The Andersen–Kashaev TQFT

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Preface

This dissertation is the culmination of my years as a PhD student at the Centre for Quantum Geometry of Moduli Spaces (QGM) at Aarhus University. I would like to thank QGM and the Department of Mathematics for an inspiring atmosphere. I would also like to thank Université de Genève and especially Rinat Kashaev for hosting me during the autumn 2012.

In particular I would like to thank my PhD supervisor Jørgen Ellegaard Andersen for introducing me to and guiding me through the topics covered in this dissertation. Furthermore I would like to say thanks for his patience in helping me out with problems and questions that have arised in the process.

I would like to thank Søren Fuglede Jørgensen and Simone Marzioni for carefully proofreading this thesis and for helpful discussions. Also I give many thanks to Birgitte Kvist who has been very helpful in making illustrations throughout the whole thesis.

Finally I would like to thank my previous and present office colleagues Troels Agerholm, Anne Marie Svane, Martin Mygind and Troels Bak Andersen for an inspiring environment.

Resume

Teorien, som vil blive diskuteret i denne afhandling går tilbage til Atiyah [7], Segal [45] og Witten [51], som var de første der opdagede og aksiomatiserede (2+1)-dimensional topologisk kvantefeltteori (TQFT). Edward Wittens studium fra 1989 [51] af Chern–Simons teori, som en $(2+1)$-dimensional kvantefeltteori giver anledning til det vi kalder for en topologisk kvantefeltteori. Mere specifikt, lad $P \to M$ være et principalbundt over en 3-mangfoldighed M med (simpel) Lie gruppe G som strukturgruppe og lad $\mathfrak{g}$ være Lie algebraen hørende til G. Virkningsfunktionalet i Chern–Simons teori er givet ved:

$$\mathrm{CS}_M(A) := \frac{1}{8\pi^2}\int_M \mathrm{Tr}(A \wedge A + \frac{2}{3} A \wedge A \wedge A),$$

hvor $A \in \mathcal{A}_P = \Omega^1(M, \mathfrak{g})$. Dette virkningsfunktional indgår i partitionsfunktionen i kvantefeltteori, der er udtrykt ved stiintegralet

$$Z_k(M) = \int_{\mathcal{A}_P/\mathcal{G}_P} e^{2\pi i k\, \mathrm{CS}_M(A)} \mathcal{D}A, \tag{1}$$

$k \in \mathbb{N}$, som dog ikke er matematisk veldefineret. På nuværende tidspunkt eksisterer der ikke nogen metode til på naturlig vis at knytte et mål $\mathcal{D}A$ til det uendeligdimensionale rum $\mathcal{A}_P/\mathcal{G}_P$. Det lykkedes dog, i tilfældet hvor G er kompakt, for Reshetikhin og Turaev ([44],[43] og [50]), at definere kvantefeltteorier, med de ønskede egenskaber foreskrevet af Witten.

I denne afhandling vil vi studere et nyt bidrag indenfor topologiske kvantefeltteorier, som er udviklet af Andersen og Kashaev [6]. Partitionsfunktionen i Andersen–Kashaev TQFT'en, som vi i nær-værende afhandling er interesseret i, forventes at være relateret til det ovenstående stiintegral i tilfældet, hvor $G = \mathrm{PSL}(2, \mathbb{C})$.

Andersen–Kashaev TQFT'en bygger på kvante-Teichmüllerteori, som den blev udviklet af Kashaev [28].

I kvantiseringen af Teichmüllerrum tog Kashaev udgangspunkt i Penners parametrisering af det dekorerede Teichmüllerrum [38, 39], hvor afbildningsklassegruppen ses eksplicit gennem rationale transformationer frembragt af sammensætninger af elementære Ptolemy-transformationer. Faddeevs kvantedilogaritme optræder som en central ingrediens i denne teori. Faddeevs kvantedilogaritme er allerede blevet brugt i tilstandsintegralkonstruktioner af perturbative invarianter af 3-mangfoldigheder af Hikami [20, 21]. Dog er de matematiske aspekter om konvergens og uafhængighed af triangulering ikke berørt i disse tilfælde. Andersen–Kashaev TQFT'en tager sig af disse spørgsmål.

For nylig har Andersen og Kashaev givet en ny formulering af Andersen–Kashaev TQFT'en [5]. Det er formodet, at denne teori er ækvivalent med teorien fra [6]. I nærværende afhandling vil vi gennem udregninger se eksempler på, hvordan de to formuleringer hænger sammen.

Vi vil desuden give en repræsentation for afbildningsklassegruppen $\Gamma_{1,1}$ af den punkterede torus . Det viser sig, at Andersen–Kashaev TQFT'en giver anledning til repræsentationer

$$\rho_{\text{A–K}} : \Gamma_{1,1} \to \mathcal{B}(\mathcal{S}(\mathbb{R})),$$

hvor $\mathcal{B}(\mathcal{S}(\mathbb{R}))$ er begrænsede operatorer på Schwartzrummet.

I den engelsksprogede introduktion giver vi en kapiteloversigt samt en oversigt over resultater indeholdt i denne afhandling.

Introduction

The theory we are about to discuss in this dissertation can be traced back to Atiyah [7], Segal [45] and Witten [51] who were the first ones to discover and axiomatize Topological Quantum Field Edward Wittens studies from 1989 [51] of Chern–Simons theory as a $(2+1)$-dimensional quantum field theory give rise to what we call a topological quantum field theory (TQFT). More specific, let G be a (simple) Lie group, let $P \to M$ be a principal G-bundle over a 3-manifold M and let $\mathfrak{g}$ be the Lie algebra corresponding to G. The action functional of Chern–Simons theory is given by:

$$\mathrm{CS}_M(A) := \frac{1}{8\pi^2} \int_M \mathrm{Tr}(A \wedge A + \frac{2}{3} A \wedge A \wedge A),$$

where $A \in \mathcal{A}_P = \Omega^1(M, \mathfrak{g})$. This action functional is a part of the partition function i quantum field theory which is given by the path integral

$$Z_k(M) = \int_{\mathcal{A}_P/\mathcal{G}_P} e^{2\pi i k \,\mathrm{CS}_M(A)} \mathcal{D}A, \tag{2}$$

$k \in \mathbb{N}$, although not mathematically well-defined. At the time of writing there is method of associating in a natural way a measure $\mathcal{D}A$ to the infinite-dimensional space $\mathcal{A}_P/\mathcal{G}_P$. However, after a few years, Reshetikhin and Turaev succeeded in defining quantum field theories with the properties prescribed by Witten's quantum theory when G is compact [44],[43],[50].

In the dissertation at hand we will study a new contribution to topological quantum field theories developed by Andersen and Kashaev [6]. In the Andersen–Kashaev TQFT the partition function is expected to be related to the path integral above in the case where $G = \mathrm{PSL}(2, \mathbb{C})$.

The Andersen–Kashaev TQFT builds on quantum Teichmüller theory developed by Kashaev [28], which produces unitary representations of centrally extended mapping class groups of punctured surfaces in infinite-dimensional Hilbert spaces.

In the quantization of Teichmüller space Kashaev started from the Penner parameterization of the (decorated) Teichmüller space [38, 39], where the mapping class group is realized explicitly through rational transformations generated by compositions of the elementary Ptolemy transformations. A central ingredient in this theory is Faddeev's quantum dilogarithm. The quantum dilogarithm has already been used in state integral constructions of perturbative invariants of 3-manifolds by Hikami in [20, 21] but the mathematical aspects of convergence and independence of triangulation have not been addressed so far. The Andersen–Kashaev TQFT addresses these problems.

Andersen and Kashaev have made a new development reformulating the Andersen–Kashaev TQFT in [5]. It is conjectured in [5] that the new formulation of the Andersen–Kashaev TQFT is equivalent to that of [6]. We will see through calculations of specific examples that this conjecture is well substantiated.

Furthermore we will give a representation of the mapping class group $\Gamma_{1,1}$ of the once punctured torus. It turns out that the Andersen–Kashaev TQFT gives rise to representations

$$\rho_{\text{A–K}} : \Gamma_{1,1} \to \mathcal{B}(\mathcal{S}(\mathbb{R})),$$

where $\mathcal{B}(\mathcal{S}(\mathbb{R}))$ is bounded operators on the Schwartz space.

Summary

The dissertation is structured as follows: It is split into 10 chapters and an Appendix, the first six of which contain relevant background material. The mathematical contents of these chapters should be well-known to most experts of the field but is included to set up notation, to ease reference, and also to provide a more gentle introduction to the field. The last four are devoted to the study of the Andersen–Kashaev TQFT.

Chapter 1, we start off gently, by introducing the fundamentals about the mapping class group. In chapter 8 we will use some of this background material for calculating a presentation of the mapping class group of the once punctured torus using the theory developed by Andersen and Kashaev.

In **Chapter 2** we give an outline of gauge theory and the study of connections in principal bundles over manifolds. Furthermore we will here look at classical Chern–Simons theory with a compact gauge group in some detail. Then we will look at Chern–Simons theory with a non-compact gauge group and recall the intimate relation between the Chern–Simons invariant and the hyperbolic volume.

In **Chapter 3** we review the theory of canonical and geometric quantization and go through pre-quantization as well as polarization.

In **Chapter 4** we look at the theory of Teichmüller space. Teichmüller space of a real topological surface R parametrizes complex structures on R up to the action of homeomorphisms that are isotopic to the identity homeomorphism. We recall how one can give global coordinates to Teichmuller space in order to get a better understanding of it. Indeed we will decompose a Riemann surface R into pairs of pants which will lead to Fenchel–Nielsen coordinates. We will recall the Penner coordinates and eventually turn to Kashaev coordinates and quantization of Teichmüller space in these.

Chapter 5 is concerned with the theory of hyperbolic geometry. We will recall how to do geometrization of knot complements. Due to Thurston we know that most 3-manifolds are hyperbolic, and since every closed 3-manifold is obtained by Dehn surgeries on knots in S^3, hyperbolic geometry and knot theory are closely related. It turns out that the hyperbolic structure is a topological property of the knot.

In **Chapter 6** we will turn our attention to the main topic of this dissertation, namely TQFTs. We will recall the historical background and state the axioms for a TQFT. We will then again turn to Chern–Simons theory. Again we will present in most detail the compact version. But we will also look at the quantum Chern-Simons theory with a non-compact gauge group.

Chapter 7 is devoted to the study of the Andersen–Kashaev TQFT. We will look at the construction of the Andersen–Kashaev TQFT in its original version.

We will turn to the new formulation of the Andersen–Kashaev TQFT in **Chapter 8**. Here we recall the definition of the partition function for closed oriented levelled shaped triangulated pseudo 3-manifolds. We will see how this extends to manifolds with boundary which gives rise to a TQFT. Further we discuss how to get mapping class group representations from TQFTs.

In **Chapter 9** we will calculate a number of examples verifying conjectures from [6], [5] and [26]. To be more precise we calculate the first examples of "ideal-" and "one vertex H-triangulations" of knot complements using the new formulation of the Andersen–Kashaev TQFT, showing that there is indeed a connection between the original and the new theory. We also do examples regarding the original version of the theory. Some of these computations were presented at a Winter School in Mathematical Physics [1]. Following this winter school, proceedings [2] will be published. In these proceedings a calculation of the Andersen–Kashaev partition function for the knot complement $(S^3, 6_1)$ is done by the author. Furthermore we see that the partition function for the complement of $(S^3, 6_1)$

[1] http://www.unige.ch/math/folks/podkopaeva/leshouches2012/
[2] Mathematical Aspects of Quantum Field Theories, Springer

given an H-triangulation is equivalent to the expression for partition function in *A TQFT of Turaev-Viro type on shaped triangulations* as conjectured in [26].

In **Chapter 10** we will do a presentation of the mapping class group for the once punctured torus $\Gamma_{1,1}$ using the theory developed in Chapter 8. We get a family of representations depending on the shape structure:

$$\rho_{\text{A-K}} : \Gamma_{1,1} \to \mathcal{B}(\mathcal{S}(\mathbb{R})).$$

Finally we end this dissertation with **Chapter 11** which is an appendix. In this appendix we prove some of the properties of Faddeev's quantum dilogarithm. We look at asymptotic expansions of Faddeevs dilogarithm both for $\mathrm{b}^2 \to 0$ and $\mathrm{b}^2 \to -\frac{1}{N}$. Furthermore we elaborate on Remark 8.2. To be precise we look at line bundles on a complex torus.

Chapter 1

Mapping Class Group

1.1 Definition and examples

We consider a compact connected orientable surface Σ. By the classification theorem of surfaces we know that the surface Σ is determined up to homeomorphisms by the number of connected components of its boundary $\partial\Sigma$

$$b := |\pi_0(\partial\Sigma)| \tag{1.1}$$

and its genus

$$g := \frac{1}{2}(\operatorname{rank} H_1(\Sigma, \mathbb{Z}) - b + 1).$$

When we want to emphasise the topological type we will write $\Sigma_{g,b}$ for a surface Σ specifying the genus and number of connected components of the boundary.

Let $\mathrm{Homeo}(\Sigma, \partial\Sigma)$ denote the group of orientation-preserving homeomorphisms restricting to the identity on the boundary $\partial\Sigma$, and let $\mathrm{Homeo}_0(\Sigma, \partial\Sigma)$ denote the normal subgroup of homeomorphisms that are isotopic to the boundary.

Definition 1.1. The *mapping class group* of Σ is the quotient group

$$\Gamma(\Sigma) := \mathrm{Homeo}(\Sigma, \partial\Sigma) / \mathrm{Homeo}_0(\Sigma, \partial\Sigma). \tag{1.2}$$

There are other common notations for the mapping class group of $\Sigma = \Sigma_{g,b}$ including $\mathrm{MCG}(\Sigma)$, $\mathcal{M}_{g,b}$ and $\Gamma_{g,b}$. Also there are different variations of the definition of the mapping class group $\Gamma(\Sigma)$ which may or may not give the exact same group.

- We could fix a smooth structure on Σ and then replace homeomorphism by diffeomorphism. This would not affect the definition of $\Gamma(\Sigma)$
- We could allow homeomorphisms not to be the identity on the boundary. Let $\Gamma(\Sigma, \partial)$ be the resulting group. We have an exact sequence of groups

  $$\mathbb{Z}^b \to \Gamma(\Sigma) \to \Gamma(\Sigma, \partial) \to \mathfrak{S}_b \to 1. \tag{1.3}$$

  The map $\mathbb{Z}^b \to \Gamma(\Sigma)$ sends the i-th canonical vector of $\mathbb{Z}^b$ to the Dehn twist along a curve parallel to the i-th component of $\partial\Sigma$, the map $\Gamma(\Sigma) \to \Gamma(\Sigma, \partial)$ is the canonical one and the map $\Gamma(\Sigma, \partial) \to \mathfrak{S}_b$ records how homeomorphisms permute the components of $\partial\Sigma$.
- We could allow homeomorphisms not to be orientation-preserving. Let us denote the resulting group by $\Gamma^{\pm}(\Sigma)$. If the boundary $\partial\Sigma$ is non-empty, then any boundary fixing homeomorphism must preserve the orientation. Hence

  $$\text{for } b > 0, \quad \Gamma^{\pm}(\Sigma) = \Gamma(\Sigma). \tag{1.4}$$

If the boundary is empty, then we have a short exact sequence of groups:

$$\text{For } b=0, \quad 1 \to \Gamma(\Sigma) \to \Gamma^{\pm}(\Sigma) \to \mathbb{Z}/2\mathbb{Z} \to 1. \tag{1.5}$$

This sequence is split since there exists an involution $\Sigma_g \to \Sigma_g$ which reverses the orientation.

Remark 1.2. If we give the set $\mathrm{Homeo}(\Sigma, \partial\Sigma)$ the compact-open topology then a continuous path $\rho : [0,1] \to \mathrm{Homeo}(\Sigma, \partial\Sigma)$ is the same thing as an isotopy between $\rho(0)$ and $\rho(1)$. We therefore have the equality $\Gamma(\Sigma) = \pi_0(\mathrm{Homeo}(\Sigma, \partial\Sigma))$.

Let us here consider a couple of relatively easy examples. We start by looking at the disk $D^2 := \{z \in \mathbb{C} \mid |z| < 1\}$, i.e. we look at a surface of genus $g = 0$ and one boundary component ∂D^2. The mapping class group of this surface $\Sigma_{0,1}$ is given by the following proposition.

Proposition 1.3. The space $\mathrm{Homeo}(D^2, \partial D^2)$ is contractible. In particular we have

$$\Gamma(D^2) = \{1\}.$$

Proof. Let $f : D^2 \to D^2$ be a homeomorphism which is the identity on the boundary. For all $t \in [0,1]$, we define a homeomorphism $f_t : D^2 \to D^2$ by

$$f_t(x) := \begin{cases} t \cdot f(x/t) & \text{if } 0 \le |x| \le t, \\ x & \text{if } t \le |x| \le 1. \end{cases} \tag{1.6}$$

Then the map $H : \mathrm{Homeo}(\Sigma, \partial\Sigma) \times [0,1] \to \mathrm{Homeo}(\Sigma, \partial\Sigma), (f,t) \mapsto f_t$ is a homotopy between the retraction of $\mathrm{Homeo}(\Sigma, \partial\Sigma)$ to $\{\mathrm{id}_{D^2}\}$ and the identity of $\mathrm{Homeo}(\Sigma, \partial\Sigma)$. Therefore $\mathrm{Homeo}(\Sigma, \partial\Sigma)$ deformation retracts to $\{\mathrm{id}_{D^2}\}$. □

From Proposition 1.3 it is fairly easy to deduce the mapping class group of the sphere S^2 (or $\Sigma_{0,0}$).

Corollary 1.4. $\Gamma(S^2) = \{1\}$.

Proof. Let $f : S^2 \to S^2$ be an orientation-preserving homeomorphism. Let γ be a simple closed oriented curve in S^2. Then $f(\gamma)$ is isotopic to γ, so WLOG we can assume that $f(\gamma) = \gamma$. Proposition 1.3 can now be applied to each of the disks which γ splits S^2 into. □

Let us consider the mapping class group of a 2-torus $\mathbb{T}^2 = S^1 \times S^1$. Recall that $H_1(S^1 \times S^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

Proposition 1.5. Let (α, β) be the basis of $H_1(T^2, \mathbb{Z})$ defined by $\alpha := [S^1 \times 1]$ and $\beta := [1 \times S^1]$. Then the map

$$M : \Gamma(\mathbb{T}^2) \to \mathrm{SL}(2, \mathbb{Z}) \tag{1.7}$$

which sends the isotopy class $[f]$ to the matrix of $f_* : H_1(\mathbb{T}^2, \mathbb{Z}) \to H_1(\mathbb{T}^2, \mathbb{Z})$ is a group isomorphism.

Proof. It is clear that we have a group homomorphism $M : \Gamma(\mathbb{T}^2) \to \mathrm{GL}(2, \mathbb{Z})$. Let us check that it takes values in $\mathrm{SL}(2, \mathbb{Z})$. Let $[f] \in \Gamma(\Sigma)$ then

$$M([f]) = \begin{pmatrix} f_*(a) \bullet b & f_*(b) \bullet b \\ -f_*(a) \bullet a & -f_*(b) \bullet a \end{pmatrix}, \tag{1.8}$$

where $\bullet$ denotes the intersection pairing $H_1(\mathbb{T}^2, \mathbb{Z}) \times H_1(\mathbb{T}^2, \mathbb{Z}) \to \mathbb{Z}$, where we use the fact that f is orientation preserving and therefore leaves the intersection pairing invariant. Hence $\det M([f]) = 1$.

The map M is surjective. Realise $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$ such that the loop $S^1 \times 1$ lifts to $[0,1] \times 1$ and $1 \times S^1$ lifts to $0 \times [0,1]$. Any matrix $T \in \mathrm{SL}(2,\mathbb{Z})$ defines a linear homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$, which globally leaves $\mathbb{Z}^2$ invariant. Therefore T induces an orientation-preserving homeomorphism $t : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, and $M([t]) = T$.

For the injectivity of the map we consider a homeomorphism $f : S^1 \times S^1 \to S^1 \times S^1$ such that $M([f])$ is trivial. Since the fundamental group $\pi_1(S^1 \times S^1)$ is abelian this implies that f acts trivially on the level of the fundamental group. The projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ gives the universal covering of $\mathbb{T}^2$. Thus we can lift f to a unique homeomorphism $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\tilde{f}(0) = 0$ and by the assumption on f we get that $\tilde{f}$ is $\mathbb{Z}^2$-equivariant. The affine homotopy

$$H : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2, (x,t) \mapsto t \cdot \tilde{f}(x) + (1-t) \cdot x$$

between $\mathrm{id}_{\mathbb{R}^2}$ and $\tilde{f}$, descends to a homotopy from $\mathrm{id}_{S^1 \times S^1}$ to f. Since homotopy coincides with isotopy in dimension two, we deduce that $[f] = 1 \in \Gamma(\mathbb{T}^2)$. □

In a similar manner one can compute the mapping class group of an annulus $S^1 \times [0,1]$. Again one uses the universal cover to deduce:

Proposition 1.6.

$$\Gamma(S^1 \times [0,1]) \approx \mathbb{Z}$$

For a proof of this fact, one should consult [13].

1.2 Dehn Twists

As we will see in this section, mapping class groups are generated by *Dehn twists*. A Dehn twist is a homeomorphism $\Sigma \to \Sigma$ having support in a regular neighbourhood of a simple closed curve. The simple closed curve does not necessarily need to be oriented. Intuitively we think of a Dehn twist on a surface as obtained by cutting the surface along a curve giving one of the boundary components a 2π left Dehn twist, and gluing the boundary components back together as indicated in Figure 1.1.

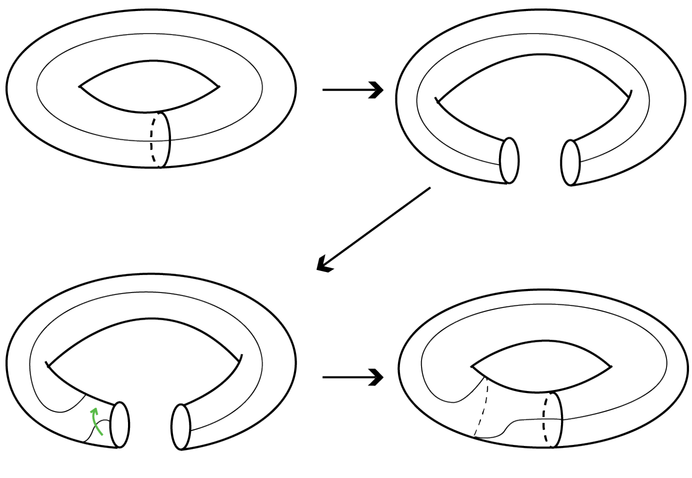


Figure 1.1: The action of the Dehn twist about a meridian on two simple closed curves in a torus.

Let α, β be two simply closed curves on Σ. We define their *geometric intersection number* or just *intersection number* to be

$$i(\alpha,\beta) := \min\{|\alpha' \cap \beta'| \mid \alpha' \text{ isotopic to } \alpha, \beta' \text{ isotopic to } \beta, \alpha' \pitchfork \beta'\}. \tag{1.9}$$

Definition 1.7. Let α be a simple closed curve on Σ. We choose a regular neighbourhood N of α in Σ and we identify it with $S^1 \times [0,1]$ in a way such that the orientation is preserved. A *Dehn twist* along α is the homeomorphism $t_\alpha : \Sigma \to \Sigma$ defined by

$$t_\alpha = \begin{cases} t_\alpha(x) = x, & \text{if } x \notin N, \\ (e^{2\pi i(\theta + r)}, r), & \text{if } x = (e^{2\pi i\theta}, r) \in N = S^1 \times [0,1]. \end{cases} \tag{1.10}$$

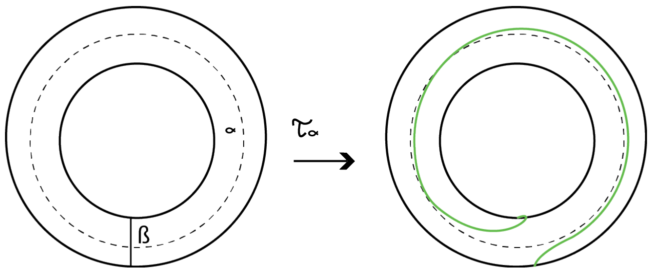


Figure 1.2: The action of the twist map $t_\alpha : N \to N$ on a horizontal line β in the annulus.

The isotopy class of t_α does only depend on the isotopy class of the curve α. The Dehn twist t_α has infinite order in $\Gamma(\Sigma)$ if $[\alpha] \neq 1 \in \pi_1(\Sigma)$. One can prove the following fact:

$$\forall \text{ simple closed curves } \beta \subset \Sigma, \quad \forall k \in \mathbb{Z}, \quad i(t_\alpha^k(\beta), \beta) = |k| \cdot i(\alpha, \beta)^2. \tag{1.11}$$

A proof of this statement can be found in [13].

The conjugate of a Dehn twist is again a Dehn twist. Indeed if $f : \Sigma \to \Sigma$ is an orientation-preserving homeomorphism, then we have the following lemma

Lemma 1.8. For $f \in \Gamma(\Sigma)$, we have

$$f \circ t_\alpha \circ f^{-1} = t_{f(\alpha)}. \tag{1.12}$$

When we write a product of mapping classes we always apply them from left to right.

Proof. Let ϕ be a representative for the mapping class f, and let γ be a representative of α. Then ϕ^{-1} takes a neighbourhood of $\phi(\gamma)$ to a regular neighbourhood of gamma. We use this neighbourhood to obtain the relation $t_{\phi(\gamma)} = \phi t_\gamma \phi^{-1}$. □

We can consider the Dehn twist along the "middle" of the annulus $S^1 \times [0,1]$. With the notation from Proposition 1.6 We see that $J(t_\alpha) = 1$. It follows that $\Gamma(S^1 \times [0,1])$ is infinite cyclic generated by t_α. More general we have the following result which goes back to Dehn.

1.3 Generators of the Mapping Class Group

Theorem 1.9 (Dehn)**.** The group $\Gamma(\Sigma)$ is generated by Dehn twists along non-separating simple closed curves and simple closed curves encircling some boundary components.

In order to prove the above theorem the following result comes in handy. We here assume that the surface Σ is endowed with an arbitrary smooth structure and a Riemannian metric. The results here go back to [10].

Theorem 1.10 (Birman's exact sequence)**.** Let Σ' be the compact oriented surface obtained from Σ by removing a disk D. Then there is an exact sequence of groups

$$\pi_1(U(\Sigma)) \xrightarrow{Push} \Gamma(\Sigma') \xrightarrow{\cup \mathrm{id}_D} \Gamma(\Sigma) \to 1 . \tag{1.13}$$

Here $U(\Sigma)$ denotes the total space of the unit tangent bundle of Σ and the $Push$ map is generated by some products of Dehn-twists along curves which are non-separating or which encircle boundary components.

Sketch of proof. We let Diffeo$(\Sigma, \partial\Sigma)$ denote the group of orientation-preserving and boundary fixing diffeomorphisms $\Sigma \to \Sigma$. In dimension two "diffeotopy groups" coincide with "homeotopy groups" and we have the equality

$$\Gamma(\Sigma) = \pi_0(\text{Diffeo}(\Sigma, \partial\Sigma)). \tag{1.14}$$

Let v be a unit tangent vector of D and consider the subgroup Diffeo$(\Sigma, \partial\Sigma, v)$ consisting of diffeomorphisms whose differential fixes v. One can the show that

$$\Gamma(\Sigma') = \pi_0(\text{Diffeo}(\Sigma, \partial\Sigma, v)). \tag{1.15}$$

The map Diffeo$(\Sigma, \partial\Sigma) \to U(\Sigma)$ defined by $f \mapsto d_p f(v)$ is a fibre bundle where the fibre is Diffeo$(\Sigma, \partial\Sigma, v)$. According to (1.14) and (1.15), the long exact sequence for homotopy groups induced by this fibration terminates with

$$\pi_1(\text{Diffeo}(\Sigma, \partial\Sigma)) \longrightarrow \pi_1(U(\Sigma)) \xrightarrow{Push} \Gamma(\Sigma') \xrightarrow{\cup \mathrm{id}_D} \Gamma(\Sigma) \longrightarrow 1\ . \tag{1.16}$$

The map $\pi_1(U(\Sigma)) \to \Gamma(\Sigma')$ is called the "Push" map because of the following description. A loop γ in $U(\Sigma)$ based at v can be seen as an isotopy of the disk $I : D^2 \times [0,1] \to \Sigma$ such that $I(\cdot, 0) = I(\cdot, 1)$ is a fixed parametrisation $D^2 \cong D$ of the disk $D \subset \Sigma$. This isotopy can be extended to an ambient isotopy $\bar{I} : \Sigma \times [0,1] \to \Sigma$ starting with $\bar{I}(\cdot, 0) = \mathrm{id}_\Sigma$. Define now

$$Push([\gamma]) := [\text{restriction of } \bar{I}(\cdot, 1) \text{ to } \Sigma' = \Sigma \backslash D] \tag{1.17}$$

Now assume that γ is the unit tangent vector field of a smooth simple closed curve α. Let N be a closed regular neighbourhood of α and let α_-, α_+ be the boundary components of N. Then we have

$$Push([\gamma]) = t_{\alpha_-} t_{\alpha_+}^{-1}, \tag{1.18}$$

as is seen in Figure 1.3.

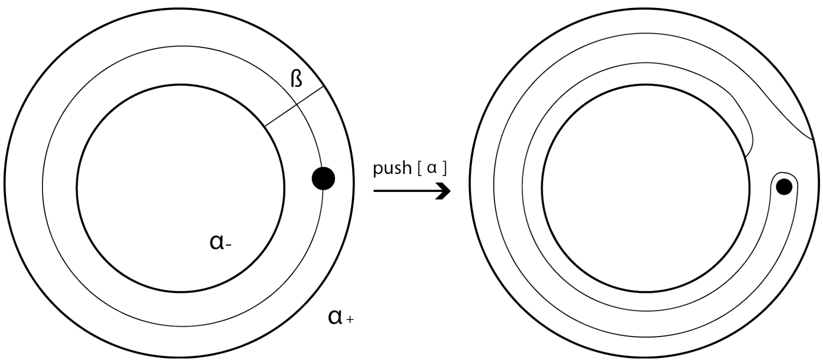


Figure 1.3: The Push map.

From the long exact sequence in homotopy for the fibration $U(\Sigma) \to \Sigma$ we get an exact sequence of groups

$$\pi_1(S^1) \to \pi_1(U(\Sigma)) \to \pi_1(\Sigma) \to 1. \tag{1.19}$$

Hereby we see that $\pi_1(U(\Sigma))$ is generated by the fiber and by unit tangent vector fields of smooth simple closed curves which are non-separating or which encircle components of the boundary $\partial\Sigma$. Since the image of the fiber S^1 by the $Push$-map is $t_{\partial D}$, we conclude that $Push(\pi_1(U(\Sigma)))$ is generated by products of Dehn twists along non-separating curves or curves which encircle boundary components. □

For a more precise version of the proof we ask the reader to consult [32].

This allows us to prove Theorem 1.9.

Proof of theorem 1.9. We deduce from Theorem 1.10 that if the statement holds at a given genus g for $b = 0$ boundary components then it holds for every $b \geq 0$. So without loss of generality we can assume that Σ is closed and the proof now goes by induction on $g \geq 0$. For $g = 0$ there is nothing to show since we have already shown that $\Gamma(S^2) = \{1\}$ in Proposition 1.4. For $g = 1$, we use Proposition 1.5. The group $\mathrm{SL}(2, \mathbb{Z})$ is generated by the two elements

$$S := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{1.20}$$

which corresponds to Dehn-twists along the curves $[S^1 \times 1]$ and $[1 \times S^1]$ respectively. So assume that $g \geq 2$.

Let $f \in \Gamma(\Sigma)$ and let α be a non-separating simple closed curve on Σ. The image $f(\alpha)$ is of course another non-separating simple closed curve on Σ. The following fact due to Lickorish [33] comes in handy. A proof can be found in [13]

Claim 1.11 (Connectedness of curves.)**.** Assume $g \geq 2$. Then for any two non-separating simple closed curves γ and γ' there exists a sequence of non-separating simple closed curves

$$\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n = \gamma'$$

such that $i(\gamma_j, \gamma_{j+1}) = 0$ for $j \in \{1, 2, \ldots, n-1\}$.

Further we make use of the following claim:

Claim 1.12. If β and γ are two non-separating closed curves on Σ such that $i(\beta, \gamma) = 0$, then there is a product of Dehn twists T along non-separating simple closed curves such that $T(\beta) = \gamma$.

We can find another non-separating simple closed and oriented curve $\alpha \subset \Sigma$ such that $i(\alpha, \gamma) = i(\alpha, \beta) = 1$. We have $t_\alpha t_\gamma \circ t_\beta t_\alpha(\beta) = t_\alpha t_\gamma(\alpha) = \gamma$. The two claims say that there is a product of Dehn twists along non-separating curves such that $T(\alpha) = f(\alpha)$. In other words we can assume that f fixes α. In this case we consider a non-separating curve β such that the intersection number $i(\alpha, \beta) = 1$. Notice that $t_\beta t_\alpha^2 t_\beta$ preserves α but reverses its orientation. Therefore after possible multiplication with $t_\beta t_\alpha^2 t_\beta$ we can assume that f preserves α with orientation. Since there is only one orientation-preserving homeomorphism of S^1 up to isotopy, we can assume that f is the identity on α. Further we can assume that f is the identity on a closed regular neighbourhood of N of α.

Let $\Sigma' := \Sigma \setminus \mathrm{int}(N)$ and we let f' be the restriction of f to Σ' The surface Σ' has genus $g' = g - 1$ and has $b' = b + 2$ boundary components. We conclude by induction hypothesis since a non-separating circle in Σ' is non-separating in Σ and a boundary curve in Σ' is either a boundary curve in Σ or is isotopic to α. □

Actually one can improve to show that only finitely many Dehn twists are required in order to generate the mapping class group. We have already mentioned that the mapping class group for the torus is generated by elements S, T which corresponds to two closed curves. This is a special case of the following theorem.

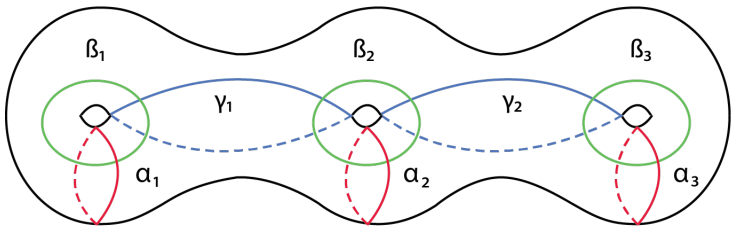


Figure 1.4: The curves appearing in the Dehn–Lickorish theorem in the case where $g = 3$.

Theorem 1.13 (Dehn–Lickorish)**.** For $g \geq 1$, the group $\Gamma(\Sigma_g)$ is generated by the Dehn twists along the following $3g - 1$ simple closed curves:

Later Humphries showed for $g \geq 2$ that $2g + 1$ Dehn twists are actually enough to generate the mapping class group $\Gamma(\Sigma_g)$. More precisely the mapping class group is generated by the Dehn twists along the curves $\beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g-1}, \alpha_1, \alpha_2$ using the notation from above. For a proof see [24].

Humphries also proved that the mapping class group cannot be generated by fewer Dehn twists when $g \geq 2$.

1.4 Presentation

We want to be able to find presentations for mapping class groups, whose generators are Dehn twists. First of all, one should find out which relations exists between *two* Dehn twists. It is intuitively clear that the relations must depend on how much the two curves intersect each other.

Lemma 1.14 (Disjointness relation)**.** Dehn twists about two simple closed curves commute if and only if the isotopy classes of the curves have zero intersection number

Proof. Let α and β be representatives for curves. It is obvious that Dehn twists of non-intersecting curves commute. It follows from Lemma 1.8 and the fact that $t_\alpha = t_\beta$ implies $\alpha = \beta$ that a given mapping class f commutes with a Dehn twists t_α if and only if f fixes α. So if $t_\alpha t_\beta = t_\beta t_\alpha$ we obtain that $t_\alpha(b) = b$ and from (1.11), we get

$$i(\alpha, \beta)^2 = i(t_\alpha(\beta), \beta) = 0.$$

□

Lemma 1.15 (Braid relations)**.** Let α and β be isotopy classes for two simple closed curves on Σ with $i(\alpha, \beta) = 1$. Then we have the braid relation $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$.

Proof. Let us first prove $t_\alpha t_\beta(\alpha) = \beta$. By using the change of coordinate principle we assume that α and β are represented by curves as in Figure 1.5 which indicates that the equality is true. It follows that $t_{t_\alpha t_\beta(\alpha)} = t_\beta$. Again we use Lemma 1.8 to conclude that $t_\alpha t_\beta t_\alpha (t_\alpha t_\beta)^{-1} = t_\beta$. □

Remark 1.16. From the classification of surfaces it follows that there exists a orientation preserving homeomorphism of Σ taking one simple closed curve to another if and only if the two results of cutting the surface along the two curves will be homeomorphic surfaces. I.e. up to homeomorphism there is only one non-separating curve and finitely many separating ones, and we may assume that α is one of the curves in Figure 1.6, see [13].

If $i(\alpha, \beta) \geq 2$ then t_α and t_β generate a free group on two generators [25]. In other words there are no relations between t_α and t_β.

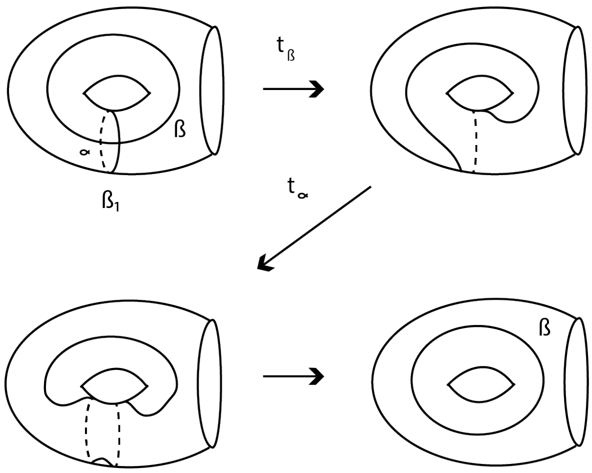


Figure 1.5: The curves α, β and the equation $t_\alpha t_\beta(\alpha) = \beta$. The last map is a simple isotopy.

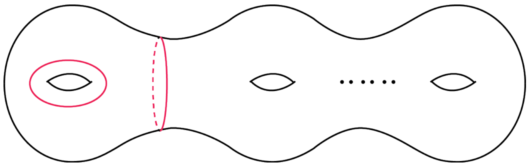


Figure 1.6: Using the change of coordinate principle to simplify a curve.

Theorem 1.17. Let α and β be isotopy classes for two simple closed curves on $\mathbb{T}^2$ with intersection number 1. Let $A := t_\alpha$ be a Dehn twist along the curve α and $B := t_\beta$ be a Dehn twist along the curve β, we have

$$\Gamma(\mathbb{T}^2) = \langle A, B \mid ABA = BAB,\ (AB)^6 = 1 \rangle . \tag{1.21}$$

Note that the first relation is the braid relation from above.

Proof. Let $\mathrm{PSL}(2,\mathbb{Z})$ be the quotient of $\mathrm{SL}(2,\mathbb{Z})$ by its order 2 subgroup $\{\pm I\}$. $\mathrm{PSL}(2,\mathbb{Z})$ is a free product group $\mathbb{Z}_2 * \mathbb{Z}_3$. As a matter of fact we have

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\langle \overline{T}, \overline{U} \mid \overline{T}^2 = 1, \overline{U}^3 = 1 \right\rangle$$

where $\overline{T}$ and $\overline{U}$ are the classes of the following matrices:

$$T := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

From the short exact sequence

$$0 \longrightarrow \{\pm I\} \longrightarrow \mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{PSL}(2,\mathbb{Z}) \longrightarrow 0 ,$$

we deduce the presentation

$$\mathrm{SL}(2,\mathbb{Z}) = \langle T, U \mid T^4 = 1, U^3 = 1, [U, T^2] = 1 \rangle .$$

If we set $V := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, and observe that $U = V^{-1}T^2$, we obtain the equivalent presentation

$$\mathrm{SL}(2,\mathbb{Z}) = \langle T, V \mid V^6 = 1,\ T^2 = V^3 \rangle .$$

Finally we set

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

so that A and B are the two mapping classes in Theorem 1.17. We make the observation that $T = ABA$ and $V = BA$ and obtain the presentation

$$\mathrm{SL}(2,\mathbb{Z}) = \left\langle A, B \mid (ABA)^2 = (BA)^3,\ (BA)^6 = 1 \right\rangle$$

which is equivalent to (1.21). $\square$

For higher genus, we consider the involution h of $\Sigma_g \subset \mathbb{R}^3$ which is a rotation around a appropriate line by the angle π. This involution can be written in terms of Lickorish's generators in the following manner

$$h = t_{\alpha_g} t_{\beta_g} t_{\gamma_{g-1}} t_{\beta_{g-1}} \dots t_{\gamma_2} t_{\beta_2} t_{\gamma_1} t_{\beta_1} t_{\alpha_1} t_{\alpha_1} t_{\beta_1} t_{\gamma_1} t_{\beta_2} t_{\gamma_2} \dots t_{\beta_{g-1}} t_{\gamma_{g-1}} t_{\beta_g} t_{\alpha_g}.$$

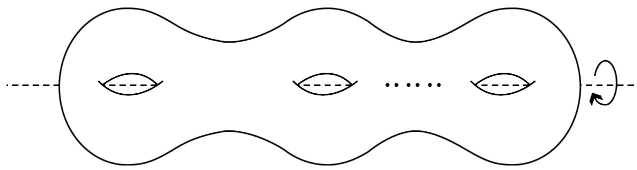


Figure 1.7: The hyperelliptic involution as a rotation of a surface.

Then we have a second relation between Lickorish's generators. The first one is obvious and the second one follows from the braid relation and the fact that $h(\alpha_g) = \alpha_g$.

Lemma 1.18 (Hyperelliptic involution). In $\Gamma(\Sigma_g)$, we have the relations $h^2 = 1$ and $[h, t_{\alpha_g}] = 1$

The hyperelliptic relations allow a presentation of $\Gamma(\Sigma_2)$ which is due to Birman and Hilden [8].

Theorem 1.19 (Birman–Hilden). Let $A := t_{\alpha_1}$, $B := t_{\beta_1}$, $C := t_{\gamma_1}$, $D := t_{\beta_2}$ and $E := t_{\alpha_2}$. Then the mapping class group for a genus 2 surface has the following presentation:

$$\Gamma(\Sigma_2) \cong \left\langle A, B, C, D, E \mid \text{disjointness, braid, } (ABC)^4 = E^2,\ [H, A] = 1,\ H^2 = 1 \right\rangle. \quad (1.22)$$

Here braid stands for the 4 possible braid relations between A, B, C, D, E and disjointness stands for relations between them and $H := EDCBA^2BCDE$.

Two particular elements of $\mathrm{SL}(2,\mathbb{Z})$ are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Theorem 1.20. The matrices S and T generate $\mathrm{SL}(2,\mathbb{Z})$.

For a proof see [30, App. A]

Chapter 2

Classical Chern–Simons theory

In this chapter we will start by recalling preliminary framework for what will be used later on. Then we will define the moduli space of flat connections of a principal G-bundle, since this is an important quantity in the study of Chern–Simons theory. In the end of this chapter we will turn to classical Chern–Simons theory with a compact gauge group. We give the definition of the Chern–Simons action which we will come back to in a later chapter.

The theory discussed in this chapter follows Kobayashi and Nomizu's book: *Foundations of differential geometry* [31], lecture notes by Himpel: *Lie groups and Chern–Simons Theory* [22] and Freed's *Classical Chern–Simons Theory, Part 1* [15].

2.1 Connections in Principal G-bundles

Definition 2.1. Let M be a manifold and G a Lie group. A **principal G-bundle** over M is a manifold P satisfying the following conditions.

1. There is a right action of G on P such that the quotient space of P under this action is M, and the quotient $\pi : P \to P/G = M$ is smooth.
2. P is locally trivializable; i.e. every point of M has a neighbourhood U with an equivariant diffeomorphism $\pi^{-1}(U) \to U \times G$ covering the identity on M.

Remark 2.2. As a consequence the transition functions $f_{\alpha\beta}$ satisfy

i) $f_{\alpha\beta} : U_\alpha \cap U_\beta \to G$,

ii) $(f_\alpha \circ f_\beta^{-1})(x, g) = (x, f_{\alpha\beta}(x)g)$ for every $x \in U_\alpha \cap U_\beta,\ g \in G$,

iii) $f_{\alpha\alpha} = e$,

iv) $f_{\alpha\beta}(x)f_{\beta\gamma}(x) = f_{\alpha\gamma}(x)$ for every $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

The action of G on the tangent bundle is denoted by $vg, v \in TP$ and $g \in G$. On the other side, the infinitesimal action of an element X of the Lie algebra $\mathfrak{g}$ at $p \in P$ is given by

$$pX := \frac{d}{dt}\bigg|_{t=0} p\exp(tX),$$

which is an element of the tangential space of P at the point p. The set

$$V_p := \{pX \in T_pP \mid X \in \mathfrak{g}\}$$

is called the *vertical space at the point p*. Because G preserves the fibers and is transitive, the vertical space at p is equal to the kernel of $d\pi(p)$ where $d\pi(p) : T_pP \to T_{\pi(p)}M$, or we could view V_p as the tangential space $T_p(\pi^{-1}(x))$ where $x = \pi(p)$. The subbundle

$$V := \{(p, pX) \in TP \mid p \in P, X \in \mathfrak{g}\} \subset TP$$

is called the *vertical space* of the principal bundle P.

Definition 2.3. A **connection** is an equivariant function $A : TP \to \mathfrak{g}$, i.e.

(i) $A(p, pX) = X \quad \forall p \in P, X \in \mathfrak{g}$,

(ii) $A(pg, vg) = g^{-1}A_p(v)g \quad \forall p \in P, \forall v \in T_pP.$

Throughout the rest of this report, let $\mathcal{A}_P$ denote the set of connections on the principal bundle $P \to M$.

Locally, a connection $A \in \mathcal{A}(P)$ is a 1-form $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$, where, for an $X_p \in T_pP$

$$A_\alpha(x, d_p\pi(X_p)) = f_\alpha^{-1}(p)A(p, X_p)f_\alpha(p) - d_pf_\alpha(X_p)f_\alpha^{-1}(p),$$

and therefore on $U_\alpha \cap U_\beta$,

$$A_\beta = f_{\alpha\beta}^* A_\alpha = f_{\alpha\beta}^{-1}A_\alpha f_{\alpha\beta} + f_{\alpha\beta}^{-1}df_{\alpha\beta}. \tag{2.1}$$

A connection can be seen as a choice of an equivariant horizontal distribution $H \subset TP$ which corresponds to the kernel of A and at each point $p \in P$ induces the short exact sequence:

$$0 \longrightarrow H_p = \ker A(p,\cdot) \xrightarrow{\iota} T_pP \longrightarrow V_p \longrightarrow 0\ , \tag{2.2}$$

where the map $\iota : H_p \to T_pP$ is just the inclusion and $H_{pg} = H_pg$. Since $V_p = \ker d\pi(p)$ and $T_pP = H_p \oplus V_p$, $d\pi(p)$ induces an isomorphism between H_p and $T_{\pi(p)}M$, hence the horizontal distribution is isomorphic to the pullback π^*TM and this implies that a vector field X on M has a unique horizontal lift $\tilde{X}$ such that $\tilde{X}(p) \in H_p$ and $d_p\pi(\tilde{X}_p) = X(\pi(p))$.

Definition 2.4. A Lie group defines on itself a **conjugation**

$$\begin{aligned} c : G &\to \mathrm{Aut}(G) \\ g &\mapsto c_g \end{aligned}$$

such that $c_g(h) = g^{-1}hg$ for all $h \in G$. The derivative at the identity acts on the Lie algebra $\mathfrak{g}$ and it is called the **adjoint representation**, i.e. $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$ and, for an element $X \in \mathfrak{g}$ we have

$$\mathrm{Ad}_g(X) = \left.\frac{d}{dt}\right|_{t=0} c_g(h(t)),$$

where $h(t)$ is a curve in the Lie group G such that $h(0) = e$ and $\frac{d}{dt}\big|_{t=0}h(t) = X$. We can choose the exponential map $\exp(tX)$ as $h(t)$ and we write $\mathrm{Ad}_g(X) = g^{-1}Xg$, where multiplication between an element g of G and an element X of $\mathfrak{g}$ is defined as the derivative at the identity of the left translation by g in the direction of X

$$gX = \left.\frac{d}{dt}\right|_{t=0} g\exp(tX) \in T_gG,$$

and the multiplication between an element of the Lie algebra $\mathfrak{g}$ and one of the group G using right translation is

$$Xg = \left.\frac{d}{dt}\right|_{t=0} \exp(tX)g \in T_gG.$$

Definition 2.5. Let $\pi : P \to M$ be a principal G-bundle, let N be a manifold and let $f : G \to \mathrm{Diffeo}(N)$ where again $\mathrm{Diffeo}(N)$ denote the diffeomorphisms on N. The **associated bundle** $P \times_f N$ is the locally trivial G-bundle with fibre N, consisting of equivalence classes $[pg, n] \equiv [p, f(g)(n)]$ and projection $\pi_1 : P \times_f N \to M$ given by $\pi_1([p, n]) = \pi(p)$.

Remark 2.6. Notice that if N is a vector space, then the associated bundle, $P \times_f N$ is a vector bundle. If for example $N = \mathfrak{g}$ then the associated bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ is denoted $\mathfrak{g}_P$ and we call this bundle the **adjoint bundle**, here we have

$$[pg, X] \equiv [p, \mathrm{Ad}_g(X)] = [p, g^{-1}Xg].$$

The set of all equivariant horizontal functions $\alpha : TP \to \mathfrak{g}$, i.e. smooth functions α where $V \subset \ker \alpha$, is denoted by $\Omega^1_{\mathrm{Ad},H}(P, \mathfrak{g})$ and we let $\Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g})$ denote the space of horizontal equivariant k-forms.

An $\omega \in \Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g})$ satisfies the conditions

$$\begin{aligned} \omega(pg; v_1 g, v_2 g, \ldots, v_k g) &= g^{-1}\omega(p; v_1, v_2, \ldots, v_k)g, \\ \omega(p, v_1, \ldots, v_k) &= 0 \quad \text{if} \quad v_i = pX \quad \text{for an } i \in \{1, \ldots, k\}. \end{aligned}$$

where $p \in P$, $g \in G$, $X \in \mathfrak{g}$, $v_i \in T_pP$. In other words the horizontal and equivariant k-forms $\Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g})$ correspond to the k-forms over M with values in the adjoint bundle $\mathfrak{g}$. In other words $\Omega^k_{\mathrm{Ad},H}(P, \mathfrak{g}) = \Omega^k(M, \mathfrak{g}_P)$.

We now fix a connection A_0. Then for every $\alpha \in \Omega^1_{\mathrm{Ad},H}(P, \mathfrak{g})$, $A_0 + \alpha$ is again a connection. In fact, $\forall p \in P, \forall X \in \mathfrak{g}, \forall v \in T_pP$ we have

$$A_0(p, pX) + \alpha(p, pX) = X + 0 = X, \tag{2.3}$$

and

$$A_0(pg, vg) + \alpha(pg, vg) = g^{-1}A_0(p, v)g + g^{-1}\alpha(p, v)g = g^{-1}(A_0(p, v) + \alpha(p, v))g. \tag{2.4}$$

Conversely we have that the difference between two connections is an element of $\Omega^1_{\mathrm{Ad},H}(P, \mathfrak{g})$. It follows that the space $\mathcal{A}(P)$ is an affine space and we can write

$$\mathcal{A}(P) = A_0 + \Omega^1_{\mathrm{Ad},H}(P, \mathfrak{g}) = A_0 + \Omega^1(M, P \times_{\mathrm{Ad}} \mathfrak{g}). \tag{2.5}$$

Definition 2.7. The Lie group $\mathcal{G}_P$ of equivariant smooth maps $u : P \to G$ is called the **gauge group** of P, i.e.

$$\mathcal{G}_P = \{u \in C^\infty(P, G) \mid u(pg) = g^{-1}u(p)g, \ \forall p \in P, \ \forall g \in G\}.$$

Because G acts on P, every element of the gauge group induces a **gauge transformation** of the bundle P, i.e.

$$\begin{aligned} \tilde{u} : P &\to P \\ p &\mapsto pu(p). \end{aligned}$$

The gauge transformation is a G**-bundle isomorphism**. Conversely, a G-bundle isomorphism comes from a gauge transformation since G acts freely. The gauge group $\mathcal{G}_P$ is isomorphic to the group of sections of the associated bundle $P \times_c G$, where we have the equivalence $[p, g] \equiv [pq, q^{-1}gq]$ for every $p \in P$ and $g, q \in G$. For an element of the gauge group, say, $u \in \mathcal{G}_P$ the section is defined as the map $M \to P \times_c G$ given by $\pi(p) \mapsto [p, u(p)]$. Conversely a section $\overline{u}$ which takes $\pi(p) \mapsto [p, \overline{u}(p)]$ induces a gauge transformation $\tilde{u}(p) = p\overline{u}(p)$. This implies that $\mathcal{G}_P = C^\infty(M, P \times_c G) = \Omega^0(M, P \times_c G)$ and therefore the Lie algebra of $\mathcal{G}_P$ is the space of equivariant, horizontal 0-forms over P;

$$T_{\mathrm{id}\,\mathcal{G}}\mathcal{G}_P = \Omega^0(M, \mathfrak{g}_P)$$

where $\mathrm{id}\,\mathcal{G} : M \to G; x \mapsto e$ is the identity of $\mathcal{G}_P$.

An element u of the gauge group $\mathcal{G}_P$ acts on a connection $A \in \mathcal{A}(P)$ in the following way: Let $X_p \in T_pP$, then, because an element of the gauge group acts as the pullback of its gauge transformation and the connection A is linear we get

$$\begin{aligned}
u^*A(p,X_p) :=& \tilde{u}^*A(p,X_p) = A(\tilde{u}(p), d_p\tilde{u}(X_p)) \\
=& A(pu(p), d_p(pu(p))(X_p)) = A(pu(p), X_pu(p) + pd_pu(X_p)) \\
=& A(pu(p), X_pu(p)) + A(pu(p), pd_pu(X_p)) \\
=& u(p)^{-1}A(p,X_p)u(p) + A(pu(p), (pu(p))u(p)^{-1}d_pu(X_p)) \\
=& u(p)^{-1}A(p,X_p)u(p) + u(p)^{-1}d_pu(X_p),
\end{aligned}$$

This means that

$$u^*A = u^{-1}Au + u^{-1}du,$$

and we can consider u as a change of trivialization. To compute the infinitesimal gauge transformation on a connection A, choose an element ϕ of the Lie algebra $\Omega^0(M, \mathfrak{g}_P)$ and set $u_t = \exp(t\phi) = 1 + t\phi + O(t^2)$, then

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0}(u_t^*A) &= -\left.\frac{d}{dt}\right|_{t=0}(u_t^{-1}Au_t + u_t^{-1}du_t) \\
&= -[A,\phi] - d\phi.
\end{aligned}$$

Choosing a connection $A \in \mathcal{A}(P)$ lets us define the **covariant derivative**

$$\begin{aligned}
d_A : \Omega^0(M, \mathfrak{g}_P) &\to \Omega^1(M, \mathfrak{g}_P) \\
\phi &\mapsto d_A\phi = d\phi + [A, \phi]
\end{aligned}$$

and the **exterior derivative**

$$\begin{aligned}
d_A : \Omega^k(M, \mathfrak{g}_P) &\to \Omega^{k+1}(M, \mathfrak{g}_P), \\
\omega &\mapsto d_A\omega = d\omega + [A \wedge \omega],
\end{aligned}$$

where $[\omega_1 \wedge \omega_2] := \omega_1 \wedge \omega_2 - (-1)^{lk}\omega_2 \wedge \omega_1$ denotes the Lie bracket operator for $\omega_1 \in \Omega^l(M, \mathfrak{g}_P)$ and $\omega_2 \in \Omega^k(M, \mathfrak{g}_P)$. Locally $(d_A\omega)_\alpha = d\omega_\alpha + [A_\alpha \wedge \omega_\alpha]$.

2.2 Holonomy

Let $A \in \mathcal{A}(P)$ and let $\gamma : [0,1] \to M$ be a C^1 curve on the base manifold. Then γ lifts to a unique horizontal curve $\Phi_A(\gamma, p) : [0,1] \to P$ for each $p \in \pi^{-1}(\gamma(0))$, i.e.

(i) $\Phi_A(\gamma,p)(0) = p$,

(ii) $\pi(\Phi_A(\gamma,p)(t)) = \gamma(t) \quad \forall t \in [0,1]$,

(iii) $\frac{d}{dt}\Phi_A(\gamma,p)(t) \in H_{\Phi_A(\gamma,p)(t)}$ and $d_{\Phi_A(\gamma,p)(t)}\pi(\frac{d}{dt}\Phi_A(\gamma,p)(t)) = \frac{d}{dt}\gamma(t)$ for every $t \in [0,1]$.

Choosing γ as a loop in M we have that $\pi(\gamma(0)) = \pi(\gamma(1))$ and we see that γ induces a homomorphism $\Psi_A(\gamma)$ on the fiber $\pi^{-1}(\gamma(0))$ as follows

$$\begin{aligned}
\Psi_A(\gamma) : \pi^{-1}(\gamma(0)) &\to \pi^{-1}(\gamma(0)), \\
p &\mapsto \Psi_A(\gamma)(p) = \Phi_A(\gamma,p)(1).
\end{aligned}$$

G acts freely on the fibers, so if we choose a point $p \in \pi^{-1}(x)$, then for each $\Psi_A(\gamma)$ there exists an element $g_p(\Psi_A(\gamma)) \in G$ such that $\Psi_A(\gamma)(p) = g_p(\Psi_A(\gamma))p$ and

$$\mathrm{Hol}_p(A) := \{g_p(\Psi_A(\gamma)) \in G \mid \gamma \in C^1([0,1], M), \gamma(0) = \gamma(1) = \pi(p)\} \tag{2.6}$$

is called the **holonomy group** of A in p. The holonomy group is a subgroup of G since

(i) $e = g_p(\Psi_A(\gamma))$ when we choose γ as the constant loop.

(ii) For every $g \in \mathrm{Hol}_p(A)$ if there exists $\Phi_A(\gamma_1, p)$ joining p and gp then $g^{-1}\Phi_A(\gamma_1, p)$ horizontally joins p and $g^{-1}p$ and therefore $g^{-1} \in \mathrm{Hol}_p(A)$.

(iii) For every $h \in \mathrm{Hol}_p(A)$, $\Phi_A(\gamma_2, p)$ going from p to hp, $g\Phi_A(\gamma_2, p)$ goes from gp to ghp and therefore $g\Phi_A(\gamma_2, p) \circ \Phi_A(\gamma_1, p)$ goes from p to ghp and hence $gh \in \mathrm{Hol}_p(A)$.

If we now let $p, q \in P$ such that there is a C^1 horizontal curve β connecting them. If $g \in \mathrm{Hol}_p(A)$ and $\Phi_A(\gamma_1, p)$ goes from p to gp, then $g\beta \circ \Phi_A(\gamma_1, p) \circ \beta^{-1}$ goes from q to gq. We conclude that p and q have the same holonomy group. Moreover $\mathrm{Hol}_{gp}(A) = g\,\mathrm{Hol}_p(A)g^{-1}$ and we can consider the holonomy group as an equivalence class of subgroups of G defined using the conjugation in G. Hence we can simply write $\mathrm{Hol}(A)$.

We define the subgroup $\mathrm{Hol}^0_p(A)$ of $\mathrm{Hol}_p(A)$ in the following way:

$$\mathrm{Hol}^0_p(A) := \{g_p(\Psi_A(\gamma)) \in \mathrm{Hol}_p(A) \mid \gamma \in C^1([0,1], M),\ \gamma(0) = \gamma(1) = p, \gamma \text{ is null-homotopic}\}.$$

Remark 2.8. Like $\mathrm{Hol}_p(A)$ the subgroup $\mathrm{Hol}^0_p(A)$ does not depend on p and we can write $\mathrm{Hol}^0(A)$.

If we choose two 0-homotopic loops $\gamma_0, \gamma_1 \subset M$, such that $\gamma_0(0) = \gamma_0(1) = \gamma_1(0) = \gamma_1(1)$, there is a continuous homotopy $h : [0,1] \times [0,1] \to M$, $h_0(s) = \gamma_0(s)$, $h_1(s) = \gamma_1(s)$. Thus, $g_p(\Psi_A(h_s))$ is a curve in $\mathrm{Hol}^0(A)$ from $g_p(\Psi_A(\gamma_0))$ to $g_p(\Psi_A(\gamma_1))$ and hence $\mathrm{Hol}^0_p(A)$ is connected and every connected subgroup of a Lie group is itself a Lie group. Next let $\gamma_0 \subset M$ be a 0-homotopic loop and γ_2 be a generic loop in M such that $\gamma_0(0) = \gamma_0(1) = \gamma_2(0) = \gamma_2(1)$, then $(\gamma_2)^{-1} \circ \gamma_0 \circ \gamma_2$ is 0-homotopic too. Therefore $\mathrm{Hol}^0_p(A)$ is normal in $\mathrm{Hol}_p(A)$, and we have

Lemma 2.9. If M is connected, $p \in P$, then $\mathrm{Hol}^0_p(A)$ is a connected Lie subgroup of G and it is a normal subgroup of $\mathrm{Hol}_p(A)$.

Further we have

Lemma 2.10. There is a surjective group homomorphism

$$\theta : \pi_1(M) \to \mathrm{Hol}(A)/\mathrm{Hol}^0(A). \tag{2.7}$$

Proof. We work with $\mathrm{Hol}_p(A)$ and $\mathrm{Hol}^0_p(A)$. Let

$$[\gamma] \mapsto g_p(\Psi_A(\gamma)).\,\mathrm{Hol}^0_p(A)$$

where γ is a loop in M, $\gamma(0) = \gamma(1) = \pi(p)$ and $[\gamma]$ is the equivalence class in $\pi_1(M)$. θ is surjective because of the definition of $\mathrm{Hol}_p(A)$ and for every two loops $\gamma_1, \gamma_2 \subset M$ with $[\gamma_1] = [\gamma_2]$, $\gamma := \gamma_2 \circ (-\gamma_1)$ is 0-homotopic and hence $g_p(\Psi_A(\gamma)) \in \mathrm{Hol}^0_p(A)$. □

If we now choose a point $p \in P$. Then

$$P(p) := \{\Phi_A(\gamma, p)(1) \in P \mid \gamma \in C^1([0,1], M), \gamma(0) = \pi(p)\}$$

is a submanifold of P and $\pi|_{P(p)} : P(p) \to M$ is a principal $\mathrm{Hol}_p(A)$-bundle with the connection $A|_{P(p)}$ because its horizontal distribution is equal to the restriction on $P(p)$ of the horizontal distribution of P with respect to A and is therefore well-defined. We have

Theorem 2.11. A principal G-bundle $\pi : P \to M$ with connection A is equivalent to $\pi|_{P(p)} \to M$ with connection $A|_{P(p)}$ for any $p \in P$.

2.3 Inner product on $\Omega^*(M, \mathfrak{g}_P)$

We now wish to construct an inner product on $\Omega^*(M, \mathfrak{g}_P)$. We first stress the fact, that on every Lie algebra $\mathfrak{g}$ of a compact Lie group G there exists an inner product which is invariant under the adjoint action of the group.

$$\langle \mathrm{Ad}_g \xi, \mathrm{Ad}_g \nu \rangle_{\mathfrak{g}} = \langle \xi, \nu \rangle_{\mathfrak{g}} \quad \forall \xi \in \mathfrak{g}, \forall g \in G.$$

This inner product on the Lie algebra $\mathfrak{g}$ can easily be constructed in terms of the Killing form.

Now an inner product on the Lie algebra induces a well-defined inner product on the fiber $\pi^{-1}(x) \times_{\mathrm{Ad}} \mathfrak{g} \subset \mathfrak{g}_P$ for every $x \in M$, namely

$$\langle [p,\xi], [p,\nu] \rangle_{\mathfrak{g}_P} = \langle \xi, \nu \rangle_{\mathfrak{g}},$$

$g \in G, p \in \pi^{-1}(x)$ and $\xi, \nu \in \mathfrak{g}$. It is easy to see that the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}_P}$ on $\mathfrak{g}_P$ does not depend on the choice of $p \in \pi^{-1}(x)$, since, for any $g \in G$ we have

$$\langle [p,\xi], [p,\nu] \rangle_{\mathfrak{g}_P} = \langle \xi, \nu \rangle_{\mathfrak{g}} = \langle \mathrm{Ad}_g \xi, \mathrm{Ad}_g \nu \rangle_{\mathfrak{g}} = \langle [p, \mathrm{Ad}_g \xi], [p, \mathrm{Ad}_g \nu] \rangle_{\mathfrak{g}_P} = \langle [gp,\xi], [gp,\nu] \rangle_{\mathfrak{g}_P}.$$

This is true since the inner product on $\mathfrak{g}$ was invariant under the adjoint action.

Let us recall the definition of the **Hodge operator** $*$.

Definition 2.12. Let (M, g) be an n-dimensional, oriented, pseudo-Riemannian manifold. Let $\mathrm{dvol_M} \in \Omega^n(M)$ be the volume form on M corresponding to g and let $\omega \in \Omega^k(M)$. In the local set $U_\alpha \subset M$ we can choose orthogonal coordinates $(e^1, \dots, e^n)$ and using Einstein sum convention we can write

$$\begin{aligned} \omega_\alpha &= \frac{1}{k!} \omega_{j_1 \dots j_k} de^{j_1} \wedge \dots \wedge de^{j_k}, \\ (\mathrm{dvol_M})_\alpha &= \frac{1}{n!} v_{i_1 \dots i_n} de^{i_1} \wedge \dots \wedge de^{i_n}. \end{aligned}$$

For all $k \in \{1, \dots n\}$ we define the Hodge operator $*$ to be the map

$$\begin{aligned} * : \Omega^k(M) &\to \Omega^{n-k}(M) \\ \omega &\mapsto *\omega, \end{aligned}$$

where in the local set U_α we have

$$(*\omega)_\alpha = \frac{1}{k!} v_{i_1 \dots i_n} g^{i_1 j_1} \dots g^{i_k j_k} \omega_{j_1 \dots j_k} de^{i_{k+1}} \wedge \dots \wedge de^{i_n}. \tag{2.8}$$

The Riemannian metric g defines an inner product on $\Omega^k(M)$. Let $\omega, \theta \in \Omega^k(M)$. Then the inner product on the space of k-forms is given by

$$\langle \omega_\alpha, \nu_\alpha \rangle_{TM} = \frac{1}{k!} \omega_{i_1 \dots i_k} g^{i_1 j_1} \dots g^{i_k j_k} \nu_{j_1 \dots j_k} \tag{2.9}$$

and therefore we get the equality $\omega \wedge *\nu = \nu \wedge *\omega = \langle \omega, \nu \rangle_{TM} \mathrm{dvol_M}$.

The Hodge star defines a dual in the sense that when it is applied twice, the result is an identity on the exterior algebra, up to sign,

$$*(*\omega) = (-1)^{k(n-k)} \operatorname{sign}(g) \omega,$$

where $\operatorname{sign}(g)$ is the signature of the metric. And we can write the adjoint operator of the exterior derivative as follows,

$$d_A^* \theta = (-1)^{(n-k+1)(k+1)} * d_A * \theta.$$

The Hodge operator acts on k-forms with values in the Lie algebra $\mathfrak{g}_P$ too. Recall that $\Omega^k(M, \mathfrak{g}_P) = \Gamma(^kT^*M \otimes \mathfrak{g}_P)$, where the latter denotes the sections of $\Lambda^kT^*M \otimes \mathfrak{g}_P \to M$. Then $\forall \omega \in \Omega^k(M), \forall \xi \in \Omega^0(M, \mathfrak{g}_P)$, we define

$$*(\omega \otimes \xi) := *\omega \otimes \xi.$$

Now we can finally use the two inner products mentioned above to construct an inner product on the k-forms $\Omega^k(M, \mathfrak{g}_P)$,

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha \wedge *\beta \rangle \quad \forall \alpha, \beta \in \Omega^k(M, \mathfrak{g}). \tag{2.10}$$

Because $T_A\mathcal{A}(P) = \Omega^1(M, \mathfrak{g}_P)$ then for every connection $A \in (P)$ the space of connections in the principal bundle $\mathcal{A}(P)$ is a symplectic manifold with symplectic form:

$$\omega_A(\alpha, \beta) = \int_M \langle \alpha \wedge \beta \rangle \quad \forall \alpha, \beta \in \Omega^1(M, \mathfrak{g}_k). \tag{2.11}$$

For two vector fields X, Y on M we have

$$\langle \alpha \wedge \beta \rangle (X, Y) = \langle \alpha(X), \beta(Y) \rangle - \langle \alpha(Y), \beta(X) \rangle .$$

Because the symplectic two form does not depend on the base connection A, it is constant and therefore closed.

The inner product satisfies the condition $\langle d_A\omega, \theta \rangle = \langle \omega, d_A^*\theta \rangle$ for every $\omega \in \Omega^k(M, \mathfrak{g}_P)$ and every $\theta \in \Omega^{k+1}(M, \mathfrak{g}_P)$.

2.4 Curvature

Let $A \in \mathcal{A}(P)$, the two form $F_A := d_A + \frac{1}{2}[A \wedge A] \in \Omega^2(M, \mathfrak{g}_P)$ is called the **curvature** of the connection A.

Let us here write down a couple of properties for the curvature F_A.

Proposition 2.13. For the curvature F_A we have the following

(i) $F_A \wedge \omega = d_A d_A \omega$.

(ii) The curvature can geometrically be seen as an obstruction to the integrability of the horizontal sub-bundle of TP.

Proof. (i) Let $\omega \in \Omega^k(M, \mathfrak{g}_P)$ then because $d[A \wedge \omega] = [dA \wedge \omega] - [A \wedge d\omega]$ we have

$$d_A d_A \omega = d^2\omega + d[A \wedge \omega] + [A \wedge d\omega] + [A \wedge [A \wedge \omega]] = [F_A \wedge \omega].$$

For (ii) we let $p \in P$ and $X_p, Y_p \in H_p$. We then have

$$F_A(p, X_p, Y_p) = d_A(p, X_p, Y_p) + \frac{1}{2}[A(p, X_p) \wedge A(p, Y_p)] = d_A(p, X_p, Y_p) = d_A(p, [X_p, Y_p]).$$

Therefore $[X_p, Y_p] \in H_p$ if and only if $F_A|_p = 0$. $\square$

A consequence of (ii) is that $F_A(p, X_p, V_p) = 0$ if either X_p or Y_p is in V_p.

With the definition of the curvature in hand we are in a position to define the space of flat connections:

$$\mathcal{F}_P = \{A \in \mathcal{A}(P) \mid F_A = 0\}.$$

For a connection $A \in \mathcal{F}_P$, since $d_A \circ d_A = 0$ the cohomology groups are well defined.

$$H^k_A(M, \mathfrak{g}_P) = \frac{\ker d_A}{\operatorname{im} d_A}\bigg|_{\Omega^k(M,\mathfrak{g}_P)} = \ker d_A \cap \ker d_A^*\bigg|_{\Omega^k(M,\mathfrak{g}_P)}, \quad \forall k \in \mathbb{N}.$$

It is not hard to see that the space of k-forms with values in the Lie algebra $\mathfrak{g}_P$ has the decomposition

$$\Omega^k(M, \mathfrak{g}_P) = d_A\Omega^{k-1}(M, \mathfrak{g}_P) \oplus H^k_A(M, \mathfrak{g}_P) \oplus d_A^*\Omega^{k+1}(M, \mathfrak{g}_P)$$

Indeed if we let $\alpha \in \Omega^{k-1}(M, \mathfrak{g}_P), \beta \in H^k_A(M, \mathfrak{g}_P)$ and $\gamma \in d_A^*\Omega^{k+1}(M, \mathfrak{g}_P)$, we have;

$$\begin{aligned}
\langle d_A\alpha, d_A^*\gamma\rangle &= \langle d_A d_A\alpha, \gamma\rangle = 0, && \text{since } d_A d_A = 0,\\
\langle d_A\alpha, \beta\rangle &= \langle \alpha, d_A^*\beta\rangle = 0, && \text{since } \beta \in \ker d_A^*,\\
\langle d_A^*\gamma, \beta\rangle &= \langle \gamma, d_A\beta\rangle, && \text{since } \beta \in \ker d_A.
\end{aligned}$$

Lemma 2.14. Let $A_0 \in \mathcal{F}_P$. Then $T_{A_0}\mathcal{F}_P = \ker d_{A_0}$.

Proof. Let $A_t = A_0 + \sum_{j=1}^{\infty} t^i\alpha_i$ be a curve in $\mathcal{F}_P$ with $\alpha_1 = \frac{d}{dt}\Big|_{t=0} A_t \in T_{A_0}\mathcal{F}_P$ for $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. Since the curvature $0 = F_{A_0} = F_{A_t}$ we have that

$$0 = \frac{d}{dt}\bigg|_{t=0} F_{A_t} = \frac{d}{dt}\bigg|_{t=0} \left(F_{A_0} + t d_{A_0}\alpha_1 + \mathcal{O}(t^2)\right) = d_{A_0}\alpha_1,$$

and hence, $\alpha_1 \in \ker d_{A_0}$. □

2.5 The moduli space of flat connections

Definition 2.15. A principal bundle homomorphism between two principal G-bundles P and P' is a G-equivariant bundle homomorphism. If $P = P'$ it is called a *gauge transformation* of the bundle. Denote by $\mathcal{G}_P$ the group of all gauge transformations $P \to P$.

Remark 2.16. To every G-equivariant map $u : P \to G$, $p \mapsto u_p$, we associate a gauge transformation $\Phi : P \to P$ by letting $\Phi(p) = p \cdot u_p$. Here, $g \in G$ acts on itself on the right by $h \mapsto g^{-1}hg$. This association is a bijection.

The group $\mathcal{G}_P$ acts on $\mathcal{A}_P$ via pullback, and the action preserves $\mathcal{F}_P$. For a G-equivariant map $u : P \to G$, we write this action $A \mapsto A \cdot u$.

The space which we are interested in quantizing is the moduli space of flat connections in a trivializable principal G-bundle.

Definition 2.17. The moduli space of flat connections on a trivializable principal G-bundle $P \to M$ is the space $\mathcal{M}_{\text{Flat}}(G, M) = \mathcal{F}_P/\mathcal{G}_P$.

This space can be given a set theoretical description using the holonomy map. The proofs of the following results can be found in [22].

Proposition 2.18. Let A be a flat connection in P and assume that M is connected. Let $x_0 \in M$, let $p_0 \in \pi^{-1}(x_0)$ and let γ be a loop in M with base point x_0. Up to conjugation in G, the association $A \mapsto g_p(\Psi_A(\gamma))$ is independent of the base point x_0, the choice of lift p_0, the gauge transformation class of the connection A and the homotopy class of γ. In short, we have a well defined map

$$\text{hol} : \mathcal{M}_{\text{Flat}}(G, M) \to \operatorname{Hom}(\pi_1(M), G)/G,$$

where G acts on $\operatorname{Hom}(\pi_1(M), G)$ on the right, $(\rho \cdot g)(\gamma) = g^{-1}\rho(\gamma)g$.

Definition 2.19. A *flat principal G-bundle* on a manifold M is a pair (P, A) consisting of a principal G-bundle $P \to M$ and a flat connection A in P. Two flat principal G-bundles (P, A) and (P', A') are called isomorphic if there is a principal bundle homomorphism $\Phi : P \to P'$ such that $A = \Phi^*(A')$. The set $\mathcal{M}^G$ of isomorphism classes is called the *moduli space of flat principal G-bundles on M.*

Theorem 2.20. The map $\mathcal{M}_{\text{Flat}}(G, M) \to \mathrm{Hom}(\pi_1(M), G)/G$ taking $[(P, A)]$ to $[\mathrm{hol}_A]$ is a bijection.

2.6 Chern–Simons Theory

Three-dimensional Chern–Simons gauge theory is an example of what we later in this thesis will view as a topological quantum field theory (TQFT). Chern–Simons theory with a compact gauge group G is a well-known and studied subject with a history going back to the 1980's. We will here review this theory following Freed [15]. Then we will briefly discuss the case, where G is no longer compact. I.e. we will discuss what happens when G is replaced by its complexification $G_{\mathbb{C}}$ the Lie algebra $\mathfrak{g}$ is replaced by $\mathfrak{g}_{\mathbb{C}}$.

For now, we will assume that G is a simple, connected, simply connected and compact Lie group. It is a well known fact that any principal G-bundle P over M where $\dim M \leq 3$ is trivializable. Let M be a compact and oriented 3-manifold with boundary Σ. Let $P \to M$ be a principal G-bundle. Trivializing $P \cong M \times G$ by using the trivialization $p \to (\pi(p), g_p)$ is equivalent to choosing a section $s : M \to P$ through the identification $p \cdot g_p = s(\pi(p))$. Using a section like this, the pull-back of a connection determines an identification between the space of connections and one forms on M with values in the Lie algebra, i.e. $\mathcal{A}_P \cong \Omega^1(M, \mathfrak{g})$ and further we can identify $\mathcal{G}_P \cong C^\infty(M, G)$.

Definition 2.21. For a connection $A \in \mathcal{A}_P$ with curvature $F_A \in \Omega^2(M, \mathfrak{g}_P)$ we define the *Chern–Simons form* $\alpha(A) \in \Omega^3(P)$ by

$$\alpha(A) = \langle A \wedge F_A \rangle - \frac{1}{6} \langle A \wedge [A \wedge A] \rangle \tag{2.12}$$

Definition 2.22. The *Chern–Simons action* of *Chern–Simons functional* of A for a trivialization $s : M \to P$ of the bundle $P \to M$ is given by

$$\mathrm{CS}_s(A) = \int_M s^*(\alpha(A)) \in \mathbb{R}.$$

Let us see how the Chern–Simons functional behaves under gauge transformation (see [15, Prop. 2.10])

Proposition 2.23. Let $\theta \in \Omega^1(G; \mathfrak{g})$ be the Mauer–Cartan form i.e. $\theta(v) = (dl_{g^{-1}})v \in \mathfrak{g}$ for $v \in T_g G$. Let $\Psi : P \to P$ be a gauge transformation with associated map $u : P \to G$, and let $\theta_u = (u \circ s)^*\theta$. Then for a connection $A \in \Omega^1(M, \mathfrak{g})$ we have

$$\begin{aligned} \mathrm{CS}_{\Psi \circ s}(A) &= \mathrm{CS}_s(\Psi^* A) \\ &= \mathrm{CS}_s(A) + \int_{\partial M} \left\langle \mathrm{Ad}_{(u \circ s)^{-1}} A \wedge \theta_u \right\rangle - \frac{1}{6} \int_M \langle \theta_u \wedge [\theta_u \wedge \theta_u] \rangle . \end{aligned}$$

Assume know that $\langle \cdot, \cdot \rangle$ is normalized such that $-\frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle$ represents an integral class in $H^3(G, \mathbb{R})$, then the last integral in Proposition 2.23 is an integer.

Definition 2.24. In the case where M is a closed 3-manifold we obtain the Chern–Simons action

$$\mathrm{CS}_M : \mathcal{A}_P / \mathcal{G}_P \to \mathbb{R}/\mathbb{Z}. \tag{2.13}$$

Here we have forgotten the subscript s since any two sections are related by a gauge transformation and by Proposition 2.23 this function is independent of s. Instead we put on the subscript M to remind the reader that the Chern Simons action depends on the manifold.

It turns out, that the Chern–Simons action can be written in the form

$$\mathrm{CS}_M(A) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A), \tag{2.14}$$

where $A \in \Omega^1(M, \mathfrak{g})$, (If $G = \mathrm{SU}(N)$ then Tr should denote the trace in the N-dimensional representation).

2.6.1 The Chern–Simons line bundle

Let us now discuss the case where M is a 3-manifold with boundary $\partial M = \Sigma$. Let $Q = P|_\Sigma$.

Lemma 2.25. For any gauge transformation $g \in C^\infty(M, G)$ the functional

$$W_\Sigma(g) = \int_M -\frac{1}{6}\langle g^*\theta \wedge [g^*\theta \wedge g^*\theta]\rangle \pmod 1 \tag{2.15}$$

only depends on the restriction of g to Σ.

This is what is called the *Wess-Zumino-Witten functional*, and a proof can be found in [15, 2.12].

It is easily seen that any two sections of $P \to M$ are related via a gauge transformation, which implies that CS_s only depends on the restriction to the boundary. This can be used to to define a principal $\mathrm{U}(1)$-bundle over $\mathcal{A}_Q$ such that

$$e^{2\pi i\,\mathrm{CS}_M(A)} \in \mathcal{L}_Q. \tag{2.16}$$

which is essentially for defining a *Lagrangian field theory*.

Remark 2.26. As we have already mentioned, principal G-bundles $P \to M$ are trivializable when $\dim M \le 3$. As sections correspond to trivializations of P, we can suppress the reference to P in (2.16).

We consider the principal G-bundle $Q \to \Sigma$. We may think of a principal $\mathrm{U}(1)$-bundle $\mathcal{L}$ over $\mathcal{A}_Q$ as the (complex) line bundle $\mathcal{L}$ associated to the defining representation $\mathrm{U}(1) \hookrightarrow \mathbb{C}^* = \mathrm{GL}(\mathbb{C})$ over $\mathcal{A}_Q$ known as the *Chern–Simons line bundle*.

As the space of connections $\mathcal{A}_Q$ is contractible, $\mathcal{L}$ will be trivializable. We could therefore describe it using one single chart. However, we need the trivialization

$$\phi_s : \mathcal{L}_Q \to \mathcal{A}_Q \times \mathbb{C}$$

to depend in a non-trivial way on the section $s : \Sigma \to Q$ in the same way the Chern–Simons function behaves, so that Equation (2.16) is satisfied.

In order for $\mathcal{L}$ to be a line bundle, the transition functions $\phi_{ss'} := \phi_s\phi_{s'}^{-1}$ must then satisfy the cocycle condition:

$$\phi_{s_3s_2}\phi_{s_2s_1} = \phi_{s_3s_1}.$$

A section $s : \Sigma \to Q$ gives identifications $s^* : \mathcal{A}_Q \to \Omega^1(\Sigma, \mathfrak{g})$ and $g^s : \mathcal{G}_Q \to C^\infty(\Sigma, G)$ determined by $\Phi \circ s(x) = s(x) \cdot g^s(\Psi)(x)$. In view of the behaviour of the Chern-Simons function under a gauge transformation Ψ it turns out that what we want to have for $A \in \mathcal{A}_Q$ is:

$$\phi_{\Psi\circ s} = c_\Sigma(s^*A, g^s(\Psi))\phi_s \tag{2.17}$$

where

$$c_\Sigma(A,g) := \exp\left(2\pi i\left(\int_\Sigma \langle \mathrm{Ad}_{g^{-1}} A \wedge g^*\theta\rangle + W_\Sigma(g)\right)\right). \tag{2.18}$$

It can be shown that c_Σ satisfies the cocycle condition

$$c_\Sigma(g^*A,h)c_\Sigma(A,g) = c_\Sigma(A,gh).$$

If $u \mapsto A_u$ is a smooth family of connections varying over a smooth manifold U. Then the transition functions $u \mapsto c_\Sigma(s^*A_u, g^s(\Psi))$ are smooth, so that $\mathcal{L}$ is a smooth vector bundle over U. Further, we constructed it so that $A \mapsto \exp(2\pi i\,\mathrm{CS}_s(A))$ is a section of $\mathcal{L}$, which is the *Chern–Simons invariant for manifolds X with boundary.*

Remark 2.27. From here it can be shown that the Chern–Simons action is the action of a local Lagrangian field theory. See e.g. [22].

To summarise what we have seen so far, we get a line bundle on the moduli space, i.e., if for the manifold M with boundary $\partial M = \Sigma$ we have

$$\begin{tikzcd} r^*\mathcal{L} \arrow[d] & \mathcal{L} \arrow[d] \\ \mathcal{M}_{\text{Flat}}(G,M) \arrow[u, bend left, "e^{2\pi i\,\mathrm{CS}_M(A)}"] \arrow[r, "r"] & \mathcal{M}_{\text{Flat}}(G,\Sigma) \end{tikzcd}$$

2.6.2 Symplectic form on the moduli space

The main purpose of this section is to construct a symplectic structure on some subspace of the moduli space. This is done through a quotient construction. The technicalities in this construction are great since we are dealing with $\mathcal{A}_Q$ which is actually a infinite-dimensional manifold, modelled on the space of 1-forms on Σ with values in $\mathfrak{g}$. The technical detail are omitted and we will simply state that for any given connection $A \in \mathcal{A}_Q$, there is an identification $T_A\mathcal{A}_Q \cong \Omega^1(\Sigma,\mathfrak{g})$. Then there is a natural symplectic form ω on $\mathcal{A}_Q$, invariant under $\mathcal{G}_Q$, defined by

$$\omega(\alpha,\beta) = -\int_\Sigma \langle \alpha\wedge\beta\rangle \tag{2.19}$$

for $\alpha,\beta \in \Omega^1(\Sigma,\mathfrak{g})$. If we use the identification $T_{\text{id}}\mathcal{G}_Q \cong C^\infty(\Sigma,\mathfrak{g})$, a moment map $\mu : \mathcal{A} \to C^\infty(\Sigma,\mathfrak{g})^*$ for the action of $\mathcal{G}_Q$ on $\mathcal{A}_Q$ is given by

$$\mu_\xi(A) = 2\int_\Sigma \langle F_A \wedge \xi\rangle\,,$$

for $\xi \in C^\infty(\Sigma,\mathfrak{g})$, and $A \in \mathcal{A}_Q$ with curvature $F_A \in \Omega^2(\Sigma,\mathfrak{g})$. The key fact is now, that the Marsden–Weinstein quotient [1]

$$\mathcal{M}_{\text{Flat}}(G,\Sigma) = \mu^{-1}(\{0\}) /\!\!/ \mathcal{G}_Q$$

is exactly the moduli space $\mathcal{M}_{\text{Flat}}(G,\Sigma)$ of flat connections on Q up to gauge transformations.

If we now consider the subspace $\mathcal{A}_Q^* \subset \mathcal{A}_Q$ consisting of flat *irreducible* connections in Q, i.e. connections A such that ∇^A (the induced connection in Ad_P) is injective, and let $\mathcal{M}_{\text{Flat}}(G,\Sigma)^* = \mathcal{A}_Q^*/\mathcal{G}_Q$. This space can be shown to be an open subset of $\mathcal{M}_{\text{Flat}}(G,M)$ and therefore one obtains the structure of a symplectic manifold through the quotient construction.

[1] Actually the infinite-dimensional analogue of the Marsden–Weinstein quotient.

We let $\tilde{\mathcal{L}}_Q = \mathcal{A}_Q \times \mathbb{C}$ be the trivial bundle over $\mathcal{A}_Q$ and we lift the action of $\mathcal{G}_Q$ to $\tilde{\mathcal{G}}_Q$ using the function c_Σ defined above. Then there exists a connection B on $\tilde{\mathcal{L}}_Q$ given in a trivialization $s : \Sigma \to Q$ by

$$(B_s)_A(\zeta) = \int_\Sigma \langle A \wedge \zeta \rangle ,$$

$A \in \mathcal{A}_Q \cong \Omega^1(\Sigma, \mathfrak{g}), \zeta \in T_A\mathcal{A}_Q \cong \Omega^1(\Sigma, \mathfrak{g})$. This connection on $\tilde{\mathcal{L}}_Q$ satisfy what in the next chapter will define as *the pre-quantum condition*. (3.1). It turns out that the connection B is preserved by the lifted action of $\mathcal{G}_Q$ and induces a connection $\overline{B}$ on the line bundle $\mathcal{L} \to \mathcal{M}_{\text{Flat}}(G, \Sigma)^*$ defined to be all equivalence classes of elements of $\mathcal{A}_Q^* \times \mathbb{C}$ under the relation

$$(A, z) \sim (g^*A, c_\Sigma(A, g) \cdot z),$$

for all gauge transformations g in $\mathcal{G}_Q$. We recall that the function c_Σ is $\mathrm{U}(1)$ and therefore the line bundle $\mathcal{L}$ carries a hermitian structure, and the connection $\overline{B}$ is compatible with this structure. Thus we can summarise and we obtain the following:

Theorem 2.28. Let Σ be a closed surface and let $Q \to \Sigma$ be a principal G-bundle. Then the moduli space $\mathcal{M}_{\text{Flat}}(G, \Sigma)^*$ of irreducible flat connections is pre-quantizable.

2.6.3 Complex Chern–Simons

Let us now complexify the action. This means that the compact Lie group G is replaced by its complexifycation $G_\mathbb{C}$, the moduli space $\mathcal{M}_{\text{Flat}}(G, M)$ is replaced by the moduli space $\mathcal{M}_{\text{Flat}}(G_\mathbb{C}, M)$, and the Chern–Simons action CS_M by $\mathrm{CS}_{M,\mathbb{C}}$:

$$\mathrm{CS}_{M,\mathbb{C}} : \mathcal{A}_{P_\mathbb{C}}/\mathcal{G}_{P_\mathbb{C}} \to \mathbb{C}/\mathbb{Z}. \tag{2.20}$$

Here $P_\mathbb{C}$ denotes a principal $G_\mathbb{C}$-bundle $P_\mathbb{C} \to M$. Again the action can be written as

$$\mathrm{CS}_{M,\mathbb{C}}(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{2.21}$$

where now $A \in \Omega^1(M, \mathfrak{g}_\mathbb{C})$.

The Chern-Simons functional is intimately related to the volume of a hyperbolic manifold in the sense that

$$\mathrm{CS}_{M,\mathbb{C}}(A) = \mathrm{Vol}(M) + i\,\mathrm{CS}_M(A).$$

We will return to this subject in Chapter 6, where we will look at the quantization of the Chern–Simons theory.

Chapter 3

Geometric Quantization

3.1 Quantization

In this section we will discuss quantization as a mathematical concept. Quantization has its roots in the world of physics, but the physical motivation for the different approaches will not be discussed. We take a more axiomatic way of reasoning. The main references are [1, 18]. First we discuss general axioms for quantization. As these lead to contradictions we turn to geometric quantization. This will be our preferred method of quantizing symplectic manifolds.

3.1.1 Canonical Quantization

Quantization is concerned with the transition from a classical physical theory to a quantum mechanical theory. In other words we seek a quantum theory that in some certain limit yields back the classical theory we started with. In the classical mechanics we consider $\mathbb{R}^n$ and we have the phase space $T^*\mathbb{R}^n$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ where the q_i's are the position coordinates and p_i's are the momentum coordinates. The standard symplectic form in these coordinates is $\omega_{\text{std}} = \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j$, the observables are the smooth functions defined on $\mathbb{R}^n$. An important operation on the observables is the Poisson bracket given by

$$\{f, g\} = \sum_j \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j},$$

described in terms of the symplectic form as $\{f, g\} = \omega_{\text{std}}(X_f, X_g)$ where X_f is the Hamiltonian vector field defined by $-df = \omega(X_f, \cdot)$. It follows that

$$[X_f, X_g] = X_{\{f,g\}}.$$

Quantization of this system is a way to assign to a class of observables f a self-adjoint operator Q_f on $L^2(\mathbb{R}^n, dq)$. The assignment should satisfy the following properties:

(i) The map $f \mapsto Q_f$ is $\mathbb{R}$-linear.

(ii) $Q_1 = \mathrm{id}$.

(iii) The functional calculus for self-adjoint operators should yield $\phi(Q_f) = Q_{\phi \circ f}$ for $\phi : \mathbb{R} \to \mathbb{R}$, where defined.

(iv) The operators corresponding to the coordinate functions should satisfy

$$Q_{q_j}\psi = q_j\psi, \quad Q_{p_j}\psi = -i\hbar\frac{\partial \psi}{\partial q_j}.$$

(v) The commutator of two operators should be $[Q_f, Q_g] = -i\hbar Q_{\{f,g\}}$, which we call the canonical commutation relation, and which of course expresses the celebrated Heisenberg uncertainty principle.

The type of quantization described is what is called *canonical quantization*. Unfortunately it turns out that the axioms (i)-(v) are not quite consistent: (i)-(iv) make it possible for us to express Q_f for the function $f(\mathbf{q}, \mathbf{p}) = q_1^2 p_1^2 = (q_1 p_1)^2$ in two different ways. See [1]. There are ways to handle this inconvenient fact. One, which we will use, is to keep the quantization axioms but quantize only few observables. This will lead us to *geometric quantization*. Another approach is based on the principle that the commutation relation should hold asymptotically as $\hbar$ goes to zero and therefore be replaced by

$$[Q_f, Q_g] = -i\hbar Q_{\{f,g\}} + \mathcal{O}(\hbar^2) \quad \text{as} \quad \hbar \to 0.$$

This procedure would lead to *deformation quantization* which we will not go deeper into.

3.1.2 Geometric Quantization

In geometric quantization, we wish to quantize a symplectic manifold (M, ω) which is usually called the phase spase by assigning a separable Hilbert space $\mathcal{H}$ and a linear map $Q : f \mapsto Q_f$ from a subspace $\mathcal{F}$ of real valued functions on M which is a Lie algebra under the Poisson bracket, into self-adjoint operators on a dense subset $D \subset \mathcal{H}$ satisfying the following axioms:

(a) The assignment $f \to Q_f$ is $\mathbb{R}$-linear.

(b) $Q_1 = \mathrm{id}$, where 1 is the constant function and id is the identity operator on $\mathcal{H}$.

(c) $[Q_f, Q_g] = -i\hbar Q_{\{f,g\}}$, for $f, g \in \mathcal{F}$.

(d) If given two symplectic manifolds (M, ω) and $(\tilde{M}, \tilde{\omega})$ and a symplectomorphism between those $\phi : (M, \omega) \to (\tilde{M}, \tilde{\omega})$, then for $f \in \tilde{\mathcal{F}}$ we require that $Q_{f\circ\phi}$ and $\tilde{Q}_f$ are conjugate by a unitary operator from $\mathcal{H}$ to $\tilde{\mathcal{H}}$.

(e) For $M = \mathbb{R}^{2n}$ with the standard symplectic form, we recover the operators Q_{q_j} and Q_{p_j} from the canonical quantization.

Let us describe the construction according to Kostant and Souriau. We start by constructing a pre-quantization by ignoring that we should recover the Schrödinger representation when $(M, \omega) = (\mathbb{R}^{2n}, \omega_{\text{std}})$. We follow Woodhouse [54].

3.1.3 Pre-quantization

Let $\mathcal{L} \to M$ be a complex Hermitian line bundle over the symplectic manifold (M, ω), let ∇ be the canonical connection induced by the Hermitian metric. Locally over an open subset $U \subset M$ let θ be the connection matrix and s_U a non-vanishing section in $\mathcal{L}$. Then,

$$\nabla_X(f s_U) = X(f) s_U + \theta(X) f s_U.$$

The pre-quantization operator $Q : C^\infty(M) \to \mathrm{OP}(D)$, is given by:

$$f \mapsto f - i\hbar \nabla_{X_f},$$

where $D = L_0^2(M, \mathcal{L}) \subset L^2(M, \mathcal{L})$, is the subset of smooth square integrable sections of $\mathcal{L}$ with compact support. The metric on $L^2(M, \mathcal{L})$ is given by

$$\langle s_1, s_2 \rangle = \int_M s_1 \overline{s}_2 \frac{\omega^n}{n!}.$$

Since the integral of the Lie derivative of a top form over a manifold M without boundary is 0, it follows that Q_f is self-adjoint when f is a smooth real function on M. The assignment of the pre-quantization operator is further linear.

With the assignment of pre-quantum operator above we get the commutator:

$$[Q_f, Q_g] = -\hbar^2[\nabla_{X_f}, \nabla_{X_g}] - 2i\hbar\{f, g\}.$$

Recall that in terms of the connection the curvature is determined by the formula:

$$F_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Using this fact we see that goving a condition on the commutator of quantum operators gives constraint on the curvature of ∇. When we have equality $i\hbar F_\nabla(X_f, X_g) = \omega(X_f, X_g)$ this is exactly the commutator relation which we want. In other word, the decided commutator appears when

$$\left[\frac{i}{2\pi}F_\nabla\right] = c_1(\mathcal{L}) = \left[\frac{1}{\hbar}\omega\right] \in \mathrm{Im}(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})).$$

Definition 3.1. A *pre-quantum line bundle* on the symplectic manifold (M, ω) is a triple $(\mathcal{L}, \nabla, (\cdot, \cdot))$ consisting of a complex line bundle $\mathcal{L} \to M$ with a Hermitian structure $(\cdot, \cdot)$, and a compatible connection ∇ satisfying the pre-quantum condition

$$F_\nabla = \frac{-i}{\hbar}\omega. \tag{3.1}$$

A symplectic manifold admitting a pre-quantum line bundle is called *pre-quantizable*. As we have seen it is definitely not every symplectic manifold which admits a pre-quantum line bundle. For further details see e.g. [54].

The cohomological investigation further reveals that, if a pre-quantum line bundle exists, the inequivalent choices of pre-quantum line bundles are parametrized by $H^1(M, \mathrm{U}(1))$.

Prequantization satisfies all the properties required of a quantization, except that it fails to reproduce canonical quantization when applied to $\mathbb{R}^{2n}$. In a sense, it produces a Hilbert space of wave functions which depend on twice as many variables as they should. Indeed, if $\omega_{\text{std}} = -d\theta$ where $\theta = \sum_j p_j dq_j$ and if X_f is a Hamiltonian vector field for a function f,

$$X_f = \sum_{j=1}^{n} \frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q_j},$$

and because $\nabla_{X_f} = X_f - \frac{i}{\hbar}X_f \cdot \theta$ the operator Q_f is given by:

$$Q_f = f + \sum_{j=1^n} p_j \frac{\partial f}{\partial p_j} - i\hbar\left(\sum_{j=1}^{n} \frac{\partial f}{\partial p_j}\frac{\partial f}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial f}{\partial p_j}\right),$$

which does not act on $L^2(\mathbb{R}^n)$ but $L^2(\mathbb{R}^{2n})$ instead. We therefore need to restrict Q to the space of functions only depending on the $q-$variables and is quadratic integrable over this variable in order to get the Schrödinger representation we want.

A standard way around this is to pick an polarization on M and consider the space of polarized sections of the line bundle.

3.1.4 Polarization

Polarizations are the geometric objects that are used to decrease the dependency to n variables. Given a symplectic manifold (M, ω) of dimension $2n$, we will choose n directions in M by a choice of a special distribution $\mathcal{P} \in TM_{\mathbb{C}}$ called a polarization. Then we say that a section s of a pre-quantum line bundle is polarized if it is constant along all vector fields X of $\mathcal{P}$, so

$$\nabla_X s = 0.$$

In general it is not sufficient to take the quantization space to be the L^2 integrable polarized sections, it still has to be modified in some way. In what follows we define polarizations, consider some special kinds, namely real and Kähler polarizations.

Definition 3.2. Let (M, ω) be a symplectic manifold. A complex polarization is a distribution $\mathcal{P}$ of $TM_{\mathbb{C}}$ satisfying the following criterions

1 $\mathcal{P}$ is Lagrangian, i.e. $\mathcal{P} = \{X \in TM_{\mathbb{C}} \mid \omega(X, Y) = 0 \text{ for all } Y \in \mathcal{P}\}$.

2 $\mathcal{P}$ is involutive, i.e. $[X, Y] \in \mathcal{P}$ fo all $X, Y \in \mathcal{P}$.

3 $\dim(\mathcal{P}_x \cap \overline{\mathcal{P}}_x \cap T_x M)$ is constant for all $x \in M$.

It is not hard to check that if $\mathcal{P}$ is a polarization then $\overline{\mathcal{P}}$ is also a polarization. The involutivity condition is equivalent to $\mathcal{P}$ being integrable by the Frobenius Criterion.

3.1.5 Real and Kähler polarizations

Given a symplectic manifold (M, ω) a polarization $\mathcal{P}$ of M is real if $\mathcal{P} = \overline{\mathcal{P}}$.

Definition 3.3. Let $\mathcal{P}$ be a complex polarization on a symplectic manifold (M, ω). The polarization $\mathcal{P}$ is called a *Kähler polarization* if the Hermitian form on $\mathcal{P}$ defined by $h(u, v) = i\omega(u, \overline{v})$ is positive definite.

With a Kähler polarization we can define a complex structure I on M by letting $\mathcal{P}$ be the $-i$-eigenspace of I and $\overline{\mathcal{P}}$ the i-eigenspace of I. Involutivity of $\mathcal{P}$ gives integrability of I, and by the Newlander-Nirenberg Theorem there exists a unique complex structure on M which induces I. The metric which we can define by the formula $g(X, Y) = \omega(X, IY)$ for vector fields X, Y on M is positive definite. Furthermore it is Hermitian and since ω is closed (M, ω, I) is a Kähler manifold. Conversely every Kähler manifold admits a Kähler polarization by choosing the polarization $\mathcal{P}$ to be the $-i$-eigenspace. To summarise we have shown:

Proposition 3.4. Given a symplectic manifold (M, ω) and a complex structure I, then $\mathcal{P}$ be the $-i$-eigenspace and $\overline{\mathcal{P}}$ be the i-eigenspace are Kähler polarizations. Conversely if (M, ω) has a Kähler polarization then there exists a compatible complex structure I on M.

With a Kähler polarization on M, the line bundle $\mathcal{L} \to M$ has a natural complex structure. A section s of $\mathcal{L}$ is called holomorphic if $\nabla_X s = 0$ for all $X \in \mathcal{P}$. If two non-vanishing sections s, s' of $\mathcal{L}$ differ by a non-vanishing function ϕ, $s = \psi s'$ and if s, s' are both holomorphic then

$$0 = \nabla_X s = \nabla_X(\phi s') = X(\phi)$$

ψ is holomorphic. By choosing a trivialization of $\mathcal{L} \to M$ of holomorphic sections, the transition functions are holomorphic.

The space $D = \{s \in L^2(M, \mathcal{L}) \mid \nabla_X s = 0 \text{ for all } X \in \mathcal{P}\}$, is a closed subspace of $L^2(M, \mathcal{L})$ and therefore a Hilbert space, see e.g. [54] Operators on D will be the target space of the quantization map. Let us check which observables we are able to quantize.

The covariant derivative of $(Q_f)s$ with respect to $X \in \mathcal{P}$ is calculated to be

$$\begin{aligned}\nabla_X((Q_f)s) &= -i\hbar \nabla_X \nabla_{X_f} s + X(f)s + f\nabla_X s \\ &= Q_f(\nabla_X s) - i\hbar \nabla_{[X, X_f]} s,\end{aligned}$$

so for $X \in \mathcal{P}$, Q_f preserves D is $[X, X_f] \in \mathcal{P}$. Hereby we have found the space of quantiziable observables

$$\tilde{D} = \{f \in C^\infty(M) \mid [X, X_f] \in \mathcal{P} \text{ for all } X \in \mathcal{P}\}.$$

Example 3.5. Let us consider two simple examples in relation to geometric quantization. Let $M = T^*Q$ with the canonical basis $\{q_i, p_i\}$ and symplectic form $\omega_{\text{std}} = \sum dq_i \wedge dp_i$. We take the polarization $\mathcal{P}$ to be the vertical vector fields, i.e. the span of $\{\frac{\partial}{\partial p_i}\}_{i=1}^n$. The polarized sections s are sections for which $\frac{\partial s}{\partial p_i} = 0$, so those which are constant along the fibers. This is the Schrödinger representation of (T^*M, ω). The quantum operators corresponding to to the observables *positions* and *momenta* are

$$Q_{q_j} = q_j \quad \text{and} \quad Q_{p_j} = -i\hbar \frac{\partial}{\partial q_j}.$$

If $Q = \mathbb{R}^n$ we could take $\mathcal{P}$ to be spanned by $\{\frac{\partial}{\partial q_i}\}_{i=1}^n$. Then we would obtain the momentum representation. The quantum operators in this case are

$$Q_{q_j} = i\hbar \frac{\partial}{\partial p_j} \quad \text{and} \quad Q_{p_j} = p_j.$$

We observe that the relation between these two representations is the Fouriertransform.

Example 3.6. If we now let $M = T^*Q$ but take as basis $z_j, \overline{z_j}$, where $z_j = p_j + iq_j$. Then the standard symplectic form becomes $\omega = \frac{1}{2} \sum d\overline{z_j} \wedge dz_j$, and the complex structure is defined by $Iz_i = iz_i$, $I\overline{z_i} = -i\overline{z_i}$. Choosing the Kähler polarization corresponding to I, that is $\mathcal{P}$ is spanned by $\{\frac{\partial}{\partial \overline{z_j}}\}_{j=1}^n$ the polarized sections s must satisfy $\frac{\partial s}{\partial \overline{z_j}} = 0$ so they are holomorphic sections. This is the *Bargman-Fock* representation. If instead we had chosen $\overline{\mathcal{P}}$ we would have obtained the anti-holomorphic sections.

3.1.6 Change of polarization

Let us here explain how a change of polarization is related to the Fourier transform. The reference for this section is [2]. For simplicity we will just consider $\mathbb{R}^{2n}$ with the standard symplectic structure

$$\omega_{\text{std}} = \sum_{i=1}^n dx_i \wedge dx_{n+i}.$$

We can define

$$\alpha = \frac{1}{2} \sum_{i=1}^n (x_i dx_{n+i} - x_{n+i} dx_i).$$

Then we have

$$\omega_{\text{std}} = d\alpha.$$

The one form α defines a connection ∇ in the trivial line bundle $\mathcal{L} = \mathbb{R}^{2n} \times \mathbb{C}$. For any polarization $\mathcal{P}$ on $\mathbb{R}^{2n}$ we can consider the space of sections of $\mathcal{L}^k$, which are covariant constant along $\mathcal{P}$:

$$\mathcal{H}_{\mathcal{P}} = \{\phi \in C^\infty(\mathbb{R}^{2n}, \mathcal{L}^k) \mid \phi(\nabla_X s) = 0 \quad \forall X \in \mathcal{P},\ s \in C^\infty(\mathbb{R}^{2n}, \mathcal{L}^k)\}.$$

For a general Lagrangian subspace $\mathcal{P}$ we can find a Lagrangian subspace $\mathcal{P}'$ of $\mathbb{R}^{2n}$ which is transversal to $\mathcal{P}$ and which induces a reducible polarization on M. We will construct an isomorphism

$$U : \mathcal{H}_{\mathcal{P}} \to \mathcal{H}_{\mathcal{P}'}$$

Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are two transverse Lagrangian subspaces of $\mathbb{R}^{2n}$. We can then find a Lagrangian subspace Y transversal to both $\mathcal{P}_1$ and $\mathcal{P}_2$. Let

$$\rho_i : \mathbb{R}^{2n} \to Y$$

be the projection $\mathbb{R}^{2n} = \mathcal{P}_i \oplus Y \to Y$. Sections of $\tilde{\mathcal{L}}^k$ covariant constant along $\mathcal{P}_i$ can be identified with sections of $\tilde{\mathcal{L}}^k|_Y$. Let s be a covariant constant section of $\tilde{\mathcal{L}}^k|_Y$. By using s we identify $C_c^\infty(\tilde{\mathcal{L}}^k|_Y)$ with $C_c^\infty(Y)$. Extend s to a section s_i of $\tilde{\mathcal{L}}^k|_Y$ by extending covariantly constant along $\mathcal{P}_i$. Assume that s is of unit length, hence so is s_i. Let (q,p) be symplectic coordinates on $\mathbb{R}^{2n}$ such that $\rho_2(q,p) = q'$. Then

$$q'_i = q_i + \sum_{i=1}^{n} S_{ij} p_j,$$

where S is a symmetric matrix and non-singular since $\mathcal{P}_1$ and $\mathcal{P}_2$ are transversal. Let $\mathcal{S}(Y)$ denote the space of Schwartz functions on Y. Consider the operator

$$U : \mathcal{S}(Y) \to \mathcal{S}(Y)$$

given by

$$U(f)(q) = \int_Y \exp(i\pi k\langle Sp, p\rangle) f(q + Sp) dp$$

for $f \in \mathcal{S}(Y)$. Since S is invertible, we get that

$$U(f)(q) = \exp(i\pi k\langle S^{-1}q, q\rangle) \int_Y \exp(i\pi k\langle p, Sp - 2q\rangle) f(Sp) dp.$$

Define operators $V, W : \mathcal{S}(Y) \to \mathcal{S}(Y)$ by

$$V(f)(p) = \exp(i\pi k\langle S^{-1}p, p\rangle) f(p),$$

$$W(f)(p) = f(Sp),$$

and the Fourier transform $\mathcal{F} : \mathcal{S}(Y) \to \mathcal{S}(Y)$ by

$$\mathcal{F}(f)(q) = \int_Y \exp(-2i\pi k\langle p, q\rangle) f(p) dp.$$

Since all these maps are isomorphisms we conclude that $U = V \circ \mathcal{F} \circ W \circ V$ is an isomorphism on $\mathcal{S}(Y)$. Actually we get an isomorphism

$$U : \mathcal{H}_{\mathcal{P}} \to \mathcal{H}_{\mathcal{P}'}$$

and we see directly that a change of polarization is related to the Fourier transform. In Chapter 4 we will an analogue of this fact when coordinates are changed in Teichmüller space. The story is not as simple as in the case described above since the coordinate transformations used are not as simple.

Chapter 4

Quantum Teichmüller theory

Teichmüller space will play an important role later on in this thesis. Therefore, we would like to identify useful global coordinates on Teichmüller space to get a better understanding of it. We construct the Fenchel–Nielsen coordinates and generalise these coordinates to a Riemann surface with punctures and holes. We look at another set of coordinates due to Penner [40]; these coordinates can be given on a surface with at least one hole or puncture. Finally in this section we will look at Kashaev coordinates for Teichmüller space and the quantization Teichmüller space in these coordinates.

4.1 Pants decomposition

Given a connected Riemann surface R of genus g we would like to decompose it into a number of building blocks. If the genus g is at least two one can show that there exist a collection $\Gamma = \{\gamma_i\}_{i=1}^{3g-3}$ of simple closed geodesics on R which decomposes R into $2g-2$ pairs of pants. The genus 2 case is illustrated in Figure 4.1

Definition 4.1. A *pair of pants* P of a Riemann surface R is a simple subsurface of R whose boundary ∂P in R consists of three simple closed geodesics.

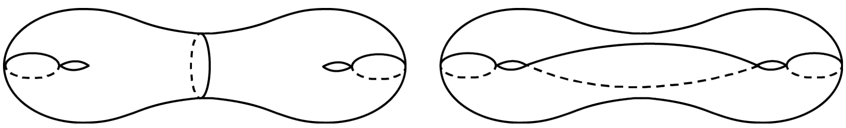


Figure 4.1: Two different pair of pants decompositions of a genus 2 surface.

The complex structure of a pair of pants P is uniquely determined by the lengths of the geodesics $\gamma_1, \gamma_2, \gamma_3$ of the boundary ∂P we denote these lengths as l_1, l_2, l_3 respectively. To see this decompose P into two right angled hexagons by cutting along three shortest geodesics with lengths d_{12}, d_{13}, d_{23} connecting the boundary components, see figure 4.2. Because the hexagons have the three edges d_{12}, d_{13}, d_{23} in common, then by elementary hyperbolic trigonometry the two hexagons must be identical. Therefore each hexagon is uniquely determined by the lengths $l_1/2, l_2/2, l_3/2$ and hence P is uniquely determined by l_1, l_2, l_3. This is illustrated in Figure 4.3.

4.2 Fenchel–Nielsen coordinates

Since the complex structure is already fixed we only need to specify how we glue the pairs of pants back together to reconstruct our Riemann surface. This is done by defining *twisting parameters* τ_i; one for each closed geodesic in Γ. Notice that after choosing an order-

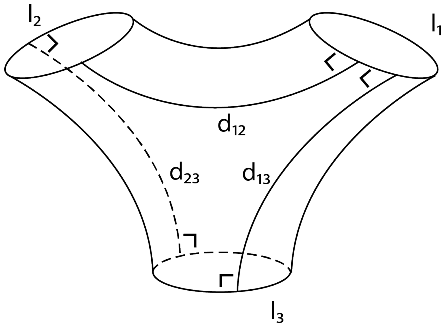


Figure 4.2: A pair of pants given by a region of the Poincaré disk.

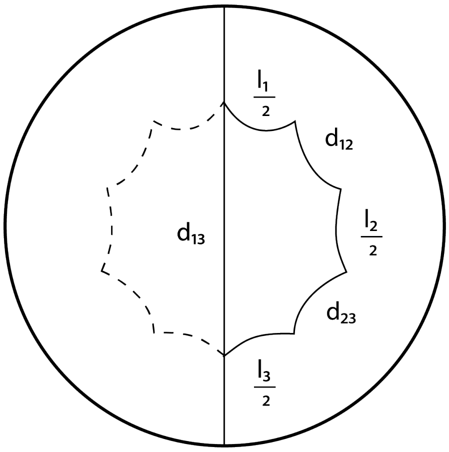


Figure 4.3: A pair of pants given by a region of the Poincaré disk.

ing of boundary components in each pair of pants, the connecting geodesics with lengths d_{12}, d_{13}, d_{23} define distinguished points on $\gamma_1, \gamma_2, \gamma_3$ respectively. We define the twisting parameter τ_i modulo γ_i to be the distance along γ_i between the two distinguished points corresponding to the two pairs of pants glued along γ_i.

Due to a result known as *Teichmüller Theorem* we know that Teichmüller space $\mathcal{T}_g$ is simply connected and therefore the parameters τ_i are allowed to run over the whole set $\mathbb{R}$. We have the following consequence:

Lemma 4.2 (Fenchel–Nielsen coordinates)**.** Given a collection Γ of decomposing simple closed curves on R, fixing the zeroes of the twisting parameters, we obtain a diffeomorphism $\Psi : \mathcal{T}_g \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$.

In other words we have a global set of coordinates on Teichmüller space which is known as the Fenchel–Nielsen coordinates.

4.3 Punctures and holes

Using the pair of pants decomposition one can easily generalise Teichmüller space to Riemann surfaces having holes or punctures. We decompose our Riemann surface R of genus g into pairs of pants such that one pair of pants looks like a handle and throw this away to obtain a Riemann surface of genus $g-1$ with a hole with geodesic boundary length equal to the corresponding Fenchel–Nielsen coordinate.

Define $\mathcal{T}_{g,s}^{l_1,\dots,l_s}$ to be the Teichmüller space of a Riemann surface of genus g with s holes of geodesic boundary lengths $\gamma_1,\dots,\gamma_s$. Considering the pair of pants decomposition leads to the fact that $\mathcal{T}_{g,s}^{\gamma_1,\dots,\gamma_s}$ is a space of dimension $6g-6+2s$ and again the Fenchel–Nielsen coordinates define global coordinates.

We would also like to allow zero boundary length, which corresponds to a Riemann surface with *punctures*. Figure 4.2 and elementary hyperbolic geometry leads to the following equations:

$$\frac{\sinh d_{23}}{\sinh \frac{1}{2}l_1} = \frac{\sinh d_{13}}{\sinh \frac{1}{2}l_2} = \frac{\sinh d_{12}}{\sinh \frac{1}{2}l_3}. \tag{4.1}$$

Therefore fixing the geodesic lengths l_2, l_3 of γ_2 and γ_3 and letting l_1 approach zero d_{12} and d_{13} will go to infinity. Punctures therefor correspond to a Riemann surface having infinitely long spikes.

If we let γ be the Möbius transformation corresponding to a path around a puncture then from the equation

$$|\mathrm{Tr}(\gamma)| = 2\cosh\left(\frac{l_\gamma}{2}\right) \tag{4.2}$$

we see that $|\mathrm{Tr}(\gamma)| = 2$ and therefore γ has to be parabolic, which means that it has a fixed point on the boundary of $\mathbb{H}$, which of course is the puncture.

Remark 4.3. In the case where punctures appear we must reformulate our notion that a Fuchsian group only consists of the identity element and hyperbolic conjugacy classes. A Fuchsian model of R with s punctures consists of the identity element and exactly s distinct parabolic conjugacy classes and hyperbolic conjugacy classes.

4.4 Penner coordinates

Dealing with Riemann surfaces having punctures there is another useful set of coordinates on Teichmüller space.

Let R be a Riemann surface of genus g with $s > 0$ punctures. There exists $6g-6+3s$ disjoint geodesics running between punctures of R which decompose R into $4g-4+2s$ triangles. Dual to this triangulation is a trivalent graph called a fat graph on R.

It would be tempting to view the lengths of the geodesic edges of the triangulation as coordinates. This is of course not possible since the geodesics connect punctures and are therefore infinitely long.

Instead we choose a horocycle around each of the punctures. A horocycle for a puncture is a path around the puncture which is perpendicular to all geodesics originating from the puncture.

Example 4.4. In the the Poincaré disc horocycles are given by circles tangent to the boundary.

The length l_e of an edge e in the triangulation is defined to be the distance along the edge e between the horocycles of the punctures which it connects. Shifting a horocycle for some puncture just corresponds to adding a constant to the lengths of all edges coming from that puncture. Modulo this symmetry the set of lengths $\{l_e\}_{e\in\Delta}$ constitute global coordinates on $\mathcal{T}_{g,s}$ which are called *Penner coordinates*.

4.5 Kashaev coordinates

Quantization of the Teichmüller space of a surface with boundary and holes was achieved by Kashaev in [28] and independently by Chekhov–Fock [14]. The main ingredient in both constructions is the very special function called the quantum dilogarithm. There is a universal setting for the construction, namely quantization of the universal Teichmüller space, which we think of as Teichmüller space of the open disk $\mathbb{D}$ with certain boundary behaviour, or of the closed unit disc with a countable number of distinguished points on the boundary. Quantization requires the choice of a coordinate system on the Teichmüller space, which depends on the choice of a certain infinite triangulation of the surface $\mathbb{D}$, which is called a Tessellation of $\mathbb{D}$. Let us now introduce the Kashaev coordinates for Teichmüller space. After this definition we will look at the quantization of Teichmüller space following the outline from [28]

4.5.1 Tessellations

In this subsection we put up a universal setting for ideal triangulations of hyperbolic surfaces. The surface we want to deal with is the open unit disc $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$ equipped with the Poincaré metric $ds^2 = \frac{|z|^2}{(1-|z|^2)^2}$.

Definition 4.5. An ideal arc connecting two fixed distinct ideal points on the unit circle $S^1 = \partial\mathbb{D}$ is a homotopy class of smooth arcs in $\mathbb{D}$ connecting the points. No orientation on arcs is imposed. A region bounded by 3 ideal arcs connecting three distinct ideal points is called an *ideal triangles*.

Definition 4.6 (Tessellation)**.** A *tessellation* τ of the unit disc $\mathbb{D}$ is a locally finite triangulation, i.e., any point of $\mathbb{D}$ admits a neighbourhood meeting only finitely many geodesics in τ, of $\mathbb{D}$ into ideal triangles. The vertices of a tessellation are the endpoints of the ideal arcs in the tessellation. The collection of vertices, ideal arcs and ideal triangles of a tessellation is denoted by $\tau^{(0)}, \tau^{(1)}$ and $\tau^{(2)}$ respectively.

Definition 4.7. By μ we denote the Cayley transform from the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ to the unit disc $\mathbb{D}$, which also extends to their boundaries:

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{RP}^1 \longleftrightarrow \overline{\mathbb{D}} \cup S^1 \tag{4.3}$$

$$x \mapsto \frac{x-i}{x+i}, \tag{4.4}$$

here we think of $\mathbb{RP}^1 = \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. When we use a Möbius transformation we will mean an element of the automorphism group $PSL(2,\mathbb{R})$ of the hyperbolic space $\mathbb{H}$ given by the fractional linear transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \in \mathrm{PSL}(2,\mathbb{R}) : x \in \overline{\mathbb{H}} \mapsto \frac{ax+b}{cx+d} \in \overline{\mathbb{D}}, \tag{4.5}$$

when we mention the action on $\overline{\mathbb{D}}$ of an element g of the automorphism group $PSL(2,\mathbb{R})$ or its subgroup $PSL(2,\mathbb{Z})$ we mean the conjugated action $\mu \circ g \circ \mu^{-1}$.

Definition 4.8. A nonzero rational number is said to be in reduced form if it's written as p/q where $\gcd(p,q) = 1$, with $p, q \in \mathbb{Z}, q > 0$. We set $\frac{0}{1}$ for the reduced expression of 0, and $\frac{1}{0}$ or $\frac{-1}{0}$ for the reduced expression for ∞. We call $\mathbb{Q} \cup \{\infty\}$ the extended rationals.

Definition 4.9. The *Farey tessellation* τ^* is the tessellation whose vertices are all the rational points of S^1, and two rational points $\mu(a/b)$ and $\mu(c/d)$ are connected by an ideal arc if and only if $|ad - bc| = 1$. Alternatively one could start with the basic ideal triangle with vertices $\mu(\frac{0}{1}), \mu(\frac{1}{0}), \mu(-\frac{1}{1}) \in S^1$ and take the orbit of its sides under the $\mathrm{PSL}(2,\mathbb{Z})$-action.

We are interested in a bit more general tessellation than just the Farey tessellation. We say that a tessellation is of *Farey-Type* if it satisfies the following definition.

Definition 4.10. A *Farey-type tessellation* is a tessellation whose vertices are the rational points of S^1, all but finitely many of whose ideal arcs are those of the Farey tessellation. In this section we let $\mathcal{F} := \{\text{Farey-type Tessellations } \tau\}$.

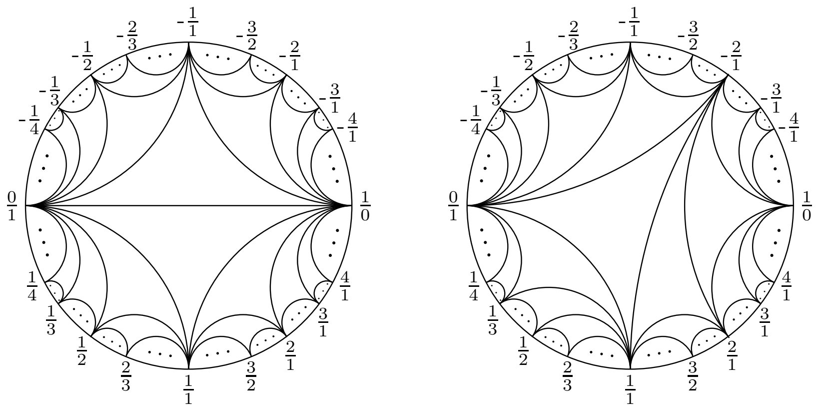


Figure 4.4: On the left the Farey tessellation, on the right a Farey-type tessellation.

4.5.2 Decorated Tessellations

It is often necessary to put some decoration on the tessellation. Different authors use different types of decorations. In [40] Penner uses a distinguished edge as decoration. Here we will use a distinguished corner in each triangle as decoration of the tessellation along with a labelling rule of the ideal triangles in the tessellation.

Definition 4.11. A tessellation τ with a choice of distinguished corner for each triangle $\tau^{(2)} \in \tau$ is called a *decorated ideal triangulation* (d.i.t.). Further we impose the condition that a decorated ideal triangulation should also have a labelling L, which is just a bijection between the triangles $\tau^{(2)} \in \tau$ and $\mathbb{Q}^* = \mathbb{Q}\backslash\{0\}$. We let

$$\mathcal{F}_{dot} := \{\text{Decorated Farey-type Tessellations } \tau\}$$

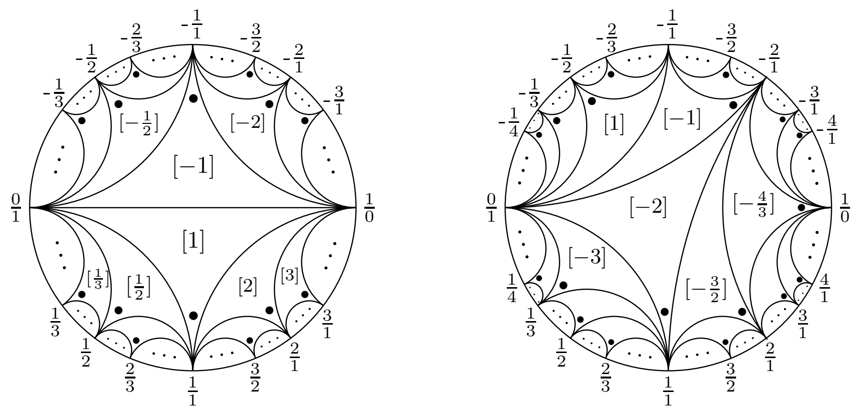


Figure 4.5: Examples of dotted Tessellations. On the left the standard d.i.t. On the right a more general d.i.t.

There is of course a natural map

$$\mathcal{F}_{dot} \to \mathcal{F},$$

which just forgets the decoration and returns the underlying tessellation.

4.5.3 Automorphisms of $\mathcal{F}_{dot}$

Now we proceed to study actions on the decorated tessellations. It turns out that it is more natural to consider a groupoid, instead of a group.

Definition 4.12. Let the Ptolemy groupoid Pt be the category whose objects are the (Farey-type) tessellations $\tau \in \mathcal{F}$, and for two objects τ, τ', there is exactly one morphism denoted by $[\tau, \tau']$. We let composition of morphisms be given as

$$[\tau', \tau''] \circ [\tau, \tau'] = [\tau, \tau''], \tag{4.6}$$

just like the composition of functions. Analogously, define the decorated Ptolemy groupoid Pt_{dot} to be the category whose objects are decorated Tessellations $\tau_{dot} \in \mathcal{F}_{dot}$ and for any two objects τ_{dot}, τ'_{dot} there is exactly one morphism denoted by $[\tau_{dot}, \tau'_{dot}]$.

Definition 4.13. For any edge e of a (Farey-type) Tessellation τ, there are exactly two ideal triangles in $\tau^{(2)}$ having e as one of their sides. These two triangles form an ideal quadrilateral which has e as a diagonal arc. Replace e with the other diagonal e' of the quadrilateral to obtain a new tessellation τ'. The morphism $[\tau, \tau']$ is called the flip of τ with respect to e.

One can easily observe:

Proposition 4.14. Any two Farey-type tessellations τ, τ' can be related through a finite number of flips.

The flips as defined in the previous definition require the choice of a tessellation τ together with one of its arcs e. In the next definition we will instead describe and give names to the elementary morphisms of Pt_{dot} in a decorated tessellation only in terms of the labels of the involved triangles.

Definition 4.15. We describe the elementary moves $A_{[j]}, T_{[j][k]}, P_{(jk)}$ of Pt_{dot} for $j, k \in \mathbb{Q}^*, j \neq k$, each representing a morphism of Pt_{dot}.

(i) Let $\tau'_{dot} \in \mathcal{F}_{dot}$ be obtained from $\tau_{dot} \in \mathcal{F}_{dot}$ by moving the distinguished corner of the triangle in $\tau^{(2)}$ labelled by $j \in \mathbb{Q}^*$ in the counterclockwise direction to the next corner in that triangle, leaving all other information intact. This morphism $\tau \cdot \tau'$ is denoted $A_{[j]}$. See Figure 4.6.

(ii) Suppose that for $\tau_{dot} \in \mathcal{F}_{dot}$ the triangles labelled by $[j]$ and $[k]$ share one common edge and that the distinguished corners of $[j], [k]$ are exactly as in Figure 4.7. If $\tau'_{dot} \in \mathcal{F}_{dot}$ is obtained from $\tau_{dot} \in \mathcal{F}_{dot}$ by replacing the common arc of the triangles labelled by $[j], [k]$ by the other diagonal arc of the ideal quadrilateral formed by these two triangles, and setting the distinguished corners according to the picture on the right of Figure 4.7, as if we rotate the diagonal arc clockwise while letting the dots $\bullet$ and triangle labels be floating and thus pushed according to the rotation arc, while leaving all other information intact, then we name the morphism $[\tau_{dot}, \tau'_{dot}]$ of Pt_{dot} by $T_{[j][k]}$.

(iii) If $\tau'_{dot} \in \mathcal{F}_{dot}$ is obtained from τ_{dot} by exchanging the labels of the two triangles labelled by $[j], [k] \in \mathbb{Q}^*$ and leaving all other information intact, then we name the morphism $[\tau_{dot}, \tau'_{dot}]$ of Pt_{dot} by $P_{(jk)}$.

Proposition 4.16. The morphism between any two objects of Pt_{dot} can be written as a finite composition of elementary morphisms and therefore can be represented as a finite composition of elementary moves.

The proposition is easily observed since two Farey-type tessellations are related by a finite number of flips [40].

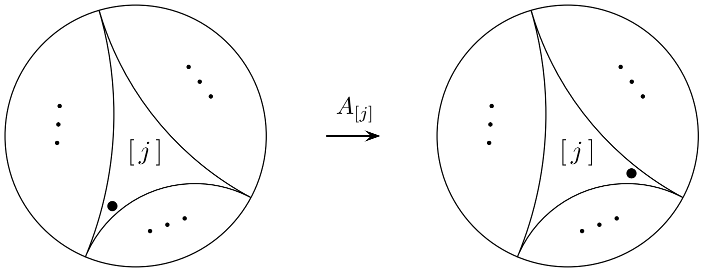


Figure 4.6: The action of $A_{[j]}$

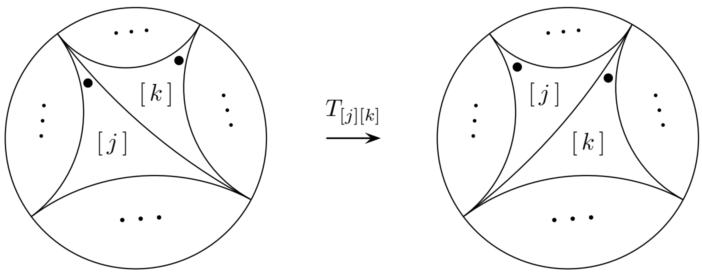


Figure 4.7: The action of $T_{[j],[k]}$

Theorem 4.17 (Kashaev)**.** All nontrivial algebraic relations among the elementary moves of Pt_{dot} are consequences of

$$
\begin{aligned}
A_{[j]}^3 &= \mathrm{id}, &(4.7)\\
T_{[k][l]}T_{[j][k]} &= T_{[j][k]}T_{[j][l]}T_{[k][l]}, &(4.8)\\
A_{[j]}T_{[j][k]}A_{[k]} &= A_{[k]}T_{[k][j]}A_{[j]}, &(4.9)\\
T_{[j][k]}A_{[j]}T_{[k][j]} &= A_{[j]}A_{[k]}P_{(jk)}, &(4.10)
\end{aligned}
$$

where $j, k, l \in \mathbb{Q}^*$ are distinct.

There are trivial relations too, subject to the index permutations

$$P_{(jk)}^2 = \mathrm{id}, \quad P_{(jk)} = P_{(kj)}, \quad P_{(jk)} f_{\ldots,j,\ldots,k,\ldots} P_{(kj)} = f_{\ldots,k,\ldots,j,\ldots},$$

where $f_{\ldots,j,\ldots,k,\ldots}$ is any composition of elementary moves (conjugation by $P_{(jk)}$ results in exchanging the subscripts j and k), and that any two words in the elementary moves whose collections of subscripts do not intersect with each other commute. See [46, 29]

It is convenient to define the following group G_I, see [16]

Definition 4.18. For any index set I, define the *Kashaev group* G_I associated to the index set $\mathbb{Q}^*$ by generators and relations, with generators $A_{[j]}, T_{[j][k]}, P_{(jk)}$, $(j, k \in I, j \neq k)$ and the relations (4.7), (4.8), (4.9), (4.10) and the commuting relations mentioned in theorem 4.17.

We think about the Kashaev group as the formal group of changes of decorated tessellations.

Let $\mathcal{T}$ denote the universal Teichmüller space. We will here briefly review Kashaev's quantization of the universal Teichmüller space. One should consult [28, 29]. Suppose we have chosen a horocycle at each puncture (vertices of τ).

Definition 4.19 (Kashaev coordinates of the universal Teichmüller space)**.** In Kashaev's quantization of the universal Teichmüller space $\mathcal{T}$ each choice of a decorated triangulation τ_{dot} gives rise to a coordinate system on $\mathcal{T}$, which to each traingle $j \in \tau^{(2)}$ assigns two coordinates p_j, q_j. I.e. we have an injective map $\mathcal{T} \to (\mathbb{R}^{\tau^{(2)}})^2$, where

$$p_j = l_{j,1} - l_{j,2}, \quad q_j = l_{j,3} - l_{j,2},$$

where $l_{j,1}, l_{j,2}, l_{j,3}$ are the geodesic lengths of the sides of the triangle j where the cyclic labelling of the three sides is determined by the choice of decoration, where we trim the sides using the chosen horocycles (lengths might be negative). These lengths are the logarithm of the *lambda lengths* of Penner. See figure 4.8.

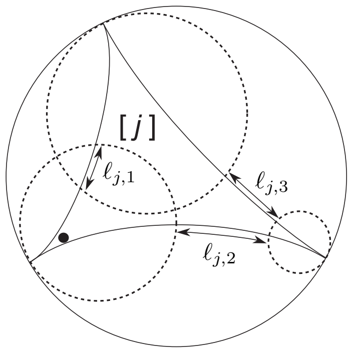


Figure 4.8: The lambda lengths for a triangle labelled by $[j]$

Before turning to the quantization of Teichmüller space let us describe the change of Kashaev variables induced by a change of trianglation. Following [28], define the following two transformations associated to the elementary moves $A_{[j]}$ and $T_{[j][k]}$ respectively:

$$A_{[j]} : (q_j, p_j) \mapsto (p_j - q_j, -q_j). \tag{4.11}$$

$$T_{[j][k]} : \begin{cases} (X_j, Y_j) \mapsto (X_j X_k, X_j Y_k + Y_j), \\ (X_k, Y_k) \mapsto ((X_k Y_j)(X_j Y_k + Y_j)^{-1}, Y_k (X_j Y_k + Y_j)^{-1}), \end{cases} \tag{4.12}$$

where we have set $X_j \equiv e^{q_j}$ and $Y_j \equiv e^{p_j}$ for all $j \in \tau^{(2)}$.

4.6 Quantization of Teichmüller space

Denote by $\{\cdot, \cdot\}$ the canonical Weil–Petersson Poisson bracket on the space of functions on $\mathcal{T}$.

Proposition 4.20. The Kashaev coordinates satisfy

$$\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \forall i, j \in \tau^{(2)}. \tag{4.13}$$

This system has a canonical quantization

$$p_j \to \hat{p}_j = 2\pi\, \mathrm{b}\, P_j, \quad q_j \to \hat{q}_j = 2\pi\, \mathrm{b}\, Q_j, \tag{4.14}$$

realised as self-adjoint operators on (a dense subspace) the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{\tau^{(1)}})$, where any element of $\mathcal{H}$ is a function in the variable $x = (x_j)_{j\in\tau^{(2)}}$ where $\mathrm{b} \in \mathbb{R}$ is the quantisation parameter, and $\mathrm{b}^2 \notin \mathbb{Q}$. The operators P_j, Q_j are given by

$$P_j f = \frac{1}{2\pi i}\frac{\partial}{\partial x_j} f, \quad Q_j f = x_j f \quad \forall f \in L^2(\mathbb{R}^{\tau^{(1)}}), \tag{4.15}$$

which satisfy $[P_i, Q_j] = \frac{1}{2\pi i}\delta_{i,j}, [P_i, P_j] = [Q_i, Q_j] = 0$ [The Heisenberg algebra]. Then one has

$$[\hat{p}_i, \hat{q}_j] = -2\pi i\, \mathrm{b}^2\, \delta_{i,j}, \quad [\hat{p}_i, \hat{p}_j] = [\hat{q}_i, \hat{q}_j] = 0. \tag{4.16}$$

Usually, quantization of $\mathcal{T}$ is described as a family of non-commutative algebras depending on a real parameter, whose generators are realised as self-adjoint operators on a Hilbert space. In this case, we use the exponents

$$\hat{X}_j = e^{\hat{q}_j}, \quad \hat{Y}_j = e^{\hat{p}_j}, \tag{4.17}$$

as the generators of the non-commutative algebra, subject to the relations (4.16).

Definition 4.21. For $\mathrm{b} \in \mathbb{R}$, $\mathrm{b}^2 \notin \mathbb{Q}$, define $q \in \mathbb{C}^*$ to be the number

$$q = e^{\pi i\, \mathrm{b}^2}. \tag{4.18}$$

For a decorated triangulation τ_{dot}, let the Kashaev algebra $\mathcal{K}^q_{\tau_{dot}}$ be the algebra generated by $\hat{X}_j, \hat{Y}_j$, $j \in \tau^{(2)}$ with the relations

$$\hat{X}_j\hat{Y}_j = q^2\hat{Y}_j\hat{X}_j, \quad [\hat{X}_j, \hat{Y}_k] = [\hat{X}_j, \hat{X}_k] = [\hat{Y}_j, \hat{Y}_k] = 0. \tag{4.19}$$

Elements of $\mathcal{K}^q_{\tau_{dot}}$ can be thought of as operators on a Hilbert space $L^2(\mathbb{R}^{\tau^{(1)}})$ via the representation π given by:

$$\pi(\hat{X}_j) = e^{2\pi i Q_j}, \quad \pi(\hat{Y}_j) = e^{2\pi i P_j}, \tag{4.20}$$

where P_j, Q_j is defined in (4.15).

The Kashaev algebra

$$\mathcal{K}^q_{\tau_{dot}} = \left\langle \hat{X}_i, \hat{Y}_i \mid j \in \tau^{(2)} \right\rangle / (\text{rel. in (4.19)})$$

is the non-commutative deformation under this quantization of the algebra of functions on $\mathcal{T}$ generated by the (exponents of the) coordinate functions $X_j = e^{q_j}, Y_j = e^{p_j}$ which depend on τ_{dot}.

For a finite type surface, i.e. surfaces isomorphic to a compact surface with a finite number of points removed, the Weil–Petersson Poisson structure on Teichmüller space is preserved under the action of the mapping class group of the surface. For the case of the universal Teichmüller space $\mathcal{T}$, each element of the universal mapping class group can be represented by an element of the Kashaev group G_{dot} (changes of dotted tessellations). In other words this means that a change of decorated tessellation of $\mathcal{T}$ yields a corresponding change of the Kashaev coordinates on $\mathcal{T}$. It turns out that the Weil–Petersson Poisson structure on $\mathcal{T}$ is preserved under the coordinate change induced by the Kashaev group G.

The quantization of Teichmüller space should therefore done in such a manner that G_{dot} still acts on the non-commutative algebra $\mathcal{K}^q_{\tau_{dot}}$ preserving the algebra structure. If we can identify $\mathcal{K}^q_{\tau_{dot}}$ for different decorated tessellations, this would correspond to G_{dot} acting on $K^q_{\tau_{dot}}$ as algebra automorphisms.

Definition 4.22. Suppose $\tau_{dot}, \tau'_{dot} \in \mathcal{F}_{dot}$, whose triangle label rules let us identify $\tau^{(2)}$ with $\{j \ : \ j \in \mathbb{Q}^*\}$ and $(\tau')^{(2)}$ with $\{j' \ : \ j \in \mathbb{Q}^*\}$. Define the map $I_{\tau_{dot},\tau'_{dot}} : \mathcal{K}^q_{\tau_{dot}} \to \mathcal{K}^q_{\tau'_{dot}}$ by

$$I_{\tau_{dot},\tau'_{dot}}(\hat{Y}_j) = \hat{Y}_{j'} \quad \text{and} \quad I_{\tau_{dot},\tau'_{dot}}(\hat{Z}_j) = \hat{Z}_{j'}, \tag{4.21}$$

which is easily seen to be an algebra isomorphism.

Before stating the main result of Kashaev on the quantization of Teichmüller space let us introduce Faddeev's quantum dilogarithm.

Definition 4.23. Faddeev's quantum dilogaritm is a function of two complex arguments z and b, defined by the formula

$$\Phi_{\mathrm{b}}(z) = \exp\left(\int_{\mathcal{C}} \frac{e^{-2iz\omega}}{4\sinh(\omega\,\mathrm{b})\sinh(\omega\,\mathrm{b}^{-1})\omega} d\omega\right),$$

where the contour $\mathcal{C}$ runs along the x-axis, deviating into the upper half plane in the vicinity of the origin, and where the parameter $\hbar$ is in $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$, and $\mathrm{b} \in \mathbb{C}$ is chosen such that $\hbar = (\mathrm{b} + \mathrm{b}^{-1})^{-2}$.

We will look much more into this function later on in this thesis. Therefore we ask the reader to see (11.9) for more information about this function.

Following [29] closely we define a projective representation of the Ptolemy groupoid in therms of the following set of unitary operators:

Theorem 4.24. Let the dotted Kashaev group G_{dot} be given as a finitely presented group $G_{dot} = F_{dot}/N_{dot}$ where F_{dot} is the group generated by $A_{[j]}, T_{[j][k]}, P_{(jk)}$, $(j, k \in I, j \neq k)$, and N_{dot} is the normal subgroup generated by the relations in Theorem 4.17. For the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^{\tau^{(1)}})$ consider the group homomorphism

$$\rho : F_{dot} \to GL(\mathcal{H})$$

given by assigning to each generator of F_{dot} a unitary operator on $\mathcal{H}$ as follows:

$$\rho(A_{[j]}) = e^{-\pi i/3} e^{3\pi i Q_j^2} e^{\pi i (P_j + Q_j)^2}, \tag{4.22}$$

$$\rho(T_{[j][k]}) = e^{2\pi i P_j Q_k} \Phi_{\mathrm{b}}(Q_j + P_k - Q_k)^{-1}, \tag{4.23}$$

$$(\rho(P_{(jk)})f)(\dots, x_j, \dots, x_k, \dots) = f(\dots, x_k, \dots, x_j, \dots), \tag{4.24}$$

where P_j, Q_j are as in (4.15). Then the quantum version of the coordinate change induced by an element g of G_{dot} is given by conjugation in $\rho(g)$ in the following way. Then for each $g \in G_{dot}$ which can be applied to τ_{dot}, we associate an algebra isomorphism

$$\Psi^q_g : \mathcal{K}^q_{\tau_{dot}} \to \mathcal{K}^q_{g.\tau_{dot}}$$

which after identification of $\mathcal{K}^q_{g.\tau_{dot}}$ with $\mathcal{K}^q_{\tau_{dot}}$ via (4.21) is as follows, in terms of the representation π of $\mathcal{K}^q_{g.\tau_{dot}}$:

$$\pi(I_{g.\tau_{dot},\tau_{dot}} \circ \Psi^q_g) : \pi(\mathcal{K}^q_{\tau_{dot}}) \to \pi(\mathcal{K}^q_{\tau_{dot}}), \tag{4.25}$$

this is the mapping $\pi(\hat{x}) \mapsto \rho(g)\pi(\hat{x})\rho(g)^{-1}$, for all $\hat{x} \in \mathcal{K}^q_{\tau_{dot}}$. The map (4.25) is well defined, and it provides an algebra isomorphism of $\pi(\mathcal{K}^q_{\tau_{dot}})$ since it is a conjugation. When $q = e^{\pi i\,\mathrm{b}^2} \to 1$ as $\mathrm{b} \to 0$ in $\mathbb{R}$, the limit of the map (4.25) recovers the classical coordinate change map induced by g.

Remark 4.25. The operators defined in Theorem 4.24 are unitary; when b is real or on the unit circle

$$(1 - |\mathrm{b}|)\mathrm{Im}\,\mathrm{b} = 0 \Rightarrow \overline{\Phi_{\mathrm{b}}(z)} = 1/\Phi_{\mathrm{b}}(\overline{z}).$$

Example 4.26 (The action of the operators $A_{[j]}$ and $T_{[j][k]}$ on L^2)**.** Let us first look at the action of $\rho(A_{[j]})$. One sees that this operator only involves use of the two operators P_j and Q_j we therefore think of this operator as an operator acting on $L^2(\mathbb{R}, dx_j)$. This operator is written in the following way: First for a $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, or in the Schwartz space. By continuity the operator extends to the whole $L^2(\mathbb{R}, dx_j)$ we denote this operator by $\hat{A}$:

$$(\hat{A}f)(x_j) = e^{-\pi i/12} \int_{\mathbb{R}} e^{2\pi i y_j x_j} e^{\pi i x_j^2} f(y_j) dy_j. \tag{4.26}$$

This operator is the unique unitary operator up to scalar multiplication by a complex number of modulus one which satisfies

$$\hat{A} Q \hat{A}^{-1} = P - Q, \quad \hat{A} P \hat{A}^{-1} = -Q. \tag{4.27}$$

where $P = \frac{1}{2\pi i} \frac{d}{dx}$ and $Q = x$ are symmetric operators on a dense subset of $L^2(\mathbb{R}, dx)$. The equations (4.27) can still be written as

$$\hat{A} e^{isQ} \hat{A}^{-1} = e^{is(P-Q)}, \quad \hat{A} e^{isP} \hat{A}^{-1} = e^{-isQ}, \quad s \in \mathbb{R}. \tag{4.28}$$

which translate to

$$\hat{A} : e^{isx} f(x) \mapsto e^{s^2/(4\pi i)} (\hat{A} f) \left(x + \frac{s}{2\pi}\right), \quad f\left(x + \frac{s}{2\pi}\right) \mapsto e^{-isx} (\hat{A} f)(x). \tag{4.29}$$

The operators that can be written as the exponential of a quadratic expression in P and Q are analogues of the Fourier transform $\mathcal{F} : f \mapsto \left(x \mapsto \int_{\mathbb{R}} e^{-2\pi i x y} f(y) dy\right)$ which is characterised up to a multiplicative constant by $\mathcal{F} Q \mathcal{F}^{-1} = -P$ and $\mathcal{F} P \mathcal{F}^{-1} = Q$.

For the operator $\rho(T_{[j][k]})$ in Theorem 4.24 we view its right hand side as an operator acting on $L^2(\mathbb{R}^2, dx_j dx_k)$. The unitary operator $e^{2\pi i P_j Q_k}$ acts in the following way:

$$(e^{2\pi i P_j Q_k}) f(x_j, x_k) = f(x_j + x_k, x_k).$$

For the remaining part of the operator we write $\Phi_{\mathrm{b}}(Q_j + P_k - Q_k)^{-1} = \hat{A}_k \Phi_{\mathrm{b}}(Q_j + Q_k) \hat{A}_k^{-1}$ using (4.27). We know how the unitary operators $\hat{A}_k, \hat{A}_k^{-1}$ act. We still need to know what the operator $\Phi_{\mathrm{b}}(Q_j + Q_k)^{-1}$ do, but this is just multiplication by $\Phi_{\mathrm{b}}(x_j + x_k)^{-1}$.

We have the result:

Proposition 4.27 (Kashaev)**.** The map ρ satisfies

$$\rho(A_{[j]})^3 = \mathrm{id}, \tag{4.30}$$

$$\rho(T_{[k][l]}) \rho(T_{[j][k]}) = \rho(T_{[j][k]}) \rho(T_{[j][l]}) \rho(T_{[k][l]}), \tag{4.31}$$

$$\rho(A_{[j]}) \rho(T_{[j][k]}) \rho(A_{[k]}) = \rho(A_{[k]}) \rho(T_{[k][j]}) \rho(A_{[j]}), \tag{4.32}$$

$$\rho(T_{[j][k]}) \rho(A_{[j]}) \rho(T_{[k][j]}) = \zeta \rho(A_{[j]}) \rho(A_{[k]}) \rho(P_{(jk)}), \tag{4.33}$$

where $\zeta = e^{-\pi i (\mathrm{b} + \mathrm{b}^{-1})^2/12}$, as well as the trivial relations:

$$\rho(P_{(jk)})^2 = \mathrm{id}, \tag{4.34}$$

$$\rho(P_{(jk)}) f_{\ldots,[j],\ldots,[k],\ldots} \rho(P_{(jk)}) = f_{\ldots,[j],\ldots,[k],\ldots}, \tag{4.35}$$

$$\rho(P_{(jk)}) = \rho(P_{(kj)}). \tag{4.36}$$

Therefore $\rho : F_{dot} \to GL(\mathcal{H})$ is an "almost G_{dot}-homomorphism" into $GL(L^2(\mathcal{H})$ in other words $\rho(R_{dot}) = \mathbb{C}^*$.

Remark 4.28. In Chapter 3 we saw that a change of polarization was related via the Fourier transform. In quantization of Teichmüller space the story is more involved. We saw in (4.11), (4.12) that elementary moves changes coordinates not in a trivial way. We need some kind of logarithm for handling the sum in equation (4.12). What saves us is the Faddeev quantum dilogarithm function which let us translate from one set of coordinates to another.

From the quantization of Teichmüller space Andersen and Kashaev build tetrahedral operators satisfying conditions related to the change of coordinates on the Teichmüller space. We will look at this in greater detail in Chapter 7.

Chapter 5

Hyperbolic geometry

5.1 Hyperbolic geometry

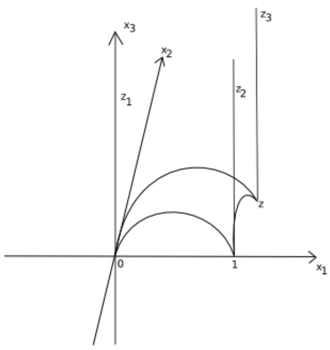
Recall that the hyperbolic 3-space $\mathbb{H}^3$ can be viewed as the upper half space $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ with metric

$$ds^2 = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2), \quad x_3 > 0, \tag{5.1}$$

of constant curvature -1. The boundary $\partial\mathbb{H}^3$, topologically a two-sphere, consists of the plane $x_3 = 0$ together with a point at infinity. The group of isometries of $\mathbb{H}^3$ is $\mathrm{PSL}(2, \mathbb{C})$ which acts on the boundary via Möbius transformations. In the upper half-space model the geodesic surfaces are spheres of any radius which intersect the boundary $\partial\mathbb{H}^3$ orthogonally.

An ideal tetrahedron Δ in $\mathbb{H}^3$ has by definition all its faces along geodesic surfaces, and all vertices lies on the boundary of $\mathbb{H}^3$. Using Möbius transformations one can always fix three of the vertices of an ideal tetrahedron to be $(0, 0, 0), (1, 0, 0)$ and ∞. The last vertex having the coordinate $(x_1, x_2, 0)$, with $x_2 \geq 0$. This fourth vertex defines a complex number $z = x_1 + ix_2$ which is usually called the *shape parameter*. At the various edges the faces of the tetrahedron form dihedral angles $\arg z_j$, $(j = 1, 2, 3)$. The invariants $z_j, j = 1, 2, 3$, are given by

$$z_1 = z, \quad z_2 = 1 - \frac{1}{z}, \quad z_3 = \frac{1}{1 - z}. \tag{5.2}$$



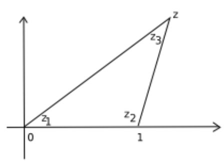


Figure 5.1: On the left hand side we see an ideal tetrahedron in upper half space. On the right hand side we se the projection of the tetrahedron onto the boundary plane.

Although all points of the ideal tetrahedron Δ lie on the boundary which implies Δ to be noncompact, the (hyperbolic) volume is finite: The hyperbolic volume of a tetrahedron

Δ_z with shape variable z, is given by

$$\operatorname{Vol}\Delta_z = D(z), \tag{5.3}$$

where $D(z)$ is the Bloch–Wigner dilogarithm function, related to the usual dilogarithm function (see Section 11) Li_2 by the relation

$$D(z) = \operatorname{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|. \tag{5.4}$$

One should here note that any z_j can be taken to be the shape parameter of Δ, and that $D(z_j) = \operatorname{Vol}(\Delta_z)$ for each j. We allow the shape parameter z to take values in $\mathbb{C}\backslash\{0,1\}$, noting that for $z \in \mathbb{R}$ the tetrahedron is degenerate and that for $\operatorname{Im} z < 0$ the tetrahedron will have negative volume due to orientation.

5.2 Geometrization of knot complements

A hyperbolic structure on a 3-manifold is a metric that is locally isometric to $\mathbb{H}^3$. Most 3-manifolds are hyperbolic. Among these are the majority of knot and link complements in S^3. We call a 3-manifold hyperbolic if it admits a hyperbolic structure that is geodesically complete and has finite volume. We say that a knot $K \in S^3$ is hyperbolic if the knot complement $S^3\backslash K$ is homeomorphic to $\mathbb{H}^3/\Gamma$, where Γ is a torsion free subgroup of the group of orientation preserving isometries of $\mathbb{H}^3$. Thurston proved that a knot complement is hyperbolic as long as the knot is not a torus or a satellite knot [47]. Every closed 3-manifold can be obtained by Dehn surgery on a knot in S^3. Employing Dehn surgery on a hyperbolic knot in S^3 yields hyperbolic manifolds for all but finitely many such surgeries [48].

It is clear that if two hyperbolic knot complements are isometric then their complements in S^3 are homeomorphic as well. What is far from clear is that the opposite should be true. However by the Mostow–Prasad Rigidity theorem this is indeed the case. In other words; two knot complements $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$ are homeomorphic if and only if Γ_1 and Γ_2 are conjugate in the isometry group.

One can also phrase the property of hyperbolicity of a knot in terms of representations of its fundamental group $\rho : \pi_1(S^3\backslash K) \to \mathrm{PSL}(2,\mathbb{C})$. Having a complete hyperbolic structure amounts to have a unique (up to conjugation) faithful representation. The importance of these observations is that the hyperbolic structure is a topological property of the knot. Hence geometric information can be used to distinguish knots. In a more mathematical language, geometric invariants become topological invariants and since the hyperbolic volume of a knot complement is a geometric invariant it also becomes a topological invariant.

It is not clear how to define the volume in the case where the knot is not hyperbolic. The standard thing to do is to extend the hyperbolic volume by stating it to be additive under connected sum so we can restrict to prime knots. So suppose we have a non-hyperbolic prime knot. By Thurston's theorem such a knot is either a torus knot or a satellite knot. We now define the volume of a torus knot to be zero and the volume of a satellite knot to be the volume of its companion plus the volume of its pattern. This provides a natural extension of the volume to all knot complements that is known to agree with the Gromov norm of knot complement.

5.3 Ideal triangulation

Any orientable hyperbolic 3-manifold M is homeomorphic to the interior of a compact manifold $\overline{M}$ with boundary consisting of finitely many tori. M itself can be viewed or thought of as $\overline{M}$ union neighbourhood of the cusps each of the neighbourhoods homeomorphic to $\mathbb{T}^2 \times [0,\infty)$. Therefore we can construct hyperbolic manifolds from knot or link complements in closed 3-manifolds. Furthermore every hyperbolic manifold has an ideal triangulation which means a finite decomposition into ideal tetrahedra where some of the tetrahedra might be degenerate. See e.g.[48]. It is believed and conjectured that non-degenerate tetrahedra are sufficient.

Given a finite set of tetrahedra $\{\Delta_i\}_{i=1}^n$ one can construct a manifold M by gluing faces of tetrahedra in pairs. Of course vertices of tetrahedra are not a part of the manifold M and that the combined boundaries of their neighbourhoods in M are tori. One can always find a triangulation of M where edges are oriented such that the boundary of each face (shared by two tetrahedra) has two edges oriented in the same direction and the last one opposite. Vertices of each tetrahedra can now canonically be labeled by numbers $0, 1, 2, 3$ corresponding to the number of edges entering the vertices. Now each tetrahedron can be identified with one of the tetrahedra in Figure 5.2.

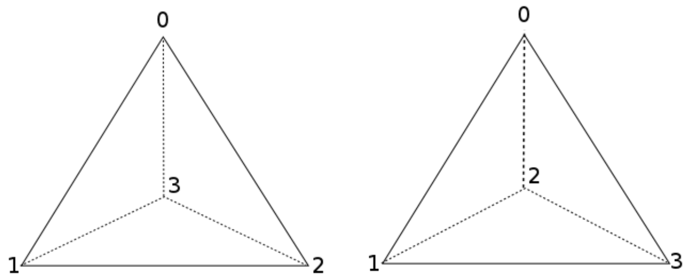


Figure 5.2: Different orientations of tetrahedra. On the left hand side a positively oriented tetrahedron, on the right hand side a negatively oriented tetrahedron.

This labelling of vertices induces an orientation of each tetrahedron. Having an orientation of each tetrahedron allows us to give shape parameters to the tetrahedra (z_1^i, z_2^i, z_3^i) running counterclockwise around each vertex if the tetrahedron is positively oriented and clockwise if the orientation is negative. See figure 5.2

For a manifold M with cusps specified by holonomy parameters u_j the shape parameters z_k^i of the tetrahedron Δ_i in its triangulation are fixed by two sets of conditions.

(i) The product of shape parameters z_j^i around every edge in the triangulation must equal 1 in order for the hyperbolic structure between adjacent tetrahedra to match.

(ii) One can compute the holonomy eigenvalues around each torus boundary in M as a product of z_j^i's by mapping out the neighbourhood of each vertex in what is called a developing map, and following the procedure illustrated in [35]. There is one distinct vertex inside each boundary torus. It is then required that the eigenvalues of the holonomy around the k-th component are equal to $e^{\pm u_k}$.

These conditions are what is usually referred to as the edge and cusp condition respectively.

5.3.1 The Bloch group and hyperbolic volume

When studying hyperbolic manifolds one of the interesting objects to consider is the volume spectrum

$$\textbf{Vol} = \{\mathrm{Vol}\, M \mid M \text{ is a hyperbolic 3-manifold}\} \subset \mathbb{R}_+.$$

From the work of Jørgensen and Thurston it is known that **Vol** is a countable and well-ordered subset of $\mathbb{R}_+$. And its exact nature is of great interest in both topology and number theory. Equation (5.3) as it stands says nothing about this since any real number can be written as a finite number of values $D(z), z \in \mathbb{C}$. However the shape parameters z_j of the tetrahedra triangulating a complete hyperbolic 3-manifold satisfy an extra relation, namely

$$\sum_{i=1}^{n} z_i \wedge (1 - z_i) = 0, \tag{5.5}$$

where the sum is taken in the abelian group $\Lambda^2\mathbb{C}^*$. Now (5.3) does give information about **Vol** because the set of numbers $\sum_{i=1}^{n} D(z_i)$ with z_i satisfying (5.5) is countable. This statement can be made more precise by introducing the *Bloch group*. Consider an abelian group of formal sums

$$[z_1] + \cdots + [z_n] \in \mathbb{C}^*\backslash\{1\}$$

satisfying (5.5). It is not hard to see that the elements

$$[x] + [1/x], \quad [x] + [1-x], \quad [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right] \tag{5.6}$$

with $x, y \in \mathbb{C}^*\backslash\{1\},\ xy \neq 1$ corresponding to the symmetry properties and the five-term relation satisfied by $D(\cdot)$, are contained in the Bloch group.

Definition 5.1. The Bloch group is defined as

$$\mathcal{B}_{\mathbb{C}} = \{[z_1] + \cdots + [z_n] \text{ satisfying (5.5)}\}/(\text{subgroup generated by the elements (5.6)}).$$

Every 3-manifold M has a well-defined class in the Bloch group. The five-term relation takes into account the fact that a polyhedron with five ideal vertices can be decomposed into ideal tetrahedra in multiple ways. The five ideal tetrahedron in this polygon, each obtained by deleting an ideal vertex, can be given the five shape parameters $x, y, \frac{1-x}{1-xy}, 1-xy, \frac{1-y}{1-xy}$ appearing in relation (5.6). The signs of the different terms correspond to orientations. Geometrically this five-term relation can be visualised as the "2-3" Pachner move, illustrated in Figure 5.3.

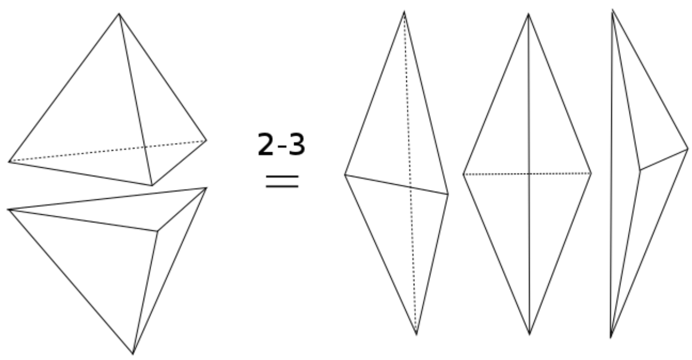


Figure 5.3: The "2-3" Pachner move.

The class $[M]$ of a hyperbolic 3-manifold M in the Bloch group can be computed by summing (with orientation) the shape parameters $[z_i]$ corresponding to any ideal triangulation, but is independent of triangulation. Hence hyperbolic invariants may be obtained from functions compatible with (5.6). And the hyperbolic volume of a 3-manifold M triangulated by $\{\Delta_i\}_{i=1}^{n}$ is given by

$$\operatorname{Vol} M = \sum_{i=1}^{n} \epsilon_i D(z^i), \tag{5.7}$$

where ϵ_i is either plus or minus 1 corresponding to the orientation of the tetrahedron in question.

Chapter 6

Topological quantum field theory

6.1 The historical background

The idea of topological invariants defined by use of path integrals was actually first introduces by A.S. Schwarz (1977) and formalised in its full power by E. Witten (1988) who introduced the notion of a Topological quantum field theory (TQFT). Such a theory, independent of Riemannian metrics, is rather rare in quantum physics. On the other hand such theories admit a rather simple axiomatic description first suggested by Atiyah [7]. This description was inspired by Segal's axioms for a 2-dimensional conformal field theory. The axiomatic formulation makes the theories suitable for purely mathematical research, which involves combining methods from topology, algebra and mathematical physics.

In the $80'$s Witten interpreted the Chern–Simons action (with compact gauge group) as the Lagrangian of a quantum field theory. In these theories the *partition function* plays an important role. This function is related to the partition function in statistical mechanics. In the case of quantum field theory the partition function is given by a path integral. Let M be a 3-manifold and G a (simple) Lie group. The quantum partition function is defined formally by the following path integral

$$Z_k(M) = \int_{\mathcal{A}_P/\mathcal{G}_P} e^{2\pi i k\, \mathrm{CS}_M(A)} \mathcal{D}A,$$

$k \in \mathbb{N}$. This is an ill-defined quantity since the space we integrate over is infinite-dimensional and therefore there is currently no canonical way to make sense of the integral. Nevertheless Witten argues on the physical level of rigour that the path integral defines a topological invariant of the 3-manifold.

Subsequently Reshetikhin and Turaev were able to define TQFTs using the representation theory of quantum groups [44],[43],[50] in the case where G is compact. We should of course also mention the skein theoretical construction of TQFT due to Blanchet, Habegger, Masbaum and Vogel [9]. Essentially equivalent to the construction of Reshetikhin and Turaev with gauge group $G = \mathrm{SU}(2)$.

6.2 TQFT from the axiomatic point of view

Let us look at the axioms for a TQFT following Turaev [49].

An $(n+1)$-dimensional TQFT (V, Z) over a scalar field k assigns to any closed oriented n-manifold Σ a finite dimensional vector space $V(\Sigma)$ over k and assigns to every cobordism (M, Σ, Σ') a k-linear map

$$Z(M) = Z(M, \Sigma, \Sigma') : V(\Sigma) \to V(\Sigma').$$

Here a cobordism (M, Σ, Σ') between Σ and Σ' is a compact oriented $(n+1)$-dimensional manifold endowed with a orientation preserving diffeomorphism $\partial M \simeq \overline{\Sigma} \sqcup \Sigma'$, where the overline indicates that the orientation is reversed. All manifolds and cobordisms are supposed to be smooth. In order for (V, Z) to be a TQFT, the following axioms must be satisfied.

(i) *Naturality.* Any orientation preserving diffeomorphism f of closed oriented n-dimensional manifolds Σ and Σ' induces an isomorphism $f_\sharp : V(\Sigma) \to V(\Sigma')$. For a diffeomorphism g between cobordisms (M, Σ_1, Σ_2) and $(M', \Sigma_1', \Sigma_2')$ the following diagram must commute:

$$\begin{tikzcd} V(\Sigma_1) \arrow[r, "(g|_{\Sigma_1})_\sharp"] \arrow[d, "Z(M)"'] & V(\Sigma_1') \arrow[d, "Z(M')"] \\ V(\Sigma_2) \arrow[r, "(g|_{\Sigma_2})_\sharp"] & V(\Sigma_2') \end{tikzcd}$$

(ii) *Functoriality.* If a cobordism (W, Σ, Σ') is obtained by gluing two cobordisms $(M, \Sigma, \tilde{\Sigma})$ and $(M', \tilde{\Sigma}', \Sigma')$ along a diffeomorphism $f : \tilde{\Sigma} \to \tilde{\Sigma}'$ then the following diagram should commute:

$$\begin{tikzcd} V(\Sigma) \arrow[r, "Z(W)"] \arrow[d, "Z(M)"'] & V(\Sigma') \arrow[d, "Z(M')"] \\ V(\tilde{\Sigma}) \arrow[r, "f_\sharp"] & V(\tilde{\Sigma}') \end{tikzcd}$$

(iii) *Normalization.* For any n-dimensional manifold Σ, the linear map

$$Z([0,1] \times \Sigma) : V(\Sigma) \to V(\Sigma)$$

is the identity.

(iv) *Multiplicativity.* There are functorial isomorphisms

$$V(\Sigma \sqcup \Sigma') \simeq V(\Sigma) \otimes V(\Sigma'),$$
$$V(\emptyset) \simeq k,$$

such that the following diagrams commute:

$$\begin{tikzcd} V((\Sigma \sqcup \Sigma') \sqcup \Sigma'') \arrow[r, phantom, "\simeq"] \arrow[d] & (V(\Sigma) \otimes V(\Sigma')) \otimes V(\Sigma'') \arrow[d] \\ V(\Sigma \sqcup (\Sigma' \sqcup \Sigma'')) \arrow[r, phantom, "\simeq"] & V(\Sigma) \otimes (V(\Sigma') \otimes V(\Sigma'')) \end{tikzcd} \qquad \begin{tikzcd} V(\Sigma \sqcup \emptyset) \arrow[r, phantom, "\simeq"] \arrow[d] & V(\Sigma) \otimes k \arrow[d] \\ V(\Sigma) \arrow[r, phantom, "="] & V(\Sigma) \end{tikzcd}$$

Here $\otimes = \otimes_k$ is the tensor product over k. The vertical maps are the ones induced by the obvious diffeomorphisms, and the standard isomorphisms of vector spaces respectively.

(v) *Symmetry.* The isomorphism

$$V(\Sigma \sqcup \Sigma') \simeq V(\Sigma' \sqcup \Sigma)$$

induced by the obvious diffeomorphism corresponds to the standard isomorphism of vector spaces

$$V(\Sigma) \otimes V(\Sigma') \simeq V(\Sigma') \otimes V(\Sigma).$$

Given a TQFT (V, Z), we obtain an action of the group of diffeomorphisms of a closed oriented n-dimensional manifold Σ on the vector space $V(\Sigma)$. This action can be used to study this group. An important feature of a TQFT (V, Z) is that it provides numerical invariants of compact oriented $(n+1)$-dimensional manifolds without boundary. This is so because such a manifold M can be considered as a cobordism between two copies of $\emptyset$. In this case $Z(M) \in \mathrm{Hom}_k(k, k) = k$.

6.3 Quantum Chern–Simons

In Chapter 2 we introduced the basics of classical Chern–Simons gauge theory. We defined the Chern–Simons action (2.13) which was done by integrating the Chern–Simons lagrangian or form (2.12) over space-time (a compact oriented 3-manifold). Since the Chern-Simons form lives on the total space of a bundle, and not on the base, we choose a section of the bundle to define the action. On closed 3-manifolds we saw, that the integral was independent of the section, up to an integer, if an appropriate normalization was done on the bilinear form. We ended up by defining a line bundle on the moduli space of flat connections, both if M was closed and if M has boundary $\partial M = \Sigma$.

The story is very different in the non-compact case. And there is still no mathematical definition of the path integral in this case. We will now follow Wittens approach to quantum Chern–Simons theory:

6.4 The complex story

We now shift gear since what we really want is the complex variant of the story. Now let G denote a compact gauge group and denote its non-compact complexification by $G_{\mathbb{C}}$, the respective Lie algebras will be denoted by $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$. We assume that the gauge groups $G, G_{\mathbb{C}}$ are reductive.

It turns out that the classical action of Chern–Simons theory with a complex gauge group is purely topological, as is in the compact case. In other words; the action is independent of the metric of the underlying 3-manifold M.

Definition 6.1. The Chern–Simons action for a complex gauge field $\mathcal{A}$ on a 3-manifold M can be written as a sum of two classically topological terms, one for $\mathcal{A}$ and one for the complex conjugate $\overline{\mathcal{A}}$:

$$L \;=\; \frac{t}{8\pi}\int_M \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)+\frac{\bar{t}}{8\pi}\int_M \mathrm{Tr}\left(\overline{\mathcal{A}}\wedge d\overline{\mathcal{A}}+\frac{2}{3}\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\right). \tag{6.1}$$

The field $\mathcal{A}$ is a locally defined $\mathfrak{g}_{\mathbb{C}}$-valued one-form on the manifold M. The two coupling constants t and $\bar{t}$ is conveniently written as

$$t = k + is, \quad \bar{t} = k - is,$$

with k, s being real and $\mathcal{A} = A + iB$ with A, B being $\mathfrak{g}$-valued one forms. The Lagrangian then takes the form

$$\begin{aligned} L = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA - B\wedge dB + \frac{2}{3}A\wedge A\wedge A - 2A\wedge B\wedge B\right) \\ - \frac{s}{2\pi}\,\mathrm{Tr}\left(A\wedge dB + 2A\wedge A\wedge B - \frac{2}{3}B\wedge B\wedge B\right). \end{aligned} \tag{6.2}$$

The parameter k is subject to the same quantization law as in Chern–Simons theory with compact gauge group G. So if " Tr " is normalized correctly (if $G = SU(N)$ then Tr should denote the trace in the N dimensional representation), then k must be an integer. In [52] it is furthermore shown that s has to be either real or imaginary in order to obtain a unitary field theory. As an example, let us look at the case, where $G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$. This is the group that describes $(2+1)$-dimensional gravity in a space-time of Lorentz signature and with a positive cosmological constant. Writing the Lagrangian (6.2) it is convenient to take G to be the real form $\mathrm{SL}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{C})$, then A can be identified with the spin connection ω of general gravity, and B with the vierbein e. Then the term of (6.2) proportional to s is the Einstein–Hilbert action with a cosmological constant, and under the resulting unitary structure the coupling s must be real.

It is explained in [52] that introducing a non-compact gauge group is a perfectly acceptable option in Chern–Simons theory. In Yang–Mills theories, a non-compact gauge group would lead to a kinetic term that is not positive definite, and hence to unbounded energy (or an ill-defined path integral). In Chern-Simons theory with complex gauge group the kinetic term is indefinite, but this is no problem: The Hamiltonian of the theory vanishes due to topological invariance, so the "energy" is always exactly zero.

Given a 3-manifold M (possibly with boundary), Chern–Simons theory associates to M a "quantum $G_{\mathbb{C}}$-invariant" which we in this section denote as $Z_k^{\text{phys}}(M)$. Physically, this quantum invariant is the partition function of the Chern–Simons gauge theory on M, defined as the path integral

$$Z_k^{\text{phys}}(M) = \int_{\mathcal{A}_P/\mathcal{G}_P} e^{iL}\mathcal{D}A. \tag{6.3}$$

From a mathematical point of view, this path integral is ill-defined. There is currently no canonical way to make sense of the integral over the infinite dimensional space $\mathcal{A}_P/\mathcal{G}_P$. Nevertheless, since the action is independent of the metric of M, one might expect that the quantum $G_{\mathbb{C}}$ invariant $Z_k^{\text{phys}}(M)$ is a topological invariant of M.

The question then rises: How does one compute the invariant $Z_k^{\text{phys}}(M)$? In Chern–Simons theory with compact gauge group the invariant is computed by cutting M into simple pieces, where the path integral can be evaluated. Then by gluing rules the invariant is assembled. A similar set of gluing rules should exist in a theory with complex gauge group, but these are expected to be much more involved than those in the compact case. The reason being that the Hilbert space of this theory is infinite dimensional. One consequence is that where finite sums which appear in gluing rules for Chern–Simons theory with compact gauge group we will now have integrals over continuous parameters.

To be modest one can try to compute $Z_k^{\text{phys}}(M)$ perturbatively, by expanding the integral (6.3) in inverse powers of t and $\bar{t}$ around a saddle point which is a classical solution. In Chern–Simons theory, the classical solutions, or extrema of the action (6.2), are flat connections. These are connections satisfying the equations

$$d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0, \quad d\overline{\mathcal{A}} + \overline{\mathcal{A}}\wedge\overline{\mathcal{A}} = 0.$$

The flat $G_{\mathbb{C}}$ connections on a 3-manifold M are determined by their holonomies. So the flat connections are determined by a homomorphism from the fundamental group of the 3-manifold into the group $G_{\mathbb{C}}$, i.e.

$$\rho : \pi_1(M) \to G_{\mathbb{C}},$$

This homomorphism is of course only defined modulo gauge transformations, which act via conjugation by elements in $G_{\mathbb{C}}$. Therefore the moduli space of classical solutions can be written as

$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}; M) = \mathrm{Hom}(\pi_1(M); G_{\mathbb{C}})/G_{\mathbb{C}}.$$

Consider a gauge equivalence class of a given flat connection $\mathcal{A} \in \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}; M)$ corresponding to the homomorphism ρ. The classical Chern–Simons action is a sum of terms for $\mathcal{A}$ and $\overline{\mathcal{A}}$, and it turns out that the perturbative expansion $Z^{\rho}(M)$ of the partition function $Z_k^{\text{phys}}(M)$ will factorize into a product of holomorphic and antiholomorphic terms:

$$Z^{\rho}(M) = Z^{\rho}(M; t)Z^{\rho}(M; \bar{t}). \tag{6.4}$$

As argued in [53] and [11] the exact non-perturbative function $Z_k^{\text{phys}}(M)$ depends in a non-trivial way on both t and $\bar{t}$, and the best hope is that it can be written in the form

$$Z_k^{\text{phys}}(M) = \sum_{\rho} Z^{\rho}(M; t)Z^{\rho}(M; \bar{t}),$$

where the sum is over all classical solutions. In [11] the perturbative function $Z^\rho(M)$ is studied. Due to the factorization into its holomorphic and antiholomorphic part, it suffices to study the holomorphic part $Z^\rho(M;t)$. The perturbative expansion of this function is in inverse powers of t, so it becomes convenient to introduce a new expansion parameter

$$\hbar = \frac{2\pi}{t},$$

which plays the role of Planck's constant.

The semiclassical limit corresponds to $\hbar \to 0$. The perturbative function $Z^\rho(M;\hbar)$ is an asymptotic power series in $\hbar$. The general form is found by doing a stationary phase approximation to the integral (6.3) and turns out to be

$$Z^\rho(M;\hbar) = \exp\left(\frac{1}{\hbar}S_0^{(\rho)} - \frac{1}{2}\delta^{(\rho)}\log\hbar + \sum_{n=0}^{\infty} S_{n+1}^{(\rho)}\hbar^n\right).$$

This is the general form of the perturbative partition function in Chern–Simons gauge theory with any gauge group, compact or non-compact.

The leading term $S_0^{(\rho)}$ in the asymptotic expansion is the value of the classical Chern–Simons functional evaluated on a flat gauge connection $\mathcal{A}^{(\rho)}$ associated with a homomorphism ρ. The coefficient of the second term $\delta^{(\rho)}$ is an integer which like all other terms depends on the manifold M, the gauge group $G_\mathbb{C}$ and the classical solution ρ. The rest of the terms $S_n^{(\rho)}$ are obtained by summing over Feynman diagrams with n loops.

6.4.1 Quantization

We now turn to the problem of quantization of the basic Lagrangian (6.1):

$$L = \frac{t}{8\pi}\int_M \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) + \frac{\bar{t}}{8\pi}\int_M \mathrm{Tr}\left(\overline{\mathcal{A}}\wedge d\overline{\mathcal{A}} + \frac{2}{3}\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\right).$$

on a 3-manifold of the form $\Sigma\times\mathbb{R}$ where Σ is an oriented closed two dimensional surface. And $\mathcal{A}$ is a connection on a principal $G_\mathbb{C}$-bundle E over M.

Canonical quantization will associate a Hilbert space $\mathcal{H}_\Sigma$ to the Riemann surface Σ. It turns out that the Hilbert space $\mathcal{H}_\Sigma$ must depend only on Σ as a topological surface, with no chosen metric or complex structure. The Hilbert space $\mathcal{H}_\Sigma$ will be obtained by quantizing an appropriate symplectic manifold. The symplectic manifold we would like to quantize is the moduli space of stationary points of the Lagrangian (6.1). The Euler–Lagrange equations derived from (6.1) are as follows. If we vary the connection in S

$$\begin{aligned}\delta L =& \frac{t}{8\pi}\int_M \mathrm{Tr}\left(\delta\mathcal{A}\wedge d\mathcal{A} + \mathcal{A}\wedge d\delta\mathcal{A} + 2\delta\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\\ &+ \frac{\bar{t}}{8\pi}\int_M \mathrm{Tr}\left(\delta\overline{\mathcal{A}}\wedge d\overline{\mathcal{A}} + \overline{\mathcal{A}}\wedge d\delta\overline{\mathcal{A}} + 2\delta\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\right).\end{aligned}$$

Using the fact that d is an odd derivation, we get the equation

$$\mathrm{Tr}(\mathcal{A}\wedge d\delta\mathcal{A}) = -d\,\mathrm{Tr}(\mathcal{A}\wedge\delta\mathcal{A}) + \mathrm{Tr}(d\mathcal{A}\wedge\delta A),$$

which implies

$$\begin{aligned}\delta L = \frac{t}{4\pi}\int_M \mathrm{Tr}(\delta\mathcal{A}\wedge(d\mathcal{A}+\mathcal{A}\wedge\mathcal{A})) + \frac{t}{8\pi}\int_{\partial M}\mathrm{Tr}(\delta\mathcal{A}\wedge\mathcal{A})&\\ + \frac{t}{4\pi}\int_M \mathrm{Tr}(\delta\overline{\mathcal{A}}\wedge(d\overline{\mathcal{A}}+\overline{\mathcal{A}}\wedge\overline{\mathcal{A}})) + \frac{t}{8\pi}\int_{\partial M}\mathrm{Tr}(\delta\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}),&\end{aligned}$$

where the boundary terms drop out because we have assumed that $\partial M = \emptyset$. In summary the Euler–Lagrange equation is the flatness of the curvature

$$F_{\mathcal{A}} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0, \quad F_{\overline{\mathcal{A}}} = d\overline{\mathcal{A}} + \overline{\mathcal{A}} \wedge \overline{\mathcal{A}} = 0.$$

This tells us that the moduli space that must be quantized is the moduli space $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}; M)$ of flat $G_{\mathbb{C}}$-connections on Σ, up to gauge transformations. The moduli space $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}; M)$ has a t-dependent symplectic structure that can be deduced from the Lagrangian,

$$\omega = \frac{i}{4\hbar} \int_{\Sigma} \mathrm{Tr}(\delta A \wedge \delta A). \tag{6.5}$$

The Hilbert space $\mathcal{H}_{\Sigma}$ is then obtained by quantizing the moduli space of flat $G_{\mathbb{C}}$-connections on Σ with symplectic structure (6.5) as described by Witten in [52], done by regarding $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}; M)$ as the symplectic quotient of the space $\mathcal{A}_{P,\mathbb{C}}$ of all $G_{\mathbb{C}}-$connections on Σ, by the action of the group of gauge transformations. Quantization of $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}; M)$ proceeds by quantizing $\mathcal{A}_{P,\mathbb{C}}$ and the picking out the $G_{\mathbb{C}}$-invariant subspace.

According to Witten, the line bundle can be characterized by saying that the commutators of covariant derivatives in $\mathcal{A}_{P,\mathbb{C}}$ acting on sections of $\mathcal{L}_{\text{pr}}$, are

$$\left[\frac{\delta}{\delta\mathcal{A}_i^a(z)}, \frac{\delta}{\delta\mathcal{A}_j^b(w)}\right] = -\frac{t}{8\pi}\delta^{ab}\varepsilon_{ij}\delta(z,w),$$
$$\left[\frac{\delta}{\delta\overline{\mathcal{A}}_i^a(z)}, \frac{\delta}{\delta\overline{\mathcal{A}}_j^b(w)}\right] = -\frac{\bar{t}}{8\pi}\delta^{ab}\varepsilon_{ij}\delta(z,w),$$

with all other components vanishing. Here ϵ_{ij} is the Levi–Civita tensor density on the oriented surface Σ and we have expand the connection $\mathcal{A} = \sum \mathcal{A}^a T_a$, where $\{T_a\}$ is a basis of the real Lie algebra $\mathfrak{g}$ whose complexification is $\mathfrak{g}_{\mathbb{C}}$.

One then defines a pre-quantum Hilbert space $\mathcal{H}_{\text{pr}}$ consisting of square integrable sections of the pre-quantum line bundle. The constraint operators, which are the generators of the gauge group, are acting on the pre-quantum Hilbert space $\mathcal{H}_{\Sigma}$ and the gauge invariant subspace of $\mathcal{H}_{\text{pr}}$ is the subspace annihilated by these operators.

The pre-quantum Hilbert space is much bigger than the desired Hilbert space and the Hilbert space is obtained from $\mathcal{H}_{\text{pr}}$ by a choice of polarization. We will not go further into details about the polarization.

We now look at a 3-manifold M with boundary $\partial M = \Sigma$, and the associated state $|M\rangle \in \mathcal{H}_{\Sigma}$. In a semi-classical theory, quantum states correspond to *Lagrangian submanifolds* of the classical phase space. In this case, the phase space is $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}; M)$ and the Lagrangian submanifold associated to a 3-manifold M with boundary $\partial M = \Sigma$ consists of the classical solutions on M, which is the moduli space of flat connections on M,

$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M) = \mathrm{Hom}(\pi_1(M), G_{\mathbb{C}})/G_{\mathbb{C}}.$$

Then

$$\mathcal{L} = \iota(\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M)),$$

under the map $\iota : \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M) \to \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ induced by the natural inclusion $\pi_1(\Sigma) \to \pi_1(M)$.

6.4.2 Perspectives

Let us try to relate the story to what we saw in Chapter 2 where we looked at Chern–Simons theory with compact gauge group. In [41] it is shown with $G = \mathrm{SU}(2)$ that

$$S^{(k)}(A) = \int_{\mathbb{A} \in \mathcal{A}/\mathcal{G}} e^{2\pi i k \, \mathrm{CS}_M(A)} \mathcal{D}A \tag{6.6}$$

where $\iota_\Sigma^* \mathbb{A} = A$ gives a holomorphic section of the bundle $\mathcal{L}^k$:

$$\begin{tikzcd} & \mathcal{L}^k \arrow[d] \\ \mathcal{M}_{\text{Flat}}(G,\Sigma) = \mu^{-1}(0)/\mathcal{G} \quad \subseteq & \mathcal{A}/\mathcal{G}. \arrow[u, bend left, "S^k"] \end{tikzcd}$$

If we let $\mathcal{T}$ be a smooth manifold parametrizing Kähler structures on M. That is, assume there is a map $I : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ mapping $\sigma \to I_\sigma$ such that for every $\sigma \in \mathcal{T}$, (M, ω, I_σ) is Kähler. We denote by M_σ the Kähler manifold (M, ω, I_σ). Then it turns out that $S_\sigma^{(k)} \in \mathrm{H}^0(M_\sigma, \mathcal{L}_\sigma^k)$. In [4] Andersen proved that one can construct a Hitchin connection ∇^H and the quantum spaces associated with different complex structures can be identified through parallel transport of the Hitchin connection. For a deeper investigation on this subject se also [17].

However in the non-compact case $G = \mathrm{SL}(2, \mathbb{C})$ one obtains smooth sections of $\mathcal{T} \times C^\infty(M, \mathcal{L}^k) \to \mathcal{T}$. And the connection needed to identify quantum spaces is even more subtle than the Hitchin connection. This work is under construction by Andersen and Gammelgaard. Who have constructed a Hitchin–Witten connection, which is needed to identify quantum spaces.

The Chern–Simons theory with non-compact gauge group is expected to be related to the Andersen–Kashaev TQFT which we are going to study in the next chapters.

Chapter 7

Andersen–Kashaev TQFT

In this section we recall the work of Jørgen Ellegaard Andersen and Rinat Kashaev in their joint work *A TQFT from quantum field theory* [6]. We will recall the setup and state the main results and conjectures regarding their work before continuing to their new formulation of the theory which is presented in their joint work *A new formulation of the Teichmüller TQFT* [5].

7.1 Preliminaries

7.1.1 Oriented triangulated pseudo 3-manifolds

This subsection contains some of the preliminaries for defining the Andersen–Kashaev TQFT. We start by defining *oriented triangulated pseudo 3-manifolds* and equip them with some extra structure, which will be of great importance later on.

Let Y be a finite union of disjoint compact 3-simplices each having totally ordered vertices, which induces an orientation on the tetrahedra, and let Ψ be a collection of affine vertex order-preserving and orientation reversing affine homeomorphisms $\{\psi_1, \ldots, \psi_r\}$ such that

1. for each ψ_i, there are two distinct codimension-1 faces τ_i and δ_i in Y for which the map $\psi_i : \tau_i \to \delta_i$ is an affine homeomorphism, and
2. $\{\tau_i, \delta i\} \cap \{\tau_j, \delta_j\} = \emptyset$ for $i \neq j$.

The quotient space $X = Y/\Psi$, obtained from Y by identifying $x \in \tau_i$ with $\psi_i(x) \in \delta_i$ for each i, is called an *oriented triangulated pseudo 3-manifold* it is a specific CW-complex with oriented edges. For $i \in \{0, 1, 2, 3\}$, we will denote by $\Delta_i(X)$ the set of cells with dimension i in X. For any $i > j$ we denote

$$\Delta_i^j(X) = \{(a, b) \mid a \in \Delta_i(X), b \in \Delta_j(a)\}$$

with a natural projection map

$$\phi^{i,j} : \Delta_i^j(X) \to \Delta_j(X).$$

We also have the canonical partial boundary maps

$$\partial_i : \Delta_j(X) \to \Delta_{j-1}(X), \ \ 0 \leq i \leq j.$$

In the case where $S = [v_0, \ldots, v_j]$ is a j-dimensional simplex with ordered vertices, the boundary map takes the form

$$\partial_i S = [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_j], \ \ i \in \{0, \ldots, j\}.$$

7.1.2 Shaped pseudo 3-manifolds

Let X be an oriented triangulated pseudo 3-manifold.

Definition 7.1. A *shape structure* on X is an assignment to each edge of each tetrahedron of X a positive number called the *dihedral angle*

$$\alpha_X : \Delta_3^1(X) \to \mathbb{R}_+,$$

so that the sum of the three angles at the edges from each vertex of each tetrahedron is π. (For a *generalized shape structure* the map α_X goes to $\mathbb{R}$) An oriented triangulated pseudo 3-manifold with a shape structure will be called a *shaped pseudo 3-manifold.*

It is straightforward to see that opposite edges of any tetrahedron must have the same dihedral angle. So each tetrahedron acquires three dihedral angles associated to three pairs of opposite edges which sum up to π. This is of course closely connected to the shape variable from section 5.1. Indeed, the usual shape variable for a tetraahedron $[v_0, v_1, v_2, v_3]$, with dihedral angles α, β, γ associated to the edges $[v_0, v_1], [v_0, v_2], [v_0, v_3]$, is

$$z = \frac{\sin\beta}{\sin\gamma} e^{i\alpha}.$$

Definition 7.2. To each shape structure on X, we associate a *Weight function*

$$\omega_X : \Delta_1(X) \to \mathbb{R}_+,$$

which to each edge e of X associates the sum of dihedral angles around it,

$$\omega_X(e) = \sum_{b \in (\phi^{3,1})^{-1}(e)} \alpha_X(b).$$

Definition 7.3. An edge e of a shaped triangulated pseudo 3-manifold X will be called *balanced* if it is internal and $\omega_X(e) = 2\pi$. An edge which is not balanced will be called *unbalanced*. A shaped 3-manifold where all edges are balanced will be called *fully balanced*.

Remark 7.4. A shape structure whose weight function takes the value 2π on each edge of a closed triangulated pseudo 3-manifold is the same as the *angle structure* introduced by Casson, Rivin and Lackenby. To study the necessary and sufficient conditions for angle structures to exist on the interior M of a compact ideal (topological) triangulated 3-manifold with non-empty boundary one should consult [34]. Feng and Tillmann uses normal surface theory to state under which condition angle structures are possible. Cason and Rivin observed that the existence of an angle structure implies that all boundary components of the manifold in question are tori or Klein bottles and that the manifold is irreducible and atoroidal.

Theorem 7.5 (Hodgson, Rubinstein, Segerman)**.** is a cusped hyperbolic 3-manifold homeomorphic to the interior of a compact 3-manifold $\overline{M}$ with torus or Klein bottle boundary components. If

$$H_1(\overline{M}; \mathbb{Z}_2) \to H_1(\overline{M}; \partial\overline{M}; \mathbb{Z}_2)$$

is the zero map then M admits an ideal triangulation with a fully balanced shape structure.

For a proof see [23].

A corollary is that if M is a hyperbolic link complement in S^3, then M admits an ideal triangulation with a fully balanced shape structure.

7.1.3 $\mathbb{Z}/3\mathbb{Z}$ action on pairs of opposite edges of tetrahedra

For an oriented triangulated pseudo 3-manifold X we let $\Delta_3^{1/p}$ denote the set of pairs of opposite edges of all tetrahedra in the triangulation of X. Set theoretically this is just the quotient of the set $\Delta_3^1(X)$ with respect to the equivalence relation given by all pairs of opposite edges of all tetrahedra. The quotient map is written as

$$p : \Delta_3^1(X) \to \Delta_3^{1/p}(X).$$

We define a skew symmetric function $\epsilon : \Delta_3^{1/p}(X) \times \Delta_3^{1/p}(X) \to \{0, \pm 1\}$ by

$$\epsilon_{e,e'} = \begin{cases} 0 & \text{if } e = e' \text{ or if } e \text{ and } e' \text{ belong to different tetrahedra} \\ 1 & \text{if } e' \text{ is right after } e \text{ in cyclic order} \\ -1 & \text{if } e' \text{ is right before } e \text{ in cyclic order} \end{cases}$$

Remark 7.6. ϵ gives us a symplectic structure on $\mathbb{R}_+^{\Delta_3^{1/p}(X)}$ and on $\mathbb{R}^{\Delta_3^{1/p}(X)}$, the latter is the set of generalized shape structures.

7.1.4 Levelled shaped 3-manifolds

What we have defined so far is a shaped oriented triangulated pseudo 3-manifold. Let us now also define what is called the *level* for our shaped 3-manifold.

Definition 7.7. A *levelled shaped 3-manifold* is a pair (X, l_X) where X is a shaped pseudo 3-manifold X and a real number $l_X \in \mathbb{R}$ called the level.

This definition extends the shape structure by a real parameter which will make the TQFT to be defined well defined.

One can show that the TQFT will enjoy a certain gauge-invariance which is as follows.

Definition 7.8. Two levelled shaped pseudo 3-manifolds (X, α_X, l_X) and (Y, α_Y, l_Y) are called *gauge equivalent* if there exists an isomorphism $h : X \to Y$ of the underlying cellular structures and a function

$$g : \Delta_1(X) \to \mathbb{R}$$

such that

$$\Delta_1(\partial X) \subset g^{-1}(0), \quad \alpha_Y(h(a)) = \alpha_X(a) + \pi \sum_{b \in \Delta_3^1(X)} \epsilon_{p(a),p(b)} g\left(\phi^{3,1}(b)\right) \quad \forall a \in \Delta_3^1(X)$$

and

$$l_Y = l_X + \sum_{e \in \Delta_1(X)} g(e) \sum_{a \in (\phi^{3,1})^{-1}(e)} \left(\frac{1}{3} - \frac{\alpha_X(a)}{\pi}\right)$$

Proposition 7.9. The weights on edges are gauge invariant in the sense that

$$\omega_X = \omega_Y \circ h,$$

if X, Y are gauge equivalent.

Proof. Let e be an edge in X. We want to show, that $\omega_X(e) = \omega_Y(h(e))$. The left hand side gives us

$$\begin{aligned}\omega_X(e) &= \sum_{a\in(\phi^{3,1})^{-1}(e)} \alpha_X(a) = \sum_{a\in(\phi^{3,1})^{-1}(e)} \left(\alpha_Y(h(a)) - \pi \sum_{b\in\Delta_3^1(X)} \epsilon_{p(a),p(b)} g(\phi^{3,1}(b))\right)\\ &= \sum_{a\in(\phi^{3,1})^{-1}(e)} \alpha_Y(h(a)) = (\omega_Y \circ h)(e),\end{aligned}$$

because the sum $\pi \sum_{b\in\Delta_3^1(X)} \varepsilon_{p(a),p(b)} g(\phi^{3,1}(b))$ disappears: Either $\phi^{3,1}(b)$ is in $\Delta_1(\partial X)$ which makes $g(\phi^{3,1}(b)) = 0$, or else the edge e is shared by different tetrahedra and $\varepsilon = 0$, or $(\phi^{3,1})^{-1}(e)$ lies in the same tetrahedron but then $p(a) = p(b)$ because of the gluing conditions and $\varepsilon = 0$. Thereby the left hand side is equal to the right hand side for every $e \in \Delta_1(X)$. □

The gauge equivalence is called *based gauge equivalence* in the case where the isomorphism $h : X \to X$ is an isomorphism.

We observe that the (based) gauge equivalence relation on leveled shaped pseudo 3-manifolds induces a (based) gauge equivalence relation on shaped pseudo 3-manifolds under the map which forgets the level. Let the set of gauge equivalence classes of based levelled shape structures on X be denoted $LS_r(X)$ and let $S_r(X)$ denote the set of gauge equivalence classes of based shape structures on X.

7.1.5 Categroid

The Andersen–Kashaev TQFT will be well defined on a certain sub-categroid of the category of levelled shaped pseudo 3-manifolds. A categroid $\mathcal{C}$ consists of a family of objects $\mathrm{Obj}(\mathcal{C})$ and for any pair of objects A, B from $\mathrm{Obj}(\mathcal{C})$ a set of morphisms $\mathrm{Mor}_{\mathcal{C}}(A, B)$ such that the following two properties are satisfied:

1. For any three objects $A, B, C \in \mathrm{Obj}(\mathcal{C})$ there is a subset

$$K^{\mathcal{C}}_{A,B,C} \subset \mathrm{Mor}_{\mathcal{C}}(A, B) \times \mathrm{Mor}_{\mathcal{C}}(B, C),$$

called the composable morphisms and a composition map

$$\circ : K^{\mathcal{C}}_{A,B,C} \to \mathrm{Mor}_{\mathcal{C}}(A, C),$$

such that composition of morphisms is associative.

2. For any object $A \in \mathrm{Obj}(\mathcal{C})$ we have an identity morphism $\mathrm{id}_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ which is composable with any morphism $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ or $g \in \mathrm{Mor}_{\mathcal{C}}(B, A)$ and we have the equations

$$\mathrm{id}_A \circ f = f \quad \text{and} \quad g \circ \mathrm{id}_A = g.$$

Let $\mathcal{B}$ be the category where the objects are triangulated surfaces and composition is gluing along the relevant parts of the boundary by edge orientation preserving and face orientation reversing CW-homeomorphisms with the obvious composition of dihedral angles and addition of levels. For $X, Y \in \mathcal{B}$ the morphisms $\mathrm{Mor}(X, Y)$ are equivalence classes of levelled shaped pseudo 3-manifolds with boundary identified with $X \sqcup Y$. i.e.

$$[(X, \alpha_X, l_X)] \circ [(Y, \alpha_Y, l_Y)] = [(X \cup_\Sigma Y, \alpha_X \cup \alpha_Y, l_X + l_Y)]$$

Remark 7.10. There are of course different ways of splitting the boundary. Therefore a levelled shaped pseudo 3-manifold can be interpreted as different morphisms in $\mathcal{B}$. The canonical choice is the following: For a tetrahedron T in $\mathbb{R}^3$ with ordered vertices $[v_0, v_1, v_2, v_3]$ we can define its sign by

$$\mathrm{sign}(T) = \mathrm{sign}(\det(v_1 - v_0, v_2 - v_0, v_3 - v_0)).$$

Furthermore we define the sign on faces by

$$\mathrm{sign}(\partial_i T) = (-1)^i \,\mathrm{sign}(T), \quad i \in \{0, 1, 2, 3\}.$$

For a triangulated pseudo 3-manifold X, the sign of faces of the tetrahedra in the triangulation of X induces a sign function on the faces of the boundary of X,

$$\mathrm{sign}_X : \Delta_2(\partial X) \to \{\pm 1\}.$$

This gives us a splitting of the boundary into two parts, one negative and one positive;

$$\partial X = \partial_- X \cup \partial_+ X, \quad \Delta_2(\partial X) = \mathrm{sign}^{-1}(\pm 1).$$

We will think of the equivalence class of levelled shaped pseudo 3-manifolds X as the $\mathcal{B}$-morphisms between the objects $\partial_- X$ and $\partial_+ X$.

Before defining the source categroid we need the notion of an admissible pseudo 3-manifold.

Definition 7.11. An oriented triangulated pseudo 3-manifold X is called *admissible* if

$$H_2(X - \Delta_0(X), \mathbb{Z}) = 0.$$

7.1.6 The source categroid

Definition 7.12. The categroid $\mathcal{B}_a$ is the subcategroid of $\mathcal{B}$ whose morphisms consist of equivalence classes of admissible levelled shaped pseudo 3-manifolds.

$$K^{\mathcal{B}_a}_{A,B,C} = \{(X_1, X_2) \in \mathrm{Mor}_{\mathcal{B}_a}(A, B) \times \mathrm{Mor}_{\mathcal{B}_a}(B, C) \mid H_2(X_1 \circ X_2 - \Delta_0(X_1 \circ X_2), \mathbb{Z}) = 0\}$$

are the composable morphisms.

7.1.7 The target categroid

Recall that the space of complex tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is the space of continuous linear functionals on the complex Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Definition 7.13. The categroid $\mathcal{D}$ has as objects finite sets and for two finite sets n, m the set of morphisms from n to m is

$$\mathrm{Hom}_{\mathcal{D}}(n, m) = \mathcal{S}'(\mathbb{R}^{n \sqcup m}).$$

In [36] we examined under which circumstances tempered distributions could be composed. We here omit proofs and refer the reader to [42, 36]. Let $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$ denote the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^m)$ and let $\phi \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$, $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^m)$. Then the element $\phi(f)(g)$ is a separately continuous bilinear function on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m)$. The Nuclear theorem [42, Theorem V.12], tells us that there exists a unique tempered distribution $\tilde{\phi}$ such that

$$\phi(f)(g) = \tilde{\phi}(f \otimes g).$$

By this formula we have established an isomorphism

$$\tilde{\cdot} : \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m)) \to \mathcal{S}'(\mathbb{R}^{n\sqcup m}).$$

Given elements $T_1 \in \mathcal{S}'(\mathbb{R}^{n\sqcup m})$ and $T_2 \in \mathcal{S}'(\mathbb{R}^{m\sqcup l})$, for positive integers n, m, l we would like to be able to compose these tempered distributions. According to Hörmander we have pull-back maps induced by the projection:

$$\pi_{n,m}^* : \mathcal{S}'(\mathbb{R}^{m\sqcup n}) \to \mathcal{S}'(\mathbb{R}^{n\sqcup m\sqcup l}) \quad \text{and} \quad \pi_{m,l}^* : \mathcal{S}'(\mathbb{R}^{m\sqcup l}) \to \mathcal{S}'(\mathbb{R}^{n\sqcup m\sqcup l}).$$

In order to make sense of a product of tempered distributions we first introduce the *wave front set* of the pull-back. Let $Z_{\mathbb{R}^n}$ denote the zero section of the cotangent bundle of $\mathbb{R}^n$.

Definition 7.14. For a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$, its *wave front set* is defined to be the following subset of the cotangent bundle of $\mathbb{R}^n$:

$$\mathrm{WF}(T) = \{(x, \xi) \in T^*(\mathbb{R}^n) - Z_{\mathbb{R}^n} \mid \xi \in \Sigma_x(T)\},$$

where

$$\Sigma_x(T) = \bigcap_{\phi \in C_x^\infty(\mathbb{R}^n)} \Sigma(\phi T).$$

Here

$$C_x^\infty(\mathbb{R}^n) = \{\phi \in C_0^\infty \mid \phi(x) \neq 0\},$$

i.e. smooth functions with compact support which do not vanish at x, and $\Sigma(S)$ are all $\eta \in \mathbb{R}^n \backslash \{0\}$ having no conic neighborhood V such that

$$|\hat{S}(\xi)| \leq C_N(1 + |\xi|)^{-N}, \quad N \in \mathbb{Z}_+, \xi \in V.$$

In here a set Γ is conic if $\xi \in \Gamma$ implies that $a\xi \in \Gamma$ for all a>0. The following result makes it easier to calculate the wave front set of some special tempered distributions.

Lemma 7.15. Suppose u is a bounded density on a smooth sub-manifold Y of $\mathbb{R}^n$, then $u \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\mathrm{WF}(u) = \{(x, \xi) \in T^*(\mathbb{R}^n) \mid x \in \mathrm{supp}(u), \xi \neq 0 \text{ and } \xi(T_x Y) = 0\}$$

Definition 7.16. Let T and S be tempered distributions on $\mathbb{R}^n$. We define

$$\mathrm{WF}(T) \oplus \mathrm{WF}(S) = \{(x, \xi_1 + \xi_2) \in T^*(\mathbb{R}^n) \mid (x, \xi_1) \in \mathrm{WF}(T), (x, \xi_2) \in \mathrm{WF}(S)\}.$$

Theorem 7.17. Let S, T be tempered distributions on $\mathbb{R}^n$ and let $Z_{\mathbb{R}^n}$ denote the zero section of the cotangent bundle of $\mathbb{R}^n$. If

$$\mathrm{WF}(S) \oplus \mathrm{WF}(T) \cap Z_{\mathbb{R}^n} = \emptyset \tag{7.1}$$

then the product of the tempered distributions exists and $ST \in \mathcal{S}'(\mathbb{R}^n)$.

This enables us to say when two morphisms of $\mathcal{D}$ can be composed.

Definition 7.18. Denote by $\mathcal{S}(\mathbb{R}^n)_m$ the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indices α and β such that if $\alpha_i = 0$ then $\beta_i = 0$ for $n - m < i \leq n$. Define $\mathcal{S}'(\mathbb{R}^n)_m$ to be the continuous dual of $\mathcal{S}(\mathbb{R}^n)_m$ with respect to these semi-norms.

Observe that if $\pi : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is the projection onto the first $n-m$ coordinates, then $\pi^*(\mathcal{S}(\mathbb{R}^{n-m})) \subset \mathcal{S}(\mathbb{R}^n)_m$, i.e., we have a well-defined push forward map

$$\pi_* : \mathcal{S}'(\mathbb{R}^n)_m \to \mathcal{S}'(\mathbb{R}^n - m).$$

Theorem 7.17 leads to the following:

Definition 7.19. For $A \in \mathrm{Hom}_{\mathcal{D}}(n,m)$ and $B \in \mathrm{Hom}_{\mathcal{D}}(m,l)$ satisfying

$$(\mathrm{WF}(\pi_{n,m}^*(A)) \oplus \mathrm{WF}(\pi_{m,l}^*(B))) \cap Z_{n\sqcup m\sqcup l} = \emptyset,$$

and such that $\pi_{n,m}^*(A)\pi_{m,l}^*(B)$ extends continuously to $\mathcal{S}(\mathbb{R}^{n\sqcup m\sqcup l})_m$, we define

$$AB = (\pi_{n,l})_*(\pi_{n,m}^*(A)\pi_{m,l}^*(B)) \in \mathrm{Hom}_{\mathcal{D}}(n,l).$$

This does indeed define a categroid if we let the composable morphisms be the set

$K_{n,m,l}^{\mathcal{D}} = \{(A,B) \in \mathcal{S}'(\mathbb{R}^{n\sqcup m}) \times \mathcal{S}'(\mathbb{R}^{m\sqcup l}) |$The conditions from the definition above are satisfied$\}$

7.2 The TQFT functor

Definition 7.20. For any $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$ we define the unique adjoint $A^* \in \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}'(\mathbb{R}^n))$ by

$$A^*(f)(g) = \overline{\overline{f}A(\overline{g})}$$

for all $f \in \mathcal{S}(\mathbb{R}^m)$ and all $g \in \mathcal{S}(\mathbb{R}^n)$.

Definition 7.21. A functor $F : \mathcal{B}_a \to \mathcal{D}$ is said to be a $*$-functor if

$$F(\overline{X}) = F(X)^*$$

where $\overline{X}$ is again X with opposite orientation, and $F(X)*$ is the dual map of $F(X)$.

Andersen and Kashaev have proven the following theorem which is the main theorem of [6]. Here $\hbar := (\mathrm{b}+\mathrm{b}^{-1})^{-2}$, and Φ_{b} is Faddeev's quantum dilogarithm, see Section 11.

Theorem 7.22 (Andersen–Kashaev)**.** For any $\hbar \in \mathbb{R}_+$ there is a unique $*$-functor $F_\hbar : \mathcal{B}_a \to \mathcal{D}$ such that

$$F_\hbar(A) = \Delta_2(A) \quad \forall A \in \mathrm{Obj}(\mathcal{B}_a),$$

and for any admissible levelled shaped pseudo 3-manifold (X, l_X),

$$F_\hbar(X, l_X) = Z_\hbar(X) e^{i\pi \frac{l_X}{4\hbar}} \in \mathcal{S}'(\mathbb{R}^{\Delta_2(\partial X)}),$$

where $Z_\hbar(X)$ for a tetrahedron T with positive sign, is given by

$$Z_\hbar(T)(x) = \delta(x_0 + x_2 - x_1) \frac{\exp\left(2\pi i(x_3 - x_2)(x_0 + \frac{\alpha_3}{2i\hbar}) + \pi i \frac{\phi_T}{4\hbar}\right)}{\Phi_{\mathrm{b}}\left((x_3 - x_2) + \frac{1-\alpha_1}{2i\hbar}\right)},$$

where $\delta(t)$ is Dirac's delta-function,

$$\phi_T = \alpha_0\alpha_2 + \frac{1}{3}(\alpha_0 - \alpha_2) - \frac{2\hbar + 1}{6}, \quad \alpha_i := \frac{1}{\pi}\alpha_T(\partial_i\partial_0 T), \quad i = 1,2,3,$$

and

$$x_i := x(\partial_i(T)), \quad x : \Delta_2(\partial T) \to \mathbb{R}.$$

For a closed oriented triangulated pseudo 3-manifold with a shape structure α, associate the function

$$Z_\hbar(X,\alpha) := \int_{\mathbb{R}^{\Delta_2(X)}} \prod_{T\in\Delta_3(X)} Z_\hbar(T,x,\alpha)dx. \tag{7.2}$$

If the 3-manifold X is admissible the quantity $|Z_\hbar(X,\alpha)|$ is well defined in the sense that the integral is absolutely convergent. It depends only on the gauge equivalence class of α and it is invariant under "$3-2$" Pachner moves.

Andersen and Kashaev remarks [6, Rem. 1]

Remark 7.23. We emphasize that for an admissible pseudo 3-manifold X, our TQFT functor provides us with the following well defined function

$$F_\hbar : LS_r(X) \to \mathcal{S}'(\mathbb{R}^{\partial X}). \tag{7.3}$$

7.2.1 Invariants of knots in 3-manifolds

By considering one-vertex ideal triangulations of complements of hyperbolic knots in compact oriented closed 3-manifolds, we obtain knot invariants. In this case, for such an X, the Andersen–Kashaev invariant is a complex valued function on the affine $\mathbb{R}$-bundle $LS_r(X)$ over $S_r(X)$, which forms an open convex (if non-empty) subset of the affine space $\tilde{S}_r(X)$, which is modelled on the real cohomology of the boundary of a tubular neighborhood of the knot.

Another possibility is to consider a one-vertex Hamiltonian triangulation (H-triangulation) of pairs (a closed 3-manifold M, a knot K in M), i.e., a one-vertex triangulation of M, where the knot is represented by one edge, with degenerate shape structures, where the weight on the knot approaches zero and where simultaneously the weights on all other edges approach the balanced value 2π. This limit by itself is divergent as a simple pole (after analytic continuation to complex angles) in the weight of the knot, but the residue at this pole is a knot invariant which is a direct analogue of Kashaev's invariants [27] which were at the origin of the hyperbolic volume conjecture.

Jørgen Ellegaard Andersen and Rinat Kashaev have set forth the following conjecture:

Conjecture 7.24. Let M be a closed oriented 3-manifold. For any hyperbolic knot $K \subset M$, there exists a smooth function $J_{M,K}(\hbar, x)$ on $\mathbb{R}_{>0}\times\mathbb{R}$ which has the following properties.

(1) For any fully balanced shaped ideal triangulation X of the complement of K in M, there exists a gauge invariant real linear combination of dihedral angles λ, a (gauge non-invariant) real quadratic polynomial of dihedral angles ϕ such that

$$Z_\hbar(X) = e^{i\frac{\phi}{\hbar}} \int_{\mathbb{R}} J_{M,K}(\hbar,x)e^{-\frac{x\lambda}{\sqrt{\hbar}}}dx.$$

(2) For any one vertex shaped H-triangulation Y of the pair (M,K) there exists a real quadratic polynomial of dihedral angles ϕ such that

$$\lim_{\omega_Y\to\tau} \Phi_{\mathrm{b}}\left(\frac{\pi-\omega_Y(K)}{2\pi i\sqrt{\hbar}}\right) Z_\hbar(Y) = e^{i\frac{\phi}{\hbar}-i\pi/12} J_{M,K}(\hbar,0),$$

where $\tau : \Delta_1(Y) \to \mathbb{R}$ takes the value 0 on the knot K and the value 2π on all other edges.

(3) The hyperbolic volume of the complement of K in M is recovered as the limit

$$\lim_{\hbar\to 0} 2\pi\hbar \log|J_{M,K}(\hbar,0)| = -\operatorname{Vol}(M\backslash K).$$

Theorem 7.25 (Andersen–Kashaev)**.** The above conjecture is true for the hyperbolic knots 4_1 and 5_2.

For a proof see [6]

Theorem 7.26. (1) and (2) in conjecture 7.24 is satisfied for the two hyperbolic knots 6_1 and 6_2.

For a proof see the calculations in Chapter 9.

Remark 7.27. The volume conjecture has until now only been approached numerically by use of the computer software mathematica. A rigorous proof is still in process.

7.3 The tetrahedral operator

In Chapter 4 we looked at Kashaev's quantization of Teichmüller space of punctured surfaces with the Weil–Petersson symplectic structure. Kashaev showed, starting from Penner's parameterization of the (decorated) Teichmüller space [38], that the Teichmüller space of marked conformal types of hyperbolic metrics on a punctured surface with the Weil–Petersson symplectic form and the action of the mapping class group can be described as the Hamiltonian reduction of a finite dimensional symplectic manifold which we know how to quantize. Moreover the action of the mapping class group is realized through the quantum dilogarithm introduced by Faddeev (4.23).

Upon canonical quantization of the cotangent bundle $T^*\mathbb{R}^n$ with the standard symplectic structure in the position representation, the Hilbert space we get is $L^2(\mathbb{R}^n)$. We consider instead the pre-Hilbert space $\mathcal{S}(\mathbb{R}^n)$ and its dual space $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions. The position coordinates q_i and momentum coordinates p_i on the cotangent bundle become operators q_i and p_i respectively acting on $\mathcal{S}(\mathbb{R}^n)$ via the formulae

$$\mathrm{q}_i(f)(x) = x_i f(x) \text{ and } \mathrm{p}_i(f)(x) = \frac{1}{2\pi i}\frac{\partial}{\partial x_i}(f)(x), \quad \forall x \in \mathbb{R}^n \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

These operators extend continuously to operators on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, still satisfying the Heisenberg commutator relations,

$$[\mathrm{p}_i, \mathrm{p}_j] = [\mathrm{q}_i, \mathrm{q}_j] = 0, \quad [\mathrm{p}_i, \mathrm{q}_j] = (2\pi i)^{-1}\delta_{ij}.$$

Fix a $\mathrm{b} \in \mathbb{C}$ such that $\mathrm{Re}(\mathrm{b}) \neq 0$. From the spectral theorem we can define operators

$$\mathrm{u}_i = e^{2\pi\,\mathrm{b}\,\mathrm{q}_i}, \quad \mathrm{v}_i = e^{2\pi\,\mathrm{b}\,\mathrm{p}_i}.$$

These operators are contained in $\mathcal{L}(S_\alpha(\mathbb{R}^n), S_{\alpha-\mathrm{Re}(b)}(\mathbb{R}^n))$, where $S_\alpha(\mathbb{R}^n) = e^{\alpha\rho}S(\mathbb{R}^n)$ for any $\alpha \in \mathbb{R}$, and where ρ is a smooth function which coincides with the function $|x|$ on the complement of compact subset of $\mathbb{R}^n$. Commutator relations between these operators are

$$[\mathrm{u}_i, \mathrm{u}_j] = [\mathrm{v}_i, \mathrm{v}_j] = 0, \quad \mathrm{u}_i\,\mathrm{v}_j = e^{2\pi i\,\mathrm{b}^2\,\delta_{ij}}\,\mathrm{v}_j\,\mathrm{u}_i\,.$$

In [28], Kashaev introduces two operations for $\mathbf{w}_i = (\mathrm{u}_i, \mathrm{v}_i), i = 1, 2$ namely

$$\mathbf{w}_1 \cdot \mathbf{w}_2 := (\mathrm{u}_1\,\mathrm{u}_2, \mathrm{u}_1\,\mathrm{v}_2 + \mathrm{v}_1),$$

$$\mathbf{w}_1 * \mathbf{w}_2 := (\mathrm{v}_1\,\mathrm{u}_2(\mathrm{u}_1\,\mathrm{v}_2 + \mathrm{v}_1)^{-1}, \mathrm{v}_2(\mathrm{u}_1\,\mathrm{v}_2 + \mathrm{v}_1)^{-1}).$$

These operations correspond exactly to the change of coordinates in the Kashaev coordinates on Teichmüller space. See (4.11) and (4.12).

Proposition 7.28. [28] Let ψ be some solution to the functional equation

$$\psi(z + i\,\mathrm{b}/2) = \psi(z - i\,\mathrm{b}/2)(1 + e^{2\pi\,\mathrm{b}\,z}) \tag{7.4}$$

Then, the operator $\mathrm{T} = \mathrm{T}_{12} := e^{2\pi\,\mathrm{p}_1\,\mathrm{q}_2}\psi(\mathrm{q}_1 + \mathrm{p}_2 - \mathrm{q}_2) = \psi(\mathrm{q}_1 - \mathrm{p}_1 + \mathrm{p}_2)e^{2\pi\,\mathrm{p}_1\,\mathrm{q}_2}$ defines an element of $\mathcal{L}(\mathcal{S}(\mathbb{R}^4), \mathcal{S}(\mathbb{R}^4))$, which satisfies the equations

$$\mathbf{w_1} \cdot \mathbf{w_2}\,\mathrm{T} = \mathrm{T}\,\mathbf{w_1}, \quad \mathbf{w_1} * \mathbf{w_2}\,\mathrm{T} = \mathrm{T}\,\mathbf{w_2}. \tag{7.5}$$

Remark 7.29. The operator T furthermore satisfies the following system of equations

$$\mathrm{T}\,\mathrm{q}_1 = (\mathrm{q}_1 + \mathrm{q}_2)\mathrm{T} \tag{7.6}$$
$$\mathrm{T}(\mathrm{p}_1 + \mathrm{p}_2) = \mathrm{p}_2\,\mathrm{T} \tag{7.7}$$
$$\mathrm{T}(\mathrm{p}_1 + \mathrm{q}_2) = (\mathrm{p}_1 + \mathrm{q}_2)\mathrm{T} \tag{7.8}$$
$$\mathrm{T}e^{2\pi\,\mathrm{b}\,\mathrm{p}_1} = (e^{2\pi\,\mathrm{b}(\mathrm{q}_1 + \mathrm{p}_2)} + e^{2\pi\,\mathrm{b}\,\mathrm{p}_1})\mathrm{T} \tag{7.9}$$

A proof of proposition 7.28 is given in [6].

Remark 7.30. We will prove later in this chapter (11.4) that the inverse of Faddeev's quantum dilogarithm satisfies the functional equation (7.4), i.e., a particular solution is

$$\psi(z) = 1/\Phi_{\mathrm{b}}(z).$$

An important fact about the operator T with ψ given by $1/\Phi_{\mathrm{b}}(z)$ is that it satisfies the pentagon identity

$$\mathrm{T}_{12}\mathrm{T}_{13}\mathrm{T}_{23} = \mathrm{T}_{23}\mathrm{T}_{12}$$

which is a consequence of the five-term identity (11.14).

7.3.1 Charged tetrahedral operators

Let $a, c \in \mathbb{R}$ be positive real numbers such that $b := \frac{1}{2} - a - c$ is also positive. Then we define charged T-operators by the formulae:

$$\mathrm{T}(a, c) = e^{-\pi i c_{\mathrm{b}}^2(4(a-c)+1)/6}e^{4\pi i c_{\mathrm{b}}(c\,\mathrm{q}_2 - a\,\mathrm{q}_1)}\mathrm{T}e^{-4\pi i c_{\mathrm{b}}(a\,\mathrm{p}_2 + c\,\mathrm{q}_2)},$$

$$\overline{\mathrm{T}}(a, c) = e^{\pi i c_{\mathrm{b}}^2(4(a-c)+1)/6}e^{-4\pi i c_{\mathrm{b}}(a\,\mathrm{p}_2 - c\,\mathrm{q}_2)}\overline{\mathrm{T}}e^{4\pi i c_{\mathrm{b}}(c\,\mathrm{q}_2 + a\,\mathrm{q}_1)},$$

where $c_{\mathrm{b}} = i(\mathrm{b} + \mathrm{b}^{-1})/2$ is purely imaginary. These charged operators $\mathrm{T}(a, c), \overline{\mathrm{T}}(a, c)$ take $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}(\mathbb{R}^2)$.

By substituting in the operator T we have the formula

$$\mathrm{T}(a, c) = e^{2\pi i\,\mathrm{p}_1\,\mathrm{q}_2}\psi_{a,c}(\mathrm{q}_1 - \mathrm{q}_2 + \mathrm{p}_2),$$

where

$$\psi_{a,c}(x) = \psi(x - 2c_{\mathrm{b}}(a + c))e^{-4\pi i c_{\mathrm{b}} a(x - c_{\mathrm{b}}(a+c))e^{-\pi i c_{\mathrm{b}}^2(4(a-c)+1)/6}}.$$

In Dirac's bra-ket notation we have for $\mathrm{T}(a, c) \in \mathcal{S}'(\mathbb{R}^4)$:

$$\langle x_0, x_2 \mid T(a, c) \mid x_1, x_3 \rangle = \delta(x_0 + x_2 - x_1)\tilde{\psi}'_{a,c}(x_3 - x_2)e^{2\pi i x_0(x_3 - x_2)}.$$

where

$$\tilde{\psi}'_{a,c}(x) := e^{-\pi i x^2}\tilde{\psi}_{a,c}(x), \quad \tilde{\psi}_{a,c}(x) = \int_{\mathbb{R}} \psi_{a,c}(y)e^{-2\pi i x y}dy. \tag{7.10}$$

Remark 7.31. The condition that the positive real numbers a, b, c must sum to $\frac{1}{2}$ is to ensure that the Fourier integral above is absolutely convergent.

7.3.1.1 Rules for the dilogarithm function

The Fourier transformation formula for the quantum dilogarithm (Appendix A in [6]) leads to the identity

$$\tilde{\psi}'_{a,c}(x) = e^{-\frac{\pi i}{12}} \psi_{c,b}(x).$$

With respect to complex cunjugation, the following formula holds:

$$\overline{\psi_{a,c}(x)} = e^{-\frac{\pi i}{6}} e^{\pi i x^2} \psi_{c,a}(-x) = e^{-\frac{\pi i}{12}} \tilde{\psi}_{b,c}(-x).$$

From these it follows that

$$\overline{\tilde{\psi}_{a,c}(x)} = e^{\frac{\pi i}{12}} \overline{\psi_{c,b}(x)} = e^{-\frac{\pi i}{12}} e^{\pi i x^2} \psi_{b,c}(-x).$$

We are now in a position to calculate and obtain a formula for $\overline{T}(a,c)$:

$$\begin{aligned}
\langle x, y | \overline{T}(a,c) | u, v \rangle &= \overline{\langle u, v | T(a,c) | x, y \rangle} \\
&= \delta(u + v - x) \overline{\tilde{\psi}'_{a,c}(y - v)} e^{-2\pi i u (y-v)} \\
&= \delta(u + v - x) \psi_{b,c}(v - y) e^{-\frac{\pi i}{12}} e^{\pi i (v-y)^2} e^{-2\pi i u (y-v)}.
\end{aligned}$$

7.3.1.2 Charged pentagon equation

Proposition 7.32. The charged tetrahedron operators satisfy a pentagon equation given by

$$\mathrm{T}_{12}(a_4, c_4) \mathrm{T}_{13}(a_2, c_2) \mathrm{T}_{23}(a_0, c_0) = e^{\pi i c_{\mathrm{b}}^2 P_e / 3} \mathrm{T}_{23}(a_1, c_1) \mathrm{T}_{12}(a_3, c_3) \tag{7.11}$$

where $P_e = 2(c_0 + a_2 + c_4) - \frac{1}{2}$ and $a_i, c_i \in \mathbb{R}, \quad 0 \le i \le 4,$ such that

$$a_1 = a_0 + a_2,\ a_3 = a_2 + a_4,\ c_1 = c_0 + c_4,\ c_3 = a_0 + c_4,\ c_2 = c_1 + c_3. \tag{7.12}$$

Proof. Write

$$\mathrm{T}(a,c) = f(a-c) \tilde{\mathrm{T}}(a,c)$$

where

$$\tilde{\mathrm{T}}(a,c) := e^{-4\pi i c_{\mathrm{b}} (a\, \mathrm{q}_1 - c\, \mathrm{q}_2)} \mathrm{T} e^{-4\pi i c_{\mathrm{b}} (a\, \mathrm{p}_2 + c\, \mathrm{q}_2)} = \xi^{(a\, \mathrm{q}_1 - c\, \mathrm{q}_2)} \mathrm{T} \xi^{(a\, \mathrm{p}_2 + c\, \mathrm{q}_2)},$$

and

$$f(x) := e^{-\pi i c_{\mathrm{b}}^2 (4x+1)/6}.$$

Under condition (7.12) we have

$$\begin{aligned}
\frac{f(a_4 - c_4) f(a_2 - c_2) f(a_0 - c_0)}{f(a_1 - c_1) f(a_3 - c_3)} &= e^{-\pi i c_{\mathrm{b}}^2 (4(a_4 - c_4) + 4(a_2 - c_2) + 4(a_0 - c_0) + 3 - 4(a_1 - c_1) - 4(a_3 - c_3) - 2)/6} \\
&= e^{-\pi i c_{\mathrm{b}}^2 (4(a_3 - a_3 + c_2 - c_2 - a_2 - c_0 - c_4) + 1)/6} \\
&= e^{\pi i c_{\mathrm{b}}^2 (2(c_0 + a_2 + c_4) - \frac{1}{2})/3} \\
&= e^{\pi i c_{\mathrm{b}}^2 P_e / 3}.
\end{aligned}$$

We see that showing the (7.11) is equivalent to showing that

$$\tilde{\mathrm{T}}_{12}(a_4, c_4) \tilde{\mathrm{T}}_{13}(a_2, c_2) \tilde{\mathrm{T}}_{23}(a_0, c_0) = \tilde{\mathrm{T}}_{23}(a_1, c_1) \tilde{\mathrm{T}}_{12}(a_3, c_3). \tag{7.13}$$

Calculating the right hand side yields

$$\begin{aligned}
\tilde{\mathrm{T}}_{23}(a_1, c_1) \tilde{\mathrm{T}}_{12}(a_3, c_3) &= (\xi^{a_1\, \mathrm{q}_2 - c_1\, \mathrm{q}_3} \mathrm{T}_{23} \xi^{a_1\, \mathrm{p}_3 + c_1\, \mathrm{q}_3})(\xi^{a_3\, \mathrm{q}_1 - c_3\, \mathrm{q}_2} \mathrm{T}_{12} \xi^{a_3\, \mathrm{p}_2 + c_3\, \mathrm{q}_2}) \\
&= \xi^{a_3\, \mathrm{q}_1} \xi^{a_1\, \mathrm{q}_2 - c_1\, \mathrm{q}_3} \mathrm{T}_{23} \xi^{-c_3\, \mathrm{q}_2} \mathrm{T}_{12} \xi^{a_1\, \mathrm{p}_3\, c_1\, \mathrm{q}_3} \xi^{a_3\, \mathrm{p}_2 + c_3\, \mathrm{q}_2} \\
&= \xi^{a_3\, \mathrm{q}_1 + (a_1 - c_3)\, \mathrm{q}_2 - (c_1 + c_3)\, \mathrm{q}_3} \mathrm{T}_{23} \mathrm{T}_{12} \xi^{a_3\, \mathrm{p}_2 + c_3\, \mathrm{q}_2 + a_1\, \mathrm{p}_3 + c_1\, \mathrm{q}_3}.
\end{aligned}$$

In the first equality we used the trivial commutativity and in the second we used the fact that $\mathrm{T}_{23}\xi^{-c_3\,\mathrm{q}_2} = \xi^{-c_3(\mathrm{q}_2+\mathrm{q}_3)}\mathrm{T}_{23}$ coming from (7.6). Likewise one can compute the left hand side of the equation. Using the rules (7.6)-(7.9) and the Heisenberg commutativity relations yields that

$$\begin{aligned}&\tilde{\mathrm{T}}_{12}(a_4,c_4)\tilde{\mathrm{T}}_{13}(a_2,c_2)\tilde{\mathrm{T}}_{23}(a_0,c_0)\\&=\xi^{-c_4\,\mathrm{q}_2+a_4\,\mathrm{q}_1}\xi^{a_2(\mathrm{q}_1+\mathrm{q}_2)-c_2\,\mathrm{q}_3}\mathrm{T}_{12}\mathrm{T}_{13}\mathrm{T}_{23}\xi^{c_\mathrm{b}(a_2c_2+a_4c_4)+a_2(\mathrm{p}_2+\mathrm{p}_3)}\xi^{a_4(\mathrm{p}_2+\mathrm{q}_3)}\xi^{c_3\,\mathrm{q}_2}\end{aligned}$$

Note that (7.12) gives us that the factors on the left hand side of the equations are equal i.e.

$$\xi^{-c_4\,\mathrm{q}_2+a_4\,\mathrm{q}_1+a_2(\mathrm{q}_1+\mathrm{q}_2)-c_2\,\mathrm{q}_3}=\xi^{(a_4+a_2)\,\mathrm{q}_1+(a_2-c_4)\,\mathrm{q}_2-c_2\,\mathrm{q}_3}=\xi^{a_3\,\mathrm{q}_1+(a_1-c_3)\,\mathrm{q}_2-(c_1+c_3)\,\mathrm{q}_3}.$$

From the terms on the right hand side of the equations we get

$$\xi^{c_\mathrm{b}(a_2c_2+a_4c_4)+a_2(\mathrm{p}_2+\mathrm{p}_3)+a_4(\mathrm{p}_2+\mathrm{q}_3)+c_3\,\mathrm{q}_2+a_0\,\mathrm{p}_3+c_0\,\mathrm{q}_3}=\xi^{a_3\,\mathrm{p}_2+c_3\,\mathrm{q}_2+a_1\,\mathrm{p}_3+c_1\,\mathrm{q}_3},$$

equivalent to the equation

$$a_2c_2+a_4c_4+a_4(a_0-c_3)=a_2(a_4+c_0+c_3),$$

clearly satisfied under the relation (7.12). □

Define now two tempered distributions $A, B \in \mathcal{S}'(\mathbb{R}^n)$ by the formulae

$$A(x,y)=\langle x,y|A\rangle=\delta(x+y)e^{i\pi(x^2+\frac{1}{12})}\quad\text{and}\quad B(x,y)=\langle x,y|B\rangle=e^{i\pi(x-y)^2}.$$

Lemma 7.33. The following three identities are satisfied.

$$\int_{\mathbb{R}^2}\overline{\langle v,s\mid A\rangle}\langle x,s\mid \mathrm{T}(a,c)\mid u,t\rangle\langle t,y\mid A\rangle dsdt=\langle x,y\mid\overline{\mathrm{T}}(a,b)\mid u,v\rangle, \tag{7.14}$$

$$\int_{\mathbb{R}^2}\overline{\langle u,s\mid A\rangle}\langle s,x\mid \mathrm{T}(a,c)\mid v,t\rangle\langle t,y\mid B\rangle dsdt=\langle x,y\mid\overline{\mathrm{T}}(b,c)\mid u,v\rangle, \tag{7.15}$$

$$\int_{\mathbb{R}^2}\overline{\langle u,s\mid B\rangle}\langle s,y\mid \mathrm{T}(a,c)\mid t,v\rangle\langle t,x\mid B\rangle dsdt=\langle x,y\mid\overline{\mathrm{T}}(a,b)\mid u,v\rangle. \tag{7.16}$$

Proof. Let us just check (7.14). In order for the product on the left hand side to make sense one has to check the wave front set condition. One does this using Lemma 7.15. The rest is straight forward computation:

$$\begin{aligned}&\int_{\mathbb{R}^2}\overline{\langle v,s\mid A\rangle}\langle x,s\mid \mathrm{T}(a,c)\mid u,t\rangle\langle t,y\mid A\rangle dsdt\\&=\int_{\mathbb{R}^2}\delta(v+s)\delta(t+y)e^{\pi i(y^2-v^2)}\langle x,s\mid\mathrm{T}(a,c)\mid u,t\rangle dsdt\\&=\langle x,-v|T(a,c)|u,-y\rangle e^{\pi i(y^2-v^2)}\\&=\delta(x-v-u)\tilde{\psi}'_{a,c}(-y+v)e^{2\pi ix(-y+v)}e^{\pi i(y^2-v^2)}\\&=\delta(u+v-x)e^{-\frac{\pi i}{12}}\psi_{c,b}(v-y)e^{i\pi(v-y)(2u+v-y)}\\&=\delta(u+v-x)e^{-\frac{\pi i}{12}}\psi_{c,b}(v-y)e^{i\pi(v-y)^2}e^{-2\pi iu(y-v)}\\&=\langle x,y|T(a,c)|u,v\rangle.\end{aligned}$$

□

Remark 7.34. The partition function for the Andersen–Kashaev TQFT defined in this chapter also satisfies gauge transformation properties. Further, convergence properties under gluing of tetrahedra is proven by Andersen and Kashaev. We will not here elaborate on these subjects but instead refer the reader to [6].

Chapter 8

New formulation of the Andersen–Kashaev TQFT

In this chapter we will describe the new version of the Andersen–Kashaev TQFT. In this description the Weil–Gel'fand–Zak transform of Faddeev's quantum dilogarithm plays an important role. Using this transform Andersen and Kashaev propose a state-integral model for the Andersen–Kashaev TQFT.

The setup for this model is analogous to the setup in the original version, so we refer to Chapter 7 for the notation.

For any map

$$x : \Delta_1(X) \to \mathbb{R},$$

define a Boltzmann weight

$$B(T, x) = g_{\alpha_1,\alpha_3}(x_{02} + x_{13} - x_{03} - x_{12}, x_{02} + x_{13} - x_{01} - x_{23})$$

if T is positively oriented and the complex conjugate if it carries the opposite orientation. The variable $x_{ij} := x(v_i v_j)$ and $\alpha_i = \alpha_T(v_0 v_i)$.

The map $g_{a,c} \in C^\infty(\mathbb{T}^2, L)$ is defined by the map:

$$g_{a,c}(s,t) = \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a,c}(s+m) e^{\pi i t(s+2m)}, \tag{8.1}$$

where the map $\tilde{\psi}'_{a,c}(s)$ is defined in section 7.3.

Theorem 8.1 (Andersen – Kashaev)**.** Let X be a closed levelled, shaped, triangulated and oriented pseudo 3-manifold. The quantity

$$Z_\hbar^{\text{new}}(X) := e^{\frac{\pi i l_X}{4\hbar}} \int_{[0,1]^{\Delta_1(X)}} \left(\prod_{T\in\Delta_3(X)} B(T, x|_{\Delta_1(T)}) \right) dx \tag{8.2}$$

admits an analytic continuation to a meromorphic function of the complex shapes which is invariant under all shaped "$2-3$" and "$3-2$" Pachner moves (along balanced edges).

Remark 8.2. The state integral in (8.2) extends to arbitrary (non-closed) levelled, shaped, triangulated and oriented pseudo 3-manifolds. In the case, where the pseudo-manifold is not closed one only has to integrate over state variables living on internal edges. The result is a meromorphic section of a line bundle over a complex torus $(\mathbb{C}^*)^{\Delta_1(X)}$

Conjecture 8.3. The model proposed in this section is equivalent to the Andersen–Kashaev TQFT of [6].

The proof of Theorem 8.1 is given in [5]. We will not go through the proof, but only state the crucial part of the theorem where we see that the Boltzmann weights satisfies a certain integral identity which is called *the pentagon identity*. This identity is a direct analogue of the charged pentagon equation 7.32.

Proposition 8.4. For any $(x, y, z, w) \in \mathbb{R}^4$ the following integral identity is satisfied:

$$\int_{[0,1]} g_{a_4,c_4}(z-v, x+w-v) g_{a_2,c_2}(v, y+w) g_{a_0,c_0}(x-v, y+z-v) dv \tag{8.3}$$

$$= e^{\frac{-\pi i}{12\hbar} P_e} g_{a_1,c_1}(x, y) g_{a_3,c_3}(z, w), \tag{8.4}$$

where $P_e := 2(c_0 + a_2 + c_4) - \frac{1}{2}$, and the set of positive reals $\{a_i, c_i \mid i = 0, 1, \ldots, 4\}$ is such that

$$b_i := \frac{1}{2} - a_i - c_i > 0, \quad i = 0, 1, \ldots, 4,$$

and

$$a_1 = a_0 + a_2, \quad a_3 = a_2 + a_4, \quad c_1 = c_0 + a_4, \quad c_3 = a_0 + c_4, \quad c_2 = c_1 + c_3.$$

Proposition 8.4 is proven in the paper [5]. It is proved there in great detail, and therefore we omit the proof in this section.

8.1 Weil–Gel'fand–Zak transformation

To a function $f \in \mathcal{S}(\mathbb{R})$ the WGZ transformation associates a smooth section of the line bundle L over the two torus corresponding to the quasi-periodicity properties:

$$g(x+1, y) = e^{-\pi i y} g(x, y),$$
$$g(x, y+1) = e^{\pi i x} g(x, y).$$

We define the Weil–Gel'fand–Zak (WGZ) transformation by the formula

$$(Wf)(x, y) = e^{\pi i x y} \sum_{m \in \mathbb{Z}} f(x+m) e^{2\pi i m y}.$$

The inverse of the WGZ transformation is given by the formula

$$(W^{-1} g)(x) = \int_0^1 g(x, y) e^{-\pi i x y}\, dy. \tag{8.5}$$

In the language above, the Boltzmann weights are given by the section $g_{a,c}$ of the line bundle L over the 2-torus with the periodicity properties mentioned above, or the complex conjugate of this section. There are several symmetry properties for the WGZ transformation of Faddeev's dilogarithm function which are proven in [5]. Together with an analytic continuation, TQFT-rules and tetrahedral symmetries Andersen and Kashaev have proven Theorem 8.1

8.2 Results of calculations via the new formulation

The results we here impart will be proven in the Chapter 9. See also this chapter for notation.

Theorem 8.5. We have proven that the new formulation of the theory correspond to the old formulation for the knots 4_1 and 5_2. The partition function for the knot complement $X = (S^3, 4_1)$ given an ideal triangulation is given by

$$Z_\hbar^{\text{new}}(X) = \nu_{c_+,b_+} \nu_{b_-,c_-} \zeta_{inv} e^{-\frac{\pi i}{6}} W(\chi_{4_1})(u, v).$$

For an H-triangulation Y of the knot complement $(S^3, 4_1)$ the renormalised partition function takes the form

$$\tilde{Z}_\hbar^{\text{new}}(Y) = \frac{e^{-\frac{\pi i}{12}}}{\nu_{c_0,0}} \chi_{4_1}(0)$$

Where the function $\chi_{4_1}(x) = \int_{\mathbb{R}-i0} \frac{\Phi_\mathrm{b}(x-y)}{\Phi_\mathrm{b}(y)} e^{2\pi i x(2y-x)} dy$, and u, v are linear combinations of dihedral angles, $u = 2c_\mathrm{b}(b_+ - b_-)$ and $v = 2b_- + c_- = 2b_+ + c_+$.

The partition function for the knot complement $V = (S^3, 5_1)$ given an ideal triangulation is given by

$$Z_\hbar^{\text{new}}(V) = \nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3} W \chi_{5_2}(u, v).$$

For an H-triangulation U of the knot complement $(S^3, 5_2)$ the renormalised partition function takes the form

$$\tilde{Z}_\hbar^{\text{new}}(U) = \frac{e^{\pi i/4}}{\nu_{c_0,0}} \chi_{5_2}(0).$$

where $\chi_{5_2}(u)$ is given by the formula

$$\chi_{5_2}(u) = e^{-\frac{\pi i}{3}} \int_{\mathbb{R}-i0} \frac{e^{\pi i(w-u)(w+u)}}{\Phi_\mathrm{b}(w+m+u)\Phi_\mathrm{b}(w-m-u)\Phi_\mathrm{b}(w)} dw.$$

where $u = 2c_\mathrm{b}(a_1 - a_3)$ and $v = 2c_\mathrm{b}(a_1 - c_1 + b_2 - a_3)$.

The proof of these facts will follow from the computations in Chapter 9.

8.3 TQFT from the new formulation

The state integral in Theorem 8.5 extends to arbitrary non-closed levelled, shaped, oriented, triangulated pseudo 3-manifolds. In this case we only integrate over the state variables living on internal edges as remarked in [5, Rem. 1].

It is evident from the axioms in Section 6.2 that in order to have a TQFT (V, Z) over the field $\mathbb{C}$ we need to specify V. To every compact surface Σ we assign the vector space $V(\Sigma)$ and to every cobordism (M, Σ, Σ') we get a linear map

$$Z(M) = Z(M, \Sigma, \Sigma') : V(\Sigma) \to V(\Sigma').$$

In our case the pseudo 3-manifold comes with a triangulation. That means that the boundary surfaces have triangulations. The edges of these triangulations are equipped with state variables. Since the Boltzmann weights above are specified by sections $g_{a,c}$ of a line bundle L over a 2-torus the vector space we associate to a triangulated surface Σ is $C^\infty(\mathbb{T}^n, \mathcal{L})$, where n here denotes the number of edges in the triangulation of Σ.

Hence, if (M, Σ, Σ') is a cobordism where $|\Delta_1(\Sigma)| = n$ and $|\Delta_1(\Sigma')| = m$ we have

$$Z(M) : C^\infty(\mathbb{T}^n, \mathcal{L}) \to C^\infty(\mathbb{T}^m, \mathcal{L}).$$

Furthermore we deal with shaped triangulations. This puts restrictions on which cobordisms we can compose. Actually we have a shape structure on the cobordism manifold (M, Σ, Σ') which to each edge induces a weight.

The constraint on dihedral angles on boundary edges is that they must sum to 2π when we compose cobordisms. We will assume that generators of mapping class elements can be constructed such that all dihedral angles are positive. We will see in an application in Chapter 10 that this is indeed the case for the generators of $\Gamma_{1,1}$.

Remark 8.6. Although we assume above that all dihedral angles are positive there is no restriction on angles in the new formulation of the Andersen–Kashaev TQFT. The tetrahedral weights in the new formulation admits analytic continuation to meromorphic sections of line bundles over complex tori. This means that the partition function can be analytically continued to arbitrary complex shapes so that the theory is well-defined without imposing positivity conditions on shapes. A consequence of this is that the "$2-3$" and "$3-2$" Pachner moves are valid without restrictions.

8.4 Mapping class group representations from TQFTs

The axioms for a TQFT given in Section 6.2 hint at how to construct representations of mapping class groups of closed surfaces from a $(2+1)$-dimensional TQFT.

Let Σ be a closed oriented surface, and let $f : \Sigma \to \Sigma$ be an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma$. Now, let $\phi \in \Gamma(\Sigma)$ denote the mapping class of f. Put $\rho(\phi) = f_\sharp : V(\Sigma) \to V(\Sigma)$, and let

$$M_f = \Sigma \times \left[0, \frac{1}{2}\right] \cup_f \Sigma \times \left[\frac{1}{2}, 1\right]$$

be the mapping cylinder of f obtained by gluing together the two copies of $\Sigma \times \left\{\frac{1}{2}\right\}$ using f.

Proposition 8.7. The map $\rho : \Gamma(\Sigma) \to \mathrm{End}(V(\Sigma))$ is a well-defined representation of $\Gamma(\Sigma)$. Furthermore, if ϕ is the mapping class of f as above, then $\rho(\phi) = Z(M_f)$.

Proof. Let $f_t : \Sigma \to \Sigma$ be an isotopy between orientation-preserving diffeomorphisms f_0 and f_1. The map $\Sigma \times I \to \Sigma \times I$ given by

$$(x, t) \mapsto (f_1 f_t^{-1}(x), t)$$

extends the map $f_1 f_0^{-1} \sqcup \mathrm{id} : \Sigma \sqcup -\Sigma \to \Sigma \sqcup -\Sigma$, and it follows from the axioms that $(f_1 f_0^{-1})_\sharp = \mathrm{id}$ and $(f_0)_\sharp = (f_1)_\sharp$. The last statement follows since $f \sqcup \mathrm{id} : \Sigma \sqcup -\Sigma \to \Sigma \sqcup -\Sigma$ extends to an orientation preserving diffeomorphism $\Sigma \times I \to M_f$. It follows that

$$Z(M_f) = f_\sharp = \rho(\phi).$$

□

The axioms for a TQFT also hint that for a 3-manifold M with boundary $\partial M = \Sigma$, the action of the mapping class $[f]$ on a vector $Z(M) \in V(\Sigma)$ is given by

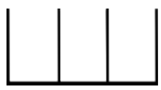
$$\rho(\phi)(Z(M)) = Z(M \cup_\Sigma M_f).$$

We will get back to specific mapping class group representations in Chapter 10.

Chapter 9

Calculations of specific knot complements

In the following examples we encode as in [6] an oriented triangulated pseudo 3-manifold X into a diagram where a tetrahedron is represented by an element



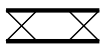
where vertical segments, ordered from left to right, correspond to the faces $\partial_0 T, \partial_1 T, \partial_2 T$ and $\partial_3 T$ respectively. When we glue tetrahedron along faces, we illustrate this by joining corresponding segments.

In the calculations we will denote:

$$v_{x,y} := e^{-\pi i c_{\mathrm{b}}^2 (4(x-y)+1)/6}.$$

9.1 The complement of the figure-8-knot

Let X be represented by the usual diagram



Choosing an orientation, the diagram consists of one positive tetrahedron T_+ and one negative T_-. $\partial X = \emptyset$ and combinatorially we have $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{e_0, e_1\}$. The gluing of the tetrahedra is vertex order preserving which means that edges are glued together in the following manner.

$$e_0 = x_{01}^+ = x_{03}^+ = x_{23}^+ = x_{02}^- = x_{12}^- = x_{13}^- =: x,$$
$$e_1 = x_{02}^+ = x_{13}^+ = x_{12}^+ = x_{01}^- = x_{03}^- = x_{23}^- =: y.$$

The topological space $X \backslash \{*\}$ is homeomorphic to the complement of the figure-eight knot. The set $\Delta_3^1(X)$ consists of elements $(T_\pm, e_{j,k})$ for $0 \leq j < k \leq 3$. We fix a shape structure

$$\alpha_X : \Delta_3^1(X) \to \mathbb{R}_+$$

by the formulae

$$\alpha_X(T_\pm, e_{0,1}) = 2\pi a_\pm, \quad \alpha_X(T_\pm, e_{0,2}) = 2\pi b_\pm, \quad \alpha_X(T_\pm, e_{0,3}) = 2\pi c_\pm,$$

where $a_\pm + b_\pm + c_\pm = \frac{1}{2}$. This result in the following weight functions

$$\omega_X(e_0) = 2a_+ + c_+ + 2b_- + c_-, \quad \omega_X(e_1) = 2b_+ + c_+ + 2a_- + c_-.$$

In the completely balanced case these equations correspond to

$$a_+ - b_+ = a_- - b_-.$$

The Boltzmann weights is given by the functions

$$\begin{aligned} B\left(T_+, x_{|\Delta_1(T_+)}\right) &= g_{a_+,c_+}(y-x, 2(y-x)), \\ B\left(T_-, x_{|\Delta_1(T_-)}\right) &= \overline{g_{a_-,c_-}(x-y, 2(x-y))}. \end{aligned}$$

We calculate the partition function for the Andersen–Kashaev TQFT using the new formulation.

$$\begin{aligned} Z_\hbar^{\text{new}}(X) &= \int_{[0,1]^2} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a_+,c_+}(y-x+m)\overline{\tilde{\psi}'_{a_-,c_-}(x-y+n)} e^{4\pi i(y-x)(m+n)}\, dxdy \\ &= \int_{[0,1]} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a_+,c_+}(y+m)\overline{\tilde{\psi}'_{a_-,c_-}(-y+n)} e^{4\pi i y(m+n)} dy \\ &= \sum_{m,n\in\mathbb{Z}} \int_{[m,m+1]} \tilde{\psi}'_{a_+,c_+}(y)\overline{\tilde{\psi}'_{a_-,c_-}(-y+m+n)} e^{4\pi i(y-m)(m+n)} dy \\ &= \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \tilde{\psi}'_{a_+,c_+}(y)\overline{\tilde{\psi}'_{a_-,c_-}(-y+p)} e^{4\pi i y p} dy \\ &= e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \psi_{c_+,b_+}(y)\psi_{b_-,c_-}(y-p) e^{\pi i (y-p)^2} e^{4\pi i y p} dy \\ &= e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \psi(y-2c_{\mathrm{b}}(c_+ + b_+))\psi(y-p-2c_{\mathrm{b}}(b_- + c_-)) e^{\pi i y^2} e^{\pi i p^2} e^{2\pi i y p} \\ &\quad \times e^{-4\pi i c_{\mathrm{b}} c_+ (y - c_{\mathrm{b}}(c_+ + b_+))} e^{-4\pi i c_{\mathrm{b}} b_- (y-p-c_{\mathrm{b}}(b_- + c_-))} \\ &\quad \times e^{-\pi i(4(c_+ - b_+)+1)/6} e^{-\pi i (4(b_- - c_-)+1)/6} dy. \end{aligned}$$

We set $Y = y - 2c_{\mathrm{b}}(c_+ + b_+)$. Assuming that we are in the completely balanced case we have

$$-b_- - c_- + c_+ + b_+ = -b_+ + b_-.$$

Furthermore we have $y^2 = Y^2 + 4c_{\mathrm{b}}^2(c_+ + b_+)^2 + 4c_{\mathrm{b}} Y(c_+ + b_+)$. Implementing this we get the following.

$$\begin{aligned} Z_\hbar^{\text{new}}(X) &= \nu_{c_+,b_+}\nu_{b_-,c_-} e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \psi(Y-p-2c_{\mathrm{b}}(b_+ - b_-))\psi(Y) \\ &\qquad \times e^{\pi i (Y^2 + 4c_{\mathrm{b}}^2(c_+ + b_+)^2 + 4c_{\mathrm{b}} Y(c_+ + b_+))} e^{\pi i p^2} \\ &\qquad \times e^{2\pi i (Y + 2c_{\mathrm{b}}(c_+ + b_+))p} \\ &\qquad \times e^{-4\pi i c_{\mathrm{b}} c_+ (Y + c_{\mathrm{b}}(c_+ + b_x))} e^{-4\pi i c_{\mathrm{b}} b_- (Y - p - c_{\mathrm{b}}(b_- + c_-) + 2c_{\mathrm{b}}(c_+ + b_+))} dY \\ &= \nu_{c_+,b_+}\nu_{b_-,c_-} e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \frac{1}{\Phi_{\mathrm{b}}(Y-p-2c_{\mathrm{b}}(b_+ - b_-))} \frac{1}{\Phi_{\mathrm{b}}(Y)} \\ &\qquad \times e^{\pi i Y^2} e^{\pi i p^2} e^{-4\pi i c_{\mathrm{b}} Y(-c_+ - b_+ + c_+ + b_-)} e^{2\pi i Y p} \\ &\qquad \times e^{-4\pi i c_{\mathrm{b}} p(-(c_+ + b_+) - b_-)} \\ &\qquad \times e^{4\pi i c_{\mathrm{b}}^2((c_+ + b_+)^2 - c_+(c_+ + b_+) - b_-(b_- + c_- - 2(c_+ + b_+))} dY. \end{aligned}$$

Now set $u = 2c_{\mathrm{b}}(b_+ - b_-)$ and $v = 2b_- + c_- = 2b_+ + c_+$, and use the formula

$$\Phi_{\mathrm{b}}(z)\Phi_{\mathrm{b}}(-z) = \zeta_{inv}^{-1} e^{\pi i z^2}.$$

along with the calculation:

$$b_- + b_+ + c_+ = b_- + b_+ - 2b_+ + 2b_- + c_- = -(b_+ - b_-) + (2b_- + c_-).$$

$$\begin{aligned} Z_\hbar^{\text{new}}(X) = & \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv}e^{-\frac{\pi i}{6}}\sum_{p\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\Phi_{\mathrm{b}}(p+u-Y)}{\Phi_{\mathrm{b}}(Y)}e^{-\pi i(Y^2+u^2+p^2-2Yu-2Yp+2up)} \\ & \times\, e^{\pi iY^2}e^{\pi ip^2}e^{2\pi iYu}e^{2\pi iYp}dY \\ & \times\, e^{-2\pi ipu}e^{2\pi ipv} \\ & \times\, e^{4\pi ic_{\mathrm{b}}^2((c_++b_+)^2-c_+(c_++b_+)-b_-(b_-+c_--2(c_++b_+))}. \end{aligned}$$

Using the balance condition and formulas for u and v we get the equality

$$\begin{aligned} & -4\pi ic_{\mathrm{b}}^2\{(c_++b_+)^2+b_-(-b_--c_-+2c_++2b_+)+c_+(c_++b_+)\} = \\ & -4\pi ic_{\mathrm{b}}^2\{(-(b_+-b_1)^2)-c_+b_++c_+b_-+b_-c_+-b_-c_-\} = \\ & -2\pi ic_{\mathrm{b}}\{-(c_++2b_-)u\}+\pi iu^2 = \\ & -2\pi ic_{\mathrm{b}}\{-(2b_-+c_-)u+2(b_+-b_-))u\}+\pi iu^2 = \pi i(uv-u^2). \end{aligned}$$

We get the following expression for the partition function:

$$\begin{aligned} Z_\hbar^{\text{new}}(X) = & \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv}e^{-\frac{\pi i}{6}}\sum_{p\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\Phi_{\mathrm{b}}(p+u-Y)}{\Phi_{\mathrm{b}}(Y)}e^{-\pi iu^2} \\ & \times\, e^{4\pi iYu}e^{4\pi iYp}e^{-4\pi ipu}e^{2\pi ipv}e^{\pi i(uv-u^2)}dY \\ = & \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv}e^{-\frac{\pi i}{6}}\sum_{p\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\Phi_{\mathrm{b}}(p+u-Y)}{\Phi_{\mathrm{b}}(Y)} \\ & \times\, e^{4\pi iYu}e^{4\pi iYp}e^{-4\pi ipu}e^{2\pi ipv}e^{\pi iuv}e^{-2\pi iu^2}dY \\ = & \ \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv}e^{-\frac{\pi i}{6}}\sum_{p\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\Phi_{\mathrm{b}}(p+u-Y)}{\Phi_{\mathrm{b}}(Y)}e^{2\pi i(u+p)(2Y-u-p)}dY\cdot\ e^{2\pi ipv}e^{\pi iuv}. \end{aligned}$$

Using the Weil-Gel'fand-Zak transform we see that the partition function has the form:

$$Z_\hbar^{\text{new}}(X) = \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv}e^{-\frac{\pi i}{6}}W(\chi_{4_1})(u,v)$$

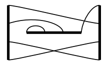
Where the function $\chi_{4_1}(x) = \int_{\mathbb{R}-i0}\frac{\Phi_{\mathrm{b}}(x-y)}{\Phi_{\mathrm{b}}(y)}e^{2\pi ix(2y-x)}dy$. The function $\chi_{4_1}(x)$ is exactly the function $J_{S^3,4_1}$ from [6, Thm. 5]. It should be noted that this result is connected to Hikami's invariant. Andersen and Kashaev observes in [6] that the expression

$$\frac{1}{2\pi\,\mathrm{b}}\chi_{4_1}\left(-\frac{u}{\pi\,\mathrm{b}},\frac{1}{2}\right),$$

where $\chi_{4_1}(x,\lambda) = \chi_{4_1}(x)e^{4\pi ic_{\mathrm{b}}\lambda}$ is equal to the formal derived expression in [21].

9.2 One vertex H-triangulation of the figure-8-knot

Let X be represented by the diagram



where the figure-eight knot is represented by the edge of the central tetrahedron connecting the maximal and next to maximal vertices. Choosing an orientation, the diagram consists of two positive tetrahedra T_1, T_3 and one negative T_2. $\partial X = \emptyset$ and combinatorially we have $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{x, y, z, x'\}$. The gluing of the tetrahedra is vertex order preserving which means that edges are glued together in the following manner.

$$\begin{aligned}
x &= x^1_{01} = x^1_{03} = x^2_{02} = x^3_{02} = x^3_{03},\\
y &= x^1_{02} = x^1_{12} = x^1_{13} = x^2_{01} = x^2_{03} = x^2_{23} = x^2_{23},\\
y &= x^1_{23} = x^2_{12} = x^2_{13} = x^3_{12} = x^3_{13},\\
x' &= x^3_{01}.
\end{aligned}$$

This result in the following equations for the dihedral angles when we balance all edges but one edge.

$$b_1 + a_3 = b_2, \quad a_1 = a_2 + a_3.$$

In the limit where we let $a_3 \to 0$ we get the equations

$$b_1 = b_2, \quad a_1 = a_2.$$

The Boltzmann weights is given by the functions

$$\begin{aligned}
B\left(T_1, x_{|\Delta_1(T_1)}\right) &= g_{a_1,c_1}(y - x, 2y - x - z),\\
B\left(T_2, x_{|\Delta_1(T_2)}\right) &= \overline{g_{a_2,c_2}(x - y, x + z - 2y)},\\
B\left(T_3, x_{|\Delta_1(T_3)}\right) &= g_{a_3,c_3}(0, x + z - x' - y).
\end{aligned}$$

$$\begin{aligned}
Z^{\text{new}}_\hbar(X) = \int_{[0,1]^4} \sum_{m,n,l\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(y - x + m) e^{\pi i(2y-x-z)(y-x+2m)}\\
& \overline{\tilde{\psi}'_{a_2,c_2}(x - y + n)} e^{-\pi i(x+z-2y)(x-y+2n)}\\
& \tilde{\psi}'_{a_3,c_3}(l) e^{2\pi i(x+z-x'-y)l}\, dxdydzdx'.
\end{aligned}$$

Integration over x' removes one of the sums since $\int_0^1 e^{-2\pi i x' l} dx' = \delta(l)$. Hence

$$\begin{aligned}
Z^{\text{new}}_\hbar(X) &= \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]^3} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y - x + m) e^{\pi i(2y-x-z)(y-x+2m)}\\
&\qquad \overline{\tilde{\psi}'_{a_2,c_2}(x - y + n)} e^{-\pi i(x+z-2y)(x-y+2n)}\, dxdydz\\
&= \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]^3} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y - x + m) \overline{\tilde{\psi}'_{a_2,c_2}(x - y + n)}\\
&\qquad e^{2\pi i(2y-x)(m+n)} e^{-2\pi i z(m+n)}\, dxdydz.
\end{aligned}$$

Now integration over z gives $\int_0^1 e^{-2\pi i z(m+n)} dz = \delta(n + m)$. So the partition function takes the form

$$Z_\hbar^{\text{new}}(X) = \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]^2} \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y-x+m)\overline{\tilde{\psi}'_{a_2,c_2}(x-y-m)}\, dxdy,$$

We make the shift $y \mapsto y + x$ to get the expression

$$\begin{aligned}
Z_\hbar^{\text{new}}(X) &= \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]^2} \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y+m)\overline{\tilde{\psi}'_{a_2,c_2}(-y-m)}\, dxdy \\
&= \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]} \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y+m)\overline{\tilde{\psi}'_{a_2,c_2}(-y-m)}\, dy \\
&= \tilde{\psi}'_{a_3,c_3}(0) \int_{\mathbb{R}} \tilde{\psi}'_{a_1,c_1}(y)\overline{\tilde{\psi}'_{a_2,c_2}(-y)}\, dy \\
&= e^{-\frac{\pi i}{6}} \tilde{\psi}'_{a_3,c_3}(0) \int_{\mathbb{R}} \psi_{c_1,b_1}(y)\psi_{b_2,c_2}(y) e^{\pi i y^2}.
\end{aligned}$$

We set $Y = y - 2c_{\mathrm{b}}(c_1 + b_1) = y - c_{\mathrm{b}}(1 - 2a_1)$. Assuming that we are in the case where all but one edge is balanced we have $a_1 = a_2$

$$y^2 = Y^2 + c_{\mathrm{b}}^2(1-2a_1)^2 + 2c_{\mathrm{b}}Y(1-2a_1).$$

Implementing this we get the following.

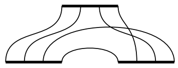
$$\begin{aligned}
Z_\hbar^{\text{new}}(X) = e^{-\frac{\pi i}{6}} \tilde{\psi}'_{a_3,c_3}(0) \int_{\mathbb{R}} &\psi(Y)\psi(Y) e^{\pi i (Y^2 + c_{\mathrm{b}}^2(1-2a_1)^2 + 2c_{\mathrm{b}}Y(1-2a_1))} \\
&e^{-4\pi i c_{\mathrm{b}} c_1 (Y + c_{\mathrm{b}}(1/2 - a_1))} \nu_{c_1,b_1} \\
&e^{-4\pi i c_{\mathrm{b}} b_2 (Y + c_{\mathrm{b}}(1/2 - a_1))} \nu_{b_2,c_2} dy \\
= e^{-\frac{\pi i}{6}} \nu_{c_1,b_1}\nu_{b_2,c_2} \tilde{\psi}'_{a_3,c_3}(0) \int_{\mathbb{R}-0i} &\frac{1}{\Phi(Y)^2} e^{\pi i Y^2}\, dy\, e^{\frac{i\phi}{\hbar}}.
\end{aligned}$$

This result corresponds exactly to the partition function in the original formulation, see [6, Chap. 11]. I.e. in the limit where $a_3 \to 0$ we get the renormalised partition function

$$\tilde{Z}_\hbar^{\text{new}}(X) := \lim_{a_3\to 0} \Phi_{\mathrm{b}}(2c_{\mathrm{b}}a_3 - c_{\mathrm{b}}) Z_\hbar^{\text{new}}(X) = \frac{e^{-\pi i/12}}{\nu(c_3)} \chi_{4_1}(0).$$

9.3 The complement of the knot 5_2

Let X be represented as the diagram



Choosing an orientation the diagram consists of three positive tetrahedra. We denote T_1, T_2, T_3 the left, the right an top tetrahedra respectively. The combinatorial data in this case are $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{e_0, e_1, e_2\}$, $\Delta_2(X) = \{f_0, f_1, f_2, f_3, f_4, f_5\}$ and $\Delta_3(X) = \{T_1, T_2, T_3\}$.

The edges are glued in the following manner:

$$\begin{aligned}
e_0 &= x^1_{02} = x^1_{12} = x^2_{13} = x^2_{23} = x^3_{01} = x^3_{23} =: x,\\
e_1 &= x^1_{03} = x^1_{23} = x^2_{02} = x^2_{03} = x^3_{03} = x^3_{13} = x^3_{12} =: y\\
e_2 &= x^1_{01} = x^1_{13} = x^2_{01} = x^2_{12} = x^3_{02} =: z.
\end{aligned}$$

We impose the condition that all edges are balanced which exactly corresponds to the two equations

$$2a_3 = a_1 + c_2, \quad b_3 = c_1 + b_2.$$

The Bolzmann weights are given by the equations

$$\begin{aligned}
B\left(T_1, x_{|_{\Delta_1(T_1)}}\right) &= g_{a_1,c_1}(z-y, x-y),\\
B\left(T_2, x_{|_{\Delta_1(T_2)}}\right) &= g_{a_2,c_2}(x-z, y-z),\\
B\left(T_2, x_{|_{\Delta_1(T_2)}}\right) &= g_{a_3,c_3}(z-y, z+y-2x).
\end{aligned}$$

We calculate the following function

$$\begin{aligned}
Z^{\mathrm{new}}_{\hbar}(X) = \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} & g_{a_1,c_1}(z-y, x-y) g_{a_2,c_2}(x-z, y-z)\\
& g_{a_3,c_3}(z-y, z+y-2x)\; dxdydz.
\end{aligned}$$

$$\begin{aligned}
Z^{\mathrm{new}}_{\hbar}(X) = \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(z-y+j) e^{\pi i(x-y)(z-y+2j)} \tilde{\psi}'_{a_2,c_2}(x-z+k) e^{\pi i(y-z)(x-z+2k)}\\
\times\, & \tilde{\psi}'_{a_3,c_3}(z-y+l) e^{\pi i(z+y-2x)(z-y+2l)}\; dxdydz.
\end{aligned}$$

Shift $x \mapsto x+z$,

$$\begin{aligned}
Z^{\mathrm{new}}_{\hbar}(X) = \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(z-y+j) e^{\pi i(x+z-y)(z-y+2j)} \tilde{\psi}'_{a_2,c_2}(x+k) e^{\pi i(y-z)(x+2k)}\\
\times\, & \tilde{\psi}'_{a_3,c_3}(z-y+l) e^{\pi i(y-2x-z)(z-y+2l)}\; dxdydz.
\end{aligned}$$

Shift $z \mapsto z + y$

$$
\begin{aligned}
Z_\hbar^{\mathrm{new}}(X) =& \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(z+j) e^{\pi i (x+z)(z+2j)} \tilde{\psi}'_{a_2,c_2}(x+k) e^{\pi i(-z)(x+2k)} \\
& \times \tilde{\psi}'_{a_3,c_3}(z+l) e^{\pi i(-2x-z)(z+2l)} \; dxdydz \\
=& \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(z+j) \tilde{\psi}'_{a_2,c_2}(x+k) \tilde{\psi}'_{a_3,c_3}(z+l) \\
& \times e^{\pi i (x+z)(z+2j)} e^{\pi i(-z)(x+2k)} e^{\pi i(-2x-z)(z+2l)} \; dxdydz \\
=& \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(z+j) \tilde{\psi}'_{a_2,c_2}(x+k) \tilde{\psi}'_{a_3,c_3}(z+l) \\
& \times e^{2\pi i (x(j-2l-z)+z(j-k-l)} \; dxdydz.
\end{aligned}
$$

Integration over y contributes nothing. We now shift $x \mapsto x - k$ and integrate over the interval $[-k, -k+1]$.

$$
\begin{aligned}
Z_\hbar^{\mathrm{new}}(X) =& \sum_{j,k,l\in\mathbb{Z}} \int_{[0,1]} \int_{[-k,-k+1]} \tilde{\psi}'_{a_1,c_1}(z+j) \tilde{\psi}'_{a_2,c_2}(x) \tilde{\psi}'_{a_3,c_3}(z+l) \\
& \times e^{2\pi i ((x-k)(j-2l-z)+z(j-k-l)} \; dxdz \\
=& \sum_{j,k,l\in\mathbb{Z}} e^{2\pi i k(2l-j)} \int_{[0,1]} \int_{[-k,-k+1]} \tilde{\psi}'_{a_1,c_1}(z+j) \tilde{\psi}'_{a_2,c_2}(x) \tilde{\psi}'_{a_3,c_3}(z+l) \\
& \times e^{2\pi i (x(j-2l-z)+z(j-l))} \; dxdz \\
=& \sum_{j,l\in\mathbb{Z}} \int_{[0,1]} \tilde{\psi}'_{a_1,c_1}(z+j) \tilde{\psi}'_{a_3,c_3}(z+l) e^{2\pi i z(j-l)} \int_{\mathbb{R}} \tilde{\psi}'_{a_2,c_2}(x) e^{-2\pi i x(z+2l-j)} dx \, dz \\
=& e^{-\frac{\pi i}{12}} \sum_{j,l\in\mathbb{Z}} \int_{[0,1]} \tilde{\psi}'_{a_1,c_1}(z+j) \tilde{\psi}'_{a_3,c_3}(z+l) e^{2\pi i z(j-l)} \int_{\mathbb{R}} \psi_{c_2,b_2}(x) e^{-2\pi i x(z+2l-j)} dx \, dz \\
=& e^{-\frac{\pi i}{4}} \sum_{j,l\in\mathbb{Z}} \int_{[0,1]} \psi_{c_1,b_1}(z+j) \psi_{c_3,b_3}(z+l) \tilde{\psi}_{c_2,b_2}(z+2l-j) e^{2\pi i z(j-l)} dz.
\end{aligned}
$$

We set $m = j - l$.

$$
\begin{aligned}
Z_\hbar^{\mathrm{new}}(X) =& e^{-\frac{\pi i}{4}} \sum_{l,m\in\mathbb{Z}} \int_{[0,1]} \psi_{c_1,b_1}(z+l+m) \psi_{c_3,b_3}(z+l) \tilde{\psi}_{c_2,b_2}(z+l-m) e^{2\pi i z m} \; dz \\
=& e^{-\frac{\pi i}{4}} \sum_{l,m\in\mathbb{Z}} \int_{[l,l+1]} \psi_{c_1,b_1}(z+m) \psi_{c_3,b_3}(z) \tilde{\psi}_{c_2,b_2}(z-m) e^{2\pi i z m} \; dz \\
=& e^{-\frac{\pi i}{3}} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \psi_{c_1,b_1}(z+m) \psi_{c_3,b_3}(z) \psi_{b_2,a_2}(z-m) e^{\pi i (z-m)^2} e^{2\pi i z m} \; dz \\
=& e^{-\frac{\pi i}{3}} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \psi_{c_1,b_1}(z+m) \psi_{c_3,b_3}(z) \psi_{b_2,a_2}(z-m) e^{\pi i (z^2+m^2)} \; dz.
\end{aligned}
$$

$$
\begin{aligned}
Z_\hbar^{\text{new}}(X) = e^{-\frac{\pi i}{3}} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} & \psi(z+m-c_{\mathrm{b}}(1-2a_1))e^{-4\pi i c_{\mathrm{b}} c_1\{(z+m)-c_{\mathrm{b}}(1/2-a_1)\}} \\
& e^{-\pi i c_{\mathrm{b}}^2(4(c_1-b_1)+1)/6} \\
& \psi(z-m-c_{\mathrm{b}}(1-2c_2))e^{-4\pi i c_{\mathrm{b}} c_1\{(z+m)-c_{\mathrm{b}}(1/2-c_2)\}} \\
& e^{-\pi i c_{\mathrm{b}}^2(4(c_1-b_1)+1)/6} \\
& \psi(z-c_{\mathrm{b}}(1-2a_3))e^{-4\pi i c_{\mathrm{b}} c_3\{(z+m)-c_{\mathrm{b}}(1/2-a_3)\}} e^{-\pi i c_{\mathrm{b}}^2(4(c_3-b_3)+1)/6} \\
& e^{\pi i z^2} e^{\pi i p^2}\, dz.
\end{aligned}
$$

Set $w = z - c_{\mathrm{b}}(1-2a_3)$

$$
\begin{aligned}
Z_\hbar^{\text{new}}(X) = e^{-\frac{\pi i}{3}} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}-i0} & \psi(w+m+2c_{\mathrm{b}}(a_1-a_3))\psi(w-m+2c_{\mathrm{b}}(c_2-a_3))\psi(w) \\
& \times e^{\pi i p^2} e^{\pi i w^2} e^{4\pi i c_{\mathrm{b}}^2(1/2-a_3)^2} e^{4\pi i c_{\mathrm{b}} w(1/2-a_3)} \\
& e^{-4\pi i c_{\mathrm{b}} c_1\{w+p+c_{\mathrm{b}}(1-2a_3)-c_{\mathrm{b}}(1/2-a_1)\}} \\
& e^{-4\pi i c_{\mathrm{b}} c_1\{w-p+c_{\mathrm{b}}(1-2a_3)-c_{\mathrm{b}}(1/2-c_2)\}} \\
& e^{-4\pi i c_{\mathrm{b}} c_1\{w+c_{\mathrm{b}}(1/2-a_3)\}} \nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}\, dw.
\end{aligned}
$$

Simplify by setting $u = 2c_{\mathrm{b}}(a_1-a_3)$. Using $c_1+b_2+c_3+a_3-1/2=0$ we are left with

$$
\begin{aligned}
Z_\hbar^{\text{new}}(X) = & e^{-\frac{\pi i}{3}} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}-i0} \psi(w+m+u)\psi(w-m-u)\psi(w) \\
& e^{\pi i w^2} e^{\pi i m^2} e^{4\pi i c_{\mathrm{b}}(b_2-c_1)m} \\
& e^{-4\pi i c_{\mathrm{b}}^2\{-b_3^2-b_3c_3+c_1(b_3+c_3)+b_2(b_3+c_3)+(c_1-b_2)(a_1-a_3)\}}\, dw \\
& \nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}.
\end{aligned}
$$

Let $v = 2c_{\mathrm{b}}(a_1-c_1+b_2-a_3)$, then Note that

$$
\begin{aligned}
4\pi i c_{\mathrm{b}}(b_2-c_1)p = 4\pi i c_{\mathrm{b}}(a_1-c_1+b_2-a_3)p - 4\pi i c_{\mathrm{b}}(a_3-a_1) = 2\pi i(vp-up), \\
-b_3^2-b_3c_3+c_1(b_3+c_3)+b_2(b_3+c_3) = 0,
\end{aligned}
$$

and

$$
-4\pi i c_{\mathrm{b}}^2((c_1-b_2)(a_1-a_3)) = 4\pi i c_{\mathrm{b}}^2((a_1-c_1+b_2-a_3)(a_1-a_3)-(a_1-a_3)(a_1-a_3)) = \pi i(vu-u^2).
$$

$$
\begin{aligned}
Z_\hbar^{\text{new}}(X) = & e^{-\frac{\pi i}{3}} e^{\pi i u v} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}-i0} \frac{e^{\pi i w^2} e^{-\pi i m^2} e^{-\pi i u^2}}{\Phi_{\mathrm{b}}(w+m+u)\Phi_{\mathrm{b}}(w-m-u)\Phi_{\mathrm{b}}(w)} dw e^{2\pi i v m} \\
& \nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3} \\
= & e^{-\frac{\pi i}{3}} e^{\pi i u v} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}-i0} \frac{e^{\pi i(w+(u+m))(w-(u+m))}}{\Phi_{\mathrm{b}}(w+m+u)\Phi_{\mathrm{b}}(w-m-u)\Phi_{\mathrm{b}}(w)} dw e^{2\pi i v m} \\
& \nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3} \\
= & W\chi_{5_2}(u,v)\nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}.
\end{aligned}
$$

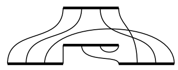
Where $\chi_{5_2}(u)$ is given by the formula

$$
\chi_{5_2}(u) = e^{-\frac{\pi i}{3}} \int_{\mathbb{R}-i0} \frac{e^{\pi i(w-u)(w+u)}}{\Phi_{\mathrm{b}}(w+m+u)\Phi_{\mathrm{b}}(w-m-u)\Phi_{\mathrm{b}}(w)}\, dw.
$$

Again the function χ_{5_2} is that of [6], which again is related to Hikami's invariant, in particular Hikami's formally derived expression in [21, (4.10)] is equal to $e^{\pi i \frac{c_{\mathrm{b}}2}{3}} \frac{1}{2\pi\,\mathrm{b}}\chi_{5_2}(\frac{-u}{\pi\,\mathrm{b}}, \frac{1}{2})$, where $\chi_{5_2} := \chi_{5_2}(x)e^{4\pi i c_{\mathrm{b}} x\lambda}$.

9.4 One vertex H-triangulation of $(S^3, 5_2)$

Let X be represented by the diagram



Choosing an orientation, the diagram consists of four positive tetrahedra T_0, T_1, T_2, T_3. $\partial X = \emptyset$ and combinatorially we have $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{x, y, z, w, x'\}$. The gluing of the tetrahedra is vertex order preserving which means that edges are glued together in the following manner.

$$\begin{aligned}
x &= x^0_{03} = x^0_{13} = x^1_{01} = x^3_{12} = x^3_{02}, \\
y &= x^1_{03} = x^1_{12} = x^1_{13} = x^2_{02} = x^2_{03} = x^3_{03} = x^3_{23}, \\
z &= x^0_{01} = x^1_{02} = x^2_{01} = x^2_{12} = x^3_{01} = x^3_{13} \\
v &= x^0_{02} = x^0_{12} = x^1_{23} = x^2_{13} = x^2_{23}, \\
x' &= x^0_{23}.
\end{aligned}$$

This result in the following equations for the dihedral angles when we balance all edges but one edge.

$$a_3 = a_1 - a_0 = c_2, \quad a_0 + b_1 = b_2 + c_3, \quad a_1 + a_2 + b_3 = \frac{1}{2} + c_1.$$

The Boltzmann weights is given by the functions

$$\begin{aligned}
B\left(T_0, x_{|\Delta_1(T_0)}\right) &= g_{a_0,c_0}(0, v + x - z - x'), \\
B\left(T_1, x_{|\Delta_1(T_1)}\right) &= g_{a_1,c_1}(z - y, z + y - x - v), \\
B\left(T_2, x_{|\Delta_1(T_2)}\right) &= g_{a_2,c_2}(v - z, y - z), \\
B\left(T_3, x_{|\Delta_1(T_3)}\right) &= g_{a_3,c_3}(z - y, x - y).
\end{aligned}$$

The partition function is represented by the integral

$$\begin{aligned}
Z^{\mathrm{new}}_\hbar(X) = \int_{[0,1]^5} \sum_{m,n,k,p\in\mathbb{Z}} & \tilde{\psi}'_{a_0,c_0}(m) e^{\pi i (v+x-z-x')(2m)} \\
& \tilde{\psi}'_{a_1,c_1}(z - y + n) e^{\pi i (z+y-x-v)(z-y+2n)} \\
& \tilde{\psi}'_{a_2,c_2}(v - z + k) e^{\pi i (y-z)(v-z+2k)} \\
& \tilde{\psi}'_{a_3,c_3}(z - y + p) e^{\pi i (x-y)(z-y+2p)}\, dx' dx dy dz dv
\end{aligned}$$

Integration over x' removes one of the sums since $\int_0^1 e^{-2\pi i x' m} dx' = \delta(m)$. Hence

$$\begin{aligned}
Z^{\mathrm{new}}_\hbar(X) = \tilde{\psi}'_{a_0,c_0}(0) \int_{[0,1]^4} \sum_{n,k,p\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(z - y + n) e^{\pi i (z+y-x-v)(z-y+2n)} \\
& \tilde{\psi}'_{a_2,c_2}(v - z + k) e^{\pi i (y-z)(v-z+2k)} \\
& \tilde{\psi}'_{a_3,c_3}(z - y + p) e^{\pi i (x-y)(z-y+2p)}\, dx' dx dy dz dv
\end{aligned}$$

Now integration over x gives $\int_0^1 e^{-2\pi i x(n-p)}dx = \delta(n-p)$. Implementing this and shifting the variable $v \mapsto v+z$, the partition function takes the form

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = \tilde{\psi}'_{a_0,c_0}(0) \int_{[0,1]^3} \sum_{n,k\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(z-y+n) e^{\pi i (y-v)(z-y+2n)} \\ & \tilde{\psi}'_{a_2,c_2}(v+k) e^{\pi i (y-z)(v+2k)} \\ & \tilde{\psi}'_{a_3,c_3}(z-y+n) e^{-\pi i y(z-y+2n)}\ dydzdv. \end{aligned}$$

We make the shift $z \mapsto z+y$ to get the expression

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = \tilde{\psi}'_{a_0,c_0}(0) \int_{[0,1]^3} \sum_{n,k\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(z+n) e^{\pi i (y-v)(z+2n)} \\ & \tilde{\psi}'_{a_2,c_2}(v+k) e^{-\pi i z(v+2k)} \\ & \tilde{\psi}'_{a_3,c_3}(z+n) e^{-\pi i y(z+2n)}\ dydzdv, \end{aligned}$$

which is independent of y so we can remove the integration over this variable. We integrate over the variable v.

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \int_{[0,1]} \tilde{\psi}'_{a_2,c_2}(v+k) e^{-2\pi i v(z+n)} dv e^{-2\pi i z k} =& \sum_{k\in\mathbb{Z}} \int_k^{k+1} \tilde{\psi}'_{a_2,c_2}(v) e^{-2\pi i v(z+n)} dv \\ & e^{-2\pi i z k} e^{2\pi i k(z+n)} \\ =& e^{-\frac{\pi i}{12}} \int_{\mathbb{R}} \psi_{c_2,b_2}(v) e^{-2\pi i z(z+n)} dv \\ =& e^{-\frac{\pi i}{12}} \tilde{\psi}_{c_2,b_2}(z+n) \\ =& e^{-\frac{\pi i}{6}} e^{\pi i (z+n)^2} \psi_{b_2,a_2}(z+n). \end{aligned}$$

We therefore get the expression

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) &= e^{-\frac{\pi i}{3}} \tilde{\psi}'_{a_0,c_0}(0) \int_{[0,1]} \sum_{n\in\mathbb{Z}} \psi_{c_1,b_1}(z+n)\psi_{b_2,a_2}(z+n)\psi_{c_3,b_3}(z+n) e^{\pi i (z+n)^2} dz \\ &= e^{-\frac{\pi i}{3}} \tilde{\psi}'_{a_0,c_0}(0) \int_{\mathbb{R}} \psi_{c_1,b_1}(z)\psi_{b_2,a_2}(z)\psi_{c_3,b_3}(z) e^{\pi i (z)^2} \end{aligned}$$

We set $Z = z - 2c_{\mathrm{b}}(c_1+b_1) = y - c_{\mathrm{b}}(1-2a_1)$. Assuming that we are in the case where all but the edge representing the knot is balanced, i.e. $a_0 \to 0$, we have $a_1 = c_2 = a_3$.

$$z^2 = Z^2 + c_{\mathrm{b}}^2(1-2a_1)^2 + 2c_{\mathrm{b}} Z(1-2a_1).$$

Implementing this we get the expression.

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = e^{-\frac{\pi i}{3}} \tilde{\psi}'_{a_0,c_0}(0) \int_{\mathbb{R}} & \psi(Z)\psi(Z)\psi(Z) e^{\pi i (Z^2 + c_{\mathrm{b}}^2(1-2a_1)^2 + 2c_{\mathrm{b}} Z(1-2a_1))} \\ & e^{-4\pi i c_{\mathrm{b}} c_1 (Z + c_{\mathrm{b}}(1/2-a_1))} \nu_{c_1,b_1} \\ & e^{-4\pi i c_{\mathrm{b}} b_2 (Z + c_{\mathrm{b}}(1/2-c_2))} \nu_{b_2,a_2} \\ & e^{-4\pi i c_{\mathrm{b}} c_2 (Z + c_{\mathrm{b}}(1/2-a_3))} \nu_{c_3,b_3} dz. \end{aligned}$$

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) &= \nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3} e^{-\frac{\pi i}{3}} e^{\frac{\phi i}{\hbar}} \tilde{\psi}'_{a_0,c_0}(0) \int_{\mathbb{R}} \psi(Z)^3 e^{\pi i Z^2}\ dz \\ &= \nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3} e^{-\frac{\pi i}{3}} e^{\frac{\phi i}{\hbar}} \tilde{\psi}'_{a_0,c_0}(0) \int_{\mathbb{R}} \frac{e^{\pi i Z^2}}{\Phi_{\mathrm{b}}(Z)^3}\ dz \end{aligned}$$

Because the combination of dihedral angles in front of Z sums to 0.

$$-4\pi i c_{\mathrm{b}} Z(c_1 + b_2 + c_3 - \frac{1}{2} + a_1) = -4\pi i c_{\mathrm{b}} Z(a_1 + b_1 + c_1 - \frac{1}{2}) = 0$$

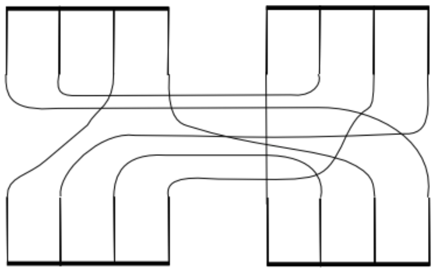
This corresponds to the partition function in the original formulation, see [6].

In this case the renormalised partition function takes the form.

$$\tilde{Z}_{\hbar}^{\mathrm{new}}(X) = \lim_{a_0 \to 0} \Phi_{\mathrm{b}} Z_{\hbar}^{\mathrm{new}}(X) = \frac{e^{i\pi/4}}{\nu_{c_0,0}} \chi_{5_2}(0).$$

9.5 The complement of the knot 6_1

We calculate the partition function of the Andersen–Kashaev TQFT for an ideal triangulation of the complement of the hyperbolic knot 6_1. Let X be represented by the diagram



Associating a shape structure as in the previous examples gives us the following equations when weights on edges are fully balanced:

$$2a_1 + a_4 = c_2 + a_3, \quad a_1 + b_1 + b_2 - c_2 + c_3 = 0, \quad c_1 + c_2 + b_3 + b_4 = 1, \quad a_3 = b_2 + b_4.$$

$$\begin{aligned} Z_\hbar(X) &= \int_{\mathbb{R}^8} \langle b, d|T_{a_1,c_1}|g, c\rangle \, \overline{\langle f, e|T_{a_2,c_2}|g, h\rangle \langle d, i|T_{a_3,c_3}|c, f\rangle} \, \langle e, i|T_{a_4,c_4}|b, h\rangle \, d\underline{x} \\ &= \int_{\mathbb{R}^8} \delta(b + d - g)\delta(f + e - g)\delta(d + i - c)\delta(e + i - b) \\ &\quad \times \tilde{\psi}'_{a_1,c_1}(c - d)e^{2\pi i b(c-d)} \overline{\tilde{\psi}'_{a_2,c_2}(h - e)} e^{-2\pi i f(h-e)} \\ &\quad \times \overline{\tilde{\psi}'_{a_3,c_3}(f - i)} e^{-2\pi i d(f-i)} \tilde{\psi}'_{a_4,c_4}(h - i)e^{2\pi i e(h-i)} d\underline{x} \end{aligned}$$

By integrating over four of the variables g, f, i, e we have the identities

$$g = b + d, \quad f = c, \quad i = c - d, \quad e = b + d - c,$$

and we are left with the intergral

$$\begin{aligned} Z_\hbar(X) = \int_{\mathbb{R}^4} & \tilde{\psi}'_{a_1,c_1}(c - d)\overline{\tilde{\psi}'_{a_2,c_2}(h + c - b - d)\tilde{\psi}'_{a_3,c_3}(d)}\tilde{\psi}'_{a_4,c_4}(h + d - c) \\ & \times e^{2\pi i\{b(c+h) - c(d+2h) + dh\}} d\underline{x} \end{aligned}$$

Integration over the variable b

$$\begin{aligned} \overline{\int_{\mathbb{R}} \tilde{\psi}'_{a_2,c_2}(h + c - b - d)e^{-2\pi i b(c+h)} \, db} &= \overline{\int_{\mathbb{R}} \tilde{\psi}'_{a_2,c_2}(-\tilde{b})e^{-2\pi i(\tilde{b}+h+c-d)(c+h)} \, d\tilde{b}} \\ &= \overline{\int_{\mathbb{R}} \tilde{\psi}'_{a_2,c_2}(-\tilde{b})e^{-2\pi i\tilde{b}(c+h)} \, d\tilde{b}} \, e^{2\pi i(h+c-d)(c+h)} \\ &= e^{\frac{\pi i}{12}} \overline{\tilde{\psi}_{c_2,b_2}(-c - h)} e^{2\pi i(h+c-d)(c+h)} \\ &= \psi_{a_2,b_2}(c + h)e^{2\pi i(h+c-d)(c+h)}. \end{aligned}$$

We continue the calculation of the partition function.

$$\begin{aligned} Z_\hbar(X) &= \int_{\mathbb{R}^3} \tilde{\psi}'_{a_1,c_1}(c - d)\psi_{a_2,b_2}(c + h)\overline{\tilde{\psi}'_{a_3,c_3}(d)}\tilde{\psi}'_{a_4,c_4}(h + d - c) \, e^{2\pi i\{c^2 - 2cd + h^2\}} d\underline{x} \\ &= e^{-\frac{\pi i}{4}} \int_{\mathbb{R}^3} \psi_{c_1,b_1}(c)\psi_{a_2,b_2}(c + h + d)\psi_{b_3,c_3}(-d)\psi_{c_4,b_4}(h - c)e^{\pi i(2c^2 - d^2 + 2h^2)} d\underline{x}. \end{aligned}$$

Now use the definition of the function $\psi_{a,c}(x)$.

$$\begin{aligned}\psi_{c_1,b_1}(c) &= \psi(c - c_{\mathrm{b}}(1-2a_1))e^{-4\pi i c_{\mathrm{b}} c_1 (c - c_{\mathrm{b}}(1/2-a_1))} e^{-\pi i c_{\mathrm{b}}^2 (4(c_1-b_1)+1)/6} \\ &= \psi(\tilde{c}) e^{-4\pi i c_{\mathrm{b}} b_3 (\tilde{c} + c_{\mathrm{b}}(1/2-a_1))} \nu_{c_1,b_1}.\end{aligned}$$

$$\begin{aligned}\psi_{b_3,c_3}(-d) &= \psi(-d - c_{\mathrm{b}}(1-2a_3))e^{-4\pi i c_{\mathrm{b}} b_3 (-d - c_{\mathrm{b}}(1/2-a_3))} e^{-\pi i c_{\mathrm{b}}^2 (4(b_3-c_3)+1)/6} \\ &= \psi(\tilde{d}) e^{-4\pi i c_{\mathrm{b}} b_3 (\tilde{d} + c_{\mathrm{b}}(1/2-a_3))} \nu_{b_3,c_3}.\end{aligned}$$

$$\begin{aligned}\psi_{c_4,b_4}(h-c) &= \psi(h - \tilde{c} - 2c_{\mathrm{b}}(1-(a_1+a_4)))e^{-4\pi i c_{\mathrm{b}} c_4 (h - \tilde{c} - c_{\mathrm{b}}(1-2a_1) - c_{\mathrm{b}}(1/2-a_4))} \nu_{c_4,b_4} \\ &= \psi(\tilde{h} - \tilde{c}) e^{-4\pi i c_{\mathrm{b}} c_4 (\tilde{h} - \tilde{c} + c_{\mathrm{b}}(1/2-a_4))} \nu_{c_4,b_4}.\end{aligned}$$

$$\begin{aligned}\psi_{a_2,c_2}(c+h+d) =& \psi(\tilde{c} + \tilde{h} - \tilde{d} + c_{\mathrm{b}}(1 + 2(a_3 + c_2 - a_4 - 2a_1)) \\ & e^{-4\pi i c_{\mathrm{b}} a_2 (\tilde{c} + \tilde{h} - \tilde{d} + c_{\mathrm{b}}(3/2-c_2))} \nu_{a_2,b_2}.\end{aligned}$$

$$\begin{aligned}Z_\hbar(X) = & \nu_{c_1,b_1}\nu_{a_2,b_2}\nu_{b_3,c_3}\nu_{c_4,b_4} e^{-\frac{\pi i}{4}} \int_{\mathbb{R}^3} \psi(\tilde{c})\psi(\tilde{c}+\tilde{h}-\tilde{d}+c_{\mathrm{b}})\psi(\tilde{d})\psi(\tilde{h}-\tilde{c}) \\ & \times e^{-4\pi i c_{\mathrm{b}} c_1 (\tilde{c}+c_{\mathrm{b}})} \\ & \times e^{-4\pi i c_{\mathrm{b}} a_2 (\tilde{c}+\tilde{h}-\tilde{d}+c_{\mathrm{b}}(3/2-c_2))} \\ & \times e^{-4\pi i c_{\mathrm{b}} b_3 (\tilde{d}+c_{\mathrm{b}}(1/2-a_3))} \\ & \times e^{-4\pi i c_{\mathrm{b}} c_4 (\tilde{h}-\tilde{c}+c_{\mathrm{b}}(1/2-a_4))} \\ & \times e^{2\pi i\{\tilde{c}^2 - \frac{1}{2}d^2 + \tilde{h}^2\}} e^{2\pi i c_{\mathrm{b}}^2((1-2a_1)^2 - \frac{1}{2}(1-2a_3)^2 + 4(1-(a_1+a_4))^2)} \\ & \times e^{2\pi i (2\tilde{c} c_{\mathrm{b}}(1-2_{a_1}) - \tilde{d} c_{\mathrm{b}}(1-2a_3) 4\tilde{h} c_{\mathrm{b}}(1-(a_1+a_4)))} d\underline{x}\end{aligned}$$

Collecting the factor of $-4\pi i c_{\mathrm{b}}\tilde{c}$ in the exponent we get

$$c_1 + a_2 - c_4 - (1-2a_1) =: \lambda,$$

Then collecting the factor of $-4\pi i c_{\mathrm{b}}\tilde{d}$ in the exponent one gets

$$\begin{aligned}b_3 - a_2 + \frac{1}{2}(1-2a_3) &= -a_2 + b_3 + 1/2 - a_3 = -a_2 + b_3 + 1/2 + c_2 - 2a_1 - a_4 \\ &= -a_2 - 2a_1 + c_4 + 1 - c_1 = -\lambda.\end{aligned}$$

The factor of $-4\pi i c_{\mathrm{b}}\tilde{h}$

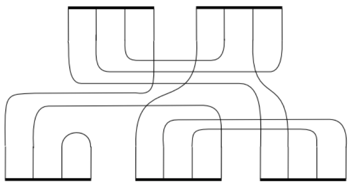
$$\begin{aligned}a_2 + c_4 - 2 + 2a_1 + 2a_4 &= a_2 + 1/2 - b_4 - 2 + 2a_1 + a_4 \\ &= -\frac{3}{2} + a_2 + c_2 + a_3 - b_4 = -1 - b_2 - b_4 + a_3 \\ &= -1.\end{aligned}$$

Now set $x = \tilde{c} - \tilde{d}$, and we can write the partition function for the knot complement of 6_1 as conjecture 7.24 suggests.

$$\begin{aligned}
Z_\hbar(X) =& e^{\frac{i\phi}{\hbar}} \int_{\mathbb{R}^3} \psi(x+\tilde{d})\psi(x+\tilde{h}+c_{\mathrm{b}})\psi(\tilde{d})\psi(\tilde{h}-x-\tilde{d}) \\
& \times e^{2\pi i(x^2+\frac{1}{2}\tilde{d}^2+\tilde{h}^2+2x\tilde{d})} e^{-4\pi i c_{\mathrm{b}}(\lambda x-\tilde{h})} d\tilde{d}\tilde{h}dx \\
=& e^{\frac{i\phi}{\hbar}} \int_{\mathbb{R}^3} \frac{1}{\Phi_{\mathrm{b}}(x+\tilde{d})\Phi_{\mathrm{b}}(x+\tilde{h}+c_{\mathrm{b}})\Phi_{\mathrm{b}}(\tilde{d})\Phi_{\mathrm{b}}(\tilde{h}-x-\tilde{d})} \\
& \times e^{2\pi i(x^2+\frac{1}{2}\tilde{d}^2+\tilde{h}^2+2x\tilde{d})} e^{-4\pi i c_{\mathrm{b}}(\lambda x-\tilde{h})} d\tilde{d}\tilde{h}dx \\
=& e^{\frac{i\phi}{\hbar}} \int_{\mathbb{R}} J_{S^3,6_1}(\hbar, x) e^{-\frac{x\lambda}{\sqrt{\hbar}}}\, dx.
\end{aligned}$$

9.6 One vertex H-triangulation of $(S^3, 6_1)$-knot

We here calculate the partition function for the H-triangulation of the knot 6_1 using the formulation from [6] Let X be represented by the diagram



This one vertex H-triangulation of $(S^3, 6_1)$ consists of 5 tetrahedra T_1 and T_3 which are negatively oriented tetrahedra and t_2, T_4, T_5 which are positively oriented tetrahedra. The tetrahedra are situated in the following way: In the bottom we have T_1, T_2, T_3 from left to right. And on top we have T_4, T_5 from right to left. The gluing pattern of faces results in the gluing of edges:

$$\begin{aligned}
x &:= x^1_{02} = x^1_{03} = x^2_{01} = x^2_{02} = x^3_{01},\\
y &:= x^2_{03} = x^2_{13} = x^3_{02} = x^3_{03} = x^3_{13} = x^4_{02} = x^4_{03} = x^5_{03},\\
z &:= x^2_{23} = x^3_{12} = x^4_{12} = x^5_{01},\\
v &:= x^1_{12} = x^1_{13} = x^3_{23} = x^4_{23} = x^5_{12} = x^5_{13},\\
w &:= x^1_{23} = x^2_{12} = x^4_{01} = x^4_{13} = x^5_{02} = x^5_{23},\\
x' &:= x^1_{01}.
\end{aligned}$$

From here we can easily get a shape structure. We balance all but one edge. This results in the following equations on the shape parameters:

$$\begin{aligned}
&a_3 = a_1 + c_2, \quad a_3 + a_4 = a_1 + a_5, \quad a_1 + c_2 = c_4 + c_5,\\
&\frac{1}{2} + b_3 + c_5 = a_2 + a_3 + a_4, \quad 1 = a_2 + c_3 + c_4 + a_5.
\end{aligned}$$

We calculate the partition function for the Andersen–Kashaev TQFT.

$$\begin{aligned}
Z_\hbar(X) = \int_{\mathbb{R}^{10}} &\overline{\langle J, E \mid T_{a_1,c_1} \mid F, E\rangle} \, \langle H, B \mid T_{a_2,c_2} \mid G, F\rangle \, \overline{\langle K, B \mid T_{a_3,c_3} \mid C, G\rangle}\\
&\langle D, L \mid T_{a_4,c_4} \mid C, H\rangle \, \langle J, K \mid T_{a_5,c_5} \mid L, D\rangle \, d\bar{x}
\end{aligned}$$

$$\begin{aligned}
Z_\hbar(X) = \int_{\mathbb{R}^{10}} &\delta(J + E - F)\delta(H + B - G)\delta(K + B - C)\delta(D + L - C)\delta(J + K - L)\\
&\overline{\tilde{\psi}'_{a_1,c_1}(0)} e^{-2\pi i J(0)}\\
&\tilde{\psi}'_{a_2,c_2}(F - B) e^{2\pi i H(F-B)}\\
&\overline{\tilde{\psi}'_{a_3,c_3}(G - B)} e^{-2\pi i K(G-B)}\\
&\tilde{\psi}'_{a_4,c_4}(H - L) e^{2\pi i D(H-L)}\\
&\tilde{\psi}'_{a_5,c_5}(D - K) e^{2\pi i J(D-K)} \, dB dC dD dE dF dG dH dJ dK dL
\end{aligned}$$

Integrating over five variables E, G, C, L, J yields the expression:

$$\begin{aligned} Z_\hbar(X) = \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^5} & \tilde{\psi}'_{a_2,c_2}(F-B)e^{2\pi i H(F-B)} \\ & \times \overline{\tilde{\psi}'_{a_3,c_3}(H)} e^{-2\pi i K H} \\ & \times \tilde{\psi}'_{a_4,c_4}(H+D-K-B)e^{2\pi i D(H+D-K-B)} \\ & \times \tilde{\psi}'_{a_5,c_5}(D-K)e^{2\pi i (B-D)(D-K)}\, dBdDdFdHdK. \end{aligned}$$

We integrate over the variable F using the Fourier transform.

$$e^{-\frac{\pi i}{12}} \int_{\mathbb{R}} \psi_{c_2,b_2}(F-B)e^{2\pi i H(F-B)} dF = e^{-\frac{\pi i}{12}} \tilde{\psi}_{c_2,b_2}(-H) = e^{-\frac{\pi i}{6}} e^{\pi i H^2} \psi_{b_2,a_2}(-H).$$

Using formulas from Section 7.3.1.1 we can write

$$\begin{aligned} Z_\hbar(X) = e^{-\frac{3\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^4} & \psi_{b_2,a_2}(-H)e^{\pi i H^2} \psi_{b_3,c_3}(-H)^{\pi i H^2} \\ & \tilde{\psi}'_{a_4,c_4}(H+D-K-B)\tilde{\psi}'_{a_5,c_5}(D-K) \\ & e^{2\pi i(DH-KH-BK)}\, dBdDdHdK. \end{aligned}$$

Integration over the variable B becomes

$$\begin{aligned} \int_{\mathbb{R}} \tilde{\psi}'_{a_4,c_4}(H+D-K-B)e^{-2\pi i BK} dB &= e^{-\frac{\pi i}{12}} \tilde{\psi}_{c_4,b_4}(-K)e^{2\pi i(K^2-HK-DK)} \\ &= e^{-\frac{\pi i}{6}} \psi_{b_4,a_4}(-K)e^{2\pi i(\frac{3}{2}K^2-HK-DK)} \end{aligned}$$

$$\begin{aligned} Z_\hbar(X) = e^{-\frac{5\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} & \psi_{b_2,a_2}(-H)\psi_{b_3,c_3}(-H)\psi_{b_4,a_4}(-K)\tilde{\psi}'_{a_5,c_5}(D-K) \\ & e^{2\pi i(DH-2KH+H^2+\frac{3}{2}K^2-DK)}\, dKdDdH \end{aligned}$$

Integration over D now gives

$$\begin{aligned} e^{-\frac{\pi i}{12}} \int_{\mathbb{R}} \psi_{c_5,b_5}(D-K)e^{-2\pi i D(K-H)} dD &= e^{-\frac{\pi i}{12}} \tilde{\psi}_{c_5,b_5}(K-H)e^{-2\pi i(K^2-KH)} \\ &= e^{-\frac{\pi i}{6}} \psi_{b_5,a_5}(K-H)e^{\pi i(K-H)^2} e^{-2\pi i(K^2-KH)}. \end{aligned}$$

So the partition function takes the form:

$$\begin{aligned} Z_\hbar(X) = e^{-\frac{7\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} & \psi_{b_2,a_2}(-H)\psi_{b_3,c_3}(-H)\psi_{b_4,a_4}(-K)\psi_{b_5,a_5}(K-H) \\ & e^{2\pi i(-2KH+\frac{3}{2}H^2+K^2)}\, dDdH. \end{aligned}$$

Set $-\tilde{H} = -H - c_{\mathrm{b}}(1-2c_2)$ and $-\tilde{K} = -K - c_{\mathrm{b}}(1-2c_4)$. Then

$$-H - c_{\mathrm{b}}(1-2a_3) = -\tilde{H} + c_{\mathrm{b}}(1-2c_2) - c_{\mathrm{b}}(1-2a_3) = -\tilde{H},$$

because $a_3 \to c_2$ in the limit where $a_1 \to 0$. Further we have

$$K - H = \tilde{K} - c_{\mathrm{b}}(1-2c_4) - \tilde{H} + c_{\mathrm{b}}(1-2c_2) - c_{bq}(1-2c_5) = \tilde{K} - \tilde{H} - c_{\mathrm{b}}$$

because

$$c_4 + c_5 - c_2 \to 0$$

when $a_1 \to 0$.

We can now write the partition function in the following way

$$
\begin{aligned}
Z_\hbar(X) = e^{-\frac{7\pi i}{12}} \int_{\mathbb{R}^2} & \psi(-\tilde{H})\psi(-\tilde{H})\psi(-\tilde{K})\psi(\tilde{K}-\tilde{H}-c_{\mathrm{b}})\overline{\tilde{\psi}'_{a_1,c_1}(0)} \\
& e^{-4\pi i(\tilde{H}-c_{\mathrm{b}}(1-2c_2))(\tilde{K}-c_{\mathrm{b}}(1-2c_4))+3\pi i(\tilde{H}-c_{\mathrm{b}}(1-2c_2))^2+2\pi i(\tilde{K}-c_{\mathrm{b}}(1-2c_4))^2} \\
& e^{-4\pi i c_{\mathrm{b}} b_2(-\tilde{H}+c_{\mathrm{b}}(1/2-c_2))}\nu_{a_2,b_2} \\
& e^{-4\pi i c_{\mathrm{b}} b_3(-\tilde{H}+c_{\mathrm{b}}(1-2c_2)-c_{\mathrm{b}}(1/2-a_3))}\nu_{b_3,c_3} \\
& e^{-4\pi i c_{\mathrm{b}} b_4(-\tilde{K}-c_{\mathrm{b}}(1/2-c_4))}\nu_{b_4,a_4} \\
& e^{-4\pi i c_{\mathrm{b}} b_5(\tilde{K}-\tilde{H}-c_{\mathrm{b}}(1-2c_4)+c_{\mathrm{b}}(1-2c_2)-c_{\mathrm{b}}(1/2-c_5))}\nu_{b_5,a_5} d\tilde{D} d\tilde{H}
\end{aligned}
$$

In front of $\tilde{H}$ we have the factor

$$
\begin{aligned}
& -4\pi i c_{\mathrm{b}}(-1+2c_4+3/2-3c_2-b_2-b_3-b_5) \\
&= -4\pi i c_{\mathrm{b}}(1/2+2c_4-2c_2-b_2-a_3-b_3-b_5) \\
&= -4\pi i c_{\mathrm{b}}(1/2+2c_4-c_2-1/2+a_2-1/2+c_3-1/2+a_5+c_5) \\
&= -4\pi i c_{\mathrm{b}}(-1+1+c_4-c_2+c_5) = 0.
\end{aligned}
$$

In front of $\tilde{K}$ we also have the factor 0 since

$$
\begin{aligned}
b_5-b_4-1+2c_2+1-2c_4 &= \frac{1}{2}-a_5-c_5-c_4-b_4-c_4+2c_2 \\
&= \frac{1}{2}-a_5-\frac{1}{2}+a_4+a_3 = 0.
\end{aligned}
$$

This gives us the partition function

$$
\begin{aligned}
Z_\hbar(X) = e^{i\frac{\phi}{\hbar}} e^{-\frac{7\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} & \psi(-\tilde{H})\psi(-\tilde{H})\psi(-\tilde{K})\psi(\tilde{K}-\tilde{H}-c_{\mathrm{b}}) \\
& e^{2\pi i(\frac{3}{2}\tilde{H}^2+\tilde{K}^2-2\tilde{K}\tilde{H})} d\tilde{K} d\tilde{H}.
\end{aligned}
$$

$$
Z_\hbar(X) = e^{i\frac{\phi'}{\hbar}} e^{-\frac{7\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} \frac{\Phi_{\mathrm{b}}(\tilde{H})}{\Phi_{\mathrm{b}}(-\tilde{H})\Phi_{\mathrm{b}}(-\tilde{K})\Phi_{\mathrm{b}}(\tilde{K}-\tilde{H}-c_{\mathrm{b}})} e^{2\pi i(\tilde{K}-\tilde{H})^2} d\tilde{K} d\tilde{H}.
$$

Let $\tilde{K} \mapsto \tilde{K}+\tilde{H}+c_{\mathrm{b}}$

$$
\begin{aligned}
Z_\hbar(X) &= e^{i\frac{\phi'}{\hbar}} e^{-\frac{7\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} \frac{\Phi_{\mathrm{b}}(\tilde{H})}{\Phi_{\mathrm{b}}(-\tilde{H})\Phi_{\mathrm{b}}(-\tilde{K}-\tilde{H}-c_{\mathrm{b}})\Phi_{\mathrm{b}}(\tilde{K})} e^{2\pi i(\tilde{K}+c_{\mathrm{b}})^2} d\tilde{K} d\tilde{H} \\
&= e^{i\frac{\phi'}{\hbar}} e^{-\frac{7\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} \frac{\Phi_{\mathrm{b}}(\tilde{H})\Phi_{\mathrm{b}}(-\tilde{K})}{\Phi_{\mathrm{b}}(-\tilde{H})\Phi_{\mathrm{b}}(-\tilde{K}-\tilde{H}-c_{\mathrm{b}})} e^{\pi i\tilde{K}^2+4\pi i c_{\mathrm{b}} K+2\pi i c_{\mathrm{b}}^2} d\tilde{K} d\tilde{H}
\end{aligned}
$$

Finally we get to the expression

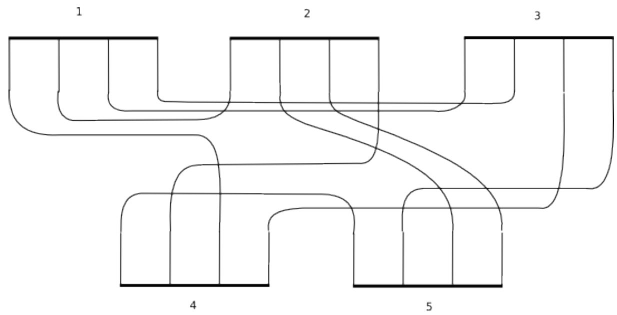
$$
Z_\hbar(X) = e^{i\frac{\phi}{\hbar}} e^{-\frac{7\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} \frac{\Phi_{\mathrm{b}}(\tilde{H})\Phi_{\mathrm{b}}(\tilde{K})}{\Phi_{\mathrm{b}}(-\tilde{H})\Phi_{\mathrm{b}}(\tilde{K}-\tilde{H}-c_{\mathrm{b}})} e^{\pi i\tilde{K}^2-4\pi i c_{\mathrm{b}}\tilde{K}} d\tilde{K} d\tilde{H}
$$

which exactly corresponds to the result for a given H-triangulation of the 6_1 knot in *A TQFT of Turaev-Viro type on shaped triangulations* [26]. Further it is easily checked the part two of conjecture 7.24 is satisfied when all but the knotted edge is balanced.

The volume conjecture has until now only been approached by use of mathematica. A rigorous proof of the conjecture in this case is still to be made.

9.7 The complement of the knot 6_2

We now let X be represented by the diagram below.



Choosing an orientation it consists of three positive tetrahedra T_1 and T_2 and T_4 and two negative tetrahedra T_3 and T_5. The diagram shows how to glue the faces of the four tetrahedra. Remember that the affine gluing homeomorphisms must be vertex order preserving and orientation reversing.

Combinatorially we have that $\partial X = \emptyset$. $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{e_1, e_2, e_3, e_4, e_5\}$, $\Delta_2(X) = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_10\}$ and $\Delta_3(X) = \{T_1, T_2, T_3, T_4, T_5\}$. The topological space $X\backslash\{*\}$ is homeomorphic to the complement of the 6_2 knot. We fix a shape structure

$$\alpha_X : \Delta_3^1(X) \to \mathbb{R}_+$$

by the formulae

$$\alpha_X(T_i, e_{0,1}) = 2\pi a_i, \quad \alpha_X(T_i, e_{0,2}) = 2\pi b_i, \quad \alpha_X(T_i, e_{0,3}) = 2\pi c_i,$$

where $a_i + b_i + c_i = \frac{1}{2}$ for $i \in \{0, 1, 2, 3\}$.

The weight function takes the values

$$\omega_X(e_1) = 2\pi(b_1 + c_1 + b_2 + a_3 + b_4 + c_4 + b_5), \quad \omega_X(e_2) = 2\pi(a_1 + c_1 + a_2 + b_2 + b_3 + a_4 + a_5),$$

$$\omega_X(e_3) = 2\pi(a_1 + c_2 + b_3 + c_3 + b_5 + c_5), \quad \omega_X(e_4) = 2\pi(b_1 + c_2 + c_3 + b_4),$$

$$\omega_X(e_5) = 2\pi(a_2 + a_3 + a_4 + c_4 + a_5 + c_5).$$

As the 6_2 knot is hyperbolic, the completely balanced case is accessible directly. This gives us the equations:

$$\frac{1}{2} + c_1 + b_2 = c_2 + c_3 + c_5, \quad a_1 + c_2 = a_3 + a_5,$$

$$a_2 + a_3 = b_4 + b_5, \quad a_1 + c_5 = b_2 + b_4, \quad 1 = b_1 + c_2 + c_3 + b_4.$$

.

We write down the partition function:

$$\begin{aligned}
Z_\hbar(X) = &\int_{\mathbb{R}^{10}} \langle \epsilon, \alpha | T(a_1, c_1) | a, d \rangle \, \langle \beta, \delta | T(a_2, c_2) | b, \alpha \rangle \overline{\langle e, \epsilon | T(a_3, c_3) | c, a \rangle} \\
&\times \langle \gamma, d | T(a_4, c_4) | \beta, c \rangle \, \overline{\langle \gamma, \delta | T(a_5, c_5) | e, b \rangle} \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \, \mathrm{d}d \, \mathrm{d}e \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma \mathrm{d}\delta \, \mathrm{d}\epsilon \\
= &\int_{\mathbb{R}^{10}} \delta(\epsilon + \alpha - a)\delta(\beta + \delta - b)\delta(e + \epsilon - c)\delta(\gamma + d - \beta)\delta(\gamma + \delta - e) \\
&\times \tilde{\psi}'_{a_1,c_1}(d - \alpha) e^{2\pi i \epsilon(d-\alpha)} \tilde{\psi}'_{a_2,c_2}(\alpha - \delta) e^{2\pi i \beta(\alpha - \delta)} \overline{\tilde{\psi}'_{a_3,c_3}(a - \epsilon) e^{2\pi i e(a-\epsilon)}} \\
&\times \tilde{\psi}'_{a_4,c_4}(c - d) e^{2\pi i \gamma(c-d)} \overline{\tilde{\psi}'_{a_5,c_5}(b - \delta) e^{2\pi i \gamma(b-\delta)}}.
\end{aligned}$$

From the δ-functions we have

$$a = \epsilon + \alpha, \quad b = \beta + \delta, \quad c = \gamma + \delta + \epsilon, \quad d = \beta - \gamma, \quad e = \gamma + \delta.$$

$$\begin{aligned}Z_\hbar(X) = \int_{\mathbb{R}^5} &\tilde\psi'_{a_1,c_1}(\beta-\gamma-\alpha)e^{2\pi i\epsilon(\beta-\gamma-\alpha)}\tilde\psi'_{a_2,c_2}(\alpha-\delta)e^{2\pi i\beta(\alpha-\delta)}\overline{\tilde\psi'_{a_3,c_3}(\alpha)e^{2\pi i(\gamma+\delta)(\alpha+\epsilon-\epsilon)}}\\ &\times \tilde\psi'_{a_4,c_4}(2\gamma+\delta+\epsilon-\beta)e^{2\pi i\gamma(2\gamma+\delta+\epsilon-\beta)}\overline{\tilde\psi'_{a_5,c_5}(\beta)e^{2\pi i\gamma\beta}}\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\mathrm{d}\delta\mathrm{d}\epsilon.\end{aligned}$$

Integrating over ϵ and δ yields

$$\begin{aligned}\int_{\mathbb{R}} \tilde\psi'_{a_4,c_4}(2\gamma+\delta+\epsilon-\beta)e^{2\pi i\epsilon(\beta-\alpha)}\mathrm{d}\epsilon &= \int_{\mathbb{R}} \tilde\psi'_{a_4,c_4}(\epsilon)e^{2\pi i(\epsilon-2\gamma-\delta+\beta)(\beta-\alpha)}\mathrm{d}\epsilon\\ &= e^{2\pi i(\beta-2\gamma-\delta)(\beta-\alpha)}\int_{\mathbb{R}} \tilde\psi'_{a_4,c_4}(\epsilon)e^{-2\pi i\epsilon(\alpha-\beta)}\mathrm{d}\epsilon\\ &= e^{-\frac{\pi i}{12}}\tilde\psi_{c_4,b_4}(\alpha-\beta)e^{2\pi i(\beta-2\gamma-\delta)(\beta-\alpha)}.\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}} \tilde\psi'_{a_2,c_2}(\alpha-\delta)e^{2\pi i\delta(\gamma-2\beta)}\mathrm{d}\delta &= \int_{\mathbb{R}} \tilde\psi'_{a_2,c_2}(\delta)e^{2\pi i(\alpha-\delta)(\gamma-2\beta)}\mathrm{d}\delta\\ &= e^{2\pi i\alpha(\gamma-2\beta)}\int_{\mathbb{R}} \tilde\psi'_{a_2,c_2}(\delta)e^{-2\pi i\delta(\gamma-2\beta)}\mathrm{d}\delta\\ &= e^{-\frac{\pi i}{12}}e^{2\pi i\alpha(\gamma-2\beta)}\tilde\psi_{c_2,b_2}(\gamma-2\beta).\end{aligned}$$

$$\begin{aligned}Z_\hbar(X) =& e^{-\frac{\pi i}{6}}\int_{\mathbb{R}^3} \tilde\psi'_{a_1,c_1}(\beta-\gamma-\alpha)\tilde\psi_{c_2,b_2}(\gamma-2\beta)\overline{\tilde\psi'_{a_3,c_3}(\alpha)}\tilde\psi_{c_4,b_4}(\alpha-\beta)\overline{\tilde\psi'_{a_5,c_5}(\beta)}\\ &\times e^{2\pi i(\beta^2+2\gamma^2+2\alpha\gamma-4\beta\gamma-2\alpha\beta)}\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\\ =& e^{-\frac{7\pi i}{12}}\int_{\mathbb{R}^3} \psi_{c_1,b_1}(\beta-\gamma-\alpha)\psi_{b_2,a_2}(\gamma-2\beta)\psi_{b_3,c_3}(-\alpha)\psi_{b_4,a_4}(\alpha-\beta)\psi_{b_5,c_5}(-\beta)\\ &\times e^{2\pi i(\alpha^2+4\beta^2+\frac{5}{2}\gamma^2-3\alpha\beta+2\alpha\gamma-6\beta\gamma)}\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma.\end{aligned}$$

Changing to parameters:

$$-\tilde\beta = -\beta - c_{\mathrm{b}}(1-2a_5), \quad -\tilde\alpha = -\alpha - c_{\mathrm{b}}(1-2a_3), \quad \tilde\gamma = \gamma + c_{\mathrm{b}}(1-2(2a_5-c_2)$$

yields the formula

$$\begin{aligned}Z_\hbar(X) = \int_{\mathbb{R}^3} & e^{-\frac{7\pi i}{12}}\psi(-\tilde\alpha)\psi(-\tilde\beta)\psi(\tilde\alpha-\tilde\beta-c_{\mathrm{b}})\psi(\tilde\gamma-2\tilde\beta)\psi(\tilde\beta-\tilde\gamma-\tilde\alpha)\\ &\times e^{-4\pi i c_{\mathrm{b}}c_1(\tilde\beta-\tilde\gamma-\tilde\alpha+c_{\mathrm{b}}(1/2-a_1))}e^{\pi i c_{\mathrm{b}}^2(4(c_1-b_1)+1)/6}\\ &\times e^{-4\pi i c_{\mathrm{b}}b_2(\tilde\gamma-2\tilde\beta+c_{\mathrm{b}}(\frac12-c_2))}e^{\pi i c_{\mathrm{b}}(4(b_2-a_2)+1)/6}\\ &\times e^{-4\pi i c_{\mathrm{b}}b_3(-\tilde\alpha+c_{\mathrm{b}}(\frac12-a_3))}e^{\pi i c_{\mathrm{b}}(4(b_3-c_3)+1)/6}\\ &\times e^{-4\pi i c_{\mathrm{b}}b_4(\tilde\alpha-\tilde\beta-c_{\mathrm{b}}+c_{\mathrm{b}}(1/2-c_4))}e^{\pi i c_{\mathrm{b}}(4(b_4-a_4)+1)/6}\\ &\times e^{-4\pi i c_{\mathrm{b}}b_5(-\tilde\beta+c_{\mathrm{b}}(\frac12-a_5))}e^{\pi i c_{\mathrm{b}}(4(b_5-c_5))/6}\\ &\times e^{2\pi i(\tilde\alpha^2+4\tilde\beta^2+\frac52\tilde\gamma^2-3\tilde\alpha\tilde\beta+2\tilde\alpha\tilde\gamma-6\tilde\beta\tilde\gamma)}\\ &\times e^{-4\pi i c_{\mathrm{b}}\tilde\alpha(1-2a_3-\frac32(1-2a_5)+1-2(2a_5-c_2))}\\ &\times e^{-4\pi i c_{\mathrm{b}}\tilde\beta(4(1-2a_5)-\frac32(1-2a_3)-3(1-2(2a_5-c_2)))}\\ &\times e^{-4\pi i c_{\mathrm{b}}\tilde\gamma(\frac52(1-2(2a_5-c_2))+1-2a_3-3(1-2a_5))}\\ &\times e^{\pi i c_{\mathrm{b}}^2(2(1-2a_3)^2+8(1-2a_5)^2+5(1+2(2a_5-c_2))^2)}\\ &\times e^{\pi i c_{\mathrm{b}}^2(4(1-2a_3)(1+2(2a_5-c_2))-12(1-2a_5)(1+2(2a_5-c_2))-6(1-2a_3)(1-2a_5)}.\end{aligned}$$

Collecting the shape variables which are multiplied onto $-4\pi i c_{\mathrm{b}}\tilde{\alpha}$, we get

$$\begin{aligned}
& -c_1 - b_3 + b_4 - \frac{3}{2}(1-2a_5) + 1 - 2(2a_5 - c_2) + (1-2a_3) \\
&= -c_1 - b_3 + b_4 + \frac{1}{2} - a_5 + 2c_2 - 2a_3 \\
&= -a_1 - c_1 + c_2 - b_3 - a_3 + b_4 + \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

Collecting the shape variables which are multiplied onto $-4\pi i c_{\mathrm{b}}\tilde{\gamma}$, we get

$$\frac{5}{2} + 1 - 3 - c_1 + b_2 - 10a_5 + 5c_2 - 2a_3 + 6a_5 = \frac{1}{2} - c_1 + b_2 - 4a_5 + 5c_2 - 2a_3 =: \lambda.$$

Collecting the shape variables which are multiplied onto $-4\pi i c_{\mathrm{b}}\tilde{\beta}$, we get

$$-\frac{1}{2} + c_1 - b_2 + 4a_5 - 5c_2 + 2a_3 - \frac{1}{2} = -\lambda - \frac{1}{2}.$$

$$Z_\hbar(X) = e^{i\frac{\phi}{\hbar}} \int_{\mathbb{R}} J_{S^3,6_2}(\hbar, x) e^{-\frac{\lambda x}{\hbar}} \mathrm{d}x$$

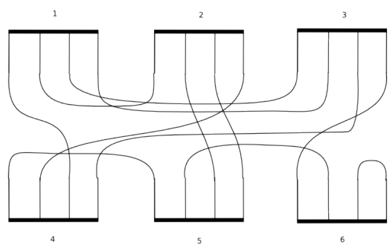
where

$$J_{S^3,6_2}(\hbar, x) = \int_{\mathbb{R}^2} \frac{1}{\Phi_{\mathrm{b}}(-\tilde{\alpha})} \frac{1}{\Phi_{\mathrm{b}}(-\tilde{\beta})} \frac{1}{\Phi_{\mathrm{b}}(-\tilde{\alpha}-\tilde{\beta}-c_{\mathrm{b}})} \frac{1}{\Phi_{\mathrm{b}}(x-\tilde{\beta})} \frac{1}{\Phi_{\mathrm{b}}(-x-\tilde{\alpha})} \\ e^{2\pi i(\tilde{\alpha}^2 + \frac{1}{2}\tilde{\beta}^2 + \frac{5}{2}x^2 - \tilde{\alpha}\tilde{\beta} + 2\tilde{\alpha}x - \tilde{\beta}x)} e^{-4\pi i c_{\mathrm{b}}(\tilde{\alpha}-\tilde{\beta})} d\tilde{\alpha} d\tilde{\beta},$$

where ϕ is the quadratic term of dihedral angles and λ is defined above.

9.8 One vertex H-triangulation of $(S^3, 6_2)$-knot

Let X be represented by the diagram



We choose an orientation of the diagram. The edge representing the knot has weight $frm{-}e\pi a_6$. In the limit $a_6 \to 0$ all edges except for the knot becomes balanced under equivalent conditions as in the case for the ideal triangulation of the same knot. The renormalised partition function takes the form

$$\tilde{Z}_\hbar(X) := \lim_{a_6 \to 0} \Phi_{\mathrm{b}}(2c_{\mathrm{b}}) Z_\hbar(X) = e^{i\frac{\phi}{\hbar}} \frac{e^{-\pi i/12}}{\nu_{c_0,0}} J_{S^3,6_2}(\hbar, 0),$$

where the function $J_{S^3,6_2}$ is defined above.

We omit the tedious calculations since they are similar to the calculations in previous examples.

The volume conjecture has until now only been approached by use of mathematica. A rigorous proof of the conjecture in this case is still to be made.

Chapter 10

A–K representation of the mapping class group $\Gamma_{1,1}$

In this chapter we give a representation for the mapping class group of the once punctured torus by use of the new formulation of the Andersen–Kashaev TQFT.

10.1 The once punctured torus

We have that $H_1(\Sigma_{1,1};\mathbb{Z}) \approx H_1(\mathbb{T}^2;\mathbb{Z}) \approx \mathrm{SL}(2,\mathbb{Z})$. Therefore there is a homomorphism $\sigma : \Gamma_{1,1} \to \mathrm{SL}(2,\mathbb{Z})$. The map is surjective since any element of $\mathrm{SL}(2,\mathbb{Z})$ can be realised as a map of $\mathbb{R}^2$ that is equivariant with respect to $\mathbb{Z}^2$ and that fixes the origin; such a map descends to a homeomorphism of $\Sigma_{1,1}$ with the desired action on homology. It is also injective: Let α and β be simple closed curves in $\Sigma_{1,1}$ that intersects at one point. If $f \in \ker\sigma$ is represented by ϕ, then $\phi(\alpha)$ and $\phi(\beta)$ are isotopic to α and β. We can then modify ψ by isotopy so that it fixes α and β pointwise. If we cut $\Sigma_{1,1}$ along $\alpha \cup \beta$, we obtain a once-punctured disk, and ϕ induces a homomorphism of this disk fixing the boundary. By Alexander's trick, this homomorphism of the punctured disk is homotopic to the identity by a homotopy that fixes the boundary. It follows that ϕ is homotopic to the identity. Recall from Theorem 1.20 that $\mathrm{SL}(2,\mathbb{Z})$ is generated by the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In spite of Section 8.4 we want to build a cobordism $(M, \mathbb{T}^2, \mathbb{T}^{2\prime})$ from one triangulation of $\mathbb{T}^2$ to the image of this triangulation under the action of S and likewise for the action of T. We triangulate the torus $\mathbb{T}^2 = S^1 \times S^1$ according to Figure 10.1. In this triangulation opposite arrows are identified and this gives us a triangulation with two triangles and three edges. We build the cobordism for S according to Figure 10.2 and the cobordism for T according to Figure 10.3. We see that on each boundary component we have three edges. The cobordisms that we build are given shaped triangulations. We can choose the dihedral angles such that they are all positive. And we are able to compose these cobordisms.

For each edge in these triangulations we assign a state variable. We abuse notation and label an edge and a state variable by the same letter. We assign a multiplier to each edge (see Section 11.3.2). As we will see below in Lemma 10.3 and Lemma 10.5 it turns out, that all internal edges each have trivial multiplier. Further we emphasise that there is a direction on each of the two boundary tori where the multiplier is trivial.

The Andersen–Kashaev TQFT gives an operator between the vector spaces associated to each of the boundary components. We will see that we get representations

$$\rho_{\text{A–K}} : \Gamma_{1,1} \to \mathcal{B}(C^\infty(\mathbb{T}^3, \mathcal{L}')),$$

of the mapping class group $\Gamma_{1,1}$ into bounded operators on the smooth sections $C^\infty(\mathbb{T}^3, \mathcal{L}')$. However, we will show below that we actually get representations into $\mathcal{B}(\mathcal{S}(\mathbb{R}))$, bounded operators on the Schwartz space $\mathcal{S}(\mathbb{R})$.

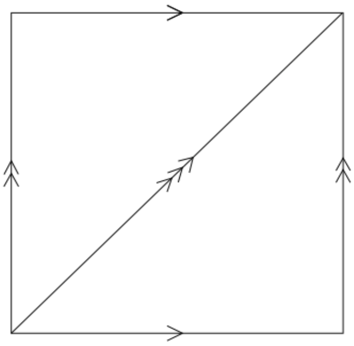


Figure 10.1: Triangulation of the torus into two triangles.

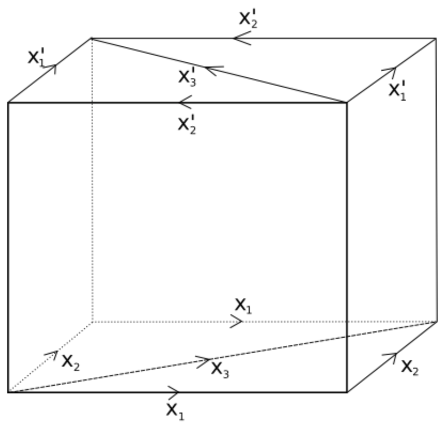


Figure 10.2: The cobordism for the operator S which we triangulate.

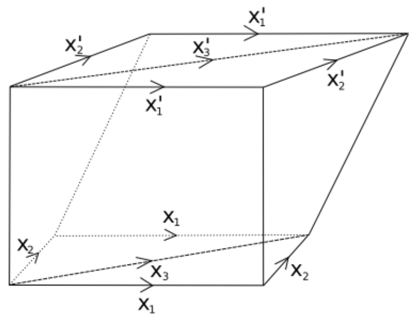


Figure 10.3: The cobordism for the operator T which we triangulate.

Theorem 10.1. The Andersen–Kashaev TQFT provides us with representations

$$\rho_{\text{A–K}} : \Gamma_{1,1} \to \mathcal{B}(\mathcal{S}(\mathbb{R}))$$

of the mapping class group $\Gamma_{1,1}$ into bounded operators on the Schwarz space $\mathcal{S}(\mathbb{R})$. In particular we get operators $\rho_{\text{A–K}}(S), \rho_{\text{A–K}}(T) : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ according to the diagram (10.1).

$$\begin{tikzcd}
\mathcal{S}(\mathbb{R}) \arrow[r, "\rho_{\text{A–K}}(S),\rho_{\text{A–K}}(T)"] \arrow[d, "W"] & \mathcal{S}(\mathbb{R}) \arrow[d, "W"] \\
C^\infty(\mathbb{T}^2, \mathcal{L}) \arrow[r] \arrow[d, "\pi^*"] & C^\infty(\mathbb{T}^2, \mathcal{L}) \arrow[d, "\pi^*"] \\
\pi^*\left(C^\infty(\mathbb{T}^2, \mathcal{L})\right) \arrow[r] & \pi^*\left(C^\infty(\mathbb{T}^2, \mathcal{L})\right) \\
\cap & \cap \\
C^\infty(\mathbb{T}^3, \mathcal{L}') \arrow[r, "\rho(S)", "\rho(T)"'] & C^\infty(\mathbb{T}^3, \mathcal{L}')
\end{tikzcd} \tag{10.1}$$

where $\mathcal{L}' = \pi^*\mathcal{L}$.

Proof. We know that the Weil–Gel′fand–Zak transformation gives an isomorphism from the Schwarz space to smooth sections of the complex line bundle $\mathcal{L}$ over the 2-torus. If a section of $C^\infty(\mathbb{T}^2, \mathcal{L})$ is pulled back to $\pi^*\left(C^\infty(\mathbb{T}^2, \mathcal{L})\right)$ we show in Lemma 10.4 and Lemma 10.6 that the operators $\rho(S), \rho(T)$ acting on $C^\infty(\mathbb{T}^3, \mathcal{L}')$ take this pull back of a section to the pull back of a section in $\pi^*\left(C^\infty(\mathbb{T}^2, \mathcal{L})\right)$. In Lemma 10.3 and 10.5 we prove that multipliers on internal edges are trivial. Further we show that the multipliers on the two boundary tori are trivial in the direction $(1, 1, 1)$. We can therefore integrate over the fibre in this direction. We then use the inverse WGZ transformation. In other words we have shown that the operators $\rho(S), \rho(T)$ induce operators $\rho_{\text{A–K}}(S), \rho_{\text{A–K}}(T) : \mathcal{S}(\mathbb{R}) \to S(\mathbb{R})$ given by

$$\rho_{\text{A–K}}(S) = W^{-1} \circ \int_{F_{z'}} \circ\, \rho(S) \circ \pi^* \circ W,$$
$$\rho_{\text{A–K}}(T) = W^{-1} \circ \int_{F_{z'}} \circ\, \rho(S) \circ \pi^* \circ W.$$

□

Remark 10.2. Above we obtained a representation for the mapping class group $\Gamma_{1,1}$. We do not in a similar manner get a representation for the mapping class group $\Gamma_{1,0}$. The reason is that not all edges in the cobordisms can be balanced without turning to negative angles.

10.2 Line bundle over the two boundary torus

Let us here describe how the line bundles we pull back looks like.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\pi(x_1, x_2, x_3) = (ax_1 + bx_2 + cx_3, \alpha x_1 + \beta x_2 + \gamma x_3)$. Recall that we have the relation on multipliers

$$e_\lambda^{\pi^*}(x, y, z) = e_{\pi(\lambda)}(\pi(x, y, z)). \tag{10.2}$$

Note that the map π sends $\lambda_{x_1} = (1, 0, 0),\ \lambda_{x_2} = (0, 1, 0),\ \lambda_{x_3} = (0, 0, 1)$ to the following elements of $\mathbb{R}^2$

$$\pi(\lambda_{x_1}) = (a, \alpha), \quad \pi(\lambda_{x_2}) = (b, \beta), \quad \pi(\lambda_{x_3}) = (c, \gamma).$$

The equation (10.2) gives the following relations:

In the λ_{x_1}-direction

$$\begin{aligned} e^{2\pi i(x_3-x_2)} =& e_{(1,0,0)(x_1,x_2,x_3)} = e_{(a,\alpha)}(ax_1+bx_2+cx_3, \alpha x_1+\beta x_2+\gamma x_3) \\ =& e_{(a,0)}(ax_1+bx_2+cx_3, \alpha x_1+\beta x_2+\gamma x_3) \\ & e_{(0,\alpha)}(ax_1+bx_2+cx_3, \alpha(x_1+1)+\beta x_2+\gamma x_3) \\ =& e^{-\pi i a(\alpha x_1+\beta x_2+\gamma x_3)} e^{\pi i(ax_1+bx_2+cx_3)} \\ =& e^{\pi i((\alpha b - a\beta)x_2+(\alpha c - a\gamma)x_3)}, \end{aligned}$$

In the λ_{x_2}-direction

$$e^{2\pi i(x_1-x_3)} = e_{(0,1,0)(x_1,x_2,x_3)} = e^{\pi i((\beta a-\alpha b)x_1+(\beta c-b\gamma)x_3)},$$

In the λ_{x_3}-direction

$$e^{2\pi i(x_2-x_1)} = e_{(0,0,1)(x_1,x_2,x_3)} = e^{\pi i((\gamma a-\alpha c)x_1+(\gamma b-c\beta)x_2)}.$$

In other words we only need to solve the three equations

$$\alpha b - a\beta = -2, \quad \alpha c - a\gamma = 2, \quad \beta c - b\gamma = -2. \tag{10.3}$$

One particular solution is $a = -2, b = 0, c = 2, \alpha = 0, \beta = -1, \gamma = 1$ which gives the map

$$\pi(x_1, x_2, x_3) = (-2x_1 + 2x_3, -x_2 + x_3).$$

10.3 The operator $\rho(S)$

The operator $\rho(S)$ can be viewed as the cobordism X_S which is triangulated into 6 tetrahedra $T_1, \ldots, T_6$ where T_1, T_3, T_4, T_6 have positive orientation and the tetrahedra T_2, T_5 have negative orientation. See the gluing pattern in figure 10.5.

In the triangulation we have ten edges $x_1, x_2 \ldots, x_7, x_1', x_2', x_3'$. To each of the edges on the boundary we associate the a weight function:

$$\begin{aligned} &\omega_{X_S}(x_1) = 2\pi(a_1+a_5+c_3), && \omega_{X_S}(x_2) = 2\pi(a_4+c_5+a_6), && \omega_{X_S}(x_3) = 2\pi(b_5+b_6), \\ &\omega_{X_S}(x_1') = 2\pi(a_1+c_2+a_3), && \omega_{X_S}(x_2') = 2\pi(a_2+c_3+a_4), && \omega_{X_S}(x_3') = 2\pi(b_2+b_3). \end{aligned}$$

and to the edges x_4, x_5, x_6, x_7 we associate the weight functions:

$$\begin{aligned} &\omega_{X_S}(x_4) = 2\pi(a_1+c_2+b_4+c_5+c_6), && \omega_{X_S}(x_5) = 2\pi(c_1+b_3+b_4+a_5+a_6), \\ &\omega_{X_S}(x_6) = 2\pi(b_1+a_2+a_3+c_4+b_6), && \omega_{X_S}(x_7) = 2\pi(b_1+c_2+c_3+c_4+b_5). \end{aligned}$$

When we balance edges x_4, x_5, x_7 and the boundary edges on the bottom torus are given weights $\omega_{X_S}(x_1) = \alpha$, $\omega_{X_S}(x_2) = \beta$, $\omega_{X_S}(x_3) = \gamma$. We then get the following restrictions on the dihedral angles:

$$
\begin{aligned}
a_1 &= \alpha + \beta + \gamma + b_4 + c_4 - c_3 + c_6, \\
a_2 &= \alpha + \beta + \gamma - b_2 + 2b_4 - c_3 + c_4 + c_5 + 2c_6 - 2, \\
a_3 &= -2\gamma - \beta - 2\alpha - b_4 + 2c_3 - 3c_5 - 3c_6 + 3, \\
a_4 &= \frac{1}{2} - b_4 - c_4, \\
a_5 &= \frac{3}{2} - \gamma - \beta - b_4 - c_4 - c_6, \\
a_6 &= \beta + b_4 + c_4 - c_5 - c_6, \\
b_1 &= \alpha + b_4 - 2c_3 - c_4 + 2c_5 + c_6 - \frac{1}{2}, \\
b_3 &= 2\gamma + \beta + 2\alpha + b_4 - 3c_3 + 3c_5 + 3c_6 - \frac{5}{2}, \\
b_5 &= \gamma + \beta + b_4 + c_4 - c_5 + c_6 - 1, \\
b_6 &= -\beta - b_4 - c_4 + c_5 - c_6 + 1, \\
c_1 &= -\gamma - \beta - 2\alpha - 2b_4 + 3c_3 - 2c_5 - 2c_6 + \frac{5}{2}, \\
c_2 &= -\gamma - \beta - \alpha - 2b_4 + c_3 - c_4 - c_5 - 2c_6 + \frac{5}{2},
\end{aligned}
$$

and $b_2, b_4, c_3, c_4, c_5, c_6$ are free variables. Since the edges x_1, x_2, x_3 form a triangle the sum of the weights $\alpha+\beta+\gamma$ must sum to $\frac{1}{2}$. It is easily checked that the variables $\tilde{\alpha} := a_1 + c_2 + a_3$, $\tilde{\beta} := a_2 + c_3 + a_4$ and $\tilde{\gamma} := b_2 + b_3$ sum to $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = \frac{1}{2}$, where now $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ represents the weights on the edges x_1', x_2', x_3' respectfully. We can choose solutions such that dihedral angles are positive.

10.3.1 Boltzmann weights

The Bolzman weights assigned to the tetrahedra are

$$
\begin{aligned}
B\left(T_1, x_{|\Delta_1(T_1)}\right) &= g_{a_1,c_1}(x_7 + x_6 - x_4 - x_5, x_7 + x_6 - x_1' - x_1), \\
B\left(T_2, x_{|\Delta_1(T_2)}\right) &= \overline{g_{a_2,c_2}(x_3' + x_4 - x_1' - x_7, x_3' + x_4 - x_2' - x_6)}, \\
B\left(T_3, x_{|\Delta_1(T_3)}\right) &= g_{a_3,c_3}(x_3' + x_5 - x_2' - x_7, x_3' + x_5 - x_1' - x_6) \\
B\left(T_4, x_{|\Delta_1(T_4)}\right) &= g_{a_4,c_4}(x_5 + x_4 - x_7 - x_6, x_5 + x_4 - x_2' - x_2) \\
B\left(T_5, x_{|\Delta_1(T_5)}\right) &= \overline{g_{a_5,c_5}(x_7 + x_3 - x_4 - x_2, x_5 + x_4 - x_5 - x_1)} \\
B\left(T_6, x_{|\Delta_1(T_6)}\right) &= g_{a_6,c_6}(x_6 + x_3 - x_4 - x_1, x_6 + x_3 - x_5 - x_2).
\end{aligned}
$$

Lemma 10.3. The multipliers corresponding to the edges are calculated to be 1 for the internal edges x_4, x_5, x_6, x_7. And the multipliers for the remaining 6 edges are calculated to be

$$
\begin{aligned}
&e_{\lambda_{x_1}}(\mathbf{x}) = e^{2\pi i(x_3 - x_2)}, \quad e_{\lambda_{x_2}}(\mathbf{x}) = e^{2\pi i(x_1 - x_3)}, \quad e_{\lambda_{x_3}}(\mathbf{x}) = e^{2\pi i(x_2 - x_1)}, \\
&e_{\lambda_{x_1'}}(\mathbf{x}) = e^{2\pi i(x_2' - x_3')}, \quad e_{\lambda_{x_2'}}(\mathbf{x}) = e^{2\pi i(x_3' - x_1')}, \quad e_{\lambda_{x_3'}}(\mathbf{x}) = e^{2\pi i(x_1' - x_2')},
\end{aligned}
$$

where $\mathbf{x}$ denotes the tuple $\mathbf{x} = (x_1, x_2, x_3, x_1', x_2', x_3')$.

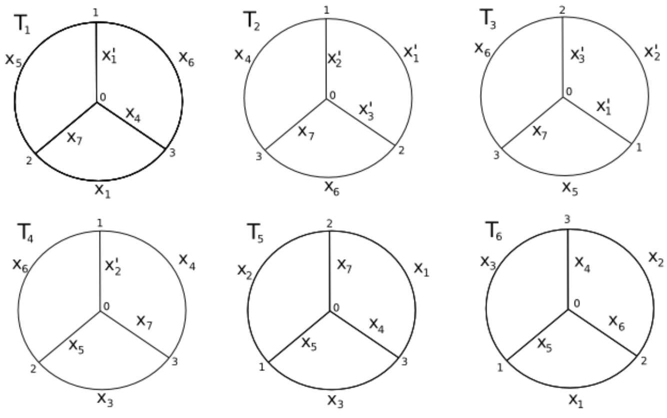


Figure 10.4: The tetrahedra of the triangulation of X_S

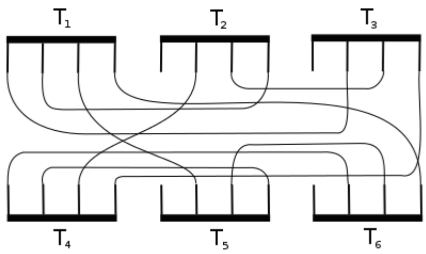


Figure 10.5: The tetrahedra of the triangulation of X_S are glued together following the rules of this diagram.

Proof. The multipliers are calculated by use of (11.20). Let us here just calculate the multiplier for the direction x_4. The rest follows by analogous calculations. The edge x_4 is an edge in the tetrahedra T_1, T_2, T_4, T_5, T_6 each contributing to the multiplier. The contribution from T_1 corresponds to the multiplier

$$\begin{aligned} e_{\lambda_{x_4}}(x_1, x_2, \ldots, x_7, x_1', x_2', x_3') &= e_{-(1,0)}(x_5 + x_4 - x_7 - x_6, x_5 + x_4 - x_2' - x_2) \\ &= e^{\pi i (x_7 + x_6 - x_1' - x_1)}. \end{aligned}$$

The contribution from T_2 corresponds is

$$\begin{aligned} e_{\lambda_{x_4}}(x_1, x_2, \ldots, x_7, x_1', x_2', x_3') &= \overline{e_{(1,1)}(x_3' + x_4 - x_1' - x_7, x_3' + x_4 - x_2' - x_6)} \\ &= -e^{-\pi i (x_2' + x_6 - x_1' - x_7)}. \end{aligned}$$

The contribution from T_4 corresponds is

$$\begin{aligned} e_{\lambda_{x_4}}(x_1, x_2, \ldots, x_7, x_1', x_2', x_3') &= e_{(1,1)}(x_5 + x_4 - x_7 - x_6, x_5 + x_4 - x_2' - x_2) \\ &= -e^{\pi i (x_2' + x_2 - x_6 - x_7)}. \end{aligned}$$

The contribution from T_5 corresponds is

$$e_{\lambda_{x_4}}(x_1, x_2, \ldots, x_7, x_1', x_2', x_3') = -e^{-\pi i (x_7 + x_3 - x_5 - x_1)}.$$

The contribution from T_6 corresponds is

$$e_{\lambda_{x_4}}(x_1, x_2, \dots, x_7, x_1', x_2', x_3') = -e^{\pi i(x_6+x_3-x_5-x_2)}.$$

Multiplying these contributions gives $e^0 = 1$. □

We remark that the multiplier on each boundary component in direction $(1,1,1)$ is trivial.

We are interested in how the operator $\rho(S)$ works. Recall from section 6.2 that a TQFT assigns to a cobordism (M, Σ, Σ') a linear map from the from the vector space assigned to Σ to the vector space assigned to Σ'. We look at the expression for the operator $\rho(S)$. This is done using 8.2. The expression we get is the kernel of an integral, we therefore express the operator $\rho(S)$ in terms of the integeral kernel K_S. The operator $\rho(S)$ acts on sections in the following manner:

$$\rho(S)(s)(x_1', x_2', x_3') = \int_{[0,1]^3} K_S(x_1', x_2', x_3', x_1, x_2, x_3) s(x_1, x_2, x_3)\, dx_1 dx_2 dx_3. \tag{10.4}$$

Let us look at the expression for the kernel K_S :

$$\begin{aligned}
K_S(x_1', x_2', x_3', x_1, x_2, x_3) = \\
\int_{[0,1]^4} \sum_{k,l,m,n,p,q} & \tilde{\psi}'_{a_1,c_1}(x_7+x_6-x_4-x_5+k) e^{\pi i(x_7+x_6-x_1'-x_1)(x_7+x_6-x_4-x_5+2k)} \\
& \overline{\tilde{\psi}'_{a_2,c_2}(x_3'+x_4-x_1'-x_7+l)} e^{-\pi i(x_3'+x_4-x_2'-x_6)(x_3'+x_4-x_1'-x_7+2l)} \\
& \tilde{\psi}'_{a_3,c_3}(x_3'+x_5-x_2'-x_7+m) e^{\pi i(x_3'+x_5-x_1'-x_6)(x_3'+x_5-x_2'-x_7+2m)} \\
& \tilde{\psi}'_{a_4,c_4}(x_5+x_4-x_7-x_6+n) e^{\pi i(x_5+x_4-x_2'-x_2)(x_5+x_4-x_7-x_6+2n)} \\
& \overline{\tilde{\psi}'_{a_5,c_5}(x_7+x_3-x_4-x_2+p)} e^{-\pi i(x_7+x_3-x_5-x_1)(x_7+x_3-x_4-x_2+2p)} \\
& \tilde{\psi}'_{a_6,c_6}(x_6+x_3-x_4-x_1+q) e^{\pi i(x_6+x_3-x_5-x_2)(x_6+x_3-x_4-x_1+2q)} \\
& dx_4 dx_5 dx_6 dx_7.
\end{aligned}$$

Making shifts $x_3' \mapsto x_3' - x_4 + x_7$, $x_2' \mapsto x_2' - x_4 + x_5$, $x_3 \mapsto x_3 + x_4 - x_7$, $x_1 \mapsto x_1 + x_6 - x_7$, $x_7 \mapsto x_7 - x_6 + x_4 + x_5$ and $x_5 \mapsto x_5 + x_6$ we arrive at the expression:

$$\begin{aligned}
K_S(x_1', x_2', x_3', x_1, x_2, x_3) = \int_{[0,1]^4} \sum_{k,l,m,n,p,q} & \tilde{\psi}'_{a_1,c_1}(x_7+k) e^{\pi i(-x_1'-x_1+2x_7+2x_4+2x_5)(x_7+2k)} \\
& \overline{\tilde{\psi}'_{a_2,c_2}(x_3'-x_1'+l)} e^{-\pi i(x_3'-x_2'+2x_4-2x_6+x_7)(x_3'-x_1'+2l)} \\
& \tilde{\psi}'_{a_3,c_3}(x_3'-x_2'+m) e^{\pi i(x_3'-x_1'+2x_5+x_7)(x_3'-x_2'+2m)} \\
& \tilde{\psi}'_{a_4,c_4}(-x_7+n) e^{\pi i(2x_4-x_2'-x_2)(-x_7+2n)} \\
& \overline{\tilde{\psi}'_{a_5,c_5}(x_3-x_2+p)} e^{-\pi i(x_3-x_1+2x_4-2x_6+x_7)(x_3-x_2+2p)} \\
& \tilde{\psi}'_{a_6,c_6}(x_3-x_1+q) e^{\pi i(x_3-x_2-2x_5-x_7)(x_3-x_1+2q)} \\
& dx_4 dx_5 dx_6 dx_7.
\end{aligned}$$

Integration over x_6 gives

$$\int_0^1 e^{2\pi i x_6(x_3'-x_1'+x_3-x_2+2(l+p))} dx_6 = \frac{e^{2\pi i(x_3'-x_1'+x_3-x_2+2(l+p))} - 1}{2\pi i(x_3'-x_1'+x_3-x_2+2(p+l))} =: I_1^{l,p}(x_2, x_3, x_1', x_3').$$

Integration over x_4 gives

$$\begin{aligned}\int_0^1 e^{2\pi i x_4(x_1'-x_3'-x_3+x_2+2(k-l+n-p))}dx_4 &= \frac{e^{2\pi i(x_1'-x_3'-x_3+x_2+2(k-l+n-p))}-1}{2\pi i(x_1'-x_3'-x_3+x_2+2(k-l+n-p))}\\ &=: I_2^{k,l,n,p}(x_2,x_3,x_1',x_3').\end{aligned}$$

We collect the terms where x_7 and x_5 appear.

$$\begin{aligned}K_S(x_1',x_2',x_3',x_1,x_2,x_3) = \sum_{k,l,m,n,p,q}\int_{[0,1]^2} &\tilde{\psi}'_{a_1,c_1}(x_7+k)\tilde{\psi}'_{a_4,c_4}(-x_7+n)\\ &e^{2\pi i x_7(x_2-x_3+x_7+x_5+2k+m-p-q)}\\ &e^{2\pi i x_5(x_3'-x_2'-x_3+x_1+k-q)}dx_5dx_7\\ &\overline{\tilde{\psi}'_{a_2,c_2}(x_3'-x_1'+l)}e^{-\pi i(x_3'-x_2')(x_3'-x_1'+2l)}\\ &\tilde{\psi}'_{a_3,c_3}(x_3'-x_2'+m)e^{\pi i(x_3'-x_1')(x_3'-x_2'+2m)}\\ &\overline{\tilde{\psi}'_{a_5,c_5}(x_3-x_2+p)}e^{-\pi i(x_3-x_1)(x_3-x_2+2p)}\\ &\tilde{\psi}'_{a_6,c_6}(x_3-x_1+q)e^{\pi i(x_3-x_2)(x_3-x_1+2q)}\\ &I_1^{l,p}(x_2,x_3,x_1',x_3')I_2^{k,l,n,p}(x_2,x_3,x_1',x_3').\end{aligned}$$

Simplifying this we end up with the integral kernel

$$\begin{aligned}K_S(x_1,x_2,x_3,x_1',x_2',x_3') := \sum_{k,l,m,n,p,q}\int_{[0,1]^2} &\tilde{\psi}'_{a_1,c_1}(x_7+k)\tilde{\psi}'_{a_4,c_4}(-x_7+n)\\ &e^{2\pi i x_7(x_2-x_3+x_7+x_5+2n-m+k-j)}\\ &e^{2\pi i(x_3'-x_2'-x_3+x_1+k-j)}dx_5dx_7\\ &\overline{\tilde{\psi}'_{a_2,c_2}(x_3'-x_1'+l)}e^{-2\pi i(x_3'-x_2')l}\\ &\tilde{\psi}'_{a_3,c_3}(x_3'-x_2'+m)e^{2\pi i(x_3'-x_1')m}\\ &\overline{\tilde{\psi}'_{a_5,c_5}(x_3-x_2+p)}e^{-2\pi i(x_3-x_1)p}\\ &\tilde{\psi}'_{a_6,c_6}(x_3-x_1+q)e^{2\pi i(x_3-x_2)q}\\ &I_1(x_2,x_3,x_1',x_3')I_2(x_2,x_3,x_1',x_3').\end{aligned}$$

We want to show that the operator S takes the pull back of a section to the pull back of a section. Using integration by parts it is enough to check that the sum of partial derivatives disappear.

Lemma 10.4. The sum of the partial derivatives of K_S disappears. I.e.

$$\frac{\partial K_S}{\partial x_1'}+\frac{\partial K_S}{\partial x_2'}+\frac{\partial K_S}{\partial x_3'}+\frac{\partial K_S}{\partial x_1}+\frac{\partial K_S}{\partial x_2}+\frac{\partial K_S}{\partial x_3}=0.$$

Proof. Let

$$\begin{aligned}I_3^{n,m,k,j}(x_1,x_2,x_3,x_2',x_3') := \int_{[0,1]^2} &\tilde{\psi}'_{a_1,c_1}(x_7+k)\tilde{\psi}'_{a_4,c_4}(-x_7+n)\\ &e^{2\pi i x_7(x_2-x_3+x_7+x_5+2n-m+k-j)}\\ &e^{2\pi i(x_3'-x_2'-x_3+x_1+k-j)}dx_5dx_7\end{aligned}$$

The partial derivatives of I_3 with respect to $x_1, x_2, x_3, x_2', x_3'$ are easily calculated to be

$$\begin{aligned}
\frac{\partial}{\partial x_1} I_3^{n,m,k,j}(x_1,x_2,x_3,x_2',x_3') &= 2\pi i x_5 I_3(x_1,x_2,x_3,x_2',x_3') =: I_3'(x_1,x_2,x_3,x_2',x_3'),\\
\frac{\partial}{\partial x_2} I_3^{n,m,k,j}(x_1,x_2,x_3,x_2',x_3') &= 2\pi i x_7 I_3(x_1,x_2,x_3,x_2',x_3') =: I_3''(x_1,x_2,x_3,x_2',x_3'),\\
\frac{\partial}{\partial x_3} I_3^{n,m,k,j}(x_1,x_2,x_3,x_2',x_3') &= -I_3'(x_1,x_2,x_3,x_2',x_3') - I_3''(x_1,x_2,x_3,x_2',x_3'),\\
\frac{\partial}{\partial x_2'} I_3^{n,m,k,j}(x_1,x_2,x_3,x_2',x_3') &= -I_3'(x_1,x_2,x_3,x_2',x_3'),\\
\frac{\partial}{\partial x_3'} I_3^{n,m,k,j}(x_1,x_2,x_3,x_2',x_3') &= I_3'(x_1,x_2,x_3,x_2',x_3').
\end{aligned}$$

The partial derivatives of I_2 with respect to the variables x_2, x_3, x_1', x_3' are

$$\begin{aligned}
\frac{\partial}{\partial x_2} I_2^{k,l,n,p}(x_2,x_3,x_1',x_3') =& \frac{e^{2\pi i(x_1'-x_3'-x_3+x_2+2(k,l,n,p))}(x_1'-x_3'-x_3+x_2+2(k,l,n,p))}{(x_1'-x_3'-x_3+x_2+2(k,l,n,p))^2}\\
&- \frac{(e^{2\pi i(x_1'-x_3'-x_3+x_2+2(k,l,n,p))}-1)}{2\pi i(x_1'-x_3'-x_3+x_2+2(k,l,n,p))^2}\\
=&: I_2'(x_2,x_3,x_1',x_3'),\\
\frac{\partial}{\partial x_3} I_2^{k,l,n,p}(x_2,x_3,x_1',x_3') =& -I_2'(x_2,x_3,x_1',x_3'),\\
\frac{\partial}{\partial x_1'} I_2^{k,l,n,p}(x_2,x_3,x_1',x_3') =& I_2'(x_2,x_3,x_1',x_3'),\\
\frac{\partial}{\partial x_3'} I_2^{k,l,n,p}(x_2,x_3,x_1',x_3') =& -I_2'(x_2,x_3,x_1',x_3').
\end{aligned}$$

The partial derivatives of I_1 with respect to the variables x_2, x_3, x_1', x_3' are

$$\begin{aligned}
\frac{\partial}{\partial x_2} I_1^{l,p}(x_2,x_3,x_1',x_3') =& -\frac{e^{2\pi i(x_3'-x_1'-x_2+x_3+2(m+q))}(x_3'-x_1'-x_2+x_3+2(m+q))}{(x_3'-x_1'-x_2+x_3+2(m+q))^2}\\
&+ \frac{(e^{2\pi i(x_3'-x_1'-x_2+x_3+2(m+q))}-1)}{2\pi i(x_3'-x_1'-x_2+x_3+2(m+q))^2}\\
=& I_1'(x_2,x_3,x_1',x_3'),\\
\frac{\partial}{\partial x_3} I_1^{l,p}(x_2,x_3,x_1',x_3') =& -I_1'(x_2,x_3,x_1',x_3'),\\
\frac{\partial}{\partial x_1'} I_1^{l,p}(x_2,x_3,x_1',x_3') =& I_1'(x_2,x_3,x_1',x_3'),\\
\frac{\partial}{\partial x_3'} I_1^{l,p}(x_2,x_3,x_1',x_3') =& -I_1'(x_2,x_3,x_1',x_3').
\end{aligned}$$

The rest of the terms in K_S all depends on pairs of the variables $x_1, x_2, x_3, x_1', x_2', x_3'$ with opposite sign, summing all contributions together therefore shows that the sum of the partial derivatives disappears. $\square$

10.4 The operator $\rho(T)$

The operator $\rho(T)$ can be seen as the cobordism Y_s which is triangulated into 6 tetrahedra $T_1, \ldots, T_6$ where T_1, T_4, T_5 have negative orientation and the tetrahedra T_2, T_3, T_6 have positive orientation.

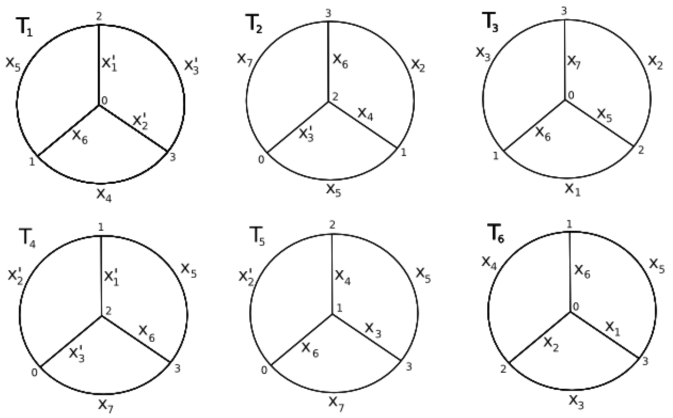


Figure 10.6: The tetrahedra of the triangulation of X_T

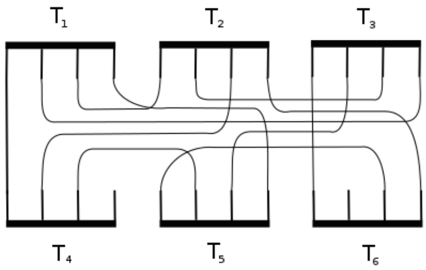


Figure 10.7: The tetrahedra of the triangulation of X_T are glued together following the rules of this diagram.

In the triangulation we have ten edges $x_1, x_2 \ldots, x_7, x_1', x_2', x_3'$. The weight functions corresponding to this triangulation for the edges $x_1, x_2, x_3, x_1', x_2', x_3'$ are

$$\omega_{Y_T}(x_1) = 2\pi(c_3 + a_6), \qquad \omega_{Y_T}(x_2) = 2\pi(b_2 + a_3 + b_6), \qquad \omega_{Y_T}(x_3) = 2\pi(b_3 + b_5 + c_6),$$
$$\omega_{Y_T}(x_1') = 2\pi(a_1 + c_4), \qquad \omega_{Y_T}(x_2') = 2\pi(b_1 + a_4 + b_5), \qquad \omega_{Y_T}(x_3') = 2\pi(c_1 + b_2 + b_4).$$

and to the edges x_4, x_5, x_6, x_7 we associate the weight functions:

$$\omega_{Y_T}(x_4) = 2\pi(a_1 + c_2 + c_5 + a_6), \qquad \omega_{Y_T}(x_5) = 2\pi(b_1 + a_2 + b_3 + b_4 + a_5 + b_6),$$
$$\omega_{Y_T}(x_6) = 2\pi(c_1 + a_2 + a_3 + a_4 + a_5 + c_6), \qquad \omega_{Y_T}(x_7) = 2\pi(b_2 + c_3 + c_4 + c_5).$$

When we balance edges x_4, x_5, x_7 and the boundary edges on the bottom torus are given weights $\omega_{X_T}(x_1) = \delta$, $\omega_{X_S}(x_2) = \varepsilon$, $\omega_{X_S}(x_3) = \zeta$. We then get the following restrictions on the dihedral angles:

$$\begin{aligned}
a_1 &= 1 - a_6 - c_5 - c_2, \\
a_2 &= \frac{1}{2} - c_2 - b_2, \\
a_3 &= \frac{1}{2} - \delta - b_3 - a_6, \\
a_4 &= -\zeta + \delta + b_1 - c_2, \\
a_5 &= -\zeta - \varepsilon - \delta + b_2 - c_5 + \frac{3}{2}, \\
b_4 &= \zeta - b_1 + c_2 - b_3 + b_2 + c_5 + a_6 - 1, \\
b_5 &= \delta + \varepsilon + \zeta - b_2 - 1, \\
b_6 &= \varepsilon + \delta + b_3 - b_2 - a_6 - \frac{1}{2}, \\
c_1 &= -b_1 + c_2 + c_5 + a_6 - \frac{1}{2}, \\
c_3 &= \delta - a_6, \\
c_4 &= -a - b_2 - c_5 - a_6 + 1, \\
c_6 &= \varepsilon - \delta - b_3 + b_2 + 1,
\end{aligned}$$

and $a_6, b_1, b_2, b_3, c_2, c_5$ are free variables. We can choose solutions such that dihedral angles are positive.

The Bolzman weights assigned to the tetrahedra are

$$\begin{aligned}
B\left(T_1, x_{|\Delta_1(T_1)}\right) &= \overline{g_{a_1,c_1}(x_5 + x_2' - x_3' - x_6, x_5 + x_2' - x_1' - x_4)}, \\
B\left(T_2, x_{|\Delta_1(T_2)}\right) &= g_{a_2,c_2}(x_3' + x_2 - x_7 - x_4, x_3' + x_2 - x_5 - x_6), \\
B\left(T_3, x_{|\Delta_1(T_3)}\right) &= g_{a_3,c_3}(x_5 + x_3 - x_7 - x_1, x_5 + x_3 - x_6 - x_2) \\
B\left(T_4, x_{|\Delta_1(T_4)}\right) &= \overline{g_{a_4,c_4}(x_3' + x_5 - x_7 - x_1', x_3 + x_5 - x_2' - x_6)} \\
B\left(T_5, x_{|\Delta_1(T_5)}\right) &= \overline{g_{a_5,c_5}(x_2' + x_3 - x_7 - x_4, x_2' + x_3 - x_6 - x_5)} \\
B\left(T_6, x_{|\Delta_1(T_6)}\right) &= g_{a_6,c_6}(x_2 + x_5 - x_3 - x_6, x_2 + x_5 - x_4 - x_1).
\end{aligned}$$

Lemma 10.5. The multipliers corresponding to the edges are calculated to be 1 for the internal edges x_4, x_5, x_6, x_7. And the multipliers for the remaining 6 edges are calculated to be

$$\begin{aligned}
&e_{\lambda_{x_1}}(\mathbf{x}) = e^{2\pi i(x_3 - x_2)}, \quad e_{\lambda_{x_2}}(\mathbf{x}) = e^{2\pi i(x_1 - x_3)}, \quad e_{\lambda_{x_3}}(\mathbf{x}) = e^{2\pi i(x_2 - x_1)}, \\
&e_{\lambda_{x_1'}}(\mathbf{x}) = e^{2\pi i(x_2' - x_3')}, \quad e_{\lambda_{x_2'}}(\mathbf{x}) = e^{2\pi i(x_3' - x_1')}, \quad e_{\lambda_{x_3'}}(\mathbf{x}) = e^{2\pi i(x_1' - x_2')},
\end{aligned}$$

where **x** denotes the tuple $\mathbf{x} = (x_1, x_2, x_3, x_1', x_2', x_3')$.

Proof. The proof is straight forward verification. The computations are analogue to the calculations in 10.3. □

We also calculate the integral kernel for the operator $\rho(T)$.

$$
\begin{aligned}
&K_T(x_1', x_2', x_3', x_1, x_2, x_3) = \\
&\quad \int_{[0,1]^4} \sum_{k,l,m,n,p,q} \overline{\tilde{\psi}'_{a_1,c_1}(x_5 + x_2' - x_3' - x_6 + k)} e^{-\pi i (x_5 + x_2' - x_1' - x_4)(x_5 + x_2' - x_3' - x_6 + 2k)} \\
&\qquad \tilde{\psi}'_{a_2,c_2}(x_3' + x_2 - x_7 - x_4 + l) e^{\pi i (x_3' + x_2 - x_5 - x_6)(x_3' + x_2 - x_7 - x_4 + 2l)} \\
&\qquad \tilde{\psi}'_{a_3,c_3}(x_5 + x_3 - x_7 - x_1 + m) e^{\pi i (x_5 + x_3 - x_6 - x_2)(x_5 + x_3 - x_7 - x_1 + 2m)} \\
&\qquad \overline{\tilde{\psi}'_{a_4,c_4}(x_3' + x_5 - x_7 - x_1' + n)} e^{-\pi i (x_3' + x_5 - x_2' - x_6)(x_3' + x_5 - x_7 - x_1' + 2n)} \\
&\qquad \overline{\tilde{\psi}'_{a_5,c_5}(x_2' + x_3 - x_7 - x_4 + p)} e^{-\pi i (x_2' + x_3 - x_6 - x_5)(x_2' + x_3 - x_7 - x_4 + 2p)} \\
&\qquad \tilde{\psi}'_{a_6,c_6}(x_2 + x_5 - x_3 - x_6 + q) e^{\pi i (x_2 + x_5 - x_4 - x_1)(x_2 + x_5 - x_3 - x_6 + 2q)} \\
&\qquad dx_4 dx_5 dx_6 dx_7.
\end{aligned}
$$

We make the following shifts in the variables: $x_2 \mapsto x_2 - x_5$, $x_3 \mapsto x_3 - x_6$, $x_2' \mapsto x_2' + x_6 + x_7 + x_4$, $x_3 \mapsto x_3' + x_4 + x_5 + x_7$, $x_1' \mapsto x_1' + x_4 + 2x_5$ and $x_1 \mapsto x_1 + x_5 - x_6 - x_7$, and we get the expression

$$
\begin{aligned}
K_T(x_1', x_2', x_3', x_1, x_2, x_3) = \int_{[0,1]^4} \sum_{k,l,m,n,p,q} &\overline{\tilde{\psi}'_{a_1,c_1}(x_2' - x_3' + k)} \\
&e^{-\pi i (x_2' - x_1' - x_4 - x_5 + x_6 + x_7)(x_2' - x_3' + 2k)} \\
&\tilde{\psi}'_{a_2,c_2}(x_3' + x_2 + l) \\
&e^{\pi i (x_3' + x_2 + x_4 - x_5 - x_6 + x_7)(x_3' + x_2 + 2l)} \\
&\tilde{\psi}'_{a_3,c_3}(x_3 - x_1 + m) \\
&e^{\pi i (2x_5 + x_3 - 2x_6 - x_2)(x_3 - x_1 + 2m)} \\
&\overline{\tilde{\psi}'_{a_4,c_4}(x_3' - x_1' + n)} \\
&e^{-\pi i (x_3' - x_2' + 2x_5 - 2x_6)(x_3' - x_1' + 2n)} \\
&\overline{\tilde{\psi}'_{a_5,c_5}(x_2' + x_3 + p)} \\
&e^{-\pi i (x_2' + x_3 + x_4 - x_5 - x_6 + x_7)(x_2' + x_3 + 2p)} \\
&\tilde{\psi}'_{a_6,c_6}(x_2 - x_3 + q) \\
&e^{\pi i (x_2 - x_1 - x_4 - x_5 + x_6 + x_7)(x_2 - x_3 + 2q)} \\
&dx_4 dx_5 dx_6 dx_7.
\end{aligned}
$$

Integration over the variable x_4 gives

$$
\int_0^1 e^{2\pi i x_4 (k + l - p - q)} dx_4 = \delta(k + l - p - q).
$$

This removes one of the sums in the expression and we are left with

$$\begin{aligned}K_T(x_1',x_2',x_3',x_1,x_2,x_3) = \int_{[0,1]^4} \sum_{k,l,m,n,p} & \overline{\tilde{\psi}'_{a_1,c_1}(x_2'-x_3'+k)} e^{-\pi i(x_2'-x_1'-x_5+x_6+x_7)(x_2'-x_3'+2k)} \\ & \tilde{\psi}'_{a_2,c_2}(x_3'+x_2+l) e^{\pi i(x_3'+x_2-x_5-x_6+x_7)(x_3'+x_2+2l)} \\ & \tilde{\psi}'_{a_3,c_3}(x_3-x_1+m) e^{\pi i(2x_5+x_3-2x_6-x_2)(x_3-x_1+2m)} \\ & \overline{\tilde{\psi}'_{a_4,c_4}(x_3'-x_1'+n)} e^{-\pi i(x_3'-x_2'+2x_5-2x_6)(x_3'-x_1'+2n)} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(x_2'+x_3+k+l-q)} \\ & e^{-\pi i(x_2'+x_3-x_5-x_6+x_7)(x_2'+x_3+2(k+l-q))} \\ & \tilde{\psi}'_{a_6,c_6}(x_2-x_3+q) e^{\pi i(x_2-x_1-x_5+x_6+x_7)(x_2-x_3+2q)} \\ & dx_5 dx_6 dx_7.\end{aligned}$$

We chance the signs of the three variables x_1, x_2, x_3 which does not affect the operator and then we integrate over the three remaining variables.

$$\begin{aligned}K_T(x_1',x_2',x_3',x_1,x_2,x_3) = \int_{[0,1]^4} \sum_{k,l,m,n,p} & \overline{\tilde{\psi}'_{a_1,c_1}(x_2'-x_3'+k)} e^{\pi i(x_2'-x_1'-x_5+x_6+x_7)(x_3'-x_2'-2k)} \\ & \tilde{\psi}'_{a_2,c_2}(x_3'-x_2+l) e^{\pi i(x_3'-x_2-x_5-x_6+x_7)(x_3'-x_2+2l)} \\ & \tilde{\psi}'_{a_3,c_3}(x_3-x_1+m) e^{\pi i(x_2-x_3+2x_5-2x_6)(x_3-x_1+2m)} \\ & \overline{\tilde{\psi}'_{a_4,c_4}(x_3'-x_1'+n)} e^{\pi i(x_3'-x_2'+2x_5-2x_6)(x_1'-x_3'-2n)} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(x_2'-x_3+k+l-q)} \\ & e^{\pi i(x_2'-x_3-x_5-x_6+x_7)(x_3-x_2'-2(k+l-q))} \\ & \tilde{\psi}'_{a_6,c_6}(x_3-x_2+q) e^{\pi i(x_1-x_2-x_5+x_6+x_7)(x_3-x_2+2q)} \\ & dx_5 dx_6 dx_7.\end{aligned}$$

Integration over the variable x_5 is

$$\int_0^1 e^{2\pi i x_5(-2x_3'-2x_3+x_1+x_1'+x_2+x_2'+2(m-n+p-l))} dx_5 \quad = \quad J_1^{m,n,p,l}(x_1,x_2,x_3,x_1',x_2',x_3').$$

Integration over the variable x_6 gives

$$\begin{aligned}\int_0^1 e^{2\pi i x_6(x_3+x_3'-x_1-x_1'+2(m+n))} dx_6 =& \frac{e^{2\pi i(x_3+x_3'-x_1-x_1'+2(m+n))}-1}{2\pi i(x_3+x_3'-x_1-x_1'+2(m+n))} \\ =& J_2^{m,n}(x_1,x_3,x_1',x_3').\end{aligned}$$

Integration over the variable x_7 gives

$$\begin{aligned}\int_0^1 e^{2\pi i x_7(x_3'-x_2-x_2'+x_3+2(l-p))} dx_6 =& \frac{e^{2\pi i(x_3'-x_2-x_2'+x_3+2(l-p))}-1}{2\pi i(x_3'-x_2-x_2'+x_3+2(l-p))} \\ =& J_3^{l,p}(x_2,x_3,x_2',x_3').\end{aligned}$$

So we have the expression for the operator K_T

$$\begin{aligned} K_T(x_1,x_2,x_3,x_1',x_2',x_3') = \sum_{k,l,m,n,p} & \overline{\tilde{\psi}'_{a_1,c_1}(x_2'-x_3'+k)}e^{\pi i(x_2'-x_1')(x_3'-x_2'-2k)} \\ & \tilde{\psi}'_{a_2,c_2}(x_3'-x_2+l)e^{\pi i(x_3'-x_2)(x_3'-x_2+2l)} \\ & \tilde{\psi}'_{a_3,c_3}(x_3-x_1+m)e^{\pi i(x_2-x_3)(x_3-x_1+2m)} \\ & \overline{\tilde{\psi}'_{a_4,c_4}(x_3'-x_1'+n)}e^{\pi i(x_3'-x_2')(x_1'-x_3'-2n)} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(x_2'-x_3+k+l-q)}e^{\pi i(x_2'-x_3)(x_3-x_2'-2(k+l-q))} \\ & \tilde{\psi}'_{a_6,c_6}(x_3-x_2+q)e^{\pi i(x_1-x_2)(x_3-x_2+2q)} \\ & J_1^{m,n,p,l}(x_1,x_2,x_3,x_1',x_2',x_3') \\ & J_2^{m,n}(x_1,x_3,x_1',x_3')J_3^{l,p}(x_2,x_3,x_2',x_3'). \end{aligned}$$

Again, in order to check that the operator $\rho(T)$ takes the pull back of a section to a pull back of a section we show the following Lemma.

Lemma 10.6. The sum of the partial derivatives of K_T disappears. I.e.

$$\frac{\partial K_T}{\partial x_1'} + \frac{\partial K_T}{\partial x_2'} + \frac{\partial K_T}{\partial x_3'} + \frac{\partial K_T}{\partial x_1} + \frac{\partial K_T}{\partial x_2} + \frac{\partial K_T}{\partial x_3} = 0$$

Proof. In each term of the expression for K_T there is an equal number of variables one half having positive coefficient and the other half having negative coefficient. Therefore the sum of the partial differentials must equal zero. □

10.4.1 Change of coordinates

Clearly the multipliers are trivial in the direction

$$z = x_1 + x_2 + x_3 \quad \text{and} \quad z' = x_1' + x_2' + x_3'.$$

We change coordinates to

$$\begin{aligned} x &= -2x_1 + 2x_3, & x' &= 2x_1' - 2x_3' \\ y &= -x_2 + x_3, & y' &= -x_2' + x_3' \\ z &= x_1 + x_2 + x_3, & z' &= x_1' + x_2 + x_3', \end{aligned}$$

In these coordinates we have

$$\begin{aligned} x_1 &= \frac{1}{3}(z+y-x), & x_1' &= \frac{1}{3}(z'+y'+x'), \\ x_2 &= \frac{1}{3}\left(z-2y+\frac{1}{2}x\right), & x_2' &= \frac{1}{3}\left(z'-2y'-\frac{1}{2}x'\right), \\ x_3 &= \frac{1}{3}\left(z+y+\frac{1}{2}x\right), & x_3' &= \frac{1}{3}\left(z'+y'-\frac{1}{2}\right). \end{aligned}$$

Note that the transformation $x = x(x_1,x_2,x_3)$, $y = y(x_1,x_2,x_3)$ and $z = z(x_1,x_2,x_3)$ changes the volume element $dxdydz = |J(x_1,x_2,x_3)|dx_1dx_2dx_3$. Here J is just the usual Jacobian. In out case $|J(x_1,x_2,x_3)| = 6$.

10.4.2 WGZ-transformation of wavelet

Let us shortly describe how a wavelet transforms under the Weil–Gel'fand–Zak transformation.

We let $f \in \mathcal{S}(\mathbb{R})$ be the wavelet function defined by $f(x) = \frac{1}{2\pi}e^{-\frac{1}{2}x^2}$. We let $T_{a,b}$ be the translation $T_{a,b}(x) = ax + b$ then we define the function $f_{a,b}(x) := f \circ T_{a,b}(x) \in \mathcal{S}(\mathbb{R})$.

Lemma 10.7. Let $f_{a,b} \in \mathcal{S}(\mathbb{R})$, then we have the following transformation rule

$$(Wf_{a,b})(x,y) = e^{\pi ixy} f_{a,b}(x) \cdot \theta\left(y - \frac{ab + a^2 x}{2\pi i}, \frac{-a^2}{2\pi i}\right), \tag{10.5}$$

where $\theta(z;\tau) \equiv \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2 + 2\pi izn}$ is Riemann's theta function.

Proof. The proof is direct computation.

$$\begin{aligned}
(Wf_{a,b})(x,y) &= e^{\pi ixy} \sum_{m\in\mathbb{Z}} f_{a,b}(x+m) \cdot e^{2\pi imy} \\
&= \frac{1}{2\pi} e^{\pi ixy} \sum_{m\in\mathbb{Z}} e^{-\frac{1}{2}(a(x+m)+b)^2} \cdot e^{2\pi imy} \\
&= \frac{1}{2\pi} e^{-\frac{b^2}{2}} e^{\pi ixy} \sum_{m\in\mathbb{Z}} e^{-\frac{1}{2}\left(a^2(x+m)^2\right)} e^{-abx} e^{-abm} \cdot e^{2\pi imy} \\
&= \frac{1}{2\pi} e^{-\frac{b^2}{2}} e^{\pi ixy} \sum_{m\in\mathbb{Z}} e^{-\frac{1}{2}\left(a^2(x^2+m^2+2mx)\right)} e^{-abx} e^{-abm} \cdot e^{2\pi imy} \\
&= e^{\pi ixy} f_{a,b}(x) \sum_{m\in\mathbb{Z}} e^{-\frac{1}{2}(am)^2 + m(2\pi iy - ab - xa^2)} \\
&= e^{\pi ixy} f_{a,b}(x) \cdot \theta\left(y - \frac{ab + a^2 x}{2\pi i}, \frac{-a^2}{2\pi i}\right).
\end{aligned}$$

□

10.4.3 Properties of the Theta function

The Theta function behaves very regularly with respect to its quasi-period τ and satisfies the functional equation

$$\theta(z + p + q\tau, \tau) = \theta(z,\tau) e^{-\pi i\tau q^2 - 2\pi iqz}$$

where p, q are integers.

$$\begin{aligned}
\theta(z + p + q\tau, \tau) &= \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2 + 2\pi i(z+p+q\tau)n} = \sum_{n\in\mathbb{Z}} e^{\pi i\tau(n+q)^2 - \pi i\tau q^2 + 2\pi izn} \\
&= \sum_{l\in\mathbb{Z}} e^{\pi i\tau l^2 - \pi i\tau q^2 + 2\pi izl - 2\pi iqz} = \theta(z,\tau) e^{-\pi i\tau q^2 - 2\pi iqz}
\end{aligned}$$

We apply this formula to the Theta function $\theta\left(y + q - \frac{ab+2a^2(\tilde{x}-m)}{2\pi i}, \frac{-a^2}{2\pi i}\right)$. Here $\tau = -\frac{a^2}{2\pi i}$. Direct calculation shows that

$$\theta\left(y + q - \frac{b}{a}\tau + 2\tau(\tilde{x} - m), \tau\right) = \theta\left(y + \frac{b}{a}\tau + 2\tau x, \tau\right) e^{+4\pi imy} e^{2m^2a^2 - 2mab - 4ma^2 x}$$

Therefore we get the much simpler expression for the product

$$\theta\left(y + q - \frac{b}{a}\tau + 2\tau(\tilde{x} - m), \tau\right) \cdot f_{a,b}(2(\tilde{x} - m)) = e^{4\pi imy} \theta\left(y - \frac{b}{a}\tau + 2\tau\tilde{x}, \tau\right) \cdot f_{a,b}(2\tilde{x}).$$

10.5 Representations of $\rho_{\mathbf{A\text{–}K}}(S), \rho_{\mathbf{A\text{–}K}}(T)$

10.5.0.1 Representation of $\rho_{\mathbf{A\text{–}K}}(T)$

The operator $\rho_{\text{A–K}}(T) = W^{-1} \circ \int_{F_{z'}} \circ \rho(T) \circ \pi^* \circ W$, where $\int_{F'_z}$ is integration over the fiber and $\rho(T)$ acts in the following way:

$$\rho(T).(s)(x'_1, x'_2, x'_3) = \int_{[0,1]^3} K_T(x'_1, x'_2, x'_3, x_1, x_2, x_3) s(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 \tag{10.6}$$

Recall that the expression for the kernel is given by

$$\begin{aligned} K_T(x'_1, x'_2, x'_3, x_1, x_2, x_3) = \int_{[0,1]^3} \sum_{k,l,m,n,q} & \overline{\tilde{\psi}'_{a_1,c_1}(x'_2 - x'_3 + k)} e^{-\pi i(x'_2 - x'_1 - x_5 + x_6 + x_7)(x'_2 - x'_3 + 2k)} \\ & \tilde{\psi}'_{a_2,c_2}(x'_3 + x_2 + l) e^{\pi i(x'_3 + x_2 - x_5 - x_6 + x_7)(x'_3 + x_2 + 2l)} \\ & \tilde{\psi}'_{a_3,c_3}(x_3 - x_1 + m) e^{\pi i(2x_5 + x_3 - 2x_6 - x_2)(x_3 - x_1 + 2m)} \\ & \overline{\tilde{\psi}'_{a_4,c_4}(x'_3 - x'_1 + n)} e^{-\pi i(x'_3 - x'_2 + 2x_5 - 2x_6)(x'_3 - x'_1 + 2n)} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(x'_2 + x_3 + p)} \\ & e^{-\pi i(x'_2 + x_3 - x_5 - x_6 + x_7)(x'_2 + x_3 + 2(k + l - q))} \\ & \tilde{\psi}'_{a_6,c_6}(x_2 - x_3 + k + l - p) \\ & e^{\pi i(x_2 - x_1 - x_5 + x_6 + x_7)(x_2 - x_3 + 2q)} \\ & dx_5 dx_6 dx_7. \end{aligned}$$

Writing this in the coordinates x', y', z', x, y, z we get the integral kernel

$$\begin{aligned} K_T(x', y', z', x, y, z) = \int_{[0,1]^3} \sum_{k,l,m,n,q} & \overline{\tilde{\psi}'_{a_1,c_1}(-y' + k)} e^{-\pi i(-y' - \frac{1}{2}x' - x_5 + x_6 + x_7)(-y' + 2k)} \\ & \tilde{\psi}'_{a_2,c_2}(\frac{1}{3}(z' + y' - \frac{1}{2}x' + z - 2y + \frac{1}{2}x) + l) \\ & e^{\pi i(\frac{1}{3}(z' + y' - \frac{1}{2}x' + z - 2y + \frac{1}{2}x) - x_5 - x_6 + x_7)(\frac{1}{3}(z' + y' - \frac{1}{2}x' + z - 2y + \frac{1}{2}x) + 2l)} \\ & \tilde{\psi}'_{a_3,c_3}(\frac{1}{2}x + m) e^{\pi i(y + 2x_5 - 2x_6)(\frac{1}{2}x + 2m)} \\ & \overline{\tilde{\psi}'_{a_4,c_4}(-\frac{1}{2}x' + n)} e^{-\pi i(y' + 2x_5 - 2x_6)(-\frac{1}{2}x' + 2n)} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(\frac{1}{3}(z' - 2y' - \frac{1}{2}x' + z + y + \frac{1}{2}x) + k + l - q)} \\ & e^{-\pi i(\frac{1}{3}(z - 2y' - \frac{1}{2}x' + z + y + \frac{1}{2}x) - x_5 - x_6 + x_7)(\frac{1}{3}(z - 2y' - \frac{1}{2}x' + z + y + \frac{1}{2}x) + 2(k + l - q))} \\ & \tilde{\psi}'_{a_6,c_6}(-y + q) e^{\pi i(\frac{1}{2}x - y - x_5 + x_6 + x_7)(-y + 2q)} \\ & dx_5 dx_6 dx_7. \end{aligned}$$

We shift $z \mapsto z - z' - y' + \frac{1}{2}x' + 2y - \frac{1}{2}x$.

$$\begin{aligned}K_T(x',y',z',x,y,z) = \int_{[0,1]^3} \sum_{k,l,m,n,q} & \overline{\tilde{\psi}'_{a_1,c_1}(-y'+k)} e^{-\pi i(-y'-\frac{1}{2}x'-x_5+x_6+x_7)(-y'+2k)} \\ & \tilde{\psi}'_{a_2,c_2}(\frac{1}{3}z+l) e^{\pi i(\frac{1}{3}z-x_5-x_6+x_7)(\frac{1}{3}z+2l)} \\ & \tilde{\psi}'_{a_3,c_3}(\frac{1}{2}x+m) e^{\pi i(y+2x_5-2x_6)(\frac{1}{2}x+2m)} \\ & \overline{\tilde{\psi}'_{a_4,c_4}(-\frac{1}{2}x'+n)} e^{-\pi i(y'+2x_5-2x_6)(-\frac{1}{2}x'+2n)} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(\frac{1}{3}z-y'+y+k+l-q)} \\ & e^{-\pi i(\frac{1}{3}z-y'+y-x_5-x_6+x_7)(\frac{1}{3}z-y'+y+2(k+l-q))} \\ & \tilde{\psi}'_{a_6,c_6}(-y+q) e^{\pi i(\frac{1}{2}x-y-x_5+x_6+x_7)(-y+2q)} \\ & dx_5 dx_6 dx_7.\end{aligned}$$

$$\begin{aligned}\rho(T)(\pi^*(Wf_{a,b}))(x',y',z') = \int_{[0,1]^6} \sum_{k,l,m,n,q} & \overline{\tilde{\psi}'_{a_1,c_1}(-y'+k)} e^{-\pi i(-y'-\frac{1}{2}x'-x_5+x_6+x_7)(-y'+2k)} \\ & \tilde{\psi}'_{a_2,c_2}(\frac{1}{3}z+l) e^{\pi i(\frac{1}{3}z-x_5-x_6+x_7)(\frac{1}{3}z+2l)} \\ & \tilde{\psi}'_{a_3,c_3}(\frac{1}{2}x+m) e^{\pi i(y+2x_5-2x_6)(\frac{1}{2}x+2m)} \\ & \overline{\tilde{\psi}'_{a_4,c_4}(-\frac{1}{2}x'+n)} e^{-\pi i(y'+2x_5-2x_6)(-\frac{1}{2}x'+2n)} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(\frac{1}{3}z-y'+y+k+l-q)} \\ & e^{-\pi i(\frac{1}{3}z-y'+y-x_5-x_6+x_7)(\frac{1}{3}z-y'+y+2(k+l-q))} \\ & \tilde{\psi}'_{a_6,c_6}(-y+q) e^{\pi i(\frac{1}{2}x-y-x_5+x_6+x_7)(-y+2q)} \\ & dx_5 dx_6 dx_7 \\ & e^{\pi iyx} f_{a,b}(x) \cdot \theta\left(y - \frac{ab+a^2x}{2\pi i}, \frac{-a^2}{2\pi i}\right) \frac{1}{6} \cdot dxdydz\end{aligned}$$

Note that this is independent of z'. Therefore integration over the fiber is trivial. For simplicity we write $\tilde{z} = \frac{1}{3}z$, $\tilde{x} = \frac{1}{2}x$ and $\tilde{x}' = \frac{1}{2}x'$. Note that $6d\tilde{x}d\tilde{z} = dxdz$

$$
\begin{aligned}
\rho_{\text{A–K}}(T)(f_{a,b})(\tilde{x}') = \sum_{k,l,m,n,q} \int_{[0,1]^3} & \overline{\tilde{\psi}'_{a_1,c_1}(-y'+k)}\tilde{\psi}'_{a_2,c_2}(\tilde{z}+l)\tilde{\psi}'_{a_3,c_3}(\tilde{x}+m)\overline{\tilde{\psi}'_{a_4,c_4}(-\tilde{x}'+n)} \\
& \overline{\tilde{\psi}'_{a_5,c_5}(\tilde{z}-y'+y+k+l-q)}\tilde{\psi}'_{a_6,c_6}(-y+q) \\
& \int_0^1 e^{2\pi i x_5(y-y'+\tilde{x}+\tilde{x}'+2(k-q+m-n))}dx_5 \\
& \int_0^1 e^{-2\pi i x_6(\tilde{x}+\tilde{x}'+2(m-n))}dx_6 \\
& \int_0^1 e^{2\pi i x_7(y'-y+2(q-k))}dx_7 \\
& e^{2\pi i\tilde{z}((y'-k)+(-y+q))}e^{2\pi i\tilde{x}q}e^{2\pi i\tilde{x}'k} \\
& e^{2\pi i y(y'+m-k-l)}e^{-2\pi i y'^2}e^{2\pi i y'(2k+l-q-n)} \\
& e^{2\pi i y\tilde{x}}f_{a,b}(2\tilde{x})\cdot\theta\left(y-\frac{ab+2a^2\tilde{x}}{2\pi i},\frac{-a^2}{2\pi i}\right) \\
& e^{-2\pi i\tilde{x}'y'}\ d\tilde{x}dyd\tilde{z}dy'.
\end{aligned}
$$

We do the following substitution in order move the summation variables to the integrals i.e. to change $\sum_m \int_0^1 f(x+m)dx$ into $\sum_m \int_m^{m+1} f(x)dx$. The substitutions look like: $y' \mapsto y'+k$, $\tilde{z} \mapsto \tilde{z}-l$, $\tilde{x} \mapsto \tilde{x}-m$ and $y \mapsto y+q$ We have the expression

$$
\begin{aligned}
\rho_{\text{A–K}}(T)(f_{a,b})(\tilde{x}') = \sum_{k,l,m,n,q} & \int_{y'=k}^{y'=k+1}\int_{\tilde{z}=-l}^{\tilde{z}=-l+1}\int_{\tilde{x}=-m}^{\tilde{x}=-m+1}\int_{y=q}^{y=q+1} \overline{\tilde{\psi}'_{a_1,c_1}(-y')}\tilde{\psi}'_{a_2,c_2}(\tilde{z}) \\
& \tilde{\psi}'_{a_3,c_3}(\tilde{x})\overline{\tilde{\psi}'_{a_4,c_4}(-\tilde{x}'+n)\tilde{\psi}'_{a_5,c_5}(\tilde{z}-y'+y)}\tilde{\psi}'_{a_6,c_6}(-y) \\
& \int_0^1 e^{2\pi i x_5(y-y'+(k-q)+\tilde{x}+\tilde{x}'+m-2n)}dx_5 \\
& \int_0^1 e^{2\pi i x_6(\tilde{x}+\tilde{x}'+m-2n)}dx_6 \\
& \int_0^1 e^{2\pi i x_7(y'-y+k-q)}dx_7 \\
& e^{2\pi i(\tilde{z}-l)(y'-y)}e^{2\pi i(\tilde{x}-m)q}e^{2\pi i\tilde{x}'k} \\
& e^{2\pi i(y+q)(y'+m-l)} \\
& e^{-2\pi i(y'+k)^2}e^{2\pi i(y'+k)(2k+l-q-n)}e^{2\pi i(y+q)(\tilde{x}-m)}e^{-2\pi i\tilde{(x')}(y'+k)} \\
& f_{a,b}(2(\tilde{x}-m))\cdot\theta\left(y+q-\frac{ab+2a^2(\tilde{x}-m)}{2\pi i},\frac{-a^2}{2\pi i}\right)\ d\tilde{x}dyd\tilde{z}dy'
\end{aligned}
$$

Reducing the phase we get

$$
\begin{aligned}
\rho_{\text{A-K}}(T)(f_{a,b})(\tilde{x}') = \sum_{k,l,m,n,q} & \int_{y'=k}^{y'=k+1} \int_{\tilde{z}=-l}^{\tilde{z}=-l+1} \int_{\tilde{x}=-m}^{\tilde{x}=-m+1} \int_{y=q}^{y=q+1} \overline{\tilde{\psi}'_{a_1,c_1}(-y')} \tilde{\psi}'_{a_2,c_2}(\tilde{z}) \\
& \tilde{\psi}'_{a_3,c_3}(\tilde{x}) \overline{\tilde{\psi}'_{a_4,c_4}(-\tilde{x}'+n) \tilde{\psi}'_{a_5,c_5}(\tilde{z}-y'+y)} \tilde{\psi}'_{a_6,c_6}(-y) \\
& \int_0^1 e^{2\pi i x_5 (y-y'+(k-q)+\tilde{x}+\tilde{x}'+m-2n)} dx_5 \\
& \int_0^1 e^{2\pi i x_6 (\tilde{x}+\tilde{x}'+m-2n)} dx_6 \\
& \int_0^1 e^{2\pi i x_7 (y'-y+k-q)} dx_7 \\
& e^{2\pi i (\tilde{z})(y'-y)} e^{4\pi i \tilde{x} q} e^{2\pi i \tilde{x}')k} e^{-2\pi i y'^2} \\
& e^{2\pi i y y'} e^{-2\pi i y' n} e^{2\pi i y \tilde{x}} e^{-2\pi i \tilde{x}' y'} e^{-2\pi i \tilde{x}' k} \\
& f_{a,b}(2(\tilde{x}-m)) \cdot \theta\left(y+q-\frac{ab+2a^2(\tilde{x}-m)}{2\pi i}, \frac{-a^2}{2\pi i} \right) d\tilde{x} dy d\tilde{z} dy'
\end{aligned}
$$

Hence

$$
\begin{aligned}
\rho_{\text{A-K}}(T)(f_{a,b})(\tilde{x}') = \sum_{k,l,m,n,q} & \int_{y'=k}^{y'=k+1} \int_{\tilde{z}=-l}^{\tilde{z}=-l+1} \int_{\tilde{x}=-m}^{\tilde{x}=-m+1} \int_{y=q}^{y=q+1} \overline{\tilde{\psi}'_{a_1,c_1}(-y')} \tilde{\psi}'_{a_2,c_2}(\tilde{z}) \\
& \tilde{\psi}'_{a_3,c_3}(\tilde{x}) \overline{\tilde{\psi}'_{a_4,c_4}(-\tilde{x}'+n) \tilde{\psi}'_{a_5,c_5}(\tilde{z}-y'+y)} \tilde{\psi}'_{a_6,c_6}(-y) \\
& \int_0^1 e^{2\pi i x_5 (y-y'+(k-q)+\tilde{x}+\tilde{x}'+m-2n)} dx_5 \\
& \int_0^1 e^{2\pi i x_6 (\tilde{x}+\tilde{x}'+m-2n)} dx_6 \\
& \int_0^1 e^{2\pi i x_7 (y'-y+k-q)} dx_7 \\
& e^{2\pi i (\tilde{z})(y'-y)} e^{4\pi i \tilde{x} q} e^{-2\pi i y'^2} \\
& e^{2\pi i y y'} e^{-2\pi i y' n} e^{2\pi i y \tilde{x}} e^{-2\pi i \tilde{x}' y'} \\
& f_{a,b}(2(\tilde{x}-m)) \cdot \theta\left(y+q-\frac{ab+2a^2(\tilde{x}-m)}{2\pi i}, \frac{-a^2}{2\pi i} \right) d\tilde{x} dy d\tilde{z} dy'.
\end{aligned}
$$

Integration over the variable z can now be carried out:

$$
\begin{aligned}
& \sum_{l\in\mathbb{Z}} \int_{-l}^{-l+1} \tilde{\psi}'_{a_2,c_2}(\tilde{z}) \overline{\tilde{\psi}'_{a_5,c_5}(\tilde{z}-y'+y)} e^{2\pi i \tilde{z}(y'-y)} \, dz \\
&= \int_{\mathbb{R}} \tilde{\psi}'_{a_2,c_2}(\tilde{z}) \psi_{b_5,c_5}((y'-y)-\tilde{z}) e^{\pi i (\tilde{z}-y'+y)} e^{-2\pi i z(y'-y)} \, dz \\
&= \int_{\mathbb{R}} e^{\pi i \tilde{z}^2} \tilde{\psi}'_{a_2,c_2}(\tilde{z}) \psi_{b_5,c_5}((y'-y)-\tilde{z}) \, dz \, e^{\pi i (y^2+y'^2)} e^{-2\pi i y' y} \\
&= (g_{a_2,c_2} * \psi_{b_5,c_5})(y'-y) \cdot e^{-2\pi i y' y} e^{\pi i (y^2+y'^2)},
\end{aligned}
$$

where $g_{a_2,c_2}(\tilde{z}) = e^{\pi i \tilde{z}^2} \cdot \tilde{\psi}'_{a_2,c_2}(z)$. The first equality follows from the properties of the charged tetrahedral operators.

We now have the expression

$$
\begin{aligned}
\rho_{\text{A–K}}(T)(f_{a,b})(\tilde{x}') = \sum_{k,m,n,q} & \int_{y'=k}^{y'=k+1} \int_{\tilde{x}=-m}^{\tilde{x}=-m+1} \int_{y=q}^{y=q+1} \overline{\tilde{\psi}'_{a_1,c_1}(-y')} \tilde{\psi}'_{a_3,c_3}(\tilde{x}) \\
& \overline{\tilde{\psi}'_{a_4,c_4}(-\tilde{x}')} \tilde{\psi}'_{a_6,c_6}(-y)(g_{a_2,c_2} * \psi_{b_5,c_5})(y'-y) \\
& \int_0^1 e^{2\pi i x_5 (y-y'+(k-q)+\tilde{x}+\tilde{x}'+m-n)} dx_5 \\
& \int_0^1 e^{2\pi i x_6 (\tilde{x}+\tilde{x}'+m-n)} dx_6 \\
& \int_0^1 e^{2\pi i x_7 (y'-y+k-q)} dx_7 \\
& e^{4\pi i \tilde{x} q} e^{-\pi i y'^2} e^{\pi i y^2} \\
& e^{-2\pi i y' n} e^{2\pi i y \tilde{x}} e^{-2\pi i \tilde{x}' y'} \\
& f_{a,b}(2(\tilde{x}-m)) \cdot \theta\left(y+q-\frac{ab+2a^2(\tilde{x}-m)}{2\pi i}, \frac{-a^2}{2\pi i} \right) \, d\tilde{x} dy dy'.
\end{aligned}
$$

Using the property of the Theta function we can write down the expression

$$
\begin{aligned}
\rho_{\text{A–K}}(T)(f_{a,b})(\tilde{x}') = \sum_{k,m,n,q} & \int_{y'=k}^{y'=k+1} \int_{\tilde{x}=-m}^{\tilde{x}=-m+1} \int_{y=q}^{y=q+1} \overline{\tilde{\psi}'_{a_1,c_1}(-y')} \tilde{\psi}'_{a_3,c_3}(\tilde{x}) \\
& \overline{\tilde{\psi}'_{a_4,c_4}(-\tilde{x}')} \tilde{\psi}'_{a_6,c_6}(-y)(g_{a_2,c_2} * \psi_{b_5,c_5})(y'-y) \\
& \int_0^1 e^{2\pi i x_5 (y-y'+(k-q)+\tilde{x}+\tilde{x}'+m-n)} dx_5 \\
& \int_0^1 e^{2\pi i x_6 (\tilde{x}+\tilde{x}'+m-n)} dx_6 \\
& \int_0^1 e^{2\pi i x_7 (y'-y+k-q)} dx_7 \\
& e^{4\pi i \tilde{x} q} e^{4\pi i \tilde{y} m} e^{-\pi i y'^2} e^{\pi i y^2} \\
& e^{-2\pi i y' n} e^{2\pi i y \tilde{x}} e^{-2\pi i \tilde{x}' y'} \\
& f_{a,b}(2\tilde{x}) \cdot \theta\left(y-\frac{ab+2a^2\tilde{x}}{2\pi i}, \frac{-a^2}{2\pi i} \right) \, d\tilde{x} dy dy'.
\end{aligned}
$$

10.5.0.2 Representation of $\hat{S}$

The operator $\rho_{\text{A–K}}(S) = W^{-1} \circ \int_{F_{z'}} \circ \rho(S) \circ \pi^* \circ W$, where again $\int_{F'_z}$ is integration over the fibre and $\rho(S)$ acts in the following way:

$$
\rho(S)(s)(x_1', x_2', x_3') = \int_{[0,1]^3} K_S(x_1', x_2', x_3', x_1, x_2, x_3) s(x_1, x_2, x_3) \, dx_1 dx_2 dx_3. \tag{10.7}
$$

Recall that the expression for the kernel is given by

$$\begin{aligned}
K_S(x_1', x_2', x_3', x_1, x_2, x_3) = \int_{[0,1]^4} \sum_{k,l,m,n,p,q} & \tilde{\psi}'_{a_1,c_1}(x_7+k) e^{\pi i(-x_1'-x_1+2x_7+2x_4+2x_5)(x_7+2k)} \\
& \overline{\tilde{\psi}'_{a_2,c_2}(x_3'-x_1'+l)} \\
& e^{-\pi i(x_3'-x_2'+2x_4-2x_6+x_7)(x_3'-x_1'+2l)} \\
& \tilde{\psi}'_{a_3,c_3}(x_3'-x_2'+m) e^{\pi i(x_3'-x_1'+2x_5+x_7)(x_3'-x_2'+2m)} \\
& \tilde{\psi}'_{a_4,c_4}(-x_7+n) e^{\pi i(2x_4-x_2'-x_2)(-x_7+2n)} \\
& \overline{\tilde{\psi}'_{a_5,c_5}(x_3-x_2+p)} \\
& e^{-\pi i(x_3-x_1+2x_4-2x_6+x_7)(x_3-x_2+2p)} \\
& \tilde{\psi}'_{a_6,c_6}(x_3-x_1+q) e^{\pi i(x_3-x_2-2x_5-x_7)(x_3-x_1+2q)} \\
& dx_4 dx_5 dx_6 dx_7.
\end{aligned}$$

We shift $x_6 \mapsto x_6 + x_4$ Which gives the expression

$$\begin{aligned}
K_S(x_1', x_2', x_3', x_1, x_2, x_3) = \int_{[0,1]^4} \sum_{k,l,m,n,p,q} & \tilde{\psi}'_{a_1,c_1}(x_7+k) e^{\pi i(-x_1'-x_1+2x_7+2x_4+2x_5)(x_7+2k)} \\
& \overline{\tilde{\psi}'_{a_2,c_2}(x_3'-x_1'+l)} e^{-\pi i(x_3'-x_2'-2x_6+x_7)(x_3'-x_1'+2l)} \\
& \tilde{\psi}'_{a_3,c_3}(x_3'-x_2'+m) e^{\pi i(x_3'-x_1'+2x_5+x_7)(x_3'-x_2'+2m)} \\
& \tilde{\psi}'_{a_4,c_4}(-x_7+n) e^{\pi i(2x_4-x_2'-x_2)(-x_7+2n)} \\
& \overline{\tilde{\psi}'_{a_5,c_5}(x_3-x_2+p)} e^{-\pi i(x_3-x_1-2x_6+x_7)(x_3-x_2+2p)} \\
& \tilde{\psi}'_{a_6,c_6}(x_3-x_1+q) e^{\pi i(x_3-x_2-2x_5-x_7)(x_3-x_1+2q)} \\
& dx_4 dx_5 dx_6 dx_7.
\end{aligned}$$

We can now do the integration

$$\int_0^1 e^{2\pi i x_4(k+n)} dx_4 = \delta(k+n),$$

which also removes one of the summations.

$$\begin{aligned}
K_S(x_1', x_2', x_3', x_1, x_2, x_3) = \int_{[0,1]^3} \sum_{k,l,m,p,q} & \tilde{\psi}'_{a_1,c_1}(x_7+k) e^{\pi i(-x_1'-x_1+2x_7+2x_5)(x_7+2k)} \\
& \overline{\tilde{\psi}'_{a_2,c_2}(x_3'-x_1'+l)} e^{-\pi i(x_3'-x_2'-2x_6+x_7)(x_3'-x_1'+2l)} \\
& \tilde{\psi}'_{a_3,c_3}(x_3'-x_2'+m) e^{\pi i(x_3'-x_1'+2x_5+x_7)(x_3'-x_2'+2m)} \\
& \tilde{\psi}'_{a_4,c_4}(-x_7-k) e^{\pi i(x_2'+x_2)(x_7+2k)} \\
& \overline{\tilde{\psi}'_{a_5,c_5}(x_3-x_2+p)} e^{-\pi i(x_3-x_1-2x_6+x_7)(x_3-x_2+2p)} \\
& \tilde{\psi}'_{a_6,c_6}(x_3-x_1+q) e^{\pi i(x_3-x_2-2x_5-x_7)(x_3-x_1+2q)} \\
& dx_5 dx_6 dx_7.
\end{aligned}$$

We collect the terms where x_5, x_7 appears.

$$\begin{aligned}K_S(x_1', x_2', x_3', x_1, x_2, x_3) = \int_{[0,1]^2} \sum_{k,l,m,p,q} & \tilde{\psi}'_{a_1,c_1}(x_7 + k)\tilde{\psi}'_{a_4,c_4}(-x_7 - k) \\ & e^{2\pi i x_7(x_7+x_5+x_2-x_3+2k-l+m-p-q)} dx_7 \\ & e^{2\pi i x_5(x_3'-x_2'+x_3-x_1+2(m+k+q))} dx_5 \\ & \int_0^1 e^{2\pi i x_6(x_3'-x_1'+x_3-x_2+2(p+l))} dx_6 \\ & \overline{\tilde{\psi}'_{a_2,c_2}(x_3' - x_1' + l)} e^{-2\pi i l(x_3'-x_2')} \\ & \tilde{\psi}'_{a_3,c_3}(x_3' - x_2' + m) e^{2\pi i m(x_3'-x_1')} \\ & e^{-2\pi i k(x_1'+x_1)} e^{2\pi i k(x_2'+x_2)} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(x_3 - x_2 + p)} e^{-2\pi i p(x_3-x_1)} \\ & \tilde{\psi}'_{a_6,c_6}(x_3 - x_1 + q) e^{2\pi i q(x_3-x_2)}.\end{aligned}$$

We wish to find an expression for the operator $\rho_{\text{A–K}}(S) : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, where

$$\rho_{\text{A–K}}(S) = W^{-1} \circ \int_{F_{z'}} \circ \rho(S) \circ \pi^* \circ W,$$

where $\rho(S)$ is the operator

$$\rho(S)(s)(x_1', x_2', x_3') = \int_{[0,1]^3} K_S(x_1', x_2', x_3', x_1, x_2, x_3) \cdot s(x_1, x_2, x_3)\, dx_1 dx_2 dx_3.$$

Again we change coordinates according to section 10.4.1.

$$\begin{aligned}\rho(S)(\pi^*(Wf_{a,b}))(x', y', z') = \int_{[0,1]^3} \int_{[0,1]^2} \sum_{k,l,m,p,q} & \tilde{\psi}'_{a_1,c_1}(x_7 + k)\tilde{\psi}'_{a_4,c_4}(-x_7 - k) \\ & e^{2\pi i x_7(x_7+x_5-y+2k-l+m-p-q)} dx_7 \\ & e^{2\pi i x_5(y'+\frac{1}{2}x+2(m+k+q))} dx_5 \\ & \int_0^1 e^{2\pi i x_6(-\frac{1}{2}x'+y+2(p+l))} dx_6 \\ & \overline{\tilde{\psi}'_{a_2,c_2}(-\frac{1}{2}x' + l)} e^{-2\pi i l(y')} \\ & \tilde{\psi}'_{a_3,c_3}(y' + m) e^{-2\pi i m(\frac{1}{2}x')} \\ & e^{-2\pi i k(\frac{1}{3}(z'+y'+x'+z+y-x))} \\ & e^{2\pi i k(\frac{1}{3}(z'-2y'-\frac{1}{2}x'+z-2y+\frac{1}{2}x))} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(y + p)} e^{-2\pi i p(\frac{1}{2}x)} \\ & \tilde{\psi}'_{a_6,c_6}(\frac{1}{2}x + q) e^{2\pi i q y} \\ & e^{\pi i x y} f_{a,b}(x) \cdot \theta\left(y - \frac{ab + 2a^2\tilde{x}}{2\pi i}, \frac{-a^2}{2\pi i}\right) \\ & \frac{1}{6} \cdot dx dy dz\end{aligned}$$

We see that this expression is independent of both z and z'. Integration over the fiber F_z' becomes trivial. We obtain the following expression for the operator $\rho_{\text{A–K}}(S)$ on a wavelet

$f_{a,b} \in \mathcal{S}(\mathbb{R})$.

$$\begin{aligned}\rho_{\text{A–K}}(S)(f_{a,b})(\tilde{x}') = \sum_{k,l,m,p,q} \int_{[0,1]^2} \int_{[0,1]^3} \int_{[0,1]^2} & \tilde{\psi}'_{a_1,c_1}(x_7+k)\tilde{\psi}'_{a_4,c_4}(-x_7-k) \\ & e^{2\pi i x_7(x_7+x_5-y+2k-l+m-p-q)}dx_7 \\ & e^{2\pi i x_5(y'+\tilde{x}+2(m+k+q))}dx_5 \\ & \int_0^1 e^{2\pi i x_6(-\tilde{x}'+y+2(p+l))}dx_6 \\ & \overline{\tilde{\psi}'_{a_2,c_2}(-\tilde{x}'+l)}e^{-2\pi i l y'} \\ & \tilde{\psi}'_{a_3,c_3}(y'+m)e^{-2\pi i m\tilde{x}'} \\ & e^{-2\pi i k(\frac{1}{3}(z'+y'+2\tilde{x}'+z+y-2\tilde{x}))} \\ & e^{2\pi i k(\frac{1}{3}(z'-2y'-\tilde{x}'+z-2y+\tilde{x}))} \\ & \overline{\tilde{\psi}'_{a_5,c_5}(y+p)}e^{-2\pi i p(\tilde{x})} \\ & \tilde{\psi}'_{a_6,c_6}(\tilde{x}+q)e^{2\pi i q y} \\ & e^{2\pi i \tilde{x} y} f_{a,b}(2\tilde{x}) \\ & \theta\left(y - \frac{ab+2a^2\tilde{x}}{2\pi i}, \frac{-a^2}{2\pi i}\right) d\tilde{x}dydzd\tilde{z}' \\ & e^{-2\pi i \tilde{x}' y'}dy'\end{aligned}$$

Rewriting this we get the expression:

$$\begin{aligned}\rho_{\text{A–K}}(S)(f_{a,b})(\tilde{x}') = \sum_{k,l,m,p,q} \int_{y'=0}^{y'=1} \int_{[0,1]^2} \int_{[0,1]^2} & \tilde{\psi}'_{a_1,c_1}(x_7+k)\tilde{\psi}'_{a_4,c_4}(-x_7-k) \\ & e^{2\pi i x_7(x_7+x_5-y+2k-l+m-p-q)}dx_7 \\ & e^{2\pi i x_5(y'+\tilde{x}+2(m+k+q))}dx_5 \\ & \int_0^1 e^{2\pi i x_6(-\tilde{x}'+y+2(p+l))}dx_6 \\ & \overline{\tilde{\psi}'_{a_2,c_2}(-\tilde{x}'+l)}\tilde{\psi}'_{a_3,c_3}(y'+m) \\ & \overline{\tilde{\psi}'_{a_5,c_5}(y+p)}\tilde{\psi}'_{a_6,c_6}(\tilde{x}+q) \\ & e^{-2\pi i l y'}e^{-2\pi i m\tilde{x}'} \\ & e^{-2\pi i k(y'+y+\tilde{x}'-\tilde{x})}e^{-2\pi i p\tilde{x}}e^{2\pi i q y} \\ & e^{2\pi i \tilde{x} y}e^{-2\pi i \tilde{x}' y'} \\ & f_{a,b}(2\tilde{x}) \cdot \theta\left(y - \frac{ab+2a^2\tilde{x}}{2\pi i}, \frac{-a^2}{2\pi i}\right) d\tilde{x}dydy'\end{aligned}$$

Unfortunately we have not yet been able to arrive at nice expressions for the representation. This is still work in progress.

Chapter 11

Appendix

11.1 Appendix A

11.1.1 Quantum dilogarithm

Since the quantum dilogarithm plays an important role in this thesis we here take time to discuss some of its properties. There are more than one function that carries the name quantum dilogarithm. They are not equal but nevertheless connected.

(i) The quantum dilogarithm function $\mathrm{Li}_2(x;q)$, studied by Fadeev–Kashaev [12] and other authors, is the function of two variables defined by the series

$$\mathrm{Li}_2(x;q) = \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^n)}, \tag{11.1}$$

where $x, q \in \mathbb{C}$, with $|x|, |q| < 1$. It is connected to the classical Euler dilogarithm Li_2 given by $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ in the sense that it is a q-deformation of the classical one in the following manner

$$\lim_{\epsilon \to 0} \left(\epsilon \, \mathrm{Li}_2(x, e^{-\epsilon}) \right) = \mathrm{Li}_2(x), \quad |x| < 1. \tag{11.2}$$

Indeed using the expansion $\frac{1}{1-e^{-t}} = \frac{1}{t} + \frac{1}{2} + \frac{t}{12} - \frac{t^3}{720} + \dots$ we obtain a complete asymptotic expansion

$$\mathrm{Li}_2(x, e^{-\epsilon}) = \mathrm{Li}_2(x)\epsilon^{-1} + \frac{1}{2}\log\left(\frac{1}{1-x}\right) + \frac{x}{1-x}\frac{\epsilon}{12} - \frac{x+x^2}{(1-x)^3}\frac{\epsilon^3}{720} + \dots \tag{11.3}$$

as $\epsilon \to 0$ with fixed $x \in \mathbb{C}$, $|x| < 1$.

(ii) The second quantum dilogarithm $(x;q)_\infty$ defined for $|q| < 1$ and all $x \in \mathbb{C}$ is given as the function

$$(x;q)_\infty = \prod_{i=0}^{\infty} (1 - xq^i). \tag{11.4}$$

This second quantum dilogarithm is related to the first by the formula

$$(x;q)_\infty = \exp(-\mathrm{Li}_2(x;q)). \tag{11.5}$$

This is easily proven by a direct calculation

$$-\log\,(x;q)_\infty = \sum_{i=0}^{\infty} \log(1-xq^i) = \sum_{i=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n} x^n q^{in} = \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^n)} = \mathrm{Li}_2(x;q). \tag{11.6}$$

Proposition 11.1. The function $(x;q)_\infty$ and its reciprocal have the Taylor expansions

$$(x;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q)_n} q^{\frac{n(n-1)}{2}} x^n, \qquad \frac{1}{(x;q)_\infty} \sum_{n=0}^{\infty} \frac{1}{(q)_n} x^n, \tag{11.7}$$

around $x = 0$, where

$$(q)_n = \frac{(q;q)_\infty}{(q^{n+1};q)_\infty} = (1-q)(1-q^2)\cdot(1-q^n).$$

The proofs of these formulas follows easily from the recursion formula $(x;q)_\infty = (1-x)(qx;x)_\infty$, which together with initial value $(0;q)_\infty = 1$ determines the power series for $(x;q)_\infty$ uniquely.

Yet another famous result for the function $(x;q)_\infty$, which can be proven by use of the Taylor expansion and the identity $\sum_{m-n=k} \frac{q^{mn}}{(q)_m(q)_n} = \frac{1}{(q)_\infty}$, is the Jacobi triple product formula

$$(q;q)_\infty (x;q)_\infty (qx^{-1};q)_\infty = \sum_{k\in\mathbb{Z}} (-1)^k q^{\frac{k(k-1)}{2}} x^k, \tag{11.8}$$

which relates the quantum dilogarithm function to the classical Jacobi theta-function.

(iii) The quantum dilogarithm functions introduced are related to yet another quantum dilogarithm function named after Faddeev.

Definition 11.2. Faddeev's quantum dilogarithm Faddeev's quantum dilogarithm function is a function in two complex arguments z and b defined by the formula

$$\Phi_{\mathrm{b}}(z) = \int_C \exp\left(\frac{e^{-2izw}dw}{4\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}\right), \tag{11.9}$$

where the contour C runs along the real axis, deviating into the upper half plane in the vicinity of the origin.

Proposition 11.3. Faddeev's quantum dilogarithm function $\Phi_{\mathrm{b}}(z)$ is related to the function $(x;q)_\infty := \prod_{i=0}^{\infty}(1-xq^i)$, where $|q| < 1$, in the following sense. When $\mathrm{Im}(\mathrm{b}^2) > 0$, the integral can be calculated explicitly

$$\Phi_{\mathrm{b}}(z) = \frac{\left(e^{2\pi(z+c_{\mathrm{b}})\mathrm{b}};q^2\right)_\infty}{\left(e^{2\pi(z-c_{\mathrm{b}})\mathrm{b}};\tilde{q}^2\right)_\infty} \tag{11.10}$$

where $q \equiv e^{i\pi \mathrm{b}^2}$ and $\tilde{q} \equiv e^{-\pi i \mathrm{b}^{-2}}$.

Proof. We collect a residue of the integrand $I(z,\mathrm{b}) = \frac{1}{4}\int_C \frac{e^{-2izw}}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw$. The integrand has poles at $w = \pi i n\,\mathrm{b}$ and $w = \pi i n\,\mathrm{b}^{-1}$. The residue at c of a fraction i.e. $f(x) = \frac{g(x)}{h(x)}$ can be calculated as $\mathrm{Res}\,f(c) = \frac{g(c)}{h'(c)}$ when c is a simple pole. Therefore we get by the residue theorem

$$
\begin{aligned}
I(z,\mathrm{b}) &= \frac{\pi i}{2}\sum_{n=1}^{\infty}\frac{e^{2\pi z\,\mathrm{b}\,n}}{\pi in\,\mathrm{b}(-1)^n\sinh(\pi in\,\mathrm{b}^2)}+\frac{e^{2\pi z\,\mathrm{b}^{-1}\,n}}{\pi in\,\mathrm{b}(-1)^n\sinh(\pi in\,\mathrm{b}^{-2})}\\
&= \sum_{n=1}^{\infty}\frac{e^{\pi in}e^{2\pi z\,\mathrm{b}\,n}}{n(e^{\pi in\,\mathrm{b}^2}-e^{-\pi in\,\mathrm{b}^2})}+\frac{e^{\pi in}e^{2\pi z\,\mathrm{b}^{-1}\,n}}{n(e^{\pi in\,\mathrm{b}^{-2}}-e^{-\pi in\,\mathrm{b}^{-2}})}\\
&= \sum_{n=1}^{\infty}-\frac{\left(e^{2\pi z\,\mathrm{b}+\pi i+\pi i\,\mathrm{b}^2}\right)^n}{n(1-e^{2\pi i\,\mathrm{b}^2\,n})}+\frac{\left(e^{2\pi z\,\mathrm{b}^{-1}-\pi i-\pi i\,\mathrm{b}^{-2}}\right)^n}{n(1-e^{-2\pi i\,\mathrm{b}^{-2}\,n})}\\
&= \sum_{n=1}^{\infty}-\frac{e^{2\pi(z+c_{\mathrm{b}})\,\mathrm{b}\,n}}{n(1-e^{2\pi i\,\mathrm{b}^2\,n})}+\frac{e^{2\pi(z-c_{\mathrm{b}})\,\mathrm{b}^{-1}\,n}}{n(1-e^{-2\pi i\,\mathrm{b}^{-2}\,n})}\\
&= \log\left(e^{2\pi(z+c_{\mathrm{b}})\,\mathrm{b}};q^2\right)_\infty-\log\left(e^{2\pi(z-c_{\mathrm{b}})\,\mathrm{b}};\tilde{q}^2\right)_\infty.
\end{aligned}
$$

The result follows by taking the exponential of both sides. □

Lemma 11.4. Faddeev's quantum dilogarithm function satisfies the two functional equations:

$$\frac{1}{\Phi_{\mathrm{b}}(z+i\mathrm{b}/2)}=\frac{1}{\Phi_{\mathrm{b}}(z-i\mathrm{b}/2)}\left(1+e^{2\pi\mathrm{b}z}\right), \tag{11.11}$$

$$\Phi_{\mathrm{b}}(z)\Phi_{\mathrm{b}}(-z)=e^{i\pi(1+2c_{\mathrm{b}}^2)/6}e^{i\pi z^2}. \tag{11.12}$$

Proof. Let us first prove (11.11). We have

$$
\begin{aligned}
\frac{\Phi_{\mathrm{b}}(z-i\mathrm{b}/2)}{\Phi_{\mathrm{b}}(z+i\mathrm{b}/2)} &= \exp\int_C\frac{e^{-2i(z-i\mathrm{b}/2)w}-e^{-2i(z+i\mathrm{b}/2)w}}{4\sinh(w\mathrm{b})\sinh(w/\mathrm{b})w}\,dw\\
&= \exp\int_C\frac{e^{-2izw}\left(e^{-\mathrm{b}w}-e^{\mathrm{b}w}\right)}{4\sinh(w\mathrm{b})\sinh(w/\mathrm{b})w}\,dw\\
&= \exp\left(-\frac{1}{2}\int_C\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\right)\,dw
\end{aligned}
$$

Let $a>0$. Let $\varepsilon=1$ if $\mathrm{Im}(-2iz)\geq 0$ and $\varepsilon=-1$ otherwise. Put $\delta_a^-=[-a,i\varepsilon a]$ and $\delta_a^-=[i\varepsilon a,a]$. The integrals $\int_{\delta_{a\pm}}\frac{e^{-2izw}}{2\sinh(w/\mathrm{b})w}\,dw$ converge to zero as $a\to\infty$. Therefore

$$\int_C\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\,dw=\epsilon 2\pi i\left(c_\epsilon+\sum_{n=1}^{\infty}\mathrm{Res}_{w=\epsilon i\pi\mathrm{b}n}\left\{\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\right\}\right),$$

where $c_1=0$ and $c_{-1}=\mathrm{Res}_{w=0}\left\{\frac{e^{2izw}}{\sinh(w/\mathrm{b})w}\right\}=-2iz\mathrm{b}$. For $n\in\mathbb{Z}\setminus\{0\}$ we have

$$\mathrm{Res}_{w=\pi in\mathrm{b}\epsilon}\left\{\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\right\}=\frac{(-1)^ne^{2z\pi\mathrm{b}\epsilon n}}{\pi in}$$

so

$$\int_C\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\,dw=(\epsilon-1)2\pi z\mathrm{b}-2\log(1+e^{2z\pi\mathrm{b}\epsilon}),$$

giving the first result.

To prove equation (11.12) let us choose the path $C=(-\infty,-\epsilon]\cup\epsilon\exp([\pi i,0])\cup[\epsilon,\infty)$ and let $\epsilon\to 0$. The rest is just calculations:

$$\log\Phi_{\mathrm{b}}(z)\Phi_{\mathrm{b}}(-z)=\frac{1}{2}\int_C\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw$$

Note that

$$\frac{1}{2}\int_{(-\infty,-\epsilon]}\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw=-\frac{1}{2}\int_{[\epsilon,\infty)}\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw.$$

i.e. it is enough to collect the half residue around $w=0$ of the remaining intergral.

$$\begin{aligned}\frac{1}{2}\int_{\epsilon([\pi i,0])}\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw &= \frac{\pi i}{2}\operatorname{Res}_{w=0}\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}\\ &= \frac{\pi i}{2}\left(\frac{b^2+b^{-2}}{6}+2z^2\right)\\ &= e^{\pi i(1+2c_{\mathrm{b}}^2)/6}e^{\pi i z^2}.\end{aligned}$$

□

The functional equation (11.11) shows that $\Phi_{\mathrm{b}}(z)$, which in its initial domain of definition has no zeroes and poles, extends (for fixed b with $\operatorname{Im}\mathrm{b}^2>0$) to a meromorphic function of z with zeroes and poles:

$$(\Phi_{\mathrm{b}}(z))^{\pm 1}=0 \iff z=\mp(c_{\mathrm{b}}+mi\,\mathrm{b}+ni\,\mathrm{b}). \tag{11.13}$$

11.1.2 Five term relation

The quantum dilogarithm function satisfy various five term relations, of which the five term relation for the dilogarithm function $\mathrm{Li}_2(x)$ is a limiting case, when the arguments are non-commuting variables. The far simplest relation is the following

$$(Y;q)_\infty(X;q)_\infty=(X;q)_\infty(-YX;q)_\infty(Y;q)_\infty,$$

where the operators X and Y satisfy the equation $XY=qYX$.

From this equation one deduces the famous **quantum pentagon identity**

$$\Phi_{\mathrm{b}}(\hat p)\Phi_{\mathrm{b}}(\hat q)=\Phi_{\mathrm{b}}(\hat q)\Phi_{\mathrm{b}}(\hat p+\hat q)\Phi_{\mathrm{b}}(\hat p), \tag{11.14}$$

where $\hat p,\hat q\in L^2(\mathbb{R})$ are selfadjoint operators satisfying

$$[\hat p,\hat q]=(2\pi i)^{-1}.$$

11.1.3 Asymptotic expansion

Proposition 11.5. For fixed x and $\mathrm{b}\to 0$ we have the following asymptotic expansion

$$\log\Phi_{\mathrm{b}}\left(\frac{x}{2\pi\,\mathrm{b}}\right)=\sum_{n=0}^{\infty}(2\pi i\,\mathrm{b})^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\frac{\partial^{2n}\,\mathrm{Li}_2(-e^x)}{\partial x^{2n}}, \tag{11.15}$$

where $B_{2n}(1/2)$ are the Bernoulli polynomials evaluated at $1/2$.

Proof. From (11.11) we have that

$$\log\left(\frac{\Phi_{\mathrm{b}}\left(\frac{x-i\pi\,\mathrm{b}^2}{2\pi\,\mathrm{b}}\right)}{\Phi_{\mathrm{b}}\left(\frac{x+i\pi\,\mathrm{b}^2}{2\pi\,\mathrm{b}}\right)}\right)=\log(1+e^x)$$

The left hand side yields

$$\log\Phi_{\mathrm{b}}\left(\frac{x-i\pi\,\mathrm{b}^2}{2\pi\,\mathrm{b}}\right)-\log\Phi_{\mathrm{b}}\left(\frac{x+i\pi\,\mathrm{b}^2}{2\pi\,\mathrm{b}}\right)=-2\sinh(i\pi b^2\partial/\partial x)\log\Phi_{\mathrm{b}}\left(\frac{x}{2\pi\,\mathrm{b}}\right)$$

where we have used the fact that

$$f(x+y) = e^{y\frac{\partial}{\partial x}}\left(f(x)\right),$$

which is just the Taylor expansion of f around x. While the right hand side can be written in the following manner

$$\log(1+e^x) = \frac{\partial}{\partial x}\int_{-\infty}^{x} \log(1+e^z)\, dz = -\frac{\partial}{\partial x}\,\mathrm{Li}_2(-e^x).$$

Using the expansion

$$\frac{z}{\sinh(z)} = \sum_{n=0}^{\infty} B_{2n}(1/2)\frac{(2z)^{2n}}{(2n)!}$$

gives exactly (11.8). □

Corollary 11.6. For fixed x and $\mathrm{b} \to 0$ one has

$$\Phi_{\mathrm{b}}\left(\frac{x}{2\pi\,\mathrm{b}}\right) = \exp\left(\frac{1}{2\pi i b^2}\,\mathrm{Li}_2(-e^x)\right)\left(1+O(b^2)\right). \tag{11.16}$$

11.1.4 Asymptotic expansion at a N'th root of unity

An open and very interesting question for the TQFT studied in this thesis is, what happens when b^2 approaches a negative rational. Only a little progress was made in this direction. Nevertheless we write down what is examined about the quantum dilogarithm $(x;q)_\infty$. We recall that Faddeev's quantum dilogarithm can be writen as a fraction of the dilogaritm function (11.10).

In the results for the partition function of the Andersen–Kashaev TQFT Faddeev's quantum dilogarithm is evident and therefore it would be nice to be able to evaluate these results when b^2 approaches a negative rational. Unfortunately this leads to integration over a function with infinitely many zeroes and poles, which the author was not able to handle.

Let $b^2 = \epsilon - 1/N$, where $\mathrm{Im}(b^2) > 0$. Note that $q = e^{2\pi i\epsilon}e^{-2\pi i/N} = wp$, where p is a primitive root of unity. i.e. $q^N = (wp)^N = w^n$. We write $n = \alpha + kN$, then

$$\begin{aligned}(x;q)_\infty &= \prod_{n=0}^{\infty}(1-q^n x) = \prod_{k=0}^{\infty}\prod_{\alpha=0}^{N-1}(1-q^{\alpha+kN}x)\\ &= \prod_{\alpha=0}^{N-1}\prod_{k=0}^{\infty}(1-q^{kN}\cdot q^\alpha x) = \prod_{\alpha=0}^{N-1}\left((wp)^\alpha x; w^N\right)_\infty.\end{aligned}$$

We want to make an expansion of each of the N factors $\left(x(wp)^\alpha; w^N\right)_\infty$ as $\epsilon \to 0$.

Let us here recall the Euler-Maclaurin formula [37].

Proposition 11.7. The Euler-Maclaurin formula reads:

$$\sum_{j=a}^{n} f(j) = \int_a^n f(x)\, dx + \frac{1}{2}f(a) - \frac{1}{2}f(n) + \sum_{s=1}^{m-1}\frac{B_{2s}}{(2s)!}\left\{f^{(2s-1)}(n) - f^{(2s-1)}(a)\right\} + R_m(n)$$

where a, m and n are arbitrary integers such that $a < n$ and $m > 0.$, and

$$R_m(n) = \frac{b_{2m}}{(2m)!}\left\{f^{(2m-1)(n)} - f^{(2m-1)}(a)\right\} - \int_a^n \frac{B_{2m}(x-[x])}{(2m)!}f^{(2m)}(x)\, dx.$$

Proposition 11.8. The quantum dilogarithm $(x;q)_\infty$ has the following expansion for $q \to 1$

$$\left(x;e^{-\tau}\right)_\infty = (1-x)^{1/2}\exp\left(-\frac{1}{\tau}\,\mathrm{Li}_2(x) + \frac{\tau}{12}\cdot\frac{x}{x-1}\right)\left(1+O(\tau^3)\right), \qquad \tau \to 0.$$

Proof. We apply the Euler-Maclaurin formula to the logarithm of $(x;q)_\infty$.

$$\begin{aligned}\log(x;q)_\infty &= \log\prod_{j=0}^{\infty}\left(1-xq^j\right) = \sum_{j=0}^{\infty}\log\left(1-xq^j\right)\\ &= \int_0^{\infty}\log\left(1-xe^{-\tau y}\right)\,dy + \frac{1}{2}\log(1-x) + \frac{\tau}{12}\cdot\frac{x}{x-1} + R_1(\infty)\\ &= -\left[\frac{\mathrm{Li}_2(xe^{-\tau y})}{\log(e^{-\tau})}\right]_0^{\infty} + \frac{1}{2}\log(1-x) + \frac{\tau}{12}\cdot\frac{x}{x-1} + R_1(\infty)\\ &= -\frac{1}{\tau}\,\mathrm{Li}_2(x) + \frac{1}{2}\log(1-x) + \frac{\tau}{12}\cdot\frac{x}{x-1} + R_1(\infty),\end{aligned}$$

Therefore we get the expansion

$$\left(x;e^{-\tau}\right)_\infty = (1-x)^{1/2}\exp\left(-\frac{1}{\tau}\,\mathrm{Li}_2(x) + \frac{\tau}{12}\cdot\frac{x}{x-1}\right)\left(1+O(\tau^3)\right).$$

□

Proposition 11.9. Let $\mathrm{b}^2 = \epsilon - \frac{1}{N}$, where $\Im\,\mathrm{b}^2 > 0$ and assume that $|x| < 1$. For $k = 0,1,\dots,N-1$ the function $\left(x(w\xi)^k;w^N\right)_\infty$ has the following expansion.

$$\left(x(w\xi)^k;w^N\right)_\infty = \sqrt{1-(w\xi)^k x}\cdot\exp\left(\frac{1}{2\pi i\epsilon N}\,\mathrm{Li}_2(x(w\xi)^k) + \frac{2\pi i\epsilon N}{12}\frac{x(w\xi)^k}{x(w\xi)^k-1}\right)\left(1+O(\epsilon^3)\right), \tag{11.17}$$

where $w = e^{2\pi i\epsilon}$ and $\xi = e^{-\frac{2\pi i}{N}}$.

Proof. This is an immediate consequence of proposition 11.8. □

Proposition 11.10. When $|x| < 1$, $q = wp = e^{-\tau/N^2}\xi$ where $\xi = e^{-2\pi i\frac{1}{N}}$ is a primitive root of unity, then for $\tau \to 0$ we have the asymptotic form

$$(x;q)_\infty = \sqrt{1-x^N}\cdot g(x)\cdot\exp\left(-\frac{1}{\tau}\,\mathrm{Li}_2(x^N) + \frac{\tau}{12N}\sum_{k=0}^{N-1}\frac{x\xi^k}{x\xi^k-1}\right)\left(1+O(\tau^3)\right), \tag{11.18}$$

where $g(x) = \prod_{k=0}^{N-1}(1-x\xi^k)^{k/N}$.

Proof. As above we write

$$(x;q)_\infty = \prod_{k=0}^{N-1}\left(xq^k;q^N\right)_\infty.$$

Now use proposition 11.9 to each factor in the product and expand this product in a series in τ this yields the following

$$(x;q)_\infty = \prod_{k=0}^{N-1}\left(1-x\xi^k\right)^{\frac{1}{2}-\frac{1}{N}}\exp\left(-\frac{N}{\tau}\sum_{k=0}^{N-1}\mathrm{Li}_2(x\xi^k) + \frac{\tau}{12N}\sum_{k=0}^{N-1}\frac{x\xi^k}{x\xi^k-1}\right)\left(1+O(\tau^3)\right).$$

Note that $\prod_{k=0}^{N-1}(1-x\xi^k) = (1-x^N)$ since ξ is a primitive root of unity. We show below in lemma 11.11 have the identity

$$\mathrm{Li}_2(x^N) = N\sum_{k=0}^{N-1}\mathrm{Li}_2(x\xi^k).$$

which gives the result. □

Lemma 11.11. When ξ is a primitive Nth root of unity we have the identity

$$\mathrm{Li}_2(x^N) = N\sum_{k=0}^{N-1}\mathrm{Li}_2(x\xi^k).$$

Proof. The proof is shown by direct computation.

$$\mathrm{Li}_2(x^N) = N^2\sum_{n=1}^{\infty}\frac{\left(x^N\right)^n}{n^2N^2} = N\sum_{n=1}^{\infty}\sum_{k=0}^{N}(\xi^N)^{kn}\frac{\left(x^N\right)^n}{n^2N^2} = N\sum_{m=1}^{\infty}\sum_{k=0}^{N-1}\xi^{km}\frac{x^m}{m^2} = N\sum_{k=0}^{N-1}\mathrm{Li}_2(x\xi^k),$$

because

$$\sum_{k=0}^{N-1}\xi^{km} = \begin{cases} 0 & \text{when} \quad N\nmid m, \\ 1 & \text{when} \quad N \mid m. \end{cases}$$

□

Remark 11.12. The function g has the following property. Again here ξ is a primitive root of unity.

$$\begin{aligned} g(x\xi) &= \prod_{k=0}^{N-1}(1-x\xi^{k+1})^{k/N} = \prod_{k=1}^{N-1}(1-x\xi^{k+1})^{k/N} = \prod_{j=0}^{N-1}(1-x\xi^k)^{\frac{k-1}{N}} \\ &= \prod_{k=1}^{N}\frac{(1-x\xi^k)^{k/N}}{(1-x\xi^k)^{1/N}} = \frac{\prod_{k=0}^{N}(1-x\xi^k)^{k/N}}{\prod_{k=1}^{N}(1-x\xi^k)^{1/N}} = \frac{(1-x\xi^N)\prod_{k=0}^{N-1}(1-x\xi^k)}{(1-x^N)^{1/N}} \\ &= \frac{(1-x)g(x)}{(1-x^N)^{1/N}}. \end{aligned}$$

In other words

$$\frac{g(x\xi)}{g(x)} = \frac{1-x}{(1-x^N)^{1/N}}.$$

From here it follows, that

$$\frac{g(x\xi^n)}{g(x)} = \frac{g(x\xi^n)}{g(x\xi^{n-1})}\frac{g(x\xi^{n-1})}{g(x\xi^{n-2})}\cdots\frac{g(x\xi)}{g(x)} = \frac{(x;\xi)_n}{(1-x^N)^{1/N}}.$$

Hopefully future studies will lead to an answer of what happens for the partition function $Z_\hbar$ from the Andersen–Kashaev TQFT when $\mathrm{b}^2 \to -\frac{1}{N}$, and thereby maybe connect the theory to Liouville theory.

11.2 Appendix B

11.3 Line bundles on a complex torus

As mentioned above the Boltzmann weights are given by sections of a certain line bundle. Therefore, we will look at line bundles on a complex torus.

Let us construct line bundles on a manifold M given by a quotient $M = V/\Lambda$ by complex functions on the universal cover satisfying some functional equations. We will also discuss the space of holomorphic sections of a line bundle over a torus and see that a basis of this space is given by theta functions. This section is based on [19] and [3].

Before dealing with the concrete case of a torus let us recall the Riemann conditions.

11.3.1 Riemann condition

Let V be a complex vector space of dimension n, $\Lambda \subset V$ a discrete lattice of maximal rank, The complex torus $M = V/\Lambda$ is called an *abelian variety* if it is a projective algebraic variety i.e. can be embedded into projective space.

We will recall the necessary and sufficient conditions for embedding M into projective space. Kodaira's embedding theorem gives such a condition, and we will use this and rewrite it for our purpose. The result we get to is the *Riemann conditions*.

We will start by looking at the cohomology of M. By an argument using harmonic forms one can show that

$$H^*(M, \mathbb{C}) = \wedge^* V \otimes \wedge^* \overline{V}. \tag{11.19}$$

Let us give a basis for $H^*(M,\mathbb{C})$ expressing the complex structure and a basis for $\wedge^* V \otimes \wedge^* \overline{V}$ expressing Λ and the rational structure of $H^1(M,\mathbb{Z})$. V has euclidian coordinates $z = (z_1, \ldots, z_n)$ given by a complex basis $(e_1, \ldots, e_n)$ and $dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n$ are global 1-forms on M. $H^*(M,\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{dz_A \wedge d\overline{z}_B\}_{A,B}$, where A, B are multi-indices. Let γ be a loop in $H_1(M,\mathbb{Z})$ with base point $[0] \in M$. γ lifts to a path $\tilde{\gamma} \in V$ which starts at 0 and ends at $\lambda \in \Lambda$. V is the universal cover of M and hence Λ is the deck transformations so $H_1(M,\mathbb{Z}) = \Lambda$. Let $\{\lambda_1, \ldots, \lambda_{2n}\}$ be a basis for Λ. It follows since Λ was of maximal rank that $\{\lambda_i\}$ is a real basis of V. Let $\{x_1, \ldots, x_{2n}\}$ be the dual coordinates on V and let $\{dx_1, \ldots, dx_{2n}\}$ be one forms on M. By definition of the coordinates, integrating dx_i around the loop λ_j gives δ_{ij} and therefore we can write $H^1(M,\mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{dx_1, \ldots, dx_{2n}\}$, and generally $H^k(M,\mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{dx_I\}_{|I|=k}$.

This gives us two different bases for the cohomology on M. $\{dz_\alpha, d\overline{z}_\alpha\}$ which reflects the complex structure on $H^*(M,\mathbb{C})$ and $\{dx_i\}$ reflecting the rational structure. Now *Kodaira embedding theorem* says that M is algebraic if and only if there exists a *Hodge form* on M. I.e. a closed, positive form of type $(1,1)$ representing a rational cohomology class.

For the remaining part of this chapter we let greek indices run from 1 to n and latin indices run from 1 to $2n$. Let $\Pi = (\pi_{i\alpha})$ be the $2n \times n$-matrix such that $\tilde{\Pi} = (\Pi, \overline{\Pi})$ changes basis from $\{dz_\alpha, d\overline{z}_\alpha\}$ to $\{dx_i\}$. Let $\Omega = (w_{\alpha i})$ be the period matrix of Λ i.e. $\lambda_i = \sum_\alpha w_{\alpha i} e_\alpha$. Finally $\omega = \frac{1}{2}\sum_{ij} q_{ij} dx_i \wedge dx_j$ a two-form with $Q = (q_{ij})$ an integral skew-symmetric $2n \times 2n$-matrix.

Proposition 11.13 (Riemann Conditions)**.** M is an abelian variety if and only if one of the following equivalent conditions are satisfied.

(i) There exists an integral skew-symmetric matrix Q such that

$$\Pi Q \Pi = 0 \quad \text{and} \quad -i\Pi^T Q \overline{\Pi} = 0$$

(ii) There exists an integral skew-symmetric matrix Q such that

$$\Omega Q^{-1} \Omega^T = 0 \quad \text{and} \quad -i\Omega Q^{-1} \overline{\Omega}^T = 0$$

(iii) There exists an integral basis $\{\lambda_1, \ldots, \lambda_{2n}\}$ for Λ and a complex basis $e_1, \ldots, e_n$ for V such that $\Omega = \Delta_\delta, Z$ with Δ_δ diagonal with integer entries and Z symmetric and $\mathrm{Im}(Z) > 0$.

Lemma 11.14. If Q is an integral skew-hermitian quadratic form on $\Lambda \simeq \mathbb{Z}^{2n}$, then there exists a basis $\lambda_1, \ldots, \lambda_{2n}$ for Λ in terms of which Q is given by the matrix

$$Q = \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix}, \quad \text{where} \quad \Delta_\delta = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix}.$$

With this lemma, the symmetry of Z follows from ΩQ^{-1} which is a necessary condition for ω to be of type (1,1). $\mathrm{Im}(Z) > 0$ follow from $-i\Omega Q^{-1}\overline{\Omega}^T = 0$, which is necessary for ω to be positive. The cohomology class $[\omega]$ is called a polarization of M, and if all δ_i are 1, M is called principal polarized.

If (M_Z, ω) is a principal polarized abelian variety, Z reflects the complex structure on V. Since ω is both non-degenerate and positive symplectic form, the metric $g(\cdot,\cdot) = \omega(\cdot, I_Z\cdot)$ is positive definite, where I_Z is the complex structure defined by Z, hence (M_Z, ω) is actually Kähler.

11.3.2 Line bundles

Let $\mathcal{L} \to M$ be a complex line bundle. If we pull back $\mathcal{L}$ to V by the projection map $\pi : V \to M$, the line bundle $\pi^*\mathcal{L}$ is trivial since V is contractible. This is an easy consequence of parallel transport.

Proposition 11.15. If $F_0, F_1 : N \to M$ are smoothly homotopic maps and E is a vector bundle over M, then F_0^*N and F_1^*N are isomorphic vector bundles over N.

Proof. Let $J_0, J_1 : N \to N \times [0,1]$ be the smooth maps defined by

$$J_0(p) = (p, 0), \quad J_1(p) = (p, 1).$$

If F_0 is smoothly homotopic to F_1, there exists a smooth map $H : N \times [0,1] \to M$ such that

$$H \circ J_0 = F_0, \quad H \circ J_1 = F_1.$$

Thus is suffices to show that if E is a vector bundle over $N \times [0,1]$, then J_0^*E is isomorphic to J_1^*E. Give E a connection and let $\tau_p : E_{(p,0)} \to E_{(p,1)}$ denote parallel transport along the curve $t \to (p,t)$. We can then define a vector bundle isomorphism $\tau : J_0^*E \to J_1^*E$ by

$$\tau(p, v) = (p, \tau_p(v)), \quad \text{for} \quad v \in E_{(p,0)} = J_0^*E.$$

□

Corollary 11.16. Every vector bundle over a contractible manifold is trivial.

Proof. If M is contractible then the identity map on M is homotopic to the constant map, and hence any vector bundle over M is isomorphic to the pullback bundle over a point via the constant map. □

Pick a global trivialization $\phi : \pi^*\mathcal{L} \to V \times \mathbb{C}$. In each fiber $(\pi^*\mathcal{L})_z$ we have an isomorphism $\phi_z : (\pi^*\mathcal{L})_z \to \mathbb{C}$. From the definition of the pullback and the periodicity of our lattice we have that $(\pi^*\mathcal{L})_z = \mathcal{L}_{\pi(z)} = \mathcal{L}_{\pi(z+\lambda)} = (\pi^*\mathcal{L})_{z+\lambda}$, for $\lambda \in \Lambda$. If we compose trivializations $\phi_{z+\lambda} \circ \phi_z^{-1} : \mathbb{C} \to \mathbb{C}$ we get an automorphism of $\mathbb{C}$. Composition of trivializations are thus multiplication by complex numbers depending on z and λ. Let us denote this number by $e_\lambda(z)$. Varying z gives a family of functions $\{e_\lambda \in \mathcal{O}^*(V)\}_{\lambda\in\Lambda}$ which we call multipliers. These must satisfy the relations

$$e_\lambda(z)e_{\lambda'}(z+\lambda) = e_{\lambda'}(z)e_\lambda(z+\lambda') = e_{\lambda'+\lambda}(z), \tag{11.20}$$

which follow from the commutativity of the diagram below.

$$
\begin{tikzcd}
(\pi^*\mathcal{L})_z \arrow[r, "\phi_z"] \arrow[d, equal] & \mathbb{C} \arrow[d, "e_\lambda(z)"] \arrow[dd, bend left, "e_{\lambda+\lambda'}(z)"] \arrow[ddd, bend left=60, "e_{\lambda'}(z)"] \\
(\pi^*\mathcal{L})_{z+\lambda} \arrow[r, "\phi_{z+\lambda}"] \arrow[d, equal] & \mathbb{C} \arrow[d, "e_{\lambda'}(z+\lambda)"] \\
(\pi^*\mathcal{L})_{z+\lambda+\lambda'} \arrow[r, "\phi_{z+\lambda+\lambda'}"] \arrow[d, equal] & \mathbb{C} \\
(\pi^*\mathcal{L})_{z+\lambda'} \arrow[r, "\phi_{z+\lambda'}"] & \mathbb{C} \arrow[u, "e_\lambda(z+\lambda')"]
\end{tikzcd}
$$

Assume given such family of non-vanishing holomorphic functions $\{e_\lambda\}_{\lambda\in\Lambda}$ satisfying the above equations. Let $\mathcal{L} \to M$ be the quotient of $V \times \mathbb{C}$ by identifying $(z, \xi) \sim (z+\lambda, e_\lambda(z)\xi)$. Then $\mathcal{L}$ is a line bundle over M with the given functions as multipliers. By the compatibility relation we can give such a collection by specifying e_{λ_α} for some basis $\{\lambda_\alpha\}$ for Λ so long as the functions e_{λ_α} satisfy the relation

$$e_{\lambda_\alpha}(z+\lambda_\beta)e_{\lambda_\beta}(z) = e_{\lambda_\beta}(z+\lambda_\alpha)e_{\lambda_\alpha}(z).$$

We want to show that any line bundle $L \to M$ can be given by multipliers of a simple character. First we construct a line bundle having arbitrary positive Chern class, using elementary functions e_λ. Then we show that any positive line bundle $L \to M$ is determined by its Chern class.

If $\{\lambda_1, \dots, \lambda_{2n}\}$ is a basis for Λ over $\mathbb{Z}$ with $\{\lambda_1, \dots, \lambda_n\}$ linearly independent over $\mathbb{C}$, we have

$$\frac{V}{\mathbb{Z}\{\lambda_1, \dots, \lambda_n\}} \cong (\mathbb{C}^*)^n$$

and we can factor our projection map $\pi : V \to M$ by

$$V \to \frac{V}{\mathbb{Z}\{\lambda_1, \dots, \lambda_n\}} \to M.$$

By Poincaré's $\bar{\partial}$-lemma

$$H^1((\mathbb{C}^*)^n, \mathcal{O}) = H^2((\mathbb{C}^*)^n, \mathcal{O}) = 0.$$

Combining this result with a long exact sequence on sheaf cohomology it turns out that

$$c_1 : H^1((\mathbb{C}^*)^n, \mathcal{O}^*) \to H^2((\mathbb{C}^*)^n, \mathbb{Z})$$

is an isomorphism. I.e. any line bundle on $(\mathbb{C}^*)^n$ is determined by its first Chern class.

For any line bundle $\mathcal{L} \to M$ we can choose our basis $\lambda_1, \dots, \lambda_{2n}$ for Λ such that in terms of the dual coordinates $x_1, \dots, x_{2n}$ on V the first Chern class is given by

$$c_1(\mathcal{L}) = \sum_{i=1}^{n} \delta_\alpha dx_\alpha \wedge dx_{\alpha+n}.$$

The functions $x_{\alpha+n}$ are well-defined global functions on $V/\mathbb{Z}\{\lambda_1, \dots, \lambda_n\}$, so we have $[dx_{\alpha+n}] = 0 \in H^1_{DR}(V/\mathbb{Z}\{\lambda_1, \dots, \lambda_n\})$. This means that $c_1(\pi_1^*\mathcal{L}) = \pi_1^*(c_1(\mathcal{L})) = 0$ and $\pi_1^*(\mathcal{L})$ is trivial. Let $\tilde{\phi} : \pi_1^*\mathcal{L} \to (\mathbb{C}^*)^n \times \mathbb{C}$ be a trivialization and choose our trivialization ϕ of $\pi^*\mathcal{L}$ to extend $\tilde{\phi}$, that is $\phi_z = \tilde{\phi}_{\pi_2(z)}$ and $\phi_{z+\lambda_\alpha} = \tilde{\phi}_{\pi_2(z+\lambda_\alpha)}$ for every $\alpha = 1, \dots, n$.

$$
\begin{tikzcd}
\mathbb{C} \arrow[d, "e_{\lambda_\alpha}(z)"'] & (\pi^*\mathcal{L})_z \arrow[l, "\phi_z"'] \arrow[r, equal] & (\pi_1^*\mathcal{L})_{\pi_2(z+\lambda_2)} \arrow[r, "\tilde{\phi}_{\pi_2(z)}"] \arrow[d, equal] & \mathbb{C} \arrow[d, "f_{\lambda_\alpha}(z)"] \\
\mathbb{C} & (\pi^*\mathcal{L})_{z+\lambda_\alpha} \arrow[l, "\phi_{z+\lambda_\alpha}"'] \arrow[r, equal] & (\pi^*\mathcal{L})_{\pi_2(z+\lambda_\alpha)} \arrow[r] & \mathbb{C}
\end{tikzcd}
$$

Since $\tilde{\phi}_{\pi_2(z)} = \tilde{\phi}_{\pi_2(z+\lambda_\alpha)}$, f_{λ_α} is forced to be constantly 1. Commutativity and the fact that ϕ extends $\tilde{\phi}$ implies that $e_{\lambda_\alpha} = 1$ for $\alpha = 1, \ldots, n$. I.e. we only need to consider multipliers with the first n being equal to 1.

Now assume ω is any invariant integral form, positive of type $(1,1)$ on V. Choose a basis $\{\lambda_1, \ldots, \lambda_{2n}\}$ for Λ over $\mathbb{Z}$ such that in terms of the dual coordinates $x_1, \ldots, x_{2n}$ on V the form ω can be written as

$$\omega = \sum_{\alpha=1}^{n} \delta_\alpha dx_\alpha \wedge dx_{\alpha+n}, \quad \delta_\alpha \in \mathbb{Z}.$$

Further we require that the first n of the $\lambda_\alpha's$ are linearly independent over $\mathbb{C}$. Because ω is non-degenerate each $\delta_\alpha \neq 0$ and we can define $e_\alpha = \delta_\alpha^{-1}\lambda_\alpha$, $\alpha = 1, \cdots, n$. We let $z_1, \cdots, z_n$ be the corresponding coordinates on V. We can write $(\lambda_1, \ldots, \lambda_{2n}) = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \Omega$, i.e. $\begin{pmatrix} dz_1 \\ \vdots \\ dz_n \end{pmatrix} = \Omega^T(dx_1, \ldots, dx_{2n})$, where $\Omega = (\Delta_z, Z)$ and the third Riemann condition implies that $Z = Z^T$ and $\operatorname{Im} Z > 0$.

Lemma 11.17. The line bundle $\mathcal{L} \to M$ defined by multipliers $e_{\lambda_\alpha} = 1$ and $e_{\lambda_{\alpha+n}}(z) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}$, $\alpha = 1, \ldots, n$ has Chern class $c_1(\mathcal{L}) = [\omega]$.

Proof. Let us first check that these multipliers satisfies the line bundle condition (11.20). We have to show that

$$e_{\lambda_\alpha}(z+\lambda_\beta)e_{\lambda_\beta}(z) = e_{\lambda_\beta}(z+\lambda_\alpha)e_{\lambda_\alpha}(z) = e_{\lambda_\alpha+\lambda_\beta}(z).$$

This is clearly satisfied for α or $\beta \leq n$ and writing $Z = (Z_{\alpha\beta})$

$$\begin{aligned}
e_{\lambda_{n+\beta}}(z+\lambda_{n+\alpha})e_{\lambda_{n+\alpha}}(z) &= e^{2\pi i(z_\beta + Z_{\beta\alpha})}e^{-\pi i Z_{\beta\beta}}e^{-2\pi i(z_\alpha)}e^{-\pi i Z_{\alpha\alpha}} \\
&= e^{2\pi i(z_\alpha + Z_{\alpha\beta})}e^{-\pi i Z_{\beta\beta}}e^{-2\pi i(z_\beta)}e^{-\pi i Z_{\alpha\alpha}} \\
&= e_{\lambda_{n+\alpha}}(z+\lambda_{n+\beta})e_{\lambda_{n+\beta}}(z).
\end{aligned}$$

as required. Now let $\phi : \pi^*\mathcal{L} \to V \times \mathbb{C}$ be a trivialization of $\pi^*\mathcal{L}$ inducing the given multipliers. Then for any section $\tilde{\theta}$ of $\mathcal{L}$ over $U \subset M$, $\theta = \phi^*(\pi^*\tilde{\theta})$ is an analytic function on $\pi^{-1}(U)$ satisfying

$$\theta(z+\lambda_\alpha) = \theta(z), \quad \text{and} \quad \theta(z+\lambda_{n+\alpha}) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}\theta(z), \tag{11.21}$$

for $\alpha = 1, \ldots, n$ and conversely any such function defines a section of $\mathcal{L}$. If $||\cdot||$ is any metric on $\mathcal{L}$ then $||\tilde{\theta}(z)||^2 = h(z)|\theta(z)|^2$ for every section $\tilde{\theta}$ of $\mathcal{L}$ where $|\cdot|$ is the usual inner product on $\mathbb{C}$. h will be a positive smooth function of z and satisfies the equation:

$$h(z)|\theta(z)|^2 = ||\tilde{\theta}(z)||^2 = h(z+\lambda)|\theta(z+\lambda)|^2 \tag{11.22}$$

for any $\lambda \in \Lambda$. It follows that

$$h(z+\lambda_\alpha) = h(z)$$
$$h(z+\lambda_{n+\alpha}) = |e^{2\pi i z_\alpha \pi i Z_{\alpha\alpha}}|^2 h(z).$$

Conversely, any h satisfying the above equations will be a metric on $\mathcal{L}$.

Now let $Z \in \mathbb{H} = \{Z \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid Z^T = Z,\ \mathrm{Im}\, Z > 0\}$. Write $Z = X + iY$, where $X; Y$ are real $n \times n$ matrices. Clearly Y is invertible and we define $W = (W_{\alpha\beta}) = Y^{-1}$. The function

$$h(z) = e^{\frac{\pi}{2}\sum W_{\alpha\beta}(z_\alpha - \bar{z}_\alpha)(z_\beta - \bar{z}_\beta)} = e^{-2\pi y\cdot Y y}, \tag{11.23}$$

where of course $z = x + Zy$, satisfies the equations above. This is straight forward verification see e.g. [19].

Now one can compute the curvature form Θ_L associated to the metric given by h.

$$\Theta_L = \partial\bar{\partial}\log\frac{1}{h} = \pi\sum_{\alpha,\beta} W_{\alpha,\beta}\, dz_\alpha \wedge d\bar{z}_\beta$$

In terms of the basis $\{dx_\alpha, dx_{n+\alpha}\}$ we have

$$dz_\alpha = \delta_\alpha dx_\alpha + \sum_\beta z_{\alpha\beta} dx_{n+\beta}$$
$$d\bar{z}_\alpha = \delta_\alpha dx_\alpha + \sum_\beta \bar{z}_{\alpha\beta} dx_{n+\beta}$$

Hence

$$\Theta_{\mathcal{L}} = -2\pi i \sum_\alpha \delta_\alpha dx_\alpha \wedge dx_{n+\alpha}.$$

and finally

$$c_1(\mathcal{L}) = \left[\frac{i}{2\pi}\Theta_{\mathcal{L}}\right] = [\omega]. \tag{11.24}$$

□

For further details the detailed proofs are written out in [19].

11.3.3 Theta functions

In the previous section we introduced a Hermitian structure $h\langle\cdot,\cdot\rangle_{\mathbb{C}}$, where $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is the standard inner product on $\mathbb{C}$. Since

$$h(z+\lambda) = \frac{1}{|e_\lambda(z)|^2} h(z),$$

we see that this Hermitian structure is $\Lambda-$invariant and induces a Hermitian structure on the line bundle $\mathcal{L}$.

If we let $\mathcal{C}$ denote the space of complex structures on V, which are compatible with the metric, that is $\mathcal{C}$ consists of the symplectomorphisms $I : V \to V$ such that the symmetric form $\omega(\cdot, I\cdot)$ is a positive definite inner product on V. If all δ_α are 1 then the triple $M_I = (M, \omega, I)$ is a principal polarized abelian variety. Using the basis $\{\lambda_1, \ldots, \lambda_{2n}\}$ one can identify the space $\mathcal{C}$ with the Siegel generalized upper half space $\mathbb{H} = \{Z \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid Z^T = Z,\ \mathrm{Im}\, Z > 0\}$ For any $I \in \mathcal{C}$ $\{\lambda_1, \ldots, \lambda_n\}$ is a basis for V over $\mathbb{C}$ with respect to I. We let $z = (z_1, \dots, z_n)$ be the dual coordinates on V relative to the basis $\{\lambda_1, \dots, \lambda_n\}$. The complex structure determines a unique $Z \in \mathbb{H}$ such that

$$z = x + Zy \tag{11.25}$$

Since any $Z \in \mathbb{H}$ gives a positive complex structure, say $I(Z)$, compatible with the symplectic form, we have a bijective map $I : \mathbb{H} \to \mathcal{C}$, given by sending $Z \in \mathbb{H}$ to $I(Z)$.

If multipliers are chosen with respect to the complex structure we get a line bundle $\mathcal{L}_I$ over M_I. The space of holomorphic sections of $\mathcal{L}_I^k$, $H^0(M, \mathcal{L}_I^k)$, has dimension k^n and they give a bundle $H^{(k)}$ over $\mathcal{C}$ by letting $H_I^{(k)} = H^0(M_I, \mathcal{L}_I^k)$. The L^2 inner product on the latter is given by

$$(s_1, s_2) = \int_M s_1(z)\overline{s_2(z)}e^{-2\pi y \cdot Y y}dxdy, \tag{11.26}$$

for $s_1, s_2 \in H^0(M_I, \mathcal{L}_I^k)$.

This space has an explicit basis given in terms of *Theta functions* of level k:

$$\Theta_{\alpha,k}(z, Z) = \sum_{l \in \mathbb{Z}^n} e^{\pi i k(l+\alpha)\cdot Z(l+\alpha)} e^{2\pi i k(l+\alpha)\cdot z}.$$

For further details one should consult [3] and [19].

Remark 11.18. If $c_1(\mathcal{L})$ is a principal polarization of M, $H^0(M, \mathcal{L})$ is one dimensional and generated by the section $\tilde{\theta}$ with corresponding

$$\theta(z) = \sum_{l \in \mathbb{Z}^n} e^{i\pi l \cdot Z l} e^{2\pi i l \cdot z}$$

satisfying the functional equation (11.21). These functions are called *Riemann theta functions* of the principal polarized abelian variety (M, ω). Since these functions depends on both z, Z one often writes $\theta(z, Z)$. These functions appear when the Weil–Gel'fand–Zak transform is used on a wavelet as we will se later.

Further we note that we have actually constructed a pre-quantum line bundle over a torus.

Example 11.19 (Complex line bundle over the Torus)**.** A complex line bundle over the torus $T^m = \mathbb{R}^m/\mathbb{Z}^m$ can as above be described by a cocycle

$$\mathbb{Z}^m \to C^\infty(R^m, S^1) : \lambda \to e_\lambda$$

which satisfies

$$e_{\lambda+\lambda'}(z) = e_{\lambda'}(z+\lambda)e_\lambda(z),$$

for $z \in \mathbb{R}^m$ and $\lambda, \lambda' \in \mathbb{Z}^m$. The associated complex line bundle is

$$\mathcal{L} := \frac{\mathbb{R}^m \times \mathbb{C}}{\mathbb{Z}^m}, \quad [z, \xi] \equiv [z+\lambda, e_\lambda(z)\xi], \quad \forall \lambda \in \mathbb{Z}^m.$$

A Hermitian connection in L has the form

$$\nabla = d + A, \quad A = \sum_{i=1}^n A(x)dx^i,$$

where the function $A : \mathbb{R}^2 \to i\mathbb{R}$ satisfy the condition

$$A(x+\lambda) - A(x) = -e_\lambda(x)^{-1}\frac{\partial e_\lambda}{\partial x^i}(x).$$

If we specify our multipliers to be $e_{(1,0)}(u_1', u_2') = e^{-\pi i u_2'}$ and $e_{(0,1)(u_1', u_2')} = e^{\pi i u_1'}$ Our connection is determined by

$$A_1((u_1', u_2') + (1,0)) - A((u_1', u_2')) = -e_{(1,0)}(u_1', u_2')^{-1} \frac{\partial}{\partial u_1'} e_{(1,0)}(u_1', u_2') = \pi i u_2'$$

and

$$A_2((u_1', u_2') + (0,1)) - A((u_1', u_2')) = -e_{(0,1)}(u_1', u_2')^{-1} \frac{\partial}{\partial u_1'} e_{(0,1)}(u_1', u_2') = -\pi i u_1'$$

And we have

$$\nabla = d + \pi i (u_1' du_2' - u_2' du_1').$$

11.3.3.1 Pull back of line bundles

In the following we want to consider the necessities for a line bundle to be a pull back of some other line bundle. In the following lemma we consider the restrictions on the multipliers for the pull back bundle.

Lemma 11.20. Let $\mathbb{T}^n = \mathbb{R}^n/\Lambda_1$ and $\mathbb{T}^m = \mathbb{R}^m/\Lambda_2$. Let $f : \mathbb{T}^m \to \mathbb{T}^n$ be a map. If $\mathcal{L}$ is a line bundle over $\mathbb{T}^n$ determined by multipliers $e^{(1)}_{\lambda'}, \lambda' \in \Lambda_1$ then the pullback bundle $f^*\mathcal{L}$ is determined by multipliers satisfying the formula

$$e^{(2)}_{\lambda}(x) = e^{(1)}_{F(\lambda)}\left(F(x)\right),$$

where F is the map covering f and $\lambda \in \Lambda_2$.

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "F"] \arrow[d, "p_2"'] & \mathbb{R}^n \arrow[d, "p_1"] \\ \mathbb{T}^m \arrow[r, "f"] & \mathbb{T}^n \end{tikzcd} .$$

Proof. Note that $f \circ p_2 = p_1 \circ F$. We look at the following diagram

$$\begin{tikzcd} p_2^*(f^*\mathcal{L}) \arrow[r] \arrow[d] & f^*\mathcal{L} \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] & p_1^*\mathcal{L} \arrow[l] \arrow[d] \\ \mathbb{R}^m \arrow[r, "p_2"] & \mathbb{T}^m \arrow[r, "f"] & \mathbb{T}^n & \mathbb{R}^n \arrow[l, "p_1"'] \end{tikzcd}$$

Since $\mathbb{R}^n$ and $\mathbb{R}^m$ both are contractible we can choose global trivializations $\phi : p_1^*\mathcal{L} \to \mathbb{R}^n \times \mathbb{C}$ and $\psi : p_2^*(f^*\mathcal{L}) \to \mathbb{R}^m \times \mathbb{C}$ for the pull back bundles. Furthermore we note that for $\lambda \in \Lambda_2$ $F(\lambda) \in \Lambda_1$. This follows since

$$p_1(F(\lambda)) = f(p_2(\lambda)) = f([0]_{(2)}) = [0]_{(1)},$$

Hence $F(\lambda) \in \ker(p_1)$ i.e. $F(\lambda) \in \Lambda_1$.

For $F(z) \in \mathbb{R}^n, F(\lambda) \in \Lambda_1$ we have equalities of fibers by definition of pull back bundle

$$(p_1^*\mathcal{L})_{F(z)} = \mathcal{L}_{p_1(F(z))} = \mathcal{L}_{p_1(F(z)+F(\lambda))} = (p_1^*\mathcal{L})_{F(z)+F(\lambda)}.$$

$\phi_{F(z)+F(\lambda)} \circ \phi^{-1}_{F(z)}$ is thus multiplication by a complex number which we denote $e^{(1)}_{F(\lambda)}(F(z))$.

Multipliers for the pull back bundle $p_2^*(f^*\mathcal{L})$ is given in the same way as above. Now it is enough to note that for $z \in \mathbb{R}^m$ we have the following equalities of fibers.

$$(p_2^*(f^*\mathcal{L}))_z = (f^*\mathcal{L})_{p_2(z)} = \mathcal{L}_{f \circ p_2(z)} = \mathcal{L}_{p_1 \circ F(z)} = (p_1^*\mathcal{L})_{F(z)}.$$

It follows that

$$e^{(2)}_{\lambda}(z) = e^{(1)}_{F(\lambda)}(F(z)).$$

□

Because the multipliers for a given line bundle depends on the global trivialization $\chi : \pi^*\mathcal{L} \to V \times \mathbb{C}$ we see that multiplication by a nowhere vanishing holomorphic function $g : V \to \mathbb{C}^*$ the set of original multipliers $\{e_\lambda\}_{\lambda\in\Lambda}$ is replaced by a new set of multipliers $\{e'_\lambda\}_{\lambda\in\Lambda}$ satisfying the relation

$$\frac{g(z)}{g(z+\lambda)} e'_\lambda(z) = e_\lambda(z).$$

Hence we assume that $E \to T^m$ is a line bundle isomorphic to the line bundle $f^*\mathcal{L} \to T^m$. Since these line bundles are bundles over the same base space it follows that an isomorphism $\Phi : E \to f^*\mathcal{L}$ locally has the form $\Phi(z,\xi) = (z, g(z)\cdot\xi)$. Where $g : \mathbb{R}^m \to \mathbb{C}^*$ is a holomorphic function.

Proposition 11.21. Let $E \to \mathbb{T}^m$ be a line bundle and let $f : T^m \to \mathbb{T}^n$ be a map having $F : \mathbb{R}^m \to \mathbb{R}^n$ as a covering map. There exists a line bundle $\mathcal{L} \to \mathbb{T}^n$ such that $E = f^*\mathcal{L}$.

Proof. We need to define multipliers for the line bundle $\mathcal{L}$ over $\mathbb{T}^n$ such that the pull back bundle $f^*\mathcal{L}$ has the multipliers of E. By the lemma above, and the observation that holomorphic line bundles differ by a cocycle we see that the restriction of multipliers must be

$$e^{(3)}_\lambda(z) = e^{(1)}_{F(\lambda)}(F(z))\frac{g(z+\lambda)}{g(z)}. \tag{11.27}$$

In other words we need to choose a holomorphic function $g : \mathbb{R}^m \to \mathbb{C}^*$ such that equation (11.27) is satisfied. For every $\lambda \notin \ker F$ we set $\frac{g(z+\lambda)}{g(z)} = 1$. Now let $\lambda \in \ker F$ be a basis vector in the lattice $\mathbb{Z}^n$. Since $F(\lambda) = 0$ the multiplier $e^{(1)}_{F(\lambda)}(F(z)) = e^{(1)}_0(F(z)) = 1$ so we rewrite (11.27) and solve the equation

$$g(z)e^{(3)}_\lambda(z) = g(z+\lambda). \tag{11.28}$$

solving this equation is equivalent to solving the equation

$$e^{\pi i\gamma(z)}e^{\pi i\alpha_\lambda(z)} = e^{\pi i\gamma(z+\lambda)},$$

hence we look for a solution to the problem

$$\alpha_\lambda(z) = \gamma(z+\lambda) - \gamma(z).$$

Writing $\alpha_\lambda(z) = \sum_{i=1}^m \alpha_i z_i$ a solution to this problem is given by the function

$$\gamma(z) = \sum_{i,j} A_{ij}x_ix_j + \sum_i B_ix_i + C$$

where A is chosen to be the symmetric matrix with entries

$$A_{ij} = A_{ji} = \begin{cases} \frac{\alpha_j}{2} & \text{for } \lambda_i = 1, j \in \{1,\dots,n\}. \\ 0 & \text{for all other entries} \end{cases}$$

and

$$B_i = \begin{cases} -\frac{\alpha_i}{2} & \lambda_i = 1, \\ 0 & \text{otherwise} \end{cases}$$

and C is just a constant. □

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