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The geometry of Deligne-Lusztig varieties; Higher-Dimensional AG codes

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Contents

Introduction

This thesis gathers the results I have obtained during my graduate studies. Through the last four years I have been working with several problems of different nature. Actually so different that I felt it necessary to give the thesis the 'double-title' it has. The areas have, however, the methods of algebraic geometry over finite fields in common. Hence this could also have been a suitable title.

Since it is my hope that this thesis also will provide a good starting point for new-comers to the subject, I have often provided examples of the various definitions, constructions and calculations. For the same reason, I have tried to provide ample references not only to closely related work, but also to papers that may have a more general interest to readers of the present thesis.

Main results

Let me briefly state the main results of the thesis:

- It is proved (Chapter 2) that Deligne-Lusztig varieties are normal and Cohen-Macaulay. A sufficient criterion for non-singularity is given.

- For any Deligne-Lusztig variety X arising from one of the classical (possibly twisted) groups, a (finite) basis for the Picard group of X is given. Further evidence concerning a conjectural behaviour of the Chow groups of Deligne-Lusztig varieties is also given (see Chapter 3).

- The canonical bundle of Deligne-Lusztig varieties has been calculated. Explicit expressions have been given, allowing one to prove ampleness of the canonical bundle in certain cases. These results have appeared in the article *Canonical bundles of Deligne-Lusztig varieties*, Manuscripta Math. **98** (1999), 363–375.

- A general framework for determining the parameters of error-correcting algebraic-geometric codes arising from higher-dimensional varieties has been developed. These results have appeared in a separate preprint *Error-correcting codes from higher-dimensional varieties*, submitted to Finite Fields and Their Applications, 1998.

Summary

Deligne-Lusztig varieties was defined in the mid-seventies by Deligne and Lusztig [**DL76**] as a mere tool in an ingenious construction of certain representations of finite groups. Until the beginning of the nineties, the study of these varieties continued to be from the representation theoretic point of view *cf.* [**Lus76a, Lus76b, Lus78, Haa84, Haa86**].

In the meantime Goppa [**Gop88**] had shown how to construct error correcting codes from algebraic curves over finite fields, and codes arising from Deligne-Lusztig curves turned out to provide excellent examples, *cf.* [**HS90, Han92, HP93**]. So suddenly Deligne-Lusztig varieties gained, so to speak, a life of their own.

Early on I became interested in constructing codes from higher-dimensional varieties; and from Deligne-Lusztig varieties in particular *cf.* [**HH95b**]. Needless to say, things got more difficult than in the curve case. Just to get started, one needed a good description of the divisors and their intersections. The concerns of this thesis are therefore primarily the geometry of Deligne-Lusztig varieties.

In Chapter 1 we start out by gathering various results from the theory of reductive groups. If G is a reductive algebraic group it is well-known that the homogeneous space G/B — the variety of Borel subgroups of G — has a decomposition into disjoint locally closed subsets $B\dot{w}B$ where w runs through the elements of the Weyl group W of G. In the positive characteristic situation we may furthermore associate to G a socalled Frobenius morphism $F : G \to G$. One way of defining Deligne-Lusztig varieties is now as follows: Let $O(w)$ denote the pairs (g_1B, g_2B) of $G/B \times G/B$ satisfying that $g_1^{-1}g_2$ is in $B\dot{w}B$. Then the Deligne-Lusztig variety $X(w)$ is the intersection of $O(w)$ and the graph Γ of F. Similar definitions for the closure $\overline{X(w)}$ of $X(w)$ in G/B and for a Demazure-type desingularisation $\bar{X}(w)$ are made. In the last sections of Chapter 1 we recollect various properties of Deligne-Lusztig varieties (and their desingularisations) and we extend some results of Lusztig [**Lus76a**] regarding their irreducible components.

In Chapter 2 I prove that Deligne-Lusztig varieties are normal and Cohen-Macaulay (Proposition 2.2). For Deligne-Lusztig varieties arising from Coxeter elements this actually implies non-singularity (Theorem 2.3). These properties are mainly derived from the corresponding properties for the varieties $\overline{O(w)}$ [**MR88**] combined with transversality arguments.

One of the main results of this thesis appears in Chapter 3 (Theorem 3.14). For any Deligne-Lusztig variety $\bar{X}(w)$ arising from one of the classical (possibly twisted) groups, I give a (finite) basis for the Picard group of $\bar{X}(w)$. The proof goes as follows: For Deligne-Lusztig varieties of classical type one may

construct a birational morphism $\pi : \bar{X}(w) \to Z$ to a complete intersection in projective space. A careful study of the singularities of Z reveals that the the divisor class group of Z equals the Picard group of Z. Since the latter equals $\mathbb{Z}$ (by the Lefschetz theorem for Picard groups [**Gro68**]) we reach the conclusion that the Picard group of $\bar{X}(w)$ is (freely) generated by the class of the hyperplane section on Z pulled back to $\bar{X}(w)$, and by the classes of the finitely many exceptional fibres [**Ful83**].

In Chapter 3 it is furthermore conjectured that this behaviour in codimension 1 is more generally true in any codimension. That is, the Chow groups of $X(w)$ consist (in positive codimension) at the most of torsion. For Deligne-Lusztig varieties of type A_n it is proved that this is indeed the case. The assertion is also proved for the G^F-invariant part of the Chow groups.

When confronted with a poorly studied variety defined over a finite field, a natural question (at least from a positive-characteristic geometer) is: "Is it Frobenius split?" In Chapter 4 I have included the paper *Canonical bundles of Deligne-Lusztig varieties* [**HH99a**] where another of my main results appears. Using the adjunction formula and results of Mehta and Ramanathan [**MR88**] regarding the canonical bundles of the varieties $\bar{O}(w)$, I prove a general formula for the canonical bundle of a Deligne-Lusztig variety. Employing techniques as sketched in the original paper by Deligne and Lusztig [**DL76**], I refined these formulas to rather explicit forms. With the aid of these formulas I were also able to prove that a certain kind of Deligne-Lusztig surfaces provides a whole class of counter-examples to the socalled Miyaoka-Yau inequality otherwise true in characteristic zero [**Miy77**, Theorem 4].

As a corollary of the results in the paper I could also give the above question a negative answer. That is, Deligne-Lusztig varieties are generally not Frobenius split. In the chapter I also give an alternative proof of this fact and extend some of the results of [**HH99a**].

Coming back to the starting point of this introduction, I will conclude by mentioning another main contribution given in the paper *Error-correcting codes from higher-dimensional varieties* [**HH98**]. An error-correcting code C with parameters $[n, k, d]_q$ is, roughly speaking, nothing but a k-dimensional subspace of an n-dimensional vector space over the finite field with q elements. Furthermore, all (non-zero) points of C are assumed to have at least d non-zero coordinates. Algebraic geometric (AG) codes are then what is obtained when the global sections of a fixed line bundle is 'evaluated' in n fixed $\mathbb{F}_q$-rational points.

In the above-mentioned paper I address the problems that arise when one wants to not only construct, but also estimate the parameters of codes coming from higher-dimensional varieties. Using intersection theory I prove general results concerning the dimension and minimum distance of error-correcting codes arising from varieties of dimension two or higher. In Chapter 5 I have reproduced the paper.

Occasionally I have needed various small constructions and lemmas that I was unable to find in the literature. I have collected these (and their proofs) in Appendix A.

As it often happens, answering one question leads to a wealth of new ones. In Appendix B I have gathered some of the questions I have not been able to answer; this may as well be ascribed lack of time as my mathematical limitations.

For the interested reader there are in Appendix C included some computations done with the computer algebra package Maple.

At the end, in Appendix D, a résumé for the University Annual Report is included.

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Parts of the work presented in this thesis was carried out while I was visiting *Institut Mittag-Leffler*. I take here the opportunity to acknowledge the support I received and the work-friendly atmosphere created by the Institute staff.

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The present version of this thesis differs from the submitted thesis mostly in lay-out. Some typos have been corrected and a couple of linguistic improvements have been made. Furthermore Section 3.2 has been slightly changed.

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Søren Have Hansen

CHAPTER 1

Preliminaries

1.1. Notation and conventions

Throughout this thesis, we will adhere to the following notation and conventions:

- Let q be a prime power. Let $\mathbb{F}_q$ be the finite field with q elements and denote by k an algebraic closure of $\mathbb{F}_q$.
- For a finite set I, $\#I$ will denote the cardinality of I whereas we shall use the notation $|H|$ for the cardinality of a finite group H.
- If H is an abelian group we shall denote $H \otimes_{\mathbb{Z}} \mathbb{Q}$ by $H_{\mathbb{Q}}$. Examples are Chow groups, Picard groups, groups of characters etc.
- Unless otherwise stated we assume diagrams of morphisms to be commutative.
- Varieties are not necessarily irreducible.

1.2. Reductive groups

In this section we review (some of) the necessary material about reductive groups. We refer to [**Bor92, Car85, Hum75, Sri79**] for further information.

1.2.1. Frobenius morphisms. Let G be a reductive (connected) linear algebraic group over k.

DEFINITION 1.1. A morphism of algebraic groups $F : G \to G$ is a *standard Frobenius morphism* if there exists an embedding $G \xrightarrow{i} \mathrm{GL}_n(k)$ and a positive integer e such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{F} & G \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathrm{GL}_n(k) & \xrightarrow{F_e} & \mathrm{GL}_n(k) \end{array}$$

commutes for $F_e\big((a_{ij})\big) = (a_{ij}^{p^e})$, $p = char(k)$. F is a *Frobenius morphism* if some positive power F^i of F is a standard Frobenius morphism. Define Q to be the unique positive real number such that $Q^i = p^e$. Let

$$G^F = \{g \in G : F(g) = g\}$$

be the subgroup of fixed-points. (These definitions can be seen to be independent of choice of embedding.)

Associated to F is the *Lang map* $L : G \to G$ taking an element $g \in G$ to $g^{-1}F(g)$. (Caution: contrary to F, L is **not** a group homomorphism, only a morphism of varieties.) By the Lang-Steinberg Theorem [**Bor92**, Theorem 16.3] this morphism of varieties is surjective with finite fibres. From this result it follows that, by conjugacy of tori and Borel subgroups, there exists F-stable maximal tori and Borel subgroups. Fix an F-stable *Borel subgroup* B containing an F-stable *maximal torus* T.

REMARK 1.2. The finite groups G^F arising as the fixed-points of a Frobenius morphism acting on a reductive, connected linear algebraic group are called *finite groups of Lie type.* It was the search for characteristic zero representations of these groups which led Deligne and Lusztig [**DL76**] to the construction of Deligne-Lusztig varieties. (G^F acts on $X(w)$ as a group of automorphisms inducing an action on the ℓ-adic cohomology vector spaces. See also [**Haa84, Haa86**].)

1.2.2. Root systems and Dynkin diagrams. Let (A_{ij}) be the *Cartan matrix* of G and let W be the *Weyl group* of G; $W = N_G(T)/T$, for the maximal torus $T \subseteq G$. W is a finite group generated by a subset $S \subseteq W$ of *simple reflections*,

$$W = \langle s_1, \dots, s_r : s_i^2 = 1, (s_i s_j)^{n_{ij}} = 1 \text{ for } i \neq j \rangle$$

where

$$n_{ij} = 2 \Leftrightarrow A_{ij}A_{ji} = 0 \qquad n_{ij} = 3 \Leftrightarrow A_{ij}A_{ji} = 1$$
$$n_{ij} = 4 \Leftrightarrow A_{ij}A_{ji} = 2 \qquad n_{ij} = 6 \Leftrightarrow A_{ij}A_{ji} = 3.$$

Similarly, the nodes of the *Dynkin diagram* corresponding to s_i and s_j are joined by $0, 1, 2$ or 3 bonds when n_{ij} equals $2, 3, 4$ or 6 respectively.

Any $w \in W$ then has a well-defined *length* $l(w)$, hence we have a partial order $\leq$ on W. Denote by w_0 *the longest element in* W. For any $w \in W$ we may assume that w consists of the first $l(w)$ reflections of a reduced expression of w_0.

Since T is F-stable we get a homomorphism $F\colon W \to W$ and an action ρ on the Dynkin diagram $\mathcal{D}$ of G. It is then a well-known result that pairs (G, F), G a simple (connected) algebraic group, F a Frobenius morphism, are classified (up to isogeny type) by triples $(\mathcal{D}, \rho, Q)$ of (connected) Dynkin diagrams with automorphisms plus the real number Q. We will henceforth refer to a pair (G, F) as being of type ${}^{\delta}\mathrm{X}_n$, X_n being the Dynkin diagram of G and δ the order of ρ. In Table C.1 (page 105) and Table C.2 (page 105) we have listed the groups where ρ acts non-trivially with one or two orbits respectively.

For $w_1, w_2 \in W$ we shall say that w_1 and w_2 are *F-conjugate* if there exists $w' \in W$ such that $w_2 = w'w_1F(w')^{-1}$. We note that w and $F(w)$ are F-conjugate for any $w \in W$. It should also be noted that two F-conjugated Weyl group elements does not necessarily have the same length.

From now on, when we speak of a reductive group G, we shall assume it has connected Dynkin diagram.

1.2.3. Character groups and their F-action. Let

$$X_0 = \mathrm{Hom}(T, \mathbb{G}_m) \qquad \text{and} \qquad Y_0 = \mathrm{Hom}(\mathbb{G}_m, T)$$

be the character and co-character groups with respect to T. We will now and then write $X(T)$ for X_0. There is a non-degenerate pairing $\langle\cdot,\cdot\rangle : X_0 \times Y_0 \to \mathbb{Z}$ and actions of W and F on both X_0 and Y_0 such that

$$\langle \chi, F(\gamma)\rangle = \langle F(\chi), \gamma\rangle \qquad \chi \in X_0,\ \gamma \in Y_0 \tag{1.1}$$

$$\langle \chi, \gamma^w\rangle = \langle {}^w\chi, \gamma\rangle \qquad \chi \in X_0,\ \gamma \in Y_0,\ w \in W \tag{1.2}$$

(see [**Car85**, p. 18+35]). Given the choice of configuration $B \supset T$ we may write

$$B = T \times \prod_{\alpha \in A} U_\alpha$$

for some $A \subset X(T)$. We define the *positive roots* Φ^+ to be A. In some papers the positive roots are chosen to be $\{-A\}$ (see for example [**Jan87**, p. 180] and [**And85**]). Therefore we will have to change signs (add or remove "anti") when citing those works. Let $M = \#\Phi^+$.

To each simple reflection s_i there corresponds a *simple root* $\alpha_i \in X$ and a *co-root* $\alpha_i^\vee \in Y$ such that $\langle \alpha_j, \alpha_i^\vee\rangle = A_{ij}$. If we identify the simple reflections with their corresponding simple roots, we may relate the action of F to ρ: F transforms each simple root α into a positive multiple of $\rho^{-1}(\alpha)$ and then $F(U_{\alpha_i}) = U_{\rho(\alpha_i)}$.

Above we introduced the real number Q defined by the relation $Q^i = p^e$ whenever F^i is the restriction of the Frobenius morphism $x \mapsto x^{p^e}$ on $\mathrm{GL}_n(\mathbb{F}_q)$. This allows us to write $F = QF_0$ with F_0 of finite order on X_0 and Y_0. The actions of W, F and F_0 extend by linearity to $X_0 \otimes \mathbb{R}$ and $Y_0 \otimes \mathbb{R}$ and may be described in terms of matrices. If the matrix M represents F_0 on $X_0 \otimes \mathbb{R}$ then by (1.1), F_0 is represented by the transpose M^t on $Y_0 \otimes \mathbb{R}$.

1.2.4. Bruhat decomposition and Schubert varieties. For any Borel subgroup B' the homogeneous space G/B' is a non-singular projective variety which may be identified with the G-conjugation orbits of B',

$$G/B' = \{gB'g^{-1} : g \in G\}$$

(the coset gB' corresponds to the orbit of $gB'g^{-1}$). By conjugacy of Borel subgroups, the quotient X is independent of choice of B', and we shall therefore think of X as the *variety of Borel subgroups of* G. We will usually depict X as the quotient of G by the F-stable subgroup B, but when convenient we may choose any other Borel subgroup.

G has a *Bruhat decomposition* into disjoint locally closed subvarieties,

$$G = \bigcup_{w \in W} B\dot{w}B \tag{1.3}$$

where $\dot{w} \in G$ is a representative of $w \in W$. This decomposition passes to X,

$$X = \bigcup_{w \in W} B\dot{w}B/B \tag{1.4}$$

where the *Bruhat cells* $B\dot{w}B/B$ may be viewed as the orbits of B's action on G/B by left translation. $B\dot{w}B/B$ is a locally closed subvariety of X of dimension $l(w)$; in fact $B\dot{w}B/B \simeq \mathbb{A}^{l(w)}$. The closure of $B\dot{w}B/B$ in X is given by

$$X_w = \overline{B\dot{w}B}/B = \bigcup_{w' \leq w} B\dot{w}'B/B. \tag{1.5}$$

The X_w are called *Schubert varieties* and are generators of the Chow ring of G/B *cf*. [**Che94, Dem74**].

1.2.5. Schubert varieties in $G/B \times G/B$. Let G act diagonally on $X \times X$ and let

$$O(w) = G.(eB, \dot{w}B) = \{(g_1B, g_2B) \in X \times X : g_1^{-1}g_2 \in B\dot{w}B\} \tag{1.6}$$

be the orbit of $(eB, \dot{w}B)$ under this action. From the decomposition (1.4) it follows that we also have a decomposition of $X \times X$,

$$X \times X = \bigcup_{w \in W} O(w). \tag{1.7}$$

Projection to the first factor $O(w) \to X$ makes $O(w)$ an affine bundle over X with fiber $B\dot{w}B/B$, hence $O(w)$ is a locally closed subvariety of $X \times X$ of dimension $\dim(X) + l(w)$. The closure of $O(w)$ in $X \times X$ is given by

$$\overline{O(w)} = \bigcup_{w' \leq w} O(w'). \tag{1.8}$$

Below (Chapter 2) we will see how Deligne-Lusztig varieties inherit (non)-singularity from X_w and $\overline{O(w)}$. As $\overline{O(w)}$ and X_w often are singular we construct desingularisations and determine when the Deligne-Lusztig varieties actually are non-singular.

1.2.6. Homogeneous line bundles. Let $\{\lambda_i\}$ (one for each generator s_i of W) be a $\mathbb{Z}$-basis of the roots $\Phi \subseteq X(T)$. We may choose the λ_i such that $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. For $\lambda \in X(T)$ we shall say that λ is *dominant* (resp. *strictly dominant*) if $\langle \lambda, \alpha^\vee \rangle \geq 0$ (resp. > 0) for all positive roots $\alpha \in \Phi^+$. Hence, if we write $\lambda = \sum_i m_i\lambda_i$, then λ is strictly anti-dominant if and only if $m_i < 0$ for all i. Let $\rho = \sum_i \lambda_i$. We state the following well-known result [**And85**],[**Jan87**, p. 231].

THEOREM 1.3. *Let $\lambda \in X(T)$ and let $\mathcal{L}(\lambda)$ be the line bundle $G \times^B k_\lambda$ on X (where the subscript λ denotes that B acts on k via λ: $b.x = \lambda(b)^{-1}x;\ x \in k,\ b \in B$). Then*

$$\mathcal{L}(\lambda) \text{ is ample if and only if } \lambda \text{ is strictly antidominant.} \tag{1.9}$$

In particular, $L(-\rho - \lambda)$ is ample for any $\lambda \geq 0$.

Now $F^*\mathcal{L}(\lambda) = L(F(\lambda))$, where $F(\lambda) = \lambda \circ F$. Suppose $\rho(s_i) = s_j$ and the corresponding roots α_i and α_j have the same length. Then $F(\lambda_j) = Q\lambda_i$ (recall that Q is the real number with the property that $Q^i = p^e$ when F^i is the standard Frobenius map raising to p^e'th powers).

REMARK 1.4. We should also mention *Kempf's vanishing theorem* [**Jan87**, p. 227] which in its most general form [**And85**, Theorem 1] may be stated as: if λ is anti-dominant (this is equivalent to $\mathrm{H}^0(X, \mathcal{L}(\lambda))$ being non-zero), then $\mathrm{H}^i(X, \mathcal{L}(\lambda)) = 0$ for any $i > 0$ (recall our convention: $X = G/B$ where B comes from the *positive* roots).

1.3. Defining Deligne-Lusztig varieties

We are now in position to define Deligne-Lusztig varieties.

DEFINITION 1.5. Let G be a reductive algebraic group with Frobenius morphism $F : G \to G$ and Weyl group W. Consider the graph Γ of the induced map $F : X \to X$. The *Deligne-Lusztig variety* $X(w)$ is then the intersection scheme $O(w) \cap \Gamma$.

PROPOSITION 1.6. *$X(w)$ has the following properties.*

a) *$X(w)$ is a locally closed non-singular (possibly reducible) subvariety of $X \times X$ of pure dimension $l(w)$.*
b) *G^F acts as a group of $\mathbb{F}_q$-rational automorphisms on $X(w)$.*
c) *The closure of $X(w)$ in $X \times X$ is given by*

$$\overline{X(w)} = \bigcup_{w' \leq w} X(w'). \tag{1.10}$$

PROOF. For a) and b) see [**DL76**, 1.4]. c) is immediate from (1.8). ☐

REMARK 1.7. Let G be a reductive group. From G we may pass to its adjoint form G_{ad} by factoring out with the centre ZG. Since $ZG = ZB$ for any Borel subgroup B of G [**Hum75**, Corollary 22.2.B] we have a canonical isomorphism of varieties $(G/ZG)/(B/ZB) \simeq G/B$. As G was of arbitrary isogeny type it follows that G, G_{ad} and G_{sc} define the same varieties of Borel subgroups G/B (*cf.* [**Che94**, Proposition 7]). So $X(w)$ only depends on the triple $(\mathcal{D}, \rho, Q)$ and *not* on the isogeny type of G. Henceforth we will therefore not specify the isogeny type of the group we are dealing with. See also [**Ive78**].

REMARK 1.8. We note some immediate properties of Deligne-Lusztig varieties.

1. From (1.6) it follows that we may also describe $X(w)$ as

$$X(w) = \{(gB, F(g)B) \in X \times X : g^{-1}F(g) \in B\dot{w}B\}.$$

   This is nothing but the image of $L^{-1}(B\dot{w}B)$ in G/B. Since the Lang map $L : G \to G$ is an isogeny with zero-dimensional fibres ($L : G^F \setminus G \to G$ is an isomorphism), it is both flat (of relative dimension zero) and proper

[**Har77**, Exercise III.10.9]. Being flat, L is an open map [**Har77**, Exercise III.9.1] so $L^{-1}(\overline{B\dot{w}B}) = \overline{L^{-1}(B\dot{w}B)}$ and $\overline{X(w)} = L^{-1}(\overline{B\dot{w}B})/B$. (Clearly $\overline{L^{-1}(B\dot{w}B)} \subseteq L^{-1}(\overline{B\dot{w}B})$ so let $y \in L^{-1}(\overline{B\dot{w}B})$ and let U be an open neighborhood of y. Then $L(U)$ is an open neighborhood of $L(y) \in \overline{B\dot{w}B}$. Hence there exists $u \in U$ such that $L(u) \in B\dot{w}B$. But then $U \cap L^{-1}(B\dot{w}B)$ is non-empty.) Since the quotient morphism $G \to G/B$ also is flat (being locally trivial) we arrive, also in this description, at the conclusion that $X(w)$ and $B\dot{w}B/B$ must have the same dimension.

2. The projection to the first factor $p : \Gamma \to X$ defines an isomorphism and we shall occasionally identify $X(w)$ and $\overline{X(w)}$ with their images in X under this map.
3. $X(e)$ is the set of F-stable Borel subgroups of G.
4. Let δ be the minimal integer such that the action of F on W is the identity. Then,

$$\begin{aligned} F(X(w)) &= \{F(g)B : g^{-1}F(g) \in B\dot{w}B\} \\ &\subseteq \{gB : g^{-1}F(g) \in F(B\dot{w}B)\} \\ &= \{gB : g \in BF(\dot{w})B\} = X(F(w)) \end{aligned}$$

giving an automorphism $F^\delta : X(w) \to X(w)$. Hence $X(w)$ is defined over $\mathbb{F}_{q^\delta}$ and we may find the $\mathbb{F}_{q^\delta}$-rational points as the fixed-points $X(w)^{F^\delta}$ [**Lus76a**, (6.1)] (in Table C.3 (page 107) these numbers are given in the curve and surface cases). Similarly, F maps the Schubert variety X_w to $X_{F(w)}$.

Note furthermore that $X(w)$ can be identified with a closed subvariety of the (larger) Deligne-Lusztig variety $X(wF(w))$ of (G, F^2): If $g^{-1}F(g) \in B\dot{w}B$ then

$$L_{F^2}(g) = g^{-1}F^2(g) = g^{-1}F(g)F(g^{-1})F^2(g)$$

is in $B\dot{w}B \cdot BF(\dot{w})B$. Hence $L_{F^2}(g) \in B\dot{w}F(\dot{w})B$. (Similarly for higher powers of F.)

5. One may define ‘generalized’ Deligne-Lusztig varieties as follows: let $P_I \subseteq G$ be an F-stable parabolic subgroup. Let $w \in W^I$ and define $X(w)_I = L^{-1}(B\dot{w}B)/P_I$ (similarly for the closure). G^F also acts on these varieties.

1.4. Desingularisations

We shall need the Demazure desingularisation (see [**DL76**, 9.3], [**Dem74**] and Section 3.3.2): let $w = s_1 \cdot \ldots \cdot s_n$ be a reduced expression for w. Let $O(s_1, \ldots, s_n)$ (resp. $\bar{O}(s_1, \ldots, s_n)) \subseteq X \times \cdots \times X$ be the space of sequences $(B_0, \ldots, B_n)$ of Borel subgroups such that $(B_{i-1}, B_i) \in O(s_i)$ (resp. $O(s_i) \cup O(e)$). Then we obtain a desingularisation

$$\varphi : \bar{O}(s_1, \ldots, s_n) \to \overline{O(w)} \tag{1.11}$$

by sending $(B_0, \dots, B_n)$ to (B_0, B_n). By composing with projection to the first factor, we get affine and projective bundles

$$\pi : O(s_1, \dots, s_n) \to X \tag{1.12}$$

$$\bar{\pi} : \bar{O}(s_1, \dots, s_n) \to X. \tag{1.13}$$

Over the affine piece $B\dot{w}'B/B$ we have isomorphisms

$$O(s_1, \dots, s_n) \simeq B\dot{w}'B/B \times B\dot{w}B/B \simeq \mathbb{A}^{l(w')+n}.$$

The morphism $\bar{\pi}$ has iterated fibre $X_{s_i} \simeq \mathbb{P}^1$, hence is flat of relative dimension n [**Ful83**, 1.7, B.2.5]. Note that for $s_{i_1} \cdot \dots \cdot s_{i_r} \leq w$ we may identify $\bar{O}(s_{i_1}, \dots, s_{i_r})$ with at least one subvariety of $\bar{O}(s_1, \dots, s_n)$.

Since the fibre of $\overline{O(w)} \to X$ over eB is X_w we obtain a desingularisation $Z_w \to X_w$ by taking the fibre over eB of the composition $\bar{O}(w) \xrightarrow{\varphi} \overline{O(w)} \to X$.

Now construct a desingularisation of $\overline{X(w)}$ as follows. Let $\bar{X}(s_1, \dots, s_n)$ be defined by the fibre product

$$\begin{tikzcd}
\bar{X}(s_1, \dots, s_n) \arrow[r, "i'"] \arrow[d, "j'"'] & \bar{O}(s_1, \dots, s_n) \arrow[d, "j"] \\
X \arrow[r, "i=(\mathrm{id},F)"] & X \times X.
\end{tikzcd} \tag{1.14}$$

By [**DL76**, Lemma 9.11] this intersection is transverse, hence $\bar{X}(s_1, \dots, s_n)$ is smooth. By construction, $\bar{X}(s_1, \dots, s_n)$ is the subvariety of $\bar{O}(w)$ consisting of the sequences $(B_0, \dots, B_n)$ of Borel subgroups of G such that $B_n = F(B_0)$ and $(B_{i-1}, B_i) \in O(e) \cup O(s_i)$. We may identify $X(w)$ with the open subvariety

$$\{(B_0, \dots, B_n) : B_n = F(B_0), \ \ (B_{i-1}, B_i) \in O(s_i)\}$$

of $\bar{X}(s_1, \dots, s_n)$ by mapping $(B_0, \dots, B_n)$ to $(B_0, F(B_0)) \in \Gamma$.

For any w' obtained by 'deleting' some of the s_i occurring in w, $\bar{X}(w')$ defines in a natural way a closed subvariety of $\bar{X}(w)$. In particular there are divisors

$$D_j = \bar{X}(s_{i_1}, \dots, \hat{s}_{i_j}, \dots, s_{i_n}) \qquad ; \ j = 0, \dots, n.$$

We shall now and then refer to these as *boundary divisors* (their union $\partial\bar{X}(w)$ is the complement of $X(w)$ in $\bar{X}(w)$).

We have the diagram,

$$\begin{tikzcd}
\bar{X}(w) \arrow[rrr, "i'", "\text{reg. emb.}"'] \arrow[dr, "\varphi'"] \arrow[ddr, bend right, "j'"'] & & & \bar{O}(w) \arrow[dl, "\varphi"'] \arrow[ddl, bend left, "j"] \\
& \overline{X(w)} \arrow[r] \arrow[d] & \overline{O(w)} \arrow[d] & \\
& X \arrow[r, "i=(\mathrm{id},F)"] & X \times X &
\end{tikzcd} \tag{1.15}$$

with φ' being an isomorphism over $\overline{X(w)} - \partial\overline{X(w)} = X(w)$ and the inverse image $\varphi'^{-1}(\partial\overline{X(w)}) = \partial\bar{X}(w)$ is a divisor with normal crossings. From [**Har77**, Proposition II.7.17] it follows that the morphisms $\varphi : \bar{O}(w) \to \overline{O(w)}$ and $\varphi' : \bar{X}(w) \to \overline{X(w)}$ may be considered as blow-ups of some coherent sheaves of ideals.

For later use we state:

LEMMA 1.9. *Let $w \in W$ have reduced the expression $w = s_{i_1} \cdot \ldots \cdot s_{i_n}$ with all i_j different. Then the resolution $\varphi : \bar{O}(w) \to \overline{O(w)}$ is bijective. Hence also the resolution $\varphi' : \bar{X}(w) \to \overline{X(w)}$ is bijective.*

PROOF. We prove the claim by induction on $l(w)$. For $l(w) = 0$ the claim is trivially true. So assume $l(w) > 0$. Since φ is an isomorphism over $O(w)$ we only need to show that φ is bijective on any $\overline{O(w')}$ with $w' \leq w$ and $l(w') = l(w) - 1$. Now, any such w' is obtained from the reduced expression of w by omitting a unique s_{i_j}. That is, there exists a unique index j such that

$$w' = s_{i_1} \cdot \ldots \cdot \hat{s}_{i_j} \cdot \ldots \cdot s_{i_n}$$

and this is a reduced expression of w' (by the special property of w). But then w' satisfies the induction hypothesis and the assertion follows. □

The reason why we have to assume that all the s_i are different is illustrated by the following example.

EXAMPLE 1.10. Let (G, F) be of type ${}^2A_3$ and let $w = s_1 s_2 s_1$. Then

$$\overline{O(w)} = O(e) \cup O(s_1) \cup O(s_2) \cup O(s_1 s_2) \cup O(s_2 s_1) \cup O(s_1 s_2 s_1).$$

The desingularisation $\bar{O}(s_1, s_2, s_1)$ has a similar decomposition

$$\begin{aligned}\bar{O}(s_1, s_2, s_1) = {} & O(e,e,e) \cup O(s_1,e,e) \cup O(e,s_2,e) \cup O(e,e,s_1) \\ & \cup O(s_1,s_2,e) \cup O(e,s_2,s_1) \cup O(s_1,e,s_1) \cup O(s_1,s_2,s_1).\end{aligned}$$

Under the projection φ taking (g_aB, g_bB, g_cB, g_dB) to (g_aB, g_dB), the strata of $\bar{O}(s_1, s_2, s_1)$ is mapped as follows:

$$\begin{tikzcd}
O(s_1,e,s_1) \arrow[dr, "\mathbb{A}^1\text{-fibration}"'] & O(e,e,e) \arrow[d, "\simeq"'] & O(s_1,s_2,e) \arrow[d, "\simeq"'] & O(e,s_2,s_1) \arrow[d, "\simeq"'] \\
& O(e) & O(s_1 s_2) & O(s_2 s_1)
\end{tikzcd}$$

$$\begin{tikzcd}
O(s_1,e,e) \arrow[dr, "\simeq"'] & O(e,e,s_1) \arrow[d, "\simeq"'] & O(s_1,s_2,s_1) \arrow[d, "\simeq"'] & O(e,s_2,e) \arrow[d, "\simeq"'] \\
& O(s_1) & O(s_1 s_2 s_1) & O(s_2).
\end{tikzcd}$$

Clearly, φ is not bijective in this case.

1.5. Irreducibility

Recall the induced action of F on $S \subseteq W$. Let S_F denote the set of orbits for this action and set $r = \#S_F$. We call r the *rank* of (G, F).

DEFINITION 1.11. If $w = s_1 \cdot \ldots \cdot s_r$ is a reduced expression for w with exactly one s_i from each of the orbits in S_F, we shall call w a *Coxeter element* [**Lus76a**, (1.7)]. When G is semi-simple with connected Dynkin diagram, there is a natural choice of Coxeter element: let $w = s_{i_1} \cdot \ldots \cdot s_{i_n}$ where $i_1 = 1$ and any pair i_j, i_{j+1} corresponds to two joined nodes. When choosing this Coxeter element, we shall refer to $\bar{X}(w)$ as being the *standard* Deligne-Lusztig variety (corresponding to (G, F)).

Deligne-Lusztig varieties are not in general irreducible contrary to the case of Schubert varieties. From [**Lus76a**, (4.8) Proposition] we have:

PROPOSITION 1.12. *$X(w)$ is irreducible if w contains at least one s_i from each orbit in S_F. In particular, if w is a Coxeter element, $X(w)$ is irreducible.*

REMARK 1.13. Let us analyze what happens when w is not necessarily a Coxeter element. Let w have reduced expression $w = s_{i_1} \cdot \ldots \cdot s_{i_n}$. Let P be the parabolic subgroup determined by these simple reflections (P is generated by B and all $BF^m(s_{i_j})B$, $m = 1, \ldots, \delta$, $j = 1, \ldots, n$). In other words, P is the smallest F-stable parabolic subgroup of G containing the cosets $B, B\dot{s}_{i_1}B, \ldots, B\dot{s}_{i_n}B$.

Let $\mathcal{P}$ denote the (finite) set of all G^F-conjugates of P. For each $P' \in \mathcal{P}$ we let $w(P')$ denote the image of w in the Weyl group of $P'/U_{P'}$ ($U_{P'}$ is the unipotent radical of P'). By [**Lus76a**, (1.17)] we then have the following decomposition of $X(w)$,

$$X(w) \xrightarrow{\sim} \coprod_{P' \in \mathcal{P}} X(w(P')) \tag{1.16}$$

where $X(w(P'))$ is a Deligne-Lusztig variety consisting of Borel subgroups of $G' = P/U_P$. $X(w(P'))$ may be identified with the closed subset of $X(w)$ consisting of the Borel subgroups contained in P'. Hence,

$$X(w) \xrightarrow{\sim} \coprod_{P' \in \mathcal{P}} L^{-1}(B\dot{w}B) \cap P'/B. \tag{1.17}$$

PROPOSITION 1.14. *The decomposition* (1.16) *is valid also for $\overline{X(w)}$ and $\bar{X}(w)$. That is,*

$$\overline{X(w)} \xrightarrow{\sim} \coprod_{P' \in \mathcal{P}} \overline{X(w(P'))} \tag{1.18}$$

and similarly for $\bar{X}(w)$.

PROOF. By the above,

$$\overline{X(w)} \xrightarrow{\sim} \overline{\coprod_{P' \in \mathcal{P}} L^{-1}(B\dot{w}B) \cap P'/B} = \bigcup_{P' \in \mathcal{P}} \overline{L^{-1}(B\dot{w}B) \cap P'/B}.$$

Since each P' is a closed subvariety of G,

$$\overline{L^{-1}(B\dot{w}B)\cap P'}/B = \overline{L^{-1}(B\dot{w}B)}\cap P'/B,$$

hence the union is disjoint (since the P''s are). For the desingularisation $\bar{X}(w)$ we have that $\bar{X}(w) = \bigcup_{P'\in\mathcal{P}} \bar{X}(w(P'))$. Since $\bar{X}(w)$ is non-singular the components cannot intersect. ☐

REMARK 1.15. We note that, for any $w \in W$, G^F acts transitively on the components of $X(w)$, $\overline{X(w)}$ and $\bar{X}(w)$.

The Dynkin diagram of P/U_P will be that part of the Dynkin diagram of G that is spanned by the nodes corresponding to the reflections $s_{i_1},\dots,s_{i_n}$ plus their iterated images under F. Obviously, if we have (at least) one reflection from each of F's orbits, P is equal to G and the decomposition (1.16) has only one member.

If $\bar{X}(w)$ is a subvariety of a larger Deligne-Lusztig variety $\bar{X}(w')$, the composition

$$\bar{X}(w') \xrightarrow{\mathrm{pr}_1} G/B \to G/P$$

will contract the irreducible components of $\bar{X}(w)$ mapping them to the points of $(G/P)^F$ (see also Lemma 3.9). It should be noted that the number $|G^F/P^F|$ of G^F-conjugates to P is rather easily calculated (see [**Lus76a**, p. 106], [**Car85**, p. 106, Proposition 3.3.5]).

We also see that in determining the Chow groups of Deligne-Lusztig varieties $X(w)$ we may assume that $X(w)$ is irreducible as

$$\mathrm{A}_*\big(X(w)\big) = \bigoplus_{P\in\mathcal{P}_I} \mathrm{A}_*\big(X(w(P))\big) \tag{1.19}$$

cf. [**Ful83**, 1.3.1]. This applies of course also to $\overline{X(w)}$ and $\bar{X}(w)$.

EXAMPLE 1.16 ($^2\mathrm{A}_4$ case). Consider the reductive group $\mathrm{SL}_5(k)$ with the F-action given by the Dynkin diagram

$$\begin{tikzcd} s_1 \arrow[r, dash] \arrow[rrr, bend left, leftrightarrow] & s_2 \arrow[r, dash] \arrow[r, bend left, leftrightarrow] & s_3 \arrow[r, dash] & s_4 \end{tikzcd}.$$

Then $\bar{X}(s_1s_2)$ is an irreducible Deligne-Lusztig variety. Let us determine what its Deligne-Lusztig subvarieties look like.

First consider $\bar{X}(s_1)$. The single reflection s_1 determines the F-stable parabolic subgroup $P_1 = \langle B, B\dot{s}_1B, B\dot{s}_4B\rangle$. Let $\mathcal{P}_1$ be the corresponding set of G^F-conjugacy classes. By [**Car85**, Chapter 2] we have

$$|\mathcal{P}_1| = |G^F/P_1^F| = (q^5+1)(q^3+1).$$

P_1/U_{P_1} is a reductive group of type $\mathrm{SL}_2(k)\times\mathrm{SL}_2(k)$ with F-action interchanging the two factors. By [**Lus76a**, (1.18)] it follows that for each $P'\in\mathcal{P}_1$, $\bar{X}(s_1(P'))$ is identified[1] with an irreducible Deligne-Lusztig variety in $\mathrm{SL}_2(k)/B \simeq \mathbb{P}^1$. So $\bar{X}(s_1)$ is a disjoint union of $(q^5+1)(q^3+1)$ rational curves.

[1] $(B_1,B_2)\in\mathrm{SL}_2(k)\times\mathrm{SL}_2(k)$ is in $\bar{X}(s_1(P'))$ if and only if $(B_1,F^2(B_1))$ is in the $\mathrm{SL}_2(k)$-orbit of $(eB,\dot{w}B)$

As for $\bar{X}(s_2)$ we find that $P_2 = \langle B, B\dot{s}_2B, B\dot{s}_3B\rangle$ determines

$$|\mathcal{P}_2| = (q^5+1)(q^2+1)$$

conjugacy classes.
P_2/U_{P_2} is a reductive group with Dynkin diagram $s_1 - s_2$. Hence the components of $\bar{X}(s_2)$ are isomorphic to the Deligne-Lusztig variety corresponding to the ${}^2\mathrm{A}_2$ case. But this is a plane Hermitian curve (see [**Han92**]) with equation $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$. We conclude that $\bar{X}(s_2)$ is a disjoint union of $(q^5+1)(q^2+1)$ Hermitian curves.

DEFINITION 1.17. Introduce the following notation:

$$\mathfrak{I} = \left\{ i \;:\; \begin{array}{c} \text{some connected component of the Dynkin} \\ \text{diagram corresponding to } D_i \text{ occurs as a subgraph} \\ \text{of the Dynkin diagram corresponding to } D_1 \end{array} \right\}.$$

REMARK 1.18. The motivation for defining $\mathfrak{I}$ is the following: Suppose the subgraph of the Dynkin diagram defined by a boundary divisor D consists of the components $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ (since we only ‘remove’ δ nodes we can only cut $\mathcal{D}$ into 3 pieces, at the most). Now, if e.g. $\mathcal{D}_2$ is a subgraph of the Dynkin diagram defined by D_1, this means geometrically that D is the direct product of the Deligne-Lusztig subvariety $D \cap D_1$ of D_1, with the other Deligne-Lusztig varieties corresponding to the diagrams $\mathcal{D}_1$ and $\mathcal{D}_3$. So, in particular, if D_1 is contracted to points, then also D_i drops in dimension for all $i \in \mathfrak{I}$.

Some examples of how the index set $\mathfrak{I}$ looks like, are listed in Table 1.1.

type of $\bar{X}(w)$	$\mathfrak{I}$ ($n \geq 2$)
A_n	$\{n-1\}$
${}^2\mathrm{A}_{2n-1}$	$\{1,2,\ldots,n-1\}$
${}^2\mathrm{A}_{2n}$	$\{1,2,\ldots,n-1\}$
${}^2\mathrm{D}_n$	$\{1,2,\ldots,n-3\}$

TABLE 1.1. The set $\mathfrak{I}$ for some standard Deligne-Lusztig varieties.

1.6. Various properties of Deligne-Lusztig varieties

1.6.1. Affinity of Deligne-Lusztig varieties. Let $U \subset G$ be the unipotent radical of B and let U^F be the fixed points under F. From [**Lus76a**, (2.7) Corollary] it follows that when w is a Coxeter element, U^F acts freely on $X(w)$ with quotient isomorphic to a torus $(\mathbb{A}^1 - \{0\})^{l(w)}$. Since this is an affine open subset of $\mathbb{A}^n$ it follows that also $X(w)$ is affine. Actually all Deligne-Lusztig varieties $X(w)$ are quasi-affine [**Haa86**].

1.6.2. The Euler characteristic of Deligne-Lusztig varieties. Let $\chi(X)$ denote the topological Euler characteristic of a variety X (the alternating sum of the Betti numbers of X; equal to the degree of the top Chern class of $\mathcal{T}_X$ when X is complete). From [**DL76**, Theorem 7.1] we then have:

PROPOSITION 1.19. *Let $X(w)$ be a Deligne-Lusztig variety. Choose (by the surjectivity of the Lang map) $g \in G$ such that $g^{-1}F(g) = \dot{w}$. Then*

$$\chi(X(w)) = (-1)^{\sigma(G)-\sigma({}^gT^F)} \frac{|G^F|}{\mathrm{St}_G(e)|{}^gT^F|}. \tag{1.20}$$

REMARK 1.20. Some explanation of (1.20) is in place. Let $P_f(x, V)$ denote the characteristic polynomial of a linear endomorphism $f : V \to V$ of a vector space V. Then

$$\sigma(G) = \dim\big(Y_0 \otimes \mathbb{R}\big)^{F_0} \tag{1.21}$$

$$\sigma({}^gT) = \dim\big(Y_0 \otimes \mathbb{R}\big)^{F_0 w^{-1}} \tag{1.22}$$

$$|{}^gT^F| = P_{F_0^{-1}w}(Q, Y_0) = |\det{}_{Y_0\otimes\mathbb{R}}(w^{-1} \circ F - \mathrm{id})| \tag{1.23}$$

cf. [**Car85**, p. 197+86]. $\mathrm{St}_G(\cdot)$ is the character of the Steinberg representation and $\mathrm{St}_G(e)$ equals q^M (recall that M denotes the number of positive roots).

Note the difference between the *topological Euler characteristic χ of X* and the *Euler characteristic of a vector bundle (or coherent sheaf) E on X*, $\chi(X, E) = \sum_i (-1)^i \dim_k \mathrm{H}^i(X, E)$. It follows from the *Hirzebruch-Riemann-Roch theorem* [**Ful83**, Corollary 15.2.1] that if X is non-singular and complete,

$$\chi(X, E) = \int_X ch(E) \cap td(\mathcal{T}_X).$$

In particular $\chi(X, \mathcal{O}_X) = \int_X td_{\dim(X)}(\mathcal{T}_X)$. If X is a non-singular complete curve of genus g (that is, $2 - 2g = \int_X c_1(\mathcal{T}_X)$) then $\chi(X, \mathcal{O}_X) = \frac{1}{2}\int_X c_1(\mathcal{T}_X) = g - 1$. For a surface we have

$$\chi(X, \mathcal{O}_X) = 1 - q + p_g(X) = \frac{1}{12}\big(K_X^2 + \chi\big)$$

where $q = \dim \mathrm{H}^1(X, \mathcal{O}_X)$.

Since χ is given by a Lefschetz trace formula it is additive with respect to any given stratification of X; that is, if X is a finite union of disjoint locally closed subvarieties X_i then $\chi(X) = \sum_i \chi(X_i)$ *cf.* [**Lus78**, (1.2.2)] (a formula of the same kind holds for $\chi(X, \mathcal{O}_X)$ *cf.* [**Ful83**, Example 15.2.10]). Using this additivity property we may calculate the topological Euler characteristic of $\overline{X}(w)$ by summing over all $\chi(X(w'))$, $w' \leq w$. Since the calculations are mostly linear algebra this is easily done with the aid of computer software like for example Maple. In Section C.1 we have done the calculations for the surface cases; the results are listed in Table C.3 (page 107).

CHAPTER 2

Singularities of Deligne-Lusztig varieties

As we have mentioned earlier, Deligne-Lusztig varieties may be singular. In this chapter we will investigate the fact that $\overline{X(w)}$ is the transverse intersection of Γ and $\overline{O(w)}$ inside $X \times X$ and therefore is smooth whenever $\overline{O(w)}$ is. (By transverse we mean that for every point P of a dense open subset of the intersection, the cotangent space $Cot_P(X \times X)$ is the direct sum of $Cot_P(X)$ and $Cot_P(\overline{O(w)})$.)

2.1. Normality and Cohen-Macaulayness

Being 'nicely constructed' subvarieties of the normal and Cohen-Macaulay varieties $\overline{O(w)}$ (see [**MR88**, Corollaries 1+2]), it is natural to ask whether also Deligne-Lusztig varieties also are normal and Cohen-Macaulay. Below we will see that in both cases the answers are affirmative.

LEMMA 2.1. *Let X be an irreducible variety regularly embedded in a variety Y (of codimension d). Let V be a closed subvariety of Y. Suppose the scheme-theoretic intersection is reduced and that $X \cdot V = [Z]$ in $\mathrm{A}_*(Z)$ ($i(Z, V \cdot X, Y) = 1$, Z irreducible). Then Z is regularly embedded in V (of codimension d).*

PROOF. Let $U \subseteq V$ be an open affine neighborhood of Z; $U = \operatorname{Spec} A$, $U \cap Z = \operatorname{Spec} A/P$ where P is a prime ideal of A (Z is irreducible and reduced). We must show that P can be generated by a regular sequence in A of length d [**Ful83**, B.7.1]. Since the multiplicity of Z in $X \cdot V$ is one, A_P is a regular local ring [**Ful83**, Proposition 7.2] of dimension d. So PA_P is generated by a regular sequence $\frac{x_1}{y_1}, \dots, \frac{x_d}{y_d}$ in A_P. Since A is a domain we may clear the denominators to get generators $x_1, \dots, x_d$ of $P \subseteq A$. Now for any i, x_i cannot be a zero-divisor on $A/(x_1 \dots, x_{i-1})$ since this then should be the case in all localisations (including localising in P). Hence $x_1, \dots, x_d$ is a regular sequence in A. ☐

PROPOSITION 2.2. *Let $\overline{X(w)}$ be an irreducible Deligne-Lusztig variety. Then $\overline{X(w)}$ is normal and Cohen-Macaulay. Hence* any *Deligne-Lusztig variety is normal and Cohen-Macaulay (*cf. *Proposition 1.14).*

PROOF. First we will see that $\overline{X(w)}$ is regular in codimension 1, so let x be a point such that $\overline{\{x\}}$ is singular. Let Y be the closure in $\overline{O(w)}$ of the orbit of x under G. Since $\overline{O(w)}$ is normal (hence regular in codimension 1) and Y is singular, we must have $\mathrm{codim}(Y) > 1$. Being the closure of a G-orbit, Y intersects Γ transversely [**DL76**, p. 151]. So $\mathrm{codim}(\Gamma \cap Y) > 1$ in $\overline{X(w)}$. Since $\overline{\{x\}} \subseteq \Gamma \cap Y$ it follows that $\overline{X(w)}$ is regular in codimension 1.[1]

It follows from Lemma 2.1 that $\overline{X(w)}$ is regularly embedded in $\overline{O(w)}$. Hence locally $\overline{X(w)}$ is given by $\operatorname{Spec} A/P$ where A is Cohen-Macaulay (since $\overline{O(w)}$ is Cohen-Macaulay) and P is generated by a regular sequence in A. But then also A/P is Cohen-Macaulay [**Har77**, Proposition II.8.21A (d)]. Then normality follows from Serre's criterion [**Har77**, Theorem II.8.22A]. $\square$

2.2. Non-singularity

In some cases Deligne-Lusztig varieties are actually non-singular. The following result was claimed without proof in a remark of [**Sri87**]. We have not seen the proof anywhere else.

THEOREM 2.3. *Let $\overline{X(w)}$ be a Deligne-Lusztig variety. If w at the most contains one s_i from each of F orbits, then $\overline{X(w)}$ is a disjoint union of its* non-singular *components. In particular, if w is a Coxeter element, then $\overline{X(w)}$ is irreducible and isomorphic to $\bar{X}(w)$ (hence non-singular).*

PROOF. From Proposition 1.14 it follows that we only have to show the last claim. When w is a Coxeter elements it follows from Lemma 1.9 that the desingularisation $\varphi' : \bar{X}(w) \to \overline{X(w)}$ is a surjective bijective morphism of projective varieties. By [**Iit82**, Theorem 2.24] φ' then has to be the normalisation of $\overline{X(w)}$ ($\bar{X}(w)$ is non-singular). But $\overline{X(w)}$ is normal by Proposition 2.2 and φ' is forced to be an isomorphism (by the uniqueness of normalisation). $\square$

REMARK 2.4. For the classical groups, it has been determined exactly when a Schubert variety X_w is non-singular [**Lak95**]. A sufficient criterion, at least in the A_n-case, is that w at the most contains each s_i once.

There are also some other (few) cases where the X_w are non-singular (thereby implying the non-singularity of $\overline{X(w)}$). For example, suppose $\overline{B\dot{w}B}$ is a parabolic subgroup P. Then X_w is a homogeneous space for P, hence non-singular (see [**Jan87**, p. 384-385]).

[1]Equivalently, $O(w) \cup \bigcup_{i=1}^{n} O(s_1 \cdot \ldots \cdot \hat{s}_i \cdot \ldots \cdot s_n)$ is non-singular by normality of $\overline{O(w)}$ (the singular locus is closed and G-stable, hence it has to be a union of orbits $O(w')$ of G). Hence so is $X(w) \cup \bigcup_{i=1}^{n} X(s_1 \cdot \ldots \cdot \hat{s}_i \cdot \ldots \cdot s_n)$ (by transversality) and $\overline{X(w)}$ is regular in codimension one.

CHAPTER 3

Intersection theory of Deligne-Lusztig varieties

3.1. Chow groups of $X(w)$

One of the original goals of this dissertation was to give a complete description of the Chow groups of Deligne-Lusztig varieties (for the general theory of Chow groups, we refer to [**Ful83**]). Below we shall see how the free part ought to behave.

3.1.1. Rephrasing the problem. Let $X(w)$ be an irreducible Deligne-Lusztig variety. We will relate the rational Chow groups of $X(w)$ to those of a quasi-projective variety $\tilde{X}(\dot{w})$ on which a finite group acts with quotient $X(w)$. $\tilde{X}(\dot{w})$ will be the base-space of a locally trivial fibre-bundle $\tilde{S}(\dot{w})$ with fibre a unipotent group (isomorphic to some affine space). Hence we may alternatively determine the Chow groups of $\tilde{S}(\dot{w})$.

Consider the unipotent subgroup $U_w = U \cap \dot{w}U\dot{w}^{-1}$ of G. As a variety,

$$U_w \simeq \mathbb{A}^{M-l(w)}.$$

Let $\tilde{S}(\dot{w}) = L^{-1}(\dot{w}U)$. Then $S(\dot{w})$ is an étale covering of $\dot{w}U$.[1] U_w acts on $\tilde{S}(\dot{w})$ by right translation: $L(su) = (su)^{-1}F(su) = u^{-1}L(s)F(u)$, which is in $\dot{w}U$ for $s \in \tilde{S}(\dot{w}), u \in U_w$. Consider the composition,

$$f : \tilde{S}(\dot{w}) \xrightarrow{L} \dot{w}U \xrightarrow{\dot{w}^{-1}} U.$$

Let U_w act on U from the right by the rule

$$s.u = \dot{w}^{-1}u\dot{w}sF(u^{-1})$$

for $s \in U$, $u \in U_w$ (see [**DL76**, 1.12]). Then, for $g \in \tilde{S}(\dot{w})$, $u \in U_w$,

$$\begin{aligned} f(g.u) &= \dot{w}^{-1}L(g.u) = \dot{w}^{-1}u^{-1}L(g)F(u) \\ &= \dot{w}^{-1}u^{-1}\dot{w}(\dot{w}^{-1}L(g))F(u) \\ &= (\dot{w}^{-1}L(g)).u^{-1} = f(g).u^{-1}. \end{aligned}$$

Hence, if $g \in \tilde{S}(\dot{w})$ is fixed by u then $f(g)$ is fixed by u^{-1}.

[1]Note that is is only in positive characteristic one finds non-trivial étale coverings of affine space. In characteristic zero affine space is simply connected (which by definition means that it has no non-trivial étale coverings; a covering $Y \to X$ with group H is trivial if $Y = X \times_k H$).

PROPOSITION 3.1. *The quotient $\tilde{S}(\dot{w})/U_w := \tilde{X}(\dot{w})$ exists and the canonical morphism $\Theta : S(\dot{w}) \to \tilde{X}(\dot{w})$ is locally trivial with fibre U_w.*

PROOF. Consider the quotient $q : G \to G/U_w$ (U_w is closed in G, hence the quotient exists). Since U_w is unipotent (hence solvable) q is locally trivial with fibre U_w [**Ser58**, Proposition 14]. Restricting to the locally closed subvariety $\tilde{S}(\dot{w})$ we get that $\Theta : S(\dot{w}) \to \tilde{X}(\dot{w})$ is locally trivial with fibre U_w. (A shorter proof: use Corollary A.18.) □

Since the fibres of Θ are of constant dimension $M - l(w)$, Θ is flat [**Har77**, Ex.III.10.9]. Let $\tilde{X}(\dot{w})$ denote the quotient of $\tilde{S}(\dot{w})$ by U_w.

THEOREM 3.2. *Let $X(w)$ be an irreducible Deligne-Lusztig variety. Then*

$$\mathrm{A}_k(X(w))_{\mathbb{Q}} = \mathrm{A}_k\left(\tilde{X}(\dot{w})\right)^{T_w^F}_{\mathbb{Q}} \hookrightarrow \mathrm{A}_{k+M-l(w)}\left(\tilde{S}(\dot{w}U)\right) \tag{3.1}$$

for $k \neq l(w)$ and $\mathrm{A}_{l(w)}\big(X(w)\big) = \mathbb{Z}$.

PROOF. The last assertion is obvious. For the first we will use that for

$$T_w^F = \{t \in T : F(t) = \dot{w}^{-1} t \dot{w}\}$$

we have $\tilde{X}(\dot{w})/T_w^F = X(w)$ (this follows from [**DL76**, 1.8+1.11]). Hence, by [**Ful83**, 1.7.6], $\mathrm{A}_k\big(X(w)\big)_{\mathbb{Q}} \simeq \mathrm{A}_k\left(\tilde{X}(\dot{w})\right)^{T_w^F}_{\mathbb{Q}}$. Then the assertion follows from Corollary A.18. □

REMARK 3.3. We have now reformulated the problem of calculating the Chow groups of $X(w)$ into that of calculating those of $\tilde{S}(\dot{w}U)$. Unfortunately we lost track of the torsion elements on our way.

3.1.2. A straight-forward case. In the simplest case we may attack the problem directly.

THEOREM 3.4. *Let $X(w)$ be a standard Deligne-Lusztig variety corresponding to the A_n case. Then*

$$\mathrm{A}_k(X(w))_{\mathbb{Q}} = 0 \tag{3.2}$$

unless $k = l(w)$ (in which case $\mathrm{A}_k(X(w)) = \mathbb{Z}$). Furthermore, for any variety Y, we have $\mathrm{A}_*(X(w) \times Y) \simeq A_*(X(w)) \otimes \mathrm{A}_*(Y) = \mathrm{A}_*(Y)$.

PROOF. [2] In this case, $X(w)$ is identified with an open affine subset of $\mathbb{P}^{l(w)}$ with complement D equal to the union of all $\mathbb{F}_q$-rational hyper-planes (see [**DL76**, 2.2]). Hence we have the short exact sequence [**Ful83**, Proposition 1.8],

$$\mathrm{A}_k(D) \to \mathrm{A}_k(\mathbb{P}^{l(w)}) \to \mathrm{A}_k(X(w)) \to 0.$$

Since D is projective, $\mathbb{Z} \subseteq \mathrm{A}_k(D)$ for $k < l(w)$, hence the rank of $\mathrm{A}_k(X(w))$ is necessarily 0. (See also [**Ful83**, Examples 1.9.3+4].)

[2]This proof could probably be replaced by one using higher Chow groups and the five-lemma.

For the last assertion we consider the commutative diagram:

$$\begin{tikzcd}
\mathrm{A}_*(D\times Y) \arrow[r, "p"] & \mathrm{A}_*(\mathbb{P}^{l(w)}\times Y) \arrow[r, "q"] & \mathrm{A}_*(X(w)\times Y) \arrow[r] & 0 \\
\mathrm{A}_*(D)\otimes \mathrm{A}_*(Y) \arrow[r, "\bar{p}"] \arrow[u, "\varphi_1"] & \mathrm{A}_*(\mathbb{P}^{l(w)})\otimes \mathrm{A}_*(Y) \arrow[r, "\bar{q}"] \arrow[u, "\varphi_2"] & \mathrm{A}_*(X(w))\otimes \mathrm{A}_*(Y) \arrow[r] \arrow[u, "\varphi_3"] & 0.
\end{tikzcd}$$

By [**Ful83**, Example 8.3.7], φ_2 is an isomorphism. Since D is a union of hyperplanes (each isomorphic to $\mathbb{P}^{l(w)-1}$) we conclude that φ_1 also is an isomorphism. Since q, φ_2 and $\bar{q}$ are surjective, commutativity of the diagram forces φ_3 to be surjective as well. Suppose $\varphi_3(\beta)=0$. Choose $\gamma\in \mathrm{A}_*(\mathbb{P}^{l(w)})\otimes \mathrm{A}_*(Y)$ such that $\bar{q}(\gamma)=\beta$. Then $q\varphi_2(\gamma)=0$, hence $\varphi_2(\gamma)=p(\delta)$ for some $\delta\in \mathrm{A}_*(D\times Y)$. But then $\bar{p}\varphi_1^{-1}(\delta)=\varphi_2^{-1}p(\delta)=\gamma$, hence $\beta=\bar{q}\bar{p}\varphi_1^{-1}(\delta)$ and $\beta=0$. ☐

LEMMA 3.5. *Suppose w and w' are two different Coxeter elements in W. Let $\bar{X}(w)$ and $\bar{X}(w')$ be the corresponding Deligne-Lusztig varieties. Then $\mathrm{A}_i(\bar{X}(w))_{\mathbb{Q}}\simeq \mathrm{A}_i(\bar{X}(w'))_{\mathbb{Q}}$ for all i.*

PROOF. Let us first consider the case where $w'=F(w)$. Since the automorphism $F^\delta:\bar{X}(w)\to\bar{X}(w)$ induces multiplication by a power of Q on $\mathrm{A}_i(\bar{X}(w))$ [**Ful83**, Example 1.7.4], each of the homomorphisms in the composite (we have $\delta=2$ since $F(w)=w'\neq w$)

$$\mathrm{A}_i(\bar{X}(w)) \xrightarrow{F_*} \mathrm{A}_i(\bar{X}(w')) \xrightarrow{F_*} \mathrm{A}_i(\bar{X}(w))$$

must be isomorphisms modulo torsion.

By [**Lus76a**, (1.8) Lemma], the only other cases we need to consider are those where w is on the form $w=w_1w_2$ and then $w'=w_2F(w_1)$. The proof now follows the lines of the proof of [**DL76**, Theorem 1.6, case 1]. For any $P=(B_0,B_1,\dots,B_{l(w_1)},\dots,F(B_0))\in\bar{X}(w)$ we have that $(B_0,B_1,\dots,B_{l(w_1)})\in\bar{O}(w_1)$ and $(B_{l(w_1)},\dots,F(B_0))\in\bar{O}(w_2)$. But then

$$\sigma(P):=(B_{l(w_1)},\dots,F(B_0),F(B_1),\dots,F(B_{l(w_1)}))\in\bar{X}(w')$$

giving a morphism $\sigma:\bar{X}(w)\to\bar{X}(w')$. In exactly the same way, we get a morphism $\tau:\bar{X}(w')\to\bar{X}(F(w))$. It follows that $F=\tau\circ\sigma$. Arguing as in the special case, it follows that $\tau_*:\mathrm{A}_i\big(\bar{X}(w')\big)_{\mathbb{Q}}\to \mathrm{A}_i\big(\bar{X}(F(w))\big)_{\mathbb{Q}}$ must be surjective. The assertion now follows by symmetry. ☐

REMARK 3.6. Since Lusztig have shown that Deligne-Lusztig varieties coming from F-conjugate Coxeter elements have the same number of rational points [**Lus76a**, (1.10) Proposition], hence the same Zeta-function and Betti-numbers, the lemma is only a natural parallel.

THEOREM 3.7. *Let $\bar{X}(w)$ a Deligne-Lusztig variety of type A_n. Let $j:D\to\bar{X}(w)$ be the inclusion of the boundary divisors. Then $\mathrm{A}_{l(w)}(\bar{X}(w))=\mathbb{Z}$ and $\mathrm{A}_k(\bar{X}(w))_Q=j_*\,\mathrm{A}_k(D)_{\mathbb{Q}}$ for $k<l(w)$.*

PROOF. By Lemma 3.5 we may assume $\bar{X}(w)$ is of standard type. Then it follows from Remark 3.8 and Theorem 3.4 that $\bar{X}(w)$ has a stratification satisfying Lemma A.2. □

Based on the known examples and the above we do not hesitate to claim the following:

CONJECTURE 1. *Let $X(w)$ be a Deligne-Lusztig variety. Then $\mathrm{A}_i(X(w))$ has rank zero for $i < l(w)$.*

Granted this conjecture, it would follow from Lemma A.2 that for all i, $\mathrm{A}_i(\bar{X}(w))_{\mathbb{Q}}$ is generated by the classes of the components of Deligne-Lusztig subvarieties $\bar{X}(w')$, where $w' \leq w$ and $l(w) = i$. In codimension one, this would agree with the results of Section 3.2 (below).

3.1.3. The G^F-invariant Chow groups. Let $\bar{X}(w)$ be of standard type. As we noted in Section 1.6.1 (page 11) there is a (finite) subgroup U^F of G^F acting on $X(w)$, with quotient $X(w)/U^F$ isomorphic to an open subset of a torus. Since the Chow groups of affine space vanish in positive codimension [**Ful83**, p. 23] the same is true for tori and therefore also for the quotient variety $X(w)/U^F$ [**Ful83**, Proposition 1.8]. Since there is a finite surjective morphism $X(w)/U^F \to X(w)/G^F$ (inducing a surjection in Chow groups with rational coefficients) it follows that the G^F-invariant Chow groups of $X(w)$ satisfies

$$\mathrm{A}_i(X(w))_{\mathbb{Q}}^{G^F} = 0 \qquad \text{for } i < l(w) \tag{3.3}$$

(see [**Ful83**, Example 1.7.6]). So the conjecture stated above holds at the least for the G^F-invariant part.

3.2. Picard Groups of 'classical' Deligne-Lusztig varieties

Say that $\bar{X}(w)$ is *of classical type* if $\bar{X}(w)$ corresponds to one of the classical groups: A_n, ${}^2\mathrm{A}_n$, B_n, C_n, D_n or ${}^2\mathrm{D}_n$.

REMARK 3.8. Suppose $\bar{X}(w)$ is of type A_n. Let $w' \leq w$. Then each irreducible component of $\bar{X}(w')$ is a product of Deligne-Lusztig varieties also of type A_n. For example: In $\bar{X}(s_1s_2s_3)$, the divisors D_1 and D_3 are disjoint unions of components of type A_2 and D_2 is a disjoint union of components of type $\mathrm{A}_1 \times \mathrm{A}_1$.

Similarly, when $\bar{X}(w)$ is of type ${}^2\mathrm{A}_n$, the divisor D_i is a disjoint union of Deligne-Lusztig varieties of type $\mathrm{A}_{i-1} \times {}^2\mathrm{A}_{n-2}$. The same remarks apply to any other Deligne-Lusztig variety of classical type. That is, if $\bar{X}(w)$ is of classical type, then so are the irreducible components of the divisors D_i.

In the preceding section we were able to give generators for the free part of all $\mathrm{A}_i(\bar{X}(w))$ whenever $\bar{X}(w)$ is a Deligne-Lusztig variety of type A_n. In this section we shall examine the ${}^2\mathrm{A}_n$, B_n, C_n, D_n and ${}^2\mathrm{D}_n$ cases. We follow the notation of [**DL76**, (2.1)] and [**Lus76b**].

Let V be an N-dimensional vector space ($N \geq 2$) over k equipped with a Frobenius morphism $F_V : V \to V$. Assume furthermore that V comes equipped with a form of one of the following kinds:

(O): Let $char(k) \neq 2$ and let $Q : V \to k$ be a non-singular quadratic form defined over $\mathbb{F}_q$ That is, $Q(F_V(x)) = Q(x)^q$ for any $x \in V$. Define the inner product

$$\langle x, y \rangle_{\mathrm{O}} = Q(x+y) - Q(x) - Q(y)$$

on V. For N even, we will distinguish between the cases where Q is *split* and *non-split* (Q is split if F_V leaves stable some subspace $V' \subseteq V$ satisfying that $V' \subseteq V'^{\perp}$ and $Q|_{V'} = 0$ and that V' is maximal with property).

To be able to do explicit calculations, we fix a standard basis for V and let $Q(x)$ be defined as follows (with respect to the chosen basis):

$$Q(x) = \begin{cases} \sum_{i=1}^{n} x_i x_{i+n} + \sum_{i=n+1}^{2n} x_i x_{i-n} & N = 2n \\ x_N^2 + \sum_{i=1}^{n} x_i x_{i+n} + \sum_{i=n+1}^{2n} x_i x_{i-n} & N = 2n+1. \end{cases}$$

With this choice, F_V acts as follows:

$$F_V(x) = \begin{cases} (x_{n+1}^q, \ldots, x_N^q, x_1^q, \ldots, x_n^q) & N = 2n \\ (x_1^q, \ldots, x_N^q) & N = 2n+1. \end{cases}$$

(Sp): Assume N is even, $N = 2n$. Let $\langle\ ,\ \rangle_{\mathrm{Sp}} : V \times V \to k$ be a non-singular symplectic form defined over $\mathbb{F}_q$, that is, $\langle F_V(x), F_V(y) \rangle_{\mathrm{Sp}} = \langle x, y \rangle_{\mathrm{Sp}}^q$ for any $x, y \in V$.

In the chosen basis, F_V takes $(x_1, \ldots, x_N)$ to $(x_1^q, \ldots, x_N^q)$ and we may write the form as

$$\langle x, y \rangle_{\mathrm{Sp}} = \sum_{i=1}^{n} x_i y_{i+n}^q - x_{i+n} y_i^q.$$

(U): Here our base field is $\mathbb{F}_{q^2}$. Let $\langle\ ,\ \rangle_{\mathrm{U}} : V \times V \to k$ be a non-singular sesquilinear form with respect to the automorphism $\lambda \mapsto \lambda^q$ of k. That is, $\langle \lambda x, y \rangle_{\mathrm{U}} = \lambda \langle x, y \rangle_{\mathrm{U}}$ and $\langle x, \lambda y \rangle_{\mathrm{U}} = \lambda^q \langle x, y \rangle_{\mathrm{U}}$ for $x, y \in V, \lambda \in k$. Furthermore assume that

$$\langle F_V(x), y \rangle_{\mathrm{U}} = \langle y, x \rangle_{\mathrm{U}}^q$$

for $x, y \in V$.

In the chosen basis F_V takes $(x_1, \ldots, x_N)$ to $(x_1^{q^2}, \ldots, x_N^{q^2})$ and we may write the form as

$$\langle x, y \rangle_{\mathrm{U}} = \sum_{i=1}^{n} x_i y_{i+n}^q + x_{i+n} y_i^q.$$

In the following we shall omit the subscripts indicating whether the form is symplectic, orthogonal or unitary when we wish to speak of any of these types of forms.

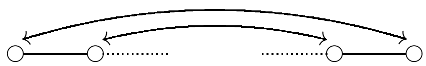
We may now give the explicit description of the classical linear algebraic groups with their Frobenius morphism $F : G \to G$. For later use we define in each case an integer a_0.

(SL): We have $G = \mathrm{SL}_N(k) = \{g \in \mathrm{GL}_N(k) : \det(g) = 1\}$. The Frobenius morphism F acts on G by raising each entry of the matrix g to the q'th power, that is, $F(g) = g \circ F_V$. The corresponding Dynkin diagram is

A_{N-1} 

($N-1$ nodes).

(U): We have $G = \mathrm{SL}_N(k)$. Let $F' : G \to G$ be defined by $\langle F'(g)x, gy\rangle_{\mathrm{U}} = \langle x, y\rangle_{\mathrm{U}}$ for any $x, y \in V$. For any $g \in G$ we have $F'^2(g) = g \circ F_V$. This gives G an $\mathbb{F}_q$-rational structure. The corresponding Dynkin diagram is

${}^2\mathrm{A}_{N-1}$ 

($N-1$ nodes). Set $a_0 = n-1$ for $N = 2n$ and $a_0 = n$ for $N = 2n+1$.

(O), $N = 2n+1$: We have

$$\begin{aligned} G &= \mathrm{SO}_N(k) \\ &= \{g \in \mathrm{GL}_N(k) : \langle g(x), g(y)\rangle_{\mathrm{O}} = \langle x, y\rangle_{\mathrm{O}} \text{ for any } x, y \in V\}. \end{aligned}$$

Let F act on G by the rule: $F(g)F_V(x) = F_V(gx)$. The corresponding Dynkin diagram is

B_n 

(n nodes, $n \geq 2$). Set $a_0 = n$.

(Sp), $N = 2n$: We have

$$\begin{aligned} G &= \mathrm{Sp}_n(k) \\ &= \{g \in \mathrm{GL}_N(k) : \langle g(x), g(y)\rangle_{\mathrm{Sp}} = \langle x, y\rangle_{\mathrm{Sp}} \text{ for any } x, y \in V\}. \end{aligned}$$

Let F act on G by the rule: $F(g)F_V(x) = F_V(gx)$. The corresponding Dynkin diagram is

C_n 

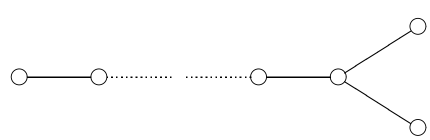
(n nodes, $n \geq 3$). Set $a_0 = n$.

(O), $N = 2n$, Q split: We have

$$\begin{aligned} G &= \mathrm{SO}_N(k) \\ &= \{g \in \mathrm{SL}_N(k) : \langle g(x), g(y)\rangle_{\mathrm{O}} = \langle x, y\rangle_{\mathrm{O}} \text{ for any } x, y \in V\}. \end{aligned}$$

Let F act on G by the rule: $F(g)F_V(x) = F_V(gx)$. The corresponding Dynkin diagram is

D_n



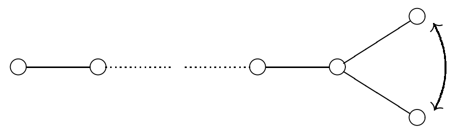
(n nodes, $n \geq 4$). Set $a_0 = n-1$.

(O), $N = 2n$, Q non-split: We have

$$\begin{aligned} G &= \mathrm{SO}_N(k) \\ &= \{g \in \mathrm{GL}_N(k) : \langle g(x), g(y)\rangle_{\mathrm{O}} = \langle x, y\rangle_{\mathrm{O}} \text{ for any } x, y \in V\}. \end{aligned}$$

Let F act on G by the rule: $F(g)F_V(x) = F_V(gx)$. The corresponding Dynkin diagram is

${}^2\mathrm{D}_n$



(n nodes, $n \geq 4$). Set $a_0 = n$.

LEMMA 3.9. *Let P be the parabolic subgroup generated by B and all Bs_iB except Bs_1B. Then the quotient map*

$$\pi : G/B \to G/P$$

sends the divisor D_1 to the points G^F/P^F. Hence, by Remark 1.18, all divisors D_i, $i \in \mathfrak{I}$ are mapped to subvarieties of codimension at least 2.

PROOF. Since $\bar{X}(w)$ may be described as

$$\begin{aligned} \bar{X}(w) = \{(g_0B, \ldots, g_nB) :\ & g_n^{-1}F(g_0) \in B; \\ & g_i^{-1}g_{i+1} \in \overline{Bs_{i+1}B}, i = 0, 1, \ldots, n-1\}, \end{aligned} \tag{3.4}$$

it follows that D_1 consists of those $(g_0B, \ldots, g_nB) \in \bar{X}(w)$ such that $g_0^{-1}g_1 \in B$. But then

$$g_0^{-1}F(g_0) = (g_0^{-1}g_1)(g_1^{-1}g_2)\ldots(g_{n-1}^{-1}g_n)(g_n^{-1}F(g_0))$$

is a product of elements from P. Hence D_1 is mapped to the (finitely many) points gP of G/P satisfying $g^{-1}F(g) \in P$. ☐

REMARK 3.10. Let $g \in P^F$, $\alpha \in \mathrm{A}_*(\bar{X}(w) - D_1)$. Then $\pi_*\alpha = \pi_*(g\alpha)$, hence $g\alpha = \alpha$ in $\mathrm{A}_*(\bar{X}(w))$ (when α is supported on $\bar{X}(w) - D_1$ we have $\pi^*(\pi_*\alpha) = \alpha$).

EXAMPLE 3.11 (${}^2\mathrm{A}_3$ case). In this case $P = \langle B, Bs_2B, Bs_3B\rangle$ with F-stable subgroup $\underline{P} = \langle \underline{B}, Bs_2B\rangle$. Consider the projection $\pi : (G/B)^3 \to G/B \to G/P \simeq \mathbb{P}^3$. We have $\pi(\bar{X}(w)) = Z(f)$, $f = X^{q+1} + Y^{q+1} + Z^{q+1}$. D_1 is the union of $(q^2 + 1)(q^3 + 1)$ lines and $D_2 = \coprod_{g \in M} g.V$ where V is the component of D_2 containing eB and M is a set of representatives of $G^F/\overline{Bs_1s_3B}^F$. We have $\#M = (q^3 + 1)(q + 1)$. A set of representatives could for example be:

$$\begin{aligned} M = eB/B \cup (Bs_2B)^F/B \cup (Bs_1s_2s_3B \cup Bs_3s_2s_1B)^F/B \\ \cup (Bs_1s_2s_3s_2B \cup Bs_3s_2s_1s_2B)^F/B \end{aligned}$$

(there are $1 + q + q^3 + q^4$ elements here). Under the projection $G/B \to G/P$, $(Bs_2B)^F/B$ is mapped to eP. The second contribution is mapped to q different points and the last to q^2 other points. Hence, M is mapped to $1 + q + q^2$ points. (For $q = 2$ this equals 7; compare Example B.7.)

To avoid confusion, let us recapitulate [**Har70**, p. 119] the following:

DEFINITION 3.12. A closed subscheme Y of $\mathbb{P}^N$ of codimension r is called an *ideal-theoretic (or strict) complete intersection* if Y is the *scheme-theoretic* intersection of r hyper-surfaces $H_1, \ldots, H_r$ in $\mathbb{P}^N$. In algebraic terms, if we let the hyper-surfaces be defined by the homogeneous polynomials $f_1, \ldots, f_r$, then $Y = \mathrm{Proj}(k[X_0, \ldots, X_N]/I)$ with $I = (f_1, \ldots, f_r)$.

A closed subset $Y \subset \mathbb{P}^N$ is said to be a *set-theoretic complete intersection* if it is the support of an ideal-theoretic complete intersection.

EXAMPLE 3.13. The 'standard' example of a set-theoretic complete intersection that is not a strict complete intersection is the image in $\mathbb{P}^3$ of the 3-uple embedding of $\mathbb{P}^1$ (a point $(t:u) \in \mathbb{P}^1$ is mapped to $(t^3 : t^2u : tu^2 : u^3) \in \mathbb{P}^3$). The curve C is given (set-theoretically) by

$$C = \{(x, y, x, w) \in \mathbb{P}^4 : y^3 + x^2w = 2xyz\} \cap \{(x, y, x, w) \in \mathbb{P}^4 : z^2 = yw\}$$

but the ideal of functions vanishing on C cannot be generated by less than three elements. The three linearly independent functions $xw - yz,\ xz - y^2$ and $yw - z^2$ obviously vanish on C, and

$$rad(y^3 + x^2w - 2xyz, z^2 - yw) = (xw - yz, xz - y^2, yw - z^2).$$

THEOREM 3.14. *Let $\bar{X}(w)$ be a standard Deligne-Lusztig variety of type* ${}^2\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n,\ \mathrm{D}_n$ *or* ${}^2\mathrm{D}_n$. *Let P be as in Lemma 3.9 and let*

$$\pi : (G/B)^{l(w)+1} \to G/P$$

be the projection. Then the image $Z = \pi(\bar{X}(w))$ is a normal strict complete intersection, possibly with singularities in $Z(\mathbb{F}_{q^{\delta a_0}})$. For $l(w) \geq 4$,

$$\mathrm{Pic}(\bar{X}(w)) = \mathbb{Z}[\{[V] : V \text{ component of some } D_i,\ i \in \mathfrak{I}\ \}] \oplus \mathbb{Z}[\pi^*H] \qquad (3.5)$$

where H is the hyperplane section of Z. (Since $\bar{X}(w)$ is non-singular we do of course have the same equality for $\mathrm{A}_{l(w)-1}(\bar{X}(w))$.*)*

In all of the mentioned cases, the rank of the Picard group is unchanged if we replace $\bar{X}(w)$ by a (not necessarily standard) Deligne-Lusztig variety arising from a Coxeter element F-conjugated to w.

PROOF. First we will handle the non-${}^2\mathrm{D}_n$ case. From Lemma 3.9 it follows that π contracts the divisor D_1 mapping it to the $\mathbb{F}_{q^\delta}$-rational points of $G/P \subseteq \mathbb{P}(V) \simeq \mathbb{P}^{N-1}$ (this inclusion is an equality in the non-orthogonal cases). Consider the hypersurfaces in $\mathbb{P}^{N-1}$:

$$H_i = \{(x_1 : x_2 : \cdots : x_N) \in \mathbb{P}^{N-1} : \langle x, F_V^i(x) \rangle = 0\}$$

where $i = 0, 1, \cdots, a_0 - 1$ (a_0 is defined as above) and $H_0 = \{(x_1 : x_2 : \cdots : x_N) \in \mathbb{P}^{N-1} : Q(x) = 0\}$ in the orthogonal case. Note that in the C_n-case $H_0 = \mathbb{P}(V)$ since $\langle\ ,\ \rangle_{\mathrm{Sp}}$ is alternating.

Lusztig shows [**Lus76b**] that Z equals the support of the scheme-theoretic complete intersection $Z' = \cap_{i=0}^{a_0-1} H_i$ with $X(w)$ mapping isomorphically onto the open subset $\langle x, F_V^{a_0}(x)\rangle \neq 0$ of Z; see also Table 3.1. We claim that Z' and

	${}^2\mathrm{A}_{2n+1}$	${}^2\mathrm{A}_{2n}$	B_n	C_n	D_n	${}^2\mathrm{D}_n$
$\dim(\mathbb{P}(V)) = N-1$	$2n+1$	$2n$	$2n$	$2n-1$	$2n-1$	$2n-1$
$\dim(\bar{X}(w)) = \dim(Z)$	$n+1$	n	n	n	n	$n-1$
a_0	n	n	n	n	$n-1$	n
#equations defining Z	n	n	n	$n-1$	$n-1$	n
equation for H_0	$\sum_j X_j^{q+1}$	$\sum_j X_j^{q+1}$	$\sum_j X_j^2$	none	$\sum_j X_j^2$	$\sum_j X_j^2$

TABLE 3.1. Data relating to Deligne-Lusztig varieties of 'classical' type. The condition $\langle x, x\rangle = 0$ is always true in the symplectic case, whence the difference in the C_n-case between a_0 and the number of defining equations. We see that in all cases Z has the 'correct' codimension in $\mathbb{P}(V)$. The equations for the hypersurfaces have in some cases been transformed to the (equivalent) diagonal form via a projective transformation (possibly with coefficients in a larger field). This allows us to use the common expression $\sum_j X_j^{q^{i\delta+1}+1} = 0$ for all H_i, $i > 0$.

Z are equal *as schemes*; that is, if we let $f_i \in k[X_1, \ldots, X_N]$ denote the form defining the hypersurface H_i, then the ideal $(f_0, \ldots, f_{a_0-1})$ is prime. Indeed, Z' is a complete intersection and is therefore Cohen-Macaulay.[3] So the problem amounts to showing that Z' is regular in codimension 1 (by Serre's Criterion for normality [**Har77**, Proposition II.8.23]). So suppose $P = (x_1 : x_2 : \cdots : x_N) \in Z'$ is a singular point. Then, for each $i = 0, 1, \ldots, a_0 - 1$, we have in the non-orthogonal cases, an equation (see Table 3.1)

$$\frac{1}{q^{i\delta+1}+1}\sum_{j=1}^{N}\frac{\partial f_i}{\partial X_j}(P)(X_j - x_j) = \sum_{j=1}^{N} x_j^{q^{i\delta+1}}(X_j - x_j) = 0 \tag{3.6}$$

[3]Additional note: By Hartshorne's Connectedness Theorem (see for example [**Eis95**, Theorem 18.12]) Z' is then connected in codimension two. That is, removing a subset of codimension at least two cannot make Z' disconnected. This also follows from the more general *Fulton-Hansen Connectedness Theorem* [**FH79**]. In [**Han83**] where this theorem was generalized from $\mathbb{P}^n$ to general flag manifolds, it was also proved that an intersection of two normal sub-varieties V (dimension v), W (dimension w) of $\mathbb{P}^n$ is $(v+w-n-1)$-connected. So in any case, if we did not already have the information that Z' is irreducible (because Z is), we could deduce this from proving that the singular locus is of codimension at least two (any two irreducible components would have to meet along a singular subvariety of codimension one).

From [**Har77**, Theorem III.7.11] it follows that $Z \subset \mathbb{P}^{N-1}$ has dualizing sheaf $\omega_Z^\circ = \mathcal{O}_Z(\sum_i deg(H_i) - (N+2))$. See also [**Har77**, Exercise II.8.4].

for the tangent-space to the hyper-plane H_i at P. Since each H_i is non-singular, P is only singular when some two of these tangent planes coincide. So, for some integers, r and s say, we have that $x_j^{q^{r\delta+1}} = x_j^{q^{s\delta+1}}$ for all j. Assuming $r < s$, we arrive at the equalities

$$x_j^{q^{r\delta+1}} = \left(x_j^{q^{r\delta+1}}\right)^{q^{(s-r)\delta}} \quad ; \qquad j = 1, \ldots, N.$$

Hence, $x_i \in \mathbb{F}_{q^{\delta a_0}}$ for all i.

In the orthogonal cases similar arguments apply since any one of the H_i $(i > 0)$ intersects the quadric $H_0 \simeq G/P$ transversely. Hence the singularities of Z' (if any) consist of the finitely many k'-rational points of Z' for some *finite* extension k' of $\mathbb{F}_q$.[4]

It follows that Z' is regular in codimension one (the singularities being of codimension $\dim(Z)$) and therefore Z and Z' are equal as schemes. This also shows that Z is normal [**Har77**, Proposition II.8.23].

From Corollary A.15 it now follows that, under the assumption $\dim(Z) = l(w) \geq 4$,

$$\mathrm{Pic}(Z) = \mathrm{A}_{l(w)-1}(Z - Z(\mathbb{F}_{q^{\delta a_0}})) = \mathrm{A}_{l(w)-1}(Z).$$

As $l(w) \geq 3$, $\mathrm{Pic}(Z) = \mathbb{Z}$ by the Lefschetz theorem for Picard groups [**Gro68**, Exposé XII, Corollaire 3.7] (see also page 85). Now the claim follows from Lemma A.8 and Lemma A.9.

The ${}^2\mathrm{D}_n$ case is only little different. Again we have a birational morphism $\pi : \bar{X}(w) \cup \bar{X}(F(w)) \to Z$ contracting the divisor $D_1 \cup F(D_1)$ to points. Since $X(w) \cup X(F(w))$ is isomorphic to an open subset of the complete intersection Z having a projective complement of codimension 1, $\mathrm{Pic}(X(w) \cup X(F(w)))$ has to have rank 0. Since both $X(w)$ and $X(F(w))$ are open in $X(w) \cup X(F(w))$ each of these varieties will also have Picard group of rank 0. Lemma A.2 then yields the wanted result.

Finally, the last assertion follows from Lemma 3.5 □

EXAMPLE 3.15. Consider $X(w)$, $l(w) \geq 4$. Suppose $X(w)$ is standard of type ${}^2\mathrm{A}_n$. From [**Lus76a**, Corollary (2.10)] and [**Jan96**, C.29-30] it follows that we have a decomposition

$$\mathrm{A}_k(X(w)/U_I^F) = \bigoplus_{i+j=k} \mathrm{A}_i(X_I(w)) \otimes \mathrm{A}_j(\mathbb{A}^1 - \{0\}).$$

By choosing I to correspond to the 'last $l(w) - 1$ orbits' we get that $X_I(w)$ is a standard Deligne-Lusztig variety of type ${}^2\mathrm{A}_{n-1}$ (of dimension $l(w) - 1$). Using

[4] Alternatively, we could have used the Jacobian Criterion for singularities: the singular locus of Z' is the zeros of the ideal J generated by the $a_0 \times a_0$-minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_j}\right)$. These polynomials are simultaneously zero at a point $P = (x_1 : x_2 : \cdots : x_N)$ only if P is $\mathbb{F}_{q^{\delta a_0}}$-rational.

this recursively we get $\mathrm{A}_{l(w)-1}(X(w))_{\mathbb{Q}} = 0$ also in the ${}^2\mathrm{A}_3$, ${}^2\mathrm{A}_4$, ${}^2\mathrm{A}_5$ and ${}^2\mathrm{A}_6$ cases: we have

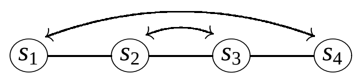
$$0 = \mathrm{A}_{l(w)}(X(w)) \supseteq \mathrm{A}_{l(w)}(X(w)/U_i^F)_{\mathbb{Q}} \simeq \mathrm{A}_{l(w)-1}(X_I(w))_{\mathbb{Q}}.$$

Similarly, we can describe the Picard group of the B_2 and B_3 cases as a quotient of the Picard group of some Deligne-Lusztig variety of dimension 4 or more. It follows that, in either case, $\mathrm{A}_1(\bar{X}(w))_{\mathbb{Q}}$ is generated by the classes of the components of the boundary divisors D_1 and D_2.

From these remarks it follows that to prove Conjecture 1 for a a given standard Deligne-Lusztig variety of classical type, it is sufficient to prove it for just one standard Deligne-Lusztig variety (of the same type, of course) of higher dimension.

REMARK 3.16. From the proof of the theorem we get that, for standard Deligne-Lusztig varieties of classical type, $X(w)$ is the complement (in Z) of the ample divisor H_{a_0}. Hence in this special case, we get a much simpler proof of the affinity of $X(w)$ *cf.* [**Har70**, Proposition II.2.1].

EXAMPLE 3.17 (${}^2\mathrm{A}_4$ case). Let us examine the ${}^2\mathrm{A}_4$-case:



with $\bar{X}(w) = \bar{X}(s_1 s_2)$. With the notation introduced above, Z is the complete intersection in $\mathbb{P}^4$ given by the hyper-surfaces

$$H_0: \quad \sum_{i=0}^{4} X_i^{q+1} = 0 \qquad \text{and} \qquad H_1: \quad \sum_{i=0}^{4} X_i^{q^3+1} = 0.$$

Earlier (Example 1.16) we found that the divisor $D_1 = \bar{X}(s_2) \subseteq \bar{X}(w)$ is a disjoint union of $(q^5+1)(q+1)$ Hermitian curves and D_2 is the disjoint union of $(q^5+1)(q^3+1)$ rational curves. From [**DL76**, Section 9] one may calculate that the ample line bundle $\mathcal{L}(-\lambda_1)$ on G/B satisfies

$$j'^*\mathcal{L}(-\lambda_1)^{\otimes(q^5+1)} \simeq \mathcal{O}_{\bar{X}(w)}((q^3+1)D_1 + (q+1)D_2).$$

Since j' is finite, it follows that $D_1 \cup D_2$ is the support of an ample divisor. Hence $D_1 \cup D_2$ is connected [**Har77**, Corollary III.7.9].

Consider the composite morphism $\rho : \bar{X}(w) \to Z \to \mathbb{P}^4$. As calculated in Example 4.24, the Hermitian curves all have self-intersection $-(q+1)$ and the rational curves have self-intersection $-q^2$. It follows that

$$D_1 \cdot [\text{fibre}] = -(q+1) \qquad \text{and} \qquad D_2 \cdot [\text{fibre}] = q^3+1.$$

So we may write [**Ful83**, 8.3.11]

$$\rho^*\mathcal{O}_{\mathbb{P}^4}(q^5+1) = (q^3+1)D_1 + (q+1)D_2$$

(which is indeed consistent with the fact that $\mathcal{L}(-\lambda_1)$ is the pull-back of $\mathcal{O}_{\mathbb{P}^4}(1)$ under the projection $G/B \to G/P \simeq \mathbb{P}^4$). We also find that

$$\deg_{j'^*\mathcal{L}(-\lambda_1)}[\text{fibre}] = \frac{1}{q^5+1}\big(-(q+1)(q^3+1)+(q+1)(q^3+1)\big) = 0$$

$$\deg_{j'^*\mathcal{L}(-\lambda_1)}[\text{rational curve}] = \frac{1}{q^5+1}\big((q^3+1)(q^2+1)-q^2(q+1)\big) = 1.$$

How are the rational and Hermitian curves configured in $\bar{X}(w)$? We know the following facts:

- As the action of G^F on $\bar{X}(w)$ permutes both the rational and the Hermitian curves, it follows that if a rational curve intersects one Hermitian curve, it must also intersect q^2 others (consider the automorphisms $g \in G^F$ fixing the Hermitian curve).
- Since the morphism $\pi : \bar{X}(w) \to Z$ collapses the Hermitian curves to points, there will pass

$$\frac{(\text{\# of components in } D_2)(\text{\# of rational points on a comp.})}{(\text{\# of rational points on } Z)} = \frac{(q^5+1)(q^3+1)(q^2+1)}{(q^5+1)(q^2+1)} = q^3+1$$

  lines through any rational point of Z.

Combined with the knowledge that the curves has to connect all components of D_1 and D_2, these properties force the curves to be configured as follows: Schematically, imagine $(q^5+1)(q^3+1)$ circles (the rational curves) distributed disjointly around the surface of a torus. Number these from 1 to $(q^5+1)(q^3+1)$. Each of these curves have (q^2+1) rational points. Through each rational point of the i'th rational curve goes a loop (Hermitian curve) around the torus connecting the point to a rational point on every $i + j(q^5+1)$'th rational curve $(j = 1, 2, \dots, q^3+1)$.[5]

From Example 3.15 it follows that the Picard group of $\bar{X}(w)$ is generated by the components of D_1 and D_2 (modulo torsion). See also Example 4.24.

REMARK 3.18. From [**Har77**, Proposition II.7.17] it follows that the birational morphism $\pi : \bar{X}(w) \to Z$ may be identified with the blow-up $\mathrm{Bl}_{\mathcal{I}} Z \to Z$ along some coherent sheaf of ideals $\mathcal{I}$. In the surface case, $\mathcal{I}$ has to have zero-locus equal to the image of D_1: By the universal property of blow-up, π factors through $\mathrm{Bl}_{\mathcal{I}} Z$ giving a bijective projective (hence finite) and surjective morphism $\bar{X}(w) \to \mathrm{Bl}_{\mathcal{I}} Z$ which has to be the normalisation of $\mathrm{Bl}_{\mathcal{I}} Z$. Since Z is normal, so is $\mathrm{Bl}_{\mathcal{I}} Z$, hence the induced morphism is an isomorphism. In higher dimensions it is not (a priori) obvious that the induced morphism should be surjective and bijective.

[5] A more concise way of stating this (communicated to the author by F. Rodier), is to say that the curves make a graph dual to the Bruhat-Tits building of the ${}^2\mathrm{A}_4$-group.

3.3. Relating the Chow ring of $\bar{X}(w)$ to that of G/B

Deligne-Lusztig varieties have some resemblance to Schubert varieties X_w. Therefore it is natural to ask if their Chow rings are related. We show that this is indeed the case: the intersection product of two properly intersecting Deligne-Lusztig subvarieties of $A_*(\bar{X}(w))_{\mathbb{Q}}$ is determined by the intersection of the two corresponding generators of the Chow ring of the Demazure desingularisation Z_w of X_w.

Introduce the notation $\mathrm{CH}^*(Y)$ for the Chow ring of a non-singular variety *cf.* [**Ful83**, 8.1]. If we only are interested in the additive structure of $\mathrm{CH}^*(Y)$ we shall just write $A_*(Y)$ as usual.

3.3.1. The Chow groups of G/B. Chow groups of flag varieties was first described in Chevalley's (unpublished) manuscript [**Che94**] and later in [**Dem74, Dem76**]. The following facts are sufficient for our purposes:

1. The action of G induced on $A_*(X)$ is trivial.
2. $\{[X_w] : w \in W\}$ is a basis of $A_*(X)$ with $[X_w] \in A_{l(w)}(X)$. Setting $Y_w = X_{w_0 w}$ we get that $\{[Y_w] : w \in W\}$ is a basis of $\mathrm{CH}^*(X)$; $[Y_w] \in \mathrm{CH}^{l(w)}(X)$. These bases are dual, in the sense that
$$[X_w] \cdot [Y_{w'}] = [X_w \cap w_0 Y_{w'}] = \begin{cases} [\{\dot{w}B\}] & w = w' \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$
3. $\mathrm{CH}^*(X)$ is generated in degree 1: any Schubert variety X_w of codimension ≥ 1 is a component in an iterated intersection of Schubert varieties of codimension 1.
4. The intersection pairing
$$\mathrm{CH}^1(X) \times \mathrm{CH}^k(X) \to \mathrm{CH}^{k-1}(X)$$
is given in terms of the Cartan matrix (A_{ij}) of G: let $\lambda_i \in X_0$ be the fundamental weight corresponding to the root α_i. These are given in terms of a base-change under the Cartan matrix (and are listed in e.g. [**Hum72**, p. 69]). Then, for $w \in W$ and $s_i \in S$,
$$[Y_{s_i}] \cdot [Y_w] = \sum_{\{\beta \in \Phi^+ : l(ws_\beta) = l(w)+1\}} \langle \lambda_i, \beta^\vee \rangle [Y_{ws_\beta}]. \tag{3.8}$$
5. Let us also mention Demazure's nice formula [**Dem76**],
$$0 = \sum_{\{w' \in W : l(w') + l(w'^{-1} w) = l(w)\}} (-1)^{l(w')} [Y_{w'}] \cdot [Y_{w^{-1} w'}] \tag{3.9}$$
for all $w \neq e$.

PROPOSITION 3.19. *The cycles* $\{[\overline{X(w)}] : w \in W\}$ *do* also *form a basis for* $A_*(X)_{\mathbb{Q}}$.

PROOF. Since the cardinality of the set in each degree is correct (being the same as that of Schubert varieties), we only need to prove that the cycles are linearly independent in $A_*(X)_{\mathbb{Q}}$. Like in the proof of the corresponding statement for Schubert varieties, it will suffice to find a set of $\mathbb{Q}$-dual elements [**Dem74**]. To this end, let $\dot{w}_0$ denote a representative of the longest element in W and let $w' \in W$ be arbitrary. Set $\overline{Y(w')} = \pi(L^{-1}(\dot{w}_0\overline{B\dot{w}'B}))$. Set-theoretically we have

$$\begin{aligned}\overline{X(w)} \cap \overline{Y(w')} &= \pi(L^{-1}(\overline{B\dot{w}B})) \cap \pi(L^{-1}(\dot{w}_0\overline{B\dot{w}'B}))\\ &= \pi(L^{-1}(\overline{B\dot{w}B} \cap \dot{w}_0\overline{B\dot{w}'B})).\end{aligned}$$

Since $\overline{B\dot{w}B} = \pi^{-1}(X_w)$ (similarly for w') it follows from the properties of Schubert varieties that

$$\overline{X(w)} \cap \overline{Y(w')} = \begin{cases} \pi(L^{-1}(\dot{w}_0)) & w' = w_0 w \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.10}$$

Since the intersection is proper when non-empty, we see that we have the wanted $\mathbb{Q}$-dual basis (X is projective). As $F(w_0) = w_0$, it follows that $L(w_0 g) = L(g)$ for all $g \in G$. Hence $\overline{X(w)} \cap \overline{Y(w)} = X(e)$. □

COROLLARY 3.20. *Let $\bar{X}(w)$ be a Deligne-Lusztig variety and let $\bar{X}(w_1)$, $\bar{X}(w_2)$ be two different Deligne-Lusztig subvarieties of $\bar{X}(w)$. Then $\bar{X}(w_1)$ and $\bar{X}(w_2)$ are linearly independent in $A_*(\bar{X}(w))$ (similarly in $\overline{X(w)}$).*

PROOF. If $\bar{X}(w_1)$ and $\bar{X}(w_2)$ are linearly dependent, then so are $\overline{X(w_1)}$ and $\overline{X(w_2)}$ [**Ful83**, Theorem 1.4]. Pushing this equivalence forward to $A_*(X)_{\mathbb{Q}}$ allows us to use Proposition 3.19. □

COROLLARY 3.21. *Let w_0 denote the longest element in W. For $k < l(w_0)$ we have $A_k(X(w_0))_{\mathbb{Q}} = 0$. More generally, for all k, n such that $k < n \leq l(w_0)$, we have that*

$$A_k\big(\cup_{l(w)\geq n} X(w)\big)_{\mathbb{Q}} = 0. \tag{3.11}$$

PROOF. From Proposition 3.19 it follows that in the short exact sequence [**Ful83**, Proposition 1.8] of finite-dimensional $\mathbb{Q}$-vector spaces,

$$\bigoplus_{i=1}^{N} A_k\big(\overline{X(w_0 s_i)}\big)_{\mathbb{Q}} \xrightarrow{\varphi} A_k(X)_{\mathbb{Q}} \to A_k(X(w_0))_{\mathbb{Q}} \to 0$$

φ has to be surjective. The first assertion then follows. For the last assertion we may argue similarly, using the exact sequence

$$\bigoplus_{l(w)=k} A_k\big(\overline{X(w)}\big)_{\mathbb{Q}} \xrightarrow{\varphi} A_k(X)_{\mathbb{Q}} \to A_k\big(\cup_{l(w)>k} X(w)\big)_{\mathbb{Q}} \to 0$$

plus the fact that the union $\cup_{l(w)\geq n} X(w)$ is open in $\cup_{l(w)>k} X(w)$. □

3.3.2. The Chow ring of Demazure desingularisations. In this section we will extract some facts from [**Dem74**] concerning the the Chow ring of the desingularisations $Z_w \to X_w$ (see also [**Mag96**]). Recall that we have reduced expressions $w = s_1 \cdot \ldots \cdot s_n$ and $w_0 = s_1 \cdot \ldots \cdot s_M$. Let $\tilde{\alpha}_1 = \alpha_1, \tilde{\alpha}_2 = s_1(\alpha_2), \ldots, \tilde{\alpha}_M = s_1 s_2 \cdots s_{M-1}(\alpha_M)$ and let $w_k = \prod_{m=1}^{k} s_i$. Furthermore, for any subset $K \subseteq [1, M]$ let $w_K = \prod_{i \in K} w_k$ Then define root systems $R_0 = \Phi^+, R_1 = w_1(\Phi^+), \ldots, R_M = w_M(\Phi^+)$. Let B_i be the Borel subgroup of G for which R_i comprise the positive roots and let P_i be the parabolic subgroup given by $R_i \cup R_{i-1}$ (that is, generated by B_i and B_{i-1}). Let

$$X_n = P_1 \times^{B_1} P_2 \times^{B_2} \cdots \times^{B_{n-1}} P_n.$$

The quotient X_n/B_n can be identified with the quotient of $P_1 \times \cdots \times P_n$ by $B_1 \times \cdots \times B_n$ acting from by the right by the rule

$$(p_1, p_2, \ldots, p_n)(b_1, b_2, \ldots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{n-1}^{-1} p_n b_n).$$

We have B_0-equivariant sections σ_i and projections f_i

$$\begin{tikzcd}
X_0 \arrow[d] & X_1 \arrow[d] & & X_n \arrow[d] & & X_M \arrow[d] \\
X_0/B_0 \arrow[r, shift right, "\sigma_1"'] & X_1/B_1 \arrow[l, shift right, "f_1"'] \arrow[r, shift right, "\sigma_2"'] & \cdots \arrow[l, shift right, "f_2"'] \arrow[r, shift right, "\sigma_n"'] & X_n/B_n \arrow[l, shift right, "f_n"'] \arrow[r, shift right, "\sigma_{n+1}"'] & \cdots \arrow[l, shift right, "f_{n+1}"'] \arrow[r, shift right, "\sigma_M"'] & X_M/B_M. \arrow[l, shift right, "f_M"']
\end{tikzcd} \tag{3.12}$$

Then $Z_w := X_n/B_n$ is the *Demazure desingularisation* of X_w. For any i let $Z_i = f_n^{-1} \ldots f_{i+1}^{-1}(Im(\sigma_i))$.[6] Then, for any $K \subseteq [1, n]$ we have a closed subvariety of Z_w,

$$Z_K = \cap_{i \in K} Z_i = \{(p_1, p_2, \ldots, p_n) : p_i \text{ represents } w_i \text{ for any } i \in K\}.$$

We then have

PROPOSITION 3.22 (DEMAZURE). *Let the notation be as above. Furthermore, let t_i denote the class of the divisor $Z_{\{i\}}$ in $\mathrm{A}_*(Z_w)$. Then,*

1. $Z_\emptyset = Z_n$, $Z_K \cap Z_L = Z_{K \cup L}$, $\mathrm{codim}(Z_K) = |K|$.
2. *For $K = [1, i]$ ($i \leq n$), Z_K is the typical fibre (over $Im(\sigma_i)$) of the projection $Z_n = X_n/B_n \to X_i/B_i$. For $K = [i+1, n]$, Z_K is the image of the immersion of X_i/B_i into X_n/B_n.*
3. *The symbols*
$$\left\{ t_K = \prod_{i \in K} t_i : K \subseteq [1, n] \right\}$$
form a basis of the $\mathbb{Z}$-module $\mathrm{A}_*(Z_w)$.

---

[6] It is not always the case that Z_i is the desingularisation of $X_{w'}$, $w' = s_1 \cdot \ldots \cdot \hat{s}_i \cdot \ldots \cdot s_n$. Actually this is only the case when $i = n$. Then $Z_i = Z_{n-1} =$ the desingularisation of $X_{s_1 \cdot \ldots \cdot \hat{s}_i \cdot \ldots \cdot s_n}$. In general we have that $Z_{[k+1,n]}$ is the desingularisation of $X_{s_1 \cdot \ldots \cdot s_k}$.

4. *As Z_w is smooth we have a multiplicative structure on* $\mathrm{A}_*(Z_w)$. *The Chow ring* $\mathrm{CH}^*(Z_w)$ *of* Z_w *is* $\mathbb{Z}[t_1,\dots,t_n]$ *modulo the relations*

$$\begin{aligned} t_1^2 &= 0 \\ t_2^2 + \langle\tilde\alpha_2,\tilde\alpha_1^\vee\rangle t_1t_2 &= 0 \\ &\vdots \\ t_n^2 + \langle\tilde\alpha_n,\tilde\alpha_1^\vee\rangle t_1t_n + \cdots + \langle\tilde\alpha_n,\tilde\alpha_{n-1}^\vee\rangle t_{n-1}t_n &= 0. \end{aligned}$$

EXAMPLE 3.23 (A_2 case). Consider the A_2 case ($G = \mathrm{SL}_3$) with $w = s_1s_2$ and $w_0 = s_1s_2s_1$. Then $t_1 = [Z_1] = [f_1^{-1}(X_0/B_0)] = [f_1^{-1}(pt)]$ and $t_2 = [Z_2] = [Im(\sigma_2)] = [Z_{s_1}]$. Since $s_1(\alpha_2) = \alpha_2 - \langle\alpha_2,\alpha_1^\vee\rangle\alpha_1$ we get $\langle\alpha_2, s_1(\alpha_2)^\vee\rangle = -1 + 2 = 1$. Hence

$$\mathrm{CH}^*(Z_w) = \mathbb{Z}[t_1,t_2]/(t_1^2, t_2^2 + t_1t_2).$$

This is not surprising since in this case Z_w actually is the blow-up of the plane in a point; $t_1 = L - E$ is the pull-back of a line in the plane minus the exceptional divisor (the proper transform of a line through the blown-up point) and t_2 is the exceptional divisor.[7]

3.3.3. Multiplication in $\mathrm{A}_*(\bar X(w))$.

PROPOSITION 3.24. *Let w be a Coxeter element and let $\bar X(w)$ be the corresponding Deligne-Lusztig variety. The intersection product of two Deligne-Lusztig subvarieties of $\bar X(w)$ is determined by the intersection product on $\bar O(w)$.*

PROOF. Consider the intersection product

$$[\bar X(w_1)]\cdot[\bar X(w_2)] \qquad ; w_i \le w.$$

Interpreting (1.14) as an intersection product, it follows [**Ful83**, Section 8.1] that there is a Gysin homomorphism

$$\gamma'^* : \mathrm{A}_k\big(\bar O(w)\big) \to \mathrm{A}_k\big(\bar X(w)\big) \tag{3.13}$$

such that for $w' \le w$ we have $\gamma'^*[\bar O(w')] = [\bar X(w')]$ in $\mathrm{A}_k\big(\bar X(w)\big)$. Since both $\bar O(w)$ and $\bar X(w)$ are non-singular, γ'^* is a ring-homomorphism [**Ful83**, Proposition 8.3], hence

$$[\bar X(w_1)]\cdot[\bar X(w_2)] = \gamma'^*\big([\bar O(w_1)]\cdot[\bar O(w_2)]\big) \tag{3.14}$$

and the assertion follows. $\square$

COROLLARY 3.25. *Let $w_1, w_2 \le w$ be such that the intersection $Z_{w_1}\cap Z_{w_1}$ is proper. Suppose*

$$[Z_{w_1}]\cdot[Z_{w_2}] = \sum_{w'\in W} n_{w'}[Z_{w'}]$$

[7] Z_w may also be realized as the ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-1))$ over $Z_{s_1}\simeq\mathbb{P}^1$ (Hartshorne's notation here). Then t_1 is the class of a fiber and t_2 is the class of a section.

in $A_*(Z_w)$. *Then*

$$[\bar{O}(w_1)]\cdot[\bar{O}(w_2)] = \sum_{w'\in W} n_{w'}[\bar{O}(w')]$$

in $A_*(\bar{O}(w))$, *hence*

$$[\bar{X}(w_1)]\cdot[\bar{X}(w_2)] = \sum_{w'\in W} n_{w'}[\bar{X}(w')] \tag{3.15}$$

in $A_*(\bar{X}(w))$.

PROOF. The first assertion follows from the local nature of the construction of the associated bundles $\bar{O}(w)$ (see Section A.2). The last equality is obvious from Proposition 3.24. ☐

EXAMPLE 3.26 (A_2 case). In this case $\bar{X}(w)$ arises as the blow-up of the projective plane $\mathbb{P}^2$ in the rational points $\mathbb{P}^2(\mathbb{F}_q)$. D_1 is equal to the exceptional divisor E and D_2 equals the pull-back of $\mathcal{O}_{\mathbb{P}^2}(q^2+q+1)$ minus $(q+1)E$ (see also Example 1 in Chapter 4). From the above it follows that

$$[D_1]\cdot[D_2] = (\text{coefficient to } Z_{s_1s_2} \text{ in } Z_{s_1}\cdot Z_{s_2})[\bar{X}(e)] = [X(e)].$$

Using the explicit description we have at hand, we may check this:

$$\begin{aligned}\deg([D_1]\cdot[D_2]) &= \deg(E\cdot(\pi^*\mathcal{O}_{\mathbb{P}^2}(q^2+q+1)-(q+1)E))\\ &= -(q+1)\deg(E^2) = (q+1)\#\mathbb{P}^2(\mathbb{F}_q)\\ &= (q+1)(q^2+q+1).\end{aligned}$$

Furthermore, $\deg(D_1^2) = \deg(E^2) = -(q^2+q+1)$ and $\deg(D_2^2) = -q(q^2+q+1)$.

REMARK 3.27. Since X is a homogeneous space, the tangent bundle $\mathcal{T}_X$ is generated by its global sections *cf.* [**Ful83**, 12.2.1]. It would also be convenient if $\mathcal{T}_{\bar{X}(w)}$ were generated by its global sections, because in that case intersections of non-negative cycles would give non-negative cycles in $A_*(\bar{X}(w))$ *cf.* [**Ful83**, Theorem 12.2]. However, as the above example shows, this cannot (in general) be the case.

REMARK 3.28. Using the explicit description of $j'^*\mathcal{L}(\lambda)$ (see Chapter 4) we may give an alternative way of intersecting two boundary components D_i and D_j. Choose $\lambda_i \in X(T)_{\mathbb{Q}}$ (resp. λ_j) such that $j'^*\mathcal{L}(\lambda_i) = \mathcal{O}_{\bar{X}(w)}(D_i)$ (similarly for j). Let $D = c_1(\mathcal{L}(\lambda_j))\cap X = \sum_{k=1}^N -\langle\lambda_j,\alpha_k^\vee\rangle X_{w_0s_k}$ be the divisor in X defined by $\mathcal{L}(\lambda_j)$. Then

$$\begin{aligned}[D_i]\cdot[D_j] &= c_1(j'^*\mathcal{L}(\lambda_i))\cap[D_j] = j'^*(c_1(\mathcal{L}(\lambda_i))\cap D) &(3.16)\\ &= j'^*\left(-\sum_{k=1}^N\left[\sum_{\alpha\in\Phi^+:l(w_0s_ks_\alpha)=N-2}\langle\lambda_j,\alpha_k^\vee\rangle\langle\lambda_i,\alpha^\vee\rangle X_{w_0s_ks_\alpha}\right]\right) &(3.17)\end{aligned}$$

(see [**Dem74**, p. 78]).

CHAPTER 4

Canonical bundles of Deligne-Lusztig varieties

In this chapter we have reproduced the body of the article *Canonical bundles of Deligne-Lusztig varieties* [**HH99a**]. Please observe that the references made in the paper refer to the bibliography at the end of this thesis. In the last section of this chapter, we make some supplementary remarks to the results of the paper.

Let us make a few introductory remarks. The paper was originally motivated by the question whether Deligne-Lusztig varieties are Frobenius split or not. Soon, however, it was apparent that this was not in general to be expected. Instead one could actually prove ampleness of the canonical bundle in some cases.

Later on it became evident from the work with the Picard groups (see Chapter 3) that Deligne-Lusztig varieties are *almost never* Frobenius split. We give the proof below.

4.1. Criteria for Frobenius splitting

General reference for this section is [**MR85**]. See also [**MR88, IM94, LT97, BTLM97, LM99, MT99**].

DEFINITION 4.1. Let $\tilde{F} : X \to X'$ denote the absolute Frobenius morphism on an algebraic variety X (see for example [**Har77**, p. 301]).

1. X is *Frobenius split* if the morphism $\mathcal{O}_{X'} \to \tilde{F}_*\mathcal{O}_X$ is a split homomorphism of $\mathcal{O}_{X'}$-modules.
2. Given a splitting $\sigma : \tilde{F}_*\mathcal{O}_X \to \mathcal{O}_{X'}$ we call a closed subscheme $Y \subseteq X$ *compatibly σ-split* if $\sigma(\tilde{F}_*(\mathcal{I}_Y)) \subseteq \mathcal{I}_{Y'}$.

The following proposition [**MR85**, Proposition 8] is a good tool for finding splittings (if possible).

PROPOSITION 4.2. *Let X be an n-dimensional non-singular projective variety over k. Suppose the inverse canonical bundle K_X^{-1} has a global section $s \in \mathrm{H}^0(X, K_X^{-1})$ such that the divisor of zeros $(s)_0$ decomposes into a sum of effective prime divisors*

$$(s)_0 = Z_1 + \cdots + Z_n + E$$

where the Z_i are non-singular and intersect transversally. If the point $P = Z_1 \cap \cdots \cap Z_n$ is not contained in the support of E then s defines a splitting $\sigma : \tilde{F}_* \mathcal{O}_X \to \mathcal{O}_{X'}$ which compatibly splits the Z_i.

PROOF. (Sketch). Duality for the finite flat morphism $\tilde{F} : X \to X'$ gives (using [**Har77**, Exercises III.6.10+III.7.2])

$$\operatorname{Hom}(\tilde{F}_* \mathcal{O}_X, \mathcal{O}_{X'}) \simeq \mathrm{H}^0(X, K_X^{\otimes 1-p}).$$

Since X is projective, any regular function is constant. So in order to get a splitting we only need to find a global section $s \in \mathrm{H}^0(X, K_X^{-1})$ and one point where F composed with the induced splitting is non-zero.

The decomposition of $(s)_0$ means that, in a neighborhood of P,

$$s = h \cdot x_1 \cdot \ldots \cdot x_n \otimes \frac{1}{dx_1 \cdot \ldots \cdot dx_n}$$

for a regular system of parameters $(x_1, \ldots, x_n)$ in P with $h \notin \mathfrak{m}_P$, that is, $h(P) \neq 0$. Then the power series expansion of s^{p-1} in $(\widehat{K_X^{\otimes p-1}})_P = K_X^{\otimes p-1})_P \otimes \widehat{\mathcal{O}_{X,P}}$ has non-zero coefficient to the $(x_1 \cdot \ldots \cdot x_n)^{\otimes p-1} \otimes \frac{1}{(dx_1 \cdot \ldots \cdot dx_n)^{\otimes p-1}}$ part. One then verifies that this is exactly what is needed to make the composite non-zero in the point P. $\square$

REMARK 4.3. Conversely, it follows from [**MR85**, Proposition 6] that if $\mathrm{H}^0(X, K_X^{\otimes 1-p}) = 0$, X cannot be Frobenius split. This is for example the case if K_X (hence also $K_X^{\otimes p-1}$) is ample (from [**Har77**, Exercise III.7.1] we have that $\mathrm{H}^0(X, L^{-1}) = 0$ whenever L is an ample line bundle on a non-singular projective variety X).

If X is a non-singular curve, a line bundle $L \simeq \mathcal{O}_X(D)$ is ample if and only if $\deg D > 0$. In particular, $K_X = \mathcal{O}_X(2g-2)$ is ample if and only if $g > 1$.

Consider the non-singular projective plane ${}^2\mathrm{A}_2$ curve $\bar{X}(w)$ given by the equation $X^{q+1} + Y^{q+1} + Z^{q+1} = 0$. For $q = 2$ this is a non-singular elliptic curve (of Hasse-invariant 0 [**Har77**, Proposition IV.4.21]), hence $K_{\bar{X}(w)} = \mathcal{O}_{\bar{X}(w)}(2g-2) = \mathcal{O}_{\bar{X}(w)}$ is neither ample or anti-ample. (A direct calculation of the homomorphism $\mathrm{H}^1(\bar{X}(w), \mathcal{O}_{\bar{X}(w)}) \to \mathrm{H}^1(\bar{X}(w), \mathcal{O}_{\bar{X}(w)}^{\otimes 2})$ shows that this map is the zero map, hence $\bar{X}(w)$ is not Frobenius split either [**MR85**, Proposition 9].)

THEOREM 4.4. *Let $\bar{X}(w)$ be a standard Deligne-Lusztig variety of type ${}^2\mathrm{A}_n$, ${}^2\mathrm{B}_2$, B_n, C_n, ${}^2\mathrm{F}_4$ or ${}^2\mathrm{G}_2$. Then $\bar{X}(w)$ is* not *Frobenius split whenever $q > 2$ or $l(w) \geq 3$.*[1]

PROOF. In all of the mentioned cases, we have (*cf.* Chapter 3) a surjective morphism of normal projective varieties $\pi : \bar{X}(w) \to Z$ with the property that $\pi_* \mathcal{O}_{\bar{X}(w)} = \mathcal{O}_Z$ [**Jan87**, p. 400]. Hence, if we had a splitting, we could push it

[1] The assumption is a bit more than sufficient. What precisely is needed is that the degrees of the hypersurfaces cutting out Z sum up to more than 1 plus the dimension of the surrounding projective space.

forward with π to obtain a splitting of Z [**MR85**, Proposition 4]. But since Z is a complete intersection in $\mathbb{P}^n$ cut out by hypersurfaces of total degree strictly larger than $n+1$, it cannot be Frobenius split [**Koc97**, Corollary 2.7] □

REMARK 4.5. Similarly, we do not in general expect the Frobenius on a Deligne-Lusztig variety to lift to the Witt vectors of length 2 as this would imply Bott vanishing for $\bar{X}(w)$ (*cf.* [**BTLM97**]): Let L be an ample line bundle on $\bar{X}(w)$. Then $\mathrm{H}^i(\bar{X}(w), \Omega^j_{\bar{X}(w)} \otimes L) = 0$ for $i > 0$ and any j.

Consider for example the ${}^2\mathrm{A}_3$ case where $\bar{X}(w)$ is the blow-up of a non-singular Fermat hypersurface $S \subset \mathbb{P}^3$ of degree $q+1$. Let $L = \mathcal{O}_{\bar{X}(w)}(1)$ and let π denote the blow-up morphism which is trivial (in the sense of Definition B.16). Since $K_{\bar{X}(w)} = \pi^* K_S \otimes \mathcal{O}_{\bar{X}(w)}(-1)$, we get $\mathrm{H}^i(\bar{X}(w), \Omega^2_{\bar{X}(w)} \otimes L) = \mathrm{H}^i(\bar{X}(w), \pi^* K_S) = \mathrm{H}^i(S, K_S)$. By shifting the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-(q+1)) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

with $K_S = \mathcal{O}_S(q-3)$ we get from the induced long exact cohomology sequence that $\mathrm{H}^2(S, K_S) \simeq k \neq 0$. Hence Bott vanishing does not apply to this particular Deligne-Lusztig variety.

Canonical bundles of Deligne-Lusztig varieties†

SØREN HAVE HANSEN

ABSTRACT. In this paper we consider Deligne-Lusztig varieties. We explicitly describe the canonical bundles of their smooth compactifications in terms of homogeneous line bundles pulled back from G/B. Using this description we show that the members (one member in each dimension) of a special family of Deligne-Lusztig varieties have ample canonical bundles. A consequence is that, unlike the closely related Schubert varieties, Deligne-Lusztig varieties are not in general Frobenius split.
Several examples are given. Among these we exhibit (Example 4.21) an infinite family of counter-examples to the Miyaoka-Yau inequality (one for each prime power).

4.2. Deligne-Lusztig varieties

Let (G, F) be a connected reductive algebraic group equipped with an $\mathbb{F}_q$-structure coming from a Frobenius morphism $F : G \to G$. Let $L : G \to G$ be the Lang map taking an element $g \in G$ to $g^{-1}F(g)$. By the Lang-Steinberg Theorem [**Bor92**, Theorem 16.3] this morphism of varieties is surjective with finite fibres. From this result it follows that, by conjugacy of tori and Borel subgroups, there exists F-stable maximal tori and Borel subgroups. Hence there are (with abuse of notation) natural endomorphisms $F : W \to W$ and $F : X \to X$ of the Weyl group of G and the variety X of Borel subgroups of G. Let W be generated by the simple reflections $s_1, \ldots, s_N$ and let $l(\cdot)$ be the length function with respect to these generators.

Fix an F-stable Borel subgroup $B \subseteq G$ containing an F-stable maximal torus T. Let $\pi : G \to G/B \simeq X$ be the projection. Let U be the unipotent radical of B (also F-stable as B and T are) and denote by Φ^+ the set of positive roots with respect to B. Let $X(T)$ be the k-valued characters of T. (All this notation is explained in e. g. [**Car85**, Chapter 1].) For any $\lambda \in X(T)$, the homogeneous line bundle $\mathcal{L}(\lambda)$ on X will be ample (very ample, in fact) if λ is strictly antidominant (negative with respect to the basis $\{\lambda_1, \ldots, \lambda_N\}$ of $X(T)$ determined by Φ^+) [**Jan87**, p. 231].

DEFINITION 4.6. Fix an element w in the Weyl group W and let $w = s_{i_1} \cdot \ldots \cdot s_{i_n}$ be a reduced expression. Call w a *Coxeter element* if there is exactly one s_i from each of the orbits of F on $\{s_1, \ldots, s_N\}$.

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1. Define the (open) *Deligne-Lusztig variety* $X(w)$ to be the image of $L^{-1}(B\dot{w}B)$ in G/B. That is,
$$X(w) = \pi(L^{-1}(B\dot{w}B)).$$
We shall think of the points of $X(w)$ as Borel subgroups of G.
2. Define the closed subvariety of X^{n+1}
$$\begin{aligned}\bar{X}(s_{i_1}, \dots, s_{i_n}) = \{&(g_0B, \dots, g_nB) \in X^{n+1} : \\ &g_k^{-1}g_{k+1} \in B \cup Bs_{i_{k+1}}B \text{ for } 0 \le k < n,\ g_n = F(g_0)\}.\end{aligned}$$
For brevity, we will write $\bar{X}(w)$ for this variety. For any $w' \le w$, $\bar{X}(w')$ defines in a natural way a closed subvariety of $\bar{X}(w)$. In particular there are divisors
$$D_j = \bar{X}(s_{i_1}, \dots, \hat{s}_{i_j}, \dots, s_{i_n}) \qquad ;\ j = 1, \dots, n.$$
3. When G is semi-simple with connected Dynkin diagram, there is a natural choice of Coxeter element: let $w = s_{i_1} \cdot \dots \cdot s_{i_n}$ where $i_1 = 1$ and any pair i_j, i_{j+1} correspond to two joined nodes. When choosing this Coxeter element, we shall refer to $\bar{X}(w)$ as being *the* Deligne-Lusztig variety corresponding to (G, F).

Since L is flat, it is open, hence $\overline{L^{-1}(B\dot{w}B)} = L^{-1}\left(\overline{B\dot{w}B}\right)$. So $X(w)$ is smooth of dimension n and the closure of $X(w)$ in X is given by the disjoint union

$$\overline{X(w)} = \bigcup_{w' \le w} X(w'), \tag{4.1}$$

where as usual $\le$ is the Bruhat order in W. This closure is singular whenever the Schubert variety $X_w = \overline{B\dot{w}B}/B$ is. But since the open subset

$$\{(g_0B, \dots, g_nB) \in X^{n+1} : g_k^{-1}g_{k+1} \in Bs_{i_{k+1}}B,\ 0 \le k < n,\ g_n = F(g_0)\}$$

of the smooth projective variety $\bar{X}(w)$ [**DL76**, 9.10] maps isomorphically onto $X(w)$ under projection to the first factor, we have a good compactification of $X(w)$. In fact the complement of $X(w)$ in $\bar{X}(w)$, which is easily seen to be the union of the D_j's, is a divisor with normal crossings [**DL76**, 9.11]. If w is a Coxeter element, then $X(w)$ and $\bar{X}(w)$ are irreducible [**Lus76a**, Proposition (4.8)].

REMARK 4.7. Groups G^F arising as the fixed-points of a Frobenius morphism acting on a reductive, connected linear algebraic group are called *finite groups of Lie type*. It was the search for a unified description of the characteristic zero representations of these groups that led Deligne and Lusztig [**DL76**] to the construction of Deligne-Lusztig varieties. (G^F acts on $X(w)$ as a group of automorphisms inducing an action on the ℓ-adic cohomology vector spaces. See also [**Haa86**].)

REMARK 4.8. There is an equivalent way of defining $X(w)$ and $\bar{X}(s_{i_1} \cdot \dots \cdot s_{i_n})$ which will be useful later on. Define the locally closed subvariety of $X \times X$,

$$O(w) = \{(g_0B, g_1B) \in X \times X : g_0^{-1}g_1 \in BwB\}$$

(the orbit of $(eB, \dot{w}B)$ under the diagonal action of G). Denote its closure by $\overline{O(w)}$. Let $i = (\mathrm{id}, F)$ be the graph map of F. Then we have the following pull-back diagram

$$\begin{tikzcd} X(w) \arrow[r] \arrow[d] & O(w) \arrow[d] \\ X \arrow[r, "i"] & X \times X. \end{tikzcd}$$

That is, $X(w)$ is the transversal intersection of $O(w)$ and the graph of F inside $X \times X$ (see [**DL76**, 1.4+(1.11.1)]). We have a Demazure-type desingularisation $\bar{O}(s_1, \ldots, s_n)$ of $\overline{O(w)}$. Intersecting with the graph of F we actually get the corresponding smooth compactification

$$\begin{tikzcd} \bar{X}(s_1, \ldots, s_n) \arrow[r] \arrow[d, "j'"'] & \bar{O}(s_1, \ldots, s_n) \arrow[d, "j"] \\ X \arrow[r, "i"] & X \times X \end{tikzcd}$$

of $X(w)$ [**DL76**, 9.10+9.11]. We shall write $\bar{O}(w)$ for $\bar{O}(s_1, \ldots, s_n)$.

LEMMA 4.9. *Let $w \in W$ have reduced expression $w = s_{i_1} \cdot \ldots \cdot s_{i_n}$ with all i_j distinct. Then the resolution $\varphi : \bar{O}(w) \to \overline{O(w)}$ is bijective. Hence also the resolution $\varphi' : \bar{X}(w) \to \overline{X(w)}$ is bijective.*

PROOF. We prove the claim by induction on $l(w)$. For $l(w) = 0$ the claim is trivially true. So assume $l(w) > 0$. Since φ is an isomorphism over $O(w)$ we only need to show that φ is bijective on any $\overline{O(w')}$ with $w' \leq w$ and $l(w') = l(w) - 1$. Now, any such w' is obtained from the reduced expression of w by omitting a unique s_{i_j}. That is, there exists a unique index j such that

$$w' = s_{i_1} \cdot \ldots \cdot \hat{s}_{i_j} \cdot \ldots \cdot s_{i_n}$$

and this is a reduced expression of w' (by the special property of w). But then w' satisfies the induction hypothesis and the assertion follows. $\square$

REMARK 4.10. Suppose L is an ample line bundle on X. Since j' is the composition of the closed immersion j of $\overline{X(w)}$ in X and the resolution φ' : $\bar{X}(w) \to \overline{X(w)}$, it follows that j'^*L is ample when φ' is finite. This is for example the case when w is a Coxeter element (Lemma 4.9).

4.3. The Canonical Bundle

Let ∂X be the sum of the codimension one Schubert varieties of X (similarly with $\bar{O}(w)$ and $\bar{X}(w)$). Since the *positive* roots correspond to B we have $K_X = \mathcal{O}_X(-2\partial X) = \mathcal{L}(2\rho)$ (ρ being half the sum of the positive roots). Since ρ is strictly dominant, $K_X^{-1} = \mathcal{L}(-2\rho)$ and $\mathcal{L}(-\rho)$ are ample line bundles on X.

Recall that the commutative diagram

$$\begin{tikzcd} \bar{X}(s_1, \dots, s_n) \arrow[r] \arrow[d, "j'"'] & \bar{O}(s_1, \dots, s_n) \arrow[d, "j"] \\ X \arrow[r, "i"] & X \times X \end{tikzcd}$$

(j maps $(B_0, \dots, B_n)$ to (B_0, B_n)) arises as the pull-back of the commutative diagram of closed embeddings

$$\begin{tikzcd} \overline{X(w)} \arrow[r] \arrow[d] & \overline{O(w)} \arrow[d] \\ X \arrow[r, "i=(\mathrm{id},F)"] & X \times X \end{tikzcd}$$

via the desingularisation $\varphi : \bar{O}(w) \to \overline{O(w)}$. Furthermore, we have at the bottom the triangle

$$\begin{tikzcd} X \arrow[r, shift left, "i=(\mathrm{id},F)"] \arrow[dr, "F"'] & X \times X \arrow[l, shift left, "\mathrm{pr}_1"] \arrow[dl, "\mathrm{pr}_2"] \\ & X. \end{tikzcd}$$

As $\mathrm{pr}_1 \circ j \circ i' = j'$ and $\mathrm{pr}_2 \circ j \circ i' = F \circ j'$, it follows that

$$i'^*\left(j^*(\mathrm{pr}_1^* \mathcal{L}(\rho) \otimes \mathrm{pr}_2^* \mathcal{L}(\rho))\right) = j'^*(\mathcal{L}(\rho) \otimes F^*\mathcal{L}(\rho)). \tag{4.2}$$

By definition, $\partial \bar{X}(w)$ is the restriction of $\partial \bar{O}(w)$ to $\bar{X}(w)$, hence we have

$$i'^* \mathcal{O}_{\bar{O}(w)}(\partial \bar{O}(w)) = \mathcal{O}_{\bar{X}(w)}(\partial \bar{X}(w)).$$

PROPOSITION 4.11. *Let $\bar{X}(w)$ be a Deligne-Lusztig variety. Then the canonical bundle $K_{\bar{X}(w)}$ is given by*

$$K_{\bar{X}(w)} = \mathcal{O}_{\bar{X}(w)}(-\partial \bar{X}(w)) \otimes j'^* \mathcal{L}(\rho - F(\rho)). \tag{4.3}$$

PROOF. By the adjunction formula [**Ful83**, B.7.2] we have

$$\begin{aligned} K_{\bar{X}(w)} &= i'^* K_{\bar{O}(w)} \otimes \wedge^r N_{\bar{X}(w)} \bar{O}(w) \\ &= i'^* K_{\bar{O}(w)} \otimes \wedge^r j'^* N_{i(X)}(X \times X) \end{aligned}$$

(the last equality since i and i' are regular closed embeddings of the same codimension). Now

$$K_{\bar{O}(w)} = \mathcal{O}_{\bar{O}(w)}(-\partial \bar{O}(w)) \otimes j^*(\mathrm{pr}_1^* \mathcal{L}(\rho) \otimes \mathrm{pr}_2^* \mathcal{L}(\rho))$$

cf. [**MR88**, Proposition 2][3] and by [**Ful83**, B.7.3]

$$\begin{aligned}\wedge^r N_{i(X)}(X \times X) &= \wedge^r F^* \mathcal{T}_X = F^* \wedge^r \mathcal{T}_X \\ &= F^* K_X^{-1} = F^* \mathcal{L}(-2\rho).\end{aligned}$$

Combining these equalities with (4.2) the proposition follows. □

4.4. Computation of the canonical bundle $K_{\bar{X}(w)}$

We will in this section refine the description of $K_{\bar{X}(w)}$. For simplicity we will assume that we are not in any of the cases ${}^2\mathrm{B}_2, {}^2\mathrm{G}_2$ and ${}^2\mathrm{F}_4$. (These cases have been treated separately in [**Han92**] and [**Rod96**].) Then F is a standard Frobenius morphism and $F(\lambda_j) = q\lambda_i$ where q is a positive *integral* power of p. In particular, $F(\rho) = q\rho$.

REMARK 4.12. Let A denote the matrix of the linear map sending the character λ to $F(w^{-1}(\lambda)) - \lambda$ with respect to the basis $\{\lambda_1, \dots, \lambda_N\}$ of $X(T)$. Let $\lambda \in X(T)$. Write $\lambda = A\mu$ for some $\mu \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ (this is always possible *cf.* [**Haa86**, 2.3]). Let m be a positive integer such that $m\mu \in X(T)$. Set $\tilde{\alpha}_1 = \alpha_1$ and

$$\tilde{\alpha}_i = s_1 \cdot \ldots \cdot s_{i-1}(\alpha_i) \qquad ; i = 2, \dots, n$$

and define the integers

$$v_i = \langle m\mu, \tilde{\alpha}_i^\vee \rangle \qquad ; i = 1, \dots, n. \tag{4.4}$$

Then (see [**DL76**, 9.5+9.6]) the restriction of the homogeneous line bundle $\mathcal{L}(\lambda)$ to $X(w) \subseteq X$ is trivial, that is, $\mathcal{L}(\lambda)|_{X(w)} \simeq \mathcal{O}_{X(w)}$. Furthermore, $j'^* \mathcal{L}(m\lambda)$ has a section $\Psi(\dot{w})$ with divisor of zeros

$$\big(\Psi(\dot{w})\big)_0 = \sum_{i=1}^n v_i D_i.$$

Hence

$$j'^* \mathcal{L}(m\lambda) \simeq \mathcal{O}_{\bar{X}(w)}\Big(\sum_{i=1}^n v_i D_i\Big). \tag{4.5}$$

Now consider the equations and inequalities in the unknown $x \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\langle x, \tilde{\alpha}_i^\vee \rangle = -1 \qquad ; i = 1, \dots, n \tag{4.6}$$

$$\langle Ax, \alpha_i^\vee \rangle < q - 1 \qquad ; i = 1, \dots, N. \tag{4.7}$$

As $\langle x, \tilde{\alpha}_i^\vee \rangle = \langle x, s_1 \cdot \ldots \cdot s_{i-1}(\alpha_i)^\vee \rangle = \langle s_{i-1} \cdot \ldots \cdot s_1(x), \alpha_i^\vee \rangle$, the first equation says that the i'th coordinate in $s_{i-1} \cdot \ldots \cdot s_1(x)$ is -1 for $i = 1, \dots, n$.

We can then give a nice description of $K_{\bar{X}(w)}$.

[3] Note that in [**MR88**] it is the *negative* roots that are chosen to correspond to B. Therefore we use ρ instead of $-\rho$.

PROPOSITION 4.13. *Keep the above notation. Suppose* $\mu \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ *solves* (4.6)*. Let* m *be a positive integer such that* $A(m\mu) \in X(T)$*. Then*

$$K_{\bar{X}(w)}^{\otimes m} = j'^{*}\mathcal{L}(m((1-q)\rho + A\mu)). \tag{4.8}$$

If μ *also solves* (4.7)*,* $\mathcal{L}(m((1-q)\rho + A\mu))$ *is a* very ample *line bundle.*

PROOF. By Proposition 4.11, $K_{\bar{X}(w)} = \mathcal{O}_{\bar{X}(w)}(-\partial \bar{X}(w)) \otimes j'^{*}\mathcal{L}((1-q)\rho)$. From Remark 4.12 and the assumption (4.6) on μ we have

$$j'^{*}\mathcal{L}(A(m\mu)) = \mathcal{O}_{\bar{X}(w)}\Big(\sum_{i=1}^{n} m\langle x, \tilde{\alpha}_i^{\vee}\rangle D_i\Big) = \mathcal{O}_{\bar{X}(w)}(-\partial \bar{X}(w))^{\otimes m}.$$

Hence $K_{\bar{X}(w)}^{\otimes m} = j'^{*}\mathcal{L}(m((1-q)\rho + A\mu))$. The second assumption (4.7) on μ assures that $\mathcal{L}(m((1-q)\rho + A\mu))$ is ample. □

In the A_N and ${}^2\mathrm{A}_N$ cases explicit calculations are amenable and we get an ever nicer description.

THEOREM 4.14. a) *Let* G *be simple of type* A_N *(of any isogeny type) and let* $\bar{X}(w)$ *be* the *corresponding Deligne-Lusztig variety. Then*

$$K_{\bar{X}(w)} = j'^{*}\mathcal{L}((1-q)\rho + q\lambda_N + \lambda_1) \tag{4.9}$$

b) *Let* G *be simple of type* ${}^2\mathrm{A}_{2n-1}$ *(of any isogeny type) and let* $\bar{X}(w)$ *be* the *corresponding Deligne-Lusztig variety. Then*

$$K_{\bar{X}(w)} = j'^{*}\mathcal{L}((1-q)\rho + q\lambda_n - q\lambda_{n-1} + \lambda_1). \tag{4.10}$$

c) *Let* G *be simple of type* ${}^2\mathrm{A}_{2n}$ *(of any isogeny type) and let* $\bar{X}(w)$ *be* the *corresponding Deligne-Lusztig variety. Then*

$$K_{\bar{X}(w)}^{\otimes 2} = j'^{*}\mathcal{L}(\sigma) \tag{4.11}$$

where

$$\begin{aligned}\sigma = {} & (4-2q)\lambda_1 + (2-2q)\lambda_2 + \ldots + (2-2q)\lambda_{n-1} + (2-q)\lambda_n \\ & + (-\lambda_{n+1}) + (2-2q)\lambda_{n+2} + \ldots + (2-2q)\lambda_N.\end{aligned}$$

For $q > 2$ *this is the pull-back of a very ample line bundle on* X*.*

PROOF. Suppose $\mu = \sum_{i=1}^{N} t_i\lambda_i \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfies (4.6). Then

$$\begin{aligned}
-1 &= \langle \mu, \alpha_1^{\vee}\rangle = t_1 \\
-1 &= \langle \mu, s_1(\alpha_2)^{\vee}\rangle = t_1 + t_2 = -1 + t_2 \\
&\ \vdots \\
-1 &= \langle \mu, s_1 \cdot \ldots \cdot s_{n-1}(\alpha_n)^{\vee}\rangle = t_1 + \ldots + t_n = -1 + 0 + \ldots + 0 + t_n.
\end{aligned}$$

It follows that μ must be on the form $\mu = -\lambda_1 + \sum_{i=n+1}^{N} t_i\lambda_i$. On the other hand, any μ on this form solves (4.6). (In the A_N case $n = N$, hence $\mu = -\lambda_1$.)

As $(s_1 \cdot \ldots \cdot s_n)^{-1}(-\lambda_1) = \lambda_n - \lambda_{n+1}$ and as $(s_1 \cdot \ldots \cdot s_n)^{-1}(\lambda_i) = \lambda_i$ for $i > n$, we get

$$w^{-1}(\mu) = \lambda_n + (t_{n+1} - 1)\lambda_{n+1} + t_{n+2}\lambda_{n+2} + \ldots + t_N\lambda_N.$$

Hence in the A_N case where F acts by multiplication with q, we have

$$\begin{aligned} K_{\bar{X}(w)} &= j'^{*}\mathcal{L}((1-q)\rho + F(w^{-1}(\mu)) - \mu) \\ &= j'^{*}\mathcal{L}((1-q)\rho + q\lambda_N + \lambda_1) \end{aligned}$$

and the assertion in *a)* follows.

We have seen that for $N \neq n$ we may write

$$A\mu = \begin{bmatrix} 1+qt_N \\ qt_{N-1} \\ \vdots \\ qt_{N-(n-3)} \\ q(t_{N-(n-2)}-1) \\ q \\ -t_{N-(n-2)} \\ \vdots \\ -t_N \end{bmatrix} \text{ and } A\mu = \begin{bmatrix} 1+qt_N \\ qt_{N-1} \\ \vdots \\ qt_{N-(n-2)} \\ q(t_{N-(n-1)}-1) \\ q - t_{N-(n-1)} \\ -t_{N-(n-2)} \\ \vdots \\ -t_N \end{bmatrix}$$

for $N = 2n-1$ and $N = 2n$, respectively. For odd N we see that the n'th coordinate of $A\mu + (1-q)\rho$ always will be $q + (1-q) = 1$, so we cannot realize $K_{\bar{X}(w)}$ as the pull back of an ample homogeneous line bundle $\mathcal{L}(\lambda)$ on X. Choosing $t_i = 0$ for $i > 0$ we obtain *b)*.

For $N = 2n$ we are in better shape. Choosing t_{n+1} equal to $3/2$ and $t_i = 0$ otherwise, we get that $\mu = -\lambda_1 + 3/2\lambda_{n+1}$ solves (4.6). Furthermore, with this choice

$$\begin{aligned} 2(A\mu + (1-q)\rho) = (4-2q)\lambda_1 &+ (2-2q)\lambda_2 + \ldots + (2-2q)\lambda_{n-1} \\ &+ (2-q)\lambda_n + (-\lambda_{n+1}) + (2-2q)\lambda_{n+2} + \ldots + (2-2q)\lambda_N \end{aligned}$$

is strictly anti-dominant and *c)* follows. □

COROLLARY 4.15. *Assume $q > 2$. Let $\bar{X}(w)$ be a Deligne-Lusztig variety corresponding to the ${}^2\mathrm{A}_N$ case (N even). Then $K_{\bar{X}(w)}$ is ample. Hence $\bar{X}(w)$ is a variety of general type.*

PROOF. Combining Theorem 4.14 and Remark 4.10 it follows that $K_{\bar{X}(w)}^{\otimes 2}$ is ample. But then so is $K_{\bar{X}(w)}$. □

COROLLARY 4.16. *Let $\bar{X}(w)$ be a Deligne-Lusztig variety corresponding to the ${}^2\mathrm{A}_N$ case (N odd). Then $K_{\bar{X}(w)}$ is* not *ample.*

PROOF. Suppose $K_{\bar{X}(w)} = j'^{*}\mathcal{L}((1-q)\rho + q\lambda_n - q\lambda_{n-1} + \lambda_1)$ is ample. Let C be one of the components of $\bar{X}(s_n)$. Then, since j' is finite and surjective, the degree of $K_{\bar{X}(w)}$ along C is the same as the degree of $\mathcal{L}((1-q)\rho + q\lambda_n +$

λ_1) along $j'(C)$. This degree should be positive [**Ful83**, Lemma 12.1]. By the projection formula [**Ful83**, Proposition 3.1.(c)] we may calculate this degree on either $j'(C)$ or X. Now, since $j'(C) \simeq \mathbb{P}^1$ is F-stable and $\mathrm{A}_1(X)$ is freely generated by the Schubert varieties $X_{s_1} \ldots, X_{s_N}$ [**Dem74**], we must have $[j'(C)] = [X_{s_n}]$ in $\mathrm{A}_1(X)$. By [**Dem74**, Corollaire 1, p. 78] we then get (using the notation of [**Ful83**])

$$\begin{aligned}\deg K_{\bar{X}(w)}|_C &= \deg \mathcal{L}((1-q)\rho + q\lambda_n - q\lambda_{n-1} + \lambda_1)|_{j'(C)} \\ &= \int_X c_1(\mathcal{L}((1-q)\rho + q\lambda_n - q\lambda_{n-1} + \lambda_1)) \cap [j'(C)] \\ &= \int_X c_1(\mathcal{L}((1-q)\rho + q\lambda_n - q\lambda_{n-1} + \lambda_1)) \cap [X_{s_n}] \\ &= \langle (1-q)\rho + q\lambda_n - q\lambda_{n-1} + \lambda_1, -\alpha_n^\vee \rangle = -1\end{aligned}$$

contradicting the positivity of $K_{\bar{X}(w)}$. □

REMARK 4.17. Since $K_{\bar{X}(w)}$ is ample in the ${}^2\mathrm{A}_N$ case (N even), it follows that $K_{\bar{X}(w)}^{-1}$ has no global sections [**Har77**, Exercise III.7.1]. Hence $\bar{X}(w)$ cannot be Frobenius split in the sense of [**MR85**, Definition 2]. For N odd, other methods give that $K_{\bar{X}(w)}^{-1}$ cannot have global sections, and we arrive at the same conclusion. So despite the similarity in definitions, Deligne-Lusztig varieties and Schubert varieties are very different from each other (the latter being Frobenius-split varieties).

4.5. Examples

In the following examples we will use the above results to describe the canonical bundles and compare this to already known results.

EXAMPLE 4.18 (A_2 case). Let $\bar{X}(w)$ correspond to the Dynkin diagram

$$s_1 \text{———} s_2.$$

From Theorem 4.14 we get that $K_{\bar{X}(w)}^{\otimes \det A} = j'^*\mathcal{L}(A\mu)$ with $\det A = q^2 + q + 1$ and

$$\mu = (\det A) A^{-1} \begin{bmatrix} 2-q \\ 1 \end{bmatrix}.$$

From Remark 4.12 it then follows that

$$K_{\bar{X}(w)}^{\otimes \det A} = \mathcal{O}_{\bar{X}(w)}((q^2 - 2q - 2)D_1 - 3D_2).$$

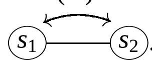
We may compare this with the following. Let $P_i = B \cup Bs_iB$, $i = 1, 2$. Restricting the composition $\pi : X \times X \times X \xrightarrow{\mathrm{pr}_1} X \to G/P_2 \simeq \mathbb{P}^2$ to $\bar{X}(w)$ realizes $\bar{X}(w)$ as the blow-up of $\mathbb{P}^2$ in the points $G^F/P_2^F \simeq \mathbb{P}^2(\mathbb{F}_q)$ (of which there are $q^2 + q + 1$). The exceptional divisor E is identified with $\bar{X}(s_2) = D_1$ and the divisor D_2 maps to a union of $q^2 + q + 1$ lines. Therefore $q^2 + q + 1$ times the hyper-plane class

h pulls back to $D_2 + (q+1)D_1$ (each copy of h has $q+1$ rational points on it). Hence

$$\begin{aligned} K_{\bar{X}(w)}^{\otimes q^2+q+1} &= \pi^* K_{\mathbb{P}^2}^{\otimes q^2+q+1} + \mathcal{O}_{\bar{X}(w)}((q^2+q+1)E) \\ &= \mathcal{O}_{\bar{X}(w)}(-3(q^2+q+1)\pi^* h + (q^2+q+1)E) \\ &= \mathcal{O}_{\bar{X}(w)}(-3D_2 - 3(q+1)D_1 + (q^2+q+1)D_1) \\ &= \mathcal{O}_{\bar{X}(w)}((q^2-2q-2)D_1 - 3D_2) \end{aligned}$$

as expected.

EXAMPLE 4.19 (${}^2\mathrm{A}_2$ case). Let $\bar{X}(w)$ correspond to the Dynkin diagram



In this example we will approach the problem in a little different manner. Take $\mu = -(q+1)\lambda_1 + (2q-1)\lambda_2$. Then $A\mu = \det(A)\rho = (q^2-q+1)(\lambda_1+\lambda_2)$ and therefore

$$\begin{aligned} j'^*\mathcal{L}(\det(A)\rho) &= \mathcal{O}_{\bar{X}(w)}(\langle \mu, s_0(\alpha_1)^\vee \rangle X(e)) \\ &= \mathcal{O}_{\bar{X}(w)}(-(q+1)X(e)). \end{aligned}$$

Similarly we find that

$$\begin{aligned} j'^*\mathcal{L}(A(\lambda_1 - 2\lambda_2)) &= j'^*\mathcal{L}(-(q+1)\lambda_1 - (q-2)\lambda_2) \\ &= \mathcal{O}_{\bar{X}(w)}(X(e)). \end{aligned}$$

So we have two different line bundles restricting to line bundles of degree $-(q+1)|X(e)|$ and $|X(e)|$ respectively. Solving for λ_1 and λ_2 (using $|X(e)| = q^3+1$) we find that $\mathcal{L}(\lambda_1)$ and $\mathcal{L}(\lambda_2)$ restricts to line bundles of degrees $-(q+1)$ and $-q(q+1)$ respectively. Hence $\mathcal{L}((1-q)\rho)$ restricts to a line bundle of degree $(1-q)(-(q+1)-q(q+1))$. Therefore

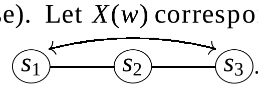
$$\begin{aligned} \deg K_{\bar{X}(w)} &= (1-q)(-(q+1)-q(q+1)) - |X(e)| \\ &= q^2 - q - 2. \end{aligned}$$

Since we know that $\bar{X}(w)$ in this case is a non-singular plane curve of degree $q+1$ (see [**Han92**]) we can check:

$$\begin{aligned} \deg K_{\bar{X}(w)} &= 2g - 2 = ((q+1)-1)((q+1)-2) - 2 \\ &= q^2 - q - 2 \end{aligned}$$

Alternatively, by Theorem 4.14, we have $K^2_{\bar{X}(w)} = j'^*\mathcal{L}((4-q)\lambda_1 - \lambda_2)$ and we find that $K_{\bar{X}(w)}^{\otimes \det A} = \mathcal{O}_{\bar{X}(w)}((q-2)|X(e)|)$ (and $(q-2)|X(e)| = \det A \cdot \deg K_{\bar{X}(w)}$). $K_{\bar{X}(w)}$ is ample if and only if $\deg K_{\bar{X}(w)} > 0$, that is, if and only if $q > 2$.

EXAMPLE 4.20 (${}^2\mathrm{A}_3$ case). Let $\bar{X}(w)$ correspond to the Dynkin diagram



$\bar{X}(w)$ is the blow-up of the smooth Fermat surface S in $\mathbb{P}^3$ (of degree $q+1$) in its $(q^3+1)(q^2+1)$ $\mathbb{F}_{q^2}$-rational points [**Rod96**]. Let E be the exceptional divisor. We have $K_S = \mathcal{O}_S(q-3)$ [**Har77**, Example II.8.20.3], hence [**Har77**, Exercise II.8.5]

$$K_{\bar{X}(w)} = \mathcal{O}_{\bar{X}(w)}(q-3) \otimes \mathcal{O}_{\bar{X}(w)}(E).$$

Since

$$\deg K^2_{\bar{X}(w)} = (q+1)(q-3)^2 - (q^3+1)(q^2+1) < 0$$

[**Har77**, Exercise V.1.5] for any q, $K_{\bar{X}(w)}$ cannot be ample.

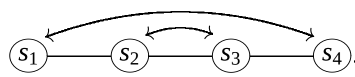
Since $(x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1})^{p-1}$ always is of degree strictly larger than p, S cannot be Frobenius split [**Koc97**, Corollary 2.7]. So neither is $\bar{X}(w)$ (if it were, we could use [**MR85**, Proposition 4] to produce a splitting of S).

Following the procedure from Theorem 4.14 we find

$$K_{\bar{X}(w)} = j'^*\mathcal{L}((2(1-q)+qt)\lambda_1 + \lambda_2 + (1-q-t)\lambda_3)$$

for any integer t. Corollary 4.16 then predicts that the restriction of $K_{\bar{X}(w)}$ to any of the exceptional divisors has degree -1 (this was of course to be expected *cf.* [**Har77**, Section V.3]).

EXAMPLE 4.21 ($^2A_4$ case). Let $\bar{X}(w)$ correspond to the Dynkin diagram



We find that

$$K^{\otimes 2}_{\bar{X}(w)} = j'^*\mathcal{L}((4-2q)\lambda_1 + (2-q)\lambda_2 - \lambda_3 + (2-2q)\lambda_4)$$

is ample for any $q > 2$. Actually this gives an infinite collection of positive characteristic counter-examples to the Miyaoka-Yau inequality [**Miy77**, Theorem 4]:

$$c_1(\bar{X}(w))^2 \leq 3c_2(\bar{X}(w))$$

where $c_i(\bar{X}(w)) = c_i(T_{\bar{X}(w)})$. Hence the inequality reads: $K^2_{\bar{X}(w)} \leq e(\bar{X}(w))$ where $e(\bar{X}(w))$ is topological Euler characteristic of $\bar{X}(w)$ (the alternating sum of the Betti numbers). Proceeding as in Example 4.19 we find that

$$K^{\otimes \det(A)}_{\bar{X}(w)} = \mathcal{O}_{\bar{X}(w)}(v_1 D_1 + v_2 D_2)$$

with $\det(A) = q^4 - q^3 + q^2 - q + 1$, $v_1 = q^4 - 2q^2 + 2q - 2$ and $v_2 = q^3 + q - 3$. D_1 is a disjoint union $(q^5+1)(q^2+1)$ nonsingular Hermitian curves C_1 of genus $g(C_1) = \frac{1}{2}q(q-1)$ and D_2 is a disjoint union $(q^5+1)(q^3+1)$ rational curves C_2 of genus $g(C_2) = 0$. By the adjunction formula [**Har77**, Proposition V.1.5] we then find

$$C_1^2 = \frac{\det(A)(2g(C_1)-2) - v_2|C_1(\mathbb{F}_{q^2})|}{\det(A)+v_1} = -(q+1)$$

$$C_2^2 = \frac{\det(A)(2g(C_2)-2) - v_1|C_2(\mathbb{F}_{q^2})|}{\det(A)+v_2} = -q^2$$

where $|C_1(\mathbb{F}_{q^2})| = q^3+1$ and $|C_2(\mathbb{F}_{q^2})| = q^2+1$. Hence

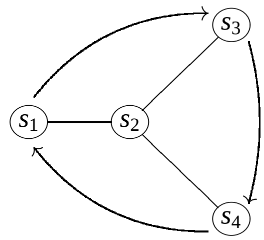
$$\begin{aligned} K^2_{\bar{X}(w)} &= \frac{(v_1D_1+v_2D_2)^2}{(\det(A))^2} \\ &= \frac{1}{(\det A)^2}\big(2v_1v_2(q^5+1)(q^3+1)(q^2+1) \\ &\qquad - v_1^2(q^5+1)(q^2+1)(q+1) - v_2^2(q^5+1)(q^3+1)q^2\big) \\ &= (q+1)(2q^8-3q^6-5q^4+5q^3-q^2-4q+8). \end{aligned}$$

From [**DL76**, Theorem 7.1] one easily calculates

$$e(\bar{X}(w)) = q^8+q^6-q^4+2q^3-q^2+2q+4$$

(using that the Euler characteristic is additive with respect to the decomposition of $\bar{X}(w)$). So for any $q>2$ we obtain a surface S defined over $\mathbb{F}_{q^2}$ such that $c_1(S)^2 > (q+1)c_2(S)$. From [**SB91**, Corollary 15] it follows that $\Omega^1_{\bar{X}(w)}$ is (Bogomolov)-unstable.

EXAMPLE 4.22 ($^3\mathrm{D}_4$ case). Let $\bar{X}(w)$ correspond to the Dynkin diagram



Using the same approach as in the proof of Theorem 4.14 we find that we may realize $K_{\bar{X}(w)}$ as

$$K_{\bar{X}(w)} = j'^*\mathcal{L}(2\lambda_1+\lambda_2-(q+1)\lambda_3-\lambda_4).$$

Again we see that $K_{\bar{X}(w)}$ has negative degree along the lines corresponding to the fixed node.

4.6. Complements

Above we saw that in order to know that we are restricting an ample bundle to $\bar{X}(w)$, we must assume $q>2$. Of course $K_{\bar{X}(w)}$ can be ample for other reasons, but we cannot in general expect Corollary 4.15 to be true also for $q=2$ *cf.* Example 4.19.

In [**Rod96**] it is mentioned that the Deligne-Lusztig surface corresponding to the $^2\mathrm{A}_4$ case (*cf.* Example 4.21) is of general type for any choice of q. The proof uses different methods than those of this paper (private communication with F. Rodier).

As Example 4.22 suggests, one may of course apply this papers methods to the non-$^*\mathrm{A}_n$ cases. I expect that Corollary 4.16 can be proved (with the same techniques, case by case) for any Deligne-Lusztig variety coming from a

Dynkin diagram where (at least) one node is fixed under the action induced by F.

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(bibliography omitted; end of article)

4.7. Further remarks regarding the ampleness of $K_{\bar{X}(w)}$

Let $\bar{X}(w)$ and $\bar{X}(w')$ come from F-conjugate Weyl group elements. From the proof of Lemma 3.5 and the description of the canonical bundles given above, it follows that a power of $K_{\bar{X}(w')}$ is the pull-back of $K_{\bar{X}(w)}$ under a finite surjective morphism. Hence $K_{\bar{X}(w)}$ is ample if and only if $K_{\bar{X}(w')}$ is [**Har77**, Exercise III.5.7]. This extends the results of [**HH99a**].

In [**HH99a**] we omitted the most trivial example of calculating $K_{\bar{X}(w)}$. For instructional purposes we give it here:

EXAMPLE 4.23 (A_1 case). For $G = \mathrm{A}_1$ we have $\bar{X}(w) = \bar{X}(s) = G/B = \mathbb{P}^1$. Let $\mu = -\frac{1}{2}\alpha_1 = -\lambda_1$. Then $\langle \mu, s_0(\alpha_1)^\vee \rangle = \langle \lambda_1, \alpha_1^\vee \rangle = -1$. Hence

$$\mathcal{O}_{\bar{X}(w)}(-\partial \bar{X}(w)) = j'^*\mathcal{L}(F(s(\mu)) - \mu) = j'^*\mathcal{L}((q+1)\lambda_1)$$

and

$$\begin{aligned} K_{\bar{X}(w)} &= \mathcal{O}_{\bar{X}(w)}(-\partial \bar{X}(w)) \otimes j'^*\mathcal{L}((1-q)\rho) \\ &= j'^*\mathcal{L}(2\lambda_1) = j'^*\mathcal{O}_{\mathbb{P}^1}(-2) = \mathcal{O}_{\bar{X}(w)}(-2). \end{aligned}$$

This is consistent with the fact that $\deg K_{\mathbb{P}^1} = 2g - 2 = -2$. (Alternatively: for $\tau = \frac{q-1}{q+1}\lambda_1$ we have $F(s(\tau)) - \tau = (1-q)\rho$, hence

$$j'^*\mathcal{L}((1-q)\rho) = \mathcal{O}_{\bar{X}(w)}(\langle \tau, s_0(\alpha_1)^\vee \rangle X(e)) = \mathcal{O}_{\bar{X}(w)}(q-1)$$

and again $K_{\bar{X}(w)} = \mathcal{O}_{\bar{X}(w)}(-2)$.)

EXAMPLE 4.24 (${}^2\mathrm{A}_4$ case). Let us now move to our favorite example, the ${}^2\mathrm{A}_4$-case (see also Example 3.17). Following [**HH99a**] we determine the matrix A representing the automorphism $\lambda \to (F(w^{1-}(\lambda)) - \lambda)$ of the character group $X(T)$. We get

$$A := \begin{bmatrix} -1 & 0 & 0 & q \\ q & q-1 & q & 0 \\ -q & -q & -1 & 0 \\ 0 & q & 0 & -1 \end{bmatrix}.$$

Now we let Maple help us a bit in calculating the canonical bundle $K_{\bar{X}(w)}$, its self-intersection and its intersection with (the components of) the divisors D_1 and D_2.

```
> D:=det(A);
```

$$D := -q + 1 + q^2 - q^3 + q^4$$

```
> v:=evalm(inverse(A) &* vector([4-2*q,2-q,-1,2-2*q]));
```

$$v := \left[2\,\frac{2\,q - 2\,q^2 - 2 + q^4}{-q+1+q^2-q^3+q^4},\ -2\,\frac{-2\,q^2 - q^3 + q + 1 + q^4}{-q+1+q^2-q^3+q^4},\right.$$
$$\left. -\frac{-5\,q + q^2 + q^3 + q^4 - 1}{-q+1+q^2-q^3+q^4},\ -2\,\frac{-4\,q^3 + q^4 + 3\,q^2 - q + 1}{-q+1+q^2-q^3+q^4}\right]$$

Hence, $\det(A)$ times $K_{\bar{X}(w)}^{\otimes 2}$ is the pull-back of the bundle $\mathcal{L}(A(\det(A)v))$. Setting

```
> a:= simplify(det(A)*v[1]/2);
```

$$a := 2\,q - 2\,q^2 - 2 + q^4$$

```
> b:=simplify(det(A)*v[2]/2+a);
```

$$b := q^3 + q - 3$$

it follows [**HH99a**] that $\det(A)$ times the canonical divisor is linearly equivalent to the divisor $aD_1 + bD_2$.

D_1 consists of $(q^5+1)(q^2+1)$ disjoint Hermitian plane curves of degree $q+1$:

```
> g(C1):=(q^2-q)/2;
```

$$\mathrm{g}(C1) := \frac{1}{2}\,q^2 - \frac{1}{2}\,q$$

D_2 consists of $(q^5+1)(q^3+1)$ disjoint rational curves:

```
> g(C2):=0;
```

$$\mathrm{g}(C2) := 0$$

We now use the adjunction formula [**Har77**, Proposition V.1.5] to compute the self-intersections:

```
> C1_square:=(det(A)*(2*g(C1)-2)-b*(q ^3+1))/(det(A)+a);
```

$$C1\_square := -q - 1$$

```
> C2_square:=(det(A)*(2*g(C2)-2)-a*(q ^2+1))/(det(A)+b);
```

$$C2\_square := -q^2$$

```
> D1_square:=simplify((q^5+1)*(q^2+1)*C1_square );
```

$$D1\_square := -(q^5+1)\,(q^2+1)\,(q+1)$$

```
> D2_square:=simplify((q^5+1)*(q^3+1)*C2_square );
```

$$D2\_square := -(q^5+1)\,(q^3+1)\,q^2$$

and the intersections with $K_{\bar{X}(w)}$:

```
> KC1:=2*g(C1)-2-C1_square;
```

$$KC1 := q^2 - 1$$

```
> KC2:=2*g(C2)-2-C2_square;
```

$$KC2 := q^2 - 2$$

We know that D_1 and D_2 intersect transversely in $(q^5+1)(q^3+1)(q^2+1)$ points, hence

$$\begin{aligned} K_X^2 &= \frac{2ab(q^5+1)(q^2+1) + a^2D_1^2 + b^2D_2^2}{\det(A)^2} \\ &= (q+1)(2q^8 - 3q^6 - 5q^4 + 5q^3 - q^2 - 4q + 8). \end{aligned}$$

In Example 3.17 we found that

$$\rho^*\mathcal{O}_{\mathbb{P}^4}(q^5+1) = (q^3+1)D_1 + (q+1)D_2 \tag{4.12}$$

with $\rho^*\mathcal{O}_{\mathbb{P}^4}(q^5+1)$ having degree zero along D_1. Hence

$$\begin{aligned} (q+1)^2D_2^2 = (q^5&+1)^2(q^3+1)(q+1) \\ &+ (q^3+1)^2D_1^2 + \deg_{\rho^*\mathcal{O}_{\mathbb{P}^4(q^5+1)}}(D_1) \\ &= -(q^5+1)(q^3+1)(q+1)((q^2+1)(q^3+1) - q^5 - 1) \\ &= -(q^5+1)(q^3+1)(q+1)^2q^2. \end{aligned}$$

So also from this calculation we get $D_2^2 = -(q^5+1)(q^3+1)q^2$.

Using (4.12) we may rewrite the formula for $K_{\bar{X}(w)}$ as

$$\begin{aligned} (q^5+1)K_{\bar{X}(w)} &= (q+1)(q^4 - 2q^2 + 2q - 2)D_1 \\ &\quad + \rho^*\mathcal{O}_{\mathbb{P}^4}((q^5+1)(q^3+q-3)) \\ &\quad - (q^3+1)(q^3+q-3)D_1 \\ &= (q^5+1)\big(\rho^*\mathcal{O}_{\mathbb{P}^4}(q^3+q-3) - (q+1)D_1\big). \end{aligned}$$

Since Z is a complete intersection of multi-degree $(q+1, q^3+1)$ we have $\omega_Z = \mathcal{O}_Z(q+1+q^3+1-4-1) = \mathcal{O}_Z(q^3+q-3)$. So pulling back K_Z to $\bar{X}(w)$ gives $K_{\bar{X}(w)}$ and a multiple of the exceptional divisor D_1 — as expected *cf.* [**Har77**, Exercise II.8.5].

CHAPTER 5

Error-correcting codes from higher-dimensional varieties

On the following pages we have reproduced the paper *Error-correcting codes from higher-dimensional varieties* [**HH98**]. Please observe that the references made in the paper refer to the bibliography at the end of this thesis.

Error-Correcting Codes from higher-dimensional varieties*

Søren Have Hansen

Abstract. We give ways to obtain information on the parameters of algebraic geometric error-correcting codes constructed from any variety over a finite field. The methods are then applied to a Deligne-Lusztig surface of general type yielding very long codes over $\mathbb{F}_{q^2}$, with parameters n,k,d satisfying $d+k \geq n - O(n^{4/5})$, $n \sim q^{10}$.
We also consider examples coming from Hermitian hyper-surfaces and ruled surfaces over Hermitian curves.

Introduction

Let $\mathbb{F}_q$ denote the finite field with q elements (q a prime power). Goppa and Manin gave in the early eighties an algebraic geometric method to construct error-correcting codes over $\mathbb{F}_q$. Most work so far have concentrated on the case where the codes arise from a non-singular projective curve. In this paper we consider the higher dimensional cases. We prove several results. Let us mention:

Theorem 5.1. *Let X be a non-singular projective surface defined over $\mathbb{F}_q$. Let L be a nef (numerically effective) line bundle on X and let H be an ample divisor, both defined over $\mathbb{F}_q$. Let $C_1,\dots,C_a$ be curves on X with $\mathbb{F}_q$-rational points $\mathcal{P} = \{P_1,\dots,P_n\}$. Assume that $L\cdot H < C_i \cdot H$ for all i. Then the code $C(L,\mathcal{P})$ has length n and minimum distance*

$$d \geq n-m \qquad ; m = \sum_i^a \deg_L(C_i).$$

For $m<n$, the code has dimension $k = \dim_{\mathbb{F}_q} \Gamma(X,L)$.

Since many ‘good’ codes arise from so-called Deligne-Lusztig curves, it is natural to apply a version of the above construction to Deligne-Lusztig surfaces. Taking one particular surface we obtain:

Theorem 5.2. *Let X be the Deligne-Lusztig surface of type ${}^2\mathrm{A}_4$ defined over the field $\mathbb{F}_{q^2}$. Then, for $t = 1,2,\dots,q$, we can construct a code on X over $\mathbb{F}_{q^2}$ with parameters*

$$\begin{aligned} n &= (q^5+1)(q^3+1)(q^2+1) \\ k &= \frac{1}{24}(t+1)(t+2)(t+3)(t+4) \\ d &\geq n - tP(q) \end{aligned}$$

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where $P(q)$ *is the monic polynomial* $P(q) = (q^3+1)(q^5+1) + (q+1)(q^3+1)(q^2 - t + 1)$ *of degree* 8 *in* q*.*

For practical purposes we need the defining equations of the Deligne-Lusztig varieties. Since any 'classical' Deligne-Lusztig variety is a blow-up of a complete intersection of Fermat hyper-surfaces (*cf.* [**HH99c, Lus76b**]) we can give the equations for any (reasonable small) choice of q. In particular, the surface used above is given by blowing up the $\mathbb{F}_{q^2}$-rational points of the intersection $H_1 \cap H_2 \subset \mathbb{P}^4$, where each hyper-surface H_i is the zero locus of $f_i = \sum_{j=0}^{4} x_j^{q^{2i-1}+1}$. Using another construction (Proposition 5.19) we are also able to determine the parameters of codes constructed from this (singular) complete intersection.

The paper is organized as follows: In Section 5.1 we review the definitions of Deligne-Lusztig varieties. In Section 5.2 we describe the algebraic-geometric way of constructing codes and we give some examples coming from Deligne-Lusztig curves. These examples should motivate considering higher-dimensional Deligne-Lusztig varieties. In Section 5.3 we prove the theoretical results which we use in Section 5.4 to give some examples.

5.1. Deligne-Lusztig varieties

In this section we shall define Deligne-Lusztig varieties. General references are [**DL76**, Sections 2 and 9] and the introduction in [**Han92**].

Let (G, F) be a connected reductive algebraic group equipped with an $\mathbb{F}_q$-structure coming from a Frobenius morphism $F : G \to G$. Let $L : G \to G$ be the Lang map taking an element $g \in G$ to $g^{-1}F(g)$. By the Lang-Steinberg Theorem [**Bor92**, Theorem 16.3] this morphism of varieties is a surjective isogeny with finite fibres. From this result it follows that, by conjugacy of tori and Borel subgroups, there exists F-stable maximal tori and Borel subgroups. Hence there are (with abuse of notation) natural endomorphisms $F : W \to W$ and $F : X \to X$ of the Weyl group of G and the variety X of Borel subgroups of G. Let W be generated by the simple reflections $s_1, \ldots, s_N$ and let $l(\cdot)$ be the length function with respect to these generators. F induces a permutation of the simple reflections, let δ denote the order of this permutation.

Choose an F-stable Borel subgroup $B \subseteq G$ containing an F-stable maximal torus T. Let $\pi : G \to G/B \simeq X$ be the projection.

DEFINITION 5.3. Fix an element w in the Weyl group W and let $w = s_{i_1} \cdot \ldots \cdot s_{i_n}$ be a reduced expression. Call w a *Coxeter element* if there is exactly one s_i from each of the orbits of F on $\{s_1, \ldots, s_N\}$.

1. Set $S(\dot{w}) = L^{-1}(B\dot{w}B)$. Then we define the *Deligne-Lusztig variety* $X(w)$ to be the image of $S(\dot{w})$ in G/B. That is,

$$X(w) = \pi(S(\dot{w})).$$

   We shall think of the points of $X(w)$ as Borel subgroups of G.

2. Define the closed subvariety of X^{n+1}

$$\bar{X}(s_{i_1},\dots,s_{i_n}) = \{(g_0B,\dots,g_nB) \in X^{n+1} : \\ g_k^{-1}g_{k+1} \in B \cup Bs_{i_{k+1}}B \text{ for } 0 \le k < n,\ g_n = F(g_0)\}.$$

For brevity, we will write $\bar{X}(w)$ for this variety. For any $w' \le w$, $\bar{X}(w')$ defines in a natural way a closed subvariety of $\bar{X}(w)$. In particular there are divisors

$$D_j = \bar{X}(s_{i_1},\dots,\hat{s}_{i_j},\dots,s_{i_n}) \qquad ;\ j = 0,\dots,n.$$

Since L is flat, it is open, hence $\overline{L^{-1}(B\dot{w}B)} = L^{-1}\big(\overline{B\dot{w}B}\big)$. So $X(w)$ is non-singular of dimension n and the closure of $X(w)$ in X is given by the disjoint union

$$\overline{X(w)} = \bigcup_{w' \le w} X(w'), \tag{5.1}$$

where as usual $\le$ is the Bruhat order in W. This closure is singular whenever the Schubert variety $X_w = \overline{B\dot{w}B}/B$ is. But since the open subset

$$\{(g_0B,\dots,g_nB) \in X^{n+1} : g_k^{-1}g_{k+1} \in Bs_{i_{k+1}}B\ (0 \le k < n),\ g_n = F(g_0)\}$$

of the non-singular projective variety $\bar{X}(w)$ maps isomorphically onto $X(w)$ under projection to the first factor, we have a good compactification of $X(w)$ [**DL76**, 9.10]. In fact the complement of $X(w)$ in $\bar{X}(w)$, which is easily seen to be the union of the D_j's, is a divisor with normal crossings [**DL76**, 9.11]. If w is a Coxeter element, then $X(w)$ and $\bar{X}(w)$ are irreducible [**Lus76a**, Proposition (4.8)] and $\bar{X}(w)$ maps bijectively to $\overline{X(w)}$; see [**HH99a**].

REMARK 5.4. The Deligne-Lusztig variety $\bar{X}(w)$ is defined over the finite field $\mathbb{F}_{q^\delta}$ and has many rational points making $\bar{X}(w)$ a good candidate for constructing long codes over relatively small fields.

The finite group G^F of fixed-points under the Frobenius morphism acts on both $X(w)$ and $\bar{X}(w)$. Hence the codes (to be constructed below) will have the structure of an $\mathbb{F}_{q^\delta}[G^F]$-module.

5.2. Constructing the codes

In the theory of linear Error-Correcting Codes one is interested in, subject to some conditions, having an easy way of constructing k-dimensional subspaces of some n-dimensional $\mathbb{F}_q$-vectorspace.

DEFINITION 5.5. A (q-ary) $[n,k,d]$-code is a k-dimensional linear subspace S of $\mathbb{F}_q^n$ such that the *minimum distance*

$$\min_{(x,y)\in S\times S} d(x,y) = \min_{(x,y)\in S\times S} \#\{i : x_i \ne y_i\}$$

equals d. We call n the *length* of the code and k the *dimension*.

DEFINITION 5.6. Tsfasman and Vlăduţ introduced the following construction [**TV91**, Chapter 3.1] (generalizing the Goppa-Manin construction [**Gop88**]): Let X be a normal projective variety over $\mathbb{F}_q$. Let L be a line bundle defined over $\mathbb{F}_q$ and let $P_1, \dots, P_n$ be distinct $\mathbb{F}_q$-rational points on X. Set $\mathcal{P} = \{P_1, \dots, P_n\}$. In each point P_i, choose isomorphisms of the fibres L_{P_i} with $\mathbb{F}_q$. The *linear code* $C(L, \mathcal{P})$ of length n associated to $(X, L, \mathcal{P})$ is the image of the germ map

$$\alpha : \Gamma(X, L) \longrightarrow \bigoplus_{i=1}^{n} L_{P_i} \simeq \mathbb{F}_q^n. \tag{5.2}$$

We shall of course from now on assume that all line-bundles considered actually have non-zero global section.

Suppose L arises as the line bundle associated to a divisor D and that the P_i are not in the support of D. Then we get the same code (up to isomorphism) as when evaluating the rational functions

$$L(D) = \{f \in k(X)^* : div(f) + D \geq 0\}$$

in the points $\mathcal{P}$.

REMARK 5.7. One is interested in codes where, for a fixed minimum distance, the *information rate* k/n is as large as possible. Therefore one introduces the parameter $\delta = \frac{d}{n}$ and considers the function

$$\alpha(\delta) = \limsup_{n\to\infty} \tfrac{1}{n} \max\{k : \text{ there exists an } [n, k, \delta n] \text{ code over } \mathbb{F}_q\}. \tag{5.3}$$

There are both upper and lower bounds on $\alpha(\delta)$. Before Tsfasman, Vlăduţ and Zink (see [**TV91**]) in 1982 applied Goppas construction to families of modular curves and improved the lower bound, it was widely believed that the Gilbert-Varshamov lower bound from the fifties was the best possible.

EXAMPLE 5.8. For a curve C with $L = \mathcal{O}_C(G)$, the points $\mathcal{P}$ defines a divisor D and Goppa's construction yields a code $C(G, D)$ with parameters

$$k = \dim\, C(G, D) = l(G) - l(G - D) \tag{5.4}$$

hence

$$k = l(G) \geq \deg(G) - g + 1 \qquad \text{if } \deg(G) < \deg(D) = n \tag{5.5}$$

with equality if $\deg(G) > 2g - 2$ (by the Riemann-Roch theorem [**Har77**, Theorem IV.1.3]). The minimum distance satisfies

$$d \geq n - \deg(G). \tag{5.6}$$

We have the following strategy for constructing good codes from algebraic curves.

1. Find a smooth algebraic curve over $\mathbb{F}_q$ of genus g with a lot of rational points, say $P_0, \dots, P_n$.
2. Set $G = mP_0$ and $D = P_1 + \dots + P_n$.

3. Compute the image of the evaluation map $\alpha : L(G) \longrightarrow \mathbb{F}_q^n$. This amounts to computing an $\mathbb{F}_q$-rational basis of $L(G)$.

The resulting linear $[n,k,d]$-code will satisfy that $d + k \geq n + 1 - g$, or equivalently

$$\frac{d}{n} + \frac{k}{n} \geq 1 + \frac{1}{n} - \frac{g}{n}. \tag{5.7}$$

By the Hasse-Weil-Serre-Oesterlé bound there are restrictions on how many points a smooth curve of fixed genus can have [**Ser85, Wei52, Wei49**]. The following Deligne-Lusztig curves are maximal with respect to these bounds and have given some of the best possible codes from curves:

Hermitian Curves: Let C be the curve arising as the Deligne-Lusztig variety corresponding to the ${}^2\mathrm{A}_2$ case (see [**Han92**]). C is a plane curve defined over $\mathbb{F}_{q^2}$ with equation

$$C: \qquad X^{q+1} + Y^{q+1} + Z^{q+1} = 0.$$

Hence C has genus $g = \frac{1}{2}q(q-1)$ and $1 + q^2 + 2gq$ rational points (over $\mathbb{F}_{q^2}$). The resulting code (over $\mathbb{F}_{q^2}$) satisfies

$$d + k \geq 1 + q^3 - \tfrac{1}{2}(q^2 - q).$$

Suzuki Curves: Let $q = 2^{2m+1}$ be an odd power of 2; set $q_0 = 2^m$. Let C be the curve arising as the Deligne-Lusztig variety corresponding to the ${}^2\mathrm{B}_2$ case (see [**Han92, HS90**]). C is defined over $\mathbb{F}_q$ with equation

$$C: \qquad X^{q_0}(Z^q + ZX^{q-1}) = Y^{q_0}(Y^q + YZ^{q-1}),$$

genus $g = q_0(q-1)$ and $1 + q^2$ rational points (over $\mathbb{F}_q$). The resulting code (over $\mathbb{F}_q$) satisfies

$$d + k \geq 1 + q^2 - q_0(q-1).$$

Ree Curves: Let $q = 3^{2m+1}$ be an odd power of 3; set $q_0 = 3^m$. Let C be the curve arising as the Deligne-Lusztig variety corresponding to the ${}^2\mathrm{G}_2$ case (see [**Han92, HP93**]). C is defined over $\mathbb{F}_q$ with genus $g = \frac{3}{2}q_0(q-1)(q+q_0+1)$ and has $q^3 + 1$ rational points (over $\mathbb{F}_q$).

These examples motivate the study of codes arising from higher-dimensional Deligne-Lusztig varieties.

5.3. General results

In this section we will investigate the germ map (5.2) further. The fundamental question is:

For a line bundle L on X, how many zeroes does a section $s \in \Gamma(X, L)$ then have along a fixed set $\mathcal{P}$ of rational points?

Using the correspondence between line bundles and (Weil) divisors on a normal variety [**Ful83**, 2.1.1], we may reformulate the question as:

> *For a fixed line bundle L, given an effective divisor D such that $L = \mathcal{O}_X(D)$, how many points from $\mathcal{P}$ is in the support of D?*

In the curve case, where the points $P \in \mathcal{P}$ happen to be divisors, one may apply the Riemann-Roch theorem to give a lower bound on d and a formula for k. In higher dimensions, however, we find ourselves trying to compare objects of different dimension. This may be remedied in two ways:

1. Make the objects have the same codimension. This can be done by blowing up.
2. Make the objects have complementary dimensions. That is, make the points into curves.

In the following we shall pursue these ideas and use intersection theory (as defined in e.g. [**Ful83**]) on the variety X to answer the questions posed above. We find lower bounds on the minimum distance, hence sufficient criteria for the injectivity of the germ map.

For a global section s of a line bundle L we shall let $Z(s)$ denote the corresponding effective divisor. If C is a curve on X, we shall denote the intersection number of L and C by $L \cdot C$. It is calculated as the degree of the divisor cut out on C by any section $Z(s)$.

The following observation is straight-forward, but useful:

PROPOSITION 5.9. *Let X be a normal projective variety defined over $\mathbb{F}_q$ of dimension at least two. Let $C_1, \dots, C_a$ be (irreducible) curves on X with $\mathbb{F}_q$-rational points $\mathcal{P} = \{P_1, \dots, P_n\}$. Assume the number of $\mathbb{F}_q$-rational points on each C_i is less than N. Let L be a line bundle on X, defined over $\mathbb{F}_q$, such that $L \cdot C_i \geq 0$ for all i. Let*

$$\ell = \sup_{s \in \Gamma(X,L)} \#\{i : Z(s) \text{ contains } C_i\}.$$

Then the code $C(L, \mathcal{P})$ has length n and minimum distance

$$d \geq n - \ell N - \sum_{i=1}^{a} L \cdot C_i.$$

If $L \cdot C_i = \eta \leq N$ for all i, then $d \geq n - \ell N - (a - \ell)\eta$.

PROOF. Let $s \in \Gamma(X, L)$ be a section and D its corresponding divisor of zeros. The vector $\alpha(s) \in \mathbb{F}_q^n$ has $\#(D \cap \cup_i C_i)(\mathbb{F}_q)$ zero coordinates. Set-theoretically we have

$$D \cap \cup_{i=1}^{a} C_i = (\cup_{C_i \subseteq D} C_i) \cup (D \cap \cup_{C_i \not\subseteq D} C_i).$$

Since the last intersection is proper it follows that $\alpha(s)$ has at the most $\ell N + \sum_{C_i \not\subseteq D} L \cdot C_i$ coordinates equal to zero. As L is non-negative along the curves C_i, the sum $\sum_{C_i \not\subseteq D} L \cdot C_i$ is bounded by the larger sum over all curves.

If each curve counts the same in the intersection product, we may correct for the possible zeros we have counted twice by subtracting $\ell\eta$ from the number of possible zeros. This yields the last formula. ∎

From the proof we immediately get:

COROLLARY 5.10. *If $n > \ell N + \sum_{i=1}^{a} L \cdot C_i$, then the germ map*

$$\alpha : \Gamma(X, L) \to \mathbb{F}_q^n$$

is injective.

COROLLARY 5.11. *Assume furthermore that X is a non-singular surface and that H is a nef divisor on X with $H \cdot C_i > 0$ for all i. Then*

$$\ell \leq \frac{L \cdot H}{\min_i \{C_i \cdot H\}}. \tag{5.8}$$

Consequently, if $L \cdot H < C_i \cdot H$ for all i, we have $\ell = 0$ and

$$d \geq n - m, \qquad ; m = \sum_{i=1}^{a} L \cdot C_i.$$

PROOF. Let D be a member of the linear system corresponding to L containing ℓ of the curves C_i. As H is nef,

$$L \cdot H = D \cdot H \geq \min_i \{C_i \cdot H\} \cdot \ell.$$

The assertion follows. □

This concludes the proof of Theorem 5.1.

EXAMPLE 5.12. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. We have $\mathrm{Pic}(X) \simeq \mathbb{Z} \times \mathbb{Z}$. Let L be a line bundle on X of type (d_1, d_2) with d_i non-negative integers (then L is nef). The divisor $H = (0, 1)$ is nef. Let $\mathcal{P}$ consist of the $\mathbb{F}_q$-rational points of X. These points are distributed equally on $q + 1$ disjoint lines C_i. We may assume that each C_i is of type $(1, 0)$, hence $C_i \cdot H = 1$. We have

$$e = L \cdot C_i = d_2 \qquad \ell = \frac{H \cdot L}{H \cdot C_i} = d_1. \tag{5.9}$$

Hence for $d_1 + d_2 < q + 1$ we obtain codes with parameters

$$\begin{aligned} n &= (q+1)^2 \\ k &= \binom{d_1+1}{d_1}\binom{d_2+1}{d_2} = (d_1+1)(d_2+1) \\ d &\geq n - (d_1+d_2)(q+1) + d_1 d_2. \end{aligned}$$

We then have $k + d \geq n - (d_1 + d_2)q + 1$. As expected, the expressions are symmetric in d_1 and d_2. The code constructed have the same parameters as the product code. (See also [**Han98**] for a description of this example in the framework of toric varieties.)

EXAMPLE 5.13. Let $X = \mathbb{P}^2$ and let $\mathcal{P}$ consist of the $q^3 + 1$ $\mathbb{F}_{q^2}$-rational points on the plane Hermitian curve C, given by the equation

$$C : \qquad X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0.$$

Let $L = \mathcal{O}_{\mathbb{P}^2}(t)$ and set $H = \mathcal{O}_{\mathbb{P}^2}(1)$. Then, for $1 \leq t < q+1$ the inequality,

$$t = L \cdot H < C \cdot H = q+1$$

is satisfied, hence by Corollary 5.11 we get codes over $\mathbb{F}_{q^2}$ with parameters

$$\begin{aligned} n &= q^3 + 1 \\ k &= \tfrac{1}{2}(t+1)(t+2) \\ d &\geq n - L \cdot C = n - t(q+1) \end{aligned}$$

Taking $q = 2, t = 2$ yields a $[9, 6, 3]_4$ code.

More generally, if C is a non-singular plane curve of degree m, then the same construction can be carried out, yielding codes with parameters $n = \#C(\mathbb{F}_q)$, $k = \frac{1}{2}(t+1)(t+2)$ and $d \geq n - tm$ (for $t < m$). Taking $t = m-1$ yields the well-known bound $k + d \geq n + 1 - g(C)$. So the construction above also accounts for the curve codes.

5.3.1. Ample line bundles, Seshadri constants and codes. Let X be a projective variety of dimension at least two. For $V \subseteq X$ a closed subvariety defined by a coherent sheaf of ideals $\mathcal{I}$ and for a line bundle L on X, one defines the *(local) Seshadri constant of L at V* as

$$\varepsilon(L, V) = \sup\{\varepsilon \in \mathbb{Q} : f^*L - \varepsilon E \text{ is nef }\}$$

(here $f : Bl_{\mathcal{I}} X \to X$ is the blow-up with exceptional divisor E). When V is a set of points $\mathcal{P} = \{x_1, \ldots, x_m\}$ we have the equality

$$\varepsilon(L, \mathcal{P}) = \inf_C \left\{ \frac{L \cdot C}{\sum_i mult_{x_i}(C)} \right\}$$

the infimum being taken over all irreducible curves C in X.

The *(global) Seshadri constant of* L is $\varepsilon(L) = \inf_{x \in X} \varepsilon(L, x)$. We state the well-known result [**Har70**, Theorem 7.1]:

THEOREM 5.14 (SESHADRI'S CRITERION FOR AMPLENESS).
Let L be a line bundle on a projective variety X. Then

$$L \textit{ is ample if and only if } \varepsilon(L) > 0.$$

REMARK 5.15. If L is very ample then $\varepsilon(L) \geq 1$.

LEMMA 5.16. *Let X be a projective variety, L a line bundle on X.*

1. *If L is generated by global sections then L is nef.*
2. *For any proper morphism $f : Y \to X$ we have that f^*L is nef (on Y) if L is nef (on X).*

PROOF. Let $s_0, s_1, \ldots, s_n \in \Gamma(X, L)$ generate L. These sections define a proper morphism $i : X \to \mathbb{P}^n$ such that $L = i^*\mathcal{O}_{\mathbb{P}^n}(1)$ [**Har77**, Theorem II.7.1]. Let $C \subset X$ be an irreducible curve. Then, by the projection formula [**Ful83**, Theorem 3.2 (c)],

$$L \cdot C = i^*\mathcal{O}_{\mathbb{P}^n}(1) \cdot C = \mathcal{O}_{\mathbb{P}^n}(1) \cdot i_*C.$$

Now either $i(C)$ is a curve and then the right hand side is the degree of this curve or $i(C)$ is a point in which case the right hand side is zero. Hence $L \cdot C \geq 0$ for any irreducible curve. The second assertion follows similarly from the projection formula: $f^*L \cdot C = L \cdot f_*C \geq 0$. $\square$

PROPOSITION 5.17. *Let X be a non-singular projective variety defined over $\mathbb{F}_q$. Let $\mathcal{I}$ be the ideal sheaf of the $\mathbb{F}_q$-rational points $\mathcal{P} = \{P_1, \dots, P_n\}$ of X. Let L be a line bundle on X.*

1. *Suppose L is ample with Seshadri constant $\varepsilon(X, \mathcal{P}) \geq \varepsilon,\ \varepsilon \in \mathbb{N}$. Then, for $n > \varepsilon^{1-\dim X} L^{\dim X}$, the germ map* (5.2) *gives a code of length n and minimum distance $d \geq n - \varepsilon^{1-\dim X} L^{\dim X}$.*
2. *Suppose $L^{\otimes \xi} \otimes \mathcal{I}$ is generated by global sections (such $\xi \in \mathbb{N}$ exists if for example L is ample). Then, for $n > \xi^{\dim X - 1} L^{\dim X}$, the germ map* (5.2) *gives a code of length n and minimum distance $d \geq n - \xi^{\dim X - 1} L^{\dim X}$.*

In both cases, the code will have dimension $k \leq h^0(X, L)$ (with equality if the bound on d is positive).

PROOF. By definition, $f^*L - \varepsilon \sum_{i=1}^n E_i$ is nef on $Bl_{\mathcal{I}}X$. Suppose there exists a non-zero section $s \in \Gamma(X, L)$ mapping to zero in the germs L_{P_i}, $i \in I$ (and non-zero otherwise). Then $s \in \Gamma(X, L \otimes \mathcal{I}_I)$ defines a non-zero section of the bundle $f^*L \otimes \mathcal{O}(\sum_{i \in I} -E_i)$ on $Bl_{\mathcal{I}}X$. Hence $f^*L - \sum_{i \in I} E_i$ is represented by an effective divisor on $Bl_{\mathcal{I}}X$. But then [**Har70**, Theorem 6.1],

$$\left(f^*L - \varepsilon \sum_{i=1}^n E_i\right)^{\dim X - 1} \cdot \left(f^*L - \sum_{i \in I} E_i\right) \geq 0.$$

Or equivalently, $\#I \leq \varepsilon^{1-\dim X} L^{\dim X}$ (we have always $(-E_i)^{\dim X} = -1$). So a section maps to zero in $\varepsilon^{1-\dim X} L^{\dim X}$ points at the most. Hence $n - d \leq \varepsilon^{1-\dim X} L^{\dim X}$.

For the second part, let $\pi : Bl_{\mathcal{I}}X \to X$ be the blow-up of $\mathcal{I}$. Then $\pi^*(L^{\otimes \xi} \otimes \mathcal{I}) = \pi^* L^{\otimes \xi} - E$ is nef by Lemma 5.16. Now argue as in the first part. $\square$

EXAMPLE 5.18. Let $X = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2}(m)$, $(m \leq q)$ and let $\mathcal{P}$ denote the $\mathbb{F}_q$-rational points of X.

The study of ample line bundles on the blow up of the projective plane in a given number of points is well studied (*cf.* [**Cop95, Xu95**] and the references given there).

Let us determine a lower bound for $\varepsilon(L, \mathcal{P})$: For any irreducible curve, we may (using methods similar to those in [**Ser91**]) bound the sum

$$\sum_{P \in \mathcal{P}} mult_P(C) \leq \sqrt{(\deg(C) - 1)(\deg(C) - 2)(\deg(C)q + 1)}$$

if $\deg(C) \leq q+1$. For $\deg(C) \geq q+1$ the sum can be bounded by $\sqrt{(\deg(C)-1)(\deg(C)-2)(q^2+q+1)}$. So, in any case,

$$\begin{aligned}\sum_{P\in\mathcal{P}} mult_P(C) &\leq \sqrt{(\deg(C)-1)(\deg(C)-2)(q^2+q+1)}\\ &\leq \deg(C)(q+1).\end{aligned}$$

So, for all irreducible curves $C \subset X$,

$$\frac{L\cdot C}{\sum_{P\in\mathcal{P}} mult_P(C)} = \frac{m\deg(C)}{\sum_{P\in\mathcal{P}} mult_P(C)} \geq \frac{m}{q+1}.$$

Hence $\varepsilon(L, \mathcal{P} \geq \frac{m}{q+1}$ and the first method of the proposition then gives the bound $n-d \leq \frac{m}{q+1}m^2 = m(q+1)$. The dimension of the code is of course $k = \frac{1}{2}(m+1)(m+2)$ and the length is $n = q^2+q+1$.

The second method gives almost the same result: a calculation shows that locally $\mathcal{O}_{\mathbb{P}^2}(\xi m)\otimes\mathcal{I}$ is generated by sections of degree $\xi m - q$. If $\xi m > q$, these sections glue to give global sections (see for example the proof of Serre's theorem [**Har77**, Theorem II.5.17]). Hence, choosing $\xi = \lceil\frac{q+1}{m}\rceil$ makes $\mathcal{O}_{\mathbb{P}^2}(\xi m)\otimes\mathcal{I}$ generated by global section. We therefore obtain the bound $n-d \leq \xi m^2 \leq m(q+1)$.

A variation of the above method is:

PROPOSITION 5.19. *Let X be a normal projective variety defined over $\mathbb{F}_q$; assume* $\dim X \geq 2$. *Let $\mathcal{P}$ be the $\mathbb{F}_q$-points of X and let $\pi : Y \to X$ be a birational proper morphism with exceptional fibres (divisors) E_i over the points of $\mathcal{P}$. Assume Y is non-singular and let H be an ample line bundle on Y. Then, for any line bundle L on X, we can construct codes with parameters*

$$\begin{aligned}
&n = \#\mathcal{P}\\
&d > n - \frac{H^{\dim X-1}\cdot\pi^*L}{\min_i\{H^{\dim X-1}\cdot E_i\}}\\
&k \leq \dim\Gamma(X,L) \qquad \textit{(with equality if the bound on } d \textit{ is positive).}
\end{aligned}$$

PROOF. Only the claim about the minimum distance is non-trivial. So let $s\in\Gamma(X,L)$ be a section of L mapping to zero in the germs L_{P_i}, $i\in I$ for some index set I. As π is dominant, this gives us a non-zero global section of the line bundle $\pi^*L\otimes\mathcal{O}_Y\big(-\sum_{i\in I}E_i\big)$. [**Iit82**, Lemma 2.35]. So this bundle is represented by an effective divisor, whence

$$\begin{aligned}0 &< H^{\dim X-1}\cdot\Big(\pi^*L\otimes\mathcal{O}_Y\big(-\sum_{i\in I}E_i\big)\Big)\\ &= H^{\dim X-1}\cdot\pi^*L - \sum_{i\in I}H^{\dim X-1}\cdot E_i\end{aligned}$$

(by the Nakai-Moishezon criterion [**Har77**, Theorem A.5.1] applied to H). But then

$$\#I < \frac{H^{\dim X-1} \cdot \pi^* L}{\min_i \{H^{\dim X-1} \cdot E_i\}}$$

(recall that H is ample). The formula for d follows. □

REMARK 5.20. As we shall see below, this method works very well for complete intersections of Hermitian hyper-surfaces, since in these cases we may take Y to be a non-singular Deligne-Lusztig variety. Giving ample line bundles on such varieties is easy (*cf.* [**HH99a**]).

5.4. Applications

5.4.1. Deligne-Lusztig surfaces. Now let X be a 2-dimensional Deligne-Lusztig variety $\bar{X}(w)$. The numbers of rational points of these surfaces are relatively large and are given in [**Rod96**] (typically we have $\#X(\mathbb{F}_{q^\delta})/\#\mathbb{P}^2(\mathbb{F}_{q^\delta}) \sim q^4$). Some examples are listed in Table 5.1. We see that the number of $\mathbb{F}_{q^\delta}$-rational

G^F	δ	number of $\mathbb{F}_{q^\delta}$-points on $\bar{X}(w)$
${}^2\mathrm{A}_3$	2	$(q^3+1)(q^2+1)^2$
${}^2\mathrm{A}_4$	3	$(q^5+1)(q^3+1)(q^2+1)$
${}^3\mathrm{D}_4$	3	$(1+q^3)^2(1+q^4+q^8)$

TABLE 5.1. Numbers of rational points for some Deligne-Lusztig surfaces

points on $\bar{X}(w)$ is given by a polynomial of high degree in q. Furthermore, any divisor of the type $G = m_1 D_1 + m_2 D_2$ will be G^F-invariant.

First we shall see Proposition 5.9 and Corollary 5.11 in use:

PROPOSITION 5.21. *Let X be the Deligne-Lusztig surface of type ${}^2\mathrm{A}_4$ defined over the field $\mathbb{F}_{q^2}$. Then for $t = 1, 2, \ldots, q$, we can construct a code on X over $\mathbb{F}_{q^2}$ with parameters*

$$n = (q^5+1)(q^3+1)(q^2+1)$$
$$k = \frac{1}{24}(t+1)(t+2)(t+3)(t+4)$$
$$d \geq n - tP(q)$$

where $P(q)$ is the monic polynomial $P(q) = (q^3+1)(q^5+1) + (q+1)(q^3+1)(q^2 - t+1)$ of degree 8 in q.

PROOF. Since the canonical divisor K_X is ample *cf.* [**HH99a, Rod96**], we could take $H = K_X$. We shall however construct our codes from another line bundle L which itself have the necessary properties for using Corollary 5.11; that is, we shall see that we may take H equal to L.

For a suitable parabolic subgroup P of G (see [**Rod96**, p. 565]), we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & G/B \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ Z & \xrightarrow{i} & G/P \simeq \mathbb{P}^4 \end{array} \tag{5.10}$$

where Z is a complete intersection of the two Fermat hyper-surfaces of degree $q+1$ and q^3+1, i is an embedding, j is finite, π is locally trivial (in the Zariski topology) and ρ is birational and surjective; see [**Rod96**, p. 565] or [**Rod98**, Section 8]. Of course all morphisms are proper.

Let $L = \rho^* i^* \mathcal{O}_{\mathbb{P}^4}(t)$. L is then nef by Lemma 5.16. Since Z is a complete intersection, and is non-singular away from its $\mathbb{F}_{q^2}$-rational points, Z is normal [**Har77**, Proposition II.8.23]. Furthermore, as ρ is birationally onto Z, we get [**Iit82**, Theorem 2.31],

$$\Gamma(X, L) \simeq \Gamma(Z, i^* \mathcal{O}_{\mathbb{P}^4}(t)).$$

For $t \leq q$ it follows from the long exact cohomology sequences that we have an isomorphism $\Gamma(Z, i^*\mathcal{O}_{\mathbb{P}^4}(t)) \simeq \Gamma(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t))$. The last term has dimension $\binom{4+t}{t}$. Hence the expression for k follows once we have shown that the germ map is injective. But this follows from the expression for d and Corollary 5.10.

We take $\mathcal{P}$ to be the $\mathbb{F}_{q^2}$-rational points on X. From [**Rod96**] we have $n = |\mathcal{P}| = (q^5+1)(q^3+1)(q^2+1)$. These points are distributed equally on the $(q^5+1)(q^3+1)$ disjoint rational curves C_i constituting the irreducible components of the divisor D_2 (see Definition 5.3).

By the commutativity of (5.10), L is equal to the restriction to X of the line bundle $\pi^*\mathcal{O}_{\mathbb{P}^4}(t)$ on G/B. Using the same technique as described in [**DL76**, 9.1] (and more detailed in [**HH99a**]), we may express L in terms of the boundary divisors D_1 and D_2,

$$L^{\otimes q^5+1} = \mathcal{O}_X(t(q^3+1)D_1 + t(q+1)D_2).$$

Furthermore, in [**HH99a**, Example 5] the intersection numbers of the (components of) D_1 and D_2 was calculated. Hence,

$$\begin{aligned} L \cdot C_i &= t \qquad ; \text{ for all } i \\ L \cdot D_2 &= \frac{t}{q^5+1} D_2 \cdot \big((q^3+1)D_1 + (q+1)D_2\big) \\ &= t(q^3+1)(q^5+1) \\ L \cdot L &= t^2(q^3+1)(q+1). \end{aligned}$$

It follows that a divisor in the linear system corresponding to L contains less than $\frac{L\cdot L}{L\cdot C_i} = t(q+1)(q^3+1)$ rational curves C_i (each containing q^2+1 points from $\mathcal{P}$). Since $L \cdot C_i = t$ is less than the number of points on each curve, we may use the last part[2] of Proposition 5.9. $\square$

Taking $t = 1$ in the proposition, we obtain:

COROLLARY 5.22. *Let X be the Deligne-Lusztig surface of type ${}^2\mathrm{A}_4$. Then we can construct a code on X over $\mathbb{F}_{q^2}$ with parameters*

$$\begin{aligned} n &= (q^5+1)(q^3+1)(q^2+1) \\ k &= 5 \\ d &\geq n - (q^5+1)(q^3+1) - (q^3+1)(q+1)q^2. \end{aligned}$$

REMARK 5.23. Since all sections of the chosen line bundle L "come from Z", it follows that the code on X is the repetition q^3+1 times of the code on Z with parameters:

$$\begin{aligned} n_Z &= (q^5+1)(q^2+1) \\ k_Z &= \frac{1}{24}(t+1)(t+2)(t+3)(t+4) \\ d_Z &\geq n_Z - t(q^5+1) - t(q+1)(q^2-t+1). \end{aligned}$$

This is a quite good code, actually (the bound on d being close to the Griesmer bound for large q).

For $t = 1$, the code equals the one constructed by Rodier [**Rod97**] from the same complete intersection. Over $\mathbb{F}_{q^2}$ he also obtains codes with parameters $n = (q^5+1)(q^2+1)$, $k = 5$ and $d = q^7 - q^3 = n - (q^5+q^3+q^2+1)$. The estimates of Rodier are based on work by [**Cha90**]. See also Example 5.26 for using Proposition 5.19 to determine bounds on these codes on Z.

The reason for choosing L to be a pull-back from Z is to be able to give an explicit formula for the dimension k. Of course we can also construct codes from other line bundles on X besides those coming from G/B (or quotients thereof). Then we will need a more detailed study of the Picard group of X. This problem is addressed in the authors forthcoming thesis [**HH99b**].

In the proposition above we may actually let t go up to q^2 (this is clear from the expression for d) but then we do no longer have that $\Gamma(Z, i^*\mathcal{O}_{\mathbb{P}^4}(t)) \simeq \Gamma(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t))$. However, we may still calculate k as the dimension of the t'th graded piece of the homogeneous coordinate ring of Z.

EXAMPLE 5.24. Let X be the Deligne-Lusztig surface of type ${}^2\mathrm{A}_4$ with $q = 4$, $t = 1$. Then we get a code over $\mathbb{F}_{16}$ with parameters

$$n = 1\,132\,625 \qquad k = 5 \qquad d \geq 1\,061\,200.$$

We see that going to higher dimensions makes it possible to obtain very long algebraic geometric codes over rather small fields.

[2] The author thanks the referee for pointing out that the last statement of Proposition 5.9 could be used here.

5.4.2. Complete intersections of Hermitian hyper surfaces.

EXAMPLE 5.25. Let $X \subseteq \mathbb{P}^3$ be the Fermat surface defined by $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$. X is defined over $\mathbb{F}_{q^2}$ with its $(q^2+1)(q^3+1)$ rational points distributed equally on $(q^3+1)(q+1)$ rational curves C_i cut out by the equation $X_0^{q^3+1} + X_1^{q^3+1} + X_2^{q^3+1} + X_3^{q^3+1} = 0$; see [**Rod98**, Section 6] or [**Lus76b**].

We will give bounds for the codes constructed from the line bundle $L = \mathcal{O}_{\mathbb{P}^3}(t)|_X$ on X. In the first approach we will use Corollary 5.11.

As $\mathcal{O}_{\mathbb{P}^3}(q^3+1)|_X \simeq \mathcal{O}_X(\sum_i C_i)$ we find that $m = \sum_i L \cdot C_i = t(q^3+1)(q+1)$. So by symmetry, $L \cdot C_i = t$.

The self-intersection L^2 may by the projection formula be calculated in the ambient space $\mathbb{P}^4$, hence $L^2 = t^2(q+1)$. This yields $\ell \leq t(q+1)$.

Again, for $t \leq q$, $\Gamma(X, L) = \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(t))$. Hence for any $t \leq q$ we obtain a code with parameters

$$\begin{aligned} n &= (q^2+1)(q^3+1) \\ k &= \tfrac{1}{6}(t+1)(t+2)(t+3) \\ d &\geq n - t(q^3+1) - \textstyle\sum_i L \cdot C_i = n - t(q^3+1)(q+2). \end{aligned}$$

This bound is actually not that good. The reason is that the curves C_i are not disjoint.

However, it happens that the blow-up Y of X in all its $\mathbb{F}_{q^2}$-rational points is nothing but the Deligne-Lusztig variety corresponding to the ${}^2\mathrm{A}_3$ case (see [**Rod96**]). As in the proof of Proposition 5.21 one determines an ample divisor,

$$M = j^*\mathcal{L}(-2\lambda_1 - (q-1)\lambda_2 - (q-1)\lambda_3) \simeq \mathcal{O}_Y(2D_1 + D_2)$$

(see [**HH99a**] for a description of the notation $\mathcal{L}(\cdot)$ and how to calculate the isomorphism). Let $\pi : Y \to X$ denote the morphism given by the blow-up. We find that $\pi^* L^{\otimes q^3+1} = \mathcal{O}_Y(t(q+1)D_1 + tD_2)$. So from Proposition 5.19 we now get another bound on d,

$$d > n - \frac{M \cdot \pi^* L}{M \cdot [\text{exc. fibre}]} = n - tq^2 \frac{(q+1)^2}{q-1}.$$

This last bound on d is almost as sharp as the one obtained by Lachaud in [**Lac96**].

EXAMPLE 5.26. Let us construct codes on the complete intersection Z from Proposition 5.21 using the line bundle $L = \mathcal{O}_{\mathbb{P}^4}(t)|_Z$. The morphism $\rho : X \to Z$ from the Deligne-Lusztig variety corresponding to the ${}^2\mathrm{A}_4$ case to Z is an isomorphism except over the $\mathbb{F}_{q^2}$-rational points of Z where the fibre is a Hermitian curve C. As in [**HH99a**, Example 4] one calculates that

$$\rho^* L \cdot K_X = t(q^3+1)(q^4+q^3+q^2-2q-3)$$

and $K_X \cdot C = q^2 - 1$. Hence, for any $t \leq q$, we obtain codes with parameters

$$n = (q^5+1)(q^2+1)$$
$$k = \tfrac{1}{24}(t+1)(t+2)(t+3)(t+4)$$
$$d > n - t\frac{(q^3+1)(q^3+q-3)}{q-1}.$$

The bound on d is not quite as good as the one obtained earlier (Remark 5.23) but it is still of the same order of magnitude, $d \geq n - tO(n^{5/7})$.

5.4.3. Ruled surfaces. Let C be a non-singular curve of genus g defined over $\mathbb{F}_q$. Let $\mathcal{F}$ be a locally free sheaf on C of rank 2. Let $e = -\deg \bigwedge^2 \mathcal{F}$. If $\mathcal{F}$ is normalized and decomposable, then $e \geq 0$. If $\mathcal{F}$ is normalized and indecomposable, then $-g \leq e \leq 2g-2$.

Let $X = \mathrm{Proj}(\mathrm{Symm}\,\mathcal{F})$ be the corresponding ruled surface. The projection $\pi : X \to C$ is a $\mathbb{P}^1$-bundle with a section and a relatively ample line bundle $\mathcal{O}_X(1)$. Let C_0 be the corresponding divisor (when $\mathcal{F}$ is normalized, C_0 is the image of a section of π). From [**Har77**, Chapter V.2] we gather the following results about the divisors on X:

LEMMA 5.27. 1. $\mathrm{Pic}(X) \simeq \mathbb{Z} \oplus \pi^* \mathrm{Pic}(C)$.

2. $\mathrm{Num}(X) \simeq \mathbb{Z}\cdot[C_0] \oplus \mathbb{Z}\cdot[f]$ *with product given by* $f^2 = 0$, $f \cdot C_0 = 1$ *and* $C_0^2 = -e$ *(here f is the numerical equivalence class of any fibre of π).*
3. *Let $D = b_1 C_0 + b_2 f$ be a divisor. Define the rational number κ depending on $p = char(\mathbb{F}_q)$, e and g by*

$$\kappa(p,e,g) = \begin{cases} e & e \geq 0 \\ \frac{1}{2}e & e < 0,\ g < 2 \\ \frac{1}{2}e + \frac{p}{g-1} & e < 0,\ g \geq 2 \end{cases}$$

   Then D is ample (resp. nef) when $b_1 > 0$ and $b_2 > b_1\kappa$ (resp. $b_1 > 0$ and $b_2 \geq b_1\kappa$).
4. *When $\mathcal{F}$ is the direct sum of two ample line bundles on C, then the divisor C_0 is ample (see* [**Har70**, Theorem III.1.1]*).*

EXAMPLE 5.28. Let C be any curve. Then, for any $e \geq 0$, the normalized rank 2 bundle $\mathcal{F} = \mathcal{O}_C \oplus \mathcal{O}_C(-e)$ gives a ruled surface with invariant $e \geq 0$. For an example where $e < 0$, see [**Har77**, V.2.11.6].

PROPOSITION 5.29. *Let C be a non-singular curve of genus g. Let $\pi : X = \mathbb{P}(\mathcal{F}) \to C$ be a ruled surface with invariant $e \geq -g$. Let f be the fibre over the point $P_0 \in C$. Set $a = \#C(\mathbb{F}_q)$. Then we can construct codes with parameters*

$$\begin{aligned} &n = (q+1)a \\ &k = h^0(C, \mathrm{Symm}^{b_1}(\mathcal{F}) \otimes \mathcal{O}_C(b_2 P_0)) && b_1 \geq 0,\ b_2 \geq 0 \\ &d \geq n - (a-\ell)b_1 - \ell(q+1) \\ &\qquad \textit{where } \ell = b_1(\lceil \kappa \rceil - e) + b_2 && (= b_2 \textit{ for } e \geq 0). \end{aligned}$$

(assuming $\ell < a$ and that the bound on d is strictly positive). If $\mathcal{F}$ is ample, we have $\ell = b_2 - e$.

PROOF. Let $f_1, \dots, f_a$ be the fibres over the $\mathbb{F}_q$-rational points of C. These disjoint lines contain all $\mathbb{F}_q$-rational points of X. Let $L = \mathcal{O}_X(b_1C_0 + b_2 f)$. From [**Har77**, Lemmas V.2.1+4, Proposition II.7.11] it follows that

$$\Gamma(X, L) \simeq \Gamma(C, \pi_* L) = \Gamma(C, \mathrm{Symm}^{b_1}(\mathcal{F}) \otimes \mathcal{O}_C(b_2 P)).$$

Now let $H = C_0 + \lceil\kappa\rceil f$. By Lemma 5.27, H is nef. We have $H \cdot f_i = 1$, $L \cdot f_i = b_1$ for all i and

$$\begin{aligned} H \cdot L &= b_1 C_0^2 + \lceil\kappa\rceil b_1 C_0 \cdot f + b_2 f \cdot C_0 + \lceil\kappa\rceil b_2 f^2 \\ &= b_1(\lceil\kappa\rceil - e) + b_2 \qquad (= b_2 \text{ for } e \geq 0). \end{aligned}$$

With the notation of Proposition 5.9 and Corollary 5.11 we have $\ell \leq b_1(\lceil\kappa\rceil - e) + b_2$. If $\mathcal{F}$ is ample, we may take H equal to C_0 and then $H \cdot L = b_2 - e$. The proposition now follows from Proposition 5.9. □

COROLLARY 5.30. *Over $\mathbb{F}_{q^2}$ there exist algebraic geometric codes with parameters*

$$\begin{aligned} &n = (q^2+1)(q^3+1) \\ &k = b_1^2(q+1) + b_2 && \text{for } q - 2 < b_1 \leq q + 1,\ 0 \leq b_2 \leq q^2 \\ &d \geq n - (q^3 + 1 - \ell)b_1 - \ell(q^2+1) && \text{where } \ell = 2b_1 + b_2 \end{aligned}$$

(assuming that the bound on d is strictly positive).

PROOF. Let C be the Hermitian curve of degree $q+1$ in $\mathbb{P}^2$. C is defined over $\mathbb{F}_{q^2}$ of genus $g = \frac{1}{2}q(q-1)$ and has the maximal number of points allowed, namely $q^3 + 1$. Let $X = \mathbb{P}(\mathcal{O}_C(1) \oplus \mathcal{O}(1))$ be the ruled surface over C with invariant $e = -2$. Then the line bundle $\mathcal{O}_X(1)$ is ample on X. Let the codes be constructed from the line bundle $L = \mathcal{O}_X(b_1) \otimes \mathcal{O}_X(b_2 f)$. Since this bundle has degree $2b_1 + b_2$ with respect to $\mathcal{O}_X(1)$, we must have $\ell = 2b_1 + b_2$.

We have $h^0(L) = h^0(\mathrm{Symm}^{b_1}(\mathcal{O}_C(1) \oplus \mathcal{O}(1)) \otimes \mathcal{O}_X(b_2\{pt\}))$. The bundle on the right hand side is a sum of b_1 copies of the line bundle $\mathcal{O}_C(b_1) \otimes \mathcal{O}_X(b_2\{pt\})$ and, for $b_1 > q - 2$, these have so large degree that the formula follows from the Riemann-Roch theorem [**Har77**, Theorem IV.1.3]. □

EXAMPLE 5.31. Taking $q = 2$ in the above corollary we obtain codes over $\mathbb{F}_4$. The (bounds on the) parameters are listed in Table 5.2.

REMARK 5.32. It should be noted that the codes constructed in Corollary 5.30 are better than those obtainable from the product code construction: On the Hermitian curve C we have codes over $\mathbb{F}_{q^2}$ with parameters

$$\begin{aligned} &n_1 = q^3 + 1 \\ &k_1 \geq \alpha - g + 1 \qquad ;\ 0 \leq \alpha \leq q^3 + 1 \\ &d_1 \geq n_1 - \alpha. \end{aligned}$$

b_1	1	1	1	2	2	2	3
b_2	0	1	2	1	2	0	0
k	3	4	5	13	14	12	27
d	28	24	20	12	9	15	6

TABLE 5.2. Parameters of some examples of surface codes over the Hermitian elliptic curve. In all cases, $n = 45$.

Similarly, on $\mathbb{P}^1$ we have (MDS) codes over $\mathbb{F}_{q^2}$ with parameters

$$\begin{aligned} n_2 &= q^2 + 1 \\ k_2 &= \beta + 1 \qquad ; 0 \leq \beta \leq q^2 + 1 \\ d_2 &\geq n_2 - \beta. \end{aligned}$$

The product code has the same length $n = n_1 n_2$ as the codes constructed above. The dimension of the product code is bounded below by $(\alpha - g + 1)(\beta + 1)$. This bound is only non-zero for $\alpha \geq g = \frac{1}{2}q(q-1)$.

The codes constructed above all have dimension of order q^3 (if we for example take $b_1 = q$, then the dimension is bounded below by $q^2(q+1)$).

To obtain product codes with comparable dimension (bound), we will have to select α and β rather large. The resulting bound on the minimum distance $d_{\text{prod}} \geq d_1 d_2 = n - \alpha n_2 - \beta n_1 + \alpha\beta$ will under these conditions always be worse than the bound given in Corollary 5.30.

Note that, when taking ruled surfaces with invariant $e = 0$, the codes constructed in Proposition 5.29 will indeed be the product codes.

REMARK 5.33. There is an obvious generalisation of Proposition 5.29: Let X be a non-singular projective curve defined over a finite field $\mathbb{F}_q$. Let $\mathcal{F}$ be a vector-bundle on X of rank e with $\pi : Y = P(\mathcal{F}) \to X$ the corresponding projective bundle over X. Then the Chow ring of Y is given by the isomorphism of graded rings,

$$A^*(Y) \simeq A^*(X)[\eta]/\left(\eta^e + c_1(\pi^* E) \cap \eta^{e-1} + \ldots + c_e(\pi^* E)\right)$$

[**Ful83**, Example 8.3.4] (η is the hyper-plane section on Y). Set-theoretically we have

$$Y(\mathbb{F}_q) = \bigcup_{P \in X(\mathbb{F}_q)} \pi^{-1}(P).$$

Let L be a line bundle on Y from which we want to construct codes. Suppose we have a section $D \in |L|$ containing ℓ fibres f_i. Then, for any ample divisor H on Y, we have

$$H^{\dim Y - 1} \cdot L \geq \sum_{f_i \subseteq D} H^{\dim Y - 1} \cdot f_i.$$

(We may assume that H is very ample. Then, on any effective divisor, H cuts out an effective divisor. Hence the assertion follows from the Nakai-Moishezon criterion [**Har77**, Therem A.5.1].) Hence, a section of L can at the most contain $\left[\frac{H^{e-1}\cdot L}{H^{e-1}\cdot f}\right]$ fibres (the fibres are algebraically equivalent as X is non-singular) each containing π_{e-1} points, where $\pi_m = \#\mathbb{P}^m(\mathbb{F}_q)$. Now, for $L = \mathcal{O}_Y(t)\otimes\mathcal{O}_Y(f)$ we have that when intersecting properly, each section of L cuts out a degree t hyper-surface in each fibre. For $t < q$, the number of rational points on this section is bounded by $tq^{e-2}+\pi_{e-3}$ (*cf.* [**Ser91**]). So the code obtained from evaluating the sections of L in the $\mathbb{F}_q$-rational germs will satisfy

$$n - d \leq \pi_{e-1}\left[\frac{H^{\dim Y-1}\cdot L}{H^{\dim Y-1}\cdot f}\right] + (tq^{e-2}+\pi_{e-3})\cdot \#X(\mathbb{F}_q).$$

REMARK 5.34. Using the above construction and the asymptotically good towers of Garcia and Stichtenoth [**GS96a**] we may easily make towers of long surface codes. An interesting problem would be to determine the asymptotic properties of these codes.

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(bibliography omitted; end of article)

APPENDIX A

General results related to Chow groups

In this chapter we state various results relating to intersection theory and the calculation of Chow groups. Some of the results are probably well-known but for lack of reference we also give the proofs here.

A.1. Some lemmas

LEMMA A.1. *Let X and Y be a schemes, let U be an unipotent algebraic group acting with finitely many orbits on X and Y. Suppose $f : X \to Y$ is a finite surjective U-equivariant morphism. Then*

1. $\mathrm{A}_*(X)$ *is a free Abelian group with basis*

$$\{[V] : V \text{ is a } U\text{-stable closed subvariety of } X\}.$$

2. $\mathrm{A}_*(Y)_{\mathbb{Q}}$ *is a free Abelian group with basis*

$$\{[f(V)] : V \text{ is a } U\text{-stable closed subvariety of } X\}.$$

In particular, if $\mathrm{A}_k(X) = 0$ *for* $k < \dim(X)$ *then also* $\mathrm{A}_k(Y)_{\mathbb{Q}} = 0$ *for* $k < \dim(Y) = \dim(X)$.

PROOF. 1) is an immediate consequence of Theorem 1 in [**FMSS95**] and 2) is a consequence of the well-known fact that proper surjective maps induce surjective maps in Chow homology groups with rational coefficients. ☐

LEMMA A.2. *Let X be an algebraic scheme (not necessarily irreducible) with a stratification*

$$X_0 \subset X_1 \subset \cdots \subset X_n = X \quad ; X_i \text{ closed subschemes of pure dimension } i$$

such that $\mathrm{A}_k(X_i - X_{i-1}) = 0$ *for* $k \neq i$. *Then for all* $k \leq n$ *we have surjections*

$$\mathrm{A}_k(X_k) \to \mathrm{A}_k(X) \to 0. \tag{A.1}$$

PROOF. For $k = n$ the assertion is trivial, and from [**Ful83**, Proposition 1.9] we have the exact sequence

$$\mathrm{A}_k(X_{n-1}) \to \mathrm{A}_k(X_n) \to \mathrm{A}_k(X_n - X_{n-1}) \to 0 \tag{A.2}$$

hence surjections $\mathrm{A}_k(X_{n-1}) \to \mathrm{A}_k(X_n) \to 0$ for all $k < n$. By induction we may assume we have surjections $\mathrm{A}_k(X_k) \to \mathrm{A}_k(X_{n-1}) \to 0$ for all $k < n-1$. Now compose these surjections. ☐

REMARK A.3. Of course, the lemma also holds for Chow groups with rational coefficients.

THEOREM A.4 ([**MS76**, Theorem 7.1]). *Let $\varphi : Y \to Y'$ be a proper surjective morphism of varieties. Then the sequence*

$$\mathrm{A}_0(Y \times_{Y'} Y) \xrightarrow{\mathrm{pr}_{1*} - \mathrm{pr}_{2*}} \mathrm{A}_0(Y) \xrightarrow{\varphi_*} \mathrm{A}_0(Y') \to 0 \tag{A.3}$$

is exact.

COROLLARY A.5. *If $\varphi : Y \to Y'$ is the quotient map for the action of a finite group G', then*

$$\mathrm{A}_0(Y') = \mathrm{A}_0(Y)/G' \tag{A.4}$$

(where the right-hand side is to be read as the Abelian group $\mathrm{A}_0(Y)$ modulo the relation induced by the natural action of G').

PROOF. Since in this case $\mathrm{A}_0(Y \times_{Y'} Y)$ is generated by

$$\{(y_1, y_2) : y_i \in Y,\ \exists g \in G' : g.y_1 = y_2\},$$

Theorem A.4 gives that the kernel of φ_* is generated by cycles of the form $y - g.y,\ y \in Y, g \in G'$. Whence the corollary. □

Theorem A.4 has a natural generalisation.

THEOREM A.6. *Let $\varphi : Y \to Y'$ be a proper surjective morphism of irreducible algebraic varieties. Then the sequence*

$$\mathrm{A}_k(Y \times_{Y'} Y)_{\mathbb{Q}} \xrightarrow{\mathrm{pr}_{1*} - \mathrm{pr}_{2*}} \mathrm{A}_k(Y)_{\mathbb{Q}} \xrightarrow{\varphi_*} \mathrm{A}_k(Y')_{\mathbb{Q}} \to 0 \tag{A.5}$$

is exact for all k.

PROOF. The surjectivity of φ_* follows easily: If $V \subseteq Y'$ is a closed subvariety, there exists a closed subvariety $W \subseteq Y$ mapping finitely onto V with $\dim(W) = \dim(V)$ ([**MS76**, Lemma 7.2]). But then the cycle $\deg(W/V)^{-1}[W]$ maps to $[V]$. Hence we have a surjection on cycles which passes to rational equivalence since φ is proper.

By definition

$$Y \times_{Y'} Y = \{(y_1, y_2) \in Y \times_k Y : \varphi(y_1) = \varphi(y_2)\}$$

(for closed points) and since Y' is separated $Y \times_{Y'} Y$ is a closed subvariety of $Y \times Y$. Indeed, we may realize $Y \times_{Y'} Y$ as the fiber product

$$\begin{tikzcd} Y \times_{Y'} Y \arrow[r] \arrow[d, "\varphi \circ \mathrm{pr}_i"'] & Y \times Y \arrow[d, "\varphi \times \varphi"] \\ Y' \arrow[r] & Y' \times Y'. \end{tikzcd} \tag{A.6}$$

Then, since φ is proper, $Y \times_{Y'} Y = (\varphi \times \varphi)^{-1}(\Delta(Y'))$ is closed if and only $\Delta(Y')$ is. But this is the case if Y' is separated. Hence $\mathrm{Z}_k(Y \times_{Y'} Y)_{\mathbb{Q}}$ is generated by the components of the cycles $\{[Z] \in \mathrm{Z}_k(Y \times Y)_{\mathbb{Q}} : \varphi \circ \mathrm{pr}_1(Z) = \varphi \circ \mathrm{pr}_2(Z)\}$. So obviously $\varphi_*(\mathrm{pr}_{1*} - \mathrm{pr}_{2*}) = 0$ on cycles. Therefore $\mathrm{coker}(\mathrm{pr}_{1*} - \mathrm{pr}_{2*}) \subseteq \ker(\varphi_*)$.

Let $C_k = \mathrm{coker}(\mathrm{pr}_{1*} - \mathrm{pr}_{2*}) = \mathrm{A}_k(Y)_{\mathbb{Q}} / \mathrm{im}(\mathrm{pr}_{1*} - \mathrm{pr}_{2*})$ and let $\bar{\varphi}_* : C_k \to \mathrm{A}_k(Y')_Q$ be the induced (surjective) map. We must show that this map also is injective. We will do this by defining a an inverse map $\xi : \mathrm{Z}_k(Y')_{\mathbb{Q}} \to C_k$ in the following way: For $z \in \mathrm{Z}_k(Y')_{\mathbb{Q}}$ choose (by the surjectivity of φ_*) $z' \in \mathrm{Z}_k(Y)_{\mathbb{Q}}$ such that $\varphi_*(z') = z$. Then let $\xi(z)$ be the image of z' in C_k. This is well-defined: If $\varphi_*(z'_1) = \varphi_*(z'_2)$ then $z'_1 \times z'_2 \in \mathrm{Z}_k(Y \times_{Y'} Y)_{\mathbb{Q}}$ hence $z'_1 - z'_2 = (\mathrm{pr}_{1*} - \mathrm{pr}_{2*})(z'_1 \times z'_2) \in \mathrm{im}(\mathrm{pr}_{1*} - \mathrm{pr}_{2*})$.

Claim: $\xi : \mathrm{Z}_k(Y')_{\mathbb{Q}} \to C_k$ factors through rational equivalence giving a homomorphism $\eta : \mathrm{A}_k(Y')_{\mathbb{Q}} \to C_k$.

Clearly, then $\bar{\varphi}_*\eta = \mathrm{id}_{\mathrm{A}_k(Y')_{\mathbb{Q}}}$ and since $z \in \mathrm{Z}_k(Y')_{\mathbb{Q}}$ is a lifting of $\varphi_*(z)$ we also have $\eta\varphi_* = \mathrm{id}_{C_k}$.

Proof of claim. Let f be a rational function on a closed subvariety $V \subseteq Y'$. We must show that $\xi(\mathrm{div}(f)) = 0$ in C_k. Again, let $i : W \hookrightarrow Y$ be a closed subvariety mapping finitely onto V. We want a lifting $z \in \mathrm{Z}_k(W')_{\mathbb{Q}}$ such that $z \sim 0$. But this is easy. Let $g = \deg(W/V)^{-1} f \circ \varphi$. Then g is a rational function on W and pushes forward to $\mathrm{div}(f)$ under the finite map $\varphi|_W : W \to V$ of degree $\deg(W/V)$. Hence $\xi(\mathrm{div}(f))$ is the class of $i_*(\mathrm{div}(g))$ modulo $\mathrm{coker}(\mathrm{pr}_{1*} - \mathrm{pr}_{2*})$. But this is zero already in $\mathrm{A}_k(Y)_{\mathbb{Q}}$ (i being a closed immersion). □

REMARK A.7. From the proof it follows that if we want to avoid introducing rational coefficients we must have $\deg(W/V) = 1$ for any pair of closed subvarieties $W \subseteq Y$, $V \subseteq Y'$ such that W maps onto V. If W and V are points, then both function fields $k(W)$ and $k(V)$ are algebraic extensions of the base field k which is assumed algebraically closed. Hence the fields are all equal and the degree is 1. This is why we in Theorem A.4 can avoid rational coefficients.

LEMMA A.8. *Let $V_1, \ldots, V_m$ be prime divisors on a non-singular projective variety X;* $\dim X \geq 2$. *Assume that the V_i are contracted to distinct points $P_1, \ldots, P_m$ under a morphism $\pi : X \to Y$ where* $\dim X = \dim Y$, *Y is projective and $\pi^{-1}(P_i) = V_i$. Then the V_i are independent in* $\mathrm{Pic}(X)$. *Hence, for any $L \in \mathrm{Pic}(Y)$, the classes $\pi^*L, V_1, \ldots, V_m$ in* $\mathrm{Pic}(X)$ *are linearly independent too.*

PROOF. A dependence relation $0 = \sum_i n_i[V_i]$, $n_i \in \mathbb{Z}$, will imply $[V_i]^2 = 0$ (as a cycle in $\mathrm{A}^2(X)$) for any i. We shall see that this cannot be the case.

Let V be any of the components, let $P = \pi(V)$. Since Y is projective we may choose a very ample (Cartier) divisor H on Z. Choose furthermore effective divisors H_0, H_1 linearly equivalent to H such that P is in H_0 but not in H_1. On an open neighborhood[1] of P, the map π looks like Figure A.8. Since π^*H_1 does not intersect V,

$$0 = [\pi^*H_1] \cdot [V] = \pi^*[H_0] \cdot [V] = \big([\tilde{H}_0] + m_P(H_0)[V]\big) \cdot [V].$$

[1]If the self-intersection is non-zero on an open subset of X, it cannot be zero in X.

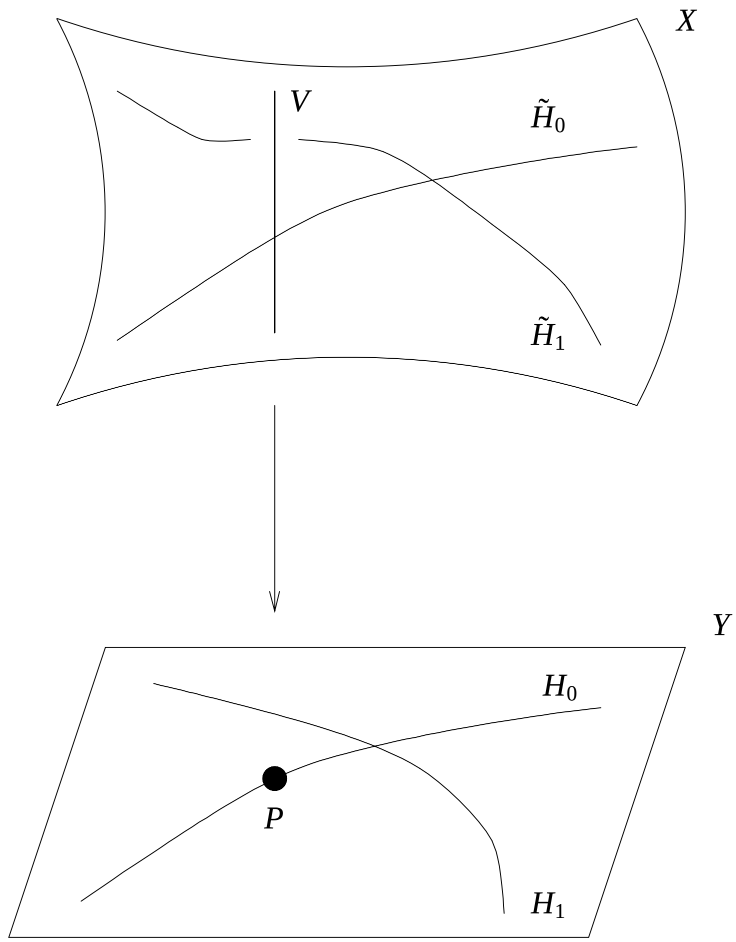


FIGURE 1. The blow-down of the divisor V

By the choice of H_0, $m_P(H_0) > 0$, hence $[V]^2$ is a (negative) non-zero multiple of the proper (non-zero) intersection $[V] \cdot [\tilde{H}_0]$.

For the last assertion. Assume $d\pi^*[L] = \sum_i n_i[V_i]$. Then, by pushing down with π we get the relation $d\pi_*\pi^*[L] = \sum_i n_i\pi_*[V_i] = 0$. Since $\pi_*\pi^*[L]$ is a multiple of $[L]$ we must have $d = 0$ and, by the above, all $n_i = 0$. $\square$

LEMMA A.9. *Let $f : X \to Y$ be a birational morphism of algebraic schemes with exceptional locus E, codim$(E) \geq 1$. Let $\alpha \in \mathrm{A}_*(X)$, $\alpha \not\subset E$. Then, if $f_*\alpha$ is zero in $\mathrm{A}_*(Y)$, so is α. That is, the kernel of $\pi_* : \mathrm{A}_*(X) \to \mathrm{A}_*(Y)$ is supported on E.*

PROOF. Obvious (restrict to the open subset where f is an isomorphism and use [**Ful83**, Proposition 1.8]). □

A.1.1. Samuel's conjecture. Since our calculation of the Picard group of Deligne-Lusztig varieties of classical type rely heavily on the now proven conjecture by Samuel, we will here give a sketch of the proof (given by Grothendieck; see [**Gro68**, p. 132]).

DEFINITION A.10. A local Noetherian ring A is *parafactorial* if, for $X = \operatorname{Spec} A$ and $\{x\}$ equal to the (unique) closed point of X,

- *depth* $A \geq 2$
- $\operatorname{Pic}(X - \{x\}) = 0$

(This is not the original definition, but an equivalent one *cf.* [**Gro68**, Proposition 3.5, p. 127].)

LEMMA A.11. *If X is a normal variety, then* $\operatorname{Pic}(X) = \mathrm{A}_{\dim X-1}(X)$ *if and only if X is locally factorial.*

PROOF. This is well-known; see [**Har77**, Corollary II.6.16]. □

LEMMA A.12 ([**Gro68**, Corollaires 3.9+10, p. 130]). *Let A be a local Noetherian ring of dimension at least* 2. *Let $X = \operatorname{Spec} A$ and $\{x\}$ be as above and set $U = X - \{x\}$. Assume*

a) *For all $y \in U$, the local ring $\mathcal{O}_{y,U}$ is factorial*
b) *A is parafactorial*

Then A is factorial.

PROOF. Using Serre's criterion for normality [**Mat89**, p. 183] it follows that under these conditions, A is normal. Since $\{x\}$ has codimension 2 or more, $\mathrm{A}_{\dim X-1}(X) = \mathrm{A}_{\dim X-1}(U)$. Now $\mathrm{A}_{\dim X-1}(U) = \operatorname{Pic}(U)$ by a) and Lemma A.11. But then, by b), $\mathrm{A}_{\dim X-1}(X) = 0$, hence A is factorial [**Har77**, Proposition II.6.2]. □

Now, let us just state the following without proof:

THEOREM A.13 ([**Gro68**, Théorème 3.13, p. 132]). *Let A be a Noetherian parafactorial local ring that is a complete intersection (A is the quotient of a regular local ring by a regular sequence). Assume* $\dim A \geq 4$. *Then A is factorial.*

The following was conjectured by Samuel and proved by Grothendieck:

THEOREM A.14 (SAMUEL–GROTHENDIECK). *Let A be a Noetherian local ring that is a complete intersection. Assume A is factorial in codimension* 3 *(that is, A_P is factorial when localising in all primes P satisfying* $\dim A_P \leq 3$*). Then A is factorial.*[2]

[2] Note that the inequality in the statement of this theorem mistakingly has been reversed in [**Fos73**].

PROOF. The proof uses induction on the dimension of A. For $\dim A \leq 3$, the claim is in the hypotheses. Assume $\dim A \geq 4$. By Theorem A.13 it follows that A is parafactorial. By induction, A_P is factorial for all primes P of height at least 1 (a localisation of a complete intersection is again a complete intersection). Then apply Lemma A.12. ☐

COROLLARY A.15. *Let X be a normal variety, such that the singular locus of X has codimension at least* 4. *Assume furthermore that X is a strict complete intersection. Then* $\mathrm{Pic}(X) = \mathrm{A}_{\dim X-1}(X)$.

PROOF. We must show that under the given assumptions it follows that X is locally factorial. Then Lemma A.11 will apply.

Let S be a local ring of X. Then S is a complete intersection ring. Let P be a prime in S such that S_P is not factorial. We must then show that $\dim S_P > 3$. But this must be the case: If not, X would have a non-regular local ring of dimension less than 3. And since X is assumed to be non-singular in codimension at least 4, all local rings in X are either regular or of dimension at least 4. ☐

EXAMPLE A.16. It is necessary to assume that the singularities only occur in codimension at least 4: For any field k of characteristic different from 2 the projective quadric hyper-surface $H: \; x_0x_1 = x_2^2 + \cdots + x_r^2$ in $\mathbb{P}^r$ has the following properties [**Har77**, Exercise II.6.5]:

- For $r \geq 2$ H is normal.
- $\mathrm{A}_{r-2}(H) = \mathrm{Cl}(H) = \begin{cases} \mathbb{Z} & r = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & r = 3 \\ \mathbb{Z} & r \geq 4. \end{cases}$

By the Lefschetz theorem for Picard groups, $\mathrm{Pic}(H) = \mathbb{Z}$ for $r \geq 4$.

The following proposition slightly extends (and clarifies) a remark of Grothendieck [**Gro58**, p. 4-35].

PROPOSITION A.17. *Let $\pi : E \to X$ be a locally trivial $\mathbb{A}^n$-bundle with affine transition functions*[3] *(i.e. for a covering $\cup_i U_i$ of X, $\pi^{-1}(u_i) \simeq U_i \times \mathbb{A}^n$ and the transition functions $\varphi_{ij} : \pi^{-1}(U_i \cap U_j) \to \pi^{-1}(U_i \cap U_j)$ induces either linear maps or translations on $\mathbb{A}^n$). Then*

$$\pi^* : \mathrm{A}_p(X) \longrightarrow \mathrm{A}_{p+n}(E) \tag{A.7}$$

is an isomorphism for all p.

PROOF. We will use the notion of higher Chow groups [**Ros96**] (thanks to G. Ellingsrud for suggesting this approach). Choose $U \subseteq X$ such that E is

[3]Maybe this condition can be omitted.

trivial over U and let $\tilde{U} = \pi^{-1}(U)$. Then we have long exact sequences

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{A}_{p+1}(U;-p) \arrow[r, "\partial"] \arrow[d, "\alpha_1"] & \mathrm{A}_p(X-Y) \arrow[r] \arrow[d, "\alpha_2"] & \mathrm{A}_p(X) \arrow[r] \arrow[d, "\alpha_3"] & \mathrm{A}_p(U) \arrow[r] \arrow[d, "\alpha_4"] & 0 \\
{} \arrow[r] & \mathrm{A}_{p+1+n}(\tilde{U};-p) \arrow[r, "\partial"] & \mathrm{A}_{p+n}(E-\tilde{U}) \arrow[r] & \mathrm{A}_{p+n}(E) \arrow[r] & \mathrm{A}_{p+n}(\tilde{U}) \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{A}_k(-;l)$ are the higher Chow groups. By homotopy invariance of higher Chow groups [**Ros96**, Proposition 8.6, Remark 5.1], α_1 and α_4 are isomorphisms. By Noetherian induction we may assume that also α_2 is an isomorphism. Then the assertion follows from the 5-lemma. □

COROLLARY A.18. *Let G be a reductive algebraic group. Let $U \subseteq G$ be a closed unipotent subgroup acting on G by right translation. Then for every U-stable locally closed subvariety $Y \subseteq G$, the quotient morphism $\pi : Y \to Y/U$ induces an isomorphism $\pi^* : \mathrm{A}_*(Y/U) \xrightarrow{\simeq} \mathrm{A}_*(Y)$.*

PROOF. Since the assertion is local, we may assume $Y = G$. The quotient $\pi : G \to G/U$ exists (U is closed in G) and is locally trivial with fibre U *cf.* [**Ser58**, Proposition 14] (U is unipotent, hence solvable). We must show that the transition functions are affine. Let $\cup_i U_i$ be an open cover of G/U. Then $\pi^{-1}(U_i \cap U_j) = (U_i \cap U_j) \cdot U$, hence the transition functions induces morphisms of the type $g \mapsto g'g$ ($g' \in U$) on U (π being the quotient map). But this is a translation of $(\mathbb{A}^n, +)$. □

A.2. Intersection theory on associated bundles

In this section G will denote a connected linear algebraic group, varieties are reduced schemes of finite type over an algebraically closed field k. All actions are algebraic actions.

DEFINITION A.19. Let $f : P \to X$ be a morphism of varieties. We will say that f is a *right principal G-bundle* if

1. G acts on P from the right stabilising the fibres of f.
2. $f : P \to X$ is locally trivial, that is, there exists an open cover $\{U_i\}_{i \in I}$ of X such that for any $i \in I$

$$\begin{tikzcd}
f^{-1}(U_i) \arrow[rr, "\simeq"] \arrow[dr] & & U_i \times G \arrow[dl] \\
& U_i &
\end{tikzcd}$$

is commutative and G-equivariant.

EXAMPLE A.20. If $H \leq G$ is a closed subgroup then G acts on the quotient G/H stabilising the fibres and the quotient morphism is locally trivial if H is connected and solvable [**Ser58**, Proposition 14].

Suppose from now on that $f : P \to X$ is a principal G-bundle.

DEFINITION A.21. Suppose G acts on a variety F from the left. Then G acts diagonally on $P \times F$ by

$$g.(y,z) = (yg, g^{-1}z) \qquad ; \; y \in P, \; z \in F.$$

Assume the quotient $G \setminus P \times F$ exists. Denote it by $P \times^G F$ and call it *the associated bundle (to $f : P \to X$) with fiber F.*

REMARK A.22. $P \times^G F$ has the following properties:

1. The diagram
$$\begin{array}{ccc} P \times F & \longrightarrow & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ P \times^G F & \xrightarrow{h} & X \end{array}$$
is a pull-back diagram. The morphism $h : P \times^G F \to X$ is locally trivial with fiber F and $f' : P \times F \to P \times^G F$ is a principal G-bundle.
2. The construction is functorial: if $F' \to F$ is a G-equivariant morphism of varieties, then there is a natural induced morphism $P \times^G F' \to P \times^G F$ of locally trivial bundles with fibres F' and F respectively. In particular, if $i : F' \to F$ is the inclusion of a G-stable subvariety, we get an inclusion of bundles over X.

EXAMPLE A.23. Let Z_w be the desingularisation of the Schubert variety X_w and let $\pi : G \to G/B$ be the locally trivial quotient morphism. The associated bundle

$$\bar{O}(w) = G \times^B Z_w$$

is a locally trivial bundle over G/B with fibre Z_w. If $w' \leq w$ is obtained from w by omitting $l(w) - l(w')$ reflections we may identify $Z_{w'}$ with a B-stable closed subvariety of Z_w. Hence there is an inclusion of locally trivial bundles $\bar{O}(w') \hookrightarrow \bar{O}(w)$.

LEMMA A.24. *Let $\pi : E \to X$ be a locally trivial bundle on the scheme X. Let $\pi_A : A \to X$ and $\pi_B : B \to X$ be locally trivial subbundles with fibres V and W respectively such that V and W are closed subvarieties of F. Then if $[A] = [B]$ in $\mathrm{A}_*(E)$ we also have $[V] = [W]$ in $\mathrm{A}_*(F)$.*

PROOF. Taking the fibre over a (closed) point of X yields a closed immersion $j : F \to E$. Since the refined Gysin map $j^! : \mathrm{A}_*(E) \to \mathrm{A}_*(F)$ preserves rational equivalence [**Ful83**, Section 6.2], the assertion follows. ☐

PROPOSITION A.25. *Let $f : P \to X$ be a principal G-bundle and $P \times^G F \to X$ the associated bundle with fibre F. Assume that F and $P \times^G F$ are nonsingular varieties, such that F and $P \times^G F$ carry intersection products. Let*

$Z_*(F)^G$ *denote the free Abelian group on the G-stable closed subvarieties of F. Then the homomorphism*

$$\varphi : Z_*(F)^G \to Z_*(P \times^G F)$$
$$[V] \mapsto [P \times^G V]$$

is a (codimension 0*) homomorphism taking proper intersections to proper intersection. In fact φ also takes transversal intersections to transversal intersections.*

PROOF. Since dimensions and intersection multiplicities for each component can be calculated locally on an open subset [**Ful83**, Theorem 6.2(c)], we may (by local triviality) assume that $\pi : E \to X$ is the projection $\mathrm{pr}_1 : X \times F \to X$. Then φ takes $[V] \in Z_*(F)$ to $[X \times V]$. As,

$$\varphi([V]) \cdot \varphi([W]) = [X \times V] \cdot [X \times W] = [X] \times ([V] \cdot [W]),$$

[**Ful83**, 8.3.7], it follows that the intersection on the left hand side is proper (transversal) whenever the intersection $V \cap W$ is. $\square$

APPENDIX B

Open problems

B.1. Chow groups and étale cohomology

For varieties defined over finite fields there are various conjectures (some of which are partially verified) relating the Chow groups to the étale cohomology groups (see [**Car85**, Appendix] for a readable introduction to étale cohomology). In this section we shall review some of these conjectures and their (possible) consequences when it comes to Deligne-Lusztig varieties.

We will change our notation slightly, letting k denote an arbitrary field, $\bar{k}$ an algebraically closed extension with $G(k'/k)$ the corresponding Galois group of automorphisms of $\bar{k}$ fixing k.

Let $V(k)$ denote the class of non-singular projective varieties defined over k (the residue field of the generic point is k). For any $X \in V(k)$ let $\int_X : \mathrm{A}_*(X) \to \mathbb{Z}$ denote the degree morphism (relative to k). Let $\mathrm{Z}^s(X)$ be the free Abelian group on the subvarieties of X of codimension s which are defined over k. Write $\bar{X}$ for $X \times_k \bar{k}$.

B.1.1. Tate's conjectures. Let $X \in V(k)$ be of dimension d. On $\mathrm{Z}^s(X)$ one may define several (potentially) different equivalence relations giving various quotients $\mathrm{Z}^s(X)/\sim$ *cf.* [**Ful83**, Section 19.3].

Say that $\alpha \in \mathrm{Z}^s(X)$ is *homologous to zero* ($\alpha \in \mathrm{Hom}(X)$) if α is in the kernel of the class map to étale cohomology

$$cl_X : \mathrm{Z}^s(X) \to \mathrm{H}^{2s}(X_{\text{ét}}, \mathbb{Q}_l(s)).$$

Let $\mathrm{B}^s(X)$ be the quotient of $\mathrm{Z}^s(X)$ by this relation. (Since X is non-singular, $\mathrm{Hom}^T(X)$ is actually an ideal in $\mathrm{A}^s(X)$, hence we have a surjection of rings $\mathrm{A}^s(X) \to \mathrm{B}^s(X)$.) Then Tate conjectured [**Tat65**]:

(a): $\mathrm{Hom}(X)$ is independent of the choice of prime $l \neq char(k)$.
(a'): $\mathrm{Hom}(X) = \mathrm{Num}(X)$, that is, $\alpha \in \mathrm{Hom}(X)$ if and only if $\int_X \alpha \cap \beta = 0$ for all $\beta \in \mathrm{A}_*(X)$.
(b): $\mathrm{B}^s(X)$ is a finitely generated Abelian group and the map

$$\varphi : \mathrm{B}^s(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \xrightarrow{cl_X \otimes 1} \mathrm{H}^{2s}(\bar{X}_{\text{ét}}, \mathbb{Q}_l(s))$$

is injective (that is, the class map is an isomorphism on l-torsion).
(1): For finitely generated k (e.g. k finite) the image of $\mathrm{Z}^s(X)$ under φ is exactly $\left(\mathrm{H}^{2s}(\bar{X}_{\text{ét}}, \mathbb{Q}_l(s))\right)^{G(\bar{k}/k)}$.

REMARK B.1. 1. In characteristic zero (a) and (b) are true.
2. In codimension one, that is, for divisors (a), (a') and (b) are true (compare [**Ful83**, 19.3.1]).
3. If there are strong vanishing theorems (see for example Section B.1.2 below) for the cohomology groups $\mathrm{H}^{2i}(X_{\text{ét}}, \mathbb{Q}_l)$ we see that $\mathrm{Hom}^{\tau}(X)$ may be quite large. Hence we may loose a lot of information when passing to the quotient $\mathrm{B}^s(X)$.
4. Recently (1994), Tate has commented on the status of the conjectures; see [**Tat94**].

There are some immediate consequences just from the existence of the class map. We note the following concerning the Neron-Severi groups (cycles modulo algebraic equivalence).

PROPOSITION B.2. *Let $X \in V(k)$.*

1. *The Neron-Severi group $\mathrm{NS}^k(X)$ is torsion-free and finitely generated.*
2. *In the case where X is a surface we actually have* [1]

$$rank\, \mathrm{NS}^1(X) \leq b_2 = \dim_{\mathbb{Q}_l} \mathrm{H}^2(\bar{X}_{ét}, \mathbb{Q}_l). \tag{B.1}$$

PROOF. By definition, for any non-zero class $\alpha \in \mathrm{NS}^k(X)$ there exists $\beta \in \mathrm{NS}^{\dim X - k}(X)$ such that $\int_X \alpha \cap \beta \neq 0$; hence any multiple of α is non-zero. So we need only show that $\mathrm{NS}^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ is finitely generated. Since the class map carries the intersection product of $\mathrm{CH}^*(X)$ into the non-degenerate cup product in $\mathrm{H}^*(\bar{X}_{\text{ét}}, \mathbb{Q}_l)$, the kernel of

$$cl_X : \mathrm{CH}^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \mathrm{H}^{2k}(\bar{X}_{\text{ét}}, \mathbb{Q}_l)$$

is contained in the kernel of

$$\Theta : \mathrm{CH}^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \mathrm{NS}^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l .$$

But then

$$Im(cl_X) \simeq \mathrm{CH}^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l / ker(cl_X) \xrightarrow{surj.} \mathrm{CH}^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l / ker(\Theta)$$

with the right hand side isomorphic to $\mathrm{NS}^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l$. As $\mathrm{H}^{2k}(\bar{X}_{\text{ét}}, \mathbb{Q}_l)$ is finite dimensional the assertion follows (compare [**Ful83**, Example 19.1.4]). For 2), see [**Mil80**, Corollary 3.28]. □

Now we specialise to the case where k is the finite field with q elements. Associated to X we then have its Zeta-function $Z(X,t)$ on the form

$$Z(X,t) = \frac{P_1(t) \cdot \ldots \cdot P_{2d-1}(t)}{P_0(t) P_2(t) \cdot \ldots \cdot P_{2d}(t)} \tag{B.2}$$

(recall that the polynomial $P_i(t)$ is the characteristic polynomial for the endomorphism of $\mathrm{H}^i(\bar{X}_{\text{ét}}, \mathbb{Q}_l)$ induced by the Frobenius morphism).

[1] See also [**Shi86**] for an algorithm for determining the rank of $\mathrm{NS}^1(X)$ when X is a Delsarte-surface (Fermat-surfaces are among the Delsarte-surfaces).

REMARK B.3. Tate showed [**Tat65**, p. 101] (assuming the Weil-conjectures which subsequently were proved by Deligne [**Del74**]) that when the Frobenius action on $\mathrm{H}^*(\bar{X}_{\text{ét}}, \mathbb{Q}_l)$ is semi-simple the $G(\bar{k}/k)$-invariant part of $\mathrm{H}^*(\bar{X}_{\text{ét}}, \mathbb{Q}_l)$ has rank equal to the order of the pole of $Z(X,t)$ at $t = q^{-i}$.

So assuming the conjectures (a) and (b) it would follow from Conjecture (1) that

$$\begin{gathered}\text{the rank of } \mathrm{B}_i(X) \text{ equals the order}\\ \text{of the pole of } Z(X,t) \text{ at } t = q^{-i}.\end{gathered} \tag{T}$$

In particular, if $P_i(t)$ is some linear form raised to the b_{2i}'th power, then the pole order of $Z(X,t)$ at $t = q^{-i}$ is $b_{2i} = \dim_{\mathbb{Q}_l} \mathrm{H}^{2i}(X_{\text{ét}}, \mathbb{Q}_l)$ and proving (T) amounts to proving that $\mathrm{H}^{2i}(X_{\text{ét}}, \mathbb{Q}_l)$ is spanned by algebraic cycles.

Conversely, Tate has shown [**Tat94**] that the conjecture (T) implies the conjectures (a), (b) and (1).

Over finite fields Milne [**Mil86**, Section 8] has shown that Conjecture (1) is independent of l and has related the conjectures to other conjectures.

EXAMPLE B.4. Let $X = V_{n,r,p}$ be the hypersurface in $\mathbb{P}^r_k$ ($char(k) = p$) defined by the equation

$$X_0^n + X_1^n + \cdots + X_r^n = 0 \tag{B.3}$$

where $p \nmid n$. For these Fermat varieties the action of F on $\mathrm{H}^*(X_{\text{ét}}, \mathbb{Q}_l(s))$ is semisimple [**Sou84**, Remarque p. 334] hence the pole-order of $Z(X,t)$ at $t = q^{-i}$ equals the dimension of the $G(\bar{k}/k)$-invariant part of $\mathrm{H}^*(X_{\text{ét}}, \mathbb{Q}_l(s))$. The zeta-functions of the $V_{n,r,p}$ were determined by Weil in [**Wei49**].

For r odd, Tate proved (see [**Tat65**, p. 102], written out in more detail in [**HM78**]) that in case $p^\nu \equiv -1 \pmod n$ for some integer ν, then Conjecture (1) is true (for X) since $\mathrm{H}^*(X_{\text{ét}}, \mathbb{Q}_l)$ in that case is spanned by algebraic cycles. In [**SK79**, Theorem II] it was established that for r odd and $n \geq 4$ the étale cohomology is spanned by algebraic cycles if and only if $p^\nu \equiv -1 \pmod n$ for some integer ν. See also [**GY95**].

PROBLEM 1. Verify Tate's conjecture (T) for Deligne-Lusztig varieties.

B.1.2. Lefschetz theorems. Let X be a d-dimensional non-singular complete intersection in $\mathbb{P}^{n+1}$. Then for $i \neq d$ we have [**Ill77**, Exposé VII]

$$\mathrm{H}^i(X_{\text{ét}}, \mathbb{Q}_l) = \begin{cases} 0 & i \text{ odd} \\ \mathbb{Q}_l \cdot [V_i] & i \text{ even} \end{cases} \tag{B.4}$$

where $[V_i] = cl_X(L^{d-i/2} \cap X)$ with L^k a k-dimensional linear subspace not containing X. Hence showing that the étale cohomology of an even-dimensional hypersurface is spanned by algebraic cycles amounts to showing that this is the case for the middle cohomology $\mathrm{H}^d(X_{\text{ét}}, \mathbb{Q}_l)$.

See [**GL98**] for generalisations of such Lefschetz theorems to singular varieties.

The Lefschetz principle for Picard groups says: for any (possibly singular) complete intersection $X \subseteq \mathbb{P}^{n+1}$ of dimension $d \geq 3$ we have $\mathrm{Pic}(X) = \mathbb{Z}$ [**Gro68**, Exposé XII, Corollaire 3.7]. In particular, if X is a non-singular hypersurface in $\mathbb{P}^{n+1}$ ($n \geq 3$), $\mathrm{A}_{n-1}(X) = \mathrm{Pic}(X) = \mathbb{Z} \cdot [L^n \cap X]$ with the above notation.[2]

B.1.3. Enter the automorphism groups. The Fermat varieties considered above (Example B.4) is acted upon by the finite unitary group $G = {}^2\mathrm{A}_2(q^2)$.

Tate's insight: composing the action of G on $\mathrm{Z}_s(X)$ with the class map to étale cohomology gives a representation of the finite group on the finite-dimensional $\mathbb{Q}_l$-vector space $\mathrm{H}^{2s}(X_{\text{ét}}, \mathbb{Q}_l)$. In this particular case G has only the trivial and one other irreducible representation on $\mathrm{H}^{2s}(X_{\text{ét}}, \mathbb{Q}_l)$ of dimension, m say. Since the class map cl_X is non-trivial, the rank of $\mathrm{B}_s(X)$ is greater than or equal to m (and therefore also $rank\, \mathrm{A}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \geq m$).

This philosophy applies particularly well to the case where X is the smooth compactification of an irreducible Deligne-Lusztig variety $X(w)$ with the (large) finite group G^F acting as automorphisms on $\bar{X}(w)$. The dimensions of the irreducible G^F-representations on $\mathrm{H}_c^*(X(w)_{\text{ét}}, \mathbb{Q}_l)$ has been determined by Lusztig [**Lus76a**], hence by the above procedure one should be able to get (at least) lower bounds on the ranks of the Chow groups $\mathrm{A}^i(\bar{X}(w))$ (after relating the cohomology with compact support of $X(w)$ to the cohomology of the smooth compactification $\bar{X}(w)$ — this may be done using the long exact $\mathrm{H}^i(\cdot)$-cohomology sequence coming from the inclusions $X(w) \xrightarrow{\text{open}} \bar{X}(w) \xleftarrow{\text{closed}} \partial \bar{X}(w)$; see [**Sri79**, p. 52]).

PROBLEM 2. Determine the character(s) of the representation(s) of G^F on $\mathrm{A}_i(\bar{X}(w))_{\mathbb{Q}}$.

REMARK B.5. It is known that the action on $\mathrm{H}_c^i(X(w)_{\text{ét}}, \mathbb{Q}_l)$ induced by Frobenius F^δ is semi-simple [**Lus76a**, (6.1) Theorem]. From the above-mentioned long exact sequence it follows by induction that this is also true for $\bar{X}(w)$.

B.1.4. Classes of 'good' varieties. Let $A(k)$ be the minimal subset of $V(k)$ containing all geometrically irreducible curves and being closed under the operations

- product,
- disjoint union,
- finite extensions of base-field and
- blowing up $X \in A(k)$ along $Y \in A(k)$, $Y \subseteq X$

cf. [**Sou84**, 3.3.1]. Note that $A(k)$ contains the Fermat varieties [**SK79**] and all abelian varieties [**Sou84**, 3.3.3]. The following theorem extracts Soulé's results [**Sou84**, Theorems 3+4] on the Chow groups of such varieties.

[2] Since $n \geq 3$, cl_X does not map $\mathrm{A}^1(X)$ to the middle cohomology ($2n - 2 = n$ implies $n = 2$), hence we can still have high-rank middle cohomology generated by algebraic cycles.

THEOREM B.6 (SOULE). *Let $X \in A(k)$ be of dimension d. Then for $i \in \{0, 1, d-1, d\}$, $\mathrm{CH}^i(X) = \mathbb{Z}^{r_i} \oplus T$ where T is a finite torsion group. Furthermore, the class map is an isomorphism modulo torsion; that is,*

$$cl_X : \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \left(\mathrm{H}^{2i}(\bar{X}_{\acute{e}t}, \mathbb{Q}_l(i)) \right)^{G(\bar{k}/k)} \tag{B.5}$$

is an isomorphism. Furthermore, for some finite extension k' of k we have

$$\mathrm{CH}^i(X \times_k k')/\,\mathrm{CH}^i(X \times_k k')_{torsion} \simeq \mathrm{CH}^i(\bar{X})/\,\mathrm{CH}^i(\bar{X})_{torsion}.$$

EXAMPLE B.7. Let $X = \bar{X}(s_1 s_2)$ be the ${}^2\mathrm{A}_3$ Deligne-Lusztig variety with $q = 2$. X is then the blow-up of the non-singular Fermat cubic surface

$$S : \qquad X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$$

in its $(q^3+1)(q^2+1) = 45$ $\mathbb{F}_{q^2}$-rational points. Being a non-singular cubic surface in $\mathbb{P}^3$, S is the projective plane blown up in 6 points in general position [**Har77**, Section V.4], hence X is the blow-up of the projective plane in 51 points. So $\mathrm{A}_1(X) = \bigoplus_{i=1}^{51} \mathbb{Z} \cdot [E_i] \oplus \mathbb{Z} \cdot [H]$ where H is the pull-back of a line in $\mathbb{P}^2$ and the E_i are the exceptional divisors *cf.* [**Ful83**, Example 8.3.10]. In [**Rod96**] the Betti numbers have determined; we find

$$b_2 = \dim_{\mathbb{Q}_l} \left(\mathrm{H}^2(X_{\acute{e}t}, \mathbb{Q}_l) \right)^{G(\bar{k}/k)} = q^5 + 2q^3 + q + 2 = 52$$

and this is also the pole-order of $Z(X, t)$ at $t = q^{-i}$. So all the cohomology is algebraic as predicted by Theorem B.6. [3]

So it seems that we in some degrees may reformulate Problem 1 as:

PROBLEM 3. Determine whether Deligne-Lusztig varieties are in $A(k)$ or not. One might start by determining if for example a complete intersection is in $A(k)$ when the intersecting hyper-surfaces are (*cf.* Chapter 3).

Define a subset $B(k)$ of $A(k)$ by:

- $B(k)$ contains products of geometrically irreducible curves.
- If $X, Y \in B(k)$, $\dim X = \dim Y$, then $X \coprod Y \in B(k)$.
- Suppose $X, Y \in V(k)$ are equi-dimensional of the same dimension and $f : X \to Y$ is a surjective morphism. Then, if X is in $B(k)$ so is Y.
- If $X \times_k k' \in B(k)$ then also $X \in B(k)$ and
- $B(k)$ is minimal with these properties.

$B(k)$ contains the Abelian varieties and is closed under products *cf.* [**Sou84**, 3.4]. We then have the following Lefschetz-type theorem [**Sou84**, Theorem 7]:

THEOREM B.8 (SOULE). *If $X \in B(k)$ and $\xi \in \mathrm{CH}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the class of an ample divisor on X, then for $2i \le d = \dim X$, the multiplication map*

$$\mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cdot \cap \xi^{d-2i}} \mathrm{CH}^{d-i}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

---

[3] Note: for $q = 2$, $\bar{X}(s_1)$ has 27 components which (probably) are the proper transforms of the 27 lines on S.

REMARK B.9. Let C be a non-singular projective curve defined over $\mathbb{F}_q$. Assume C h as a rational point P over $\mathbb{F}_{q^\delta}$. Then the degree map $\mathrm{A}_0(C) \to \mathbb{Z}$ has image $\delta\mathbb{Z}$ and kernel $\mathrm{Alb}(C) = \mathrm{A}_0(C)_0$. Since the kernel is the (finitely many) closed points of an Abelian variety, it is all torsion, hence $\mathrm{A}_0(C)_{\mathbb{Q}} = \mathbb{Q}$ with generator $\frac{1}{\delta}[P]$. For example, if C is an elliptic curve with a rational point P, then all $[P'] \in \mathrm{Pic}^0(C)$ are torsion elements.

From the above remark (or from Theorem B.8) we have:

PROPOSITION B.10. *Let $\bar{X}(w)$ be a one-dimensional Deligne-Lusztig variety. Then* $\mathrm{A}_1(\bar{X}(w)) = \mathbb{Z}$ *and* $\mathrm{A}_0(\bar{X}(w))_{\mathbb{Q}} = \mathbb{Q}$.

B.1.5. Deligne-Lusztig surfaces. Let $w \in W$ be a standard Coxeter element of length 2, $w = s_1 s_2$. So we are considering pairs $(\mathcal{D}, \rho)$ where ρ has two orbits on $\mathcal{D}$. From [**Car85**, p. 37] we see that restricting ourselves to non-trivial graph automorphisms ρ we have the possibilities listed in Table C.2 (page 105). These surfaces have been examined in [**Rod96**] where their number of rational points (over their natural field of definition) is being compared to their Betti numbers *cf.* [**LT95, Tsf94**].

Using the results of Rodier [**Rod96**] and the explicit descriptions given in Chapter 3 we will now examine the A_2, C_2, ${}^2\mathrm{A}_3$ and ${}^2\mathrm{A}_4$ standard Deligne-Lusztig varieties.

EXAMPLE B.11 (A_2 case). Restricting the composite map

$$X \times X \times X \xrightarrow{\mathrm{pr}_1} X \to G/P_{I_1} \simeq \mathbb{P}^2$$

to $\bar{X}(w)$ realizes $\bar{X}(w)$ as the blow-up of $\mathbb{P}^2$ in its $q^2 + q + 1$ $\mathbb{F}_q$-rational points. $\mathrm{A}^*(\bar{X}(w))$ is given by [**Ful83**, Example 8.3.10] and by Theorem B.6 Tate's conjecture (T) is verified in this case.

REMARK B.12. Let $\bar{X}(w)$ be an irreducible Deligne-Lusztig variety with zeta function $Z(t)$ over $\mathbb{F}_{q^\delta}$,

$$t\frac{d}{dt}\ln(Z(t)) = \sum_{s=1}^{\infty} |\bar{X}(w)^{F^{\delta s}}| t^s.$$

Then it follows from the Weil conjectures [**Del74**] that we may write $Z(t)$ as a rational function

$$Z(t) = \frac{P_1(t) \cdot \ldots \cdot P_{2d-1}(t)}{P_0(t)P_2(t) \cdot \ldots \cdot P_{2d}(t)}$$

with $\deg P_i = b_i$ (the i'th Betti number of $\bar{X}(w)$). In [**Lus76a**, Theorem 6.1] the series $\sum_{s=1}^{\infty} |X(w)^{F^{\delta s}}| t^2$ is given for a Coxeter stratum $X(w)$. Summing over these we get for example in the A_2 case:

$$t\frac{d}{dt}\ln(Z(t)) = \frac{q^3(q^2-1)(q^3-1)}{q^2+q+1}t^3\frac{1}{(1-t)(1-qt)(1-q^2t)} + 2(q^2+q+1)\frac{q(q^2-1)}{q+1}t^2\frac{1}{(1-qt)(1-t)} + (q+1)(q^2+q+1)\frac{t}{1-t}$$

(the first term is from the open stratum, the next is from the two codimension 1 strata and the last term comes from $X(e)$). Manipulating this expression one finds that

$$Z(t) = \frac{1}{(1-t)(1-qt)^{q^2+q+2}(1-q^2t)}$$

hence $b_1 = b_3 = 0$, $b_0 = b_4 = 1$ and $b_2 = q^2+q+2$.

EXAMPLE B.13 (C_2 case). Let $V = k^4$ be equipped with the form $\langle\cdot,\cdot\rangle$ given by

$$\langle x, y\rangle = (y_1x_3 - x_1y_3) + (y_2x_4 - x_2y_4).$$

F_V acts on V by raising the coordinates to the q'th power. Then $X(w)$ is given by

$$X(w) = \{x \in \mathbb{P}^3 : \langle x, F_V(x)\rangle = 0, \langle x, F_V^2(x)\rangle \neq 0\}$$

and by blowing up the non-singular surface $S = \{x \in \mathbb{P}^3 : \langle x, F_V(x)\rangle = 0\}$ in its $\mathbb{F}_q$-rational points (the points for which $F_V(x) = x$), we obtain $\bar{X}(w)$. Hence

$$\begin{aligned} A^0(\bar{X}(w)) &= \mathbb{Z}[\bar{X}(w)] \\ A^1(\bar{X}(w)) &= A^1(S) \oplus \bigoplus_{i=1}^{q^3+q^2+q+1} \mathbb{Z}[E_i] \\ A^2(\bar{X}(w)) &= A^2(S). \end{aligned}$$

From [**Rod96**, Proposition 1] we get that S has Betti numbers $b_1 = b_3 = 0$, $b_0 = b_4 = 1$ and $b_2 = q^3 - q^2 + q + 1$. The pole-order of $Z(t)$ at $t = q^{-1}$ is $\frac{1}{2}(q^3+q+2)$. From Proposition B.2 we have

$$rank\,\mathrm{NS}^k(S) \leq q^3 - q^2 + q + 1$$

and from [**SK79**] it follows that $S \in A(k)$ (see Section B.1.4) hence $A^1(S) = \mathbb{Z}^r \oplus (torsion)$; $r = \frac{1}{2}(q^3+q+2)$. (Hence Tate's conjecture (T) is OK also in this case.)

EXAMPLE B.14 (${}^2A_3$ case). Restricting the projection

$$X \times X \times X \xrightarrow{\mathrm{pr}_1} X \to G/P \simeq \mathbb{P}^3$$

to $\bar{X}(w)$ realizes $\bar{X}(w)$ as the blow-up of the Fermat surface $S \subseteq \mathbb{P}^3$,

$$S: \qquad x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0.$$

in its $\mathbb{F}_{q^2}$-rational points (of which there are $(q^3+1)(q^2+1)$). From Theorem B.8 it follows that $\mathrm{A}^i(S)$ has rank b_{2i}. Again we have $b_1 = b_3 = 0$, $b_0 = b_4 = 1$. By counting the number of rational points over extensions of $\mathbb{F}_q$ one may determine b_2 (as in Remark B.12) to be equal to $q^3 - q^2 + q + 1$. Hence

$$\begin{aligned}
\mathrm{A}^0(\bar{X}(w)) &= \mathbb{Z}[\bar{X}(w)] \\
\mathrm{A}^1(\bar{X}(w)) &= \mathrm{A}^1(S) \oplus \bigoplus_{i=1}^{(q^3+1)(q^2+1)} \mathbb{Z}[E_i]\,; \;\; \mathrm{A}^1(S) = \mathbb{Z}^{b_2} \oplus (\text{torsion}) \\
\mathrm{A}^2(\bar{X}(w)) &= \mathrm{A}^2(S) = \mathbb{Z} \oplus (\text{torsion}).
\end{aligned}$$

EXAMPLE B.15 ($^2\mathrm{A}_4$ case). As seen in Example 3.17, restricting the composite map

$$X \times X \times X \xrightarrow{\mathrm{pr}_1} X \to G/P \simeq \mathbb{P}^3$$

to $\bar{X}(w)$ gives a birational map to the intersection of two Fermat hypersurfaces of degree $q+1$ and q^3+1. As seen earlier (Example 1.16) the exceptional fibers are Deligne-Lusztig varieties of type $^2\mathrm{A}_2$, that is, Hermitian curves. $\bar{X}(w)$ has Betti numbers $b_1 = b_3 = q(q-1)(q^2-1)$, $b_0 = b_4 = 1$ and $b_2 = q^8 + q^6 + q^4 + q^2 + 2$ (see [**Rod96**, Proposition 2]). From Proposition B.2 we have

$$\mathit{rank}\, \mathrm{NS}^1(\bar{X}(w)) \leq q^8 + q^6 + q^4 + q^2 + 2.$$

On the other hand, it follows from Lemma A.8 that $\mathit{rank}\, \mathrm{A}^1(\bar{X}(w)) \geq (q^5+1)(q^2+1)$.

B.2. Cohomology of homogeneous line bundles

DEFINITION B.16. A morphism of varieties (or schemes) $f : X \to Y$ is *trivial* with image $f(Y)$ if $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, $f(Y) = \operatorname{Spec} f_*\mathcal{O}_X$ and $\mathrm{R}^i f_*\mathcal{O}_X = 0$ for $i > 0$.

A closed embedding $i : X \to Y$ is trivial with image $i(X)$ [**Ram85**, Remark 4]. For a trivial morphism we have $\mathrm{H}^j(X, f^*M) = \mathrm{H}^j(Y, M)$ for all $j \geq 0$ and for any locally free $\mathcal{O}_Y$-module M of finite rank [**Har77**, Exercise III.8.3]. (And for f affine we have for any quasi-coherent sheaf N on X that $\mathrm{H}^i(X, N) = \mathrm{H}^i(Y, f_*N)$ [**Har77**, Exercise III.8.2].)

PROBLEM 4. For a Schubert variety X_w it is known that the cohomology groups $\mathrm{H}^i(X_w, \mathcal{L}(\lambda)|_{X_w})$ vanishes for $i > 0$ whenever λ is strictly anti-dominant. Under the same condition, the restriction map

$$\mathrm{H}^i(X, \mathcal{L}(\lambda)) \to \mathrm{H}^i(X_w, \mathcal{L}(\lambda)|_{X_w})$$

is surjective. Prove (or dis-prove) similar statements for the restriction of such line bundles to Deligne-Lusztig varieties.

The proof in the Schubert variety case uses that the Demazure desingularisations $Z_w \xrightarrow{\varphi_e} X_w$ and $\bar{O}(w) \xrightarrow{\varphi} \overline{O(w)}$ are surjective and trivial [**Dem74, MR88**] and an induction argument.

REMARK B.17. It is *not* true that the contraction map $\pi : \bar{X}(w) \to Z$ is trivial in general (in the sense of Definition B.16). A calculation in the ${}^2\mathrm{A}_4$ case shows that $\mathrm{H}^1(C, \mathcal{O}_C) = \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(q+2)) \neq 0$, hence the limit

$$\mathrm{R}^1 f_* \mathcal{O}_{\bar{X}(w)} = \varprojlim{}_n \mathrm{H}^1(D_{1,n}, \mathcal{O}_{D_{1,n}})$$

has as its first term a number of copies of $\mathrm{H}^1(C, \mathcal{O}_C)$ (C is one of the components of D_1 and $D_{1,i}$ is the subscheme of $\bar{X}(w)$ defined by the i'th power of the ideal defining D_1). Hence $\mathrm{R}^1 f_* \mathcal{O}_{\bar{X}(w)} \neq 0$.

B.3. The finer structure of the Picard group

Given the results of Chapter 3 it is natural to pose:

PROBLEM 5. Let D be an *effective* divisor (or line bundle) on a Deligne-Lusztig variety $\bar{X}(w)$ of classical type. We know that there exists unique rational numbers m, n_i allowing us to write $D = m\pi^*[H] + \sum_i n_i V_i$ (the V_i are the components of D_1). Determine which m and n_i can occur and what further restrictions need to be imposed to ensure that D is *ample*.

One might start out by calculating the coordinates for $K_{\bar{X}(w)}$ (which we know is ample in some cases) and see if any pattern shows up.

PROBLEM 6. In Chapter 3 we saw that the rank of the Picard group is the same for Deligne-Lusztig varieties arising from F-conjugate Weyl group elements (Lemma 3.5). Determine how the generators change under this transformation.

Similarly, given $\lambda \in X(T)$ and integers v_i such that the pull-back of the homogeneous line bundle $\mathcal{L}(\lambda)$ to $\bar{X}(w)$ equals $\mathcal{O}_{\bar{X}(w)}(\sum_i v_i D_i)$, how does the v_i change?[4]

[4]For each specific example one may follow the methods of [**HH99a**], but it would be nice with a general formula.

APPENDIX C

Maple calculations

In this chapter we have included some of the calculations carried out with the computer algebra program Maple. The code should be rather self-explanatory.

C.1. The surface case

Each of the cases in Table C.2 (page 105) have been examined for determining the numbers of rational points and the Euler characteristics.

```
> with(linalg):

  Warning, new definition for norm

  Warning, new definition for trace
```

First some constants:

```
> epsilon := exp(2*Pi*I/3);
```

$$\varepsilon := -\frac{1}{2} + \frac{1}{2} I \sqrt{3}$$

```
> G_2A3 := Q^6*(Q^2-1)*(Q^3+1)*(Q^4-1);
```

$$G\_2A3 := Q^6 (Q^2-1)(Q^3+1)(Q^4-1)$$

```
> G_2A4 := Q^10*(Q^2-1)*(Q^3+1)*(Q^4-1)*(Q^5+1);
```

$$G\_2A4 := Q^{10} (Q^2-1)(Q^3+1)(Q^4-1)(Q^5+1)$$

```
> G_3D4 :=factor(expand(Q^12*(Q^2-1)*(Q^4-epsilon)*(Q^4-epsilon^2)*(Q^6-1)));
```

$$G\_3D4 := Q^{12} (Q^4 - Q^2 + 1)(Q-1)^2 (Q+1)^2 (Q^2+Q+1)^2 (Q^2-Q+1)^2$$

(Here we made some mumbo-jumbo to get rid of the square-roots)

```
> G_2F4 := Q^24*(Q^2-1)*(Q^6+1)*(Q^8-1)*(Q^12+1);
```

$$G\_2F4 := Q^{24} (Q^2-1)(Q^6+1)(Q^8-1)(Q^{12}+1)$$

```
> phi_plus_2A3:=6:phi_plus_2A4:=10:phi_plus_3D4 :=12:phi_plus_2F4:=24:
```

Matrix representations of F and F_0 acting on Y_0 (tensored with $\mathbb{R}$):

```
> F_2A3:=matrix([[0,0,Q],[0,Q,0],[Q,0,0]]):
> F0_2A3:=matrix([[0,0,1],[0,1,0],[1,0,0]]):
> F_2A4:=matrix([[0,0,0,Q],[0,0,Q,0],[0,Q,0,0], [Q,0,0,0]]):
> F0_2A4:=matrix([[0,0,0,1],[0,0,1,0],[0,1,0,0] ,[1,0,0,0]]):
> F_3D4:=matrix([[0,0,Q,0],[0,Q,0,0],[0,0,0,Q], [Q,0,0,0]]):
> F0_3D4:=matrix([[0,0,1,0],[0,1,0,0],[0,0,0,1] ,[1,0,0,0]]):
```

```
> F_2F4:=matrix([[0,0,0,Q],[0,0,sqrt(2)*Q,0],[0 ,sqrt(1/2)*Q,0,0],[Q,0,0,0]]):
> F0_2F4:=matrix([[0,0,0,1],[0,0,sqrt(2),0],[0, sqrt(1/2),0,0],[1,0,0,0]]):
```

Matrix representations of s_1, s_2 etc. acting (from the right) on Y_0 (tensored with $\mathbb{R}$):

```
> s1_2A3:=matrix([[-1,0,0],[1,1,0],[0,0,1]]):
> s1_2A4:=matrix([[-1,0,0,0],[1,1,0,0],[0,0,1,0 ],[0,0,0,1]]):
> s1_3D4:=matrix([[-1,0,0,0],[1,1,0,0],[0,0,1,0 ],[0,0,0,1]]):
> s1_2F4:=matrix([[-1,0,0,0],[1,1,0,0],[0,0,1,0 ],[0,0,0,1]]):
> s2_2A3:=matrix([[1,1,0],[0,-1,0],[0,1,1]]):
> s2_2A4:=matrix([[1,1,0,0],[0,-1,0,0],[0,1,1,0 ],[0,0,0,1]]):
> s2_3D4:=matrix([[1,1,0,0],[0,-1,0,0],[0,1,1,0 ],[0,1,0,1]]):
> s2_2F4:=matrix([[1,1,0,0],[0,-1,0,0],[0,1,1,0 ],[0,0,0,1]]):
> s1s2_2A3:= evalm(s1_2A3 &* s2_2A3):
> s1s2_2A4:= evalm(s1_2A4 &* s2_2A4):
> s1s2_3D4:= evalm(s1_3D4 &* s2_3D4):
> s1s2_2F4:= evalm(s1_2F4 &* s2_2F4):
> F0_s1inv_2A3:=evalm(F0_2A3 &* inverse(s1_2A3)):
> F0_s1inv_2A4:=evalm(F0_2A4 &* inverse(s1_2A4)):
> F0_s1inv_3D4:=evalm(F0_3D4 &* inverse(s1_3D4)):
> F0_s1inv_2F4:=evalm(F0_2F4 &* inverse(s1_2F4)):
> F0_s2inv_2A3:=evalm(F0_2A3 &* inverse(s2_2A3)):
> F0_s2inv_2A4:=evalm(F0_2A4 &* inverse(s2_2A4)):
> F0_s2inv_3D4:=evalm(F0_3D4 &* inverse(s2_3D4)):
> F0_s2inv_2F4:=evalm(F0_2F4 &* inverse(s2_2F4)):
> F0_s1s2inv_2A3:=evalm(F0_2A3 &* inverse((s1_2A3 &* s2_2A3))):
> F0_s1s2inv_2A4:=evalm(F0_2A4 &* inverse((s1_2A4 &* s2_2A4))):
> F0_s1s2inv_3D4:=evalm(F0_3D4 &* inverse((s1_3D4 &* s2_3D4))):
> F0_s1s2inv_2F4:=evalm(F0_2F4 &* inverse((s1_2F4 &* s2_2F4))):
> F_s1s2inv_2A3:=evalm(F_2A3 &* inverse((s1_2A3 &* s2_2A3)));
```

$$F\_s1s2inv\_2A3 := \begin{bmatrix} Q & Q & Q \\ -Q & -Q & 0 \\ 0 & Q & 0 \end{bmatrix}$$

```
> F_s1s2inv_2A4:=evalm(F_2A4 &* inverse((s1_2A4 &* s2_2A4)));
```

$$F\_s1s2inv\_2A4 := \begin{bmatrix} 0 & 0 & 0 & Q \\ Q & Q & Q & 0 \\ -Q & -Q & 0 & 0 \\ 0 & Q & 0 & 0 \end{bmatrix}$$

We need the number of eigenvalues equal to 1:

```
> eigenvals(F0_2A3);
```

$$-1, 1, 1$$

```
> sigma2A3:=2;
```

$$sigma2A3 := 2$$

```
> eigenvals(F0_2A4);
```

$$1,\, 1,\, -1,\, -1$$

```
> sigma2A4:=2;
```

$$sigma2A4 := 2$$

```
> eigenvals(F0_3D4);
```

$$-\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3},\, -\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3},\, 1,\, 1$$

```
> sigma3D4:=2;
```

$$sigma3D4 := 2$$

```
> eigenvals(F0_2F4);
```

$$1,\, 1,\, -1,\, -1$$

```
> sigma2F4:=2;
```

$$sigma2F4 := 2$$

```
> eigenvals(F0_s1inv_2A3);
```

$$1,\, I,\, -I$$

```
> sigma_s1T_2A3:=1;
```

$$sigma\_s1T\_2A3 := 1$$

```
> eigenvals(F0_s1inv_2A4);
```

$$1,\, -1,\, I,\, -I$$

```
> sigma_s1T_2A4:=1;
```

$$sigma\_s1T\_2A4 := 1$$

```
> eigenvals(F0_s1inv_3D4);
```

$$1,\, -1,\, \frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3},\, \frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}$$

```
> sigma_s1T_3D4:=1;
```

$$sigma\_s1T\_3D4 := 1$$

```
> eigenvals(F0_s1inv_2F4);
```

$$1,\, -1,\, I,\, -I$$

```
> sigma_s1T_2F4:=1;
```

$$sigma\_s1T\_2F4 := 1$$

```
> eigenvals(F0_s2inv_2A3);
```

$$1,\, -1,\, -1$$

```
> sigma_s2T_2A3:=1;
```

$$sigma\_s2T\_2A3 := 1$$

```
> eigenvals(F0_s2inv_2A4);
```

$$1,\ -1,\ \frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3},\ \frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}$$

```
> sigma_s2T_2A4:=1;
```

$$sigma\_s2T\_2A4 := 1$$

```
> eigenvals(F0_s2inv_3D4);
```

$$1,\ -1,\ -\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3},\ -\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3}$$

```
> sigma_s2T_3D4:=1;
```

$$sigma\_s2T\_3D4 := 1$$

```
> eigenvals(F0_s2inv_2F4);
```

$$1,\ -1,\ \frac{1}{2}\,\sqrt{2}+\frac{1}{2}\,I\,\sqrt{2},\ \frac{1}{2}\,\sqrt{2}-\frac{1}{2}\,I\,\sqrt{2}$$

```
> sigma_s2T_2F4:=1;
```

$$sigma\_s2T\_2F4 := 1$$

```
> eigenvals(F0_s1s2inv_2A3);
```

$$-1,\ \frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3},\ \frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3}$$

```
> sigma_s1s2T_2A3:=0;
```

$$sigma\_s1s2T\_2A3 := 0$$

```
> eigenvals(F0_s1s2inv_2A4);
```

$$\frac{1}{4}\,\sqrt{5}+\frac{1}{4}-\frac{1}{4}\,I\,\sqrt{2}\,\sqrt{5+\sqrt{5}},\ \frac{1}{4}\,\sqrt{5}+\frac{1}{4}-\frac{1}{4}\,I\,\sqrt{2}\,\sqrt{5-\sqrt{5}},$$
$$\frac{1}{4}\,\sqrt{5}+\frac{1}{4}+\frac{1}{4}\,I\,\sqrt{2}\,\sqrt{5-\sqrt{5}}, -\frac{1}{4}\,\sqrt{5}+\frac{1}{4}+\frac{1}{4}\,I\,\sqrt{2}\,\sqrt{5+\sqrt{5}}$$

```
> sigma_s1s2T_2A4:=0;
```

$$sigma\_s1s2T\_2A4 := 0$$

```
> eigenvals(F0_s1s2inv_3D4);
```

$$\frac{1}{2}\,\sqrt{2-2\,I\,\sqrt{3}},\ -\frac{1}{2}\,\sqrt{2-2\,I\,\sqrt{3}},\ \frac{1}{2}\,\sqrt{2+2\,I\,\sqrt{3}},\ -\frac{1}{2}\,\sqrt{2+2\,I\,\sqrt{3}}$$

```
> sigma_s1s2T_3D4:=0;
```

$$sigma\_s1s2T\_3D4 := 0$$

```
> eigenvals(F0_s1s2inv_2F4);
```

$$\mathrm{RootOf}(-\_Z^3\,\sqrt{2}+\_Z^4+\_Z^2-\sqrt{2}\,\_Z+1)$$

```
> sigma_s1s2T_2F4:=0;
```

$$sigma\_s1s2T\_2F4 := 0$$

The numbers of fixed points on the Coxeter tori:

```
> fixed_points_T_2A3:=sort(charpoly(inverse(F0_ 2A3),Q));
```

$$fixed\_points\_T\_2A3 := Q^3 - Q^2 - Q + 1$$

```
> fixed_points_T_2A4:=sort(charpoly(inverse(F0_ 2A4),Q));
```

$$fixed\_points\_T\_2A4 := Q^4 - 2\,Q^2 + 1$$

```
> fixed_points_T_3D4:=sort(charpoly(inverse(F0_ 3D4),Q));
```

$$fixed\_points\_T\_3D4 := Q^4 - Q^3 - Q + 1$$

```
> fixed_points_T_2F4:=sort(charpoly(inverse(F0_ 2F4),Q));
```

$$fixed\_points\_T\_2F4 := Q^4 - 2\,Q^2 + 1$$

```
> fixed_points_g1T_2A3:=sort(charpoly(inverse(F 0_2A3) &* s1_2A3,Q));
```

$$fixed\_points\_g1T\_2A3 := Q^3 - Q^2 + Q - 1$$

```
> fixed_points_g1T_2A4:=sort(charpoly(inverse(F 0_2A4) &* s1_2A4,Q));
```

$$fixed\_points\_g1T\_2A4 := Q^4 - 1$$

```
> fixed_points_g1T_3D4:=sort(charpoly(inverse(F 0_3D4) &* s1_3D4,Q));
```

$$fixed\_points\_g1T\_3D4 := Q^4 - Q^3 + Q - 1$$

```
> fixed_points_g1T_2F4:=sort(charpoly(inverse(F 0_2F4) &* s1_2F4,Q));
```

$$fixed\_points\_g1T\_2F4 := Q^4 - 1$$

```
> fixed_points_g2T_2A3:=sort(charpoly(inverse(F 0_2A3) &* s2_2A3,Q));
```

$$fixed\_points\_g2T\_2A3 := Q^3 + Q^2 - Q - 1$$

```
> fixed_points_g2T_2A4:=sort(charpoly(inverse(F 0_2A4) &* s2_2A4,Q));
```

$$fixed\_points\_g2T\_2A4 := Q^4 - Q^3 + Q - 1$$

```
> fixed_points_g2T_3D4:=sort(charpoly(inverse(F 0_3D4) &* s2_3D4,Q));
```

$$fixed\_points\_g2T\_3D4 := Q^4 + Q^3 - Q - 1$$

```
> fixed_points_g2T_2F4:=sort(charpoly(inverse(F 0_2F4) &* s2_2F4,Q));
```

$$fixed\_points\_g2T\_2F4 := Q^4 - \sqrt{2}\,Q^3 + \sqrt{2}\,Q - 1$$

```
> fixed_points_g1g2T_2A3:=sort(charpoly(inverse (F0_2A3) &* s1s2_2A3,Q));
```

$$fixed\_points\_g1g2T\_2A3 := Q^3 + 1$$

```
> fixed_points_g1g2T_2A4:=sort(charpoly(inverse (F0_2A4) &* s1s2_2A4,Q));
```

$$fixed\_points\_g1g2T\_2A4 := Q^4 - Q^3 + Q^2 - Q + 1$$

```
> fixed_points_g1g2T_3D4:=sort(charpoly(inverse (F0_3D4) &* s1s2_3D4,Q));
```

$$fixed\_points\_g1g2T\_3D4 := Q^4 - Q^2 + 1$$

```
> fixed_points_g1g2T_2F4:=sort(charpoly(inverse (F0_2F4) &* s1s2_2F4,Q));
```

$$fixed\_points\_g1g2T\_2F4 := Q^4 - \sqrt{2}\,Q^3 + Q^2 - \sqrt{2}\,Q + 1$$

No we proceed to find the Euler-characteristics of the strata:

```
> chi_2A3_X(e) := simplify((-1)^(sigma2A3-sigma2A3) *
> (G_2A3) / (Q^phi_plus_2A3 * fixed_points_T_2A3));
```

$$\text{chi\_2A3\_X}(e) := (Q^3 + Q^2 + Q + 1)(Q^3 + 1)$$

```
> chi_2A4_X(e) := simplify((-1)^(sigma2A4-sigma2A4) *
> (G_2A4) / (Q^phi_plus_2A4 * fixed_points_T_2A4));
```

$$\text{chi\_2A4\_X}(e) := (Q^5 + 1)(Q^3 + Q^2 + Q + 1)(Q^2 - Q + 1)$$

```
> chi_3D4_X(e) := simplify((-1)^(sigma3D4-sigma3D4) *
> (G_3D4) / (Q^phi_plus_3D4 * fixed_points_T_3D4));
```

$$\text{chi\_3D4\_X}(e) := (Q^2 - Q + 1)^2 (Q^2 + Q + 1)(Q + 1)^2 (Q^4 - Q^2 + 1)$$

```
> chi_2F4_X(e) := simplify((-1)^(sigma2F4-sigma2F4) *
> (G_2F4) / (Q^phi_plus_2F4 * fixed_points_T_2F4));
```

$$\text{chi\_2F4\_X}(e) := (Q^{12} + 1)(Q^6 + Q^4 + Q^2 + 1)(Q^6 + 1)$$

```
> chi_2A3_X(s1) := simplify((-1)^(sigma2A3-sigma_s1T_2A3) *
> (G_2A3) / (Q^phi_plus_2A3 * fixed_points_g1T_2A3));
```

$$\text{chi\_2A3\_X}(s1) := -(Q^2 - 1)(Q^3 + 1)(Q + 1)$$

```
> chi_2A4_X(s1) := simplify((-1)^(sigma2A4-sigma_s1T_2A4) *
> (G_2A4) / (Q^phi_plus_2A4 * fixed_points_g1T_2A4));
```

$$\text{chi\_2A4\_X}(s1) := -(Q^2 - 1)(Q^3 + 1)(Q^5 + 1)$$

```
> chi_3D4_X(s1) := simplify((-1)^(sigma3D4-sigma_s1T_3D4) *
> (G_3D4) / (Q^phi_plus_3D4 * fixed_points_g1T_3D4));
```

$$\text{chi\_3D4\_X}(s1) := -(Q^2 - Q + 1)(Q^2 + Q + 1)^2 (Q + 1)(Q - 1)(Q^4 - Q^2 + 1)$$

```
> chi_2F4_X(s1) := simplify((-1)^(sigma2F4-sigma_s1T_2F4) *
> (G_2F4) / (Q^phi_plus_2F4 * fixed_points_g1T_2F4));
```

$$\text{chi\_2F4\_X}(s1) := -(Q^{12} + 1)(Q^8 - 1)(Q^4 - Q^2 + 1)$$

```
> chi_2A3_X(s2) := simplify((-1)^(sigma2A3-sigma_s2T_2A3) *
> (G_2A3) / (Q^phi_plus_2A3 * fixed_points_g2T_2A3));
```

$$\text{chi\_2A3\_X}(s2) := -(Q^4 - 1)(Q^2 - Q + 1)$$

```
> chi_2A4_X(s2) := simplify((-1)^(sigma2A4-sigma_s2T_2A4) *
> (G_2A4) / (Q^phi_plus_2A4 * fixed_points_g2T_2A4));
```

$$\text{chi\_2A4\_X}(s2) := -(Q^5 + 1)(Q^4 - 1)(Q + 1)$$

```
> chi_3D4_X(s2) := simplify((-1)^(sigma3D4-sigma_s2T_3D4) *
> (G_3D4) / (Q^phi_plus_3D4 * fixed_points_g2T_3D4));
```

$$\text{chi\_3D4\_X}(s2) := -(Q^2 - Q + 1)^2 (Q^2 + Q + 1)(Q + 1)(Q - 1)(Q^4 - Q^2 + 1)$$

```
> chi_2F4_X(s2) := simplify((-1)^(sigma2F4-sigma_s2T_2F4) *
> (G_2F4) / (Q^phi_plus_2F4 * fixed_points_g2T_2F4));
```

$$\text{chi\_2F4\_X}(s2) := -\frac{(Q^{12} + 1)(Q^8 - 1)(Q^6 + 1)}{Q^2 - \sqrt{2}\,Q + 1}$$

```
> chi_2A3_X(s1s2) := simplify((-1)^(sigma2A3-sigma_s1s2T_2A3) *
> (G_2A3) / (Q^phi_plus_2A3 * fixed_points_g1g2T_2A3));
```

$$\text{chi\_2A3\_X}(s1s2) := (Q^2-1)(Q^4-1)$$

```
> chi_2A4_X(s1s2) := simplify((-1)^(sigma2A4-sigma_s1s2T_2A4) *
> expand(G_2A4) / (Q^phi_plus_2A4 * fixed_points_g1g2T_2A4));
```

$$\text{chi\_2A4\_X}(s1s2) := Q^{10}+Q^9-Q^8-2\,Q^5-Q^2+Q+1$$

```
> chi_3D4_X(s1s2) := simplify((-1)^(sigma3D4-sigma_s1s2T_3D4) *
> expand(G_3D4) / (Q^phi_plus_3D4 * fixed_points_g1g2T_3D4));
```

$$\text{chi\_3D4\_X}(s1s2) := Q^{12}-2\,Q^6+1$$

```
> chi_2F4_X(s1s2) := simplify((-1)^(sigma2F4-sigma_s1s2T_2F4) *
> (G_2F4) / (Q^phi_plus_2F4 * fixed_points_g1g2T_2F4));
```

$$\text{chi\_2F4\_X}(s1s2) := \frac{(Q^2-1)(Q^6+1)(Q^8-1)(Q^{12}+1)}{Q^4-\sqrt{2}\,Q^3+Q^2-\sqrt{2}\,Q+1}$$

and now we sum over the strata:

```
> chi_2A3_X(es1):=simplify(chi_2A3_X(e)+chi_2A3 _X(s1));
>
```

$$\text{chi\_2A3\_X}(es1) := 2\,Q^3+2\,Q^4+2\,Q+2$$

$$\text{chi\_2A3\_X}(es1) := 2\,Q^3+2\,Q^4+2\,Q+2$$

```
> chi_2A3_X(es2):=simplify(chi_2A3_X(e)+chi_2A3 _X(s2));
```

$$\text{chi\_2A3\_X}(es2) := 2\,Q^3+2\,Q^5+2\,Q^2+2$$

```
> chi_2A3:=simplify(chi_2A3_X(e)+chi_2A3_X(s1)+ chi_2A3_X(s2)+chi_2A3_X(s1s2));
```

$$chi\_2A3 := 2\,Q^3+Q^5+Q+4$$

```
> chi_2A4_X(es1):=simplify(chi_2A4_X(e)+chi_2A4 _X(s1));
```

$$\text{chi\_2A4\_X}(es1) := 2\,Q^8+2\,Q^5+2\,Q^3+2$$

```
> chi_2A4_X(es2):=simplify(chi_2A4_X(e)+chi_2A4 _X(s2));
```

$$\text{chi\_2A4\_X}(es2) := Q^8+Q^7+2\,Q^5+Q^3+Q^2+2-Q^9+Q^6-Q^4+Q$$

```
> chi_2A4:=simplify(chi_2A4_X(e)+chi_2A4_X(s1)+ chi_2A4_X(s2)+chi_2A4_X(s1s2));
```

$$chi\_2A4 := 4+2\,Q-Q^2+2\,Q^3-Q^4+Q^6+Q^8$$

```
> chi_3D4_X(es1):=simplify(chi_3D4_X(e)+chi_3D4 _X(s1));
```

$$\text{chi\_3D4\_X}(es1) := 2+2\,Q+2\,Q^4+2\,Q^5+2\,Q^8+2\,Q^9$$

```
> chi_3D4_X(es2):=simplify(chi_3D4_X(e)+chi_3D4 _X(s2));
```

$$\text{chi\_3D4\_X}(es2) := 2+2\,Q^3+2\,Q^4+2\,Q^8+2\,Q^{11}+2\,Q^7$$

```
> chi_3D4:=simplify(chi_3D4_X(e)+chi_3D4_X(s1)+ chi_3D4_X(s2)+chi_3D4_X(s1s2));
```

$$chi\_3D4 := 4+Q+Q^3+2\,Q^4+Q^5-2\,Q^6+2\,Q^8+Q^{11}+Q^7+Q^9$$

```
> chi_2F4_X(es1):=simplify(chi_2F4_X(e)+chi_2F4 _X(s1));
```

$$\text{chi\_2F4\_X}(es1) := 2\,Q^{18}+2\,Q^{22}+2\,Q^{16}+2\,Q^{12}+2\,Q^6+2\,Q^{10}+2\,Q^4+2$$

```
> chi_2F4_X(es2):=simplify(chi_2F4_X(e)+chi_2F4 _X(s2));
```

$$\text{chi\_2F4\_X}(es2) := -\frac{(Q^{12}+1)(Q^6+1)(Q^7\sqrt{2}-2\,Q^6+Q^5\sqrt{2}-2\,Q^4+\sqrt{2}\,Q^3-2\,Q^2+\sqrt{2}\,Q-2)}{Q^2-\sqrt{2}\,Q+1}$$

```
> chi_2F4:=simplify(chi_2F4_X(e)+chi_2F4_X(s1)+ chi_2F4_X(s2)+chi_2F4_X(s1s2));
```

$$\begin{aligned} chi\_2F4 := {} & (Q^{12}+1)(4+9\,Q^2+10\,Q^4+14\,Q^6+15\,Q^8+10\,Q^{12}+11\,Q^{10}-12\,Q^7\sqrt{2} \\ & -8\,Q^5\sqrt{2}-6\sqrt{2}\,Q^3-6\sqrt{2}\,Q+Q^{16}+6\,Q^{14}-6\,Q^{13}\sqrt{2}-8\,Q^{11}\sqrt{2}-8\,Q^9\sqrt{2} \\ & -2\,Q^{15}\sqrt{2})\,/((Q^2-\sqrt{2}\,Q+1)(Q^4-\sqrt{2}\,Q^3+Q^2-\sqrt{2}\,Q+1)) \end{aligned}$$

C.2. Presenting the canonical bundle

In the ${}^2\mathrm{A}_n$ and A_n cases, the following Maple code determines the canonical bundle. These calculations proved very helpful in proving the results of [**HH99a**].

```
> with(linalg):
```

A general procedure for describing the canonical bundle of a (possibly twisted) A_n type Deligne-Lusztig variety is the pull-back of a (possibly ample) homogeneous line bundle L on G/B.

First we define constants and procedures:

The Kronecker delta function:

```
> delta :=proc(x,i) local p; p:=0; if x=i then p:=1 fi; RETURN(p); end:
```

A function describing the entries of the matrices corresponding to the action of the simple reflections on $X(T)$:

```
> extra_delta:= proc(i,j,k) local p; p:=0; if i=j and j=k then p:=-1 fi ;
> if i=j and k<>j then p:=1 fi ; if i=k+1 and i-j=1 then p:=1 fi;
> if i=k-1 and j-i=1 then p:=1 fi;RETURN(p) end;
```

$$\begin{array}{l} extra\_delta := \mathbf{proc}(i,\, j,\, k) \\ \quad \mathbf{local}\, p; \\ \quad\quad p := 0; \\ \quad\quad \mathbf{if}\, i = j \;\mathbf{and}\; j = k \,\mathbf{then}\, p := -1 \,\mathbf{fi}; \\ \quad\quad \mathbf{if}\, i = j \;\mathbf{and}\; k \neq j \,\mathbf{then}\, p := 1 \,\mathbf{fi}; \\ \quad\quad \mathbf{if}\, i = k+1 \;\mathbf{and}\; i - j = 1 \,\mathbf{then}\, p := 1 \,\mathbf{fi}; \\ \quad\quad \mathbf{if}\, i = k-1 \;\mathbf{and}\; j - i = 1 \,\mathbf{then}\, p := 1 \,\mathbf{fi}; \\ \quad\quad \mathrm{RETURN}(p) \\ \quad \mathbf{end} \end{array}$$

The following procedure takes the data (x=order of F on the Dynkin diagram, N=number of simple reflections generating W, q=a prime power or 'q') and returns a description of the canonical bundle of the corresponding Deligne-Lusztig variety as a pull-back of a homogeneous line bundle on G/B. Usually there are many possible choices. Then the procedure returns the free variables.

```
> isOK :=proc(x,N,q)
> global InverseEqs;
> local n,IN,F,tmpF,i,j,k,a,b,c,d,e,deltaq,s,w,Eq,A,C,
> vect,SmallEqs,SetEq,LastVect,
> SmallConstraints,MoreConstraints,Constraints;
> deltaq :=proc(x,i) local p; p:=0; if x=i then p:=q fi; RETURN(p); end:
> n:=ceil(N/x);printf(`\nExamining the `);
```

```
> print(x,N); printf(`case.`);IN:=matrix(N,N,delta):

> F:=matrix(N,N,deltaq): tmpF:=copy(F):

> if x=2 then for i from 1 to N do for j from 1 to N do F[i,j]:=tmpF[N+1-i,j]
od;od; fi; evalm(F);

> for i from 1 to n do b[i]:=vector(N,[]) : b[i][i]:=-1: od:

> c:=vector(N,[]): for i from 1 to N do c[i]:=1-q od:

> for i from 1 to n do e[i]:=vector(N,0) : e[i][i]:= 1: evalm(e[i]): od:

> for k from 1 to N do: s[k]:=matrix(N,N) : for i from 1 to N do

> for j from 1 to N do s[k][i,j]:=extra_delta(i,j,k) od: od: od:

> w[0]:=evalm(IN); for k from 1 to n do: w[k]:=matrix(N,N): w[k]:=evalm(w[k-1]
&* s[k]) od;

> for k from 1 to n+1 do Eq[k]:=evalm(inverse(w[k-1])) od;

> A:=evalm(F &* inverse(w[n]) - IN);

> if type(N,odd) and x<>1 then printf(`\nThe determinant of A is`);

> print(det(A)) ; printf(`\nThe coordinates (with respect to the boundary divisors)
for the inverse canonical divisor are:`);


> InverseEqs:=array(1..n);

> for i from 1 to n do C[i]:=matrix(N,N): C[i]:=evalm(inverse(w[i-1]) &* inverse(A)):

> InverseEqs[i]:=multiply(transpose(multiply(C[ i] , vector(N,1))),e[i]) :

> InverseEqs[i] := 1+simplify(InverseEqs[i] *(q-1)) : print(InverseEqs[i]) :

> od: fi;

> for k from 1 to n do vect[k]:=linsolve(Eq[k],b[k]) ;

> d[k]:=evalm(multiply(A,vect[k])) od;

> SmallEqs:=matrix(n,N): for k from 1 to n do for j from 1 to k-1 do

> for i from 1 to N do SmallEqs[k,i]:=vect[k][i]-vect[j][i] od;od;od;

> SetEq:=convert(SmallEqs,set);

> SmallConstraints:=solve(SetEq);assign(SmallCo nstraints);

> LastVect:=evalm(A &* linsolve(Eq[1],b[1]));

> printf(`\nThe canonical bundle is the`);

> printf(`\npull-back of the line bundle on G/B corresponding`);

> printf(` to the character with coordinates:`);print(evalm(LastVect + c));


> end;
```

```
isOK := proc(x, N, q)
    local n, IN, F, tmpF, i, j, k, a, b, c, d, e, deltaq, s, w, Eq, A, C, vect, SmallEqs, SetEq,
    LastVect, SmallConstraints, MoreConstraints, Constraints;
    global InverseEqs;
        deltaq := proc(x, i) local p; p := 0; if x = i then p := q fi; RETURN(p) end;
        n := ceil(N/x);
        printf(`\nExamining the `); print(x, N); printf(`case.`);
        IN := matrix(N, N, δ); F := matrix(N, N, deltaq); tmpF := copy(F);
        if x = 2 then for i to N do for j to N do F[i, j] := tmpF[N+1-i, j] od od fi;
        evalm(F);
        for i to n do b[i] := vector(N, []); b[i][i] := -1 od;
        c := vector(N, []);
        for i to N do c[i] := 1 - q od;
        for i to n do e[i] := vector(N, 0); e[i][i] := 1; evalm(e[i]) od;
        for k to N do
            s[k] := matrix(N, N); for i to N do for j to N do s[k][i, j] := extra_delta(i, j, k) od od
        od;
        w[0] := evalm(IN);
        for k to n do w[k] := matrix(N, N); w[k] := evalm(w[k-1] `&*` s[k]) od;
        for k to n + 1 do Eq[k] := evalm(inverse(w[k-1])) od;
        A := evalm((F `&*` inverse(w[n])) - IN);
        if type(N, odd) and x ≠ 1 then
            printf(`\nThe determinant of A is`); print(det(A));
            printf(`\nThe coordinates (with respect to the boundary divisors) for
                the inverse canonical divisor are:`);
            InverseEqs := array(1..n);
            for i to n do
                C[i] := matrix(N, N);
                C[i] := evalm(inverse(w[i-1]) `&*` inverse(A));
                InverseEqs[i] := multiply(transpose(multiply(C[i], vector(N, 1))), e[i]);
                InverseEqs[i] := 1 + simplify(InverseEqs[i] × (q - 1));
                print(InverseEqs[i])
            od
        fi;
        for k to n do vect[k] := linsolve(Eq[k], b[k]); d[k] := evalm(multiply(A, vect[k])) od;
        SmallEqs := matrix(n, N);
        for k to n do for j to k - 1 do for i to N do SmallEqs[k, i] := vect[k][i] - vect[j][i] od od
        od;
        SetEq := convert(SmallEqs, set); SmallConstraints := solve(SetEq);
        assign(SmallConstraints);
        LastVect := evalm(A `&*` linsolve(Eq[1], b[1]));
        printf(`\nThe canonical bundle is the pull-back of the line bundle on G/B`);
        printf(`corresponding to the character with coordinates:`);
        print(evalm(LastVect + c))
    end
```

```
> for t from 1 to 2 do for k from 2 to 5 do isOK(t,k,q) od; od;
```

Examining the

$$1, 2$$

case. The canonical bundle is the pull-back of the line bundle on G/B corresponding to the character with coordinates:

$$[2-q, 1]$$

Examining the

$$1, 3$$

case. The canonical bundle is the pull-back of the line bundle on G/B corresponding to the character with coordinates:

$$[2-q, 1-q, 1]$$

Examining the

$$1, 4$$

case. The canonical bundle is the pull-back of the line bundle on G/B corresponding to the character with coordinates:

$$[2-q, 1-q, 1-q, 1]$$

Examining the

$$1, 5$$

case. The canonical bundle is the pull-back of the line bundle on G/B corresponding to the character with coordinates:

$$[2-q, 1-q, 1-q, 1-q, 1]$$

Examining the

$$2, 2$$

case. The canonical bundle is the pull-back of the line bundle on G/B corresponding to the character with coordinates:

$$[2-2\,q+q\,b_{12}, -b_{12}+1]$$

Examining the

$$2, 3$$

case. The determinant of A is

$$-1 - q^3$$

The coordinates (with respect to the boundary divisors) for the inverse canonical divisor are:

$$1 - \frac{(1 + 2\,q)\,(q - 1)}{q^2 - q + 1}$$
$$1 - \frac{(q - 1)\,(q^2 + q + 2)}{1 + q^3}$$

The canonical bundle is the pull-back of the line bundle on G/B corresponding to the character with coordinates:

$$\left[2 - 2\,q + q\,b_{13},\; 1,\; -b_{13} + 1 - q\right]$$

Examining the

$$2, 4$$

case. The canonical bundle is the pull-back of the line bundle on G/B corresponding to the character with coordinates:

$$\left[2 + q\,b_{14} - q,\; -2\,q + q\,b_{13} + 1,\; -b_{13} + 1,\; -b_{14} + 1 - q\right]$$

Examining the

$$2, 5$$

case. The determinant of A is

$$-1 - q^5$$

The coordinates (with respect to the boundary divisors) for the inverse canonical divisor are:

$$1 - \frac{(q^2 + 1 + 3\,q^3)\,(q - 1)}{q^4 - q^3 + q^2 - q + 1}$$
$$1 - \frac{(q^2 + 3\,q + 2 + 2\,q^3)\,(q - 1)}{q^4 - q^3 + q^2 - q + 1}$$
$$1 - \frac{(q - 1)\,(q^4 + q^3 + 2\,q^2 + 2\,q + 3)}{1 + q^5}$$

The canonical bundle is the pull-back of the line bundle on G/B corresponding to the character with coordinates:

$$\left[2 + q\,b_{15} - q,\; -2\,q + q\,b_{14} + 1,\; 1,\; -b_{14} + 1 - q,\; -b_{15} + 1 - q\right]$$

C.3. Tables

In this section we have collected some results from both the literature and from the Maple calculations in the previous sections.

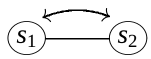
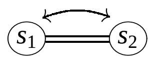
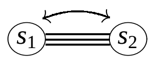
G^F	$(\mathcal{D},\rho)$	Restrictions on Q
${}^2\mathrm{A}_2(Q^2)$		Q any integral power of p
${}^2\mathrm{B}_2(Q^2)$		We must have $Q^2 = p^{2n+1}$ and $p = 2$
${}^2\mathrm{G}_2(Q^2)$		We must have $Q^2 = p^{2n+1}$ and $p = 3$

TABLE C.1. Diagrams giving irreducible Deligne-Lusztig varieties of dimension 1.

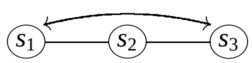
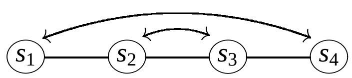
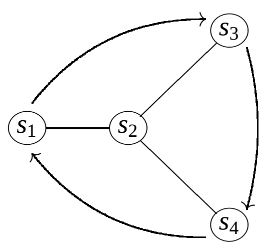
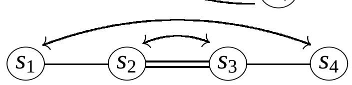
G^F	$(\mathcal{D},\rho)$	Restrictions on Q
${}^2\mathrm{A}_3(Q^2)$		Q any integral power of p
${}^2\mathrm{A}_4(Q^2)$		Q any integral power of p
${}^3\mathrm{D}_4(Q^3)$		Q any integral power of p
${}^2\mathrm{F}_4(Q^2)$		$Q^2 = p^{2n+1}$ and $p = 2$

TABLE C.2. Diagrams giving irreducible Deligne-Lusztig varieties of dimension 2.

On the following two pages you find various properties arising in connection with finite groups of Lie type giving irreducible Deligne-Lusztig varieties of dimension 2. The restrictions on Q are the same as those in Table C.2.

The elements $g_1, g_2, g \in G$ are chosen such that they are pre-images of $s_1, s_2, s_1 s_2$ respectively under the Lang-map (*cf.* Remark 1.20). Note that $\chi(X(e)) = |G^F|/(q^{|\Phi^+|}|T^F|)$.

G^F	${}^2\mathrm{A}_3(Q^2)$	${}^2\mathrm{A}_4(Q^2)$
$\lvert G^F \rvert$	$Q^6(Q^2-1)(Q^3+1)(Q^4-1)$	$Q^{10}(Q^2-1)(Q^3+1)(Q^4-1)(Q^5+1)$
$\lvert\Phi^+\rvert$	6	10
h_0	3	5
$\sigma(G)$	2	2
$\sigma({}^{g_1}T)$	1	1
$\sigma({}^{g_2}T)$	1	1
$\sigma({}^{g}T)$	0	0
$\lvert T^F \rvert$	Q^3-Q^2-Q+1	Q^4-2Q^2+1
$\lvert {}^{g_1}T^F \rvert$	Q^3-Q^2+Q-1	Q^4-1
$\lvert {}^{g_2}T^F \rvert$	Q^3+Q^2-Q-1	Q^4-Q^3+Q-1
$\lvert {}^{g}T^F \rvert$	Q^3+1	$Q^4-Q^3+Q^2-Q+1$
$\chi(X(e))$	$(Q^3+Q^2+Q+1)(Q^3+1)$	$(Q^5+1)(Q^3+Q^2+Q+1)(Q^2-Q+1)$
$\chi(X(s_1))$	$-(Q^2-1)(Q^3+1)(Q+1)$	$-(Q^2-1)(Q^3+1)(Q^5+1)$
$\chi(X(s_2))$	$-(Q^4-1)(Q^2-Q+1)$	$-(Q^5+1)(Q^4-1)(Q+1)$
$\chi(X(s_1s_2))$	$(Q^2-1)(Q^4-1)$	$Q^{10}+Q^9-Q^8-2Q^5-Q^2+Q+1$
$\lvert \overline{X(s_1s_2)}^{F^\delta} \rvert$	$(Q^3+1)(Q^2+1)^2$	$(Q^2+1)(Q^3+1)(Q^5+1)$

TABLE C.3

G^F	${}^3\mathrm{D}_4(Q^3)$	${}^2\mathrm{F}_4(Q^2)$
$\lvert G^F \rvert$	$Q^{12}(Q^4-Q^2+1)(Q-1)^2(Q+1)^2(Q^2+Q+1)^2(Q^2-Q+1)^2$	$Q^{24}(Q^2-1)(Q^6+1)(Q^8-1)(Q^{12}+1)$
$\lvert \Phi^+ \rvert$	12	24
h_0	4	12
$\sigma(G)$	2	2
$\sigma({}^{g_1}T)$	1	1
$\sigma({}^{g_2}T)$	1	1
$\sigma({}^{g}T)$	0	0
$\lvert T^F \rvert$	Q^4-Q^3-Q+1	Q^4-2Q^2+1
$\lvert {}^{g_1}T^F \rvert$	Q^4-Q^3+Q-1	Q^4-1
$\lvert {}^{g_2}T^F \rvert$	Q^4+Q^3-Q-1	$Q^4-\sqrt{2}Q^3+\sqrt{2}Q-1$
$\lvert {}^{g}T^F \rvert$	Q^4-Q^2+1	$Q^4-\sqrt{2}Q^3+Q^2-\sqrt{2}Q+1$
$\chi(X(e))$	$(Q^2-Q+1)^2(Q^2+Q+1)(Q+1)^2(Q^4-Q^2+1)$	$(Q^{12}+1)(Q^6+Q^4+Q^2+1)(Q^6+1)$
$\chi(X(s_1))$	$-(Q^2-Q+1)(Q^2+Q+1)^2(Q+1)(Q-1)(Q^4-Q^2+1)$	$-(Q^{12}+1)(Q^8-1)(Q^4-Q^2+1)$
$\chi(X(s_2))$	$-(Q^2-Q+1)^2(Q^2+Q+1)(Q+1)(Q-1)(Q^4-Q^2+1)$	$-\dfrac{(Q^{12}+1)(Q^8-1)(Q^6+1)}{Q^2-\sqrt{2}Q+1}$
$\chi(X(s_1s_2))$	$Q^{12}-2Q^6+1$	$-\dfrac{(Q^2-1)(Q^6+1)(Q^8-1)(Q^{12}+1)}{-Q^4+\sqrt{2}Q^3-Q^2+\sqrt{2}Q-1}$
$\lvert \overline{X(s_1s_2)}^{F^\delta} \rvert$	$(1+Q^3)^2(1+Q^4+Q^8)$	$(1+Q^2)(1+Q^4)(1+Q^6)(1+Q^{12})$

TABLE C.3. (continued)

APPENDIX D

Résumé

Deligne-Lusztig varieties was defined in the mid-seventies by P. Deligne and G. Lusztig as a mere tool in an ingenious construction of certain representations of finite groups. Until the beginning of the nineties, the study of these varieties continued to be from the representation theoretic point of view. In the meantime Goppa had shown how to construct error correcting codes from algebraic curves over finite fields, and codes arising from Deligne-Lusztig curves turned out to provide excellent examples. So suddenly Deligne-Lusztig varieties gained, so to speak, a life of their own. Early on I became interested in constructing codes from higher-dimensional varieties; and from Deligne-Lusztig varieties in particular. Needless to say, things got more difficult than in the curve case. Just to get started, one needed a good description of the divisors and their intersections. The concerns of the thesis are therefore primarily the geometry of Deligne-Lusztig varieties.

In Chapter 1 we start out by gathering various results from the theory of reductive groups and we define Deligne-Lusztig varieties.

In Chapter 2 I prove that Deligne-Lusztig varieties are normal and Cohen-Macaulay. For Deligne-Lusztig varieties arising from Coxeter elements this actually implies non-singularity. These properties are mainly derived from the corresponding properties for the Schubert varieties in $G/B \times G/B$ combined with transversality arguments.

One of the main results of the thesis appears in Chapter 3. For any Deligne-Lusztig variety $\overline{X}(w)$ arising from one of the classical (possibly twisted) groups I give a (finite) basis for the Picard group of $\overline{X}(w)$. The proof goes as follows: For Deligne-Lusztig varieties of classical type one may construct a birational morphism to a complete intersection Z in projective space. A careful study of the singularities of Z reveals that the the divisor class group of Z equals the Picard group of Z. Since the latter is generated by the hyper plane section H (by the Lefschetz theorem for Picard groups) we reach the conclusion that the Picard group of $\overline{X}(w)$ is (freely) generated by the class of the hyperplane section on Z pulled back to $\overline{X}(w)$, and by the classes of the finitely many exceptional fibres. In Chapter 3 it is furthermore conjectured that this behaviour in codimension 1 is more generally true in any codimension. That is, the Chow groups of $X(w)$ consist (in positive codimension) at the most of torsion. For Deligne-Lusztig varieties of type A_n it is proved that this is indeed the case. The assertion is also proved for the G^F-invariant part of the Chow groups.

When confronted with a poorly studied variety defined over a finite field, a natural question (at least from a positive-characteristic geometer) is: "Is it Frobenius split?" In Chapter 4 my paper *Canonical bundles of Deligne-Lusztig varieties* is included. Using the adjunction formula and results of Mehta and Ramanathan regarding the canonical bundles of the varieties $\overline{O}(w)$ I proved in that paper a general formula for the canonical bundle of a Deligne-Lusztig variety. Employing techniques as sketched in the original paper by Deligne and Lusztig, I refined these formulas to rather explicit forms. With the aid of these formulas I were also able to prove that a certain kind of Deligne-Lusztig surfaces provides a whole class of counter-examples to the socalled Miyaoka-Yau inequality otherwise true in characteristic zero. As a corollary of the results in the paper I could also give the above question a negative answer. That is, Deligne-Lusztig varieties are generally not Frobenius split.

Coming back to the starting point of this introduction, I will conclude by mentioning another main contribution given in the paper *Error-correcting codes from higher-dimensional varieties*. An error-correcting code C with parameters $[n,k,d]_q$ is, roughly speaking, nothing but a k-dimensional subspace of an n-dimensional vector space over the finite field with q elements. Furthermore, all (non-zero) points of C are assumed to have at least d non-zero coordinates. Algebraic geometric (AG) codes are then what is obtained when the global sections of a fixed line bundle is 'evaluated' in n fixed $\mathbb{F}_q$-rational points. In the above-mentioned paper I address the problems that arise when one wants to not only construct, but also estimate the parameters of codes coming from higher-dimensional varieties. Using intersection theory I prove general results concerning the dimension and minimum distance of error-correcting codes arising from varieties of dimension two or higher. In Chapter 5 of the thesis I have reproduced the paper.

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