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Abstract

The invariator principle is a measure decomposition that was rediscovered in local stereology in 2005 and has since been used widely in the stereological literature. We give an exposition of invariator related results where existing formulae are generalized and new ones proposed. In particular, we look at rotational Crofton-type formulae that are obtained by combining the invariator principle and classical Crofton formulae. This results in geometrical quantities represented as averages over weighted Crofton-type integrals in linear sections. We refer to these weighted integrals as measurement functions and derive several, more explicit representations of these functions. In particular, we use Morse theory to write the measurement functions in terms of critical values of the sectioned object. This is very useful for surface area estimation.

Keywords: Local stereology; invariator principle; rotational Crofton-type formulae; Morse theory; Hadwiger's index; surface area estimation

1 Introduction

The invariator (Cruz-Orive, 2005) is a powerful principle for generating a hyperplane in an isotropic random subspace that is motion invariant in n-dimensions. It is a special case of a classical result (Petkantschin, 1936) that was rediscovered in local stereology and used for applications in (Cruz-Orive, 2005). Since then it has received much interest in the stereological literature. It was generalized in (Gual-Arnau and Cruz-Orive, 2009) to Riemannian manifolds with constant sectional curvature. In (Gual-Arnau and Cruz-Orive, 2009) and, independently, in (Auneau and Jensen, 2010), the invariator principle was combined with the classical Crofton formula to obtain new rotational Crofton-type formulae which yield new stereological estimators of geometrical quantities. The purpose of this survey is to give an overview of invariator related results in Euclidean space and to include natural generalizations that apparently have not been treated in the literature yet.

Crofton's formula is an important result of integral geometry as it relates properties on flat sections of a spatial structure to geometrical quantities of the original structure. Rotational versions of Crofton's formula only use sections with linear subspaces, that is, subspaces through a fixed reference point, which usually is assumed to be the origin. They express certain geometrical quantities as averages of measurements in the linear sections. The average is taken with respect to a rotation invariant measure, therefore the word *rotational.* Techniques that are based on sections through a fixed reference point are often called *local*; see the monograph (Jensen, 1998) on local stereology. In integral geometry the term 'local' is used when so-called local versions of the intrinsic volumes are considered, that is when also normals and position of boundary points of an object of interest are taken into account. In the following we use the latter notion of the word.

The major new contribution of the present paper is to combine the invariator and concepts from Morse theory for obtaining a new rotational Crofton formula. What is different and appealing with this new formula is that the measurement functional on the section of the object is written entirely in terms of so-called critical points. This proves to be very useful for applications where the surface area is sought for.

The paper is written self-contained. It is organized as follows. In Section 2 we introduce the notation and recall some important concepts. The first result, Proposition 2 in Section 3, is a rotational Crofton formula for the support measures obtained by combining the invariator principle and a local Crofton formula. It has as a special case the rotational Crofton formula for intrinsic volumes derived in (Gual-Arnau and Cruz-Orive, 2009) and (Auneau and Jensen, 2010). The combination of the invariator principle and the classical Crofton formula does not yield an explicit form of the functional to be measured on the section. This functional will be called the *measurement function* from now on, and we will present more explicit representations of this measurement function in Section 4. We start by generalizing the results of (Auneau and Jensen, 2010). In (Auneau and Jensen, 2010) the measurement function involving the intrinsic volumes is written as an integral over the object's boundary and we extend this to curvature measures. Then we show that when the object of interest is convex, the measurement function can be written in terms of the radial function of the sectioned object and an angle in the section plane. The main result can be found in Section 4.3. Here we give a very basic introduction to Mose theory before presenting the new rotational Crofton formula for smooth manifolds in Theorem 6. This theorem is formulated for smooth manifolds as we want to apply classical Morse theory. As shown in Theorem 7 an analogous result holds for polyconvex sets, where Hadwiger's index (Hadwiger, 1955), an index closely related to the Morse index, is used for determining critical points. When the geometrical quantity of interest is the surface area of a topologically regular set with smooth boundary, the two theorems coincide. In Section 4.4 we discuss the formal analogy of the new formula with Kubota's formula and give a simple computational formula for the measurement function when the object of interest is a polytope. We conclude the paper with a discussion on stereological applications of these rotational formulae, both old and new.

2 Preliminaries

Throughout, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and O its origin. The Euclidean scalar product is denoted by $\langle \cdot, \cdot \rangle$ and the Euclidean norm by $\|\cdot\|$. For a topological space E we let $\mathcal{B}(E)$ be the Borel σ-algebra in E. We furthermore write $\mathcal{H}_n^d$ for the d-dimensional *Hausdorff measure* in $\mathbb{R}^n$ (Schneider and Weil, 2008, p. 634). When n is clear from the context, $\mathcal{H}_n^d(\mathrm{d}u)$ is abbreviated to $\mathrm{d}u^d$. For a set $Y \subseteq \mathbb{R}^n$, we define

$$Y + x = \{y + x \mid y \in Y\},\ x \in \mathbb{R}^n, \qquad \alpha Y = \{\alpha y \mid y \in Y\},\ \alpha > 0.$$

We use ∂Y for the boundary, int$Y$ for the interior, cl$Y$ for the closure and $\mathbf{1}_Y$ for the indicator function of Y. When we want to emphasize the geometric meaning, we write $S(Y) = \mathcal{H}_n^{n-1}(Y)$ for the *surface area* of a Borel-set Y. Whenever defined, $\chi(Y)$ denotes the *Euler characteristic* of Y. If $Y \subseteq \mathbb{R}^1$ is compact $\chi(Y)$ is the number of connected components of Y. The unit ball in $\mathbb{R}^n$ is $B_n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and the boundary of it is the unit sphere (in $\mathbb{R}^n$) $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. The volume of B_n is given by

$$\kappa_n = \pi^{n/2}\Gamma(1 + \tfrac{n}{2})^{-1}$$

and the surface area of its boundary by

$$\sigma_n = n\kappa_n = 2\pi^{n/2}\Gamma(\tfrac{n}{2})^{-1}.$$

To simplify later expressions, we write

$$c_{j_1,j_2\ldots,j_k}^{i_1,i_2,\ldots,i_{k'}} = \frac{\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{k'}}}{\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_k}}. \tag{2.1}$$

For $\alpha, \beta, \gamma \in \mathbb{R}, \gamma \notin \{0, -1, -2, \ldots\}$, we write $F(\alpha, \beta, \gamma; \cdot)$ for the hypergeometric function

$$F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}, \qquad z \in [-1, 1],$$

where $(x)_k$ is the Pochhammer symbol

$$(x)_k = \begin{cases} \frac{\Gamma(x+k)}{\Gamma(x)}, & x > 0, \\ (-1)^k \frac{\Gamma(-x+1)}{\Gamma(-x-k+1)}, & x \leq 0. \end{cases}$$

Let $X \subseteq \mathbb{R}^n$ be a nonempty, compact set which is *star-shaped* at O (i.e. every line through O that hits X does so in a (possibly degenerate) line segment). The *radial function* of X, ρ_X, is defined by

$$\rho_X(x) = \sup\{\alpha \in \mathbb{R} \mid \alpha x \in X\},$$

for $x \in \mathbb{R}^n \setminus \{O\}$. The set X is uniquely determined by ρ_X. We use $\mathcal{K}^n$ for the family of all *convex bodies* (compact, convex sets) of $\mathbb{R}^n$; cf. (Schneider, 1993) for the theory of convex bodies. If $X \in \mathcal{K}^n$ and ∂X does not contain any line segment X is called *strictly convex*. For $X \in \mathcal{K}^n$ its *support function*, h_X, is given by

$$h_X(u) = \max_{x \in X} \langle u, x \rangle, \qquad u \in S^{n-1}.$$

The value $h_X(u)$ is the signed distance from O to the supporting hyperplane to X with outer unit normal vector u. For $q > 0$ we define the *q-flower set* H_X^q of $X \in \mathcal{K}^n$ by

$$\rho_{H_X^q}(u) = \mathrm{sgn}(h_X(u))|h_X(u)|^q, \qquad u \in S^{n-1}, \tag{2.2}$$

where sgn$(\cdot)$ is the *signum function*. Note that the right hand side of (2.2) is always the radial function of some set, as $-h_X(-u) \leq h_X(u)$ for all $u \in S^{n-1}$. When $q = 1$, H_X^1 is the set whose radial function is the support function of X and is referred to as the support set of X in (Cruz-Orive, 2005). When X is a planar polygon, H_X^1 is a union of finitely many disks and resembles slightly a flower and was called the flower of X in (Cruz-Orive, 2011). As already mentioned, we extend that terminology and speak of a q-flower set, see Section 4.4 for its relevance in connection with the invariator principle.

We let $\mathcal{R}^n$ be the family of all *polyconvex sets* (sets that can be expressed as finite unions of convex bodies) of $\mathbb{R}^n$. A support element of $\emptyset \neq X \in \mathcal{K}^n$ is a pair $(x, u) \in \Sigma = \mathbb{R}^n \times S^{n-1}$, where $x \in \partial X$ and u is an outer unit normal vector of X at x. More formally $(x, u) \in \Sigma$ is a support element of X if and only if $x \in X$ satisfies $h_X(u) = \langle u, x\rangle$. We let Nor$X$ be the set of all support elements of X. As in (Glasauer, 1997, p. 109) we extend this definition to polyconvex sets. For $X \in \mathcal{R}^n$ let $R(X)$ be the set of all sequences $(X_i)_{i\in\mathbb{N}}$ in $\mathcal{K}^n$ with $X = \cup_{i=1}^{\infty} X_i$ and $X_i = \emptyset$ for almost all $i \in \mathbb{N}$ and let $S(\mathbb{N})$ be the set of all nonempty subsets of $\mathbb{N}$. Then we define

$$\mathrm{Nor}X = \cap_{(X_i)\in R(X)} \cup_{v\in S(\mathbb{N})} \mathrm{Nor}(\cap_{i\in v} X_i). \tag{2.3}$$

We follow the notation in (Schneider, 1993, 4.2) and write $\Xi_m(X, \cdot)$, $0 \leq m \leq n-1$, for the support measures of $X \in \mathcal{R}^n$ on $\mathcal{B}(\Sigma)$. They are concentrated on Nor$X$. We obtain the curvature measures by the specialization $\Phi_m(X, A) = \Xi_m(X, A \times S^{n-1})$, $A \in \mathcal{B}(\mathbb{R}^n)$, and the area measures by $\Psi_m(X, B) = \Xi_m(X, \mathbb{R}^n \times B)$, $B \in \mathcal{B}(S^{n-1})$, $m \in \{0, 1, \ldots, n-1\}$. For $m = n$, only the curvature measure is defined. We put $\Phi_n(X, \cdot) = \mathcal{H}_n^n(X \cap \cdot)$, so $\Phi_n(X, \cdot)$ is the restriction of the Lebesgue measure to X The intrinsic volumes are the total measures $V_m(X) = \Xi_m(X, \Sigma)$. Of special interest will be the volume V_n, the surface area $2V_{n-1}$ and the Euler characteristic $V_0 = \chi$. For $X \in \mathcal{K}^n$, integers $r, s \geq 0$ and $m \in \{0, 1, \ldots, n-1\}$, we write

$$\Phi_{m,r,s}(X) = \frac{\sigma_{n-m}}{r!s!\sigma_{n-m+s}} \int_\Sigma x^r u^s \Xi_m(X, \mathrm{d}(x, u)),$$

for the *Minkowski tensors*. Here $x^r u^s$ is the symmetric tensor product of rank $r + s$ of the symmetric tensors x^r and u^s. For $s = 0$ we obtain the volume tensor of rank r

$$\Phi_{m,r,0}(X) = \frac{1}{r!} \int_X x^r \Phi_m(X, \mathrm{d}x),$$

which is also defined for $m = n$. Note also that $\Phi_{m,0,0}(X) = V_m(X)$. For an introduction to Minkowski tensors see (Hug et al., 2008) and references therein.

For $j = 0, 1, \ldots, n$ we let

$$\begin{aligned}
\mathcal{L}_{j[O]}^n &= \{L_{j[O]}^n \subseteq \mathbb{R}^n \mid L_{j[O]}^n \text{ is a } j\text{-dim. linear subspace}\},\\
\mathcal{L}_j^n &= \{L_j^n \subseteq \mathbb{R}^n] \mid L_j^n \text{ is a } j\text{-dim. affine subspace}\}
\end{aligned}$$

be the families of all j-dimensional linear and affine subspaces of $\mathbb{R}^n$, respectively. For $0 \leq j \leq r \leq n$ and a fixed $L_{r[O]}^n \in \mathcal{L}_{r[O]}^n$ we write $\mathcal{L}_j^r$ for the family of all j-dimensional affine subspaces L_j^r within this linear subspace, despite the fact that this notation does not reflect the surrounding linear space. These spaces are equipped with their standard topologies and endowed with their natural invariant measures; see (Schneider and Weil, 2008). We write $\mathrm{d}L^n_{j[O]}$, and $\mathrm{d}L^r_j$, respectively, when integrating with respect to these invariant measures. We use the same normalization as in (Schneider and Weil, 2008):

$$\int_{\mathcal{L}^n_{j[O]}} \mathrm{d}L^n_{j[O]} = 1 \quad \text{and} \quad \int_{\{L^r_j \in \mathcal{L}^r_j : L^r_j \cap B_n \neq \emptyset\}} \mathrm{d}L^r_j = \kappa_{r-j}.$$

A random subspace $L^n_{j[O]}$ is called *isotropic random* (IR) if and only if its distribution is given by

$$\mathbb{P}_{L^n_{j[O]}}(A) = \int_{\mathcal{L}^n_{j[O]}} \mathbf{1}_A \, \mathrm{d}L^n_{j[O]}, \quad A \in \mathcal{B}(\mathcal{L}^n_{j[O]}).$$

Similarly, a random flat $L^n_j \in \mathcal{L}^n_j$ is called *isotropic uniform random* (IUR) *hitting a compact object* Y if and only if its distribution is given by

$$\mathbb{P}_{L^n_j}(A) = c \int_{\mathcal{L}^n_j} \mathbf{1}_{A \cap \{L^n_j \in \mathcal{L}^n_j : L^n_j \cap Y \neq \emptyset\}} \, \mathrm{d}L^n_j, \qquad A \in \mathcal{B}(\mathcal{L}^n_j),$$

where c is a normalizing constant. We write $(L^n_j)^\perp \in \mathcal{L}^n_{n-j[O]}$ for the linear subspace orthogonal to $L^n_{j[O]} \in \mathcal{L}^n_{j[O]}$ and $x|L^n_{j[O]}$ for the orthogonal projection of $x \in \mathbb{R}^n$ onto $L^n_{j[O]} \in \mathcal{L}^n_{j[O]}$. We furthermore adopt the convention of writing $u^\perp$ for the orthogonal complement of the line through O with direction $u \in S^{n-1}$. For $\eta \subseteq \Sigma$ and $L^n_j \in \mathcal{L}^n_j$, $j \in \{0, \dots, n-1\}$, we define

$$\begin{aligned} \eta \wedge L^n_j = \{(x,u) \in \Sigma \mid {} & \text{there are } u_1, u_2 \in S^{n-1} \text{ with} \\ & (x, u_1) \in \eta, x \in L^n_j, u_2 \in (L^n_j)^\perp, u \in \mathrm{pos}\{u_1, u_2\}\}, \end{aligned}$$

where $\mathrm{pos}\{u_1, u_2\} = \{\lambda_1 u_1 + \lambda_2 u_2 \mid \lambda_1, \lambda_2 \geq 0\}$ is the positive hull of the set $\{u_1, u_2\}$. For $B \in \mathcal{B}(S^{n-1})$ we let

$$\begin{aligned} B \wedge L^n_j = \{u \in S^{n-1} \mid {} & \text{there are } u_1, u_2 \in S^{n-1} \\ & \text{with } u_1 \in B, u_2 \in (L^n_j)^\perp, u \in \mathrm{pos}\{u_1, u_2\}\}. \end{aligned}$$

A generalization of the classical Crofton formula is the following local Crofton formula for polyconvex sets.

Proposition 1 (Glasauer (1997, Theorem 3.4))**.** *Let* $X \in \mathcal{R}^n$ *and* j, m *be integers satisfying* $0 \leq m < j \leq n-1$. *Then for* $\eta \in \mathcal{B}(NorX)$

$$\Xi_{n-j+m}(X, \eta) = c^{j+1, n-j+m+1}_{m+1, n+1} \int_{\mathcal{L}^n_j} \Xi_m(X \cap L^n_j, \eta \wedge L^n_j) dL^n_j. \tag{2.4}$$

We use the word *smooth* to mean differentiable of class C^∞ and refer to (Bredon, 1993, 2.1. Definition p. 68) for a definition of an m-dimensional smooth manifold in $\mathbb{R}^n$. For a manifold $X \subseteq \mathbb{R}^n$ of class C^1 let $T_x(X)$ be the vector space of all tangent vectors to X at a point $x \in X$. For two manifolds $X_1, X_2 \subseteq \mathbb{R}^n$ of class C^1 we write $X_1 \pitchfork X_2$ in $\mathbb{R}^n$, and say that X_1 intersects X_2 *transversely* in $\mathbb{R}^n$, if whenever $x \in X_1 \cap X_2$, we have $T_x(X_1)+T_x(X_2) = T_x(\mathbb{R}^n)$. This is standard notation in differential geometry (Bredon, 1993, 7.6. Definition p. 84). Correspondingly, for $X \in \mathcal{R}^n$ and $L^n_{j[O]} \in \mathcal{L}^n_{j[O]}$ we write $\partial X \pitchfork L^n_{j[O]}$ in $\mathbb{R}^n$ if any supporting hyperplane of X at any point in $\partial X \cap L^n_{j[O]}$, together with $L^n_{j[O]}$, spans $\mathbb{R}^n$, that is

$$(x, u) \in \mathrm{Nor}X, x \in L^n_{j[O]} \Rightarrow u \not\perp L^n_{j[O]}.$$

When $X \in \mathcal{R}^n$ and $O \notin \partial X$, we have $\partial X \pitchfork L^n_{j[O]}$ in $\mathbb{R}^n$ for almost all $L^n_{j[O]} \in \mathcal{L}^n_{j[O]}$. This was shown for $X \in \mathcal{K}^n$ in (Jensen and Rataj, 2008, Prop. 1) and generalizes to polyconvex sets using (2.3). Furthermore, if X_1 and X_2 are embedded submanifolds of $\mathbb{R}^n$, in the sense of (Bredon, 1993, 5.7. Definition p. 79), and $X_1 \pitchfork X_2$ in $\mathbb{R}^n$, then $X_1 \cap X_2$ is a submanifold of $\mathbb{R}^n$ of dimension $\dim(X_1) + \dim(X_2) - n$ (Bredon, 1993, 7.7. Theorem p. 84).

3 Invariator principle and rotational Crofton formulae

The goal of rotational integral geometry is to find analogs of (2.4) where the motion invariant integration over all affine flats is replaced by rotation invariant integration over all linear subspaces. In its most general form, a rotational Crofton formula is thus

$$\beta(X) = \int_{\mathcal{L}^n_{j+1[O]}} \alpha(X \cap L^n_{j+1[O]}) \mathrm{d}L^n_{j+1[O]}, \tag{3.1}$$

$j = 0, 1, \ldots, n-1$, for suitable X and functionals $\alpha(\cdot)$ and $\beta(\cdot)$. We consider here only the stereologically motivated question how $\alpha(\cdot)$ should be chosen in order to obtain a desired geometric characteristic $\beta(X)$ of X. For the question of how $\beta(X)$ can be explicitly calculated, when $\alpha(\cdot)$ is given (e.g. an intrinsic volume) see (Jensen and Rataj, 2008).

In (Auneau and Jensen, 2010, Proposition 1) and (Gual-Arnau et al., 2010, Theorem 3.1 with $\lambda = 0$) a functional $\alpha(\cdot)$ was given such that (3.1) holds where $\beta(X) = V_m(X)$, $m = n-j, \ldots, n$. The key idea is to combine the classical Crofton formula with a Blaschke-Petkantschin-type result, which is often called the *invariator principle* in stereology. In stereological terminology, this relation states how a j-dimensional flat in an isotropic $(j+1)$-dimensional subspace must be chosen in order to obtain an IUR flat in $\mathbb{R}^n$. For all non-negative measurable functions f on $\mathcal{L}^n_j$ and $j \in \{0, 1, \ldots, n-1\}$

$$\int_{\mathcal{L}^n_j} f(L^n_j) \mathrm{d}L^n_j = c_1^{n-j} \int_{\mathcal{L}^n_{j+1[O]}} \int_{\mathcal{L}^{j+1}_j} f(L^{j+1}_j) d(O, L^{j+1}_j)^{n-j-1} \mathrm{d}L^{j+1}_j \mathrm{d}L^n_{j+1[O]}, \tag{3.2}$$

where $d(O, L^{j+1}_j)$ is the Euclidean distance from O to L^{j+1}_j and the constant c_1^{n-j} is given by (2.1). This follows from (Gual-Arnau and Cruz-Orive, 2009, Corollary 3.1 when $\lambda = 0$) where different normalizations of the invariant measures have been used. The same approach leads also to a rotational Crofton formula for support measures by combining Proposition 1 with (3.2).

Proposition 2. *Let $X \in \mathcal{R}^n$, $j \in \{1,\dots,n-1\}$, $m \in \{0,\dots,j-1\}$ and $\eta \in \mathcal{B}(NorX)$. For $\beta(X) = \Xi_{n-j+m}(X,\eta)$, equation* (3.1) *holds with*

$$\alpha(\cdot) = c_{m+1,n+1,1}^{j+1,n-j+m+1,n-j} \int_{\mathcal{L}_j^{j+1}} \Xi_m(\cdot \cap L_j^{j+1}, \eta \wedge L_j^{j+1}) d(O, L_j^{j+1})^{n-j-1} dL_j^{j+1},$$

with the leading constant given by (2.1).

The proposition holds in particular for the marginal measures of the support measures and their total masses, the intrinsic volumes. Explicitly, taking $\eta = A \times S^{n-1}$ in Proposition 2, with $A \in \mathcal{B}(\mathbb{R}^n)$, it follows that for $\beta(X) = \Phi_{n-j+m}(X, A)$, equation (3.1) holds with

$$\alpha(\cdot) = c_{m+1,n+1,1}^{j+1,n-j+m+1,n-j} \int_{\mathcal{L}_j^{j+1}} \Phi_m(\cdot \cap L_j^{j+1}, A \cap L_j^{j+1}) d(O, L_j^{j+1})^{n-j-1} \mathrm{d}L_j^{j+1}, \quad (3.3)$$

$0 \le m \le j \le n-1$. Similarly for $\beta(X) = \Psi_{n-j+m}(X, B)$, $B \in \mathcal{B}(S^{n-1})$, equation (3.1) holds with

$$\alpha(\cdot) = c_{m+1,n+1,1}^{j+1,n-j+m+1,n-j} \int_{\mathcal{L}_j^{j+1}} \Psi_m(\cdot \cap L_j^{j+1}, B \wedge L_j^{j+1}) d(O, L_j^{j+1})^{n-j-1} \mathrm{d}L_j^{j+1}, \quad (3.4)$$

$0 \le m < j \le n-1$. As already stated in (Auneau and Jensen, 2010, Proposition 1) and (Gual-Arnau et al., 2010, Theorem 3.1 with $\lambda = 0$) for more general set classes, taking $\eta = \Sigma$, equation (3.1) with $\beta(X) = V_{n-j+m}(X)$ holds for

$$\alpha(\cdot) = c_{m+1,n+1,1}^{j+1,n-j+m+1,n-j} \int_{\mathcal{L}_j^{j+1}} V_m(\cdot \cap L_j^{j+1}) d(O, L_j^{j+1})^{n-j-1} \mathrm{d}L_j^{j+1}, \quad (3.5)$$

$0 \le m \le j \le n-1$. This relation will be of particular interest when the Euler characteristic occurs on the right hand side. Taking $m = 0$ in (3.5), applying the duality result (Jensen, 1998, Proposition 3.3) and an invariance argument, we note that $\beta(X) = V_{n-j}(X)$ and

$$\alpha(\cdot) = c_{n+1,1,1}^{n-j+1,n-j} \int_{S^{n-1} \cap L_{j+1[O]}^n} \int_{-\infty}^{\infty} \chi(\cdot \cap (ru + u^\perp)) |r|^{n-j-1} \mathrm{d}r \mathrm{d}u^j \quad (3.6)$$

satisfy (3.1). When the section profile $Y = X \cap L_{j+1[O]}^n$ is convex, the Euler characteristic of $Y \cap (ru + u^\perp)$ equals one if the hyperplane $ru + u^\perp$ hits Y, and zero otherwise. Clearly, for a given $u \in S^{n-1} \cap L_{j+1[O]}^n$, $ru + u^\perp$ hits Y if and only if $-h_{X \cap L_{j+1[O]}^n}(-u) \le r \le h_{X \cap L_{j+1[O]}^n}(u)$. Hence, we can calculate the inner integral in (3.6) explicitly and obtain, using the reflection invariance of the Hausdorff measure,

$$\alpha(\cdot) = \frac{c_{n+1,1}^{n-j+1,n-j}}{n-j} \int_{S^{n-1} \cap L_{j+1[O]}^n} \operatorname{sgn}(h_{(\cdot)}(u)) |h_{(\cdot)}(u)|^{n-j} \mathrm{d}u^j. \quad (3.7)$$

If furthermore X contains O, the expression becomes

$$\alpha(\cdot) = \frac{c_{n+1,1}^{n-j+1,n-j}}{n-j} \int_{S^{n-1} \cap L_{j+1[O]}^n} h_{(\cdot)}^{n-j}(u) \mathrm{d}u^j. \quad (3.8)$$

For a smooth manifold $X \subseteq \mathbb{R}^n$ of dimension m, equation (3.1) holds with $\beta(X) = \mathcal{H}_n^m(X)$ and

$$\alpha(\cdot) = c_{j+m-n+1,n+1,1}^{m+1,j+1,n-j} \int_{\mathcal{L}_j^{j+1}} \mathcal{H}_n^{m-n+j}(\cdot \cap L_j^{j+1}) d(O, L_j^{j+1})^{n-j-1} \mathrm{d}L_j^{j+1}, \tag{3.9}$$

where j is an integer satisfying $n - m \leq j \leq n - 1$. This follows by combining (3.2) with the Crofton formula for manifolds (Jensen, 1998, Proposition 3.7).

When $X \in \mathcal{K}^n$ a classical Crofton formula for Minkowski tensors (Hug et al., 2008, Theorem 2.2) (see also (Schneider and Schuster, 1999) for special cases) can be combined with the invariator principle to obtain rotational Crofton formulae for Minkowski tensors. The derivation of these formulae is straightforward, but we do not report them here as the function $\beta(\cdot)$ occurring in these formulae is typically a linear combination of several Minkowski tensors also involving the metric tensor and complicated coefficients. In (Auneau-Cognacq et al., 2012) the notion of Minkowski tensors was extended to so-called integrated Minkowski tensors obtained as certain tensor averages of flat sections of X. This extended class has the appealing property to be closed under rotational Crofton integrals: if $\alpha(\cdot)$ in (3.1) is an integrated Minkowski tensor, then $\beta(\cdot)$ is an integrated Minkowski tensor, too (Auneau-Cognacq et al., 2012, Proposition 4.1). The proof is based on a measure decomposition that generalizes (3.2); see (3.12) below. A special case of this result, particularly important for applications, is obtained in (Auneau-Cognacq et al., 2012, Corollary 4.4 with $q = 1$): For $r \in \mathbb{N}_0$ and $s \in \{0, 1\}$ equation (3.1) holds with $\beta(X) = \Phi_{n+m-j-1,r,s}(X)$ and

$$\alpha(\cdot) = c \int_{\mathcal{L}_j^{j+1}} \Phi_{m-1,r,s}^{(L_j^{j+1})}(\cdot \cap L_j^{j+1}) d(O, L_j^{j+1})^{n-j-1} \mathrm{d}L_j^{j+1}. \tag{3.10}$$

Here $0 < m \leq j \leq n - 1$, $\Phi_{m-1,r,s}^{(L_j^{j+1})}$ is the Minkowski tensor relative to L_j^{j+1} and

$$c = \tfrac{(m-1)!(n-1)!}{(j-1)!(n+m-1-j)!} c_{n-j+m+s+1,j,1}^{n,s+m+1,n-j}.$$

Furthermore, for $j \in \{0, 1, \ldots, n-1\}$ and any non-negative integer r, equation (3.1) with $\beta(X) = \Phi_{n,r,0}(X)$ holds for

$$\alpha(\cdot) = c_1^{n-j} \int_{\mathcal{L}_j^{j+1}} \Phi_{j,r,0}^{(L_j^{j+1})}(\cdot \cap L_j^{j+1}) d(O, L_j^{j+1})^{n-j-1} \mathrm{d}L_j^{j+1}. \tag{3.11}$$

It is not a limitation of the results of this section that they are obtained using (3.2) instead of the more general measure decomposition (Schneider and Weil, 2008, p. 285)

$$\int_{\mathcal{L}_r^n} f(L_r^n) \mathrm{d}L_r^n = c(n, j, r) \int_{\mathcal{L}_{j+1[O]}^n} \int_{\mathcal{L}_r^{j+1}} f(L_r^{j+1}) d(O, L_r^{j+1})^{n-j-1} \mathrm{d}L_r^{j+1} \mathrm{d}L_{j+1[O]}^n, \tag{3.12}$$

where $f \geq 0$ is a measurable function on $\mathcal{L}_r^n$, $0 \leq r \leq j \leq n - 1$ and $c(n, j, r)$ is a constant depending on n, j and r. Combining this measure decomposition with

Crofton's formula produces expressions of the form (3.1) where the measurement functions are integrals over $\mathcal{L}_r^{j+1}$ instead of $\mathcal{L}_j^{j+1}$. These functionals do though not depend on r, as was shown for the intrinsic volumes in (Auneau-Cognacq, 2010, Proposition 2). The proofs for the cases where $\beta(\cdot)$ is a support measure (in $\eta \subseteq \mathrm{Nor}X$), a Hausdorff measure or $\Phi_{k,r,s}$, where $1 \leq k \leq n-1$, r a non-negative integer and $s \in \{0,1\}$, are almost identical to the one given there and are based on an application of Crofton's formula in L_r^{j+1}.

4 Representations of the measurement function

The measurement function $\alpha(\cdot)$ in Proposition 2 and the special cases given in (3.3)–(3.5), as well as (3.9), are difficult to evaluate as they involve a weighted Crofton-type integration in the section plane $L^n_{j+1[O]}$. More explicit representations for the measurement function are known, in particular when X has a C^2 boundary or is a polytope. We will now give different representations of the measurement function, which all play a role when applying rotational formulae in stereology.

A particularly simple representation is obtained for the volume functional $\beta(\cdot) = \mathcal{H}^n_n(\cdot)$. In this case, no assumptions on X, apart from measurability, are required.

Proposition 3. *For any $X \in \mathcal{B}(\mathbb{R}^n)$, $\beta(X) = \mathcal{H}^n_n(X)$ and*

$$\alpha(\cdot) = c^n_{j+1} \int_{(\cdot)} \|z\|^{n-j-1} dz^{j+1}$$

satisfy (3.1) *for any $0 \leq j \leq n-1$.*

Proposition 3 follows from a twofold application of spherical coordinates and an invariance argument; see also (Auneau and Jensen, 2010, Proposition 2) for an alternative proof. It implies in particular that $\beta(X) = \Phi_n(X, A)$ and

$$\alpha(\cdot) = c^n_{j+1} \int_{(\cdot)\cap A} \|z\|^{n-j-1} \mathrm{d}z^{j+1} \tag{4.1}$$

satisfy (3.1) for any $X \in \mathcal{R}^n$, $A \in \mathcal{B}(\mathbb{R}^n)$.

4.1 The measurement function as an integral over the profile boundary

We show that the measurement function associated to the curvature measures can be written as an integral over the boundary of the section profile. As this integral involves principal curvatures, we assume that $X \in \mathcal{K}^n$ has a boundary of class C^2. For $L^n_j \in \mathcal{L}^n_j$, $j \in \{1, \ldots, n-1\}$, let $\partial'(X \cap L^n_j)$ be the relative boundary of $X \cap L^n_j$, i.e. its boundary as a subset of L^n_j. As $\partial X \pitchfork L^n_j$ for almost all $L^n_j \in \mathcal{L}^n_j$, the principal curvatures $\kappa'_1(x), \ldots, \kappa'_{j-1}(x)$ of $\partial'(X \cap L^n_j) \subseteq L^n_j$ at $x \in \partial'(X \cap L^n_j)$, as well as the normalized elementary symmetric functions of the principal curvatures of $\partial'(X \cap L^n_j)$, $H_0 = 1$,

$$H_m(x, L^n_j) = \binom{j-1}{m}^{-1} \sum_{1 \leq i_1 < \cdots < i_m \leq j-1} \kappa'_{i_1}(x) \cdots \kappa'_{i_m}(x),$$

$m = 1, \dots, j-1$, exist almost surely. In addition, we write $n'(x)$ for the (almost surely unique) outer unit normal of $X \cap L_j^n$ at $x \in \partial'(X \cap L_j^n)$.

Proposition 4. *Let $X \in \mathcal{K}^n$ with boundary of class C^2, $A \in \mathcal{B}(\mathbb{R}^n)$ and let j, m be integers with $0 \le m < j \le n-1$. Then $\beta(X) = \Phi_{n-j+m}(X, A)$ and*

$$\alpha(X \cap L_{j+1[O]}^n) = c_{m+1,n+1,j-m,1}^{j+1,n-j+m+1,n-j} \int_{\partial'(X \cap L_{j+1[O]}^n) \cap A} h_m(X \cap L_{j+1[O]}^n, x)\, dx^j \tag{4.2}$$

satisfy (3.1)*, where*

$$\begin{aligned} &h_m(X \cap L_{j+1[O]}^n, x) \\ &\quad = \binom{j-1}{m} \int_{\mathcal{L}_{j[O]}^{j+1}} H_{j-m-1}(x, L_{j[O]}^{j+1} + x) \| n'(x) | L_{j[O]}^{j+1} \| \| x | (L_{j[O]}^{j+1})^\perp \|^{n-j-1} \, dL_{j[O]}^{j+1}. \end{aligned}$$

The proof of Proposition 4 uses the representation (Schneider and Weil, 2008, p. 607) of $\Phi_{n-j+m}(X, \cdot)$ as integral involving principal curvatures and follows otherwise the proof of (Auneau and Jensen, 2010, Proposition 3) where (4.2) is shown for $A = \mathbb{R}^n$ without the convexity assumption.

For $m = j-1$, the function $h_{j-1}(X \cap L_{j+1[O]}^n, \cdot)$ does not depend on the principal curvatures, and was determined in (Auneau and Jensen, 2010, Proposition 4). In view of (4.2) and using this simplification, $\beta(X) = \Phi_{n-1}(X, A)$ and

$$\begin{aligned} &\alpha(X \cap L_{j+1[O]}^n) \\ &\quad = c_{j+1,1}^n \int_{\partial'(X \cap L_{j+1[O]}^n) \cap A} \|z\|^{n-j-1} F(-\tfrac{1}{2}, -\tfrac{n-j-1}{2}; \tfrac{j}{2}; \sin^2 \angle(n'(z), z)) \mathrm{d}z^j \end{aligned} \tag{4.3}$$

satisfy (3.1). The special cases $A = \mathbb{R}^n$ of (4.2) and (4.3) yield the known rotational Crofton formula (Auneau and Jensen, 2010, Proposition 3 and p. 6) for intrinsic volumes.

Note that (4.3) can be written using the j-th support measure $\Xi_j'(Y, \cdot)$ of $Y = X \cap L_{j+1[O]}^n$ with respect to $L_{j+1[O]}^n$; see for instance (Hug et al., 2008, p. 488). Hence $\beta(X) = \Phi_{n-1}(X, A)$ and

$$\begin{aligned} \alpha(X \cap L_{j+1[O]}^n) = c_{j+1,1}^n \int_{(A \cap L_{j+1[O]}^n) \times (S^{n-1} \cap L_{j+1[O]}^n)} &\|z\|^{n-j-1} \\ &F(-\tfrac{1}{2}, -\tfrac{n-j-1}{2}; \tfrac{j}{2}; \sin^2 \angle(u, z)) \Xi_j'(X \cap L_{j+1[O]}^n, \mathrm{d}(z, u)) \end{aligned} \tag{4.4}$$

satisfy (3.1). As support measures are weakly continuous (Schneider, 1993, Theorem 4.2.1) and any convex body can be approximated by a decreasing sequence of convex bodies with boundary of class C^2 (Schneider, 1993, pp. 59–60), equation (4.4) is a solution of (3.1) with $\beta(X) = \Phi_{n-1}(X, A)$ for arbitrary convex bodies, as long as $\Phi_{n-1}(X, \partial A) = 0$. In particular, the choice $A = \mathbb{R}^n$ gives a rotational integral formula for $V_{n-1}(X)$ for all $X \in \mathcal{K}^n$.

4.2 The measurement function as an integral over the sphere

When $\beta(X)$ in (3.1) is the surface area of $X \in \mathcal{K}^n$, the measurement function can be written in terms of the radial function of the section profile and an angle in the section plane. This is obtained by using representation (4.4) derived in the preceding section for $m = j - 1$ and the coarea formula.

Proposition 5. *For $X \in \mathcal{K}^n$ with $O \in intX$ and $L^n_{j+1[O]} \in \mathcal{L}^n_{j+1[O]}$ let $Y = X \cap L^n_{j+1[O]}$. Then $\beta(X) = V_{n-1}(X)$ and*

$$\alpha(Y) = c^n_{j+1,1} \int_{S^{n-1} \cap L^n_{j+1[O]}} \rho_Y^{n-1}(u) \frac{1}{\cos\alpha} F(-\tfrac{1}{2}, -\tfrac{n-j-1}{2}; \tfrac{j}{2}; \sin^2\alpha)\, du^j \tag{4.5}$$

satisfy (3.1)*, where α is the angle between the (almost surely unique) outer unit normal of Y in $L^n_{j+1[O]}$ at $\rho_Y(u)u$ and the line connecting this boundary point with O.*

Proof. We assume first that X has a unique outer unit normal in every boundary point. This is equivalent to saying that ∂X is a C^1-surface; see e.g. (Schneider, 1993, p. 104). Then (4.3) with $A = \mathbb{R}^n$ gives

$$\alpha(Y) = c^n_{j+1,1} \int_{\partial' Y} \|z\|^{n-j-1} F(-\tfrac{1}{2}, -\tfrac{n-j-1}{2}; \tfrac{j}{2}; \sin^2\alpha) \mathrm{d}z^j.$$

In the following, we identify $L^n_{j+1[O]}$ with $\mathbb{R}^{j+1}$ (and hence assume $Y \subseteq \mathbb{R}^{j+1}$). The claim then follows for X with boundary of class C^1 if we can show the transformation formula

$$\int_{S^j} g(f(u)) Jf(S^j; u) \mathrm{d}u^j = \int_{\partial Y} g(z) \mathrm{d}z^j \tag{4.6}$$

with

$$\begin{aligned} f : \mathbb{R}^{j+1} \setminus \{O\} &\to \partial Y \\ x &\mapsto \rho_Y(x) x, \end{aligned}$$

and Jacobian $Jf(S^j; u) = \rho_Y^j(u)/\cos\alpha$, for arbitrary measurable $g \geq 0$.

Equation (4.6) follows from an application of the coarea formula (Jensen, 1998, Theorem 2.1) by calculation of the Jacobian. We have

$$Df(x) = \Big(\frac{\partial f_i}{\partial x_k}\Big)_{i,k=1,\dots,j+1} = x(\nabla\rho_Y(x))^t + \rho_Y(x) I_{j+1},$$

where I_k is the $(k \times k)$-identity matrix. If u denotes the outer unit normal of Y at $\rho_Y(x)x$, the directional derivative of f in direction $y \neq O$ must be a vector in the tangent space $\rho_Y(x)x + u^\perp$, so $(Df(x)y)^t u = 0$. More explicitly, $y^t \nabla\rho_Y(x) x^t u + \rho_Y(x) y^t u = 0$. Choosing $y \in u^\perp$ arbitrary, and then $y = u$ gives $\nabla\rho_Y(x) = -\frac{\rho_Y(x)}{\cos\alpha} u$, so

$$Df(x) = \rho_Y(x)(I_{j+1} - \tfrac{1}{\cos\alpha} x u^t)$$

when $x \in S^j$. The Jacobian is given by

$$Jf(S^j; x) = \sqrt{\det(EDf(x)^t (EDf(x)^t)^t)}$$

where the rows of the matrix E consist of an orthonormal basis of $x^{\perp}$. This gives

$$\begin{aligned} Jf(S^j;x) &= \rho_Y^j(x)\sqrt{\det(I_{j+1}+\tfrac{1}{\cos^2\alpha}Eu(Eu)^t)} \\ &= \rho_Y^j(x)(1+\tfrac{\|Eu\|^2}{\cos^2\alpha})^{1/2} \\ &= \frac{\rho_Y^j(x)}{\cos\alpha}, \end{aligned}$$

as required. Using the continuity of the intrinsic volumes and Lebesgue's dominated convergence theorem the C^1-assumption can be omitted, as outlined in the following. If Y has a unique outer unit normal in $L^n_{j+1[O]}$ at $y \in \partial Y$ we say that y is a regular point of Y. Let $\mathrm{reg}Y$ be the set of regular points of Y, $S_Y = \{u \in S^{n-1} \cap L^n_{j+1[O]} \mid \rho_Y(u)u \notin \mathrm{reg}Y\}$ and define

$$\begin{aligned} g : \partial Y \setminus \mathrm{reg}Y &\to S_Y \\ x &\mapsto \frac{x}{\|x\|}. \end{aligned}$$

As (Schneider, 1993, Theorem 2.2.4.) implies $\mathcal{H}^j_{j+1}(\partial Y \setminus \mathrm{reg}Y) = 0$ and g is a Lipschitz mapping, we have $\mathcal{H}^j_{j+1}(S_Y) = 0$. We therefore only consider $u \in S^{n-1} \cap L^n_{j+1[O]} \setminus S_Y$ in the following.

Let $X_i = X + i^{-1}B_n$ be the parallel body of X at distance i^{-1}. Then ∂X_i is a C^1-surface. As $X_i \searrow X$ for $i \to \infty$ we have that $Y_i = X_i \cap L^n_{j+1[O]}$ converges to $Y = X \cap L^n_{j+1[O]}$ and, by continuity of the radial function, $\rho_{Y_i}(u)u \to \rho_Y(u)u$, as $i \to \infty$. By the fact that $\rho_Y(u)u \in \mathrm{reg}Y$, we conclude that $\alpha_i \to \alpha$ for $i \to \infty$, where α_i is the angle between the outer unit normal of Y_i at $\rho_{Y_i}(u)u$ and the line connecting this boundary point with O. As $O \in \mathrm{int}X$, there exists α' such that $\alpha_i < \alpha' < \pi/2$ for all i and hence $\cos\alpha_i > \cos\alpha' > 0$. This implies that the hypergeometric function $F(-\frac{1}{2}, -\frac{n-j-1}{2}; \frac{j}{2}; \sin^2\alpha_i)$ can be written as an absolutely convergent power series in $\sin^2\alpha_i$ and is therefore a continuous function on $[0,\alpha']$. Therefore, there exists a finite constant $C = C(Y,n,j)$ such that

$$\rho^{n-1}_{Y_i}(u)\frac{1}{\cos\alpha_i}F(-\tfrac{1}{2}, -\tfrac{n-j-1}{2}; \tfrac{j}{2}; \sin^2\alpha_i) < C$$

for all $u \in S^{n-1} \cap L^n_{j+1[O]}$ and all i. Furthermore, we have shown pointwise convergence of the integrand

$$\rho^{n-1}_{Y_i}(u)\frac{1}{\cos\alpha_i}F(-\tfrac{1}{2}, -\tfrac{n-j-1}{2}; \tfrac{j}{2}; \sin^2\alpha_i) \to \rho^{n-1}_{Y}(u)\frac{1}{\cos\alpha}F(-\tfrac{1}{2}, -\tfrac{n-j-1}{2}; \tfrac{j}{2}; \sin^2\alpha)$$

for $u \in S^{n-1} \cap L^n_{j+1[O]} \setminus S_Y$. Hence Lebesgue's dominated convergence theorem can be applied and the result follows without assuming that ∂X is a C^1-surface. □

We remark that the proposition also holds without assuming $O \in \mathrm{int}X$ but then the assumptions that X is strictly convex and ∂X a C^1-surface have to be added. Then (4.5) becomes

$$\alpha(Y) = c^n_{j+1,1}\int_{\{u \in S^{n-1}\cap L^n_{j+1[O]} \mid\ \exists \beta\in\mathbb{R}:\beta u \in Y\}} \rho^{n-1}_Y(u)\frac{1}{\cos\alpha}F(-\tfrac{1}{2}, -\tfrac{n-j-1}{2}; \tfrac{j}{2}; \sin^2\alpha)\mathrm{d}u^j.$$

The hypergeometric function in (4.5) simplifies when $n = 3$, $j = 1$ (Jensen, 1998, Example 5.10)

$$F(-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2\alpha) = \cos\alpha + \alpha\sin\alpha. \tag{4.7}$$

Hence, according to Proposition 5, for $X \in \mathcal{K}^3$ with $O \in \mathrm{int}X$, equation (3.1) with $\beta(X) = V_2(X)$ is satisfied by

$$\alpha(X \cap L^3_{2[O]}) = \int_{S^2 \cap L^3_{2[O]}} \rho^2_{X\cap L^3_{2[O]}}(u)(1 + \alpha\tan\alpha)\mathrm{d}u.$$

This integral equation is the basis of the well-known surfactor, see Section 5.

4.3 Morse type representation

In the derivation of (3.7), we have seen that a measurement function depending on the Euler characteristic of hyperplane sections can be expressed by means of the support function when X is convex. The values of the support function can be thought of as critical values of the section profiles. We now show that the use of critical values of the section profiles can be extended to more general sets. We first formulate the result for smooth manifolds and then for polyconvex sets.

In order to obtain the counting measure on the right hand side of (3.9) we consider an $(n-j)$-dimensional manifold X. If we assume that $X \pitchfork L^n_{j+1[O]}$ in $\mathbb{R}^n$ for almost all $L^n_{j+1[O]} \in \mathcal{L}^n_{j+1[O]}$, then $Y = X \cap L^n_{j+1[O]}$ is almost surely a one-dimensional smooth manifold; see the discussion at the end of Section 2. To discuss critical values of the manifold Y we use classical Morse theory. This theory studies the topology of manifolds in terms of functions defined on the manifolds. For the convenience of the reader we give here the basics of Morse theory for one-dimensional manifolds and refer to (Milnor, 1963) for more general results. We describe Morse theory in $\mathbb{R}^n$, but will later apply it to the section plane $L^n_{j+1[O]}$. For the purposes of stating results from Morse theory we introduce *CW-complexes*. The notion of a CW-complex is due to (Whitehead, 1949). We will assume that X is compact and therefore only need to consider finite CW-complexes. A finite CW complex Y is a topological space such that there is $n \in \mathbb{N}_0$ and a finite nested sequence

$$\emptyset \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n = Y, \tag{4.8}$$

such that the following two conditions hold

(i) Y_0 is finite,

(ii) for each $d \in \{1, \ldots, n\}$, Y_d is obtained from Y_{d-1} by attaching finitely many d-cells, as described in (Lundell and Weingram, 1969, p. 47), where a d-cell is the image of a continuous function $\phi : B_d \to X$ that is injective on $\mathrm{int}B_d$.

The number n in the above nested sequence is the *dimension* of the CW-complex Y. If n_0 is the number of elements in Y_0, and n_d is the number of d-cells attached to Y_{d-1} to obtain Y_d, the Euler characteristic of Y is given by (Lee, 2000, p. 373)

$$\chi(Y) = \sum_{d=0}^{n} (-1)^d n_d. \tag{4.9}$$

In the present work only CW-complexes of dimension one play a role, and they will be used only as a tool to determine the Euler-characteristic of hyperplane sections and sublevel sets of one-dimensional smooth manifolds.

Let $Y \subseteq \mathbb{R}^n$ be a compact smooth manifold of dimension one and let $f : Y \to \mathbb{R}$ be a smooth function. A point $p \in Y$ is a *critical point* of f if there is a local coordinate system $\phi : U \to Y$, where U is a neighbourhood of O, $\phi(O) = p$, such that $\bar{f} = f \circ \phi$ has a usual critical point at O:

$$\frac{\mathrm{d}\bar{f}}{\mathrm{d}x}(O) = 0.$$

If furthermore

$$\frac{\mathrm{d}^2\bar{f}}{\mathrm{d}x^2}(O) \neq 0$$

we say that p is a *non-degenerate* critical point. The definition of a critical point and non-degeneracy does not depend on the choice of the local coordinate system ϕ. If p is a critical point of f then $f(p)$ is called a *critical value* of f. We say that a function f is a *Morse function* if all of its critical points are non-degenerate and with different critical values. It is shown in (Fu, 1989, Section 5) that the height function $f_u(y) = \langle y, u\rangle$ is a Morse function for almost all $u \in S^{n-1}$, even under the weaker assumption that Y is a set of positive reach. A set $X \subseteq \mathbb{R}^n$ is said to be of positive reach if there exists $r > 0$ such that for all $x \in X + rB_n$ there exists a unique point of X nearest to x.

If the second derivative at a non-degenerate critical point is negative, the critical point is said to have index one, otherwise it has index zero. Again, the index does not depend on the local coordinate system chosen. According to the Morse Lemma (Milnor, 1963, Lemma 2.2) the behaviour of f in a neighbourhood of a non-degenerate critical point p can be completely described by its index: There exists a chart y in a neighbourhood U of p with $y(p) = 0$ and such that

$$f = f(p) + (-1)^\lambda y^2$$

holds throughout U, where $\lambda \in \{0, 1\}$ is the index of f at p. From this it follows that a non-degenerate critical point is isolated.

In the following we will apply Morse theory only to height functions f_u, where $u \in S^{n-1}$ is chosen such that f_u is a Morse function for a given manifold Y. For a given $r \in \mathbb{R}$ and $u \in S^{n-1}$ we define the sub- and superlevel sets

$$Y_{\leq r} = \{y \in Y : f_u(y) \leq r\},$$
$$Y_{\geq r} = \{y \in Y : f_u(y) \geq r\}.$$

According to (Bredon, 1993, 7.4. Corollary p. 84) (with θ the height function on Y) and the fact that f_u is a Morse function for almost all u, the set $Y \cap (ru + u^\perp)$ is an embedded submanifold of Y for almost all $r \in \mathbb{R}$ and almost all $u \in S^{n-1}$. Therefore we can use the additivity of the Euler characteristic for manifolds, to write

$$\chi(Y \cap (ru + u^\perp)) = \chi(Y_{\leq r}) + \chi(Y_{\geq r}) - \chi(Y). \tag{4.10}$$

Let $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ and assume that the set $\{y \in Y : r_1 \leq f_u(y) \leq r_2\}$ contains no critical points of f_u. Then, by (Milnor, 1963, Theorem 3.1), $Y_{\leq r_2}$ and $Y_{\leq r_1}$

are homotopy equivalent. Furthermore if f_u has no degenerate critical points, Y has the homotopy type of a CW-complex with one cell of dimension λ for each critical point of index λ (Milnor, 1963, Theorem 3.5). According to (Milnor, 1963, Remark on p. 24), for all $r \in \mathbb{R}$ the set $Y_{\le r}$ has the homotopy type of a finite CW-complex, with one cell of dimension λ for each critical point of index λ in $Y_{\le r}$. This holds even if r is a critical value. In particular, both have the same Euler characteristic. Let now $m = m(u)$ be the number of critical points of the height function f_u on Y and $r_i = r_i(u)$, $i = 1, \dots, m$, their critical values. We assume without loss of generality that the critical points are enumerated such that $r_1 < r_2 < \cdots < r_m$. Then, if $\lambda_i = \lambda_i(u)$, $i = 1, \dots, m$, are the indices of the respective critical points, it follows from (4.9) with $n = 1$ that

$$\chi(Y_{\le r}) = \sum_{i:r_i \le r} (-1)^{\lambda_i}. \tag{4.11}$$

Applying the same argument with the function f_{-u} and constant $-r$, we obtain the Euler characteristic of the superlevel sets

$$\chi(Y_{\ge r}) = \sum_{i:r_i \ge r} (-1)^{1-\lambda_i}. \tag{4.12}$$

From (4.11) and (4.12) we get

$$\chi(Y) = \sum_{i=1}^{m} (-1)^{\lambda_i} = \sum_{i=1}^{m} (-1)^{1-\lambda_i}, \tag{4.13}$$

so $\chi(Y) = 0$. Inserting (4.11)–(4.13) into (4.10) gives

$$\begin{aligned} \chi(Y \cap (ru + u^\perp)) &= \sum_{i:r_i \le r} (-1)^{\lambda_i} - \sum_{i:r_i \ge r} (-1)^{\lambda_i} \\ &= 2 \sum_{i:r_i \le r} (-1)^{\lambda_i} - \sum_{i:r_i = r} (-1)^{\lambda_i}. \end{aligned} \tag{4.14}$$

We are now equipped with the necessary terminology and results for writing the measurement function associated to the Hausdorff measures in (3.9) with $m = n - j$ in terms of critical points of the height function on the section profile.

Theorem 6. *Let $X \subseteq \mathbb{R}^n$ be a compact smooth manifold of dimension $n - j$, where $j \in \{0, 1, \dots, n-1\}$. Assume that $X \pitchfork L^n_{j+1[O]}$ for almost all $L^n_{j+1[O]} \in \mathcal{L}^n_{j+1[O]}$. Then for $\beta(X) = \mathcal{H}^{n-j}_n(X)$, equation (3.1) holds with*

$$\alpha(\cdot) = \frac{c^{n-j+1,n-j}_{n+1,1}}{n-j} \int_{S^{n-1} \cap L^n_{j+1[O]}} M(\cdot, u)\, du^j, \tag{4.15}$$

where

$$M(Y, u) = \sum_{k=2}^{m} (\operatorname{sgn}(r_k)|r_k|^{n-j} - \operatorname{sgn}(r_{k-1})|r_{k-1}|^{n-j}) \sum_{i=1}^{k-1} v_i \tag{4.16}$$

depends on all the critical values $r_1 < r_2 < \cdots < r_m$ of the smooth one-dimensional manifold $Y \subseteq L^n_{j+1[O]}$ with respect to the function $f_u(x) = \langle x, u \rangle$. The respective Morse indices are $\lambda_1, \dots, \lambda_m$ and we abbreviated $v_i = (-1)^{\lambda_i}$, $i = 1, \dots, m$.

Proof. Taking $m = n - j$ in (3.9) and using the results leading to (3.6) ((Jensen, 1998, Proposition 3.3) and an invariance argument), the expression becomes

$$\alpha(X \cap L^n_{j+1[O]}) = c^{n-j+1,n-j}_{n+1,1,1} \int_{S^{n-1} \cap L^n_{j+1[O]}} \int_{-\infty}^{\infty} \mathcal{H}^0_n(X \cap L^n_{j+1[O]} \cap (ru + u^\perp)) \\ |r|^{n-j-1} \mathrm{d}r \mathrm{d}u^j. \tag{4.17}$$

Due to the assumption of transversality, $X \cap L^n_{j+1[O]}$ is a one-dimensional embedded submanifold of $\mathbb{R}^n$ for almost all $L^n_{j+1[O]} \in \mathcal{L}^n_{j+1[O]}$; see the discussion at the end of Section 2. As f_u is a Morse function on $X \cap L^n_{j+1[O]}$ for almost all $u \in S^{n-1} \cap L^n_{j+1[O]}$, $(X \cap L^n_{j+1[O]}) \cap (ru + u^\perp)$ is a finite set for almost all $u \in S^{n-1} \cap L^n_{j+1[O]}$ and $r \in \mathbb{R}$. Hence, the counting measure in (4.17) can be replaced by the Euler characteristic and the theorem follows by inserting (4.14) into (4.17) and calculating the inner integral explicitly. □

We remark that it might be possible to generalize Theorem 6 to sets of positive reach by using (Fu, 1989), where the classical Morse theory is extended to sets of positive reach. We do not consider this here but give an analogous result for not necessarily smooth polyconvex sets using Hadwiger's index, an index closely related to the Morse index. For $Y \in \mathcal{R}^n$ and $u \in S^{n-1}$ let

$$g^*_u(Y; r) = \lim_{\varepsilon \to 0+} (\chi(Y \cap (ru + u^\perp)) - \chi(Y \cap ((r - \varepsilon)u + u^\perp))), \tag{4.18}$$

$r \in \mathbb{R}$, be the index function given by (Hadwiger, 1955, Eq. (9)). The index function is non-zero for only finitely many r. Hadwiger (Hadwiger, 1955) showed that

$$\chi(Y) = \sum_r g^*_u(Y; r) \tag{4.19}$$

holds for all $u \in S^{n-1}$. We use this index to represent the measurement function (3.6) associated to the intrinsic volumes entirely in terms of critical values in the section profile. For $u \in S^{n-1}$ let

$$g_u(r) = g^*_u(Y; r) - g^*_{-u}(Y; r), \quad r \in \mathbb{R}. \tag{4.20}$$

We note that g_u also depends on Y but decided not to overload the notation. For a given $u \in S^{n-1}$ we say that $r \in \mathbb{R}$ is a *critical value* of Y in direction u if $g_u(r) \neq 0$.

In order to parallel the formulation to Theorem 6 in the following result for polyconvex sets, we choose $\beta(X) = 2V_{n-j}(X)$ for the left hand side of (3.1). That the factor two is natural here, can be seen in the case $j = 1$, as $2V_{n-1}(X) = \mathcal{H}^{n-1}_n(\partial X)$ for any convex body X with interior points.

Theorem 7. *Let $X \in \mathcal{R}^n$ and $j \in \{0, 1, \ldots, n-1\}$. Then for $\beta(X) = 2V_{n-j}(X)$, equation* (3.1) *holds with*

$$\alpha(\cdot) = \frac{c^{n-j+1,n-j}_{n+1,1}}{n-j} \int_{S^{n-1} \cap L^n_{j+1[O]}} M(\cdot, u)\, du^j, \tag{4.21}$$

where

$$M(Y,u)=\sum_{k=2}^{m}(\operatorname{sgn}(r_k)|r_k|^{n-j}-\operatorname{sgn}(r_{k-1})|r_{k-1}|^{n-j})\sum_{i=1}^{k-1}v_i \tag{4.22}$$

depends on all the critical values $r_1<r_2<\cdots<r_m$ of $Y\subseteq L^n_{j+1[O]}$ in direction u with respective indices $v_i=g_u(r_i)$, $i=1,\dots,m$, where g_u is given by (4.20).

Proof. For $L^n_{j+1[O]}\in\mathcal{L}^n_{j+1[O]}$ the set $Y=X\cap L^n_{j+1[O]}$ is polyconvex. Fix $u\in S^{n-1}\cap L^n_{j+1[O]}$. Using (4.19), we have $\chi(Y)=\sum_{i=1}^m g_u^*(Y;r_i)$. Furthermore, as the sublevel set $Y_{\le r}$, $r\in\mathbb{R}$, is polyconvex and

$$g_u^*(Y;r')=g_u^*(Y_{\le r};r')$$

for $r'\le r$, its Euler characteristic can be written as

$$\chi(Y_{\le r})=\sum_{i:r_i\le r}g_u^*(Y;r_i),\qquad r\in\mathbb{R}.$$

Similarly, we can write the Euler characteristic of the superlevel set

$$\chi(Y_{\ge r})=\sum_{i:r_i\ge r}g_{-u}^*(Y;r_i),\qquad r\in\mathbb{R}.$$

As (4.10) also holds when Y is a polyconvex set, this gives

$$\chi(Y\cap(ru+u^\perp))=\sum_{i:r_i\le r}(g_u^*(Y;r_i)-g_{-u}^*(Y;r_i))+\sum_{i:r_i=r}g_{-u}^*(Y;r_i).$$

Inserting this into (3.6) and calculating the inner integral explicitly, the result follows. □

As already noted in (3.7) and at the beginning of this section, when $X\in\mathcal{K}^n$ there are two critical values for any given direction $u\in S^{n-1}\cap L^n_{j+1[O]}$, and these are $h_{X\cap L^n_{j+1[O]}}(u)$ and $-h_{X\cap L^n_{j+1[O]}}(-u)$. Using that $-h_{X\cap L^n_{j+1[O]}}(-u)\le h_{X\cap L^n_{j+1[O]}}(u)$, that $v_1=1$ for all $u\in S^{n-1}\cap L^n_{j+1[O]}$ and all $L^n_{j+1[O]}\in\mathcal{L}^n_{j+1[O]}$, and that the Hausdorff measure is reflection invariant, it follows that (4.21) simplifies to (3.7) and furthermore to (3.8) if X contains O. This shows in particular that when $X\in\mathcal{K}^n$ has a smooth boundary, the M-functions in (4.16) and (4.22) with $j=1$ coincide, which implies that Theorem 6, applied to ∂X, is equivalent to Theorem 7. The M-functions agree for more general classes of sets than smooth convex sets. Let X be a compact, topologically regular set, i.e. $X=\operatorname{cl}(\operatorname{int}X)$. If ∂X is a smooth manifold of dimension $n-1$ and $\partial X\pitchfork L^n_{2[O]}$, then the boundary of $Y=X\cap L^n_{2[O]}$ is a one-dimensional smooth manifold. We formulate the result for Y and identify $L^n_{2[O]}$ with $\mathbb{R}^2$.

Proposition 8. *Let $Y\subseteq\mathbb{R}^2$ be compact, topologically regular and such that ∂Y is a one-dimensional smooth manifold. Then, for almost all $u\in S^1$, r is a critical value of ∂Y with respect to the height function f_u, in the sense of classical Morse theory, if and only if $g_u(r)\ne0$ for Y. Furthermore, if λ is the Morse index of a non-degenerate critical point with critical value r, then*

$$g_u(r)=(-1)^\lambda. \tag{4.23}$$

Proof. Let $u \in S^1$ be such that f_u is a Morse function. If r is not a critical value in the sense of Morse theory, (4.14) implies that $\chi(X \cap (tu + u^\perp))$ is constant for all t in a neighbourhood of r. This implies $g_u(r) = 0$, so r is not a critical value in the Hadwiger sense.

Now assume that r is a critical value in the sense of Morse theory. To simplify notation, we assume that $u = (0, 1)$ and $r = 0$ holds. Hence, the x-axis is a tangent to ∂Y at some point p, which we may assume to be O. As f_u is a Morse function, the origin is non-degenerate and isolated from all other critical points. Assume first that the index of O is $\lambda = 0$. Then there is an $\varepsilon > 0$ and a neighbourhood U of O in the x-axis such that $\partial Y \cap \varepsilon B_2 = \mathrm{graph}\gamma$ for some convex function $\gamma : U \to \mathbb{R}$. As Y is topologically regular, either $M_+ = \mathrm{epi}\gamma \cap \varepsilon B_2 = \{(x, y) \in \varepsilon B_2 \mid x \in U, \gamma(x) \le y\}$ or $M_- = \mathrm{cl}(\varepsilon B_2 \setminus M_+)$ coincides with $Y \cap \varepsilon B_2$.

Consider the case $Y \cap \varepsilon B_2 = M_-$. As all other critical values are at positive distance from $r = 0$, (4.14) shows that $\chi(X \cap (tu + u^\perp))$ does not change for small $t \le 0$ implying $g^*_u(Y; 0) = 0$. However, for small $t > 0$, $\chi(X \cap (tu + u^\perp)) = \chi(X \cap u^\perp) + 1$ and $g^*_{-u}(Y; 0) = -1$. This gives $g_u(0) = 1 = (-1)^\lambda$, as required. The case $Y \cap \varepsilon B_2 = M_+$ is treated in a similar way, and the case $\lambda = 1$ can be reduced to the above by replacing u with $-u$. Summarizing, the definition of critical value is the same for both, Morse and Hadwiger theory, and (4.23) holds. □

4.4 The generalized flower volume and projection formulae

The invariator principle was first used in (Cruz-Orive, 2005) to estimate volume and surface area of objects in $\mathbb{R}^3$ from 2-dimensional flat sections. Up to a factor 2, the surface area of $X \in \mathcal{K}^3$ is $V_2(X)$ and it follows from (3.7) and the definition (2.2) of the q-flower set that

$$V_2(X) = 2 \int_{\mathcal{L}^3_{2[O]}} V_2(H^1_{X \cap L^3_{2[O]}}) \mathrm{d}L^3_{2[O]}. \tag{4.24}$$

For $O \in X$ this was observed in (Cruz-Orive, 2005). An analogous result holds in all dimensions.

Lemma 9. *Let $X \in \mathcal{K}^n$ be given. Then, for $j \in \{1, \ldots, n-1\}$,*

$$V_{n-j}(X) = c^{n-j+1,n-j}_{n+1,1} \tfrac{j+1}{n-j} \int_{\mathcal{L}^n_{j+1[O]}} V_{j+1}(H^{\frac{n-j}{j+1}}_{X \cap L^n_{j+1[O]}}) dL^n_{j+1[O]}. \tag{4.25}$$

Proof. Equation (3.7) and the definition (2.2) of the q-flower set imply

$$\begin{aligned} V_{n-j}(X) =& \frac{c^{n-j+1,n-j}_{n+1,1}}{n-j} \int_{\mathcal{L}^n_{j+1[O]}} \Big(\int_{\{u \in S^{n-1} \cap L^n_{j+1[O]} | h_{X \cap L^n_{j+1[O]}}(u) \ge 0\}} \rho^{j+1}_{H^q_{X \cap L^n_{j+1[O]}}}(u) \mathrm{d}u^j \\ & - \int_{\{u \in S^{n-1} \cap L^n_{j+1[O]} | h_{X \cap L^n_{j+1[O]}}(u) \le 0\}} |\rho_{H^q_{X \cap L^n_{j+1[O]}}}(u)|^{j+1} \mathrm{d}u^j \Big) \mathrm{d}L^n_{j+1[O]}, \end{aligned}$$

where $q = (n-j)/(j+1)$. Introducing spherical coordinates in $L^n_{j+1[O]}$ shows that the inner integrals yield the $(j+1)$-dimensional volume of $H^q_{X \cap L^n_{j+1[O]}}$ up to a factor $j + 1$. □

In (Cruz-Orive, 2012, Section 4.3) the formal analogy of (4.24) with Kubota's formula (Schneider and Weil, 2008, Eq. (6.11)) was remarked. Kubota's formula expresses intrinsic volumes of $X \in \mathcal{K}^n$ by orthogonal projections $X|L^n_{n-j[O]}$ of X onto $L^n_{n-j[O]} \in \mathcal{L}^n_{n-j[O]}$:

$$V_{n-j}(X) = c^{n-j+1,j+1}_{n+1,1} \int_{\mathcal{L}^n_{n-j[O]}} V_{n-j}(X|L^n_{n-j[O]})\mathrm{d}L^n_{n-j[O]}. \tag{4.26}$$

The special case $n = 3, j = 1$ reads

$$V_2(X) = 2 \int_{\mathcal{L}^3_{2[O]}} V_2(X|L^3_{2[O]})\mathrm{d}L^3_{2[O]}, \tag{4.27}$$

so the surface area of X is proportional to the average area of all its projections on isotropic hyperplanes. Similarly (4.24) expresses $V_2(X)$ as average of areas associated to sections with isotropic hyperplanes, where now, areas of the associated 1-flower set $H^1_{X \cap L^3_{2[O]}}$ have to be taken. Lemma 9 shows that this analogy breaks down in general dimensions for two reasons: the $(n-j)$th intrinsic volume of X requires $(j+1)$-dimensional sections, and a $(n-j)/(j+1)$-flower set has to be considered instead of a 1-flower set. Only when n is odd and $j = (n-1)/2$ the formal analogy between (4.26) and (4.25) holds, like in the special case $n = 3, j = 1$. It is thus questionable if (4.24) should be considered as a 'dual' of (4.27) in the spirit of the dual theory of convex geometry. It appears that this analogy is a coincidence due to a special choice of dimensions.

It should also be noted that formulae like (4.25) trivially hold for *some* associated set replacing the $(n-j)/(j+1)$-flower set of X, as any non-negative number α is the volume of e.g. a $(j+1)$-dimensional ball with radius $(\alpha/\kappa_{j+1})^{1/(j+1)}$.

Due to the relevance for applications, we return to the analogy of the special cases (4.24) and (4.27). In (Schneider, 1988) it was shown (in arbitrary dimension) that (4.27) still holds for $X \in \mathcal{R}^3$, if the integrand on the right hand side of

$$V_2(X|L^3_{2[O]}) = \int_{L^3_{2[O]}} \mathbf{1}_{X|L^3_{2[O]}}(z)\mathrm{d}z^2$$

is replaced by the integral of the orthogonal projections of X on $L^3_{2[O]}$ with multiplicities. This is also true for (4.24), if the indicator in

$$V_2(H^1_{X \cap L^3_{2[O]}}) = \int_{L^3_{2[O]}} \mathbf{1}_{H^1_{X \cap L^3_{2[O]}}}(z)\mathrm{d}z^2$$

is replaced by $\chi(X \cap L^3_{2[O]} \cap (z + z^\perp))$. The latter function only takes integer values and could be interpreted as 'indicator function of $H^1_{X \cap L^3_{2[O]}}$ with multiplicities'. The next proposition determines this function more explicitly when X is a finite union of polytopes in $\mathcal{K}^3$. We write $Y = X \cap L^3_{2[O]}$ and identify $L^3_{2[O]}$ with $\mathbb{R}^2$. We restrict attention to topologically regular sets $Y \subseteq \mathbb{R}^2$.

Proposition 10. *Let $Y \subseteq \mathbb{R}^2$ be topologically regular, bounded and polygonal. Then ∂Y consists of finitely many closed polygonal Jordan paths $p^{(1)}, p^{(2)}, \dots, p^{(k)} \subseteq \mathbb{R}^2$ such that $p^{(i)} \cap p^{(j)}$ is empty or finite for all $1 \leq i < j \leq k$. If $y_1^{(i)}, \dots, y_{m_i}^{(i)}$ are the consecutive vertices when walking along $p^{(i)}$, then*

$$\chi(Y \cap (z + z^{\perp})) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \mathbf{1}_{B(y_j^{(i)}) \setminus B(y_{j+1}^{(i)})}(z)$$

for $\mathcal{H}_2^2$-almost all $z \in \mathbb{R}^2$, where $B(y) = \frac{y}{2} + \frac{\|y\|}{2} B_2$ and $y_{m_i+1}^{(i)} := y_1^{(i)}$.

Proof. For $\mathcal{H}_2^2$-almost all $z \in \mathbb{R}^2$ we have

$$\begin{aligned}
\chi(Y \cap (z + z^{\perp})) &= \tfrac{1}{2} \chi(\partial Y \cap (z + z^{\perp})) \\
&= \sum_{i=1}^{k} \frac{1}{2} \sum_{j=1}^{m_i} \chi\big([y_j^{(i)}, y_{j+1}^{(i)}] \cap (z + z^{\perp})\big),
\end{aligned} \tag{4.28}$$

where $[y, y']$ is the line segment with endpoints $y, y' \in \mathbb{R}^2$. By Pythagoras' theorem, we have $[0, y] \cap (z + z^{\perp}) \neq \emptyset$ if and only if $z \in B(y)$ and hence, for $\mathcal{H}_2^2$-almost all $z \in \mathbb{R}^2$,

$$[y, y'] \cap (z + z^{\perp}) \neq \emptyset \Leftrightarrow z \in (B(y) \setminus B(y')) \cup (B(y') \setminus B(y)).$$

Thus, for almost all z,

$$\begin{aligned}
\frac{1}{2} \sum_{j=1}^{m_i} \chi\big([y_j^{(i)}, y_{j+1}^{(i)}] \cap (z + z^{\perp})\big) &= \frac{1}{2} \sum_{j=1}^{m_i} (\mathbf{1}_{B(y_j^{(i)}) \setminus B(y_{j+1}^{(i)})}(z) + \mathbf{1}_{B(y_{j+1}^{(i)}) \setminus B(y_j^{(i)})}(z)) \\
&= \sum_{j=1}^{m_i} \mathbf{1}_{B(y_j^{(i)}) \setminus B(y_{j+1}^{(i)})}(z).
\end{aligned}$$

Inserting this into (4.28) gives the assertion. □

When $Y \subseteq \mathbb{R}^2$ is a simply connected, polygonal set with interior points, and $y_1, \dots, y_m$ are its consecutive vertices, then

$$\chi(Y \cap (z + z^{\perp})) = \sum_{i=1}^{m} \mathbf{1}_{B(y_i) \setminus B(y_{i+1})}(z)$$

for $\mathcal{H}_2^2$-almost all $z \in \mathbb{R}^2$. In other words, $\chi(Y \cap (z + z^{\perp}))$ can be read from the vector $v = (\mathbf{1}_{B(y_1)}(x), \mathbf{1}_{B(y_2)}(x), \dots, \mathbf{1}_{B(y_m)}(x)) \in \{0, 1\}^m$ by counting the number of blocks with consecutive 1's (in a cyclic manner). For instance, when $v = (1, 1, 0, 1, 1, 0, 1)$, the number of such blocks is $\chi(Y \cap (x + x^{\perp})) = 2$. A combination of Proposition 10 and (3.6), together with an explicit calculation gives the following corollary.

Corollary 11. *Let $X \subseteq \mathbb{R}^3$ be a simply connected set with interior points that can be represented as the union of finitely many polytopes in $\mathcal{K}^3$. Then, equation* (3.1) *holds with $\beta(X) = V_2(X)$ and*

$$\alpha(X \cap L_{2[O]}^3) = 2 \sum_{i=1}^{m} V_2(B(y_i) \setminus B(y_{i+1})), \tag{4.29}$$

where $B(y) = \frac{y}{2} + \frac{\|y\|}{2} B_2$ *and* $y_1, \dots, y_m$*,* $y_{m+1} = y_1$ *are the consecutive vertices of* $X \cap L^3_{2[O]}$*. Equivalently,*

$$\alpha(X \cap L^3_{2[O]}) \\ = \frac{1}{2} \sum_{i=1}^{m} \left(\pi \|y_i\|^2 - \gamma\left(\|y_i\|, \tfrac{\|y_i\|^2 - \langle y_i, y_{i+1}\rangle}{\|y_i - y_{i+1}\|}\right) - \gamma\left(\|y_{i+1}\|, \tfrac{\|y_{i+1}\|^2 - \langle y_i, y_{i+1}\rangle}{\|y_i - y_{i+1}\|}\right)\right), \tag{4.30}$$

where $\gamma(r, x) = r^2 \arccos \frac{x}{r} - x\sqrt{r^2 - x^2}$*.*

Proof. The measurement function (3.6) associated to the intrinsic volumes with $n = 3$ and $j = 1$ can be written as

$$\alpha(X \cap L^3_{2[O]}) = 2 \int_{L^3_{2[O]}} \chi(X \cap L^3_{2[O]} \cap (z + z^\perp)) \mathrm{d}z^2.$$

Therefore, using Proposition 10, equation (4.29) is evident.

The latter representation (4.30) is obtained from the first one by direct calculation. We consider the triangle whose vertices are O and the midpoints of the circles $B(y_i)$ and $B(y_{i+1})$. Let ϕ be the angle between the line segments $[O, y_i/2]$ and $[y_{i+1}/2, y_i/2]$ and ϕ' the angle between $[O, y_{i+1}/2]$ and $[y_{i+1}/2, y_i/2]$. Furthermore, let $r = \|y_i\|/2$ and $r' = \|y_{i+1}\|/2$ be the radii of the circles and $m' = \frac{1}{2}\|y_i - y_{i+1}\|$ be the lenght of the line segment connecting the midpoints of the circles. Draw the line orthogonal to the line connecting $y_{i+1}/2$ and $y_i/2$ and passing through O and let

$$x = \frac{1}{2}\Big(m' + \frac{r^2 - (r')^2}{m'}\Big), \quad x' = \frac{1}{2}\Big(m' - \frac{r^2 - (r')^2}{m'}\Big).$$

Applying Pythagoras' theorem, we find

$$V_2(B(y_i) \setminus B(y_{i+1})) = \pi r^2 - (r^2\phi - x\sqrt{r^2 - x^2}) - ((r')^2\phi' - x'\sqrt{(r')^2 - (x')^2}).$$

Again using Pythagoras' theorem, we can write $\phi = \arccos \frac{x}{r}$, $\phi' = \arccos \frac{x'}{r'}$,

$$x = \frac{\|y_i\|^2 - \langle y_i, y_{i+1}\rangle}{2\|y_i - y_{i+1}\|} \quad \text{and} \quad x' = \frac{\|y_{i+1}\|^2 - \langle y_i, y_{i+1}\rangle}{2\|y_i - y_{i+1}\|}$$

and the result follows. □

When $X \in \mathcal{K}^3$ and $O \in \mathrm{int}X$ in Corollary 11, alternatives to (4.30) can be found in (Cruz-Orive, 2011, Proposition 3) and (Cruz-Orive, 2012, Corollary 2).

5 Stereological applications

There are various applications in local stereology of the different representations of the measurement function given in Section 4. We mention some of them here, with an emphasis on surface area estimation. We start by a review on existing methods and then present applications of the new rotational Crofton formulae given in Section 4.3.

Choosing $m = j - 1$ in (3.5) and mulitplying by two, a measurement function for the surface area of the boundary of $X \in \mathcal{R}^n$ is obtained. Similarly, with $m = j$ we obtain a measurement function for the volume of X. We assume $j = 1$ in the following, which gives the relations

$$S(\partial X) = 2V_{n-1}(X) = (n-1)c_1^n \int_{\mathcal{L}_{2[O]}^n} \int_{\mathcal{L}_1^2} V_0(X \cap L_{2[O]}^n \cap L_1^2) d(O, L_1^2)^{n-2} \mathrm{d}L_1^2 \mathrm{d}L_{2[O]}^n,$$

$$V_n(X) = c_1^{n-1} \int_{\mathcal{L}_{2[O]}^n} \int_{\mathcal{L}_1^2} V_1(X \cap L_{2[O]}^n \cap L_1^2) d(O, L_1^2)^{n-2} \mathrm{d}L_1^2 \mathrm{d}L_{2[O]}^n.$$

Identical relations can be obtained for a smooth manifold by choosing $m = n - j$ and $j = 1$ or $j = 0$, respectively, in (3.9). As mentioned at the beginning of Section 4.4, the above relations were first applied in stereology in (Cruz-Orive, 2005) for bounded objects in $\mathbb{R}^3$ with piecewise smooth boundary of class C^1. As evident, an unbiased estimator for the surface area of ∂X is obtained by taking an IR two-dimensional subspace and then generating an IUR line (hitting a reference set) within this subspace, weighting that line by a power of its distance from O and counting how often the weighted line hits the profile section. In (Cruz-Orive, 2005) for $n = 3$ a line obtained in this way is referred to as an r-weighted line, where r is its distance from O. Similarly, by measuring the length of the intersection of an r-weighted line and the profile section, an unbiased estimator for the volume of X is obtained. An r-weighted line in a two-dimensional plane in $\mathbb{R}^3$ can be generated by choosing a uniformly distributed point z in the section plane intersected with the reference set and taking a line through that point that is orthogonal to the line connecting z with O. Formally this follows by introducing polar coordinates in the section plane. An application of these estimators was illustrated in (Cruz-Orive et al., 2010), where they are denoted *invariator estimators* (in (Cruz-Orive, 2008) the estimators are referred to as *pivotal estimators*).

Already in (Cruz-Orive, 2005) improved surface area estimators were suggested for three-dimensional convex objects containing O. A first approach is to measure the support function for a given angle in a given IR subspace instead of generating an r-weighted line. If the support function can be measured in *all* directions in the subspace the flower estimator is obtained

$$\hat{S}_{\text{flo}} = 2 \int_{S^2 \cap L_{2[O]}^3} h^2_{X \cap L_{2[O]}^3}(u) \mathrm{d}u^1, \tag{5.1}$$

which is (3.8) with $n = 3, j = 1$, up to a factor 2. This is the area of the 1-flower set $H^1_{X \cap L_{2[O]}^3}$, called flower area in (Cruz-Orive, 2011), up to a factor four. In (Cruz-Orive, 2011), both the flower estimator for convex bodies and the wedge estimator for volume based on the invariator principle, were studied. In particular, simple formulae for calculating the flower area when the object of interest is either an ellipsoid or a convex polygon, were given. We already referred to the latter case in Section 4.4. As anticipated in (Cruz-Orive, 2005) a good compromise between accuracy and effort might be not to measure the whole flower area but apply angular systematic random sampling in the plane, measuring the support function for N angles for a discrete approximation of the flower area. In (Dvořák and Jensen, 2013) it was shown that the flower estimator for three-dimensional ellipsoids with O in the interior is identical to its discretization when the support function is measured at four perpendicular directions. There, a semi-automatic estimation of the flower estimator was proposed and studied, in analogy to the approach in (Hansen et al., 2011) for the nucleator volume estimator.

Using the new rotational Crofton-type formulae derived in Section 4.3, we obtain analogues of these improved estimators in general dimension and without assuming convexity of the object of interest. We state these in the following. We assume that $X \in \mathcal{R}^n$ or that X is a compact, topologically regular set with ∂X an $(n-1)$-dimensional smooth manifold satisfying $\partial X \pitchfork L^n_{2[O]}$ in $\mathbb{R}^n$ for almost all $L^n_{2[O]} \in \mathcal{L}^n_{2[O]}$. According to Theorem 7 and Proposition 8 we can use Hadwiger's index to write the surface area of ∂X as

$$S(\partial X) = c^n_{2,1} \int_{\mathcal{L}^n_{2[O]}} \int_{S^{n-1} \cap L^n_{2[O]}} M(X \cap L^n_{2[O]}, u) \mathrm{d}u^1 \mathrm{d}L^n_{2[O]}, \tag{5.2}$$

where M is given by (4.22) with $j = 1$. Then an unbiased estimator for the surface area of ∂X is given by

$$\hat{S}_1 = c^n_1 M(X \cap L^n_{2[O]}, U), \tag{5.3}$$

where U is uniformly distributed in $S^{n-1} \cap L^n_{2[O]}$ and $L^n_{2[O]} \in \mathcal{L}^n_{2[O]}$ is IR. When $X \in \mathcal{K}^n$ the determination of $M(X \cap L^n_{2[O]}, U)$ is equivalent to measuring the support function in the two opposite directions U and $-U$. The estimator can be improved further by finding the critical points in all directions in the two-dimensional IR subspace. The estimator

$$\hat{S}_{\text{flo}} = c^n_{2,1} \int_{S^{n-1} \cap L^n_{2[O]}} M(X \cap L^n_{2[O]}, u) \mathrm{d}u^1, \tag{5.4}$$

where $L^n_{2[O]} \in \mathcal{L}^n_{2[O]}$ is IR, is an unbiased estimator for the surface area of X. For $X \in \mathcal{K}^3$ containing O, this is equivalent to the flower estimator (5.1) for surface area. A discretization of the generalized flower estimator gives the following unbiased estimator

$$\hat{S}_N = \frac{c^n_1}{N} \sum_{l=0}^{N-1} M(X \cap L^n_{2[O]}, u_{\alpha_0 + l\frac{\pi}{N}}), \tag{5.5}$$

where u_α is a unit vector making an angle α with a fixed axis in the IR section plane $L^n_{2[O]} \in \mathcal{L}^n_{2[O]}$, α_0 is uniformly distributed in the interval $[0, \pi/N)$ and N is the number of sampled angles. Choosing $N = 1$ in (5.5) gives (5.3). We refer to these estimators as the *Morse type surface area estimators*. As $M(\cdot, U) = M(\cdot, U + \pi)$, the Cauchy-Schwarz inequality implies

$$\mathbb{V}\mathrm{ar}(\hat{S}_{2N}) \le \mathbb{V}\mathrm{ar}(\hat{S}_N) \le \mathbb{V}\mathrm{ar}(\hat{S}_1)$$

for all $N \in \mathbb{N}$. This was shown for $X \in \mathcal{K}^3$ in (Dvořák and Jensen, 2013, p. 145). Furthermore from the law of total variance $\mathbb{V}\mathrm{ar}(\hat{S}_{\text{flo}}) \le \mathbb{V}\mathrm{ar}(\hat{S}_N)$ for all $N \in \mathbb{N}$. The drawback of $\hat{S}_{\text{flo}}$ is that it requires finding critical points in *all* directions in the section plane, which is usually not feasible in practice (unless the object of interest is a simply connected polytope as then Corollary 11 can be used).

In a separate work (Thórisdóttir et al., In preparation, 2013) we will apply a semi-automatic procedure based on these Morse type formulae to estimate the average surface area of the nuclei of giant-cell glioblastoma from microscopy images. Also the precision gain in terms of variance reduction compared to earlier approaches is discussed in (Thórisdóttir et al., In preparation, 2013).

The different representations of the measurement functions in Section 4 all stem from the invariator expressions in Section 3 and are therefore equivalent, for a given n, j, m and $\beta(\cdot)$. This shows in particular that estimators based on these expressions, some of which were originally derived independently of each other, coincide. We make this more precise for the intrinsic volumes in the following.

In (Gual-Arnau and Cruz-Orive, 2009) it was asked if equation (3.1) with $\beta(X) = V_{n-j+m}(X)$ holds only if the measurement function is of the invariator form (3.5). Some light was shed on this uniqueness conjecture in (Cruz-Orive, 2012) by showing that the integrated versions of the classical estimators of volume and surface area, the nucleator and the surfactor, respectively, coincide with the invariator estimators. More specifically, it was shown that the integrated nucleator (Hansen et al., 2011, Section 2.1.2) coincides with the mean wedge volume estimator (Cruz-Orive, 2012, Eq. (10)) and that for a strictly convex object with O in its interior and C^2 boundary, or a convex polygonal object containing O, the flower estimator coincides with the integrated surfactor (Cruz-Orive, 2012, Eq. (24)); see (Jensen, 1998, Section 5.6) for a derivation of the classical surfactor. The proofs rely on the use of figures and differentials and are restricted to three-dimensional objects. Section 4 presents alternative proofs of these results in arbitrary dimension, as (4.1) with $A = \mathbb{R}^n$ is the integrated nucleator and (4.5) essentially the integrated surfactor. In particular, as (4.5) and (3.8) with $j = 1$ are both derived from (3.5) with $m = 0$, $j = 1$, we have for $X \in \mathcal{K}^n$ with $O \in \mathrm{int}X$

$$\int_{S^{n-1}\cap L^n_{2[O]}} h^{n-1}_{X\cap L^n_{2[O]}}(u)\mathrm{d}u^1$$
$$= \int_{S^{n-1}\cap L^n_{2[O]}} \rho^{n-1}_{X\cap L^n_{2[O]}}(u)\frac{1}{\cos\alpha}F\Big(-\frac{1}{2}, -\frac{n-2}{2}; \frac{1}{2}; \sin^2\alpha\Big)\mathrm{d}u^1,$$

which is a generalization of (Cruz-Orive, 2012, Propositions 2 and 3) to arbitrary dimension and without assuming strict convexity of X. This relation even holds with arbitrary power of the support and radial functions. We formulate the result for two-dimensional convex bodies.

Proposition 12. *Let $Y \in \mathcal{K}^2$ with $O \in intY$. Then for $i \in \{1, 2, \dots\}$*

$$\int_{S^1} h^i_Y(u)\,du^1 = \int_{S^1} \rho^i_Y(u)\frac{1}{\cos\alpha}F(-\tfrac{1}{2}, -\tfrac{i-1}{2}; \tfrac{1}{2}; \sin^2\alpha)\,du^1, \tag{5.6}$$

where α is the angle between the (almost surely unique) outer unit normal of Y at $\rho(u)u$ and the line connecting this boundary point with O.

Proof. The derivation of (3.8) from (3.5) with $m = 0$ can be repeated with an arbitrary power of the distance, leading to

$$\int_{S^1} h^i_Y(u)\mathrm{d}u^1 = i\int_{S^1}\int_0^\infty \chi(Y\cap(ru+u^\perp))r^{i-1}\mathrm{d}r\mathrm{d}u^1.$$

The rest of the proof follows the one of Proposition 5 word by word, where only the power $n-j-1$ has to be replaced with $i-1$. $\square$

As for Proposition 5, we obtain an expression analogous to (5.6) without assuming $O \in \text{int}Y$ if Y is strictly convex and ∂Y is a C^1-curve

$$\int_{S^1} \text{sgn}(h_Y(u))|h_Y(u)|^i \mathrm{d}u^1 = \int_{\{u\in S^1 |\ \exists \beta \in \mathbb{R}: \beta u \in Y\}} \rho_Y^i(u) \frac{1}{\cos\alpha} F(-\tfrac{1}{2}, -\tfrac{i-1}{2}; \tfrac{1}{2}; \sin^2\alpha) \mathrm{d}u^1.$$

When $i=1$ in Proposition 12 we find a formula for the boundary length of $Y \in \mathcal{K}^2$

$$2V_1(Y) = \int_{S^1} \rho_Y(u) \frac{1}{\cos\alpha} \mathrm{d}u^1,$$

which is essentially the Horvitz-Thompson estimator for length (Jensen, 1998, p. 122).

Stereological estimators of Minkowski tensors follow directly from the Minkowski tensor relations (3.10) and (3.11) as shown in (Jensen and Ziegel, 2013, Proposition 1). In (Jensen and Ziegel, 2013, Section 5) a detailed account of all the estimators obtained for $n=3$ and $r+s \leq 2$ is given. These include the classical estimators of volume and surface area but also new local stereological estimators of centres of gravity and tensors of rank two. As an example, choosing $n=3$ and $r=j=0$ in (3.11) gives the nucleator estimator for volume while for $j=1$ it is the integrated nucleator. Similarly, letting $n=3$, $r=s=0$ and $j=m=1$ in (3.10) we obtain the flower estimator for surface area.

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References

J. Auneau and E.B.V. Jensen. Expressing intrinsic volumes as rotational integrals. *Adv. Appl. Math.*, 45(1):1–11, 2010.

J. Auneau-Cognacq. A rotational Crofton formula for flagged intrinsic volumes of sets of positive reach. Technical report, CSGB, Aarhus University, 2010.

J. Auneau-Cognacq, J. Ziegel, and E.B.V. Jensen. Rotational integral geometry of tensor valuations. *Adv. Appl. Math.*, 50(3):429–444, 2012.

G.E. Bredon. *Topology and Geometry*, volume 139. Springer, New York, 1993.

L.M. Cruz-Orive. A new stereological principle for test lines in three-dimensional space. *J. Microsc.*, 219(1):18–28, 2005.

L.M. Cruz-Orive. Comparative precision of the pivotal estimators of particle size. *Image Anal. Stereol.*, 27(1):17–22, 2008.

L.M. Cruz-Orive. Flowers and wedges for the stereology of particles. *Journal of Microscopy*, 243(1):86–102, 2011.

L.M. Cruz-Orive. Uniqueness properties of the invariator, leading to simple computations. *Image Anal. Stereol.*, 31(2):89–98, 2012.

L.M. Cruz-Orive, M.L. Ramos-Herrera, and E. Artacho-Pérula. Stereology of isolated objects with the invariator. *J. Microsc.*, 240(2):94–110, 2010.

J. Dvořák and E.B.V. Jensen. On semiautomatic estimation of surface area. *J. Microsc.*, 250(2):142–157, 2013.

J.H.G. Fu. Curvature measures and generalized Morse theory. *J. Differ. Geom.*, 30: 619–642, 1989.

S. Glasauer. A generalization of intersection formulae of integral geometry. *Geom. Dedicata*, 68(1):101–121, 1997.

X. Gual-Arnau and L.M. Cruz-Orive. A new expression for the density of totally geodesic submanifolds in space forms, with stereological applications. *Differ. Geom. Appl.*, 27:124–128, 2009.

X. Gual-Arnau, L.M. Cruz-Orive, and J.J. Nuño Ballesteros. A new rotational integral formula for intrinsic volumes in space forms. *Adv. Appl. Math.*, 44(3): 298–308, 2010.

H. Hadwiger. Eulers Charakteristik und kombinatorische Geometrie. *J. Reine Angew. Math.*, 194:101–110, 1955.

L.V. Hansen, J.R. Nyengaard, J.B. Andersen, and E.B.V. Jensen. The semi-automatic nucleator. *J. Microsc.*, 242(2):206–215, 2011.

D. Hug, R. Schneider, and R. Schuster. Integral geometry of tensor valuations. *Adv. Appl. Math.*, 41(4):482–509, 2008.

E.B.V. Jensen. *Local Stereology.* World Scientific, Singapore, 1998.

E.B.V. Jensen and J. Rataj. A rotational integral formula for intrinsic volumes. *Adv. Appl. Math.*, 41:530–560, 2008.

E.B.V. Jensen and J. F. Ziegel. Local stereology of tensors of convex bodies. *Methodology and Computing in Applied Probability*, 2013.

J.M. Lee. *Introduction to Smooth Manifolds, Second Edition.* Springer, New York, 2000.

A.T. Lundell and S. Weingram. *The Topology of CW Complexes, The University Series in Higher Mathematics.* Van Nostrand Reinhold Company, New York, 1969.

J.W. Milnor. *Morse Theory.* Princeton University Press, Princeton, 1963.

B. Petkantschin. Integralgeometrie 6. Zusammenhänge zwischen den Dichten der linearen Unterräume im n-dimensionalen Raum. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 11:249–310, 1936.

R. Schneider. Curvature measures and integral geometry of convex bodies, III. *Rend. Sem. Mat. Univ. Politec. Torino*, 46:111–123, 1988.

R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44. Cambridge University Press, Cambridge, 1993.

R. Schneider and R. Schuster. *Tensor valuations on convex bodies and integral geometry.* Albert-Ludwigs-Univ., Math. Fak., 1999.

R. Schneider and W. Weil. *Stochastic and Integral Geometry.* Springer Verlag, 2008.

Ó. Thórisdóttir, A.H. Rafati, and M. Kiderlen. Estimating the surface area of non-convex particles from central planar sections. In preparation, 2013.

J.H.C. Whitehead. Combinatorial homotopy I. *Bull. Am. Math. Soc.*, 55(3):213–245, 1949.