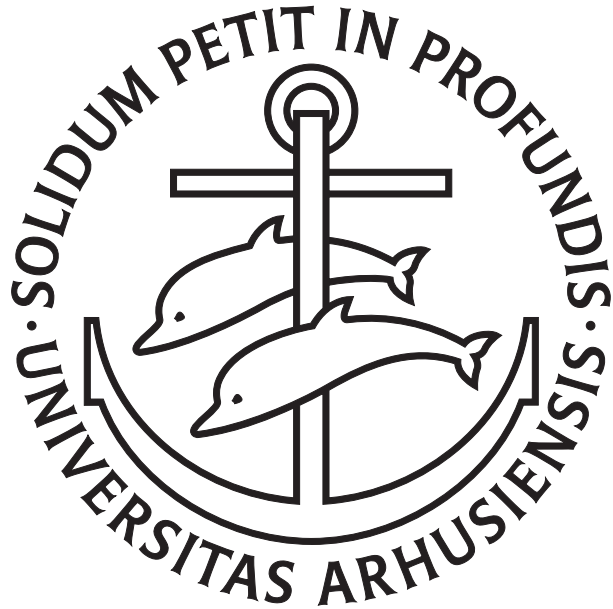# THE CORE OF C*-ALGEBRAS ASSOCIATED WITH CIRCLE MAPS

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Abstract

Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be any (surjective) continuous and piecewise monotone circle map. We consider the principal and locally compact Hausdorff étale groupoid R_φ^+ from [50]. Already $C_r^*(R_\varphi^+)$ is a unital separable direct limit of Elliott–Thomsen building blocks. A characterization of simplicity of $C_r^*(R_\varphi^+)$ is given assuming surjectivity in addition. We also prove that $C_r^*(R_\varphi^+)$ has a unique tracial state and real rank zero when simple. As a consequence $C_r^*(R_\varphi^+)$ has slow dimension growth in the sense of [36] when simple. This means that $C_r^*(R_\varphi^+)$ are classified by their graded ordered K-theory due to [58]. We compute $K_0(C_r^*(R_\varphi^+))$ for a subclass of circle maps. In general $K_1(C_r^*(R_\varphi^+)) \simeq \mathbb{Z}$. A counterexample yields non-semiconjugate circle maps with isomorphic K-theory.

We give a classification of transitive critically finite circle maps up to conjugacy. This class of circle maps contains the surjective circle maps for which $C_r^*(R_\varphi^+)$ is simple. A transitive circle map is always conjugate to a uniformly piecewise linear circle map. We offer a constructive approach to this fact, which also implies a uniqueness result.

Resumé

Lad $\varphi\colon \mathbb{T} \to \mathbb{T}$ være en (surjektiv) kontinuert stykkevist monoton cirkelafbildning. Vi betragter her den principale og lokalkompakte Hausdorff étale groupoid R_φ^+ fra [50]. Vi ved at $C_r^*(R_\varphi^+)$ er en unital separabel direkte grænse af Elliott–Thomsen byggesten. Der gives en karakterisering af simplicitet for $C_r^*(R_\varphi^+)$ (under antagelse af surjektivitet). Vi beviser at $C_r^*(R_\varphi^+)$ har en entydig sportilstand og reel rang nul når den er simpel. Det følger heraf, at $C_r^*(R_\varphi^+)$ har langsom dimensionsvækst når den er simpel, jf. [36]. Det betyder at $C_r^*(R_\varphi^+)$ er klassificerede ved deres graduerede ordnede K-teori jf. [58]. Vi finder $K_0(C_r^*(R_\varphi^+))$ for en klasse af cirkelafbildninger. Vi har altid $K_1(C_r^*(R_\varphi^+)) \simeq \mathbb{Z}$. Et modeksempel giver ikke-semikonjugerede cirkelafbildninger med samme K-teori.

Vi klassificerer de transitive kritisk endelige cirkelafbildninger op til konjugering. Denne klasse indeholder de surjektive cirkelafbildninger for hvilke $C_r^*(R_\varphi^+)$ er simpel. En transitiv cirkelafbildning er altid konjugeret til en der er uniformt stykkevist lineær. Vi giver en konstruktiv tilgang til dette faktum, som også giver et entydighedsresultat.

Preface

The relationship between dynamical systems and operator algebras is one that has been fruitful and mutually beneficial and is by now both well-established, -aged and -matured. A testament to the well-being of this relationship which is also a significant benchmark is the complete classification of minimal dynamics on a Cantor set up to orbit equivalence by Thierry Giordano, Ian F. Putnam, Christian F. Skau in [18].

This thesis delves into an aspect of the relationship between dynamics on the unit circle and the associated operator algebras, and a short introduction seems to be appropriate. The infant stages in the study of circle dynamics goes back to the work of Henri Poincaré. In an attempt to classify flows on the torus he ended up classifying circle homeomorphisms. He introduced the rotation number $\rho(\varphi) \in \mathbb{R}$ for any circle homeomorphism $\varphi \colon \mathbb{T} \to \mathbb{T}$ which is irrational if and only if there are no periodic points for φ and demonstrated that any such map must be semiconjugate to an irrational rotation $\varphi_\alpha \colon \mathbb{T} \to \mathbb{T}$ with $\rho(\varphi_\alpha) = \alpha$, by proving that the points in their orbits in fact adhere to the same order on the circle. The semiconjugacy is essentially unique in the aforementioned result of Poincaré's work. The rotation number behaves particularly well with respect to conjugation as it turns out, as $\rho(\varphi) = \rho(\psi)$ when $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ are homeomorphisms of the circle which are conjugate through a conjugacy $h \colon \mathbb{T} \to \mathbb{T}$ which is an orientation-preserving circle homeomorphism, and $\rho(\varphi) = 1 - \rho(\psi)$ when the conjugacy is orientation-reversing.

In the world of operator algebras the irrational rotation C*-algebra was studied early on. The irrational rotation C*-algebra is the transformation group C*-algebra $A_\alpha = C(\mathbb{T}) \rtimes_{\varphi_\alpha} \mathbb{Z}$ for the integer action of an irrational rotation $\varphi_\alpha \colon \mathbb{T} \to \mathbb{T}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ on the unit circle. It was shown that A_α are always simple with a unique normalized trace τ, and eventually Marc A. Rieffel in [45], and Mihai V. Pimsner and Dan V. Voiculesco in [40], [39] proved $A_\alpha \simeq A_\beta$ if and only if $\alpha = \beta$ or $\alpha = 1 - \beta$, as $K_0(A_\alpha) \simeq \mathbb{Z} + \alpha \mathbb{Z}$ as ordered abelian groups. An important ingredient in the argument is the construction of a projection $p \in A_\alpha$ with $\tau(p) = \gamma$ for any prescribed $\gamma \in (\mathbb{Z} + \alpha \mathbb{Z}) \cap [0, 1]$. This is now known as a Rieffel projection. George A. Elliott and David E. Evans in [17] subsequently managed to prove that in fact A_α is always isomorphic to a direct limit of circle algebras and also has real rank zero. Another study was done by Ian F. Putnam, Klaus Schmidt, Christian F. Skau in [42], where they considered any homeomorphism of the circle $\varphi \colon \mathbb{T} \to \mathbb{T}$ without periodic points, i.e. with irrational rotation number, which is just not conjugate to an irrational rotation. Any such homeomorphism is called a Denjoy homeomorphism, and it is never transitive. Instead there exists a unique minimal closed invariant set $X \subseteq \mathbb{T}$ and it is a Cantor set. They studied $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ and $C(X) \rtimes_\varphi \mathbb{Z}$ and – from the K-theory of these C*-algebras – were able to recover the complete invariant for Denjoy homeomorphisms from [26].

In the seminal work [43] about full and reduced groupoid C*-algebras, Jean N. Renault introduced the transformation group groupoid which was in turn later expanded on by Valentin Deaconu in [15] and Claire Anantharaman-Delaroche in [5], and as a consequence the transformation group groupoid is well-behaved for any surjective local homeomorphism on a locally compact Hausdorff space, and in particular for any homeomorphism.

The introduction of the transformation group groupoid paved the way for developments in the relationship between dynamical systems (and now groupoids) and operator algebras. Kasper K. S. Andersen og Klaus E. Thomsen in [6] used the transformation group groupoid to study the dynamics of various maps including any local homeomorphism $\varphi\colon \mathbb{T} \to \mathbb{T}$.

Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be any piecewise monotone circle map (it could even be non-surjective). Thomas L. Schmidt and Klaus E. Thomsen in [50] used the construction of a groupoid from a pseudo-group $\mathcal{P}$ on $\mathbb{T}$ and introduced local transfers $\mathcal{T}(\varphi)$ for φ (as in [56] and [55]) to construct a locally compact Hausdorff étale groupoid $\Gamma_\varphi(\mathcal{P})$, and for a certain $\mathcal{P} = \mathcal{P}^+$, $\Gamma_\varphi(\mathcal{P}^+)$ can be identified with a subgroupoid Γ_φ^+ of the transformation group groupoid Γ_φ. The groupoid Γ_φ^+ contains an open subgroupoid R_φ^+ which is an equivalence relation on $\mathbb{T}$. They showed in particular that $C_r^*(R_\varphi^+)$ is a direct limit of Elliott–Thomsen building blocks. The simplicity of $C_r^*(R_\varphi^+)$ is now characterized by conditions on φ in the current thesis, when φ is surjective, and when these conditions are met, then $C_r^*(R_\varphi^+)$ is shown to have a unique tracial state and real rank zero, and classification results apply, and subsequently $K_0(C_r^*(R_\varphi^+))$ is a countable simple dimension group and it is shown that $K_1(C_r^*(R_\varphi^+)) \simeq \mathbb{Z}$. These results together with results in [49] implies that $C_r^*(R_\varphi^+)$ is the fixed point algebra of an order-two $*$-automorphism on an approximately finite dimensional algebra whereas $C_r^*(R_\varphi^+)$ is not approximately finite dimensional but a direct limit of circle algebras, in fact it is isomorphic to the crossed product of a minimal homeomorphism on a Cantor set.

Another aspect of this thesis are the results solely about dynamics on the unit circle. It is not unusual for the dynamics of circle maps and interval maps to go hand-in-hand, and many results – though not all – may be transferred directly from one area to the other. A prominent example of this connection is the striking key result by William Parry in [33] which entails that any transitive piecewise monotone interval map is conjugate through an orientation-preserving conjugacy to an interval map which is uniformly piecewise linear. The proof of this statement in this general version is unfortunately not very constructive. A similar statement proved by similar methods is applicable for circle maps, cf. [3] or [50]. The results obtained in this thesis on circle dynamics are in the last half of Chapter 3, where we will restrict our attention to the class of transitive critically finite circle maps, for which a complete classification is attained using an idea resembling that of Poincaré's. This classification is in fact based on the ideas of similar results for interval maps, cf. [11]. In the aftermath of this classification for the class of transitive critically finite circle maps, Parry's result for this specific class of circle maps reemerges and is established by using the Perron–Frobenius theorem to construct a uniformly piecewise linear circle map which is essentially unique in the conjugacy class of the transitive critically finite circle map.

The reader is supposed to be well versed in C*-algebras and K-theory (and classification). The following is a list of general references ordered in level of complexity:

$$[30], [14], [9], [10], [62], [47], [48].$$

The reader is not assumed to be familiar with dynamical system theory nor with groupoids. An introduction to each of these two topics is given in Chapter 1 and Chapter 2 respectively. The introductions will both be short of references and omit arguments for concrete results. The material covered in each topic will be viewed as background knowledge throughout. The appendices will also be invoked occasionally:

- Chapter A gives a very short introduction to selected singular topics on C*-algebras, most notably the Elliott–Thomsen building blocks and some applications hereof.
- Chapter B contains miscellaneous themes on non-negative matrices and integer matrices, namely the Perron–Frobenius theorem and the Schmidt normal form.

There are segments of the thesis that have been copied more or less directly from my Qualifying Examination Progress Report and subsequently altered if changes were needed.

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Part I

Preliminaries

Topological dynamics

In this chapter we will cover some of the basic concepts and results of topological dynamics, and in particular we discuss special concepts and results related to topological transitivity, as these will be relevant later on; it is natural that this summary of concepts and results will not be particularly comprehensive nor detailed and only add references and arguments for results that require depths or ideas that will not be provided here.

Let X and Y be topological spaces and $\varphi\colon X \to X$ and $\psi\colon Y \to Y$ continuous maps.

Definition 1.1. The pair (X, φ) is called a (topological) dynamical system, and we write $\varphi^0 = \mathrm{id}_X$ and $\varphi^n = \varphi \circ \varphi^{n-1}$ for any $n \in \mathbb{N}$ (and for any $n \in \mathbb{Z}$ if φ is invertible).

We will not distinguish between the map $\varphi\colon X \to X$ and the dynamical system (X, φ).

Definition 1.2. Let $h\colon X \to Y$ be a continuous map. Then

(i) h is a (topological) semiconjugacy from φ to ψ if h is surjective and $h \circ \varphi = \psi \circ h$,

(ii) h is a (topological) conjugacy from φ to ψ if h is a homeomorphism and $h \circ \varphi = \psi \circ h$.

We say φ is (topologically) semiconjugate to ψ if (i) for some continuous map $h\colon X \to Y$. If this is the case, then φ is called an extension of ψ, and also ψ is called a factor of φ. We say φ and ψ are (topologically) conjugate if (ii) for some map $h\colon X \to Y$.

As one should expect, many properties are preserved by (semi)conjugacy.

Definition 1.3. A point $x \in X$ is a preperiodic point if $\varphi^n(x) = \varphi^m(x)$ for some $n, m \in \mathbb{N}$, a periodic point if $\varphi^n(x) = x$ for some $n \in \mathbb{N}$ and a fixed point if $\varphi(x) = x$.

Definition 1.4. Define the following sets:

(i) $\mathcal{O}_\varphi^+(x) = \{\varphi^n(x) : n \in \mathbb{N}_0\}$ is the forward orbit of $x \in X$.

(ii) $\mathcal{O}_\varphi^-(x) = \{\varphi^{-n}(x) : n \in \mathbb{N}_0\}$ is the backward orbit of $x \in X$.

(iii) $\mathcal{O}_\varphi(x) = \mathcal{O}_\varphi^+(x) \cup \mathcal{O}_\varphi^-(x)$ is the (full) orbit of $x \in X$, i.e. $\mathcal{O}_\varphi(x) = \{\varphi^n(x) : n \in \mathbb{Z}\}$.

We will reserve the term trajectory for the time-ordered sequences corresponding to orbits. We say $A \subseteq X$ is forward invariant if $\varphi(A) \subseteq A$ and backward invariant if $\varphi^{-1}(A) \subseteq A$.

Remark 1.5. A set is forward invariant if and only if the complement is backward invariant.

(a) The closure of a forward invariant set is forward invariant.

(b) The interior of a backward invariant set is backward invariant.

Proposition 1.6. *The following properties are equivalent:*

 (i) *For any non-empty open $U, V \subseteq X$ there exists $n \in \mathbb{N}$ such that $\varphi^n(U) \cap V \neq \emptyset$.*

 (ii) *Any backward invariant subset of X either has empty interior or is dense in X.*

(iii) *Any forward invariant subset of X is either dense or nowhere dense in X.*

(iv) *Any non-empty open backward invariant subset of X is dense in X.*

 (v) *Any proper closed forward invariant subset of X is nowhere dense in X.*

(vi) *For any non-empty open $U \subseteq X$ the set $\bigcup_{n \in \mathbb{N}} \varphi^n(U)$ is dense in X.*

Remark 1.7. Make the following observations:

 (a) For any $A, B \subseteq X$ and $n \in \mathbb{N}$ we have $\varphi^n(A) \cap B \neq \emptyset$ if and only if $A \cap \varphi^{-n}(B) \neq \emptyset$.

 (b) For any $A \subseteq X$ and $m, n = 0, 1, \ldots$ we have $\varphi^{-n}(\varphi^{-m}(A)) = \varphi^{-(n+m)}(A)$.

It follows that the set $\mathbb{N}$ in Proposition 1.6 (i) and (vi) may equivalently be replaced by either of the sets $\{m, m + 1, \ldots\}$ or $\{\ldots, -(m + 1), -m\}$ for any $m = 0, 1, \ldots$.

Definition 1.8. We say that φ is (topologically) transitive if for any non-empty open sets $U \subseteq X$ and $V \subseteq X$ there exists $n \in \mathbb{N}$ such that $\varphi^n(U) \cap V \neq \emptyset$.

We refer the reader to [1] for a comprehensive treatment of (topological) transitivity. It includes other versions of transitivity and a discussion of when these are equivalent.

Remark 1.9. The following is a selection of conclusions from [1] about transitivity:

 (a) If X is a perfect non-meager second countable Hausdorff space then all conditions in Definition 1.1 in [1] are equivalent, cf. Theorem 1.4 in [1] for more details.

 (b) If X is a perfect compact Hausdorff space and φ is transitive, then φ is surjective, since $\varphi(X)$ is compact and dense in X, cf. Corollary 4.4 in [1].

In particular under the assumptions of (a), transitivity is equivalent to point transitivity, i.e. the existence of a point with dense forward orbit, and adding the assumption of (b), all of these versions of transitivity necessitate surjectivity.

For details about the following remark see [8]:

Remark 1.10. Let $n, m \in \mathbb{N}$ then φ^{nm} is transitive if and only if φ^n and φ^m are transitive. Thus the minimal $n \in \mathbb{N}$ such that φ^n is not transitive must be prime or one if it exists.

Definition 1.11. We say that φ is totally transitive if φ^n is transitive for all $n \in \mathbb{N}$.

There is an abundance of maps which are transitive but which are not totally transitive. In other words assuming total transitivity is strictly stronger than assuming transitivity. In order to better understand maps which are transitive but not totally transitive we take a closer look at the notion of a periodic decomposition (cf. [8]):

A periodic decomposition of X of length $n \in \mathbb{N}$ is a collection $\mathcal{X} = \{X_0, X_1, \ldots, X_{n-1}\}$ consisting of subsets of X with non-empty interior such that $X = X_0 \cup X_1 \cup \cdots \cup X_{n-1}$ with X_i not intersecting the interior of X_j whenever $i \neq j$ and with $\varphi(X_i) \subseteq X_{i+1 \mod n}$. A periodic decomposition $\mathcal{X}$ of X is called connected/regular if all the elements of $\mathcal{X}$ are connected/regular closed subsets of X. The periodic decomposition $\mathcal{X} = \{X\}$ is trivial. A periodic decomposition $\mathcal{X}$ is called a refinement of another periodic decomposition $\mathcal{Y}$ if for any $A \in \mathcal{X}$ there exists $B \in \mathcal{Y}$ with $A \subseteq B$. If we assume that $\mathcal{X}$ is a refinement of $\mathcal{Y}$, then for any $A \in \mathcal{X}$ the set $B \in \mathcal{Y}$ with $A \subseteq B$ is unique, and $\#\{A \in \mathcal{X} : A \subseteq B\}$ is the same for all $B \in \mathcal{Y}$, and therefore $\#\mathcal{X}$ is a multiple of $\#\mathcal{Y}$.

Remark 1.12. If X has a periodic decomposition of length $n \neq 1$, then φ^n is not transitive. In particular there are no non-trivial periodic decomposition of X if φ is totally transitive.

Remark 1.13. If $\mathcal{X} = \{X_0, X_1, \ldots, X_{n-1}\}$ is a regular periodic decomposition of X, then $X_i \cap X_j$ is nowhere dense whenever $i \neq j$, and $\varphi(X_i) = X_{i+1 \mod n}$ when φ is also closed. If φ is transitive there is at most one regular periodic decomposition of X of length $n \in \mathbb{N}$.

Details about the above remarks can be found in [8] as well.

Theorem 1.14. *If X has a regular periodic decomposition $\mathcal{X} = \{X_0, X_1, \ldots, X_{n-1}\}$, then φ is transitive if and only if $\varphi^n|_{X_i}$ is transitive for all $i = 0, \ldots, n-1$.*

The following result is a converse of Remark 1.12 (see [8] for details):

Theorem 1.15. *If φ is transitive and φ^n is not transitive for some $n \in \mathbb{N}$, $n \neq 1$, then there exists a regular periodic decomposition of X of some length $m \neq 1$ dividing n.*

The next result is about a special case that we will return to later (see [8] for details):

Theorem 1.16. *If X is locally connected and φ is transitive (not totally transitive), then any regular periodic decomposition of X has a connected regular refinement.*

A regular periodic decomposition of maximal length is called terminal (see [8] for details).

Theorem 1.17. *Any given regular periodic decomposition $\mathcal{X} = \{X_0, X_1, \ldots, X_{n-1}\}$ of X is terminal if and only if $\varphi^n|_{X_i}$ is totally transitive for all $i = 0, \ldots, n-1$.*

The concepts and results related to periodic decompositions will resurface later on where the situation simplifies to a combination of Theorem 1.16 and Theorem 1.17.

A couple of assumptions which are both strictly stronger than transitivity and in fact strictly stronger than total transitivity are the following two assumptions:

Definition 1.18. We say that

(i) φ is (topologically) mixing if for any non-empty open sets $U \subseteq X$ and $V \subseteq X$ there exists $N \in \mathbb{N}$ such that $\varphi^n(U) \cap V$ for all $n \geq N$.

(ii) φ is weakly (topologically) mixing if $\varphi \times \varphi \colon X \times X \to X \times X$ is transitive.

Remark 1.19. The map $\varphi \times \psi \colon X \times Y \to X \times Y$ is mixing whenever φ and ψ are mixing. In particular any mixing map is also weakly mixing as the naming seemingly suggests. However a map need not be weakly mixing when it is transitive or even totally transitive. We do have $\varphi \times \cdots \times \varphi \colon X \times \cdots \times X \to X \times \cdots \times X$ is transitive when φ is weakly mixing. This in turn now implies that any weakly mixing map is in fact also totally transitive. Conversely any totally transitive map with a dense set of periodic points is weakly mixing. For details about these results see [8].

The relationship between being totally transitive and mixing also simplifies later on.

Proposition 1.20. *The following properties are equivalent:*

(i) *For any non-empty open $U \subseteq X$ we have $\bigcup_{n \in \mathbb{N}} \varphi^n(U) = X$.*

(ii) *Any non-empty backward invariant subset of X is dense in X.*

(iii) *For any $x \in X$ the set $\bigcup_{n \in \mathbb{N}} \varphi^{-n}(x)$ is dense in X.*

An assumption that is also strictly stronger than transitivity is either of the following:

Definition 1.21. We say that

(i) φ is strongly (topologically) transitive if for any non-empty open set $U \subseteq X$ we have $\bigcup_{n=1}^{\infty} \varphi^n(U) = X$, and φ is very strongly (topologically) transitive if for any non-empty open set $U \subseteq X$ there exists $N \in \mathbb{N}$ such that $\bigcup_{n=1}^{N} \varphi^n(U) = X$.

(ii) φ is (topologically) exact or locally eventually onto if for any non-empty open set $U \subseteq X$ there exists $N \in \mathbb{N}$ such that $\varphi^N(U) = X$.

Remark 1.22. If φ is exact, then φ is obviously also very strongly transitive and mixing. If X is compact and φ is open and strongly transitive, then φ is very strongly transitive. It also easily follows that φ is surjective whenever φ is strongly transitive, and also that φ is strongly transitive whenever φ is very strongly transitive.

Proposition 1.23. *The following properties are equivalent:*

(i) *The forward orbit of any point is dense in X.*

(ii) *There are no non-empty proper closed forward invariant subsets of X.*

(iii) *There are no non-empty proper open backward invariant subsets of X.*

(iv) *For any non-empty open $U \subseteq X$ we have that $\bigcup_{n \in \mathbb{N}} \varphi^{-n}(U) = X$.*

Definition 1.24. We say that φ is minimal if any forward orbit is dense in X.

Remark 1.25. If X is compact and φ is minimal, then φ is also very strongly transitive. If φ is minimal, then φ is also transitive and surjective.

The set $\mathbb{N}$ in both Proposition 1.20(i), Proposition 1.20(iii) and Proposition 1.23(iv) may equivalently be replaced by $\{m, m+1, \ldots\}$ for any $m = 0, 1, \ldots$.

Assume throughout the rest of this chapter that X is also compact.

Definition 1.26. Define the common refinement

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$$

for open covers $\mathcal{U}$ and $\mathcal{V}$ of X, and $N(\mathcal{U})$ as the smallest cardinality of a subcover of $\mathcal{U}$.

Definition 1.27. Define the (topological) entropy of φ as

$$h(\varphi) = \sup \lim_{n \to \infty} \tfrac{1}{n} \log N(\mathcal{U} \vee \varphi^{-1}(\mathcal{U}) \vee \cdots \vee \varphi^{-n+1}(\mathcal{U}))$$

where the supremum is taken over all open covers $\mathcal{U}$ of X.

Lemma 1.28. *We have that $h(\varphi^n) = nh(\varphi)$ and*

(i) $h(\varphi) \geq h(\psi)$ *whenever φ is semiconjugate to ψ;*

(ii) $h(\varphi) = h(\psi)$ *whenever φ and ψ are conjugate.*

C*-algebras associated with groupoids

In this chapter we review concepts and results on C*-algebras associated with groupoids that will become relevant later on without going into too many details and discussions. The reader is referred to [29], [38], [41], [43], [44] for some helpful treatments and references and details that are not provided here.

Definition 2.1. A groupoid G is a set along with a map $G \ni x \mapsto x^{-1} \in G$ (inversion) and a map $G^{(2)} \ni (x, y) \mapsto xy \in G$ (composition) where $G^{(2)} \subseteq G \times G$ such that

  (i) For any $x \in G$, $(x^{-1})^{-1} = x$.

  (ii) If $(x, y), (y, z) \in G^{(2)}$, then $(xy, z), (x, yz) \in G^{(2)}$ and $(xy)z = x(yz)$.

  (iii) For any $x \in G$, $(x^{-1}, x) \in G^{(2)}$. If $(x, y) \in G^{(2)}$, then $x^{-1}(xy) = y$.

  (iv) For any $x \in G$, $(x, x^{-1}) \in G^{(2)}$. If $(y, x) \in G^{(2)}$, then $(yx)x^{-1} = y$.

Define the source map $s : G \to G$ and the range map $r : G \to G$ by

$$s(x) = x^{-1}x \quad \text{and} \quad r(x) = xx^{-1}.$$

A pair $(x, y) \in G^{(2)}$ is called a composable pair, and $G^{(0)} = r(G) = s(G)$ is the unit space.

Definition 2.2. A subgroupoid H of a given groupoid G is a (non-empty) subset $H \subseteq G$ such that $xy \in H$ for all $(x, y) \in (H \times H) \cap G^{(2)}$ and $z^{-1} \in H$ for all $z \in H$, and

$$H^{(2)} = (H \times H) \cap G^{(2)} \quad \text{and} \quad H^{(0)} = H \cap G^{(0)}.$$

The operations in H are inherited from G.

Definition 2.3. A groupoid homomorphism is a map $\varphi \colon G \to H$ between two groupoids G and H such that $(\varphi(x), \varphi(y)) \in H^{(2)}$ for all $(x, y) \in G^{(2)}$, and $\varphi(xy) = \varphi(x)\varphi(y)$.

Remark 2.4. A very good way to think of groupoids intuitively is in terms of categories, since groupoids are characterized as small categories in which all morphisms are invertible. The following diagram visualizing a groupoid expresses this perspective:
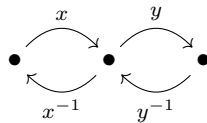
This point of view can be utilized in obtaining various algebraic expressions.

Definition 2.5. The set $G|_u = \{x \in G : r(x) = s(x) = u\}$ in a groupoid G with $u \in G^{(0)}$ is called the isotropy group at $u \in G^{(0)}$, and $u \in G^{(0)}$ has trivial isotropy if $G|_u = \{u\}$.

A groupoid G is principal if each $u \in G^{(0)}$ has trivial isotropy.

Example 2.6. Let $R \subseteq X \times X$ be (the graph of) an equivalence relation on some set X. Then R is a groupoid with

$$(x, y)^{-1} = (y, x) \quad \text{and} \quad (x, y)(y, z) = (x, z);$$
$$r(x, y) = x \quad \text{and} \quad s(x, y) = y.$$

We will always make the natural identification between $(x, x) \in R^{(0)}$ and $x \in X$ as above. A groupoid is principal if and only if it is (the graphs of) an equivalence relation on a set.

Definition 2.7. Let G be a groupoid and $U \subseteq G^{(0)}$.

(i) The set $G|_U = r^{-1}(U) \cap s^{-1}(U)$ is called the reduction of G by U.
(ii) The set U is called G-invariant if $r(s^{-1}(U)) \subseteq U$.

The set $\mathcal{O}_G(u) = r(s^{-1}(\{u\}))$ is called G-orbit of $u \in G^{(0)}$.

Remark 2.8. Let G be a groupoid and $U \subseteq G^{(0)}$.

(a) The reduction $H = G|_U$ of G by U is a subgroupoid of G with $H^{(0)} = U$.
(b) The set U is G-invariant if and only if $F = G^{(0)} \setminus U$ is G-invariant.

Definition 2.9. A groupoid G equipped with a topology is called a topological groupoid if the inversion map and the composition map are both continuous maps.

(i) A topological groupoid G is called étale if r and s are local homeomorphisms.
(ii) A subset $W \subseteq G$ is called a bisection if $r|_W$ and $s|_W$ are injective.

Remark 2.10. Let G be a locally compact Hausdorff groupoid and $U, F \subseteq G^{(0)}$.

(a) The sets $G^{(2)} \subseteq G \times G$ and $G^{(0)} \subseteq G$ are both closed.
(b) If G is étale, then $G^{(0)}$ is open, and $s^{-1}(\{u\})$, $r^{-1}(\{u\})$ are discrete for all $u \in G^{(0)}$.

If U is open and F is closed (e.g. $F = G^{(0)} \setminus U$), then $G|_U$ is open and $G|_F$ is closed, and $G|_U$ and $G|_F$ are locally compact Hausdorff groupoids, which are étale if G is étale.

Definition 2.11. A locally compact Hausdorff groupoid G is called topologically principal whenever the set which consists of all points in $G^{(0)}$ with trivial isotropy is dense in $G^{(0)}$, and G is minimal whenever there are no non-trivial open G-invariant subsets of $G^{(0)}$.

Remark 2.12. Any locally compact Hausdorff étale groupoid G is topologically principal if and only if any open G-invariant subset of $G^{(0)}$ contains a point with trivial isotropy, and G is minimal if and only if the G-orbit $\mathcal{O}_G(u)$ is dense in $G^{(0)}$ for all $u \in G^{(0)}$.

It follows from either of these two characterizations that G is topologically principal whenever G is minimal and $G^{(0)}$ contains at least one point $u \in G^{(0)}$ with trivial isotropy, because $G|_{r(x)} = \{xyx^{-1} : y \in G|_{s(x)}\}$ for any $x \in G$.

Definition 2.13. Let R and S be principal locally compact Hausdorff étale groupoids. Example 2.6 now enable us to view $R \subseteq X \times X$ and $S \subseteq Y \times Y$ as equivalence relations on locally compact Hausdorff spaces $X = R^{(0)}$ and $Y = S^{(0)}$.

(i) R and S are orbit equivalent if $h \colon X \to Y$ is a homeomorphism with $h \times h(R) = S$.

(ii) R and S are isomorphic if $h \times h \colon R \to S$ is a homeomorphism.

Example 2.14. We define the transformation (group) groupoid introduced in [43], [15], [5]. Let X be a locally compact Hausdorff space and $\varphi \colon X \to X$ a continuous map. Define

$$\Gamma_\varphi = \{(x, k, y) \in X \times \mathbb{Z} \times X : k = n - m, \varphi^n(x) = \varphi^m(y) \text{ for some } n, m \in \mathbb{N}\}$$

The set Γ_φ is a groupoid with

$$(x, k, y)^{-1} = (y, -k, x) \quad \text{and} \quad (x, k, y)(y, l, z) = (x, k + l, z);$$
$$r(x, k, y) = x \quad \text{and} \quad s(x, k, y) = y.$$

We will describe a topology on Γ_φ, whenever φ is a surjective local homeomorphism. Assume $\varphi^n|_U \colon U \to W$, $\varphi^m|_V \colon V \to W$ are locally defined homeomorphisms. Define

$$\Omega_{(n,m)}(U, V) = \{(z, n - m, (\varphi^m|_V)^{-1} \circ \varphi^n(z)) : z \in U\}$$

Then Γ_φ is a locally compact Hausdorff étale groupoid with a basis consisting of all $\Omega_{(n,m)}(U, V)$ where $\varphi^n|_U \colon U \to W$, $\varphi^m|_V \colon V \to W$ are locally defined homeomorphisms.

There is a distinguished subgroupoid $R_\varphi = \{(x, k, y) \in \Gamma_\varphi : k = 0\}$ which is principal. In fact $R_\varphi = \bigcup_{n \in \mathbb{N}} R_\varphi(n)$ is (the graph of) an equivalence relation on X, where

$$R_\varphi(n) \simeq \{(x, y) \in X \times X : \varphi^n(x) = \varphi^n(y)\}$$

It can be shown that $\Gamma_\varphi = \bigsqcup_{k \in \mathbb{Z}} \Gamma_\varphi(k)$ actually has the disjoint union topology where $\Gamma_\varphi(k) = \bigcup_{m \in \mathbb{N} \cap (\mathbb{N} - k)} \Gamma_\varphi(k, m)$ has the inductive limit topology and

$$\Gamma_\varphi(k, m) = \{(x, k, y) \in X \times \mathbb{Z} \times X : \varphi^{m+k}(x) = \varphi^m(y)\} \subseteq X \times \mathbb{Z} \times X.$$

In particular $\Gamma_\varphi(k, m)$ is open in $\Gamma_\varphi(k + 1, m)$, and $\Gamma_\varphi(k)$ is open in Γ_φ, and

$$R_\varphi = \Gamma_\varphi(0) \quad \text{and} \quad R_\varphi(n) = \Gamma_\varphi(0, n).$$

The following is from [49].

Example 2.15. Let X be a locally compact Hausdorff space, $\varphi \colon X \to X$ a continuous map, and $\mathcal{P}$ a collection of locally defined homeomorphisms $\eta \colon U \to V$ between open sets in X. Assume that $\mathcal{P}$ is a pseudo-group on X, i.e.

(a) $\eta^{-1} \in \mathcal{P}$, $\eta^{-1} \colon V \to U$ for any $\eta \in \mathcal{P}$, $\eta \colon U \to V$.

(b) $\mathrm{id}\,|_U \in \mathcal{P}$, $\mathrm{id}\,|_U \colon U \to U$ for any open subset U of X.

(c) $\eta' \circ \eta \in \mathcal{P}$, $\eta' \circ \eta \colon U \to \eta'(V)$ for any $\eta, \eta' \in \mathcal{P}$, $\eta \colon U \to V$, $\eta' \colon U' \to V'$ with $V \subseteq U'$.

It follows that $\eta|_W \in \mathcal{P}$, $\eta|_W \colon W \to \eta(W)$ for any $\eta \in \mathcal{P}$, $\eta \colon U \to V$ and $W \subseteq U$ open, and if $\eta, \eta' \in \mathcal{P}$, $\eta \colon U \to V$, $\eta' \colon U' \to V'$ then $\eta' \circ \eta \in \mathcal{P}$, $\eta' \circ \eta \colon U \cap \eta^{-1}(V \cap U') \to \eta'(V \cap U')$.

The germ of $\eta \in \mathcal{P}$, $\eta \colon U \to V$ at $x \in U$ is the equivalence class $[\eta]_x$ of all $\eta' \in \mathcal{P}$, $\eta' \colon U' \to V'$ with $x \in U'$ such that $\eta|_W = \eta'|_W$ for some neighbourhood W of $x \in U \cap U'$.

The collection $\mathcal{T}_k(\varphi)$ consist of all $\eta \in \mathcal{P}$, $\eta \colon U \to V$ for which there exists $n, m \in \mathbb{N}$ with $k = n - m$ such that $\varphi^n(z) = \varphi^m(\eta(z))$ for all $z \in U$, and $\mathcal{T}(\varphi) = \bigcup_{k \in \mathbb{Z}} \mathcal{T}_k(\varphi)$ is called the set of local transfers for φ. Define

$$\mathcal{G}_\varphi(\mathcal{P}) = \{(x, k, \eta, y) \in X \times \mathbb{Z} \times \mathcal{P} \times X : \eta \in \mathcal{T}_k(\varphi), \eta(x) = y\}.$$

Define the equivalence relation on $\Gamma_\varphi(\mathcal{P})$ with $(x, k, \eta, y) \sim (x', k', \eta', y')$ if and only if $x = x'$, $y = y'$, $k = k'$, and $[\eta]_x = [\eta]_x$. The equivalence class of $(x, k, \eta, y) \in \mathcal{G}_\varphi(\mathcal{P})$ is denoted $[x, k, \eta, y]$, and $\Gamma_\varphi(\mathcal{P}) = \mathcal{G}_\varphi(\mathcal{P})/\!\sim$.

The set $\Gamma_\varphi(\mathcal{P})$ is a groupoid with

$$[x, k, \eta, y]^{-1} = [y, -k, \eta^{-1}, x] \quad \text{and} \quad [x, k, \eta, y][y, l, \zeta, z] = [x, k + l, \zeta \circ \eta, z];$$
$$r([x, k, \eta, y]) = [x, 0, \mathrm{id}_U, x] \quad \text{and} \quad s([x, k, \eta, y]) = [y, 0, \mathrm{id}_V, y].$$

The set $\Gamma_\varphi(\mathcal{P})$ is a topological groupoid with a base for the topology given by

$$\Omega_k(\eta) = \{[z, k, \eta, \eta(z)] : z \in U\}$$

where $k \in \mathbb{Z}$ and $\eta \in \mathcal{T}_k(\varphi)$, $\eta \colon U \to V$, and $\Gamma_\varphi(\mathcal{P})$ is étale.

Also $\Gamma_\varphi(\mathcal{P})$ is locally compact Hausdorff if $[\eta]_x = [\zeta]_x$ for $\eta, \zeta \in \mathcal{T}_k(\varphi)$ with $\eta(x) = \zeta(x)$.

Definition 2.16. A locally compact Hausdorff étale groupoid G is locally contracting when we have that for any non-empty open set $U \subseteq G^{(0)}$ there exists an open set $V \subseteq U$ and an open bisection $W \subseteq G$ such that $\overline{V} \subseteq r(W)$ and $s((r|_{W^{-1}})^{-1}(\overline{V})) \subsetneq V$.

Remark 2.17. We will now briefly describe the classic construction of $C_r^*(G)$ and $C^*(G)$ for a locally compact Hausdorff étale groupoid G. This was originally done in [43].

The space $C_c(G)$ of compactly supported continuous complex-valued functions on G is a $*$-algebra with multiplication and involution given by

$$fg(z) = \sum_{xy=z} f(x)g(y) \quad \text{and} \quad f^*(z) = \overline{f(z^{-1})}.$$

Define the universal norm and the reduced norm on $C_c(G)$ by

$$\|f\|_u = \sup \|\pi(f)\| \quad \text{and} \quad \|f\|_r = \sup_{u \in G^{(0)}} \|\pi_u(f)\|$$

where the first supremum above is taken over all $*$-representations π on Hilbert spaces, and $\pi_u \colon C_c(G) \to B(\ell^2(s^{-1}(\{u\})))$ is the $*$-representation given by

$$\pi_x(f)\xi(z) = \sum_{xy=z} f(x)\xi(y).$$

The statements made explicitly or implicitly above are not trivial but will not be discussed. This can also be said for the statement that $\|f\|_u = \|f\|_r = \|f\|_\infty$ for all $f \in C_c(G^{(0)})$.

Definition 2.18. Let G be a locally compact Hausdorff étale groupoid.

(i) The C*-algebra $C^*(G)$ is the completion of $C_c(G)$ with respect to the norm $\|\cdot\|_u$ and it is called the full groupoid C*-algebra of G.

(ii) The C*-algebra $C_r^*(G)$ is the completion of $C_c(G)$ with respect to the norm $\|\cdot\|_r$ and it is called the reduced groupoid C*-algebra of G.

Proposition 2.19. *Let G be a locally compact Hausdorff étale groupoid.*

(i) *If $G^{(0)}$ is compact, then $C_r^*(G)$ is unital.*

(ii) *If G is second countable, then $C_r^*(G)$ is separable.*

Lemma 2.20. *Let G be a locally compact Hausdorff étale groupoid.*

(i) *If H is an open subgroupoid of G, then the naturally given embedding $C_c(H) \to C_c(G)$ extends to an injective $*$-homomorphism $\iota\colon C_r^*(H) \to C_r^*(G)$.*

(ii) *If H is a closed subgroupoid of G, then the naturally given restriction $C_c(G) \to C_c(H)$ extends to a surjective positive linear map $\pi\colon C_r^*(G) \to C_r^*(H)$.*

Remark 2.21. We make some comments about Lemma 2.20:

(a) If $H^{(0)} = G^{(0)}$ and is compact, then the extensions in (i) and (ii) are unital.

(b) If H is open and closed then the extension in (ii) is a conditional expectation.

Denote by $P^H\colon C_r^*(G) \to C_r^*(H)$ the conditional expectation of $C_r^*(G)$ onto $C_r^*(H)$ in (b). We note $G^{(0)}$ is an open and closed subgroupoid of G and $\|f\|_r = \|f\|_\infty$ for all $f \in C_c(G^{(0)})$. Lemma 2.20(i) consequently provides an isometric $*$-homomorphism $\iota\colon C_0(G^{(0)}) \to C_r^*(G)$. Lemma 2.20(ii) gives a conditional expectation $P = \pi\colon C_r^*(G) \to C_0(G^{(0)})$ which is faithful. The conditional expectation $P\colon C_r^*(G) \to C_0(G^{(0)})$ is unique if G is topologically principal.

The following two results follow from [4] and [60] and they both concern amenability of second countable locally compact Hausdorff étale groupoids, which we will not define now. We settle for the characterization of amenability given in Proposition 2.23.

Proposition 2.22. *Let G be a second countable locally compact Hausdorff étale groupoid. If G is amenable, then $C_r^*(G) \simeq C^*(G)$ and satisfies the universal coefficient theorem.*

Proposition 2.23. *Let G be a second countable locally compact Hausdorff étale groupoid. Then G is amenable if and only if $C_r^*(G)$ is nuclear (and so if and only if $C^*(G)$ is nuclear).*

The following result is Proposition 2.4 in [5].

Proposition 2.24. *Let G be a second countable locally compact Hausdorff étale groupoid. Assume G is topologically principal and locally contractive, then $C_r^*(G)$ has the property that any non-zero heriditary C*-subalgebra of $C_r^*(G)$ contains an infinite projection.*

Lemma 2.25. *Let G be a locally compact Hausdorff étale groupoid and $U \subseteq G^{(0)}$ is open. Assume that U is G-invariant and $F = G^{(0)} \setminus U$.*

(i) *The natural embedding $C_c(G|_U) \to C_c(G)$ extends to an injective $*$-homomorphism $\iota\colon C_r^*(G|_U) \to C_r^*(G)$ and $\iota(C_r^*(G|_U))$ is an ideal in $C_r^*(G)$.*

(ii) *The natural restriction $C_c(G) \to C_c(G|_F)$ extends to a surjective $*$-homomorphism $\pi\colon C_r^*(G) \to C_r^*(G|_F)$ and $\iota(C_r^*(G|_U)) \subseteq \ker \pi$.*

Remark 2.26. We have that $\iota(C_c(G|_U)) = \{f \in C_c(G) : f|_{G|_F} = 0\}$ in Lemma 2.25.

The association $\mathcal{O}(G) \ni U \mapsto C_r^*(G|_U) \in \mathcal{I}(C_r^*(G))$ is an order preserving injection between the lattice of open G-invariant subsets of $G^{(0)}$ and the lattice of ideals in $C_r^*(G)$.

Lemma 2.27. *Let G be a locally compact Hausdorff étale groupoid which is also principal. The association $\mathcal{O}(G) \ni U \mapsto C_r^*(G|_U) \in \mathcal{I}(C_r^*(G))$ is an order preserving bijection between the lattice of open G-invariant subsets of $G^{(0)}$ and the lattice of ideals in $C_r^*(G)$.*

Lemma 2.28. *Let G be a topologically principal locally compact Hausdorff étale groupoid. Then $C_r^*(G)$ is simple if and only if G is minimal.*

Remark 2.29. A locally compact Hausdorff étale groupoid G is minimal if $C_r^*(G)$ is simple, and G need not be topologically principal for this implication to hold (cf. Lemma 2.25), and conversely $C_r^*(G)$ is simple if G is minimal and there is some point with trivial isotropy, which is a consequence of Lemma 2.28 and Remark 2.12, which is Corollary 2.18 in [54].

Proposition 2.30. *Let G be a second countable locally compact Hausdorff étale groupoid. Assume G is amenable, and that U is an open G-invariant subset of $G^{(0)}$ and $F = G^{(0)} \setminus U$. Then $C_r^*(G|_F) \simeq C_r^*(G)/\iota(C_r^*(G|_U))$ and we have the short exact sequence*

$$0 \longrightarrow C_r^*(G|_U) \overset{\iota}{\longrightarrow} C_r^*(G) \overset{\pi}{\longrightarrow} C_r^*(G|_F) \longrightarrow 0 \ .$$

Remark 2.31. Let G be a locally compact Hausdorff étale groupoid and consider $c \colon G \to \mathbb{Z}$ a continuous groupoid homomorphism, i.e. $c(xy) = c(x) + c(y)$ for all $x, y \in G$.

Define $\sigma^c \colon \mathbb{R} \to \mathrm{Aut}(C_c(G))$ on $C_c(G)$ by $t \mapsto \sigma_t$ where for each $f \in C_c(G)$ and $x \in G$,

$$\sigma_t(f)(x) = e^{2\pi i t c(x)} f(x).$$

The one-parameter automorphism group $\sigma^c \colon \mathbb{R} \to \mathrm{Aut}(C_c(G))$ on $C_c(G)$ extends to a strongly continuous one-parameter automorphism group $\sigma^c \colon \mathbb{R} \to \mathrm{Aut}(C_r^*(G))$ on $C_r^*(G)$. As $\sigma = \sigma^c$ is periodic it can be regarded as an action of either $\mathbb{R}$ or $\mathbb{T}$ called the gauge action. Define a faithful conditional expectation $Q \colon C_r^*(G) \to C_r^*(G)^\sigma$ of $C_r^*(G)$ onto $C_r^*(G)^\sigma$ by $Q(a) = \int_0^1 \sigma_t(a)\, dt$ for $a \in C_r^*(G)$ where $C_r^*(G)^\sigma = \{a \in C_r^*(G) : \sigma_t(a) = a \text{ for all } t \in \mathbb{R}\}$.

Proposition 2.32. *Let $\sigma = \sigma^c \colon \mathbb{R} \to \mathrm{Aut}(C_r^*(G))$ be as above, and $H = c^{-1}(\{0\}) \subseteq G$. Then $Q|_{C_c(G)} = P^H|_{C_c(G)}$ and $C_r^*(G)^\sigma \simeq C_r^*(H)$, and G is amenable if H is amenable.*

The C*-algebra $C_r^*(G)^\sigma$ is called the core of $C_r^*(G)$.

Remark 2.33. Let G be a locally compact Hausdorff étale groupoid, μ a measure on $G^{(0)}$. A measure μ on $G^{(0)}$ is G-invariant if $\mu(r(W)) = \mu(s(W))$ for any open bisection $W \subseteq G$. Assume μ is a regular Borel probability measure.

Define a state ω on $C_r^*(G)$ such that for all $a \in C_r^*(G)$,

$$\omega(a) = \int_{G^{(0)}} P(a)\, d\mu \tag{2.1}$$

Proposition 2.34. *Let μ be a regular Borel probability measure on $G^{(0)}$ just as above. Then the state ω on $C_r^*(G)$ given by (2.1) is tracial if and only if μ is G-invariant.*

Proposition 2.35. *Let G be a locally compact Hausdorff étale groupoid that is principal. Then any tracial state ω on $C_r^*(G)$ is given by (2.1) for some G-invariant regular Borel probability measure μ on $G^{(0)}$ as in Remark 2.33 and Proposition 2.34.*

Remark 2.36. There is a number of results which hold true both for $C_r^*(G)$ and $C^*(G)$, e.g. Lemma 2.20, Lemma 2.25, Proposition 2.32 but sometimes amendments are needed, e.g. Remark 2.21 where the last statements about $P\colon C^*(G) \to C_0(G^{(0)})$ are not true and Proposition 2.30 which is true without the assumption of amenability on G provided that $C^*(G)$, $C^*(G|_U)$ and $C^*(G|_F)$ takes the place of $C_r^*(G)$, $C_r^*(G|_U)$ and $C_r^*(G|_F)$.

Part II

The core content

Topological dynamics of a circle map

We consider the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ equipped with positive orientation. Furthermore we will make the identifications $\mathbb{T} \cong \mathbb{R}/\mathbb{Z} \cong [0,1]/\sim$, where $x \sim y$ for $x, y \in \mathbb{R}$ if and only if $x \equiv y \pmod 1$, i.e. $x - y \in \mathbb{Z}$; we will use these identifications throughout. We occasionally refer to $\pi \colon \mathbb{R} \to \mathbb{T}$ given by $\pi(t) = e^{2\pi i t}$ for all $t \in \mathbb{R}$ as the quotient map. An interval $I \subseteq \mathbb{T}$ is the image of a corresponding interval $J \subseteq \mathbb{R}$ under the quotient map. Let λ denote both the Lebesgue measure on $\mathbb{R}$ and also (abusing notation slightly) the normalized Lebesgue measure on $\mathbb{T}$ (the pushforward measure under the quotient map).

For any continuous circle map $\varphi \colon \mathbb{T} \to \mathbb{T}$ there is a continuous map $\Phi \colon \mathbb{R} \to \mathbb{R}$ with $\varphi(e^{2\pi i t}) = e^{2\pi i \Phi(t)}$ for all $t \in \mathbb{R}$ which is essentially unique in the sense that if $\Psi \colon \mathbb{R} \to \mathbb{R}$ is continuous then $\varphi(e^{2\pi i t}) = e^{2\pi i \Psi(t)}$ for all $t \in \mathbb{R}$ if and only if $\Psi = \Phi + K$ for some $K \in \mathbb{Z}$. The map Φ is called the lift for φ and it is completely determined by the values on $[0,1]$, so sometimes we might as well consider the restricted lift $\Phi|_{[0,1]}$ instead of the lift Φ itself. Alternatively we may represent $\varphi \colon \mathbb{T} \to \mathbb{T}$ and $\Phi \colon \mathbb{R} \to \mathbb{R}$ via $\Upsilon \colon [0,1) \to [0,1)$ given by $\Upsilon(t) = \{\Phi(t)\}$ where $\{s\} = s - \lfloor s \rfloor \in [0,1)$ is the fractional part of $s \in \mathbb{R}$.
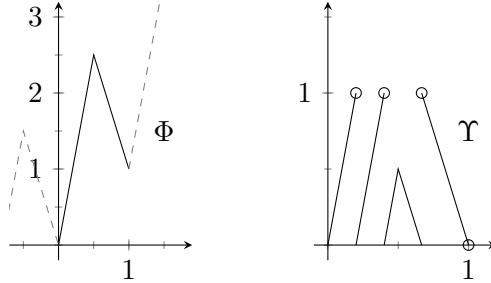
We have that $\Phi(1) - \Phi(0) = \Phi(t+1) - \Phi(t) \in \mathbb{Z}$ for all $t \in \mathbb{R}$ as the lift Φ is continuous, and $\deg(\varphi) = \Phi(1) - \Phi(0)$ is called the degree of φ, and so for any $t \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$\Phi(t + k) = \Phi(t) + k \deg(\varphi).$$

Lemma 3.1. *Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ and $h \colon \mathbb{T} \to \mathbb{T}$ be continuous circle maps with $\deg(h) \neq 0$. Then $\deg(\varphi \circ \psi) = \deg(\varphi) \deg(\psi)$; $\deg(\varphi) = \deg(\psi)$ if $h \circ \varphi = \psi \circ h$; and $\deg(\varphi^n) = \deg(\varphi)^n$.*

Proof. Let $\Phi \colon \mathbb{R} \to \mathbb{R}$ and $\Psi \colon \mathbb{R} \to \mathbb{R}$ be lifts for $\varphi \colon \mathbb{T} \to \mathbb{T}$ and $\psi \colon \mathbb{T} \to \mathbb{T}$ respectively. Then $\Phi \circ \Psi \colon \mathbb{R} \to \mathbb{R}$ is a lift for $\varphi \circ \psi \colon \mathbb{T} \to \mathbb{T}$, and

$$\Phi \circ \Psi(1) - \Phi \circ \Psi(0) = \Phi(\Psi(0) + \deg(\psi)) - \Phi(\Psi(0)) = \deg(\varphi) \deg(\psi).$$

Therefore $\deg(\varphi \circ \psi) = \deg(\varphi) \deg(\psi)$ and the other properties follows from this. $\qquad\square$

The following result is Theorem 6.7 in [12].

Theorem 3.2. *Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ be continuous circle maps with $\deg(\varphi) > 0$, $\deg(\psi) > 1$. Then φ is not semiconjugate to ψ when $\deg(\varphi) \neq \deg(\psi)$.*

Remark 3.3. Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ be continuous circle maps, $\deg(\varphi) \neq 0$, $\deg(\psi) \notin \{-1, 0, 1\}$. Then φ is not semiconjugate to ψ when $\deg(\varphi) \neq \deg(\psi)$, since $\deg(\varphi^2) > 0$, $\deg(\psi^2) > 1$. Any semiconjugacy from φ to ψ is also a semiconjugacy from φ^n to ψ^n for any $n \in \mathbb{N}$.

Corollary 3.4. *Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ be continuous circle maps, $\deg(\varphi), \deg(\psi) \notin \{-1, 0, 1\}$. Then there is no semiconjugacy between φ and ψ.*

Definition 3.5. Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map with lift $\Phi \colon \mathbb{R} \to \mathbb{R}$.

(i) We say φ is (strictly) increasing respectively decreasing on an interval $I = \pi(J) \subseteq \mathbb{T}$ with $\lambda(I) \leq 1$ whenever Φ is (strictly) increasing respectively decreasing on $J \subseteq \mathbb{R}$.

(ii) We say φ is piecewise (strictly) monotone if there exists $0 = c_0 < c_1 < \cdots < c_N = 1$ such that Φ is either (strictly) increasing or (strictly) decreasing on each (c_{i-1}, c_i).

We will always by piecewise monotone mean piecewise *strictly* monotone.

(iii) We say φ is (uniformly) piecewise linear if there exists $0 = c_0 < c_1 < \cdots < c_N = 1$ such that Φ is linear on each (c_{i-1}, c_i) (with constant absolute value of the slopes).

We will only consider piecewise linear maps which are *not locally constant*.

Example 3.6. The simplest continuous circle maps $\varphi_\alpha \colon \mathbb{T} \to \mathbb{T}$ are rotations given by $\varphi_\alpha(x) = e^{2\pi i \alpha} x$ for all $x \in \mathbb{T}$ with $\alpha \in \mathbb{Q}$ or $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $\Phi_\alpha(t) = t + \alpha$ for all $t \in \mathbb{R}$. The continuous circle maps $\psi_\beta \colon \mathbb{T} \to \mathbb{T}$ called tent maps of slope $2\beta \in \mathbb{R}^+$ are given by $\psi_\beta(\pi(t + n)) = \pi(\Psi_\beta(t + n))$ and $\Psi_\beta(t + n) = \beta(1 - 2|t - \frac{1}{2}|)$ for all $t \in [0, 1]$ and $n \in \mathbb{Z}$. The two local homeomorphisms $\lambda_n \colon \mathbb{T} \to \mathbb{T}$ and $\lambda_n^* \colon \mathbb{T} \to \mathbb{T}$ where $n \in \mathbb{N}$ are given by $\lambda_n(x) = x^n$ and $\lambda_n^*(x) = (\overline{x})^n$ for all $x \in \mathbb{T}$ with $\Lambda_n(t) = nt$ and $\Lambda_n^*(t) = -nt$ for all $t \in \mathbb{R}$. In the following illustration $\alpha = \frac{\sqrt{5}-1}{2}$ and $\beta = n = 2$:
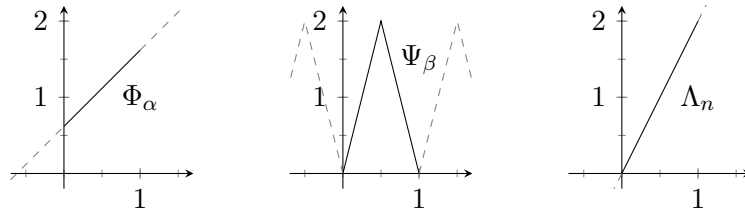
The two maps φ_α and ψ_β are uniformly piecewise linear with slope 1 and 2β respectively. The two maps λ_n and λ_n^* are incidentally both uniformly piecewise linear with slope $n \in \mathbb{N}$. Also $\deg(\varphi_\alpha) = 1$, $\deg(\psi_\beta) = 0$ and $\deg(\lambda_n) = n$, $\deg(\lambda_n^*) = -n$.

For any continuous circle map $\varphi \colon \mathbb{T} \to \mathbb{T}$ which is piecewise monotone map, consider

$$\mathcal{V} = \{(+,+), (-,-), (+,-), (-,+)\} \quad \text{and} \quad \mathcal{W} = \{(+,+), (-,-)\}.$$

We define the valency $\mathrm{val}(\varphi, x) \in \mathcal{V}$ of φ at $x \in \mathbb{T}$ by

- $\mathrm{val}(\varphi, x) = (+, +)$ if φ is strictly increasing on an open neighbourhood of x.
- $\mathrm{val}(\varphi, x) = (-, -)$ if φ is strictly decreasing on an open neighbourhood of x.
- $\mathrm{val}(\varphi, x) = (+, -)$ if φ has a strict local maximum at x.
- $\mathrm{val}(\varphi, x) = (-, +)$ if φ has a strict local minimum at x.

The following figure illustrate the valencies $(-, -)$, $(+, +)$, $(+, -)$, $(-, +)$:
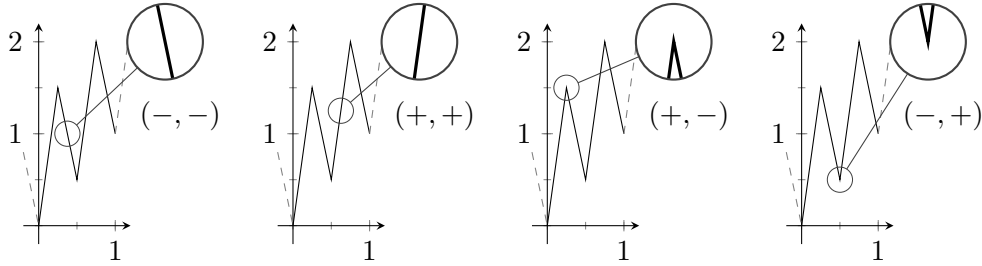
The set of valencies $\mathcal{V}$ is equipped with a composition turning it into a monoid (Table 3.1).

$v \bullet w$	$w = (+, +)$	$w = (+, -)$	$w = (-, +)$	$w = (-, -)$
$v = (+, +)$	$(+, +)$	$(+, -)$	$(-, +)$	$(-, -)$
$v = (+, -)$	$(+, -)$	$(+, -)$	$(+, -)$	$(+, -)$
$v = (-, +)$	$(-, +)$	$(-, +)$	$(-, +)$	$(-, +)$
$v = (-, -)$	$(-, -)$	$(-, +)$	$(+, -)$	$(+, +)$

Table 3.1: The composition table for $\bullet$

This composition of valencies is compatible with composition of piecewise monotone maps, in the sense that the valency of a composition of piecewise monotone maps decomposes into a composition of valencies in the following way:

Lemma 3.7. *Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ be continuous piecewise monotone circle maps and $x \in \mathbb{T}$. Then the following composition rule applies:*

$$\mathrm{val}(\psi \circ \varphi, x) = \mathrm{val}(\psi, \varphi(x)) \bullet \mathrm{val}(\varphi, x)$$

Proof. This is an elementary and easy case-by-case verification. $\square$

In particular for all $n, m \in \mathbb{N}$ with $n > m$,

$$\mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi, \varphi^{n-1}(x)) \bullet \mathrm{val}(\varphi, \varphi^{n-2}(x)) \bullet \cdots \bullet \mathrm{val}(\varphi, x)$$
$$= \mathrm{val}(\varphi^{n-m}, \varphi^m(x)) \bullet \mathrm{val}(\varphi^m, x).$$

All (or almost all) questions related to valency in the following will be resolved by referring to the table and the corresponding composition rule either directly or indirectly.

It is convenient to note that $\mathrm{val}(\varphi^0, x) = (+, +)$ for any $x \in \mathbb{T}$.

Remark 3.8. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone. The following is a dense compilation of notation and terminology and some observations:

Denote by $\mathcal{C}_n = \{x \in \mathbb{T} : \mathrm{val}(\varphi^n, x) \in \{(+,-),(-,+)\}\}$ the critical points for φ^n, and in particular $\mathcal{C} = \mathcal{C}_1$ denotes the critical points for φ, and observe that $\mathcal{C}_n = \bigcup_{l=0}^{n-1} \varphi^{-l}(\mathcal{C})$. Denote by $\mathcal{C}_\infty = \bigcup_{n=1}^{\infty} \mathcal{C}_n = \bigcup_{l=0}^{\infty} \varphi^{-l}(\mathcal{C})$ the set of pre-critical points and note that

$$\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots \subseteq \mathcal{C}_\infty.$$

Also $\mathcal{D}^1 = \bigcup_{l=1}^{\infty} \varphi^l(\mathcal{C})$ is the set of post-critical points and $\varphi(\mathcal{C})$ is the set of critical values. In particular the critical points are always pre-critical but they are not always post-critical. In general we will use the notation $\mathcal{D}^n = \bigcup_{l=n}^{\infty} \varphi^l(\mathcal{C})$ for any $n \in \mathbb{Z}$ and $\mathcal{D}^\infty = \bigcup_{l \in \mathbb{Z}} \varphi^l(\mathcal{C})$. It is also worth noting that $\varphi^n(\mathcal{C}_n) \subseteq \varphi^{n+1}(\mathcal{C}_{n+1})$, because $\varphi^{-1}(\mathcal{C}_n) = \bigcup_{l=1}^{n} \varphi^{-l}(\mathcal{C}) \subseteq \mathcal{C}_{n+1}$. In fact $\varphi^n(\mathcal{C}_n) = \bigcup_{l=0}^{n-1} \varphi^{n-l}(\mathcal{C})$ when φ is surjective, and then $\bigcup_{l=1}^{\infty} \varphi^l(\mathcal{C}) = \bigcup_{n=l}^{\infty} \varphi^n(\mathcal{C}_n)$. If $\varphi(\mathcal{C}) \subseteq \mathcal{C}$ then $\varphi(\mathcal{C}_{n+1}) \subseteq \mathcal{C}_n$ and so $\varphi^{n+1}(\mathcal{C}_{n+1}) \subseteq \varphi^n(\mathcal{C}_n)$ in particular $\varphi^n(\mathcal{C}_n) = \varphi(\mathcal{C})$. Also $\mathcal{C}_\infty$ and $\mathcal{D}^1$ are always both countable, and the set of pre-periodic points is countable when φ is not only piecewise monotone but uniformly piecewise linear with slope $s > 1$. We say that φ is (post-)critically finite whenever all the critical points are pre-periodic, i.e. the number of post-critical points is finite.

We use the notation $\mathcal{C}^\varphi = \mathcal{C}$ if there are any ambiguity.

The following result from [50] will be relevant later when relating to Example 2.15.

Lemma 3.9. *Let $\varphi, \psi\colon \mathbb{T} \to \mathbb{T}$ be continuous circle maps which are piecewise monotone. Then $\varphi(x) = \psi(y)$ and $\mathrm{val}(\varphi, x) = \mathrm{val}(\psi, y)$ for $x, y \in \mathbb{T}$ if and only if there exists an orientation-preserving locally defined homeomorphism $\eta\colon U \to V$ of $\mathbb{T}$ (see Example 2.15) with $x \in U \subseteq \mathbb{T}$ and $y \in V \subseteq \mathbb{T}$ such that $\eta(x) = y$ and $\varphi(z) = \psi(\eta(z))$ for all $z \in U$.*

Moreover if this is the case, then the germ of η at x is independent of the choice of η.

Proof. This is an elementary and easy case-by-case verification. $\qquad\square$

Remark 3.10. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map. Then

(a) φ is locally injective if and only if φ is a local homeomorphism,

(b) φ is a local homeomorphism if and only if φ is a covering map.

3.1 An exposition of transitivity for circle maps

We are now going to describe what it means for a continuous circle map to be transitive, as transitivity and related properties are fundamental in dealing with dynamical systems and will play a crucial role later on.

Lemma 3.11. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map.*

(i) *If φ is piecewise monotone and transitive, then φ is (very) strongly transitive.*

(ii) *If φ is transitive without periodic points, then φ is conjugate to an irrational rotation.*

(iii) *If φ is transitive with periodic points, then the set of periodic points is dense in $\mathbb{T}$.*

In particular we have φ is minimal if and only if φ is conjugate to an irrational rotation. In addition φ is transitive if and only if φ is point transitive.

Proof. The reader will find (i) follows from Corollary 4.2 in [63], (ii) is Corollary 2 in [7], (iii) is Corollary 3.4 in [13], and the last follows from Lemma 3.11(ii) and Remark 1.9. $\square$

Remark 3.12. The following classic result goes back to the work of Poincaré (cf. e.g. [33]): Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a homeomorphism without periodic points (irrational rotation number), then there exists an orientation-preserving continuous surjective circle map $h \colon \mathbb{T} \to \mathbb{T}$ (being essentially unique) such that $h \circ \varphi = \psi \circ h$ with $\psi \colon \mathbb{T} \to \mathbb{T}$ an irrational rotation, and the map $h \colon \mathbb{T} \to \mathbb{T}$ is injective (i.e. a homeomorphism) if and only if φ is transitive. The result is proved in Theorem 11.2.7 in [22] and Theorem 3.1, Corollary 3.2 in [42].

We also recommend the reader to see [23], [26], [27] along with Theorem 5.10 in [61]: Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a homeomorphism. Then φ is transitive if and only if φ is minimal.

As a common outcome of both Lemma 3.11(ii) and Remark 3.12 we obtain the result that any circle homeomorphism without periodic points is conjugate to an irrational rotation. Remark 3.12 highlights that the conjugation map can be chosen to be orientation-preserving, which in turn also applies to Lemma 3.11(ii), which is not confined to homeomorphisms. This common outcome can also be seen as a special case of the following result:

Lemma 3.13. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone. If φ is transitive, then there exists an orientation-preserving homeomorphism $h \colon \mathbb{T} \to \mathbb{T}$ such that $\psi = h \circ \varphi \circ h^{-1}$ is a uniformly piecewise linear circle map with slope $s \geq 1$.*

Proof. The statement follows from a slight variation of the proof of Theorem 4.1 in [50]. The proof in [50] relies on Proposition 3.6 and Proposition 4.3 and Corollary 4.4 in [51]. The results in [51] in turn expands on Theorem 5 among other of the results in [33]. Alternatively the statement follows directly from a combination of Theorem 4.1 in [50], Theorem 4.4 in [6] and Poincaré's classic result, cf. Remark 3.12.

The result is also proved in [3] where it is placed into the context of entropy. □

Remark 3.14. Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a transitive local homeomorphism and $d = |\deg(\varphi)|$.

(a) If $d = 1$, then φ is conjugate to φ_α for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\deg(\varphi) = 1$.
As a consequence φ is totally minimal but not exact.

(b) If $d \geq 2$, then φ is conjugate to λ_n if $\deg(\varphi) > 0$ and λ_n^* if $\deg(\varphi) < 0$.
As a consequence φ is exact but not minimal.

These implications follow from Lemma 3.11, Remark 3.12 and Lemma 3.13.

The following two results follows from Theorem 3' in [28] and Theorem 9.5 in [24]:

Lemma 3.15. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be continuous uniformly piecewise linear with slope $s \geq 1$. Then the topological entropy of φ is determined as $h(\varphi) = \log(s)$.*

Lemma 3.16. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map which is additionally transitive. Then φ conjugate to an irrational rotation if and only if $h(\varphi) = 0$.*

Remark 3.17. The reader is encouraged to consult [3] for more on entropy for circle maps. In particular it is interesting to take note of Theorem 4.6.8 and Proposition 4.6.9 in [3].

The following two results are excerpts of Theorem C in [13]:

Lemma 3.18. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map (maybe not piecewise monotone). Then φ is totally transitive with periodic points if and only if φ is mixing.*

Lemma 3.19. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone. Then φ is totally transitive with periodic points if and only if φ is exact.*

The following result is a consequence of results in [50] and seems to have gone unnoticed:

Lemma 3.20. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone. If φ is transitive with $\deg(\varphi) \notin \{-1, 0, 1\}$, then φ is totally transitive with periodic points.*

Proof. This follows from Proposition 5.8, Theorem 5.21 in [50] and Theorem 4.4 in [6]. $\square$

The following concerns the situation, when a map is transitive but not totally transitive. This result is the marriage of Theorem 1.16 and Theorem 1.17.

Lemma 3.21. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map which is additionally transitive. If φ is not totally transitive, then there exists a periodic decomposition $\mathcal{X} = \{X_1, \ldots, X_N\}$ which consists of closed intervals such that $\varphi^N|_{X_n}$ is totally transitive for all $n = 1, \ldots, N$. If $\mathcal{E} = \{x \in \mathbb{T} : x \in X_n \cap X_m \text{ for some } n, m \in \{1, \ldots, N\}, n \neq m\}$, then $\varphi^{-1}(\mathcal{E}) \setminus \mathcal{C} = \mathcal{E}$.*

Proof. This follows from Theorem 2.3 in [63] and the more general Corollary 2.7 in [2]. $\square$

Remark 3.22. Lemma 3.21 provides directions for constructing continuous circle maps of various degrees which are either transitive but not totally transitive, or totally transitive. A forward invariant subset containing a non-empty open subset and with a complement also containing a non-empty open subset is helpful when looking for a way to construct surjective continuous circle maps of various degrees which are not transitive.

Example 3.23. We consider uniformly piecewise linear circle maps with periodic points:

(a) The following continuous circle maps are not transitive:
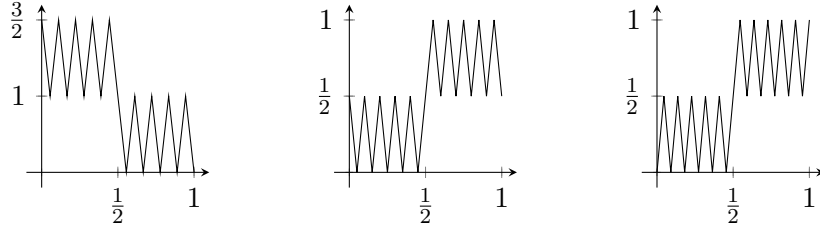
(b) The following continuous circle maps are transitive but not totally transitive:
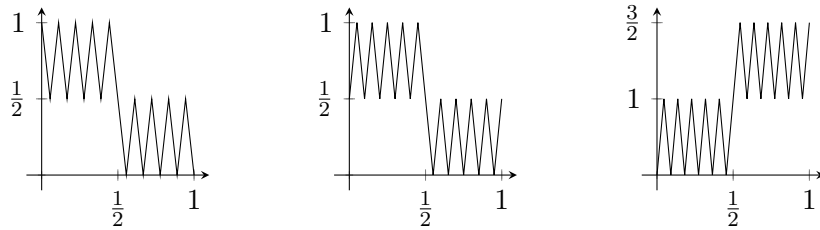
(c) The following continuous circle maps are totally transitive:

We also note φ_α is minimal if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, ψ_β is exact if $\beta \geq 1$, λ_n and λ_n^* are exact if $n \neq 1$, and non of the maps φ_α, ψ_β, λ_n and λ_n^* are transitive when $\alpha \in \mathbb{Q}$, $0 < \beta < 1$ and $n = 1$. The maps φ_α, ψ_β, λ_n and λ_n^* were introduced in Example 3.6.

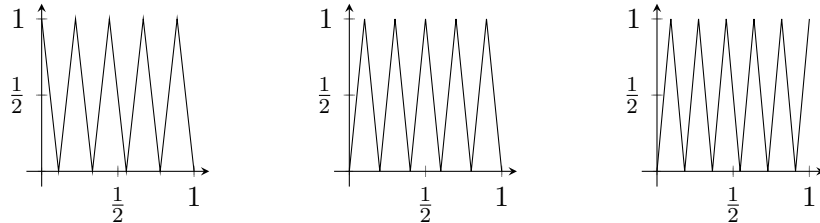*Remark 3.24.* Example 3.23 exhibits a wide variety of maps with properties worth noting. We observe that there are continuous surjective uniformly piecewise linear circle maps, some transitive, some not transitive, some of positive entropy, and some of entropy zero, and all possible combinations of these various properties actually occur in Example 3.23. It is also worth noting that the only occurrence of a transitive circle map of entropy zero is the irrational rotation as established by Lemma 3.16. Let $d \in \{-1, 0, 1\}$.

(a) There exists a continuous surjective uniformly piecewise linear circle map $\varphi\colon \mathbb{T} \to \mathbb{T}$ such that $\deg(\varphi) = d$ and $h(\varphi) > 0$ which is not transitive.

(b) There exists a continuous surjective uniformly piecewise linear circle map $\varphi\colon \mathbb{T} \to \mathbb{T}$ such that $\deg(\varphi) = d$ and $h(\varphi) > 0$ which is transitive but not totally transitive.

(c) There exists a continuous surjective uniformly piecewise linear circle map $\varphi\colon \mathbb{T} \to \mathbb{T}$ such that $\deg(\varphi) = d$ and $h(\varphi) > 0$ which is totally transitive.

It follows from Lemma 3.20 that any transitive piecewise monotone circle map $\varphi\colon \mathbb{T} \to \mathbb{T}$ with $\deg(\varphi) \notin \{-1, 0, 1\}$ is totally transitive with periodic points. Let $d \in \mathbb{Z} \setminus \{-1, 0, 1\}$.

(d) There exists a continuous surjective uniformly piecewise linear circle map $\varphi\colon \mathbb{T} \to \mathbb{T}$ such that $\deg(\varphi) = d$ and $h(\varphi) > 0$ which is not transitive.

The reader is referred to Theorem 9.6 and Theorem 9.7 in [24] for details pertaining to the entropy of transitive circle maps of various degrees.

The following concepts makes sense for any metric space (not just the circle):

Definition 3.25. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map. We say that

(i) φ has sensitive dependence when there exists $r > 0$ such that for all $x \in \mathbb{T}$ and $\varepsilon > 0$ there exists $y \in \mathbb{T}$ with $d(x, y) < \varepsilon$ and $d(\varphi^n(x), \varphi^n(y)) \geq r$ for some $n \in \mathbb{N}$.

(ii) φ is chaotic when φ is transitive and the set of periodic points are dense in $\mathbb{T}$.

A periodic point $x \in \mathbb{T}$ of period $n \in \mathbb{N}$ is repelling whenever there exists $r > 1$ and $\varepsilon > 0$ such that $d(x, \varphi^n(y)) \geq rd(x, y)$ for all $y \in \mathbb{T}$ with $d(x, y) < \varepsilon$.

The following holds for any metric space with more than one point (not just the circle):

Lemma 3.26. *The $\varphi\colon \mathbb{T} \to \mathbb{T}$ a continuous map. Then we have the following:*

(i) *If φ is mixing, then φ has sensitive dependence (on initial conditions).*

(ii) *If φ is chaotic, then φ has sensitive dependence (on initial conditions).*

Proof. These results are Theorem 7.2.12 and Theorem 7.2.14 in [21]. □

Remark 3.27. The above notion of chaos in a dynamical system is not generally accepted. A distinctly different but related notion of chaos is determined by having positive entropy. Lemma 3.11(iii) ensures any transitive continuous circle map with periodic points is chaotic.

Another specific result that follows from [50]:

Lemma 3.28. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be continuous uniformly piecewise linear with slope $s > 1$. Then every periodic point is repelling.*

3.2   A classification result for critically finite circle maps

The main results of this section are mostly inspired by results about interval maps in [11].

Remark 3.29. We briefly recapitulate the emergence of the invariant in [11]. Put $I = [0, 1]$. Let $\varphi, \psi \colon I \to I$ be continuous piecewise monotone interval maps which are critically finite. The set of homeomorphisms of the unit interval is particularly restricted in the sense that $h(0) = 0$ and $h(1) = 1$ or $h(0) = 1$ and $h(1) = 0$ for any given homeomorphism $h \colon I \to I$. We have $\varphi(d_i^\varphi) = d_{\tau_\varphi(i)}^\varphi$ for a unique map $\tau_\varphi \colon \{1, \ldots, N_\varphi\} \to \{1, \ldots, N_\varphi\}$ as $\varphi(\mathcal{D}^\varphi) \subseteq \mathcal{D}^\varphi$ where again $\mathcal{D}^\varphi = \bigcup_{n=0}^\infty \varphi^n(\mathcal{C}^\varphi) = \{d_1^\varphi, \ldots, d_{N_\varphi}^\varphi\}$ and $\mathcal{C}^\varphi$ is the set of critical points for φ, where $\mathcal{C}^\varphi$ includes the endpoints 0 and 1 and we have the ordering $0 = d_1^\varphi < \cdots < d_N^\varphi = 1$. Assume there exists a homeomorphism $h \colon I \to I$ such that $h \circ \varphi = \psi \circ h$. We have that

$$N = N_\varphi = N_\psi.$$

Also $N + 1 - \tau_\varphi(N + 1 - n) = \tau_\psi(n)$ for all $n \in \{1, \ldots, N\}$ when h is orientation-reversing, and $\tau_\varphi(n) = \tau_\psi(n)$ for all $n \in \{1, \ldots, N\}$ when h is orientation-preserving.

In other words $\tau \colon \varphi \mapsto \tau_\varphi$ is an invariant in this sense.

Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ be continuous piecewise monotone circle maps which are critically finite. The reason for restricting our attention to critically finite maps (besides the convenience) will become apparent in Theorem 4.43 but at present it is merely a matter of convenience. We will assume throughout this section that $\mathcal{C}^\varphi \neq \emptyset$ and $\mathcal{C}^\psi \neq \emptyset$.

Put $\mathcal{D}^\varphi = \bigcup_{n=0}^\infty \varphi^n(\mathcal{C}^\varphi) = \{d_1^\varphi, \ldots, d_{N_\varphi}^\varphi\} \subseteq \mathbb{T}$ where $\mathcal{C}^\varphi$ is the set critical points for φ, and $\mathcal{E}^\varphi = \pi^{-1}(\mathcal{D}^\varphi) = \{e_{i+jN_\varphi}^\varphi : i \in \{1, \ldots, N_\varphi\}, j \in \mathbb{Z}\} \subseteq \mathbb{R}$ where $0 \leq e_1^\varphi < \cdots < e_{N_\varphi}^\varphi < 1$ with $e_{i+jN_\varphi}^\varphi = e_i^\varphi + j$ and $\pi(e_i^\varphi) = d_i^\varphi$ and $e_{i+jN_\varphi}^\varphi = e_i^\varphi + j$ for all $i = 1, \ldots, N_\varphi$ and $j \in \mathbb{Z}$.

Remark 3.30. Assume there exists a homeomorphism $h \colon \mathbb{T} \to \mathbb{T}$ such that $h \circ \varphi = \psi \circ h$. We may deduce as in Remark 3.29 together with Lemma 3.1 that

$$N = N_\varphi = N_\psi \quad \text{and} \quad d = \deg(\varphi) = \deg(\psi).$$

We also have that $\varphi(d_i^\varphi) = d_{\tau_\varphi(i)}^\varphi$ since $\varphi(\mathcal{D}^\varphi) \subseteq \mathcal{D}^\varphi$ and $h(d_i^\varphi) = d_{\sigma(i)}^\psi$ since $h(\mathcal{D}^\varphi) = \mathcal{D}^\psi$, where $\tau_\varphi, \sigma \colon \{1, \ldots, N\} \to \{1, \ldots, N\}$ are uniquely determined and σ is a permutation, which is cyclic if h is orientation-preserving and is anti-cyclic if h is orientation-reversing. In addition we have that $h^{-1}(d_i^\psi) = d_{\sigma^{-1}(i)}^\varphi$ and $\psi(d_i^\psi) = d_{\sigma \circ \tau_\varphi \circ \sigma^{-1}(i)}^\psi$, so $\tau_\psi = \sigma \circ \tau_\varphi \circ \sigma^{-1}$. In the following we are going to reach a similar conclusion with lifts instead of circle maps and retrace our steps in order to attain a complete invariant culminating in Theorem 3.36.

Define $T_\Phi \colon \mathbb{Z} \to \mathbb{Z}$ by $\Phi(e_n^\varphi) = e_{T_\Phi(n)}^\varphi$ as $\Phi(\mathcal{E}^\varphi) \subseteq \mathcal{E}^\varphi$ for a lift $\Phi \colon \mathbb{R} \to \mathbb{R}$ of $\varphi \colon \mathbb{T} \to \mathbb{T}$. We say that T_Φ is the extended pattern for φ. For any $n, l \in \mathbb{Z}$ and $K \in \mathbb{Z}$ we have

$$T_\Phi(n + lN_\varphi) = T_\Phi(n) + lN_\varphi \deg(\varphi) \quad \text{and} \quad T_{\Phi+K} = T_\Phi + N_\varphi K. \qquad (3.1)$$

This follows from Φ and $\Phi + K$ being lifts of the circle map φ.

We will explore the association $T : \varphi \mapsto T_\Phi$ in the following.
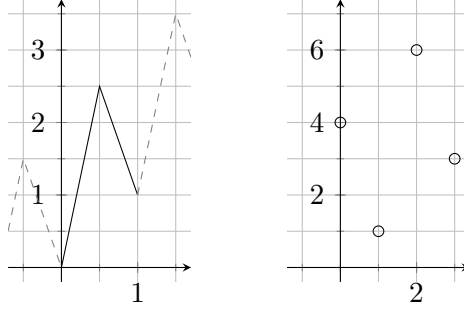
Remark 3.31. Assume there exists a homeomorphism $h \colon \mathbb{T} \to \mathbb{T}$ such that $h \circ \varphi = \psi \circ h$. Denote by $\Phi, \Psi \colon \mathbb{R} \to \mathbb{R}$ the lifts of $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ and by $H \colon \mathbb{R} \to \mathbb{R}$ a lift of $h \colon \mathbb{T} \to \mathbb{T}$. Then $H \circ \Phi \circ H^{-1} + K = \Psi$ for some $K \in \mathbb{Z}$, and $H(\mathcal{E}^\varphi) = \mathcal{E}^\psi$, so

$$H(e_n^\varphi) = e_{\Sigma(n)}^\psi \quad \text{and} \quad H \circ \Phi \circ H^{-1}(e_n^\psi) = e_{\Sigma \circ T_\Phi \circ \Sigma^{-1}(n)}^\psi.$$

The map $\Sigma \colon \mathbb{Z} \to \mathbb{Z}$ is a uniquely determined bijection. We note that

- H and Σ respectively are strictly increasing if and only if h is orientation-preserving, in which case Σ is a translation, i.e. $\Sigma(n) = n + L$ for all $n \in \mathbb{Z}$ for some $L \in \mathbb{Z}$,
- H and Σ respectively are strictly decreasing if and only if h is orientation-reversing in which case Σ is a reflection, i.e. $\Sigma(n) = L - n$ for all $n \in \mathbb{Z}$ for some $L \in \mathbb{Z}$.

Let us consider the two possibilities; h is orientation-preserving or orientation-reversing:

(a) If Σ is a translation, i.e. $\Sigma(n) = n + L$ for all $n \in \mathbb{Z}$ for some $L \in \mathbb{Z}$, then

$$T_\Psi(n) = \Sigma \circ T_\Phi \circ \Sigma^{-1}(n) + NK = T_\Phi(n - L) + L + NK. \tag{3.2}$$

(b) If Σ is a reflection, i.e. $\Sigma(n) = L - n$ for all $n \in \mathbb{Z}$ for some $L \in \mathbb{Z}$, then

$$T_\Psi(n) = \Sigma \circ T_\Phi \circ \Sigma^{-1}(n) + NK = L - T_\Phi(L - n) + NK. \tag{3.3}$$

In the wake of the deductions made above we introduce some notation:

Definition 3.32. Let $S, T \colon \mathbb{Z} \to \mathbb{Z}$. Write $S \sim T$ if $S \overset{+}{\sim} T$ or $S \overset{-}{\sim} T$ where

(i) $S \overset{+}{\sim} T$ if there exists $K, L \in \mathbb{Z}$ such that $S(n - L) + L + NK = T(n)$,
(ii) $S \overset{-}{\sim} T$ if there exists $K, L \in \mathbb{Z}$ such that $L - S(L - n) + NK = T(n)$.

Remark 3.33. The relation $\overset{+}{\sim}$ is an equivalence relation on the set of integer functions, whereas the relation $\overset{-}{\sim}$ is symmetric but it is incidentally neither transitive nor reflexive, but there is a connection between the two relations $\overset{+}{\sim}$ and $\overset{-}{\sim}$:

(a) If $S \overset{+}{\sim} T$, then $T \overset{+}{\sim} S$. If $S \overset{-}{\sim} T$, then $T \overset{-}{\sim} S$.
(b) If $R \overset{+}{\sim} S \overset{+}{\sim} T$, then $R \overset{+}{\sim} T$. If $R \overset{-}{\sim} S \overset{-}{\sim} T$, then $R \overset{+}{\sim} T$. If $R \overset{-}{\sim} S \overset{+}{\sim} T$, then $R \overset{-}{\sim} T$.

In particular the relation $\sim$ is also an equivalence relation.

Remark 3.34. We make the following observations about the association $T: \varphi \mapsto T_\Phi$.

(a) Assume now again there exists a homeomorphism $h: \mathbb{T} \to \mathbb{T}$ such that $h \circ \varphi = \psi \circ h$.

 - If h is orientation-preserving, then $T_\Phi \overset{+}{\sim} T_\Psi$ due to (3.2).

 - If h is orientation-reversing, then $T_\Phi \overset{-}{\sim} T_\Psi$ due to (3.3).

Thus $T_\Phi \sim T_\Psi$ in any case. In other words $T: \varphi \mapsto T_\Phi$ is an invariant in this sense.

(b) Let $T: \mathbb{Z} \to \mathbb{Z}$ such that for some $d \in \mathbb{Z}$ and $N \in \mathbb{N}$ and for all $n \in \mathbb{N}$ we have

$$T(n + lN) = T(n) + lNd, \tag{3.4a}$$
$$T(n) \neq T(n+1). \tag{3.4b}$$

Let $\mathcal{D} \subseteq [0,1)$ with $\#\mathcal{D} = N$, i.e. $\mathcal{D} = \{e_1 < \cdots < e_N\}$. Put $\mathcal{E} = \mathcal{D} + \mathbb{Z}$; $e_{i+jN} = e_i + j$. Define $\Phi_T: \mathbb{R} \to \mathbb{R}$ by $\Phi_T(e_n) = e_{T(n)}$ and linear between e_n and e_{n+1} for all $n \in \mathbb{N}$. Then $\Phi_T: \mathbb{R} \to \mathbb{R}$ is in fact the lift of a piecewise linear circle map $\varphi_T: \mathbb{T} \to \mathbb{T}$, and φ_T is critically finite and $\deg(\varphi_T) = d$, and $T_\Phi = T$ if Φ is such a linearization of T. The requirement (3.4b) is only there to ensure that the piecewise linear circle map φ_T is also piecewise monotone according to our definition, i.e. not locally constant.

Define the standard linearization of T to be determined by

$$\mathcal{D} = \{0, \tfrac{1}{N}, \dots, \tfrac{N-1}{N}\} \quad \text{and} \quad \Phi_T(\tfrac{k}{N}) = \tfrac{T(k+1)-1}{N}.$$

It follows in particular that any integer map $T: \mathbb{Z} \to \mathbb{Z}$ satisfying (3.4) arise as the invariant for a critically finite piecewise monotone circle map $\varphi: \mathbb{T} \to \mathbb{T}$.

We also have the following expression

$$d = \tfrac{T(n_0 + l_0 N) - T(n_0)}{l_0 N}$$

for any two fixed numbers $n_0 \in \mathbb{Z}$, $l_0 \in \mathbb{Z} \setminus \{0\}$ which can be substituted into (3.4a). This means that we are able recover d from N and the values of T at n_0 and $n_0 + l_0 N$. This in turn leads us to the following conclusion:

 - If d is known we may recover T from $T|_{\{1,\dots,N\}}$ together with (3.4a).

 - If d is unknown we may recover T from $T|_{\{0,1,\dots,N\}}$ together with (3.4a).

There is nothing special about the sets $\{1, \dots, N\}$ or $\{0, 1, \dots, N\}$:

(c) The property (3.4a) allow us to consider a restriction of T instead of the map itself: Let $A \subseteq \mathbb{Z}$ and $S: A \to \mathbb{Z}$ with $S = T|_A$ where $T: \mathbb{Z} \to \mathbb{Z}$ satisfies (3.4a).

We say that S is a pattern for T. Assume either of the following:

 - $\#A \cap a = 1$ for each $a \in \mathbb{Z}/N\mathbb{Z}$, and $N \in \mathbb{N}$ and $d \in \mathbb{Z}$ are fixed.

 - $\#A = N + 1$, $\#A \cap a \geq 1$ for each $a \in \mathbb{Z}/N\mathbb{Z}$, but only $N \in \mathbb{N}$ is fixed, and

$$S(n + lN) = S(n) + lNd$$

for some $d \in \mathbb{Z}$ whenever $A \cap a = \{n, n + lN\}$ for some $a \in \mathbb{Z}/N\mathbb{Z}$.

In either case we may recover T from S by using (3.4a).

The maps $T_\Phi|_{\{1,\dots,N\}}$ and $T_\Phi|_{\{0,1,\dots,N\}}$ are called the standard patterns.

Definition 3.35. Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ be continuous circle maps.

(i) Write $\varphi \overset{+}{\sim} \psi$ when there exists a homeomorphism $h \colon \mathbb{T} \to \mathbb{T}$ such that $h \circ \varphi = \psi \circ h$, which is additionally orientation-preserving.

(ii) Write $\varphi \overset{-}{\sim} \psi$ when there exists a homeomorphism $h \colon \mathbb{T} \to \mathbb{T}$ such that $h \circ \varphi = \psi \circ h$, which is additionally orientation-reversing.

(iii) Write $\varphi \sim \psi$ when there exists a homeomorphism $h \colon \mathbb{T} \to \mathbb{T}$ such that $h \circ \varphi = \psi \circ h$, i.e. $\varphi \sim \psi$ if and only if $\varphi \overset{+}{\sim} \psi$ or $\varphi \overset{-}{\sim} \psi$.

The following result is reminiscent of Theorem 2.1 in [11].

Theorem 3.36. *Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ be continuous circle maps that are piecewise monotone. Assume that φ and ψ are transitive and critically finite. Assume in addition that $N_\varphi = N_\psi$. Then $\varphi \overset{\pm}{\sim} \psi$ if and only if $T_\Phi \overset{\pm}{\sim} T_\Psi$. In particular $\varphi \sim \psi$ if and only if $T_\Phi \sim T_\Psi$.*

Proof. Denote by (a, b) the numbers strictly between $a \in \mathbb{R}$ and $b \in \mathbb{R}$ (even if $a > b$).

Assume first that $T_\Phi \overset{+}{\sim} T_\Psi$, i.e. $T_\Psi(n) = T_\Phi(n - L) + L + NK$ for some $K, L \in \mathbb{Z}$. Define $\mathcal{E}_l^\varphi = \Phi^{-l}(\mathcal{E}^\varphi)$ and $\mathcal{E}_l^\psi = \Psi^{-l}(\mathcal{E}^\psi)$ and notice that

$$\mathcal{E}^\varphi = \mathcal{E}_0^\varphi \subseteq \mathcal{E}_1^\varphi \subseteq \mathcal{E}_2^\varphi \subseteq \cdots \subseteq \mathcal{E}_\infty^\varphi = \bigcup_{l=0}^\infty \mathcal{E}_l^\varphi,$$
$$\mathcal{E}^\psi = \mathcal{E}_0^\psi \subseteq \mathcal{E}_1^\psi \subseteq \mathcal{E}_2^\psi \subseteq \cdots \subseteq \mathcal{E}_\infty^\psi = \bigcup_{l=0}^\infty \mathcal{E}_l^\psi.$$

It follows from Lemma 3.11(i) that φ and ψ are strongly transitive as φ and ψ are transitive. As a consequence of this Proposition 1.20 implies that $\mathcal{E}_\infty^\varphi$ and $\mathcal{E}_\infty^\psi$ are both dense in $\mathbb{R}$. Define an increasing bijection $H_0 \colon \mathcal{E}^\varphi \to \mathcal{E}^\psi$ by $H_0(e_n^\varphi) = e_{n+L}^\psi$ for all $e_n^\varphi \in \mathcal{E}^\varphi$. Then

$$H_0 \circ \Phi \circ H_0^{-1}(e_n^\psi) + K = e_{T_\Phi(n-L)+L+NK}^\psi = e_{T_\Psi(n)}^\psi = \Psi(e_n^\psi).$$

Assume $H_{l-1} \colon \mathcal{E}_{l-1}^\varphi \to \mathcal{E}_{l-1}^\psi$ is an increasing bijection such that for all $e_n^\varphi \in \mathcal{E}^\varphi$ and $e \in \mathcal{E}_{l-1}^\psi$,

$$H_{l-1}(e_n^\varphi) = e_{n+L}^\psi \quad \text{and} \quad H_{l-1} \circ \Phi \circ H_{l-1}^{-1}(e) + K = \Psi(e). \tag{3.5}$$

For any consecutive $e, f \in \mathcal{E}_{l-1}^\varphi$ with $e < f$ consider the numbers

$$P(e, f) = \#\mathcal{E}_{l-1}^\varphi \cap (\Phi(e), \Phi(f)), \qquad Q(e, f) = \#\mathcal{E}_{l-1}^\psi \cap (\Psi \circ H_{l-1}(e), \Psi \circ H_{l-1}(f)),$$
$$R(e, f) = \#\mathcal{E}_l^\varphi \cap (e, f), \qquad\qquad S(e, f) = \#\mathcal{E}_l^\psi \cap (H_{l-1}(e), H_{l-1}(f)).$$

We know that $\Phi|_{(e,f)}$ and $\Psi|_{(H_{l-1}(e), H_{l-1}(f))}$ are injective which maps respectively

$$\mathcal{E}_l^\varphi \cap (e, f) \quad \text{onto} \quad \mathcal{E}_{l-1}^\varphi \cap (\Phi(e), \Phi(f)), \text{ and}$$
$$\mathcal{E}_l^\psi \cap (H_{l-1}(e), H_{l-1}(f)) \quad \text{onto} \quad \mathcal{E}_{l-1}^\psi \cap (\Psi \circ H_{l-1}(e), \Psi \circ H_{l-1}(f)).$$

We also know that H_{l-1} is a bijection which by (3.5) maps

$$\Phi(e) \text{ to } \Psi(H_{l-1}(e)) \quad \text{and} \quad \Phi(f) \text{ to } \Psi(H_{l-1}(f)).$$

Therefore $R(e, f) = P(e, f)$, $S(e, f) = Q(e, f)$, and $P(e, f) = Q(e, f)$, so $R(e, f) = S(e, f)$. This yields an increasing bijection $H_l \colon \mathcal{E}_l^\varphi \to \mathcal{E}_l^\psi$ such that for all $e_n^\varphi \in \mathcal{E}^\varphi$ and $e \in \mathcal{E}_{l-1}^\varphi$,

$$H_l(e_n^\varphi) = e_{n+L}^\psi \quad \text{and} \quad H_l(e) = H_{l-1}(e).$$

This map is unique and for all $e \in \mathcal{E}_l^\varphi$,

$$H_l \circ \Phi \circ H_l^{-1}(e) + K = \Psi(e).$$

In this way we recursively obtain the sequence of functions $H_0, H_1, \ldots$. Define an increasing bijection $H_\infty \colon \mathcal{E}_\infty^\varphi \to \mathcal{E}_\infty^\psi$ by $H_\infty(e) = H_l(e)$ for all $e \in \mathcal{E}_l^\varphi$, and notice that for all $e \in \mathcal{E}_l^\psi$,

$$H_\infty \circ \Phi \circ H_\infty^{-1}(e) + K = \Psi(e).$$

The sets $\mathcal{E}_\infty^\varphi$ and $\mathcal{E}_\infty^\psi$ are both dense in $\mathbb{R}$ when φ and ψ are transitive, so $H_\infty \colon \mathcal{E}_\infty^\varphi \to \mathcal{E}_\infty^\psi$ extends to an increasing homeomorphism $H \colon \mathbb{R} \to \mathbb{R}$ with $H \circ \Phi \circ H^{-1} + K = \Psi$ which is a lift of an orientation-preserving circle homeomorphism $h \colon \mathbb{T} \to \mathbb{T}$ such that $h \circ \varphi = \psi \circ h$.

Assume similarly that $T_\Phi \stackrel{-}{\sim} T_\Psi$, i.e. $T_\Psi(n) = L - T_\Phi(L - n) + NK$ for some $K, L \in \mathbb{Z}$. We get a decreasing homeomorphism $H \colon \mathbb{R} \to \mathbb{R}$ with $H \circ \Phi \circ H^{-1} + K = \Psi$ which is a lift of an orientation-reversing circle homeomorphism $h \colon \mathbb{T} \to \mathbb{T}$ such that $h \circ \varphi = \psi \circ h$. $\square$

Definition 3.37. Put $I_n = (e_n^\varphi, e_{n+1}^\varphi)$ for any $n \in \mathbb{Z}$. Define for any $i, j \in \mathbb{Z}$,

$$A_{ij} = \#\{k \in \mathbb{Z} : \Phi(I_i) \supseteq I_j + k\}. \tag{3.6}$$

Define $A_\varphi = A(\Phi) = A$ as the $N_\varphi \times N_\varphi$-matrix that is given by (3.6) with $i, j \in \{1, \ldots, N_\varphi\}$. We call A_φ the transition matrix of φ. It does not depend on the choice of lift of φ.

Remark 3.38. The following observations can be made about the transition matrix.

(a) For each $i, j \in \mathbb{Z}$ and $l \in \mathbb{Z}$ we have the following identity: $A_{ij} = A_{i+lN_\varphi, j} = A_{i, j+lN_\varphi}$.

(b) For any linearization $\psi = \varphi_T \colon \mathbb{T} \to \mathbb{T}$ of $T = T_\Phi$ we have that $A_\varphi = A_\psi$.

(c) For any $n \in \mathbb{N}$ we have that $A(\Phi^n) = A(\Phi)^n$.

The transition matrix A_φ is not a permutation matrix as $\mathcal{C}^\varphi \neq \emptyset$.

Remark 3.39. Let $S, T \colon \mathbb{Z} \to \mathbb{Z}$ be any two integer maps which satisfy (3.4a) and (3.4b). If $S \sim T$ then $A(\Phi_S)$ is permutation-similar to $A(\Phi_T)$. In fact we have the following:

(a) If $S \stackrel{+}{\sim} T$ then $A(\Phi_S)$ is permutation-similar to $A(\Phi_T)$ via a cyclic permutation.

(b) If $S \stackrel{-}{\sim} T$ then $A(\Phi_S)$ is permutation-similar to $A(\Phi_T)$ via an anti-cyclic permutation.

In particular taking $S = T_\Phi$ and $T = T_\Psi$ we have the following:

(c) If $T_\Phi \stackrel{+}{\sim} T_\Psi$, then $A(\Phi)$ is permutation-similar to $A(\Psi)$ via a cyclic permutation.

(d) If $T_\Phi \stackrel{-}{\sim} T_\Psi$, then $A(\Phi)$ is permutation-similar to $A(\Psi)$ via an anti-cyclic permutation.

In particular $A(\Phi) \stackrel{\pm}{\sim} A(\Psi)$ whenever $T_\Phi \stackrel{\pm}{\sim} T_\Psi$ where we use the following notation:

Definition 3.40. Let $A, B \in M_n$. Write $A \sim B$ if $A \stackrel{+}{\sim} B$ or $A \stackrel{-}{\sim} B$ where

(i) $A \stackrel{+}{\sim} B$ if A is permutation-similar to B via a cyclic permutation.

(ii) $A \stackrel{-}{\sim} B$ if A is permutation-similar to B via an anti-cyclic permutation.

The reader is referred to Appendix B for an introduction to the Perron–Frobenius theorem and related notions like irreducible matrices and primitive matrices.

Lemma 3.41. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone. Assume that φ is critically finite (but not necessarily piecewise linear). Then*

(i) *A_φ is irreducible if φ is transitive,*

(ii) *A_φ is primitive if φ is exact.*

Proof. Let $I_i = (e_i^\varphi, e_{i+1}^\varphi)$, $I_j = (e_j^\varphi, e_{j+1}^\varphi)$ for any $i,j \in \{1, \ldots, N_\varphi\}$.

If φ is transitive, then φ is (very) strongly transitive by Lemma 3.11(i); for some $L \in \mathbb{N}$,

$$\bigcup_{n=1}^{L} \varphi^n(\pi(I_i)) = \mathbb{T} \supseteq \pi(I_j), \quad \text{so} \quad \bigcup_{n=1}^{L} \varphi^n(\pi(I_i)) \setminus \mathcal{D}^\varphi \supseteq \pi(I_j).$$

In particular $\Phi^n(I_i) \supseteq I_j + k$ for some $k \in \mathbb{Z}$ and $n \in \{1, \ldots, L\}$.

If φ is exact, then there exists $L \in \mathbb{N}$ such that for all $i,j \in \{1, \ldots, N_\varphi\}$,

$$\varphi^L(\pi(I_i)) = \mathbb{T} \supseteq \pi(I_j) \quad \text{so} \quad \varphi^L(\pi(I_i)) \setminus \mathcal{D}^\varphi \supseteq \pi(I_j)$$

In particular $\Phi^L(I_i) \supseteq I_j + k$ for some $k \in \mathbb{Z}$ for each $i,j \in \{1, \ldots, N_\varphi\}$. $\qquad\square$

We note that the proof of Lemma 3.42 is inspired by the proof of Lemma 2.5 in [11]:

Lemma 3.42. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone. Assume that φ is critically finite and linear on each connected component of $\mathbb{T} \setminus \mathcal{D}^\varphi$. Then*

(i) *A_φ is irreducible if and only if φ is transitive,*

(ii) *A_φ is primitive if and only if φ is exact.*

Proof. Let $\mathcal{I}$ denote the set of connected components of $\mathbb{T} \setminus \mathcal{D}^\varphi$, and $\mathcal{K} = \{\overline{I} \subseteq \mathbb{T} : I \in \mathcal{I}^\varphi\}$. Then $\#\mathcal{K} \geq 2$. Let $J \subseteq \mathbb{T}$ be any closed interval. Let $I_i = (e_i^\varphi, e_{i+1}^\varphi)$ for all $i \in \{1, \ldots, N_\varphi\}$. Assume that A_φ is irreducible or primitive. Let us rule out some possibilities:

- Assume $\#\varphi^n(J) \cap \mathcal{D}^\varphi = 0$ for all $n \in \mathbb{N}$, and choose $K_n \in \mathcal{K}$ such that $\varphi^n(J) \subseteq K_n$. We note that φ is linear on $\varphi^n(J)$ for all $n \in \mathbb{N}$, since $\#\varphi^n(J) \cap \mathcal{D}^\varphi = 0$ for all $n \in \mathbb{N}$. If $\varphi(K_n)$ is an element in $\mathcal{K}$, then $\lambda(\varphi^{n+1}(J))/\lambda(\varphi^n(J)) = \lambda(K_{n+1})/\lambda(K_n)$, and so

$$1 \geq \lambda(\varphi^{n+1}(J))/\lambda(K_{n+1}) = \lambda(\varphi^n(J))/\lambda(K_n). \tag{3.7}$$

If $\varphi(K_n)$ is a union of at least two elements of $\mathcal{K}$, then since $K_{n+1} \subseteq \varphi(K_n)$ we have

$$1 \geq \lambda(\varphi^{n+1}(J))/\lambda(K_{n+1}) \geq c\lambda(\varphi^n(J))/\lambda(K_n), \tag{3.8}$$

where $c = \min \lambda(\varphi(K))/\lambda(K') > 1$, the minimum is taken over all (K, K') with $K, K' \in \mathcal{K}$, $K' \subseteq \varphi(K)$ such that $\varphi(K)$ is the union of at least two elements of $\mathcal{K}$. As a result of (3.7) and (3.8) there exists $m \in \mathbb{N}$ such that $\varphi(K_n) \in \mathcal{K}$ for all $n \geq m$, since (3.8) only for finitely many $n \in \mathbb{N}$. In particular $\varphi(K_n) = K_{n+1}$ for all $n \geq m$. In conclusion $K_n = K_{n+k}$ for some $n \geq m$ and $k \in \mathbb{N}$, and φ^{2k} is the identity on K_n. This is impossible when A_φ is irreducible or primitive and not a permutation matrix.

- Assume $\#\varphi^n(J) \cap \mathcal{D}^\varphi \leq 1$ for all $n \in \mathbb{N}$, and choose $m \in \mathbb{N}$ with $\#\varphi^m(J) \cap \mathcal{D}^\varphi = 1$. As $\#\varphi^m(J) \cap \mathcal{D}^\varphi = 1$ and $\mathcal{D}^\varphi$ is forward invariant we have that for all $n \geq m$,

$$\#\varphi^n(J) \cap \mathcal{D}^\varphi = 1.$$

We now consider a closed subinterval H of $\varphi^m(J)$ with one of the endpoints in $\mathcal{D}^\varphi$. The same arguments as above can be applied to H to reach a similar contradiction.

Thus for some $m \in \mathbb{N}$ we have $\#\varphi^m(J) \cap \mathcal{D}^\varphi \geq 2$, so $\varphi^m(J) \supseteq \pi(I_i)$ for some $i \in \{1, \ldots, N\}$.

If A_φ is irreducible, then there exists $L \in \mathbb{N}$ such that

$$\textstyle\bigcup_{n=1}^{L} \varphi^n(\pi(I_i)) \supseteq \mathbb{T} \setminus \mathcal{D}^\varphi.$$

In particular $\bigcup_{n=1}^{L+m} \varphi^n(J) = \mathbb{T}$ so φ is (very) strongly transitive.

If A_φ is primitive, then there exists $L \in \mathbb{N}$ such that

$$\varphi^L(\pi(I_i)) \supseteq \mathbb{T} \setminus \mathcal{D}^\varphi.$$

In particular $\varphi^{L+m}(J) = \mathbb{T}$ so φ is exact. $\qquad\square$

Corollary 3.43. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map that is piecewise monotone. Assume that φ is transitive and critically finite. Let $\varphi_{T_\Phi}\colon \mathbb{T} \to \mathbb{T}$ be a linearization of T_Φ. Then φ_{T_Φ} is also transitive and critically finite, and $\varphi \overset{+}{\sim} \varphi_{T_\Phi}$.*

Proof. Lemma 3.41 together with Remark 3.38(b) implies that $A_\varphi = A_{\varphi_{T_\Phi}}$ is irreducible. Thus φ_{T_Φ} is transitive by Lemma 3.42; $\varphi \overset{+}{\sim} \varphi_{T_\Phi}$ by Remark 3.34(b) and Theorem 3.36.$\square$

Remark 3.44. Let $T\colon \mathbb{Z} \to \mathbb{Z}$ be any integer map.

(a) $T = T_\Phi$ for some continuous piecewise monotone critically finite circle map $\varphi\colon \mathbb{T} \to \mathbb{T}$ if and only if (3.4) for all $n \in \mathbb{N}$ for some $d \in \mathbb{Z}$, $N \in \mathbb{N}$; if so $d = \deg(\varphi)$ and $N = N_\varphi$.

(b) $A_{\varphi_{T_\Phi}}$ is irreducible if and only if φ_{T_Φ} is transitive, when φ_{T_Φ} is a linearization of T_Φ.

We note that (a) follows from Remark 3.34(b) and (3.1) and (b) follows from Lemma 3.42. A combination of (a) and (b) now characterizes the range of the invariant in Theorem 3.36.

Remark 3.45. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone. Assume that φ is transitive and critically finite. Then A_φ is irreducible by Lemma 3.41.

- Denote by $v \in (\mathbb{R}^+)^N$ the Perron–Frobenius probability eigenvector of A_φ.
- Denote by $s \in \mathbb{R}^+$ the Perron–Frobenius eigenvalue of A_φ.

Define $e_{1+jN}^\psi = j$ and $e_{i+jN}^\psi = e_i^\psi + j = \sum_{l=1}^{i-1} v_l + j$ for each $i \in \{2, \ldots, N\}$ and all $j \in \mathbb{Z}$. Define $\Psi\colon \mathbb{R} \to \mathbb{R}$ by $\Psi(e_n^\psi) = e_{T_\Phi(n)}^\psi$ and linearly between any e_{n-1}^ψ and e_n^ψ for all $n \in \mathbb{Z}$.

Then $\Psi\colon \mathbb{R} \to \mathbb{R}$ is the lift of $\psi\colon \mathbb{T} \to \mathbb{T}$ which is in fact uniformly piecewise linear as
$\Psi(e_{i+1}^\psi) - \Psi(e_i^\psi) = e_{T_\Phi(i+1)}^\psi - e_{T_\Phi(i)}^\psi = (A_\varphi v)_i = sv_i = s(e_{i+1}^\psi - e_i^\psi)$ when $T_\Phi(i+1) > T_\Phi(i)$,
$\Psi(e_i^\psi) - \Psi(e_{i+1}^\psi) = e_{T_\Phi(i)}^\psi - e_{T_\Phi(i+1)}^\psi = (A_\varphi v)_i = sv_i = s(e_{i+1}^\psi - e_i^\psi)$ when $T_\Psi(i+1) < T_\Psi(i)$.
These calculations involve Definition 3.37. Also ψ is critically finite and $T_\Phi = T_\Psi$, $A_\varphi = A_\psi$.

It follows from Lemma 3.42 and Theorem 3.36 that ψ is transitive and $\varphi \overset{+}{\sim} \psi$.

Lemma 3.46. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone. Assume that φ is transitive and critically finite. Then $\varphi \sim \psi$ as just above in Remark 3.45. Let $\theta\colon \mathbb{T} \to \mathbb{T}$ be a continuous uniformly piecewise linear circle map such that $\theta \sim \varphi \sim \psi$. Then there exists a rotation or a reflection $h\colon \mathbb{T} \to \mathbb{T}$ such that $h \circ \theta = \psi \circ h$.*

Proof. If $\theta \overset{\pm}{\sim} \psi$, then $N = N_\varphi = N_\psi$, $T_\Theta \overset{\pm}{\sim} T_\Psi$, $A_\theta \overset{\pm}{\sim} A_\psi$, and $\exp(h(\theta)) = \exp(h(\psi)) = s$. Put $w_i = e_{i+1}^\theta - e_i^\theta$ for $i \in \{1, \ldots, N\}$. It follows that $A_\theta w = sw$
$(A_\theta w)_i = e_{T_\Theta(i+1)}^\theta - e_{T_\Theta(i)}^\theta = \Theta(e_{i+1}^\theta) - \Theta(e_i^\theta) = s(e_{i+1}^\theta - e_i^\theta) = sw_i$ when $T_\Theta(i+1) > T_\Theta(i)$,
$(A_\theta w)_i = e_{T_\Theta(i)}^\theta - e_{T_\Theta(i+1)}^\theta = \Theta(e_i^\theta) - \Theta(e_{i+1}^\theta) = s(e_{i+1}^\theta - e_i^\theta) = sw_i$ when $T_\Theta(i+1) < T_\Theta(i)$.

The Perron–Frobenius probability eigenvector of an irreducible matrix is in fact unique, so since $A_\theta \sim A_\psi$ there exists a cyclic or anti-cyclic permutation $\sigma\colon \{1,\dots,N\} \to \{1,\dots,N\}$ such that $e^\theta_{i+1} - e^\theta_i = w_i = v_{\sigma(i)} = e^\psi_{\sigma(i)+1} - e^\psi_{\sigma(i)}$ for all $i \in \{1,\dots,N\}$.

Thus there exists a rotation or a reflection $h\colon \mathbb{T} \to \mathbb{T}$ such that $h \circ \theta = \psi \circ h$. $\qquad\square$

Corollary 3.47. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map that is piecewise monotone. Assume that φ is transitive and critically finite. We adopt the notation from Remark 3.45. In particular $s \in \mathbb{R}^+$ is the Perron–Frobenius eigenvalue of A_φ. Then $h(\varphi) = \log(s)$.*

Proof. This follows directly from Remark 3.45 and Lemma 3.15. $\qquad\square$

Remark 3.48. We should acknowledge the importance of Remark 3.45 and Lemma 3.46:

(a) Remark 3.45 essentially provides an alternative constructive proof of Lemma 3.13, when we restrict our attention to transitive critically finite circle maps.

(b) Lemma 3.46 proves that any continuous piecewise monotone circle map $\psi\colon \mathbb{T} \to \mathbb{T}$, which is uniformly piecewise linear such that $\varphi \sim \psi$ is essentially unique.

C*-algebras associated with circle maps

We construct C*-algebras from continuous piecewise monotone circle maps just as in [50]. Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ be continuous piecewise monotone maps.

Definition 4.1. Define the groupoid

$$\Gamma_\varphi^+ = \{(x, k, y) \in \mathbb{T} \times \mathbb{Z} \times \mathbb{T} : x \overset{k}{\sim} y\},$$

where $x \overset{k}{\sim} y$ for $x, y \in \mathbb{T}$ and $k \in \mathbb{Z}$ if and only if there exists $n, m \in \mathbb{N}$ with $k = n - m$ such that $\varphi^n(x) = \varphi^m(y)$ and $\mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^m, y)$. The groupoid operations are defined by

$$(x, k, y)^{-1} = (y, -k, x) \quad \text{and} \quad (x, k, y)(y, l, z) = (x, k + l, z);$$
$$r(x, k, y) = x \quad \text{and} \quad s(x, k, y) = y.$$

The collection of orientation-preserving locally defined homeomorphisms of $\mathbb{T}$ is called $\mathcal{P}^+$. This is a pseudo-group of locally defined homeomorphisms of $\mathbb{T}$ and we refer Example 2.15. It follows from Example 2.15 and the last part of Lemma 3.9 that

$$\Gamma_\varphi(\mathcal{P}^+) = \{(x, k, \eta, y) \in \mathbb{T} \times \mathbb{Z} \times \mathcal{P}^+ \times \mathbb{T} : \eta \in \mathcal{T}_k(\varphi), \eta(x) = y\}/\sim$$

is an étale locally compact Hausdorff groupoid. It also follows from Lemma 3.9 that $[x, k, \eta, y] \mapsto (x, k, y)$ is a bijection between $\Gamma_\varphi(\mathcal{P}^+)$ and Γ_φ^+, and we equip Γ_φ^+ with the topology provided in Example 2.15 generated by

$$\{\Omega_k(\eta) : k \in \mathbb{Z}, \eta \in \mathcal{T}_k(\varphi)\}$$

where $k \in \mathbb{Z}$ and $\eta \in \mathcal{T}_k(\varphi)$.

For any $k \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $n = m + k \in \mathbb{N}$ put

$$\Gamma_\varphi^+(k, m) = \{(x, k, y) \in \mathbb{T} \times \mathbb{Z} \times \mathbb{T} : \varphi^n(x) = \varphi^m(y), \mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^m, y)\}$$

and put $\Gamma_\varphi^+(k) = \bigcup_{m \in \mathbb{N} \cap (\mathbb{N} - k)} \Gamma_\varphi^+(k, m)$. Then $\Gamma_\varphi^+ = \bigsqcup_{k \in \mathbb{Z}} \Gamma_\varphi^+(k)$.

Lemma 4.2. *The set $\Gamma_\varphi^+(k, m)$ is a locally closed subset of $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$ and consequently $\Gamma_\varphi^+(k, m)$ is a locally compact Hausdorff space in the relative topology.*

Proof. This follows directly from Lemma 3.3 in [50]. $\square$

If $\Gamma_\varphi^+(k,m)$ inherits the product topology from $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$, then $\Gamma_\varphi^+(k)$ could be equipped with the inductive limit topology, and $\Gamma_\varphi^+ = \bigsqcup_{k \in \mathbb{Z}} \Gamma_\varphi^+(k)$ with the disjoint union topology. As it turns out, this topology is no different from the topology already introduced on Γ_φ^+:

Lemma 4.3. *Let $\Gamma_\varphi^+(k,m)$ be equipped with the relative topology inherited from $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$. Then a subset $W \subseteq \Gamma_\varphi^+(k,m)$ is open if and only if*

$$\forall (x,k,y) \in W \, \exists \eta \in \mathcal{T}_k(\varphi) : (x,k,y) \in \Omega_k(\eta) \subseteq W.$$

Corollary 4.4. *The étale locally compact Hausdorff groupoid Γ_φ^+ is second countable.*

It follows from Lemma 3.9 and Lemma 4.3 that $\Gamma_\varphi^+(k,m)$ is open in $\Gamma_\varphi^+(k,m+1)$, and $\Gamma_\varphi^+(k)$ is open and closed in Γ_φ^+. Moreover $\Gamma_\varphi^+(0)$ is a principal subgroupoid of Γ_φ^+.

Definition 4.5. Put $R_\varphi^+(n) = \Gamma_\varphi^+(0,n)$ and $R_\varphi^+ = \Gamma_\varphi^+(0) = \bigcup_{n \in \mathbb{N}} R_\varphi^+(n)$.

Remark 4.6. We can think of the groupoid Γ_φ^+ as consisting of layers $\Gamma_\varphi^+(k)$ where $k \in \mathbb{Z}$, and each layer $\Gamma_\varphi^+(k)$ with $k \in \mathbb{Z}$ is build in steps $\Gamma_\varphi^+(k,m)$ where $m \in \mathbb{N} \cap (\mathbb{N} - k)$ varies, and Γ_φ^+ is symmetric as $\Gamma_\varphi^+(k,m)$ is a reflection of $\Gamma_\varphi^+(-k,m+k)$ through the diagonal, and so $R_\varphi^+(n) = \Gamma_\varphi^+(0,n)$ is symmetric for each $n \in \mathbb{N}$ and so is $R_\varphi^+ = \Gamma_\varphi^+(0)$.

We may also view each layer $\Gamma_\varphi^+(k)$ of Γ_φ^+ and each step $\Gamma_\varphi^+(k,m)$ as a subset of $\mathbb{T} \times \mathbb{T}$, and also identify the unit spaces of Γ_φ^+ and R_φ^+ as well as $R_\varphi^+(n)$ for any $n \in \mathbb{N}$ with $\mathbb{T}$, being the diagonal of $R_\varphi^+(n)$ for each $n \in \mathbb{N}$, i.e. $\{(x,k,x) \in \mathbb{T} \times \mathbb{Z} \times \mathbb{T} : k = 0\} \simeq \mathbb{T}$.

We will employ the following identification throughout:

$$R_\varphi^+(n) \simeq \{(x,y) \in \mathbb{T} \times \mathbb{T} : \varphi^n(x) = \varphi^n(y), \mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^n, y)\}.$$

These are $\Gamma_\varphi^+(0,1), \Gamma_\varphi^+(0,2), \Gamma_\varphi^+(0,3), \Gamma_\varphi^+(1,1), \Gamma_\varphi^+(1,2)$ for $\varphi = \psi_2$, cf. Example 3.6:
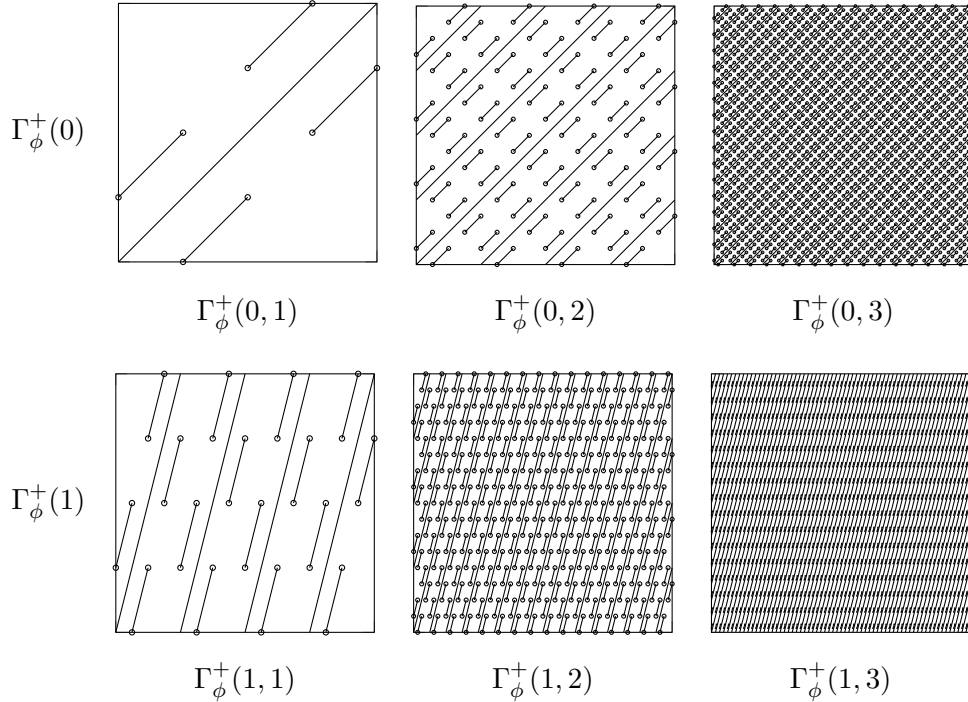
Remark 4.7. If φ is a homeomorphism, then $R_\varphi^+ \simeq \mathbb{T}$ and $\mathcal{O}_{R_\varphi^+}(x) = \{x\}$ for all $x \in \mathbb{T}$.

The collection of all possible locally defined homeomorphisms of $\mathbb{T}$ is denoted by $\mathcal{P}^{\pm}$. The topological groupoid which is described in Example 2.15 is denoted by $\Gamma_\varphi^{\pm} = \Gamma_\varphi(\mathcal{P}^{\pm})$. The groupoid Γ_φ is described in Example 2.14. We will not focus too much on $\Gamma_\varphi^{\pm}$ or Γ_φ. The results for Γ_φ^+ for the most part transfer directly to $\Gamma_\varphi^{\pm}$.

Remark 4.8. We make the following observations on isomorphisms of groupoids:

(a) If φ is a local homeomorphism, then $\Gamma_\varphi^+ \simeq \Gamma_\psi$ and $R_\varphi^+ \simeq R_\psi$ where $\psi = \varphi^2$.

(b) If φ is an orientation-preserving local homeomorphism, then $\Gamma_\varphi^+ \simeq \Gamma_\varphi$ and $R_\varphi^+ \simeq R_\varphi$.

(c) If φ and ψ are conjugate, then $\Gamma_\varphi^+ \simeq \Gamma_\psi^+$ and $R_\varphi^+ \simeq R_\psi^+$.

(d) If $\psi = \varphi^n$ for some $n \in \mathbb{N}$, then $R_\varphi^+ \simeq R_\psi^+$.

Example 4.9. These are $\Gamma_\varphi^+(0,1)$ and $\Gamma_\varphi(0,1)$ for $\varphi = \psi_2$, cf. Example 3.6, Example 2.14:

$$\Gamma_\varphi^+(0,1) \qquad\qquad \Gamma_\varphi(0,1)$$
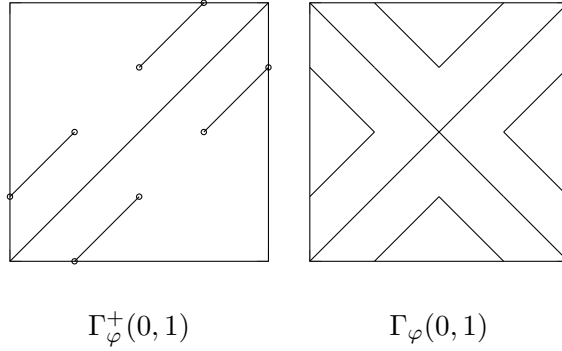
It is worth noting that $\Gamma_\varphi(0,1)$ is not étale with the usual topology inherited from $\mathbb{T} \times \mathbb{T}$. This observation sheds some light on the difficulty in getting a suitable topology for Γ_φ.

4.1 The core of C*-algebras associated with circle maps

The construction of $C_r^*(\Gamma_\varphi^+)$, $C_r^*(R_\varphi^+)$ and $C_r^*(R_\varphi^+(n))$ follows Definition 2.18.

Definition 4.10. The gauge action $C_r^*(\Gamma_\varphi^+)$ is a strongly continuous action of $\mathbb{T}$ given by

$$\sigma_z(f)(x,k,y) = z^{c(x,k,y)} f(x,k,y) = z^k f(x,k,y)$$

for all $f \in C_c(\Gamma_\varphi^+)$, $z \in \mathbb{T}$ and $(x,k,y) \in \Gamma_\varphi^+$, where $c \colon \Gamma_\varphi^+ \to \mathbb{Z}$ is given by $c(x,k,y) = k$.

It follows from Proposition 2.32 that $C_r^*(\Gamma_\varphi^+)^\sigma \simeq C_r^*(R_\varphi^+)$ which is called the core of $C_r^*(\Gamma_\varphi^+)$.

Remark 4.11. As $R_\varphi^+ = \bigcup_{n \in \mathbb{N}} R_\varphi^+(n)$ has the given topology,

$$C_r^*(R_\varphi^+) \simeq \overline{\bigcup_{n \in \mathbb{N}} C_r^*(R_\varphi^+(n))} = \varinjlim (C_r^*(R_\varphi^+(n)), \rho_n). \tag{4.1}$$

In the directed system $\{(C_r^*(R_\varphi^+(n)), \rho_n)\}_{n \in \mathbb{N}}$ the map $\rho_n \colon C_r^*(R_\varphi^+(n)) \to C_r^*(R_\varphi^+(n+1))$ is the $*$-homomorphism extending the embedding $C_c(R_\varphi^+(n)) \to C_c(R_\varphi^+(n+1))$.

For any subset $\mathcal{S}$ of $\mathbb{T}$ we clearly have that $\varphi^{-n}(\mathcal{S})$ is an $R_\varphi^+(n)$-invariant subset of $\mathbb{T}$. Let $\mathcal{D}$ be a finite subset of $\mathbb{T}$ such that $\mathcal{C}_n \subseteq \mathcal{D}_n$ where $\mathcal{D}_n = \varphi^{-n}(\mathcal{D})$, e.g. $\mathcal{D} = \varphi^n(\mathcal{C}_n)$. Let $\mathcal{I}$ and $\mathcal{I}_n$ denote the connected components of $\mathbb{T} \setminus \mathcal{D}$ and $\mathbb{T} \setminus \mathcal{D}_n$ respectively and

$$\mathcal{D}_n^{(2)} = \{(x,y) \in \mathcal{D}_n \times \mathcal{D}_n : \varphi^n(x) = \varphi^n(y), \operatorname{val}(\varphi^n, x) = \operatorname{val}(\varphi^n, y)\},$$
$$\mathcal{I}_n^{(2)} = \{(I,J) \in \mathcal{I}_n \times \mathcal{I}_n : \varphi^n(I) = \varphi^n(J), \operatorname{val}(\varphi^n, I) = \operatorname{val}(\varphi^n, J)\}.$$

It will be convenient once and again to use the notation $\mathcal{D} = \{d_l\}_{l=1}^N$ and $\mathcal{I} = \{I_l\}_{l=1}^N$ where $d_1 < d_2 < \cdots < d_N = d_0$ and $I_1 < I_2 < \cdots < I_N$ and $I_l = (d_{l-1}, d_l)$, $l = 1, \ldots, N$.

Remark 4.12. $I \in \mathcal{I}_n$ if and only if I is an open interval with $I \cap \mathcal{C}_n = \emptyset$ and $\varphi^n(I) \in \mathcal{I}$.

Let $\mathbb{A}_n$ and $\mathbb{B}_n$ be the finite-dimensional C*-algebras generated by the sets of matrix units $\{e_{x,y} : (x,y) \in \mathcal{D}_n^{(2)}\}$ and $\{e_{I,J} : (I,J) \in \mathcal{I}_n^{(2)}\}$ respectively. It is a trivial observation that

$$\mathbb{A}_n = \bigoplus_{(d,v)\in\mathcal{D}(\pm)_n} M_{a_n(d,v)} \quad \text{and} \quad \mathbb{B}_n = \bigoplus_{(I,w)\in\mathcal{I}(\pm)_n} M_{b_n(I,w)}.$$

where $\mathcal{D}(\pm)_n \subseteq \mathcal{D} \times \mathcal{V}$ and $\mathcal{I}(\pm)_n \subseteq \mathcal{I} \times \mathcal{W}$ are given by

$$\mathcal{D}(\pm)_n = \{(d,v) \in \mathcal{D} \times \mathcal{V} : \varphi^n(x) = d, \mathrm{val}(\varphi^n, x) = v \text{ for some } x \in \mathcal{D}_n\} \quad \text{and}$$
$$\mathcal{I}(\pm)_n = \{(I,w) \in \mathcal{I} \times \mathcal{W} : \varphi^n(J) = I, \mathrm{val}(\varphi^n, J) = w \text{ for some } J \in \mathcal{I}_n\},$$

and $a_n(d,v)$ and $b_n(I,w)$ for any $(d,v) \in \mathcal{D}(\pm)_n$ and $(I,w) \in \mathcal{I}(\pm)_n$ are given as

$$a_n(d,v) = \#\{x \in \mathcal{D}_n : \varphi^n(x) = d, \mathrm{val}(\varphi^n, x) = v\} \quad \text{and}$$
$$b_n(I,w) = \#\{J \in \mathcal{I}_n : \varphi^n(J) = I, \mathrm{val}(\varphi^n, J) = w\}.$$

Remark 4.13. We always have the upper bounds $\#\mathcal{D}(\pm)_n \le 4\#\mathcal{D}$ and $\#\mathcal{I}(\pm)_n \le 2\#\mathcal{I}$. We have the lower bounds $\#\mathcal{D} \le \#\mathcal{D}(\pm)_n$ and $\#\mathcal{I} \le \#\mathcal{I}(\pm)_n$ whenever φ is surjective. It is also worth noting that $\min_{(I,w)\in\mathcal{I}(\pm)_n} b_n(I,w) \ge \min_{(d,v)\in\mathcal{D}(\pm)_n} a_n(d,v)$.

Remark 4.14. In the following we introduce notation that will be used occasionally.

For each $I_l = (d_{l-1}, d_l) \in \mathcal{I}$ consider a homeomorphism $\kappa_{I_l} : (0,1) \to I_l$ with

$$\lim_{t\to 0} \kappa_{I_l}(t) = d_{l-1} \quad \text{and} \quad \lim_{t\to 1} \kappa_{I_l}(t) = d_l,$$

e.g. the linear homeomorphism given by $\kappa_{I_l}(t) = (d_l - d_{l-1})t + d_{l-1}$ and $\kappa_{I_l}^{-1}(t) = \frac{t - d_{l-1}}{d_l - d_{l-1}}$, which is the homeomorphism we will consider in the following.

For each $I \in \mathcal{I}_n$ with $\varphi^n(I) = I_l \in \mathcal{I}$ consider $\kappa_I^n = (\varphi^n|_I)^{-1} \circ \kappa_{I_l} : (0,1) \to I$, and note

$$\lim_{t\to 0} \kappa_I^n(t) = \lim_{t\to d_{l-1}} (\varphi^n|_I)^{-1}(t) \quad \text{and} \quad \lim_{t\to 1} \kappa_I^n(t) = \lim_{t\to d_l} (\varphi^n|_I)^{-1}(t)$$

both exist in $\mathcal{D}_n$ as the two endpoints of I.

For any $f \in C(R_\varphi^+(n))$ and $(I,J) \in \mathcal{I}_n^{(2)}$ note $(\kappa_I^n(t), \kappa_J^n(t)) \in R_\varphi^+(n)$ and all $t \in (0,1)$, and consider $f_{I,J} \in C((0,1))$ given by $f_{I,J}(t) = f(\kappa_I^n(t), \kappa_J^n(t))$.

Lemma 4.15. *We have* $C_r^*(R_\varphi^+(n)|_{\mathcal{D}_n}) \simeq \mathbb{A}_n$ *and* $C_r^*(R_\varphi^+(n)|_{\mathbb{T}\backslash\mathcal{D}_n}) \simeq C_0((0,1), \mathbb{B}_n)$.

Proof. Define $a_n : C_c(R_\varphi^+(n)|_{\mathcal{D}_n}) \to \mathbb{A}_n$ and $b_n : C_c(R_\varphi^+(n)|_{\mathbb{T}\backslash\mathcal{D}_n}) \to C_c((0,1), \mathbb{B}_n)$ by

$$a_n(f) = \sum_{(x,y)\in\mathcal{D}_n^{(2)}} f(x,y)e_{x,y} \quad \text{and} \quad b_n(f) = \sum_{(I,J)\in\mathcal{I}_n^{(2)}} f_{I,J}e_{I,J}.$$

Then a_n and b_n are bijective $*$-homomorphisms and extends to isomorphisms, since

$$\|\pi_c(f)\| = \left\| \sum_{(x,y)\in\mathcal{A}_c} f(x,y)e_{x,y} \right\| \quad \text{and} \quad \|\pi_s(f)\| = \left\| \sum_{(I,J)\in\mathcal{B}_s} f_{I,J}(t)e_{I,J} \right\|$$

for any $c \in \mathcal{D}_n$ and $s \in K \in \mathcal{I}_n$ with $\varphi^n(s) = \kappa_{\varphi^n(K)}(t) \subseteq \mathbb{T} \setminus \mathcal{D}$ as is the case in [50], where $\mathcal{A}_c = \{(x,y) \in \mathcal{D}_n^{(2)} : (x,c), (y,c) \in \mathcal{D}_n^{(2)}\}$ and $\mathcal{B}_s = \{(I,J) \in \mathcal{I}_n^{(2)} : (I,K), (J,K) \in \mathcal{I}_n^{(2)}\}$. Indeed we have the following matrix representations

$$\pi_c(f) = (f(x,y))_{x,y \in \mathcal{A}_c} \quad \text{and} \quad \pi_s(f) = (f_{I,J}(t))_{I,J \in \mathcal{B}_s},$$

and $\mathcal{D}_n^{(2)} = \bigcup_{c \in \mathcal{D}_n} \mathcal{A}_c$ and $\mathcal{I}_n^{(2)} = \bigcup_{s \in \mathbb{T} \setminus \mathcal{D}_n} \mathcal{B}_s$ where for any $c, d \in \mathcal{D}_n$ and $r, s \in \mathbb{T} \setminus \mathcal{D}_n$ either $\mathcal{A}_c = \mathcal{A}_d$ or $\mathcal{A}_c \cap \mathcal{A}_d = \emptyset$ and either $\mathcal{B}_s = \mathcal{B}_r$ or $\mathcal{B}_s \cap \mathcal{B}_r = \emptyset$. $\square$

For $x, y \in \mathbb{T}$ and $I \subseteq \mathbb{T}$ an interval write $x < I < y$ if $x < z < y$ for $z \in I$ and $x, y \in \partial I$. Define $I_n, U_n \colon \mathbb{A}_n \to \mathbb{B}_n$ by

$$I_n(e_{x,y}) = \sum_{(I,J)} e_{I,J} \tag{4.2}$$

where the sum is over all $(I,J) \in \mathcal{I}_n^{(2)}$ with either $x < I$, $y < J$, and $\text{val}(\varphi^n, I) = (+, +)$ or $I < x$, $J < y$, and $\text{val}(\varphi^n, I) = (-, -)$, and

$$U_n(e_{x,y}) = \sum_{(I,J)} e_{I,J} \tag{4.3}$$

where the sum is over all $(I,J) \in \mathcal{I}_n^{(2)}$ with either $I < x$, $J < y$, and $\text{val}(\varphi^n, I) = (+, +)$ or $x < I$, $y < J$, and $\text{val}(\varphi^n, I) = (-, -)$. Note I_n and U_n are unital $*$-homomorphisms. Note also the sums on right-hand sides of (4.2) and (4.3) each contains at most two terms. Define the Elliott–Thomsen building block:

$$\mathbb{D}_n = \{(a, f) \in \mathbb{A}_n \oplus C([0,1], \mathbb{B}_n) : I_n(a) = f(0), U_n(a) = f(1)\}.$$

The following result was obtained in [50]:

Lemma 4.16. *We have that $C^*(R_\varphi^+(n)) \simeq \mathbb{D}_n$.*

Proof. This was shown in [50] and is essentially an extension of the idea in Lemma 4.15. For any $f \in C_c(R_\varphi^+(n))$ and $(I,J) \in \mathcal{I}_n^{(2)}$,

$$a = \lim_{t \to 0} \kappa_I^n(t) \quad \text{and} \quad b = \lim_{t \to 1} \kappa_I^n(t),$$
$$c = \lim_{t \to 0} \kappa_J^n(t) \quad \text{and} \quad d = \lim_{t \to 1} \kappa_J^n(t),$$

all exist in $\mathcal{D}_n$ as the endpoints of I and J respectively, and

- $f_{I,J}(t) \to 0$ as $t \to 0$ if $(a,c) \notin R_\varphi^+(n)$; $f_{I,J}(t) \to f(a,c)$ as $t \to 0$ if $(a,c) \in R_\varphi^+(n)$;
- $f_{I,J}(t) \to 0$ as $t \to 1$ if $(b,d) \notin R_\varphi^+(n)$; $f_{I,J}(t) \to f(b,d)$ as $t \to 1$ if $(b,d) \in R_\varphi^+(n)$.

In this way $f_{I,J}$ extends uniquely to $\tilde{f}_{I,J} \in C([0,1])$.

Define $*$-homomorphisms $\tilde{a}_n \colon C_c(R_\varphi^+(n)) \to \mathbb{A}_n$ and $\tilde{b}_n \colon C_c(R_\varphi^+(n)) \to C([0,1], \mathbb{B}_n)$ by

$$\tilde{a}_n(f) = \sum_{(x,y) \in \mathcal{D}_n^{(2)}} f(x,y) e_{x,y} \quad \text{and} \quad \tilde{b}_n(f) = \sum_{(I,J) \in \mathcal{I}_n^{(2)}} \tilde{f}_{I,J} e_{I,J}.$$

We have that $I_n(a_n(f)) = b_n(f)(0)$ and $U_n(a_n(f)) = b_n(f)(1)$, and so $(a_n(f), b_n(f)) \in \mathbb{D}_n$. It follows from an argument presented [50] similar to the argument in the proof Lemma 4.15 that $C_c(R_\varphi^+) \to \mathbb{D}_n$, $f \mapsto (a_n(f), b_n(f))$ is an isometric $*$-homomorphism and accordingly we have that $C_c(R_\varphi^+) \to \mathbb{D}_n$, $f \mapsto (a_n(f), b_n(f))$ extends to an injective $*$-homomorphism $\mu_n \colon C_r^*(R_\varphi^+(n)) \to \mathbb{D}_n$ which is also surjective by Lemma 6.6 in [50]. $\square$

Remark 4.17. We note the following in the light of Lemma 4.15 and Lemma 4.16:

(a) The short exact sequence (4.6) corresponds to

$$0 \longrightarrow S\mathbb{B}_n \xrightarrow{\iota_n} \mathbb{D}_n \xrightarrow{\pi_n} \mathbb{A}_n \longrightarrow 0 \tag{4.4}$$

where $\iota_n(f) = (0, f)$ and $\pi_n(a, f) = a$, as $\iota_n \circ b_n(g) = \mu_n(g)$ and $\pi_n \circ \mu_n(h) = a_n(h|_{\mathcal{D}_n})$ for all $g \in C_c(R_\varphi^+(n)|_{\mathbb{T}\setminus\mathcal{D}_n})$ and $h \in C_c(R_\varphi^+(n))$.

(b) It follows from Lemma 4.16 that

$$C_r^*(R_\varphi^+) \simeq \varinjlim(\mathbb{D}_n, \Phi_n). \tag{4.5}$$

where $\Phi_n \colon \mathbb{D}_n \to \mathbb{D}_{n+1}$ is the uniquely determined $*$-homomorphism corresponding to the $*$-homomorphism $\rho_n \colon C_r^*(R_\varphi^+(n)) \to C_r^*(R_\varphi^+(n+1))$, i.e. $\mu_{n+1} \circ \rho_n = \Phi_n \circ \mu_n$.

(c) The building block $\mathbb{D}_n$ is a recursive subhomogenous algebra by Proposition A.2, and $C_r^*(R_\varphi^+)$ is approximately subhomogenous with no dimension growth (cf. [36]).

It follows from (c) is stably finite (cf. [9]).

The following result was proved in [50] when $\varphi(\mathcal{C}) \subseteq \mathcal{C}$.

Proposition 4.18. *Assume φ is critically finite, and $\mathcal{D}$ is forward invariant and $\mathcal{C} \subseteq \mathcal{D}$. There are $*$-homomorphisms*

$$\zeta_n \colon \mathbb{A}_n \to \mathbb{A}_{n+1}, \quad \eta_n \colon C([0,1], \mathbb{B}_n) \to \mathbb{A}_{n+1}$$
$$\text{and} \quad \xi_n \colon C([0,1], \mathbb{B}_n) \to C([0,1], \mathbb{B}_{n+1})$$

such that ζ_n and η_n have orthogonal ranges and

$$\Phi_n \colon \mathbb{D}_n \to \mathbb{D}_{n+1}, \quad (a, f) \mapsto (\zeta_n(a) + \eta_n(f), \xi_n(f))$$

is the unique $*$-homomorphism corresponding to ρ_n, i.e. $\Phi_n \circ \mu_n = \mu_{n+1} \circ \rho_n$.

Proof. We will repeat the definitions of ζ_n, η_n and ξ_n so that they can be utilized later. Let $f \in C([0,1])$ and $(I, J) \in \mathcal{I}_n^{(2)}$ with $\varphi^n(I) = \varphi^n(J) = I_i \in \mathcal{I}$ for some $i \in \{1, \dots, N\}$. Define $\zeta_n(e_{x,y}) = e_{x,y}$ for all $(x, y) \in \mathcal{D}_n^{(2)}$;

$$\eta_n(f e_{I,J}) = \sum_{(x,y)\in\mathcal{R}_n(I,J)} f(\kappa_{I_i}^{-1}(c_{x,y})) e_{x,y}$$

where $\mathcal{R}_n(I, J)$ consists of all $(x, y) \in I \times J$ with $c_{x,y} = \varphi^n(x) = \varphi^n(y)$ and $\varphi(c_{x,y}) \in \mathcal{D}$;

$$\xi_n(f e_{I,J}) = \sum_{(I^\sigma, J^\sigma)\in\mathcal{S}_n(I,J)} f_\sigma e_{\sigma(I,J)}$$

where $\mathcal{S}_n(I, J)$ consists of all $\sigma(I, J) = (I^\sigma, J^\sigma) \in \mathcal{I}_{n+1}^{(2)}$ satisfying $I^\sigma \subseteq I$, $J^\sigma \subseteq J$ and with $\varphi^{n+1}(I^\sigma) = \varphi^{n+1}(J^\sigma) = I_{i_\sigma} \in \mathcal{I}$ for some $i_\sigma \in \{1, \dots, N\}$, and where $f_\sigma \in C([0,1])$ is the unique continuous extension of

$$t \mapsto f \circ \kappa_{I_i}^{-1} \circ (\varphi|_{I_i \cap \varphi^{-1}(I_{i_\sigma})})^{-1} \circ \kappa_{I_{i_\sigma}}(t), \quad t \in]0, 1[.$$

All ζ_n, η_n and ξ_n are $*$-homomorphisms, with ζ_n and η_n having orthogonal ranges, and

$$\Phi_n \circ \mu_n(g) = \mu_{n+1} \circ \rho_n(g)$$

for all $g \in C_c(R_\varphi^+(n))$ since

$$\zeta_n(a_n(g)) + \eta_n(b_n(g)) = \sum_{(x,y)\in\mathcal{D}_n^{(2)}} g(x,y)e_{x,y} + \sum_{(I,J)\in\mathcal{I}_n^{(2)}} \sum_{(x,y)\in\mathcal{R}_n(I,J)} g(x,y)e_{x,y}$$

$$= \sum_{(x,y)\in\mathcal{D}_{n+1}^{(2)}} g(x,y)e_{x,y} = a_{n+1}(\rho_n(g))$$

and

$$\xi_n(b_n(g)) = \sum_{(I,J)\in\mathcal{I}_n^{(2)}} \sum_{(I^\sigma,J^\sigma)\in\mathcal{S}_n(I,J)} (g_{I,J}^n)_\sigma e_{\sigma(I,J)} = \sum_{(I,J)\in\mathcal{I}_n^{(2)}} \sum_{(I^\sigma,J^\sigma)\in\mathcal{S}_n(I,J)} g_{I^\sigma,J^\sigma}^{n+1} e_{I^\sigma,J^\sigma}$$

$$= \sum_{(I,J)\in\mathcal{I}_{n+1}^{(2)}} g_{I,J}^{n+1} e_{I,J} = b_{n+1}(\rho_n(g)). \qquad \square$$

Remark 4.19. Assume φ is critically finite, and $\mathcal{D}$ is forward invariant but not that $\mathcal{C} \subseteq \mathcal{D}$. The only part of Proposition 4.18 requiring that $\mathcal{C} \subseteq \mathcal{D}$ is the part concerning the map ξ_n. Thus we still have for any $g \in C_c(R_\varphi^+(n))$ that

$$\pi_{n+1}(\Phi_n(\mu_n(g))) = a_{n+1}(\rho_n(g)) = \zeta_n(a_n(g)) + \eta_n(b_n(g)).$$

Remark 4.20. It follows from Proposition 6.2 in [29], Proposition 2.1.2 in [48], Lemma 4.15, Proposition 2.23, and Proposition 2.22 that $R_\varphi^+(n)$ is amenable and $C_r^*(R_\varphi^+(n))$ is nuclear. We could also use Lemma 4.16 instead of Lemma 4.15 to infer that $C_r^*(R_\varphi^+(n))$ is nuclear. Proposition 6.2 in [29] is in fact the analog of Proposition 2.30 mentioned in Remark 2.36. As a consequence we obtain from Proposition 2.30 the short exact sequence

$$0 \longrightarrow C_r^*(R_\varphi^+(n)|_{\mathbb{T}\setminus\mathcal{D}_n}) \longrightarrow C_r^*(R_\varphi^+(n)) \longrightarrow C_r^*(R_\varphi^+(n)|_{\mathcal{D}_n}) \longrightarrow 0 . \qquad (4.6)$$

It follows from the above together with (4.1) that R_φ^+ is amenable and $C_r^*(R_\varphi^+)$ is nuclear. It also follows from Proposition 2.30 that we have the short exact sequence

$$0 \longrightarrow C_r^*(R_\varphi^+|_U) \longrightarrow C_r^*(R_\varphi^+) \longrightarrow C_r^*(R_\varphi^+|_F) \longrightarrow 0$$

for any open R_φ^+-invariant subset U of $\mathbb{T}$ where $F = \mathbb{T}\setminus U$.

It follows from Proposition 2.32 and the above Γ_φ^+ is amenable and $C_r^*(\Gamma_\varphi^+)$ is nuclear. It also follows from Proposition 2.22 that $C_r^*(\Gamma_\varphi^+)$ satisfies the universal coefficient theorem. As a consequence we obtain from Proposition 2.30 the short exact sequence

$$0 \longrightarrow C_r^*(\Gamma_\varphi^+|_U) \longrightarrow C_r^*(\Gamma_\varphi^+) \longrightarrow C_r^*(\Gamma_\varphi^+|_F) \longrightarrow 0$$

for any open Γ_φ^+-invariant subset U of $\mathbb{T}$ where $F = \mathbb{T}\setminus U$.

Let us briefly review some consequences of transitivity and existence of periodic points:

Remark 4.21. It is easy to see that any point with non-trivial isotropy is also pre-periodic. If φ is transitive, then there exists a point with dense forward orbit and so trivial isotropy. In fact the set of points with trivial isotropy is dense, whenever φ is transitive:

Lemma 4.22. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous piecewise monotone circle map.*

   (i) *If φ is transitive and has periodic points, then Γ_φ^+ is locally contractive.*

   (ii) *If φ is transitive, then Γ_φ^+ is topologically principal.*

The reader is referred to Definition 2.16 and Definition 2.11.

Proof. The first statement is a combination of Lemma 3.11(iii), Lemma 3.13, Lemma 3.28. If φ is transitive, then the set of pre-periodic points is countable as a result of Lemma 3.13. It follows in particular that the set of points with non-trivial isotropy has empty interior, and consequently Γ_φ^+ is topologically principal. This proves the second statement.

    These statements are Corollary 4.3 and Lemma 4.4 in [50]. $\square$

Proposition 4.23. *If φ is transitive and has periodic points, then $C_r^*(\Gamma_\varphi^+)$ has the property that any non-zero heriditary C^*-subalgebra of $C_r^*(\Gamma_\varphi^+)$ contains an infinite projection.*

Proof. This is Proposition 4.5 in [50] and follows from Proposition 2.24 and Lemma 4.22.$\square$

    The following expands on the consequences of Proposition 2.35:

Remark 4.24. Let μ be a regular Borel probability measure on $\mathbb{T}$, then

$$\omega_\mu(f) = \int_\mathbb{T} f(x)\,\mu(dx)$$

where $f \in C_c(R_\varphi^+)$ describes a tracial state ω on $C_r^*(R_\varphi^+)$ if and only if μ is R_φ^+-invariant, and any tracial state on $C_r^*(R_\varphi^+)$ is completely determined this way as in Proposition 2.35.

Remark 4.25. It follows from Corollary V.2.1.16 in [9] there is a tracial state on $C_r^*(R_\varphi^+)$. Thus there always is an R_φ^+-invariant regular Borel probability measure μ on $\mathbb{T}$.

If φ is transitive, then there exists an orientation-preserving homeomorphism $g\colon \mathbb{T} \to \mathbb{T}$ such that $\psi = g^{-1} \circ \varphi \circ g$ is uniformly piecewise linear with slope $s \geq 1$ by Lemma 3.13. Let $\mu = g_*(\lambda)$ be the pushforward of the normalized Lebesgue measure λ on $\mathbb{T}$.

Lemma 4.26. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone. If $\psi = g^{-1} \circ \varphi \circ g$ is a uniformly piecewise linear circle map with slope $s \geq 1$ for some orientation-preserving homeomorphism $g\colon \mathbb{T} \to \mathbb{T}$, then $\mu = g_*(\lambda)$ is R_φ^+-invariant.*

Proof. Any open bisection W in R_φ^+ is an increasing union of sets $W_n = W \cap R_\varphi^+(n)$ and

$$\mu(r(W)) = \lim_{n\to\infty} \mu(r(W_n)) \quad \text{and} \quad \mu(s(W)) = \lim_{n\to\infty} \mu(s(W_n))$$

Also each W_n is a countable union of sets $\Omega(\eta) = \{(z, \eta(z)) \in \mathbb{T} \times \mathbb{T}\}$ where $\eta\colon U \to V$ are orientation-preserving locally defined homeomorphisms of $\mathbb{T}$ such that $\varphi^n(z) = \varphi^n(\eta(z))$. Also each open set $U \subseteq \mathbb{T}$ is an at most countable union of disjoint open intervals $I \subseteq \mathbb{T}$, and $\mu(I \setminus \mathcal{C}_n) = \mu(I)$, and $\eta|_I = (\varphi^n|_J)^{-1} \circ \varphi^n|_I$ when $\eta|_I\colon I \to \eta(I) = J$ with $I \cap \mathcal{C}_n = \emptyset$. Thus it suffices to show that $\mu(r(\Omega)) = \mu(s(\Omega))$ for any set of the form

$$\Omega = \{(g(z), g \circ \eta_\psi(z)) \in \mathbb{T} \times \mathbb{T} : z \in I\} = \{(g(z), \eta_\varphi \circ g(z)) \in \mathbb{T} \times \mathbb{T} : z \in I\} \subseteq R_\varphi^+(n)$$

for some open interval $I \subseteq \mathbb{T} \setminus \mathcal{C}_n$ and $n \in \mathbb{N}$ where $\eta_\psi\colon I \to \eta_\psi(I)$ and $\eta_\varphi\colon g(I) \to \eta_\varphi \circ g(I)$ are orientation-preserving homeomorphisms, and $\psi^n|_I\colon I \to \psi^n(I)$ and $\psi^n|_J\colon J \to \psi^n(J)$ are homeomorphisms where $J = \eta_\psi(I)$, and

$$\eta_\psi = (\psi^n|_J)^{-1} \circ \psi^n|_I \quad \text{and} \quad \eta_\varphi = (\varphi^n|_{g(J)})^{-1} \circ \varphi^n|_{g(I)}.$$

We have that $\mu(r(\Omega)) = \lambda(I) = \lambda(J) = \mu(s(\Omega))$. $\square$

Corollary 4.27. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be continuous uniformly piecewise linear with slope $s \geq 1$. The normalized Lebesgue measure λ on $\mathbb{T}$ is R_φ^+-invariant.*

We give the following introduction to Rieffel projections in $C_c(R_\varphi^+(n))$ for any $n \in \mathbb{N}$. This construction of projections was communicated to the author by Ian F. Putnam:

Remark 4.28. Let $a < b \leq c < d$, $d - a \leq 1$ with $I = \pi(a, b)$, $J = \pi(c, d)$ and $(I, J) \in \mathcal{I}_n^{(2)}$. We consider the locally defined homeomorphism $\gamma\colon I \to J$ given by $\gamma = (\varphi^n|_J)^{-1} \circ \varphi^n|_I$. Let $f_+\colon (a, b) \to [0, 1]$ and $f_-\colon (a, b) \to [0, 1]$ be continuous maps with $f_- = 1 - f_+$ and

$$\lim_{t \to a^+} f_+(t) = 0 \quad \text{and} \quad \lim_{t \to b^-} f_+(t) = 1.$$

Assume also in the following that $\{t \in (a, b) : f_+(t) \notin \{0, 1\}\} \subseteq [a + \varepsilon, b - \varepsilon]$ for some $\varepsilon > 0$. We are going to employ the identifications $(x, 0, y) \mapsto (x, y)$ and $(x, 0, x) \mapsto (x, x) \mapsto x$. Write $x = \pi(t)$ with $t \in (a, b)$ when $x \in I$. Define $f \in C_c(R_\varphi^+(n))$ by

$$f(x) = f_+(t) \quad \text{and} \quad f(\gamma(x)) = f_-(t)$$

for all $x \in I$; $f(x) = 1$ for all $x \in \pi([b, c])$ and $f(x) = 0$ for all $x \in \pi([d, a + 1])$;

$$f(x, \gamma(x)) = f(\gamma(x), x) = \sqrt{f_+(t) - f_+(t)^2} = \sqrt{f_-(t) - f_-(t)^2} = g(t)$$

for all $x \in I$ as $(x, \gamma(x)) \in R_\varphi^+(n)$; and $f(x, y) = 0$ when $x \neq y$ and $x \notin I \cup J$ or $y \notin I \cup J$.

This results in the function illustrated below with underlying groupoid.
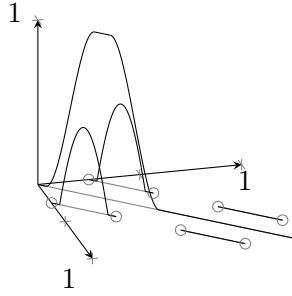
It is easy to verify case by case that $f^* = f$, and for each $x \in I$,

$$f^2(x) = f(x)^2 + f(x, \gamma(x))f(\gamma(x), x)$$
$$= f_+(t) = f(x),$$
$$f^2(\gamma(x)) = f(\gamma(x))^2 + f(\gamma(x), x)f(x, \gamma(x))$$
$$= f_-(t) = f(\gamma(x)),$$
$$f^2(x, \gamma(x)) = f(x)f(x, \gamma(x)) + f(x, \gamma(x))f(\gamma(x))$$
$$= (f_+(t) + f_-(t))g(t) = g(t) = f(x, \gamma(x)).$$

In addition $f^2(x) = f(x)^2$ for all $x \notin I \cup J$ and $f^2(x, y) = 0 = f(x, y)$ whenever $x \neq y$ and $x \notin I \cup J$ or $y \notin I \cup J$; i.e. $f^2 = f$. Thus f is a projection in $C_c(R_\varphi^+(n))$. Put $p_{I,J} = f$.

Definition 4.29. The function $p_{I,J} \in C_c(R_\varphi^+(n))$ is called a Rieffel projection.

Remark 4.30. Let ω_μ be the tracial state on $C_r^*(R_\varphi^+)$ which was described in Remark 4.24 where μ is a R_φ^+-invariant regular Borel probability measure on $\mathbb{T}$. Then

$$\omega_\mu(p_{I,J}) = \int_I f(x)\,\mu(dx) + \mu(\pi([b, c])) + \int_J f(x)\,\mu(dx)$$

If φ is uniformly piecewise linear and $\mu = \lambda$, then $\omega_\mu(p_{I,J}) = c - a = d - b$ as $b - a = d - c$. This property will be utilized several times later.

4.2   A characterization of simplicity

A first step in characterizing simplicity for $C_r^*(\Gamma_\varphi^+)$ and $C_r^*(R_\varphi^+)$ is to utilize Lemma 2.28.

Lemma 4.31. *Let* $\varphi\colon \mathbb{T} \to \mathbb{T}$ *be a continuous piecewise monotone circle map. Then*

(i) $C_r^*(\Gamma_\varphi^+)$ *is simple if and only if* Γ_φ^+ *is minimal,*
(ii) $C_r^*(R_\varphi^+)$ *is simple if and only if* R_φ^+ *is minimal.*

Proof. The second statement follows immediately from Lemma 2.28 since R_φ^+ is principal. The first is Lemma 5.3 in [50] and follows from Lemma 5.2 in [50] and Remark 2.29.   □

We do not offer a comprehensive analysis of Γ_φ^+-invariant or R_φ^+-invariant subsets of $\mathbb{T}$ but only broach some of the most essential conclusions made in [50] related to simplicity.

Remark 4.32. If R_φ^+ is minimal, then Γ_φ^+ is minimal, since $\mathcal{O}_{R_\varphi^+}(x) \subseteq \mathcal{O}_{\Gamma_\varphi^+}(x)$ for all $x \in \mathbb{T}$.

Lemma 4.33. *Let* $\varphi\colon \mathbb{T} \to \mathbb{T}$ *be a continuous surjective piecewise monotone circle map. If* Γ_φ^+ *is minimal, then* φ *is totally transitive.*

Proof. This is implied in [50] since otherwise φ is just transitive by Lemma 5.4 in [50] and Lemma 3.21 yields a non-empty, finite set $\mathcal{E}$ which is Γ_φ^+-invariant by Lemma 5.10 in [50].□

Theorem 4.39 and Theorem 4.43 both concern circle maps that are not locally injective. This restriction hinges on the following result, which is Lemma 5.5 in [50].

Lemma 4.34. *Let* $\varphi\colon \mathbb{T} \to \mathbb{T}$ *be a continuous piecewise monotone circle map with* $\mathcal{C} \neq \emptyset$. *If* φ *is transitive, then any* Γ_φ^+-*invariant set is dense or consists of finitely many points which are all post-critical and not pre-critical.*

The following is a dichotomy of sufficient conditions for minimality of Γ_φ^+ by degree of φ. The crux of the implications in this dichotomy is Lemma 4.34.

Lemma 4.35. *Let* $\varphi\colon \mathbb{T} \to \mathbb{T}$ *be a continuous piecewise monotone circle map with* $\mathcal{C} \neq \emptyset$. *A point* $x \in \mathbb{T}$ *is called exposed if* $\mathcal{O}_{\Gamma_\varphi^+}(x)$ *is finite. Put* $d = |\deg(\varphi)|$.

(i) *If* $d = 1$ *and* φ *is totally transitive, there is at most one exposed point.*
(ii) *If* $d = 0$ *and* φ *is totally transitive, then* Γ_φ^+ *is minimal.*
(iii) *If* $d \geq 2$ *and* φ *is transitive, then* Γ_φ^+ *is minimal.*

Proof. This is a combination of Lemma 5.15, Lemma 5.19 and Lemma 5.7 in [50].   □

Remark 4.36. If there exists an exposed point $x \in \mathbb{T}$, when $d = 1$ and φ is totally transitive, then it is the only exposed point and $\varphi^{-1}(\{x\}) \setminus \mathcal{C} = \{x\}$ as noted in Lemma 5.15 in [50]. If $x \in \mathbb{T}$ is a point satisfying $\varphi^{-1}(\{x\}) \setminus \mathcal{C} = \{x\}$, then $\{x\}$ is Γ_φ^+-invariant.

If φ is transitive but not totally transitive and $d = 1$, then the set $\mathcal{E}$ from Lemma 3.21 is Γ_φ^+-invariant by Lemma 5.14 in [50] and $\varphi^{-1}(\mathcal{E}) \setminus \mathcal{C} = \mathcal{E}$ by Lemma 5.10 in [50].

Definition 4.37. A point $x \in \mathbb{T}$ is an exceptional fixed point if $\varphi^{-1}(\{x\}) \setminus \mathcal{C} = \{x\}$.

Remark 4.38. The following continuous circle maps each have one exceptional fixed point; and respectively not transitive, transitive but not totally transitive, and totally transitive.
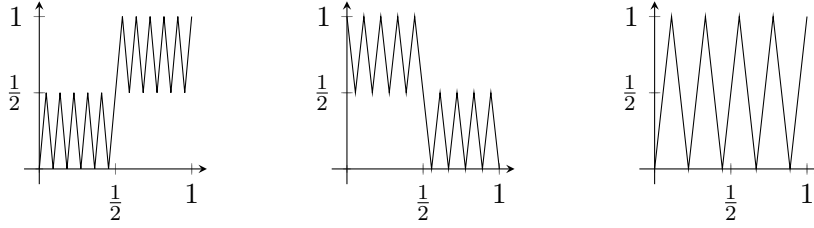
If φ is totally transitive, then the only possible exposed point is an exceptional fixed point. If φ is transitive but not totally transitive, then other invariant sets come into consideration. If φ is not transitive, then situation is completely different and partially untreated.

Lemma 5.16, Lemma 5.20 in [50] describes extensions that arise from exposed points, when φ is either transitive and not totally transitive, or totally transitive.

The following result is the culmination of results and discussions of this section so far. This corresponds to Theorem 5.21 in [50].

Theorem 4.39. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map as in Lemma 4.35 which is also surjective. Then Γ_φ^+ is minimal if and only if φ is totally transitive without exceptional fixed points.*

Proof. This follows from Lemma 4.35, Lemma 4.33, Lemma 4.34 and Remark 4.36. $\square$

Remark 4.40. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map as in Lemma 4.35 which is also surjective. Then φ is totally transitive if and only if φ is exact by Lemma 3.11(ii) and Lemma 3.19: Transitivity and the existence of critical points implies the existence of periodic points.

Remark 4.41. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map as in Lemma 4.35 which is also surjective. Assume $\varphi(\mathcal{C}) \subseteq \mathcal{C}$. Then Γ_φ^+ is minimal if and only if φ is transitive by Lemma 7.1 in [50]. This also applies to local homeomorphisms, i.e. $\mathcal{C} = \emptyset$, cf. Proposition 4.45(i).

We will now establish necessary and sufficient conditions for the simplicity of the core. This is a departure from [50] but draws inspiration particularly from Lemma 6.1 in [50].

Lemma 4.42. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous piecewise monotone circle map with $\mathcal{C} \neq \emptyset$. If R_φ^+ is minimal, then φ must be critically finite.*

Proof. We note $\mathcal{C}$ is finite. If there exists a critical point $x \in \mathbb{T}$ which is not pre-periodic we may assume that there are no other critical points in the forward orbit $\mathcal{O} = \mathcal{O}_\varphi^+(x)$. Let $c_1, \ldots, c_L$ be the critical points which meet $\mathcal{O}$ after $m_1, \ldots, m_L$ iterations respectively in that $\varphi^{m_i}(c_i) = \varphi^{n_i}(x) \in \mathcal{O}$ and $\varphi^l(c_i) \notin \mathcal{O}$ for all $0 < l < m_i$ and for all $i = 1, \ldots, L$. We note that the natural numbers $n_1, \ldots, n_L$ are all unique because x is not pre-periodic. Let $y \in \mathcal{O}_{R_\varphi^+}(x)$ and take $n \in \mathbb{N}$ minimal with $\varphi^n(x) = \varphi^n(y)$ and $\mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^n, y)$. We note that $y \in \varphi^{-k}(\{c\})$ for some $c \in \mathcal{C}$ and some $0 \le k < n$ for the valency to match. This means that $y \in \bigcup_{i=1}^L \varphi^{m_i - n_i}(\{c_i\})$, and so we have that $\mathcal{O}_{R_\varphi^+}(x) \subseteq \bigcup_{i=1}^L \varphi^{m_i - n_i}(\{c_i\})$. Thus $\mathcal{O}_{R_\varphi^+}(x)$ is finite and hence not dense in $\mathbb{T}$.

Theorem 4.43. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be circle map as in Lemma 4.35 which is also surjective. Then R_φ^+ is minimal if and only if Γ_φ^+ is minimal (Theorem 4.39) and φ is critically finite, i.e. if and only if φ is critically finite and exact and without exceptional fixed points.*

Proof. It follows from Remark 4.32, Lemma 4.42, Theorem 4.39, and also Lemma 3.19 that φ is exact without exceptional fixed points and critically finite, when R_φ^+ is minimal and φ is surjective and not locally injective. It remains to prove the converse implication. Assume that φ is exact and critically finite and has no exceptional fixed points.

The strategy in the following will be to show that $\mathcal{O}_{R_\varphi^+}(x)$ is dense in $\mathbb{T}$ for all $x \in \mathbb{T}$ by imitating the proof of Lemma 6.1 in [50]. Let $x \in \mathbb{T}$ and $I \subseteq \mathbb{T}$ an open interval.

(1) If $x \in \mathbb{T}$ is pre-critical with $\varphi^n(x) = c \in \mathcal{C}$, it follows that x is pre-periodic to a periodic orbit $\mathcal{O}$ of period p, since φ is critically finite. Take $k \in \mathbb{N}$ such that $\varphi^k(c) \in \mathcal{O}$. As φ is exact we may take $N \in \mathbb{N}$ such that $\varphi^N(I) = \mathbb{T}$ and $r \in 2\,\mathbb{N}$ such that $n + rp \geq N$. Notice that $\varphi^{n+rp}(I) = \mathbb{T}$ as φ is surjective and take $y \in I$ such $\varphi^{n+rp}(y) = c$. Then

$$\varphi^{n+k+(r+1)p}(x) = \varphi^{k+p}(c) = \varphi^{n+k+(r+1)p}(y),$$

$$\mathrm{val}(\varphi^{n+k+(r+1)p}, x) = \mathrm{val}(\varphi^{k+(r+1)p}, c) \bullet \mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^{k+(r+1)p}, c)$$

$$= \mathrm{val}(\varphi^{rp}, \varphi^k(c)) \bullet \mathrm{val}(\varphi^p, \varphi^k(c)) \bullet \mathrm{val}(\varphi^k, c)$$

$$= \mathrm{val}(\varphi^p, \varphi^k(c)) \bullet \mathrm{val}(\varphi^k, c) = \mathrm{val}(\varphi^{k+p}, c)$$

$$= \mathrm{val}(\varphi^{k+p}, c) \bullet \mathrm{val}(\varphi^{n+rp}, y)$$

$$= \mathrm{val}(\varphi^{n+k+(r+1)p}, y).$$

Therefore $y \in \mathcal{O}_{R_\varphi^+}(x)$, which finishes the case where $x \in \mathbb{T}$ is assumed to be pre-critical.

(2) If $x \in \mathbb{T}$ is not pre-critical, then we subdivide and obtain two separate cases assuming in addition that **(2a)** $x \in \mathbb{T}$ is pre-periodic, or **(2b)** $x \in \mathbb{T}$ is not pre-periodic. Before we do this notice that since φ is exact there exists $N \in \mathbb{N}$ such that $\varphi^{N-1}(I) = \mathbb{T}$, and $\varphi^N(I) = \mathbb{T}$ since φ is surjective, and notice that since φ is piecewise monotonous and not locally injective it follows that $\varphi^N|_I$ is piecewise monotonous and not locally injective, i.e. $\mathcal{C}_N \cap I \neq \emptyset$, as $\varphi^{N-1}(I) = \mathbb{T}$ so $\varphi^{N-1}(I) \cap \mathcal{C} = \mathcal{C} \neq \emptyset$ hence $\mathcal{C}_N \cap I \supseteq \varphi^{1-N}(\mathcal{C}) \cap I \neq \emptyset$. Therefore there are intervals $I_+, I_- \subseteq I$ and an interval J such that $\varphi^N(I_-) = \varphi^N(I_+) = J$ and $\mathrm{val}(\varphi^N, I_\pm) = (\pm, \pm)$, and we can choose $M \in N$ such that $\varphi^M(J) = \mathbb{T}$ as φ is exact. Put $L = N + M$ and note that if $b \in \mathbb{T}\setminus\varphi^L(\mathcal{C}_L)$ then there exists $y_\pm \in I_+ \cup I_-$ such that $b = \varphi^L(y_\pm)$ and $\mathrm{val}(\varphi^L, y_\pm) = (\pm, \pm)$.

(2a) If in addition to not being pre-critical $x \in \mathbb{T}$ is assumed to be pre-periodic to a periodic orbit $\mathcal{O}$ of period p, then we want to find $a \in \mathcal{O}$ such that for all $j \in \mathbb{N}$,

$$A_j = \varphi^{-j}(\{a\}) \setminus \left(\mathcal{C}_j \cup \textstyle\bigcup_{k=0}^{j-1} \varphi^{-k}(\mathcal{O})\right) \neq \emptyset.$$

Note that there are no $K \in \mathbb{N}$ such that $\varphi^{-j}(\mathcal{O}) \subseteq \mathcal{C}_j \cup \bigcup_{k=0}^{j-1} \varphi^{-k}(\mathcal{O})$ for all $j > K$, since otherwise $\mathcal{O}_{\Gamma_\varphi^+}(x) \subseteq \bigcup_{k=0}^{K} \varphi^{-k}(\mathcal{O})$ is finite, but $\mathcal{O}_{\Gamma_\varphi^+}(x)$ is dense in $\mathbb{T}$ for all $x \in \mathbb{T}$ by Lemma 4.31(i) and Theorem 4.39 as φ is both exact and has no exceptional fixed points. Take a sequence $j_1 < j_2 < \cdots$ such that $\varphi^{-j_l}(\mathcal{O}) \not\subseteq \mathcal{C}_{j_l} \cup \bigcup_{k=0}^{j_l-1} \varphi^{-k}(\mathcal{O})$ for all $l \in \mathbb{N}$, and for each $l \in \mathbb{N}$ choose $y_l \in \varphi^{-j_l}(\mathcal{O}) \setminus (\mathcal{C}_{j_l} \cup \bigcup_{k=0}^{j_l-1} \varphi^{-k}(\mathcal{O}))$. Furthermore since $\mathcal{O}$ is finite, there exists some $a \in \mathcal{O}$ such that $\varphi^{j_l}(y_l) = a$ for infinitely many l, and we may assume without loss of generality that this holds for all l by considering suitable subsequences. Therefore $A_{j_l} \neq \emptyset$ for all $l \in \mathbb{N}$, but in fact we have $A_j \neq \emptyset$ for all $j \in \mathbb{N}$ as claimed above, because if $A_j \neq \emptyset$ for some $1 < j \leq j_l$ we may choose $x_j \in A_j$ and note $\varphi(x_j) \in A_{j-1} \neq \emptyset$. Now we have found $a \in \mathcal{O}$ such that $A_j = \varphi^{-j}(\{a\}) \setminus (\mathcal{C}_j \cup \bigcup_{k=0}^{j-1} \varphi^{-k}(\mathcal{O})) \neq \emptyset$ for all $j \in \mathbb{N}$.

Notice that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, and so since $\varphi^L(\mathcal{C}_L)$ is finite there exists $K \in \mathbb{N}$ such that $\varphi^L(\mathcal{C}_L) \cap A_j = \emptyset$ for all $j \geq K$, in particular $B_j = \varphi^{-j}(\{a\}) \setminus (\mathcal{C}_j \cup \varphi^L(\mathcal{C}_L)) \neq \emptyset$. Now take $q, r \in \mathbb{N}$ such that $\varphi^q(a) = \varphi^{rp}(x)$, and take $s \geq r$ such that $sp - L - q \geq K$. Take any $b \in B_{sp-L-q}$ and $y_\pm \in I_+ \cup I_-$ such that $b = \varphi^L(y_\pm)$ and $\mathrm{val}(\varphi^L, y_\pm) = (\pm, \pm)$. Thus $\varphi^{sp}(y_\pm) = \varphi^{sp-L}(b) = \varphi^q(a) = \varphi^{rp}(x) = \varphi^{sp}(x)$ since $\varphi^{rp}(x) = \varphi^q(a) \in \mathcal{O}$ and $s \geq r$, so either $y_- \in \mathcal{O}_{R_\varphi^+}(x)$ or $y_+ \in \mathcal{O}_{R_\varphi^+}(x)$.

(2b) If $x \in \mathbb{T}$ is not pre-periodic, then it is not pre-critical since φ is critically finite. We want to find $K \in \mathbb{N}$ such that for all $j \geq K$,

$$A_j = \varphi^{-L}(\{\varphi^j(x)\}) \setminus (\mathcal{C}_L \cup \textstyle\bigcup_{k \in \mathbb{N}}\{\varphi^k(x)\}) \neq \emptyset.$$

Note there are only finitely many $j \in \mathbb{N}$ for which $\varphi^{-L}(\{\varphi^j(x)\}) \setminus \varphi^{j-L}(\{x\}) \subseteq \mathcal{C}_L$ since $\#\varphi^{-L}(\{\varphi^j(x)\}) > 1$ for all $j \in \mathbb{N}$, otherwise $\varphi^j(x) \in \varphi^L(\mathcal{C}_L)$ for infinitely many $j > L$, but $\varphi^L(\mathcal{C}_L)$ is finite and $\varphi^i(x) \neq \varphi^j(x)$ for all $i, j \in \mathbb{N}$ when $i \neq j$ as x is not pre-periodic. Thus there exists $K > L$ such that $A_j \neq \emptyset$ for all $j \geq K$, and we may choose $x_j \in A_j$. Define $B_j = \bigcup_{k \in \mathbb{N}} \varphi^{-kL}(\{x_j\}) \cup \{x_j\}$ for each $j \geq K$, and notice $B_i \cap B_j = \emptyset$ when $i \neq j$, because otherwise there would be $y \in \varphi^{-k_i L}(x_i) \cap \varphi^{-k_j L}(x_j)$ which is not possible since $\varphi^i(x) = \varphi^{(k+1)L}(y) = \varphi^j(x)$ if $k = k_i = k_j$ and $x_j = \varphi^{k_j L}(y) = \varphi^{(k_j - k_i - 1)L + i}(x)$ if $k_i < k_j$, and since $\varphi^L(\mathcal{C}_L)$ is finite there exists $m > 2$ such that $mL > K$ and $B_{mL} \cap \varphi^L(\mathcal{C}_L) = \emptyset$. Take $b \in \varphi^{(2-m)L}(x_{mL})$ and $y_\pm \in I_+ \cup I_-$ such that $b = \varphi^L(y_\pm)$ and $\mathrm{val}(\varphi^L, y_\pm) = (\pm, \pm)$. Notice that $\varphi^{jL}(b) \in \varphi^{-(m-j-2)L}(x_{mL}) \subseteq B_{mL}$ for all $j \in \{1, \ldots, m-2\}$ in particular $\varphi^{(j-1)L}(b) \notin \mathcal{C}_L$ so $b \notin \mathcal{C}_{(m-2)L}$ since $\mathrm{val}(\varphi^{(m-2)L}, b) = \mathrm{val}(\varphi^L, \varphi^{(m-3)L}(b)) \bullet \cdots \bullet \mathrm{val}(\varphi^L, b)$. Thus $\varphi^{mL}(y_\pm) = \varphi^{(m-1)L}(b) = \varphi^L(x_{mL}) = \varphi^{mL}(x)$ and $\mathrm{val}(\varphi^{mL}, y_\pm) \in \{(+, +), (-, -)\}$ since $\mathrm{val}(\varphi^{mL}, y_\pm) = \mathrm{val}(\varphi^L, x_{mL}) \bullet \mathrm{val}(\varphi^{(m-2)L}, b) \bullet \mathrm{val}(\varphi^L, y_\pm)$, $x_{mL} \notin \mathcal{C}_L$, $b \in \mathcal{C}_{(m-2)L}$ and $\mathrm{val}(\varphi^L, y_\pm) = (\pm, \pm)$, so $y_- \in \mathcal{O}_{R_\varphi^+}(x)$ or $y_+ \in \mathcal{O}_{R_\varphi^+}(x)$. $\qquad\square$

Remark 4.44. Theorem 3.36 yields a classification of transitive critically finite circle maps. The exact critically finite circle maps fits into this classification.

We now consider minimality of Γ_φ^+ and R_φ^+ when $\varphi\colon \mathbb{T} \to \mathbb{T}$ is a local homeomorphisms, because this was not done in Lemma 4.34, Lemma 4.35, Theorem 4.39 and Theorem 4.43, and because [50] does not cover Γ_φ^+ or R_φ^+ when $\varphi\colon \mathbb{T} \to \mathbb{T}$ is a local homeomorphisms, which could be justified as in [50] by referring the reader to [6], cf. Remark 4.8.

Proposition 4.45. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a local homeomorphism. Then*

(i) *Γ_φ^+ is minimal if and only if φ is transitive,*
(ii) *R_φ^+ is minimal if and only if φ is exact.*

Proof. If Γ_φ^+ is minimal, then φ is transitive because of Lemma 4.33, since φ is surjective. If φ is transitive, then φ^2 is strongly transitive by Remark 3.14 and Remark 3.11(i), and $\mathcal{O}_{\varphi^2}^-(x) \subseteq \mathcal{O}_{\Gamma_\varphi^+}(x)$ is dense in $\mathbb{T}$ for all $x \in \mathbb{T}$ by Proposition 1.20, so Γ_φ^+ is minimal

If R_φ^+ is minimal, then Γ_φ^+ is minimal by Remark 4.32, φ is transitive by Lemma 4.33, and in fact φ is exact by Remark 3.14 and Remark 4.7. If on the other hand φ is exact, then R_φ^+ is minimal by Remark 3.14 since for any open interval $I \subseteq \mathbb{T}$ and for any $x \in \mathbb{T}$ there exists $N \in \mathbb{N}$ such that $\varphi^{2N}(I) = \mathbb{T}$, and so $\varphi^{2N}(x) \in \varphi^{2N}(I)$ and $I \cap \mathcal{O}_{R_\varphi^+}(x) \neq \emptyset$, which means that $\mathcal{O}_{R_\varphi^+}(x)$ is dense in $\mathbb{T}$ for any $x \in \mathbb{T}$. $\qquad\square$

The reader may compare Proposition 4.45 to Proposition 4.3 and Proposition 4.1 in [16].

Remark 4.46. Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a local homeomorphism and $d = |\deg(\varphi)|$.

(a) If $d = 1$, then any fixed point is exceptional and R_φ^+ is not minimal by Remark 4.7,

(b) If $d \geq 2$, then none of the fixed points are exceptional.

This means that Proposition 4.45 is compatible with Theorem 4.39 and Theorem 4.43.

Remark 4.47. It turns out that Γ_φ^+ and R_φ^+ can be minimal even for non-surjective maps. Example 3.6 yields $\varphi = \psi_{1/2}^2$ which is not surjective, but for which Γ_φ^+ and R_φ^+ are minimal. The characterizations of minimality must be amended accordingly, when φ is not surjective.

4.3   The structure of a simple core

If φ is critically finite: Let $\mathcal{D} \subseteq \mathbb{T}$ such that $\mathcal{C}_n \subseteq \mathcal{D}_n$ for all $n \in \mathbb{N}$, e.g. $\mathcal{D} = \bigcup_{l=1}^\infty \varphi^l(\mathcal{C})$.
   The following result is an analogues of Lemma 7.2 in [50].

Lemma 4.48. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous piecewise monotone circle map with $\mathcal{C} \neq \emptyset$. If φ is exact, then there exists $k \in \mathbb{N}$ such that for any $l \geq k$ and $x \in \mathbb{T} \setminus \varphi^l(\mathcal{C}_l)$ there exists $y_\pm \in \mathbb{T}$ with $\varphi^l(y_\pm) = x$ and $\mathrm{val}(\varphi^l, y_\pm) = (\pm, \pm)$.*

Proof. As φ is not locally injective, there exist open intervals I_+, I_- and I such that $\mathrm{val}(\varphi, I_\pm) = (\pm, \pm)$ and $\varphi(I_\pm) = I$. As φ is exact, there exists $k \in \mathbb{N}$ such that $\varphi^{k-1}(I) = \mathbb{T}$. As φ is surjective, there exists $z \in I$ such that $\varphi^{l-1}(z) = x$, and $z \notin \mathcal{C}_{l-1}$ since $x \notin \varphi^l(\mathcal{C}_l)$. We may now take $y_\pm \in I_+ \cup I_-$ with $\varphi(y_\pm) = z$ and $\mathrm{val}(\varphi^l, y_\pm) = (\pm, \pm)$. □

Remark 4.49. Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous piecewise monotone circle map with $\mathcal{C} \neq \emptyset$. Define $k_\varphi \in \mathbb{N}$ to be the smallest number $k \in \mathbb{N}$ (whenever φ is exact) such that for any $l \geq k$ and $x \in \mathbb{T} \setminus \varphi^l(\mathcal{C}_l)$ there exists $y_\pm \in \mathbb{T}$ with $\varphi^l(y_\pm) = x$ and $\mathrm{val}(\varphi^l, y_\pm) = (\pm, \pm)$. Define the following numbers (whenever φ is critically finite) for each $c \in \mathcal{C}$:

- $l_c = \min\{l \in \mathbb{N}_0 : \varphi^{l+p}(c) = \varphi^l(c) \text{ for some } p \in \mathbb{N}\}$ and
- $q_c = \max\{q \in \mathbb{N}_0 : q \leq 2p_c, \varphi^l(c) \notin \mathcal{C}_q \text{ for some } l \geq l_c\}$ where
- $p_c = \min\{p \in \mathbb{N} : \varphi^{l_c+p}(c) = \varphi^{l_c}(c)\}$.

Define in addition the number $r_\varphi = \max\{l_c + q_c : c \in \mathcal{C}, \{\varphi(c), \ldots, \varphi^{l_c+p_c-1}(c)\} \nsubseteq \mathcal{C}\}$. Define $\mathrm{ord}(\varphi) = r_\varphi + k_\varphi + 1$ to be the order of φ when φ is exact and critically finite.
   These numbers are chosen to make way for Lemma 4.51, Remark 4.52 and Lemma 4.53. They have been chosen carefully to be as small as possible while still fitting the arguments. The arguments likely simplifies under further assumptions or by choosing bigger numbers. Almost all the numbers have an intuitive meaning:

- l_c is the *distance* from c to the period orbit $\mathcal{O}_\varphi^+(\varphi^{l_c}(c))$ in $\mathcal{O}_\varphi^+(c)$,
- q_c is the *shortest non-critical path* in $\mathcal{O}_\varphi^+(\varphi^{l_c}(c))$ of *length* less than $2p_c$,
- p_c is the period of the periodic orbit $\mathcal{O}_\varphi^+(\varphi^{l_c}(c))$.

Remark 4.50. If $\varphi(\mathcal{C}) \subseteq \mathcal{C}$, then $\mathrm{ord}(\varphi) = k_\varphi + 1$. In addition we have the following:

(a) If $\{\varphi^{l_c}(c), \ldots, \varphi^{l_c+p_c-1}(c)\} \nsubseteq \mathbb{T} \setminus \mathcal{C}$, then $q_c < p_c$.

(b) If $\{\varphi^{l_c}(c), \ldots, \varphi^{l_c+p_c-1}(c)\} \subseteq \mathbb{T} \setminus \mathcal{C}$, then $q_c = 2p_c$.

The following result is a generalization of Lemma 7.3 in [50] and Lemma 7.8 in [50]. For formulations which are more similar to the results in [50] see [49] or Remark 4.52.

Lemma 4.51. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous piecewise monotone circle map with $\mathcal{C} \neq \emptyset$. If φ is both exact and critically finite and without exceptional fixed points (Theorem 4.43), then we have the following properties related to Remark 4.52 and Lemma 4.53:*

(i) *For any $x \in \mathbb{T}$ there exists $y \in \mathbb{T}$ such that $(x, 1, y) \in \Gamma_\varphi^+(1, \mathrm{ord}(\varphi))$.*
(ii) *For any $y \in \mathbb{T}$ there exists $x \in \mathbb{T}$ such that $(x, 1, y) \in \Gamma_\varphi^+(1, \mathrm{ord}(\varphi))$.*

Proof. Let $x \in \mathbb{T}$. If $\varphi(x) \in \mathcal{C}_{\mathrm{ord}(\varphi)}$, then $\mathrm{val}(\varphi^{\mathrm{ord}(\varphi)}, y) = \mathrm{val}(\varphi^{\mathrm{ord}(\varphi)+1}, x)$ for $y = \varphi(x)$. If $x \in \mathcal{C}$ but $\varphi(x) \notin \mathcal{C}_{\mathrm{ord}(\varphi)}$, then $2p_x \leq r_\varphi$ and now for any $y \in \mathbb{T}$ with $\varphi^{2p_x-1}(y) = x$ we have that $\varphi^{\mathrm{ord}(\varphi)}(y) = \varphi^{\mathrm{ord}(\varphi)-(2p_x-1)}(x) = \varphi^{\mathrm{ord}(\varphi)+1}(x)$ and

$$\mathrm{val}(\varphi^{\mathrm{ord}(\varphi)}, y) = \mathrm{val}(\varphi^{p_x}, \varphi^{\mathrm{ord}(\varphi)+p_x}(y)) \bullet \mathrm{val}(\varphi^{p_x}, \varphi^{\mathrm{ord}(\varphi)}(y)) \bullet \mathrm{val}(\varphi^{\mathrm{ord}(\varphi)}, y)$$

$$= \mathrm{val}(\varphi^{\mathrm{ord}(\varphi)+1}, \varphi^{2p_x-1}(y)) \bullet \mathrm{val}(\varphi^{2p_x-1}, y) = \mathrm{val}(\varphi^{\mathrm{ord}(\varphi)+1}, x).$$

If $x \notin \mathcal{C}_{\mathrm{ord}(\varphi)+1}$ whereas $\varphi^{\mathrm{ord}(\varphi)+1}(x) \in \varphi^{\mathrm{ord}(\varphi)}(\mathcal{C}_{\mathrm{ord}(\varphi)})$ and $\{\varphi^{\mathrm{ord}(\varphi)+1+l}(x)\}_{l=0}^\infty \not\subseteq \mathbb{T} \setminus \mathcal{C}$, then $\varphi^{k_\varphi+1}(x) \notin \varphi^{k_\varphi}(\mathcal{C}_{k_\varphi})$ because $\varphi^{k_\varphi+1}(x) \notin \mathcal{C}_{r_\varphi+1}$ and by Lemma 4.48 there exists $y \in \mathbb{T}$ such that $\varphi^{k_\varphi}(y) = \varphi^{k_\varphi+1}(x)$ and $\mathrm{val}(\varphi^{k_\varphi}, y) = \mathrm{val}(\varphi^{k_\varphi+1}, x)$.
If $x \notin \mathcal{C}_{\mathrm{ord}(\varphi)+1}$ whereas $\varphi^{\mathrm{ord}(\varphi)+1}(x) \in \varphi^{\mathrm{ord}(\varphi)}(\mathcal{C}_{\mathrm{ord}(\varphi)})$ and $\{\varphi^{\mathrm{ord}(\varphi)+1+l}(x)\}_{l=0}^\infty \subseteq \mathbb{T} \setminus \mathcal{C}$, then there exists $z \in \mathbb{T}$ with $z \notin \varphi^{k_\varphi}(\mathcal{C}_{k_\varphi})$ and $z \notin \mathcal{C}_{r_\varphi+1}$ such that $\varphi^{r_\varphi+1}(z) = \varphi^{\mathrm{ord}(\varphi)+1}(x)$, since otherwise the Γ_φ^+-orbit of $\varphi^{\mathrm{ord}(\varphi)+1}(x)$ is finite, and by Lemma 4.48 there exists $y \in \mathbb{T}$ such that $\varphi^{k_\varphi}(y) = z$ and $\mathrm{val}(\varphi^{\mathrm{ord}(\varphi)}, y) = \mathrm{val}(\varphi^{r_\varphi+1}, z) \bullet \mathrm{val}(\varphi^{k_\varphi}, y) = \mathrm{val}(\varphi^{\mathrm{ord}(\varphi)+1}, x)$. If $x \notin \mathcal{C}_{\mathrm{ord}(\varphi)+1}$ and $\varphi^{\mathrm{ord}(\varphi)+1}(x) \notin \varphi^{\mathrm{ord}(\varphi)}(\mathcal{C}_{\mathrm{ord}(\varphi)})$, then by Lemma 4.48 there exists $y \in \mathbb{T}$ such that $\varphi^{\mathrm{ord}(\varphi)}(y) = \varphi^{\mathrm{ord}(\varphi)+1}(x)$ and $\mathrm{val}(\varphi^{\mathrm{ord}(\varphi)}, y) = \mathrm{val}(\varphi^{\mathrm{ord}(\varphi)+1}, x)$.

Let $y \in \mathbb{T}$. If $y \in \mathcal{C}_{\mathrm{ord}(\varphi)}$, then $\mathrm{val}(\varphi^{\mathrm{ord}(\varphi)+1}, x) = \mathrm{val}(\varphi^{\mathrm{ord}(\varphi)}, y)$ for any $x \in \varphi^{-1}(\{y\})$. If $y \notin \mathcal{C}_{\mathrm{ord}(\varphi)}$ whereas $\varphi^{\mathrm{ord}(\varphi)}(y) \in \varphi^{\mathrm{ord}(\varphi)}(\mathcal{C}_{\mathrm{ord}(\varphi)})$ and $\{\varphi^{\mathrm{ord}(\varphi)+l}(y)\}_{l=0}^\infty \not\subseteq \mathbb{T} \setminus \mathcal{C}$ this time, then $\varphi^{k_\varphi}(y) \notin \varphi^{k_\varphi+1}(\mathcal{C}_{k_\varphi+1})$ because $\varphi^{k_\varphi}(y) \notin \mathcal{C}_{r_\varphi+1}$ and by Lemma 4.48 there exists $x \in \mathbb{T}$ such that $\varphi^{k_\varphi+1}(x) = \varphi^{k_\varphi}(y)$ and $\mathrm{val}(\varphi^{k_\varphi+1}, x) = \mathrm{val}(\varphi^{k_\varphi}, y)$.
If $y \notin \mathcal{C}_{\mathrm{ord}(\varphi)}$ whereas $\varphi^{\mathrm{ord}(\varphi)}(y) \in \varphi^{\mathrm{ord}(\varphi)}(\mathcal{C}_{\mathrm{ord}(\varphi)})$ and $\{\varphi^{\mathrm{ord}(\varphi)+l}(y)\}_{l=0}^\infty \subseteq \mathbb{T} \setminus \mathcal{C}$ this time, then there is a $z \in \mathbb{T}$ with $z \notin \varphi^{k_\varphi+1}(\mathcal{C}_{k_\varphi+1})$ and $z \notin \mathcal{C}_{r_\varphi+1}$ such that $\varphi^{r_\varphi+1}(z) = \varphi^{\mathrm{ord}(\varphi)}(y)$, since otherwise the Γ_φ^+-orbit of $\varphi^{\mathrm{ord}(\varphi)}(y)$ is finite and by Lemma 4.48 there exists $x \in \mathbb{T}$ with $\varphi^{k_\varphi+1}(x) = z$ and $\mathrm{val}(\varphi^{\mathrm{ord}(\varphi)+1}, x) = \mathrm{val}(\varphi^{r_\varphi+1}, z) \bullet \mathrm{val}(\varphi^{k_\varphi+1}, x) = \mathrm{val}(\varphi^{\mathrm{ord}(\varphi)}, y)$. If $y \notin \mathcal{C}_{\mathrm{ord}(\varphi)}$ and $\varphi^{\mathrm{ord}(\varphi)}(y) \notin \varphi^{\mathrm{ord}(\varphi)}(\mathcal{C}_{\mathrm{ord}(\varphi)})$, then again by Lemma 4.48 there exists $x \in \mathbb{T}$ such that $\varphi^{\mathrm{ord}(\varphi)+1}(x) = \varphi^{\mathrm{ord}(\varphi)}(y)$ and $\mathrm{val}(\varphi^{\mathrm{ord}(\varphi)+1}, x) = \mathrm{val}(\varphi^{\mathrm{ord}(\varphi)}, y)$. $\square$

Remark 4.52. We make the same assumptions as in Lemma 4.51. Put $l = \mathrm{ord}(\varphi)$.

Lemma 4.51(i) gives Lemma 7.3 in [50]: Let $(x, k, y) \in \Gamma_\varphi^+(k, m)$ with $n = k + m \geq l + 1$. There exists $z \in \mathbb{T}$ such that $\varphi^l(z) = \varphi^{l+1}(x)$ and $\mathrm{val}(\varphi^l, z) = \mathrm{val}(\varphi^{l+1}, x)$, and

$$\varphi^{n-1}(z) = \varphi^n(x) = \varphi^m(y)$$
$$\mathrm{val}(\varphi^{n-1}, z) = \mathrm{val}(\varphi^{n-1-l}, \varphi^l(z)) \bullet \mathrm{val}(\varphi^l, z)$$
$$= \mathrm{val}(\varphi^{n-1-l}, \varphi^{l+1}(x)) \bullet \mathrm{val}(\varphi^{l+1}, x)$$
$$= \mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^m, y)$$

This version of Lemma 7.3 in [50] in turn implies Lemma 7.4 in [50] and Lemma 4.51(i).

Lemma 4.51(ii) gives Lemma 7.8 in [50]: Let $(x, k, y) \in \Gamma_\varphi^+(k, m)$ with $n = k + m \geq l$. There exists $z \in \mathbb{T}$ such that $\varphi^{l+1}(z) = \varphi^l(x)$ and $\mathrm{val}(\varphi^{l+1}, z) = \mathrm{val}(\varphi^l, x)$, and

$$\varphi^{n+1}(z) = \varphi^n(x) = \varphi^m(y)$$
$$\mathrm{val}(\varphi^{n+1}, z) = \mathrm{val}(\varphi^{n-l}, \varphi^{l+1}(z)) \bullet \mathrm{val}(\varphi^{l+1}, z)$$
$$= \mathrm{val}(\varphi^{n-l}, \varphi^l(x)) \bullet \mathrm{val}(\varphi^l, x)$$
$$= \mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^m, y)$$

The above version of Lemma 7.8 in [50] in turn implies Lemma 4.51(ii).

Lemma 4.53. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous piecewise monotone circle map with $\mathcal{C} \neq \emptyset$. If φ is both exact and critically finite and without exceptional fixed points (Theorem 4.43), then $\mathcal{D}(\pm)_n = \mathcal{D}(\pm)_{\mathrm{ord}(\varphi)}$ and $\mathcal{I}(\pm)_n = \mathcal{I}(\pm)_{\mathrm{ord}(\varphi)}$ for all $n \geq \mathrm{ord}(\varphi)$.*

Proof. Let $(d, v) \in \mathcal{D}(\pm)_n$ with $n \geq \mathrm{ord}(\varphi)$, and $x \in \mathcal{D}_n$ such that $\varphi^n(x) = d$ and $\mathrm{val}(\varphi^n, x)$. It follows from Remark 4.51 (with $k = 0$ and $y = x$) that there exists $z \in \mathbb{T}$ such that

$$\varphi^{n+1}(z) = \varphi^n(x) \quad \text{and} \quad \mathrm{val}(\varphi^{n+1}, z) = \mathrm{val}(\varphi^n, x).$$

This means that $(d, v) \in \mathcal{D}(\pm)_{n+1}$. We may conclude that $\mathcal{D}(\pm)_n \subseteq \mathcal{D}(\pm)_{n+1}$.

Let $(d, v) \in \mathcal{D}(\pm)_n$ with $n \geq \mathrm{ord}(\varphi) + 1$, and $x \in \mathcal{D}_n$ such that $\varphi^n(x) = d$ and $\mathrm{val}(\varphi^n, x)$. It follows from Remark 4.51 (with $k = 0$ and $y = x$) that there exists $z \in \mathbb{T}$ such that

$$\varphi^{n-1}(z) = \varphi^n(x) \quad \text{and} \quad \mathrm{val}(\varphi^{n-1}, z) = \mathrm{val}(\varphi^n, x).$$

This means that $(d, v) \in \mathcal{D}(\pm)_{n-1}$. We may conclude that $\mathcal{D}(\pm)_n \subseteq \mathcal{D}(\pm)_{n-1}$.

A very argument shows that $\mathcal{I}(\pm)_n = \mathcal{I}(\pm)_{n+1}$ for all $n \geq \mathrm{ord}(\varphi)$. $\qquad\square$

Remark 4.54. Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be as in Lemma 4.53. We write

$$\mathcal{D}(\pm) = \mathcal{D}(\pm)_{\mathrm{ord}(\varphi)} \quad \text{and} \quad \mathcal{I}(\pm) = \mathcal{I}(\pm)_{\mathrm{ord}(\varphi)}.$$

Lemma 4.55. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map that is also piecewise monotone. If R_φ^+ is minimal, then $\min_{x \in \mathbb{T}} \# \mathcal{O}_{R_\varphi^+(n)}(x) \to \infty$ as $n \to \infty$.*

Proof. Assume the sequence $\{\min_{x \in \mathbb{T}} \# \mathcal{O}_{R_\varphi^+(n)}(x)\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ is bounded by some $K \in \mathbb{N}$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{T}$ such that $\{\# \mathcal{O}_{R_\varphi^+(n)}(x_n)\}_{n \in \mathbb{N}}$ is bounded by K. We consider a subsequence $\{x_{n_l}\}_{l \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ with $x_{n_l} \to x$ as $l \to \infty$ for some $x \in \mathbb{T}$. We have that

$$\mathcal{O}_{R_\varphi^+(0)}(x) \subseteq \mathcal{O}_{R_\varphi^+(1)}(x) \subseteq \cdots \quad \text{and} \quad \mathcal{O}_{R_\varphi^+}(x) = \bigcup_{l \in \mathbb{N}} \mathcal{O}_{R_\varphi^+(n_l)}(x).$$

Also $\mathcal{O}_{R_\varphi^+}(x)$ is infinite as R_φ^+ is minimal, so $\# \mathcal{O}_{R_\varphi^+(n_L)}(x) \geq K + 1$ for some $L \in \mathbb{N}$.

We may take an open neighbourhood W_y of (y, x) in $R_\varphi^+(n_L)$ for each $y \in \mathcal{O}_{R_\varphi^+(n_L)}(x)$. We can choose the sets W_y such that all $r(W_y)$ are disjoint, since $\# \mathcal{O}_{R_\varphi^+(n_L)}(x)$ is finite, and we can find $l > L$ such that $x_{n_l} \in s(W_y)$ for all $y \in \mathcal{O}_{R_\varphi^+(n_L)}(x)$, so we may choose $\gamma_y \in W_y$ all distinct such that $s(\gamma_y) = x_{n_l}$ since all $r(W_y)$ are disjoint. This means that

$$\# \mathcal{O}_{R_\varphi^+(n_l)}(x_{n_l}) \geq \# \mathcal{O}_{R_\varphi^+(n_L)}(x_{n_l}) \geq \# \mathcal{O}_{R_\varphi^+(n_L)}(x) \geq K + 1$$

This contradict $\# \mathcal{O}_{R_\varphi^+(n_l)}(x_{n_l}) \leq K$. $\qquad\square$

Remark 4.56. If R_φ^+ is minimal, then Lemma 1.8 and Corollary 1.9 in [36] implies that

$$\min_{(d,v)\in\mathcal{D}(\pm)_n} a_n(d,v) \to \infty \quad \text{as} \quad n \to \infty. \tag{4.7}$$

This hinges on Lemma 4.31(ii). If R_φ^+ is minimal, then (4.7) alternatively follows from

$$\min_{(d,v)\in\mathcal{D}(\pm)_n} a_n(d,v) \geq \min_{x\in\mathcal{D}_n} \#\mathcal{O}_{R_\varphi^+(n)}(x) \geq \min_{x\in\mathbb{T}} \#\mathcal{O}_{R_\varphi^+(n)}(x)$$

since $\min_{x\in\mathbb{T}} \#\mathcal{O}_{R_\varphi^+(n)}(x) \to \infty$ as $n \to \infty$ by Lemma 4.55.

The following result is Lemma 3.2 in [57].

Lemma 4.57. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous piecewise monotone circle map with $\mathcal{C} \neq \emptyset$. Let μ and ν be any two non-atomic R_φ^+-invariant regular Borel probability measures on $\mathbb{T}$. If φ is exact, then $\mu = \nu$. As a consequence $\omega_\mu = \omega_\nu$.*

Lemma 4.58. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous piecewise monotone circle map with $\mathcal{C} \neq \emptyset$. If φ is both exact and critically finite and without exceptional fixed points (Theorem 4.43), then there exists in fact a unique R_φ^+-invariant regular Borel probability measure μ_φ on $\mathbb{T}$. As a consequence there exists a unique tracial state $\omega_\varphi = \omega_\mu$ on $C_r^*(R_\varphi^+)$.*

Proof. There exists an R_φ^+-invariant regular Borel probability measure by Remark 4.25. We must show for any R_φ^+-invariant regular Borel probability measure μ on $\mathbb{T}$ and $x \in \mathbb{T}$,

$$\mu(\{x\}) = 0.$$

This is where we apply Lemma 4.57 which states that there is at most one such measure. Theorem 4.43 implies that R_φ^+ is minimal. Therefore $\mathcal{O}_{R_\varphi^+}(x)$ is infinite in $\mathbb{T}$ for all $x \in \mathbb{T}$.

Assume μ is an R_φ^+-invariant regular Borel probability measure on $\mathbb{T}$ and $x_0 \in \mathbb{T}$ with

$$\mu(\{x_0\}) > 0.$$

Let $x \in \mathcal{O}_{R_\varphi^+}(x_0)$. Take an open bisection $W \subseteq R_\varphi^+$ and $\xi \in W$ with $s(\xi) = x_0$ and $r(\xi) = x$. Consider a sequence of open sets $W \supseteq W_1 \supseteq W_2 \supseteq \ldots$ such that $\bigcap_{n\in\mathbb{N}} W_n = \{\xi\}$. Then

$$\mu(\{x\}) = \lim_{n\to\infty} \mu(r(W_n)) = \lim_{n\to\infty} \mu(s(W_n)) = \mu(\{x_0\})$$

As $\mathcal{O}_{R_\varphi^+}(x_0)$ is infinite we may take a sequence $\{x_n\}_{n=1}^\infty$ of distinct elements $x_n \in \mathcal{O}_{R_\varphi^+}(x_0)$, so $1 = \mu(\mathbb{T}) \geq \mu(\bigcup_{n\in\mathbb{N}}\{x_n\}) = \sum_{n=1}^\infty \mu(\{x_n\}) = \infty$ which is a contradiction. $\qquad\square$

Lemma 4.59. *Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map that is also piecewise monotone. If φ is both exact and critically finite and without exceptional fixed points (Theorem 4.43), then for any $\varepsilon > 0$ there exists a projection $p \in C_r^*(R_\varphi^+)$ such that $\omega_\varphi(p) < \varepsilon$.*

Proof. We may assume that φ is a uniformly piecewise linear map due to Lemma 3.13. Let $\varepsilon > 0$ and $[0,1) = [0,\frac{1}{L}) \cup [\frac{1}{L},\frac{2}{L}) \cup \cdots \cup [\frac{L-1}{L},1)$ with $\frac{3}{L} < \varepsilon$, and choose $n \in \mathbb{N}$ and $(d,v) \in \mathcal{D}(\pm)_n$ such that $a_n(d,v) \geq L+1$ by Remark 4.56, and choose $l \in \{1,\ldots,L\}$ and $x,y \in [\frac{l-1}{L},\frac{l}{L})$, $x \neq y$ and such that $\varphi^n(x) = \varphi^n(y) = d$ and $\mathrm{val}(\varphi^n,x) = \mathrm{val}(\varphi^n,y) = v$. For $I,J \in \mathcal{I}_n$ with $x < I$ and $y < J$ we have $(I,J) \in \mathcal{I}_n^{(2)}$, and

$$\lambda(I) = \lambda(J) \leq \mathrm{dist}(x,y) \leq \tfrac{1}{L} \quad \text{and} \quad \mathrm{dist}(I,J) \leq \tfrac{1}{L}, \qquad\qquad \square$$

Thus $\min\{\omega_\varphi(p_{I,J}),\omega_\varphi(p_{J,I})\} \leq \frac{3}{L} < \varepsilon$ by Remark 4.30.

Remark 4.60. Another line of reasoning for Lemma 4.59 is also presented in Appendix A. This is based on Elliott–Thomsen building blocks and especially Lemma A.5.

In the following we consider some of the formal consequences of results obtained above. We will refer to [35] and [36] though some of these consequences may be deduced otherwise. The reader is referred to [9] and [48] along with [36] for introductions to these concepts.

Remark 4.61. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map that is also piecewise monotone. If φ is both exact and critically finite and without exceptional fixed points (Theorem 4.43), then the following structural properties applies to $C_r^*(R_\varphi^+)$:

(a) $C_r^*(R_\varphi^+)$ has strict slow dimension growth by Corollary 1.9 in [36] and Remark 4.17(c).

(b) $C_r^*(R_\varphi^+)$ topological stable rank one by Theorem 3.6 in [36].

(c) $C_r^*(R_\varphi^+)$ has real rank zero by Theorem 4.2 in [35] and Lemma 4.58, Lemma 4.59.

(d) $C_r^*(R_\varphi^+)$ has the order on projections determined by traces by Theorem 2.3 in [36].

(e) $C_r^*(R_\varphi^+)$ has the cancellation property by Theorem 2.2 in [36].

In particular $C_r^*(R_\varphi^+) \in \mathfrak{A}$ and $C_r^*(R_\varphi^+) \in \mathfrak{B}$, cf. Theorem A.17 and Theorem A.18.

CHAPTER 5

A classification of simple cores

Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be continuous piecewise monotone circle maps. We consider the invariant

$$F(C_r^*(R_\varphi^+)) = (K_0(C_r^*(R_\varphi^+)), K_0(C_r^*(R_\varphi^+))^+, [1]_0, K_1(C_r^*(R_\varphi^+)))$$

This invariant turns out to be complete when restricting the class of circle maps:

Remark 5.1. Let $\varphi, \psi\colon \mathbb{T} \to \mathbb{T}$ be circle maps which are continuous and piecewise monotone. If φ and ψ are exact and critically finite and without exceptional fixed points, then

$$C_r^*(R_\varphi^+) \simeq C_r^*(R_\psi^+) \Leftrightarrow F(C_r^*(R_\varphi^+)) \simeq F(C_r^*(R_\psi^+))$$

This classification result follows from Theorem 4.43, Remark 4.17 and Remark 4.61, since

- $C_r^*(R_\varphi^+)$ is unital, separable, infinite dimensional and simple,
- $C_r^*(R_\varphi^+)$ is approximately subhomogeneous with no dimension growth,
- $C_r^*(R_\varphi^+)$ has a unique tracial state and real rank zero.

The same properties are of course satisfied for $C_r^*(R_\psi^+)$.

It follows from (4.1) that for $i = 0, 1$,

$$K_i(C_r^*(R_\varphi^+)) \simeq \bigcup_{n \in \mathbb{N}} K_i(C_r^*(R_\varphi^+(n))) = \varinjlim(K_i(C_r^*(R_\varphi^+(n))), (\rho_n)_i). \qquad (5.1)$$

We make the following initial observations:

Remark 5.2. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be a circle map which is continuous and piecewise monotone.

(a) $K_0(C_r^*(R_\varphi^+))$ is a countable ordered abelian group, cf. [10] or [47].
(b) $K_0(C_r^*(R_\varphi^+))$ is torsion-free since $K_0(C_r^*(R_\varphi^+(n)))$ are torsion-free.
(c) $K_0(C_r^*(R_\varphi^+))$ is a simple ordered abelian group when R_φ^+ is minimal, cf. [10] or [47].

If φ is exact and critically finite and without exceptional fixed points, then

(d) $K_0(C_r^*(R_\varphi^+))$ is weakly unperforated by Theorem 2.4 in [36].
(e) $K_0(C_r^*(R_\varphi^+))$ is an interpolation group by Theorem 4.2 in [36].
(f) $K_0(C_r^*(R_\varphi^+))$ has a unique state $(\omega_\varphi)_*$ by Lemma 4.58 and Remark A.22.

In conclusion $K_0(C_r^*(R_\varphi^+)))$ is a countable simple dimension group, and by Remark 4.61(d),

$$K_0(C_r^*(R_\varphi^+))^+ = \{g \in K_0(C_r^*(R_\varphi^+)) : \omega_*(g) > 0\} \cup \{0\}.$$

It follows from Theorem 2.1 in [36] that

$$K_1(C_r^*(R_\varphi^+)) \simeq \mathcal{U}(C_r^*(R_\varphi^+))/\mathcal{U}_0(C_r^*(R_\varphi^+)).$$

The potential range of the invariant is reduced significantly by the above observations.

5.1   The building blocks and connecting maps

We have already observed that (5.1) for $i = 0, 1$. It follows from (4.5) that for $i = 0, 1$,

$$K_i(C_r^*(R_\varphi^+)) \simeq \varinjlim(K_i(\mathbb{D}_n), (\Phi_n)_i).$$

It is important to understand the building blocks $K_i(\mathbb{D}_n)$ and the connecting maps $(\Phi_n)_i$.

Remark 5.3. We make the following deductions from Remark A.12:

(a) It follows from (4.4) that we have the exact sequence:

$$0 \longrightarrow K_0(\mathbb{D}_n) \xrightarrow{(\pi_n)_0} K_0(\mathbb{A}_n) \xrightarrow{(\delta_n)_0} K_1(S\mathbb{B}_n) \xrightarrow{(\iota_n)_1} K_1(\mathbb{D}_n) \longrightarrow 0 . \qquad (5.2)$$

(b) In particular $K_0(\mathbb{D}_n) \simeq \ker(\delta_n)_0$ and $K_1(\mathbb{D}_n) \simeq \operatorname{coker}(\delta_n)_0$ where

$$K_0(\mathbb{A}_n) \simeq \mathbb{Z}^{\mathcal{D}(\pm)_n} \quad \text{and} \quad K_0(\mathbb{B}_n) \simeq K_1(S\mathbb{B}_n) \simeq \mathbb{Z}^{\mathcal{I}(\pm)_n} . \qquad (5.3)$$

The two finitely generated free abelian groups $\mathbb{Z}^{\mathcal{D}(\pm)_n}$ and $\mathbb{Z}^{\mathcal{I}(\pm)_n}$ have ordered bases $\{[d, v] : (d, v) \in \mathcal{D}(\pm)_n \subseteq \mathcal{D} \times \mathcal{V}\}$ and $\{[I, w] : (I, w) \in \mathcal{I}(\pm)_n \subseteq \mathcal{I} \times \mathcal{W}\}$ respectively so, where the orders are induced by the inherited lexicographical orders on $\mathcal{D}(\pm)_n$ and $\mathcal{I}(\pm)_n$, where $\mathcal{D}$ and $\mathcal{I}$ could be ordered cyclically and $\mathcal{V}$ and $\mathcal{W}$ could be ordered by

$$(-, +) < (+, -) < (+, +) < (-, -). \qquad (5.4)$$

Remark 5.4. Let $\beta_{\mathbb{B}_n} : K_0(\mathbb{B}_n) \to K_1(S\mathbb{B}_n)$ be the Bott map. Then by Proposition A.13,

$$(\delta_n)_0 = \beta_{\mathbb{B}_n} \circ ((I_n)_0 - (U_n)_0). \qquad (5.5)$$

We invoke the isomorphisms in (5.3) and abuse notation by writing

- $(I_n)_0, (U_n)_0 \colon \mathbb{Z}^{\mathcal{D}(\pm)_n} \to \mathbb{Z}^{\mathcal{I}(\pm)_n}$ as $K_0(\mathbb{A}_n) \simeq \mathbb{Z}^{\mathcal{D}(\pm)_n}$ and $K_0(\mathbb{B}_n) \simeq \mathbb{Z}^{\mathcal{I}(\pm)_n}$,
- $\beta_{\mathbb{B}_n} \colon \mathbb{Z}^{\mathcal{I}(\pm)_n} \to \mathbb{Z}^{\mathcal{I}(\pm)_n}$ as $K_0(\mathbb{B}_n) \simeq \mathbb{Z}^{\mathcal{I}(\pm)_n}$ and $K_1(S\mathbb{B}_n) \simeq \mathbb{Z}^{\mathcal{I}(\pm)_n}$,
- $(\delta_n)_0 \colon \mathbb{Z}^{\mathcal{D}(\pm)_n} \to \mathbb{Z}^{\mathcal{I}(\pm)_n}$ as $K_0(\mathbb{A}_n) \simeq \mathbb{Z}^{\mathcal{D}(\pm)_n}$ and $K_1(S\mathbb{B}_n) \simeq \mathbb{Z}^{\mathcal{I}(\pm)_n}$.

It follows that $\beta_{\mathbb{B}_n}([I, w]) = [I, w]$ for all $(I, w) \in \mathcal{I}(\pm)_n$, and so $(\delta_n)_0 = (I_n)_0 - (U_n)_0$. Thus $K_0(C_r^*(R_\varphi^+(n))) \simeq \ker((I_n)_0 - (U_n)_0)$ and $K_1(C_r^*(R_\varphi^+(n))) \simeq \operatorname{coker}((I_n)_0 - (U_n)_0)$.

Remark 5.5. An exact sequence similar to (5.2) is obtained from (4.6) with

$$(\delta_n^!)_0 \colon K_0(C_r^*(R_\varphi^+(n)|_{\mathcal{D}_n})) \to K_1(C_r^*(R_\varphi^+(n)|_{\mathbb{T}\setminus\mathcal{D}_n})).$$

It follows from Lemma 4.15 that for $i = 0, 1$,

$$K_i(C_r^*(R_\varphi^+(n)|_{\mathcal{D}_n})) \simeq K_i(\mathbb{A}_n) \quad \text{and} \quad K_i(C_r^*(R_\varphi^+(n)|_{\mathbb{T}\setminus\mathcal{D}_n})) \simeq K_i(S\mathbb{B}_n).$$

It follows from Lemma 4.16 in a similar way that $K_i(C_r^*(R_\varphi^+(n))) \simeq K_i(\mathbb{D}_n)$ for $i = 0, 1$. We may now conclude that $K_0(C_r^*(R_\varphi^+(n))) \simeq \ker(\delta_n^!)_0$ and $K_1(C_r^*(R_\varphi^+(n))) \simeq \operatorname{coker}(\delta_n^!)_0$. Also $(\delta_n)_0 \circ (a_n)_0 = (b_n)_1 \circ (\delta_n^!)_0$, and for all $(d, v) \in \mathcal{D}(\pm)_n$ and $(I, w) \in \mathcal{I}(\pm)_n$,

$$(a_n)_0([d, v]) = [d, v] \quad \text{and} \quad (b_n)_1([I, w]) = (-1)^w [I, w].$$

This is a consequence of κ_J^n possibly being orientation-reversing for $J \in \mathcal{I}_n$.

We proceed by giving a careful description of the maps $(I_n)_0, (U_n)_0 \colon \mathbb{Z}^{\mathcal{D}(\pm)_n} \to \mathbb{Z}^{\mathcal{I}(\pm)_n}$.

Proposition 5.6. *Let* $(d, v) \in \mathcal{D}(\pm)_n$ *with* $I > d > J$ *where* $I, J \in \mathcal{I}$ *with* $d \in \overline{I} \cap \overline{J}$. *Then*

$$(I_n)_0([d, v]) = \begin{cases} [I, (\pm, \pm)], & v = (-, +) \\ 0, & v = (+, -) \\ [I, (+, +)], & v = (+, +) \\ [I, (-, -)], & v = (-, -) \end{cases} \quad and \quad (U_n)_0([d, v]) = \begin{cases} 0, & v = (-, +) \\ [J, (\pm, \pm)], & v = (+, -) \\ [J, (+, +)], & v = (+, +) \\ [I, (-, -)], & v = (-, -) \end{cases}$$

where $[I, (\pm, \pm)] = [I, (+, +)] + [I, (-, -)]$ *and* $[J, (\pm, \pm)] = [J, (+, +)] + [J, (-, -)]$.

Proof. This is an elementary and easy case-by-case verification. $\qquad\qquad\square$

The formulas for $(I_n)_0$ and $(U_n)_0$ above do not depend on the number $n \in \mathbb{N}$.

Remark 5.7. Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be as in Lemma 4.53 and Remark 4.54. Then for all $n \geq \mathrm{ord}(\varphi)$,

$$K_0(\mathbb{A}_n) \simeq \mathbb{Z}^{\mathcal{D}(\pm)} \quad and \quad K_0(\mathbb{B}_n) \simeq \mathbb{Z}^{\mathcal{I}(\pm)} \,.$$

In this case use the notation $I_0 = (I_n)_0$ and $U_0 = (U_n)_0$ for all $n \geq \mathrm{ord}(\varphi)$, and

$$K_0(C_r^*(R_\varphi^+(n))) \simeq \ker(I_0 - U_0) \quad and \quad K_1(C_r^*(R_\varphi^+(n))) \simeq \mathrm{coker}(I_0 - U_0).$$

This notation will only be used, when the assumptions of Lemma 4.53 are satisfied.

In the following Remark 5.8(b) we will use the notation from Appendix A.

Remark 5.8. We abuse notation and write $(I_n)_0$ and $(U_n)_0$ for the matrices representing $(I_n)_0$ and $(U_n)_0$ with respect to the bases of $K_0(\mathbb{A}_n) \simeq \mathbb{Z}^{\mathcal{D}(\pm)_n}$ and $K_0(\mathbb{B}_n) \simeq \mathbb{Z}^{\mathcal{I}(\pm)_n}$.

(a) Note $(I_n)_0$ and $(U_n)_0$ are integer $\#\mathcal{I}(\pm)_n \times \#\mathcal{D}(\pm)_n$-matrices, in fact binary matrices, i.e. each entry is equal to either 0 or 1, and the matrix $(I_n)_0 - (U_n)_0$ is a sign matrix, i.e. each entry is equal to either -1, 0 or 1.

(b) Note $(I_k)_0$ and $(U_k)_0$ are also permutation equivalent to the multiplicity matrices $(M(I_k^{ij}))_{i,j}$ and $(M(U_k^{ij}))_{i,j}$ respectively, which are partly described in Remark A.7. Any such permutation equivalence corresponds to an appropriate reordering of the bases of $K_0(\mathbb{A}_k) \simeq \mathbb{Z}^{\mathcal{D}(\pm)_k}$ and $K_0(\mathbb{B}_k) \simeq \mathbb{Z}^{\mathcal{I}(\pm)_k}$:

- Each column is indexed by an element in $\mathcal{D}(\pm)_k$.
- Each row is indexed by an element in $\mathcal{I}(\pm)_k$.

(c) Note $(I_n)_0$ and $(U_n)_0$ can be realized as submatrices obtained by deleting (the same) columns and rows of the $2\#\mathcal{I} \times 4\#\mathcal{D}$-matrices I and U respectively, where

$$I = \begin{pmatrix} I^0 & 0 & 0 & \cdots & 0 \\ 0 & I^0 & 0 & \cdots & 0 \\ 0 & 0 & I^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I^0 \end{pmatrix} \quad and \quad U = \begin{pmatrix} 0 & U^0 & 0 & \cdots & 0 \\ 0 & 0 & U^0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U^0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$I^0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad and \quad U^0 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The matrices I and U comes from extending the formulas in Proposition 5.6 to $\mathcal{D} \times \mathcal{V}$. We may view $(I_n)_0$ and $(U_n)_0$ as block matrices just as I and U by grouping valencies corresponding to the same element in $\mathcal{D}$ or the same element in $\mathcal{I}$:

- Each block-column (i.e. column of blocks) is indexed by an element in $\mathcal{D}$.

- Each block-row (i.e. row of blocks) is indexed by an element in $\mathcal{I}$.

This is where we use the lexicographical ordering on $\mathcal{D}(\pm)_n$ and $\mathcal{I}(\pm)_n$.

Any reordering of the elements in $\mathcal{D}$ and $\mathcal{I}$ respectively induces a permutation of block-columns and block-rows respectively of the two block-matrices $(I_n)_0$ and $(U_n)_0$. In particular reordering the elements in $\mathcal{D}$ and $\mathcal{I}$ by some cyclic permutation induces the same cyclic permutation of block-columns and block-rows resulting in matrices $(I_n)_0$ and $(U_n)_0$ which are then still submatrices of I and U.

(d) Note by (A.3) that the entries in each column of $(I_n)_0 + (U_n)_0$ always sums to 2, and therefore each non-zero column in $(I_n)_0 - (U_n)_0$ has exactly two non-zero entries. Moreover the rows of $(I_n)_0 - (U_n)_0$ can be partioned into sets of rows R_+ and R_- which are indexed by $(I, w) \in \mathcal{I}(\pm)_n$ with $w = (+, +)$ and $w = (-, -)$ respectively. We may use Proposition 5.6 to realize the following:

  - If we are given any two non-zero entries with the same sign in the same column, then one of the entries is in a row from R_+ while the other is in a row from R_-.

  - If we are given any two non-zero entries with opposite signs in the same column, then both entries are in rows from R_+ or both entries are in rows from R_-.

Thus $x \in \mathbb{Z}^{\mathcal{I}(\pm)_n}$ with $x_{(I,w)} = (-1)^w$ is orthogonal to all the columns in $(I_n)_0 - (U_n)_0$. As a consequence $(I_n)_0 - (U_n)_0$ is not surjective. In particular

$$K_1(C_r^*(R_\varphi^+(n))) \simeq \operatorname{coker}((I_n)_0 - (U_n)_0) \neq 0.$$

Proposition 5.6 in itself of course also directly implies that

$$((I_n)_0 - (U_n)_0)(\mathbb{Z}^{\mathcal{D}(\pm)_n}) \subseteq \{x \in \mathbb{Z}^{\mathcal{I}(\pm)_n} : \textstyle\sum_{(I,w)\in\mathcal{I}(\pm)_n} (-1)^w x_{(I,w)} = 0\}$$

It follows from Lemma B.9 that $(I_n)_0 - (U_n)_0$ is totally unimodular (Definition B.7). In particular $K_1(C_r^*(R_\varphi^+(n))) \simeq \operatorname{coker}((I_n)_0 - (U_n)_0)$ is a torsion-free abelian group. We will see in Proposition 5.18 that actually $K_1(C_r^*(R_\varphi^+(n)) \simeq \mathbb{Z}$ for all $n \in \mathbb{N}$.

The following two results were proved in [50] when $\varphi(\mathcal{C}) \subseteq \mathcal{C}$:

Proposition 5.9. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map that is piecewise monotone. Assume φ is critically finite, and also that $\mathcal{D}$ is forward invariant, e.g. $\mathcal{D} = \bigcup_{l=1}^{\infty} \varphi^l(\mathcal{C})$. For each $(d, v) \in \mathcal{D}(\pm)_n$ take $I \in \mathcal{I}$ such that $d < I$. Define $A_n \colon \mathbb{Z}^{\mathcal{D}(\pm)_n} \to \mathbb{Z}^{\mathcal{D}(\pm)_{n+1}}$ by*

$$A_n([d,v]) = \begin{cases} [\varphi(d), \mathrm{val}(\varphi, d) \bullet v], & v = (+,-) \\ [\varphi(d), \mathrm{val}(\varphi, d) \bullet v] + \sum_{z \in I \cap \varphi^{-1}(\mathcal{D})} [\varphi(z), \mathrm{val}(\varphi, z) \bullet (+,+)], & v = (+,+) \\ [\varphi(d), \mathrm{val}(\varphi, d) \bullet v] + \sum_{z \in I \cap \varphi^{-1}(\mathcal{D})} [\varphi(z), \mathrm{val}(\varphi, z) \bullet (-,-)], & v = (-,-) \\ [\varphi(d), \mathrm{val}(\varphi, d) \bullet v] + \sum_{z \in I \cap \varphi^{-1}(\mathcal{D})} [\varphi(z), \mathrm{val}(\varphi, z) \bullet (-,-)] \\ \qquad + \sum_{z \in I \cap \varphi^{-1}(\mathcal{D})} [\varphi(z), \mathrm{val}(\varphi, z) \bullet (+,+)], & v = (-,+) \end{cases}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} K_0(\mathbb{D}_n) & \xrightarrow{(\pi_n)_0} & K_0(\mathbb{A}_n) \\ {\scriptstyle (\Phi_n)_0} \downarrow & & \downarrow {\scriptstyle A_n} \\ K_0(\mathbb{D}_{n+1}) & \xrightarrow{(\pi_{n+1})_0} & K_0(\mathbb{A}_{n+1}) \end{array} \qquad (5.6)$$

Proof. The two $*$-homomorphisms ζ_n and η_n were defined in the proof Proposition 4.18. Define $g_s \colon C([0,1], \mathbb{B}_n) \to C([0,1], \mathbb{B}_n)$ and $h_s \colon \mathbb{D}_n \to \mathbb{A}_{n+1}$ for each $s \in [0,1]$ by

$$g_s(f)(t) = f(st) \quad \text{and} \quad h_s(a, f) = \zeta_n(a) + \eta_n(g_s(f))$$

The map $s \mapsto h_s(a, f)$ is continuous for each $(a, f) \in \mathbb{D}_n$; $h_0 = \Psi_n \circ \pi_n$ and $h_1 = \pi_{n+1} \circ \Phi_n$ where $\Psi_n = \zeta_n + \eta_n \circ I_n$ and $\pi_l \colon \mathbb{D}_l \to \mathbb{A}_l$, $l \in \mathbb{N}$ is given by $\pi_l(a, f) = a$ for all $(a, f) \in \mathbb{D}_l$. Indeed we have that $\pi_{n+1} \circ \Phi_n(a, f) = \zeta_n(a) + \eta_n(f)$ by Remark 4.19 and Proposition 4.18. Also h_s is a $*$-homomorphism for each $s \in [0,1]$. It follows that the diagram

$$
\begin{array}{ccc}
\mathbb{D}_n & \xrightarrow{\pi_n} & \mathbb{A}_n \\
\Phi_n \downarrow & & \downarrow \Psi_n \\
\mathbb{D}_{n+1} & \xrightarrow{\pi_{n+1}} & \mathbb{A}_{n+1}
\end{array}
$$

commutes up to homotopy, so (5.6) commutes as stated since $(\Psi_n)_0 = A_n$. $\qquad\square$

Remark 5.10. An alternative route can be taken in Proposition 5.9 and its proof by using $\zeta_n + \eta_n \circ U_n$ instead of $\zeta_n + \eta_n \circ I_n$ and $g_s(f)(t) = f(1 - s(1-t))$ instead of $g_s(f)(t) = f(st)$. Also $(\zeta_n)_0 + (\eta_n) \circ (U_n)_0$ and $(\zeta_n)_0 + (\eta_n) \circ (I_n)_0$ agree on $\ker((I_n)_0 - (U_n)_0) \simeq K_0(\mathbb{D}_n)$.

Remark 5.11. It follows that A_n describes $(\Phi_n)_0$, because $(\pi_l)_0$ is injective for all $l \in \mathbb{N}$. Let $C_n \colon G_n \to G_{n+1}$ denote the restriction of $A_n \colon \mathbb{Z}^{\mathcal{D}(\pm)_n} \to \mathbb{Z}^{\mathcal{D}(\pm)_{n+1}}$ to $G_n \subseteq \mathbb{Z}^{\mathcal{D}(\pm)_n}$ where $G_n = \ker((I_n)_0 - (U_n)_0)$ and $G_{n+1} = \ker((I_{n+1})_0 - (U_{n+1})_0)$.

Proposition 5.12. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map that is piecewise monotone. Assume φ is critically finite, and $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{D}$ is forward invariant, e.g. $\mathcal{D} = \bigcup_{l=0}^{\infty} \varphi^l(\mathcal{C})$. For each $(I, w) \in \mathcal{I}(\pm)_n$ choose an open interval $J \subseteq I$ such that $J \cap \mathcal{C} = \emptyset$ and $\varphi(J) \in \mathcal{I}$. Define $B_n \colon \mathbb{Z}^{\mathcal{I}(\pm)_n} \to \mathbb{Z}^{\mathcal{I}(\pm)_{n+1}}$ by*

$$B_n([I, w]) = (-1)^{\mathrm{val}(\varphi, J)}[\varphi(J), \mathrm{val}(\varphi, J) \bullet w].$$

Then we have the following commutative diagram

$$
\begin{array}{ccc}
K_1(S\mathbb{B}_n) & \xrightarrow{(\iota_n)_1} & K_1(\mathbb{D}_n) \\
B_n \downarrow & & \downarrow (\Phi_n)_1 \\
K_1(S\mathbb{B}_{n+1}) & \xrightarrow{(\iota_{n+1})_1} & K_1(\mathbb{D}_{n+1})
\end{array}
\qquad (5.7)
$$

Proof. It is possible to choose an open interval $J \subseteq I$ such that $J \cap \mathcal{C} = \emptyset$ and $\varphi(J) \in \mathcal{I}$ with $\mathrm{val}(\varphi, J) = v \in \{(+, +), (-, -)\}$, because $\mathcal{D}$ is both forward invariant and $\varphi(\mathcal{C}) \subseteq \mathcal{D}$. Take $I' \in \mathcal{I}_n$ and an open interval $J' \subseteq I'$ such that $\varphi^n(I') = I$ and $\mathrm{val}(\varphi^n, I') = w$, and

$$\varphi^n(J') = J \quad \text{and} \quad \mathrm{val}(\varphi^{n+1}, J') = v \bullet w.$$

Take any positively oriented path $\omega \in C_c(J', \mathbb{T} - 1) \subseteq C_c(R_\varphi^+(n))$ which is of degree one. It follows from the choices that $(\omega \circ \kappa_{I'}^n)e_{I',I'} \in S\mathbb{B}_n$ represents $(-1)^w[I, w] \in \mathbb{Z}^{\mathcal{I}(\pm)_n}$, and $(\omega \circ \kappa_{J'}^{n+1})e_{J',J'} \in S\mathbb{B}_{n+1}$ represents $(-1)^w(-1)^v[\varphi(J), v \bullet w] = B_n((-1)^w[I, w]) \in \mathbb{Z}^{\mathcal{I}(\pm)_{n+1}}$, and $\Phi_n \circ \iota_n((\omega \circ \kappa_{I'}^n)e_{I',I'}) = \iota_{n+1}((\omega \circ \kappa_{J'}^{n+1})e_{J',J'})$. $\qquad\square$

Remark 5.13. It follows that B_n describes $(\Phi_n)_1$, because $(\iota_l)_1$ is surjective for all $l \in \mathbb{N}$. Let $D_n \colon H_n \to H_{n+1}$ denote the quotient map induced by $B_n \colon \mathbb{Z}^{\mathcal{I}(\pm)_n} \to \mathbb{Z}^{\mathcal{I}(\pm)_{n+1}}$ where $H_n = \mathrm{coker}((I_n)_0 - (U_n)_0)$ and $H_{n+1} = \mathrm{coker}((I_{n+1})_0 - (U_{n+1})_0)$.

Remark 5.14. The formulas for A_n and B_n above do not depend on the number $n \in \mathbb{N}$. In general there may be several ways in which we can choose maps just like A_n and B_n which make (5.6) and (5.7) commute as evidenced by Remark 5.10 and Proposition 5.12. In some cases $B_n([I, w]) = [I, w]$ for $(I, w) \in \mathcal{I}(\pm)_n$ makes the diagram (5.7) commute. This may not always be the case, but as it turns out there are certain segments of the directed system which will always simplify, and consequently the direct limit will collapse. We will get to this in Proposition 5.16 and Remark 5.17.

Remark 5.15. We make the following observations about the positive cones:

(a) It follows from Lemma 4.16 and Proposition A.15 that

$$(\mu_n)_0(K_0(C_r^*(R_\varphi^+(n)))^+) = K_0(\mathbb{D}_n)^+ \quad \text{and} \quad (\pi_n)_0(K_0(\mathbb{D}_n)^+) = G_n^+,$$

where $G_n = \ker((I_n)_0 - (U_n)_0)$ and $G_n^+ = G_n \cap (\mathbb{Z}^{\mathcal{D}(\pm)_n})^+$.

(b) Let $C_n \colon G_n \to G_{n+1}$ such that $C_n \circ (\pi_n)_0 = (\pi_{n+1})_0 \circ (\Phi_n)_0$. Then

$$\mu \colon K_0(C_r^*(R_\varphi^+)) \to \varinjlim(K_0(\mathbb{D}_n), (\Phi_n)_0)) \quad \text{and} \quad \pi \colon \varinjlim(K_0(\mathbb{D}_n), (\Phi_n)_0)) \to \varinjlim(G_n, C_n)$$

are order isomorphisms induced by $(\mu_n)_0$ and $(\pi_n)_0$.

5.2 A reduction of the invariant

We will prove that $K_1(C_r^*(R_\varphi^+)) \simeq \mathbb{Z}$ making it a redundant ingredient in $F(C_r^*(R_\varphi^+))$.

Proposition 5.16. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map that is piecewise monotone. We have $K_1(C_r^*(R_\varphi^+)) \simeq K_1(C_r^*(R_\varphi^+(n)))$ for all $n \in \mathbb{N}$.*

We will provide a complete proof of Proposition 5.16 later.

Remark 5.17. Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous piecewise monotone circle map with $\mathcal{C} \neq \emptyset$. If φ is exact and critically finite and without exceptional fixed points, then we can show $K_1(C_r^*(R_\varphi^+)) \simeq K_1(C_r^*(R_\varphi^+(n)))$ for all $n \geq \text{ord}(\varphi)$ using the ideas in Proposition 5.12:

Put $(\Phi_{n+l,n})_1 = (\Phi_{n+l-1})_1 \circ \cdots \circ (\Phi_{n+1})_1 \circ (\Phi_n)_1 \colon K_1(\mathbb{D}_n) \to K_1(\mathbb{D}_{n+l})$ for any $l \in \mathbb{N}$. For each $(I, w) \in \mathcal{I}(\pm)_n$ we take an open interval $J \subseteq I$ with $J \cap \mathcal{C}_l = \emptyset$ and $\varphi^l(J) \in \mathcal{I}$. Define $B_{n+l,n} \colon K_1(S\mathbb{B}_n) \to K_1(S\mathbb{B}_{n+l})$ by

$$B_{n+l,n}([I, w]) = (-1)^{\text{val}(\varphi^l, J)}[\varphi^l(J), \text{val}(\varphi^l, J) \bullet w].$$

Then we have the following commutative diagram

$$
\begin{array}{ccc}
K_1(S\mathbb{B}_n) & \xrightarrow{(\iota_n)_1} & K_1(\mathbb{D}_n) \\
{\scriptstyle B_{n+l,n}}\downarrow & & \downarrow{\scriptstyle (\Phi_{n+l,n})_1} \\
K_1(S\mathbb{B}_{n+l}) & \xrightarrow{(\iota_{n+l})_1} & K_1(\mathbb{D}_{n+l})
\end{array}
$$

The proof of this fact is very similar to the proof of Proposition 5.12. We will not repeat it. We note $\mathcal{I}(\pm) = \mathcal{I}(\pm)_{\text{ord}(\varphi)} = \mathcal{I}(\pm)_n$ for all $n \geq \text{ord}(\varphi)$ by Lemma 4.53 and Remark 4.54. The formula for $B_{n+l,n}$ does not depend on $n \in \mathbb{N}$. Let $n \geq \text{ord}(\varphi)$.

It follows that there exists $L \in \mathbb{N}$ such that $\varphi^L(I) = \mathbb{T}$ for all $I \in \mathcal{I}$, since φ is exact. Thus for any $(I, w) \in \mathcal{I}(\pm)$ we may take $J \subseteq I$ such that $J \cap \mathcal{C}_L = \emptyset$ and $\varphi^L(J) = I$, and

$$B_{n+L,n}([I, w]) = (-1)^{\text{val}(\varphi^L, J)}[I, \text{val}(\varphi^L, J) \bullet w],$$
$$B_{n+2L,n+L}([I, w]) = (-1)^{\text{val}(\varphi^L, J)}[I, \text{val}(\varphi^L, J) \bullet w].$$

This entails $B_{n+2L,n+L} \circ B_{n+L,n}([I, w]) = [I, w]$ for all $(I, w) \in \mathcal{I}(\pm)_n$ when $n \geq \text{ord}(\varphi)$. We may conclude from the above that $K_1(C_r^*(R_\varphi^+)) \simeq K_1(C_r^*(R_\varphi^+(n)))$ for all $n \geq \text{ord}(\varphi)$, as the connected maps in a directed subsystem of the directed system are identity maps.

Proposition 5.18. *Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a continuous circle map that is piecewise monotone. We have $K_1(C_r^*(R_\varphi^+(n))) \simeq \text{coker}((I_n)_0 - (U_n)_0) \simeq \mathbb{Z}$ for all $n \in \mathbb{N}$.*

In the following we will go through two very different ways of proving Proposition 5.18. The first proof is somewhat long and requires some preparations but it is fairly elementary. The second proof is short and direct and only uses well-known but non-elementary results. We will start with the elementary proof where we use the notation from Appendix A.

Remark 5.19. The proof of Proposition 5.18 can be reduced as follows:

The Smith normal form of $(I_k)_0 - (U_k)_0$ is a binary diagonal matrix $\text{diag}(\alpha_1, \ldots, \alpha_{n_k})$ since $(I_k)_0 - (U_k)_0$ is totally unimodular, and $\alpha_{n_k} \neq 0$ since $(I_k)_0 - (U_k)_0$ is not surjective. These properties were established in Remark 5.8(d). It remains to prove that

$$\alpha_1 = \cdots = \alpha_{n_k-1} = 1.$$

It suffices to prove the existence of a non-zero $l \times l$-minor for each $1 \leq l \leq n_k - 1$. In order to do this we will find an $(n_k - 1) \times (n_k - 1)$-submatrix of $(I_k)_0 - (U_k)_0$ which is permutation equivalent to an invertible upper triangular $(n_k - 1) \times (n_k - 1)$-matrix, and by deleting the first $n_k - 1 - l$ columns and rows of the invertible upper triangular $(n_k - 1) \times (n_k - 1)$-matrix for any $1 \leq l \leq n_k - 1$ we obtain an invertible upper triangular $l \times l$-matrix which is permutation equivalent to an $l \times l$-submatrix of $(I_k)_0 - (U_k)_0$.

For any $d \in \mathcal{D}$ and $I \in \mathcal{I}$ write

$$\mathcal{V}_d^n = \{v \in \mathcal{V} : (d, v) \in \mathcal{D}(\pm)_n\} \quad \text{and} \quad \mathcal{W}_I^n = \{w \in \mathcal{W} : (I, w) \in \mathcal{I}(\pm)_n\}.$$

Lemma 5.20. *For any $d = \pi(\Phi^n(c)) \in \mathcal{D}$ with $c \in [0, 1]$,*

- $\Phi^n([0, c]) \subseteq [\Phi^n(c), \infty)$ *if* $\text{val}(\varphi^n, \pi(c)) \in \{(-, +), (-, -)\}$ *and* $\mathcal{V}_d^n \subseteq \mathcal{V} \setminus \{(+, +)\}$,
- $\Phi^n([0, c]) \subseteq (-\infty, \Phi^n(c)]$ *if* $\text{val}(\varphi^n, \pi(c)) \in \{(+, -), (+, +)\}$ *and* $\mathcal{V}_d^n \subseteq \mathcal{V} \setminus \{(-, -)\}$,
- $\Phi^n([c, 1]) \subseteq [\Phi^n(c), \infty)$ *if* $\text{val}(\varphi^n, \pi(c)) \in \{(-, +), (+, +)\}$ *and* $\mathcal{V}_d^n \subseteq \mathcal{V} \setminus \{(-, -)\}$,
- $\Phi^n([c, 1]) \subseteq (-\infty, \Phi^n(c)]$ *if* $\text{val}(\varphi^n, \pi(c)) \in \{(+, -), (-, -)\}$ *and* $\mathcal{V}_d^n \subseteq \mathcal{V} \setminus \{(+, +)\}$.

In particular whenever $\mathcal{V}_d^n \subseteq \{(-, +), (+, -)\}$,

- $\Phi^n([0, 1]) \subseteq [\Phi^n(c), \infty)$ *if* $\text{val}(\varphi^n, \pi(c)) = (-, +)$,
- $\Phi^n([0, 1]) \subseteq (-\infty, \Phi^n(c)]$ *if* $\text{val}(\varphi^n, \pi(c)) = (+, -)$,

and for any $c' \in [0, 1]$ with $\Phi^n(c) = \Phi^n(c')$ we have $\text{val}(\varphi^n, \pi(c)) = \text{val}(\varphi^n, \pi(c'))$.

Proof. Assume $\mathrm{val}(\varphi^n, \pi(c)) \in \{(-,+),(-,-)\}$ and $\mathcal{V}_d^n \subseteq \mathcal{V} \setminus \{(+,+)\}$ for $c \in [0,1]$, $c \neq 0$. If $\Phi^n(t) < \Phi^n(c)$ for some $t \in [0,c]$, then there exists $y_1 \in [0,1] \setminus \{c\}$ such that

$$\Phi^n(y_1) = \Phi^n(c), \quad \text{and so} \quad \mathrm{val}(\varphi^n, \pi(y_1)) \in \mathcal{V} \setminus \{(+,+)\}.$$

This in turn yields $y_2 \in [0,1] \setminus \{c, y_1\}$ such that

$$\Phi^n(y_2) = \Phi^n(c) \quad \text{and} \quad \mathrm{val}(\varphi^n, \pi(y_2)) \in \mathcal{V} \setminus \{(+,+)\}.$$

This procedure may be repeated indefinitely, and so we arrive at an infinite sequence $\{y_l\}_{l \in \mathbb{N}} \subseteq \Phi^{-n}(\{\Phi^n(c)\}) \cap [0,1]$ which yields a contradiction.

A similar argument applies to the other statements. $\qquad\square$

Corollary 5.21. *Assume* $\{d \in \mathcal{D} : \mathcal{V}_d^n \subseteq \{(-,+),(+,-)\}\} = \emptyset$. *Then*

$$\{d \in \mathcal{D} : (+,+) \in \mathcal{V}_d^n\} = \mathcal{D} \quad or \quad \{d \in \mathcal{D} : (-,-) \in \mathcal{V}_d^n\} = \mathcal{D}.$$

Lemma 5.22. *Assume* $\{d \in \mathcal{D} : \mathcal{V}_d^n \subseteq \{(-,+),(+,-)\}\} \neq \emptyset$. *Put* $d^\pm = \pi(e^\pm)$ *where*

$$e^+ = \max \Phi^n([0,1]) \quad and \quad e^- = \min \Phi^n([0,1]).$$

Then $\deg(\varphi^n) = \deg(\varphi) = 0$ *and* $\mathrm{val}(\varphi^n, \pi(c^\pm)) = (\pm, \mp)$ *for any* $c^\pm \in \Phi^{-n}(\{e^\pm\})$, *and*

$$\lambda(\Phi^n([0,1])) = e^+ - e^- \leq 1.$$

Moreover $\{(+,+),(-,-)\} \subseteq \mathcal{V}_d^n$ *for all* $d \in \mathcal{D} \setminus \{d^-, d^+\}$.

Corollary 5.23. *Assume* $\{d \in \mathcal{D} : \mathcal{V}_d^n \subseteq \{(-,+),(+,-)\}\} \neq \emptyset$. *Put* $d^\pm = \pi(e^\pm)$ *where*

$$e^+ = \max \Phi^n([0,1]) \quad and \quad e^- = \min \Phi^n([0,1]).$$

Then $\mathcal{V}_{d^-}^n = \{(-,+)\}$ *and* $\mathcal{V}_{d^+}^n = \{(+,-)\}$ *and* $\lambda(\Phi^n([0,1])) < 1$ *if* φ *is not surjective, and* $\mathcal{V}_{d^0}^n = \{(-,+),(+,-)\}$ *where* $d^0 = d^- = d^+$ *and* $\lambda(\Phi^n([0,1])) = 1$ *if* φ *is surjective.*

Remark 5.24. We note that Lemma 5.20, Corollary 5.21, Lemma 5.22 and Corollary 5.23 are valid for any $n \in \mathbb{N}$ and do not depend on the choice of finite set $\mathcal{D} \subseteq \mathbb{T}$ with $\mathcal{C}_n \subseteq \mathcal{D}_n$. Lemma 5.22 and Corollary 5.23 happens to generalize and expand on Lemma 5.17 in [50] while Corollary 5.21 generalizes and expands on Lemma 5.12 in [50].

Proof of Lemma 5.22 and Corollary 5.23. Let $d \in \mathcal{D}$ such that $\mathcal{V}_d^n \subseteq \{(-,+),(+,-)\}$. Lemma 3.1 implies that $\deg(\varphi^n) = \deg(\varphi)^n$. Also φ is surjective, whenever $\deg(\varphi) \neq 0$.

Assume φ is surjective and $\mathrm{val}(\varphi^n, \pi(c)) = (-,+)$ where $\pi(\Phi^n(c)) = d$ with $c \in [0,1]$. It follows from Lemma 5.20 that $\Phi^n([0,1]) \subseteq [\Phi^n(c), \infty)$ as $\mathrm{val}(\varphi^n, \pi(c)) = (-,+)$, and $\Phi^n(c) + 1 \in \Phi^n([0,1])$ since φ (and so φ^n) is surjective, so $[\Phi^n(c), \Phi^n(c) + 1] \subseteq \Phi^n([0,1])$. Thus $\Phi^n(c) + 1 = \Phi^n(c')$ for some $c' \in [0,1]$, and $\mathrm{val}(\varphi^n, \pi(c')) \neq (-,+)$ by Lemma 5.20, so $\mathrm{val}(\varphi^n, \pi(c')) = (+,-)$ and $\Phi^n([0,1]) \subseteq (-\infty, \Phi^n(c) + 1]$ by Lemma 5.20. Hence

$$\Phi^n([0,1]) = [\Phi^n(c), \Phi^n(c) + 1].$$

In particular $\deg(\varphi^n) \in \{-1, 0, 1\}$, but $\deg(\varphi^n) \neq 1$ and $\deg(\varphi^n) \neq -1$ because otherwise $\{\Phi^n(0), \Phi^n(1)\} = \{\Phi^n(c), \Phi^n(c) + 1\}$ and both $\mathrm{val}(\varphi^n, 1) = (-,+)$ and $\mathrm{val}(\varphi^n, 1) = (+,-)$. If $e \in (\Phi^n(c), \Phi^n(c) + 1)$ with $\pi(e) = d' \in \mathcal{D}$, then $\mathcal{V}_{d'}^n \not\subseteq \{(-,+),(+,-)\}$ by Lemma 5.20, and because $\deg(\varphi^n) = 0$ we have that either $\Phi^n(0) = \Phi^n(1) \neq e$ or $\Phi^n(0) = \Phi^n(1) = e$, and either way $\{(+,+),(-,-)\} \subseteq \mathcal{V}_{d'}^n$ by Lemma 5.20 since $\{(+,+),(-,-)\} \cap \mathcal{V}_{d'}^n \neq \emptyset$.

Assume φ is surjective and $\mathrm{val}(\varphi^n, \pi(c)) = (+, -)$ where $\pi(\Phi^n(c)) = d$ with $c \in [0, 1]$. It follows that $\Phi^n(c') = \Phi^n(c) - 1$ and $\mathrm{val}(\varphi^n, \pi(c')) = (-, +)$ for some $c' \in [0, 1]$, and

$$\Phi^n([0, 1]) = [\Phi^n(c) - 1, \Phi^n(c)]$$

In particular we have that $\deg(\varphi^n) \in \{-1, 0, 1\}$, but again $\deg(\varphi^n) \neq 1$ and $\deg(\varphi^n) \neq -1$. If $e \in (\Phi^n(c) - 1, \Phi^n(c))$ with $\pi(e) = d' \in \mathcal{D}$, then $\{(+, +), (-, -)\} \subseteq \mathcal{V}_{d'}^n$ by Lemma 5.20.

Assume φ is not surjective. It follows that

$$\lambda(\Phi^n([0, 1])) = e^+ - e^- < 1$$

In particular $\mathrm{val}(\varphi^n, \pi(c^\pm)) = (\pm, \mp)$ for any $c^\pm \in \Phi^{-n}(\{e^\pm\})$. As a result $\mathcal{V}_{d^\pm}^n = \{(\pm, \mp)\}$. If $e \in (e^-, e^+)$ with $\pi(e) = d' \in \mathcal{D}$, then $\{(+, +), (-, -)\} \subseteq \mathcal{V}_{d'}^n$ again by Lemma 5.20. $\square$

Remark 5.25. We note that φ is surjective when $\{d \in \mathcal{D} : \mathcal{V}_d^n \subseteq \{(-, +), (+, -)\}\} = \emptyset$. Put

$$\mathcal{V}^n = \{v \in \mathcal{V} : v \in \mathcal{V}_d^n \text{ for some } d \in \mathcal{D}\} \quad \text{and} \quad \mathcal{W}^n = \{w \in \mathcal{W} : w \in \mathcal{W}_d^n \text{ for some } I \in \mathcal{I}\}.$$

Then either $\mathcal{V}^n = \mathcal{W}^n = \{(+, +)\}$ or $\mathcal{V}^n = \mathcal{W}^n = \{(-, -)\}$ when φ is locally injective, and $\{(-, +), (+, -)\} \subseteq \mathcal{V}^n$ when φ is not locally injective. Assume φ is not locally injective. Assume $\{d \in \mathcal{D} : \mathcal{V}_d^n \subseteq \{(-, +), (+, -)\}\} = \emptyset$. Then $(d^\pm, (\pm, \mp)) \in \mathcal{D}(\pm)_n$ for some $d^\pm \in \mathcal{D}$. This choice of $d^\pm \in \mathcal{D}$ is more arbitrary than the choice in Lemma 5.22 and Corollary 5.23. We also want to consider $I^- \in \mathcal{I}$ with $d^- < I^-$, $d^- \in \overline{I^-}$ later on.

The proof of Proposition 5.18 is divided into a few cases, and we will take a look at some examples to foreshadow and shed light on each of these cases.

Example 5.26. Assume that φ is surjective and $\{d \in \mathcal{D} : \mathcal{V}_d^n \subseteq \{(-, +), (+, -)\}\} \neq \emptyset$. Then for some $d^0 \in \mathcal{D}$ and all $d \in \mathcal{D} \setminus \{d^0\}$ by Lemma 5.22 and Corollary 5.23,

$$\mathcal{V}_{d^0}^n = \{(-, +), (+, -)\} \quad \text{and} \quad \{(+, +), (-, -)\} \subseteq \mathcal{V}_d^n.$$

We may permute the block-columns and block-rows so that the block corresponding to $d^0 \in \mathcal{D}$ and $I^0 \in \mathcal{I}$ with $d^0 < I^0$ is moved to the upper left corner as illustrated below:

$$
\left(\begin{array}{cccc|cc}
1 & 0 & 1 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 & 0 & -1 \\
\hline
0 & -1 & -1 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & 1 & 0
\end{array}\right)
\rightsquigarrow
\left(\begin{array}{cc|cccc}
1 & 0 & 0 & -1 & -1 & 0 \\
1 & 0 & 0 & -1 & 0 & -1 \\
\hline
0 & -1 & 1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 1
\end{array}\right)
$$

We may then delete the second column and second row, and subsequently delete the excess columns in the other block-columns and permute the columns and rows if needed to obtain an invertible upper triangular $(n_k - 1) \times (n_k - 1)$-matrix as illustrated below:

$$
\left(\begin{array}{cc|cccc}
1 & 0 & 0 & -1 & -1 & 0 \\
1 & 0 & 0 & -1 & 0 & -1 \\
\hline
0 & -1 & 1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 1
\end{array}\right)
\rightsquigarrow
\left(\begin{array}{c|cc}
1 & -1 & 0 \\
\hline
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

It is worth noting that $d^0 = d^- = d^+$ in Lemma 5.22 and Corollary 5.23.

Example 5.27. Assume that φ is not surjective and $\{d \in \mathcal{D} : \mathcal{V}_d^n \subseteq \{(-,+),(+,-)\}\} \neq \emptyset$. Then for some $d^-, d^+ \in \mathcal{D}$ and all $d \in \mathcal{D} \setminus \{d^-, d^+\}$ by Lemma 5.22 and Corollary 5.23,

$$\mathcal{V}_{d^\pm}^n = \{(\pm, \mp)\} \quad \text{and} \quad \{(+,+),(-,-)\} \subseteq \mathcal{V}_d^n.$$

We may permute the block-columns and block-rows so that the block corresponding to $d^- \in \mathcal{D}$ and $I^- \in \mathcal{I}$ with $d^- < I^-$ is moved to the upper left corner and the block-column corresponding to $d^+ \in \mathcal{D}$ is moved right next to the first block-column as illustrated below:

$$\left(\begin{array}{ccc|c|c} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ \hline -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{c|c|ccc} 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

We may then delete the second column and second row, and subsequently delete the excess columns in the other block-columns and permute the columns and rows if needed to obtain an invertible upper triangular $(n_k - 1) \times (n_k - 1)$-matrix as illustrated below:

$$\left(\begin{array}{c|c|ccc} 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{c|cc} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

Example 5.28. Assume φ is not locally injective and $\{d \in \mathcal{D} : \mathcal{V}_d^n \subseteq \{(-,+),(+,-)\}\} = \emptyset$. Then for some $d^\pm \in \mathcal{D}$ by Remark 5.25,

$$(\pm, \mp) \subseteq \mathcal{V}_{d^\pm}^n \quad \text{as} \quad \{(-,+),(+,-)\} \in \mathcal{V}^n.$$

We may permute the block-columns and block-rows so that the block corresponding to $d^- \in \mathcal{D}$ and $I^- \in \mathcal{I}$ with $d^- < I^-$ is moved to the upper left corner as illustrated below:

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

We may then delete all columns in the first block-column except for the first, and delete any excess columns in the other block-columns and permute the columns and rows if needed to obtain an invertible upper triangular $(n_k - 1) \times (n_k - 1)$-matrix as illustrated below:

$$\left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{c|cc} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{c|cc} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

Proof of Proposition 5.18. We will continue where Remark 5.19 left off and elaborate on the ideas presented in Example 5.26, Example 5.27, and Example 5.28.

Assume φ is locally injective. Then

$$(I_k)_0 - (U_k)_0 = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

We can just delete the first row and the first column to obtain an invertible upper triangular $(n_k - 1) \times (n_k - 1)$-submatrix of $(I_k)_0 - (U_k)_0$.

Assume through the rest of the proof that φ is not locally injective. Then either

(1) $\{d \in \mathcal{D} : \mathcal{V}_d \subseteq \{(-, +), (+, -)\}\} \neq \emptyset$ as in Example 5.26 and Example 5.27, or

(2) $\{d \in \mathcal{D} : \mathcal{V}_d \subseteq \{(-, +), (+, -)\}\} = \emptyset$ as in Example 5.28.

(1) If $\{d \in \mathcal{D} : \mathcal{V}_d \subseteq \{(-, +), (+, -)\}\} \neq \emptyset$ we apply Lemma 5.22 and Corollary 5.23. It follows from Corollary 5.23 that for $I^- \in \mathcal{I}$ with $d^- < I^-$,

$$(I^-, (\pm, \pm)) \in \mathcal{I}(\pm)_k \quad \text{and} \quad (d^-, (-, +)) \in \mathcal{D}(\pm)_k.$$

We reorder the elements $\mathcal{D}$ and $\mathcal{I}$ such that $d^- < d$ and $I^- < I$ for all $d \in \mathcal{D}$ and $I \in \mathcal{I}$, and rearrange the elements of the bases for $\mathbb{Z}^{\mathcal{D}(\pm)_k}$ and $\mathbb{Z}^{\mathcal{I}(\pm)_k}$ according to this and (5.4). This corresponds to a permutation of the block-columns and block-rows of $(I_k)_0 - (U_k)_0$ such that the first block-column corresponds to d^-, the first block-row corresponds to I^-. We now delete the second row which corresponds to the element $(I^-, (-, -)) \in \mathcal{I}(\pm)_k$ and the second column which in this case (1) corresponds to the element $(d^+, (+, -)) \in \mathcal{D}(\pm)_k$. This leaves us with an $(n_k - 1) \times (m_k - 1)$-matrix in which the first block-column contains only one column which by choice corresponds to the element $(d^-, (-, +)) \in \mathcal{D}(\pm)_k$ and the only non-zero entry in this column is in the upper left corner of the matrix and is 1. We go on by deleting the excess columns in the last $\#\mathcal{D} - 1$ block-columns of this matrix.

It follows from Lemma 5.22 that for any $d \in \mathcal{D} \setminus \{d^-, d^+\}$ and $I \in \mathcal{I}$ with $d < I$,

$$(I, (\pm, \pm)) \in \mathcal{I}(\pm)_k \quad \text{and} \quad (d, (\pm, \pm)) \in \mathcal{D}(\pm)_k$$

We now delete all other columns (if any) which corresponds to the elements $(d, v) \in \mathcal{D}(\pm)_k$ for $d \in \mathcal{D} \setminus \{d^-, d^+\}$ and $v \in \{(-, +), (+, -)\}$ and subsequently we may use Remark 5.8 to permute columns and rows to obtain a unit upper triangular $(n_k - 1) \times (n_k - 1)$-matrix.

(2) If $\{d \in \mathcal{D} : \mathcal{V}_d \subseteq \{(-, +), (+, -)\}\} = \emptyset$ we adopt the notation from Remark 5.25. We note that $(I^-, (\pm, \pm)) \in \mathcal{I}(\pm)_k$ since $(d^-, (-, +)) \in \mathcal{D}(\pm)_k$ because of Remark 5.25. We order the elements of $\mathcal{D}$ and $\mathcal{I}$ such that $d^- < d$ and $I^- < I$ for all $d \in \mathcal{D}$ and $I \in \mathcal{I}$, and rearrange the elements of the bases for $\mathbb{Z}^{\mathcal{D}(\pm)_k}$ and $\mathbb{Z}^{\mathcal{I}(\pm)_k}$ according to this and (5.4). This corresponds to a permutation of the block-columns and block-rows of $(I_k)_0 - (U_k)_0$ such that the first block-column corresponds to d^-, the first block-row corresponds to I^-. We now delete the second row which corresponds to the element $(I^-, (-, -)) \in \mathcal{I}(\pm)_k$ and all columns (if any) which corresponds to the elements $(d^-, v) \in \mathcal{D}(\pm)_k$ for $v \neq (-, +)$. This leaves us with an $(n_k - 1) \times (m_k - \#\mathcal{V}_{d^-}^n + 1)$-matrix and the first block-column contains only one column which by choice corresponds to the element $(d^-, (-, +)) \in \mathcal{D}(\pm)_k$ and the only non-zero entry in this column is in the upper left corner of the matrix and is 1. We go on by deleting the excess columns in the last $\#\mathcal{D} - 1$ block-columns of this matrix.

It follows from $\{d \in \mathcal{D} : \mathcal{V}_d \subseteq \{(-, +), (+, -)\}\} = \emptyset$ that for any $d \in \mathcal{D} \setminus \{d^-\}$ and $I \in \mathcal{I}$ with $d < I$ there exists $u_d \in \{(+, +), (-, -)\}$ such that

$$(d, u_d) \in \mathcal{D}(\pm)_k \quad \text{and} \quad (I, u_d) \in \mathcal{I}(\pm)_k.$$

If $\mathcal{W}_I^n \neq \{u_d\}$ there exists $v_d \in \mathcal{V} \setminus \{u_d, (+, -)\}$ and $w_d \in \mathcal{V} \setminus \{u_d\}$ such that

$$(d, v_d) \in \mathcal{D}(\pm)_k \quad \text{and} \quad (I, w_d) \in \mathcal{I}(\pm)_k.$$

We now delete all other columns (if any) which corresponds to the elements $(d, v) \in \mathcal{D}(\pm)_k$ for each $d \in \mathcal{D} \setminus \{d^-\}$ and $v \in \mathcal{V} \setminus \{u_d, v_d\}$ and subsequently we may use Remark 5.8 to permute columns and rows to obtain a unit upper triangular $(n_k - 1) \times (n_k - 1)$-matrix.

This concludes the proof. $\qquad\square$

The above proof of Proposition 5.18 is presented in this chapter and not in Appendix A, only since it is an elementary proof that could serve as an alternative to the proof below. The technical details of the proof above are hidden in (5.5) or rather Proposition A.13.

   The following proof was suggested to the author by Ian F. Putnam.

Proof of Proposition 5.18. We more or less repeat a part of the proof of Proposition A.13: We are going to compute $(\delta_n^!)_0$ directly by using a result like Proposition 12.2.2(ii) in [47] and by utilizing the identifications and isomorphisms in Remark A.10 and Remark A.11. Let $(d,v) \in \mathcal{D}(\pm)_n$ and $c \in \mathcal{D}_n$ with

$$\varphi^n(c) = d \quad \text{and} \quad \text{val}(\varphi^n, c) = v.$$

Let $I = (s,c), J = (c,t) \in \mathcal{I}_n$, and $f_+ \colon I \to \mathbb{R}$ increasing and $f_- \colon J \to \mathbb{R}$ decreasing with

$$\lim_{x \to s^+} f_+(x) = \lim_{x \to t^-} f_-(x) = 0 \quad \text{and} \quad \lim_{x \to c^-} f_+(x) = \lim_{x \to c^+} f_-(x) = 1.$$

It can be verified easily that $a = 1_I f_+ + 1_{\{c\}} + 1_J f_- \in C(\mathbb{T})$ is self-adjoint in $C_r^*(R_\varphi^+(n))$, and $p = a|_{\mathcal{D}_n} = 1_{\{c\}} \in C(\mathcal{D}_n)$ is then a projection in $C_r^*(R_\varphi^+(n)|_{\mathcal{D}_n})$ for which $[p]_0 = [d,v]$, and $u = \exp(2\pi i a)$ is a unitary in $C_r^*(R_\varphi^+(n)|_{\mathbb{T}\setminus\mathcal{D}_n})^\sim$ such that

$$(\delta_n^!)_0([p]_0) = -[u]_1 = \begin{cases} [\varphi^n(J),(+,+)] - [\varphi^n(I),(-,-)] & v = (-,+) \\ [\varphi^n(J),(-,-)] - [\varphi^n(I),(+,+)] & v = (+,-) \\ [\varphi^n(J),(+,+)] - [\varphi^n(I),(+,+)] & v = (+,+) \\ [\varphi^n(J),(-,-)] - [\varphi^n(I),(-,-)] & v = (-,-) \end{cases}.$$

We have that $\langle [d,v] \rangle_{(d,v)\in\mathcal{D}(\pm)_n} = \mathbb{Z}^{\mathcal{D}(\pm)_n}$ so by elementary algebra,

$$(\delta_n^!)_0(\mathbb{Z}^{\mathcal{D}(\pm)_n}) = \langle (\delta_n^!)_0([d,v]) \rangle_{(d,v)\in\mathcal{D}(\pm)_n} = \{x \in \mathbb{Z}^{\mathcal{I}(\pm)_n} : \textstyle\sum_{(I,w)\in\mathcal{I}(\pm)_n} x_{(I,w)} = 0\}.$$

The map $x \mapsto \sum_{(I,w)\in\mathcal{I}(\pm)_n} x_{(I,w)}$ induces an isomorphism $\mathbb{Z}^{\mathcal{I}(\pm)_n}/(\delta_n^!)_0(\mathbb{Z}^{\mathcal{D}(\pm)_n}) \to \mathbb{Z}$, so

$$K_1(C_r^*(R_\varphi^+(n))) \simeq \text{coker}(\delta_n^!)_0 = \mathbb{Z}^{\mathcal{I}(\pm)_n}/(\delta_n^!)_0(\mathbb{Z}^{\mathcal{D}(\pm)_n}) \simeq \mathbb{Z}. \qquad \square$$

Remark 5.29. It follows from Proposition 5.18 and Remark 5.8(d) that

$$(\delta_n)_0(\mathbb{Z}^{\mathcal{D}(\pm)_n}) = \{x \in \mathbb{Z}^{\mathcal{I}(\pm)_n} : \textstyle\sum_{(I,w)\in\mathcal{I}(\pm)_n} (-1)^w x_{(I,w)} = 0\}.$$

The isomorphism $\mathbb{Z}^{\mathcal{I}(\pm)_n}/(\delta_n)_0(\mathbb{Z}^{\mathcal{D}(\pm)_n}) \to \mathbb{Z}$ is induced by $\mathbb{Z}^{\mathcal{I}(\pm)_n} \to \mathbb{Z}$ which is given by

$$x \mapsto \textstyle\sum_{(I,w)\in\mathcal{I}(\pm)_n} (-1)^w x_{(I,w)}.$$

Proposition 5.16 follows if a generator in $K_1(S\mathbb{B}_n)$ is mapped to a generator in $K_1(S\mathbb{B}_{n+1})$. This will be done below in a more direct way to make the argument less convoluted.

Remark 5.30. It follows from the last proof of Proposition 5.18 that

$$(\delta_n^!)_0(\mathbb{Z}^{\mathcal{D}(\pm)_n}) = \{x \in \mathbb{Z}^{\mathcal{I}(\pm)_n} : \textstyle\sum_{(I,w)\in\mathcal{I}(\pm)_n} x_{(I,w)} = 0\}.$$

The isomorphism $\mathbb{Z}^{\mathcal{I}(\pm)_n}/(\delta_n^!)_0(\mathbb{Z}^{\mathcal{D}(\pm)_n}) \to \mathbb{Z}$ is induced by $\mathbb{Z}^{\mathcal{I}(\pm)_n} \to \mathbb{Z}$ which is given by

$$x \mapsto \textstyle\sum_{(I,w)\in\mathcal{I}(\pm)_n} x_{(I,w)}.$$

We are now in position to tackle Proposition 5.16 in general without much effort.

Proof of Proposition 5.16. Let $I \in \mathcal{I}_n$ and $J \in \mathcal{I}_{n+1}$ with $I \cap J \neq \emptyset$.
Take a positively oriented path $\omega \in C_c(I \cap J, \mathbb{T} - 1) \subseteq C_c(R_\varphi^+(n))$ which is of degree one.
Then $\omega \in C_c(R_\varphi^+(n)|_{\mathbb{T} \backslash \mathcal{D}_n}) \cap C_c(R_\varphi^+(n+1)|_{\mathbb{T} \backslash \mathcal{D}_{n+1}})$ and represents

$$x = [\varphi^n(I), \mathrm{val}(\varphi^n, I)] \in \mathbb{Z}^{\mathcal{I}(\pm)_n} \simeq K_1(C_r^*(R_\varphi^+(n)|_{\mathbb{T} \backslash \mathcal{D}_n})) \quad \text{and}$$
$$y = [\varphi^{n+1}(J), \mathrm{val}(\varphi^{n+1}, J)] \in \mathbb{Z}^{\mathcal{I}(\pm)_{n+1}} \simeq K_1(C_r^*(R_\varphi^+(n+1)|_{\mathbb{T} \backslash \mathcal{D}_{n+1}})).$$

Remark 5.30 implies that $[x]$ is a generator for $\mathbb{Z}^{\mathcal{I}(\pm)_n}/(\delta_n^!)_0(\mathbb{Z}^{\mathcal{I}(\pm)_n}) \simeq K_1(C_r^*(R_\varphi^+(n)))$.
We also obtain the generator $[y]$ for $\mathbb{Z}^{\mathcal{I}(\pm)_{n+1}}/(\delta_{n+1}^!)_0(\mathbb{Z}^{\mathcal{I}(\pm)_{n+1}}) \simeq K_1(C_r^*(R_\varphi^+(n+1)))$.
  Finally $[x]$ is mapped to $[y]$. Thus $\varinjlim K_1(C_r^*(R_\varphi^+(n))) \simeq K_1(C_r^*(R_\varphi^+(n))) \simeq \mathbb{Z}$. $\square$

Remark 5.31. We briefly use the notation of Appendix A again:

$$\mathbb{A}_k = M_{a_k(1)} \oplus \cdots \oplus M_{a_k(m_k)} \quad \text{and} \quad \mathbb{B}_k = M_{b_k(1)} \oplus \cdots \oplus M_{b_k(n_k)}.$$

We have $K_0(\mathbb{A}_k) \simeq \mathbb{Z}^{m_k}$, $K_1(S\mathbb{B}_k) \simeq K_0(\mathbb{B}_k) \simeq \mathbb{Z}^{n_k}$, and $K_1(\mathbb{D}_k) \simeq \mathbb{Z}$ by Proposition 5.18.
The exact sequence (5.2) now yields the following exact sequence

$$0 \longrightarrow K_0(\mathbb{D}_k) \xrightarrow{(\pi_k)_0} \mathbb{Z}^{m_k} \xrightarrow{(\delta_k)_0} \mathbb{Z}^{n_k} \xrightarrow{(\iota_k)_1} \mathbb{Z} \longrightarrow 0 \ .$$

It follows that $\mathbb{Z}^{n_k} \simeq \ker(\iota_k)_1 \oplus \mathbb{Z}$ so $\ker(\iota_k)_1 \simeq \mathbb{Z}^{n_k-1}$ and $\mathbb{Z}^{m_k} \simeq \ker(\delta_k)_0 \oplus \mathbb{Z}^{n_k-1}$ so

$$K_0(C_r^*(R_\varphi^+(k))) \simeq K_0(\mathbb{D}_k) \simeq \ker(\delta_k)_0 \simeq \mathbb{Z}^{m_k-n_k+1} \ .$$

5.3   The consequences of classification

Assume in this section φ is exact and critically finite and without exceptional fixed points.
We compare $C_r^*(R_\varphi^+)$ to C*-algebras in $\mathfrak{A}$ and $\mathfrak{B}$ from Theorem A.17 and Theorem A.18.
Any C*-algebra in $\mathfrak{A}$ or $\mathfrak{B}$ with the same K-theory as $C_r^*(R_\varphi^+)$ is isomorphic to $C_r^*(R_\varphi^+)$.
  The reader is referred to [48] for a general reference on classification results.

Remark 5.32. We consider the subclass of $\mathfrak{B}$ which are direct limit of circle algebras:

(a) If K_0 is a simple dimension group with positive cone $K_0^+ \subseteq K_0$ and order unit $u \in K_0$, and K_1 a torsion free abelian group, and K_0 and K_1 are also both countable group, then there exists $A \in \mathfrak{B}$ which is a direct limit of circle algebras such that

$$(K_0(A), K_0^+(A), [1_A]_0, K_1(A)) \simeq (K_0, K_0^+, u, K_1).$$

(b) If $A \in \mathfrak{B}$ is a direct limit of circle algebras, then $K_1(A)$ is a torsion-free abelian group and $K_0(A)$ is a simple dimension group, and $K_0(A)$ and $K_1(A)$ are both countable.

It follows from Remark 5.8(d) that $K_1(C_r^*(R_\varphi^+))$ is a countable torsion-free abelian group.
It follows from Remark 5.2 that $K_0(C_r^*(R_\varphi^+))$ is a countable simple dimension group.
  We may conclude that $C_r^*(R_\varphi^+)$ is a direct limit of circle algebras.

Remark 5.33. Let X be a Cantor set and $\psi\colon X \to X$ a minimal homeomorphism on X. We have that $C(X) \rtimes_\psi \mathbb{Z} \in \mathfrak{A}$ is a direct limit of circle algebras.

(a) If K_0 is a countable simple dimension group with $K_0^+ \subseteq K_0$ and order unit $u \in K_0$, then there exists a minimal homeomorphism $\psi\colon X \to X$ on a Cantor set such that

$$(K_0(C(X) \rtimes_\psi \mathbb{Z}), K_0^+(C(X) \rtimes_\psi \mathbb{Z}), [1]_0) \simeq (K_0, K_0^+, u).$$

(b) If $\psi\colon X \to X$ is a minimal homeomorphism on a Cantor set, then $K_1(C(X) \rtimes_\psi \mathbb{Z}) \simeq \mathbb{Z}$, and $K_0(C(X) \rtimes_\psi \mathbb{Z})$ is a countable simple dimension group.

It follows from Remark 5.2 again that $K_0(C_r^*(R_\varphi^+))$ is a countable simple dimension group. It follows from Proposition 5.16 and Proposition 5.18 that $K_1(C_r^*(R_\varphi^+)) \simeq \mathbb{Z}$.

We conclude there is a minimal homeomorphism $\varphi\colon X \to X$ on a Cantor set such that

$$C_r^*(R_\varphi^+) \simeq C(X) \rtimes_\psi \mathbb{Z}.$$

5.4 A general case study

In the following example we will take a page out of [25].

Example 5.34. Assume $\mathcal{D} = \bigcup_{l=1}^\infty \varphi^l(\mathcal{C}) = \{z\}$ and $\deg(\varphi) \notin \{-1, 1\}$. Define

$$\begin{aligned}
a &= \#\{x \in \varphi^{-1}(\{z\}) : \mathrm{val}(\varphi, x) = (+, +)\}, \\
b &= \#\{x \in \varphi^{-1}(\{z\}) : \mathrm{val}(\varphi, x) = (-, -)\}, \\
c &= \#\{x \in \varphi^{-1}(\{z\}) : \mathrm{val}(\varphi, x) = (-, +)\}, \\
&= \#\{x \in \varphi^{-1}(\{z\}) : \mathrm{val}(\varphi, x) = (+, -)\}.
\end{aligned} \tag{5.8}$$

Put $d = a - b$ and $e = 2c + a + b$. Then $d = \deg(\varphi)$, and when φ is transitive $e = \exp(h(\varphi))$. We note $\#\bigcup_{l=1}^\infty \varphi^l(\mathcal{C}) = 1$ and $\#(\varphi^{-1}(\{x\}) \setminus \mathcal{C}) \neq 1$ for all $x \in \mathbb{T}$, and $\deg(\varphi) \notin \{-1, 1\}$. It follows that φ is critically finite and φ has no exceptional fixed points and in addition we have that φ is exact if and only if φ is conjugate to a uniformly piecewise linear map, because φ is conjugate to uniformly piecewise linear map when φ is exact by Lemma 3.13 and φ is exact when φ is conjugate to a uniformly piecewise linear map by Lemma 3.42 because the transition matrix has strictly positive entries or alternatively by Lemma 3.21. Also notice $\mathcal{V}^n = \mathcal{V} \setminus \mathcal{W} = \{(-, +), (+, -)\}$ or $\mathcal{V}^n = \mathcal{V} = \{(-, +), (+, -), (+, +), (-, -)\}$ for all $n > 1$ by Corollary 5.21 and Corollary 5.23 where $\mathcal{V}^n = \{v \in \mathcal{V} : (z, v) \in \mathcal{D}(\pm)_n\}$.

Let $G_n = \ker((I_n)_0 - (U_n)_0)$ and $G = \varinjlim(G_n, C_n)$ where C_n is given in Remark 5.11. Let $n \in \mathbb{N}$ with $n > 1$. We will occasionally write $C = C_n$.

Assume $\deg(\varphi) \in 2\mathbb{Z}$ and $z \in \mathcal{C}$. Then we have that

$$(G, G^+, [1]_0) \simeq \begin{cases} (\mathbb{Z}[\tfrac{1}{e}], \mathbb{Z}[\tfrac{1}{e}]^+, 1), & d = 0 \\ (\mathbb{Z}[\tfrac{1}{d}] \oplus \mathbb{Z}[\tfrac{1}{e}], (\mathbb{Z}[\tfrac{1}{d}] \oplus \mathbb{Z}[\tfrac{1}{e}])^+, (1, 1)), & d \neq 0 \end{cases}$$

where $(\mathbb{Z}[\tfrac{1}{d}] \oplus \mathbb{Z}[\tfrac{1}{e}])^+ = \{(x, y) \in \mathbb{Z}[\tfrac{1}{d}] \oplus \mathbb{Z}[\tfrac{1}{e}] : y > 0\} \cup \{(0, 0)\}$.

We will verify this statement in the following. We compute C_n with Proposition 5.9. Let $C_{\infty,n}\colon G_n \to G$ denote one of the canonical maps for the directed system $(G_n, C_n)_{n \in \mathbb{N}}$.

Assume $\mathcal{V}^n = \mathcal{V} \setminus \mathcal{W}$. This must mean that $a = b = 0$. Put $u = [z, (-, +)] + [z, (+, -)]$. It follows now that $G_n = \mathbb{Z}u$ and $G_n^+ = \mathbb{N}_0 u$, and $C(u) = 2cu = eu$ by Proposition 5.9. This induces a group isomorphism $\gamma\colon G \to \mathbb{Z}[\tfrac{1}{e}]$ which is given by $\gamma(C_{\infty,n}(x_u u)) = \frac{2x_u}{e^n}$ and satisfies both $\gamma(G^+) = \mathbb{Z}[\tfrac{1}{e}]^+$ and $\gamma([1]_0) = 1$.

Assume $\mathcal{V}^n = \mathcal{V}$. Put $u = [z, (-, +)] + [z, (+, -)]$, $v = [z, (+, +)]$ and $w = [z, (-, -)]$. It follows now that $\mathcal{B} = \{u, v, w\}$ is a basis for G_n, with $G_n = \mathrm{span}_{\mathbb{Z}} \mathcal{B}$ and $G_n^+ = \mathrm{span}_{\mathbb{N}_0} \mathcal{B}$, and by Proposition 5.9,

$$C = \begin{pmatrix} 2c & c & c \\ a+b & a & b \\ a+b & b & a \end{pmatrix}.$$

We have that $CP = PD$ and $QC = DQ$ where $D = \mathrm{diag}(0, d, e)$, $Q = DP^{-1}$, and

$$P = \begin{pmatrix} -1 & 0 & c \\ 1 & -1 & \frac{a+b}{2} \\ 1 & 1 & \frac{a+b}{2} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{b-a}{2} & \frac{a-b}{2} \\ 2 & 1 & 1 \end{pmatrix}.$$

Also P and Q are integer 3×3-matrices because $d = \deg(\varphi) \in 2\mathbb{Z}$, and $PQ = C$, $QP = D$. This yields the following commutative diagram as in Example 7.3.11 in [25]:

$$\begin{array}{ccccccc}
\mathbb{Z}^3 & \xrightarrow{C} & \mathbb{Z}^3 & \xrightarrow{C} & \mathbb{Z}^3 & \xrightarrow{C} & \cdots \\
\uparrow{\scriptstyle P} & \searrow{\scriptstyle Q} & \uparrow{\scriptstyle P} & \searrow{\scriptstyle Q} & \uparrow{\scriptstyle P} & \searrow{\scriptstyle Q} & \\
\mathbb{Z}^3 & \xrightarrow{D} & \mathbb{Z}^3 & \xrightarrow{D} & \mathbb{Z}^3 & \xrightarrow{D} & \cdots
\end{array}$$

It follows from this commutative diagram that $G = \varinjlim(G_n, C) \simeq \varinjlim(\mathbb{Z}^3, C) \simeq \varinjlim(\mathbb{Z}^3, D)$. These isomorphisms induce group isomorphisms $\alpha \colon G \to \mathbb{Z}[\frac{1}{d}] \oplus \mathbb{Z}[\frac{1}{e}]$ whenever $d \neq 0$ and $\beta \colon G \to \mathbb{Z}[\frac{1}{e}]$ when $d = 0$, which for any $x = x_u u + x_v v + x_w w$ are given by

$$\alpha(C_{\infty,n}(x)) = \left(\frac{x_v - x_w}{d^n}, \frac{2x_u + x_v + x_w}{e^n} \right) \quad \text{and} \quad \beta(C_{\infty,n}(x)) = \frac{2x_u + x_v + x_w}{e^n}. \tag{5.9}$$

In particular we have that $\alpha([1]_0) = (1, 1)$ and $\beta([1]_0) = 1$ respectively.

Assume for the sake of simplicity that φ is uniformly piecewise linear with slope $e > 1$. Let $(I, J) \in \mathcal{I}_n^{(2)}$ and $x, y \in \mathcal{D}_n$ such that $I < x$ and $y < J$. We will now use Definition 4.29. If $x = y$ and $\mathrm{val}(\varphi^n, x) = (\pm, \pm)$ then we have $[p_{I,J}]_0 = [z, \pm, \pm]$ and $(\omega_n)_0([p_{I,J}]_0) = \frac{1}{e^n}$. If $\mathrm{val}(\varphi^n, x) = (-, +)$ and $\mathrm{val}(\varphi^n, y) = (+, -)$ such that $\mathrm{val}(\varphi^n, q) = (+, +)$ for any $q \in \mathcal{D}_n$ with $x < q < y$ then $[p_{I,J}] = [z, (-, +)] + [z, (+, -)] + l[z, (+, +)]$ and $(\omega_n)_0(p_{I,J}) = \frac{l+2}{e^n}$. Therefore for any $x = x_u u + x_v v + x_w w$ we have that

$$(\omega_n)_0(C_{\infty,n}(x)) = \frac{2x_u + x_v + x_w}{e^n}. \tag{5.10}$$

It follows from Remark 5.2 by comparing (5.10) with the two formulas in (5.9) that $\alpha(G^+) = (\mathbb{Z}[\frac{1}{d}] \oplus \mathbb{Z}[\frac{1}{e}])^+ = \{(x, y) \in \mathbb{Z}[\frac{1}{d}] \oplus \mathbb{Z}[\frac{1}{e}] : y > 0\} \cup \{(0, 0)\}$ and $\beta(G^+) = \mathbb{Z}[\frac{1}{e}]^+$. This becomes evident from the formulas in (5.9) after a closer inspection.

The observant reader notes that local homeomorphisms do not figure into the above. Let $\varphi \colon \mathbb{T} \to \mathbb{T}$ be a circle map which is a local homeomorphism but not a homeomorphism. We note that $\deg(\varphi) \notin \{-1, 0, 1\}$ and there exists at least one fixed point $z \in \mathbb{T}$ for φ. Define $\mathcal{D} = \{z\}$ and (5.8). Again $d = \deg(\varphi)$, and when φ is transitive $e = \exp(h(\varphi))$. Also note that $e = d$ or $e = -d$. If $d \geq 2$, then $a = c = 0$. If $d \leq -2$, then $b = c = 0$.

We have $\mathcal{V}^n = \{(-, -)\}$ when $d \leq -2$ and $n \in 2\mathbb{N} - 1$ and $\mathcal{V}^n = \{(+, +)\}$ otherwise. Put $v = [z, (+, +)]$ and $w = [z, (-, -)]$. We will apply the exact same strategy as above. It follows now just as above that $G_n = \mathbb{Z}w$ and $G_n^+ = \mathbb{N}_0 w$ when $d \leq -2$ and $n \in 2\mathbb{N} - 1$, and $G_n = \mathbb{Z}v$ and $G_n^+ = \mathbb{N}_0 v$ otherwise, and by applying Proposition 5.9 again we have $C(v) = av = ev$ when $d \geq 2$, and $C(v) = bw = ew$ and $C(w) = bv = ev$ when $d \leq -2$. This induces a group isomorphism $\delta \colon G \to \mathbb{Z}[\frac{1}{e}]$ with $\delta(G^+) = \mathbb{Z}[\frac{1}{e}]^+$ and $\delta([1]_0) = 1$ where $\delta(C_{\infty,n}(x_w w)) = \frac{x_w}{e^n}$ when $d \leq -2$ and $n \in 2\mathbb{N} - 1$, and $\delta(C_{\infty,n}(x_v v)) = \frac{x_v}{e^n}$ otherwise, i.e.

$$(G, G^+, [1]_0) \simeq (\mathbb{Z}[\tfrac{1}{e}], \mathbb{Z}[\tfrac{1}{e}]^+, 1).$$

Remark 5.35. Given $d = a - b$ and $e = 2c + a + b$ with $a, b \in \mathbb{N}_0$ and $c \in \mathbb{N}$ and $z \in \mathbb{T}$ there is a uniformly piecewise linear $\varphi \colon \mathbb{T} \to \mathbb{T}$ with $\bigcup_{l=1}^{\infty} \varphi^l(\mathcal{C}) = \{z\} \subseteq \mathcal{C}$ and (5.8).
Also $\{(a - b, 2c + a + b) \in \mathbb{Z} \times \mathbb{N} : a, b \in \mathbb{N}_0, c \in \mathbb{N}\} = \{(d, e) \in \mathbb{Z} \times \mathbb{N} : e + d, e - d \in 2\mathbb{N}\}$,
and $d \in 2\mathbb{Z}$ if and only if $e \in 2\mathbb{N}$ for any $(d, e) \in \{(d, e) \in \mathbb{Z} \times \mathbb{N} : e + d, e - d \in 2\mathbb{N}\}$.
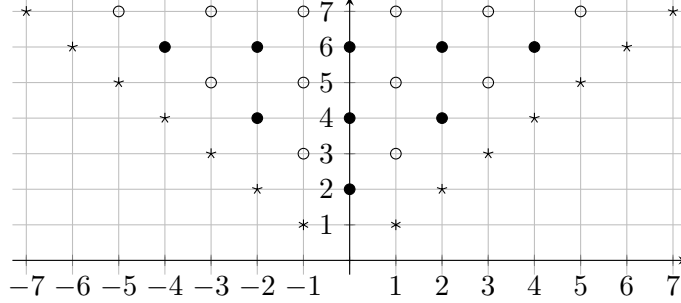

Also $\deg(\lambda_n) = -\deg(\lambda_n^*) = \exp(h(\lambda_n)) = \exp(h(\lambda_n^*)) = n$ for $n \in \mathbb{N}$, cf. Example 3.6.

Let $\mathcal{P}(n)$ denote the set of prime factors for any $n \in \mathbb{Z}$.

Proposition 5.36. *Let $\varphi, \psi \colon \mathbb{T} \to \mathbb{T}$ be continuous piecewise monotone circle maps which are both exact and which also both satisfy that $\# \bigcup_{l=1}^{\infty} \varphi^l(\mathcal{C}) = 1$ and $\# \bigcup_{l=1}^{\infty} \psi^l(\mathcal{C}) = 1$. Assume also $d_\varphi = \deg(\varphi), d_\psi = \deg(\psi) \in 2\mathbb{Z}$, i.e. $e_\varphi = \exp(h(\varphi)), e_\psi = \exp(h(\psi)) \in 2\mathbb{N}$. Let $\mathcal{R}$ and $\mathcal{S}$ be finite sets of prime numbers with $2 \in \mathcal{R}$ and $2 \in \mathcal{S}$. Then*

(i) $C_r^*(R_\varphi^+) \simeq C_r^*(R_\psi^+)$ *whenever $\mathcal{P}(d_\varphi) = \mathcal{P}(d_\psi)$ and $\mathcal{P}(e_\varphi) = \mathcal{P}(e_\psi)$.*

(ii) $\mathcal{R} = \mathcal{P}(d_\theta)$ *and $\mathcal{S} = \mathcal{P}(e_\theta)$ for some continuous piecewise monotone circle map $\theta \colon \mathbb{T} \to \mathbb{T}$ which is exact with $\# \bigcup_{l=1}^{\infty} \theta^l(\mathcal{C}) = 1$, $d_\theta \in 2\mathbb{Z}$ and $e_\theta \in 2\mathbb{N}$.*

Example 5.37. We have $C_r^*(R_\varphi^+) \simeq C_r^*(R_\psi^+)$ for $\varphi = \psi_2$ and $\psi = \psi_1^2$, cf. Example 3.6:
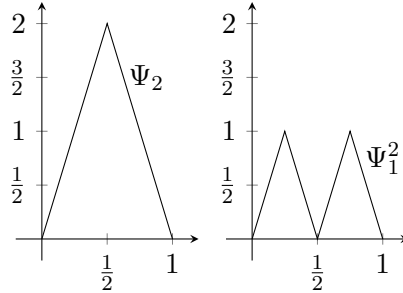
We have φ and ψ are not conjugate, though $d_\varphi = d_\psi$ and $e_\varphi = e_\psi$ and $C_r^*(R_\varphi^+) \simeq C_r^*(R_\psi^+)$, but φ is semiconjugate to ψ via the doubling map and also via the tent map.

Example 5.38. We have $C_r^*(R_\varphi^+) \simeq C_r^*(R_\psi^+)$ for φ and ψ below:
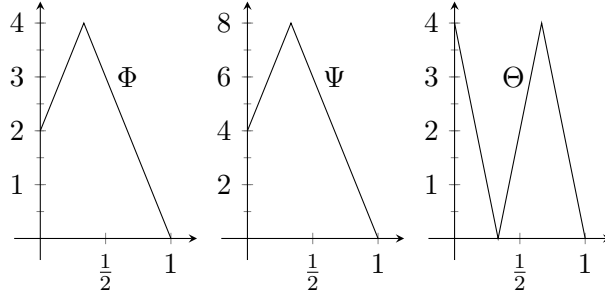
It follows from Theorem 3.2 implies that there is no semiconjugacy maps between φ and ψ.

Remark 5.39. We have the following from earlier results and comments:

(a) We have that $C_r^*(R_\varphi^+) \simeq C_r^*(R_\psi^+)$ when $\varphi = \theta_{2n}$ and $\psi = \psi_n$ from Example 3.6.

(b) We have that $C_r^*(R_\varphi^+) \simeq C_r^*(R_\psi^+)$ when $\psi = \varphi^n$ because $R_\varphi^+ \simeq R_\psi^+$ by Remark 4.8. We note $\deg(\psi) = \deg(\varphi)^n$ by Lemma 3.1 and $h(\psi) = nh(\varphi)$ by Lemma 1.28, and

$$\textstyle\bigcup_{l=1}^\infty \psi^l(\mathcal{C}^\psi) \subseteq \bigcup_{l=1}^\infty \varphi^l(\mathcal{C}^\varphi).$$

Part III

Appendices

Elliott-Thomsen building blocks

A C*-algebra A is homogeneous if and only if there are compact Hausdorff spaces $X_1, \ldots, X_l$ and projections $p_i \in M_{n(i)}(C(X_i))$, $i = 1, 2, \ldots, l$ for some $n(1), \ldots, n(l) \in \mathbb{N}$ such that

$$A \simeq p_1 M_{n(1)}(C(X_1))p_1 \oplus p_2 M_{n(2)}(C(X_2))p_2 \oplus \cdots \oplus p_l M_{n(l)}(C(X_l))p_l,$$

and subhomogeneous if and only if it is a C*-subalgebra of a homogeneous algebra, cf. [48]. A certain class of subhomogeneous algebras are the recursive subhomogeneous algebras, which are iterated pullbacks of a particular form (cf. [35], [36] and [37]).

The class of recursive subhomogeneous algebras is the smallest class of C*-algebras which contains $C(X, M_n)$ for any compact Hausdorff space X and $n \in \mathbb{N}$ and

$$A \oplus_{C(Y^{(0)}, M_m)} C(Y, M_m) = \{(a, f) \in A \oplus C(Y, M_m) : \varphi(a) = \rho(f)\}$$

for any recursive subhomogeneous algebra A and any locally compact Hausdorff space Y, and any unital $*$-homomorphism $\varphi \colon A \to C(Y^{(0)}, M_m)$ where $Y^{(0)} \subseteq Y$ is a closed subset, and $\rho \colon C(Y, M_m) \to C(Y^{(0)}, M_m)$ is the $*$-homomorphism given by restriction.

We allow $Y^{(0)} = \emptyset$ which just yields an ordinary direct sum.

Remark A.1. Any unital separable approximately subhomogeneous algebra is actually a direct limit of recursive subhomogeneous algebras, cf. [31] for details.

We will consider recursive subhomogeneous algebras of the form

$$\mathbb{D} = \{(a, f) \in \mathbb{A} \oplus C([0, 1], \mathbb{B}) : I(a) = f(0), U(a) = f(1)\}$$

with $\mathbb{A}$ and $\mathbb{B}$ finite dimensional C*-algebras and $I, U \colon \mathbb{A} \to \mathbb{B}$ (unital) $*$-homomorphisms. These algebras are known to some as the Elliott–Thomsen algebras (or Thomsen algebras) and they will serve as building blocks of unital approximately subhomogeneous algebras. We note that $\mathbb{D}$ is unital if I and U are unital. In the following we will use the notation

$$\mathbb{A} = M_{a(1)} \oplus \cdots \oplus M_{a(m)} \quad \text{and} \quad \mathbb{B} = M_{b(1)} \oplus \cdots \oplus M_{b(n)}.$$

Denote also by $I^{ij}, U^{ij} : M_{a(j)} \to M_{b(i)}$, and $I^{i\bullet}, U^{i\bullet} : \mathbb{A} \to M_{b(i)}$ and $I^{\bullet j}, U^{\bullet j} : M_{a(j)} \to \mathbb{B}$ the partial maps of $I, U : \mathbb{A} \to \mathbb{B}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Proposition A.2. *The Elliott-Thomsen algebras are recursive subhomogeneous algebras.*

Proof. We note that $\mathbb{D}^0 = \mathbb{A} = M_{a(1)} \oplus \cdots \oplus M_{a(m)}$ is a recursive subhomogeneous algebra. Put $Y = [0, 1]$ and $Y^{(0)} = \{0, 1\}$. Define $\varphi^i \colon \mathbb{D}^{i-1} \to C(Y^{(0)}, M_{b(i)})$ for each $i = 1, \ldots, n$: Put $\mathbb{D}^1 = \{(a, f_1) \in \mathbb{D}^0 \oplus C([0, 1], M_{b(1)}) : \varphi^1(a) = \rho(f_1)\}$ where $\varphi^1(a) = (I^{1\bullet}(a), U^{1\bullet}(a))$. Put $\varphi^i(a, f_1, \ldots, f_{i-1}) = (I^{i\bullet}(a), U^{i\bullet}(a))$ recursively for $i = 2, \ldots, n$ and

$$\mathbb{D}^i = \{(a, f_1, \ldots, f_i) \in \mathbb{D}^{i-1} \oplus C([0, 1], M_{b(i)}) : \varphi^i(a, f_1, \ldots, f_{i-1}) = \rho(f_i)\}.$$

Then $\mathbb{D}^i$ is a recursive subhomogeneous algebra for each $i = 1, \ldots, n$, and $\mathbb{D} = \mathbb{D}^n$. $\qquad \square$

For any projection $p \in C([0,1], M_n)$ note $\mathrm{Tr}(p(s)) = \mathrm{Tr}(p(t)) \in \mathbb{N}$ for all $s, t \in [0,1]$. Denote by $\mathrm{rank}\, p = \mathrm{Tr}(p(0))$ the rank of the projection $p \in C([0,1], M_n)$.

Proposition A.3. *Let $p \in C([0,1], M_n)$ be a projection and suppose that $\mathrm{rank}\, p = l \leq n$. There is a unitary $w \in C([0,1], M_n)$ such that*

$$wpw^* = \mathrm{diag}(d_1, \ldots, d_n) = q$$

where $d_i = 1$ for all $i = 1, \ldots, l$ and $d_j = 0$ for all $j = l+1, \ldots, n$.

Proof. As $p(t)$ and $q(t)$ have exactly the same eigenvalues for all $t \in [0,1]$ it follows from Corollary 1.3 in [52] that there is a unitary $u \in C([0,1], M_n)$ such that $\|upu^* - q\| < 1$, hence there is a unitary $v \in C([0,1], M_n)$ such that $(vu)p(vu)^* = q$. $\square$

Lemma A.4. *Let $p \in C([0,1], M_n)$ be a projection with $l = \mathrm{rank}\, p < n$. Put $p^\perp = 1 - p$. Let $x \in p^\perp(s_0) M_n p^\perp(s_0)$ and $y \in p^\perp(t_0) M_n p^\perp(t_0)$ be rank-one projections with $s_0 < t_0$. There is a rank-one projection $q \in p^\perp C([0,1], M_n) p^\perp$ such that $q(s_0) = x$ and $q(t_0) = y$.*

Proof. We note that $m = \mathrm{rank}\, p^\perp = n - l > 0$. In particular the projection $p^\perp$ is non-zero. As $p^\perp$ is a projection in $C([0,1], M_n)$, there is a unitary $w \in C([0,1], M_n)$ such that

$$wp^\perp w^* = \mathrm{diag}(d_1, \ldots, d_n)$$

where $d_i = 1$ for all $i = 1, \ldots, m$ and $d_j = 0$ for all $j = m+1, \ldots, n$, accordingly

$$p^\perp C([0,1], M_n) p^\perp = w^* C([0,1], M_m \oplus 0_l) w \simeq C([0,1], M_m).$$

Take rank-one projections $q'_x, q'_y \in C([0,1], M_m)$ and $q_x, q_y \in p^\perp C([0,1], M_n) p^\perp$ with

$$q'_x(s_0) = w(s_0) x w^*(s_0) \qquad \text{and} \qquad q'_y(t_0) = w(t_0) y w^*(t_0);$$
$$q_x = w^* q'_x w \qquad \text{and} \qquad q_y = w^* q'_y w.$$

Then $q_x, q_y \in p^\perp C([0,1], M_n) p^\perp$ are rank-one projections with $q_x(s_0) = x$ and $q_y(t_0) = y$. We apply Proposition A.3 again and obtain two unitaries $u, v \in C([0,1], M_m)$ such that $u q'_x u^* = \mathrm{diag}(1, 0, \ldots, 0)$ and $v q'_y v^* = \mathrm{diag}(1, 0, \ldots, 0)$, hence

$$q'_x = (u^* v) q'_y (u^* v)^* \quad \text{and} \quad q_x = (w^* u^* v w) q_y (w^* u^* v w)^*.$$

It is quite easy to verify that the unitary group of $C([0,1], M_m)$ is path-connected: Indeed consider any unitary $\omega \in C([0,1], M_m)$ and define a path by $\omega_s(t) = \omega(s(1-t) + t)$ between $\omega_0 = \omega$ and $\omega_1 \equiv \omega(1)$, then define a path between $\omega(1)$ and I_m using the fact that the unitary group of M_m is path-connected.

Take a path of unitaries ω_r, $r \in [s_0, t_0]$ in $C([s_0, t_0], M_m)$ with $\omega_{s_0} = v^* u$ and $\omega_{t_0} = I_m$ and extend this to a path in $C([0,1], M_m)$ by $\omega_s = \omega_{s_0}$ for $s < s_0$ and $\omega_t = \omega_{t_0}$ for $t > t_0$. Define a rank-one projection $q \in p^\perp C([0,1], M_n) p^\perp$ by $q(r) = (w^* \omega_r(r) w) q_y (w^* \omega_r(r) w)^*$. Then $q(s_0) = q_x(s_0) = x$ and $q(t_0) = q_y(t_0) = y$. $\square$

Denote by $M(I^{ij}), M(U^{ij})$ the multiplicities of the partial maps $I^{ij}, U^{ij} : M_{a(j)} \to M_{b(i)}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We note that $b(i) \geq a(j)$ if $U^{ij} \neq 0$ or $I^{ij} \neq 0$. Denote by $\mathrm{rank}(a, f) = \max_{i,j} \{\mathrm{rank}\, a_j, \mathrm{rank}\, f_i\}$ the rank of a projection $(a, f) \in \mathbb{D}$ where $a = (a_1, \ldots, a_m)$ and $f = (f_1, \ldots, f_n)$ with $a_j \in M_{a(j)}$ and $f_i \in C([0,1], M_{b(i)})$.

Lemma A.5. *Any building block* $\mathbb{D} = \{(a, f) \in \mathbb{A} \oplus C([0, 1], \mathbb{B}) : I(a) = f(0), U(a) = f(1)\}$ *satisfying* $\min_j a(j) \geq 2n + 1$ *and* $M(I^{ij}), M(U^{ij}) \in \{0, 1\}$,

$$\sum_{l=1}^{m} M(I^{il}) + \sum_{l=1}^{m} M(U^{il}) \geq 1 \tag{A.1}$$

$$\sum_{l=1}^{n} M(I^{lj}) + \sum_{l=1}^{n} M(U^{lj}) \geq 2 \tag{A.2}$$

for all $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$, *contains a non-zero projection of rank at most* $2n$.

Proof. The proof emulates the proof of Lemma 3.3 in [53]:

Define a labelled oriented graph $\mathcal{G} = (V, E)$, where $V = \{(i, *_i) : i = 1, \ldots, n, *_i\{\rightarrow, \leftarrow\}\}$ is the set of vertices and the set of edges E is determined by the following rules:

- $[(i, \rightarrow) \overset{j}{\rightsquigarrow} (i', \rightarrow)] \in E$ if $U^{ij} \neq 0$ and $I^{i'j} \neq 0$,
- $[(i, \rightarrow) \overset{j}{\rightsquigarrow} (i', \leftarrow)] \in E$ if $i \neq i'$, $U^{ij} \neq 0$ and $U^{i'j} \neq 0$,
- $[(i, \leftarrow) \overset{j}{\rightsquigarrow} (i', \leftarrow)] \in E$ if $I^{ij} \neq 0$ and $U^{i'j} \neq 0$, and
- $[(i, \leftarrow) \overset{j}{\rightsquigarrow} (i', \rightarrow)] \in E$ if $i \neq i'$, $I^{ij} \neq 0$ and $I^{i'j} \neq 0$.

Let $p \in C([0, 1], M_{b(i)})$ and $p' \in C([0, 1], M_{b(i')})$ and $q \in M_{a(j)}$ be rank-one projections. We say that an edge $(i, *_i) \overset{j}{\rightsquigarrow} (i', *_{i'})$ in $\mathcal{G}$ can be realized in the following way:

- $p \overset{q}{\rightsquigarrow} p'$ realizes $(i, \rightarrow) \overset{j}{\rightsquigarrow} (i', \rightarrow)$ if $p(1) = U^{ij}(q)$ and $p'(0) = I^{i'j}(q)$,
- $p \overset{q}{\rightsquigarrow} p'$ realizes $(i, \rightarrow) \overset{j}{\rightsquigarrow} (i', \leftarrow)$ if $p(1) = U^{ij}(q)$ and $p'(1) = U^{i'j}(q)$,
- $p \overset{q}{\rightsquigarrow} p'$ realizes $(i, \leftarrow) \overset{j}{\rightsquigarrow} (i', \leftarrow)$ if $p(0) = I^{ij}(q)$ and $p'(1) = U^{i'j}(q)$, and
- $p \overset{q}{\rightsquigarrow} p'$ realizes $(i, \leftarrow) \overset{j}{\rightsquigarrow} (i', \rightarrow)$ if $p(0) = I^{ij}(q)$ and $p'(0) = I^{i'j}(q)$.

Furthermore a path $c = (i_1, *_1) \overset{j_1}{\rightsquigarrow} \cdots \overset{j_{d-1}}{\rightsquigarrow} (i_d, *_d)$ in $\mathcal{G}$ is realized by $p_1 \overset{q_1}{\rightsquigarrow} \cdots \overset{q_{d-1}}{\rightsquigarrow} p_d$ if $p_l \overset{q_l}{\rightsquigarrow} p_{l+1}$ realizes $(i_l, *_l) \overset{j_l}{\rightsquigarrow} (i_{l+1}, *_{l+1})$ for all $l = 1, \ldots, d$ and $p_1, \ldots, p_d$ are mutually orthogonal in $C([0, 1], \mathbb{B})$ and $q_1, \ldots, q_{d-1}$ are mutually orthogonal in $\mathbb{A}$.

Any path in $\mathcal{G}$ of length at most $d - 1$ can be realized whenever $\min_i a(i) \geq d$. This is proved by induction in d. First notice that $U^{ij}(q)$, $U^{i'j}(q)$, $I^{ij}(q)$, $I^{i'j}(q)$ are all rank-one projections, when q is a rank-one projection and $U^{ij} \neq 0$, $U^{i'j} \neq 0$, $I^{ij} \neq 0$, $I^{i'j} \neq 0$ respectively, since all the partials map have multiplicities either 0 or 1, hence any path of length one $(i_1, *_1) \overset{j_1}{\rightsquigarrow} (i_2, *_2)$ is trivially realized. Assume then that any path in $\mathcal{G}$ of length at most $l - 1$ can be realized when $\min_i a(i) \geq l$ for some $l \geq 2$, and suppose that $\min_i a(i) \geq l + 1$. Then any path in $\mathcal{G}$ of length at most $l - 1$ can be realized by the induction hypothesis. Let

$$c = (i_1, *_1) \overset{j_1}{\rightsquigarrow} \cdots \overset{j_{l-1}}{\rightsquigarrow} (i_l, *_l) \overset{j_l}{\rightsquigarrow} (i_{l+1}, *_{l+1})$$

be a path in $\mathcal{G}$ of length l, and let $(i_1, *_1) \overset{j_1}{\rightsquigarrow} \cdots \overset{j_{l-1}}{\rightsquigarrow} (i_l, *_l)$ be realized by $p_1 \overset{q_1}{\rightsquigarrow} \cdots \overset{q_{l-1}}{\rightsquigarrow} p_l$. We divide into the cases

(a) $*_l = \rightarrow$ and $*_{l+1} = \rightarrow$; (b) $*_l = \rightarrow$ and $*_{l+1} = \leftarrow$;

(c) $*_l = \leftarrow$ and $*_{l+1} = \leftarrow$; (d) $*_l = \leftarrow$ and $*_{l+1} = \rightarrow$.

Consider case (a) $*_l =\to$ and $*_{l+1} =\to$. Then $q_1, \ldots, q_{l-1}$ are mutually orthogonal, and we can find a rank-one projection $q_l \in q^\perp M_{a(j_l)} q^\perp$ where $q^\perp = 1 - (q_1 + \cdots + q_{l-1})$, since $a(j_l) \geq l + 1$. It follows that q_l is orthogonal to $q_1, \ldots, q_{l-1}$ in $\mathbb{A}$, and moreover $U^{i_l j_l}(q_l)$ is orthogonal to $p_2(1), \ldots, p_{l-1}(1)$, but $U^{i_l j_l}(q_l)$ may not be orthogonal to $p_1(1)$ (when $*_1 =\leftarrow$ and $i_1 = i_l$). If $*_1 =\leftarrow$ and $i_1 = i_l$, then we can change p_1 to ensure that $U^{i_l j_l}(q_l)$ is also orthogonal to $p_1(1)$. We change p_1 as follows: Take a rank-one projection $y \in p^\perp(1) M_{b(i_1)} p^\perp(1)$ which is orthogonal to $U^{i_l j_l}(q_l)$ in $M_{b(i_1)}$ where $p^\perp = 1 - (p_2 + \cdots + p_l)$. This is possible because $b(i_1) \geq \min_i a(i) \geq l + 1$. Since $x = p_1(\frac{1}{2}) \in p^\perp(\frac{1}{2}) M_{b(i_1)} p^\perp(\frac{1}{2})$ and $y \in p^\perp(1) M_{b(i_1)} p^\perp(1)$ are rank-one projections in $M_{b(i_1)}$ there is by Lemma A.4 a rank-one projection $p_1' \in p^\perp C([0,1], M_{b(i_1)}) p^\perp$ such that $p_1'(\frac{1}{2}) = x$ and $p_1'(1) = y$, and we may replace $p_1(t)$ by $p_1'(t)$ for all $\frac{1}{2} \leq t \leq 1$ thereby obtaining p_1 which is a rank-one projection still orthogonal to $p_2, \ldots, p_l$ in $C([0,1], M_{b(i_1)})$ with $p_1(0) = I^{i_1 j_1}(q_1)$ such that $p_1(1)$ is also orthogonal to $U^{i_l j_l}(q_l)$ in $M_{b(i_1)}$, so now $U^{i_l j_l}(q_l)$ is orthogonal to $p_1(1), \ldots, p_{l-1}(1)$. Since $x = p_l(\frac{1}{2}) \in p^\perp(\frac{1}{2}) M_{b(i_l)} p^\perp(\frac{1}{2})$ and $y = U^{i_l j_l}(q_l) \in p^\perp(1) M_{b(i_l)} p^\perp(1)$ are rank-one projections in $M_{b(i_l)}$ where $p^\perp = 1 - (p_1 + \cdots + p_{l-1})$ and because $b(i_l) \geq \min_i a(i) \geq l + 1$ there is by Lemma A.4 a rank-one projection $p_l' \in p^\perp C([0,1], M_{b(i_l)}) p^\perp$ with $p_l'(\frac{1}{2}) = p_l(\frac{1}{2})$ and $p_l'(1) = U^{i_l j_l}(q_l)$, and we may replace $p_l(t)$ by $p_l'(t)$ for all $\frac{1}{2} \leq t \leq 1$ thereby obtaining p_l which is a rank-one projection still orthogonal to $p_1, \ldots, p_{l-1}$ in $C([0,1], M_{b(i_l)})$ with $p_l(0) = I^{i_l j_{l-1}}(q_{l-1})$ such that $p_l(1) = U^{i_l j_l}(q_l)$. Similarly $I^{i_{l+1} j_l}(q_l)$ is orthogonal to $p_2(0), \ldots, p_l(0)$ but $I^{i_{l+1} j_l}(q_l)$ may not be orthogonal to $p_1(0)$ (when $*_1 =\to$ and $i_1 = i_{l+1}$). If $*_1 =\to$ and $i_1 = i_{l+1}$ (and not $*_1 =\leftarrow$ as assumed above), then we can change p_1 to ensure that $I^{i_{l+1} j_l}(q_l)$ is also orthogonal to $p_1(0)$. We change p_1 in the following way: Take a rank-one projection $x \in p^\perp(0) M_{b(i_1)} p^\perp(0)$ which is orthogonal to $I^{i_{l+1} j_l}(q_l)$ in $M_{b(i_1)}$ where $p^\perp = 1 - (p_2 + \cdots + p_l)$. This is possible because $b(i_1) \geq \min_i a(i) \geq l + 1$. Since $x \in p^\perp(0) M_{b(i_1)} p^\perp(0)$ and $y = p_1(\frac{1}{2}) \in p^\perp(\frac{1}{2}) M_{b(i_1)} p^\perp(\frac{1}{2})$ are rank-one projections in $M_{b(i_1)}$ there is by Lemma A.4 a rank-one projection $p_1' \in p^\perp C([0,1], M_{b(i_1)}) p^\perp$ such that $p_1'(0) = x$ and $p_1'(\frac{1}{2}) = y$, and we may replace $p_1(t)$ by $p_1'(t)$ for all $0 \leq t \leq \frac{1}{2}$ thereby obtaining p_1 which is a rank-one projection still orthogonal to $p_2, \ldots, p_l$ in $C([0,1], M_{b(i_1)})$ with $p_1(1) = U^{i_1 j_1}(q_1)$ such that $p_1(0)$ is also orthogonal to $I^{i_{l+1} j_l}(q_l)$ in $M_{b(i_1)}$, so now $I^{i_{l+1} j_l}(q_l)$ is orthogonal to $p_1(0), \ldots, p_l(0)$. Finally ready to define p_{l+1}: Take a rank-one projection $y \in p^\perp(1) M_{b(i_{l+1})} p^\perp(1)$ where $p^\perp = 1 - (p_1 + \cdots + p_l)$. This is possible and again it is because $b(i_{l+1}) \geq \min_i a(i) \geq l + 1$. Since $x = I^{i_{l+1} j_l}(q_l) \in p^\perp(0) M_{b(i_{l+1})} p^\perp(0)$ and $y = p^\perp(1) M_{b(i_{l+1})} p^\perp(1)$ are rank-one projections in $M_{b(i_{l+1})}$ there is by Lemma A.4 a rank-one projection $p_{l+1} \in p^\perp C([0,1], M_{b(i_{l+1})}) p^\perp$ such that $p_{l+1}(0) = x$ and $p_{l+1}(1) = y$ thereby obtaining p_{l+1} a rank-one projection orthogonal to $p_1, \ldots, p_l$ in $C([0,1], M_{b(i_{l+1})})$ with $p_{l+1}(0) = I^{i_{l+1} j_l}(q_l)$. Now $p_1 \overset{q_1}{\rightsquigarrow} \cdots \overset{q_{l-1}}{\rightsquigarrow} p_l \overset{q_l}{\rightsquigarrow} p_{l+1}$ realizes c.

The other cases (b), (c), and (d) follow in the same way completing the induction.

It follows from (A.1) and (A.2) that the graph $\mathcal{G}$ has no sinks: Indeed if $(i, *) \overset{j}{\rightsquigarrow} (i', *')$ where $*, *' \in \{\to, \leftarrow\}$, $i, i' \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, then by (A.1) there exists $j' \in \{1, \ldots, m\}$ such that $U^{i' j'} \neq 0$ or $I^{i' j'} \neq 0$ and by (A.2) there exists $i'' \in \{1, \ldots, n\}$ such that altogether $(i', *') \overset{j'}{\rightsquigarrow} (i'', *'')$. Since $\mathcal{G}$ has no sinks, there is a loop

$$ c = (i_1, *_1) \overset{j_1}{\rightsquigarrow} \cdots \overset{j_{d-1}}{\rightsquigarrow} (i_d, *_d) \overset{j_d}{\rightsquigarrow} (i_1, *_1) $$

in $\mathcal{G}$ of length $d \leq 2n$. As $\min_i a(i) \geq 2n + 1$ we have c is realized by $p_1 \overset{q_1}{\rightsquigarrow} \cdots \overset{q_{d-1}}{\rightsquigarrow} p_d \overset{q_d}{\rightsquigarrow} p_{d+1}$. If $*_1 =\to$, then as before we change p_1 such that $p_1(0) = I^{i_1 j_d}(q_d)$ applying Lemma A.4, as $b(i_1) \geq \min_i a(i) \geq d + 1$, keeping $p_1(1) = U^{i_1 j_1}(q_1)$ and $p_1, \ldots, p_d$ mutually orthogonal. If $*_1 =\leftarrow$, change p_1 such that $p_1(1) = U^{i_1 j_d}(q_d)$ applying Lemma A.4, keeping

$p_1(0) = I^{i_1 j_1}(q_1)$ and $p_1, \ldots, p_d$ mutually orthogonal, as $b(i_1) \geq \min_i a(i) \geq d + 1$. Then $\pi = (\sum_{i=1}^{d} q_i, \sum_{i=1}^{d} p_i)$ is a non-zero projection in $\mathbb{D}$ with rank $\pi = d \leq 2n$. $\qquad \square$

Remark A.6. We note that $\min_j a(j) \leq \min_i b(i)$ if (A.1) is satisfied.

Lemma 4.16 states that $C_r^*(R_\varphi^+(n))$ is isomorphic to the Elliott–Thomsen algebra $\mathbb{D}_n$. We will explore some implications of this relationship in the following. We will write

$$\mathbb{A}_k = M_{a_k(1)} \oplus \cdots \oplus M_{a_k(m_k)} \quad \text{and} \quad \mathbb{B}_k = M_{b_k(1)} \oplus \cdots \oplus M_{b_k(n_k)}.$$

We will even prove some statements that were also proved in Chapter 4 using other methods. In order to do this, some preparations are needed.

Remark A.7. The maps $I_k, U_k \colon \mathbb{A}_k \to \mathbb{B}_k$ satisfy $M(I_k^{ij}), M(U_k^{ij}) \in \{0, 1\}$, and

$$\sum_{l=1}^{m_k} M(I_k^{il}) + \sum_{l=1}^{m_k} M(U_k^{il}) \geq 1$$

$$\sum_{l=1}^{n_k} M(I_k^{lj}) + \sum_{l=1}^{n_k} M(U_k^{lj}) \geq 2$$

for all $i = 1, \ldots, n_k$ and $j = 1, \ldots, m_k$. Indeed for $x \in \mathcal{D}_k$ with $e_{x,x} \in M_{a_k(j)}$ it follows that if $I_k^{ij} \neq 0$ then there is a unique $I \in \mathcal{I}_k$ such that $I_k^{ij}(e_{x,x}) = e_{I,I}$ and so $M(I_k^{ij}) = 1$, and if $U_k^{ij} \neq 0$ then there is a unique $J \in \mathcal{I}_k$ such that $U_k^{ij}(e_{x,x}) = e_{J,J}$ and so $M(U_k^{ij}) = 1$. If $I \in \mathcal{I}_k$ with $e_{I,I} \in M_{b_k(i)}$ we have $x < I < y$ with $e_{x,x} \in M_{a_k(l_x)}$ and $e_{y,y} \in M_{a_k(l_y)}$ so

$$\sum_{l=1}^{m_k} M(I_k^{il}) \geq \text{Tr}(I_k^{il_x}(e_{x,x}) + I_k^{il_y}(e_{y,y})) = \text{Tr}(e_{I,I}) = 1,$$

$$\sum_{l=1}^{m_k} M(U_k^{il}) \geq \text{Tr}(U_k^{il_x}(e_{x,x}) + U_k^{il_y}(e_{y,y})) = \text{Tr}(e_{I,I}) = 1.$$

If $x \in \mathcal{D}_k$ with $e_{x,x} \in M_{a_k(j)}$ we have $I < x < J$ so

$$\sum_{l=1}^{n_k} M(I_k^{lj}) + \sum_{l=1}^{n_k} M(U_k^{lj}) \geq \text{Tr}(e_{I,I} + e_{J,J}) = 2.$$

In fact we also have upper bounds

$$\sum_{l=1}^{n_k} M(I_k^{lj}) + \sum_{l=1}^{n_k} M(U_k^{lj}) \leq 2,$$

$$\sum_{l=1}^{m_k} M(I_k^{il}) \leq 2 \quad \text{and} \quad \sum_{l=1}^{m_k} M(U_k^{il}) \leq 2$$

for all $i = 1, \ldots, n_k$ and $j = 1, \ldots, m_k$, hence

$$\sum_{l=1}^{n_k} M(I_k^{lj}) + \sum_{l=1}^{n_k} M(U_k^{lj}) = 2, \tag{A.3}$$

$$1 \leq \sum_{l=1}^{m_k} M(I_k^{il}) \leq 2 \quad \text{and} \quad 1 \leq \sum_{l=1}^{m_k} M(U_k^{il}) \leq 2$$

for all $i = 1, \ldots, n_k$ and $j = 1, \ldots, m_k$.

Assume that R_φ^+ is minimal. Then $\min_j a_k(j) \geq 2n_k + 1$ for all $k \geq K$ for some $K \in \mathbb{N}$, because of Remark 4.13 and Remark 4.56. Thus $\mathbb{D}_k$ satisfies the conditions of Lemma A.5 for all $k \geq K$ and so contains a non-zero projection of rank at most $2n_k$.

Proposition A.8. *The extremal tracial states on a building block $\mathbb{D}$ are*

$$(a, f) \mapsto \frac{1}{a(j)} \operatorname{Tr}(a_j) \quad and \quad (a, f) \mapsto \frac{1}{b(i)} \operatorname{Tr}(f_i(t))$$

where $t \in (0, 1)$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

The proof of this result is omitted, and we will not use this.

Remark A.9. We presently assume that φ is uniformly piecewise linear with slope $s > 1$, and denote by ω the tracial state on $C_r^*(R_\varphi^+)$ by the normalized Lebesgue measure on $\mathbb{T}$. We consider $\omega_k = \omega|_{C_r^*(R_\varphi^+(k))}$. Recall that for $f \in C_c(R_\varphi^+)$,

$$\omega(f) = \int_{\mathbb{T}} f(x) \, \lambda(dx)$$

We note that there is a unique tracial state τ_k on the building block $\mathbb{D}_k$ such that $\omega_k = \tau_k \circ \mu_k$. Proposition A.8 describes tracial states on building blocks. We aim at a description of τ_k.

Recall that $\mu_k(f) = (a_k(f), b_k(f))$ for all $f \in C_c(R_\varphi^+(k))$ and

$$\tilde{b}_k(f) = \sum_{(I,J) \in \mathcal{I}_k^{(2)}} \tilde{f}_{I,J} e_{I,J},$$

where $\tilde{f}_{I,J} \colon [0, 1] \to \mathbb{C}$ is the unique continuous extension of $t \mapsto f(\kappa_I^k(t), \kappa_J^k(t))$, $t \in (0, 1)$ and $\varphi^k(I) = \varphi^k(J) = I_i = (c_{i-1}, c_i) \in \mathcal{I}$, cf. Lemma 4.16.

We note that as φ is uniformly piecewise linear with slope $s > 1$ it then follows that φ^k is uniformly piecewise linear with slope $s^k > 1$ and $(\varphi^k|_I)^{-1}$ is linear with slope $\pm s^{-k}$. Define

$$\tau_k(a, f) = \sum_{I \in \mathcal{I}_k} s_I \int_0^1 f_{I,I}(t) \, dt$$

for all $(a, f) \in \mathbb{D}_k$ where $f = \sum_{I,J} f_{I,J} e_{I,J}$ with $f_{I,J} \in C([0, 1])$ and $s_I = s^{-k}(c_i - c_{i-1})$, and notice that $\tau_k \circ \mu_k = \omega_k$ and $\sum_{I \in \mathcal{I}_k} s_I = 1$. Alternatively,

$$\tau_k(a, f) = \sum_{i=1}^{n_k} s_i \int_0^1 \operatorname{Tr} f_i(t) \, dt = \sum_{i=1}^{n_k} b_k(i) s_i \int_0^1 \operatorname{tr} f_i(t) \, dt$$

for all $(a, f) \in \mathbb{D}_k$ where $f = (f_1, \ldots, f_{n_k})$ with $f_i = \sum_{I,J} f_{I,J} e_{I,J} \in C([0, 1], M_{b_k(i)})$ where the sum is over all $(I, J) \in \mathcal{I}_k^{(2)}$ with $\varphi^k(I) = \varphi^k(J) = I_{l_i}$ and $\operatorname{val}(\varphi^k, I) = \operatorname{val}(\varphi^k, J)$ and $s_i = s^{-k}(c_{l_i} - c_{l_i-1})$, and notice that $\sum_{i=1}^{n_k} b_k(i) s_i = 1$.

For any projection $(a, f) \in \mathbb{D}_k$ notice that $\tau_k(a, f) = \sum_{i=1}^{n_k} s_i r_i$ where $r_i = \operatorname{rank} f_i \in \mathbb{N}$.

Proof of Lemma 4.59. This proof is based on the above.

Let $\varepsilon > 0$ be given. It follows from Remark A.6 that $\min_j a_k(j) \leq \min_i b_k(i)$ for all $k \in \mathbb{N}$ and by Lemma A.5 there is a projection $p_k \in C_r^*(R_\varphi^+(k))$ of rank at most $2n_k$ for all $k \geq K$ for some $K \in \mathbb{N}$, in particular

$$\omega_k(p_k) \leq \frac{2n_k}{\min_j a_k(j)}.$$

Recall that $n_k \leq 2\#\mathcal{I}$ and $\min_j a_k(j) \geq \min_{x \in \mathbb{T}} R_\varphi^+(k)x$ for all $k \in \mathbb{N}$. It follows from Lemma 4.55 that we can pick $k \geq K$ such that $\frac{4\#\mathcal{I}}{\min_j a_k(j)} < \varepsilon$, and therefore there is a projection $p = p_k \in C_r^*(R_\varphi^+(k))$ such that $\omega(p) \leq \frac{4\#\mathcal{I}}{\min_j a_k(j)} < \varepsilon$. $\qquad\square$

Remark A.10. We have the following list of isomorphisms:

$$C_0((0,1), M_n) \simeq \{f \in C([0,1], M_n) : f(0) = f(1) = 0\}, \tag{A.4}$$

$$C_0((0,1), M_n)^\sim \simeq \{f \in C([0,1], M_n) : f(0) = f(1) \in \mathbb{C}\}, \tag{A.5}$$

$$C(\mathbb{T}, M_n) \simeq \{f \in C([0,1], M_n) : f(0) = f(1)\}. \tag{A.6}$$

The following list of results can be found in [47] among other places.

$$K_1(C(\mathbb{T}, M_n)) \simeq K_1(C_0((0,1), M_n)^\sim) \simeq K_1(C_0((0,1), M_n)) \simeq \mathbb{Z}. \tag{A.7}$$

We can now use (A.4), (A.5) and (A.6) to describe the isomorphisms from (A.7) explicitly. For any $u, v \in \mathcal{U}_\infty(C(\mathbb{T}, M_n))$ and $l \in \mathbb{N}$,

$$w(\det \circ u) = w(\det \circ (u \oplus 1_l)),$$

$$w(\det \circ (u \oplus v)) = w(\det \circ u) + w(\det \circ v).$$

Therefore $[u]_1 \mapsto w(\det \circ u)$ is a well-defined group homomorphism from respectively $K_1(C(\mathbb{T}, M_n))$, $K_1(C_0((0,1), M_n)^\sim)$ and $K_1(C_0((0,1), M_n))$ into $\mathbb{Z}$, which is surjective. Any surjective group homomorphism $\mathbb{Z} \to \mathbb{Z}$ is injective. Also (A.7) is already established. In this way we obtain a group isomorphism in each case.

Remark A.11. The group isomorphisms in Remark A.10 induce an isomorphism

$$K_1(C_0((0,1), \mathbb{B})) \simeq \mathbb{Z}^n.$$

Define $u_i = (u_i^1, \ldots, u_i^n) \in C_0((0,1), \mathbb{B})^\sim$ by $(u_i^j)_{kl}(t) = \delta_{kl}(1 + \delta_{ij}(e^{2\pi i t \delta_{lm_{ij}}} - 1))$ for all $i, j \in \{1, \ldots, n\}$ and $k, l \in \{1, \ldots, b(j)\}$ where $m_{ij} \in \{1, \ldots, b(j)\}$ can be chosen arbitrary. Then $[u_i]_1 = e_i$ since $w(\det \circ u_i^j) = \delta_{ij}$.

Remark A.12. We note that $\iota \colon S\mathbb{B} \to \mathbb{D}$ given by $\iota(f) = (0, f)$ is injective, and $\pi \colon \mathbb{D} \to \mathbb{A}$ given by $\pi(a, f) = a$ is surjective since $\pi(a, f) = a$ when $f(t) = (1 - t)I(a) + tU(a)$.

$$0 \longrightarrow S\mathbb{B} \overset{\iota}{\longrightarrow} \mathbb{D} \overset{\pi}{\longrightarrow} \mathbb{A} \longrightarrow 0 .$$

We obtain the following six-term exact sequence

$$
\begin{array}{ccccc}
K_0(S\mathbb{B}) & \overset{\iota_0}{\longrightarrow} & K_0(\mathbb{D}) & \overset{\pi_0}{\longrightarrow} & K_0(\mathbb{A}) \\
\uparrow & & & & \downarrow {\scriptstyle \delta_0} \\
K_1(\mathbb{A}) & \overset{\pi_1}{\longleftarrow} & K_1(\mathbb{D}) & \overset{\iota_1}{\longleftarrow} & K_1(S\mathbb{B})
\end{array}
\tag{A.8}
$$

where $K_1(\mathbb{A}) = 0$, $K_0(S\mathbb{B}) \simeq K_1(\mathbb{B}) = 0$, $K_0(\mathbb{A}) \simeq \mathbb{Z}^m$, $K_1(S\mathbb{B}) \simeq K_0(\mathbb{B}) \simeq \mathbb{Z}^n$, so

$$K_0(\mathbb{D}) \simeq \ker \delta_0 \quad \text{and} \quad K_1(\mathbb{D}) \simeq \operatorname{coker} \delta_0.$$

Define $\bar{\iota}\colon S\mathbb{B}^{\sim} \to \mathbb{D}$ by $\bar{\iota}(f) = \iota(f - f(0)) + f(0)(1,1) = (f(0), f)$.

Proposition A.13. *Let $\beta_{\mathbb{B}}\colon K_0(\mathbb{B}) \to K_1(S\mathbb{B})$ be the Bott map. Then $\delta_0 = \beta_{\mathbb{B}} \circ (I_0 - U_0)$.*

Proof. We are going to apply Proposition 12.2.2(ii) in [47]. Let $p \in M_l(\mathbb{A})$ be a projection. Define a self-adjoint $h \in C([0,1], M_l(\mathbb{B}))$ by $h(t) = tU(p) + (1-t)I(p)$ for all $t \in [0,1]$. Then $a = (p,h) \in M_l(\mathbb{D})$; $\pi(a) = p$. Define a unitary $C([0,1], M_l(\mathbb{B}))$ by $u = \exp(2\pi i h)$. Then $\bar{\iota}(u) = (1,u) = \exp(2\pi i a)$. It follows from Proposition 12.2.2(ii) in [47] that

$$\delta_0([p]_0) = -[u]_1.$$

The map $\beta_{\mathbb{B}}\colon K_0(\mathbb{B}) \to K_1(S\mathbb{B})$ is given by $\beta_{\mathbb{B}}([q]_0) = [f_q]_1$ for any projection $q \in M_l(\mathbb{B})$. The unitary $f_q \in C([0,1], M_l(\mathbb{B}))$ is given for each $t \in [0,1]$ by

$$f_q(t) = \exp(2\pi i t)q + (1-q) = \exp(2\pi i t q),$$
$$f_q^*(t) = \exp(-2\pi i t)q + (1-q) = \exp(-2\pi i t q).$$

Then $\beta_{\mathbb{B}} \circ (I_0 - U_0)([p]_0) = \beta_{\mathbb{B}}([I(p)]_0) - \beta_{\mathbb{B}}([U(p)]_0) = [f_{I(p)}]_1 - [f_{U(p)}]_1 = -[f_{U(p)}f_{I(p)}^*]_1$. It remains to show that $[u]_1 = [f_{U(p)}f_{I(p)}^*]_1$. Define $g_s(t) = \exp(2\pi i \alpha_{s,t})$ for all $s, t \in [0,1]$;

$$\alpha_{s,t} = \begin{pmatrix} tU(p) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos(\frac{\pi s}{2}) & -\sin(\frac{\pi s}{2}) \\ \sin(\frac{\pi s}{2}) & \cos(\frac{\pi s}{2}) \end{pmatrix} \begin{pmatrix} (1-t)I(p) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi s}{2}) & \sin(\frac{\pi s}{2}) \\ -\sin(\frac{\pi s}{2}) & \cos(\frac{\pi s}{2}) \end{pmatrix}.$$

Then $g\colon [0,1] \to M_{2l}(S\mathbb{B}^{\sim})$ given by $s \mapsto g_s$ is continuous and

$$\alpha_{0,t} = \begin{pmatrix} tU(t) + (1-t)I(p) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \alpha_{1,t} = \begin{pmatrix} tU(t) & 0 \\ 0 & (1-t)I(p) \end{pmatrix}.$$

In conclusion $[u]_1 = [g_0]_1 = [g_1]_1 = [f_{U(p)} \oplus f_{I(p)}^*]_1 = [f_{U(p)}f_{I(p)}^*]_1$.   □

Remark A.14. As a consequence of Proposition A.13 we have that

$$K_0(\mathbb{D}) \simeq \ker(I_0 - U_0) \quad \text{and} \quad K_1(\mathbb{D}) \simeq \operatorname{coker}(I_0 - U_0).$$

Proposition A.15. *We have that $\pi_0(K_0(\mathbb{D})^+) = \ker(I_0 - U_0) \cap K_0(\mathbb{A})^+$.*

Proof. Let $[a] \in \ker(I_0 - U_0) \cap K_0(\mathbb{A})^+$ with $a \in M_l(\mathbb{A})$. Then

$$[I(a)]_0 = I_0([a]_0) = U_0([a]_0) = [U(a)]_0.$$

As $M_l(\mathbb{B})$ is finite-dimensional, there exists a projection $f \in C([0,1], \mathbb{B})$ such that

$$f(0) = I(a) \quad \text{and} \quad f(1) = U(a).$$

Thus $(a,f) \in \mathbb{D}$ and $\pi_0([a,f]) = [a]$. It follows that $\pi_0(K_0(\mathbb{D})^+) \supseteq \ker(I_0 - U_0) \cap K_0(\mathbb{A})^+$. We also have that $\pi_0(K_0(\mathbb{D})) \subseteq \ker(I_0 - U_0) \cap K_0(\mathbb{A})^+$ by (A.8).   □

Remark A.16. Let A be a unital C*-algebra with $K_*(A) = K_0(A) \oplus K_1(A)$. Define

$$K_*(A)^+ = \{([p],[u]) \in K_*(A) : p \in \mathcal{P}(M_\infty(A)), u \in \mathcal{U}(pM_\infty(A)p)\}.$$

If A is simple with topological stable rank one, then

$$K_*(A)^+ = \{(x,y) \in K_*(A) : x > 0\} \cup \{(0,0)\}.$$

In other words the order on $K_*(A)$ is determined by the order on $K_0(A)$ (cf. [9]).

Let $\mathfrak{A}$ be the class of unital separable simple approximately subhomogeneous algebras with no dimension growth which both have at most countably many extremal tracial states and real rank zero and are infinite dimensional.

Theorem A.17. *If $A, B \in \mathfrak{A}$ and $\Sigma\colon K_*(A) \to K_*(B)$ is a graded order isomorphism such that $\Sigma([1_A]_0) = [1_B]_0$, then there exists a $*$-isomorphism $\sigma\colon A \to B$ such that $\sigma_* = \Sigma$.*

Let $\mathfrak{B}$ be the class of unital separable simple approximately subhomogeneous algebras with slow dimension growth in which projections separate tracial states.

Theorem A.18. *If $A, B \in \mathfrak{B}$ and $\Sigma\colon K_*(A) \to K_*(B)$ is a graded order isomorphism, then there exists a $*$-isomorphism $\sigma\colon A \to B$ such that $\sigma_* = \Sigma$.*

Let $\mathfrak{C}$ be the class of unital separable nuclear purely infinite simple C*-algebras which satisfies the universal coefficient theorem.

Theorem A.19. *If $A, B \in \mathfrak{C}$ and $\Sigma\colon K_*(A) \to K_*(B)$ is a graded group isomorphism such that $\Sigma([1_A]_0) = [1_B]_0$, then there exists a $*$-isomorphism $\sigma\colon A \to B$ such that $\sigma_* = \Sigma$.*

Theorem A.17 is Theorem 4.5 in [35] and Theorem A.18 is exactly Corollary 1.4 in [58]. Theorem A.19 is Theorem 4.2.4 in [34]. We will not go into details about these results.

Remark A.20. Let A be a unital C*-algebra. Write

$$F(A) = (K_0(A), K_0(A)^+, [1_A]_0, K_1(A))$$

Then Theorem A.17, Theorem A.18 and Theorem A.19 can be phrased using this notation.

Remark A.21. We emphasize two classic results:

(a) Any non-empty metrizable Choquet simplex is affinely homeomorphic to the state space of a countable simple dimension group by Theorem 14.12 in [20].

(b) Any non-empty Choquet simplex is affinely homeomorphic to the state space of a simple dimension group by Corollary 14.9 in [20].

The reader is referred to [9] or [48] for the following results:

Remark A.22. Let A be a unital C*-algebra. Define $r_A\colon T(A) \to S(K_0(A))$ by $r_A(\tau) = \tau_*$, where $\tau_*\colon K_0(A) \to \mathbb{R}$ is given for any projections $p, q \in M_\infty(A)$ by

$$\tau_*([p]_0 - [q]_0) = \tau(p) - \tau(q)$$

The map r_A is injective if and only if projections separate traces, i.e. $\tau_1 = \tau_2$ whenever $\tau_1, \tau_2 \in T(A)$ with $\tau_1(p) = \tau_2(p)$ for all $p \in M_\infty(A)$. The map r_A is surjective if A is exact. The map r_A is a homeomorphism if A is a stably finite exact and has real rank zero.

Advanced linear algebra

We review the Perron–Frobenius theorem and the Smith normal form for integer matrices. Details can be found in [25] and [32].

Proposition B.1. *Let $A \in M_n$. The following statements are equivalent:*

(i) *A is not permutation-similar to a properly block upper triangular matrix.*
(ii) *For any partition $I \sqcup J = \{1, \ldots, n\}$ there exist $i \in I$, $j \in J$ such that $A_{ij} \neq 0$.*
(iii) *For each $i, j \in \{1, \ldots, n\}$ there exists $k \in \mathbb{N}$ such that $(A^k)_{ij} > 0$.*
(iv) *The directed graph associated with A is strongly connected.*

If A and B are permutation-similar, then A is irreducible if and only if B is irreducible.

Definition B.2. Let $A \in M_n$. We say that

(i) A is primitive if for some $k \in \mathbb{N}$ such that $(A^k)_{ij} > 0$ for all $i, j \in \{1, \ldots, n\}$.
(ii) A is reducible if it is permutation-similar to a properly block upper triangular matrix.
(iii) A is irreducible whenever A is not reducible.

Remark B.3. Let $A \in M_n$ and $v \in \mathbb{C}^n$ an eigenvector corresponding to eigenvalue $\lambda \in \mathbb{C}$. If $A \neq 0$, $A_{ij} \geq 0$ and $v_i > 0$ for all $i, j \in \{1, \ldots, n\}$, then $\lambda > 0$ and there exist $c, d > 0$ such that for any $k \in \mathbb{N}$,

$$c\lambda^k \leq \sum_{i,j=1}^{n} (A^k)_{ij} \leq d\lambda^k.$$

If in addition $w \in \mathbb{C}^n$ is an eigenvector corresponding to an eigenvalue $\mu \in \mathbb{C}$, then $|\mu| \leq \lambda$, so in particular $\mu = \lambda$ when $w_i > 0$ for all $i \in \{1, \ldots, n\}$.

The following result is known as the Perron–Frobenius theorem for irreducible matrices.

Theorem B.4. *Let $A \in M_n$ is irreducible with $A \neq 0$, $A_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$.*

(i) *There is an eigenvector $v \in \mathbb{C}^n$ with $v_i > 0$ for all $i \in \{1, \ldots, n\}$ with $Av = \lambda v$.*
(ii) *The eigenvalue $\lambda > 0$ is a simple root in the characteristic polynomial of A.*

As a consequence the eigenspace associated with λ is one-dimensional.

Then λ is called the Perron–Frobenius eigenvalue and v a Perron–Frobenius eigenvector, and v is the unique Perron–Frobenius probability eigenvector of A when $\sum_{i=1}^{n} v_i = 1$.

Proposition B.5. *Let S be an $n \times m$-matrix with integer entries and $r = \min\{n, m\}$. Then there are $n \times n$- and $m \times m$-matrices R and T with integer entries which are invertible such that $RST = D = \mathrm{diag}(\alpha_1, \ldots, \alpha_r)$ with $\alpha_l \mid \alpha_{l+1}$ for all $1 \leq l \leq r-1$ when $\alpha_{l+1} \neq 0$, and $\alpha_1, \ldots, \alpha_r$ are unique up to multiplication by ± 1.*

Then the matrix $D = \mathrm{diag}(\alpha_1, \ldots, \alpha_r)$ is called the Smith normal form of the matrix S, and the numbers $\alpha_1, \ldots, \alpha_r$ are called elementary divisors or invariant factors.

Remark B.6. Let $d_l(S)$ be the greatest common divisor of all $l \times l$-minors and $d_0(S) = 1$. If $d_l(S) \neq 0$ then $\alpha_l = \frac{d_l(S)}{d_{l-1}(S)}$ and $\alpha_l = 0$ when $d_l(S) = 0$.

Definition B.7. Let S be an $n \times m$-matrix with integer entries.

(i) S is unimodular if $n = m$ and $\det(S) = \pm$.

(ii) S is totally unimodular if each $l \times l$-minor is -1, 0 or 1 for all $1 \leq l \leq \min\{n, m\}$.

(iii) S is a sign matrix if each entry is -1, 0 or 1.

(iv) S is a binary matrix if all entry is 0 or 1.

Remark B.8. Let S be a totally unimodular $n \times m$-matrix.

(a) Any invertible square submatrix of S is unimodular.

(b) The Smith normal form of S is a binary diagonal matrix

(c) S is a sign matrix.

Lemma B.9. *Assume that S is a sign matrix in which the rows may be partitioned into two sets of rows R_+ and R_- such that*

(i) *Each column has at most two non-zero entries.*

(ii) *Any two non-zero entries in a column are in the same set of rows if they have different signs and in different set of rows if they have the same sign.*

Then S is totally unimodular.

Proof. This is a well known result, cf. Theorem 219 in [19]. Here we present the proof:

Any submatrix of S also satisfies the conditions of the statement, so we may assume without loss of generality that S is a square matrix and prove that $\det S \in \{-1, 0, 1\}$.

Suppose that S is an $l \times l$-matrix. The statement holds for $l = 1$, as S is a sign matrix. Assume for $l \geq 2$ that $\det S' \in \{-1, 0, 1\}$ for any $(l-1) \times (l-1)$-submatrix S' of S (or for any sign $(l-1) \times (l-1)$-matrix S' satisfying the two conditions).

If S has a zero column, then $\det S = 0$, so we may assume without loss of generality that all columns of S are non-zero columns.

If S has a column with exactly one non-zero entry, then by cofactor expanding $\det S$ along this column yields $\det S = \pm \det S' \in \{-1, 0, 1\}$ where S' is the cofactor of the non-zero entry (which is ± 1) in the column.

If all columns in S has exactly two non-zero entries, then for all $1 \leq j \leq l$,

$$\sum_{i \in R_+} S_{ij} = \sum_{i \in R_-} S_{ij}.$$

Hence the rows of S are linearly dependent so $\det S = 0$. $\square$

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