ and $\{2,3,6,7,9\}$ among the facets of R. This implies that

$$v_8 + v_9 \in \operatorname{span}\{v_1,v_3,v_6,v_7\} \cap \operatorname{span}\{v_2,v_3,v_6,v_7\} = \mathbb{R}v_3 + \mathbb{R}v_6 + \mathbb{R}v_7.$$

As $v_8 + v_9 = (a_1 - 1)v_1 + (a_2 - 1)v_2 + a_3v_3 + (a_6 - 2)v_6 + (a_7 - 1)v_7$, we must have $a_1 = a_2 = 1$.
Since $a_1 + a_2 + a_3 - 1 + a_7 = 0$ we must have that $a_3 < 0$ or $a_7 < 0$, which is a contradiction.
We conclude that the smooth Fano 5-polytope R does not exist. □

3.2 The SFP-algorithm

In this section we present an algorithm that can classify smooth Fano d-polytopes (up to isomorphism) for any given d. We have decided to name the algorithm SFP (for Smooth Fano Polytopes). The author has implemented it in C++, and used it to classify smooth Fano d-polytopes for $d \leq 8$.
There are (at least) three advantages of the algorithm described here:

1. The input is the positive integer d. Nothing else is needed. Compare this with the input needed in the algorithm by Nill and Kreuzer described in section 3.2.1.

2. The algorithm is quite fast: It takes about 30 seconds on a standard home computer (spring 2007) to classify smooth Fano 5-polytopes. Once again, compare this with the computation time needed in the algorithm by Nill and Kreuzer.

3. The algorithm needs almost no computer memory.

Most of the material presented here can be found in the preprint [30] by the author.

3.2.1 The algorithm by Kreuzer and Nill

In this subsection we will briefly describe another approach to construct smooth Fano d-polytopes, that has recently been used by Kreuzer and Nill to classify smooth Fano 5-polytopes ([19])
Here is the idea: Consider the projection along a vertex v of a smooth Fano d-polytope P.

$$\pi_v : \mathbb{R}^d \to \mathbb{R}^d / v\mathbb{R} \ \cong \mathbb{R}^{d-1}, \ \pi_v(x) = [x]$$

The image $\pi_v(P)$ of P under this projection turns out to be a reflexive $(d-1)$-polytope ([4] proposition 2.4.4) with certain properties (see [19] 1.3-1.8).
The algorithm developed and implemented by Kreuzer and Nill can recover smooth Fano d-polytopes from the classification list of reflexive $(d-1)$-polytopes. Using the classification of reflexive 4-polytopes they obtained the classification of smooth Fano 5-polytopes. It takes less than one hour to perform the classification of smooth Fano 5-polytopes, once the relevant subset of reflexive 4-polytopes has been determined (however, the paper doesn't inform how long this takes).

3.2.2 Special embeddings

When we proved that there are only finitely many isomorphism classes of smooth Fano d-polytopes for every d (theorem 1.16) we used the fact that $\mathcal{V}(P)$ of any smooth Fano d-polytope P can be embedded in a certain finite subset of $\mathbb{Z}^d$. We will now explicitely determine this subset, which we denote $\mathcal{W}_d$. As smooth Fano d-polytopes have at most $3d$ vertices, we can classify smooth Fano d-polytopes simply by checking every subset of $\mathcal{W}_d$ containing at most $3d$ elements. If we perform this check in a clever way, we can actually construct a quite effective algorithm.
Recall, $e_1, \dots, e_d$ is a fixed basis of $\mathbb{Z}^d$. Let I denote the $(d-1)$-simplex $\operatorname{conv}\{e_1, \dots, e_d\} \subset \mathbb{R}^d$, which we call the *initial simplex*.

Definition 3.10. *Let P be a smooth Fano d-polytope. Any smooth Fano d-polytope Q, with I as a special facet, is called a* special embedding *of P, if P and Q are isomorphic.*

Each smooth Fano d-polytope P has at least one special facet (lemma 1.12). By applying an appropriate unimodular transformation of P we see that at least one special embedding of P exists.
Now we define a concrete finite subset of $\mathbb{Z}^d$.

Definition 3.11. *Let $\mathcal{W}_d$ be the subset of $\mathbb{Z}^d$ given by: $x \in \mathcal{W}_d$ if and only if*

1. $x \neq 0$
2. *The greatest common divisor of $\langle u_I^{e_1}, x\rangle, \dots, \langle u_I^{e_d}, x\rangle$ is 1, i.e. x is a primitive lattice point.*
3. $-d \leq \langle u_I, x\rangle \leq 1$
4. *For all $1 \leq i \leq d$:* $\langle u_I^{e_i}, x\rangle \geq \begin{cases} 0 & \langle u_I, x\rangle = 1 \\ -1 & \langle u_I, x\rangle = 0 \\ \langle u_I, x\rangle & \langle u_I, x\rangle < 0 \end{cases}$

The set $\mathcal{W}_d$ is indeed finite, and it has the following important property

Theorem 3.12. *Let P be an arbitrary smooth Fano d-polytope, and Q any special embedding of P. Then $\mathcal{V}(Q)$ is contained in the set $\mathcal{W}_d$.*

Proof. Follows directly from lemma 1.9.(4), lemma 1.14 and the definition of $\mathcal{W}_d$. □

Thus, for each smooth Fano d-polytope P there is a special embedding Q of P in the (gigantic) set

$$\{\operatorname{conv}(V) \mid \{e_1, \dots, e_d\} \subseteq V \subseteq \mathcal{W}_d\}.$$

3.2.3 Defining a total order

We now define a total order on $\mathbb{Z}^d$ and use it to define a total order on subsets of $\mathcal{W}_d$. The latter will be used to define a total order on isomorphism classes of smooth Fano d-polytopes.

Definition 3.13. *Let $x = x_1e_1 + \ldots + x_de_d$, $y = y_1e_1 + \ldots + y_de_d$ be two lattice points in $\mathbb{Z}^d$. We define $x \preceq y$ if and only if*

$$(-x_1 - \ldots - x_d, x_1, \ldots, x_d) \leq_{lex} (-y_1 - \ldots - y_d, y_1, \ldots, y_d),$$

where $\leq_{lex}$ is the lexicographical ordering on the product of $d+1$ copies of the ordered set $(\mathbb{Z}, \leq)$.

The order $\preceq$ on $\mathbb{Z}^d$ is indeed a total order.

Example. If (a, b) denotes the point $ae_1 + be_2$, then

$$(0,1) \prec (-1,1) \prec (1,-1) \prec (-1,0).$$

Let V be any nonempty finite subset of lattice points in $\mathbb{Z}^d$. We define $\max V$ to the maximal element in V with respect to the ordering $\preceq$. Similarly, $\min V$ is defined to be the minimal element in V.
A important property of the ordering $\preceq$ on $\mathbb{Z}^d$ is shown in the following lemma.

Lemma 3.14. *Let P be a special embedding of a smooth Fano d-polytope. Then*

$$\min\{v \in \mathcal{V}(P) \mid \langle u_I^{e_i}, v\rangle < 0\} = n(I, e_i)$$

for every $1 \leq i \leq d$.

Proof. By lemma 1.5 the neighboring vertex $n(I, e_i)$ is in the set $\{v \in \mathcal{V}(P) \mid \langle u_I^{e_i}, v\rangle < 0\}$, and by lemma 1.9.(5) and the definition of the ordering $\preceq$, $n(I, e_i)$ is the minimal element in this set. $\square$

In fact, we chose the ordering $\preceq$ to obtain the property of lemma 3.14, and any other total order on $\mathbb{Z}^d$ having this property could have been used in what follows.

The order of a smooth Fano d-polytope

We can now define an ordering on finite subsets of $\mathcal{W}_d$. The ordering is defined recursively.

Definition 3.15. *Let X and Y be finite subsets of $\mathcal{W}_d$. We define $X \preceq Y$ if and only if one of the following conditions hold*

1. $X = \emptyset$
2. $X, Y \neq \emptyset$ *and* $\min X \prec \min Y$
3. $X, Y \neq \emptyset$ *and* $\min X = \min Y$ *and* $X \setminus \{\min X\} \preceq Y \setminus \{\min Y\}$

Lemma 3.16. *The ordering $\preceq$ on subsets of $\mathcal{W}_d$ is indeed a total order.*

Proof. We prove this by induction in the number n of elements in the largest of the sets X and Y.
Consider the statement

$\preceq$ *is a total order on subsets of $\mathcal{W}_d$ having at most n elements.*

For $n = 0$ and $n = 1$ the statement holds.
Assume we have proven it for $n - 1$, $n \geq 1$. We need to show the statement for n. Let X, Y and Z be subsets of $\mathcal{W}_d$ having at most n elements.

Reflexivity $X \preceq X$ because of condition 3 in definition 3.15 and the induction hypothesis.

Transitivity Suppose $X \preceq Y$ and $Y \preceq Z$. Is $X \preceq Z$? If one of the sets X, Y, Z is empty, then we're done. If $\min X \prec \min Y$ or $\min Y \prec \min Z$, then we're also done. If $x = \min X = \min Y = \min Z$, then $X \setminus \{x\} \preceq Y \setminus \{x\}$ and $Y \setminus \{x\} \preceq Z \setminus \{x\}$, and by the induction hypothesis we have $X \setminus \{x\} \preceq Z \setminus \{x\}$, and we're done.

Anti-symmetry Suppose $X \preceq Y$ and $Y \preceq X$. Is $X = Y$? If one is the empty set, then both are empty and we're done. If $\min X \prec \min Y$ then Y cannot be less than or equal to X. So $x = \min X = \min Y$. Then $X \setminus \{x\} \preceq Y \setminus \{x\}$ and $Y \setminus \{x\} \preceq X \setminus \{x\}$, and by the induction hypothesis we're done.

Totality Can X and Y be compared? Yes, if X or Y is the empty set. Suppose both are nonempty. If $\min X \neq \min Y$, then we're done by condition 2 in definition 3.15. If $x = \min X = \min Y$, then $X \setminus \{x\} \preceq Y \setminus \{x\}$ or $Y \setminus \{x\} \preceq X \setminus \{x\}$ by induction hypothesis.

□

Using the total order on subsets of $\mathcal{W}_d$ we define a total order on isomorphism classes of smooth Fano d-polytopes. When $\{X_1, \ldots, X_n\}$ is a set of finite subsets of $\mathcal{W}_d$ we define $\min\{X_1, \ldots, X_n\}$ to be the smallest subset X_i with respect to the ordering $\preceq$ defined in definition 3.15.

Definition 3.17. *Let P be a smooth Fano d-polytope. The* order *of P,* $\operatorname{ord}(P)$*, is defined as*

$$\operatorname{ord}(P) := \min\{\mathcal{V}(Q) \mid Q \text{ a special embedding of } P\}.$$

The set on the right hand side is non-empty and finite, so $\operatorname{ord}(P)$ *is well-defined.*
Let $[P_1]$ and $[P_2]$ be two isomorphism classes of smooth Fano d-polytopes, represented by the polytopes P_1 and P_2. Then we define $[P_1] \preceq [P_2]$ if and only if $\operatorname{ord}(P_1) \preceq \operatorname{ord}(P_2)$.

Lemma 3.18. *$\preceq$ is a total order on isomorphism classes of smooth Fano d-polytopes.*

Proof. When P_1 and P_2 are two smooth Fano d-polytopes, $\operatorname{ord}(P_1) = \operatorname{ord}(P_2)$ if and only if P_1 and P_2 are isomorphic, and the statement follows. □

Permutation of basisvectors and presubsets

The group S_d of permutations of d elements acts on $\mathcal{W}_d$ is the obvious way by permuting the basisvectors:

$$\sigma.(a_1e_1 + \ldots + a_de_d) := a_1e_{\sigma(1)} + \ldots + a_de_{\sigma(d)} \quad , \; \sigma \in S_d.$$

Similarly, S_d acts on subsets X of $\mathcal{W}_d$:

$$\sigma.X := \{\sigma.x \mid x \in X\}.$$

As a permutation of the basis vectors corresponds to a unimodular transformation of $\mathbb{R}^d$, we clearly have for any special embedding P of a smooth Fano d-polytope

$$\operatorname{ord}(P) \preceq \min\{\sigma.\mathcal{V}(P) \mid \sigma \in S_d\}.$$

Let $V = \{v_1, \ldots, v_n\}$ be a subset of $\mathcal{W}_d$, $v_1 \prec \ldots \prec v_n$. Any subset $\{v_1, \ldots, v_k\}$, $0 \leq k \leq n$, is called a *presubset*[1] of V.

Example. $\{(0,1), (-1,1)\}$ is a presubset of $\{(0,1), (-1,1), (1,-1)\}$, while $\{(0,1), (1,-1)\}$ is not.

Lemma 3.19. *Let P be a smooth Fano d-polytope. Then every presubset V of* $\operatorname{ord}(P)$ *is the minimal element in $\{\sigma.V \mid \sigma \in S_d\}$.*

[1] Every $\{v_1, \ldots, v_k\}$ preceeds V in the ordering $\preceq$ and is a subset of V. Hence the term *presubset*.

Proof. Let $\mathrm{ord}(P) = \{v_1, \ldots, v_n\}$, $v_1 \prec \ldots \prec v_n$. Suppose there exists a permutation $\sigma \in S_d$ and a k, $1 \leq k \leq n$, such that

$$\sigma.\{v_1, \ldots, v_k\} = \{w_1, \ldots, w_k\} \prec \{v_1, \ldots, v_k\},$$

where $w_1 \prec \ldots \prec w_k$. Then there is a number j, $1 \leq j \leq k$, such that $w_i = v_i$ for every $1 \leq i < j$ and $w_j \prec v_j$.
Let σ act on $\{v_1, \ldots, v_n\}$.

$$\sigma.\{v_1, \ldots, v_n\} = \{x_1, \ldots, x_n\} \quad , \; x_1 \prec \ldots \prec x_n.$$

Then $x_i \preceq v_i$ for every $1 \leq i < j$ and $x_j \prec v_j$. So $\sigma.\mathrm{ord}(P) \prec \mathrm{ord}(P)$, but this contradicts the definition of $\mathrm{ord}(P)$. □

3.2.4 How subsets are generated

A standard algorithm to produce subsets of a finite totally ordered set is the following. The SFP-algorithm will generate subsets in the same way.

`GenSubs`

Input A subset $V \subseteq \mathcal{W}_d$.

Output A finite sequence $(V_1, \ldots, V_n)$ of subsets of $\mathcal{W}_d$ such that

1. $V_i \prec V_{i+1}$ for every $1 \leq i \leq n-1$.
2. V is a presubset of every V_i.
3. Every subset having V as a presubset is equal to some V_i.

Pseudo code

1. Output V
2. For every $x \in \mathcal{W}_d$, $v \prec x \; \forall v \in V$, in increasing order with respect to $\preceq$:
   (a) Call `GenSubs`$(V \cup \{x\})$.
3. Return.

Lemma 3.20 justifies that the algorithm `GenSubs` works.

Lemma 3.20. *The algorithm* `GenSubs` *produces the promised output.*

Proof. Number the elements in $\mathcal{W}_d$ in increasing order:

$$\mathcal{W}_d = \{m_1, m_2, \dots, m_n\} \ , \ m_i \prec m_{i+1} \ \forall i$$

We wish to prove the following statement

$\texttt{GenSubs}(V)$ *produces the promised output, when* $V \neq \emptyset$ *and* $m_i \preceq \max V$

by (descending) induction in i.
For $i = n$ the statement is obvious: The output of a call $\texttt{GenSubs}(V)$, where $V \neq \emptyset$ and $\max(V) = m_n$ is a sequence of a single element, namely V. And this sequence has the desired properties.
Suppose the statement has been proved for some $i \geq 2$, and consider the statement for $i - 1$. A call $\texttt{GenSubs}(V)$, where $V \neq \emptyset$ and $\max V = m_{i-1}$ results in the following output and recursive calls:

$$\begin{gathered} \text{Output } V \\ \text{Call } \texttt{GenSubs}(V \cup \{m_i\}) \\ \text{Call } \texttt{GenSubs}(V \cup \{m_{i+1}\}) \\ \vdots \\ \text{Call } \texttt{GenSubs}(V \cup \{m_n\}) \end{gathered}$$

By the induction hypothesis a call $\texttt{GenSubs}(V)$ will produce the promised output.
The case $V = \emptyset$ must be treated seperately (as $\max \emptyset$ is not defined), but is quite clear. □

3.2.5 The algorithm

Now, we are ready to describe the SFP-algorithm that produces the complete list of isomorphism classes of smooth Fano d-polytopes for any given $d \geq 1$. The algorithm works by going through certain finite subsets of $\mathcal{W}_d$ in increasing order (with respect to the ordering defined in definition 3.15). It will output a subset V if and only if $\mathrm{conv} V$ is a smooth Fano d-polytope P and $\mathrm{ord}(P) = V$.

A tour of the SFP-algorithm

Here we describe the SFP-algorithm in words. Pseudo code and proofs justifying the algorithm will follow.
The SFP-algorithm consists of three functions,

`SFP`, `AddPoint` and `CheckSubset`.

Finite subsets of $\mathcal{W}_d$ are constructed by the function `AddPoint`, which takes a subset $V \subseteq \mathcal{W}_d$, $\{e_1, \dots, e_d\} \subseteq V$, together with a finite set $\mathcal{F}$ of $(d-1)$-simplices in $\mathbb{R}^d$ as input. It then goes through every v in the set

$$\{v \in \mathcal{W}_d \mid \max V \prec v\}$$

in increasing order, and recursively calls itself with input $V \cup \{v\}$ and some set $\mathcal{F}'$ of $(d-1)$-simplices of $\mathbb{R}^d$, $\mathcal{F} \subseteq \mathcal{F}'$ (more about the set $\mathcal{F}'$ below). In this way all subsets of $\mathcal{W}_d$ having V as a presubset are considered in increasing order (lemma 3.20).

Whenever `AddPoint` is called, it checks if the input set V is the order $\mathrm{ord}(P)$ of some smooth Fano d-polytope P, in which case the polytope $\mathrm{conv}V$ is outputted.

For any given integer $d \geq 1$ the function `SFP` calls the function `AddPoint` with input $\{e_1, \dots, e_d\}$ and $\{I\}$. In this way a call `SFP`(d) will make the algorithm go through every finite subset of $\mathcal{W}_d$ containing $\{e_1, \dots, e_d\}$, and smooth Fano d-polytopes are outputted in strictly increasing order, such that each smooth Fano d-polytope is isomorphic to exactly one outputted polytope.

It is vital for the effectiveness of the SFP-algorithm, that there is some efficient way to check if a subset $V \subseteq \mathcal{W}_d$ is a presubset of $\mathrm{ord}(P)$ for some smooth Fano d-polytope P. The function `AddPoint` should perform this check before the recursive call `AddPoint`$(V, \mathcal{F}')$.

This check is made by the function `CheckSubset`: It takes a subset $V \subseteq \mathcal{W}_d$, $\{e_1, \dots, e_d\} \subseteq V$, as input together with a finite set of $(d-1)$-simplices $\mathcal{F}$, $I \in \mathcal{F}$, and returns a set $\mathcal{F}'$ of $(d-1)$-simplices containing $\mathcal{F}$, if there exists a special embedding P of a smooth Fano d-polytope, such that

1. V is a presubset of $\mathrm{ord}(P)$
2. $\mathcal{F}$ is a subset of the facets of $\mathrm{conv}(\mathrm{ord}(P))$

If no such special embedding exists, then `CheckSubset` returns false in many cases, but not always!

The algorithm in pseudo code

We are now ready to give part of the algorithm in pseudo code. To make the flow of the algorithm easier to understand we postpone the pseudo code of `CheckSubset` to the next subsection.

SFP

Input A positive integer d.

Output A sequence of smooth Fano d-polytopes, such that

- Any smooth Fano d-polytope P in the sequence satisfies $\mathcal{V}(P) = \operatorname{ord}(P)$.
- If P_1 and P_2 are two non-isomorphic smooth Fano d-polytopes in the output sequence and P_1 preceeds P_2 in the output sequence, then $\operatorname{ord}(P_1) \prec \operatorname{ord}(P_2)$.
- Any smooth Fano d-polytope is isomorphic to exactly one polytope in the sequence.

Pseudo code

1. Construct the set $V = \{e_1, \ldots, e_d\}$ and the simplex $I = \operatorname{conv} V$.
2. Call the function `AddPoint`$(V, \{I\})$.
3. End program.

`AddPoint`

Input A subset $V \subseteq \mathcal{W}_d$, $\{e_1, \ldots, e_d\} \subseteq V$, and a set of $(d-1)$-simplices $\mathcal{F}$ in $\mathbb{R}^d$, $I \in \mathcal{F}$, with the property:

For any smooth Fano d-polytope P: If V is a presubset of $\operatorname{ord}(P)$, then $\mathcal{F}$ is a subset of the facets of $\operatorname{conv}(\operatorname{ord}(P))$.

Output A sequence of smooth Fano d-polytopes, such that

- Any smooth Fano d-polytope P in the sequence satisfies $\mathcal{V}(P) = \operatorname{ord}(P)$.
- If P_1 and P_2 are two non-isomorphic smooth Fano d-polytopes in the output sequence and P_1 preceeds P_2 in the output sequence, then $\operatorname{ord}(P_1) \prec \operatorname{ord}(P_2)$.
- A smooth Fano d-polytope P is isomorphic to a polytope in the sequence if and only if V is a presubset of $\operatorname{ord}(P)$.

Pseudo code

1. If $P = \operatorname{conv}(V)$ is a smooth Fano d-polytope and $V = \operatorname{ord}(P)$, then output P.
2. For every $v \in \mathcal{W}_d$, $\max V \prec v$, in increasing order with respect to $\preceq$:

(a) If `CheckSubset`$(V \cup \{v\}, \mathcal{F})$ returns a set $\mathcal{F}'$ of simplices, then call `AddPoint`$(V \cup \{v\}, \mathcal{F}')$ recursively.

3. Return.

CheckSubset

Input A subset $V \subseteq \mathcal{W}_d$, $\{e_1, \dots, e_d\} \subseteq V$, and a set $\mathcal{F}$ of $(d-1)$-simplices in $\mathbb{R}^d$, $I \in \mathcal{F}$.

Output There are two possible output: A set $\mathcal{F}'$ of $(d-1)$-simplices in $\mathbb{R}^d$, $\mathcal{F} \subseteq \mathcal{F}'$, or `false`.

The pseudo code of `CheckSubset` will be given in a later subsection, together with the proof of the following lemma.

Lemma 3.21. *Let $V \subseteq \mathcal{W}_d$, $\{e_1, \dots, e_d\} \subseteq V$ and $\mathcal{F}$ a set of $(d-1)$-simplices in $\mathbb{R}^d$ with $I \in \mathcal{F}$.*
Suppose there exists a smooth Fano d-polytope P, such that V is a presubset of $\operatorname{ord}(P)$ *and $\mathcal{F}$ is a subset of the facets of* $\operatorname{conv}(Porder(P))$.
Then `CheckSubset`$(V, \mathcal{F})$ *returns a set $\mathcal{F}'$ of $(d-1)$-simplices, $\mathcal{F} \subseteq \mathcal{F}'$, such that $\mathcal{F}'$ is a subset of the facets of* $\operatorname{conv}(\operatorname{ord}(P))$.

With this lemma we can now justify the SFP-algorithm.

Theorem 3.22. *Let V be a subset of $\mathcal{W}_d$, $\{e_1, \dots, e_d\} \subseteq V$. Let $\mathcal{F}$ be a set of $(d-1)$-simplices in $\mathbb{R}^d$, $I \in \mathcal{F}$, with the following property:*

For any smooth Fano d-polytope P: If V is a presubset of $\operatorname{ord}(P)$, *then $\mathcal{F}$ is a subset of the facets of* $\operatorname{conv}(\operatorname{ord}(P))$.

Then a call ***`AddPoint`***$(V, \mathcal{F})$ *will produce the promised output.*

Proof. Clearly, any polytope P in the output sequence satisfies $\mathcal{V}(P) = \operatorname{ord}(P)$.
By lemma 3.20 subsets of $\mathcal{W}_d$ having V as a presubset are considered in increasing order (with respect to $\preceq$). Hence the second claim on the output of `AddPoint`$(V, \mathcal{F})$ holds.
Now, we need to prove the third claim. Let P be a smooth Fano d-polytope, and let $Q = \operatorname{conv}(\operatorname{ord}(P))$. The claim is that Q is in the output sequence if and only if V is a presubset of $\mathcal{V}(Q)$.
Suppose Q is in the output sequence. As `AddPoint`$(V, \mathcal{F})$ only considers subsets having V as a presubset, we certainly have that V is a presubset of $\mathcal{V}(Q)$.

Conversely, suppose V is a presubset of $\mathcal{V}(Q)$. Number the elements in $\mathcal{V}(Q) \setminus V$ in increasing order.

$$\mathcal{V}(Q) \setminus V = \{q_1, \dots, q_n\}.$$

The call `AddPoint`$(V, \mathcal{F})$ is made. As $\mathcal{F}$ is a subset of the facets of Q (by the assumptions), lemma 3.21 states that `CheckSubset`$(V \cup \{q_1\}, \mathcal{F})$ returns a subset $\mathcal{F}_1$ of the facets of Q. And then the recursive call `AddPoint`$(V \cup \{q_1\}, \mathcal{F}_1)$ is made.
Continue like this to see that the call `AddPoint`$(V \cup \{q_1, \dots, q_n\}, \mathcal{F}_n)$ is made, where $\mathcal{F}_n$ is a subset of the facets of Q. Once this call has been made the polytope Q will be output. □

So to classify smooth Fano d-polytopes up to isomorphism we simply call `SFP`(d).

Corollary 3.23. *A call* `SFP`(d) *will produce the promised output.*

Proof. `SFP` will make the call `AddPoint`$(\{e_1, \dots, e_d\}, \{I\})$, which by theorem 3.22 will produce the claimed output. □

The function `CheckSubset`

If V is a presubset of $\operatorname{ord}(P)$ for some smooth Fano d-polytope P, then there are certain requirements V must fulfill. We state and prove three of these in the lemma below.

Lemma 3.24. *Let V be a subset of $\mathcal{W}_d$, $\{e_1, \dots, e_d\} \subseteq V$, and let $\nu = \sum_{v \in V} v$ be the sum of the lattice points in V.*
If there exists a smooth Fano d-polytope P, such that V is a presubset of $\operatorname{ord}(P)$*, then*

1. $\langle u_I, \nu \rangle \geq 0$.
2. $\langle u_I^{e_i}, \nu \rangle \leq \langle u_I, \nu \rangle$ *for every* $1 \leq i \leq d$.
3. *There does not exist a permutation σ, such that $\sigma.V \prec V$.*

Proof. Let P be a smooth Fano d-polytope, whose order $\operatorname{ord}(P)$ has V as a presubset. Let $\{q_1, \dots, q_n\} = \operatorname{ord}(P) \setminus V$, $q_1 \prec \dots \prec q_n$. As I is a special facet of $Q = \operatorname{conv}(\operatorname{ord}(P))$, we must have

$$0 \leq \sum_{v \in V} \langle u_I, v \rangle + \sum_{i=1}^{n} \langle u_I, q_i \rangle$$

The set $\{e_1, \ldots, e_d\}$ is contained in V, so $\langle u_I, q_i\rangle \leq 0$ for all i. Hence $\langle u_I, \nu\rangle \geq 0$.
To prove the second claim, suppose $\langle u_I^{e_i}, \nu\rangle > \langle u_I, \nu\rangle$ for some i. Clearly, $\langle u_I, \nu\rangle$ cannot be greater than d. If $\langle u_I, \nu\rangle = d$, then $\langle u_I, v\rangle \in \{0, 1\}$ for every $v \in V$. It is then easy to see that $\langle u_I^{e_i}, \nu\rangle \leq d$.
So we may assume $\langle u_I, \nu\rangle < d$, i.e. there is a vertex $v \in V$, such that $\langle u_I, v\rangle < 0$. As V is a presubset of ord(P), $\langle u_I, q_j\rangle < 0$ for all $1 \leq j \leq n$. By lemma 1.9.(4) we then have $\langle u_I^{e_i}, q_j\rangle \geq \langle u_I, q_j\rangle$. Adding all this together we get

$$\begin{aligned}
\langle u_I, \nu_Q\rangle &= \sum_{v\in\mathcal{V}(Q)} \langle u_I, v\rangle \\
&= \langle u_I, \nu\rangle + \sum_{i=1}^{n} \langle u_I, q_i\rangle \\
&< \langle u_I^{e_i}, \nu\rangle + \sum_{i=1}^{n} \langle u_I^{e_i}, q_i\rangle \\
&= \langle u_I^{e_i}, \nu_Q\rangle.
\end{aligned}$$

As the sum of all the e_k-coordinates of ν_Q equals $\langle u_I, \nu_Q\rangle$, there must be a negative e_k-coordinate of ν_Q contradicting the fact that Q is a special embedding.
We have already proved the last statement in lemma 3.19. □

But the requirements of lemma 3.24 are not the only ones. We will make the function `CheckSubset` perform a far more sophisticated check on the subset V.
The best way to grasp the idea is to look at an example of how the function `CheckSubset` works.

An example of the reasoning in `CheckSubset`

Let $d = 5$ and $V = \{v_1, \ldots, v_8\}$, where

$$v_1 = e_1\ ,\ v_2 = e_2\ ,\ v_3 = e_3\ ,\ v_4 = e_4\ ,\ v_5 = e_5$$
$$v_6 = -e_1 - e_2 + e_4 + e_5\ ,\ v_7 = e_2 - e_3 - e_4\ ,\ v_8 = -e_4 - e_5.$$

Suppose P is a special embedding of a smooth Fano 5-polytope, such that V is a presubset of $\mathcal{V}(P)$.
The question: What can we say about the face lattice of P?
The answer: Surprisingly much! And the example given here is not unique in this respect.

This is how we can deduce facets of P: Certainly, the simplex I is a facet of P. For simplicity we denote any k-simplex $\mathrm{conv}\{v_{i_1}, \ldots, v_{i_k}\}$ by $\{i_1, \ldots, i_k\}$. Since $\langle u_I, v_6\rangle = 0$, the simplices $F_1 = \{2, 3, 4, 5, 6\}$ and $F_2 = \{1, 3, 4, 5, 6\}$ are facets of P (lemma 1.7).

There are exactly two facets of P containing the ridge $\{1, 2, 4, 5\}$. One of them is I. Suppose the other one is $\{1, 2, 4, 5, 9\}$, where v_9 is some lattice point not in V, $v_9 \in \mathcal{V}(P)$. Then $\langle u_I, v_9\rangle > \langle u_I, v_7\rangle$ by lemma 1.9.(5) and then $v_9 \prec v_7$ by the definition of the ordering of lattice points $\mathbb{Z}^d$. But then V is not a presubset of $\mathcal{V}(P)$. This is the nice property of the ordering of $\mathbb{Z}^d$, and the reason why we chose it as we did. We conclude that $F_3 = \{1, 2, 4, 5, 7\}$ is a facet of P, and by similar reasoning $F_4 = \{1, 2, 3, 5, 7\}$ and $F_5 = \{1, 2, 3, 4, 8\}$ are facets of P.

Now, for each of the facets F_i and every point $v_j \in V$, we check if $\langle u_{F_i}, v_j\rangle = 0$. If this is the case, then by lemma 1.7 the $(d-1)$-simplex $\mathrm{conv}(\{v_j\} \cup \mathcal{V}(F_i) \setminus \{w\})$ is a facet of P for every $w \in \mathcal{V}(F_i)$ where $\langle u^w_{F_i}, v_j\rangle < 0$. In this way we get that

$$\{2, 4, 5, 6, 7\}\ ,\ \{1, 4, 5, 6, 7\}\ ,\ \{1, 2, 3, 7, 8\}\ ,\ \{1, 3, 5, 7, 8\}$$

are facets of P.

We continue in this way, until we cannot deduce any new facet of P. Each time we find a new facet F we check that v is beneath F (that is $\langle u_F, v\rangle \leq 1$) and that the bounds in lemma 1.9.(4) hold for any $v \in V$. If not, then the polytope P cannot exist and $\texttt{CheckSubset}(V, \{I\})$ should return false.

If no contradiction arises, $\texttt{CheckSubset}(V, \{I\})$ returns the set of deduced facets.

Pseudo code and justification of `CheckSubset`

As promised, we give the pseudo code of `CheckSubset`.

`CheckSubset`

Input A subset $V \subseteq \mathcal{W}_d$, $\{e_1, \ldots, e_d\} \subseteq V$, and a set $\mathcal{F}$ of $(d-1)$-simplices in $\mathbb{R}^d$, $I \in \mathcal{F}$.

Output There are two possible output: A set $\mathcal{F}'$, $\mathcal{F} \subseteq \mathcal{F}'$, of $(d-1)$-simplices or `false`.

Pseudo code

1. If V does not have the properties of lemma 3.24, then return `false`.

2. Let $\mathcal{F}' = \mathcal{F}$.
3. For every $i \in \{1, \ldots, d\}$:
   (a) If the set $R_i = \{v \in V | \langle u_I^{e_i}, v\rangle < 0\}$ is non-empty, consider the simplex $F_i = \operatorname{conv}(\{\min R_i\} \cup \mathcal{V}(I) \setminus \{e_i\})$. If $\mathcal{V}(F_i)$ is not a basis of $\mathbb{Z}^d$, return `false`. Otherwise, add F_i to $\mathcal{F}'$.
4. If there exists $F \in \mathcal{F}'$ and $v \in V$ such that $\langle u_F, v\rangle > 1$, then return `false`.
5. If there exists $F \in \mathcal{F}'$, $v \in V$ and $w \in \mathcal{V}(F)$, such that
$$\langle u_F^w, v\rangle < \begin{cases} 0 & \langle u_F, v\rangle = 1 \\ -1 & \langle u_F, v\rangle = 0 \\ \langle u_F, v\rangle & \langle u_F, v\rangle < 0 \end{cases}$$
   then return `false`.
6. If there exists $F \in \mathcal{F}'$, $v \in V$ and $w \in \mathcal{V}(F)$, such that $\langle u_F, v\rangle = 0$ and $\langle u_F^w, v\rangle = -1$, then consider the simplex $F' = \operatorname{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{w\})$. If $F' \notin \mathcal{F}'$, then add F' to $\mathcal{F}'$ and go back to step 4.
7. Return $\mathcal{F}'$.

And now for the justification of the function `CheckSubset`, or more precisely: The proof of lemma 3.21.

Proof of lemma 3.21. The assumptions are: The subset $V \subseteq \mathcal{W}_d$, where $\{e_1, \ldots, e_d\} \subseteq V$, is a presubset of $\operatorname{ord}(P)$ for some smooth Fano d-polytope P. $\mathcal{F}$ is a set of $(d-1)$-simplices, $I \in \mathcal{F}$, which is a subset of the facets of $Q = \operatorname{conv}(\operatorname{ord}(P))$.
As V is a presubset of $\mathcal{V}(Q)$, V has the properties of lemma 3.24.
Our aim is to prove that every simplex in $\mathcal{F}'$ is a facet of Q. If this is the case, then by lemma 1.9 `CheckSubset` will not return `false`, but return $\mathcal{F}'$.
First we show that the simplices F_i we add to $\mathcal{F}'$ in step 3a are indeed neighboring facets of I.

$$F_i = \operatorname{conv}(\{\min R_i\} \cup \mathcal{V}(I) \setminus \{e_i\}).$$

By lemma 3.14 the vertex $\min R_i$ is indeed the neighboring vertex $n(I, e_i)$, and then F_i is the neighboring facet $N(I, e_i)$.
Next, we consider the addition of simplices in step 6: If F is a facet of Q, then by lemma 1.7 the simplex $\operatorname{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{w\})$ is a facet of Q. By induction we conclude, that every simplex in $\mathcal{F}'$ is a facet of Q.
This proves the lemma. $\square$

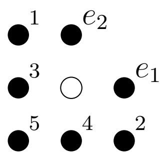


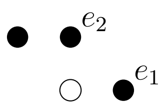
Figure 3.3: The points in the set $\mathcal{W}_2$. The points (apart from e_1 and e_2) have been numbered according to the ordering $\prec$.

3.2.6 Classification of smooth Fano 2-polytopes

In this subsection we explain in detail how the algorithm obtains the classification of smooth Fano d-polytopes for $d = 2$. We do this to become more familiar with the way the algorithm works.

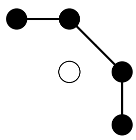
To begin with we call the function `SFP`(2). This results in a function call `AddPoint`($\{e_1, e_2\}, \{I\}$), where $I = \operatorname{conv}\{e_1, e_2\}$. The set $\mathcal{W}_2$ is shown on figure 3.3.

The function `AddPoint` will go through the points in $\mathcal{W}_2$ in increasing order. First the point $-e_1 + e_2$ is added.

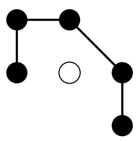


`CheckSubset` returns a set of 1-simplices, $\{I, \operatorname{conv}\{-e_1 + e_2, e_2\}\}$, and the function `AddPoint` is called recursively.

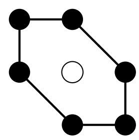
The point $e_1 - e_2$ is added, and `CheckSubset` returns a set of 1-simplices (closed line segments) as shown on the figure.



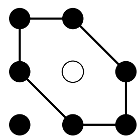
Then the point $-e_1$ is added. 1-simplices are deduced by `CheckSubset` and no contradiction arises.



The above point set is the order of a smooth Fano 2-polytope (a pseudo del Pezzo 2-polytope), and the convex hull of the point set is outputted.
Now the point $-e_2$ is added. When deducing the 1-simplices no contradiction arises in `CheckSubset`.



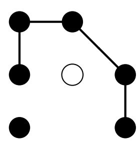
This is also the order $\mathrm{ord}(P)$ of a smooth Fano 2-polytope P (a del Pezzo 2-polytope), so output $\mathrm{conv}(\mathrm{ord}(P))$.
Adding the point $-e_1 - e_2$ results in a false return of `CheckSubset`,



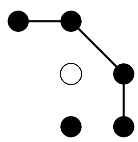
as the sum ν of all the points is equal to $-e_1 - e_2$,

$$\nu = e_1 + e_2 + (-e_1 + e_2) + (e_2 - e_1) + (-e_1) + (-e_2) + (-e_1 - e_2) = -e_1 - e_2.$$

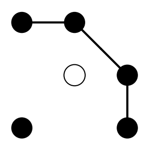
So $\langle u_I, \nu \rangle < 0$ and by statement 1 in lemma 3.24 the considered subset of points is NOT a presubset of $\mathrm{ord}(P)$ for any smooth Fano 2-polytope. Line 1 in pseudo code of `CheckSubset` results in a false return.
There are no more points to add, so go back and try to add $-e_1 - e_2$ instead of $-e_2$.



`CheckSubset` returns false (line 1).
Try to add $-e_2$ instead of $-e_1$.



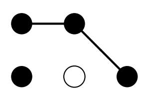
There is a permutation (swap e_1 and e_2) that will map this subset of $\mathcal{W}_2$ to a strictly smaller subset of $\mathcal{W}_2$, so the set shown above cannot be a presubset of $\mathrm{ord}(P)$ for any smooth Fano 2-polytope P (lemma 3.24 statement 3). `CheckSubset` returns false (line 1).
Try to add $-e_1 - e_2$ instead of $-e_2$.



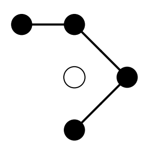
Then `CheckSubset` returns false: To see this write $-e_1 - e_2$ in the basis the simplex $F = \mathrm{conv}\{e_2, -e_1 + e_2\}$ provides.

$$-e_1 - e_2 = 1 \cdot (-e_1 + e_2) + (-2) \cdot e_2$$

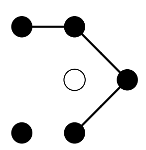
Then $-2 = \langle u_F^{e_2}, -e_1 - e_2 \rangle < \langle u_F, -e_1 - e_2 \rangle = -1$ and line 5 of `CheckSubset` results in a false return.
Add $-e_1$ instead of $e_1 - e_2$.



The sum of these points is $-e_1 + 2e_2$. So by lemma 3.24 (statement 2), `CheckSubset` returns false (line 1).
Add $-e_2$ instead. This will make `CheckSubset` return a set of 1-simplices as seen below.

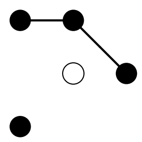


The convex hull of these points will be the third polytope outputted.
Add $-e_1 - e_2$.

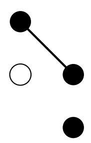


But this results in a false return from `CheckSubset` (line 1).

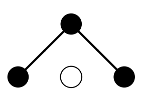
Add $-e_1 - e_2$ instead of $-e_2$.



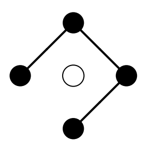
False return from `CheckSubset` (line 1 and statement 2 in lemma 3.24).
Add $e_1 - e_2$ instead of $-e_1 + e_2$.



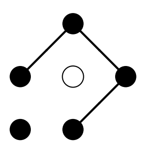
There is a permutation (swap e_1 and e_2) that changes this subset to a strictly smaller subset, so `CheckSubset` returns false (line 1).
Add $-e_1$ instead of $e_1 - e_2$. In this case `CheckSubset` returns a set of 1-simplices.



Now, add $-e_2$ and `CheckSubset` returns the simplices as seen below.

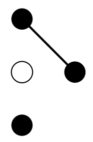


The convex hull of the above subset is the fourth smooth Fano 2-polytope, which is being output.
Add $-e_1 - e_2$.

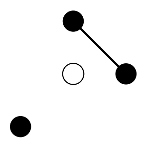


But then `CheckSubset` returns false (line 1), because of lemma 3.24 statement 1.

Add $-e_2$ instead of $-e_1$.



Due to lemma 3.24 statement 3, `CheckSubset` returns false (line 1).
Add $-e_1 - e_2$ instead of $-e_2$.



The convex hull of these points is the last smooth Fano 2-polytope, that the algorithm will output.

3.2.7 Implementation and classification results

A modified version of the SFP-algorithm has been implemented in C++, and used to classify smooth Fano d-polytopes for $d \leq 8$. On a fast computer (spring 2007) our program needs a couple of hours to construct the classification list of smooth Fano 7-polytopes, and about 2 weeks for the dimension 8 case. All these lists can be downloaded from the authors homepage: `http://home.imf.au.dk/oebro`
An advantage of the SFP-algorithm is that it requires almost no memory: When the algorithm has found a smooth Fano d-polytope P, it needs not consult the output list to decide whether to output the polytope P or not. The construction guarentees that $\mathcal{V}(P) = \min\{\sigma.\mathcal{V}(P) \mid \sigma \in S_d\}$ and it remains to check if $\mathcal{V}(P) = \mathrm{ord}(P)$. Thus there is no need of storing the output list.
The table in appendix A shows the number of isomorphism classes of smooth Fano d-polytopes with n vertices.
In appendix B one can find the C++ code.

3.2.8 Why the SFP-algorithm is not hopelessly slow

One might wonder: The set $\mathcal{W}_d$ is big, even for small d. So the set of subsets of $\mathcal{W}_d$ the SFP-algorithm has to consider is gigantic! We will now loosely discuss why the SFP-algorithm is not hopelessly slow.

Recall the counter example to Sato's conjecture (subsection 3.1.3): We were given a subset of lattice points, and assumed that it was a subset of $\mathcal{V}(P)$ of a smooth Fano 5-polytope P. As a consequence of this assumption we could say a lot about the facets of P. In fact, we were able to deduce 14 facets of P using a variant (lemma 3.7) of lemma 1.7.
Each deduced facet will put restrictions on the coordinates of additional vertices of the polytope. In the examined counter example to Sato's conjecture we could show that no additional vertex could exist (theorem 3.9).
So the effectiveness of the SFP-algorithm depends heavily on the number of facets we can deduce. Since we add vertices with weakly increasing distance to the initial facet I, we can identify the neighboring facet $N(I, e_i)$ immediately after adding the vertex $n(I, e_i)$. This is the reason for our choice of ordering on $\mathcal{W}_d$.
The more vertices a smooth Fano d-polytope P has, the closer these vertices are to any special facet F, and the more we can say about the face lattice of P by using lemma 1.7 and the more restrictions we can pose on additional vertices.
In this way the expected combinatorial explosion of the computations is avoided.

Chapter 4

Further research

In this final chapter we look at some conjectures and ideas for further research. A new result obtained by the author can also be found here (theorem 4.2).

4.1 Ewalds conjecture

There is a famous conjecture by Ewald:

Conjecture 4.1 ([11])**.** *Any smooth Fano d-polytope is isomorphic to one contained in the cube* $[-1,1]^d$*.*

Using the available classification lists conjecture 4.1 has been shown to hold for $d \leq 7$.[1] In fact, the author has noticed that an even stronger version of Ewalds conjecture holds for $d \leq 7$: *For any smooth Fano d-polytope P and any vertex $v \in \mathcal{V}(P)$, there exists a unimodular transformation φ such that $\varphi(P) \subset [-1,1]^d$ and $\varphi(v) = e_1 + \ldots + e_d$.*
It is easy to show that a smooth Fano d-polytope P can be embedded in the cube $[-1,1]^d$ if and only if the polytope $P^* \cap (-P^*)$ contains a lattice basis $\{u_1, \ldots, u_d\} \subset \mathbb{Z}^d$. The strong version of Ewalds conjecture then states: *For any smooth Fano d-polytope P and any facet F of the dual polytope P^*, there exists a lattice basis $\{u_1, \ldots, u_d\} \subset F$, such that $\{-u_1, \ldots, -u_d\} \subset P^*$.*
Conjecture 4.1 is obviously true (by theorem 2.1) for pseudo-symmetric smooth Fano polytopes, but it is wrong for arbitrary simplicial reflexive polytopes (see figure 4.1). However, it holds for pseudo-symmetric simplicial reflexive polytopes ([21] corollary 4.8), but not for general pseudo-symmetric reflexive polytopes ([21] remark 4.9).

[1] The computer has not yet checked all the smooth Fano 8-polytopes (august 2007), but so far it has found no counter examples.

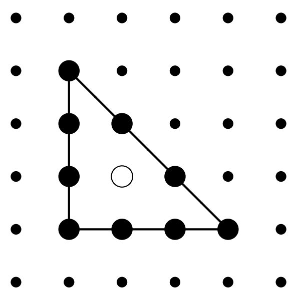


Figure 4.1: Ewald's conjecture does not hold for arbitrary simplicial reflexive polytopes: This is a simplicial reflexive 2-polytope with 9 lattice points on the boundary. But there are only 8 lattice points on the boundary of the cube $[-1,1]^2$.

To prove Ewalds conjecture one should look for lattice points in $P^* \cap -(P^*)$ for a smooth Fano polytope P. Since $\operatorname{conv}(P \cup -P) = \operatorname{conv}(\mathcal{V}(P) \cup \mathcal{V}(-P))$ is the dual of $P^* \cap -(P^*)$ ([14] p.49 exercise 5.(x)), a better understanding of the polytope $\operatorname{conv}(P \cup -P)$ might yield a proof (or a counter example) of Ewald's conjecture.
Here we shall take a babystep and show that any lattice point in the polytope $\operatorname{conv}(P \cup -P)$ is either the origin or a vertex. This is a new result.

Theorem 4.2. *Let P be a smooth Fano polytope.*
Then $\operatorname{conv}(P \cup -P)$ *is terminal.*

Proof. Suppose there exists a lattice point $x \in \operatorname{conv}(P \cup -P)$, such that $x \neq 0$ and x is not a vertex of $\operatorname{conv}(P \cup -P)$. Then x has a representation

$$x = \sum_{v \in \mathcal{V}(P)} a_v v + \sum_{w \in \mathcal{V}(-P) \setminus \mathcal{V}(P)} b_w w,$$

where $0 \leq a_v, b_w < 1$ for all v and w, and

$$\sum_{v \in \mathcal{V}(P)} a_v + \sum_{w \in \mathcal{V}(-P) \setminus \mathcal{V}(P)} b_w = 1.$$

As P is terminal we have $b_w > 0$ for some $w \in \mathcal{V}(-P) \setminus \mathcal{V}(P)$. Similarly, $a_v > 0$ for some $v \in \mathcal{V}(P)$.
Consider the two points $p \in P$ and $q \in -P$:

$$p = \frac{1}{\sum_{v \in \mathcal{V}(P)} a_v} \sum_{v \in \mathcal{V}(P)} a_v v \quad , \quad q = \frac{1}{1 - \sum_{v \in \mathcal{V}(P)} a_v} \sum_{w \in \mathcal{V}(-P) \setminus \mathcal{V}(P)} b_w w.$$

So $x \in \operatorname{conv}\{p, q\}$, which is either a point ($p = q$) or a line segment ($p \neq q$).

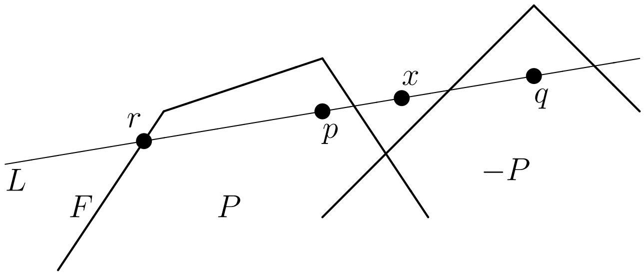


Figure 4.2: The lattice point x lies in the relative interior of the line segments $[p, q]$ and $[r, q]$. The points p, q, r do not necessarily belong to the lattice $\mathbb{Z}^d$.

If $p = q$, then $x = p = q$, which cannot be the case. So $p \neq q$. Then x is in the relative interior of the line segment $[p, q]$. As P is terminal $q \notin P$.
The intersection of the affine line $L = \operatorname{aff}\{p, q\}$ and the polytope P is a closed line segment. As $q \notin P$, there is a unique endpoint r of the closed line segment $L \cap P$, that is farthest from q. The point x is then in the relative interior of the line segment $[r, q]$, so there exists a $0 < \beta < 1$, such that $x = \beta r + (1 - r)q$.
As P is the intersection of finitely many facet-defining closed halfspaces, there is at least one facet F of P, such that $\{r\} = F \cap L$. Then x lies strictly beneath F, i.e. $\langle u_F, x\rangle < 1$. We may assume that $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$. As $-q \in P$, we have $\langle u_F, q\rangle \geq -1$. The point x is a lattice point strictly between r and q on the line segment $[r, q]$, hence $\langle u_F, x\rangle = 0$ and $0 < \langle u_F, -q\rangle \leq 1$.
Furthermore, x is not the origin, so x has at least one positive e_i-coordinate, say $\langle u_F^{e_1}, x\rangle \geq 1$. Then

$$1 \leq \langle u_F^{e_1}, x\rangle = \beta\langle u_F^{e_1}, r\rangle + (1 - \beta)\langle u_F^{e_1}, q\rangle.$$

As $r \in F$, we must have $0 \leq \langle u_F^{e_1}, r\rangle \leq 1$, and by lemma 1.6 $\langle u_F^{e_1}, -q\rangle \geq \langle u_F, -q\rangle - 1 > -1$. Hence

$$1 \leq \beta\langle u_F^{e_1}, r\rangle + (1 - \beta)\langle u_F^{e_1}, q\rangle < \beta + (1 - \beta) = 1,$$

which is a contradiction.
So every lattice point in $\operatorname{conv}(P \cup -P)$ is either the origin or a vertex of $\operatorname{conv}(P \cup -P)$. $\square$

The author suspects that theorem 4.2 does not hold for terminal simplicial reflexive polytopes.

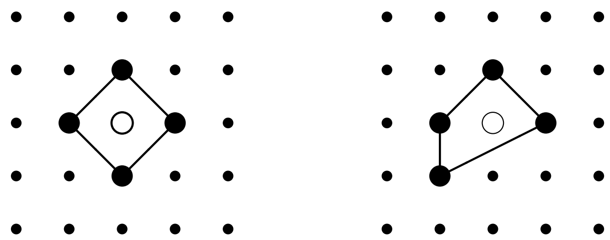


Figure 4.3: Conjecture 4.3 does not hold for simplicial reflexive polytopes: Here we have two simple 2-polytopes with isomorphic decorated graphs, but their dual polytopes are not isomorphic.

4.2 Recovery from the dual edge graph

Let P be a smooth Fano d-polytope, and consider the edge graph $\mathfrak{G}(P^*)$ of the dual polytope P^*: That is, the graph $\mathfrak{G}(P^*)$ consisting of the vertices and the edges of P^* (i.e. facets and ridges of P). Define a decoration Γ of the edges: $\Gamma(E) = |\{\mathbb{Z}^d \cap E\}|$ for each edge E of P^*. In other words, if F is a facet of P and v a vertex of F, then $\Gamma(E) = \langle u_F, n(F,v)\rangle + 2$ for the edge E corresponding to the ridge $\operatorname{conv}(\mathcal{V}(F) \setminus \{v\})$ (lemma 1.10).

Maybe it is possible to recover the polytope P (up to isomorphism of course) from the abstract graph $\mathfrak{G}(P^*)$ with the edge decoration Γ? Or in other words:

Conjecture 4.3. *Two smooth Fano polytopes P_1 and P_2 are isomorphic if and only if the decorated graphs $\mathfrak{G}(P_1^*)$ and $\mathfrak{G}(P_2^*)$ are isomorphic.*

The face lattice of a simple polytope can be recovered from the edge graph ([27] theorem 3.12). Thus two smooth Fano polytopes with non-isomorphic face lattices give non-isomorphic decorated graphs. Corollary 2.4 and theorem 2.7 implies that conjecture 4.3 holds when one of the graphs $\mathfrak{G}(P_i^*)$ has constant decoration.

But does the proposed conjecture 4.3 hold in general?

4.3 Improving and generalizing the SFP-algorithm

In this section we briefly discuss some possible improvements and generalizations of the SFP-algorithm.

4.3.1 Improvements

One way to improve the SFP-algorithm is to find new ways of concluding that a given subset V is NOT a presubset of $\mathrm{ord}(P)$ for any smooth Fano polytope P. This could decrease computation time. Some obvious properties a subset $V \subseteq \mathcal{W}_d$ must satisfy are the following

Lemma 4.4. *Suppose $V \subseteq \mathcal{W}_d$ is a presubset of* $\mathrm{ord}(P)$ *for some smooth Fano d-polytope P. Then*

1. $\forall v \in V \ : \ v \notin \mathrm{conv}\{\{0\} \cup V \setminus \{v\}\}$
2. $\mathrm{conv} V \cap \mathbb{Z}^d \setminus \{0\} = V$.

As it appears now, the SFP-algorithm does not take this into account.
And one could probably come up with other properties.

4.3.2 Generalizations

It would be interesting to see if the SFP-algorithm could be generalized to arbitrary simplicial reflexive d-polytopes. Or maybe only a subclass of these, say the class of simplicial reflexive d-polytopes having a smooth special facet (i.e. a special facet whose vertex set is a basis of the lattice $\mathbb{Z}^d$). In many cases researchers just need a lot of higher dimensional examples to test their conjectures on, and not necessarily the complete classification list.
Here we sketch a possible generalization of the SFP-algorithm to classify simplicial reflexive d-polytopes having a smooth special facet. As in the previous chapter $I := \mathrm{conv}\{e_1, \dots, e_d\}$.

Find some finite subset $\mathcal{W}_d^s \subset \mathbb{Z}^d$. The subset $\mathcal{W}_d^s$ ("s" for simplicial) should have the property that each simplicial reflexive d-polytope with a smooth special facet, is isomorphic to a polytope P, where $\mathcal{V}(P) \subset \mathcal{W}_d^s$ and I is a special facet of P.
We know that $-d \leq \langle u_F, v \rangle \leq 1$ for each special facet F of a simplicial reflexive d-polytope P and each vertex $v \in \mathcal{V}(P)$. For each $w \in \mathcal{V}(F)$ there is a lower bound on the w-coordinate of $v \in \mathcal{V}(P)$: $\langle u_F^w, v \rangle \geq \langle u_F, v \rangle - 1$. If

one assumes the special facet F is smooth, then $\langle u_F^w, v\rangle \in \mathbb{Z}$ and a desired finite set is thus determined: $x \in \mathcal{W}_d^s \subset \mathbb{Z}^d$ if and only if

1. $x \neq 0$, x is a primitive lattice point.
2. $-d \leq \langle u_I, x\rangle \leq 1$
3. $\langle u_I^{e_i}, x\rangle \geq \langle u_I, x\rangle - 1$ for every $1 \leq i \leq d$.

Find a suitable total order on $\mathcal{W}_d^s$. The ordering should have the property

$$n(I, e_i) = \min\{v \in \mathcal{V}(P) \mid \langle u_I^{e_i}, v\rangle < 0\}$$

for every $1 \leq i \leq d$, when I is a (special) facet of a simplicial reflexive d-polytope P. This was the essential property in the SFP-algorithm that allowed us to deduce the neighboring facets of the initial facet I.

Make an algorithm. Make an algorithm much like the SFP-algorithm that can classify simplcial reflexive d-polytopes having at least one smooth special facet. Use lemma 1.7 and lemma 1.6 to deduce facets and bound coordinates.

4.4 Smooth Fano polytopes with many vertices

Here we mention a few ideas for further research in the subject of smooth Fano d-polytopes with many vertices.

4.4.1 Containing del Pezzo 2-polytopes

By studying the classification lists the author has noticed that the following conjecture holds for $d \leq 8$.

Conjecture 4.5. *Let P be a smooth Fano d-polytope, such that $2|\mathcal{V}(P)| > 5d$. Then P contains a del-Pezzo 2-polytope V_2.*

If the conjecture holds, we would have the following classification result (by Casagrandes theorem 5 in [8]).

Conjecture 4.6. *Let X be a smooth Fano toric d-fold. If the number of toric divisors on X is strictly greater than $\frac{5}{2}d$, then X is a toric bundle over a lower dimensional smooth toric Fano variety, with fiber a product of del Pezzo varieties.*

4.4.2 Maximal number of facets

As mentioned earlier, it has been conjectured by Benjamin Nill, that $|\mathcal{V}(P)| \leq 6^{\frac{d}{2}}$ for every reflexive d-polytope P, with equality if and only if d is even and P^* splits into $\frac{d}{2}$ copies of del Pezzo 2-polytopes (i.e. P^* is a smooth Fano d-polytope with $3d$ vertices). If one could prove an upper bound on the number of facets $f_{d-1}(P)$ of a smooth Fano d-polytope P, a proof of the upper bound on the number of vertices of reflexive d-polytopes might follow.

4.4.3 Classification of smooth Fano d-polytopes with $3d - 2$ vertices

By appendix A the number $\mu(d)$ of isomorphism classes of smooth Fano d-polytopes with $3d - 2$ vertices grows like this.

d	2	3	4	5	6	7	8
$\mu(d)$	2	4	10	5	11	5	11

The table indicates that an explicit classification of these polytopes should be possible.
As a first step one could modify the C++ code of the SFP-algorithm to make the algorithm construct only smooth Fano d-polytopes with at least $3d - 2$ vertices. This modification can be done be making the algorithm check that there is "room" for at least $3d - 2$ vertices each time it has added a new point: For instance, a special embedding of a smooth Fano d-polytope with $3d - 2$ vertices has at least $d - 2$ vertices in $H(I, 0)$. So a point like $-e_1 - e_2 - e_3 - e_4 + 4e_5$ cannot be a vertex of such a special embedding (for $d \geq 5$).
Running the modified code might yield some additional numbers $\mu(d)$, $d > 8$. By inspecting the constructed polytopes one could probably guess the correct classification.

Appendix A

Classification results

The number of isomorphism classes of smooth Fano d-polytopes with n vertices is shown in the table below.

n	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
1								
2	1							
3		1						
4		2	1					
5		1	4	1				
6		1	7	9	1			
7			4	28	15	1		
8			2	47	91	26	1	
9				27	268	257	40	1
10				10	312	1318	643	62
11				1	137	2807	5347	1511
12				1	35	2204	19516	19453
13					5	771	26312	114697
14					2	186	14758	253950
15						39	4362	226867
16						11	1013	98657
17						1	214	26831
18						1	43	6281
19							5	1286
20							2	243
21								40
22								11
23								1
24								1
Total	1	5	18	124	866	7622	72256	749892

Appendix B

The C++ code

```
#include<fstream>
#include<vector>
#include<string>
#include<iostream>
#include<sstream>
#include<numeric>

using namespace std;

typedef vector < vector<int> > matrix;

// The dimension d

int d;

// Global variables for facets

matrix facets;
matrix normals;
vector < matrix > basechanges;
matrix neighboring_facets;

// Global variables for vertices

matrix vertices;
vector < int > neighboring_vertices;
vector < int > neighbors_in_hyperplane;

// Global constants

matrix zero_matrix;
vector < int > minus_vector;
vector < int > zero_vector;
```

```
// Output to files

vector < int > number_of_equivalence_classes;
ofstream fanovert;
ofstream msg;

// PRINTING FUNCTIONS

void print_vector(vector < int > v)
{
  for(int i=0;i<v.size();++i)
    msg << v[i] << " ";
}

// OUTPUT TO FILE

void output_vertices()
{
  for(int i=0;i<vertices.size();++i)
    {
      for(int j=0;j!=d;++j)
        fanovert << vertices[i][j] << " ";
      fanovert << "\n";
    }
  fanovert << endl;
  ++number_of_equivalence_classes[vertices.size()];
}

// LINEAR ALGEBRA FUNCTIONS

inline int dotprod ( vector <int> &v , vector <int> &w )
{
  int result=0;
  for(int i=0;i!=d;++i)
    result+=v[i]*w[i];
  return result;
}

inline int vect_mult_col ( vector < int > &v , matrix &A , int col )
{
  int res=0;
  for(int i=0;i<d;++i)
    res+=v[i]*A[i][col];
  return res;
}

inline void vect_mult_matr ( vector < int > &v , matrix &A , vector < int
> &result )
{
```

```
  result.resize(d);
  for(int j=0;j<d;++j)
    result[j]=vect_mult_col(v,A,j);
}

inline void matr_mult_matr ( matrix &A , matrix &B , matrix &C )
{
  C.resize(A.size(),zero_vector);
  for(int i=0;i!=A.size();++i)
    vect_mult_matr(A[i],B,C[i]);
}

// Add the rows of matrix

inline void add_rows ( const matrix &M , vector < int > &sum )
{
  sum.resize(d,0);
  for(int col=0;col<d;++col)
    for(int row=0;row<M.size();++row)
      sum[col]+=M[row][col];
}

// Add the entries of a vector

inline int add_entries ( vector < int > &v )
{
  return accumulate(v.begin(),v.end(),0);
}


//  FACET ADDING FUNCTIONS

// Is ridge in facet?

inline int ridge_in_facet ( int facet , vector < int > &ridge )
{
  int opposite=-1;
  for(int i=0;i<d;++i)
    {
      if(find(ridge.begin(),ridge.end(),facets[facet][i])==ridge.end())
        {
          // Return -1 if ridge not in facet

          if(opposite!=-1)
            return -1;
          opposite=i;
        }
    }
  // Return placement of opposite in facet_vertex_vector
```

```
  return opposite;
}

// Close ridges of a facet just added

inline bool close_ridges ()
{
  for(int i=0;i<d;++i)
    {
      // Create ridge

      vector < int > ridge;
      for(int j=0;j<d;++j)
        if(j!=i)
          ridge.push_back((facets.back())[j]);

      // Find a facet containing this ridge
      // (not the one just added)

     for(int facet=0;facet<facets.size()-1;++facet)
        {
          int placement=ridge_in_facet(facet,ridge);
          if(placement!=-1)
            {
              // Return false, if neighboring facet not unique

              if(neighboring_facets[facet][placement]!=-1)
                return false;

              // Join facets and close ridge

              neighboring_facets[facet][placement]=facets.size()-1;
              (neighboring_facets.back())[i]=facet;
              break;
            }
        }
    }
  return true;
}

bool vertex_facet_check ( int facet , int vertex_number );

bool make_new_facet ( int oldfacet , int neighboring_vertex , int opposite
_number )
{
  // Add new facet

  vector <int> new_facet(facets[oldfacet]);
```

```
  new_facet[opposite_number]=neighboring_vertex;
  facets.push_back(new_facet);

  // New basechange matrix

  vector < int > relation;
  vect_mult_matr(vertices[neighboring_vertex],basechanges[oldfacet],relati
on);
  relation[opposite_number]=-2;
  basechanges.push_back(basechanges[oldfacet]);
  for(int i=0;i<d;++i)
    for(int j=0;j<d;++j)
      (basechanges.back())[i][j]+=basechanges[oldfacet][i][opposite_number
]*relation[j];

  // New normal

  vector < int > new_normal(d,0);
  for(int i=0;i<d;++i)
    new_normal[i]=add_entries((basechanges.back())[i]);
  normals.push_back(new_normal);

  // New neighbors

  neighboring_facets.push_back(minus_vector);

  // Close ridges

  if(close_ridges()==false)
    return false;

  // Consequences of new facet

  int f_end=facets.size()-1;
  for(int v=0;v!=vertices.size();++v)
    if(vertex_facet_check(f_end,v)==false)
      return false;

  return true;
}

bool vertex_facet_check ( int facet , int vertex )
{
  // If vertex is in facet , then no need to check

  if(find(facets[facet].begin(),facets[facet].end(),vertex)==facets[facet]
.end())
    {
      // In which hyperplane is vertex?
```

```
      int hyperplane=dotprod(normals[facet],vertices[vertex]);

      // If vertex is not strictly beneath facet, return false

      if(hyperplane>0)
        return false;
      for(int j=0;j<d;++j)
        {
          if(neighboring_facets[facet][j]==-1)
            {
              // Calculate coeficient to basis vector j
              // with respect to basis V(facet)

              int coef=vect_mult_col(vertices[vertex],basechanges[facet],j
);

              // Add a facet , if vertex is in zero-hyperplane
              // or a neighboring vertex of initial facet

              if(hyperplane==0 || facet==0)
                {
                  if(coef<-1 || (coef==-1 && make_new_facet(facet,vertex,j
)==false))
                    return false;
                }
              else

                // Return false, if coef exceeds lower bound

                if(coef<hyperplane)
                  return false;
            }
        }
    }
  return true;
}

// Could the vertex set be a presubset of a smooth Fano d-polytope?

bool checksubset ()
{
  // Is new vertex beneath every known facet?

  int f_end=facets.size();
  for(int facet=1;facet<f_end;++facet)
    if(dotprod(normals[facet],vertices.back())>0)
      return false;
```

```
  // Consequences of new vertex

  for(int facet=0;facet<f_end;++facet)
    if(vertex_facet_check(facet,vertices.size()-1)==false)
      return false;

  return true;
}

bool add_neighboring_facet ( int facet , int opposite_number )
{
  // Find hyperplane and coeficient for each vertex

  vector < int > h;
  vector < int > c;
  for(int v=0;v<vertices.size();++v)
    {
      h.push_back(dotprod(vertices[v],normals[facet]));
      c.push_back(vect_mult_col(vertices[v],basechanges[facet],opposite_nu
mber));
    }

  // Find neighboring vertex

  int neighboring_vertex=-1;
  for(int v=0;v<vertices.size();++v)
    {
      if(c[v]==-1)
        {
          if(neighboring_vertex==-1 || h[neighboring_vertex]<h[v])
            neighboring_vertex=v;
        }
    }

  // If no neighboring vertex, return false

  if(neighboring_vertex==-1)
    return false;

  // If another vertex is not beneath the coming
  // neighboring facet, then return false

  for(int v=0;v<vertices.size();++v)
    {
      if(neighboring_vertex!=v && c[v]<0)
        {
          if(h[v]+c[v]*(h[neighboring_vertex]-1)>0)
            return false;
        }
```

```
    }

  // Create new facet

  return make_new_facet(facet,neighboring_vertex,opposite_number);
}

bool conv_is_fano ()
{
  // For every facet: Try to add missing neighboring facets

  int f=0;
  while(facets.size()>f)
    {
      for(int j=0;j<d;++j)
        if(neighboring_facets[f][j]==-1 && add_neighboring_facet(f,j)==fal
se)
          return false;
      ++f;
    }
  // Every facet had neighboring facets

  return true;
}

//  UNDO ADDED FACETS

void remove_facets( int new_size )
{
  if(new_size!=facets.size())
    {
      // Erase some facets

      facets.resize(new_size);
      normals.resize(new_size);
      basechanges.resize(new_size);
      neighboring_facets.resize(new_size);

      // Open ridges

      for(int f=0;f!=new_size;++f)
        {
          for(int j=0;j<d;++j)
            if(neighboring_facets[f][j]>=new_size)
              neighboring_facets[f][j]=-1;
        }
    }
}
```

```
// COMPARISON FUNCTIONS

// All function returns
//  -1   if first input smallest
//   1   if second input smallest
//   0   if equal input

inline int compare_points ( vector < int > &v1 , vector < int > &v2 )
{
  int h1=accumulate(v1.begin(),v1.end(),0);
  int h2=accumulate(v2.begin(),v2.end(),0);
  if(h1>h2)
    return -1;
  if(h1<h2)
    return 1;
  for(int i=0;i<d;++i)
    {
      if(v1[i]<v2[i])
        return -1;
      if(v1[i]>v2[i])
        return 1;
    }
  return 0;
}

// Swaps two columns of the matrix from a given row to the bottom

inline void swap_columns ( matrix &M , int column1 , int column2 , int row
 )
{
  int x;
  for(int v=row;v<M.size();++v)
    {
      x=M[v][column1];
      M[v][column1]=M[v][column2];
      M[v][column2]=x;
    }
}

// If two columns are equal from a given row to the top, return true
// Otherwise return false

inline bool equal_column_vectors ( matrix &M , int column1 , int column2 ,
 int row )
{
  for(int i=0;i<row;++i)
    if(M[i][column1]!=M[i][column2])
      return false;
  return true;
```

```
}

// Minimizes a row (with respect to lex order) fixing the matrix
// from a given row to the top

void minimize_row ( matrix &M , int row )
{
  for(int column1=0;column1<d;++column1)
    {
      // Find a column to swap with, to obtain a smaller row vector

      int column2=column1;
      for(int i=column1+1;i<d;++i)
        if(M[row][i]<M[row][column2] && equal_column_vectors(M,column1,i,r
ow))
          column2=i;
      if(column2!=column1)
        swap_columns(M,column1,column2,row);
    }
}

// Minimize the matrix from a given row to the bottom.
// Compares with matrix vertices.

int minimize_below ( matrix &M , int present_row , vector < int > *positio
ns , bool true_if_equal )
{
  // If past the bottom row, return 0

  if(present_row==M.size())
    return 0;

  // Snapshot of positions

  vector < int > snapshot(*positions);

  // Try every vector in the given row

  for(int i=present_row;i<M.size();++i)
    {
      // Is it necessary to permute some rows?

      if(snapshot[i]!=(*positions)[present_row])
        {
          // Yes it is. Put the right row vector in
          // the present row.

          for(int j=present_row+1;j<M.size();++j)
            {
```

```
              if(snapshot[i]==(*positions)[j])
                {
                  // Swap rows

                  vector < int > dummy_vector(M[present_row]);
                  M[present_row]=M[j];
                  M[j]=dummy_vector;
                  int dummy_int=(*positions)[present_row];
                  (*positions)[present_row]=(*positions)[j];
                  (*positions)[j]=dummy_int;
                  break;
                }
            }
        }
      // Minimize present row (fixing the rows above)

      minimize_row(M,present_row);

      // Compare present row with same row in vertices.

      int c=compare_points(vertices[d+present_row],M[present_row]);

      // If the obtained represention is smaller, return 1

      if(c==1)
        return 1;

      // If equal up to now, consider the rest.

      if(c==0)
        {
          int remain=minimize_below(M,present_row+1,positions,true_if_equa
l);

          // If the rest is larger, return 1

          if(remain==1)
            return 1;

          // If representations are equal, return true if
          // we know vertices are in the minimal representation.
          // (i.e. true_if_equal=true)

          if(remain==0 && true_if_equal)
            return 0;
        }
      // The obtained represention is larger. Try another.
   }
  return -1;
```

```
}

// Compare two special embeddings
// (other_vertices are not sorted by order)

int compare_representation ( matrix &other_vertices , bool true_if_equal )
{
  vector < int > positions;
  matrix reduced_vertices;
  for(int i=0;i<other_vertices.size();++i)
    {
      // Reduced vertices are without vertices in hyperplane one

      if(add_entries(other_vertices[i])<1)
        {
          positions.push_back(i);
          reduced_vertices.push_back(other_vertices[i]);
        }
    }
  // Minimize reduced_vertices from the top row

  return minimize_below(reduced_vertices,0,&positions,true_if_equal);
}

// Returns true if face is contained in facet, and returns false if not.

bool special_face_in_facet (vector < int > &special_face , vector < int >
&facet_vertices)
{
  for(int i=0;i<special_face.size();++i)
    if(find(facet_vertices.begin(),facet_vertices.end(),special_face[i])==
facet_vertices.end())
      return false;
  return true;
}

// Returns true if ord( conv(vertices) ) = vertices
// Otherwise false is returned.

bool it_is_new ( vector < int > &sum_of_vertices )
{
  // Find special face

  vector < int > special_face;
  for(int i=0;i<d;++i)
    {
      if(sum_of_vertices[i]<0)
        return false;
      if(sum_of_vertices[i]>0)
```

```
        special_face.push_back(i);
    }
  // Try every special embedding to see if ord(convV)=V

  for(int facet_number=1;facet_number<facets.size();++facet_number)
    {
      if(special_face_in_facet(special_face,facets[facet_number]))
        {
          matrix new_representation;
          matr_mult_matr(vertices,basechanges[facet_number],new_representa
tion);
          if(compare_representation(new_representation,true)==1)
            return false;
        }
    }
  // No strictly smaller special embedding

  return true;
}

// GREATEST COMMON DIVISOR FUNCTIONS

int gcd_pair ( int v1 , int v2 )
{
  while (v2)
    {
      int k=v2;
      v2=v1 % v2;
      v1=k;
    }
  return v1;
}

int gcd ( vector <int> &v )
{
  if(v.empty())
    return 0;
  int sfd=(v[0]<0 ? -v[0] : v[0]);
  for(int i=1;i!=v.size();++i)
    {
      int e=(v[i]<0 ? -v[i] : v[i]);
      sfd=gcd_pair(sfd,e);
    }
  return sfd;
}

// False if lower bound exceeds upperbound. True otherwise.

inline bool lower_below ( vector < int > &lowerbounds , vector < int > &up
```

```
perbounds )
{
  for(int i=0;i<d;++i)
    if(lowerbounds[i]>upperbounds[i])
      return false;
  return true;
}

// AddPoint and its related functions

inline bool first_entries_ok ( vector < int > &new_vertex , int h , vector
 < int > &lowerbounds , vector < int > &upperbounds , int position )
{
  // The first position entries have been chosen.
  // Do the first entries violate the bounds?

  for(int i=0;i<position;++i)
    if(new_vertex[i]<lowerbounds[i] || new_vertex[i]>upperbounds[i])
      return false;

  // Check bounds

  if(lower_below(lowerbounds,upperbounds)==false)
    return false;

  // The first position entries have been chosen. Is it possible to hit hy
perplane h
  // staying inside the lower and upper bounds? If not, return false.

  int partial_sum=accumulate(new_vertex.begin(),new_vertex.begin()+positio
n+1,0);
  if(accumulate(lowerbounds.begin()+position+1,lowerbounds.end(),0)+partia
l_sum>h)
    return false;
  if(accumulate(upperbounds.begin()+position+1,upperbounds.end(),0)+partia
l_sum<h)
    return false;

  // Yes it is.

  return true;
}

bool next_vertex ( vector < int > &new_vertex , int h , int position , vec
tor < int > &lowerbounds , vector < int > &upperbounds )
{
  // If all entries have been chosen, return true if gcd=1.

  if(position==d)
```

```
    {
      if(gcd(new_vertex)==1)
        return true;
      else
        return false;
    }

  // Find the interval [coord_begin,coord_end] the entry in position can b
e in.

  int coord_begin=max(lowerbounds[position],new_vertex[position]);
  int partial_sum=accumulate(new_vertex.begin(),new_vertex.begin()+positio
n,0);
  int lower_tail=accumulate(lowerbounds.begin()+position+1,lowerbounds.end
(),0);
  int coord_end=min(upperbounds[position],h-partial_sum-lower_tail);

  // Run through this interval.

  for(int coord=coord_begin;coord<=coord_end;++coord)
    {
      // If the entry in position is equal to coord,
      // how do the upper and lower bounds change?

      vector < int > new_lowerbounds(lowerbounds);
      if(neighboring_vertices[position]!=-1)
        {
          int k=h+coord*(neighbors_in_hyperplane[position]-1);
          for(int i=0;i<d;++i)
            {
              if(i!=position)
                {
                  int c=(k==0 ? -1 : k)-coord*vertices[neighboring_vertice
s[position]][i];
                  if(c>lowerbounds[i])
                    new_lowerbounds[i]=c;
                }
            }
        }
      // If the entry in position is increased,
      // then set the last entries to the lower bound.

      if(coord>new_vertex[position])
        for(int i=position+1;i<d;++i)
          new_vertex[i]=new_lowerbounds[i];

      // Set the entry in position equal to coord.

      new_vertex[position]=coord;
```

```
      // Are the first entries ok?

      if(first_entries_ok(new_vertex,h,new_lowerbounds,upperbounds,positio
n))
        {
          // Can the last entries be chosen?

          if(next_vertex(new_vertex,h,position+1,new_lowerbounds,upperboun
ds))
            return true;
        }
    }
  // It is not possible to chose the last entries.

  return false;
}

void addpoint()
{
  // Compute the sum of the vertices

  vector < int > sum_vertices;
  add_rows(vertices,sum_vertices);
  int sum_in_hyperplane=add_entries(sum_vertices);

  // If convex hull is smooth Fano and
  // special embedding minimal, output.

  int number_of_facets=facets.size();
  if(conv_is_fano() && it_is_new(sum_vertices))
    output_vertices();
  remove_facets(number_of_facets);

  // Determine which hyperplanes to place the next vertex in.

  int h_begin=0;
  if(vertices.size()>d)
    h_begin=accumulate((vertices.back()).begin(),(vertices.back()).end(),0
);
  int h_end=-sum_in_hyperplane;

  // Run through these hyperplanes.

  for(int h=h_begin;h+sum_in_hyperplane>=0;--h)
    {
      // Determine lower bounds on entries of the next vertex

      vector < int > lowerbound(d,h-1);
```

```
      for(int i=0;i<d;++i)
        {
          int k=h+lowerbound[i]*(neighbors_in_hyperplane[i]-1);
          while(k>1 || (k==1 && lowerbound[i]<-1) || (k==1 && neighboring_
facets[0][i]!=-1) || (sum_in_hyperplane+2*h<0 && lowerbound[i]+sum_vertice
s[i]<0))
            {
              ++lowerbound[i];
              k+=neighbors_in_hyperplane[i]-1;
            }
        }

      // Determine upper bounds on entries of the next vertex

      vector < int > upperbound;
      for(int i=0;i<d;++i)
        {
          if(sum_in_hyperplane+2*h<0 && neighboring_facets[0][i]==-1)
            upperbound.push_back(-1);
          else
            upperbound.push_back(h+sum_in_hyperplane-sum_vertices[i]+(h==0
 && neighboring_facets[0][i]==-1 ? 1 : 0));
        }

      // The lower bound cannot be greater than upper bound.

      if(lower_below(lowerbound,upperbound)==false)
        return;

      // New vertex is set to lower bounds....

      vector < int > new_vertex(lowerbound);

      // ... or previous vertex with last entry increased.
      // (this happens if new vertex is in the same hyperplane
      // as the previous)

      if(h==accumulate((vertices.back()).begin(),(vertices.back()).end(),0
))
        {
          new_vertex.assign((vertices.back()).begin(),(vertices.back()).en
d());
          ++new_vertex[d-1];
        }

      // Run through possible vertices in the hyperplane h
      // in strictly increasing order.

      while(next_vertex(new_vertex,h,0,lowerbound,upperbound))
```

```
        {
          // Add new vertex to vertex set

          vertices.push_back(new_vertex);

          // Is it a neighboring vertex to the initial facet?

          for(int i=0;i<d;++i)
            {
              if(new_vertex[i]==-1 && neighboring_vertices[i]==-1)
                {
                  neighboring_vertices[i]=vertices.size()-1;
                  neighbors_in_hyperplane[i]=h;
                }
            }

          // If the new vertex set is compatible and no permutation of
          // the set results in a smaller set, call addpoint recursively.

          if(checksubset() && compare_representation(vertices,false)!=1)
            addpoint();

          // Undo changes regarding neighboring vertices

          for(int i=0;i<d;++i)
            {
              if(neighboring_vertices[i]==vertices.size()-1)
                {
                  neighboring_vertices[i]=-1;
                  neighbors_in_hyperplane[i]=0;
                }
            }

          // Undo changes to vertex set and facet

          vertices.pop_back();
          remove_facets(number_of_facets);

          // Increase last entry to ensure the next considered vertex
          // is greater than the previous.

          ++new_vertex[d-1];
        }
    }
}

// INITIALIZATION

void sfp()
```

```
{
  // Add initial facet to set of facets

  vector < int > initial_facet;
  for(int i=0;i<d;++i)
    initial_facet.push_back(i);
  facets.push_back(initial_facet);

  // Add normal

  vector < int > one_vector(d,1);
  normals.push_back(one_vector);

  // Add identity matrix to basechanges

  zero_vector.assign(d,0);
  matrix unitmatrix(d,zero_vector);
  for(int i=0;i<d;++i)
    unitmatrix[i][i]=1;
  basechanges.push_back(unitmatrix);

  // No neighboring facets to begin with

  minus_vector.assign(d,-1);
  neighboring_facets.push_back(minus_vector);

  // Set vertices to the standard basis

  vertices.assign(unitmatrix.begin(),unitmatrix.end());

  // No neighboring vertices to begin with

  neighboring_vertices.assign(d,-1);
  neighbors_in_hyperplane.assign(d,0);

  // Define constants

  zero_matrix.assign(d,zero_vector);

  // Call addpoint to begin the construction.

  addpoint();
}

// MAIN

int main (int argc,char *argv[])
{
  // Read arguments.
```

```
  // Only one argument allowed: an integer between 2 and 9.

  if(argc!=2)
    return 0;
  string st(argv[1]);
  if(st.size()!=1)
    return 0;
  if(isdigit(st[0])==false)
    return 0;
  int d_char=st[0];
  d=d_char-48;
  if(d<2)
    return 0;

  // Open output files

  char outfile1[]="fanovertd";
  outfile1[8]=d_char;
  fanovert.open(outfile1);
  char outfile2[]="msgd";
  outfile2[3]=d_char;
  msg.open(outfile2);
  msg << endl << "SFP-algorithm: dimension " << d << endl << endl;
  number_of_equivalence_classes.assign(3*d+1,0);

  // Initialize and commence construction

  sfp();

  // Final output

  msg << "Construction finished!" << endl << endl;
  msg << "Up to isomorphism, the program found" << endl << endl;
  for(int i=d+1;i<3*d+1;++i)
    msg << number_of_equivalence_classes[i] << " with " << i << " vertices
." << endl << endl;
  msg << endl;
  msg << "Total number of smooth Fano " << d << "-polytopes: " << (int) ac
cumulate(number_of_equivalence_classes.begin(),number_of_equivalence_class
es.end(),0) << endl << endl;

  // Close files

  fanovert.close();
  msg.close();
}
```

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Index

SMOOTH FANO POLYTOPES CAN NOT BE INDUCTIVELY CONSTRUCTED

MIKKEL ØBRO

Abstract

We examine a concrete smooth Fano 5-polytope P with 8 vertices with the following properties: There does not exist a smooth Fano 5-polytope Q with 7 vertices such that P contains Q, and there does not exist a smooth Fano 5-polytope R with 9 vertices such that R contains P. As the polytope P is not pseudo-symmetric, it is a counter example to a conjecture proposed by Sato.

1 Introduction

Many papers have been concerned about the classification of smooth Fano polytopes (among these, e.g., [2, 4, 5, 8] and references therein). These polytopes have been completely classified up to dimension 4 modulo unimodular equivalence. Recently the classification of smooth Fano 5-polytopes has been announced ([7]).

One approach is to attempt to construct smooth Fano d-polytopes inductively from simpler or already known ones by adding and removing vertices according to some rule, while staying inside the realm of smooth Fano d-polytopes for some fixed $d \geq 1$.

This idea is behind the notion of F-equivalence, due to Sato in [8]. By $\mathcal{V}(P)$ we denote the set of vertices of a polytope P.

DEFINITION 1.1 (equivalent to Definitions 1.1 and 6.1 in [8]). Two smooth Fano d-polytopes P and Q are called *F-equivalent* if there exists a sequence

$$P_0\ ,\ P_1\ ,\ \ldots\ ,\ P_{k-1}\ ,\ P_k\ ,\ k \geq 0$$

of smooth Fano d-polytopes P_i satisfying the following:

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1. P and Q are unimodular equivalent to P_0 and P_k, respectively.

2. For every $1 \leq i \leq k$ either $\mathcal{V}(P_{i-1}) = \mathcal{V}(P_i) \cup \{w\}$ or $\mathcal{V}(P_i) = \mathcal{V}(P_{i-1}) \cup \{w\}$ for some lattice point $w \neq 0$.

3. If $w \in \mathcal{V}(P_i) \setminus \mathcal{V}(P_{i-1})$, then there exists a proper face F of P_{i-1} such that $w = \sum_{v \in \mathcal{V}(F)} v$ and the set of facets of P_i containing w is equal to

$$\{\mathrm{conv}(\{w\} \cup (\mathcal{V}(F') \setminus \{v\})) \mid F' \text{ facet of } P_{i-1},\ F \subseteq F',\ v \in \mathcal{V}(F)\}.$$

If $w \in \mathcal{V}(P_{i-1}) \setminus \mathcal{V}(P_i)$, a similar condition holds.

The third requirement in the definition above is the rule of vertex adding and removal. It has an equivalent formulation in terms of the corresponding smooth toric Fano varieties: The toric variety corresponding to P_i is an equivariant blow-up or blow-down of the toric variety corresponding to P_{i-1}.

Clearly, F-equivalence is an equivalence relation on the set of smooth Fano d-polytopes. The problem is now: Find a set of representatives, so that every smooth Fano d-polytope is F-equivalent to one of these representatives.

Sato proposes the following conjecture. Recall that a smooth Fano polytope P is called pseudo-symmetric if there exists a facet F of P, such that $-F$ is also a facet. The notion of pseudo-symmetry is due to Ewald and pseudo-symmetric smooth Fano d-polytopes have been classified completely for every $d \geq 1$ (see [5]).

Conjecture 1.2 ([8, Conjectures 1.3 and 6.3]). *Any smooth Fano d-polytope is either pseudo-symmetric or F-equivalent to the simplex*

$$T_d := \mathrm{conv}\{e_1, \ldots, e_d, -e_1 - \ldots - e_d\},$$

where (e_i) is the standard integral basis of the lattice $\mathbb{Z}^d$.

The conjecture is known to hold for $d \leq 4$ ([8, Theorems 7.1 and 8.1]). Indeed, every smooth Fano 3-polytope is F-equivalent to the simplex T_3, and there are only 2 smooth Fano 4-polytopes not F-equivalent to the simplex T_4: They are the *del Pezzo 4-polytope* V^4 and the *pseudo del Pezzo 4-polytope* $\tilde{V}^4$, where

$$\begin{aligned} V^{2k} &= \mathrm{conv}\{\pm e_1, \ldots, \pm e_{2k}, \pm(e_1 + \ldots + e_k - e_{k+1} - \ldots - e_{2k})\}, \\ \tilde{V}^{2k} &= \mathrm{conv}\{\pm e_1, \ldots, \pm e_{2k}, e_1 + \ldots + e_k - e_{k+1} - \ldots - e_{2k}\}. \end{aligned}$$

Both V^4 and $\tilde{V}^4$ are alone in their F-equivalence class. However, notice that

$$V^{2k} = \mathrm{conv}\left(\mathcal{V}(\tilde{V}^{2k}) \cup \{-e_1 - \ldots - e_k + e_{k+1} + \ldots + e_{2k}\}\right)$$

and

$$\tilde{V}^{2k} = \mathrm{conv}\left(\{\pm e_1, \ldots, \pm e_{2k}\} \cup \{e_1 + \ldots + e_k - e_{k+1} - \ldots - e_{2k}\}\right).$$

Since $\mathrm{conv}\{\pm e_1, \ldots, \pm e_{2k}\}$ is a smooth Fano $2k$-polytope F-equivalent to T_{2k} ([8, Theorem 6.7]), one might be tempted to define a new equivalence relation, say *I-equivalence* (I for inductive), by requiring only 1 and 2 in Definition 1.1, meaning that there are no restrictions on vertex adding and removal. Then by the classification of pseudo-symmetric smooth Fano polytopes ([5]) and Theorem 6.7 in [8], any pseudo-symmetric smooth Fano d-polytope is I-equivalent to the simplex T_d. Inspired by Sato's conjecture one might then suspect: *Every smooth Fano d-polytope is I-equivalent to T_d.* This would indeed hold for $d \leq 4$.

The result of this paper is that Conjecture 1.2 is not true. We show this by means of an explicit counter example. More precisely, we examine a smooth Fano 5-polytope P with 8 vertices with the following properties:

(i) P is not pseudo-symmetric.

(ii) There does not exist a smooth Fano 5-polytope Q with 7 vertices, such that $Q \subset P$ (Proposition 4.1).

(iii) There does not exist a smooth Fano 5-polytope R with 9 vertices, such that $P \subset R$ (Proposition 4.2).

Furthermore, the example shows the existence of 'isolated' smooth Fano d-polytopes: It is not possible to obtain P from another smooth Fano 5-polytope by adding or removing a vertex, no matter what rule one uses for the inductive construction.

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2 Notation

We begin by fixing the notation and recalling some basic facts.

By $\mathrm{conv}K$ we denote the *convex hull* of the set K. When P is any polytope, i.e. the convex hull of a finite set of points, $\mathcal{V}(P)$ denotes the set of vertices of P.

A simplicial convex lattice polytope in $\mathbb{R}^d$ is called a *smooth Fano d-polytope* if the origin is contained in the interior of P and the vertices $\mathcal{V}(F)$ of every facet F of P is a $\mathbb{Z}$-basis of the integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Two smooth Fano d-polytopes $P_1, P_2 \subset \mathbb{R}^d$ are called *unimodular equivalent*, if there exists a bijective linear map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, such that $\varphi(\mathbb{Z}^d) = \mathbb{Z}^d$ and $\varphi(P_1) = P_2$. Unimodular equivalence classes of smooth Fano d-polytopes correspond to isomorphism classes of smooth Fano toric d-folds ([2, Theorem 2.2.4]).

When P is a smooth Fano d-polytope and F is any facet of P, there exists a unique linear map $u_F : \mathbb{R}^d \to \mathbb{R}$ such that $u_F(v) = 1$ for every $v \in \mathcal{V}(F)$. Clearly, $u_F(x) \leq 1$ for any $x \in P$ with equality if and only if $x \in F$. Every vertex v of P is a $\mathbb{Z}$-linear combination of $\mathcal{V}(F)$, so $u_F(v) \in \mathbb{Z}$. In particular, $u_F(v) \leq 0$ if and only if $v \notin \mathcal{V}(F)$.

Recall that $(d-2)$-dimensional faces of a d-polytope are called *ridges*. Every ridge is the intersection of precisely two facets of the polytope.

Lemma 2.1. *Let P be a smooth Fano d-polytope. Let F_1 and F_2 be two facets of P such that $F_1 \cap F_2$ is a ridge G of P. Let v_i, $i = 1, 2$, be the vertex of F_i which is not contained in G.*

Then

$$v_1 + v_2 = \sum_{w \in \mathcal{V}(G)} a_w w,$$

for some integers a_w.

Every point $x \in \mathbb{Z}^d$ is a unique $\mathbb{Z}$-linear combination of the vertices of F_1

$$x = \sum_{w \in \mathcal{V}(F_1)} b_w w, \ b_w \in \mathbb{Z}$$

and

$$u_{F_2}(x) = u_{F_1}(x) + b_{v_1} \left(u_{F_1}(v_2) - 1 \right).$$

If $x \in \mathcal{V}(P)$, $x \neq v_2$ and $b_{v_1} < 0$, then $u_{F_1}(v_2) > u_{F_1}(x)$.

Proof. Both $\mathcal{V}(F_1)$ and $\mathcal{V}(F_2)$ are lattice bases of $\mathbb{Z}^d$ and the first assertion follows. The second statement is clear for all $x \in \mathcal{V}(F_2)$, and then for all $x \in \mathbb{Z}^d$. Suppose $x \in \mathcal{V}(P) \setminus (\mathcal{V}(F_1) \cup \mathcal{V}(F_2))$ and $b_{v_1} < 0$. Then $u_{F_2}(x) \leq 0$ and

$$u_{F_1}(x) \leq b_{v_1}(1 - u_{F_1}(v_2)) < u_{F_1}(v_2),$$

which proves the last inequality. □

3 Primitive relations

Now we recall the concepts of primitive collections and relations, which are due to Batyrev in [1]. These are excellent tools for representation and classification of smooth Fano polytopes (see [2, 4, 8]).

Let $C = \{v_1, \ldots, v_k\}$ be a subset of $\mathcal{V}(P)$, where P is a smooth Fano polytope. The set C is called a *primitive collection* if $\operatorname{conv}(C)$ is not a face of P, but $\operatorname{conv}(C \setminus \{v_i\})$ is a face of P for every $1 \leq i \leq k$. Consider the lattice point $x = v_1 + \ldots + v_k$. There exists a unique face $\sigma(C) \neq P$ of P, called the *focus* of C, such that x is a positive $\mathbb{Z}$-linear combination of vertices of $\sigma(C)$, that is

$$x = a_1 w_1 + \ldots + a_m w_m,\ a_i \in \mathbb{Z}_+,$$

where $\{w_1, \ldots, w_m\} = \mathcal{V}(\sigma(C))$. The linear relation

$$v_1 + \ldots + v_k = a_1 w_1 + \ldots + a_m w_m \tag{1}$$

is called a *primitive relation*. The integer $k - a_1 - \ldots - a_m$ is called the *degree* of the primitive relation (1) and is always positive ([2, Proposition 2.1.10]).

Lemma 3.1 ([3, Corollary 4.4]). *Let*

$$v_1 + \ldots + v_k = a_1 w_1 + \ldots + a_m w_m \tag{2}$$

be a linear relation of vertices of a smooth Fano polytope P such that $a_i \in \mathbb{Z}_+$ and $\{v_1, \ldots, v_k\} \cap \{w_1, \ldots, w_m\} = \emptyset$. Suppose $k - a_1 - \ldots - a_m = 1$ and that $\operatorname{conv}\{w_1, \ldots, w_m\}$ *is a face of P. Then (2) is a primitive relation, and whenever $\{w_1, \ldots, w_m\}$ is contained in a face F, $(F \cup \{v_1, \ldots, v_k\}) \setminus \{v_i\}$ is a face of P for every $1 \leq i \leq k$.*

We recall the well-known classification of smooth Fano d-polytopes with $d+2$ vertices.

Theorem 3.2 ([6, Theorem 1]). *Let P be a smooth Fano d-polytope with $d+2$ vertices, $\mathcal{V}(P) = \{v_1, \ldots, v_{d+2}\}$. Then the primitive relations of P are (up to renumeration of the vertices)*

$$v_1 + \ldots + v_k = 0,\ 2 \leq k \leq d$$

and

$$v_{k+1} + \ldots + v_{d+2} = a_1 v_1 + \ldots + a_k v_k,\ a_1, \ldots, a_k \geq 0,\ a_1 + \ldots + a_k < d + 2 - k.$$

4 A counter example to Conjecture 1.2

Let $e_1, \ldots, e_5$ be the standard basis of the integral lattice $\mathbb{Z}^5 \subset \mathbb{R}^5$. Consider the smooth Fano 5-polytope P with 8 vertices, $\mathcal{V}(P) = \{v_1, \ldots, v_8\}$.

$$v_1 = e_1\ ,\ v_2 = e_2\ ,\ v_3 = e_3\ ,\ v_6 = e_4\ ,\ v_7 = e_5\ ,$$
$$v_4 = -e_1 - e_2 - e_3 - 3e_4\ ,\ v_5 = -e_4\ ,\ v_8 = -e_1 - e_2 - 2e_4 - e_5.$$

The primitive relations are given by

$$v_1 + v_2 + v_3 + v_4 = 3v_5, \tag{3}$$
$$v_5 + v_7 + v_8 = v_3 + v_4, \tag{4}$$
$$v_3 + v_4 + v_6 = v_7 + v_8, \tag{5}$$
$$v_5 + v_6 = 0, \tag{6}$$
$$v_1 + v_2 + v_7 + v_8 = 2v_5. \tag{7}$$

When F is a face of P, $\mathcal{V}(F)$ is a subset of $\mathcal{V}(P) = \{v_1, \dots, v_8\}$. For simplicity we write $\{i_1, \dots, i_k\}$ to denote the polytope $\operatorname{conv}\{v_{i_1}, \dots, v_{i_k}\}$. In this notation the facets of P are

$$\begin{array}{lllll} \{1,2,3,5,7\}, & \{1,2,3,5,8\}, & \{1,2,4,5,7\}, & \{1,2,4,5,8\}, & \{1,3,4,5,7\}, \\ \{1,3,4,5,8\}, & \{1,3,4,7,8\}, & \{1,3,6,7,8\}, & \{1,4,6,7,8\}, & \{2,3,4,5,7\}, \\ \{2,3,4,5,8\}, & \{2,3,4,7,8\}, & \{2,3,6,7,8\}, & \{2,4,6,7,8\}, & \{1,2,3,6,7\}, \\ \{1,2,3,6,8\}, & \{1,2,4,6,7\}, & \{1,2,4,6,8\}. & & \end{array}$$

We will now show that it is not possible to add or remove a lattice point from the vertex set $\mathcal{V}(P)$ and obtain another smooth Fano 5-polytope. As P is not pseudo-symmetric, it is a counter example to Conjecture 1.2.

PROPOSITION 4.1. *There does not exist a smooth Fano 5-polytope Q with 7 vertices such that $Q \subset P$.*

PROOF. Suppose there does exist a smooth Fano 5-polytope Q with 7 vertices such that $\mathcal{V}(P) = \mathcal{V}(Q) \cup \{v_i\}$ for some i, $1 \le i \le 8$. By the existing classification (Theorem 3.2) we know that Q has exactly two primitive relations of positive degree

$$v_{i_1} + \ldots + v_{i_k} = 0, \; v_{j_1} + \ldots + v_{j_{d-k}} = c_1 v_{i_1} + \ldots + c_k v_{i_k}.$$

There are two possibilities: Either $i \in \{5,6\}$ or $i \in \{1,2,3,4,7,8\}$.

Let $i \in \{5,6\}$. Then there must be a primitive collection of vertices of Q with empty focus. But for both possible i, no non-empty subset of $\mathcal{V}(P) \setminus \{v_i\}$ add to 0.

Let $i \in \{1,2,3,4,7,8\}$. Then $v_5 + v_6 = 0$ is a primitive relation of Q, and the other primitive collection is $C = \{v_1, v_2, v_3, v_4, v_7, v_8\} \setminus \{v_i\}$. The vertices in C must add up to cv_5, where $|c| \le 4$. It is now easy to check for every possible i, that this is not the case.

Hence we are done. $\square$

PROPOSITION 4.2. *There does not exist a smooth Fano 5-polytope R with 9 vertices, such that $P \subset R$.*

PROOF. Suppose there does exist a smooth Fano 5-polytope R with 9 vertices such that $\mathcal{V}(R) = \mathcal{V}(P) \cup \{v_9\}$ for some lattice point v_9.

As v_5 is a vertex of R, Relation (3) is a primitive relation of R (Lemma 3.1). Then $\{3,4\}$ is a face of R. Relation (4) ensures that $\{7,8\}$ is also a face of R. This means that Relations (3)–(5) are primitive relations of R.

As Relations (3)–(5) all have degree one, we can deduce a lot of the combinatorial structure of R: The set $\{3,4\}$ is a face of R. Thus

$$\{3,4,5,7\}, \{3,4,5,8\}, \{3,4,7,8\}$$

are faces of R (Relation (4)). Relation (3) implies that

$$\{1,2,3,5,7\}, \{1,2,4,5,7\}, \{1,3,4,5,7\}, \{2,3,4,5,7\},$$
$$\{1,2,3,5,8\}, \{1,2,4,5,8\}, \{1,3,4,5,8\}, \{2,3,4,5,8\}$$

are facets of R. By using Relation (4), we get 2 facets of R:

$$\{1,3,4,7,8\}, \{2,3,4,7,8\}.$$

Relation (5) gives us 4 more facets of R such as

$$\{1,3,6,7,8\}, \{1,4,6,7,8\}, \{2,3,6,7,8\}, \{2,4,6,7,8\}.$$

Among the original 18 facets of P, 14 are also facets of R. The remaining 4 facets are:

$$\{1,2,3,6,7\}, \{1,2,3,6,8\}, \{1,2,4,6,7\}, \{1,2,4,6,8\}.$$

So v_9 is in a cone over one of these four facets of P, i.e., v_9 is a non-negative $\mathbb{Z}$-linear combination of vertices of one of the four facets. Without loss of generality we can assume that $v_9 \in \operatorname{cone}(v_1, v_2, v_3, v_6, v_7)$ (if this is not the case, apply an appropriate renumbering of the vertices of P, which fixes the primitive relations):

$$v_9 = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_6 v_6 + a_7 v_7, \ a_i \geq 0 \text{ for all } i \in \{1,2,3,6,7\}.$$

Then $\{1,2,3,6,7\}$ is not a facet of R. But $F = \{1,2,3,5,7\}$ is a facet of R, so on the other side of the ridge $\{1,2,3,7\}$, there must be the facet $F' = \{1,2,3,7,9\}$. By Lemma 2.1 and Relation (6), $a_6 = 1$ and $1 > u_F(v_9) > u_F(v_6) = u_F(-v_5) = -1$. So $0 = u_F(v_9) = a_1 + a_2 + a_3 - 1 + a_7$.

Since $\{1,3,6,7,8\}$ and $\{2,3,6,7,8\}$ are facets of R, we must have $\{1,3,6,7,9\}$ and $\{2,3,6,7,9\}$ among the facets of R. This implies that

$$v_8 + v_9 \in \operatorname{span}\{v_1, v_3, v_6, v_7\} \cap \operatorname{span}\{v_2, v_3, v_6, v_7\} = \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

As $v_8 + v_9 = (a_1 - 1)v_1 + (a_2 - 1)v_2 + a_3 v_3 + (a_6 - 2)v_6 + (a_7 - 1)v_7$, we must have $a_1 = a_2 = 1$.

Since $a_1 + a_2 + a_3 - 1 + a_7 = 0$, we must have that $a_3 < 0$ or $a_7 < 0$, which is a contradiction. We conclude that the smooth Fano 5-polytope R does not exist. □

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Classification of terminal simplicial reflexive d-polytopes with $3d - 1$ vertices

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Abstract. We classify terminal simplicial reflexive d-polytopes with $3d - 1$ vertices. They turn out to be smooth Fano d-polytopes. When d is even there is one such polytope up to isomorphism, while there are two when d is uneven.

1. Introduction

Let $N \cong \mathbb{Z}^d$ be a d-dimensional lattice, $d \geq 1$, and let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$. Let M be the dual lattice of N and $M_{\mathbb{R}}$ the dual of $N_{\mathbb{R}}$. A *reflexive* d-polytope P in $N_{\mathbb{R}}$ is a fully-dimensional convex lattice polytope, such that the origin is contained in the interior and such that the dual polytope $P^* := \{x \in M_{\mathbb{R}} | \langle x, y \rangle \leq 1 \ \forall y \in P\}$ is also a lattice polytope. The notion of a reflexive polytope was introduced in [3]. Two reflexive polytopes P and Q are called *isomorphic*, if there exists a bijective linear map $\varphi : N_{\mathbb{R}} \to N_{\mathbb{R}}$, such that $\varphi(N) = N$ and $\varphi(P) = Q$. For every $d \geq 1$ there are finitely many isomorphism classes of reflexive d-polytopes, and for $d \leq 4$ they have been completely classified using computer algorithms [11, 12].

Simplicial reflexive d-polytopes have at most $3d$ vertices ([6] Theorem 1). This upper bound is attained if and only if d is even and P splits into $\frac{d}{2}$ copies of *del Pezzo* 2*-polytopes* $V_2 = \mathrm{conv}\{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$, where $\{e_1, e_2\}$ is a basis of a 2-dimensional lattice.

A reflexive polytope P is called *terminal*, if $N \cap P = \{0\} \cup \mathcal{V}(P)$. An important subclass of terminal simplicial reflexive polytopes is the class of *smooth* reflexive polytopes, also known as *smooth Fano* polytopes: a reflexive polytope P is called *smooth* if the vertices of every proper face F of P is a part of a basis of the lattice N. Smooth Fano polytopes have been intensively studied and completely classified up to dimension 5 [1, 4, 10, 15, 17]. In higher dimensions not much is known. There are classification results valid in any dimension, when the polytopes have few vertices [2, 9] or if one assumes some extra symmetries [5, 8, 16]. Some of these results have been generalized to simplicial reflexive polytopes [14].

In this paper we classify terminal simplicial reflexive d-polytopes with $3d - 1$ vertices for arbitrary d. It turns out that these are in fact smooth Fano d-polytopes.

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Table 1. The three possible distributions of $3d-1$ vertices, when F is a special facet

	Case 1	Case 2	Case 3
$\lvert\mathcal{V}(P)\cap H(F,1)\rvert$	d	d	d
$\lvert\mathcal{V}(P)\cap H(F,0)\rvert$	d	d	$d-1$
$\lvert\mathcal{V}(P)\cap H(F,-1)\rvert$	$d-1$	$d-2$	d
$\lvert\mathcal{V}(P)\cap H(F,-2)\rvert$	0	1	0
$\lvert\mathcal{V}(P)\rvert$	$3d-1$	$3d-1$	$3d-1$

Theorem 1. *Let $P \subset N_{\mathbb{R}}$ be a terminal simplicial reflexive d-polytope with $3d-1$ vertices. Let $e_1,\dots,e_d$ be a basis of the lattice N.*

If d is even, then P is isomorphic to the convex hull of the points

$$\begin{gathered} e_1\ ,\ \pm e_2,\dots,\pm e_d \\ \pm(e_1-e_2),\dots,\pm(e_{d-1}-e_d). \end{gathered} \tag{1}$$

If d is uneven, then P is isomorphic to either the convex hull of the points

$$\begin{gathered} \pm e_1,\dots,\pm e_{d-1},\ e_d \\ \pm(e_1-e_2),\dots,\pm(e_{d-2}-e_{d-1}),\quad e_1-e_d. \end{gathered} \tag{2}$$

or the convex hull of the points

$$\begin{gathered} \pm e_1,\dots,\pm e_d \\ \pm(e_2-e_3),\dots,\pm(e_{d-1}-e_d). \end{gathered} \tag{3}$$

In particular, P is a smooth Fano d-polytope.

A key concept in this paper is the notion of a special facet: a facet F of a simplicial reflexive d-polytope P is called *special*, if the sum of the vertices $\mathcal{V}(P)$ of P is a non-negative linear combination of vertices of F. In particular, $\langle u_F, \sum_{v\in\mathcal{V}(P)} v\rangle \geq 0$, where $u_F \in M_{\mathbb{R}}$ is the unique element determined by $\langle u_F, F\rangle = \{1\}$. The polytope P is reflexive, so $\langle u_F, v\rangle$ is an integer for every $v \in \mathcal{V}(P)$. As $\langle u_F, v\rangle \leq 1$ with equality if and only if $v \in F$, there are at most d vertices v of P, such that $\langle u_F, v\rangle \leq -1$. For simplicity, let $H(F,i) := \{x \in N \mid \langle u_F, x\rangle = i\}$, $i \in \mathbb{Z}$. It is well-known that at most d vertices of P are situated in $H(F,0)$ for any facet F of P ([7] Sect. 2.3, Remarks 5(2), [13] Lemma 5.5). If P has $3d-1$ vertices and F is a special facet of P, then

$$d-1 \leq |\mathcal{V}(P)\cap H(F,0)| \leq d,$$

and there are only three possibilities for the placement of the $3d-1$ vertices of P in the hyperplanes $H(F,i)$ as shown in Table 1. We prove Theorem 1 by considering these three cases separately for terminal simplicial reflexive d-polytopes.

The paper is organised as follows: in Sect. 2 we define some notation and prove some basic facts about simplicial reflexive polytopes. In Sect. 3 we define the notion of special facets. In Sect. 4 we prove some lemmas needed in Sect. 5 for the proof of Theorem 1.

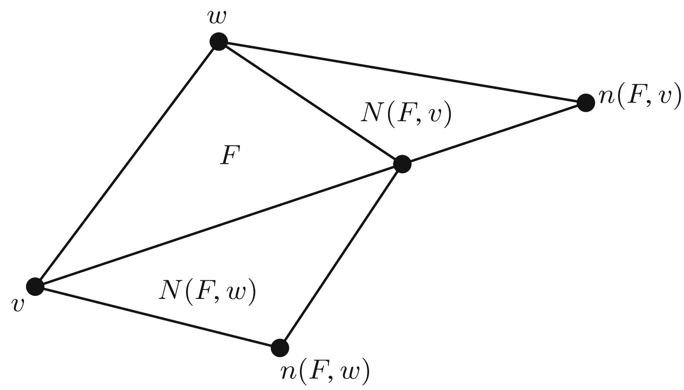

Fig. 1. This illustrates the concepts of neighboring facets and neighboring vertices

2. Notation and basic facts

In this section we fix the notation and prove some basic facts about simplicial reflexive d-polytopes.

From now on N denotes a d-dimensional lattice, $N \cong \mathbb{Z}^d$, $d \geq 1$, and M denotes the dual lattice. Let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and let $M_{\mathbb{R}}$ denote the dual of $N_{\mathbb{R}}$.

By $\operatorname{conv} K$ we denote the convex hull of a set K. A polytope is the convex hull of finitely many points, and a k-polytope is a polytope of dimension k. Recall that faces of a polytope of dimension 0 are called *vertices*, while codimension 1 and 2 faces are called *facets* and *ridges*, respectively. The set of vertices of any polytope P is denoted by $\mathcal{V}(P)$.

2.1. Simplicial polytopes containing the origin in the interior

A polytope P in $N_{\mathbb{R}}$ is called *simplicial* if every proper face of P is a simplex.

In this section P will be a simplicial d-polytope in $N_{\mathbb{R}}$ with $0 \in \operatorname{int} P$.

For any facet F of P, we define u_F to be the unique element in $M_{\mathbb{R}}$ where $\langle u_F, x\rangle = 1$ for every point $x \in F$. Certainly for any vertex v and any facet F of P, $\langle u_F, v\rangle \leq 1$ with equality if and only if v is a vertex of F.

We also define some points $u_F^v \in M_{\mathbb{R}}$ for any facet F of P and any vertex $v \in \mathcal{V}(F)$: u_F^v is the unique element where $\langle u_F^v, v\rangle = 1$ and $\langle u_F^v, w\rangle = 0$ for every $w \in \mathcal{V}(F) \setminus \{v\}$. In other words, $\{u_F^v | v \in \mathcal{V}(F)\}$ is the basis of $M_{\mathbb{R}}$ dual to the basis $\mathcal{V}(F)$ of $N_{\mathbb{R}}$.

When F is a facet of P and $v \in \mathcal{V}(F)$, there is a unique ridge $R = \operatorname{conv}(\mathcal{V}(F) \setminus \{v\})$ of P and a unique facet F' of P, such that $F \cap F' = R$. We denote this facet by $N(F, v)$ and call it a *neighboring facet of* F. The set $\mathcal{V}(N(F, v))$ consists of the vertices $\mathcal{V}(R)$ of the ridge R and a unique vertex v', which we call a *neighboring vertex of* F and denote it by $n(F, v)$. See Fig. 1.

The next lemma shows how u_F and $u_{F'}$ are related, when F' is a neighboring facet of the facet F.

Lemma 1. *Let $P \subset N_{\mathbb{R}}$ be a simplicial d-polytope containing the origin in the interior. Let F be a facet of P and $v \in \mathcal{V}(F)$. Let F' be the neighboring facet $N(F, v)$.*

Then for any point $x \in N_{\mathbb{R}}$,

$$\langle u_{F'}, x\rangle = \langle u_F, x\rangle + (\langle u_{F'}, v\rangle - 1)\langle u_F^v, x\rangle.$$

In particular,

1. $\langle u_F^v, x\rangle < 0$ *iff* $\langle u_{F'}, x\rangle > \langle u_F, x\rangle$.
2. $\langle u_F^v, x\rangle > 0$ *iff* $\langle u_{F'}, x\rangle < \langle u_F, x\rangle$.
3. $\langle u_F^v, x\rangle = 0$ *iff* $\langle u_{F'}, x\rangle = \langle u_F, x\rangle$.

Proof. The claimed equality holds for every $x \in \mathcal{V}(F)$, and then for every $x \in N_{\mathbb{R}}$ as $\mathcal{V}(F)$ is a basis of $N_{\mathbb{R}}$. The vertex v is not on the facet F', and then the term $\langle u_{F'}, v\rangle - 1$ is negative. From this the equivalences follow.

2.2. *Simplicial reflexive polytopes*

A polytope $P \subset N_{\mathbb{R}}$ is called a *lattice* polytope if $\mathcal{V}(P) \subset N$. A lattice polytope P is called *reflexive*, if $0 \in \operatorname{int} P$ and $\mathcal{V}(P^*) \subset M$, where

$$P^* := \{x \in M_{\mathbb{R}} \mid \langle x, y\rangle \leq 1 \ \forall y \in P\}$$

is the dual of P.

Reflexivity guarentees that $u_F \in M$ for every facet F of a simplicial reflexive polytope $P \subset N_{\mathbb{R}}$. Every vertex of P lies in one of the lattice hyperplanes

$$H(F, i) := \{x \in N \mid \langle u_F, x\rangle = i\}, \quad i \in \{1, 0, -1, -2, \dots\}$$

In particular, for every facet F and every vertex v of P: $v \notin F$ iff $\langle u_F, v\rangle \leq 0$. This can put some restrictions on the points of P.

Lemma 2. *Let P be a simplicial reflexive polytope. For every facet F of P and every vertex $v \in \mathcal{V}(F)$ we have*

$$\langle u_F, x\rangle - 1 \leq \langle u_F^v, x\rangle$$

for any $x \in P$. In case of equality, x is on the facet $N(F, v)$.

Proof. The inequality is obvious, when $\langle u_F^v, x\rangle > 0$. So assume $\langle u_F^v, x\rangle \leq 0$. Let F' be the neighboring facet $N(F, v)$. Since $x \in P$, $\langle u_{F'}, x\rangle \leq 1$ with equality iff $x \in F'$. From Lemma 1 we then have

$$\langle u_F, x\rangle - 1 \leq (1 - \langle u_{F'}, v\rangle)\langle u_F^v, x\rangle \leq \langle u_F^v, x\rangle$$

as $\langle u_{F'}, v\rangle \leq 0$.

The next lemma concerns an important property of simplicial reflexive polytopes.

Lemma 3. ([7] Sect. 2.3, Remarks 5(2), [13] Lemma 5.5) *Let F be a facet and $x \in H(F, 0)$ a vertex of a simplicial reflexive d-polytope P. Then x is a neighboring vertex of F.*

More precisely, for every $w \in \mathcal{V}(F)$ where $\langle u_F^w, x\rangle < 0$, x is equal to $n(F, w)$.

In particular, for every $w \in \mathcal{V}(F)$ there is at most one vertex $x \in H(F, 0) \cap \mathcal{V}(P)$, with $\langle u_F^w, x\rangle < 0$.

As a consequence, there are at most d vertices of P in $H(F, 0)$.

Proof. Since $\langle u_F, x\rangle = \sum_{w\in\mathcal{V}(F)}\langle u_F^w, x\rangle = 0$ and $x \neq 0$, there is at least one $w \in \mathcal{V}(F)$ for which $\langle u_F^w, x\rangle < 0$. Choose such a w and consider the neighboring facet $F' = N(F, w)$. By Lemma 1 we get that $0 < \langle u_{F'}, x\rangle \leq 1$. As P is reflexive, $\langle u_{F'}, x\rangle = 1$ and then $x = n(F, w)$. The remaining statements follow immediately.

3. Special facets

Now we define the notion of special facets, which will be of great use to us in the proof of Theorem 1.

P is a simplicial reflexive d-polytope in this section.

Consider the sum of all the vertices of P,

$$\nu_P := \sum_{v\in\mathcal{V}(P)} v.$$

There exists at least one facet F of P such that ν_P is a non-negative linear combination of vertices of F, i.e. $\langle u_F^w, \nu_P\rangle \geq 0$ for every $w \in \mathcal{V}(F)$. We call facets with this property *special*.

Let F be a special facet of P. In particular

$$0 \leq \langle u_F, \nu_P\rangle,$$

which implies that

$$0 \leq \sum_{v\in\mathcal{V}(P)} \langle u_F, v\rangle = \sum_{i\leq 1} i|H(F, i) \cap \mathcal{V}(P)| = d + \sum_{i\leq -1} i|H(F, i) \cap \mathcal{V}(P)|. \tag{4}$$

As there are at most d vertices in $H(F, 0)$ we can easily see that $|\mathcal{V}(P)| \leq 3d$, which was first proved in [6] Theorem 3 using a similar argument. Notice that $\langle u_F, v\rangle \geq -d$ for every vertex v of P. Notice also, that when $|\mathcal{V}(P)|$ is close to $3d$, the vertices of P tend to be in the hyperplanes $H(F, i)$ for $i \in \{1, 0, -1\}$.

4. Many vertices in $H(F, 0)$

We now study some cases of many vertices in $H(F, 0)$, where F is a facet of a simplicial reflexive d-polytope. The lemmas proven here are ingredients in the proof of Theorem 1.

Lemma 4. *Let F be a facet of a simplicial reflexive d-polytope P. Suppose there are at least $d-1$ vertices $v_1, \dots, v_{d-1}$ in $\mathcal{V}(F)$, such that $n(F, v_i) \in H(F, 0)$ and $\langle u_F^{v_i}, n(F, v_i)\rangle = -1$ for every $1 \leq i \leq d-1$.*

Then $\mathcal{V}(F)$ is a basis of the lattice N.

Proof. Follows from statement 3 in [13] Lemma 5.5.

Lemma 5. *Let F be a facet of a simplicial reflexive d-polytope P, $d \geq 2$. Let $v_1, v_2 \in \mathcal{V}(F)$, $v_1 \neq v_2$, and set $y_1 = n(F, v_1)$ and $y_2 = n(F, v_2)$. Suppose $y_1 \neq y_2$, $y_1, y_2 \in H(F, 0)$ and $\langle u_F^{v_1}, y_1\rangle = \langle u_F^{v_2}, y_2\rangle = -1$.*

Then there are no vertex $x \in \mathcal{V}(P)$ in $H(F, -1)$ with $\langle u_F^{v_1}, x\rangle = \langle u_F^{v_2}, x\rangle = -1$.

Proof. Suppose the statement is not true. For simplicity, let $G = \operatorname{conv}(\mathcal{V}(F) \setminus \{v_1, v_2\})$. The vertex x written as a linear combination of $\mathcal{V}(F)$ is then

$$x = -v_1 - v_2 + \sum_{w \in \mathcal{V}(G)} \langle u_F^w, x\rangle w.$$

The vertices of the facet $F_1 = N(F, v_1)$ are $\{y_1\} \cup (\mathcal{V}(F) \setminus \{v_1\})$, where

$$y_1 = -v_1 + \langle u_F^{v_2}, y_1\rangle v_2 + \sum_{w \in \mathcal{V}(G)} \langle u_F^w, y_1\rangle w.$$

In the basis (of $N_{\mathbb{R}}$) F_1 provides we have

$$x = y_1 + (-1 - \langle u_F^{v_2}, y_1\rangle) v_2 + \sum_{w \in \mathcal{V}(G)} \langle u_F^w, x - y_1\rangle w$$

The vertex x is in $H(F_1, 0)$ by Lemma 1. Certainly, $\langle u_F^{v_2}, y_1\rangle \leq 0$, otherwise we would have a contradiction to Lemma 2. On the other hand, $\langle u_F^{v_2}, y_1\rangle \geq 0$, as $n(F, v_2) \neq y_1$. So $\langle u_F^{v_2}, y_1\rangle = 0$ and $x = n(F_1, v_2)$. Similarly, $\langle u_F^{v_1}, y_2\rangle = 0$.

$$y_2 = -v_2 + \sum_{w \in \mathcal{V}(G)} \langle u_F^w, y_2\rangle w.$$

But then y_2 and x are both in $H(F_1, 0)$ and both have negative v_2-coordinate. This is a contradiction to Lemma 3.

4.1. *The terminal case*

If we assume that the simplicial reflexive d-polytope P is terminal, we can sharpen our results in case of d vertices in $H(F, 0)$ for some facet F of P. Recall, that a reflexive polytope is called *terminal* if $\mathcal{V}(P) \cup \{0\} = P \cap N$.

Lemma 6. *Let P be a terminal simplicial reflexive d-polytope. If there are d vertices of P in $H(F, 0)$ for some facet F of P, then*

$$\mathcal{V}(P) \cap H(F, 0) = \{-y + z_y \mid y \in \mathcal{V}(F)\}$$

where $z_y \in \mathcal{V}(F)$ for every y.

In particular $\mathcal{V}(F)$ is a basis of the lattice N.

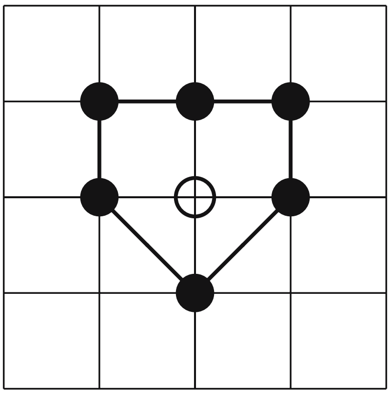

Fig. 2. Terminality is important in Lemma 6: this is a simplicial reflexive (self-dual) 2-polytope with five vertices. Consider the facet F containing three lattice points. The two vertices in $H(F,0)$ are not on the form $-y+z_y$ for vertices $y, z_y \in \mathcal{V}(F)$

Proof. Let $y \in \mathcal{V}(F)$. By Lemma 3 there exists exactly one vertex $x \in H(F,0)$, such that $x = n(F,y)$. Conversely, there are no vertex $y' \neq y$ of F, such that $x = n(F,y')$. So x is on the form

$$x = -by + a_1 w_1 + \cdots + a_k w_k, \quad 0 < b \leq 1,\ 0 < a_i \text{ and } w_i \in \mathcal{V}(F) \setminus \{y\}\ \forall i,$$

where $b = \sum_{i=1}^{k} a_i$.

Suppose there exists a facet G containing both x and y. Then

$$1 + b = \langle u_G, x + by \rangle = \langle u_G, a_1 w_1 + \cdots + a_k w_k \rangle \leq \sum_{i=1}^{k} a_i = b.$$

Which is a contradiction. So there are no such facets.

Consider the lattice point $z_y = x + y$. For any facet G of P, $\langle u_G, z_y \rangle \leq 1$ as both $\langle u_G, x \rangle, \langle u_G, y \rangle \leq 1$ and both cannot be equal to 1. So z_y is a lattice point in P. Since P is terminal, z_y is either a vertex of P or the origin.

As $1 = \langle u_F, x + y \rangle = \langle u_F, z_y \rangle$, z_y must be a vertex of F and $y \neq z_y$. And then we're done.

The vertex set $\mathcal{V}(F)$ is a basis of N by Lemma 4.

The proof of Lemma 6 is inspired by Proposition 4.1 in [13].

Lemma 7. *Let F be a facet of a terminal simplicial reflexive d-polytope P, such that $|H(F,0) \cap \mathcal{V}(P)| = d$. If $x \in H(F,-1) \cap P$, then $-x \in \mathcal{V}(F)$.*

Proof. The vertex set $\mathcal{V}(F)$ is a basis of the lattice N, and every vertex in $H(F,0)$ is of the form $-y+z$ for some $y, z \in \mathcal{V}(F)$ (Lemma 6).

Let x be vertex of P in $H(F,-1)$.

$$x = \sum_{w \in \mathcal{V}(F)} \langle u_F^w, x \rangle w,$$

where $\langle u_F^w, x \rangle \in \mathbb{Z}$ for every $w \in \mathcal{V}(F)$. If $\langle u_F^w, x \rangle \leq -2$ for some $w \in \mathcal{V}(F)$, then $x = n(F,w)$ (Lemma 2), which is not the case. So $\langle u_F^w, x \rangle \geq -1$ for every

$w \in \mathcal{V}(F)$. Furthermore, by Lemma 5 x is only allowed one negative coordinate with respect to the basis $\mathcal{V}(F)$. The only possibility is then $x = -w$, for some $w \in \mathcal{V}(F)$.

5. Proof of main result

In this section we will prove Theorem 1.

Throughout the section P is a terminal simplicial reflexive d-polytope in $N_{\mathbb{R}}$ with $3d-1$ vertices, whose sum is ν_P, and $\{e_1, \ldots, e_d\}$ is a basis of the lattice N.

$$\nu_P := \sum_{v \in \mathcal{V}(P)} v.$$

By the existing classification we can check that Theorem 1 holds for $d \leq 2$ ([13] Proposition 2.1). So we may assume that $d \geq 3$.

Let F be a special facet of P, i.e. $\langle u_F^w, \nu_P \rangle \geq 0$ for every $w \in \mathcal{V}(F)$. Of course, there are d vertices of P in $H(F, 1)$. The remaining $2d-1$ vertices are in the hyperplanes $H(F, i)$ for $i \in \{0, -1, -2, \ldots, -d\}$, such that

$$0 \leq \langle u_F, \nu_P \rangle = d + \sum_{i \leq -1} i \cdot |\mathcal{V}(P) \cap H(F, i)|.$$

So there are three cases to consider as shown in Table 1. We will consider these cases separately.

Case 1. There are d vertices in $H(F, 0)$, so by Lemma 6 $\mathcal{V}(F)$ is a basis of N. We may then assume that $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$. The sum of the vertices is a lattice point on F, since $\langle u_F, \nu_P \rangle = 1$. As P is terminal, this must be a vertex e_i of F, say $\nu_P = e_1$. Then a facet F' of P is a special facet iff $e_1 \in \mathcal{V}(F')$.
There are $d-1$ vertices in $H(F, -1)$, so by Lemma 7 we get

$$\mathcal{V}(P) \cap H(F, -1) = \{-e_1, \ldots, -e_{j-1}, -e_{j+1}, \ldots, -e_d\},$$

for some $1 \leq j \leq d$. Now, there are two possibilities: $j = 1$ or $j \neq 1$, that is $-e_1 \notin \mathcal{V}(P)$ or $-e_1 \in \mathcal{V}(P)$.

$-e_1 \notin \mathcal{V}(P)$. Then $-e_i \in \mathcal{V}(P)$ for every $2 \leq i \leq d$. There are d vertices in $H(F, 0)$, so by Lemma 6 there is a vertex $-e_1 + e_{a_1}$, which we can assume to be $-e_1 + e_2$.
Consider the facet $F' = N(F, e_2)$. This is a special facet, so we can show that

$$\mathcal{V}(P) \cap H(F', -1) = \mathcal{V}(-F') \setminus \{-e_1\}.$$

The vertex $-e_1 + e_2$ is in the hyperplane $H(F', -1)$. So $e_1 - e_2$ is a vertex of F' (Lemma 7), and then of P.
For every $3 \leq i \leq d$ we use the same procedure to show that $-e_i + e_{a_i}$ and $-e_{a_i} + e_i$ are vertices of P. This shows that d is even and that P is isomorphic to the convex hull of the points in (1).

$-e_1 \in \mathcal{V}(P)$. We may assume $-e_d \notin \mathcal{V}(P)$. The sum of the vertices $\mathcal{V}(P)$ is e_1, so there are exactly two vertices in $H(F,0)$ of the form $-e_k + e_1$ and $-e_l + e_1$, $k \neq l$. We wish to show that $k = d$ or $l = d$. This is obvious for $d = 3$. So suppose $d \geq 4$ and $k, l \neq d$, that is $-e_k, -e_l \in \mathcal{V}(P)$.

Consider the facet $F' = N(F, e_k)$, which is a special facet. So by the arguments above we get that

$$\mathcal{V}(P) \cap H(F', -1) = \mathcal{V}(-F') \setminus \{-e_d\}.$$

As $\mathcal{V}(F') = \{e_1, \ldots, e_{k-1}, e_{k+1}, e_d, -e_k + e_1\}$, we have that $-e_1 + e_k$ must be a vertex of P.

In a similar way we get that $-e_1 + e_l$ is a vertex of P. But this is a contradiction. So k or l is equal to d, and without loss of generality, we can assume that $k = 2$ and $l = d$.

For $3 \leq i \leq d-1$ we proceed in a similar way to get that both $-e_i + e_{a_i}$ and $-e_{a_i} + e_i$ are vertices of P, and $a_i \neq d$.

And so we have showed that d must be uneven and that P is isomorphic to the convex hull of the points in (2).

Case 2. Since $\langle u_F, \nu_P \rangle = 0$, the sum of the vertices is the origin, so every facet of P is special. There are d vertices in $H(F,0)$, so $\mathcal{V}(F)$ is a basis of N (Lemma 6). Without loss of generality, we can assume $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$. By Lemma 7

$$x \in \mathcal{V}(P) \cap H(F, -1) \implies x = -e_i \text{ for some } 1 \leq i \leq d.$$

Consider the single vertex v in the hyperplane $H(F, -2)$. If $\langle u_F^{e_j}, v \rangle > 0$ for some j then $\langle u_{F'}, v \rangle < -2$ for the facet $F' = N(F, e_j)$ (Lemma 1), which is not the case as F' is special. So $\langle u_F^{e_j}, v \rangle \leq 0$ for every $1 \leq j \leq d$. As v is a primitive lattice point we can without loss of generality assume $v = -e_1 - e_2$. There are d vertices in $H(F,0)$, so there is a vertex of the form $-e_1 + e_j$ for some $j \neq 1$. If $j = 2$, then $-e_1 \in \operatorname{conv}\{-e_1 + e_2, -e_1 - e_2\}$ which is not the case as P is terminal. So we may assume $j = 3$. In $H(F,0)$ we also find the vertex $-e_2 + e_i$ for some i. A similar argument yields $i \neq 1$.

Let $G = N(F, e_1)$. Then $\mathcal{V}(G)$ is a basis of the lattice N. Write v in this basis.

$$v = (-e_1 + e_3) - e_3 - e_2.$$

As $i \neq 1$, $-e_2 + e_i$ is in $H(G,0)$ and is equal to $n(G, e_2)$ (Lemma 3).

Suppose $v \neq n(G, e_3)$. As there are no vertices of P in $H(G, -2)$, there are only three possibilities for $n(G, e_3)$.

1. $n(G, e_3) \in H(G, 0)$ and $\langle u_G^{e_3}, n(G, e_3) \rangle = -1$
2. $n(G, e_3) \in H(G, -1)$ and $\langle u_G^{e_3}, n(G, e_3) \rangle = -1$
3. $n(G, e_3) \in H(G, -1)$ and $\langle u_G^{e_3}, n(G, e_3) \rangle = -2$

The first possibility cannot occur by Lemma 5. As v is not on the facet $N(G, e_3)$ we can rule out the second possibility. Vertices in $P \cap H(G, -1)$ are of the form: $-e_k$, $-e_1 - e_2$ or $-e_l + e_1 = -(-e_1 + e_3) + e_3 - e_l$ for some k, l. None of these have -2 as e_3-coordinate with respect to the basis $\mathcal{V}(G)$. Hence the third possibility does not occur.

Therefore $v = n(G, e_3)$, and $\operatorname{conv}\{v, -e_1 + e_3, e_2\}$ is a face of P.

As e_3 and $-e_1 + e_3$ are vertices of P, there are at least two vertices of P with positive e_3-coordinate (with respect to the basis F provides). There is exactly one vertex in $H(F, 0)$ with negative e_3-coordinate, namely $-e_3 + e_k$ for some k. Any other has to be in $H(F, -1)$. The vertices of P add to 0, so the point $-e_3$ must be a vertex of P.

But $-e_3 = -(-e_1 + e_3) + v + e_2$, which cannot be the case as P is simplicial. We conclude that case 2 is not possible.

Case 3. In this case we also have $\langle u_F, v_P \rangle = 0$, so every facet is special. Case 2 was not possible, so $-1 \leq \langle u_G, v \rangle \leq 1$ for any facet G and any vertex v of P. Then P is centrally symmetric and d must be uneven. By [13] Theorem 5.9 P is isomorphic to the convex hull of the points in (3).

This ends the proof of Theorem 1.

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An algorithm for the classification of smooth Fano polytopes

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Abstract

We present an algorithm that produces the classification list of smooth Fano d-polytopes for any given $d \geq 1$. The input of the algorithm is a single number, namely the positive integer d. The algorithm has been used to classify smooth Fano d-polytopes for $d \leq 7$. There are 7622 isomorphism classes of smooth Fano 6-polytopes and 72256 isomorphism classes of smooth Fano 7-polytopes.

1 Introduction

Isomorphism classes of smooth toric Fano varieties of dimension d correspond to isomorphism classes of socalled smooth Fano d-polytopes, which are fully dimensional convex lattice polytopes in $\mathbb{R}^d$, such that the origin is in the interior of the polytopes and the vertices of every facet is a basis of the integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Smooth Fano d-polytopes have been intensively studied for the last decades. They have been completely classified up to isomorphism for $d \leq 4$ ([1], [18], [3], [15]). Under additional assumptions there are classification results valid in every dimension.
To our knowledge smooth Fano d-polytopes have been classified in the following cases:

- When the number of vertices is $d+1$, $d+2$ or $d+3$ ([9],[2]).
- When the number of vertices is $3d$, which turns out to be the upper bound on the number of vertices ([6]).
- When the number of vertices is $3d-1$ ([19]).
- When the polytopes are centrally symmetric ([17]).
- When the polytopes are pseudo-symmetric, i.e. there is a facet F, such that $-F$ is also a facet ([8]).
- When there are many pairs of centrally symmetric vertices ([5]).

- When the corresponding toric d-folds are equipped with an extremal contraction, which contracts a toric divisor to a point ([4]) or a curve ([16]).

Recently a complete classification of smooth Fano 5-polytopes has been announced ([12]). The approach is to recover smooth Fano d-polytopes from their image under the projection along a vertex. This image is a *reflexive* $(d-1)$-polytope (see [3]), which is a fully-dimensional lattice polytope containing the origin in the interior, such that the dual polytope is also a lattice polytope. Reflexive polytopes have been classified up to dimension 4 using the computer program PALP ([10],[11]). Using this classification and PALP the authors of [12] succeed in classifying smooth Fano 5-polytopes.

In this paper we present an algorithm that classifies smooth Fano d-polytopes for any given $d \geq 1$. We call this algorithm SFP (for Smooth Fano Polytopes). The input is the positive integer d, nothing else is needed. The algorithm has been implemented in C++, and used to classify smooth Fano d-polytopes for $d \leq 7$. For $d = 6$ and $d = 7$ our results are new:

Theorem 1.1. *There are 7622 isomorphism classes of smooth Fano* 6*-polytopes and 72256 isomorphism classes of smooth Fano* 7*-polytopes.*

The classification lists of smooth Fano d-polytopes, $d \leq 7$, are available on the authors homepage: `http://home.imf.au.dk/oebro`
A key idea in the algorithm is the notion of a special facet of a smooth Fano d-polytope (defined in section 3.1): A facet F of a smooth Fano d-polytope is called *special*, if the sum of the vertices of the polytope is a non-negative linear combination of vertices of F. This allows us to identify a finite subset $\mathcal{W}_d$ of the lattice $\mathbb{Z}^d$, such that any smooth Fano d-polytope is isomorphic to one whose vertices are contained in $\mathcal{W}_d$ (theorem 3.6). Thus the problem of classifying smooth Fano d-polytopes is reduced to the problem of considering certain subsets of $\mathcal{W}_d$.
We then define a total order on finite subsets of $\mathbb{Z}^d$ and use this to define a total order on the set of smooth Fano d-polytopes, which respects isomorphism (section 4). The SFP-algorithm (described in section 5) goes through certain finite subsets of $\mathcal{W}_d$ in increasing order, and outputs smooth Fano d-polytopes in increasing order, such that any smooth Fano d-polytope is isomorphic to exactly one in the output list.
As a consequence of the total order on smooth Fano d-polytopes, the algorithm needs not consult the previous output to check for isomorphism to decide whether or not to output a constructed polytope.

2 Smooth Fano polytopes

We fix a notation and prove some simple facts about smooth Fano polytopes.

The convex hull of a set $K \in \mathbb{R}^d$ is denoted by conv$K$. A *polytope* is the convex hull of finitely many points. The dimension of a polytope P is the dimension of the affine hull, aff$P$, of the polytope P. A *k-polytope* is a polytope of dimension k. A *face* of a polytope is the intersection of a supporting hyperplane with the polytope. Faces of polytopes are polytopes. Faces of dimension 0 are called *vertices*, while faces of codimension 1 and 2 are called *facets* and *ridges*, respectively. The set of vertices of a polytope P is denoted by $\mathcal{V}(P)$.

Definition 2.1. *A convex lattice polytope P in $\mathbb{R}^d$ is called a* smooth Fano d-polytope*, if the origin is contained in the interior of P and the vertices of every facet of P is a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$.*

We consider two smooth Fano d-polytopes P_1, P_2 to be *isomorphic*, if there exists a bijective linear map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, such that $\varphi(\mathbb{Z}^d) = \mathbb{Z}^d$ and $\varphi(P_1) = P_2$.
Whenever F is a $(d-1)$-simplex in $\mathbb{R}^d$, such that $0 \notin \text{aff}F$, we let $u_F \in (\mathbb{R}^d)^*$ be the unique element determined by $\langle u_F, F\rangle = \{1\}$. For every $w \in \mathcal{V}(F)$ we define $u_F^w \in (\mathbb{R}^d)^*$ to be the element where $\langle u_F^w, w\rangle = 1$ and $\langle u_F^w, w'\rangle = 0$ for every $w' \in \mathcal{V}(F)$, $w' \neq w$. Then $\{u_F^w | w \in \mathcal{V}(F)\}$ is the basis of $(\mathbb{R}^d)^*$ dual to the basis $\mathcal{V}(F)$ of $\mathbb{R}^d$.
When F is a facet of a smooth Fano polytope and $v \in \mathcal{V}(P)$, we certainly have $\langle u_F, v\rangle \in \mathbb{Z}$ and

$$\langle u_F, v\rangle = 1 \iff v \in \mathcal{V}(F) \quad \text{and} \quad \langle u_F, v\rangle \leq 0 \iff v \notin \mathcal{V}(F).$$

The lemma below concerns the relation between the elements u_F and $u_{F'}$, when F and F' are adjacent facets.

Lemma 2.2. *Let F be a facet of a smooth Fano polytope P and $v \in \mathcal{V}(F)$. Let F' be the unique facet which intersects F in a ridge R of P, $v \notin \mathcal{V}(R)$. Let $v' = \mathcal{V}(F') \setminus \mathcal{V}(R)$.*
Then

1. $\langle u_F^v, v'\rangle = -1$.
2. $\langle u_F, v'\rangle = \langle u_{F'}, v\rangle$.
3. $\langle u_{F'}, x\rangle = \langle u_F, x\rangle + \langle u_F^v, x\rangle(\langle u_F, v'\rangle - 1)$ *for any* $x \in \mathbb{R}^d$.
4. *In particular,*
   - $\langle u_F^v, x\rangle < 0$ *iff* $\langle u_{F'}, x\rangle > \langle u_F, x\rangle$.
   - $\langle u_F^v, x\rangle > 0$ *iff* $\langle u_{F'}, x\rangle < \langle u_F, x\rangle$.
   - $\langle u_F^v, x\rangle = 0$ *iff* $\langle u_{F'}, x\rangle = \langle u_F, x\rangle$.

   for any $x \in \mathbb{R}^d$.

5. *Suppose $x \neq v'$ is a vertex of P where $\langle u_F^v, x\rangle < 0$. Then $\langle u_F, v'\rangle > \langle u_F, x\rangle$.*

Proof. The sets $\mathcal{V}(F)$ and $\mathcal{V}(F')$ are both bases of the lattice $\mathbb{Z}^d$ and the first statement follows.
We have $v + v' \in \operatorname{span}(F \cap F')$, and then the second statement follows.
Use the previous statements to calculate $\langle u_{F'}, x\rangle$.

$$\begin{aligned}
\langle u_{F'}, x\rangle &= \langle u_{F'}, \sum_{w\in\mathcal{V}(F)} \langle u_F^w, x\rangle w\rangle \\
&= \sum_{w\in\mathcal{V}(F)\setminus\{v\}} \langle u_F^w, x\rangle + \langle u_F^v, x\rangle\langle u_{F'}, v\rangle \\
&= \langle u_F, x\rangle + \langle u_F^v, x\rangle\big(\langle u_{F'}, v\rangle - 1\big) \\
&= \langle u_F, x\rangle + \langle u_F^v, x\rangle\big(\langle u_F, v'\rangle - 1\big).
\end{aligned}$$

As $\langle u_F, v'\rangle - 1 < 0$ the three equivalences follow directly.
Suppose there is a vertex $x \in \mathcal{V}(P)$, such that $\langle u_F^v, x\rangle < 0$ and $\langle u_F, v'\rangle \leq \langle u_F, x\rangle$. Then

$$\langle u_{F'}, x\rangle = \langle u_F, x\rangle + \langle u_F^v, x\rangle(\langle u_F, v'\rangle - 1) \geq \langle u_F, x\rangle - (\langle u_F, v'\rangle - 1) \geq 1.$$

Hence x is on the facet F'. But this cannot be the case as $\mathcal{V}(F') = \{v'\} \cup \mathcal{V}(F) \setminus \{v\}$. Thus no such x exists.
And we're done. $\square$

In the next lemma we show a lower bound on the numbers $\langle u_F^w, v\rangle$, $w \in \mathcal{V}(F)$, for any facet F and any vertex v of a smooth Fano d-polytope.

Lemma 2.3. *Let F be a facet and v a vertex of a smooth Fano polytope P. Then*

$$\langle u_F^w, v\rangle \geq \begin{cases} 0 & \langle u_F, v\rangle = 1 \\ -1 & \langle u_F, v\rangle = 0 \\ \langle u_F, v\rangle & \langle u_F, v\rangle < 0 \end{cases}$$

for every $w \in \mathcal{V}(F)$.

Proof. When $\langle u_F, v\rangle = 1$ the statement is obvious.
Suppose $\langle u_F, v\rangle = 0$ and $\langle u_F^w, v\rangle < 0$ for some $w \in \mathcal{V}(F)$. Let F' be the unique facet intersecting F in the ridge $\operatorname{conv}\{\mathcal{V}(F) \setminus \{w\}\}$. By lemma 2.2 $\langle u_{F'}, v\rangle > 0$. As $\langle u_{F'}, v\rangle \in \mathbb{Z}$ we must have $\langle u_{F'}, v\rangle = 1$. This implies $\langle u_F, v\rangle = -1$.
Suppose $\langle u_F, v\rangle < 0$ and $\langle u_F^w, v\rangle < \langle u_F, v\rangle \leq -1$ for some $w \in \mathcal{V}(F)$. Let $F' \neq F$ be the facet containing the ridge $\operatorname{conv}\{\mathcal{V}(F) \setminus \{w\}\}$, and let w' be the unique vertex in $\mathcal{V}(F') \setminus \mathcal{V}(F)$. Then by lemma 2.2

$$\langle u_{F'}, v\rangle = \langle u_F, v\rangle + \langle u_F^w, v\rangle(\langle u_F, w'\rangle - 1) \geq \langle u_F, v\rangle - \langle u_F^w, v\rangle.$$

If $\langle u_F, v\rangle - \langle u_F^w, v\rangle > 0$, then v is on the facet F'. But this is not the case as $\langle u_F^w, v\rangle < -1$. We conclude that $\langle u_F^w, v\rangle \geq \langle u_F, v\rangle$. $\square$

When F is a facet and v a vertex of a smooth Fano d-polytope P, such that $\langle u_F, v\rangle = 0$, we can say something about the face lattice of P.

Lemma 2.4 ([7] section 2.3 remark 5(2), [13] lemma 5.5)**.** *Let F be a facet and v be vertex of a smooth Fano polytope P. Suppose $\langle u_F, v\rangle = 0$. Then $\operatorname{conv}\{\{v\} \cup \mathcal{V}(F) \setminus \{w\}\}$ is a facet of P for every $w \in \mathcal{V}(F)$ with $\langle u_F^w, v\rangle = -1$.*

Proof. Follows from the proof of lemma 2.3. □

3 Special embeddings of smooth Fano polytopes

In this section we find a concrete finite subset $\mathcal{W}_d$ of $\mathbb{Z}^d$ with the nice property that any smooth Fano d-polytope is isomorphic to one whose vertices are contained in $\mathcal{W}_d$. The problem of classifying smooth Fano d-polytopes is then reduced to considering subsets of $\mathcal{W}_d$.

3.1 Special facets

The following definition is a key concept.

Definition 3.1. *A facet F of a smooth Fano d-polytope P is called* special*, if the sum of the vertices of P is a non-negative linear combination of $\mathcal{V}(F)$, that is*

$$\sum_{v \in \mathcal{V}(P)} v = \sum_{w \in \mathcal{V}(F)} a_w w \quad , \; a_w \geq 0.$$

Clearly, any smooth Fano d-polytope has at least one special facet.
Let F be a special facet of a smooth Fano d-polytope P. Then

$$0 \leq \langle u_F, \sum_{v \in \mathcal{V}(P)} v\rangle = d + \sum_{v \in \mathcal{V}(P), \langle u_F, v\rangle < 0} \langle u_F, v\rangle,$$

which implies $-d \leq \langle u_F, v\rangle \leq 1$ for any vertex v of P. By using the lower bound on the numbers $\langle u_F^w, v\rangle$, $w \in \mathcal{V}(F)$ (see lemma 2.3), we can find an explicite finite subset of the lattice $\mathbb{Z}^d$, such that every $v \in \mathcal{V}(P)$ is contained in this subset. In the following lemma we generalize this observation to subsets of $\mathcal{V}(P)$ containing $\mathcal{V}(F)$.

Lemma 3.2. *Let P be a smooth Fano polytope. Let F be a special facet of P and let V be a subset of $\mathcal{V}(P)$ containing $\mathcal{V}(F)$, whose sum is ν.*

$$\nu = \sum_{v \in V} v.$$

Then

$$\langle u_F, \nu\rangle \geq 0$$

and

$$\langle u_F^w, \nu \rangle \leq \langle u_F, \nu \rangle + 1$$

for every $w \in \mathcal{V}(F)$*.*

Proof. For convenience we set $U = \mathcal{V}(P) \setminus V$ and $\mu = \sum_{v \in U} v$. Since F is a special facet we know that

$$0 \leq \langle u_F, \sum_{v \in \mathcal{V}(P)} v \rangle = \langle u_F, \nu \rangle + \langle u_F, \mu \rangle.$$

The set $\mathcal{V}(F)$ is contained in V so $\langle u_F, v \rangle \leq 0$ for every v in U, hence $\langle u_F, \nu \rangle \geq 0$.
Suppose that for some $w \in \mathcal{V}(F)$ we have $\langle u_F^w, \nu \rangle > \langle u_F, \nu \rangle + 1$. By lemma 2.3 we know that

$$\langle u_F^w, v \rangle \geq \begin{cases} -1 & \langle u_F, v \rangle = 0 \\ \langle u_F, v \rangle & \langle u_F, v \rangle < 0 \end{cases}$$

for every vertex $v \in \mathcal{V}(P) \setminus \mathcal{V}(F)$. There is at most one vertex v of P, $\langle u_F, v \rangle = 0$, with negative coefficient $\langle u_F^w, v \rangle$ (lemma 2.4). So

$$\langle u_F^w, \mu \rangle \geq \langle u_F, \mu \rangle - 1.$$

Now, consider $\langle u_F^w, \sum_{v \in \mathcal{V}(P)} v \rangle$.

$$\langle u_F^w, \sum_{v \in \mathcal{V}(P)} v \rangle = \langle u_F^w, \nu \rangle + \langle u_F^w, \mu \rangle > \langle u_F, \nu \rangle + \langle u_F, \mu \rangle = \langle u_F, \sum_{v \in \mathcal{V}(P)} v \rangle.$$

But this implies that $\langle u_F^x, \sum_{v \in \mathcal{V}(P)} v \rangle$ is negative for some $x \in \mathcal{V}(F)$. A contradiction. □

Corollary 3.3. *Let* F *be a special facet and* v *any vertex of a smooth Fano* d*-polytope. Then* $-d \leq \langle u_F, v \rangle \leq 1$ *and*

$$\left.\begin{array}{c} 0 \\ -1 \\ \langle u_F, v \rangle \end{array}\right\} \leq \langle u_F^w, v \rangle \leq \left\{\begin{array}{cl} 1 & , \langle u_F, v \rangle = 1 \\ d-1 & , \langle u_F, v \rangle = 0 \\ d + \langle u_F, v \rangle & , \langle u_F, v \rangle < 0 \end{array}\right.$$

for every $w \in \mathcal{V}(F)$*.*

Proof. For $\langle u_F, v \rangle = 1$ the statement is obvious. When $\langle u_F, v \rangle = 0$ the coefficients of v with respect to the basis $\mathcal{V}(F)$ is bounded below by -1 (lemma 2.3), so no coefficient exceeds $d-1$.
So the case $\langle u_F, v \rangle < 0$ remains. The lower bound is by lemma 2.3. Use lemma 3.2 on the subset $V = \mathcal{V}(F) \cup \{v\}$ to prove the upper bound. □

3.2 Special embeddings

Let $(e_1, \ldots, e_d)$ be a fixed basis of the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$.

Definition 3.4. *Let P be a smooth Fano d-polytope. Any smooth Fano d-polytope Q, with* $\operatorname{conv}\{e_1, \ldots, e_d\}$ *as a special facet, is called a* special embedding *of P, if P and Q are isomorphic.*

Obviously, for any smooth Fano polytope P, there exists at least one special embedding of P. As any polytope has finitely many facets, there exists only finitely many special embeddings of P.
Now we define a subset of $\mathbb{Z}^d$ which will play an important part in what follows.

Definition 3.5. *By $\mathcal{W}_d$ we denote the maximal set (with respect to inclusion) of lattice points in $\mathbb{Z}^d$ such that*

1. *The origin is not contained in $\mathcal{W}_d$.*

2. *The points in $\mathcal{W}_d$ are primitive lattice points.*

3. *If $a_1 e_1 + \ldots + a_d e_d \in \mathcal{W}_d$, then $-d \leq a \leq 1$ for $a = a_1 + \ldots + a_d$ and*

$$\left.\begin{array}{c} 0 \\ -1 \\ a \end{array}\right\} \leq a_i \leq \left\{\begin{array}{cl} 1 & , a = 1 \\ d-1 & , a = 0 \\ d+a & , a < 0 \end{array}\right.$$

for every $i = 1, \ldots, d$.

The next theorem is one of the key results in this paper. It allows us to classify smooth Fano d-polytopes by considering subsets of the explicitely given set $\mathcal{W}_d$.

Theorem 3.6. *Let P be an arbitrary smooth Fano d-polytope, and Q any special embedding of P. Then $\mathcal{V}(Q)$ is contained in the set $\mathcal{W}_d$.*

Proof. Follows directly from corollary 3.3 and the definition of $\mathcal{W}_d$. □

4 Total ordering of smooth Fano polytopes

In this section we define a total order on the set of smooth Fano d-polytopes for any fixed $d \geq 1$.
Throughout the section $(e_1, \ldots, e_d)$ is a fixed basis of the lattice $\mathbb{Z}^d$.

4.1 The order of a lattice point

We begin by defining a total order $\preceq$ on $\mathbb{Z}^d$.

Definition 4.1. *Let $x = x_1e_1 + \ldots + x_de_d$, $y = y_1e_1 + \ldots + y_de_d$ be two lattice points in $\mathbb{Z}^d$. We define $x \preceq y$ if and only if*

$$(-x_1 - \ldots - x_d, x_1, \ldots, x_d) \leq_{lex} (-y_1 - \ldots - y_d, y_1, \ldots, y_d),$$

where $\leq_{lex}$ is the lexicographical ordering on the product of $d + 1$ copies of the ordered set $(\mathbb{Z}, \leq)$.
The ordering $\preceq$ is a total order on $\mathbb{Z}^d$.

Example. $(0, 1) \prec (-1, 1) \prec (1, -1) \prec (-1, 0)$.

Let V be any nonempty finite subset of lattice points in $\mathbb{Z}^d$. We define $\max V$ to the maximal element in V with respect to the ordering $\preceq$. Similarly, $\min V$ is defined to be the minimal element in V.
A important property of the ordering is shown in the following lemma.

Lemma 4.2. *Let P be a smooth Fano d-polytope, such that $\operatorname{conv}\{e_1, \ldots, e_d\}$ is a facet of P. For every $1 \leq i \leq d$, let $v_i \neq e_i$ denote the vertex of P, such that $\operatorname{conv}\{e_1, \ldots, e_{i-1}, v_i, e_{i+1}, \ldots, e_d\}$ is a facet of P.*
Then $v_i = \min\{v \in \mathcal{V}(P) \mid \langle u_F^{e_i}, v\rangle < 0\}$.

Proof. By lemma 2.2.(1) the vertex v_i is in the set $\{v \in \mathcal{V}(P) \mid \langle u_F^{e_i}, v\rangle < 0\}$, and by lemma 2.2.(5) and the definition of the ordering $\preceq$, v_i is the minimal element in this set. $\square$

In fact, we have chosen the ordering $\preceq$ to obtain the property of lemma 4.2, and any other total order on $\mathbb{Z}^d$ having this property can be used in what follows.

4.2 The order of a smooth Fano d-polytope

We can now define an ordering on finite subsets of $\mathbb{Z}^d$. The ordering is defined recursively.

Definition 4.3. *Let X and Y be finite subsets of $\mathbb{Z}^d$. We define $X \preceq Y$ if and only if $X = \emptyset$ or*

$$Y \neq \emptyset \wedge (\min X \prec \min Y \vee (\min X = \min Y \wedge X\backslash\{\min X\} \preceq Y\backslash\{\min Y\})).$$

Example. $\emptyset \prec \{(0, 1)\} \prec \{(0, 1), (-1, 1)\} \prec \{(0, 1), (1, -1)\} \prec \{(-1, 1)\}$.

When W is a nonempty finite set of subsets of $\mathbb{Z}^d$, we define $\max W$ to be the maximal element in W with respect to the ordering of subsets $\preceq$. Similarly, $\min W$ is the minimal element in W.
Now, we are ready to define the order of a smooth Fano d-polytope.

Definition 4.4. *Let P be a smooth Fano d-polytope. The order of P,* $\operatorname{ord}(P)$*, is defined as*

$$\operatorname{ord}(P) := \min\{\mathcal{V}(Q) \mid Q \text{ a special embedding of } P\}.$$

The set is non-empty and finite, so $\operatorname{ord}(P)$ *is well-defined.*
Let P_1 and P_2 be two smooth Fano d-polytopes. We say that $P_1 \leq P_2$ if and only if $\operatorname{ord}(P_1) \preceq \operatorname{ord}(P_2)$*. This is indeed a total order on the set of isomorphism classes of smooth Fano d-polytopes.*

4.3 Permutation of basisvectors and presubsets

The group S_d of permutations of d elements acts on $\mathbb{Z}^d$ is the obvious way by permuting the basisvectors:

$$\sigma.(a_1e_1 + \ldots + a_de_d) := a_1e_{\sigma(1)} + \ldots + a_de_{\sigma(d)} \quad , \ \sigma \in S_d.$$

Similarly, S_d acts on subsets of $\mathbb{Z}^d$:

$$\sigma.X := \{\sigma.x \mid x \in X\}.$$

In this notation we clearly have for any special embedding P of a smooth Fano d-polytope

$$\operatorname{ord}(P) \preceq \min\{\sigma.\mathcal{V}(P) \mid \sigma \in S_d\}.$$

Let V and W be finite subsets of $\mathbb{Z}^d$. We say that V is a *presubset* of W, if $V \subseteq W$ and $v \prec w$ whenever $v \in V$ and $w \in W \setminus V$.

Example. $\{(0,1),(-1,1)\}$ is a presubset of $\{(0,1),(-1,1),(1,-1)\}$, while $\{(0,1),(1,-1)\}$ is not.

Lemma 4.5. *Let P be a smooth Fano polytope. Then every presubset V of* $\operatorname{ord}(P)$ *is the minimal element in $\{\sigma.V \mid \sigma \in S_d\}$.*

Proof. Let $\operatorname{ord}(P) = \{v_1, \ldots, v_n\}$, $v_1 \prec \ldots \prec v_n$. Suppose there exists a permutation σ and a k, $1 \leq k \leq n$, such that

$$\sigma.\{v_1, \ldots, v_k\} = \{w_1, \ldots, w_k\} \prec \{v_1, \ldots, v_k\},$$

where $w_1 \prec \ldots \prec w_k$. Then there is a number j, $1 \leq j \leq k$, such that $w_i = v_i$ for every $1 \leq i < j$ and $w_j \prec v_j$.
Let σ act on $\{v_1, \ldots, v_n\}$.

$$\sigma.\{v_1, \ldots, v_n\} = \{x_1, \ldots, x_n\} \quad , \ x_1 \prec \ldots \prec x_n.$$

Then $x_i \preceq v_i$ for every $1 \leq i < j$ and $x_j \prec v_j$. So $\sigma.\operatorname{ord}(P) \prec \operatorname{ord}(P)$, but this contradicts the definition of $\operatorname{ord}(P)$. $\square$

5 The SFP-algorithm

In this section we describe an algorithm that produces the classification list of smooth Fano d-polytopes for any given $d \geq 1$. The algorithm works by going through certain finite subsets of $\mathcal{W}_d$ in increasing order (with respect to the ordering defined in the previous section). It will output a subset V iff conv$V$ is a smooth Fano d-polytope P and $\mathrm{ord}(P) = V$.
Throughout the whole section $(e_1, \ldots, e_d)$ is a fixed basis of $\mathbb{Z}^d$ and I denotes the $(d-1)$-simplex conv$\{e_1, \ldots, e_d\}$.

5.1 The SFP-algorithm

The SFP-algorithm consists of three functions,

$$\texttt{SFP}, \texttt{AddPoint} \text{ and } \texttt{CheckSubset}.$$

The finite subsets of $\mathcal{W}_d$ are constructed by the function `AddPoint`, which takes a subset V, $\{e_1, \ldots, e_d\} \subseteq V \subseteq \mathcal{W}_d$, together with a finite set $\mathcal{F}$, $I \in \mathcal{F}$, of $(d-1)$-simplices in $\mathbb{R}^d$ as input. It then goes through every v in the set

$$\{v \in \mathcal{W}_d \mid \max V \prec v\}$$

in increasing order, and recursively calls itself with input $V \cup \{v\}$ and some set $\mathcal{F}'$ of $(d-1)$-simplices of $\mathbb{R}^d$, $\mathcal{F} \subseteq \mathcal{F}'$. In this way subsets of $\mathcal{W}_d$ are considered in increasing order.
Whenever `AddPoint` is called, it checks if the input set V is the vertex set of a special embedding of a smooth Fano d-polytope P such that $\mathrm{ord}(P) = V$, in which case the polytope $P = \mathrm{conv}V$ is outputted.
For any given integer $d \geq 1$ the function `SFP` calls the function `AddPoint` with input $\{e_1, \ldots, e_d\}$ and $\{I\}$. In this way a call `SFP`(d) will make the algorithm go through every finite subset of $\mathcal{W}_d$ containing $\{e_1, \ldots, e_d\}$, and smooth Fano d-polytopes are outputted in strictly increasing order.
It is vital for the effectiveness of the SFP-algorithm, that there is some efficient way to check if a subset $V \subseteq \mathcal{W}_d$ is a presubset of $\mathrm{ord}(P)$ for some smooth Fano d-polytope P. The function `AddPoint` should perform this check before the recursive call `AddPoint`$(V, \mathcal{F}')$.
If P is any smooth Fano d-polytope, then any presubset V of $\mathrm{ord}(P)$ is the minimal element in the set $\{\sigma.V | \sigma \in S_d\}$ (by lemma 4.5). In other words, if there exists a permutation σ such that $\sigma.V \prec V$, then the algorithm should not make the recursive call `AddPoint`(V).
But this is not the only test we wish to perform on a subset V before the recursive call. The function `CheckSubset` performs another test: It takes a subset V, $\{e_1, \ldots, e_d\} \subseteq V \subseteq \mathcal{W}_d$ as input together with a finite set of $(d-1)$-simplices $\mathcal{F}$, $I \in \mathcal{F}$, and returns a set $\mathcal{F}'$ of $(d-1)$-simplices containing $\mathcal{F}$, if there exists a special embedding P of a smooth Fano d-polytope, such that

1. V is a presubset of $\mathcal{V}(P)$

2. $\mathcal{F}$ is a subset of the facets of P

This is proved in theorem 5.1. If no such special embedding exists, then `CheckSubset` returns false in many cases, but not always! Only when `CheckSubset`$(V, \mathcal{F})$ returns a set $\mathcal{F}'$ of simplices, we allow the recursive call `AddPoint`$(V, \mathcal{F}')$.
Given input $V \subseteq \mathcal{W}_d$ and a set $\mathcal{F}$ of $(d-1)$-simplices of $\mathbb{R}^d$, the function `CheckSubset` works in the following way: Suppose V is a presubset of $\mathcal{V}(P)$ for some special embedding P of a smooth Fano d-polytope and $\mathcal{F}$ is a subset of the facets of P. Deduce as much as possible of the face lattice of P and look for contradictions to the lemmas stated in section 2. The more facets we know of P, the more restrictions we can put on the vertex set $\mathcal{V}(P)$, and then on V. If a contradiction arises, return false. Otherwise, return the deduced set of facets of P.
The following example illustrates how the function `CheckSubset` works.

5.2 An example of the reasoning in `CheckSubset`

Let $d = 5$ and $V = \{v_1, \ldots, v_8\}$, where

$$v_1 = e_1 \ , \ v_2 = e_2 \ , \ v_3 = e_3 \ , \ v_4 = e_4 \ , \ v_5 = e_5$$
$$v_6 = -e_1 - e_2 + e_4 + e_5 \ , \ v_7 = e_2 - e_3 - e_4 \ , \ v_8 = -e_4 - e_5 .$$

Suppose P is a special embedding of a smooth Fano 5-polytope, such that V is a presubset of $\mathcal{V}(P)$. Certainly, the simplex I is a facet of P.
Notice, that V does not violate lemma 3.2.

$$v_1 + \ldots + v_8 = e_2 + e_5 .$$

If V did contradict lemma 3.2, then the polytope P could not exist, and `CheckSubset`$(V, \{I\})$ should return false.
For simplicity we denote any k-simplex $\operatorname{conv}\{v_{i_1}, \ldots, v_{i_k}\}$ by $\{i_1, \ldots, i_k\}$.
Since $\langle u_I, v_6 \rangle = 0$, the simplices $F_1 = \{2,3,4,5,6\}$ and $F_2 = \{1,3,4,5,6\}$ are facets of P (lemma 2.4).
There are exactly two facets of P containing the ridge $\{1,2,4,5\}$. One of them is I. Suppose the other one is $\{1,2,4,5,9\}$, where v_9 is some lattice point not in V, $v_9 \in \mathcal{V}(P)$. Then $\langle u_I, v_9 \rangle > \langle u_I, v_7 \rangle$ by lemma 2.2.(5) and then $v_9 \prec v_7$ by the definition of the ordering of lattice points $\mathbb{Z}^d$. But then V is not a presubset of $\mathcal{V}(P)$. This is the nice property of the ordering of $\mathbb{Z}^d$, and the reason why we chose it as we did. We conclude that $F_3 = \{1,2,4,5,7\}$ is a facet of P, and by similar reasoning $F_4 = \{1,2,3,5,8\}$ and $F_5 = \{1,2,3,4,8\}$ are facets of P.

Now, for each of the facets F_i and every point $v_j \in V$, we check if $\langle u_{F_i}, v_j \rangle = 0$. If this is the case, then by lemma 2.4 $\mathrm{conv}(\{v_j\} \cup \mathcal{V}(F_i) \setminus \{w\})$ is a facet of P for every $w \in \mathcal{V}(F_i)$ where $\langle u^w_{F_i}, v_j \rangle < 0$. In this way we get that

$$\{2,4,5,6,7\} \ , \ \{1,4,5,6,7\} \ , \ \{1,2,3,7,8\} \ , \ \{1,3,5,7,8\}$$

are facets of P.
We continue in this way, until we cannot deduce any new facet of P. Every time we find a new facet F we check that v is beneath F (that is $\langle u_F, v \rangle \leq 1$) and that lemma 2.3 holds for any $v \in V$. If not, then `CheckSubset`$(V, \{I\})$ should return false.
If no contradiction arises, `CheckSubset`$(V, \{I\})$ returns the set of deduced facets.

5.3 The SFP-algorithm in pseudo-code

Input: A positive integer d.
Output: A list of special embeddings of smooth Fano d-polytopes, such that

1. Any smooth Fano d-polytope is isomorphic to one and only one polytope in the output list.
2. If P is a smooth Fano d-polytope in the output list, then $\mathcal{V}(P) = \mathrm{ord}(P)$.
3. If P_1 and P_2 are two non-isomorphic smooth Fano d-polytopes in the output list and P_1 preceeds P_2 in the output list, then $\mathrm{ord}(P_1) \prec \mathrm{ord}(P_2)$.

`SFP` (an integer $d \geq 1$)

1. Construct the set $V = \{e_1, \ldots, e_d\}$ and the simplex $I = \mathrm{conv} V$.
2. Call the function `AddPoint`$(V, \{I\})$.
3. End program.

`AddPoint` (a subset V where $\{e_1, \ldots, e_d\} \subseteq V \subseteq \mathcal{W}_d$, a set of $(d-1)$-simplices $\mathcal{F}$ in $\mathbb{R}^d$ where $I \in \mathcal{F}$)

1. If $P = \mathrm{conv}(\mathcal{V}(V))$ is a smooth Fano d-polytope and $\mathcal{V}(V) = \mathrm{ord}(P)$, then output P.
2. Go through every $v \in \mathcal{W}_d$, $\max \mathcal{V}(V) \prec v$, in increasing order with respect to the ordering $\prec$:
   (a) If `CheckSubset`$(V \cup \{v\}, \mathcal{F})$ returns false, then goto (d). Otherwise let $\mathcal{F}'$ be the returned set of $(d-1)$-simplices.

(b) If $V \cup \{v\} \neq \min\{\sigma.(V \cup \{v\}) \mid \sigma \in S_d\}$, then goto (d).

(c) Call the function `AddPoint`$(V \cup \{v\}, \mathcal{F}')$.

(d) Let v be the next element in $\mathcal{W}_d$ and go back to (a).

3. Return

`CheckSubset` (a subset V where $\{e_1, \ldots, e_d\} \subseteq V \subseteq \mathcal{W}_d$, a set of $(d-1)$-simplices $\mathcal{F}$ in $\mathbb{R}^d$ where $I \in \mathcal{F}$)

1. Let $\nu = \sum_{v \in V} v$.
2. If $\langle u_I, \nu \rangle < 0$, then return false.
3. If $\langle u_I^{e_i}, \nu \rangle > 1 + \langle u_I, \nu \rangle$ for some i, then return false.
4. Let $\mathcal{F}' = \mathcal{F}$.
5. For every $i \in \{1, \ldots, d\}$: If the set $\{v \in V | \langle u_I^{e_i}, v \rangle < 0\}$ is equal to $\{\max V\}$, then add the simplex $\operatorname{conv}(\{\max V\} \cup \mathcal{V}(I) \setminus \{e_i\})$ to $\mathcal{F}'$.
6. If there exists $F \in \mathcal{F}'$ such that $\mathcal{V}(F)$ is not a $\mathbb{Z}$-basis of $\mathbb{Z}^d$, then return false.
7. If there exists $F \in \mathcal{F}'$ and $v \in V$ such that $\langle u_F, v \rangle > 1$, then return false.
8. If there exists $F \in \mathcal{F}'$, $v \in V$ and $w \in \mathcal{V}(F)$, such that
$$\langle u_F^w, v \rangle < \begin{cases} 0 & \langle u_F, v \rangle = 1 \\ -1 & \langle u_F, v \rangle = 0 \\ \langle u_F, v \rangle & \langle u_F, v \rangle < 0 \end{cases}$$
then return false.
9. If there exists $F \in \mathcal{F}'$, $v \in V$ and $w \in \mathcal{V}(F)$, such that $\langle u_F, v \rangle = 0$ and $\langle u_F^w, v \rangle = -1$, then consider the simplex $F' = \operatorname{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{w\})$. If $F' \notin \mathcal{F}'$, then add F' to $\mathcal{F}'$ and go back to step 6.
10. Return $\mathcal{F}'$.

5.4 Justification of the SFP-algorithm

The following theorems justify the SFP-algorithm.

Theorem 5.1. *Let P be a special embedding of a smooth Fano d-polytope and V a presubset of $\mathcal{V}(P)$, such that $\{e_1, \ldots, e_d\} \subseteq V$. Let $\mathcal{F}$ be a set of facets of P.*
Then `CheckSubset`$(V, \mathcal{F})$ *returns a subset $\mathcal{F}'$ of the facets of P and $\mathcal{F} \subseteq \mathcal{F}'$.*

Proof. By lemma 3.2 the subset V will pass the tests in step 2 and 3 in `CheckSubset`.
The function `CheckSubset` constructs a set $\mathcal{F}'$ of $(d-1)$-simplices containing the input set $\mathcal{F}$. We now wish to prove that every simplex F in $\mathcal{F}'$ is a facet of P: By the assumptions the subset $\mathcal{F} \subseteq \mathcal{F}'$ consists of facets of P.
Consider the addition of a simplex F_i, $1 \leq i \leq d$, in step 5:

$$F_i = \operatorname{conv}(\{\max V\} \cup \mathcal{V}(I) \setminus \{e_i\}).$$

As $\max V$ is the only element in the set $\{v \in V | \langle u_I^{e_i}, v\rangle < 0\}$ and V is a presubset of $\mathcal{V}(P)$, F_i is a facet of P by lemma 4.2.
Consider the addition of simplices in step 9: If F is a facet of P, then by lemma 2.4 the simplex $\operatorname{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{w\})$ is a facet of P.
By induction we conclude, that every simplex in $\mathcal{F}'$ is a facet of P. Then any simplex $F \in \mathcal{F}'$ will pass the tests in steps 6–8 (use lemma 2.3 to see that the last test is passed).
This proves the theorem. □

Theorem 5.2. *The SFP-algorithm produces the promised output.*

Proof. Let P be a smooth Fano d-polytope. Clearly, P is isomorphic to at most one polytope in the output list.
Let Q be a special embedding of P such that $\mathcal{V}(Q) = \operatorname{ord}(P)$. We need to show that Q is in the output list. Let $\mathcal{V}(Q) = \{e_1, \dots, e_d, q_1, \dots, q_k\}$, where $q_1 \prec \dots \prec q_k$, and let $V_i = \{e_1, \dots, e_d, q_1, \dots, q_i\}$ for every $1 \leq i \leq k$.
Certainly the function `AddPoint` has been called with input $\{e_1, \dots, e_d\}$ and $\{I\}$.
By theorem 5.1 the function call `CheckSubset`$(V_1, \{I\})$ returns a set $\mathcal{F}_1$ of $(d-1)$-simplices which are facets of Q, $I \subset \mathcal{F}_1$. By lemma 4.5 the set V_1 passes the test in 2b in `AddPoint`. Then `AddPoint` is called recursively with input V_1 and $\mathcal{F}_1$.
The call `CheckSubset`$(V_1, \mathcal{F}_1)$ returns a subset $\mathcal{F}_2$ of facets of Q, and the set V_2 passes the test in 2b in `AddPoint`. So the call `AddPoint`$(V_2, \mathcal{F}_2)$ is made.
Proceed in this way to see that the call `AddPoint`$(V_k, \mathcal{F}_k)$ is made, and then the polytope $Q = \operatorname{conv} V_k$ is outputted in step 1 in `AddPoint`. □

6 Classification results and where to get them

A modified version of the SFP-algorithm has been implemented in C++, and used to classify smooth Fano d-polytopes for $d \leq 7$. On an average home computer our program needs less than one day (january 2007) to construct the classification list of smooth Fano 7-polytopes. These lists can be downloaded from the authors homepage: `http://home.imf.au.dk/oebro`

An advantage of the SFP-algorithm is that it requires almost no memory: When the algorithm has found a smooth Fano d-polytope P, it needs not consult the output list to decide whether to output the polytope P or not. The construction guarentees that $\mathcal{V}(P) = \min\{\sigma.\mathcal{V}(P) \mid \sigma \in S_d\}$ and it remains to check if $\mathcal{V}(P) = \mathrm{ord}(P)$. Thus there is no need of storing the output list.
The table below shows the number of isomorphism classes of smooth Fano d-polytopes with n vertices.

n	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
1							
2	1						
3		1					
4		2	1				
5		1	4	1			
6		1	7	9	1		
7			4	28	15	1	
8			2	47	91	26	1
9				27	268	257	40
10				10	312	1318	643
11				1	137	2807	5347
12				1	35	2204	19516
13					5	771	26312
14					2	186	14758
15						39	4362
16						11	1013
17						1	214
18						1	43
19							5
20							2
Total	1	5	18	124	866	7622	72256

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