Complex Continued Fractions

Theoretical Aspects of Hurwitz's Algorithm

PhD Thesis

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September 2018

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by

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Denmark
2018

Ya, ya me está gustando más de lo normal,
Todos mis sentidos van pidiendo más,
Esto hay que tomarlo sin ningún apuro.

–Ramón Luis Ayala Rodríguez, *Despacito*, 2017

Abstract

Inspired by their remarkable properties, mathematicians have sought in several contexts objects similar to regular continued fractions. Hurwitz Continued fractions (HCF), proposed by Adolf Hurwitz in 1887, give a natural analogue in the complex plane. The HCF of a complex number ζ is a sequence of Gaussian integers obtained from a specific algorithm. Among the similarities between HCF and regular continued fractions we can find the irrationality of a complex number as a necessary and sufficient condition for its HCF to be infinite. In this case, there even are combinatorial conditions on the HCF implying transcendence. The usual characterization of quadratic surds in terms of periodic continued fractions also holds. However, É. Galois' theorem on purely periodic continued fractions is not true unless additional restrictions are imposed.

HCF can be understood through a dynamical perspective. With a suitably defined Gauss map, HCF give rise to an ergodic dynamical system. Although the corresponding shift space has a complicated structure, some classical Hausdorff dimension results still hold. For example, the set of complex numbers with a bounded HCF has full Hausdorff dimension (proved by V. Jarník in 1929 for $\mathbb{R}$) and the Hausdoff dimension of the set of complex numbers with a HCF tending to ∞ is half of that of the ambient space's (proved by I.J. Good in 1941 for $\mathbb{R}$). The theory of HCF is far from being completed, and its similarities and discrepancies with their real counterpart stem new problems.

The present thesis is a survey of new and old results on HCF. In the preface, a detailed list of all the novel aspects is given. Additionally, three original research papers are included as appendices. The first two deal with HCF theory. The third one — applying the geometry of regular continued fractions— contains a simple proof of P. Bengoechea's quantitative improvement of Serret's Theorem on equivalent numbers.

Resumé

Inspireret af deres bemærkelsesværdige egenskaber har matematikere i flere sammenhænge søgt objekter svarende til regulære kædebrøker. Hurwitz-kædebrøker (HCF), der blev introduceret af Adolf Hurwitz i 1887, giver en naturlig analog i den komplekse plan. HCF af et komplekst tal ζ er en følge af gaussiske heltal udregnet ved en specifik algoritme. Blandt lighederne mellem HCF og regulære kædebrøker kan vi finde, at irrationalitet af et komplekst tal er en nødvendig og tilstrækkelig begtingelse for, at HCF er uendelig. I dette tilfælde er der endda kombinatoriske betingelser på tallets HCF der garanterer tallets transcendens. Den sædvanlige karakterisering af kvadratiske irrationale som periodiske kædebrøker holder ligeledes. På den anden side er É. Galois sætning om fuldstændigt periodiske kædebrøker ikke sand, med mindre vi indfører yderligere restriktioner.

HCF kan forstås fra et dynamisk synspunkt. Med en passsende defineret Gauss-afbildning, giver HCF anledning til et ergodisk dynamisk system. Selvom det tilhørende skift-rum har en kompliceret struktur, holder visse klassiske resultater om Hausdorff-dimension fortsat. For eksempel har mængden af komplekse tal med en begrænset HCF maksimal Hausdorff-dimension (vist af V. Jarnk i 1929 for $\mathbb{R}$), og Hausdorff-dimensionen af mængden af komplekse tal for hvilke HCF går mod ∞ er halvdelen af det omgivende rums dimension (vist af I. J. Good i 1941 for $\mathbb{R}$). Teorien om HCF er langt fra fuldstændig, og fra dennes ligheder og uoverensstemmelser med dens reelle modpart vokser nye problemer.

Den foreliggende afhandling er en undersøgelse af nye og gamle resultater om HCF. I forordet findes en detajeret liste over alle nye aspekter. Desuden er tre originale forskningsartikler inkluderet som appendices. De to første omhandler HCF-teori. Den tredje, der anvender teorien om geometriske regulære kædebrøker, indeholder et simpelt bevis for P. Bengochea's kvantitative forbedring af Serrets sætning om ækvivalente tal.

Preface

Regular continued fractions have a long standing tradition of noble service to number theory; in particular, to diophantine approximation. Among other properties, they provide an algorithmic solution to the problem of finding the best rational approximation of a given real number. Unlike other representations, like b-ary or β-expansions, regular continued fractions do not depend on any number other than the represented one. Regular continued fractions also express right away intrinsic properties of the numbers they represent, like being badly approximable or a quadratic irrational, for example. In view of these nice properties, it is natural to ask for the existence of a similar process in other contexts. Several answers have been given depending on the features that are intended to extend. This text focuses on a particular continued fraction algorithm for complex numbers suggested by Adolf Hurwitz in 1887, the Hurwitz Continued Fractions (HCF). This document is a result of my work in the PhD Programme in Mathematics at Aarhus University under the supervision of Dr. Simon Kristensen.

Although Hurwitz Continued Fractions are an old subject, they have not got much attention by themselves until 2006, when D. Hensley constructed algebraic numbers of degree 4 with bounded partial quotients. Unfortunately, D. Hensley's work sheds no new light on the corresponding conjecture in the real numbers. A considerable amount of research has been done on the ergodic theory of some dynamical systems that include Hurwitz Continued Fractions. One of our main goals is to survey new and old results, generalizations, and common techniques and strategies under a unified approach. Since we contrast constantly the theory of regular and Hurwitz continued fractions, we assume the reader is acquainted with regular continued fractions.

The thesis is divided into six chapters and five appendices. Each chapter starts with an abstract, a comment on the notation and a citation of the main references. In Chapter I, the framework of complex continued fractions introduced by S.G. Dani and A. Nogueira is presented as well as some basic properties (like representation of complex numbers and convergence). Within this framework, Hurwitz Continued Fractions are properly defined and their approximation properties are studied. In Chapter 2, we discuss the algebraic properties of HCF through analogues of Euler-Lagrange Theorem, Galois Theorem on purely periodic continued fractions, and Y. Bugeaud's construction of transcendental numbers through automatic sequences. We also reproduce W. Bosma and D. Gruenewald extension of D. Hensley's result. In Chapter 3, we discuss the ergodic theory of HCF. In particular, we associate a natural dynamical system and construct a measure μ equivalent to the Lebesgue measure making the system ergodic. We discuss briefly two notions that generalize HCF, one of them is Iwasawa Continued Fractions and the other one is Fibred Systems. In Chapter 4, we explore the Hausdorff dimension of some sets defined by imposing restrictions on the HCF expansion. We stress along the chapter the importance of the bounded distortion of the functions involved in the dynamical system associated to HCF. In Chapter 5, we give a short overview of the complex continued fraction algorithm proposed by J. Hurwitz, also within the framework of Dani and Nogueira. Chapter 6 contains a short list of related problems and a brief description of ongoing research.

In the first two appendices, we just set the notation and record some basic results. The last three appendices are the research papers I finished during the PhD programme:

1. "Purely Periodic and Transcendental Complex Continued Fractions" (submitted),
2. "Good's Theorem for Hurwitz Continued Fractions" (submitted),
3. "Geometric Continued Fractions and Equivalent Numbers".

The third paper is not related with the rest of the thesis. It is the result of a seminar on the geometry of continued fractions held with Dr. Simon Kristensen and Dr. Morten Hein Tiljeset.

Besides suggesting a unified notation for the treatment of HCF, the following are the innovations of the thesis:

1. A small gap in the proof of Theorem 1.2.4 is filled for HCF. A minor mistake was also corrected in the proof of Theorem 2.2.3.
2. We introduce the notion of irregular numbers and sequences. Although the idea is clearly present in previous works, like the ones of D. Hensley, S. Tanaka, H. Nakada, and more, they were completely dismissed for being a null set.
3. Although simple and natural, we have not found Theorem 2.1.3 in the literature.

4. Theorems 2.3.2 and 2.4.1 are new (see [Gon18-01]).

5. The argument of the proof of the First Lakein Theorem (Theorem 1.2.6) is the original one, but the exposition is drastically modified. Some details are added and, hopefully, a much clearer proof is attained.

6. We have not found the first part of Proposition 3.2.6 in the available literature.

7. The Generalized Jarník Lemmas, Good's Theorem for Hurwitz Continued Fractions, and Theorem 4.5.2 are new.

8. The convergence of the Hurwitz-Tanaka continued fraction of every complex irrational is obtained as an immediate consequence of a result by S.G. Dani and A. Nogueira. The proof of Theorem 5.1.1 is significantly shorter and simpler than the previously known (see [Os16]).

All the figures were done with TiKz.

Acknowledgements. I thank Dr. Simon Kristensen for his supervision and enormous help with mathematical and non-mathematical issues even before my arrival to Denmark. I thank Dr. Yann Bugeaud for receiving me during a very fruitful semester at the Institut de Recherche Mathématique Avancée in Strasbourg, France. I want to thank Dr. Tomas Persson for his comments and, as Dr. Oleg German and Dr. Oleg Karpenkov, for his disposition. I want to thank everyone at Aarhus University's Mathematics Department for providing an unbeatable working environment with a warm human touch.

My research would not have been possible without the funding of the Consejo Nacional de Ciencia y Tecnología (grant: 410695), and the additional grants given by the Secretaría de Educación Pública and Aarhus University (Projekt: 25763, Aktivitet: 26248).

I thank Christina for her great support, particularly while writing this thesis. I thank the friends and family I got to know at Aarhus University and at Resedavej for making these years unforgettable. I thank Flavie and Yann Kaision and their family for the nice time in Strasbourg. Finally, I thank my friends and family from Mexico for their backing.

Contents

Chapter 1

Basic Properties

In this chapter, we introduce the complex continued fraction framework developed by S. G. Dani and A. Nogueira in their 2014 paper [DaNo14]. Within their framework, we discuss some general properties on representation of complex numbers. Afterwards, we consider the complex continued fraction algorithm proposed by Adolf Hurwitz as a particular case. We explore some approximation properties of Hurwitz's algorithm and conclude with a characterization of badly approximable complex numbers (defined below).

Main references. For the theory of complex continued fractions: [Da15], [DaNo14], [La73], [La74a], [La74b], [H87], [He06]. For the classical theory of regular continued fractions: [Kh].

Notation. For any $A \subseteq \mathbb{C}$ and $z \in \mathbb{C}$ we write

$$z + A := \{z + a : a \in A\}, \quad zA := \{za : a \in A\}.$$

For $A \subseteq \mathbb{C}$, A° is the interior of A, $\operatorname{Cl} A$ is the closure of A, $A^* := A \smallsetminus \{0\}$, and $A' := A \smallsetminus \mathbb{Q}(i)$.

Denote the unit disc by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} = \operatorname{Cl} \mathbb{D}$. For every $z \in \mathbb{C}$ and $r > 0$ we write

$$\mathbb{D}(z,r) := z + r\mathbb{D}, \quad \overline{\mathbb{D}}(z,r) := z + r\overline{\mathbb{D}}, \quad \mathbb{D}(z) := z + \mathbb{D}, \quad \overline{\mathbb{D}}(z) := z + \overline{\mathbb{D}},$$
$$\mathbb{E}(z,r) := \mathbb{C} \smallsetminus \overline{\mathbb{D}}(z,r), \; \overline{\mathbb{E}}(z,r) := \mathbb{C} \smallsetminus \mathbb{D}(z,r), \quad \mathbb{E}(z) := \mathbb{C} \smallsetminus \mathbb{D}(x),$$
$$\overline{\mathbb{E}}(z) := \mathbb{C} \smallsetminus \mathbb{D}(z).$$

The Golden Ratio is denoted by $\phi = \frac{1+\sqrt{5}}{2}$. For $m, n \in \mathbb{Z}$, $[m..n] = \{j \in \mathbb{Z} : m \leq j \leq n\}$.

1.1 Complex Continued Fractions

Let z be a complex number. A sequence $(w_n)_{n\geq 0}$ in $\mathbb{C}$ (possibly finite) is an **iteration sequence** of z if

$$w_0 = z, \qquad \forall n \in \mathbb{N} \qquad |w_n| \geq 1, \quad w_n - \frac{1}{w_{n+1}} \in \mathbb{Z}[i] \smallsetminus \{0\}, \tag{1.1}$$

We call $(w_n)_{n\geq 0}$ **degenerate** if $|w_n| = 1$ for some $n \in \mathbb{N}$ and **non-degenerate** otherwise. The **partial quotients** or **elements** is the sequence $(a_n)_{n\geq 0}$ in $\mathbb{Z}[i] \smallsetminus \{0\}$ given by

$$\forall n \in \mathbb{N}_0 \quad a_n = w_n - \frac{1}{w_{n+1}}. \tag{1.2}$$

Following [DaNo14], we call the sequences $(p_n)_{n\geq 0}, (q_n)_{n\geq 0}$ given by

$$\begin{pmatrix} p_{-2} & p_{-1} \\ q_{-2} & q_{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \forall n \in \mathbb{N}_0 \quad \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} \\ 1 \end{pmatrix}. \tag{1.3}$$

the $\mathcal{Q}$**-pair** of z. For $a_0, a_1, \ldots, a_n \in \mathbb{Z}[i]$ we will write

$$\langle a_0; a_1, a_2, \ldots, a_n \rangle = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.$$

We reserve the notation $[a_0; a_1, a_2, \ldots]$ for Hurwitz Continued Fractions (defined below). Note that if $z \in \mathbb{C}$ has a finite iteration sequence $(w_n)_{n=0}^N$ and elements $(a_n)_{n=0}^N$, then $z = \langle a_0; a_1, \ldots, a_n \rangle$.

Several classical identities of continued fractions depend on the recurrence relations defining $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$. For convenience, we recall them without a proof.

Proposition 1.1.1. *Let z be a complex number, $(w_n)_{n\geq 0}$ an iteration sequence of z, $(a_n)_{n\geq 0}$ its sequence of elements, $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ its $\mathcal{Q}$-pair, and $\mathcal{N} \subseteq \mathbb{N}_0$ the set of indices where the iteration sequence is defined. Then, for every $n \in \mathcal{N}$ we have*

$$\frac{p_n}{q_n} = \langle a_0; a_1, \ldots, a_n \rangle \tag{1.4}$$

$$q_n p_{n-1} - q_{n-1} p_n = (-1)^n \tag{1.5}$$

$$q_n z - p_n = \frac{(-1)^n}{w_1 w_2 \cdots w_n}, \tag{1.6}$$

$$z(q_n w_{n+1} - q_{n-1}) = p_n w_{n+1} - p_{n-1}. \tag{1.7}$$

$$\frac{q_n}{q_{n-1}} = \langle a_n; a_{n-1}, \ldots, a_1 \rangle \qquad \text{if } q_n \neq 0. \tag{1.8}$$

We refer to (1.8) as the **mirror formula**.

Remark. Whenever $|w_n| > 1$, (1.6) forces $q_m \neq 0$ for $m \geq n$.

Suppose $z \in \mathbb{C}$ is a complex irrational, that is $z \in \mathbb{C} \setminus \mathbb{Q}(i)$. Then, any iteration sequence of z has to be infinite. However, we cannot say *a priori* if the equality

$$z = \langle a_0; a_1, a_2, a_3, \ldots \rangle \tag{1.9}$$

holds (the right side understood as the limit when $n \to \infty$ of $\langle a_0; a_1, a_2, \ldots, a_n \rangle$). Under mild conditions, the infinite continued fraction arising from an iteration sequence actually represents z; i.e., (1.9) is true.

Theorem 1.1.2 (Dani and Nogueira (2011))**.** *Let z be a complex irrational. If $(z_n)_{n\geq 0}$ is a non-degenerate iteration sequence of z, then*

$$\lim_{n\to\infty} \frac{p_n}{q_n} = z.$$

Proof. In view of (1.6), it suffices to show that $\limsup_n |w_n| > 1$. We shall assume for contradiction that the inequality fails, this is equivalent to $\lim_n |w_n| = 1$. Define the set

$$R = \{z \in \mathbb{C} : |z| = 1,\ \exists w \in \mathbb{Z}[i]\ d(z, w) = 1\} = \{z \in \mathbb{C} : z^{12} = 1\},$$

then, the recurrence satisfied by $(w_n)_{n\geq 0}$ yields

$$|w_n - a_n| = \frac{1}{|w_{n+1}|} \to 1 \quad \text{as } n \to \infty,$$

so $d(w_n, R) \to 0$ when $n \to \infty$. Let $(\rho_n)_{n\geq 0}$ be a sequence in R satisfying

$$\lim_{n\to\infty} |w_n - \rho_n| = 0; \tag{1.10}$$

hence,

$$\lim_{n\to\infty} |\rho_n - a_n| = 1,$$

Consider the sequences $(\zeta_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ given by

$$\forall n \in \mathbb{N} \quad \zeta_n := w_n - \rho_n, \quad \beta_n := \rho_n - a_n.$$

Note that $\beta_n \in R$ for sufficiently large n, because R is finite, $\beta_n \to 1$ as $n \to \infty$, and $a_n \in \mathbb{Z}[i]$. Then, the identities $w_{n+1}^{-1} = w_n - a_n = \zeta_n + \beta_n$, (1.10), and the definitions of ζ_n and β_n give

$$w_{n+1} = \frac{1}{\beta_n} + \frac{-\zeta_n}{\beta_n(\beta_n + \zeta_n)} = \rho_{n+1} + \varepsilon_{n+1} = \beta_{n+1} + \zeta_{n+1}, \tag{1.11}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. For large n we have $\beta_n \in R$ and

$$\left| \frac{\zeta_n}{\beta_n(\beta_n + \zeta_n)} \right| \le \frac{|\zeta_n|}{1 - |\zeta_n|} \le |\zeta_n| \to 0 \quad \text{when } n \to \infty. \tag{1.12}$$

Thus, the finiteness of R forces the next equalities to eventually hold

$$\beta_{n+1} = \rho_{n+1} = \frac{1}{\beta_n}, \quad \zeta_{n+1} = \frac{-\zeta_n}{\beta_n(\beta_n + \zeta_n)}.$$

By (1.12), for large n we have $|\zeta_{n+1}| \ge |\zeta_n|$ and, since $\zeta_n \to 0$, the sequence $(\zeta_n)_{n\ge 0}$ ends up being constant and equal to 0. Therefore, in view of (1.11), $w_n = \rho_n$ and $|w_n| = 1$ for large n, contradicting the non-degeneracy hypothesis. Hence, $\limsup_n |w_n| > 1$ and $z = \langle a_0; a_1, a_2 \ldots \rangle$. □

A simple way to construct iteration sequences is via choice functions. We call $f : \mathbb{C}^* \to \mathbb{Z}[i]$ a **choice function** if

$$\forall z \in \mathbb{C}^* \quad f(z) \in \overline{\mathbb{D}}(z).$$

The **fundamental set** of f, F_f, is

$$F_f := \mathrm{Cl}\{z - f(z) : z \in \mathbb{C}^*\}.$$

For any $z \in \mathbb{C}^*$ we define the iteration sequence $(w_n)_{n\ge 0} = (w_n(f,z))_{n\ge 0}$ by

$$w_0 = z, \qquad \forall n \in \mathbb{N} \quad w_n = \frac{1}{w_{n-1} - f(w_{n-1})}, \tag{1.13}$$

as long as $w_{n-1} \ne f(w_{n-1})$, in which case we only keep $(w_0, \ldots, w_{n-1})$ and say that the algorithm **terminates**. The **f-sequence** of z is the sequence of elements of z associated to $(w_n)_{n\ge 0}$, they satisfy $a_n = f(w_n)$.

Theorem 1.1.3. *Let f be a choice function such that $F_f \subseteq \mathbb{D}(0,r)$ for some $0 < r < 1$. Let ζ be a complex number and $(a_n)_{n\ge 0}$ its f-sequence.*

i. If $\zeta \in \mathbb{Q}(i)$, then $(a_n)_{n\ge 0}$ is finite, say $(a_n)_{n\ge 0} = (a_0, a_1, \ldots, a_N)$, and

$$\zeta = \langle a_0; a_1, \ldots, a_N \rangle .$$

ii. If $\zeta \in \mathbb{C} \smallsetminus \mathbb{Q}(i)$, then $(a_n)_{n\ge 0}$ is infinite and

$$\zeta = \lim_{n\to\infty} \langle a_0; a_1, a_2, \ldots, a_n \rangle .$$

Proof. i. Write $\zeta = \frac{a}{b} \in \mathbb{Q}(i)$ with $a, b \in \mathbb{Z}[i]$ coprime. For any $n \in \mathbb{N}$ such that a_n is defined and $p_n/q_n \neq \zeta$, we have that

$$\frac{1}{|b|} \leq \left| q_n \frac{a}{b} - p_n \right| < r^{n+1},$$

so n must be bounded and $(a_n)_{n\geq 0}$ is finite. Since the algorithm terminates, for some $N \in \mathbb{N}$ we have $a_N = f(z_N)$ and $\zeta = \langle a_0; a_1, \ldots, a_N \rangle$ by (1.13).

ii. It follows directly from Theorem 1.1.3.

□

Let f be a choice function. The family of sets $\{f^{-1}[a] : a \in \mathbb{Z}[i]\}$ is a countable partition of $\mathbb{C}^*$ satisfying

$$\bigcup_{a \in \mathbb{Z}[i]} -a + G_a \subseteq \overline{\mathbb{D}}. \tag{1.14}$$

On the other hand, we can construct a choice function starting from a countable partition $\mathfrak{G} = \{G_a : a \in \mathbb{Z}[i]\}$ such that (1.14) holds. We just consider $f : \mathbb{C}^* \to \mathbb{Z}[i]$ to be given by the condition $f(z) \in G_{f(z)}$. Suppose that $\mathfrak{G}$ additionally satisfies

$$\forall z \in \mathbb{C}^* \quad z - f(z) \in G_0 \cup \{0\}. \tag{1.15}$$

We define the **f-Gauss map** $T_f : G_0^* \to G_0$ by

$$\forall z \in G_0 \quad T_f(z) = \frac{1}{z} - f\left(\frac{1}{z}\right).$$

In this case, to each $z \in \mathbb{C}'$ we represent its orbit under T_f by $(z_n)_{n\geq 0}$. That is, $z_0 = z$ and $z_n = T_f(z_{n-1})$ for each $n \in \mathbb{N}$.

Although restrictive, the condition (1.15) is not artificial. It holds, for example, if we consider a sub-lattice $\Lambda \subseteq \mathbb{Z}[i]$ with $[\mathbb{Z}[i] : \Lambda] \leq 2$ and take as $\mathfrak{G}$ the translations of any fundamental domain of $\mathbb{C}/\Lambda$ contained in $\overline{\mathbb{D}}$. In fact, the two algorithms that will concern us satisfy (1.15).

1.2 Hurwitz Continued Fractions

Adolf Hurwitz proposed in 1887 a complex continued fraction algorithm which shares some non-trivial and highly desirable properties with the real regular continued fractions. In a few words, his algorithm is a straight forward generalization of the nearest integer continued fraction to the complex plane. With a suitable choice function, A. Hurwitz's algorithm can be understood as an example of the iteration sequence theory.

The **nearest Gaussian integer** function $[\cdot] : \mathbb{C} \to \mathbb{Z}[i]$ associates to each complex number z the Gaussian integer closest to z. In case of a tie, it takes the one with greatest real or imaginary part. Thus, the inverse image of 0 under $[\cdot]$ is

$$\mathfrak{F} := \left\{ z \in \mathbb{C} : -\frac{1}{2} \leq \Re z, \Im z < \frac{1}{2} \right\}, \tag{1.16}$$

For any $z \in \mathfrak{F}^*$ define— as long as the operations make sense— the sequences $(a_n)_{n\geq 0}$, $(z_n)_{n\geq 1}$ by

$$\begin{aligned} z_1 &:= z, \quad \forall n \in \mathbb{N} \quad z_{n+1} = T(z_n), \\ a_0 &:= 0, \quad \forall n \in \mathbb{N} \quad a_n = \left[z_n^{-1} \right], \end{aligned}$$

where T is the corresponding f-Gauss map. The sequence $(a_n)_{n\geq 0}$ is the Hurwitz continued fraction of z.

Definition 1.2.1. *The **Hurwitz Continued Fraction** (HCF) of $z \in \mathbb{C}$ is its $[\cdot]$-sequence. If $(a_n)_{n\geq 0}$ is the HCF of z, we write $z = [a_0; a_1, a_2, \ldots]$ when the sequence is infinite and $z = [a_0; a_1, a_2, \ldots, a_n]$ when it is finite and comprises $n+1$ elements.*

From now on, except when explicitly stated, all continued fractions considered are HCF. Let z be a complex irrational and $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ its $\mathcal{Q}$-pair. The hypothesis of Theorem 1.1.3 are clearly met ($r = 2^{-\frac{1}{2}}$), so $z = [a_0; a_1, a_2, \ldots]$ is indeed true. Moreover, by $|z_n^{-1}| \geq \sqrt{2}$ and (1.6), we even know the convergence is exponentially fast:

$$\forall n \in \mathbb{N} \qquad |q_n z - p_n| \leq 2^{-\frac{n}{2}}.$$

It is not clear, though, whether $|q_n|$ tends to ∞ or not, or even if it is strictly increasing. In order to show these important features, we need to study the associated space of sequences of Gaussian integers associated to the HCF process.

1.2.1 Laws of Succession

Regular continued fractions give a topological conjugacy between the dynamical systems $([0,1] \setminus \mathbb{Q}, T_{\mathbb{R}})$ and $(\mathbb{N}^{\mathbb{N}}, \sigma)$. Here and in the following, $T_{\mathbb{R}}$ is the classical Gauss map $x \mapsto x^{-1} - [x^{-1}]$ where $[x^{-1}]$ is the integral part of x^{-1} and σ is the shift map: $\sigma((x_n)_{n\geq 1}) = (x_n)_{n\geq 2}$. In the HCF context, the structure of the the sequence space is much more complicated.

A **cylinder of depth** $n \in \mathbb{N}$ is a set of the form

$$\mathcal{C}_n(\mathbf{a}) = \{z = [0; b_1, b_2, \dots] : a_1 = b_1, \dots, b_n = a_n\},$$

where $\mathbf{a}$ is a sequence in $\mathbb{Z}[i]$. In Figure 1.1 we show the partition of $\mathfrak{F}$ induced by cylinders of depth 1. Note that for $a \in \mathbb{Z}[i]$ we have $\mathcal{C}_1(a) \neq \varnothing$ if and only if a belongs to

$$I := \{a \in \mathbb{Z}[i] : |a| \geq \sqrt{2}\}.$$

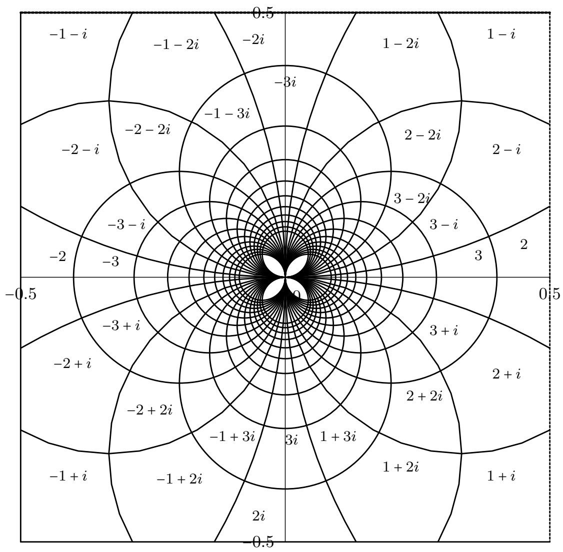


Figure 1.1: Partition $\{\mathcal{C}_1(a) : a \in I\}$ of $\mathfrak{F}$.

Terminology

A sequence $(a_n)_{n \geq 0}$ in $\mathbb{Z}[i]$ is **valid** or **admissible** if it is the Hurwitz Continued Fraction of some $z \in \mathfrak{F}$. The space of admissible sequences is denoted by Ω^{HCF}. Clearly, a sequence $\mathbf{a}$ in $\mathbb{Z}[i]$ belongs to Ω^{HCF} if and only if $\mathcal{C}_n(\mathbf{a}) \neq \varnothing$ for all n. We will call the rules that determine the membership of a sequence $\mathbf{a} \in \mathbb{Z}[i]^{\mathbb{N}}$ to Ω^{HCF} the **laws of succession**. We admit that the definition is rather loose, but no confusion shall arise from it.

A non-empty set is a **maximal feasible set** if it has the form $T^n[\mathcal{C}_n(\mathbf{a})]$ for some $\mathbf{a} \in \Omega^{\text{HCF}}$ and some $n \in \mathbb{N}$. A maximal feasible set is **regular** if it has non-empty interior, otherwise it is **irregular** and **strongly irregular** if it has only one point. A sequence $\mathbf{a} \in \Omega^{\text{HCF}}$ is **regular** (resp. **irregular**, **strongly irregular**) if $T^n[\mathcal{C}_n(\mathbf{a})]$ is regular (resp. irregular, strongly irregular). A complex irrational $z \in \mathfrak{F}$ is **regular** (resp. **irregular**, **strongly irregular**) if its HCF is regular (resp. irregular, strongly irregular). The set of regular sequences (resp. irregular sequences, regular irrationals in $\mathfrak{F}$, irregular irrationals in $\mathfrak{F}$) is denoted by Ω_R^{HCF} (resp. Ω_I^{HCF}, $\mathfrak{F}_R$, $\mathfrak{F}_I$).

Determination of the laws

The laws of succession are determined by the behaviour of complex inversion. Observe that $\mathfrak{F}^{-1} = \{z^{-1} : z \in \mathfrak{F}^*\}$ can be expressed as

$$\mathfrak{F}^{-1} = \mathbb{E}(1) \cap \overline{\mathbb{E}}(i) \cap \overline{\mathbb{E}}(-1) \cap \mathbb{E}(-i)$$

(see Figure 1.2). It is clear that for $a \in \mathbb{Z}[i]$ we have $(a+\mathfrak{F}) \cap \mathfrak{F}^{-1} \neq \varnothing$ if and only if $a \in I$. So $\mathfrak{F} = \bigcup_{a \in I} \mathcal{C}_1(a)$. There are fifteen maximal feasible sets that arise from cylinders of depth 1 and all of them are regular. Moreover, whenever $a \in I$ satisfies $|a| \geq 8$, $T[\mathcal{C}_1(a)] = \mathfrak{F}$ is true and we can show inductively that

$$\left\{(a_n)_{n\geq 0} \in I^{\mathbb{N}} : \forall n \in \mathbb{N} \quad |a_n| \geq \sqrt{8}\right\} \subseteq \Omega_R^{\text{HCF}}.$$

The other fourteen maximal feasible are $\{T[\mathcal{C}_1(a)] : \sqrt{2} \leq |a| \leq \sqrt{5}\}$; if we discard their boundary, we end up with thirteen regular maximal feasible regions. Using and inductive argument and the formulae for inversion of lines and circles (see Appendix A), we conclude that these thirteen regions are essentially all of the regular maximal feasible regions.

As an example, let us show that

$$T_1\left[\mathcal{C}_1(2)\right] = \mathfrak{F} \setminus \overline{\mathbb{D}}(-1).$$

If $z \in [\mathcal{C}_1(2)]$, then $T(z) = z^{-1} - 2 \in \mathfrak{F}$, so $z^{-1} \in \mathfrak{F}^{-1} \cap (2+\mathfrak{F})$ and $T(z) \in \mathfrak{F}^{-1} \cap (2+\mathfrak{F}) = \mathfrak{F} \setminus \overline{\mathbb{D}}(-1)$. On the other hand, when w belongs to $\mathfrak{F} \setminus \overline{\mathbb{D}}(-1)$, we have that $2+w \in \mathfrak{F}^{-1} \cap (2+\mathfrak{F})$, so $(2+w)^{-1} \in \mathcal{C}_1(2)$ and $w \in T[\mathcal{C}_1(2)]$. Very similar arguments show the following equalities:

For $a \in I$ with $|a| = \sqrt{2}$, $a \in \{1+i, -1+i, -1-i, 1-i\}$ and

$$T[\mathcal{C}_1(1+i)] = \mathfrak{F} \setminus \left(\mathbb{D}(-1) \cup \overline{\mathbb{D}}(-i)\right), \quad T[\mathcal{C}_1(-1+i)] = \mathfrak{F} \setminus \left(\mathbb{D}(1) \cup \mathbb{D}(-i)\right),$$
$$T[\mathcal{C}_1(-1-i)] = \mathfrak{F} \setminus \left(\mathbb{D}(i) \cup \overline{\mathbb{D}}(1)\right), \quad T[\mathcal{C}_1(1-i)] = \mathfrak{F} \setminus \left(\overline{\mathbb{D}}(-1) \cup \overline{\mathbb{D}}(i)\right).$$

For $a \in I$ with $|a| = 2$, $a \in \{2, 2i, -2, -2i\}$

$$T[\mathcal{C}_1(2)] = \mathfrak{F} \setminus \overline{\mathbb{D}}(-1), \quad T[\mathcal{C}_1(2i)] = \mathfrak{F} \setminus \mathbb{D}(-i),$$
$$T[\mathcal{C}_1(-2)] = \mathfrak{F} \setminus \mathbb{D}(1), \quad T[\mathcal{C}_1(-2i)] = \mathfrak{F} \setminus \overline{\mathbb{D}}(i).$$

For $a \in I$ with $|a| = \sqrt{5}$, $a \in \{1+2i, 2+i, -1+2i, -2+i, -1-2i, -2-i, 1-2i, 2-i\}$ and

$$\begin{aligned} T[\mathcal{C}_1(1+2i)] = \mathfrak{F} \setminus \overline{\mathbb{D}}(-1-i), \quad T[\mathcal{C}_1(2+i)] &= \mathfrak{F} \setminus \mathbb{D}(-1-i), \\ T[\mathcal{C}_1(-1+2i)] = T[\mathcal{C}_1(-2+i)] &= \mathfrak{F} \setminus \mathbb{D}(1-i), \\ T[\mathcal{C}_1(-1-2i)] = \mathfrak{F} \setminus \mathbb{D}(1+i), \quad T[\mathcal{C}_1(-2-i)] &= \mathfrak{F} \setminus \overline{\mathbb{D}}(1+i), \\ T[\mathcal{C}_1(1-2i)] = T[\mathcal{C}_1(2-i)] &= \mathfrak{F} \setminus \mathbb{D}(-1+i). \end{aligned}$$

Finally, as mentioned before, $T[\mathcal{C}_1(a)] = \mathfrak{F}$ when $a \in I$ has $|a| \geq \sqrt{8}$.

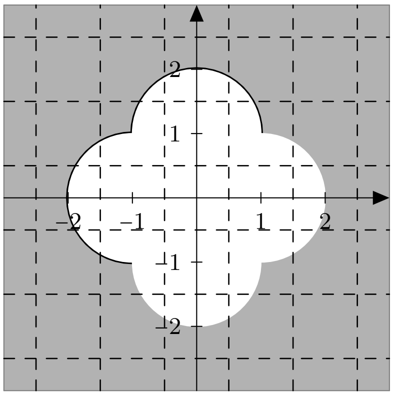


Figure 1.2: The set $\mathfrak{F}^{-1}$ and the lines $x = k$, $y = l$ for all $k, l \in \mathbb{Z}$.

We must deal with cylinders of level 2 to obtain an irregular maximal feasible set. As before, we can show that for any $a \in \mathbb{Z}$, $|a| \geq 2$, we have that

$$(T[\mathcal{C}_1(-2))^{-1} \cap (1 + ia + \mathfrak{F}) = 1 + ia + \left[\frac{-1-i}{2}, \frac{-1+i}{2}\right),$$

where $\left[\frac{-1-i}{2}, \frac{-1+i}{2}\right) = \{z \in \mathfrak{F} : \exists t \in [0,1)\ z = \frac{-1-i}{2} + ti\}$, so

$$T^2[\mathcal{C}_2(-2, 1+ia)] = \left[\frac{-1-i}{2}, \frac{-1+i}{2}\right).$$

Similarly, for $a \in \mathbb{Z}$ with $|a| \geq 2$,

$$\begin{aligned} T^2[\mathcal{C}_2(-2, 1+ai)] =& T^2[\mathcal{C}_2(-1+i, 1-i|a|)] = \left[\frac{-1-i}{2}, \frac{-1+i}{2}\right), \\ T^2[\mathcal{C}_2(2i, a+i))] =& T^2[\mathcal{C}_2(-1+i, -|a|+i)] = \left[\frac{-1-i}{2}, \frac{1-i}{2}\right). \end{aligned}$$

Proposition 1.2.1. *The set of irregular numbers has Lebesgue measure* 0*; in fact, its Hausdorff dimension is* 1.

See Chapter 4 for a longer discussion on Hausdorff dimension.

Proof. The previous discussion and the monotonicity of the Hausdorff dimension give $\dim_H \mathfrak{F}_I \geq 1$. On the other hand, an inductive argument shows that $\mathfrak{F}_I$ is contained in a countable union of arcs of circles, thus $\dim_H \mathfrak{F}_I \leq 1$. □

Lack of Markovian Property

Let us study first the HCF algorithm to the real numbers. For any $n \in \mathbb{N}$ with $n \geq 3$ we have

$$\mathfrak{F}_{\mathbb{R}} \coloneqq (\mathfrak{F} \cap \mathbb{R}) \smallsetminus \mathbb{Q} = T\left[\left[\frac{1}{n+1}, \frac{1}{n}\right]\right] = T\left[\left[-\frac{1}{n}, -\frac{1}{n+1}\right]\right]$$

However, when $n = 2$ or $n = -2$ we have

$$T\left[\left[\frac{1}{3}, \frac{1}{2}\right]\right] = \left(0, \frac{1}{2}\right) \quad T\left[\left[-\frac{1}{2}, -\frac{1}{3}\right]\right] = \left(-\frac{1}{2}, 0\right).$$

Starting from these equalities— which should be read excluding rational numbers— we can show recursively that the associated shift space can be characterized by the matrix

$$M_{i,j} = \begin{cases} 0, & \text{if } i = 2 \text{ and } j < 0, \\ 0, & \text{if } i = -2 \text{ and } j > 0, \\ 1, & \text{in other case.} \end{cases}$$

This means that an infinite sequence $(a_n)_{n\geq 1}$ in $\mathbb{Z} \smallsetminus \{-1, 0, 1\}$ is the HCF of some $z \in \mathfrak{F}_{\mathbb{R}}$ if and only if

$$\forall n \in \mathbb{N} \quad M_{a_n, a_{n+1}} = 1.$$

In general, we cannot describe Ω^{HCF} with a matrix as above. The core of the problem is that for $\mathbf{a} = (a_n)_{n\geq 1} \in \Omega^{\mathrm{HCF}}$ and $n \in \mathbb{N}$, the set $T^n[\mathcal{C}_n(\mathbf{a})]$ does not only depend on a_n if $\sqrt{5} \leq |a_n| \leq \sqrt{8}$. For example, while $T^n[\mathcal{C}_n(\mathbf{a})] = \mathfrak{F}$ whenever $|a_n| \geq 3$, we have that

$$T^2[\mathcal{C}_2(2 + 2i, -2 + 2i)] = \mathfrak{F} \neq T[\mathcal{C}_1(-2 + i)] = T^2[\mathcal{C}_2(1 + 2i, -2 + 2i)].$$

We can extend this phenomenon to arbitrarily long prefixes:

$$\begin{aligned} \mathfrak{F} &= T^{2n+2}[\mathcal{C}_{2n}(2 + 2i, -2 + 2i, \ldots, 2 + 2i, -2 + 2i)] \\ T[\mathcal{C}_1(-2 + i)] &= T^{2n+2}[\mathcal{C}_{2n}(1 + 2i, -2 + 2i, 2 + 2i, \ldots, -2 + 2i, 2 + 2i, -2 + 2i)]. \end{aligned}$$

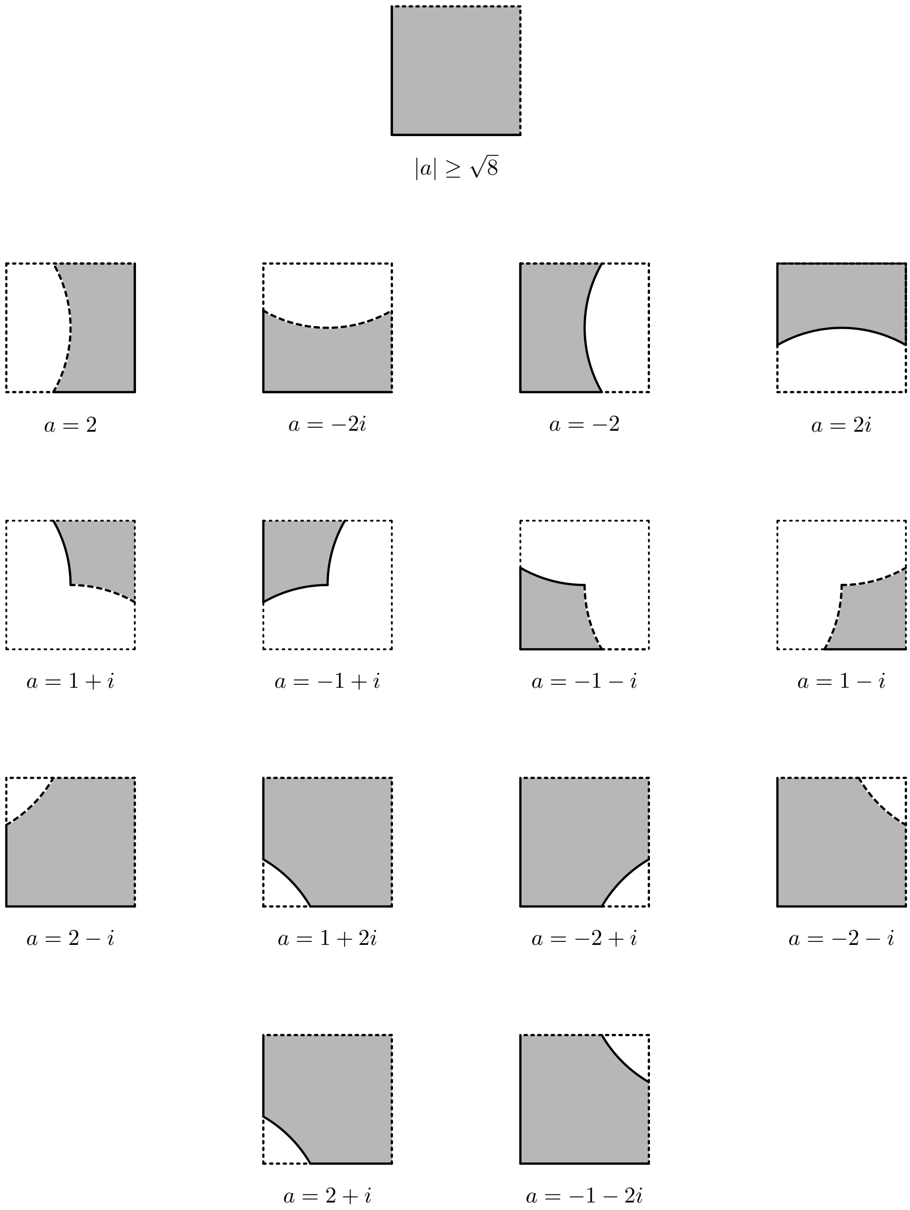
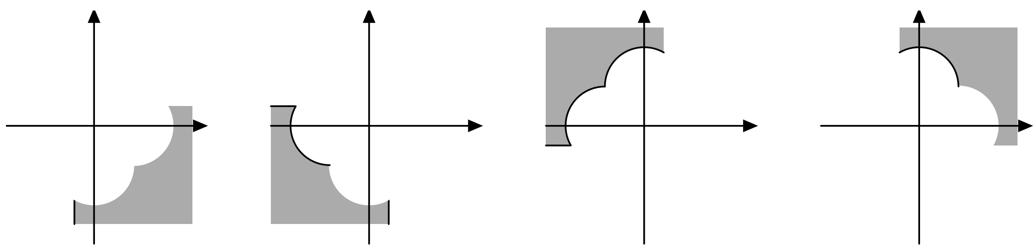


Figure 1.3: Maximal feasible sets $T[\mathcal{C}_1(a)]$.

In Figure 1.4 we show the sets $(T[\mathcal{C}_1(a)])^{-1}$ for $|a| = \sqrt{2}$. Note that for $-1+i$ the boundary makes the segments $(-1+i, 1-ia)$ and $(-1+i, -a+i)$ admissible for $a \in \mathbb{N}$, $a \geq 2$. Figure 1.5 displays the sets $(T[\mathcal{C}_1(a)])^{-1}$ for $|a| = 2$, and Figure 1.6 shows the same sets for $a = 1 + 2i$ and $a2 + i$.

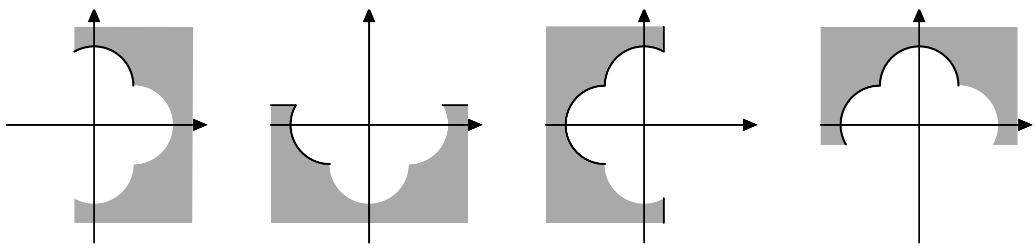
Proposition 1.2.2. *There is no infinite matrix* $M = (M_{i,j})_{i,j \in I}$ *such that* $\mathbf{a} \in \Omega^{\mathrm{HCF}}$ *if and only if* $M_{a_n, a_{n+1}} = 1$ *for every* $n \in \mathbb{N}$.

Proof. Assume such a matrix M existed. On the one hand, the segment $1+2i, -2+2i, 1+i$ is not admissible, so $M_{-2+2i,1+i} = 0$. However, $0, -2+2i, 1+i$ is an admissible prefix, which would imply $M_{-2+2i,1+i} = 1$, a contradiction. $\square$



$a_n = 1+i$ $a_n = -1+i$ $a_n = -1-i$ $a_n = 1-i$

Figure 1.4: $T[\mathcal{C}_1(a)]^{-1}$ for $|a| = \sqrt{2}$.

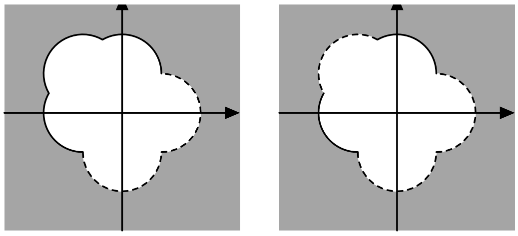


$a_n = 2$ $a_n = 2i$ $a_n = -2$ $a_n = -2i$

Figure 1.5: $T[\mathcal{C}_1(a)]^{-1}$ for $|a| = 2$.

1.2.2 Growth of Denominators

Take $\mathbf{a} = (a_n)_{n \geq 0} \in \Omega^{\mathrm{HCF}}$. So far, we know that the sequence denominators, $(q_n)_{n \geq 0}$, never vanishes. However, we have not said a word about its growth. The monotonicity of $(|q_n|)_{n \geq 0}$ goes back to A. Hurwitz in [H87]. More than a century later, Doug Hensley showed in [He06] that $(|q_n|)_{n \geq 0}$ in fact grows exponentially. Some years later, S.G. Dani and A. Nogueira improved in [DaNo14] D. Hensley's result.



$a_n = 1 + 2i$ $a_n = 2 + i$

Figure 1.6: $T[\mathcal{C}_1(a)]^{-1}$ for some $|a| = \sqrt{5}$.

Theorem 1.2.3 (Hurwitz, 1887). *If $(q_n)_{n\geq 0}$ is the sequence of denominators of a valid sequence $(a_n)_{n\geq 0}$, then*

$$1 = |q_0| < |q_1| < |q_2| < \dots .$$

Proof. Suppose that the result is false. Let $(a_n)_{n\geq 0}$ be a valid sequence with $\mathcal{Q}$-pair $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ and let $m \in \mathbb{N}$ be minimal with respect to the property $|q_m| \leq |q_{m-1}|$, hence

$$|q_0| < |q_1| < \dots < |q_{m-1}|, \quad |q_{m-1}| \geq |q_m|.$$

We certainly have $m > 1$, because $1 = |q_0| < |q_1|$, for $q_0 = 1$ and $q_1 = a_1 \in I$. Define the sequence $(k_n)_{n=1}^m$ by $k_n = q_n/q_{n-1}$, so

$$|k_m| \leq 1 < |k_1|, \dots, |k_{m-1}|.$$

From $|k_{m-1}| > 1$ and

$$k_m = a_m + \frac{q_{m-2}}{q_{m-1}} = a_m + \frac{1}{k_{m-1}} \in \mathbb{D}(a_m) \cap \mathbb{D}.$$

we obtain that $a_m \in \{1+i, -1+i, -1-i, 1-i\}$. The possible regions where k_m can lie are shown in Figure 1.7. The four options for a_m are handled in a similar fashion, so let's assume that $a_m = 1 + i$. We thus have $k_{m-1}^{-1} \in \mathbb{D} \cap \mathbb{D}(-1-i)$ and

$$k_{m-1} = a_{m-1} + \frac{1}{k_{m-2}} \in \overline{\mathbb{E}}(1,1) \cap \mathbb{D}(-1+i).$$

In Figure 1.8 we can see that any choice of a_{n-1} other than $a_{m-1} = -2 + 2i$ leads to a feasible region for w_m entailing $a_m \neq 1+i$. Therefore, $a_{m-1} = -2+2i$ and $k_{m-2}^{-1} = k_{m-1} - a_m \in \mathbb{D} \cap \mathbb{D}(-1+i)$. An analogous argument tells us that $a_{m-2} = 2 + 2i$ and we end up dealing with either of the sequences

$$(a_0, \dots, a_{m-1}, a_m) = (2 + 2i, -2 + 2i, \dots, 2 + 2i, -2 + 2i, 1 + i),$$
$$(a_0, \dots, a_{m-1}, a_m) = (-2 + 2i, 2 + 2i, \dots, 2 + 2i, -2 + 2i, 1 + i),$$

depending on the parity of m.

Suppose that m is even (m odd is treated similarly). Consider the sequence $(h_n)_{n\geq 1}$ is given by

$$h_1 = 2+2i, \qquad \forall n \in \mathbb{N} \quad h_{2n} = -2+2i+\frac{1}{h_{2n-1}},\ h_{2n+1} = 2+2i+\frac{1}{h_{2n}}.$$

It can be shown inductively that $h_n = r_n((-1)^{n+1}+i)$ where $(r_n)_{n\geq 1}$ is given by $r_1 = 2$, $r_{n+1} = 2-(2r_n)^{-1}$, and that $r_n \searrow 1+1/\sqrt{2}$ as $n \to \infty$. As a consequence, we have that

$$k_m = 1+i+\frac{1}{h_{m-1}} = 1+i+\frac{-1-i}{2r_n} = (1+i)\left(1-\frac{1}{2r_n}\right) = (1+i)(r_{n+1}-1).$$

Taking absolute values we contradict the definition of m:

$$|k_m| = |(1+i)(r_{m+1}-1)| > \sqrt{2}\frac{1}{\sqrt{2}} = 1.$$

□

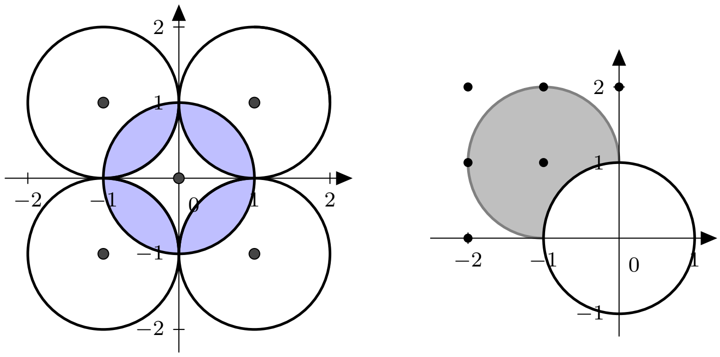
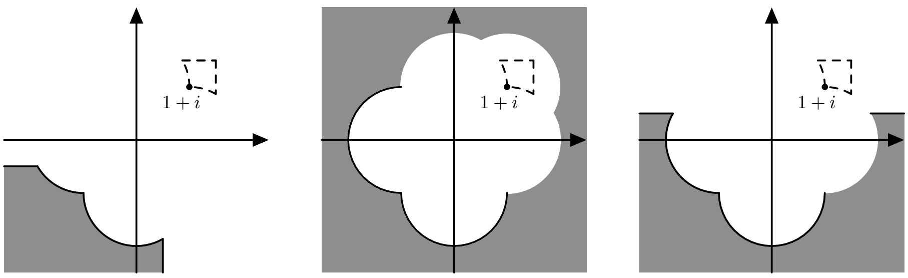


Figure 1.7: Left: Possible locations for k_m. Right: Possible choices for a_m.



$a_m = -1 + i$ $a_m \in \{-1+2i, -2i+1\}$ $a_m = 2i$

Figure 1.8: Feasible regions for w_{m-1} when $a_m = -1+i$, $a_m \in \{-1+2i, -2i+1\}$, and $a_m = 2i$.

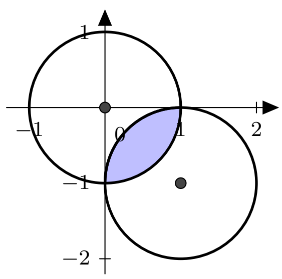


Figure 1.9: Region for k_{m-1}^{-1}.

Theorem 1.2.4 (Dani, Nogueira (2011)). *Let $\zeta = [a_0; a_1, a_2, a_3, \dots]$ be an element of $\mathbb{C} \in \mathbb{Q}(i)$ and $(p_n)_{n\geq 0}, (q_n)_{n\geq 0}$ its $\mathcal{Q}$-pair. The following statements hold.*

i. For every $n \in \mathbb{N}_0$ we have either $\phi|q_n| \leq |q_{n+1}|$ or $\phi|q_{n+1}| \leq |q_{n+2}|$.

ii. For every $n, k \in \mathbb{N}$ we have

$$|q_{n+k}| > \phi^{[\frac{k}{2}]}|q_n|.$$

The theorem is a direct conclusion of the following lemma.

Lemma 1.2.5. *Let $(a_j)_{j=0}^m$ be valid and $(q_n)_{n\geq 0}$ the associated sequence of denominators. Suppose that for some $n \in [1..m-1]$ and some σ, $\sqrt{2} < \sigma < \phi$ we have $|q_{n-1}| > \sigma|q_{n-2}|$. If $|a_n| > 2$, then $|q_n| > \sigma|q_{n-1}|$. If $|a_n| \leq 2$, then $|q_{n+1}| > \sigma|q_n|$.*

Proof. **Case $|a_n| > 2$.** On the one hand, the recurrence of $(q_n)_{n\geq 0}$ gives

$$|a_n| = \left|\frac{q_n - q_{n-2}}{q_{n-1}}\right| \leq \left|\frac{q_n}{q_{n-1}}\right| + \left|\frac{q_{n-2}}{q_{n-1}}\right| \leq \left|\frac{q_n}{q_{n-1}}\right| + \frac{1}{\sigma}.$$

On the other hand, since $t \mapsto t + t^{-1}$ is increasing for $t > 1$,

$$|a_n| \geq \sqrt{5} = \frac{1+\sqrt{5}}{2} + \left(\frac{1+\sqrt{5}}{2}\right)^{-1} \geq \sigma + \frac{1}{\sigma}.$$

The result follows.

Case $|a_n| \leq 2$. Suppose that a_n, a_{n+1} is regular. It can be checked directly that

$$|a_n| \leq 2 \quad \Longrightarrow \quad \Re(a_n a_{n+1}) \geq 2\chi_{\{\sqrt{2}\}}(|a_{n+1}|). \tag{1.17}$$

Let us write $r_{j+1} = \frac{q_j}{q_{j+1}}$ for any valid j and $r = \sigma^{-1}$. In order to conclude that $r_{n+1} \in r\mathbb{D}$ we find $z \in \mathbb{C}$ and $s > 0$ such that

$$r_{n+1} \in \mathbb{D}\left(\frac{\overline{z}}{|z|^2 - s^2}, \frac{s}{|z|^2 - s^2}\right) \subseteq r\mathbb{D}.$$

The hypothesis $|q_{n-1}| > \sigma|q_{n-2}|$ translates into $r_{n-1} \in r\mathbb{D}$, so

$$a_n + r_{n-1} \in \mathbb{D}(a_n, r) \quad \Longrightarrow \quad r_n = \frac{1}{a_n + r_{n-1}} \in \mathbb{D}\left(\frac{\overline{a_n}}{|a_n|^2 - r^2}, \frac{1}{|a_n|^2 - r^2}\right).$$

Writing $y = (|a_n|^2 - r^2)^{-1}$, $z = a_{n+1} + \overline{a_n}y$, $s = ry$ we have

$$\begin{aligned} r_n \in \mathbb{D}(\overline{a_n}y, ry) \quad &\Longrightarrow \quad a_{n+1} + r_n \in \mathbb{D}(a_{n+1} + \overline{a_n}y, ry) \\ &\Longrightarrow \quad r_{n+1} = \frac{1}{a_{n+1} + r_n} \in \mathbb{D}\left(\frac{\overline{z}}{|z|^2 - s^2}, \frac{1}{|z|^2 - s^2}\right). \end{aligned} \tag{1.18}$$

In order to verify the last implication we note that $|z|^2 - s^2 > 0$, because from

$$y(|a_n| + r) = \frac{1}{|a_n| - r} < \frac{1}{\sqrt{2} - r} < \frac{1}{\sqrt{2} - \frac{1}{\sqrt{2}}} = \sqrt{2} \le |a_{n+1}|$$

we obtain $|z| = |y\overline{a_n} + a_{n+1}| \ge |a_{n+1}| - y|a_n| > yr$.

The rest of the proof is to show that the disc in (1.18) is contained in $r\mathbb{D}$, which happens if

$$\frac{|\overline{z}|}{|z|^2 - s^2} + \frac{s}{|z|^2 - s^2} < r \quad \Longleftrightarrow \quad r(|z| - s) > 1.$$

Setting $x = y^{-1} = |a_n|^2 - r^2$, the last inequality is equivalent to $|a_n|^2 < r|\overline{a_n} + xa_{n+1}|$ and squaring

$$|a_n|^4 < r^2(\overline{a_n} + xa_{n+1})(a_n + x\overline{a_{n+1}}).$$

Taking $\alpha = |a_n a_{n+1}|^2$ and $\beta = 2\Re(a_n a_{n+1})$, we can reduce the previous equation to

$$x^2|a_{n+1}|^2 + (\beta - \alpha)x + |a_n|^2(1 - \beta) < 0. \tag{1.19}$$

The discriminant Δ of the quadratic is positive, because $\alpha \ge 4$, $\beta > 0$, and

$$\Delta = (\beta - \alpha)^2 - 4\alpha(1 - \beta) = (\alpha + \beta)^2 - 4\alpha > 0.$$

Denote by λ_- and λ_+ the real roots of (1.19):

$$\lambda_- = \frac{|a_n|^2((\alpha - \beta) - \sqrt{\Delta})}{2\alpha}, \quad \lambda_+ = \frac{|a_n|^2((\alpha - \beta) + \sqrt{\Delta})}{2\alpha}.$$

The inequality in (1.19) holds if and only if $\lambda_- < x < \lambda_+$ or, equivalently, if $|a_n|^2 - \lambda_+ < r^2 < |a_n|^2 - \lambda_-$ which is

$$|a_n|^2\left(\frac{\alpha + \beta - \sqrt{\Delta}}{2\alpha}\right) < r^2 < |a_n|^2\left(\frac{\alpha + \beta + \sqrt{\Delta}}{2\alpha}\right). \tag{1.20}$$

The upper bound does not require much work: $r^2 < 1 < \frac{|a_n|^2}{2} = |a_n|^2\left(\frac{\alpha+\beta+\sqrt{\Delta}}{2\alpha}\right)$. Using $\Delta = (\alpha + \beta)^2 - 4\alpha$, the lower bound can be seen to be equivalent to

$$\sigma^2 < \frac{2\alpha}{|a_n|^2(\alpha + \beta - \sqrt{\Delta})} = \frac{\alpha + \beta + \sqrt{\Delta}}{2|a_n|^2}.$$

We consider now two sub-cases: $|a_{n+1}| \geq 2$ and $|a_{n+1}| = \sqrt{2}$.

Case $|\mathbf{a_{n+1}}| \geq \mathbf{2}$. Since $\beta \geq 0$, then

$$\begin{aligned}\frac{\alpha+\beta-\sqrt{\Delta}}{2\alpha|a_n|^2} &\geq \frac{4|a_n|^2+\sqrt{16|a_n|^4-16|a_n|^2}}{2|a_n|^2} \\ &= 2+2\sqrt{1-\frac{1}{|a_n|^2}} \geq 2+\sqrt{2} > \sigma^2\end{aligned}$$

Case $|\mathbf{a_{n+1}}| = \sqrt{\mathbf{2}}$. By (1.17), $\Re(a_n a_{n+1}) \geq 2$ and the conclusion is gotten verifying each case separately. For instance, $a_n \in \{1-i, 2, -2i\}$ when $a_{n+1} = 1+i$ and in each case the inequality holds.

Suppose now that $a_n a_{n+1}$ is irregular. Since $|a_n| \leq 2$, $a_n \in \{-1+i, -2, 2i\}$ and a_{n+1} must have a specific form:

$$\begin{aligned} a_n = -1+i &\implies \exists m \in \mathbb{Z} \quad m \geq 2 \quad a_{n+1} = 1 - im \\ a_n = -2 &\implies \exists m \in \mathbb{Z} \quad |m| \geq 2 \quad a_{n+1} = 1 + im \\ a_n = 2i &\implies \exists m \in \mathbb{Z} \quad |m| \geq 2 \quad a_{n+1} = m + i. \end{aligned}$$

Let us first take $a_n = -2$, so $a_{n+1} = 1 + im$ for some integer m, $|m| \geq 2$. By the formula for inversion of circles,

$$\frac{q_{n+1}}{q_n} = 1 + im + \frac{1}{-2 + r_{n-1}} \in (1+im) + \mathbb{D}\left(\frac{-2}{4-r^2}, \frac{r}{4-r^2}\right).$$

Hence, for some $\varepsilon \in \mathbb{D}$ we have that

$$\frac{q_{n+1}}{q_n} = \left(1 - \frac{2}{4-r^2}\right) + im + \varepsilon\frac{r}{4-r^2}.$$

It can be checked that $\frac{1}{r} + \frac{r}{4-r^2} < 2$ whenever $\phi^{-1} < r < 2^{-\frac{1}{2}}$, so

$$\frac{1}{r} + \frac{r}{4-r^2} < 2 < \left|\left(1 - \frac{2}{4-r^2}\right) + im\right|, \tag{1.21}$$

which implies $|\frac{q_{n+1}}{q_n}| > r^{-1} = \sigma$. An analogous argument clearly holds for $a_n = 2i$ and a similar one implies $a_n = -1+i$, for which the inequality corresponding to (1.21) is

$$\frac{1}{r} + \frac{r}{2-r^2} < 2 < \left|\left(1 - \frac{1}{2-r^2}\right) - i\left(m + \frac{1}{2-r^2}\right)\right|.$$

□

Proof of Theorem 1.2.4. Take $\sigma < \phi$. The previous lemma along with an inductive argument and $|q_{-1}| = 0 < 1 = |q_0|$ give that either $|q_n| > \sigma|q_{n-1}|$ or $|q_{n+1}| > \sigma|q_n|$ hold. The theorem follows by taking $\sigma \to \phi$. □

1.2.3 Quality of Approximation

A straight forward application of the Pigeon Hole Principle argument tells us that for any $\zeta \in \mathbb{C}' := \mathbb{C} \smallsetminus \mathbb{Q}(i)$ there are infinitely many Gaussian integers $p, q,\ q \neq 0$, such that

$$\left|\zeta - \frac{p}{q}\right| \leq \sqrt{2}\frac{1}{|q|^2}. \tag{1.22}$$

On the ground of Minkowski's Second Theorem of Convex bodies, the constant $\sqrt{2}$ can be reduced to $4/\pi$. In 1925, Lester Ford showed that the least constant valid for all complex irrationals is $3^{-1/2}$. In other words, a complex number ζ is irrational if and only if there are infinitely many complex rationals p/q whose approximation to ζ is $\mathcal{O}(|q|^{-2})$.

We would like the convergents $(p_n/q_n)_{n\geq 0}$ of a complex irrational to provide an approximation of order $\mathcal{O}(|q_n|^{-2})$. We would even like to have some sort of optimality of the approximation in terms of $(q_n)_{n\geq 0}$. Richard Lakein worked in these problems during the 1970's. His results, which we shall name First and Second Lakein Theorem, are a satisfactory answer to our inquiries.

Although both of Lakein's Theorems are valid for every complex number, we will only state and discuss them for irrationals belonging to $\mathfrak{F}$. With these simplifications we do not lose generality and we avoid notational nuisance.

First Lakein's Theorems

Let $\zeta = [0; a_1, a_2, \ldots]$ with $\mathcal{Q}$-pair $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ be any member of $\mathfrak{F}'$. We aim to study the approximation error

$$\zeta - \frac{p_n}{q_n} = \frac{p_n\zeta_{n+1}^{-1} + p_{n-1}}{q_n\zeta_{n+1}^{-1} + q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^2}{q_n^2\left(\zeta_{n+1}^{-1} + \frac{q_{n-1}}{q_n}\right)} = \frac{(-1)^2}{q_n^2\left(a_{n+1} + \zeta_{n+2} + \frac{q_{n-1}}{q_n}\right)}.$$

Let $(m_n(\zeta))_{n\geq 0}$ and $(M_n(\zeta))_{n\geq 0}$ be given by

$$\forall n \in \mathbb{N} \qquad m_n(\zeta) = a_{n+1} + \zeta_{n+2} + \frac{q_{n-1}}{q_n}, \quad M_n(\zeta) = |m_n(\zeta)|.$$

Define

$$B := \inf\{M_n(\zeta) : \xi \in \mathfrak{F}', \quad n \in \mathbb{N}\}.$$

If B happens to be a positive real number, then we would have the quadratic approximation of HCF we long for:

$$\forall n \in \mathbb{N} \quad \left|\xi - \frac{p_n}{q_n}\right| \leq \frac{1}{B|q_n|^2}.$$

The First Lakein Theorem gives us precise information about B.

Theorem 1.2.6 (First Lakein Theorem, 1972)**.** $B = 1$.

Proof. First, we see that $M_n(\zeta) \geq 1$ always holds. Then, we give a pair of sequences $(\zeta^{(j)})_{j\geq 0}$ and $(n_j)_{j\geq 0}$ such that $M_{n_j}(\zeta^{(j)}) \to 1$ as $j \to \infty$.

1. Take $\zeta \in \mathbb{C} \setminus \mathbb{Q}(i)$ and $n \in \mathbb{N}$. We consider four cases depending on $|a_{n+1}|$.

   §. $|\mathbf{a_{n+1}}| \geq \sqrt{\mathbf{8}}$. The triangle inequality, $\zeta_{n+2} \in \mathfrak{F}$, and $|q_{n-2}| < |q_{n-1}|$ yield $M_n(\zeta) > 1$.

   §. $|\mathbf{a_{n+1}}| = \mathbf{2}$. By symmetry, we may assume that $a_{n+1} = 2$. In order to show that $m_n(\zeta) = 2 + \zeta_{n+2} + \frac{q_{n-1}}{q_n}$ lies outside the unit circle, note first that

$$2 + \zeta_{n+2} \in (2 + \mathfrak{F}) \cap \mathfrak{F}^{-1} = (2 + \mathfrak{F}) \setminus \overline{\mathbb{D}}(1), \tag{1.23}$$

$$\frac{q_{n-1}}{q_n} \in \iota[\mathbb{D}(a_n)] = \mathbb{D}\left(\frac{\overline{a_n}}{|a_n|^2 - 1}, \frac{1}{|a_n|^2 - 1}\right) \tag{1.24}$$

(see Figure 1.10). By elementary means (like the law of cosines, for example), we can see that

$$\inf\{|z| : z \in (2 + \mathfrak{F}) \cap \mathfrak{F}^{-1}\} = |1 + e^{i\frac{\pi}{6}}| > 1.9.$$

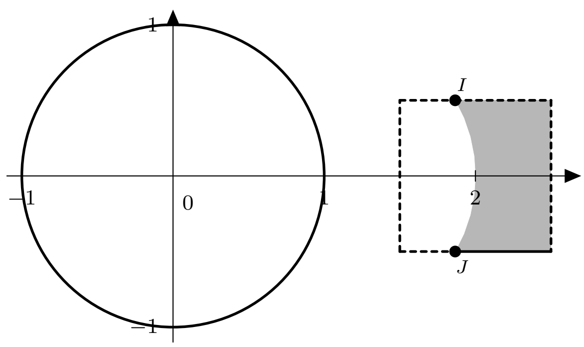


Figure 1.10: $2 + \zeta_{n+2} \in (2 + \mathfrak{F}) \cap \mathfrak{F}^{-1}$.

By (1.24), we have that

$$\left|\frac{q_{n-1}}{q_n}\right| \leq \frac{|a_n| + 1}{|a_n|^2 - 1} = \frac{1}{|a_n| - 1},$$

which is < 0.85 whenever $|a_n| \geq \sqrt{5}$ and in all of these cases, we would obtain $M_n(\zeta) > 1$. We only have to examine what happens when $|a_n| \in \{\sqrt{2}, 2\}$.

The Laws of Succession exclude the possibilities $a_n \in \{-1+i, -1-i, -2\}$. By (1.24), $\Re(a_n) \geq 1$ yields $\Re(\frac{q_{n-1}}{q_n}) > 0$ and $\Re(m_n(\zeta)) > 1$, which implies $M_n(\zeta) > 1$. Hence, the conclusion holds for $a_n \in \{1+i, 1-i, 2\}$. The remaining cases are $-2i$ and $2i$. Suppose that $a_n = 2i$. If $M_n(\zeta) \leq 1$ were true, we would have for some $\varepsilon \in \mathbb{D}$ that

$$\begin{aligned} \left|2 + \frac{2}{3}i + \zeta_{n+2} + \frac{1}{3}\varepsilon\right| < 1 \quad &\Longrightarrow \quad \left|2 + \frac{2}{3}i + \zeta_{n+2}\right| < \frac{4}{3} \\ &\Longrightarrow \quad \left|2 + \frac{2}{3}i\right| \leq \frac{4}{3} + \frac{\sqrt{2}}{2}, \end{aligned}$$

which is false. The same computations work for $-2i$ too.

§. $|\mathbf{a_{n+1}}| = \sqrt{\mathbf{5}}$. An argument similar to the case $|a_{n+1}| = 2$ does the trick. Some care must be taken, however. The set where ζ_{n+2} belongs can have the form of either $T[\mathcal{C}_1(2+i)]$ or $T[\mathcal{C}_1(2)]$.

§. $|\mathbf{a_{n+1}}| = \sqrt{\mathbf{2}}$. By symmetry, we may assume that $a_{n+1} = 1+i$. Suppose for contradiction that $M_n(\zeta) < 1$. Then, $\frac{q_{n-1}}{q_n} \in \mathbb{D}(0,1) \cap \mathbb{D}(-1-i)$ and— as in the proof of Theorem 1.2.3— a_n must be $-2+2i$. Repeating the argument, we conclude the absurdity

$$\frac{q_{n-1}}{q_n} = [0; \overline{-2+2i, 2+2i}] \in \mathbb{C} \smallsetminus \mathbb{Q}(i).$$

Therefore, $M_n(\zeta) \geq 1$.

Putting all the cases together we obtain $B \geq 1$.

2. Define the number $\xi = [0; \overline{-2+2i, 2+2i}]$ (the bar means periodicity). We can verify directly that

$$|1 + i + \xi| = 1. \tag{1.25}$$

Define $(\zeta^{(j)})_{n\geq 1}$ by

$$\forall j \in \mathbb{N} \quad \zeta^{(j)} = [0; \underbrace{-2+2i, 2+2i, \ldots, -2+2i, 2+2i}_{n \text{ times } (-2+2i, 2+2i)}, 1+i, w_j],$$

where $\Re w_j \geq 2$, $\Im w_j < -2$ and $|w_j| \nearrow \infty$ when $j \to \infty$. Then, for every $j \in \mathbb{N}$

$$m_{2j+1}(\zeta^{(j)}) = 1 + i + \frac{1}{w_n} + [0; \underbrace{2+2i, -2+2i, \ldots, 2+2i, -2+2i}_{n \text{ times } (2+2i, -2+2i)}].$$

Finally, in view of (1.25), $M_{2j+1}(\zeta^{(j)}) \to 1$ as $j \to \infty$ and $B = 1$.

□

Corollary 1.2.7. *If $\zeta \in \mathbb{C}'$ has $\mathcal{Q}$-pair $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$, then*

$$\forall n \in \mathbb{N} \qquad \left|\zeta - \frac{p_n}{q_n}\right| < \frac{1}{|q_n|^2}.$$

Second Lakein Theorem

Let ζ be a complex number. A rational complex number p/q is a **good approximation** to ζ if

$$\forall q' \in \mathbb{Z}[i] \quad \forall p' \in \mathbb{Z}[i] \quad |q'| \leq |q| \implies |qz - p| \leq |q'z - p'|. \tag{1.26}$$

and p/q is a **best approximation** to ζ if

$$\forall q' \in \mathbb{Z}[i] \quad \forall p' \in \mathbb{Z}[i] \quad |q'| < |q| \implies |qz - p| < |q'z - p'|. \tag{1.27}$$

Theorem 1.2.8 (Second Lakein Theorem, 1972)**.** *Every HCF convergent of any $\zeta \in \mathbb{C}$ is a good approximation to ζ. For Lebesgue almost all $\zeta \in \mathbb{C}$ every HCF convergent is a best approximation.*

Proof. Fix $\zeta = [0; a_1, a_2, \ldots] \in \mathfrak{F}'$ with $\mathcal{Q}$-pair $(p_n)_{n\geq 0}, (q_n)_{n\geq 0}$. Since $q_{n-1}p_n - q_np_{n-1} = (-1)^{n-1}$ holds for all n, the vectors (q_{n-1}, p_{n-1}), (q_n, p_n) always form a base of $\mathbb{Z}[i]^2$. Assume $(p, q) \in \mathbb{Z}[i]^2$ are given and take $(s, t) \in \mathbb{Z}[i]^2$ such that

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \qquad \text{so} \qquad \begin{pmatrix} s \\ t \end{pmatrix} = (-1)^{n-1} \begin{pmatrix} q_{n-1} & -p_{n-1} \\ -q_n & p_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Let $n \in \mathbb{N}$ satisfy

$$|q| < |q_n|.$$

For brevity, we write $M := M_n(\zeta)$ and consider two cases: $|t| > 2/M$ and $|t| \leq 2/M$.

§ **Case I.** $|t| > 2/M$. If we had $|q_nz - p_n| \geq |qz - p|$, then

$$\begin{aligned} \frac{2}{M}\frac{1}{|q_nq|} < \frac{|t|}{|q_nq|} &= \left|\frac{p}{q} - \frac{p_n}{q_n}\right| \\ &\leq \left|\frac{p}{q} - \zeta\right| + \left|\frac{p_n}{q_n} - \zeta\right| \\ &\leq \frac{|q_n\zeta - p_n|}{|q|} + \frac{1}{M}\frac{1}{|q_n|^2} = \frac{1}{M|q_nq|}\left(\frac{|q_n| + |q|}{|q_n|}\right). \end{aligned}$$

and we would obtain $|q_n| < |q|$, contradicting our choice of n. Therefore, $|q_n\zeta - p_n| < |q\zeta - p|$ and p_n/q_n is a best approximation to ζ.

§ **Case II.** $|t| \leq 2/M$. In view of the First Lakein Theorem, we may suppose that $|t| \leq 2$. We aim to write (1.26) (or (1.27)) in terms of s, t. Note that

$$q = sq_n + tq_{n-1} = q_n\left(s + t\frac{q_{n-1}}{q_n}\right) \implies \frac{|q|}{|q_n|} = \left|s + t\frac{q_{n-1}}{q_n}\right|.$$

Using $\zeta = \frac{p_{n-1}\zeta_{n+1}^{-1} + p_n}{q_{n-1}\zeta_{n+1}^{-1} + q_n}$ we can write ζ_{n+1}^{-1} in terms of ζ as $\zeta_{n+1}^{-1} = \frac{q_n\zeta - p_n}{-q_{n-1}\zeta - p_{n-1}}$. Therefore,

$$\begin{aligned} |q\zeta - p| &= |(sq_n + tq_{n-1})\zeta - (sp_n + tp_{n-1})| \\ &= |s(q_n\zeta - p_n) + t(q_{n-1}\zeta - p_{n-1})| = |q_n\zeta - p_n||s - t\zeta_{n+1}^{-1}|. \end{aligned}$$

The pending implication is, thus,

$$\left|s + t\frac{q_{n-1}}{q_n}\right| \leq 1 \quad \& \quad |t| < 2 \quad \implies \quad 1 \leq |s - tz_{n+1}|. \tag{1.28}$$

The restrictions $|t| < 2$ and $t \in \mathbb{Z}[i]$ imply $|t| \in \{0, 1, \sqrt{2}\}$. We may discard $t = 0$, for it gives $p = up_n$, $q = uq_n$ for some unit $u \in \mathbb{U} := \{1, i, -1, -i\}$. We will check the two remaining options separately.

Proposition 1.2.9. *Assume that* $|s + tq_{n+1}/q_n| \leq 1$. *If* $|t| = 1$, *then* $|s| < 2$. *If* $|t| = \sqrt{2}$, *then* $s \leq 2$.

Proof. The triangle inequality applied to the hypothesis gives

$$|s| < 1 + |t|.$$

The first implication is now trivial. In order to conclude the second one, we must show that $|t| = \sqrt{2}$ and $|s| = \sqrt{5}$ cannot hold simultaneously. All the possibilities are treated in a similar way, so let us show that $t = 1 + i$, $s = 1 + 2i$ is not a valid choice.

By the formula of inversion of circles,

$$\begin{aligned} \left|s + t\frac{q_{n-1}}{q_n}\right| = \frac{1}{\sqrt{2}}\left|\frac{3+i}{2} + \frac{q_{n-1}}{q_n}\right| &\implies \frac{q_{n-1}}{q_n} \in \mathbb{D}\left(\frac{3+i}{2}, \frac{1}{\sqrt{2}}\right) \\ &\implies \frac{q_n}{q_{n-1}} \in \mathbb{D}\left(\frac{3-i}{4}, \frac{1}{2\sqrt{2}}\right), \end{aligned}$$

which gives $a_n = 1 - i$ and forces $n > 1$. As a consequence, the previous membership and the equality $\frac{q_{n-2}}{q_{n-1}} = -(1-i) + \frac{q_n}{q_{n-1}}$ tell us that for some $\varepsilon_{n-1} \in \mathbb{D}$

$$a_{n-1} + \varepsilon_{n-1} = \frac{q_{n-1}}{q_{n-2}} \in \iota\left[-(1-i) + \mathbb{D}\left(\frac{3-i}{4}, \frac{1}{2\sqrt{2}}\right)\right] = \mathbb{D}\left(-\frac{1-3i}{2}, \frac{1}{\sqrt{2}}\right) =: D_1.$$

The set of Gaussian integers a, $\sqrt{2} \leq |a|$, for which $\mathbb{D}(a) \cap D_1 \neq \varnothing$ is

$$\{1-i, 1-2i, -1-i, -2-i, -2i, -1-2i, -2-2i, -3\}.$$

(see Figure 1.11). Since $a_n = 1-i$, we can immediately discard $\{1-i, -1-i, -1-2i, -2i\}$ by the laws of succession. The remaining cases are verified individually. If $a_{n-1} = 1-2i$, then a_{n-2} exists and it must belong to $\{-1-i, -2\}$. However, the segments $(-2, 1-2i)$ and $(-1-i, 1-2i)$ are not valid, so $a_{n-1} = 1-2i$ is not a valid choice. If $a_{n-1} = -2-2i$, then $a_{n-2} \in \{2, 1-i\}$, but $(2, -2-2i)$ and $(1-i, -2-2i)$ are not valid, so $a_{n-1} \neq -2-2i$. Finally, $a_{n-1} = -3$ forces $a_{n-2} \in \{-1-i, -2i, 1-i\}$ but $(1-i, -3), (-2i, -3)$, and $(-1-i, -3)$ are not valid. Hence, $|t| = \sqrt{2}$ and $|s| = \sqrt{5}$ is impossible and $|t| = \sqrt{2}$ implies $|s| \leq 2$. □

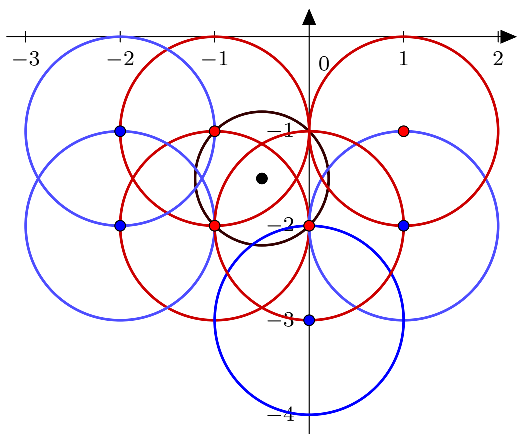


Figure 1.11: Red discs are discarded directly by the Laws of Succession; blue discs are discarded two steps. The black circle is D_1.

We finish the proof by verifying the conclusion separately for $|t| = \sqrt{2}$ and $|t| = 1$. But first, we record the next inequality, which follows from $\zeta_{n+1} \in \mathfrak{F}$,

$$|\zeta_{n+1}^{-1}| \geq 2, \quad \forall u \in \mathbb{U} \quad |\zeta_{n+1}^{-1} - u| \geq 1. \tag{1.29}$$

If $|t| = \sqrt{2}$, then $|s| \in \{0, 1, \sqrt{2}\}$ by Proposition 1.2.9.

i. If $|s| = 0$, then $|s - t\zeta_{n+1}^{-1}| = |t\zeta_{n+1}^{-1}| > 1$.

ii. If $|s| = 1$, then $|s - t\zeta_{n+1}^{-1}| \geq 2 - 1 = 1$. The equality holds if and only if $t\zeta_{n+1}^{-1} = 2u$ and $s = -u$ for some $u \in \mathbb{U}$, which happens for only finitely many choices of t, u, ζ_{n+1}.

iii. If $|s| = \sqrt{2}$, then $s = ut$ for some $u \in \mathbb{U}$, and

$$|s - t\zeta_{n+1}^{-1}| = |ut - t\zeta_{n+1}^{-1}| = |t||u - \zeta_{n+1}^{-1}| \geq \sqrt{2} > 1.$$

If $|t| = 1$, then $|s| \in \{0, 1\}$ by Proposition 1.2.9. By the symmetries of the construction, we can assume that $t = 1$.

i. If $|s| = 0$, then $|s - t\zeta_{n+1}^{-1}| > 1$.

ii. If $|s| = 1$, then $|s - t\zeta_{n+1}^{-1}| = |s - z_{n+1}| \geq 1$ by (1.29). The equality happens if and only if $\zeta_{n+1}^{-1} = s(1+i)$.

iii. If $|s| = \sqrt{2}$, we can take $s = 1 + i$. We argue by contradiction. Suppose we had

$$\left|1 + i + \frac{q_{n-1}}{q_n}\right| \leq 1, \quad |1 + i - \zeta_{n+1}^{-1}| < 1.$$

As in Theorem 1.2.3, we can obtain recursively the absurdity $q_{n-1}/q_n = [0; \overline{-2 + 2i, 2 + 2i}] \notin \mathbb{Q}(i)$. Therefore, we must have $|1 + i - z_{n+1}| \geq 1$.

In a few words, every HCF convergent of any $\zeta \in \mathbb{C}$ is a good approximation of ζ. The above argument shows also that the equality in the consequent of (1.28) is satisfied only by numbers belonging to countably many circles. Thus, the almost everywhere assertion follows. □

Badly Approximable Complex Numbers

A complex number is badly approximable whenever the approximation rate given by Dirichlet's (or Minkowski's, or Ford's or Lakein's) inequality (1.22) cannot be improved significantly.

Definition 1.2.2. *A complex number $\zeta \in \mathbb{C}$ if* ***badly approximable*** *if there is some $C > 0$ such that every $p, q \in \mathbb{Z}[i]$, $q \neq 0$, satisfy*

$$\left|\zeta - \frac{p}{q}\right| > \frac{C}{|q|^2}.$$

The set of badly approximable complex numbers is denoted $\mathbf{Bad}_{\mathbb{C}}$.

$\mathbf{Bad}_{\mathbb{C}}$ shares some properties with its well known real counterpart. For instance, we will show that the Lebesgue measure of $\mathbf{Bad}_{\mathbb{C}}$ is 0, and that $\dim_H \mathbf{Bad}_{\mathbb{C}} = 2$. In fact, the set $\mathbf{Bad}_{\mathbb{C}}$ is $\frac{1}{2}$-winning in the sense of Schmidt Games (see Chapter 4). Before addressing these properties, we characterize $\mathbf{Bad}_{\mathbb{C}}$ in terms of HCF. The Second Lakein Theorem allows us to extend the usual argument (See Theorem 23 in [Kh], p.36) to the complex context. We give it for completeness sake.

Lemma 1.2.10. *Let $\zeta = [0; a_1, a_2, \ldots] \in \mathfrak{F}$ be an irrational complex number with $\mathcal{Q}$-pair $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$, then*

$$\forall n \in \mathbb{N} \quad \frac{1}{\left(|\zeta_{n+1}^{-1}| + 1\right)|q_n|^2} < \left|\zeta - \frac{p_n}{q_n}\right| \leq \frac{2}{|q_n q_{n+1}|}. \tag{1.30}$$

Proof. Keep the notation as in the statement. The upper bound follows from the First Lakein First Theorem and the growth of $(|q_n|)_{n\geq 0}$:

$$\left|\zeta - \frac{p_n}{q_n}\right| \leq \left|\zeta - \frac{p_{n+1}}{q_{n+1}}\right| + \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{1}{|q_{n+1}|^2} + \frac{1}{|q_n q_{n+1}|} \leq \frac{2}{|q_n q_{n+1}|}.$$

By Proposition 1.1.1, we have for every $n \in \mathbb{N}$ that

$$\zeta = \frac{p_n \zeta_{n+1}^{-1} + p_{n-1}}{q_n \zeta_{n+1}^{-1} + q_{n-1}} = \frac{p_n}{q_n} + \frac{(-1)^n}{q_n^2\left(\zeta_{n+1}^{-1} + \frac{q_{n-1}}{q_n}\right)}.$$

The lower bound follows from the growth of $(|q_n|)_{n\geq 0}$ and $|\zeta_n^{-1}| \geq \sqrt{2}$. □

For our purposes, the coefficient 2 in the upper bound in (1.30) is irrelevant as long as it is constant. With this in mind, we could even avoid the use of the First Lakein Theorem. Indeed, keeping the above notation,

$$\zeta - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(q_n \xi_{n+1} + q_{n-1})} = \frac{(-1)^n}{q_n\left(q_{n+1} + q_n \frac{1}{\zeta_{n+2}}\right)} = \frac{(-1)^n}{q_n q_{n+1}\left(1 + \frac{q_n}{q_{n+1}} \frac{1}{\zeta_{n+2}}\right)}. \tag{1.31}$$

Since $(|q_n|)_{n\geq 0}$ is strictly increasing and $\zeta_{n+2}^{-1} \in \mathfrak{F} \subseteq \overline{\mathbb{D}}(0, 2^{-\frac{1}{2}})$, we obtain

$$\left|1 + \frac{q_n}{q_{n+1}} \frac{1}{\zeta_{n+2}}\right| \geq 1 - \frac{\sqrt{2}}{2} > 0.$$

The desired bound comes now from plugging the previous inequality into (1.31).

Theorem 1.2.11. $\mathbf{Bad}_{\mathbb{C}} = \{z = [a_0; a_1, a_2, \ldots] \in \mathbb{C} : \exists M > 0 \quad \forall n \in \mathbb{N}_0 \quad |a_n| \leq M\}$.

Proof. As before, we focus on $\mathfrak{F}$. We start with $\supseteq$. Suppose that $\zeta = [0; a_1, a_2, \ldots] \in \mathfrak{F}'$ has elements bounded by $M > 0$. Then,

$$\forall n \in \mathbb{N} \quad |\zeta_n^{-1}| = |a_n + \zeta_{n+1}| \leq M + 1.$$

By the left inequality of Lemma 1.30, we have for $c_1 = (M+2)^{-1}$ that

$$\forall n \in \mathbb{N} \qquad \frac{c_1}{|q_n|^2} \leq \left|\zeta - \frac{p_n}{q_n}\right|.$$

Now, take $p/q \in \mathbb{Q}(i)$ in lowest terms and $n \in \mathbb{N}$ such that $|q_{n-1}| < |q| \leq |q_n|$. By Theorem 1.2.8, $|q_n\zeta - p_n| \leq |q\zeta - p|$ and our choice of n gives

$$\left|\zeta - \frac{p}{q}\right| \geq \frac{|q_n|}{|q|}\left|\zeta - \frac{p_n}{q_n}\right| \geq \frac{c_1}{|q_n|^2} > \frac{c_1}{|q|^2}\frac{|q_{n-1}|^2}{|q_n|^2} = \frac{c_1}{|q|^2}\frac{1}{\left|a_n + \frac{q_{n-2}}{q_{n-1}}\right|^2} \geq \frac{c_1}{|q|^2(M+1)^2}.$$

Therefore, $\zeta \in \mathbf{Bad}_{\mathbb{C}}$ with constant $C = (M+1)^{-1}(M+2)^{-2}$.

In order to show the other contention, take $\zeta \in \mathbf{Bad}_{\mathbb{C}}$ and let $C = C(\zeta) > 0$ be the constant from the definition. By Lemma 1.30, for some constant $c > 0$ and any $n \in \mathbb{N}$ we have

$$\frac{C}{|q_n|^2} \leq \frac{c}{|q_n q_{n+1}|} \quad\Longrightarrow\quad \left|a_{n+1} + \frac{q_{n-1}}{q_n}\right| = \left|\frac{q_{n+1}}{q_n}\right| \ll_\zeta 1 \quad\Longrightarrow\quad |a_{n+1}| \ll_\zeta 1.$$

The last implication follows from $|q_{n-1}| < |q_n|$. □

Recently, another proof of Theorem 1.2.11 was given by Robert Hines (Theorem 1 in [Hi17]). His argument also relies on R. Lakein's work but avoids Lemma 1.30.

1.2.4 Serret's Theorem

The complicated structure of Ω^{HCF} is not the only difference between regular (or nearest integer) continued fractions and the Hurwitz algorithm. We shall see that a classical theorem on the action of $\mathrm{PGL}(2,\mathbb{Z})$ over $\mathbb{R}$ via Möbius transformations does not extend to HCF.

Theorem 1.2.12 (J.A. Serret, 1866)**.** *Two real numbers $\alpha = [a_0; a_1, \ldots], \beta = [b_0; b_1, \ldots]$ are equivalent under the action of $\mathrm{PGL}(2,\mathbb{Z})$ if and only if either they are both rational or if for some $j, k \in \mathbb{N}$ we have $a_{j+n} = b_{k+n}$ for all $n \in \mathbb{N}$.*

It is natural to ask for a similar result in the complex plane replacing regular continued fractions by HCF and $\mathrm{PGL}(2,\mathbb{Z})$ by $\mathrm{PGL}(2,\mathbb{Z}[i])$. Clearly, if ζ and ξ have infinte HCF expansion and their tails coincide, then they are equivalent under the action of $\mathrm{PGL}(2,\mathbb{Z}[i])$. However, R. Lakein ([La74a]) proved that the converse fails. Indeed, define

$$\Xi = \frac{i + (43+28i)^{\frac{1}{2}}}{2}, \quad A = \frac{5 - i - \Xi}{4 - i}, \quad B = \frac{3 + 2i + \Xi}{4},$$

the numbers A and B are equivalent under the action of $\mathrm{PGL}(2,\mathbb{Z}[i])$: $A = (2B - i)/(B - i)$, but their HCF expansions are

$$A = [\overline{2+i, 3i, -1+2i, -1+2i, 3, -2-i}],$$
$$B = [\overline{2+i, -2+i, -2+i, 1-2i, -1-2i, 1+2i}].$$

1.3 Notes and Comments

1. Our iteration sequence notation differs from the one proposed by S.G. Dani and A. Nogueira. They use $(z_n)_{n\geq 0}$ for iteration sequences, while we use it for the orbit of z under an f-Gauss map.

2. R. Lakein showed that Serret's Theorem did not hold in general. However, A. Lukyanenko and J. Vandehey proved recently in [LuVa18] that Serret's Theorem holds almost everywhere.

3. The original proof of Theorem 1.2.4 in [DaNo14] did not consider irregularities. In the original exposition of Theorem 1.2.6 in [La73], the cases $|a_n| \in \{\sqrt{2}, \sqrt{5}\}$ are disregarded.

Chapter 2

Algebraic properties

In this chapter, we survey some algebraic properties of Hurwitz Continued Fractions. We start by discussing an Euler-Lagrange-type theorem for iteration sequences. Afterwards, we see that Galois' result on purely periodic regular continued fractions holds under additional restrictions and that a full analogue is impossible. Later, we give a full proof of W. Bosma's and D. Gruenewald's extension of D. Hensley's work showing that there exist algebraic complex numbers of arbitrary large degree and bounded HCF expansion. We end the chapter by giving combinatorial conditions on infinite HCF (in the spirit of Y. Bugeaud) implying the transcendence of their limit.

Notation. For a given sequence $(a_n)_{n\geq 0}$ in $\mathbb{Z}[i]$, $[a_0; a_1, a_2, \ldots]$ represents a HCF and $\langle a_0; a_1, a_2, \ldots\rangle$ represents a complex continued fraction (not necessarily Hurwitz). If $I \subseteq \mathbb{R}$ is a compact interval and $\gamma : I \to \mathbb{C}$ is a closed path, the winding number of γ around z is $\operatorname{Ind}_\gamma(z) \coloneqq (2\pi i)^{-1} \int_\gamma \frac{\mathrm{d}w}{(w-z)}$. For $m, n \in \mathbb{Z}$ we write $[m..n] = \{j \in \mathbb{Z} : m \leq j \leq n\}$.

References. For the Euler-Lagrange-type Theorem, see [DaNo14] and [Da15]. For the construction of algebraic numbers of degree greater than 2 over $\mathbb{Q}(i)$ and bounded HCF elements, see [He06] and [BoGu12]. For purely periodic and transcendental HCF, see [Gon18-01].

2.1 Complex Algebraic Numbers of Degree 2

A very well-known result on real continued fractions is the Euler-Lagrange characterization of quadratic irrationals: an irrational number is quadratic if and only if its regular (nearest integer) continued fraction expansion is periodic. A. Hurwitz ([H87]) showed the corresponding result for quadratic complex numbers (over $\mathbb{Z}(i)$) and his continued fractions. More than a century later, in 2011, S.G. Dani and A. Nogueira ([DaNo14]) gave a similar result for more general complex continued fractions algorithms. S.G. Dani extended afterwards ([Da15]) the result to Eisenstein integers and their field of fractions.

Theorem 2.1.1 (S. G. Dani, A. Nogueira, 2011)**.** *Let $z \in \mathbb{C}$ be a quadratic surd over $\mathbb{Q}(i)$. For any $C > 0$ there is a finite set $F \subseteq \mathbb{C}$ such that whenever $\mathbf{w} = (w_n)_{n\geq 0}$ is a non-degenerate iteration sequence of z and*

$$\mathcal{N}(\mathbf{w}, C) := \left\{ n \in \mathbb{N} : |p_{n-2} - zq_{n-2}| < \frac{C}{|q_{n-2}|},\ |p_{n-1} - zq_{n-1}| < \frac{C}{|q_{n-1}|} \right\}$$

is infinite, then $\{w_n : n \in \mathcal{N}(\mathbf{w}, C)\} \subseteq F$.

Proof. Let $P \in \mathbb{Z}[i][X]$ be the minimal polynomial of z, $P(X) = aX^2 + bX + c$. Take $C > 0$, $\mathbf{w} = (w_n)_{n\geq 1}$, and $\mathcal{N} = \mathcal{N}(\mathbf{w}, C)$ as in the statement. In view of Proposition 1.1.1 in Chapter I, for every $n \in \mathbb{N}$ we have

$$a\left(\frac{w_n p_{n-1} + p_{n-2}}{w_n q_{n-1} + q_{n-2}}\right)^2 + b\left(\frac{w_n p_{n-1} + p_{n-2}}{w_n q_{n-1} + q_{n-2}}\right) + c = 0.$$

Expanding and simplifying the previous expression, we obtain that

$$\forall n \in \mathbb{N} \quad A_n w_n^2 + B_n w_n + C_n = 0, \tag{2.1}$$

where

$$\begin{aligned}
A_n &= ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2, \\
B_n &= 2ap_{n-1}p_{n-2} + b(p_{n-1}q_{n-2} + q_{n-1}p_{n-2}) + 2cq_{n-1}q_{n-2}, \\
C_n &= ap_{n-2}^2 + bp_{n-2}q_{n-2} + cq_{n-2}^2.
\end{aligned}$$

Since the roots of P are irrational, $A_n \neq 0$ always hold. Moreover, it can be verified directly that for all $n \in \mathbb{N}$

$$A_n = (p_{n-1}z - q_{n-1})(a(p_{n-1} - zq_{n-1}) + (2az + b)q_{n-1}).$$

The non-degeneracy hypothesis implies $\limsup_{n\to\infty} |w_n| > 1$ (cfr. Theorem 1.1.2), so $|p_{n-1} - zq_{n-1}| \to 0$ as $n \to \infty$ (cfr. Proposition 1.1.1) and for any large n

$$|(p_{n-1}z + q_{n-1})^2 a| \leq 1.$$

Thus, for any large $n \in \mathcal{N}$

$$|A_n| \leq |(p_{n-1}z - q_{n-1})q_{n-1}(2az + b)| + 1 \leq C(2az + b) + 1.$$

The previous bound also holds for C_n, because $C_n = A_{n-1}$ and, since the discriminant is invariant (i.e. $B_n^2 - 4A_nC_n = b^2 - 4ac$ for all n), $(B_n)_{n\in\mathcal{N}}$ is also bounded. Therefore, the sequences $(A_n)_{n\in\mathcal{N}}, (B_n)_{n\in\mathcal{N}}, (C_n)_{n\in\mathcal{N}}$ take finitely many values, so the set of polynomials $\{P_n(Z) = A_nZ^2 + B_nZ + C_n : n \in \mathcal{N}\}$, and hence the set of their roots

$$F = \{w \in \mathbb{C} : \exists n \in \mathcal{N} \quad P_n(w) = 0\}$$

are finite. The condition $\{w_n : n \in \mathcal{N}\} \subseteq F$ follows from (2.1). $\square$

In general, we cannot conclude the periodicity of an iteration sequence of a quadratic irrational. However, for certain choice functions f the associated f-sequences are periodic. HCF are an example.

Corollary 2.1.2 (A. Hurwitz, 1897)**.** *A complex irrational $z \in \mathbb{C}$ satisfies $[\mathbb{Q}(i,z) : \mathbb{Q}(i)] = 2$ if and only if its HCF is eventually periodic.*

Proof. Let $z \in \mathbb{C}'$ satisfy $z = [a_0; \ldots, a_{n-1}, \overline{a_{n+1}, \ldots, a_{n+m}}]$ where the bar denotes periodicity and let $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ be its $\mathcal{Q}$-pair. By Proposition 1.1.1 and since $q_n \neq 0$, we have that $[\overline{a_n; \ldots, a_{n+m}}]$ is a quadratic irrational over $\mathbb{Q}(i)$ and the same holds for z.

Now assume that $z = [a_0; a_1, \ldots]$ is a quadratic irrational and let $(w_n)_{n\geq 0}$ be the iteration sequence related to its HCF. By Theorem 2.1.1, for two different natural numbers n, m we have $w_n = w_m$. Hence, their HCF coincide:

$$w_n = [a_n; a_{n+1}, a_{n+2}, \ldots], \quad w_m = [a_m; a_{m+1}, a_{m+2}, \ldots],$$

and $(a_n)_{n\geq 0}$ is eventually periodic. $\square$

2.1.1 Purely Periodic HCF

A well known result by Évariste Galois ([Ga29]) states a real quadratic irrational $\alpha > 1$ has a purely periodic regular continued fraction if and only if its Galois Conjugate β satisfies $-1 < \beta < 0$. Such quadratic irrationals are called **reduced**. Under some extra restrictions, we can obtain the corresponding result for HCF.

Theorem 2.1.3. *Let $\xi = [a_0; a_1, a_2, \ldots]$ be a quadratic irrational over $\mathbb{Q}(i)$ and $\eta \in \mathbb{C}$ its Galois conjugate over $\mathbb{Q}(i)$.*

1. *If $(a_n)_{n\geq 0}$ is purely periodic, then $|\eta| < 1$.*
2. *If $|\xi| > 1$, $\eta \in \mathfrak{F}$ and $|a_n| \geq \sqrt{8}$ for every $n \in \mathbb{N}_0$, then ξ has purely periodic expansion.*

3. *The conditions $\eta \in \mathfrak{F}$ and $(\forall n \in \mathbb{N}\ |a_n| \geq \sqrt{8})$ cannot be removed from the second point. In fact, there are infinitely many pairs ξ, η such that*

   i. *$\eta \notin \mathfrak{F}$, $|a_n| < \sqrt{8}$ for some n, and $(a_n)_{n\geq 0}$ is not purely periodic,*

   ii. *$\eta \notin \mathfrak{F}$, $|a_n| \geq \sqrt{8}$ for all n, and $(a_n)_{n\geq 0}$ is not purely periodic,*

   iii. *$\eta \in \mathfrak{F}$, $|a_n| < \sqrt{8}$ for some n, and $(a_n)_{n\geq 0}$ is not purely periodic.*

We shall only mention the important aspects of the argument, the details are in [Gon18-01]. The core of the first and third points is that the Galois conjugate (over $\mathbb{Q}(i)$) η of a purely periodic HCF $\alpha = [\overline{a_0; a_1, \ldots, a_{m-1}}]$ can be written as

$$\eta = -\left\langle 0; \overline{a_{m-1}, \ldots, a_1, a_0} \right\rangle. \tag{2.2}$$

The mirror formula and the strict growth of the corresponding $(q_n)_{n\geq 0}$ give $|\eta| \leq 1$. The strict inequality is shown using an alternate sequence argument (similar to the one of Theorem 1.2.3. Although some new technicalities arise, most of the real argument can be used to show the second point (cfr. Theorem 7.2 in [NMZ]). In order to prove the third point, we just force the continued fraction in (2.2) to break the Laws of Succession.

There are several reasons for which the continued fraction (2.2) might fail to be a HCF. When these reasons are simple (in some sense), we can compute the HCF of η with a singularization[1] identity like

$$a + \cfrac{1}{2 + \cfrac{1}{b}} = a + 1 + \cfrac{1}{-2 + \cfrac{1}{b+1}}, \quad a + \cfrac{1}{1 + i + \cfrac{1}{b}} = a + (1-i) + \cfrac{1}{-(1+i) + \cfrac{1}{b+1-i}}.$$

for any $a, b \in \mathbb{C}$. A concrete example is given by

$$\zeta = [\overline{5 + 6i; -3 + 2i, 2, 9 + 4i}]$$

and its Galois Conjugate η. The sequence obtained by repeating the reversed period of the HCF of ζ is not valid, but using the first singularization formula we get

$$\eta = -[0; \overline{10 + 4I, -2, -2 + 2i, 5 + 6i}].$$

Once we know that the period cannot be reversed, it is not hard to determine the process to obtain a valid expansion. The difficulty is in determining the source of non-reversibility.

[1] We borrow the term *singularization* from [IoKr02].

2.2 Complex Algebraic Numbers of Degree Greater than 2

No matter how deep and extensive the theory of regular continued fractions has become, some problems remain open. It is still unknown if $2^{\frac{1}{3}}$ has a bounded continued fraction expansion, for example. More generally, the following conjecture has not been solved.

Conjecture 2.2.1 (Folklore Conjecture)**.** *The only badly approximable numbers that are algebraic are the quadratic irrationals.*

A strong related result was proven by Yann Bugeaud in [Bu13] on the basis of his joint work with Boris Adamczewski in [AdBu05]. We discuss below his result in further detail. In the mean time, it is enough to say that they gave sufficient combinatorial conditions on the continued fraction expansion for the transcendence of the limit.

In [He06], Doug Hensley showed what we consider to be the most impressive result in the Hurwitz Continued Fraction Theory: a negative answer to the Folklore Conjecture for HCF. Hensley gave explicitly complex irrationals $z \in \mathbb{C}$ with bounded HCF satisfying $[\mathbb{Q}(i,z) : \mathbb{Q}(i)] = 4$. Some years later, in [BoGu12], W. Bosma and D. Gruenewald extended D. Hensley's result to algebraic numbers of arbitrary even degree over $\mathbb{Q}(i)$.

The proof of the result of Hensley and the later extension rely on the study of some circles naturally associated to each complex irrational.

2.2.1 Generalized Circles

By a **generalized circle** we mean the set of solutions $w \in \mathbb{C}$ of an equation

$$\mathfrak{C} : A|w|^2 + Bw + \overline{B}\overline{w} + D = 0, \tag{2.3}$$

where $A, D \in \mathbb{R}$ are given and satisfy $B \in \mathbb{C}$, $|B|^2 - AD \geq 0$. We represent $\mathfrak{C}$ by the matrix

$$\mathfrak{C} = \begin{pmatrix} A & \overline{B} \\ B & D \end{pmatrix}. \tag{2.4}$$

Generalized circles are either circles or lines in the complex plane, but all of them are just circles in the Riemann sphere. The next properties are verified through straight forward computations.

Proposition 2.2.1. *Let $\mathfrak{C}$ be a generalized circle represented by* (2.4).

i. *$\mathfrak{C}$ is a line in the plane if and only if $A = 0$. In this case, it is the line determined by*

$$2\Re(wB) = -D.$$

Otherwise, $\mathfrak{C}$ is the circle

$$\mathfrak{C} = C\left(-\frac{\overline{B}}{A}, \frac{\sqrt{|B|^2 - AD}}{|A|}\right).$$

ii. $0 \in \mathfrak{C}$ *if and only if* $D = 0$.

iii. $\mathrm{Ind}_{\mathfrak{C}}(0) = 0$ *if and only if* $AD > 0$.

iv. $\mathrm{Ind}_{\mathfrak{C}}(0) = 1$ *if and only if* $AD < 0$.

Recall that $\iota : \mathbb{C}^* \to \mathbb{C}^*$ is given by $\iota(z) = z^{-1}$. For any $\alpha \in \mathbb{C}$ denote by τ_α the translation by α, $\tau_\alpha(z) = z + \alpha$ for any $z \in \mathbb{C}$.

Proposition 2.2.2. *Let $\mathfrak{C}$ be a generalized circle given by* (2.4)*, then the inversion of the circle is*

$$\iota[\mathfrak{C}] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A & \overline{B} \\ B & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} D & B \\ \overline{B} & A \end{pmatrix},$$

and the translation by α,

$$\tau_\alpha[\mathfrak{C}] = \begin{pmatrix} 1 & 0 \\ -\overline{\alpha} & 1 \end{pmatrix}\begin{pmatrix} A & \overline{B} \\ B & D \end{pmatrix}\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & -A\alpha + B \\ -\overline{\alpha}A + B & A|\alpha|^2 - \overline{B}\overline{\alpha} - B\alpha + D \end{pmatrix}.$$

The previous proposition can be verified directly. Note that it gives matrix representations of $\iota[\mathfrak{C}]$ and $\tau_\alpha[\mathfrak{C}]$ where the determinant is invariant.

2.2.2 Circles of a complex irrational

Take a complex irrational $z = [a_0; a_1, a_2, \ldots] \in \mathbb{C}$ and define the sequence of functions $(H_n)_{n\geq 1}$ by $H_n = \tau_{-a_n} \circ \iota$ for $n \in \mathbb{N}$, that is

$$\forall w \in \mathbb{C}^* \quad H_n(w) = \frac{1}{w} - a_n.$$

We also associate to z a sequence of generalized circles $(\mathfrak{C}_j)_{j\geq 0}$ given by

$$\mathfrak{C}_0 := \begin{pmatrix} A_0 & \overline{B}_0 \\ B_0 & D_0 \end{pmatrix} := \begin{pmatrix} 1 & \overline{a}_0 \\ a_0 & |a_0|^2 - |z_0|^2 \end{pmatrix},$$

$$\forall n \in \mathbb{N} \quad \mathfrak{C}_n := \begin{pmatrix} A_{n+1} & \overline{B}_{n+1} \\ B_{n+1} & D_{n+1} \end{pmatrix} := H_n[\mathfrak{C}_{n-1}].$$

Even if their definition is somehow involved, the sequence $(\mathfrak{C}_j)_{j\geq 0}$ arise in a natural way. To see it, define $(z_n)_{n\geq 1}$ by $z_0 = z - a_0$ and $z_n = T^{n-1}(z_0) = [0; a_n, a_{n+1}, \ldots]$ for $n \in \mathbb{N}$. Hence, by definition $z_1 \in \mathfrak{C}_0(-a_0, |z|)$, so $z_2 = H_1(z_1) \in H_1[\mathfrak{C}_0] = \mathfrak{C}_1$. In general, we may verify inductively that

$$\forall n \in \mathbb{N} \quad z_{n+1} = H_n(z_n) \in H_n[\mathfrak{C}_{n-1}] = \mathfrak{C}_n.$$

By Proposition 2.2.2, the computation of the determinant gives

$$\forall n \in \mathbb{N} \quad A_n D_n - |B_n|^2 = A_0 D_0 - |B_0|^2 = -|z|^2. \tag{2.5}$$

The main result of W. Bosma and D. Gruenewald is the following.

Theorem 2.2.3 (W. Bosma, D. Gruenewald, 2011)**.** *For any $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ with $|z|^2 = n \in \mathbb{N}$, the collection $\{\mathfrak{C}_j : j \geq 0\}$ is finite.*

Proof. The finiteness is obtained by showing that the sequences of Gaussian integers $(A_j)_{j\geq 0}$, $(B_j)_{j\geq 0}$, and $(C_j)_{j\geq 0}$ take only finitely many values. The conditions

$$\forall j \in \mathbb{N}_0 \quad |B_j|^2 - A_j D_j = n, \quad \mathfrak{C}_j \cap \mathfrak{F} \neq \varnothing.$$

lie in the heart of the argument. We start working with the simpler case of $\mathfrak{C}_j$ being a line and later we assume that it is a circle (in the plane).

§. $\mathfrak{C}_j$ is a line. Because of $A_j = 0$, we have $|B_j|^2 = n$ and there are only finitely many possibilities for B_j. To bound D_j, note that $\mathfrak{C}_j$ is the line given by

$$\mathfrak{R}(B_j w) = -\frac{D_j}{2}.$$

Therefore, since $\mathfrak{C}_j \cap \mathfrak{F} \neq \varnothing$ we can take $w \in \mathfrak{C}_j \cap \mathfrak{F}$ to obtain

$$\frac{|D_j|}{2} = |\mathfrak{R}(B_j w)| \leq |B_j||w| \leq \frac{\sqrt{2}}{2}|B_j| < \frac{\sqrt{2n}}{2}.$$

And conclude that there are only finitely many options for D_j.

§. $\mathfrak{C}_j$ is a circle. Let $R_j = \frac{\sqrt{|B|^2 - AD}}{|A|}$ be the radius of $\mathfrak{C}_j$. The conclusion follows from $R_j > 1/\sqrt{8}$. Indeed, in the presence of such inequality, Proposition 2.2.1 would give

$$|A| \leq \frac{|B_J|^2 - A_j D_j}{8} = \frac{n}{8}.$$

Again, by Proposition 2.2.1 and by $\mathfrak{C}_j \cap \mathfrak{F} \neq \varnothing$,

$$\left|\frac{-\overline{B_j}}{A_j}\right| \leq \frac{\sqrt{2}}{2} + R_j = \frac{\sqrt{2}}{2} + \frac{\sqrt{n}}{A_j},$$

which implies $|B_j| \leq |A_j|(\sqrt{n}+2) \leq K_1(n)$. Since A_j and B_j determine D_j and both A_j and B_j can take finitely many values, the same holds for D_j. Therefore, we could conclude the finiteness of the family of circles $\{\mathfrak{C}_j : j \in \mathbb{N}_0\}$.

We show by induction on j that $R_j > 1/\sqrt{8}$. Since $\mathfrak{C}_j$ is a line if $\mathfrak{C}_{j-1}$ goes through the origin, we consider the next three possibilities:

1. $\mathfrak{C}_{j-1}$ is a line not containing 0,
2. $\mathfrak{C}_{j-1}$ is a circle and $\operatorname{Ind}_{\mathfrak{C}_{j-1}}(0) = 1$,
3. $\mathfrak{C}_{j-1}$ is a circle and $\operatorname{Ind}_{\mathfrak{C}_{j-1}}(0) = 0$.

In the three cases we use a simple observation: the radius of $\mathfrak{C}_j$ is the same as the radius of $\iota[\mathfrak{C}_{j-1}]$. We denote by $|\mathfrak{C}_j| = 2R_j$ the diameter of $\mathfrak{C}_j$.

1. Assume that $\mathfrak{C}_{j-1}$ is a line not containing 0. Let $P \in \mathfrak{C}_{j-1}$ be the nearest point to 0, so $|P| \leq 1/\sqrt{2}$. Since $\{0, P^{-1}\} \subseteq \iota[\mathfrak{C}_{j-1}]$, we conclude that
$$2R_j = |\mathfrak{C}_j| \geq |P^{-1}| \geq \frac{1}{\sqrt{2}} \quad \Longrightarrow \quad R_j > \frac{1}{\sqrt{8}}.$$
2. Suppose that $\mathfrak{C}_{j-1}$ is a circle around 0. By part iv of Proposition 2.2.1 and the formula for inversion in Proposition 2.2.2, we get $\operatorname{Ind}_{\iota[\mathfrak{C}_j]}(0) = 1$. Also, by $z_j \in \mathfrak{C}_{j-1}$, $z_j^{-1} \in \iota[\mathfrak{C}_{j-1}]$ and, calling c the center of $\iota[\mathfrak{C}_{j-1}]$,
$$2R_j > |z_j^{-1} - c| + |c - 0| \geq |z_j^{-1}| \geq \sqrt{2} \quad \Longrightarrow \quad R_j > \frac{1}{\sqrt{8}}.$$
3. Assume that $\mathfrak{C}_{j-1}$ is a circle and $\operatorname{Ind}_{\mathfrak{C}_{j-1}}(0) = 0$. Let $P \in \mathfrak{C}_{j-1}$ be the point in $\mathfrak{C}_{j-1}$ closest to 0 and define $p := |P| < 1/\sqrt{2}$. Let Q be the antipodal point of P and write $|Q| = p + 2R_{j-1} = p + |\mathfrak{C}_{j-1}|$. Then, by the Induction Hypothesis $R_{j-1} > 1/\sqrt{8}$ and since $t \mapsto \frac{t}{p+t}$ is increasing,
$$\begin{aligned} 2R_j &= |\mathfrak{C}_j| \geq |\iota(P) - \iota(Q)| \\ &\geq \frac{1}{p} - \frac{1}{p + |\mathfrak{C}_{j-1}|} \\ &= \frac{|\mathfrak{C}_{j-1}|}{p(p+|\mathfrak{C}_{j-1}|)} = \frac{2R_{j-1}}{p(p+2R_{j-1})} \\ &> \frac{1/\sqrt{2}}{p(p+1/\sqrt{2})} > \frac{p}{p(p+1/\sqrt{2})} > \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

so $R_j > 1/\sqrt{8}$.

$\square$

Corollary 2.2.4 (W. Bosma, D. Gruenewald, 2011)**.** *If $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ satisfies $|z|^2 = n \in \mathbb{N}$ and $x^2 + y^2 = n$ does not have integer solutions, then z has a bounded HCF.*

Proof. None of the circles $\mathfrak{C}_j$ passes through the origin, because otherwise we would have $|a_j|^2 = |z_0|^2$ for $a_j \in \mathbb{Z}[i]$, which is impossible. Therefore,

$$\inf_{j \in \mathbb{N}_0} \min_j d(0, \mathfrak{C}_j) > 0,$$

and $(|z_n^{-1}|)_{n\geq 0}$ stays bounded away from 0. This implies that the numbers $z_n^{-1} = [a_n; a_{n+1}, \ldots]$ are bounded, so $(a_j)_{j=0}^{\infty}$ is bounded too. $\square$

By virtue of Corollary 2.2.4, we can construct infinitely many algebraic numbers in $\mathbf{Bad}_{\mathbb{C}}$ of degree greater than 2. We need to recall a result on elementary number theory. Namely, that for a given $n \in \mathbb{N}$ the equation $n = x^2 + y^2$ has integer solutions if and only if in the prime factorization of n

$$n = 2^{\gamma} \left(\prod_{p \equiv 1 (4)} p^{\alpha_p} \right) \left(\prod_{q \equiv 3 (4)} q^{\beta_q} \right).$$

all the exponents β_q are even (Theorem 2.15. in [NMZ], p.55). For example, the number $w = \sqrt{3} + i\sqrt{8}$ is algebraic of degree 4 over $\mathbb{Q}(i)$ and, since $|w|^2 = 11$, its HCF is bounded. More generally, let $n \in \mathbb{N}$ be a prime number such that $n \equiv 3 \pmod 4$. For any $m \in \mathbb{N}$ the numbers $\zeta_m = \sqrt[m]{2} + i\sqrt{n - \sqrt[m]{4}}$ have bounded HCF elements and satisfy $[\mathbb{Q}(i, \zeta), \mathbb{Q}(i)] = 2m$.

2.3 Transcendental numbers

Corollary 2.2.4, makes it particularly easy to construct transcendental numbers with bounded HCF. For instance, just take any transcendental $\zeta \in [0, 1]$ and consider $\sqrt{\zeta} + i\sqrt{7 - \zeta}$. In this section, we give another way to construct infinitely many transcendental complex numbers by imposing some combinatorial properties on their HCF expansion.

2.3.1 Combinatorics on words

Let us recall some terminology. An **alphabet** $\mathcal{A}$ is a non-empty set. A **word** in an alphabet $\mathcal{A}$ is a sequence, finite or infinite, taking values on $\mathcal{A}$. We denote the set of finite words on $\mathcal{A}$ by $\mathcal{A}^*$ and the set of infinite words (to the right) on $\mathcal{A}$ by $\mathcal{A}^\omega$. Take $\mathbf{a} \in \mathcal{A}^*$ and $\mathbf{b} \in \mathcal{A}^* \cup \mathcal{A}^\omega$, we say that $\mathbf{a} = a_1 \dots a_n$ is a **sub-word** of $\mathbf{b} = b_1 b_2 \dots$, and denote it $\mathbf{a} \leq \mathbf{b}$, if $b_{i+1} \dots b_{i+n} = a_1 \dots a_n$ for some $i \in \mathbb{N}$. The **length** of a word $\mathbf{a} = a_1 \dots a_n$ is the number of terms it comprises, we denote it by $|\mathbf{a}| = n$. For $\mathbf{a} \in \mathcal{A}^\omega$ and $m, n \in \mathbb{N}$, $m \leq n$, we write $\mathbf{a}_{[m..n]} = a_m a_{m+1} \dots a_n$.

Let $\mathbf{a} \in \mathcal{A}^\omega$ be an infinite word over a finite $\mathcal{A}$. A way to measure how sophisticated is the structure of $\mathbf{a}$ is through the **complexity function**, $p_{\mathbf{a}}(\cdot, \mathcal{A}) : \mathbb{N} \to \mathbb{N}$,

$$\forall n \in \mathbb{N} \quad p_{\mathbf{a}}(n, \mathcal{A}) = \# \left\{ \mathbf{x} \in \mathcal{A}^* : |\mathbf{x}| = n, \quad \mathbf{x} \leq \mathbf{a} \right\}.$$

Y. Bugeaud and D. H. Kim defined in [BuKi15] another useful device: the repetition exponent. Broadly speaking, it tells us how long do we have to wait in average to see the first repetition of any sub-word of length n in $\mathbf{a}$.

Definition 2.3.1. *Let $\mathbf{x} = x_1 x_2 x_3 \dots \in \mathcal{A}^\omega$. For every positive integer n consider*

$$r(n, \mathbf{x}) = \min\{ m \in \mathbb{N} : \exists i \in [1..m-n] \quad \mathbf{x}_{[i..i+n-1]} = \mathbf{x}_{[m-n+1..m]} \}.$$

*The **repetition exponent** of* $\mathbf{x}$*,* $\operatorname{rep} \mathbf{x}$*, is*

$$\operatorname{rep} \mathbf{x} = \liminf_{n \to \infty} \frac{r(n, \mathbf{x})}{n}.$$

Theorem 2.3.1 (Y. Bugeaud, 2011)**.** *Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a non-periodic sequence of natural numbers such that*

$$\operatorname{rep} \mathbf{a} < +\infty.$$

Then, $[0; a_1, a_2, a_3, \dots]$ is transcendental.

Let $\mathbf{a}$ be a sequence of natural numbers taking finitely many values. Unravelling the definition of the repetition exponent, we can show that

$$\forall n \in \mathbb{N} \quad r(n, \mathbf{a}) \leq p(n, \mathbf{a}) + n. \tag{2.6}$$

Therefore, if we had $p(n, \mathbf{a}) \ll_{\mathbf{a}} n$, the number $[0; a_1, a_2, a_3, \dots]$ would be either a quadratic irrational or trascendental. An important class of sequences whose complexity function has an at most linear growth is the family of automatic sequences (Corollary 10.3.2, p. 304 in [AlSh03]). Thus, Theorem 2.3.1 tells us that automatic regular continued fractions are either quadratic or transcendental.

2.3.2 Complex Transcendental Numbers

Let $\mathcal{A} \neq \varnothing$ be a finite alphabet and $\mathbf{a}$ an infinite word over $\mathcal{A}$. As noted in [BuKi15], the statement $\operatorname{rep}(\mathbf{a}) < +\infty$ is equivalent to the existence of three sequences of finite words in $\mathcal{A}$, $(W_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$, and $(V_n)_{n\geq 1}$, such that

i. For every n the word $W_n U_n V_n U_n$ is a prefix of $\mathbf{a}$,

ii. The sequence $(|W_n| + |V_n|)/|U_n|$ is bounded above,

iii. The sequence $(|U_n|)_{n\geq 1}$ is strictly increasing.

The behaviour of $(W_n)_{n\geq 1}$ prompts two different cases:

$$\liminf_{n\to\infty} |W_n| < +\infty \quad \vee \quad \liminf_{n\to\infty} |W_n| = +\infty.$$

In the first one, we can obtain satisfactory approximations of $[0; a_1, a_2, \ldots]$ by purely periodic HCF. In the second case, we can approximate by eventually periodic HCF.

The following result corresponds to Theorem 2.3.1 in the HCF context. Unfortunately, the slight strengthening of the hypothesis in the second point leads to a much less elegant statement.

Theorem 2.3.2. *Let $\mathbf{a} = (a_j)_{j\geq 1} \in \Omega^{\mathrm{HCF}}$ be non-periodic and such that*

$$\operatorname{rep} \mathbf{a} < +\infty.$$

Call $\zeta = [0; a_1, a_2, a_3, \ldots]$ and let $(W_n)_{n\geq 0}, (U_n)_{n\geq 0}, (V_n)_{n\geq 0}$ be as above.

i. If $\liminf_{n\to\infty} |W_n| < +\infty$, then ζ is transcendental.

ii. If $\liminf_{n\to\infty} |W_n| = +\infty$ and $|a_n| \geq \sqrt{8}$ for every $n \in \mathbb{N}$, then ζ is transcendental.

The lengthy proof of Theorem 2.3.2, found in [Gon18-01], follows the original strategy used on Theorem 2.3.1. Accordingly, a fundamental tool is a suitable version of Schmidt's Subspace Theorem. Wolfgang Schmidt showed in [Sch76] his Subspace Theorem for number fields. It reads as follows when restricted to $\mathbb{Q}(i)$.

Theorem 2.3.3 (Schmidt's Subspace Theorem for Number Fields)**.** *Let $\mathscr{L}_1, \ldots, \mathscr{L}_m$ and $\mathscr{M}_1, \ldots, \mathscr{M}_m$ be two sets of m linearly independent linear forms with algebraic coefficients.*

For any $\varepsilon > 0$ there exists $T_1, \ldots, T_k$ proper subspaces of $\mathbb{Q}(i)^m$ such that for every $\boldsymbol{\beta} = (b_1, \ldots, b_k) \in \mathbb{Z}[i]$, $\boldsymbol{\beta} \neq \mathbf{0}$,

$$\left|\prod_{j=1}^{m} \mathscr{L}_j(\boldsymbol{\beta})\right| \left|\prod_{j=1}^{m} \mathscr{M}_j(\overline{\boldsymbol{\beta}})\right| \leq \frac{1}{\|\boldsymbol{\beta}\|_\infty^\varepsilon} \quad \Longrightarrow \quad \boldsymbol{\beta} \in \bigcup_{j=1}^{k} T_j,$$

where $\|\boldsymbol{\beta}\|_\infty = \max\{|b_1|, \ldots, |b_k|\}$, and $\overline{\boldsymbol{\beta}} = (\overline{b_1}, \ldots, \overline{b_m})$.

Examples of Complex Transcendental Numbers

In order to use Theorem 2.3.2, we recall some notions related to k-uniform morphisms. Let $\mathcal{A}$ be an alphabet. The set $\mathcal{A}^*$ along with the concatenation operation has a monoid structure, it is called the **free monoid** over $\mathcal{A}$. By a **morphism** mean a monoid morphism $\varphi : \mathcal{A}^* \to \mathcal{B}^*$, where $\mathcal{B}$ is another alphabet. Clearly, such an φ is determined by its action on each $a \in \mathcal{A}$. For any $k \in \mathbb{N}$, φ is k**-uniform** if $|\varphi(a)| = k$ for all $a \in \mathcal{A}$, a **coding** is a 1-uniform morphism. A morphism is **uniform** if it is k-uniform for some k. A k-uniform morphism $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ is **prolongable** on $a \in \mathcal{A}$ if $\varphi(a) = ax$ for some $x \in \mathcal{A}^*$, and the **fixed point** of φ based at a is the infinite word

$$ax\varphi(x)\varphi^2(x)\varphi^3(x)\varphi^4(x)\ldots,$$

where, naturally, $\varphi^1 = \varphi$ and $\varphi^{n+1} = \varphi \circ \varphi^n$.

We can give right away two famous examples. Set $\mathcal{A} = \{\mathtt{a}, \mathtt{b}\}$, where $\mathtt{a}$, $\mathtt{b}$ are two different symbols. Let $\mu : \mathcal{A}^* \to \mathcal{A}^*$ be the morphism determined by $\mu(\mathtt{a}) = \mathtt{ab}$, $\mu(\mathtt{b}) = \mathtt{ba}$. The **Thue-Morse** sequence on $\mathcal{A}$, $\mathbf{t}$, is the fixed point (based on $\mathtt{a}$) of μ:

$$\mathbf{t} = \mathtt{abbabaabbaababba}\ldots.$$

Now let $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ be the morphism determined by $\varphi(\mathtt{a}) = \mathtt{aab}$, $\varphi(\mathtt{b}) = \mathtt{bba}$. The **Mephisto Waltz** sequence $\mathcal{A}$, $\mathbf{s}$, is the fixed point (based on $\mathtt{a}$) of φ:

$$\mathbf{s} = \mathtt{aabaabbbaaabaabbba}\ldots.$$

The following proposition is an immediate conclusion— almost a mere restatement— of well-known results by A. Cobham (see Theorem 6.8.2, p.190, Theorem 6.3.2, p. 175, and Corollary 10.3.2, p.304, in [AlSh03]).

Proposition 2.3.4 (A. Cobham, 1972)**.** *Let $\mathcal{A}$ be a finite alphabet and $\mathbf{a} \in \mathcal{A}^*$. Assume that one of the following the conditions hold:*

1. *If $\mathbf{a}$ is the fixed point of a uniform morphism,*
2. *There exist a pair of positive integers k, m such that for all $r \in [0..m-1]$ the sequence $(a_{nm+r})_{n\geq 0}$ is the image under a coding of the fixed point of some k-uniform morphism.*

Then, $\operatorname{rep}(\mathbf{a}) < +\infty$*.*

Regular transcendental numbers. Let $a, b \in \mathbb{Z}[i]$, $a \neq b$, be such that $|a|, |b| \geq \sqrt{5}$ and let $\mathbf{t}$ be the Thue-Morse sequence on $\{a, b\}$ starting at a. Since $\mathbf{t}$ is not periodic (cfr. Theorem 1.6.1., p.15., in [AlSh03]), Theorem 2.3.2 tells us that $\zeta = [0; t_1, t_2, t_3, \ldots]$ is transcendental.

Irregular transcendental numbers. Unlike in the regular case, we must be careful while constructing an irregular sequence $\mathbf{a}$ satisfying $\operatorname{rep}(\mathbf{a}) < +\infty$. Not every fixed point of a k-uniform morphism over a suitable alphabet will do the trick (or, more generally, not any k-automatic sequence will work). We must keep in mind the following observation.

Proposition. *Let $(a_n)_{n\geq 1} \in \Omega_I^{\mathrm{HCF}}$, then $\liminf_n |a_n| \leq 2$. In fact, one of the following two inequalities must hold*

$$\limsup_{n\to\infty} |a_{2n}| \leq 2, \quad \vee \quad \limsup_{n\to\infty} |a_{2n+1}| \leq 2.$$

Take two different rational integers m, n with $|m|, |n| \geq 3$ and let $\mathbf{s} = (s_j)_{j\geq 1}$ be the corresponding Mephisto Waltz sequence. Define the word $\mathbf{a} = a_1 a_2 a_3 \ldots$ by

$$\forall n \in \mathbb{N} \quad a_{2n-1} = -2, \quad a_{2n} = s_n.$$

Since $\mathbf{s}$ is not periodic (cfr. Exercise 16, p.25, in [AlSh03]), $\mathbf{a}$ is not periodic either. Moreover, the sequences $(W_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ as in Theorem 2.3.2 can be taken to be $W_n = V_n = \epsilon$ for all n. Hence, $\mathbf{a} \in \Omega_I^{\mathrm{HCF}}$ and by Proposition 2.3.2 and Theorem 2.3.2, $[0; a_1, a_2, a_3, \ldots]$ is transcendental.

2.4 Further results

Other transcendence results can be translated into our context with the pertinent modifications. An example is Theorem 2.4.1 (cfr. [Bu15], Theorem 11.2).

We say that an infinite word $\mathbf{a} \in \mathcal{A}$ satisfies condition (♣) if there are sequences $(W_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ of finite words in $\mathcal{A}$ such that

1. $W_n U_n V_n \widehat{U_n}$ is a prefix of $\mathbf{a}$ for every $n \in \mathbb{N}$, where $\widehat{U_n}$ is the word obtained by reversing U_n,

2. $((|W_n| + |V_n|)/|U_n|)_{n\geq 1}$ is bounded,

3. $(|U_n|)_{n\geq 1}$ tends to infinity when n does.

Theorem 2.4.1. *Let $\mathbf{a} \in \Omega^{\mathrm{HCF}}$ satisfy Condition (♣) and $|a_n| \geq \sqrt{5}$ for every n. If $\mathbf{a}$ is not periodic, then $\zeta = [0; a_1, a_2, \ldots]$ is transcendental.*

2.5 Notes and Comments

1. The restriction $|a_n| \geq \sqrt{5}$ in Theorems 2.3.2 and 2.4.1 are needed for the same reason. In $\mathbb{R}$, if the regular continued fractions of α and β are bounded, say by M, then $|\alpha - \beta| \asymp_{m,M} 1$ where m is the length of the longest prefix of α which is also a prefix of β. This is no longer true for HCF in general. However, if $F = \bigcup_a \mathcal{C}_1(a)$, where a runs along the set $\{w \in \mathbb{Z}[i] : |w| \geq \sqrt{5}\}$. There is some positive $\varepsilon > 0$ such that $d(b + F, c + F) > \varepsilon$ for any $b, c \in \mathbb{Z}[i]$ (see Figure 1.1, Chapter 1.).

2. We avoided a discussion on k-automatic sequences, because it is far simpler to introduce the notion of fixed points of uniform morphisms. However, the following is an immediate consequence of Theorem 2.3.2.

   Corollary. *Let $\mathcal{A} \subseteq \mathbb{Z}[i]$ be a finite set such that* $\min\{|a| : a \in A\} \geq \sqrt{5}$*. Then, any k-automatic sequence $(a_n)_{n\geq 1}$ over $\mathcal{A}$ is the HCF of a transcendental complex number.*

   Our omission is irrelevant in view of Cobham's characterization of automatic sequences as the image under codings of fixed points of uniform morphisms.

Chapter 3

Ergodic Theory of Hurwitz Continued Fractions

In this chapter, we present basic results on the ergodic theory of the dynamical system $(\mathfrak{F}, T)$. In particular, we show the existence of a measure μ equivalent to the Lebesgue measure making $(\mathfrak{F}, T, \mu)$ ergodic. We mention some consequences of Birkhoff's Ergodic Theorem including a formula for estimating the entropy of $(\mathfrak{F}, T, \mu)$, and an application on binary forms. We finish the chapter by mentioning two general frameworks containing HCF. One of them has been treated by F. Schweiger, H. Nakada, among others, and in a slightly modified version by A. Berechet. The second one is treated in a recent paper by A. Lukyanenko and J. Vandehey.

Notation. Whenever the Vinogradov symbol $\ll$ and in $\asymp$ appears without any parameters, the implied constants are absolute. The σ-algebra of Borel subsets of $\mathfrak{F}$ (resp. $\mathfrak{F}_n(\mathbf{a})$) is denoted by $\mathfrak{B}$ (resp. $\mathfrak{B}_{\mathfrak{F}_n(\mathbf{a})}$). The n-th term of the HCF of a number z regarded as a function of z is denoted $a_n(z)$ and similarly for $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$. If Ω is a set of sequences in $\mathbb{Z}[i]$ and $m \in \mathbb{N}$,

$$\Omega(m) := \left\{ (a_j)_{j=1}^m \in \mathbb{Z}[i]^m : \exists (b_j)_{j\geq 1} \in \Omega \quad b_1 = a_1, \ldots, b_n = a_n \right\}.$$

The Lebesgue measure on $\mathfrak{F}$ is $\mathfrak{m}$. If $(X, \mathfrak{M}, \mu)$ is a measure space, $\forall_\mu \, x \in X$ means for μ-almost all $x \in X$.

Main references. For general ergodic theory see [CFS82]. For the ergodic theory of HCF and larger settings including them see [Na76], [Sc00b], [Be01], and [NaNa03]. For Iwasawa Continued fractions see [LuVa18].

3.1 Geometric observations

We start by recalling a classical measure theoretic result. Its proof can be found in almost any measure theory book, e.g. [Bo07], Vol. 1., Theorem 3.7.1., p. 194.

Theorem 3.1.1 (Change of Variable). *Let $U \subseteq \mathbb{R}^n$ be open and let $\phi : U \to \mathbb{R}^n$ be differentiable with derivative $D\phi$. If ϕ is injective, then for any measurable $A \subseteq U$ and any Borel function $f \in L^1(\mathbb{R}^n)$ we have*

$$\int_{\phi[A]} f(x)\mathrm{d}x = \int_A (f \circ \phi)(t)\,|\det D\phi(t)|\mathrm{d}t.$$

The present and the next chapter rely heavily on a geometrical feature of the construction; loosely stated: cylinders look a lot like maximal feasible regions. This is rather unsurprising considering that T^n restricted to any regular cylinder $\mathcal{C}_n(\mathbf{a})$ is bi-holomorphic. Let us elaborate in this aspect.

We adopt the following notation:

$$\forall \mathbf{a} \in \Omega_R^{\mathrm{HCF}} \qquad \forall n \in \mathbb{N} \quad \mathfrak{F}_n(\mathbf{a}) := T^n[\mathcal{C}_n(\mathbf{a})],$$

and extend it in the obvious way to finite sequences.

Take $n \in \mathbb{N}$ and $\mathbf{a} \in \Omega_R^{\mathrm{HCF}}$ with $\mathcal{Q}$-pair $((p_m)_{m\geq 0}, (q_m)_{m\geq 0})$. The function $T^n : \mathcal{C}_n(\mathbf{a}) \to \mathfrak{F}_n(\mathbf{a})$ is bijective, its inverse $t_{\mathbf{a},n}$ is given by

$$\forall z \in \mathfrak{F}_n(\mathbf{a}) \quad t_{\mathbf{a},n}(z) = \frac{p_{n-1}z + p_n}{q_{n-1}z + q_n} = \frac{p_{n-1}}{q_{n-1}} + \frac{(-1)^n}{q_{n-1}(q_{n-1}z + q_n)}.$$

Since $|q_n| < |q_{n+1}|$, $t_{\mathbf{a},n}$ is holomorphic on its domain and

$$\forall z \in \mathfrak{F}_n(\mathbf{a}) \quad t'_{\mathbf{a},n}(z) = \frac{(-1)^{n+1}}{(q_{n-1}z + q_n)^2}.$$

Recall that if $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic, then it is real differentiable and $|Df(x,y)| = |f'(x+iy)|^2$ for every $z = x + iy \in U$ (see [Lan99], p.33). In particular, for $t_{\mathbf{a},n}$ we have

$$\forall z = x + iy \in \mathfrak{F}_n(\mathbf{a}) \qquad |\det Dt_{\mathbf{a},n}(x,y)| = \frac{1}{|q_{n-1}z + q_n|^4} = \frac{1}{|q_n|^4|\frac{q_{n-1}}{q_n}z + 1|^4}.$$

As a consequence, we have the bounds

$$\forall z \in \mathfrak{F}_n(\mathbf{a}) \qquad 2^{-4}\frac{1}{|q_n|^4} \leq |\det Dt_{\mathbf{a},n}(x,y)| \leq \left(1 - \frac{\sqrt{2}}{2}\right)^{-1}\frac{1}{|q_n|^4};$$

or more briefly,

$$|\det Dt_{\mathbf{a},n}| \asymp \frac{1}{|q_n(\mathbf{a})|^4}. \tag{3.1}$$

The previous expression is known as a Rényi condition.

Since there are only finitely many regular maximal feasible regions (thirteen if we discard null sets), the Theorem of Change of Variable and (3.1).

$$\forall B \in \mathfrak{B}_{\mathfrak{F}_n(\mathbf{a})} \quad \mathfrak{m}(t_{\mathbf{a},n}[B \cap \mathfrak{F}_n(\mathbf{a})]) \asymp \frac{\mathfrak{m}(B \cap \mathfrak{F}_n(\mathbf{a}))}{|q_n|^4}. \tag{3.2}$$

A regular cylinder $\mathcal{C}_n(\mathbf{a})$ is **full** if $\mathfrak{F}_n(\mathbf{a}) = \mathfrak{F}$. Computing the proportion of $\mathfrak{F}$ that is occupied by full cylinders of depth 1 (see Figure 1.1 in Chapter I) is an exercise in Riemann integration. There are no additional complications if we replace $\mathfrak{F}$ by any regular maximal feasible region. Since the restrictions of T^n to regular cylinders do not entail too much distortion (cfr. (3.1)), it is reasonable to believe that the proportion of a regular cylinder of depth n occupied by full cylinders of depth $n+k$ can be estimated. This phenomenon is precisely stated in Lemma 3.1.2.

We remind the reader that $I = \mathbb{Z}[i] \smallsetminus \{0, 1, i, -1, -i\}$ and write

$$\forall n \in \mathbb{N} \quad \mathsf{A}(n)^\circ = \{\mathbf{a} \in I^n : T^n[\mathcal{C}_n(\mathbf{a})] = \mathfrak{F}\}.$$

Given $E \subseteq \mathfrak{F}$, $\mathbf{a} \in \Omega_R^{\mathrm{HCF}}$ and $m, k \in \mathbb{N}$, set

$$\Upsilon(E, m) := \{\mathbf{b} \in I^m : T^m[\mathcal{C}_m(\mathbf{b}) \cap E] = \mathfrak{F}\},$$
$$\Upsilon(\mathbf{a}, k, m) := \{\mathbf{b} \in I^m : a_1 \ldots a_k \mathbf{b} \in \mathsf{A}(m+k)^\circ\}.$$

Lemma 3.1.2. *For all $m, k \in \mathbb{N}$ and any $\mathbf{a} \in \Omega_R^{\mathrm{HCF}}(k)$*

$$\sum_{\mathbf{b} \in \Upsilon(\mathbf{a},k,m)} \mathfrak{m}\left(\mathcal{C}_{m+k}(\mathbf{ab})\right) \gg \mathfrak{m}\left(\mathcal{C}_k(\mathbf{a})\right). \tag{3.3}$$

Proof. Let $\mathfrak{F}' \neq \varnothing$ be the interior of a regular maximal feasible set, the above discussion gives

$$\sum_{b \in \Upsilon(\mathfrak{F}',1)} \mathfrak{m}\left(\mathcal{C}_1(b)\right) \gg \mathfrak{m}(\mathfrak{F}'). \tag{3.4}$$

For any $\mathbf{a} \in \Omega_R^{\mathrm{HCF}}$ with $\mathcal{Q}$-pair $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ and $k \in \mathbb{N}$, the estimate (3.2) translates into

$$\mathfrak{m}\left(\mathcal{C}_k(\mathbf{a})\right) \asymp \frac{\mathfrak{m}\left(\mathfrak{F}_k(\mathbf{a})\right)}{|q_k|^4}, \qquad \forall b \in \Upsilon(\mathbf{a}, k, 1) \quad \mathfrak{m}\left(\mathcal{C}_{k+1}(\mathbf{a}b)\right) \asymp \frac{\mathfrak{m}\left(\mathcal{C}_1(b)\right)}{|q_k|^4}.$$

These bounds along with (3.4) yield

$$\sum_{\mathbf{b} \in \Upsilon(\mathbf{a},k,1)} \mathfrak{m}\left(\mathcal{C}_{k+1}(\mathbf{a}b)\right) \gg \mathfrak{m}\left(\mathcal{C}_k(\mathbf{a})\right). \tag{3.5}$$

Take $m \in \mathbb{N}$, $m \geq 2$, and put $B = \{\mathbf{b}' \in I^{m-1} : \mathcal{C}_{m-1}(\mathbf{b}') \cap \mathfrak{F}' \neq \varnothing\}$, then

$$\begin{aligned}\sum_{\mathbf{b} \in \Upsilon(\mathfrak{F}',m)} \mathfrak{m}\left(\mathcal{C}_m(\mathbf{b})\right) &= \sum_{\mathbf{b}' \in B} \sum_{b \in \Upsilon(\mathbf{b}',m-1,1)} \mathfrak{m}\left(\mathcal{C}_m(\mathbf{b}'b)\right) \\ &\gg \sum_{\mathbf{b}' \in B} \mathfrak{m}\left(\mathcal{C}_{m-1}(\mathbf{b}')\right) \geq \mathfrak{m}(\mathfrak{F}'). \end{aligned} \tag{3.6}$$

When $\mathbf{a} \in \Omega_R^{\mathrm{HCF}}$, $m, k \in \mathbb{N}$ are given, we can obtain (3.3) from (3.6) just as we got (3.5) from (3.4). □

3.2 An ergodic measure

We devote this section to construct an analogue of the Gauss measure for regular continued fractions. Unfortunately, the proof says nothing about the exact form of its density.

Recall that given a measure space $(X, \mathfrak{S}, \lambda)$ and a measurable map $\tau : X \to X$, we say that λ is **τ-invariant** if $\lambda(E) = \lambda(\tau^{-1}[E])$ for all $E \in \mathfrak{S}$. The measure λ is **τ-irreducible** (or just **irreducible** if τ has been fixed) if $\tau^{-1}[E] = E$ yields $\lambda(E) = 0$ or $\lambda(E) = 1$ for any measurable E. We say that λ is **ergodic** (with respect to τ) if it is τ-invariant and τ-irreducible. We say that τ is **non-singular** if $\lambda(\tau^{-1}[E]) = 0$ implies $\lambda(E) = 0$ for all $E \in \mathfrak{S}$.

Theorem 3.2.1 (H. Nakada, 1976). *There is a unique Borel measure μ which is equivalent to $\mathfrak{m}$ and such that the system $(\mathfrak{F}, T, \mu)$ is ergodic.*

We will require two previous results. The first one— due to N. Dunford and D.S. Miller— is shown in [DuMi46] as Theorem 1. The proof of the second one can be found in V. Bogachev's measure theory book ([Bo07], Vol. 1., Theorem 4.6.3., p. 275).

Lemma 3.2.2 (N. Dunford, Miller 1946). *Let $(X, \mathfrak{S}, \nu)$ be a measure space such that $\nu(X) < +\infty$ and let $\varphi : X \to X$ be non-singular. If there is some $M > 0$ such that*

$$\forall A \in \mathfrak{S} \quad \forall n \in \mathbb{N} \quad \frac{1}{n} \sum_{r=0}^{n-1} \nu(\varphi^r[A]) \leq M \nu(A),$$

then, for all $f \in L_1(X, \nu)$ the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} f(\varphi^r t).$$

exists ν-almost everywhere in X.

Lemma 3.2.3. *Let $(X, \mathfrak{S})$ be a measurable space and $(\mu_n)_{n\geq 1}$ a sequence of measures on X such that the following limits exist*

$$\forall E \in \mathfrak{S} \quad \mu(E) := \lim_{n\to\infty} \mu_n(E).$$

If $\mu(X) < +\infty$, then μ is a measure on $(X, \mathfrak{S})$.

Proof of Theorem 3.2.1. To construct μ, we first show that for some absolute constants

$$\forall k \in \mathbb{N} \quad \forall E \in \mathfrak{B} \quad \mathfrak{m}(E) \asymp \mathfrak{m}\left(T^{-k}[E]\right). \tag{3.7}$$

Lemma 3.2.2 applied to the functions $\{\chi_E\}_{E\in\mathfrak{B}}$ along with Lemma 3.2.3 tell us that we can define a T-invariant measure μ via

$$\forall E \in \mathfrak{B} \quad \mu(E) := \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathfrak{m}(T^{-j}[E]). \tag{3.8}$$

By (3.7), μ and $\mathfrak{m}$ are equivalent and the ergodicity of μ would follow from the irreducibility of $\mathfrak{m}$. We conclude that $\mathfrak{m}$ is irreducible by showing that any measurable E with $\mathfrak{m}(E) > 0$ satisfies

$$T^{-1}[E] = E \quad \Longrightarrow \quad \forall F \in \mathfrak{B} \quad \mathfrak{m}(E \cap F) \gg \mathfrak{m}(E)\, \mathfrak{m}(F). \tag{3.9}$$

Taking $F = \mathfrak{F} \setminus E$ we conclude $\mathfrak{m}(\mathfrak{F} \setminus E) = 0$. The uniqueness of μ follows from Birkhoff's Ergodic Theorem (see Appendix A).

Proving Theorem 3.2.1 amounts now to show (3.7) and (3.9).

Proof of (3.7). For every $k \in \mathbb{N}$ define

$$c_k := \min\left\{\mathfrak{m}(\mathfrak{F}_k(\mathbf{a})) : k \in \mathbb{N},\ \mathbf{a} \in \Omega_R^{\mathrm{HCF}}(k)\right\}.$$

Since $\mathfrak{m}\left(\mathcal{C}_k(\mathbf{a})\right) \asymp \mathfrak{m}\mathfrak{F}_k(\mathbf{a})|q_k(\mathbf{a})|^{-4}$ for any $\mathbf{a} \in I^{\mathbb{N}}$,

$$\sum_{\mathbf{a}\in\Omega_R^{\mathrm{HCF}}(k)} \frac{c_k}{|q_n(\mathbf{a})|^4} \ll \sum_{\mathbf{a}\in\Omega_R^{\mathrm{HCF}}(k)} \mathfrak{m}\left(\mathcal{C}_k(\mathbf{a})\right) = \mathfrak{m}\mathfrak{F} = 1$$

By the Theorem of Change of Variable, for all $\mathbf{a} \in \Omega_R^{\mathrm{HCF}}$ and any k

$$\begin{aligned} \mathfrak{m}(T^{-k}[E] \cap \mathcal{C}_k(\mathbf{a})) &= \mathfrak{m}(t_{\mathbf{a},k}[E \cap \mathfrak{F}_k(\mathbf{a})]) \\ &= \int_{\mathfrak{F}_k(\mathbf{a})} \chi_E\, |\det Dt_{\mathbf{a},k}| \mathrm{d}\mathfrak{m} \\ &\asymp \mathfrak{m}(E \cap \mathfrak{F}_k(\mathbf{a}))\, \mathfrak{m}\left(\mathcal{C}_k(\mathbf{a})\right). \end{aligned} \tag{3.10}$$

In particular, whenever $\mathbf{a} \in \mathsf{A}(k)^\circ$,

$$\mathfrak{m}(T^{-k}[E] \cap \mathcal{C}_k(\mathbf{a})) \asymp \mathfrak{m}(E)\, \mathfrak{m}\left(\mathcal{C}_k(\mathbf{a})\right). \tag{3.11}$$

In view of Lemma 3.1.2, (3.11) yields

$$\begin{aligned} \mathfrak{m}\left(T^{-k}[E]\right) &= \sum_{\mathbf{a}\in I^k} \mathfrak{m}(T^{-k}[E] \cap \mathcal{C}_k(\mathbf{a})) \\ &\geq \sum_{\mathbf{a}\in\mathsf{A}(k)^\circ} \mathfrak{m}(T^{-k}[E] \cap \mathcal{C}_k(\mathbf{a})) \asymp \mathfrak{m}(E) \sum_{\mathbf{a}\in\mathsf{A}(k)^\circ} \mathfrak{m}\left(\mathcal{C}_k(\mathbf{a})\right) \gg \mathfrak{m}(E). \end{aligned}$$

The other inequality also follows directly from (3.10):

$$\begin{aligned} \forall k \in \mathbb{N} \quad \mathfrak{m}\left(T^{-k}[E]\right) &= \sum_{\mathbf{a}} \mathfrak{m}(T^{-k}[E] \cap \mathcal{C}_k(\mathbf{a})) \\ &\asymp \sum_{\mathbf{a}} \mathfrak{m}(E \cap \mathfrak{F}(\mathbf{a}))\mathfrak{m}\left(\mathcal{C}_k(\mathbf{a})\right) \leq \mathfrak{m}(E), \end{aligned}$$

where $\mathbf{a}$ runs along $\Omega_R^{\mathrm{HCF}}(k)$.

Proof of (3.9) Since the family of cylinders generate the Borel σ-algebra, it suffices to establish (3.9) for cylinders. Thus, we must show that for a given set $E \in \mathfrak{B}$ such that $T^{-1}[E] = E$ and $\mathfrak{m}E > 0$ the following property holds:

$$\forall \mathbf{a} \in \Omega_R^{\mathrm{HCF}} \quad \forall m \in \mathbb{N} \qquad \mathfrak{m}(E \cap \mathcal{C}_m(\mathbf{a})) \gg \mathfrak{m}(E)\, \mathfrak{m}\left(\mathcal{C}_m(\mathbf{a})\right). \tag{3.12}$$

Take $\mathbf{a} \in \Omega_R^{\mathrm{HCF}}(m)$ and $m \in \mathbb{N}$. By the Theorem of Change of Variable and noting that the T-invariance of E yields $\chi_E \circ t_{\mathbf{c},k} = \chi_E$ for any $\mathbf{c} \in \mathsf{A}(k)^\circ$ we have

$$\begin{aligned} \mathfrak{m}(E \cap \mathcal{C}_m(\mathbf{a})) &= \sum_{b\in I} \int_{\mathcal{C}_{m+1}(\mathbf{a}b)} \chi_E \,\mathrm{d}\mathfrak{m} \\ &\geq \sum_{b\in\Upsilon(\mathbf{a},1)} \int_{\mathcal{C}_{m+1}(\mathbf{a}b)} \chi_E \,\mathrm{d}\mathfrak{m} \\ &= \sum_{b\in\Upsilon(\mathbf{a},1)} \int_{\mathfrak{F}} (\chi_E \circ t_{\mathbf{a},b})\, |\det Dt_{\mathbf{a},b}| \mathrm{d}\mathfrak{m} \\ &= \sum_{b\in\Upsilon(\mathbf{a},1)} \int_{E} |\det Dt_{\mathbf{a},b}| \mathrm{d}\mathfrak{m} \\ &\gg \sum_{b\in\Upsilon(\mathbf{a},1)} \frac{\mathfrak{m}(E)}{|q_{m+1}(\mathbf{a}b)|^4} \\ &\gg \mathfrak{m}(E) \sum_{b\in\Upsilon(\mathbf{a},1)} \mathfrak{m}\left(\mathcal{C}_{m+1}(\mathbf{a}b)\right) \gg \mathfrak{m}(E)\, \mathfrak{m}(\mathcal{C}_m(\mathbf{a})). \end{aligned}$$

□

It is well known that the density function of the real Gauss measure is given by $f:[0,1]\to\mathbb{R}$, $f(x)=(2\log x)^{-1}$. As today, it is still unknown if the density function of μ with respect to $\mathfrak{m}$ has a simple explicit form. However, using elements of thermodynamic formalism, Doug Hensley could show that ρ is real analytic almost everywhere on $\mathfrak{F}$.

Theorem 3.2.4 (D. Hensley, 2006). *Let μ be the measure from Theorem 3.2.1. The density of μ, h, is real analytic in the twelve regions determined by the boundaries of the regular maximal feasible regions (see Figure 3.1). Moreover, the symmetries $\mu(iA)=\mu(A)$ and $\mu(\overline{A})=\mu(A)$ hold for every Borel set $A\subseteq\mathfrak{F}$.*

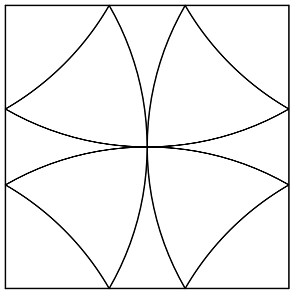


Figure 3.1: The sets of continuity of ρ.

We delay the discussion on the mixing properties of T. We content ourselves for the moment to state a theorem that is an immediate conclusion of the Nakada - Natsui Theorem discussed below.

Theorem 3.2.5. *The map T is strong mixing with respect to μ; that is*

$$\forall A,B\in\mathfrak{B}\quad \lim_{n\to\infty}\mu(T^{-n}[A]\cap B)=\mu(A)\mu(B).$$

3.2.1 Consequences and related results

Let us start with two natural observations providing a non-trivial difference and a similarity between regular and Hurwitz continued fractions. Both results follow from Birkhoff's Ergodic Theorem.

Proposition 3.2.6. *i. There are constants $\mathfrak{K},\mathfrak{h}$ such that*

$$\forall_{\mathfrak{m}} z=[a_0;a_1,a_2,\ldots]\in\mathfrak{F}\quad \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}a_j=\mathfrak{K},\quad \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}|a_j|=\mathfrak{h}.$$

ii. *There exists a constant* $\mathfrak{c}$ *such that*

$$\forall_{\mathfrak{m}} z = [0; a_1, a_2, \dots] \in \mathfrak{F} \quad \lim_{n\to\infty} |q_n|^{\frac{1}{n}} = \mathfrak{c};$$

in fact, $\mathfrak{c} = \exp(-\int_{\mathfrak{F}} \log|z| \mathrm{d}\mu)$.

For regular continued fractions, the first limit is infinite almost everywhere and a_1 is not integrable. The second part is the analogue of the Khinchin-Lévy Constant.

Proof. i. In virtue of Birkhoff Ergodic Theorem, it suffices to show that $a_1(z) : \mathfrak{F} \to I$ (the first element of the HCF) belongs to $L_1(\mu)$. By (3.1), we have that

$$\forall a \in I \quad \int_{\mathcal{C}_1(a)} |a_1(z)| \mathrm{d}\mu = \int_{\mathcal{C}_1(a)} |a| \mathrm{d}\mu \asymp \frac{1}{|a|^3};$$

as a consequence, summing over I and writing $\|a\|_\infty = \max\{|\Re a|, |\Im a|\}$,

$$\int_{\mathfrak{F}} |a_1| \mathrm{d}\mu \asymp \sum_{a\in I} \frac{1}{|a|^3} = \sum_{j\geq 2} \sum_{\|a\|_\infty = j} \frac{1}{|a|^3} + \sqrt{8} \leq 8 \sum_{j\geq 2} \frac{1}{j^2} + \sqrt{8} \leq 8 + \sqrt{8}$$

and $a_1 \in L_1(\mu)$. The constants

$$\mathfrak{h} = \int a_1 \mathrm{d}\mu, \quad \mathfrak{K} = \int |a_1| \mathrm{d}\mu,$$

give the desired conclusion.

ii. Let $z = [0; a_1, a_2, a_3, \dots] \in \mathfrak{F}$ be irrational. Since the convergents of z are in minimal expression, the identity

$$\frac{p_n(z)}{q_n(z)} = \frac{1}{a_1 + [0; a_2, a_3, \dots]} = \frac{1}{a_1 + \frac{p_{n-1}(Tz)}{q_{n-1}(Tz)}} = \frac{q_{n-1}(Tz)}{a_n p_{n-1}(Tz) + q_{n-1}(Tz)}$$

implies that $p_n(z) = u_n q_{n-1}(T(z))$ for some unit $u_n \in \mathbb{Z}[i]$ and any n. Then, we have that

$$\forall n \in \mathbb{N} \quad \left| \frac{p_n(z)}{q_n(z)} \frac{p_{n-1}(Tz)}{q_{n-1}(Tz)} \cdots \frac{p_1(T^{n-1}z)}{q_1(T^{n-1}z)} \right| = \frac{1}{|q_n(z)|}.$$

As a consequence, for all $n \in \mathbb{N}$

$$\begin{aligned} -\frac{1}{n} \log|q_n(z)| &= \frac{1}{n} \sum_{k=0}^{n-1} \log\left| \frac{p_{n-k}}{q_{n-k}}(T^k z) \right| \\ &= \left(\frac{1}{n} \sum_{j=0}^{n-1} \log|T^j z| \right) + \frac{1}{n} \left(\sum_{k=0}^{n-1} \log\left| \frac{p_{n-k}}{q_{n-k}}(T^k z) \right| - \log|T^k z| \right). \end{aligned} \tag{3.13}$$

In view of (3.7), the density of μ, h, satisfies $0 < \operatorname{ess\,inf} h$ and $\operatorname{ess\,sup} h < +\infty$. Let $\varepsilon > 0$. Then, changing into polar coordinates and recalling that $r \log r \to 0$ as $r \to 0$, we get

$$\int_{\mathfrak{F}\setminus\varepsilon\mathbb{D}} |\log|z||\mathrm{d}\mu = \int_{\mathfrak{F}\setminus\varepsilon\mathbb{D}} |\log|z||h\mathrm{d}\mathfrak{m} = \int_{\mathfrak{F}\setminus\varepsilon\mathbb{D}} r|\log r|h(r,\theta)\mathrm{d}\mathfrak{m}(r,\theta) \ll 1.$$

By The Monotone Convergence Theorem, $z \mapsto \log|z|$ belongs to $L_1(\mu)$ and, by Birkhoff's Ergodic Theorem, the first term in (3.13) tends to $\int \log|z|\mathrm{d}\mu$.

For the second term, let us observe that any irrational $w \in \mathfrak{F}$ with $\mathcal{Q}$-pair $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ satisfies for any $n \in \mathbb{N}$

$$\left| \frac{|w|}{\left|\frac{p_n}{q_n}\right|} - 1 \right| \leq \left| \frac{w}{\frac{p_n}{q_n}} - 1 \right| = \frac{|q_n|}{|p_n|} \left| w - \frac{p_n}{q_n} \right| \ll \frac{1}{|p_n q_n|} = \frac{1}{|q_{n-1} q_n|} \leq \frac{1}{\phi^n}.$$

We have used First Lakein's Theorem and Theorem 1.2.4 of Chapter 1.

The Mean Value Theorem tells us that

$$\forall n \in \mathbb{N} \quad \left| \log|w| - \log\left|\frac{p_n}{q_n}\right| \right| \ll \frac{1}{\phi^n}.$$

Hence, the co-factor of $\frac{1}{n}$ in the second term of (3.13) is bounded and the quotient tends to 0 as n grows.

□

Entropy of the HCF system

In the course of proving Proposition 3.2.6 we established that $f : \mathfrak{F} \to \mathbb{R}$, $f(z) = \log|z|$, is integrable. The real number $\|f\|_{1,\mu} = \int |f|\mathrm{d}\mu$ is strongly related to the entropy of the HCF dynamical system.

Theorem 3.2.7. *Let $f : \mathfrak{F} \to \mathbb{R}$ be given by $f(z) = \log|z|$. The entropy of $(\mathfrak{F}, \mathfrak{B}, \mu, T)$ is*

$$h_\mu(T) = 4\|f\|_{1,\mu}.$$

Proof. Let $\mathcal{P}$ be the partition formed by the cylinders of depth 1, $\mathcal{P} := \{\mathcal{C}_1(b)\}_{b\in I}$. The construction of μ guarantees the existence of two constants c_1, c_2 such that

$$\forall b \in I \quad \log \mathfrak{m}(\mathcal{C}_1(b)) + c_1 \leq \log \mu(\mathcal{C}_1(b)) \leq \log \mathfrak{m}(\mathcal{C}_1(b)) + c_2.$$

And, by (3.2), there are absolute constants c_1', c_2' for which

$$4\log|b| + c_1' \leq -\log \mu(\mathcal{C}_1(b)) \leq 4\log|b| + c_2',$$

holds for all $b \in I$. Adding over I we obtain the entropy of $\mathcal{P}$

$$H_\mu(\mathcal{P}) = -\sum_{b\in I} \mu(\mathcal{C}_1(b)) \log \mu(\mathcal{C}_1(b)) \ll \sum_{b\in I} \frac{\log|b|}{|b|^4} + \frac{1}{|b|^4} \ll \sum_{b\in I} \frac{1}{|b|^3} < +\infty.$$

The Shannon-McMillan-Breiman Theorem yields

$$\forall_\mu\, z = [0; a_1, a_2, \ldots] \in \mathfrak{F} \quad \lim_{n\to\infty} \frac{1}{n} \log \mu(\mathcal{C}_n(\mathbf{a})) = h_\mu(T);$$

and in light of (3.7) we can replace μ by $\mathfrak{m}$ in the previous limit. Finally, by the the second part of Proposition 3.2.6, for $\mathfrak{m}$-almost any $z \in \mathfrak{F}$ we have

$$-\frac{1}{n} \log \mathfrak{m}(\mathcal{C}_n(\mathbf{a})) = \frac{\log|q_n(z)|}{n} + \frac{1}{n}\mathcal{O}(1) \xrightarrow[n\to\infty]{} \|f\|_{1,\mu}.$$

□

Complex quadratic forms in two variables

We say that a sequence $\mathbf{a} \in \Omega_R^{\mathrm{HCF}}$ is **HCF-normal** if every regular prefix is a sub-block of $\mathbf{a}$. A consequence of the ergodicity of $(\mathfrak{F}, \mathfrak{B}, T, \mu)$ is the following corollary pointed out in [DaNo14].

Corollary 3.2.8. *With respect to* $\mathfrak{m}$*, almost every complex number is HCF-normal.*

Proof. Let $\mathbf{b} \in I^m$ be a regular prefix. By Birkhoff's Ergodic Theorem,

$$\forall_\mu\, z \in \mathfrak{F} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{\mathcal{C}_m(\mathbf{b})}(T^j(z)) = \mu(\mathcal{C}_m(\mathbf{b})) > 0.$$

Since there are countably many regular prefixes and μ is equivalent to $\mathfrak{m}$, every regular prefix appears in the HCF expansion of μ-almost every complex number z. □

For any pair of complex numbers $\zeta, \xi \in \mathbb{C}$ define the quadratic form $Q_{\zeta,\xi} : \mathbb{C}^2 \to \mathbb{C}$ by

$$\forall (w, z) \in \mathbb{C}^2 \quad Q_{\zeta,\xi}(w, z) = (w - \zeta)(z - \xi).$$

On the basis of the work of S. G. Dani and A. Nogueira, we can obtain the next theorem.

Theorem 3.2.9. *For almost every pair* $\zeta, \xi \in \mathbb{C}$*, the set* $Q_{\zeta,\xi}[\mathbb{Z}[i]^2]$ *is dense in* $\mathbb{C}$*.*

This theorem is the complex analogue of an earlier result also obtained by S.G. Dani and A. Nogueira too.

Theorem 3.2.10. *For almost every pair $\alpha_1, \alpha_2 \in \mathbb{R}$, the set $Q_{\alpha,\beta}[\mathbb{N}^2]$ is dense in $\mathbb{R}$.*

The complete proofs of Theorems 3.2.9 and 3.2.10 can be found, respectively, in [DaNo14] as Corollary 8.8. and in [DaNo02] as Corollary 5.3..

The Gauss-Kuzmin Theorem

The genesis of metrical diophantine approximation can be traced back to a letter from C. F. Gauss to P. S. Laplace where he claimed (in modern notation) that

$$\forall x \in (0,1) \quad \lim_{n\to\infty} \mathfrak{m}_{\mathbb{R}}\left(T_{\mathbb{R}}^{-n}[[0,x)\right) = \frac{\ln(1+x)}{\ln 2} = \mu_{\mathbb{R}}([0,x]),$$

where $T_{\mathbb{R}}$ is the Gauss map. Gauss did not publish any proof. It was until 1928 when it was finally shown, by R. Kuzmin and a year later by P. Lévy, that Gauss' claim was indeed true. A detailed proof (in English) of Kuzmin's Theorem is in [Kh]. The error term obtained by P. Lévy and later by E. Wirsing (based on the work of P.Szüsz) improves Kuzmin's.

Theorem 3.2.11 (R. Kuzmin, 1928)**.** *Let $(f_n)_{n\geq 0}$ be a sequence of real-valued and integrable functions on $(0,1)$. Suppose that*

$$\forall n \in \mathbb{N}_0 \quad f_{n+1}(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} f_n\left(\frac{1}{(k+x)}\right).$$

Assume the existence of $M, \mu > 0$ such that

$$\forall x \in (0,1) \quad 0 < f_0(x) < M, \quad |f_0'(x)| < \mu.$$

Then, there are constants $\lambda > 0$, $A = A(M,\mu) > 0$ and a function θ, $|\theta| < 1$ such that

$$\forall n \in \mathbb{N} \quad \forall x \in (0,1) \quad f_n(x) = \frac{\|f_0\|_1}{1+x} + \theta(x) A e^{-\lambda\sqrt{n}}.$$

A more modern approach to Theorem 3.2.11 requires the notion of transfer operator.

Definition 3.2.1. *Let $(X,\mathfrak{S},\nu)$ a probability space and $\tau : X \to X$ measurable non-singular map. The* ***transfer operator*** *or* ***Perron - Frobenius operator*** *of τ,*

$$P_\tau : L_1(\nu) \to L_1(\nu), \quad f \mapsto P_\tau f,$$

assigns to each $f \in L_1(\mu)$ the Radon - Nikodym derivative of $f\mathrm{d}(\nu\circ\tau^{-1})$ with respect to ν. That is, $P_\tau f \in L^1(\nu)$ is defined by the condition

$$\forall E \in \mathfrak{S} \qquad \int_E P_\tau f \mathrm{d}\nu = \int_{\tau^{-1}[E]} f \mathrm{d}\nu.$$

Remark. The non-singularity of τ is needed to ensure the absolute continuity of the signed measure $E \mapsto \int_{\tau^{-1}[E]} f\mathrm{d}\nu$ with respect to ν (enabling the use of the Radon-Nikodym Theorem).

Remark. In general, the transfer operator is linear and positive.

Adopt for a moment the notation of Definition 3.2.1. It is an exercise in measure theory to show for any $f \in L^1(\mu)$ that $P_\tau f$ is the only element of $L^1(\mu)$ verifying

$$\forall \varphi \in L^\infty(\mu) \qquad \int \varphi P_\tau f \mathrm{d}\mu = \int (\varphi\circ\tau) f\mathrm{d}\mu.$$

In particular, for the Gauss map in $[0,1]$, $T_{\mathbb{R}}$, with the Lebesgue measure we have

$$\forall f \in L^1(\mathfrak{m}) \quad P_{T_{\mathbb{R}}} f(x) = \sum_{n\geq 1} \frac{1}{(n+x)^2} f\left(\frac{1}{n+x}\right)$$

(the equality should be read $\mathfrak{m}$-almost everywhere). Theorem 3.2.11 is, thus, a result on the exponential convergence of the iterations of the transfer operators on certain elements of $L^1(\mathfrak{m})$.

The Kuzmin-type theorem we will state for HCF is restricted to a class of functions strictly smaller than $L^1(\mathfrak{m})$. Let $\mathcal{H} := \{H_1,\ldots,H_{12}\}$ be the twelve regions determined by the boundaries of regular maximal feasible regions. Let $\mathcal{L}(\mathcal{H})$ be the space of functions from $\mathfrak{F}$ to $\mathbb{R}_{>0}$ satisfying

$$\begin{aligned} &\exists\, m, M > 0 \quad m \leq \operatorname{ess\,inf} f \leq \operatorname{ess\,sup} f \leq M,\\ &\forall j \in [1..12] \quad f|_{H_j^\circ} \text{ is Lipschitz.}\end{aligned}$$

For $n\in\mathbb{N}$ set $\sigma(n) = \sup\{|\mathcal{C}_n(\mathbf{a})| : \mathbf{a} \in \Omega_R^{\mathrm{HCF}}(n)\}$.

Theorem 3.2.12 (F. Schweiger, 2000)**.** *There is an absolute constant $0 < \alpha < 1$ such that*

$$\forall f \in \mathcal{L}(\mathfrak{F}) \quad \forall n\in\mathbb{N} \qquad \left\| P_T^n f - h\int_{\mathfrak{F}} f\mathrm{d}\mathfrak{m}\right\|_\infty \ll_f \alpha^{\sqrt{n}} + \sigma[\sqrt{n}].$$

The full proof of Theorem 3.2.12 can be found in [Sc00b]. We will content ourselves to point out some important features of the argument. First, note that Schweiger's Theorem would follow if we had the uniform convergence on the closure of each bounded set H_i.

Besides restricting the study to the aforementioned compact sets, the following proposition is also required.

Proposition 3.2.13. *A positive and almost everywhere integrable function f satisfies $P_T h$ if and only if $f = \alpha h$ for some $\alpha \in \mathbb{R}_{>0}$.*

Proof. By the definition of the transfer operator and the T-invariance of μ that $P_T h = h$. Now, let $f \in L^1(\mu)$ be an almost everywhere positive such that $P_h f = f$ and assume that $\int f \mathrm{d}\mathfrak{m} = 1$. Then, the measure $\nu_f(E) := \int_E f \mathrm{d}\mathfrak{m}$ is T-invariant and, since $\mathfrak{m}$ is irreducible, it is also ergodic. Since $\nu \ll \mu$ and both are ergodic, we have $\nu = \mu$ and $f = h$ almost everywhere. □

Take any $f \in \mathcal{L}(\mathcal{H})$. Clearly, $f \asymp_f h$ and, by the positivity of the transfer operator, $P_T^n f \asymp_f P_T^n h = h$ holds for all n. Let H be the closure of some $H_i \in \mathcal{H}$. Suppose, without loss of generality, that f is continuous on H. Then, from $n^{-1} \sum_{j=0}^{n-1} P_T^j f \asymp_f h$ and the Arzelá - Ascoli Theorem, there is a uniformly convergent subsequence of $n^{-1} \sum_{j=0}^{n-1} P_T^j f$. The same argument on the other elements of $\mathcal{H}$ guarantee the existence of a subsequence of $n^{-1} \sum_{j=0}^{n} P_T^j f$ almost everywhere uniformly convergent, say, to F. The positive function F would satisfy $F = P_T F$, so $F = h$ almost everywhere. We can even take F to be continuous when restricted to the interior of each H_i. With further computations, we obtain the following proposition.

Proposition 3.2.14. *There exists some $f \in \mathcal{L}(\mathcal{H})$ such that $f = h$ a.e..*

The rest of the proof of Theorem 3.2.12 is getting a lower bound of $P_T \varphi$ for functions $\varphi \in \mathcal{L}(\mathcal{H})$. Once again, the argument focuses on full cylinders via Lemma 3.1.2.

3.3 General frameworks

Even though the ergodic theory of Hurwitz Continued Fractions is an active subject, most of the related research has been done in more general frameworks. First, we discuss the spaces where F. Schweiger got his Kuzmin-type theorem. Afterwards, we describe the very recent work of A. Lukyanenko and J. Vandehey. We already saw that the analogue of Serret's Theorem on equivalent fractions does not hold for HCF. However, a consequence of Lukyanenko and Vandehey's work is that it does hold almost everywhere.

3.3.1 Fibred Systems

A pair (B, τ) where B is a non-empty set and $\tau : B \to B$ is a **fibred system** if there is an at most countable partition of B, $\{B_i\}_{i \in I}$ such that the restrictions $\tau|_{B_i} : B_i \to B$ are injective. We call I the set of **digits**. For any $i_1 \in I$ write $B_1(i_1) = B_{i_1}$ and define for any $n \in \mathbb{N}_{\geq 2}$ and any $(i_1, \ldots, i_n) \in I^n$

$$B_n(i_1, \ldots, i_n) := B_1(i_1) \cap \tau^{-1}[B_{n-1}(i_2, \ldots, i_{n-1})].$$

A **cylinder** of depth n is a set of the form $B_n(\mathbf{i})$ for some $\mathbf{i} \in I^n$. A finite sequence $(i_1, \ldots, i_n) \in I^n$ is **valid** if $B_n(i_1, \ldots, i_n) \neq \varnothing$. A cylinder $B_n(i_1, \ldots, i_n)$ is **full** if $\tau^n[B_n(i_1, \ldots, i_n)] = B$. For any valid $\mathbf{i} = (i_1, \ldots, i_n) \in I^n$ define the function

$$V(\mathbf{i}) : \tau^n[B_n(\mathbf{i})] \to B, \quad z \mapsto V(\mathbf{i}; z)$$

to be the inverse of τ^n restricted to $B_n(\mathbf{i})$.

Let (B, τ) be a fibred system such that $B \subseteq \mathbb{R}^n$ is compact, connected and $\mathfrak{m}_n(B) = 1$ ($\mathfrak{m}_n$ is the n-dimensional Lebesgue measure); each B_i is measurable and connected, and τ is non-singular (with respect to $\mathfrak{m}_n$). We will call a finite sequence $\mathbf{i} \in I^m$ **regular** whenever $\mathfrak{m}_n(B_m(\mathbf{i})) > 0$. By the Radon-Nikodym Theorem and the non-singularity of τ, for every regular $\mathbf{i} \in I^m$ we can define $\omega(\mathbf{i}) : B_n(\mathbf{i}) \to \mathbb{R}$ by the condition

$$\forall E \in \mathfrak{B}(B) \quad \int_E \omega(\mathbf{i}; x) \mathrm{d}\mathfrak{m}_n(x) = \int_{\tau^{-m}[E] \cap B_m(\mathbf{i})} 1 \mathrm{d}\mathfrak{m}.$$

A fibred system (B, τ) is a **Schweiger Fibred System** if $B \subseteq \mathbb{R}^n$ is compact, connected and $\mathfrak{m}_n(B) = 1$; each B_i is measurable and connected, τ is non-singular (with respect to $\mathfrak{m}_n$), and conditions (S_1) to (S_7) below hold.

S_1. The sequence $\sigma(n) := \sup_{\mathbf{i} \in I^n} |B_n(\mathbf{i})|$ satisfies $\sigma(n) \to 0$ as $n \to \infty$.

S_2. (Finite range condition) There is a finite measurable collection $\mathcal{U} = \{U_1, \ldots, U_N\}$ such that for every $n \in \mathbb{N}$

$$\forall \mathbf{i} \in I^n \quad \mathbf{i} \text{ is regular} \quad \Longrightarrow \quad \exists j \in [1..N] \quad \tau^n[B_n(\mathbf{i})] = U_j.$$

The previous equality is understood up to a null set. We call $\mathcal{F}$ the partition of $B \mod 0$ induced by the intersections of the members of $\mathcal{U}$.

S_3. (Rényi Condition) There exists $C > 0$ such that for any regular $\mathbf{i} \in I^n$

$$\operatorname*{ess\,sup}_z \omega(\mathbf{i}; z) \leq C \operatorname*{ess\,inf}_w \omega(\mathbf{i}; w).$$

S_4. Every $U \in \mathcal{U}$ contains a full cylinder.

S_5. Every $\omega(\mathbf{i})$ is equal to a Lipschitz continuous function almost everywhere.

S_6. For every regular $\mathbf{i} \in I^n$, $V(\mathbf{i})$ is Lipschitz continuous.

S_7. Define for every $m \in \mathbb{N}$

$$\mathcal{L}_m := \{\mathbf{a} \in I^m : \forall F \in \mathcal{F} \quad B_m(\mathbf{a}) \notin F\}, \quad \gamma(m) := \sum_{\mathbf{i} \in \mathcal{L}_m} \mathfrak{m}_n\left(B_m(\mathbf{i})\right).$$

Then, $\gamma(m) \to 0$ as $m \to \infty$.

Conditions S_1, S_2, S_3, S_4 imply the existence of a τ-invariant measure μ equivalent to $\mathfrak{m}_n$. The referred four conditions also guarantee the existence of a function h which is Lipschitz continuous on each $F \in \mathcal{F}$ and such that $\mathrm{d}\mu = h \mathrm{d}\mathfrak{m}_n$.

The seven conditions $S_1 - S_7$ allowed F. Schweiger to show his Kuzmin-type theorem (Theorem 3.3.1 below). Let $\mathcal{L}(\mathcal{F})$ be the space of functions $f : B \to \mathbb{R}_{>0}$ such that

$$\exists m, M > 0 \quad m < \inf_{x \in B} f(x) \leq \sup_{x \in B} f(x) < M,$$
$$\forall F \in \mathcal{F} \quad f|_F \text{ is Lipschitz continuous.}$$

Theorem 3.3.1 (F. Schweiger, 2001)**.** *Let (B, τ) be a Schweiger Fibred System with τ-ergodic measure μ and call h its density. Then, there is some $0 < \alpha < 1$ for which*

$$\forall f \in \mathcal{L}(\mathcal{H}) \quad \forall n \in \mathbb{N} \quad \left\| P_\tau^n f - h \int_B f \mathrm{d}\mathfrak{m}_n \right\|_\infty \ll_f \alpha^{\sqrt{n}} + \sigma([\sqrt{n}]) + \gamma([\sqrt{n}]).$$

The norm $\|\cdot\|_\infty$ refers to $L^\infty(B, \mathfrak{m}_n)$ and $P_\tau : L^1(B, \mathfrak{m}_n) \to L^1(B, \mathfrak{m}_n)$ is the transfer operator of τ with respect to $\mathfrak{m}_n$.

As an application of the previous theorem, H. Nakada and H. Natsui proved a mixing property of Schweiger Fibred Systems.

Theorem 3.3.2 (H. Nakada, R. Natsui, 2003)**.** *Let (B, τ) be a Schweiger Fibred System and μ its ergodic measure. Define for every $t \in \mathbb{N}$*

$$\Psi(t) := \sup\left\{ \frac{|\mu(B_s(\mathbf{i}) \cap \tau^{-(s+t)}[E] - \mu B_s(\mathbf{i})\, \mu E|}{\mu B_s(\mathbf{i})\, \mu E} : s \in \mathbb{N},\ \mathfrak{m} B_s(\mathbf{i}) > 0,\ \mu E > 0 \right\}.$$

Then, $\Psi(t) \to 0$ as $t \to \infty$ and $\sup_t \Psi(t) < +\infty$.

Any fibred system satisfying the conclusion of Theorem 3.3.2 is called **continued-fraction mixing**. Some statistical properties, such as Laws of Large Numbers for stationary processes arising from Schweiger Fibred Systems, were also studied by H. Nakada and R. Natsui in [NaNa03].

3.3.2 Iwasawa Continued Fractions

Recently, A. Lukyanenko and J. Vandehey obtained the ergodicity of the Gauss map for a large family of continued fractions algorithms. Their work include Hurwitz Continued Fractions and Julius' Hurwitz Continued Fractions (discussed in the last chapter). We shall only state their results. The proofs can be found in [LuVa18].

First, let k be either the real, the complex, or the quaternion numbers and $n \in \mathbb{N}$. The **Iwasawa inversion space** $\mathbb{X} = \mathbb{X}_k^n$ is the group given by the set $k^n \times \mathfrak{I}(k)$ with the operation

$$(z,t) * (w,s) = (z+w, t+s+2\mathfrak{I}(\langle z,w\rangle)),$$

with $\langle z,w\rangle = \sum_j \overline{z_j} w_j$. We turn $\mathbb{X}$ into a metric space with the gauge function $|\cdot|$ and the Cygan metric d defined by

$$\forall (z,t),(w,s) \in \mathbb{X} \quad |(z,t)| = \left\| \|z\|^2 + t \right\|^2, \quad d((z,t),(w,s))) = |(-z,-t)*(w,s)|.$$

A conformal mapping $\iota : \mathbb{X} \to \mathbb{X}$ satisfying

$$|\iota h| = \frac{1}{|h|}, \quad d(\iota h, \iota h') = \frac{d(h,h')}{|h||h'|}$$

is an **inversion**. The notions of lattice, fundamental domain, and nearest-integer mapping can be extended to metric spaces with a measure invariant under its isometries, in particular to $\mathbb{X}$. An Iwasawa **Continued Fraction Algorithm** is defined by the next objects

i. An associative real algebra k and $n \in \mathbb{N}$.

ii. The Iwasawa inversion space $\mathbb{X} = \mathbb{X}_k^n$.

iii. An inversion.

iv. A discrete subgroup $\mathcal{Z}$ of the isometries of $\mathbb{X}$, the associated fundamental domain $K \subseteq \mathbb{X}$, and a nearest integer map $\lfloor\cdot\rceil : \mathbb{X} \to \mathcal{Z}$. The associated Gauss map $T : K \to K$ is defined by $T(0) = 0$ and $T(x) = \lfloor \iota(x) \rceil^{-1} * \iota(x)$.

As in the usual way, we can consider a continued fraction expansion. We call the algorithm **convergent** if for almost every $x \in \mathbb{X}$ the associated continued fraction converges to x.

An Iwasawa Continued Fraction is **proper** if the distance from K to the unit sphere is positive. Each Iwasawa Continued Fraction has associated a modular group $\mathcal{M}$ and whenever it is discrete, the Iwasawa Continued Fraction is **discrete**. Moreover, the continued fraciton is **complete** if the stabilizer (over $\mathcal{M}$) of a certain element is $\mathcal{Z}$.

Theorem 3.3.3 (A. Lukyanenko, J. Vandehey, 2018)**.** *If an Iwasawa Continued Fraction is discrete and proper, it is convergent. If it is complete, then it is ergodic.*

The almost everywhere version Serret's Theorem is the following result.

Theorem 3.3.4 (A. Lukyanenko, J. Vandehey, 2018)**.** *Let $\mathbb{X}$ be an Iwasawa inversion space and consider a complete, discrete, and proper Iwasawa Continued Fraction algorithm. For almost every pair $x, y \in \mathbb{X}$ the tails continued fraction expansions coincide if and only if they belong to the same $\mathcal{M}$-orbit.*

In particular, as pointed out in [LuVa18], we have the following result for HCF.

Corollary 3.3.5. *For almost every $z = [0; a_1, a_2, \ldots], w = [0; b_1, b_2, \ldots] \in \mathfrak{F}$ there exists some $M, N \in \mathbb{N}$ such that $a_{N+j} = b_{M+j}$ for all $j \in \mathbb{N}$ if and only if $Az = w$ for some $A \in \mathrm{SL}(2, \mathbb{Z}[i])$ acting via Möbius transformations.*

3.4 Notes and Comments

1. The meaning we have established for *regular*, *valid*, and *admissible* differs from that in the fibred systems literature. The standard terminology does not include the word *regular*, instead, only *admissible* is used. While avoiding the notion of regularity and irregularity works perfectly fine from a (Lebesgue) measure theoretic perspective, it must be taken into consideration when studying Hausdorff dimensions.

2. The ergodic theory of HCF is an active research field. For example, recently J. Vandehey and R. Hiary ([HiVa18]) gave some conjectures on h based on their numerical experiments.

3. We have not found (3.2.6) in the available literature. However, we believe that it is very unlikely that this has not been shown before in more general contexts.

4. In his work on multi-dimensional continued fractions, F. Schweiger obtained a result analogous to Theorem 3.2.7. Nevertheless, as he pointed out, Hurwitz Continued Fractions are not part of this framework.

   The approach on Theorem 3.2.7 is the one adopted by S. Tanaka while studying J. Hurwitz continued fractions (see Chapter 5).

5. S. G. Dani and A. Nogueira used the term *generic* instead of *normal* for naming sequences having all regular prefixes as sub-blocks. We chose *HCF-normal* in analogy to regular continued fractions.

Chapter 4

Hausdorff Dimension Theory

In this chapter, we estimate the Hausdorff dimension of sets given by HCF restrictions. First, we shift our attention to the real numbers. After recalling the definition of badly approximable real numbers, $\mathbf{Bad}_{\mathbb{R}}$, we discuss why is it a null set. Motivated by our discussion, we introduce the notion of Hausdorff dimension. Afterwards, we discuss Jarník's argument for showing $\dim_H \mathbf{Bad}_{\mathbb{R}} = 1$. Later, we review some generalizations and extensions of Jarník's work. In the following section, we fully focus again on the complex plane. First, we use a result of W.J. Leveque to establish that $\mathfrak{m}\mathbf{Bad}_{\mathbb{C}} = 0$ and then we use the Kristensen-Thorne-Velani Theorem to show that $\dim_H \mathbf{Bad}_{\mathbb{C}} = 2$. We later discuss a full HCF analogue of a theorem by I.J. Good. We finish the chapter by computing the dimension of subsets of $\mathbf{Bad}_{\mathbb{C}}$ where the laws of succession can somehow be overlooked.

Notation. During Section 1 and Section 2, we consider only regular continued fractions. In Section 3, the continued fractions used are Hurwitz. Although we use the same notation for both types of continued fractions, there will be no room for ambiguity. Recall that for any $j, k \in \mathbb{Z}$ with $j < k$ we write

$$[j..k] := \{n \in \mathbb{Z} : j \leq n \leq k\}.$$

When $\ll$ or $\asymp$ appear without any additional parameter, the implied constants are absolute.

Main references. For general theory of Hausdorff Dimension, see [Mat], [BiPe17]. For the theory of real continued fractions, Jarník's Theorem and its generalizations, see [Kh], [Ja28], [Goo41], [KTV06], [KlWe10]. For the related theory for HCF: [Gon18-02]. For Schmidt Games, see [Sch66] and [BHNS18].

4.1 Hausdorff Dimension. Definitions and Elementary Properties

Regular continued fractions provide for any irrational number $\alpha \in \mathbb{R}$ infinitely many pairs $(p,q) \in \mathbb{Z} \times \mathbb{N}$ satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

The number α is badly approximable if the previous inequality is best possible; that is, $\alpha \in \mathbb{R}$ is **badly approximable** if

$$\exists C > 0 \quad \forall (p,q) \in \mathbb{Z} \times \mathbb{N} \quad \left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}.$$

$\mathbf{Bad}_{\mathbb{R}}$ denotes the set of badly approximable real numbers. As we mentioned before, a classical theorem characterizes $\mathbf{Bad}_{\mathbb{R}}$ in terms of their continued fraction expansion:

$$\mathbf{Bad}_{\mathbb{R}} \coloneqq \{\alpha = [a_0; a_1, a_2, a_3, \ldots] \in \mathbb{R} \smallsetminus \mathbb{Q} : \limsup a_n < +\infty\}$$

(see Theorem 23, p. 36, in [Kh]). Thus, we can express $\mathbf{Bad}_{\mathbb{R}}$ as a countable union of increasing sets: $\mathbf{Bad}_{\mathbb{R}} = \bigcup_M F_M$, where

$$\forall M \in \mathbb{N} \quad F_M = \{[0; a_1, a_2, a_3, \ldots] : \forall n \in \mathbb{N} \quad a_n \leq M\}.$$

In 1909, É. Borel showed that $\mathfrak{m}_1 \mathbf{Bad}_{\mathbb{R}} = 0$ (see [Bo09]). Today we know several arguments showing Borel's equality. We briefly sketch one particularly illustrative to our ends.

i. It suffices to show $\mathfrak{m}(F_M) = 0$ for any $M \in \mathbb{N}$. Fix $M \in \mathbb{N}$. For any $\mathbf{a} \in \mathbb{N}^n$, let $\mathcal{C}_n(\mathbf{a})$ be the closure of the set of irrationals in $[0,1]$ whose regular continued fraction starts with $\mathbf{a}$. We call these sets **cylinders of depth** n. Define the collection of compact sets

$$\forall n \in \mathbb{N} \quad F_M^n \coloneqq \bigcup_{\mathbf{a} \in [1..M]^n} \mathcal{C}_n(\mathbf{a}).$$

ii. Evidently, an upper bound on $\mathfrak{m}(\mathcal{C}_n(\mathbf{a}))$, $\mathbf{a} \in [1..M]^n$, entails an upper estimate of $\mathfrak{m}(F_M^n)$. Using the estimate

$$\forall \mathbf{a} \in \mathbb{N}^n \quad \forall j \in \mathbb{N} \qquad \frac{\mathfrak{m}(\mathcal{C}_{n+1}(\mathbf{a}j))}{\mathfrak{m}(\mathcal{C}_n(\mathbf{a}))} \asymp \frac{1}{j^2}, \tag{4.1}$$

we may bound $\mathfrak{m}(F_M^{n+1})$ in terms of $\mathfrak{m}(F_M^n)$. Indeed, for any $\mathbf{a} \in [1..M]^n$

$$\begin{aligned}\mathfrak{m}\left(\mathcal{C}_n(\mathbf{a})\right) &= \sum_{j=1}^{M} \mathfrak{m}\left(\mathcal{C}_{n+1}(\mathbf{a},j)\right) + \sum_{j=M+1}^{\infty} \mathfrak{m}\left(\mathcal{C}_{n+1}(\mathbf{a},j)\right) \\ &\geq \sum_{j=1}^{M} \mathfrak{m}\left(\mathcal{C}_{n+1}(\mathbf{a},j)\right) + \frac{1}{3}\left(\sum_{j=M+1}^{\infty} \frac{1}{j^2}\right) \mathfrak{m}\left(\mathcal{C}_n \mathbf{a}\right).\end{aligned}$$

Adding along $\mathbf{a} \in [1..M]^n$, we deduce the existence of some $0 < \tau = \tau(M) < 1$ such that that $\mathfrak{m}_1(F_M^{n+1}) \leq \tau \mathfrak{m}_1(F_M^n)$ for every n. As a consequence, $\mathfrak{m}_1 F_M^{n+1} \to 0$ as $n \to \infty$ and $\mathfrak{m} F_M = 0$.

To sum up, we concluded $\mathfrak{m} F_M = 0$ thanks to a uniform relation among $\mathfrak{m}\left(\mathcal{C}_n(\mathbf{a})\right)$ and $\{\mathfrak{m}\left(\mathcal{C}_{n+1}(\mathbf{a}j)\right) : 1 \leq j \leq M\}$.

4.1.1 Definitions

Even though the Lebesgue Measure has undoubtedly proven to be extremely useful, it has some drawbacks. For instance, the structure of null sets is completely overlooked. In particlar, while F_1 (as above) contains only one point, **Bad** is— although meager— an uncountable dense subset of $[0,1]$.

A slight modification of Carathéodory's construction of Lebesgue's measure leads to the s-dimensional Hausdorff measures and to the Hausdorff dimension. These objects help to refine dramatically the notion of null sets.

Recall that for a metric space (X,d) and $A \subseteq X$, we denote by $|A|$ the diameter of A: $|A| = \sup\{d(a,b) : a, b \in X\}$ if $A \neq \varnothing$ and $|\varnothing| := 0$.

The s-Hausdorff measures are constructed with countable coverings formed by *small* sets. To be more precise, for any $\delta > 0$ a **δ-cover** of A is a countable family of subsets of X, $(A_j)_{j\geq 1}$, such that

$$A \subseteq \bigcup_{j=1}^{\infty} A_j, \quad \forall j \in \mathbb{N} \quad |A_j| \leq \delta.$$

We define Hausdorff measures and dimension as in [Mat], p.54.

Definition 4.1.1. *Let $\mathcal{B}$ be a family of subsets of a separable and complete metric space (X,d), and $\zeta : \mathcal{B} \to \mathbb{R}_{\geq 0}$. Suppose that the following conditions hold:*

i. for any $\delta > 0$ there is a δ-covering of X contained in $\mathcal{B}$,

ii. for every $\delta > 0$ there is some $B \in \mathcal{B}$ such that $\max\{|B|, \zeta(B)\} < \delta$.

For any $\delta > 0$, define $\psi_\delta : 2^X \to [0, +\infty]$ by

$$\forall A \subseteq X \quad \psi_\delta(A) := \inf\left\{\sum_{j=1}^{\infty} \zeta(A_j) : \{A_j\}_{j=1}^{\infty} \subseteq \mathcal{B} \text{ is a } \delta - \text{covering of } A\right\}.$$

The outer measure $\psi = \psi(\mathcal{B}, \zeta)$ is given by

$$\forall A \subseteq X \quad \psi(A) := \lim_{\delta \to 0} \psi_\delta(A).$$

*For $s \geq 0$, the s-**dimensional Hausdorff measure** $\mathcal{H}^s$ is the outer measure obtained by setting $\mathcal{B} = 2^X$, and $\zeta(A) = |A|^s$ for any $A \in \mathcal{B}$.*

We adopt V. Jarník's convenient notation Λ_s. If $\mathfrak{A}$ is a countable family of subsets of X and $s \geq 0$, we write

$$\Lambda_s(\mathfrak{A}) := \sum_{A \in \mathfrak{A}} |A|^s.$$

For example, for any $A \subseteq X$ we have

$$\forall \delta > 0 \quad \mathcal{H}^s_\delta(A) = \inf\{\Lambda_s(\mathfrak{A}) : \mathfrak{A} \text{ is a } \delta - \text{cover of } A\}.$$

Fix, for a moment, a set $A \subseteq X$ and take $0 \leq s < t$, $0 < \delta < 1$. Let $\mathfrak{A} = (A_j)_{j \geq 1}$ be a countable δ-cover of A, then

$$\Lambda_t(\mathfrak{A}) = \sum_{j \geq 1} |A_j|^t = \sum_{j \geq 1} |A_j|^s |A_j|^{t-s} \leq \delta^{t-s} \sum_{j \geq 1} |A_j|^s = \delta^{t-s} \Lambda_s(\mathfrak{A}).$$

By taking infima, $\mathcal{H}^t_\delta(A) \leq \delta^{t-s} \mathcal{H}^s_\delta(A)$, then $\mathcal{H}^s(A) < +\infty$ implies $\mathcal{H}^t(A) = 0$; equivalently, $\mathcal{H}^s(A) = +\infty$ whenever $\mathcal{H}^t(A) > 0$. Hence, $s \mapsto \mathcal{H}^s(A)$ takes at most three values. Moreover, $s \mapsto \mathcal{H}^s(A)$ is first constant and equal to $+\infty$, a jump happens, and afterwards it is equal to 0. We are interested in the point where the jump occurs, called the Hausdorff dimension of A.

For our goals, it suffices to heed subsets of $\mathbb{R}^n$.

Definition 4.1.2. *Let $A \subseteq \mathbb{R}^n$. The **Hausdorff dimension** of A is*

$$\dim_H A = \inf\{s : \mathcal{H}^s A = 0\}.$$

Proposition 4.1.1. *Let $A, B \subseteq \mathbb{R}^n$.*

i. $0 \leq \dim_H A \leq n$.

ii. $A \subseteq B \implies \dim_H A \leq \dim_H B$

iii. If $(A_n)_{n=1}^{\infty}$ *is a sequence of subsets of* X*, then*

$$\dim_H\left(\bigcup_{n=1}^{\infty} A_n\right) = \sup\{\dim_H A_n : n \in \mathbb{N}\}$$

iv. Let j, n *be integers with* $1 \le j \le n$*. There is a constant* $k(j)$ *such that* $\mathcal{H}^j = k(j)\mathfrak{m}_j$*, where* $\mathfrak{m}_j$ *is the* j*-th dimensional Lebesgue measure.*

The previous proposition is well known and can be found, for instance, in the fourth chapter of [Mat].

4.2 Jarník's Theorem

In 1909, É. Borel's Theorem showed that $\mathbf{Bad}_{\mathbb{R}}$ is small; in 1929, V. Jarník showed that not that much: $\dim_H \mathbf{Bad}_{\mathbb{R}} = 1$. As with $\mathfrak{m}\mathbf{Bad}_{\mathbb{R}} = 0$, there have been several improvements on the techniques required for showing Jarník's result. Most notably, Wolfgang Schmidt introduced in 1965 a technique known today as *Schmidt games* (discussed below). W. Schmidt obtained a sufficient condition (*positive winning dimension*) for a set to have the same dimension as the ambient space. Rather than the more modern machinery, we study Jarník's original argument.

Theorem 4.2.1 (V. Jarník, 1929)**.** *For any* $M \in \mathbb{N}$ *at least* 9

$$1 - \frac{4}{M\log 2} \le \dim F_M \le 1 - \frac{1}{8M\log M}.$$

4.2.1 Jarník's Strategy

After nearly a century from its publication, Jarník's approach still stands as a paradigm for Hausdorff dimension estimation. Let $M \ge 9$ be a natural number. We divide the argument into two major parts. First, we replace arbitrary coverings by coverings formed by sets $\mathcal{C}_n(\mathbf{a})$. Afterwards, by relating $|\mathcal{C}_n(\mathbf{a})|$ with $\{|\mathcal{C}_{n+1}(\mathbf{a}j)| : 1 \le j \le M\}$ in a uniform way (on $n \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{N}^n$), we obtain upper and lower bounds for $\dim_H F_M$.

A natural measure

Let $M > 1$ be a natural number. Define the family of compact sets $\{K_{\mathbf{a}}^n : n \in \mathbb{N}, \mathbf{a} \in [1..M]^n\}$ via

$$\forall n \in \mathbb{N} \quad \forall \mathbf{a} \in [1..M]^n \quad K_{\mathbf{a}}^n := \bigcup_{j=1}^{M} \mathcal{C}_{n+1}(\mathbf{a}j).$$

From the elementary continued fraction theory,

$$|\mathcal{C}_n(\mathbf{a})| \asymp_M d(K^{n+1}_{\mathbf{a}j}, K^{n+1}_{\mathbf{a}k}). \tag{4.2}$$

holds for any $n \in \mathbb{N}$, $\mathbf{a} \in [1..M]^n$ and $j, k \in [1..M]$, $j \neq k$.

Take $0 < \delta < \min\{|\mathcal{C}_1(a)| : 1 \le a \le M\}$ and let $\mathcal{G}$ be a finite open δ-covering of the compact set F_M. To each $G \in \mathcal{G}$ associate the cylinder $\mathcal{C} = \mathcal{C}(G)$ of maximum depth containing G. The maximality of $\mathcal{C}(G)$ and (4.2) imply $|G| \asymp_s |\mathcal{C}(G)|$. Call $\mathcal{C}(\mathcal{G})$ the cover obtained by replacing each $G \in \mathcal{G}$ by the corresponding $\mathcal{C}(G)$. Then, adding over $\mathcal{G}$, $\Lambda_s(\mathcal{G}) \asymp_s \Lambda_s(\mathcal{C}(\mathcal{G}))$. Taking the infimum over the family of open finite coverings and letting δ tend to 0, we obtain the left inequality in Lemma 4.2.2 below. The right inequality follows from the definition of the outer measures.

Lemma 4.2.2. *Take $s \ge 0$ and define*

$$\mathcal{B} = \{\mathcal{C}_n(\mathbf{a}) : n \in \mathbb{N}, \mathbf{a} \in \mathbb{N}^n\}, \qquad \forall A \in \mathcal{B} \quad \zeta_s(A) = |A|^s.$$

Let $\widetilde{\mathcal{H}}^s = \psi(\mathcal{B}, \zeta_s)$ be the resulting outer measure (as in Definition 4.1.1). Then, for some $B(s) > 0$

$$B(s)\widetilde{\mathcal{H}}^s(F_M) \le \mathcal{H}^s(F_M) \le \widetilde{\mathcal{H}}^s(F_M), \tag{4.3}$$

and $\dim_H F_M = \inf\{s \ge 0 : \widetilde{\mathcal{H}}^s(F_M) = 0\}$.

A lower bound

Lemma 4.2.3 (First Jarník Lemma)**.** *Let $0 < s < 1$ and $M \ge 2$ be given. If for every $\mathbf{a} \in [1..M]^{n-1}$ we have*

$$|\mathcal{C}_{n-1}(\mathbf{a})|^s \le \sum_{k=1}^{M} |\mathcal{C}_n(\mathbf{a}k)|^s, \tag{4.4}$$

then $\dim_H F_M \ge s$.

Proof. Given $\delta > 0$, let $\mathcal{G}$ be a finite δ-covering of F_M formed by pairwise disjoint cylinders. Let n be the maximal integer N such that $\mathcal{G}$ contains a cylinder of depth N. Take $\mathbf{a} \in [1..M]^{n-1}$ and $j \in [1..M]$ such that $\mathcal{C}_n(\mathbf{a}j) \in \mathcal{G}$. We must have $\{\mathcal{C}_n(\mathbf{a}k) : k \in [1..M]\} \subseteq \mathcal{G}$, since $\mathcal{G}$ covers F_M and its elements are pairwise disjoint. Let $\mathcal{G}'$ be the covering obtained by replacing in $\mathcal{G}$ the sets $\{\mathcal{C}_n(\mathbf{a}k) : k \in [1..M]\}$ by $\mathcal{C}_{n-1}(\mathbf{a})$; hence, (4.4) gives $\Lambda_s(\mathcal{G}') \le \Lambda_s(\mathcal{G})$. After iterating the argument we eventually arrive at $0 < \Lambda_s([0,1]) \le \Lambda_s(\mathcal{G})$, so $\widetilde{\mathcal{H}}^s(F_M) > 0$ and, by Lemma 4.2.2, $\dim_H F_M \ge s$. □

An upper bound

Lemma 4.2.4 (Second Jarník Lemma). *Let $0 < s < 1$ and $M \geq 2$ be given. If for every $n \in \mathbb{N}$ and all $\mathbf{a} \in [1..M]^{n-1}$ we have*

$$\sum_{k=1}^{M} |\mathcal{C}_n(\mathbf{a}k)|^s \leq |\mathcal{C}_{n-1}(\mathbf{a})|^s, \tag{4.5}$$

then $\dim_H F_M \leq s$.

Proof. Take $\delta > 0$. Since $|\mathcal{C}_n(\mathbf{a})| \leq 2^{-\frac{n}{2}}$, the family $\mathcal{G}_n = \{\mathcal{C}_n(\mathbf{b}) : \mathbf{b} \in [1..M]^n\}$ is a δ-covering of F_M for any large n. Fix such an n. Let $\mathcal{G}'_n$ be the covering of F_M obtained by replacing $\{\mathcal{C}_n(\mathbf{a}j) : 1 \leq j \leq M\}$ by $\mathcal{C}_{n-1}(\mathbf{a})$ for each $\mathbf{a} \in [1..M]^{n-1}$. By (4.5), $\Lambda_s(\mathcal{G}) \leq \Lambda_s(\mathcal{G}')$ and repeating the process we eventually obtain $\Lambda_s(\mathcal{G}) \leq \Lambda_s([0,1]) = 1$. Hence, $\widetilde{\mathcal{H}}^s(F_M) < +\infty$ and, by Lemma 4.2.2, $\dim_H F_M \leq s$. □

Proof of Theorem 4.2.1

Proof of Theorem 4.2.1. Let $M, n \in \mathbb{N}$ satisfy $n \geq 2$, $M \geq 9$. Take $\mathbf{a} \in [1..M]^n$ and let $(q_n)_{n\geq 0}$ be the associated sequence. In view of $|\mathcal{C}_{n-1}(\mathbf{a})| = (q_{n-1}(q_{n-1} + q_{n-2}))^{-1}$, there is some τ, $2^{-1} < \tau < 2$, such that

$$\sum_{k=1}^{M} |\mathcal{C}_n(\mathbf{a}k)| = |\mathcal{C}_{n-1}(\mathbf{a})| \left(1 - \frac{\tau}{M}\right)$$

Writing $|\mathcal{C}_n(\mathbf{a}k)|^s = |\mathcal{C}_n(\mathbf{a}k)||\mathcal{C}_n(\mathbf{a}k)|^{s-1}$ and calculating, we get

$$\sum_{k=1}^{M} |\mathcal{C}_n(\mathbf{a}k)|^s \geq 2^{1-s} \left(1 - \frac{2}{M}\right) |\mathcal{C}_{n-1}(\mathbf{a})|^s.$$

The hypothesis of the First Jarník Lemma (Lemma 4.2.3) are met if the coefficient of $|\mathcal{C}_{n-1}(\mathbf{a})|^s$ exceeds 1, which occurs if and only if $(1-s)\log 2 \geq -\log(1 - 2/M)$. With the aid of the Taylor expansion of the logarithm, we see that $s = 1 - (M \log 2)^{-1}$ works.

The upper bound is treated in a similar fashion with Lemma 4.2.4 instead of Lemma 4.2.3. □

4.2.2 Extensions, abstractions, and generalizations

Except for $M = 1$, the sets F_M are are homeomorphic to a countable product of finite topological spaces. Moreover, cylinders provide the appropriate compact sets to obtain each F_M through a process resembling the construction of the middle-third Cantor set. In both, Borel's Theorem and Jarník's

Theorem, this feature was heavily exploited by relating two consecutive iterations (cfr. (4.1) and (4.4),(4.5)). This is the key for the extensions we regard.

Generalized Jarník Lemmas

The first extension of Jarník's Lemmas we will consider take place in complete metric spaces. We will estimate the Hausdorff dimension of sets obtained by slightly modifying the notion of strongly tree-like sets as defined in [KlWe10].

Definition 4.2.1. *Let (X, d) be a complete metric space. A family of compact sets $\mathcal{A}$ is* ***diametrically strongly tree-like*** *if $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ where each $\mathcal{A}_k$ is finite, $\#\mathcal{A}_0 = 1$, and*

i. $\forall A \in \mathcal{A} \quad |A| > 0$,

ii. $\forall n \in \mathbb{N} \quad \forall A, B \in \mathcal{A}_n \quad (A = B) \vee (A \cap B = \varnothing)$,

iii. $\forall n \in \mathbb{N} \quad \forall B \in \mathcal{A}_n \quad \exists A \in \mathcal{A}_{n-1} \quad B \subseteq A$,

iv. $\forall n \in \mathbb{N} \quad \forall A \in \mathcal{A}_{n-1} \quad \exists B \in \mathcal{A}_n \quad B \subseteq A$,

v. $d_n(\mathcal{A}) := \max\{|A| : A \in \mathcal{A}_n\} \to 0$ *as* $n \to \infty$.

The n***-th stage diameter*** *is $d_n(\mathcal{A})$. The* ***limit set*** *of $\mathcal{A}$, $\mathbf{A}_\infty$, is*

$$\mathbf{A}_\infty := \bigcap_{n=0}^{\infty} \bigcup_{A \in \mathcal{A}_n} A.$$

The First Generalized Jarník Lemma (Lemma 4.2.5 below) gives a lower bound on the limit set of a family of diametrically strong-tree like compact sets. In the statement we have merged the conditions for enabling the generalization of Lemma 4.2.2 and 4.2.3. We also consider a restriction weaker than (4.4). An upper bound is given by the Second Generalized Jarník Lemma (Lemma 4.2.6 below). On the basis of our discussion of Jarník's argument, the proofs Lemmas 4.2.5 and Lemma 4.2.6 are now a simple exercise (for the details, see [Gon18-02]).

Lemma 4.2.5 (First Generalized Jarník Lemma)**.** *Let $\mathcal{A}$ be a diametrically strongly tree-like family of compact sets. Suppose that for some $\kappa > 1$*

$$\forall n \in \mathbb{N}_0 \quad d_n(\mathcal{A}) \leq \frac{1}{\kappa^n}. \tag{4.6}$$

Assume the existence of a sequence $(B_n)_{n\geq 1}$ *with* $0 < B_n \leq 1$ *for all* n*,*

$$\limsup_{n\to\infty} \frac{\log\log(B_n^{-1})}{\log n} < 1, \tag{4.7}$$

and such that

$$\forall n \in \mathbb{N} \quad \forall A \in \mathcal{A}_n \quad \forall Y, Z \in \mathcal{A}_{n+1} \quad \big(Y \cup Z \subseteq A \quad \& \quad Y \neq Z \implies d(Y,Z) \geq B_n|A|\big). \tag{4.8}$$

If for $s > 0$ *there exists some* $c > 0$ *such that*

$$\forall \mathfrak{X} \in 2^{\mathcal{A}} \quad \left(\#\mathfrak{X} < +\infty \quad \& \quad \mathbf{A}_\infty \subseteq \bigcup_{A\in\mathfrak{X}} A \quad \implies \quad \Lambda_s\mathfrak{X} > c\right), \tag{4.9}$$

then $\dim_H A_\infty \geq s$*.*

Kepping the lemma's notation, we can state an analogue of (4.4):

$$\forall n \in \mathbb{N} \quad \forall A \in \mathcal{A}_n \qquad \sum_{\substack{A'\subseteq A \\ A'\in\mathcal{A}_{n+1}}} |A'|^s \geq |A|^s. \tag{4.10}$$

An iterative replacement argument, like the ones in Jarník's lemmas, tells us that (4.10) implies (4.9).

Lemma 4.2.6 (Second Generalized Jarník Lemma)**.** *Let* $\mathcal{A}$ *be a family of compact sets satisfying conditions i., iii., iv., v. of the definition of strongly tree-like structure and such that the each* $\mathcal{A}_k$*,* $k \geq 1$*, is at most countable. Let* $s > 0$*.*

If for every $k \in \mathbb{N}$ *and every* $A \in \mathcal{A}_k$

$$\sum_B |B|^s \leq |A|^s,$$

where the sum runs along the sets $B \in \mathcal{A}_{k+1}$ *satisfying* $B \subseteq A$*, then* $\dim_H A_\infty \leq s$*.*

The Kristensen-Thorn-Velani Theorem

Jarník's Theorem was established in a quite abstract context by Simon Kristensen, Rebecca Thorn, and Sanju Velani ([KTV06]). In general, computing the Hausdorff Dimension of a given set can be a daunting task. While upper estimates can be obtained by choosing the right coverings, lower bounds can be much more difficult to get. A useful strategy— followed also by the aforementioned authors— for estimating the Hausdorff dimension of a specific set is constructing an appropriate probability measure.

Theorem 4.2.7 (Mass Distribution Principle)**.** *Let (X,d) be a complete metric space and $F \subseteq X$. If F supports a measure μ such that for some $s \geq 0$, and $c, \rho > 0$*

$$\mu B(x,r) \leq cr^s,$$

then $\dim_H F \geq s$.

Proof. Take $0 < 2r < \rho$ and let $(E_n)_{n=1}^{\infty}$ be an r-cover of F. For each E_n take a ball $B_n = B(x_n, \operatorname{diam} E_n)$ containing E_n, then

$$\mu F \leq \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu B_n \leq c\sum_{n=1}^{\infty} (\operatorname{diam} E_n)^s \implies \sum_{n=1}^{\infty} (\operatorname{diam} E_n)^s \geq \frac{\mu F}{c} > 0.$$

Hence, the infimum over all r-coverings is positive and $\dim_H F \geq s$. $\square$.

Let us establish the framework for the Kristensen-Thorne-Velani Theorem. Let (X,d) be a complete metric space, $\Omega \subseteq X$ a compact subset, and m a non-atomic Borel measure supported in Ω. Consider a countable family of subsets of X, called **resonant sets**, $\mathcal{R} := \{R_\alpha : \alpha \in J\}$. Let $\beta : J \to \mathbb{R}_{>0}$ satisfy

$$\forall M > 0 \quad \#\beta^{-1}[(0,M]] < +\infty,$$

and $\rho : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be such that

$$\lim_{r\to\infty} \rho(r) = 0, \qquad \exists R > 0 \quad \forall r,s \in \mathbb{R}_{>R} \quad \big(r < s \implies \rho(s) < \rho(r)\big).$$

The badly approximable elements, $\mathbf{Bad}^*(\mathcal{R},\beta,\rho)$, are the points which stay away (in terms of ρ) from resonant sets:

$$\mathbf{Bad}^*(\mathcal{R},\beta,\rho) := \{x \in \Omega : \exists c > 0 \quad \forall \alpha \in J \quad c\rho(\beta_\alpha) \leq d(x,R_\alpha).\}$$

In our scenario, the measure m and the function ρ satisfy the following conditions:

(A) There are some $r_0, \delta > 0$ such that for some absolute constants

$$\forall c \in \Omega \qquad \forall r \in (0, r_0] \qquad m(B(c,r)) \asymp r^\delta.$$

(B) There are functions $\lambda^l, \lambda^u : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\lambda^l(k) \to \infty$ as $k \to \infty$ and some $K > 0$ with

$$\forall k \in \mathbb{R}_{\geq K} \qquad \forall n \in \mathbb{N} \qquad \lambda^l(k) \leq \frac{\rho(k^n)}{\rho(k^{n+1})} \leq \lambda^u(k).$$

If $k > 1$ is fixed, define

$$\forall n \in \mathbb{N} \quad J(n) = \beta^{-1}[[k^{n-1}, k^n)]$$

Whenever $B \subseteq X$, $c(B) \in X$ will denote a center of B; i.e., for some $r > 0$ we have $B = B(c(B), r)$.

Theorem 4.2.8 (S. Kristensen, R. Thorn, S. Velani (2006))**.** *Let* $(X, d, \Omega, m, \beta, \rho)$ *be as above. Suppose that* m *satisfies Condition (A) and that* ρ *satisfies Condition (B). Assume there is some* $k_0 > 1$ *such that for every* $k \geq k_0$ *there exists* $\theta = \theta(k) > 0$ *satisfying the following:*

For every $n \in \mathbb{N}$ *and all* $c \in X$, $B_n := B(c; \rho(k^n))$, *there is a collection* $\mathcal{C}(B_n)$ *of pairwise disjoint balls of radius* $2\theta\rho(k^{n+1})$ *such that*

$$\bigcup_{B \in \mathcal{C}(B_n)} B \subseteq B(c, \theta\rho(k^n)),$$

and for some numbers $0 < \kappa_1 < \kappa_2$ *independent of* k *and* n *we have*

$$\kappa_1 \left(\frac{\rho(k^n)}{\rho(k^{n+1})} \right)^{\delta} \leq \#\mathcal{C}(B_n) \tag{4.11}$$

and

$$\#\left\{ B \in \mathcal{C}(B_n) : \min_{\alpha \in J(n+1)} d(c(B), R_\alpha) \leq 2\theta\rho(k^{n+1}) \right\} \leq \kappa_2 \left(\frac{\rho(k^n)}{\rho(k^{n+1})} \right)^{\delta}. \tag{4.12}$$

Additionally, suppose that for the δ *of Condition (A) we have* $\dim_H \bigcup_{\alpha \in J} R_\alpha < \delta$, *then*

$$\dim_H \mathbf{Bad}^*(\mathcal{R}, \beta, \rho) = \delta.$$

It can be checked (see Example 1 in [KTV06]), that Theorem 4.2.8 implies $\dim_H \mathbf{Bad}_{\mathbb{R}} = 1$.

4.3 A word on Schmidt Games

Wolfgang M. Schmidt introduced in [Sch66] a useful game to estimate the Hausdorff dimension of some sets. We comment them only in their simplest form. To start, fix a set $S \subseteq \mathbb{R}^n$. The game is played between two players, A and B, and exactly one of them will win. Assign to A and B, respectively, a number α, β in $(0, 1)$. The game starts with B choosing a compact ball B_1 of radius $\rho_1 > 0$. Then, A chooses a compact ball $B_2 \subseteq B_1$ of radius $\rho_2 := \alpha\rho_1$. Afterwards, B chooses a compact ball $B_3 \subseteq B_2$ of radius $\rho_3 = \beta\rho_2$ and continue infinitely. The sequence of radii, $(\rho_n)_{n\geq 1}$, tends to 0, hence intersection $\bigcap_n B_n$ contains only one point s. A wins if $s \in S$; otherwise, B wins.

A set $S \subseteq \mathbb{R}^n$ is **(α,β)-winning** if no matter how B plays, A can find a way to win. For a given $0 < \alpha < 1$, we say that S is **α-winning** if it is (α,β)-winning for every $0 < \beta < 1$. It can be shown (Lemma 11 in [Sch66]) that if S is α-winning, then it is α'-winning for every $0 < \alpha' < \alpha$. It is thus natural to define the winning dimension of S, $\operatorname{win}\dim S$, as

$$\operatorname{win}\dim S := \sup\{\alpha > 0 : S \text{ is } \alpha\text{-winning}\}$$

when the referred set is non-empty and $\operatorname{win}\dim S := 0$ otherwise. We call a set with positive winning dimension **winning**.

Winning sets possess desirable closure properties.

Proposition 4.3.1. *Let S, $(S_j)_{j\geq 1}$ be subsets of $\mathbb{R}^n$ and $\alpha > 0$.*

i. If $\operatorname{win}\dim S > 0$, then $\dim_H S = n$.

ii. If every S_n is α-winning, then $\bigcap_n S_n$ is α-winning.

iii. If S is winning, then it is imcompressible; that is, for every non-empty open set $U \subseteq \mathbb{R}^n$ and every sequence of uniformly bi-Lipschitz maps $(f_n)_{n\geq 1}$, $f_n : U \to \mathbb{R}^n$,

$$\dim_H \left(\bigcap_{n\geq 1} f_n^{-1}[S] \right) = n.$$

The first two statements are proven in [Sch66] as Corollary 2 and Theorem 2, respectively. The third one is shown in [Da89] as Theorem 1.1.

W. M. Schmidt showed that $\operatorname{win}\dim \mathbf{Bad}_{\mathbb{R}} = \frac{1}{2}$. In the complex plane, the analogous result was obtained by M. Dodson and S. Kristensen for $\mathbf{Bad}_{\mathbb{C}}$ (Theorem 5.2. in [DoKr]). A further extension was obtained by R. Esdahl-Schou and S. Kristensen in [EK10]. For each

$$D \in \{1, 2, 3, 5, 7, 11, 19, 43, 67, 163\}$$

define

$$\mathbf{Bad}_D := \left\{ z \in \mathbb{C} : \exists K > 0 \quad \forall (p,q) \in \mathbb{Z}\left[\sqrt{-D}\right] \times \mathbb{Z}\left[\sqrt{-D}\right]^* \ \left| z - \frac{p}{q} \right| \geq \frac{K}{|q|^2} \right\}.$$

Theorem 4.3.2 (R. Esdahl-Schou, S. Kristensen, 2009)**.** *Let $K \subseteq \mathbb{C}$ be a compact supporting a measure μ satisfying*

$$\forall z \in K \quad \forall r \in (0, r_0) \quad ar^\delta \leq \mu(\mathbb{D}(z,r)) \leq br^\delta,$$

where $a, b, \delta, r_0 > 0$ *are constant. Then, any*

$$\varnothing \neq E \subseteq \{1, 2, 3, 5, 7, 11, 19, 43, 67, 163\}$$

verifies

$$\operatorname{win} \dim \left(K \cap \bigcap_{D \in E} \mathbf{Bad}_D \right) > 0.$$

It trivially follows from Theorem 4.3.2 that

$$\dim_H \mathbf{Bad}_{\mathbb{C}} = \dim_H \mathbf{Bad}_1 = 2.$$

A recent survey on some variations of Schmidt games and their relation can be found in [BHNS18].

4.4 Hausdorff Dimension for HCF Sets

Let us return, after our short digression, to the complex plane and HCF. Once again, $[a_0; a_1, a_2, \ldots]$ will represent a HCF.

4.4.1 Measure and Dimension of $\mathbf{Bad}_{\mathbb{C}}$

W. J. Leveque developed a complex continued fraction algorithm in [Le52a] and [Le52b]. In the second paper, he could show an analogue of a famous result by A. Khinchin (Theorem 1.1 in [Bu04], or Theorem 32 in [Kh]). We need some terminology for stating Leveque's result. For any continuous and non-increasing function $\Psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, the set of **$\Psi$-well approximable numbers** is

$$W(\Psi) = \left\{ z \in \mathfrak{F} \smallsetminus \mathbb{Q}(i) : \left| z - \frac{p}{q} \right| < \Psi(q) \text{ for infinitely many } (p, q) \in \mathbb{Z}[i] \times \mathbb{Z}[i]^* \right\}.$$

Although we can surely have defined $W(\Psi)$ as a subset of $\mathbb{C}$, for notational convenience we opted not to. It should be clear that there is absolutely no loss of generality by doing this.

Theorem 4.4.1 (W. J. Leveque (1952))**.** *Let* $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *be a continuous function such that* $t \mapsto t^2 f(t)$ *is non-increasing. Then,*

$$\mathfrak{m} W_{\mathbb{C}}(\Psi) = \begin{cases} 0, & \text{if } \sum_{n \geq 1} n^3 \Psi(n)^2 < +\infty, \\ 1, & \text{if } \sum_{n \geq 1} n^3 \Psi(n)^2 = +\infty. \end{cases}$$

Proof. See Theorem 7, [Le52b]. □

The convergence part is an easy exercise on the Borel-Cantelli Lemma. The usual argument for the real version of Khinchin's Theorem (Theorem 1.10 in [Bu04]) relies on the existence of the Khinchin-Lévy constant and the Borel-Bernstein Theorem. In [Le52a] and [Le52b], Leveque obtained similar propositions that permitted him to conclude Theorem 4.4.1.

Let $\Psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be a continuous function such that $\Psi(x) = (x^2\sqrt{\log x})^{-1}$ for $x \geq 2$. Since $\mathfrak{F} \setminus W_{\mathbb{C}}(\Psi) \supseteq \mathbf{Bad}_{\mathbb{C}} \cap \mathfrak{F}$, Theorem 4.4.1 yields

$$\mathfrak{m}\mathbf{Bad}_{\mathbb{C}} = 0.$$

Thus, asking for the Hausdorff dimension of $\mathbf{Bad}_{\mathbb{C}}$ is not futile. We have already obtained from Theorem 4.3.2 that $\dim_H \mathbf{Bad}_{\mathbb{C}} = 2$. However, we give another proof, already pointed out in [KTV06], based on Theorem 4.2.8.

Theorem 4.4.2. $\dim_H \mathbf{Bad}_{\mathbb{C}} = 2$.

Proof. Recall that $\operatorname{Cl} A$ denotes the closure of a set A. Define

$$X := \Omega := \operatorname{Cl}\mathfrak{F}, \quad J = \mathbb{Z}[i] \times \mathbb{Z}[i]^*, \quad \forall \alpha = (p,q) \in J \quad R_\alpha := \left\{\frac{p}{q}\right\},$$
$$m = \mathfrak{m}, \quad \forall (p,q) \in J \quad \beta(p,q) = q, \quad \forall k > 0 \quad \rho(k) = k^{-2},$$
$$\kappa_1 = \frac{1}{25}, \quad \kappa_2 = \frac{1}{16}, \quad \delta = 2, \quad k_0 = 6.$$

Consider the metric $\tilde{d}_\infty$ given by

$$\forall z, w \in \mathbb{C} \quad \tilde{d}_\infty(z,w) = \max\{|\mathfrak{R}(z-w)|, |\mathfrak{I}(z-w)|\}$$

and denote by d_∞ its restriction to $\Omega = X$.

We now verify Condition (A). Let $B(z,r)$ denote the ball with center z and radius r with respect to the metric d_∞. Take $z \in X$, $r \in (0, \frac{1}{2})$ and divide $B_{\tilde{d}_\infty}(z,r)$ into four equal squares in the natural way. Note that $B_{d_\infty}(z,r)$ contains at least one of this quarters, so

$$\forall z \in \Omega \quad \forall r \in (0, 0.5) \quad \frac{\mathfrak{m}B(z,r)}{r^\delta} = \frac{\mathfrak{m}B(z,r)}{r^2} \asymp 1,$$

with the implied constants absolute (in fact, 4^{-1} and 4 work). Condition (B) is evident by taking $\lambda^l(k) = \lambda^u(k) = k^2$ for any $k > k_0$.

We now check the additional hypothesis of Theorem 4.2.8. First, note that for $k > k_0$ and any $n \in \mathbb{N}$ we clearly have

$$\left(\frac{\rho(k^n)}{\rho(k^{n+1})}\right)^\delta = k^4.$$

Let $\alpha = (p, q)$ and $\alpha' = (p', q')$ belong to $J(n+1)$ with $R_\alpha \neq R_{\alpha'}$, then

$$d_\infty(R_\alpha, R_{\alpha'}) \geq \frac{1}{\sqrt{2}}\left|\frac{p}{q} - \frac{p'}{q'}\right| \geq \frac{1}{\sqrt{2}}\frac{1}{|qq'|} > \frac{1}{\sqrt{2}}\frac{1}{k^{2n+2}} > \frac{1}{4k^{2n+2}}. \tag{4.13}$$

For any $k > k_0$ define $\theta(k) = \frac{1}{4}k^{-2}$. Thus, for any $c \in \Omega$ the ball $B_n = B(c, \theta\rho(k^n))$ contains at most one point of $J(n+1)$. Suppose we had

$$B_n \supseteq \left\{c + is + t \in \mathbb{C} : 0 \leq s, t < \frac{k^{-2n-2}}{4}\right\},$$

(if this did not hold, we just take the appropriate square by considering $\pm s$, $\pm t$). By direct computations, we see that the set to the right contains at least $\frac{k^4}{16} = \kappa_1\left(\frac{\phi(k^n)}{\phi(k^{n+1})}\right)^\delta$ squares with sides of length $2\theta\rho(k^{n+1})$. Also, in view of (4.13), the term at the left in (4.12) is at most $1 \leq \kappa_2 k^4$.

Finally, since $\dim\left(\bigcup_\alpha R_\alpha\right) = 0 < \delta$, Theorem 4.2.8 guarantees

$$\dim_H \mathbf{Bad}_{\mathbb{C}} = \dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho) = \delta = 2.$$

□

4.4.2 Good's Theorem for HCF

The equivalence between the boundedness of the HCF expansion and belonging to $\mathbf{Bad}_{\mathbb{C}}$ allows us to rewrite Theorem 4.4.2 as

$$\dim_H \left\{z = [a_0; a_1, a_2, a_3, \ldots] \in \mathbb{C} : \limsup_{n\to\infty} |a_n| < +\infty\right\} = 2.$$

In 1941, I. J. Good obtained that the Hausdorff dimension of the set of irrationals whose regular continued fraction tends to infinity is $\frac{1}{2}$. We can modify Good's argument– which is itself a modification of Jarník's– to deduce a full analogue of his theorem for HCF.

Theorem 4.4.3.

$$\dim_H \left\{z = [a_0; a_1, a_2, a_3, \ldots] \in \mathbb{C} : \lim_{n\to\infty} |a_n| = +\infty\right\} = 1.$$

We merely sketch the proof, the details are in [Gon18-02]. As noted before, it is enough to work in $\mathfrak{F}$. Define the set

$$E = \{\zeta = [0; a_1, a_2, \ldots] \in \mathfrak{F} : \lim_{n\to\infty} |a_n| = +\infty\}.$$

In order to obtain an upper bound of $\dim_H E$, we consider

$$\forall L \in \mathbb{N} \quad E'_L := \{\zeta = [0; a_1, a_2, \ldots] \in \mathfrak{F} : \liminf_{n\to\infty} |a_n| \geq L\},$$
$$\forall L, k \in \mathbb{N} \quad E'_L(k) := \{\zeta = [0; a_1, a_2, \ldots] \in \mathfrak{F} : \forall n \in \mathbb{N}_0 \quad |a_{n+k}| \geq L\}.$$

Lemma 4.4.4. $\lim_{L\to\infty} \dim_H E'_L(1) \le 1$.

Lemma 4.4.5. *Let $f, g : \mathbb{N} \to \mathbb{R}_{>0}$ be functions such that for some fixed c with $0 < c < 1$ we have $\sqrt{8} \le f(n) < (1-c)g(n)$ and $g(n)^2 \le e^n$ for all $n \in \mathbb{N}$. Then,* $\dim_H E_{f,g} = 1$.

For every L, $(E'_L(k))_{k\ge1}$ is an increasing sequence converging to E'_L, so

$$\lim_{k\to\infty} \dim_H E'_L(k) = \dim_H E'_L.$$

However, by the invariance of the Hausdorff dimension under bi-Lipschitz maps, $\dim_H E'_L(k) = \dim_H E'_L(1)$ and, thus, $\dim_H E'_L = \dim_H E'_L(1)$ for all k. Taking limits, $\dim_H E_k(1)' = \dim_H E'_M$. As a consequence, by $E = \bigcap_M E'_M$, $\dim_H E \le \dim_H E'_M$ for all M. Lemma 4.4.4, thus, gives

$$\dim_H E' \le 1.$$

The lower bound, as usual, is trickier. The idea is to approximate E' from the inside by sets where we can control the growth of $|a_n|$. To be more precise, given $f, g : \mathbb{N} \to \mathbb{R}_{>0}$ with $f \le g$, define

$$E_{f,g} = \{\zeta = [0; a_1, a_2, a_3, \ldots] : \forall n \in \mathbb{N} \quad f(n) \le \|a_n\| \le g(n)\},$$

where $\|z\| = \max\{|\Re z|, |\Im z|\}$ for any $z \in \mathbb{C}$.

Clearly, $f(n) \to \infty$ as $n \to \infty$ implies $E_{f,g} \subseteq E$. Lemma 4.4.5 tells us that under certain restrictions, $\dim_H E_{f,g} = 1$, so

$$1 \le \dim_H E.$$

The Generalized Jarník Lemmas lie in the heart of the proof of Lemmas 4.4.4 and 4.4.5.

4.5 Independent Blocks

Throughout this work we have seen that similarities between regular and complex continued fractions abound. We have pointed out important differences too. Most notably, we have contrasted the intricate structure of Ω_R^{HCF} against $\mathbb{N}^{\mathbb{N}}$. The complications in Ω^{HCF} disappear when we consider only sequences $(a_n)_{n\ge1}$ satisfying $\min_n |a_n| \ge \sqrt{8}$. It is natural to investigate the subsets of $\mathbf{Bad}_{\mathbb{C}}$ where uniform lower bounds are imposed. In this direction, the proof of Theorem 4.4.3 yields the next corollary (see [Gon18-02] for details).

Corollary 4.5.1. *For any $L \in \mathbb{N}$ with $L \geq \sqrt{8}$ define*

$$E_L := \{z = [0; a_1, a_2, a_3, \ldots] \in \mathfrak{F} : \forall n \in \mathbb{N} \quad L \leq |a_n|\},$$

then

$$\lim_{L \to \infty} \dim_H \left(E_L \cap \mathbf{Bad}_{\mathbb{C}}\right) = 1.$$

Working with blocks, rather than individual terms, we obtain larger subsets of $\mathbf{Bad}_{\mathbb{C}}$. For $z \in \mathbb{C}$ write $\mathscr{M}(z) := \min\{|\Re z|, |\Im z|\}$ and for any $j_0, M, l \in \mathbb{N}$ define

$$\mathcal{A}(j_0, M, l) := \left\{(a_n)_{n \geq 1} \in \Omega_R^{\mathrm{HCF}} : \forall n \in \mathbb{N} \quad |a_n| \leq M,\ j_0 \leq \mathscr{M}(a_{n(l+1)})\right\},$$
$$\mathcal{F}(j_0, M, l) := \Phi[\mathcal{A}(j_0, M, l)].$$

We devote the rest of the chapter to show the next result. Note that, in this case, we can naturally interpret the blocks of length $l+1$ as independent identically distributed random variables.

Theorem 4.5.2. *If $j_0 \in \mathbb{N}$ satisfies $j_0 \geq 3$, then*

$$\sup_{l, M \in \mathbb{N}} \dim_H \mathcal{F}(j_0, M, l) = 2.$$

4.5.1 Preliminary Lemmas

The mere definition of $\mathcal{F}(j_0, M, l)$ makes clear how to define the appropriate diameter tree-like structure. Our first preliminary result, Lemma 4.5.3, is a particular case of the First Generalized Jarník Lemma. The verification of the hypothesis hides some delicate details persuading us to give thorough exposition.

Lemma 4.5.3. *Fix $j_0, m, M \in \mathbb{N}$ with $j_0 \geq 3$. Let $s > 0$ be such that for $r \in \mathbb{N}$ and any $\mathbf{a} \in \mathcal{A}(j_0, m, M)_{[1..r(m+1)]}$ we have that*

$$\sum_{\mathbf{c}} \left|\mathcal{C}_{(r+1)(m+1)}(\mathbf{a}; \mathbf{c})\right|^s \geq \left|\mathcal{C}_{r(m+1)}(\mathbf{a})\right|^s > 0, \tag{4.14}$$

where $\mathbf{c}$ runs along $\mathcal{A}(j_0, m, M)[1..m+1]$. Then, $\dim_H \mathcal{F}(j_0, m, M) \geq s$.

Recall that the chordal metric ρ is a metric on $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, given by

$$\forall w, z \in \mathbb{C} \quad \rho(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}};$$

if $w = \infty$ and $z \in \mathbb{C}$

$$\rho(z, w) = \frac{2}{\sqrt{1 + |z|^2}},$$

and the conditions $\rho(w, z) = \rho(z, w)$, $\rho(\infty, \infty) = 0$. The proof of the next proposition can be found in A. Beardon's book ([Be91], Theorem 2.3.2., p.33).

Proposition 4.5.4. *The transformation $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given by*

$$Tz = \frac{az+b}{cz+d}$$

with $ad-bc=1$ is a Lipschitz function with respect to the chordal metric with Lipschitz constant $\|T\|^2 := |a|^2+|b|^2+|c|^2+|d|^2$.

Proof of Lemma 4.5.3. Let $j_0, M, l \in \mathbb{N}$ be such that $\mathcal{A}(j_0, M, l) \neq \varnothing$ and $j_0 \geq 3$. For $n \in \mathbb{N}$ and $\mathbf{a} \in \mathcal{A}(j_0, M, n(l+1))$ define

$$H_n(\mathbf{a}) := \mathcal{C}_{(l+1)n}(\mathbf{a}).$$

Lemma 4.5.3 follows from the First Jarník Lemma (Lemma 4.2.5) using the closure of the sets $H_n(\mathbf{a})$ as the family of compact sets once we have established the existence of some $B = B(j_0, M, l) > 0$ such that for $n \in \mathbb{N}$ and $\mathbf{a} \in \mathcal{A}(j_0, M, l+1)[1..n(l+1)]$ and $\mathbf{b}, \mathbf{c} \in \mathcal{A}(j_0, M, l+1)[1..l+1]$

$$d(H_{n+1}(\mathbf{ab}), H_{n+1}(\mathbf{ac})) > B|H_n(\mathbf{a})|. \tag{4.15}$$

First, we show that for some $\kappa = \kappa(j_0, M, l)$ we have

$$\forall \mathbf{b}, \mathbf{c} \in \mathcal{A}(j_0, M, l)[1..l+1] \qquad \mathbf{b} \neq \mathbf{c} \implies d(H_1(\mathbf{b}), H_1(\mathbf{c})) > \kappa. \tag{4.16}$$

If it happened otherwise, there would be two different $\mathbf{a}, \mathbf{b}$ in the finite set $\mathcal{A}(j_0, M, l)_{[1..l+1]}$ such that $d(H_1(\mathbf{a}), H_1(\mathbf{b})) = 0$. We claim that

$$\forall z \in \mathcal{C}_l(\mathbf{a})^\circ \quad \forall \varepsilon > 0 \quad d(z, H_1(\mathbf{b})) < \varepsilon \implies d(z, \operatorname{Fr} H_1(\mathbf{a})) \leq \varepsilon. \tag{4.17}$$

To see it, take $z \in H_1(\mathbf{a})^\circ$, $\varepsilon, \varepsilon' > 0$, $w' \in H_1(\mathbf{b})$, and $w \in H_1(\mathbf{b})$ we have $d(z,w) < \varepsilon + \varepsilon'$. By continuity of the function $f_1 : [0,1] \to \mathbb{C}$, $f(t) = z + t(w-z)$, and the connectedness of $[0,1]$, $f_1(t') \in Fr H_1(\mathbf{a})$ for some $0 < t' < 1$. The same argument holds when $H_1(\mathbf{a})$ and $H_1(\mathbf{b})$ are replaced by two cylinders of the same depth.

Call $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ the $\mathcal{Q}$-pair of $\mathbf{a}$. Let f be the Möbius transformation extending $T^l|\mathcal{C}_l(\mathbf{a})$ to the entire plane. By uniform continuity of f on $H_1(\mathbf{a})$ (its pole is $\frac{p_{l-1}}{q_{l-1}} \notin \mathcal{C}_l(\mathbf{a})$), for every $(z,w) \in \mathcal{C}_l(\mathbf{a})^\circ \times \operatorname{Fr} \mathcal{C}_l(\mathbf{a})$

$$\forall \delta > 0 \quad \exists \delta' > 0 \quad d(z,w) < \delta' \quad \implies \quad d(T^l(z), f(w)) < \delta.$$

Thus, by the continuity of the complex inversion ι, for any $z \in H_1(\mathbf{a})$ there is some $\delta > 0$ such that

$$\forall w \in \mathbb{C}^* \quad d(T^l z, w) < \delta \implies d((T^l(z))^{-1}, w^{-1}) < \frac{1}{4}.$$

Therefore, if $z \in H_1(\mathbf{a})^\circ$ and $w \in FrH_1(\mathbf{a})$ could be arbitrarily close, we would conclude

$$d\left((T^l(z))^{-1}, \operatorname{Fr}\iota[f[\mathcal{C}_l(\mathbf{a})]]\right) < \frac{1}{4}.$$

As a consequence, we would obtain the contradiction

$$a_{l+1} = \left[\frac{1}{T^l(z)}\right] \in \left\{a \in \mathbb{Z}[i] : |\Re a| = 1 \ \vee \ |\Im a| = 1 \ \vee \ |a| \leq \sqrt{8}\right\}.$$

Therefore, (4.16) must hold.

Now, take $n \in \mathbb{N}$, $\mathbf{a} \in \mathcal{A}(j_0, M, l)[1..n(l+1)]$ with $\mathcal{Q}$-pair $(p_n)_{n\geq 0}, (q_n)_{n\geq 0}$ and $\mathbf{b}, \mathbf{c} \in \mathcal{A}(j_0, M, l)[1..l+1]$. Let R be the Möbius transformation extending the inverse of $T^{n(l+1)}$ restricted to $\mathcal{C}_{n(l+1)}(\mathbf{a})$:

$$R(z) = \frac{p_{n(l+1)}z + p_{n(l+1)-1}}{q_{n(l+1)}z + q_{n(l+1)-1}};$$

then,

$$\begin{aligned} d(H_{n+1}(\mathbf{ab}), H_{n+1}(\mathbf{ac})) &= d(R[H_1(\mathbf{b})], R[H_1(\mathbf{c})]) \\ &\geq \frac{1}{2}\rho(R[H_1(\mathbf{b})], R[H_1(\mathbf{c})]) \\ &\geq \frac{1}{2\|R\|^2}\rho(R[H_1(\mathbf{b})], R[H_1(\mathbf{c})]) \\ &\geq \frac{1}{2\|R\|^2}d(R[H_1(\mathbf{b})], R[H_1(\mathbf{c})]). \end{aligned}$$

As a consequence, for a constant depending on j_0, M, l (κ_1),

$$d(H_{n+1}(\mathbf{ab}), H_{n+1}(\mathbf{ac})) \gg \frac{1}{\|R\|^2} \geq |\mathcal{C}_{n(l+1)}(\mathbf{a})| = |H_n(\mathbf{a})|.$$

□

An immediate outcome of the Isodiametric Inequality (Theorem 1 in [EvGa15], p. 69) and Lakein's First Theorem from Chapter 1 is the following lemma.

Lemma 4.5.5. *There exist absolute constants such that for any regular sequence* $\mathbf{a} \in \Omega^{HCF}$

$$\forall n \in \mathbb{N} \quad \mathfrak{m}\mathcal{C}_n(\mathbf{a}) \asymp |\mathcal{C}_n(\mathbf{a})|^2.$$

On the basis of equation (3.2) in Chapter 3, and an inductive argument, we can deduce the following lemma.

Lemma 4.5.6. *For any $j_0 \in \mathbb{N}$ and sufficiently large m, M there exists $K = K(j_0, m, M) > 0$ of the form $K = R^m r$ such that*

1. *$r = r(j_0) > 0$ and $R = R(M) \nearrow 1$ when $M \to \infty$ for every $j_0 > 0$.*
2. *For any $k \in \mathbb{N}$ and any $\mathbf{a} \in \mathcal{A}(j_0, m, M)[1..r(m+1)]$ we have*

$$\sum_{\mathbf{c}} \mathfrak{m}\mathcal{C}_{(k+1)(m+1)}(\mathbf{a}; \mathbf{c}) \geq K \mathfrak{m}\mathcal{C}_{k(m+1)}(\mathbf{a})$$

where the sum runs over $\mathcal{A}(j_0, m, M)[1..m+1]$.

4.5.2 Proof of Theorem 4.4.3

Proof of Theorem 4.4.3. Fix $j_0, M, l \in \mathbb{N}$ such that $\mathcal{A}(j_0, M, l) \neq \varnothing$. Define $k_1 := k_1(l, M)$ by

$$k_1(l, M) := \left(2 - \frac{1}{M+1}\right)^{-4} \phi^{-4(l+1)},$$

and take the $K = K(l, M)$ provided by Lemma 4.5.6; hence,

$$\begin{aligned}
\forall t \in \mathbb{N} \qquad & k_1 \mathfrak{m}\mathcal{C}_{t(l+1)}(\mathbf{a}) \geq \mathfrak{m}\mathcal{C}_{(t+1)(l+1)}(\mathbf{a}; \mathbf{c}), \\
\forall t \in \mathbb{N} \qquad & \sum_{\mathbf{c}} \mathfrak{m}\mathcal{C}_{(t+1)(l+1)}(\mathbf{a}; \mathbf{c}) \geq K \mathfrak{m}\mathcal{C}_{t(l+1)}(\mathbf{a}),
\end{aligned}$$

where $\mathbf{c}$ runs along $\mathcal{A}(j_0, M, l)_{[1..l+1]}$. Let $0 < \alpha < \beta$ be the implied constants in Lemma 4.5.5, then

$$\begin{aligned}
\sum_{\mathbf{c}} \left|\mathcal{C}_{(k+1)(l+1)}(\mathbf{a}; \mathbf{c})\right|^{2s} &\geq \alpha^s \sum_{\mathbf{c}} \left(\mathfrak{m}\mathcal{C}_{(k+1)(l+1)}(\mathbf{a}; \mathbf{c})\right)^s \\
&= \alpha^s \sum_{\mathbf{c}} \mathfrak{m}\mathcal{C}_{(k+1)(l+1)}(\mathbf{a}; \mathbf{c}) \frac{1}{\left(\mathfrak{m}\mathcal{C}_{(k+1)(l+1)}(\mathbf{a}; \mathbf{c})\right)^{1-s}} \\
&\geq \alpha^s \frac{1}{k_1^{1-s} \left(\mathfrak{m}\mathcal{C}_{k(l+1)}(\mathbf{a}; \mathbf{c})\right)^{1-s}} \sum_{\mathbf{c}} \mathfrak{m}\mathcal{C}_{(k+1)(l+1)}(\mathbf{a}; \mathbf{c}) \\
&\geq \alpha^s \frac{K}{k_1^{1-s} \left(\mathfrak{m}\mathcal{C}_{k(l+1)}(\mathbf{a}; \mathbf{c})\right)^{1-s}} \mathfrak{m}\mathcal{C}_{k(l+1)}(\mathbf{a}; \mathbf{c}) \\
&= \alpha^s \frac{K}{k_1^{1-s}} \mathfrak{m}\mathcal{C}_{k(l+1)}(\mathbf{a}; \mathbf{c})^s \\
&\geq \left(\frac{\alpha}{\beta}\right)^s \frac{K}{k_1^{1-s}} \left|\mathcal{C}_{k(l+1)}(\mathbf{a}; \mathbf{c})\right|^{2s}.
\end{aligned}$$

By Lemma 4.5.3, $\dim_H \mathcal{F}(j_0, M, l) \geq 2s$ whenever

$$\gamma^s \frac{K}{k_1^{1-s}} \geq 1,$$

where $\gamma = \alpha\beta^{-1}$, which holds if and only if

$$s\log\gamma + \log K - (1-s)\log k_1 \geq 0 \iff (1-s) \geq \frac{\log K + \log\gamma}{\log k_1 + \log\gamma} > 0.$$

Thus, it suffices to show that the last expression can be as small as we please. Expanding k_1 and K, keeping the notation of Lemma 4.5.6, we get

$$0 < \frac{\log K + \log\gamma}{\log k_1 + \log\gamma} = \frac{\log\left(R(M)\right) + \frac{\log r + \log\gamma}{l+1}}{\log\phi + \frac{\log\gamma}{l+1}},$$

which can be arbitrarily small for suitable l, M. Hence, by taking $s \to 1$ we conclude that

$$\sup_{M,l\in\mathbb{N}} \dim_H \mathcal{F}(j_0, M, l) = 2.$$

$\square$

4.6 Notes and comments

1. Theorem 4.5.2 is new. The omitted details are part of a paper in process.

Chapter 5

Hurwitz - Tanaka Continued Fractions

In this chapter, we discuss the ergodic theory of the continued fraction algorithm suggested by Julius Hurwitz (Adolph Hurwitz's older brother) in 1895 and re-discovered by S. Tanaka about a century later ([Ta85]). We define the Hurwitz-Tanaka Continued Fractions within the framework of Dani and Nogueira and see that they provide a counterexample to Theorem 1.1.2. A more dramatic feature of the Hurwitz-Tanaka algorithm is that the resulting continued fractions might not even make sense, let alone represent the original complex number. However, discarding a suitable null set, a satisfactory ergodic theory can be developed.

Notation. The ideal of $\mathbb{Z}[i]$ generated by $1+i$ is $(1+i)$. The set of non-zero elements of $(1+i)$ is $(1+i)^*$. The complex inversion $z \mapsto z^{-1}$ is denoted by ι. $\vee$ is the logic disjunction connector. If A is a finite set and $B \subseteq A^{\mathbb{N}}$, then $\sigma : B \to A^{\mathbb{N}}$ is the shift map $\sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$. The same notation σ is used for two-sided sequences.

Main references. For the ergodic theoretical aspects of the Hurwitz-Tanaka continued fractions, see [Ta85]. For more details and historical aspects, see [Os16].

5.1 Definition and convergence

Identify $\mathbb{R}^2$ with $\mathbb{C}$ in the natural way: $x + iy \sim (x, y)$. Let Λ be the lattice generated by $\alpha := (1,1) \sim 1+i$ and $\overline{\alpha} := (1,-1) \sim 1-i$. Since the square $\{r\alpha + s\overline{\alpha} : 0 \leq r, s < 1\}$ is a complete set of representatives of $\mathbb{R}^2/\Lambda$, so it is its translate

$$\mathfrak{T} := \left\{ r\alpha + s\overline{\alpha} : -\frac{1}{2} \leq r, s < \frac{1}{2} \right\}.$$

With the partition $\{a + \mathfrak{T} : a \in (1+i)\}$ we can naturally define a choice function $f_T : \mathbb{C} \to (1+i) \subseteq \mathbb{Z}[i]$ by the condition

$$\forall z \in \mathfrak{T} \quad z \in f_T(z) + \mathfrak{T}.$$

and the corresponding f_T-Gauss map by $\tau : \mathfrak{T}^* \to \mathfrak{T}^*$, $\tau(z) = z^{-1} - f_T(z^{-1})$.

Definition 5.1.1. *The* ***Hurwitz-Tanaka continued fraction*** *(TCF) of a complex number z is the f_T-sequence of z. If $(a_n)_{n \geq 0}$ is the TCF of z, we adopt the formal notation $[0; a_1, a_2, \ldots]_T$.*

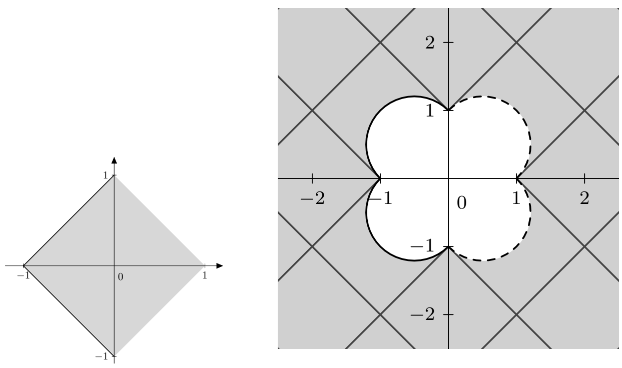


Figure 5.1: Left: $\mathfrak{T}$. Right: Inverse of $\mathfrak{T}$.

Straight forward computations tell us that the TCF of -1 is the constant sequence $(0; 0, 0, \ldots)$. Clearly, $[0; 0, 0, \ldots]_T$ cannot converge to -1. It is natural to ask under what conditions is the TCF of a complex rational convergent or even finite. J. Hurwitz asked this to himself too and found the following answer.

Theorem 5.1.1 (J. Hurwitz, 1895)**.** *A number $z \in \mathbb{Q}(i)$ has infinite TCF if and only if $\tau^n(z) = -1$ for some $n \in \mathbb{N}_0$; moreover,*

$$\bigcup_{n \geq 0} \tau^{-n}[-1] = \left\{ \frac{h}{k} \in \mathbb{Q}(i) : h, k \in \mathbb{Z}[i] \quad h \equiv k \equiv 1 \pmod{1+i} \right\}. \tag{5.1}$$

Proof. Clearly, the TCF of a complex rational $z \in \mathfrak{T}$ is infinite when $\tau^n(z) = -1$ for some n. Assume that $z = \frac{h}{k} \in \mathfrak{T}$ is rational, $h, k \in \mathbb{Z}[i]$ co-prime, and that $\tau^n \neq -1$ for all $n \in \mathbb{N}_0$. For some co-prime $h_1, k_1 \in \mathbb{Z}[i]$ and some $a \in (1+i)$ we have

$$\frac{h_1}{k_1} = \tau\left(\frac{h}{k}\right) = \frac{k}{h} - a \in \mathfrak{T}, \tag{5.2}$$

and $|h_1| \leq |k_1|$. If the equality held, then we would have $\frac{h_1}{k_1} = -1$; hence, $|h_1| < |k_1| = |h|$. Repeating the argument we eventually get $h_n = 0$, the algorithm terminates, and the TCF is finite. This shows the first assertion.

We start proving the second part by noting that for $h, k \in \mathbb{Z}[i]$ co-prime, h_1, k_1 co-prime, and $a \in (1+i)$ such that (5.2) is true, we have that

$$hh_1 \equiv kk_1 \pmod{1+i}. \tag{5.3}$$

Let us start with $\subseteq$. Take $z = \frac{h}{k} \in \mathfrak{T} \cap \mathbb{Q}(i)$ with $h, k \in \mathbb{Z}[i]$ co-prime. If $\tau(z) = -1$, then $h_1 = -k_1$ and $k \equiv -h \equiv h \pmod{1+i}$, so $h \equiv k \equiv 1 \pmod{1+i}$ (otherwise $h, k \in (1+i)$). A recursive argument now shows that $h \equiv k \equiv 1 \pmod{1+i}$ if $\tau^n(\frac{h}{k}) = -1$ for some $n \in \mathbb{N}_0$.

To prove $\supseteq$, let $z \in \mathfrak{T} \setminus \bigcup_{n\geq 0} T^{-n}[-1]$ be a complex rational, then $T^N(z) = 0$ for some N and $z = \frac{p_N(z)}{q_N(z)}$. It follows directly from the definition of $\mathcal{Q}$-pair that

$$\forall n \in \mathbb{N}_0 \qquad p_{2n+1} \equiv q_{2n} \equiv 1 \pmod{1+i}, \quad p_{2n} \equiv q_{2n+1} \equiv 0 \pmod{1+i}.$$

Therefore, z is not the quotient of $h, k \in \mathbb{Z}[i]$ satisfying $h \equiv k \equiv 1 \pmod{1+i}$. □

The convergence situation is by far simpler in the irrational case. Indeed, take $z \in \mathfrak{T}'$. Since $\mathfrak{T} \cap C_1(0) = \{-1\}$, the orbit of z under τ, $(\tau^n(z))_{n\geq 0}$, is entirely contained in the unit disc. Therefore, the iteration sequence of z is non-degenerate (cfr. Theorem 1.1.2).

Theorem 5.1.2. *If z is any number in $\mathfrak{T}'$ and $(a_n)_{n\geq 0}$ its TCF, then $z = [a_0; a_1, \dots]_T$.*

We shall use freely the notions of cylinders, maximal feasible sets, regular and irregular objects. They are defined just as in the HCF context. The set of admissible sequences (for irrational numbers) is denoted by Ω^{TCF} and we define $\Omega^{\mathrm{TCF}}(n)$, Ω^{TCF}_R as in Chapter 3. The set of regular sequences is denoted by Ω^{TCF}_R and the set of irregular sequences by Ω^{TCF}_I. If $\mathbf{b}$ is a sequence in $(1+i)^*$, $\mathcal{C}^T_n(\mathbf{b})$ is the cylinder of depth n.

The next lemma can be shown using the dual algorithm (defined below) (see [Ta85]) or with an inductive argument (see [Os16]). Note that the exponent of $|q_n|$ is -1, not -2.

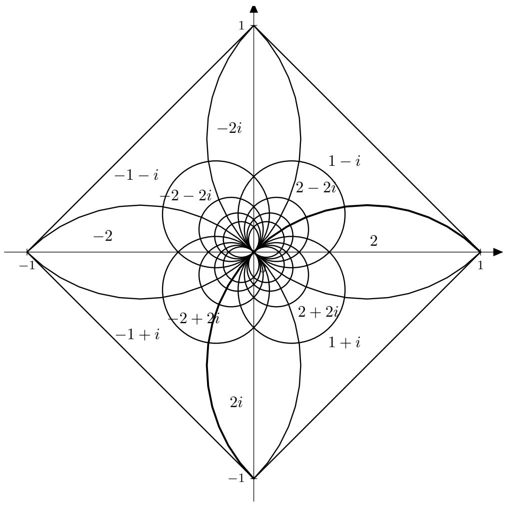


Figure 5.2: Partition of $\mathfrak{T}$ after a_1

Lemma 5.1.3. *Let* $z = [0; b_1, b_2, \ldots] \in \mathfrak{T}'$ *have* $\mathcal{Q}$*-pair* $(p_n)_{n\geq 0}, (q_n)_{n\geq 0}$*, then*

1. $(|q_n|)_{n\geq 1}$ *is strictly increasing,*
2. *For all* $n \in \mathbb{N}$ *we have* $|z - \frac{p_n}{q_n}| < |q_n|^{-1}$*,*
3. *For all* $n \in \mathbb{N}$ *we have* $|\mathcal{C}_n^T(\mathbf{b})| \ll |q_n|^{-1}$.

5.2 Laws of Succession

The structure of the space Ω^{TCF} is simpler than the one of Ω^{HCF}. To see it, let us start by writing

$$\forall b \in (1+i) \quad \mathfrak{T}(b) := \tau[\mathcal{C}_1(b)] = -b + (\iota[\mathfrak{T}] \cap (b + \mathfrak{T})).$$

Thus, $\mathfrak{T}(b) = \mathfrak{T}$ whenever $|b| \geq 2$ and, except for the boundaries, the sets $\mathfrak{T}(1+i), \mathfrak{T}(-1+i), \mathfrak{T}(-1-i), \mathfrak{T}(1-i)$ are rotations of each other (see Figure 5.3). We also have $\mathfrak{T}(0) = \{-1\}$, so

$$\Omega_R^{TCF} \subseteq \{(a_n)_{n\geq 0} : \forall n \in \mathbb{N}\ a_n \in (1+i)^*\}.$$

Note now that for any $\mathbf{b} \in \Omega_R^{\mathrm{TCF}}(n)$, $\mathbf{b} = (b_1, \ldots, b_n)$, we have

$$\tau[\mathcal{C}_n(b_1, \ldots, b_n)] = \mathcal{C}_{n-1}(b_2, \ldots, b_n) \cap \mathfrak{T}(b_1)$$

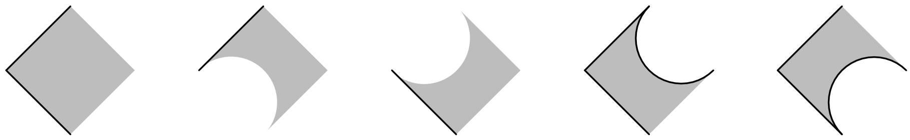


Figure 5.3: Maximal feasible regions

(the first term in the right is omitted when $n = 1$). Therefore, whenever $|b_1| \geq 2$,

$$\tau[\mathcal{C}_n(b_1,\ldots,b_n)] = \mathcal{C}_{n-1}(b_2,\ldots,b_n) \cap \mathfrak{T} == \mathcal{C}_{n-1}(b_2,\ldots,b_n). \tag{5.4}$$

Assume now that $|b| = \sqrt{2}$, $b = -1-i$ say (the other possibilities are treated similarly). From

$$\mathfrak{T}(-1-i)^{-1} = \mathfrak{T}^{-1} \cap \{z \in \mathbb{C} : \mathfrak{I}z \geq -1 + \mathfrak{R}z\}$$

we obtain for every $b \in (1+i)^*$ that

$$\mathfrak{T}(-1-i)^{-1} \cap (b + \mathfrak{T}(b)) = b + \mathfrak{T}(b) \quad \vee \quad \mathfrak{m}(\mathfrak{T}(-1-i)^{-1} \cap (b + \mathfrak{T}(b))) = 0.$$

Therefore, when $(b_1,b_2) \in \Omega_R^{\mathrm{TCF}}(2)$ we have $\tau[\mathcal{C}_2(b_1,b_2)] = \mathcal{C}_1(b_2)$. As a conse-

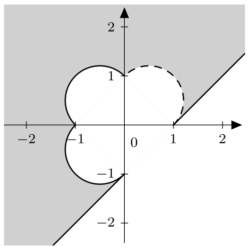


Figure 5.4: The region $\mathfrak{T}(-1-i)^{-1}$

quence, since τ restricted to any cylinder is injective, for all $\mathbf{b} = (b_1,\ldots,b_n) \in \Omega_R^{\mathrm{TCF}}(n)$ we have

$$\begin{aligned}\tau[\mathcal{C}_n(\mathbf{b})] &= \tau[\mathcal{C}_n(\mathbf{b}) \cap \mathcal{C}_1(b_1)] \\ &= \mathcal{C}_{n-1}(b_2,\ldots,b_n) \cap \mathfrak{T}(b_1) \cap \mathcal{C}_1(b_2) \\ &= \mathcal{C}_{n-1}(b_2,\ldots,b_n) \cap \mathcal{C}_1(b_2) \\ &= \mathcal{C}_{n-1}(b_2,\ldots,b_n).\end{aligned}$$

A recursive application of the previous identities yields

$$\forall n \in \mathbb{N} \quad \forall \mathbf{b} \in \Omega_R^{\mathrm{TCF}} \quad \tau^n[\mathcal{C}_n(\mathbf{b})] = \mathfrak{T}(b_n).$$

Thus, given $\mathbf{b} \in \Omega_R^{\mathrm{TCF}}(n)$ and $b \in (1+i)^*$ the membership of $\mathbf{b}b$ to $\Omega_R^{\mathrm{TCF}}(n+1)$ depends on b_n and b.

We summarize the discussion in the next result.

Theorem 5.2.1. *The dynamical system* (Ω_R^{TCF}, σ) *is a Markov shift.*

Call M the infinite matrix describing Ω_R^{TCF}, $M : (1+i)^* \times (1+i)^* \to \{0,1\}$. Put $\mathcal{J} = \{z \in (1+i)^* : \mathfrak{I}z \geq -1 + \mathfrak{R}z\}$. Then M is given by

$$\forall b, c \in (1+i)^* \quad M(b,c) = \begin{cases} 1, \text{ if } |b| \geq 2, \\ 1, \text{ if } b = i^r(-1-i),\ c \in i^r\mathcal{J},\ r \in [0..4] \\ 0, \text{ if } b = i^r(-1-i),\ c \notin i^r\mathcal{J},\ r \in [0..4]. \end{cases}$$

Irregular objects. Let us point out that irregular numbers do exist. Indeed, the segments $(-1-i, m\alpha + \overline{\alpha})$ for any $m \in \mathbb{Z} \setminus \{0\}$ are irregular (see Figure 5.4), as well as $(-1+i, \alpha + m\overline{\alpha})$ for any $m \in \mathbb{Z} \setminus \{0\}$. We leave to the reader to show that $\mathfrak{m}\mathfrak{T}_I = 0$ and $\dim_H \mathfrak{T} = 1$.

5.3 The Ergodic Theory

The simpler structure of the associated shift space allowed S. Tanaka to construct the natural extension of $(\mathfrak{T}', \tau)$ and even to obtain a measure ν equivalent to $\mathfrak{m}$ making $(\mathfrak{T}, \mathfrak{B}(\mathfrak{T}), \tau, \nu)$ ergodic.

5.3.1 The Dual Algorithm

Recall that the natural extension of a one-sided shift space is a two-sided shift space (cfr. Appendix A). Then, in order to build the natural extension of $(\Omega_R^{\mathrm{TCF}}, \sigma)$ we must determine the space of sequences $(b_n)_{n\geq 1}$ in $(1+i)^*$ such that

$$\forall n \in \mathbb{N} \quad b_n b_{n-1} \dots b_1 \in \Omega_R^{\mathrm{TCF}}(n).$$

From a symbolic point of view, it is not hard to determine the desired sequence space, for we already know the matrix determining Ω^{TCF}. Moreover, it can be shown that the space is also described by an infinite matrix $\tilde{M}$. For example, if $b \in (1+i)^*$ satisfies $\mathfrak{I}b > 1 + |\mathfrak{R}b|$, then

$$\forall c \in (1+i)^* \quad \tilde{M}(b,c) = \begin{cases} 0, & c \in \{-1+i, 1+i\}, \\ 1, & \text{otherwise.} \end{cases}$$

It turns out that the sequence space we are looking for is also generated by a complex continued fractions algorithm included in the framework of Chapter I. We refer to this process as the dual algorithm.

The dual algorithm can be defined with a suitable partition of $\mathbb{C}$. First, consider the sets

$$\mathfrak{S}_0 := \overline{\mathbb{D}}(0),$$
$$\mathfrak{S}_1 := \overline{\mathbb{D}}(0) \setminus \overline{\mathbb{D}}(-1-i), \quad \mathfrak{S}_2 := i\mathfrak{S}_1, \quad \mathfrak{S}_3 := -\mathfrak{S}_1, \quad \mathfrak{S}_4 = -i\mathfrak{S}_1,$$
$$\mathfrak{S}_5 := \overline{\mathbb{D}}(0) \setminus \left(\overline{\mathbb{D}}(-1+i) \cup \overline{\mathbb{D}}(-1-i)\right), \quad \mathfrak{S}_6 := i\mathfrak{S}_5, \quad \mathfrak{S}_7 := -\mathfrak{S}_5, \quad \mathfrak{S}_8 := -i\mathfrak{S}_5.$$

Rather than describe meticulously the translates of the sets $\mathfrak{S}_j$ partitioning $\mathbb{C}$, we content ourselves with Figure (5.6). To each $a \in (1+i)$ denote by $\mathfrak{S}(a)$ the set $\mathfrak{S}_j$ such that $a + \mathfrak{S}(a)$ belongs to the partition. The choice function $f_S : \mathbb{C} \to (1+i)$ is given by the condition $z \in f_S(a) + \mathfrak{S}(a)$ and the corresponding f_S-Gauss map by $S : \mathfrak{S}^* \to \mathfrak{S}$, $S(z) = z^{-1} - f_S(z^{-1})$.

The partition of $\mathfrak{S}$ induced by the cylinders of depth 1 is depicted in Figure 5.5.

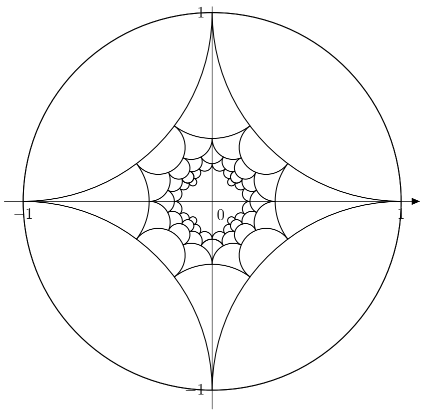


Figure 5.5: Partition of $\mathfrak{S}_1$ after $a_1(z)$.

From now on, we exclude the null set $\bigcup_{n\geq 0} S^{-n}[C(0;1)]$ and denote by Δ^{TCF} the resulting sequence space. Our omission forbids the appearances of 0 as a term of any sequence in Δ^{TCF}. Based on the geometry of the dual algorithm, it can be shown that $(\Delta^{\mathrm{TCF}}, T_d)$ is precisely the sequence space that we have been after.

Theorem 5.3.1. *Let* $\mathbf{b} = (b_n)_{n\geq 1}$ *take values in* $(1+i)^*$. *Then,* $\mathbf{b} \in \Delta^{\mathrm{TCF}}$ *if and only* $b_{n+1}b_n \in \Omega^{\mathrm{TCF}}(2)$ *for every* n.

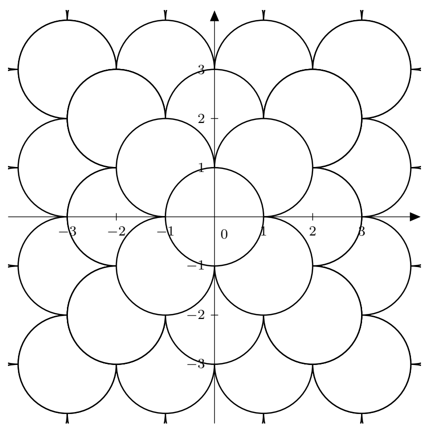


Figure 5.6: Partition of $\mathbb{C}$ for the dual algorithm.

5.3.2 The Natural Extension

The space underlying the natural extension of $(\mathfrak{I}_R, \tau)$ (with the suitable ergodic measure) is given by

$$Z = \left\{(w, z) \in \mathfrak{S} \times \mathfrak{T} : b_1(w)a_1(z) \in \Omega^{\mathrm{TCF}}(2)\right\},$$

$$R : Z \to Z \quad \forall (w, z) \in Z \quad R(w, z) = \left(\frac{1}{a_1(z) + w}, \tau(z)\right).$$

Exploring the condition $b_1(w)a_1(z) \in \Omega^{\mathrm{TCF}}(2)$ we can actually write

$$Z = \bigcup_{j=1}^{8} \mathfrak{S}_j \times X_j,$$

where $X_1, \ldots, X_8$ is the partition of $\mathfrak{T}$ shown in Figure 5.7 (the involved arcs have their center in $\frac{\alpha}{2}, i\frac{\alpha}{2}, -\frac{\alpha}{2}, -i\frac{\alpha}{2}$ and radius $2^{-1/2}$).

Theorem 5.3.2 (S. Tanaka, 1985)**.** *Let $h : Z \to \mathbb{R}_{>0}$ be given by*

$$\forall (w, z) \in Z \quad h(w, z) = \frac{1}{|1 + wz|^4}.$$

Then, the measure ρ, $\mathrm{d}\rho = h\mathrm{d}\mathfrak{m}$, is finite and R-invariant.

The Theorem of Change of Variable plays an important rôle again. Indeed, the R invariance follows from the equality

$$\forall (w, z) \in Z \quad |DR(w, z)|h(R(w, z)) = h(w, z),$$

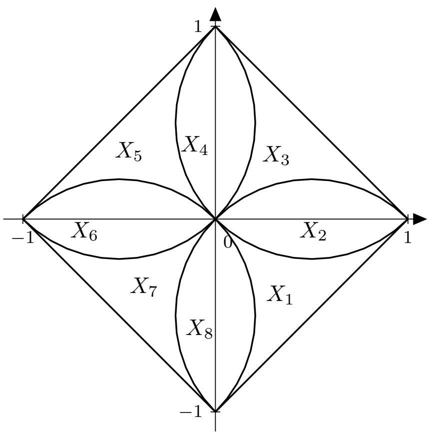


Figure 5.7: Partition of $\mathfrak{T} = \bigcup_j X_j$

where DR is the Jacobian of R (regarded as a map from a subset of $\mathbb{R}^4$ into itself). Although harder to prove, the finiteness of the measure verified by bounding separately each term of the sum

$$\rho_1(Z) = \int_Z h(w,z)\mathrm{d}\mathfrak{m}(w,z) = \sum_{j=1}^{8} \iint_{\mathfrak{S}_j \times X_j} h(w,z)\mathrm{d}\mathfrak{m}(w)\mathrm{d}\mathfrak{m}(z).$$

5.3.3 The Ergodic Measure

Let $\pi_2 : Z \to \mathfrak{T}$ denote the projection on the second coordinate, i.e. $\pi(w,z) = z$. Since $\pi \circ R = \tau \circ \pi$, the measure ν given by

$$\forall B \in \mathfrak{B}(\mathfrak{T}) \quad \nu(B) := \rho(\pi^{-1}[B])$$

is τ-invariant. The irreducibility of ν— and thus its ergodicity— is proven essentially the same way as for HCF. Some care must be taken, though. Take $z = [0; b_1, b_2, \ldots]_T \in \mathfrak{T}'$ with $\mathcal{Q}$-pair $(p_n)_{n\geq 0}, (q_n)_{n\geq 0}$. The inverse of the restriction of τ^n to $\mathcal{C}_n^T(\mathbf{b})$ is

$$\psi_{\mathbf{b},n}(z) = \frac{p_{n-1}z + p_n}{q_{n-1}z + q_n}, \quad |\psi_{\mathbf{b},n}(z)| = \frac{1}{|q_{n-1}z + q_n|^2} = \frac{1}{|q_n|^2 \left|1 + z\frac{q_{n-1}}{q_n}\right|}.$$

Since $|z|$ can be arbitrarily close to 1, it is not entirely clear that a Rényi-type condition (such as (3.1)) holds. However, $3|q_{n-1}(z)| \leq 2|q_n(z)|$ is true for all $n \in \mathbb{N}$ almost everywhere (with respect to $\mathfrak{m}$).

Proposition 5.3.3. *For almost every $z = [0; b_1, b_2, b_3, \ldots]_T \in \mathfrak{T}$ we have that*

$$\forall n \in \mathbb{N} \quad \sup_{z \in \mathfrak{T}(\mathbf{b},n)} \left|\psi'_{\mathbf{b},n}(z)\right|^2 \ll \inf_{z \in \mathfrak{T}(\mathbf{b},n)} \left|\psi'_{\mathbf{b},n}(z)\right|^2, \tag{5.5}$$

where $\mathfrak{T}(\mathbf{b}, n) = \tau^n[\mathcal{C}_n^T(\mathbf{b})]$.

From the explicit formula for the density of ρ and $Z = \bigcup_j \mathfrak{S}_j \times X_j$, S. Tanaka could find an explicit formula for the density of ν. First, define the map $A : \mathbb{R}^2_{>0} \to \mathbb{R}$ by

$$A(r, s) = r^2 \arctan \frac{s}{r} + s^2 \arctan \frac{r}{s} - rs.$$

Considering $\alpha_1 = \alpha$, $\alpha_2 = \overline{\alpha}$, $\alpha_3 = -\alpha$, and $\alpha_4 = -\overline{\alpha}$ define the functions f_1, f_2, f_3, f_4 by

$$f_0(z) := \frac{1}{\rho_1(Z)} \frac{\pi}{(1 - |z|^2)^2},$$
$$\forall j \in [1..4] \quad f_j(z) := \frac{1}{\rho_1(Z)} A\left(\frac{1}{1 - |z|^2}, \frac{1}{|z - \overline{\alpha}_j|^2 - 1}\right).$$

Theorem 5.3.4. *The measure ν in $(\mathfrak{T}, \tau)$ is ergodic and equivalent to the Lebesgue's measure. The density function of ν, f, is given by*

$$f(z) = \begin{cases} f_0(z) - f_j(z) & \text{if } z \in X_j \text{ for } j \in [1..4], \\ f_0(z) - f_1(z) - f_2(z) & \text{if } z \in X_5, \\ f_0(z) - f_2(z) - f_3(z) & \text{if } z \in X_6, \\ f_0(z) - f_3(z) - f_4(z) & \text{if } z \in X_7, \\ f_0(z) - f_4(z) - f_1(z) & \text{if } z \in X_8. \end{cases}$$

A Khinchin-Lévy type result and the entropy of $(\mathfrak{J}_R, \tau, \nu)$ can be obtained with the arguments used for HCF.

Theorem 5.3.5. *Let $k : \mathfrak{T} \to \mathbb{R}$ be given by $k(z) = \log|z|$, then $k \in L^1(\nu)$ and for almost every $z = [0; b_1, b_2, \ldots] \in \mathfrak{T}$ we have*

$$\lim_{n \to \infty} \lim_{n \to \infty} \frac{\log|q_n|}{n} = -\|k\|_1.$$

Moreover, the entropy $h_\nu(\tau)$ of the system $(\mathfrak{T}, \mathfrak{B}(\mathfrak{T}), \tau, \nu)$ is $h_\nu(\tau) = 4\|k\|_1$.

Similar results also hold for the dual system $(\mathfrak{S}', S)$.

5.4 Notes and Comments

1. S. Tanaka defined in a slightly different way the $\mathcal{Q}$-pair $((h_n)_{n\geq 0}, (k_n)_{n\geq 0})$ of his continued fractions. While the recurrences are the same, his initial conditions are

$$h_{-2} = 0,\ h_{-1} = \alpha,\ k_{-2} = \alpha,\ k_{-1} = 0.$$

   If $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ is the usual $\mathcal{Q}$-pair, we have $\alpha p_n = h_n$ and $\alpha q_n = k_n$ for every n. The discrepancy is irrelevant. We changed the notation of S. Nakada's paper aiming for consistency.

2. As noted before, Hurwitz - Tanaka continued fractions are a particular case of Iwasawa continued fractions. Thus, the general ergodic theory developed by A. Lukyanenko and J. Vandehey, although they provide no calculations of the entropy.

3. Our proof of Theorem 5.1.1 seems to be new. It is much simpler than the ones given in [Os16].

4. We have made no attempt in working with the approximation properties of the Hurwitz-Tanaka continued fractions, because the approximation rate is worse than HCF. This is not surprising, for the set where the elements can belong is significantly reduced.

Chapter 6

Current and Future Research

In this brief chapter, we discuss some problems related to Hurwitz. The first two are the subject of ongoing research.

Notation. If μ is a measure on a subset of $\mathbb{R}^n$, then $\hat{\mu}$ is its Fourier transform.

References. We add some references to each problem separately.

6.1 Current Research

6.1.1 A Kaufman Measure in the Complex Plane

In 1980, R. Kaufman published the construction of a measure supported in a particular compact set with a polynomially decreasing Fourier transform. His result has had implications, for example, in research related to the Littlewood Conjecture (cfr. [HJK]). Some hypothesis imposed by R. Kaufman were later weakened by M. Quéffelec and O. Ramaré in [QuRa03].

For each $N \in \mathbb{N}$ let F_N be the set of real numbers in $[0,1]$ whose regular continued fraction is bounded by N.

Theorem 6.1.1 (R. Kaufman, 1980)**.** *Let $N \in \mathbb{N}$ be such that $\dim_H F_N > \frac{1}{2}$. Then, F_N supports a probability measure $\lambda = \lambda_N$ such that for some $\eta > 0$ we have*

$$\forall u \in \mathbb{R} \quad \left|\hat{\lambda}(u)\right| \ll_N |u|^{-\eta}.$$

We can split R. Kaufman's proof into two parts. First, the measure is built. Then, the Fourier transform conditions are verified. We give a very broad overview of both steps.

Construction. Let N be as in the statement. We can naturally identify F_N with $[1..N]^{\mathbb{N}}$. Using the Hausdorff dimension hypothesis, Kaufman defined a probability measure μ on $[1..N]^n$. A sequence of independent identically distributed random variables $(f_n)_{n\geq 1}$ with finite mean is defined on this space. By the Weak Law of Large numbers, for large N the probability of $N^{-1}(f_1(\mathbf{a}_1) + \ldots + f_N(\mathbf{a}_N))$ being close to the mean of f_1 is greater than $\frac{1}{2}$. The probability measure λ is defined by conditioning μ^N to this set and extending it to the countable product.

Bounding the Fourier transform. By construction, λ can be written as $\lambda = \nu \otimes \lambda$. Then, by applying Fubini's Theorem and relating λ with the Lebesgue measure, Kaufman reduced the problem to bound some oscillatory integrals in one dimension.

The complex analogue. We express our first question as follows:

Problem 1. *Is there an analogue Kaufman-like measure for complex plane?*

We decided to adapt Kaufman's construction using Hurwitz Continued Fractions. Although a significant part of the work is already done, the extra dimension poses new difficulties. The first important problem is encountered in the construction. The (HCF) Laws of Succession make an straight forward adaptation of Kaufman's argument impossible. However, by Theorem 4.5.2 we can obtain sets with Hausdorff Dimension arbitrarily close to 2 formed by numbers whose HCF can be understood as a sequence of independent blocks.

The multidimensional treatment of oscillatory integrals is considerably harder than the one-dimensional. In our case, the complications come from obtaining a satisfactory lower bound of

$$z + \frac{q_N(\mathbf{a}) + q_N(\mathbf{b})}{q_{N-1}(\mathbf{b})},$$

where $z \in \mathfrak{F}$, and $\mathbf{a}, \mathbf{b}$ are valid segments. For regular continued fractions the previous expression can be easily bounded, because $x \in (0,1)$ and continuants are natural numbers and grow exponentially.

For details on the real Kaufman's measure, see [K80], [QuRa03], and [JoSa16]. The complex Kaufman measure is a work in progress.

6.1.2 A Generalization of Good's Theorem

In view of the successful extension of Good's Theorem to HCF, we can ask for a generalizations.

Problem 2. *Can Good's Theorem be extended to Schweiger Fibred Systems?*

We must consider only those Schweiger Fibred Systems with infinitely many digits. Moreover, we have to define how to interpret $a_n \to \infty$ for the regular continued fractions and $|a_n| \to \infty$ for HCF. We suggest to work with functions $\delta : I \to \mathbb{R}_{>0}$ such that for any $M > 0$ the inverse image of $[0, M]$ under δ is finite. On the basis of the proofs of the real and complex versions of Good's theorems, we suspect that some growth conditions must be imposed on δ.

In both versions of Good's Theorem irregular elements can be overlooked. This is unlikely to be the case on higher dimensions. For example, if we extend directly the construction of HCF to three dimensions (changing the complex inversion for the inversion with respect to the unit sphere), then the corresponding irregular numbers might form a set of dimension $3 - 1 = 2$. Thus, we might have to consider separately regular and irregular numbers.

Rather than obtaining that the Hausdorff dimension of a set is half of the ambient space's, we expect to obtain only estimates. We suspect that irregular elements are the key factor that will allow the Hausdorff dimension of the set of interest to vary.

6.2 Future Research

We now list a few problems that might be of interest.

Problem 3. *Can we get rid of the lower bound of the elements in Theorem 2.3.2?*

Problem 4. *Is there a Gauss-Kuzmin theorem for Iwasawa Continued fractions?*

Problem 5. *Is there a Borel-Bernstein Theorem for HCF?*

A. Nogueira gave in [No01] a Borel-Bernstein Theorem for a family multidimensional continued fractions (defined in the geometric way). However, it is not entirely clear that his work includes HCF.

Problem 6. *Is* $(\mathfrak{F}, T, \mu)$ *an* α*,* ρ*-mixing system?*

The previous problem was posed by Dr. Poj Lertchoosakul.

Problem 7. *Are there numbers* $\zeta \in \mathbb{C} \setminus \mathbb{Q}(i)$ *with an automatic HCF and such that* $|\zeta|^2 = n \in \mathbb{N}$*?*

Appendix A

In this appendix, we recall some elementary formulæon the inversion of circles in the complex plane. The proofs of the propositions below can be safely omitted, for they are just elementary exercises.

Inversion Formulæ

Let $\iota : \mathbb{C}^* \to \mathbb{C}^*$ be the complex inversion; i.e. $\iota(z) = z^{-1}$ for all $z \in \mathbb{C}^*$. For any $z_0 \in \mathbb{C}$ and any $\rho > 0$ let $C(z_0, \rho)$ be the circle of radius ρ and center z_0:

$$C(z_0, \rho) = \{w \in \mathbb{C} : |z - w| = \rho\}.$$

Proposition (Inversion of lines and circles)**.**

1. *Let* $z_0 \in \mathbb{C}$ *and* $\rho > 0$. *Then,*

$$\iota[C(z_0,\rho)] = \begin{cases} C\left(\frac{\overline{z}_0}{|z_0|^2-\rho^2}, \frac{\rho}{|\rho^2-|z_0|^2|}\right) & \text{if } \rho \neq |z_0| \\ L : 2\mathfrak{R}(z\overline{z}_0) = 1 & \text{if } \rho = |z_0|, \end{cases}$$

   where $L : 2\mathfrak{R}(z\overline{z}_0) = 1$ *is the line* $\{z \in \mathbb{C} : 2\mathfrak{R}(z\overline{z}_0) = 1\}$.

2. *Let* $a, b \in \mathbb{R}$, $(a, b) \neq (0, 0)$ *and consider the lines* $L_{a,b}$, $L'_{a,b}$

$$L_{a,b} : ax + by = 1, \quad L'_{a,b} : ax + by = 0.$$

   Then,

$$\iota[L_{a,b}] = C\left(\frac{a}{2} - i\frac{b}{2}, \frac{\sqrt{a^2+b^2}}{2}\right), \quad \iota[L'_{a,b}] = \{z = u + iv : au - bv = 0\}.$$

Appendix B
Ergodic Theory

In this appendix, we establish notation regarding ergodic theory. Although most proofs are omitted, we do provide the appropriate references.

Definitions

Let $\mathfrak{X} = (X, \mathfrak{M})$ be a measurable space and $\tau : X \to X$ a measurable function.

i. Let μ be a measure on $\mathfrak{X}$. The function τ is **measure preserving** or an **endomorphism** if

$$\forall E \in \mathfrak{M} \quad \mu(E) = \mu(\tau^{-1}[E]).$$

The quadruplet $(X, \mathfrak{M}, \mu, \tau)$ is a **measure preserving system** and when μ is a probability measure, it is a **probability measure preserving system**.

ii. A measure μ in $\mathfrak{X}$ is **irreducible** if

$$\forall E \in \mathfrak{M} \quad \tau^{-1}[E] = E \quad \Longrightarrow \quad \mu E = 0 \ \vee \ \mu(X \setminus E) = 0.$$

iii. The system $(X, \mathfrak{M}, \mu, \tau)$ is **ergodic** if it is a measure preserving system and τ is irreducible. If the measurable space $(X, \mathfrak{M})$ and the measurable map τ are given, a measure μ is **ergodic** if the system $(X, \mathfrak{M}, \mu, \tau)$ is ergodic.

iv. The function τ is **non-singular** if

$$\forall E \in \mathfrak{M} \qquad \mu(\tau^{-1}[E]) = 0 \quad \Longrightarrow \quad \mu(E) = 0.$$

Let $(X, \mathfrak{M}, \mu, \tau)$ be a (probability) measure preserving system.

1. τ is **weak-mixing** if

$$\forall A, B \in \mathfrak{M} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(\tau^{-i}[A] \cap B) - \mu A \mu B| = 0.$$

2. τ is **strong-mixing** if

$$\forall A, B \in \mathfrak{M} \quad \lim_{n\to\infty} \mu(\tau^{-i}[A] \cap B) = \mu A \mu B| = 0.$$

Ergodic Theorems

The following is a fundamental result in ergodic theory. Its proof can be found, for example, in the Walters' monograph [Wa82] as Theorem 1.14, p.34.

Theorem (Birkhoff Ergodic Theorem)**.** *Suppose $(X, \mathfrak{B}, \mu, T)$ is a finite or σ-finite measure preserving system. Then, for every $f \in L^1(\mu)$ there exists a function $f^* \in L^1(\mu)$ such that $f^* \circ T = f^*$ and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i = f^*$$

holds μ-almost everywhere. If $\mu X < +\infty$, then $\int f \mathrm{d}\mu = \int f^* \mathrm{d}\mu$. In particular, if μ is a probability measure and T is ergodic, then $f^* = \int f \mathrm{d}\mu$.

An application of Birkhoff's Ergodic Theorem to indicator functions gives the following corollary.

Corollary. *Let $(X, \mathfrak{B})$ be a measurable space and $T : X \to X$ a measurable map. If μ and ν are probability measures such that both systems $(X, \mathfrak{B}, \mu, T)$ and $(X, \mathfrak{B}, \nu, T)$ are ergodic, and $\nu \ll \mu$, then $\mu = \nu$.*

Partitions

Let $(X, \mathfrak{M}, \mu)$ be a probability space. By a **partition** we mean an at most countable collection $\{A_n\}_n \subseteq \mathfrak{M}$ such that

$$\forall n, m \in \mathbb{N},\ \mu(A_n \cap A_m) = \delta_{n,m}, \quad \mu\left(\mathfrak{X} \setminus \bigcup_n A_n\right) = 0,$$

where $\delta_{n,m}$ is the Kronecker symbol. Let $\mathcal{P}$ be a partition of $\mathfrak{X}$. For $x \in X$, $\mathcal{P}(x) \in \mathcal{P}$ is an element of the partition such that $x \in \mathcal{P}(x)$. It poses no threat to us that $x \mapsto \mathcal{P}(x)$ is uniquely defined only μ-almost everywhere. If X is also a metric space, the **diameter** of $\mathcal{P}$ is

$$\operatorname{diam} \mathcal{P} := \sup\{|P| : P \in \mathcal{P}\}.$$

We define a partial order on the partitions of $\mathfrak{X}$ by writing $\mathcal{P} \le \mathcal{Q}$ for two partitions $\mathcal{P}$, $\mathcal{Q}$ whenever each member of $\mathcal{P}$ is the union (up to a μ-null set) of elements of $\mathcal{Q}$. The **sum** of $\mathcal{P}$ and $\mathcal{Q}$ is

$$\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}.$$

Note that $\mathcal{P}, \mathcal{Q} \le \mathcal{P} \vee \mathcal{Q}$ and that the sum is minimal with respect to this property. If $\{\mathcal{P}_n\}_n$ is an at most countable family of partitions, we define

$$\bigvee_{n \in \mathbb{N}} \mathcal{P}_n$$

to be the minimal partition $\mathcal{P}$ such that $\mathcal{P}_n \le \mathcal{P}$ for every n.

Suppose that $\tau : X \to X$ is an endomorphism. For any partition $\mathcal{P}$ we define $\tau^{-1}[\mathcal{P}] = \{\tau^{-1}[P] : P \in \mathcal{P}\}$ and

$$\mathcal{P}_\tau^- := \bigvee_{k=0}^{\infty} \tau^{-k}[\mathcal{P}],$$

where $\tau^{-(n+1)}[\mathcal{P}] := \tau^{-1}[\tau^{-n}[\mathcal{P}]]$. If τ is invertible (almost everywhere) with measurable inverse, we can similarly define

$$\mathcal{P}_\tau^+ := \bigvee_{k \in \mathbb{N}} \tau^k \mathcal{P}.$$

A measurable partition $\mathcal{P}$ is a **generator** of τ if $\mathcal{P}_\tau^- = \varepsilon$, where ε is the trivial partition $\varepsilon = \{X\}$. If τ is an automorphism, we say that $\mathcal{P}$ is **exhaustive** if $\mathcal{P}_\tau^+ = \varepsilon$ and $\tau^{-1}[\mathcal{P}] \le \mathcal{P}$.

When τ has been fixed, we define for $\mathcal{P}$ the sequence $(\mathcal{P}_n)_{n \ge 1}$ by

$$\forall n \in \mathbb{N} \quad \mathcal{P}_n := \bigvee_{k=0}^{n} \tau^{-k} \mathcal{P},$$

Entropy

The notion of entropy, broadly, speaking tells us about the uncertainty of an experiment. A nice discussion can be found in [OlVi16].

Let $(X, \mathfrak{M}, \mu)$ be a finite measure space. The **entropy** of a partition $\mathcal{P}$ is

$$H_\mu(\mathcal{P}) = \sum_{P \in \mathcal{P}} \mu P \log(\mu P).$$

It can be shown that $n \mapsto H_\mu(\mathcal{P}^n)$ is sub-additive, hence the following limit exists:

$$h_\mu(\tau, \mathcal{P}) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P}^n).$$

The **entropy** of the system $(X, \mathfrak{M}, \mu, \tau)$ is

$$h_\mu(\tau) := \sup\{h_\mu(f, \mathcal{P}) : h_\mu(\tau, \mathcal{P}) < \infty\}.$$

The next theorem appears in [OlVi16] as Theorem 9.3.1, p.262.

Theorem (Shannon - McMillan - Breiman)**.** *Let $(X, \mathfrak{M}, \mu)$ be a probability space and let $\mathcal{P}$ be an at most countable and measurable partition. Suppose that $H_\mu(\mathcal{P}) < +\infty$. Then, if $\tau : \mathfrak{X} \to \mathfrak{X}$ is a measure preserving transformation, for μ almost all $x \in X$ the following limit exists*

$$h_\mu(\tau, \mathcal{P}, x) := -\lim_{n\to\infty} \frac{\log \mu(\mathcal{P}^n(x))}{n}.$$

Moreover, $h_\mu \in L^1(\mu)$ and

$$\int h_\mu(\tau, \mathcal{P}, x) \mathrm{d}\mu(x) = h_\mu(\tau, \mathcal{P}).$$

If $(X, \mathfrak{M}, \mu, \tau)$ is ergodic, then

$$\forall_\mu x \in X \quad h_\mu(\tau, \mathcal{P}, x) = h_\mu(\tau, \mathcal{P}).$$

The following result can be found as Corollary 9.2.10., p. 257 in [OlVi16].

Theorem 6.2.1. *If $\mathcal{P}$ is a partition such that* $\operatorname{diam} \mathcal{P}^n(x) \to 0$ *as $n \to \infty$ for μ almost all $x \in X$. Then, $h_\mu(\tau) = h_\mu(\tau, \mathcal{P})$.*

6.3 Rokhlin's Natural Extension

The full proofs of the statements in this sections are in V.A. Rokhlin's papers [Ro64] and [Ro67].

Let $\mathfrak{X} = (X, \mathfrak{B}, \mu, \tau)$ be a measure preserving system. We say that a partition $\mathcal{P}$ is **invariant** if $\tau^{-1}[\mathcal{P}] \leq \mathcal{P}$. Let $\mathcal{P}$ be an invariant partition and consider the quotient space $X/\mathcal{P}$. Let $f : X \to X/\mathcal{P}$ be the factor map given by $f(x) = \mathcal{P}(x)$. We say that $Z \subseteq X/\mathcal{P}$ is **measurable** if $f^{-1}[Z] \in \mathfrak{B}$ and, in this case, we write

$$\mu(Z) := \mu\left(f^{-1}[Z]\right).$$

We can further consider the factor endomorphism $\tau_{\mathcal{P}} : M/\mathcal{P} \to M/\mathcal{P}$ as the map satisfying $f \circ T = T_{\mathcal{P}} \circ f$.

Definition. *Let $\mathfrak{X} = (X, \mathfrak{B}_X, \mu_X, \tau)$ be a measure preserving system. A measure preserving system $\mathfrak{X}' = (X', \mathfrak{B}', \mu', \tau')$ is a* ***natural extension*** *of $\mathfrak{X}$ if $\mathfrak{X}'$ has an exhaustive measurable partition ζ such that τ'_ζ is isomorphic to τ.*

Theorem (V. A. Rokhlin, 1961)**.** *Any measure preserving system $\mathfrak{X}'$ has a unique natural extension $\mathfrak{X}'$. Furthermore, $\mathfrak{X}$ is ergodic if and only if $\mathfrak{X}'$ is.*

Corollary. *The natural extension of a one-sided Bernoulli shift space (over a metric space) is a two-sided Bernoulli shift on the same space of states.*

Sketch of proof. We consider a simpler context: assume that E is a complete and separable metric space. Let $\mathfrak{B}_E$ be its Borel σ-algebra and μ_E a Borel probability. By the Daniel-Kolmogorov Theorem, the system $\mathfrak{X} = (X, \mathfrak{B}_X, \mu_X, \sigma_X)$ where $X = E^{\mathbb{N}}$, σ_X the right-shift map and the rest of the objects are the usual products is well defined. Similarly, consider $\mathfrak{Y} = (Y, \mathfrak{B}_Y, \mu_Y, \sigma = \sigma_Y)$ for $Y = E^{\mathbb{Z}}$.

Let π be the projection $\pi : Y \to E$ given by $\pi(x_n)_{n\in\mathbb{Z}} = x_1$ and define the partition

$$\mathcal{P} = \{\pi^{-1}[\mathbf{x}] : \mathbf{x} \in Y\}.$$

Note that $(x_n)_{n\in\mathbb{Z}} \sim (y_n)_{n\in\mathbb{Z}}$ with respect to $\mathcal{R} := \mathcal{P}^-_{\sigma_Y}$ if and only if $x_n = y_n$ for all $n \in \mathbb{N}$. It can be shown using the definitions that $\mathcal{R}$ is exhaustive, $\sigma^{-1}[\mathcal{R}] = \mathcal{R}$ and that $\sigma_{\mathcal{R}}$ and σ_X are isomorphic. □

Rokhlin's natural extension has an important universal property. Namely, it factors every invertible extension of $\mathfrak{X}$. That is, if $\mathfrak{Y} = (Y, \mathfrak{B}_Y, \nu, S)$ is an invertible extension of X with factor map $\varphi : Y \to X$, then, there is a measure preserving map $\tilde{\varphi} : Y \to X'$ such that $\varphi = \pi\tilde{\varphi}$.

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Papers

Purely Periodic and Transcendental Complex Continued Fractions

Gerardo González Robert

Abstract

Adolf Hurwitz proposed in 1887 a continued fraction algorithm for representing complex numbers. Among other similarities with regular continued fractions, quadratic irrationals can be characterized in terms of periodic expansions. In this paper we give necessary and sufficient conditions for pure periodicity with respect to Hurwitz algorithm. We also show a complex analogue of a theorem by Y. Bugeaud ([Bu13]) which establishes a combinatorial condition for transcendence.

1 Introduction

Regular continued fractions are a remarkably useful tool in number theory. The structure of a regular continued fraction expansion sometimes helps to determine algebraic or analytic properties of the number it represents. A famous result in this direction is the Euler-Lagrange Theorem. It says that an irrational number has an ultimately periodic continued fraction if and only if it is a quadratic irrational. Another important theorem, due to Liouville, allows us to construct transcendental numbers by simply taking continued fractions whose terms grow fast enough. A well known conjecture relating transcendence and boundedness of continued fraction is the following.

Conjecture 1.1 (Folklore Conjecture)**.** *If an irrational real algebraic number has a bounded continued fraction, then it is a quadratic irrational.*

Although the conjecture remains widely open, there are important partial results. On the base of Roth's Theorem, Alan Baker showed that whenever the continued fraction of a real irrational α satisfies certain combinatorial condition, then α has to be transcendental. In 1998, Martine Quéffelec showed the transcendence of numbers whose regular continued fraction is the Thue-Morse sequence over an alphabet $\{a,b\} \subseteq \mathbb{N}$. Afterwards, she showed the transcendence of a larger class of automatic continued fractions. Based on their joint work with Florian Luca on b-ary expansions, Boris Adamczewski and Yann Bugeaud generalized in 2005 some of Quéffelec's work. They showed that palindromic continued fractions and other combinatorial conditions imply transcendence. These results were crowned by Y. Bugeaud in [Bu13]. The main result of [Bu13] implies that real numbers with an automatic continued fraction are either quadratic irrationals or transcendental.

In a recent paper ([BuKi15]), Y. Bugeaud and Dong Han Kim defined a function giving another notion of complexity of an infinite word. With their function, it is possible to state the main theorem of ([Bu13]) in an extremely neat fashion (Theorem 1.1).

We must recall first some definitions.

Definition 1.1. *Let $\mathcal{A} \neq \varnothing$ be a finite set and* $\mathbf{a}$ *an infinite word on $\mathcal{A}$. The* ***repetition function****, $r(\cdot,\mathbf{a}) : \mathbb{N} \to \mathbb{N}$, is*

$$\forall n \in \mathbb{N} \quad r(n,\mathbf{a}) = \min\{m \in \mathbb{N} : \exists i \in [1..m-n] \quad a_i \cdots a_{i+n-1} = a_{m-n+1} \cdots a_m\}.$$

The ***repetition exponent*** *of* $\mathbf{a}$*,* $\operatorname{rep}(\mathbf{a})$*, is*

$$\operatorname{rep}(\mathbf{a}) := \liminf_{n\to\infty} \frac{r(n,\mathbf{a})}{n}.$$

Theorem 1.1 (Bugeaud, 2011)**.** . *Let* $\mathbf{a} = a_1a_2\ldots$ *be a non-periodic infinite word in* $\mathbb{N}$*. If*

$$\operatorname{rep}(\mathbf{a}) < +\infty,$$

then $\alpha = [0; a_1, a_2, a_3, \ldots]$ *is transcendental.*

By Cobham's Theorem on the complexity of automatic sequences ([AlSh03], Corollary 10.3.2), every automatic sequence has a finite repetition exponent.

Theorem 1.2 (Bugeaud, 2011)**.** *Let* $(a_n)_{n=1}^{\infty}$ *be an automatic sequence of positive integers. If* $(a_n)_{n=1}^{\infty}$ *is not periodic, then* $\alpha = [0; a_1, a_2, \ldots]$ *is transcendental.*

Given the importance of a regular continued fraction, it is natural to look for a similar tool in other contexts. In the complex plane, $\mathbb{C}$, there have been several attempts. An outstanding example was given by Asmus Schmidt([ASch]). His sophisticated construction focuses on the quality of approximation. A much simpler algorithm was proposed by Adolf Hurwitz ([Hu87]), it is just the straight forward generalization of the nearest integer continued fraction (see Section 2 for details). Some other expansions and their associated ergodic theory were studied by Julius Hurwitz, William Leveque, Hitoshi Nakada, Georges Poitu, among others. Lately, some families of complex continued fractions have been studied by Shrikrishna Gopalrao Dani and Arnaldo Nogueira ([DaNo14], [Da15]).

In this paper, we restrict ourselves to Hurwitz Continued Fractions. While similarities between Hurwitz and regular continued fractions abound, there important differences. For example, Serret's theorem states that two real numbers belong to the same orbit of $\mathrm{PSL}(2,\mathbb{Z})/\mathbb{R}$ (acting via Möbius transformations) if and only if the numbers are both rational or if they are both irrational and the tails of their continued fraction expansion coincide. The analogue fails in $\mathbb{C}$ with Hurwitz Continued Fractions. Richard Lakein gave in [La74b] a pair of complex numbers, ζ and ξ, and an element $\gamma \in \mathrm{PSL}(2,\mathbb{Z}[i])$ such that $\zeta = \gamma\xi$ but whose Hurwitz continued fractions never coincide.

The most striking results was obtained by Doug Hensley ([He06]) and extended by Wieb Bosma and David Gruenewald ([BoGu12]). It is a negative answer to the Folklore Conjecture for complex numbers and Hurwitz Continued Fractions.

Theorem 1.3 (Bosma, W., Gruenewald, D. (2012))**.** *Let* n *be a natural number. There exists an algebraic number* $\alpha \in \mathbb{C}$*, whose Hurwitz Continued Fraction expansion has bounded elements, such that*

$$[\mathbb{Q}(i,\alpha), \mathbb{Q}(i)] = 2n.$$

Despite Theorem 1.3, we can also conclude certain properties about the repetition exponent of the continued fractions of algebraic numbers. A trivial consequence of one of our main result, Theorem 5.1, is a weaker analogue of Theorem 1.1.

Theorem 1.4. *Let* $\mathbf{a} = (a_1, a_2, \ldots)$ *be the Hurwitz continued fraction of a number* $\zeta \in \mathbb{C}$*. If* $\mathbf{a}$ *is not periodic and*

$$\sqrt{8} \le \liminf_{n\to\infty} |a_n|, \quad \operatorname{rep}(\mathbf{a}) < +\infty,$$

then $\zeta = [0; a_1, a_2, \ldots]$ *is transcendental.*

Corollary 1.5. *Let* $\mathbf{a} = (a_1, a_2, \ldots)$ *be the Hurwitz continued fraction of a number* $\zeta \in \mathbb{C}$*. If* $|a_n| \ge \sqrt{8}$ *for* $n \in \mathbb{N}$*, and* $\mathbf{a}$ *is automatic, then* $\zeta = [0; a_1, a_2, \ldots]$ *is quadratic over* $\mathbb{Q}(i)$ *or transcendental.*

In Theorem 3.2 we study purely periodic continued fractions. As in Theorem 1.4 and Corollary 1.5, a lower bound appears. We show that, for our result on purely continued fractions, the lower bound is- while unpleasant- best possible.

Our main results give necessary and sufficient conditions for purely periodic HCF expansions (Theorem 3.2), characterize badly approximable complex numbers (Theorem 4.1), and give necessary conditions for transcendence of complex numbers (Theorem 5.1).

The paper is organized as follows. In the second part we define properly Hurwitz Continued Fractions and we discuss the associated shift space. In the third section, we state and prove sufficient and necessary conditions on purely periodic continued fractions. In the fourth section, we characterize badly approximable in terms on Hurwitz Continued Fractions (Theorem 4.1). This result was recently shown by Robert Hines ([Hi17]), but our proof is slightly different. In the last section we state and show Theorem 5.1, our main transcendence result.

Notation. We will reserve $[a_0; a_1, a_2, \ldots]$ for Hurwitz Continued Fractions. We will also need complex continued fractions which may fail to be Hurwitz, we represent them by $\langle a_0; a_1, a_2, \ldots\rangle$. By natural numbers, $\mathbb{N}$, we mean the set of positive integers and we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given two functions $f, g : \mathbb{N}_0 \to \mathbb{R}$, the Vinogradov symbol $f \ll g$ means that for some $C > 0$ and every $n \in \mathbb{N}$ we have $f(n) \leq Cg(n)$. Finally, for any $z \in \mathbb{C}$ and $A \subseteq \mathbb{C}$ we consider $z + A := \{z + a : a \in A\}$.

2 Hurwitz Continued Fractions

Denote by $[\cdot] : \mathbb{C} \to \mathbb{Z}[i]$ the function which assigns to each complex number its nearest Gaussian integers rounding up to break ties; thus

$$\forall z \in \mathbb{C} \quad z - [z] \in \mathfrak{F} := \left\{ z \in \mathbb{C} : \frac{1}{2} \leq \Re z, \Im z < \frac{1}{2} \right\}.$$

In analogy to the Gauss map, we define the function

$$T : \mathfrak{F}^* \to \mathfrak{F}, \quad T(z) = \frac{1}{z} - \left[\frac{1}{z}\right],$$

where $\mathfrak{F}^* = \mathfrak{F} \setminus \{0\}$. Writing $T^1 := T$ and $T^{n+1} := T^n \circ T$ for $n \in \mathbb{N}$ we give to any $z \in \mathbb{C}$ a pair of sequences of complex numbers $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ through

$$\begin{aligned} z_0 &:= z, \quad a_0 := [z_0], \\ z_n^{-1} &= T^{n-1}(z_0 - a_0), \quad a_n := [z_n], \end{aligned}$$

as long as the iterations of T make sense.

Considering a_1 as a function from $\mathfrak{F}$ to $\mathbb{Z}[i]$, we get a partition of $\mathfrak{F}$ induced by the pre-images of a_1 (see Figure 1). It is clear that

$$\mathfrak{K} := \mathrm{Cl}\left\{ z \in \mathfrak{F} : |a_1(z)| \geq \sqrt{8} \right\} \subseteq \mathfrak{F}^\circ, \tag{1}$$

where Cl denotes the closure and $\circ$, the interior with respect to the usual topology.

Definition 2.1. *The* ***Hurwitz Continued Fraction*** *(HCF) of a complex number z is the sequence $(a_n)_{n\geq 0}$ obtained by the above procedure. As defined in [DaNo14], the* ***$\mathcal{Q}$-pair*** *of z is the pair of sequences $(p_n)_{n=0}^{\infty}$, $(q_n)_{n=0}^{\infty}$ given by*

$$\begin{pmatrix} p_{-2} & p_{-1} \\ q_{-2} & q_{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \forall n \in \mathbb{N}_0 \quad \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n \\ 1 \end{pmatrix}. \tag{2}$$

The terms of the sequence $(\frac{p_n}{q_n})_{n\geq 0}$ are called the ***convergents*** *of z.*

As expected, the HCF of a complex number ζ is infinite if and only if $\zeta \in \mathbb{C} \setminus \mathbb{Q}(i)$. In this case, we naturally have that the convergents converge to ζ ([DaNo14], [Hu87]).

Not every sequence in $\mathbb{Z}[i]$ is the HCF of a complex number. We say that an infinite sequence of Gaussian integers is **valid** if it is the HCF of some complex irrational number and we will denote the set of valid sequences by Ω^{HCF}. By a **valid prefix** we mean a finite sequence on $\mathbb{Z}[i]$ which is the prefix of a valid sequence. A valid segment is defined similarly.

Some necessary conditions for a sequence $(a_n)_{n=0}^{\infty}$ to belong in Ω^{HCF} follow immediately from the algorithm.

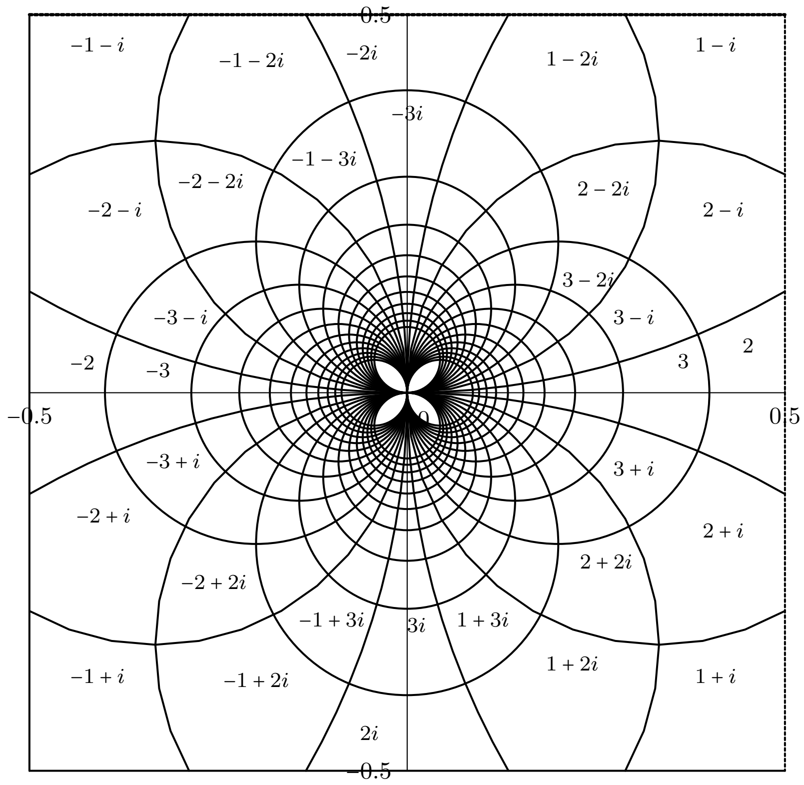

Figure 1: Partition of $\mathfrak{F}$ induced by $a_1(z)$.

Proposition 2.1. *Let $(a_n)_{n\geq 0} \in \Omega^{\mathrm{HCF}}$ be the HCF of z. The following inequalities hold*

$$\forall n \in \mathbb{N} \quad |z_n| \geq \sqrt{2}, \quad |a_n| \geq \sqrt{2}.$$

Moreover, we have that

$$\left\{(a_n)_{n\geq 0} \in \mathbb{Z}[i]^{\mathbb{N}_0} : \forall n \in \mathbb{N} \quad |a_n| \geq \sqrt{8}\right\} \subseteq \Omega^{\mathrm{HCF}}. \tag{3}$$

Take $z \in \mathfrak{F}^*$, then $z_1 \in \mathfrak{F}^{-1} := \{w^{-1} : w \in \mathfrak{F}\}$ (depicted in Figure 3). Suppose that z_2 exists. If $|a_1| \geq \sqrt{8}$, then the feasible maximal region for z_2 is again $\mathfrak{F}^{-1}$. However, if $a_1 = 1+i$–for example– z_2 belongs to the set

$$\left\{z \in \mathfrak{F}^{-1} : \left(\frac{1}{2} \leq \Re z\right) \& \left(\leq \Re z < \frac{1}{2}\right)\right\}.$$

In general, the maximal feasible region for z_{n+1}^{-1} has one of the forms drawn in Figure 2 or one of the rotations obtained by multiplying them times an integral power of i (in the referred figure we neglect the boundary). Thus, determining whether a sequence is valid or not is more complicated than just checking a uniform lower bound.

With a slight notational abuse, we see that $(T, \Omega^{\mathrm{HCF}})$ is a shift space which cannot be modelled as a Markov shift. Indeed, assume there was an infinite the matrix A characterizing[1] Ω^{HCF}. On the one hand, the segment $1+2i, -2+2i, 1+i$ is not valid, so $A_{-2+2i,1+i} = 0$. However, $0, -2+2i, 1+i$ is a valid prefix, which would imply $A_{-2+2i,1+i} = 1$, a contradiction. We can extend

[1] In the following sense: a sequence $(a_n)_{n\geq 1}$ in $\mathbb{Z}[i]$ belongs to Ω^{HCF} if and only if $A_{a_n a_{n+1}} = 1$ for every $n \in \mathbb{N}$.

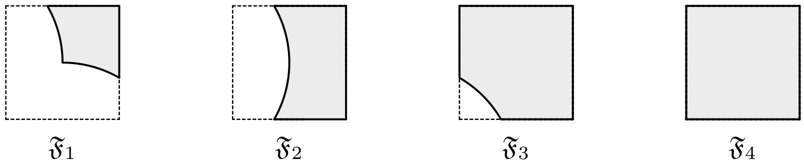

Figure 2: Feasible regions for z_{n+1}^{-1}.

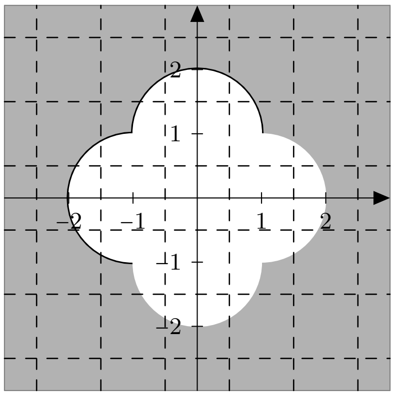

Figure 3: The set $\mathfrak{F}^{-1}$.

this observation to prefixes of arbitrary length. While for any $n \in \mathbb{N}$ the sequences

$$\begin{gathered}(\underbrace{-2+2i, 2-2i, \ldots, 2-2i, -2+2i}_{n \text{ repetitions of } -2+2i, 2-2i}, 1+i), \\ (2-2i, \underbrace{-2+2i, 2-2i, \ldots, 2-2i, , -2+2i}_{n \text{ repetitions of } -2+2i, 2-2i}, 1+i)\end{gathered} \tag{4}$$

are valid segments, the sequences

$$\begin{gathered}(1+2i, \underbrace{-2+2i, 2-2i, \ldots, 2-2i, -2+2i}_{n \text{ repetitions of } -2+2i, 2-2i}, 1+i), \\ (-1+2i, 2-2i, \underbrace{-2+2i, 2-2i, \ldots, 2-2i, , -2+2i}_{n \text{ repetitions of } -2+2i, 2-2i}, 1+i)\end{gathered} \tag{5}$$

are not. Obviously, the conclusion holds if we replace $1+2i$ and $-1+2i$ by $2+i$ and $-2+i$. We can obtain more examples using the symmetries of the HCF process.

The lack of Markov structure is a significant difference between the Hurwitz Continued Fractions and its direct real analogue, the Nearest Integer Continued Fraction (NICF). It can be easily shown that the set of valid sequences of integers with respect to the NICF algorithm can be characterized in terms of a transition matrix. This feature complicates the study of finite sequences of the form $(a_j, a_{j-1}, \ldots, a_1)$ where $(a_1, \ldots, a_{j-1}, a_j)$ is a valid prefix. Such sequences appear naturally since

$$\forall n \in \mathbb{N} \quad \frac{q_{n+1}}{q_n} = \langle a_{n+1}; a_n, \ldots, a_1 \rangle .$$

Nevertheless, the denominators $(q_n)_{n=0}^{\infty}$ have some useful properties. We borrow the following results from [DaNo14] (the third part is not stated in that paper, but it is a direct consequence of the second part).

Lemma 2.1 (Dani, Nogueira (2011)). *Let* $\zeta = [a_0; a_1, a_2, a_3, \ldots]$ *be an element of* $\mathbb{C} \smallsetminus \mathbb{Q}(i)$ *and* $(p_n)_{n=0}^{\infty}, (q_n)_{n=0}^{\infty}$. *The following statements hold.*

i. The sequence $(|q_n|)_{n\geq 0}$ *is strictly increasing.*

ii. Set $\phi = \frac{1+\sqrt{5}}{2}$ *, then*

$$\forall n \in \mathbb{N}_0 \quad \frac{|q_{n+1}|}{|q_n|} > \phi \quad \vee \quad \frac{|q_{n+2}|}{|q_{n+1}|} > \phi.$$

iii. For every $n, k \in \mathbb{N}$ *we have*

$$|q_{n+k}| > \phi^{\left[\frac{k}{2}\right]} |q_n|.$$

In the following, we adopt the notation of [La73]. For every $z \in \mathbb{C}$ and $\rho > 0$, $\mathbb{D}(z,\rho)$ is the open disc with radius ρ and centered at z, $\overline{\mathbb{D}}(z,\rho) = \operatorname{Cl}\mathbb{D}(z,\rho)$, and $\mathbb{E}(z,\rho) = \mathbb{C} \setminus \overline{\mathbb{D}}(z,\rho)$.

3 Periodic continued fractions

A. Hurwitz proved an analogue to the Euler Lagrange theorem for his continued fractions. More than a century later, S.G. Dani and A. Nogueira showed that it still holds for a larger family of complex continued fractions ([DaNo14]).

Theorem 3.1 (Hurwitz (1887))**.** *Let z be an irrational complex numbers. The Hurwitz Continued Fraction expansion of z is ultimately periodic if and only if z is an irrational quadratic number over $\mathbb{Q}(i)$.*

A well known result by Évariste Galois (1828) states a real number $\alpha > 1$ with a purely periodic continued fraction expansion is a quadratic irrational whose Galois Conjugate β satisfies $-1 < \beta < 0$ and vice versa. Such quadratic irrationals are called **reduced**. We provide a similar result for HCF.

Theorem 3.2. *Let $\xi = [a_0; a_1, a_2, \ldots]$ be a quadratic irrational over $\mathbb{Q}(i)$ and $\eta \in \mathbb{C}$ its Galois conjugate over $\mathbb{Q}(i)$.*

1. *If $(a_n)_{n=0}^{\infty}$ is purely periodic, then $|\eta| < 1$.*
2. *If $|\xi| > 1$, $\eta \in \mathfrak{F}$ and $|a_n| \geq \sqrt{8}$ for every $n \in \mathbb{N}_0$, then ξ has purely periodic expansion.*
3. *The conditions $\eta \in \mathfrak{F}$ and $(\forall n \in \mathbb{N}\ |a_n| \geq \sqrt{8})$ cannot be removed from the second point. In fact, there are infinitely many pairs ξ, η such that*
   - *i. $\eta \notin \mathfrak{F}$, $|a_n| < \sqrt{8}$ for some n, and $(a_n)_{n=0}^{\infty}$ is not purely periodic,*
   - *ii. $\eta \notin \mathfrak{F}$, $|a_n| \geq \sqrt{8}$ for all n, and $(a_n)_{n=0}^{\infty}$ is not purely periodic,*
   - *iii. $\eta \in \mathfrak{F}$, $|a_n| < \sqrt{8}$ for some n, and $(a_n)_{n=0}^{\infty}$ is not purely periodic.*

If $(a_n)_{n=0}^{\infty}$ is a purely periodic sequence, we denote it by $(a_n)_{n=0}^{\infty} = (\overline{a_0, \ldots, a_{m-1}})$. We will tacitly assume that $m \in \mathbb{N}$ is minimal with respect to the condition $a_n = a_{n+m}$ for every $n \in \mathbb{N}$. We will call a valid purely periodic sequence $(\overline{a_0, a_1, \ldots, a_{m-1}})$ **reversible** if $(\overline{a_{m-1}, \ldots, a_1, a_0})$ is valid. Note that every purely periodic sequence whose terms have absolute value at least $\sqrt{8}$ is reversible.

Proof. 1. Suppose that $\xi = [\overline{a_0; a_1, \ldots, a_{m-1}}]$. For any $j \in \mathbb{N}$ we have

$$\xi = \frac{p_{mj-1}\xi + p_{mj-2}}{q_{mj-1}\xi + q_{mj-2}} \quad \Longrightarrow \quad q_{mj-1}\xi^2 + (q_{mj-2} - p_{mj-1})\xi - p_{mj-2} = 0.$$

Dividing by q_{mj-1} we obtain a monic polynomial $P \in \mathbb{Q}(i)[X]$ satisfied by the irrational ξ, hence

$$\xi^2 - \left(\frac{p_{mj-1}}{q_{mj-1}} - \frac{q_{mj-2}}{q_{mj-1}}\right)\xi - \frac{p_{mj-2}}{q_{mj-2}} = 0 \quad \Longrightarrow \quad \eta = \frac{p_{mj-1}}{q_{mj-1}} - \xi - \frac{q_{mj-2}}{q_{mj-1}}.$$

And we conclude that

$$\lim_{j\to\infty} \frac{q_{mj-2}}{q_{mj-1}} = -\eta, \quad |\eta| \leq 1. \tag{6}$$

In order to show that $|\eta| < 1$ we check two cases: $|a_{m-1}| \geq 2$ and $|a_{m-1}| = \sqrt{2}$.

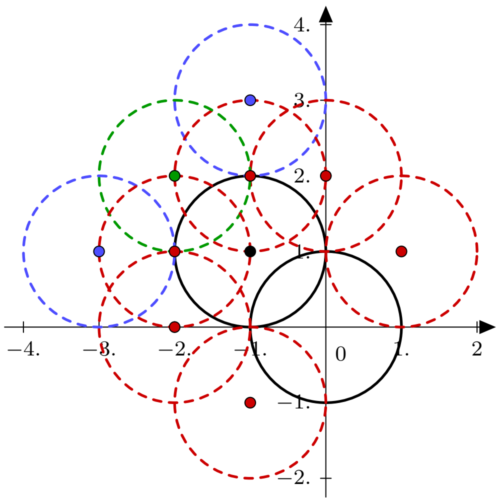

Figure 4: Exclusion by laws of succession (red) and rationality (blue).

§First case. $|a_{m-1}| \geq 2$. Assume that $|q_{jm-1}/q_{jm-2}| < \phi$ holds for large j. For such j, Lemma 2.1 implies that $\phi \leq |q_{jm-2}/q_{jm-3}|$, so $|q_{jm-3}/q_{jm-2}| \leq \phi^{-1}$ and

$$\forall j \in \mathbb{N} \quad \left|\frac{q_{jm-1}}{q_{jm-2}}\right| = \left|a_{m-1} + \frac{q_{jm-3}}{q_{jm-2}}\right| \geq 2 - \frac{1}{\phi} > 1,$$

so $|\eta^{-1}| > 1$. When $|q_{jm-1}/q_{jm-2}| \geq \phi > 1$, we also get $|\eta^{-1}| \geq \phi > 1$.

§Second case. $|a_{m-1}| = \sqrt{2}$. By symmetry, we may suppose that $a_{m-1} = 1+i$. The element a_{m-2} must exist, for $(i+1, i+1)$ is not a valid segment. Since $(|q_n|)_{n\geq 0}$ is strictly increasing,

$$\frac{q_{jm-1}}{q_{jm-2}} = 1 + i + \frac{q_{jm-3}}{q_{jm-2}} \in \mathbb{E}(0,1) \cap \mathbb{D}(1+i,1) \implies \frac{q_{jm-2}}{q_{jm-3}} = a_{m-2} + \frac{q_{jm-4}}{q_{jm-3}} \in \mathbb{E}(-1+i,1) \cap \mathbb{E}(0,1).$$

Assume for contradiction that $|\eta| = 1$. Then, since $\overline{\mathbb{D}}(a_{m-2},1)$ has to intersect the boundary of $\mathbb{E}(1+i,1) \cup \mathbb{E}(0,1)$ and $|a_{m-2}| \geq \sqrt{2}$, the only possibilities for a_{m-2} are

$$1+i,\ 2i,\ -1-i,\ -1+2i,\ -1+3i,\ -2,\ -2+i,\ -2+2i,\ -3+i. \tag{7}$$

(see Figure 4.) The laws of succession of the HCF exclude the options $1+i$, $2i, -1-i\,, -1+2i$, -2, $-2+i$. The intersection $\overline{\mathbb{D}}(-1+i,1) \cap \overline{\mathbb{D}}(-1+3i,1)$ is $-1+2i$, so by taking limits, we would have

$$\eta = 1 + i + \frac{1}{-1+2i} \in \mathbb{Q}(i),$$

which is impossible. Similarly, we can dismiss $-3+i$. Hence, a_{m-2} has to be $-2+2i$ and

$$\frac{q_{jm-2}}{q_{jm-3}} = -2 + 2i + \frac{q_{jm-4}}{q_{jm-3}} \in \mathbb{D}(0,1) \cap \mathbb{E}(-1+i,1) \implies \frac{q_{jm-4}}{q_{jm-3}} \in \mathbb{D}(0,1) \cap \mathbb{E}(1-i,1).$$

With a similar argument and the observations made in (4) and (5), we conclude that a_{m-3} exists and has to be equal to $a_{m-3} = 2+2i$. Continuing in this way, we obtain that the period has to alternate between $-2+2i$ and $2+2i$. By pure periodicity, $1+i$ eventually appears. However, the previous argument shows that $1+i$ and $1-i$ are never valid options. Therefore, we can conclude $|\eta| < 1$.

2. Take the notation as in the statement. Conjugate over $\mathbb{Q}(i)$ the sequence $(\xi_j)_{j\geq 0}$ given by

$$\xi_0 := \xi, \qquad \forall n \in \mathbb{N}_0 \quad \xi_{n+1} = \frac{1}{\xi_n - a_n} \in \mathfrak{F}$$

to obtain

$$\zeta_0 := \eta, \qquad \forall n \in \mathbb{N}_0 \quad \zeta_{n+1} = \frac{1}{\zeta_n - a_n}.$$

Let $(k_j)_{j\geq 0}$ be given by

$$k_0 = 1, \qquad \forall n \in \mathbb{N}_0 \quad k_{n+1} = \sqrt{8} - \frac{1}{k_n}.$$

It is not hard to show inductively that $k_n \to \sqrt{2}+1$ as $n \to \infty$, that $k_n > 2$ for $n \geq 2$, and that $|\zeta_n| \leq k_n^{-1}$ for all n.

Assume that ξ was not purely periodic. Write

$$\xi = [a_0; a_1, \ldots, a_n, \overline{a_{n+1}, \ldots, a_{m+n}}]$$

with $a_n \neq a_{n+m}$. Define $\xi' = [\overline{a_{n+1}, \ldots, a_{m+n}}] = \xi_{n+1}$ and call $\eta' = \zeta_{n+1}$ its Galois conjugate. The proof of the first point tells us that we have

$$-\frac{1}{\eta'} = [\overline{a_{m+n}; a_{m+n-1}, \ldots, a_{n+1}}].$$

If we had $n \geq 2$, the left-most term in

$$\zeta_n - \zeta_{n+m} = a_n + \frac{1}{\zeta_{n+1}} - a_{n+m} - \frac{1}{\zeta_{n+m+1}} = a_n - a_{n+m}$$

would belong to $\mathbb{D}(0,1)$, while the right-most term would be a non-zero Gaussian integer. Therefore, we must have either $n = 0$ or $n = 1$. Suppose $n = 1$. Conjugate $\xi_1 = a_1 + 1/\xi'$ to get $\zeta_1 = a_1 + 1/\eta'$ and

$$-\zeta_1 = -\frac{1}{\eta'} - a_1 = [a_{m+1} - a_1; a_m, \ldots, a_2, \overline{a_{m+1}, \ldots, a_2}] \in (a_{m+1} - a_1) + \mathfrak{F}. \tag{8}$$

The inequality $|\zeta_1| \leq k_1^{-1} = (\sqrt{8}-1)^{-1}$ along with $a_{m-1} \neq a_1$ and the previous contention give $|a_{m+1} - a_1| = 1$. Assume that $a_{m+1} - a_1 = 1$ (the other cases are treated similarly). Then, we have

$$-\zeta_1 - 1 \in \mathbb{D}(-1, (\sqrt{8}+1)^{-1}) \cap \mathfrak{F} \implies \frac{1}{-\zeta_1 - 1} \in \mathbb{D}\left(\frac{(\sqrt{8}-1)^2\sqrt{2}}{-8(\sqrt{2}-1)}, \frac{(\sqrt{8}-1)\sqrt{2}}{8(\sqrt{2}-1)}\right) \cap \mathfrak{F}^{-1}.$$

Call c_0 and ρ_0, respectively, the center and the radius of the last disc. Direct calculations give $0.7 < \rho_0 < 0.8$ and $-1.5 < c_0 < 1.4$, so

$$\{a \in \mathbb{Z}[i]^* : \mathbb{D}(c_0; \rho_0) \cap \mathfrak{F}^{-1} \cap (a + \mathfrak{F}) \neq \varnothing\} = \{-2+i, -2, -2-i\}.$$

Therefore, in view of (8), we would have $|a_m| \leq \sqrt{5}$, contradicting the lower bound hypothesis. The only possibility left is $n = 0$, but in this case we would have $\zeta_1 = \eta'$ and

$$\eta = a_0 + \frac{1}{\zeta_1} = a_0 - \frac{-1}{\eta'} = a_0 - a_m - [0; a_{m-1}, \ldots, a_0, \overline{a_m, \ldots, a_0}] \in \mathfrak{F},$$

which implies– because the last term belongs to $\mathfrak{F}$– that $a_m = a_0$, a contradiction. Therefore, $(a_n)_{n=0}^{\infty}$ is purely periodic.

3. i. Take $M = M_1 - iM_2, N = N_1 - iN_2 \in \mathbb{Z}[i]$ with $N_1, M_1 > 0$ and $|M|, |N|$ large. Since $(a_n)_{n\geq 1} = (\overline{M, 1+i, N, 2+4i})$ is valid and reversible, we can consider the Galois conjugate irrational quadratics ξ' and η' with HCF expansion

$$\xi' = [\overline{M; 1+i, N, 2+4i}], \qquad -\frac{1}{\eta'} = [\overline{2+4i; N, 1+i, M}].$$

Define $\xi = 3+4i+1/\xi'$ whose Galois conjugate is

$$\eta = 3+4i+\frac{1}{\eta'} = 3+4i - [\overline{2+4i; N, 1+i, M}]$$
$$= [1; -N, -1-i, -M, \overline{-2-4i, -N, -1-i, -M}].$$

We have a quadratic irrational ξ, $|\xi|>1$, whose HCF is not purely periodic and that has elements with absolute value less that $\sqrt{8}$. The Galois conjugate of ξ, η, belongs to $\mathbb{D}(0,1)\setminus\mathfrak{F}$ because $\Re(-N)<0$.

The core of the construction is to chose a reversible sequence $(a_n)_{n\geq 1} = (\overline{a_1,\dots,a_m})$ with $\Re(a_{m-1})<0$ and pick $a_0 = a_m+1$.

ii. The argument of the previous point can be easily adapted.

iii. The classical theory of regular continued fractions provide the first examples. Since reduced quadratic irrationals are dense in $(1,\infty)$, they are dense in $(1, 2.5)$. Take a reduced quadratic irrational α with $1<\alpha<2.5$. The HCF of $\alpha = [a_0; a_1, a_2, \dots]$ cannot be purely periodic, because $1<\alpha<1.5$ implies $a_0 = 1$ and $1.5<\alpha<2.5$ implies $a_0 = 2, a_1<0$. However, the Galois conjugate of α in $\mathbb{Q}$, and hence in $\mathbb{Q}(i)$, lies in $(-1,0)\subseteq\mathbb{D}$. An explicit example is the Golden Ratio ϕ

$$\phi = \frac{1+\sqrt{5}}{2} = [2; \overline{-3,3}], \quad \frac{-1}{\phi} = \frac{1-\sqrt{5}}{2} = [-1; \overline{3,-3}].$$

Although these examples do not say that $\sqrt{8}$ is best possible, they do provide a strategy to build infinitely many numbers showing it. For any $M = M_1+iM_2\in\mathbb{Z}[i]\cap\mathfrak{F}^{-1}$ with $M_1>0$ define

$$\nu := \nu(M) := \langle\overline{2+i; -2+i, M}\rangle$$

The purely periodic sequence $(\overline{2+i; -2+i, M})$ is not valid, because any valid prefix of the form $(2+i; -2+i, M, 2+i, -2+i, N)$ has $\Re N<0$. The convergence of the continued fraction defining ν follows from the validity of $(\overline{M, -2+i, 2+i})$ and the symmetry of the continuats $K_n(a_1,\dots,a_n) = K_n(a_n,\dots,a_1)$ (See Proposition 1.1, [He06]). We claim that the HCF of ν is

$$\nu = [2+i; -2+i, M+1, \overline{-2+i, 2+i, M}].$$

To verify the equality, take $\xi := [\overline{-2+i; 2+i, M}]$. The purely periodic expansions of ξ and ν yield

$$((-2+I)M+1)\nu^2 + (-4+4M)\nu + 4 = 0, \quad ((2+I)M+1)\xi^2 + (4+4M)\xi + 4 = 0.$$

Therefore, we can obtain a polynomial over $\mathbb{Z}[i]$ solved by $1+\xi^{-1}$ and ν^{-1}. After locating both points in the complex, we can conclude that $1+\xi^{-1} = \nu^{-1}$. Finally, the Galois conjugate η of ν is

$$\eta = -[0; \overline{M, -2+i, 2+i}] \in \mathfrak{F}\subseteq\mathbb{D}. \qquad \square$$

Remark. The Galois Conjugate η of $\zeta = [\overline{a_0; a_1,\dots,a_m}]$ satisfies

$$\eta = -\langle 0; \overline{a_m,\dots,a_1,a_0}\rangle,$$

which is not a surprise at all. This expansion, however, may fail to be a HCF. When $(a_n)_{n=0}^{\infty}$ has simple motives for not to be reversible, we can compute the HCF of η with a singularization[2] identities like

$$a + \cfrac{1}{2+\cfrac{1}{b}} = a+1+\cfrac{1}{-2+\cfrac{1}{b+1}}, \quad a+\cfrac{1}{1+i+\cfrac{1}{b}} = a+(1-i)+\cfrac{1}{-(1+i)+\cfrac{1}{b+1-i}}.$$

[2] We borrow the term *singularization* from [IoKr02].

for any $a, b \in \mathbb{C}$. A concrete example is given by

$$\zeta = [\overline{5 + 6i; -3 + 2i, 2, 9 + 4i}]$$

and its Galois Conjugate η. The continued fraction of ζ is not reversible. Nevertheless, we can get the HCF of $-\eta$ by reversing the period and applying the first singularization formula

$$\eta = -[0; \overline{10 + 4I, -2, -2 + 2i, 5 + 6i}].$$

Once we know why a sequence is not reversible, it is not hard to determine the process to obtain a valid expansion. The difficulty is in determining the source of non-reversibility.

4 Bounded Hurwitz Continued Fractions

Minkowski's First Convex Body Theorem implies a complex version of a famous corollary to Dirichlet's Theorem on Diophantine Approixmation. Namely, a complex number ζ is irrational if and only if there are infinitely many co-prime Gaussian integers p and q such that

$$\left|\zeta - \frac{p}{q}\right| < \frac{4}{\pi}\frac{1}{|q|^2}. \tag{9}$$

Lester Ford showed in 1920- in a similar spirit of Hurwitz Theorem on diophantine approximation- that the constant $4/\pi$ can be replaced by $(\sqrt{3})^{-1}$ and that this is best possible. As in the real case, a complex number is badly approximable if (9) cannot be improved, in some sense.

Definition 4.1. *A complex number $z \in \mathbb{C}$ if **badly approximable** if there is a $C > 0$ such that every $p, q \in \mathbb{Z}[i]$, $q \neq 0$, satisfy*

$$\left|z - \frac{p}{q}\right| > \frac{C}{|q|^2}.$$

The set of badly approximable complex numbers is denoted $\mathbf{Bad}_{\mathbb{C}}$.

$\mathbf{Bad}_{\mathbb{C}}$ shares some properties with its real counterpart. For instance, it has Lebesgue measure 0, it is $\frac{1}{2}$-winning in the sense of Schmidt games ([DoKr03]), and so it is of full Hausdorff dimension. We can also characterize $\mathbf{Bad}_{\mathbb{C}}$ in terms of Hurwitz Continued Fractions.

Theorem 4.1. *The set of badly approximable complex number is*

$$\mathbf{Bad}_{\mathbb{C}} = \{z = [a_0; a_1, a_2, \ldots] \in \mathbb{C} : \exists M > 0 \quad \forall n \in \mathbb{N}_0 \quad |a_n| \leq M\}.$$

Most of the standard argument ([Kh]) in the real version of Theorem 4.1 works in our context too. However, we need to be sure that HCF convergents do provide good approximations. Although now convergents are not best approximations, they still have some desirable properties. We say that $p/q \in \mathbb{Q}(i)$ is a **good approximation** to $\zeta \in \mathbb{C}$ if

$$|q\zeta - p| = \min\left\{|q'x - p'| : q', p' \in \mathbb{Z}[i] \quad |q'| \leq |q|\right\}.$$

We say that $p/q \in \mathbb{Q}(i)$ is a **best approximation** to $\zeta \in \mathbb{C}$ if

$$\forall p', q' \in \mathbb{C} \qquad |q'| < |q| \qquad \Longrightarrow \qquad |q\zeta - p| < |q'\zeta - p'|.$$

The next result is Theorem 1 of [La73].

Theorem 4.2 (Lakein, 1973)**.** *Let ζ be a complex number. Every convergent of ζ is a good approximation to ζ. Moreover, for almost every $\zeta \in \mathbb{C}$ (with respect to the Lebesgue measure) every convergent is a best approximation to ζ.*

Proposition 4.1. *Let $\xi = [a_0; a_1, a_2, \ldots]$ be an irrational complex number and $(p_n)_{n=0}^{\infty}$, $(q_n)_{n=0}^{\infty}$ its $\mathcal{Q}$-pair. Then, for some absolute constant we have*

$$\forall n \in \mathbb{N} \quad \frac{1}{(|z_{n+1}|+1)\,|q_n|^2} < \left|\xi - \frac{p_n}{q_n}\right| \ll \frac{1}{|q_n q_{n+1}|}. \tag{10}$$

Proof of Proposition 4.1. An inductive argument (Proposition 1, [DaNo14]) gives us that $q_n z_{n+1} + q_{n-1} = z(p_n z_{n+1} + p_{n-1})$ for every $n \in \mathbb{N}$, therefore

$$z = \frac{p_n z_{n+1} + p_{n-1}}{q_n z_{n+1} + q_{n-1}} = \frac{p_n}{q_n} + \frac{(-1)^n}{q_n^2\left(z_{n+1} + \frac{q_{n-1}}{q_n}\right)} = \frac{p_n}{q_n} + \frac{(-1)^n}{q_n^2\left(a_{n+1} + \frac{1}{z_{n+2}} + \frac{q_{n-1}}{q_n}\right)}$$

The left inequality in (10) follows immediately from the strict monotonicity of $(|q_n|)_{n\geq 0}$. For the right inequality, we use $z_{n+1} = a_{n+1} + z_{n+2}^{-1}$ to obtain

$$z - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(q_n z_{n+1} + q_{n-1})} = \frac{(-1)^n}{q_n\left(q_{n+1} + q_n \frac{1}{z_{n+2}}\right)} = \frac{(-1)^n}{q_n q_{n+1}\left(1 + \frac{q_n}{q_{n+1}}\frac{1}{z_{n+2}}\right)}. \tag{11}$$

Since $(|q_n|)_{n\geq 0}$ is strictly increasing and $z_{n+2}^{-1} \in \mathfrak{F} \subseteq \overline{\mathbb{D}}(0, 2^{-\frac{1}{2}})$, we can uniformly bound by below $\left|1 + \frac{q_n}{q_{n+1}}\frac{1}{z_{n+2}}\right|$ as follows

$$\left|1 + \frac{q_n}{q_{n+1}}\frac{1}{z_{n+2}}\right| \geq 1 - \frac{\sqrt{2}}{2} > 0.$$

The previous inequality plugged into (11) yields the result. □

Proof of Theorem 4.1. We start by showing $\supseteq$. Let $z = [a_0; a_1, a_2, \ldots]$ be a complex number whose elements are bounded by $M > 0$. Then,

$$\forall n \in \mathbb{N} \quad |z| = \left|a_n + \frac{1}{\zeta_{n+1}}\right| \leq M + 1$$

and, by the left inequality of Lemma 10, we have for $c_1 = (M+2)^{-1}$ that

$$\forall n \in \mathbb{N} \qquad \frac{c_1}{|q_n|^2} \leq \left|z - \frac{p_n}{q_n}\right|.$$

Now, take $p/q \in \mathbb{Q}(i)$ in lowest terms and $n \in \mathbb{N}$ such that $|q_{n-1}| < |q| \leq |q_n|$. By Theorem 4, $|q_n z - p_n| \leq |qz - p|$, so our choice of n gives

$$\left|z - \frac{p}{q}\right| \geq \frac{|q_n|}{|q|}\left|z - \frac{p_n}{q_n}\right| \geq \frac{c_1}{|q_n|^2} > \frac{c_1}{|q|^2}\frac{|q_{n-1}|^2}{|q_n|^2} = \frac{c_1}{|q|^2}\frac{1}{\left|a_n + \frac{q_{n-2}}{q_{n-1}}\right|^2} \geq \frac{c_1}{|q|^2(M+1)^2}.$$

Therefore, $z \in \mathbf{Bad}_{\mathbb{C}}$ satisfies with $C = (M+1)^{-1}(M+2)^{-2}$.

In order to show the other contention, take $z \in \mathbf{Bad}_{\mathbb{C}}$ and let $C = C(z) > 0$ be the constant from the definition. By Lemma 10, for some constant $c > 0$ and any $n \in \mathbb{N}$ we have

$$\frac{C}{|q_n|^2} \leq \frac{c}{|q_n q_{n+1}|} \qquad \Longrightarrow \qquad \left|a_{n+1} + \frac{q_{n-1}}{q_n}\right| = \left|\frac{q_{n+1}}{q_n}\right| \ll 1 \qquad \Longrightarrow \qquad |a_{n+1}| \ll 1.$$

The last implication follows from $|q_{n-1}| < |q_n|$. The constants in $\ll$ depend on z. □

This characterization of $\mathbf{Bad}_{\mathbb{C}}$ allows us to restate Theorem 1.3 as follows.

Theorem 4.3. *There are complex, badly approximable, algebraic numbers of arbitrary even degree over $\mathbb{Q}(i)$.*

Recently, Robert Hines ([Hi17]) gave a different proof of Theorem 4.1. His argument also relies on R. Lakein's work but avoids Lemma 10.

5 Construction of Transcendental Complex Numbers

Let us recall that the **length** of a finite word x is the number of terms it comprises and it is denoted by $|x|$. In general, let $\mathcal{A} \neq \varnothing$ be a finite alphabet and $\mathbf{a}$ an infinite word over $\mathcal{A}$. As noted in [BuKi15], $\operatorname{rep}(\mathbf{a}) < +\infty$ is equivalent to the existence of three sequences of finite words in $\mathcal{A}$, $(W_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$, and $(V_n)_{n\geq 1}$, such that

i. For every n the word $W_nU_nV_nU_n$ is a prefix of $\mathbf{a}$,

ii. The sequence $(|W_n| + |V_n|)/|U_n|$ is bounded above,

iii. The sequence $(|U_n|)_{n\geq 1}$ is strictly increasing.

The behaviour of $(W_n)_{n\geq 1}$ prompts two different cases:

$$\liminf_{n\to\infty} |W_n| < +\infty \quad \vee \quad \liminf_{n\to\infty} |W_n| = +\infty.$$

Our main result is the following theorem.

Theorem 5.1. *Let $\mathbf{a} = (a_j)_{j\geq 1}$ be a non-periodic valid sequence such that*

$$\operatorname{rep} \mathbf{a} < +\infty.$$

Call $\zeta = [0; a_1, a_2, a_3, \ldots]$ and let $(W_n)_{n\geq 0}, (U_n)_{n\geq 0}, (V_n)_{n\geq 0}$ be as above.

i. If $\liminf_{n\to\infty} |W_n| < +\infty$, then ζ is transcendental.

ii. If $\liminf_{n\to\infty} |W_n| = +\infty$ and $|a_n| \geq \sqrt{8}$ for every $n \in \mathbb{N}$, then ζ is transcendental.

A drawback of our analogue of Theorem 1.1 is that the slight weakening leads us to a lot less elegant statement.

5.1 Preliminary Lemmas

Keep the notation of Theorem 5.1 and call $(p_n)_{n=0}^{\infty}$, $(q_n)_{n=0}^{\infty}$ the $\mathcal{Q}$-pair of ζ. For simplicity's sake, consider the sequences of non-negative integers $(w_n)_{n\geq 1}$, $(u_n)_{n\geq 1}$, and $(v_n)_{n\geq 1}$ given by

$$\forall n \in \mathbb{N} \quad w_n = |W_n|, \quad u_n = |U_n|, \quad v_n = |V_n|.$$

We can control the growth of $(q_n)_{n\geq 0}$ throught the combinatorial conditions on $(W_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$, and $(V_n)_{n\geq 1}$.

Lemma 5.2. *There exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$*

$$\psi^{u_n} \geq |q_{w_n} q_{w_n+u_n+v_n}|^{\varepsilon},$$

with $\psi := \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{3}}$.

Proof. The boundedness of $(a_j)_{j\geq 1}$ and $((v_n + w_n)/u_n)_{n\geq 0}$ allows us to define

$$M := 1 + \limsup_{n\to\infty} |q_n|^{\frac{1}{n}}, \quad N := 2 + \limsup_{n\to\infty} \frac{2w_n + v_n}{u_n}.$$

By Lemma 2.1,

$$\psi = \left(M^{w_n} M^{w_n+u_n+v_n}\right)^{\frac{\log_M \psi}{2w_n+u_n+v_n}} > |q_{w_n} q_{u_n+v_n+w_n}|^{\frac{\log_M \psi}{2w_n+u_n+v_n}},$$

and $\varepsilon = \frac{\log \psi}{N \log M}$ works:

$$\psi^{u_n} > \left(|q_{w_n}|^{\frac{u_n}{2w_n+u_n+v_n}} |q_{u_n+v_n+w_n}|^{\frac{u_n}{2w_n+u_n+v_n}}\right)^{\log_M \psi} > |q_{w_n} q_{u_n+v_n+w_n}|^{\frac{\log \psi}{N \log M}}. \qquad \square$$

As in [Bu15], our main tool is an adequate version of Schmidt's Subspace Theorem. In [Sch76], Wolfgang Schmidt obtained his Subspace Theorem for number fields, but we will consider just consider a special case.

Theorem 5.3 (Schmidt's Subspace Theorem for Number Fields). *Let $\mathscr{L}_1,\ldots,\mathscr{L}_m$ and $\mathscr{M}_1,\ldots,\mathscr{M}_m$ be two sets of m linearly independent linear forms in m variables with algebraic coefficients.*

For any $\varepsilon > 0$ there are proper subspaces of $\mathbb{Q}(i)^m$, $T_1,\ldots,T_k$, such that for every $\boldsymbol{\beta} = (b_1,\ldots,b_k) \in \mathbb{Z}[i]$ with $\boldsymbol{\beta} \neq \mathbf{0}$

$$\left|\prod_{j=1}^{m} \mathscr{L}_j(\boldsymbol{\beta})\right| \left|\prod_{j=1}^{m} \mathscr{M}_j(\overline{\boldsymbol{\beta}})\right| \leq \frac{1}{\|\boldsymbol{\beta}\|_\infty^{\varepsilon}} \qquad \Longrightarrow \qquad \boldsymbol{\beta} \in \bigcup_{j=1}^{k} T_j,$$

where $\|\boldsymbol{\beta}\|_\infty = \max\{|b_1|,\ldots,|b_k|\}$, and $\overline{\boldsymbol{\beta}} = (\overline{b_1},\ldots,\overline{b_m})$.

5.2 Proof of Theorem 5.1

We can restate the two cases in a much simpler way after taking, if necessary, appropriate sub-sequences. Instead of $\liminf_n w_n < +\infty$ we will think of $(w_n)_{n\geq 1}$ as constant. And when $\liminf_n w_n < +\infty$, we will suppose that $(w_n)_{n\geq 1}$ is strictly increasing and that $a_{w_n} \neq a_{w_n+u_n+v_n}$. We do not lose generality with the later restriction. Indeed, if we had $a_{w_n} = a_{w_n+u_n+v_n} = a$, we could take finite words (possibly empty) W', V' such that $W = W'a$ and $UV = UV'a$. Hence, $W_nU_nV_nU = W'aU_nV'aU_n = \widetilde{W}\widetilde{U}\widetilde{V}\widetilde{U}$ where $\widetilde{W} = W'$, $\widetilde{U} = aU_n$, and $\widetilde{V} = V'$.

Throughout the proof, the constants implied by $\ll$ will depend on ζ.

§.$(w_n)_{n\geq 1}$ is constant. Write $k = w_1 \in \mathbb{N}_0$ and $W = W_1$. We can assume that $k = 0$, because the transcendence of any element in $\{[a_n; a_{n+1}, a_{n+2}, \ldots] : n \in \mathbb{N}\}$ implies the transcendence of all the others.

i. Write $s_n = u_n + v_n$ and define the sequence of irrational quadratic numbers $\left(\zeta^{(n)}\right)_{n\geq 1}$ by

$$\forall n \in \mathbb{N} \qquad \zeta^{(n)} = [0; b_1, b_2, b_3, \ldots] = [0; U_nV_nU_nV_nU_nV_n \ldots],$$

where the second term means

$$\forall j \in \mathbb{N} \quad b_j = (U_nV_n)_r, \quad 0 \leq r \leq s_n - 1, \quad r \equiv j \pmod{s_n}.$$

The validity of the sequences defining $\zeta^{(n)}$ is not quite evident; in fact, some of them might not be. Nevertheless, after taking a subsequence, we can assume that they are valid. We give a broad sketch of the argument and leave the details to the reader. In general, suppose that for $X = x_1\ldots x_m$, $Y = y_1\ldots y_n$ the word XYX is a valid prefix but $0XYXY$ is not. Without loss of generality, we can assume that $0XYXy_1$ is not a valid prefix. Since x_my_1 appears once but it is forbidden the second time, $|x_m| \in \{\sqrt{5}, \sqrt{8}\}$. Therefore, the maximal feasible region for a residue following x_m either has the form $\mathfrak{F}_2$ or $\mathfrak{F}_3$ (see Figure 2). After verifying separately each case for $|x_m|$, we conclude that the first $m-1$ terms of X must alternate between $i^k(2+2i)$ and $i^k(-2+2i)$ for some fixed $k \in \{1,2,3,4\}$. Moreover, we must have $|y_j| = \sqrt{5}$ for some j. This observation, $u_n \to \infty$, and $\limsup_{n\to\infty} v_n/u_n < +\infty$ imply that the sequences $0\overline{U_nV_n}$ are valid for sufficiently large n.

The first $u_n + s_n$ terms of the HCF of ζ and $\zeta^{(n)}$ coincide, then (cfr. Proposition 4.1)

$$\left|\zeta - \zeta^{(n)}\right| < \frac{1}{|q_{u_n+s_n}|^2}.$$

As in the real case, each $\zeta^{(n)}$ satisfies the polynomial

$$P_n(X) := q_{s_n-1}X^2 + (q_{s_n} - p_{s_n-1})\,X - p_{s_n}.$$

Therefore, we can bound $P_n(\zeta)$ as follows

$$\begin{aligned}|P_n(\zeta)| &= \left|P_n(\zeta) - P_n(\zeta^{(n)})\right| \\ &= \left|q_{s_n}\left(\zeta - \zeta^{(n)}\right)\left(\zeta + \zeta^{(n)}\right) + (q_{s_n} - p_{s_n-1})\left(\zeta - \zeta^{(n)}\right)\right| \\ &\ll |q_{s_n}|\left|\zeta - \zeta^{(n)}\right| + 2|q_{s_n}|\left|\zeta - \zeta^{(n)}\right| \ll \frac{|q_{s_n}|}{|q_{u_n+s_n}|^2}.\end{aligned}$$

ii. *First application of the Subspace Theorem.* Consider the sequence of points in $\mathbb{Z}[i]^4$, $(\mathbf{x}_n)_{n=1}^{\infty}$, given by

$$\forall n \in \mathbb{N} \quad \mathbf{x}_n = (q_{s_n-1}, -p_{s_n-1}, q_{s_n}, -p_{s_n}), \quad \|\mathbf{x}_n\|_\infty = |q_{s_n}|.$$

Define two collections of independent linear forms in $\mathbb{C}$, $\mathscr{L}^1 = \{\mathscr{L}_1^1, \mathscr{L}_2^1, \mathscr{L}_3^1, \mathscr{L}_4^1\}$ and $\mathscr{M}^1 = \{\mathscr{M}_1^1, \mathscr{M}_2^1, \mathscr{M}_3^1, \mathscr{M}_4^1\}$ in the variables $\mathbf{X} = (X_1, X_2, X_3, X_4)$ and $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3, \widetilde{X}_4)$, respectively, by

$$\begin{aligned}\mathscr{L}_1^1(\mathbf{X}) &= \zeta^2 X_1 - \zeta(X_2 + X_3) + X_4, & \mathscr{M}_1^1(\widetilde{\mathbf{X}}) &= \overline{\zeta}^2 \widetilde{X}_1 - \overline{\zeta}(\widetilde{X}_2 + \widetilde{X}_3) + \widetilde{X}_4, \\ \mathscr{L}_2^1(\mathbf{X}) &= \zeta X_1 - X_2, & \mathscr{M}_2^1(\widetilde{\mathbf{X}}) &= \overline{\zeta}\widetilde{X}_1 - \widetilde{X}_2, \\ \mathscr{L}_3^1(\mathbf{X}) &= \zeta X_1 - X_3, & \mathscr{M}_3^1(\widetilde{\mathbf{X}}) &= \overline{\zeta}\widetilde{X}_1 - \widetilde{X}_3, \\ \mathscr{L}_4^1(\mathbf{X}) &= X_1, & \mathscr{M}_4^1(\widetilde{\mathbf{X}}) &= \widetilde{X}_1.\end{aligned} \tag{12}$$

Evaluating the product of the forms $\mathscr{L}^1$ in $\mathbf{x}_n$ we have that

$$\begin{aligned}\left|\mathscr{L}_1^1\mathscr{L}_2^1\mathscr{L}_3^1\mathscr{L}_4^1(\mathbf{x}_n)\right| &= |P_n(\zeta)||\zeta q_{s_n-1} - p_{s_n-1}||\zeta q_{s_n-1} - q_{s_n}||q_{s_n-1}| \\ &\ll \frac{|q_{s_n}|^2}{|q_{s_n+u_n}|^2} < \frac{1}{\psi^{2u_n}}.\end{aligned}$$

Hence, by Lemma 5.2, there exists $\varepsilon > 0$ such that

$$\forall n \in \mathbb{N} \quad \left|\mathscr{L}_1^1\mathscr{L}_2^1\mathscr{L}_3^1\mathscr{L}_4^1(\mathbf{x}_n)\right| \ll \frac{1}{\|\mathbf{x}_n\|_\infty^\varepsilon}. \tag{13}$$

By construction, $|\mathscr{M}_j^1(\overline{\mathbf{x}}_n)| = |\mathscr{L}_j^1(\mathbf{x}_n)|$ for all n and $j \in \{1, \dots, 4\}$, where the bar denotes complex conjugation component-wise. From Theorem 5.3 we conclude the existence of a vector $\mathbf{0} \neq \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{Z}[i]^4$ and an infinite set $\mathcal{N}_1 \subseteq \mathbb{N}$ such that

$$\forall n \in \mathcal{N}_1 \quad x_1 q_{s_n-1} + x_2 p_{s_n-1} + x_3 q_{s_n} + x_4 p_{s_n} = 0.$$

Dividing the last expression by q_{s_n-1} we get

$$\forall n \in \mathcal{N}_1 \quad x_1 + x_2 \frac{p_{s_n-1}}{q_{s_n-1}} + x_3 \frac{q_{s_n}}{q_{s_n-1}} + x_4 \frac{p_{s_n}}{q_{s_n}} \frac{q_{s_n}}{q_{s_n-1}} = 0.$$

Since the Gaussian integers x_2 and x_4 cannot both be 0 (ζ is irrationals), we can define

$$\xi = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}_1}} \frac{q_{s_n}}{q_{s_n-1}} = -\frac{\zeta x_2 + x_1}{\zeta x_4 + x_3}.$$

The irrationality of ξ follows after obtaining infinitely many complex rational numbers approaching to ξ at a certain rate. For every $n \in \mathcal{N}_1$ we have that

$$\begin{aligned}\left|\xi - \frac{q_{s_n-1}}{q_{s_n}}\right| &= \left|\frac{x_1 + \zeta x_3}{x_2 + \zeta x_4} - \frac{x_1 + \frac{p_{s_n}}{q_{s_n}} x_3}{x_2 + \frac{p_{s_n-1}}{q_{s_n-1}} x_4}\right| \\ &\leq |x_1|\left|\frac{1}{x_1 + \zeta x_4} - \frac{1}{x_2 + x_4 \frac{p_{s_n-1}}{q_{s_n-1}}}\right| + |x_3|\left|\frac{\zeta}{x_2 + x_1\zeta} - \frac{\frac{p_{s_n}}{q_{s-n}}}{x_2 + \frac{p_{s_n-1}}{q_{s_n-1}} x_4}\right| \\ &\ll \left|\zeta - \frac{p_{s_n}}{q_{s_n}}\right| + \left|\zeta - \frac{p_{s_n-1}}{q_{s_n-1}}\right| + \left|\frac{p_{s_n-1}}{q_{s_n-1}} - \frac{p_{s_n}}{q_{s_n}}\right| \ll \frac{1}{|q_{s_n} q_{s_n-1}|}.\end{aligned} \tag{14}$$

If ξ were rational, $\xi = a/b$ for some co-prime $a, b \in \mathbb{Z}[i]$, we would have

$$\frac{1}{|bq_{v_n-1}|} \leq \left|\xi - \frac{q_{v_n-1}}{q_{v_n}}\right| \ll \frac{1}{|q_{v_n-1}q_{v_n}|} \qquad \Longrightarrow \qquad |q_{v_n-1}| \ll |b|,$$

for q_{v_n-1} and q_{v_n} co-prime. In other words, $(|q_{v_n-1}|)_{n\geq 0}$ would be bounded, which is absurd. Hence, $\xi \in \mathbb{Q}(i,\zeta)$ is irrational.

iii. *Second Application of the Subspace Theorem.* Consider the linear forms $\mathscr{L}^2 = \{\mathscr{L}_1^2, \mathscr{L}_2^2, \mathscr{L}_3^2\}$ and $\mathscr{M}^1 = \{\mathscr{M}_1^2, \mathscr{M}_2^2, \mathscr{M}_3^2\}$ in the variables $\mathbf{Y} = (Y_1, Y_2, Y_3)$ and $\widetilde{\mathbf{Y}} = (\widetilde{Y}_1, \widetilde{Y}_2, \widetilde{Y}_3)$ given by

$$\begin{aligned} \mathscr{L}_1^2(\mathbf{Y}) &= \xi Y_1 - Y_2, & \mathscr{M}_1^2(\widetilde{\mathbf{Y}}) &= \overline{\xi}\widetilde{Y}_1 - \widetilde{Y}_2,\\ \mathscr{L}_2^2(\mathbf{Y}) &= \zeta Y_1 - Y_3, & \mathscr{M}_2^2(\widetilde{\mathbf{Y}}) &= \overline{\zeta}\widetilde{Y}_1 - \widetilde{Y}_3,\\ \mathscr{L}_3^2(\mathbf{Y}) &= Y_2, & \mathscr{M}_3^2(\widetilde{\mathbf{Y}}) &= \widetilde{Y}_2. \end{aligned}$$

Define sequence $(\mathbf{y}_n)_{n\geq 0}$ in $\mathbb{Z}[i]^3$ by

$$\forall n \in \mathbb{N} \quad \mathbf{y}_n := (q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n}), \qquad \|\mathbf{y}_n\|_\infty = |q_{u_n+v_n}|.$$

Since

$$\begin{aligned} \left|\mathscr{M}_1^2\mathscr{M}_2^2\mathscr{M}_3^2(\overline{\mathbf{y}}_n)\right| &= \left|\mathscr{L}_1^2\mathscr{L}_2^2\mathscr{L}_3^2(\mathbf{y}_n)\right| \\ &= |(q_{u_n+v_n}\xi - q_{u_n+v_n-1})(q_{u_n+v_n}\zeta - p_{u_n+v_n})q_{u_n+v_n-1}| \\ &\leq \frac{|q_{u_n+v_n-1}|}{|q_{u_n+v_n}q_{u_n+v_n-1}|} = \frac{1}{\|\mathbf{y}_n\|_\infty}, \end{aligned}$$

by Lemma 5.2, there is an $\varepsilon > 0$ such that

$$\forall n \in \mathcal{N}_1 \quad \left|\prod_{j=1}^{3}\mathscr{L}_j^2(\mathbf{y}_n)\right|\left|\prod_{k=1}^{3}\mathscr{M}_k^2(\overline{\mathbf{y}_n})\right| \leq \frac{1}{\|\mathbf{y}_n\|^{\varepsilon}}.$$

Schmidt's Subspace Theorem yields the existence of a non-zero $(y_1, y_2, y_3) \in \mathbb{Z}[i]^3$ and an infinite set $\mathcal{N}_2 \subseteq \mathcal{N}_1$ such that

$$\forall n \in \mathcal{N}_2 \quad q_{v_n}y_1 + q_{v_n-1}y_2 + p_{v_n}y_3 = 0.$$

Dividing by q_{v_n} and taking limits along $\mathcal{N}_2$ we obtain

$$y_1 + \xi y_2 + \zeta y_3 = 0, \tag{15}$$

and, since ζ is irrational, $y_2y_3 \neq 0$.

iv. *Third application of the Subspace Theorem.* Define $\mathscr{L}^3 = \{\mathscr{L}_1^3, \mathscr{L}_2^3\mathscr{L}_3^3\}$ and $\mathscr{M}^3 = \{\mathscr{M}_1^3, \mathscr{M}_2^3\mathscr{M}_3^3\}$ in the variables $\mathbf{Z} = (Z_1, Z_2, Z_3)$ and $\widetilde{\mathbf{Z}} = (\widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3)$ by

$$\begin{aligned} \mathscr{L}_1^3(\mathbf{Z}) &= \xi Z_1 - Z_2, & \mathscr{M}_1^3(\widetilde{\mathbf{Z}}) &= \overline{\xi}\widetilde{Z}_1 - \widetilde{Z}_2,\\ \mathscr{L}_2^3(\mathbf{Z}) &= \zeta Z_2 - Z_3, & \mathscr{M}_2^3(\widetilde{\mathbf{Z}}) &= \overline{\zeta}\widetilde{Z}_2 - \widetilde{Z}_3,\\ \mathscr{L}_3^3(\mathbf{Z}) &= Z_2, & \mathscr{M}_3^3(\widetilde{\mathbf{Z}}) &= \widetilde{Z}_2, \end{aligned}$$

and the sequence $(\mathbf{z}_n)_{n\geq 1}$ by

$$\forall n \in \mathbb{N} \quad \mathbf{z}_n = (q_{s_n}, q_{s_n-1}, p_{s_n}), \qquad \|\mathbf{z}_n\|_\infty = |q_{s_n}|.$$

We get from (14) and the upper bound in (10) that

$$\begin{aligned} \left|\mathscr{M}_1^3\mathscr{M}_2^3\mathscr{M}_3^3(\overline{\mathbf{z}}_n)\right| &= \left|\mathscr{L}_1^3\mathscr{L}_2^3\mathscr{L}_3^3(\mathbf{z}_n)\right| \\ &= |(q_{s_n}\xi - q_{s_n-1})(\zeta q_{s_n-1} - p_{s_n-1})q_{s_n-1}| \\ &\ll \frac{|q_{s_n-1}|}{|q_{s_n-1}q_{s_n}|} = \frac{1}{\|\mathbf{z}_n\|_\infty}. \end{aligned}$$

Once again, by Schmidt's Subspace Theorem, there is a non-zero triplet $(z_1, z_2, z_3) \in \mathbb{Z}[i]^3$ and $\mathcal{N}_3 \subseteq \mathcal{N}_2$ infinite for which

$$\forall n \in \mathcal{N}_3 \quad z_1 q_{s_n} + z_2 q_{s_n-1} + z_3 p_{s_n} = 0.$$

Dividing by q_{s_n-1} and taking the limit when $n \to \infty$ along $\mathcal{N}_3$, we get

$$\frac{1}{\xi} z_1 + z_2 + \zeta z_3 = 0. \tag{16}$$

Since ζ and ξ are irrational, $z_1 z_3 \neq 0$. Combining (15) and (16) we get

$$z_1 y_2 = (z_2 + \zeta z_3)(y_1 + \zeta y_3),$$

with $z_3 y_3 = 0$. This contradicts $[\mathbb{Q}(i,\zeta) : \mathbb{Q}(i)] \geq 3$. Therefore, ζ is transcendental.

§.$(w_n)_{n\geq 1}$ is strictly increasing. As before, we start by approximating ζ with quadratic irrationals. However, this time they will not have purely periodic expansions.

i. Define the sequence $\left(\zeta^{(n)}\right)_{n\geq 1}$ by

$$\forall n \in \mathbb{N} \quad \zeta^{(n)} = [0; W_n U_n V_n U_n V_n U_n V_n \ldots].$$

Tedious but straight forward computations show that each $\zeta^{(j)}$ satisfies the polynomial

$$P_n(X) := \begin{vmatrix} q_{w_n-1} & q_{w_n+u_n+v_n-1} \\ q_{w_n} & q_{w_n+u_n+v_n} \end{vmatrix} X^2 - \left(\begin{vmatrix} q_{w_n-1} & p_{w_n+u_n+v_n-1} \\ q_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix} + \begin{vmatrix} p_{w_n-1} & q_{w_n+u_n+v_n-1} \\ p_{w_n} & q_{w_n+u_n+v_n} \end{vmatrix} \right) X + \\ + \begin{vmatrix} p_{w_n-1} & p_{w_n+u_n+v_n-1} \\ p_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix}$$

The first $w_n + 2u_n + v_n$ elements of ζ and $\zeta^{(n)}$ coincide, so

$$\forall n \in \mathbb{N} \quad \left|\zeta - \zeta^{(n)}\right| \ll \frac{1}{|q_{w_n+2u_n+v_n}|^2}.$$

The previous equality, the strict increase of $(|q_k|)_{k\geq 0}$, and $|p_k| < |q_k|$ for any k (because $\zeta \in \mathfrak{F}$) imply that

$$\forall n \in \mathbb{N} \quad |P_n(\zeta)| = \left|P_n(\zeta) - P_n\left(\zeta^{(n)}\right)\right| \ll \frac{|q_{w_n+u_n+v_n}|}{|q_{w_n} q^2_{w_n+2u_n+v_n}|}. \tag{17}$$

ii. *First application of the Subspace Theorem.* Define the forms $\mathscr{L}^1$ and $\mathscr{M}^1$ as in (12). Let $\mathbf{x}_n = (x_{n,1}, x_{n,2}, x_{n,3}, x_{n,4})$ in $\mathbb{Z}[i]^4$ be the sequence

$$x_{n,1} := \begin{vmatrix} q_{w_n-1} & q_{w_n+u_n+v_n-1} \\ q_{w_n} & q_{w_n+u_n+v_n} \end{vmatrix}, \quad x_{n,2} := \begin{vmatrix} q_{w_n-1} & p_{w_n+u_n+v_n-1} \\ q_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix},$$
$$x_{n,3} := \begin{vmatrix} p_{w_n-1} & q_{w_n+u_n+v_n-1} \\ p_{w_n} & q_{w_n+u_n+v_n} \end{vmatrix}, \quad x_{n,4} := \begin{vmatrix} p_{w_n-1} & p_{w_n+u_n+v_n-1} \\ p_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix}.$$

It can be easily proven that

$$\forall n \in \mathbb{N} \quad \|\mathbf{x}_n\|_\infty \ll |q_{w_n} q_{w_n+u_n+v_n}|. \tag{18}$$

For example, since $(|q_j|)_{j\geq 0}$ is strictly increasing and $|p_j| \leq |q_j|$ for every j, we have

$$|x_{n,2}| = |q_{w_n-1} p_{w_n+u_n+v_n} - p_{w_n+u_n+v_n-1} q_{w_n}| \leq |q_{w_n}| \left(|p_{w_n+u_n+v_n}| + |p_{w_n+u_n+v_n-1}|\right) \\ \ll |q_{w_n} q_{w_n+u_n+v_n}|.$$

Let $\epsilon > 0$ be as in Lemma 5.2. For every $\mathbf{x}_n$

$$\begin{aligned}\prod_{j=1}^{4}\left|\mathscr{M}_j^1(\overline{\mathbf{x}_n})\right| = \prod_{j=1}^{4}\left|\mathscr{L}_j^1(\mathbf{x}_n)\right| &= |P_n(\zeta)|\,|\zeta x_{n,1}-x_{n,2}|\,|\zeta x_{n,1}-x_{n,3}|\,|x_{n,1}| \\ &\ll \left|\frac{q_{w_n+u_n+v_n}}{q_{w_n}q^2_{w_n+2u_n+v_n}}\frac{q_{w_n}}{q_{w_n+u_n+v_n}}\frac{q_{w_n+u_n+v_n}}{q_{w_n}}q_{w_n}q_{w_n+u_n+v_n}\right| \\ &= \left|\frac{q^2_{w_n+u_n+v_n}}{q^2_{w_n+2u_n+v_n}}\right| \ll \frac{1}{|q_{w_n}q_{w_n+u_n+v_n}|^{\varepsilon}} = \frac{1}{\|\mathbf{x}_n\|^{\varepsilon}}.\end{aligned}$$

Therefore, there exist $\mathbf{0} \neq \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{Z}[i]^4$ and an infinite set $\mathcal{N}_1' \subseteq \mathbb{N}$ such that

$$\begin{aligned}\forall n \in \mathcal{N}_1' \quad 0 = &x_1\begin{vmatrix} q_{w_n-1} & q_{w_n+u_n+v_n-1} \\ q_{w_n} & q_{w_n+u_n+v_n}\end{vmatrix} + x_2\begin{vmatrix} q_{w_n-1} & p_{w_n+u_n+v_n-1} \\ q_{w_n} & p_{w_n+u_n+v_n}\end{vmatrix} \\ &+ x_3\begin{vmatrix} p_{w_n-1} & q_{w_n+u_n+v_n-1} \\ p_{w_n} & q_{w_n+u_n+v_n}\end{vmatrix} + x_4\begin{vmatrix} p_{w_n-1} & p_{w_n+u_n+v_n-1} \\ p_{w_n} & p_{w_n+u_n+v_n}\end{vmatrix}. \qquad (19)\end{aligned}$$

Let $(Q_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ be

$$\forall n \in \mathbb{N} \qquad Q_n := \frac{q_{w_n-1}q_{w_n+u_n+v_n}}{q_{w_n}q_{w_n+u_n+v_n-1}}, \quad R_n := \zeta - \frac{p_n}{q_n}.$$

Divide (19) by $q_{w_n}q_{w_n+u_n+v_n-1}$ to obtain for every $n \in \mathcal{N}_1$ the equation

$$\begin{aligned}0 = &\, x_1(Q_n - 1) + x_2\left(Q_n(\zeta - R_{w_n+u_n+v_n}) - (\zeta - R_{w_n+u_n+v_n-1})\right) + \\ &+ x_3\left(Q_n(\zeta - R_{w_n-1}) - (\zeta - R_{w_n})\right) + \\ &+ x_4\left(Q_n(\zeta - R_{w_n-1})(\zeta - R_{w_n+u_n+v_n}) - (\zeta - R_{w_n})(\zeta - R_{w_n+u_n+v_n-1})\right). \qquad (20)\end{aligned}$$

From (20) we obtain

$$0 = (Q_n - 1)(x_1 + (x_2 + x_3)\zeta + \zeta x_4) + \eta(n),$$

where $\eta(n) \to 0$ when $n \to \infty$ along $\mathcal{N}_1'$, because

$$\begin{aligned}\eta(n) = &\, x_2((Q_n - 1)R_{w_n+u_n+v_n} + R_{w_n+u_n+v_n-1} - R_{w_n+u_n+v_n}) + \\ &+ x_3((-Q_n + 1)R_{w_n-1} + R_{w_n} - R_{w_n-1}) + \\ &+ x_4(Q_n - 1)(-(R_{w_n-1} + R_{w_n+u_n+v_n}) - R_{w_n-1}R_{w_n+u_n+v_n}) + \\ &+ (\zeta - R_{w_n-1})(\zeta - R_{w_n+u_n+v_n}) - (\zeta - R_{w_n})(\zeta - R_{w_n+u_n+v_n-1}).\end{aligned}$$

We conclude that

$$\lim_{n\to\infty}(Q_n - 1)\left(x_1 + (x_2 + x_3)\zeta + \zeta^2 x_4\right) = 0$$

The boundedness of $(a_n)_{n=0}^{\infty}$ allows us to assume the existence of the following limits

$$\alpha = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}}\frac{q_{w_n}}{q_{w_n-1}}, \quad \beta = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}}\frac{q_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n-1}}, \qquad (21)$$

and that $a = a_{w_n} \neq a_{w_n+u_n+v_n} = b$ are constant. We can assure that (cfr. (1) and Proposition 2.1)

$$\begin{aligned}\frac{q_{w_n}}{q_{w_n-1}} &= [a; a_{w_n-1}, \ldots, a_0] \in a + \mathfrak{K}, \\ \frac{q_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n-1}} &= [b; a_{w_n+u_n+v_n-1}, \ldots, a_0] \in b + \mathfrak{K},\end{aligned}$$

and– because $a + \mathfrak{K}$ and $b + \mathfrak{K}$ are disjoint compact sets– $\alpha \neq \beta$ and $Q_n \not\to 1$ as $n \to \infty$ along $\mathcal{N}_2'$. As a consequence, we get

$$\left(x_1 + (x_2 + x_3)\zeta + \zeta^2 x_4\right) = 0. \qquad (22)$$

Since ζ is neither quadratic nor rational, we must have that $x_1 = x_4 = 0$, $x_2 = -x_3$, and the polynomials P_n become

$$P_n(X) = \begin{vmatrix} q_{w_n-1} & q_{w_n+u_n+v_n-1} \\ q_{w_n} & q_{w_n+u_n+v_n} \end{vmatrix} X^2 - 2\begin{vmatrix} q_{w_n-1} & p_{w_n+u_n+v_n-1} \\ q_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix} X + \begin{vmatrix} p_{w_n-1} & p_{w_n+u_n+v_n-1} \\ p_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix}.$$

Consider the sets of linear forms $\mathscr{L}^4 = \{\mathscr{L}_1^4, \mathscr{L}_2^4, \mathscr{L}_3^4\}$ and $\mathscr{M}^4 = \{\mathscr{M}_1^4, \mathscr{M}_2^4, \mathscr{M}_3^4\}$ in the variables $\mathbf{Y} = (Y_1, Y_2, Y_3)$ and $\widetilde{\mathbf{Y}} = (\widetilde{Y}_1, \widetilde{Y}_2, \widetilde{Y}_3)$ given by

$$\begin{aligned} \mathscr{L}_1^4(\mathbf{Y}) &= \zeta^2 Y_1 - 2\zeta Y_2 + Y_3, & \mathscr{M}_1^4(\widetilde{\mathbf{Y}}) &= \overline{\zeta}^2 \widetilde{Y}_1 - 2\overline{\zeta}\widetilde{Y}_2 + \widetilde{Y}_3 \\ \mathscr{L}_1^4(\mathbf{Y}) &= \zeta Y_1 - Y_2, & \mathscr{M}_1^4(\widetilde{\mathbf{Y}}) &= \overline{\zeta}2\widetilde{Y}_1 - \widetilde{Y}_2 \\ \mathscr{L}_1^4(\mathbf{Y}) &= Y_1, & \mathscr{M}_1^4(\widetilde{\mathbf{Y}}) &= \widetilde{Y}_1 \end{aligned}$$

For $n \in \mathcal{N}_1'$ define

$$\mathbf{v}_n := \left(\begin{vmatrix} q_{w_n-1} & q_{w_n+u_n+v_n-1} \\ q_{w_n} & q_{w_n+u_n+v_n} \end{vmatrix}, \begin{vmatrix} q_{w_n-1} & p_{w_n+u_n+v_n-1} \\ q_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix}, \begin{vmatrix} p_{w_n-1} & p_{w_n+u_n+v_n-1} \\ p_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix} \right).$$

Using the new expression for P_n and an idea similar to the one proving (17), we obtain that for some $\varepsilon > 0$

$$\begin{aligned} \left| \prod_{j=1}^{3} \mathscr{M}_j^4(\overline{\mathbf{v}_n}) \right| &= \left| \prod_{j=1}^{3} \mathscr{L}_j^4(\mathbf{v}_n) \right| \\ &= |P_n(\zeta)(\zeta v_{n,1} - v_{n,2}) v_{n,1}| \ll \frac{|q_{w_n} q_{w_n+u_n+v_m}|}{|q_{w_n+2u_n+v_n}^2|} \ll \frac{1}{|q_{w_n} q_{w_n+u_n+v_n}|^{\varepsilon}}. \end{aligned}$$

Schmidt's Subspace Theorem guarantees the existence of $\mathbf{0} \neq (t_1, t_2, t_3) \in \mathbb{Z}[i]^3$ such that for some infinite $\mathcal{N}_2' \subseteq \mathcal{N}_1'$

$$t_1 \begin{vmatrix} q_{w_n-1} & q_{w_n+u_n+v_n-1} \\ q_{w_n} & q_{w_n+u_n+v_n} \end{vmatrix} + t_2 \begin{vmatrix} q_{w_n-1} & p_{w_n+u_n+v_n-1} \\ q_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix} + t_3 \begin{vmatrix} p_{w_n-1} & p_{w_n+u_n+v_n-1} \\ p_{w_n} & p_{w_n+u_n+v_n} \end{vmatrix} = 0.$$

Dividing by $q_{w_n} q_{w_n+u_n+v_n-1}$, we get

$$\begin{aligned} t_1(Q_n - 1) + t_2 \left(Q_n \frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}} \right) + \\ + t_3 \left(Q_n \frac{p_{w_n-1}}{q_{w_n-1}} \frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n}}{q_{w_n}} \frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}} \right) = 0. \end{aligned}$$

Hence, since $Q_n \not\to 1$ when $n \to \infty$ along $\mathcal{N}_2'$,

$$t_3 \zeta^2 + t_2 \zeta + t_1 = 0.$$

The previous equation contradicts $[\mathbb{Q}(i, \zeta) : \mathbb{Q}(i)] \geq 3$. Therefore, ζ is transcendental.

6 Further results

Other transcendence results can be translated into our context with the pertinent modifications. An example is Theorem 6.1 (cfr. [Bu15], Theorem 11.2).

In general, we say that an infinite word $\mathbf{a} \in \mathcal{A}$ satisfies condition (♣) if there are sequences $(W_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$ of finite words in $\mathcal{A}$ such that

1. $W_n U_n V_n \widehat{U_n}$ is a prefix of $\mathbf{a}$ for every $n \in \mathbb{N}$, where $\widehat{U_n}$ is the word obtained by reversing U_n,
2. $((|W_n| + |V_n|)/|U_n|)_{n\geq 1}$ is bounded,

3. $(|U_n|)_{n\geq 1}$ tends to infinity when n does.

Theorem 6.1. *Let* $\mathbf{a} \in \Omega^{\mathrm{HCF}}$ *satisfy Condition* ($\clubsuit$) *and* $|a_n| \geq \sqrt{8}$ *for every* n. *If* $\mathbf{a}$ *is not periodic, then* $\zeta = [0; a_1, a_2, \ldots]$ *is transcendental.*

Remark. i. The proof of Theorem 6.1 needs an inequality involving continuants (the denominators of the convergents, denoted K_n). For any valid segments $\mathbf{a}$, $\mathbf{b}$ of length n and m respectively such that $\mathbf{ab}$ is a valid prefix, we have for some absolute constant

$$|K_{n+m}(\mathbf{ab})| \ll |K_n(\mathbf{a})K_n(\mathbf{b})|.$$

The inequality follows from well known continued fraction identity. If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ are two vectors of variables, then

$$K_{n+m}(\mathbf{xy}) = K_n(\mathbf{x})K_m(\mathbf{y}) \left(1 + \langle 0; x_n, \ldots x_1\rangle \langle 0; y_1, \ldots y_m\rangle\right).$$

For a proof, see Proposition 1.1. in [He06].

ii. Getting rid of the condition $|a_n| \geq \sqrt{8}$ for every n in Theorem 6.1 poses essentially the same problem as the corresponding bound in Theorem 5.1.

7 Final comments

Our argument for $Q_n \not\to 1$ along $\mathcal{N}'$ in the second case of Theorem 5.1 is simpler than the original one. Unfortunately, it does not help to omit the lower bound and get a full analogue of Theorem 1.1. We can understand why with the help of Figure 3. Unlike regular continued fractions, for every $M > \sqrt{8}$ we can find arbitrarily close complex numbers $z = [a_0; a_1, \ldots]$ and $w = [b_0; b_1, \ldots]$ such that $a_0 \neq b_0$ and $\sup |a_n|, \sup |b_n| < M$.

Acknowledgements.

Thanks to Yann Bugeaud for his invaluable comments. This research was supported by CONACYT (grant: 410695) and Aarhus University (St.: 2012, Pr.: 25763, Ak.: 26248).

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Good's Theorem for Hurwitz Continued Fractions

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Abstract

Good's Theorem for regular continued fraction states that the set of real numbers $[a_0; a_1, a_2, \ldots]$ such that $\lim_{n\to\infty} a_n = \infty$ has Hausdorff dimension $\frac{1}{2}$. We show an analogous result for the complex plane and Hurwitz Continued Fractions. The set of complex numbers whose Hurwitz Continued fraction $[a_0; a_1, a_2, \ldots]$ satisfies $\lim_{n\to\infty} |a_n| = \infty$ has Hausdorff dimension 1, half of the ambient space's dimension.

1 Introduction

A basic result on regular continued fractions states that every irrational number α admits infinitely many rational approximations p/q, $p, q \in \mathbb{Z}$ and $q \geq 1$, for which

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

For some real numbers the previous inequality cannot be improved in terms of the exponent of q. These numbers are called badly approximable numbers and the set of badly approximable real numbers is denoted by $\mathbf{Bad}_{\mathbb{R}}$. To be more precise, $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ is **badly approximable** if there exists $c > 0$ such that[1]

$$\forall (p,q) \in \mathbb{Z} \times \mathbb{N} \quad \left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^2}.$$

It is well known that $\mathbf{Bad}_{\mathbb{R}}$ is precisely the set of irrationals whose regular continued fraction is given by a bounded sequence. The characterization in terms of continued fraction allows us to show in a simple way that $\mathfrak{m}_1 \mathbf{Bad}_{\mathbb{R}} = 0$, where $\mathfrak{m}_1$ denotes the Lebesgue measure in $\mathbb{R}$. However small in terms of Lebesgue measure, V. Jarník published in 1928 (Satz 4, [Ja28]) an influential result saying that $\mathbf{Bad}_{\mathbb{R}}$ is rather large.

Theorem 1.1 (V. Jarník,1928)**.** *The set* $\mathbf{Bad}_{\mathbb{R}}$ *has full Hausdorff dimension; that is*

$$\dim_H \left\{ [0; a_1, a_2, \ldots] \in \mathbb{R} : \limsup_{n\to\infty} a_n < +\infty \right\} = 1.$$

A result related to Jarník's theorem was published by I.J. Good in 1941 (Theorem 1, [Goo41]).

[1] $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$ is the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Theorem 1.2 (I. G. Good, 1941)**.** *The following equality holds*

$$\dim_H \left\{ [0; a_1, a_2, \ldots] \in \mathbb{R} : \lim_{n\to\infty} a_n = +\infty \right\} = \frac{1}{2}.$$

Extensions of continued fractions and of Jarník's Theorem to other contexts have been successfully carried out (v. gr. [KTV06] or [Sc] and the references therein). Recently, important work has been done in the complex plane using a complex continued fraction algorithm suggested by Adolf Hurwitz in 1887 ([Hu87]) ([DaNo14]).

The First Minkowski's Theorem on Convex Bodies (or Dirichlet's Pigeonhole Principle) tells us that for some absolute constant $C > 0$ and any complex irrational ζ (i.e. $\zeta \in \mathbb{C} \smallsetminus \mathbb{Q}(i)$) there are infinitely many pairs $(p, q) \in \mathbb{Z}[i] \times \mathbb{Z}[i]^*$ such that

$$\left| \zeta - \frac{p}{q} \right| \le \frac{C}{|q|^2}.$$

As in the real case, we say that a complex irrational ζ is **badly approximable** if for some $c > 0$ we have

$$\forall (p, q) \in \mathbb{Z}[i] \times \mathbb{Z}[i]^* \qquad \left| \zeta - \frac{p}{q} \right| \ge \frac{c}{|q|^2}.$$

We denote by $\mathbf{Bad}_{\mathbb{C}}$ the set of badly approximable complex numbers. Some properties of $\mathbf{Bad}_{\mathbb{C}}$ are well known. For example, if $\mathfrak{m}_2$ denotes the Lebesgue measure on $\mathbb{C}$, then $\mathfrak{m}_2 \mathbf{Bad}_{\mathbb{C}} = 0$ and $\mathbf{Bad}_{\mathbb{C}}$ is $\frac{1}{2}$-winning in the sense of Schmidt games and so, it is of full Hausdorff dimension (Theorem 5.2, [DoKr03]). $\mathbf{Bad}_{\mathbb{C}}$ can also be characterized in terms of Hurwitz Continued Fractions: a complex irrational ζ belongs to $\mathbf{Bad}_{\mathbb{C}}$ if and only if its Hurwitz Continued fraction is bounded (Theorem 4.1, [Gon18]). Our main result establishes another similarity between Hurwitz and regular continued fractions.

Theorem 1.3. *For any $z \in \mathbb{C} \smallsetminus \mathbb{Q}(i)$ denote its Hurwitz Continued Fraction expansion by $z = [a_0; a_1, a_2, \ldots]$, then*

$$\dim_H \left\{ z = [a_0; a_1, a_2, a_3, \ldots] \in \mathbb{C} : \lim_{n\to\infty} |a_n| = +\infty \right\} = 1.$$

Although similarities between real and complex continued fractions abound, some differences have to be considered. While regular continued fractions establish an homeomorphism between the irrationals in $[0, 1]$ and $\mathbb{N}^{\mathbb{N}}$ the space of sequences associated to the HCF process is much more complicated. Since the difficulty arises from sequences $(a_n)_{n\ge1}$ with $\min_{n\ge1} |a_n| \le \sqrt{8}$, it is natural to ask how large are the subsets of $\mathbf{Bad}_{\mathbb{C}}$ where a uniform lower bound is imposed on the absolute value of the terms of the HCF expansion. In this direction, the proof of Theorem 1.3 gives us the next corollary (See Section 2 for the definition of $\mathfrak{F}$).

Corollary 1.4. *For any $L \in \mathbb{N}$ with $L \ge \sqrt{8}$ define*

$$E_L := \{ z = [0; a_1, a_2, a_3, \ldots] \in \mathfrak{F} : \forall n \in \mathbb{N} \quad L \le |a_n| \},$$

then

$$\dim_H \left(E_L \cap \mathbf{Bad}_{\mathbb{C}} \right) \searrow 1 \quad \textit{as} \quad L \to \infty.$$

The organization of the text is as follows. In Section 2, we define precisely the Hurwitz Continued Fraction algorithm and some of its properties. In Section 3, we give two lemmas for estimating the Hausdorff dimension of a class of Cantor sets. In Section 4 we state some preliminary lemmas concerning the Hausdorff dimension of sets obtained by imposing restrictions on the Hurwitz Continued Fraction expansions. In Section 5 we show Theorem 1.3 and Corollary 1.4. Finally, in Section 6 we prove the preliminary lemmas of Section 4.

Notation.

i. $\mathbb{N}$ is the set of natural numbers, considered as the set of positive integers,

ii. if $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are two sequences of non-negative numbers and there exists some constant $C > 0$ such that $x_n \leq C y_n$ for every large n, we write $x_n \ll y_n$. By $x_n \asymp y_n$ we mean $x_n \ll y_n \ll x_n$. Whenever the implied constants depend on a parameter ε, say, we write $x_n \ll_\varepsilon y_n$ and similarly for $\asymp$.

iii. if (X, d) is a metric space and $A \subseteq X$, the diameter of A is $|A| := \sup\{d(x,y) : x, y \in A\}$.

iv. for any complex number z we write $\mathbb{D}(z) = \{w \in \mathbb{C} : |z - w| < 1\}$, $\overline{\mathbb{D}}(z)$ is the closure of $\mathbb{D}(z)$ and $C(z)$ is its boundary.

v. let $A \subseteq \mathbb{C}$. A° is the interior of A and $A^{-1} = \{z^{-1} : z \in A\}$.

2 General Properties of Hurwitz Continued Fractions

Let $[\cdot] : \mathbb{C} \to \mathbb{Z}[i]$ be the function that assigns to each complex number the nearest Gaussian integer (choosing the one with greater real and imaginary parts to break ties). Denote by $\mathfrak{F}$ the inverse image of 0 under $[\cdot]$,

$$\mathfrak{F} := \left\{ z \in \mathbb{C} : -\frac{1}{2} \leq \Re z, \Im z < \frac{1}{2} \right\}. \tag{1}$$

Let $T : \mathfrak{F}^* \to \mathfrak{F}^*$, with $\mathfrak{F}^* := \mathfrak{F} \setminus \{0\}$, be given by

$$\forall z \in \mathfrak{F}^* \quad T(z) = \frac{1}{z} - \left[\frac{1}{z}\right].$$

For any $z \in \mathfrak{F}^*$ define– as long as the operations make sense– the sequences $(a_n)_{n\geq 0}$, $(z_n)_{n\geq 1}$ by

$$\begin{aligned} z_1 &:= z, \quad \forall n \in \mathbb{N} \quad z_{n+1} = T(z_n), \\ a_0 &:= 0, \quad \forall n \in \mathbb{N} \quad a_n = \left[z_n^{-1}\right]. \end{aligned}$$

The **Hurwitz Continued Fraction** (HCF) of z is the sequence $(a_n)_{n\geq 1}$. We can easily extend the definition to an arbitrary complex number w directly. The Hurwitz

Continued Fraction of $w \in \mathbb{C}$ is the sequence $(a_n)_{n\geq 0}$ where $a_0 = [w]$ and $(a_n)_{n\geq 1}$ is the HCF of $w - a_0$. We refer to the numbers a_n as **elements**. Following [DaNo14], we call the sequences $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$ given by

$$\begin{pmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \forall n \in \mathbb{N} \quad \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n \\ 1 \end{pmatrix}.$$

the $\mathcal{Q}$**-pair** of z.

We summarize some properties of HCF in the next proposition.

Proposition 2.1. *Let $z \neq 0$ belong to $\mathfrak{F}$ and denote by $(a_n)_{n\geq 0}$, $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$ be its associated sequences.*

i. For every $n \in \mathbb{N}$

$$\frac{p_n}{q_n} = [0; a_1, a_2, \ldots, a_n].$$

ii. The sequence $(a_n)_{n\geq 0}$ is infinite if and only if $z \in \mathfrak{F} \smallsetminus \mathbb{Q}(i)$. In this case, we have that

$$\lim_{n\to\infty} [0; a_1, a_2, \ldots, a_n] = z.$$

As a consequence, HCF give an injection from $\mathfrak{F} \smallsetminus \mathbb{Q}(i)$ into $\mathbb{Z}[i]^{\mathbb{N}}$.

iii. The sequence $(|q_n|)_{n\geq 0}$ is strictly increasing. Moreover, there exists a number $\psi > 1$ such that $|q_n| \geq \psi^n$ for all $n \in \mathbb{N}_0$.

iv. For every $n \in \mathbb{N}$

$$\left| z - \frac{p_n}{q_n} \right| \leq \frac{1}{|q_n|^2}.$$

Proof. Part i is trivial, ii is Theorem 6.1. of [DaNo14], iii is Corollary 5.3. of [DaNo14], and iv is Theorem 1 of [La73]. □

Although our main results are stated for general irrational complex numbers, there will be no loss of generality if we only work in $\mathfrak{F}$. Also, in view of Part ii of Proposition 2.1, we will abuse our notation and denote the set $\mathfrak{F} \smallsetminus \mathbb{Q}(i)$ by $\mathfrak{F}$.

F. Schweiger defined in [Sc] a **fibred system** to be a pair (B, T) where B is a non-empty set and $T : B \to B$ is a function such that there exist an at most countable partition of B, $\{B(i)\}_{i\in I}$, such that for all $i \in I$ the restriction of T to $B(i)$ is injective. We borrow the terminology from the fibred system theory to better describe HCFs.

Define $I = \{a \in \mathbb{Z}[i] : |a| \geq \sqrt{2}\}$. For every $a \in \mathbb{Z}[i]$ the **cylinder of level** 1 is

$$\mathcal{C}_1(a) = \left\{ z \in \mathfrak{F} : [z^{-1}] = a \right\}.$$

Note that $\mathcal{C}_1(a) \neq \varnothing$ if and only if $a \in I$ and that $\{\mathcal{C}_1(a)\}_{a\in I}$ gives a countable partition of $\mathfrak{F}$ (see Figure 1). Now, we can formally define the elements of the HCF as functions from $\mathfrak{F}$ onto I as follows

$$\forall n \in \mathbb{N} \quad a_n(z) = a \quad \iff \quad T^{n-1}(z) \in \mathcal{C}_1(a),$$

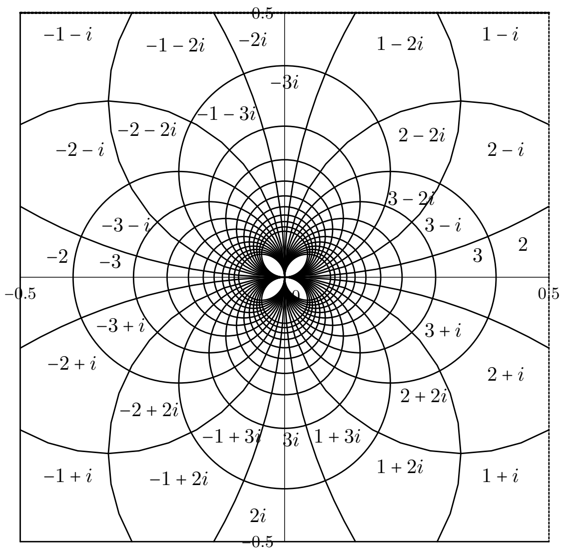

Figure 1: Partition $\{\mathcal{C}_1(a) : a \in I\}$ of $\mathfrak{F}$.

where $T^0 : \mathfrak{F} \to \mathfrak{F}$ is the identity and $T^n = T \circ T^{n-1}$ for $n \in \mathbb{N}$. For $\mathbf{a} = (a_1, \dots, a_n) \in I^n$ or $\mathbf{a} \in I^{\mathbb{N}}$ the **cylinder of level** n, $\mathcal{C}_n(\mathbf{a})$, is the set

$$\mathcal{C}_n(\mathbf{a}) = \{z \in \mathfrak{F} : a_1(z) = a_1, \dots, a_n(z) = a_n\}.$$

A sequence $\mathbf{a} \in I^{\mathbb{N}}$ is **admissible** or **valid** if it is the HCF of some $z \in \mathfrak{F}$ or, equivalently, if $\mathcal{C}_n(\mathbf{a}) \neq \varnothing$ for every $n \in \mathbb{N}$. We denote the set of admissible sequences by Ω^{HCF} and define $\Phi : \Omega^{\text{HCF}} \to \mathfrak{F}$ by[2]

$$\forall \mathbf{a} \in \Omega^{\text{HCF}} \quad \Phi(\mathbf{a}) = [0; a_1, a_2, a_3, \dots].$$

A **maximal feasible set** is a set $\mathfrak{F}' \subseteq \mathfrak{F}$ such that for some $\mathbf{a} \in \Omega^{\text{HCF}}$ and $n \in \mathbb{N}$ we have

$$T^n\left[\mathcal{C}_n(\mathbf{a})\right] = \mathfrak{F}'.$$

A maximal feasible set is **regular** if its interior is non-empty. A sequence $\mathbf{a} \in \Omega^{\text{HCF}}$ is regular if $T^n[\mathcal{C}_n(\mathbf{a})]$ is regular for every n and $z \in \mathfrak{F}$ is regular if its HCF is regular. For instance, if $w \in T_1[\mathcal{C}_1(-2)]$, then $(2+w)^{-1} \in \mathfrak{F}$, so

$$w \in \left(-2 + \mathfrak{F}^{-1}\right) \cap \mathfrak{F} = \mathfrak{F}^{-1} \cap (2 - \mathfrak{F}) + 2 = \mathfrak{F} \setminus \mathbb{D}(1).$$

On the other hand, if $w \in \mathfrak{F} \setminus \mathbb{D}(1)$, then $-2 + w \in \mathfrak{F}^{-1} \cap (2 - \mathfrak{F})$, so $[-2+w] = -2$ and $w \in T[\mathcal{C}_1(-2)]$. Similarly, we can show that $T[\mathcal{C}_1(a)] = \mathfrak{F}$ if and only if $|a| \geq \sqrt{8}$. Figure

[2]Our function Φ deviates from from Schweiger's. For him, Φ goes from the set B to the sequence space.

2 illustrates $\mathfrak{F}^{-1}$ and, with the help of the grid, the sets $a+T[\mathcal{C}_1(a)]$ for $|a| \le \sqrt{8}$ can be read.

After dismissing the boundary, all the non-empty sets $T[\mathcal{C}_1(a)]$ are of the form $i^j\mathfrak{F}_k$ for some $j,k \in \{1,2,3,4\}$, where

$$\mathfrak{F}_1 := (\mathfrak{F} \setminus (\mathbb{D}(-1) \cup \mathbb{D}(-i)))^\circ, \quad \mathfrak{F}_2 := (\mathfrak{F} \setminus \mathbb{D}(-1))^\circ,$$
$$\mathfrak{F}_3 := (\mathfrak{F} \setminus \mathbb{D}(-1-i))^\circ, \quad \mathfrak{F}_4 = \mathfrak{F}^\circ.$$

Evidently, $i^j\mathfrak{F}_4 = \mathfrak{F}_4$ for any $j \in \mathbb{Z}$. By computing directly the inversion of some circles (v.gr. $C(1+i)^{-1} = C(1-i)$), it can be shown inductively that these thirteen sets exhaust all the possibilities for the shapes that the interior of a regular maximal feasible set may assume (cfr. [HiVa18]).

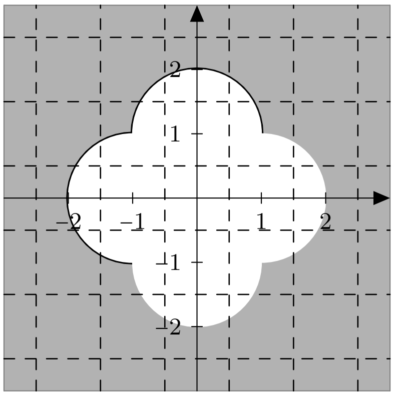

Figure 2: The set $\mathfrak{F}^{-1}$ and the lines $x = k$, $y = l$ for all $k, l \in \mathbb{Z}$.

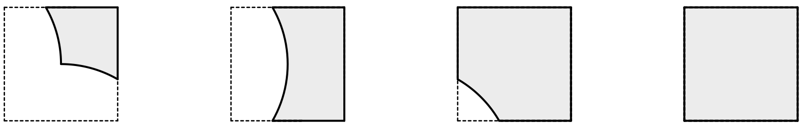

Figure 3: Sets $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$, $\mathfrak{F}_4$.

The previous observations and an iterative argument gives a simple necessary condition for a sequence to be admissible

$$E_{\sqrt{8}}(1) := \left\{ \mathbf{a} = (a_n)_{n\ge1} \in \mathbb{Z}[i]^{\mathbb{N}} : \forall n \in \mathbb{N} \quad \sqrt{8} \ge |a_n| \right\} \subseteq \Omega^{\mathrm{HCF}}.$$

Moreover, $T^n[\mathcal{C}_n(\mathbf{a})] = \mathfrak{F}$ holds for every $n \in \mathbb{N}$ whenever $\mathbf{a} \in E_{\sqrt{8}}(1)$.

Hurwitz Continued Fractions share approximation properties with the regular continued fractions. In particular, they provide quadratic approximations in terms of the denominators $(q_n)_{n\geq 0}$.

Proposition 2.2. *There are absolute constants such that for any regular* $\mathbf{a} \in \Omega^{\mathrm{HCF}}$ *with* $\mathcal{Q}$*-pair* $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ *the following estimate holds*

$$\forall n \in \mathbb{N} \quad |\mathcal{C}_n(\mathbf{a})| \asymp \frac{1}{|q_n|^2}. \tag{2}$$

Proof. Take $\mathbf{a} \in \Omega^{\mathrm{HCF}}$ and $n \in \mathbb{N}$. The inequality $|\mathcal{C}_n(\mathbf{a})| \ll |q_n|^{-2}$ follows from Proposition 2.1, iv. For $|q_n|^{-2} \ll |\mathcal{C}_n(\mathbf{a})|$, take $z, w \in \mathcal{C}_n(\mathbf{a})$ and $z', w' \in \mathbb{C}$ such that

$$z = [0; a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots], \quad w = [0; a_1, a_2, \ldots, a_n, b_{n+1}, b_{n+2} \ldots],$$
$$z' = [a_{n+1}; a_{n+2}, a_{n+3} \ldots], \quad w' = [b_{n+1}; b_{n+2}, b_{n+3} \ldots].$$

By regularity, we can choose z', w' such that $z' = 2w'$ and $|w'| \leq 10$, say, then

$$|z - w| = \frac{1}{|q_n|^2} \frac{|z' - w'|}{\left|\left(z' + \frac{q_{n-1}}{q_n}\right)\left(w' + \frac{q_{n-1}}{q_n}\right)\right|} \geq \frac{1}{|q_n|^2} \frac{|w'|}{(2|w'|+1)^2} \gg \frac{1}{|q_n|^2}. \qquad \square$$

In what follows, we will only be concerned with regular sequences without further comment.

For any $\mathbf{a} = (a_n)_{n\geq 1} \in \Omega^{\mathrm{HCF}}$ define the family of Möbius transformations, $(t_{\mathbf{a},n})_{n\geq 1}$, by

$$\forall n \in \mathbb{N} \quad \forall z \in \mathbb{C} \quad t_{\mathbf{a},n}(z) = \frac{p_{n-1}z + p_n}{q_{n-1}z + q_n},$$

where $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$ is the $\mathcal{Q}$-pair of $\mathbf{a}$. The maps $t_{\mathbf{a},n}$ are local inverses of T^n, as $t_{\mathbf{a},n} \circ T^n$ is the identity on $\mathcal{C}_n(\mathbf{a})$ and $T^n \circ t_{\mathbf{a},n}$ is the identity on $T^n[\mathcal{C}_n(\mathbf{a})]$. From a symbolic point of view, $t_{\mathbf{a},n}$ maps a word $\mathbf{b}$ to $a_1 \ldots a_n \mathbf{b}$. Note that the restrictions $t_{\mathbf{a},n} : T^n[\mathcal{C}_n(\mathbf{a})] \to \mathcal{C}_n(\mathbf{a})$ are bi-Lipschitz, because they are bi-holomorphic. For reference, we state the following lemma.

Lemma 2.1. *For any* $\mathbf{a} \in \Omega^{\mathrm{HCF}}$ *(or any* $\mathbf{a} = (a_1, \ldots, a_n)$ *that can be extended to an element of* Ω^{HCF}*) and any* $n \in \mathbb{N}$ *the map* $t_{\mathbf{a},n} : \mathcal{C}_n(\mathbf{a}) \to \mathfrak{F}$*, which acts via*

$$[0; a_1, \ldots, a_n, x_1, x_2, \ldots] \mapsto [0; x_1, x_2, \ldots],$$

is bi-Lipschitz. Moreover, if $\mathbf{b} \in \Omega^{\mathrm{HCF}}$*,* $m \in \mathbb{N}$*, and* $T^n[\mathcal{C}_n(\mathbf{a})] = T^m[\mathcal{C}_m(\mathbf{b})]$*, then the map from* $\mathcal{C}_n(\mathbf{a})$ *to* $\mathcal{C}_m(\mathbf{b})$ *given by*

$$[0; a_1, \ldots, a_n, x_1, x_2, \ldots] \mapsto [0; b_1, \ldots, b_m, x_1, x_2, \ldots]$$

is bi-Lipschitz.

For any $L > 0$ and any $N \in \mathbb{N}$ define the sets

$$E_L(N) := \{z = [0; a_1, a_2, \dots] \in \mathfrak{F} : \ \forall n \in \mathbb{N}_0 \quad |a_{n+N}| \geq I\},$$
$$E'_L := \left\{z = [0; a_1, a_2, \dots] \in \mathfrak{F} : \ \liminf_{n\to\infty} |a_n| \geq I\right\}.$$

Lemma 2.2. *For any $L \in \mathbb{R}$, $L \geq 3$, we have* $\dim_H E_L(1) = \dim_H E'_L$.

Proof. Let $L \in \mathbb{R}$ be at least 3. Since $(E_L(N))_{N\geq 1}$ is increasing and $E'_L = \bigcup_{N\geq 1} E_L(N)$,

$$\dim_H E'_L = \lim_{N\to\infty} \dim_H E_L(N).$$

Thus, it suffices to show that for all N the relation $\dim_H E_L(N) \leq \dim_H E_L(1)$ holds. Let $N \in \mathbb{N}$ be at least 2. Define $G_L := \{a \in \mathbb{Z}[i] : |a| \geq L\}$. Then,

$$E_L(1) = \Phi\left[G_L^{\mathbb{N}}\right],$$

and we can write $E_L(N)$ as the countable union

$$E_L(N) = \bigcup_{\mathbf{a}\in I^{N-1}} \Phi\left[(\{\mathbf{a}\} \times G_L^{\mathbb{N}}) \cap \Omega^{HCF}\right]$$

(some terms are empty). As a consequence, we obtain

$$\dim_H E_L(N) = \sup_{\mathbf{a}\in I^{N-1}} \dim_H \Phi\left[(\{\mathbf{a}\} \times G_L^{\mathbb{N}}) \cap \Omega^{HCF}\right],$$

Take $\mathbf{a} \in I^{N-1}$ such that $\mathcal{C}_{N-1}(\mathbf{a}) \neq \varnothing$. In view of Lemma 2.1, the map

$$s : \Phi\left[(\{\mathbf{a}\} \times G_L^{\mathbb{N}}) \cap \Omega^{\mathrm{HCF}}\right] \to T^{N-1}[\mathcal{C}_{N-1}(\mathbf{a})] \cap \Phi[G_L^{\mathbb{N}}]$$

given by

$$s([0; a_1, \dots, a_n, x_1, x_2, \dots]) = [0; x_1, x_2, x_3, \dots].$$

is bi-Lipschitz. The Hausdorff dimension invariance under bi-Lipschitz maps (Proposition 3.3. of [Fa14]) gives

$$\dim_H \Phi\left[(\{\mathbf{a}\} \times G_L^{\mathbb{N}}) \cap \Omega^{\mathrm{HCF}}\right] \leq \dim_H \Phi[G_L^{\mathbb{N}}] = \dim E_L(1).$$

Hence, by taking the supremum over $\mathbf{a} \in I^{N-1}$, $\dim_H E_L(N) \leq \dim_H E_L(1)$. □

3 Generalized Jarník Lemmas

The main tools of the proof of Theorem 1.3 are two propositions, which we shall call Generalized Jarník Lemmas. They give bounds on the Hausdorff dimension of some Cantor sets. In this section, our framework is a more restrictive version of strongly tree-like sets as defined in [KlWe10] (though we keep the name).

Definition 3.1. *Let (X, d) be a complete metric space. A family of compact sets $\mathcal{A}$ is* ***strongly tree-like*** *if $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ where each $\mathcal{A}_k$ is finite, $\#\mathcal{A}_0 = 1$, and*

i. $\forall A \in \mathcal{A} \quad |A| > 0$,

ii. $\forall n \in \mathbb{N} \quad \forall A, B \in \mathcal{A}_n \quad (A = B) \vee (A \cap B = \varnothing)$,

iii. $\forall n \in \mathbb{N} \quad \forall B \in \mathcal{A}_n \quad \exists A \in \mathcal{A}_{n-1} \quad B \subseteq A$,

iv. $\forall n \in \mathbb{N} \quad \forall A \in \mathcal{A}_{n-1} \quad \exists B \in \mathcal{A}_n \quad B \subseteq A$,

v. $d_n(\mathcal{A}) := \max\{|A| : A \in \mathcal{A}_n\} \to 0$ *as* $n \to \infty$.

The quantity $d_n(\mathcal{A})$ is the ***n-th stage diameter****. The* ***limit set*** *of $\mathcal{A}$, $\mathbf{A}_\infty$, is*

$$\mathbf{A}_\infty := \bigcap_{n=0}^{\infty} \bigcup_{A \in \mathcal{A}_n} A.$$

As in [Ja28], if $\mathfrak{Y} = \{Y_j\}_{j \in J}$ is an at most countable family of subsets of X and $s \geq 0$, we write

$$\Lambda_s(\mathfrak{Y}) := \sum_{j \in J} |Y_j|^s.$$

For $X_1, X_2 \subseteq X$ we denote the distance between them by

$$d(X_1, X_2) := \inf\{d(y_1, y_2) : y_1 \in Y_1, y_2 \in Y_2\}.$$

Lemma 3.1 (First Generalized Jarník Lemma)**.** *Let $\mathcal{A}$ be a tree-like family of compact sets (as defined above). Suppose that for some $\kappa > 1$*

$$\forall n \in \mathbb{N}_0 \quad d_n(\mathcal{A}) \leq \frac{1}{\kappa^n}. \tag{3}$$

Assume the existence of a sequence $(B_n)_{n \geq 1}$ with $0 < B_n \leq 1$ for all n,

$$\limsup_{n \to \infty} \frac{\log \log (B_n^{-1})}{\log n} < 1, \tag{4}$$

and such that

$$\forall n \in \mathbb{N} \quad \forall A \in \mathcal{A}_n \quad \forall Y, Z \in \mathcal{A}_{n+1} \quad \big(Y \cup Z \subseteq A \quad \& \quad Y \neq Z \implies d(Y, Z) \geq B_n |A|\big). \tag{5}$$

If for $s > 0$ there exists some $c > 0$ such that

$$\forall \mathfrak{X} \in 2^{\mathcal{A}} \quad \left(\#\mathfrak{X} < +\infty \quad \& \quad \mathbf{A}_\infty \subseteq \bigcup_{A \in \mathfrak{X}} A \quad \implies \quad \Lambda_s \mathfrak{X} > c \right), \tag{6}$$

then $\dim_H A_\infty \geq s$.

Proof. Keep the statement's notation. Let $\mathcal{G}$ be a finite open cover of the compact set $\mathbf{A}_\infty$. First, we associate to each element of $\mathcal{G}$ a compact $A_G \in \mathcal{A}$. For any $G \in \mathcal{G}$ let $n \in \mathbb{N}$ be maximal with respect to the property

$$\exists A \in \mathcal{A}_m \quad G \subseteq A,$$

and let A_G be the corresponding $A \in \mathcal{A}_n$. By definition of A_G, there are $A', A'' \in \mathcal{A}_{n+1}$ such that $A' \cap G \neq \varnothing$ and $A'' \cap G \neq \varnothing$, so

$$|G| \geq d(A', A'') \geq B_n |A_G|.$$

Second, take $A \in \mathcal{A}_n$ such that $A = A_G$ for some $G \in \mathcal{G}$ and consider $\{G \in \mathcal{G} : A_G = A\} = \{G_1, \ldots, G_r\}$. Given $\varepsilon > 0$, we can take $\max\{|G| : G \in \mathcal{G}\}$ so small that forces n to be so large that $\kappa^{\varepsilon n} B_n < 1$ (by (4)), and then

$$|A|^s \leq \frac{1}{B_n^s} \sum_{j=1}^{r} |G_j|^s < \frac{1}{\kappa^{\varepsilon n} B_n^s} \sum_{j=1}^{r} |G_j|^{s-\varepsilon} < \sum_{j=1}^{r} |G_j|^{s-\varepsilon}.$$

In view of (5), we have

$$c < \Lambda_s \mathfrak{X} \leq \Lambda_{s-\varepsilon} \mathcal{G},$$

and taking infima, $\dim_H A_\infty > s - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\dim_H A_\infty \geq s$. □

Rather than a direct application of Lemma 3.1, we will change (6) for the stronger condition

$$\forall n \in \mathbb{N} \quad \forall A \in \mathcal{A}_n \qquad \sum_{\substack{A' \subseteq A \\ A' \in \mathcal{A}_{n+1}}} |A'|^s \geq |A|^s. \tag{7}$$

To see that (7) implies (6) take a finite cover $\mathfrak{X} \subseteq \mathcal{A}$ of $\mathbf{A}_\infty$. Without loss of generality, assume that the elements of $\mathfrak{X}$ are disjoint by pairs. Let n be the maximal integer satisfying $\mathcal{A}_n \cap \mathfrak{X} \neq \varnothing$ and $B \in \mathcal{A}_n \cap \mathfrak{X}$. By pairwise disjointness, if $A \in \mathcal{A}_{n-1}$ satisfies $B \subseteq A$, then all the sets $B' \in \mathcal{A}_n$ with $B' \subseteq A$ also belong to $\mathfrak{X}$. Then, after replacing these compact sets $B' \in \mathcal{A}_n$ with A, we obtain a new finite cover $\mathfrak{X}'$ such that $\Lambda_s \mathfrak{X} \geq \Lambda_s \mathfrak{X}'$. Repeating the argument we eventually arrive to $\mathfrak{X}' = \mathcal{A}_0$, so (6) holds.

Lemma 3.2 (Second Generalized Jarník Lemma)**.** *Let $\mathcal{A}$ be a family of compact sets satisfying conditions i., iii., iv., v. of the definition of strongly tree-like structure and such that the each $\mathcal{A}_k$, $k \geq 1$, is at most countable. Let $s > 0$.*

If for every $k \in \mathbb{N}$ and every $A \in \mathcal{A}_k$

$$\sum_B |B|^s \leq |A|^s,$$

where the sum runs along the sets B satisfying $B \in \mathcal{A}_{k+1}$ and $B \subseteq A$, then $\dim_H A_\infty \leq s$.

Proof. For any $k \geq 1$ we have $\Lambda_s(\mathcal{A}_k) \leq \Lambda_s(\mathcal{A}_0) < +\infty$, because

$$\Lambda_s(\mathcal{A}_k) = \sum_{A \in \mathcal{A}_k} |A|^s = \sum_{B \in \mathcal{A}_{k-1}} \sum_{\substack{A \in \mathcal{A}_k \\ A \subseteq B}} |A|^s \leq \sum_{B \in \mathcal{A}_{k-1}} |B|^s \leq \Lambda_s(\mathcal{A}_{k-1}).$$

Therefore, since every $\mathcal{A}_k$ covers $\mathbf{A}_\infty$ and $d_k(\mathcal{A}) \to 0$ as $k \to \infty$, $\dim_H A_\infty \leq s$. □

4 Preliminary Lemmas

The Generalized Jarník Lemmas allow us to bound some sets built through HCF restrictions. With these sets, we will be able to approximate

$$E := \{z = [0; a_1, a_2, a_3, \dots] \in \mathfrak{F} : \lim_{n\to\infty} |a_n| = \infty\}$$

from the inside and from the outside. The outer approximation is obtained taking the numbers in $\mathfrak{F}$ whose elements have absolute value satisfying a uniform lower bound. The inner approximation is done by considering complex numbers whose elements tend to infinity at a certain rate.

For any $L, M \in \mathbb{R}_{>0}$ define

$$E_L^M := \{z = [0; a_1, a_2, a_3, \dots] \in \mathfrak{F} : \forall n \in \mathbb{N} \quad L \le |a_n| \le M\}.$$

Lemma 4.1. *1. For every $L \ge \sqrt{8}$*

$$1 \le \lim_{M\to\infty} \dim_H E_L^M.$$

2. For any positive number I let E_L be defined as in Corollary 1.4, then

$$\lim_{I\to\infty} \dim_H E_L \le 1.$$

For any $z \in \mathbb{C}$ write $\|z\| = \max\{|\Re z|, |\Im z|\}$, so $|z| \asymp \|z\|$.

Lemma 4.2. *Let $f, g : \mathbb{N} \to \mathbb{R}_{>0}$ such that $3 \le f(n) \le g(n)$ for every positive integer n. For any $n \in \mathbb{N}$, any $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}[i]^n$ such that*

$$\forall j \in [1..n] \quad f(j) \le \|a_j\| \le g(j),$$

and any $b \in \mathbb{Z}[i]$ with $f(n+1) \le |b| \le g(n+1)$ we have that

$$\dim_H \left(\mathcal{C}_n(\mathbf{a}) \cap E_{f,g}\right) = \dim_H \left(\mathcal{C}_{n+1}(\mathbf{a}b) \cap E_{f,g}\right).$$

Proof. Keep the theorem's notation and let $c \in \mathbb{Z}[i]$ satisfy $f(n+1) \le \|c\| \le g(n+1)$. Using Lemma 2.1 can see that $\left(\mathcal{C}_{n+1}(\mathbf{a}b) \cap E_{f,g}\right)$ and $\left(\mathcal{C}_{n+1}(\mathbf{a}c) \cap E_{f,g}\right)$ have the same Hausdorff dimension. We arrive at the desired conclusion by writing $\mathcal{C}_n(\mathbf{a}) \cap E_{f,g}$ as a countable union, like we did in the proof of Lemma 2.1. The details are left to the reader. □

Lemma 4.3. *Let $f, g : \mathbb{N} \to \mathbb{R}_{>0}$ be functions such that for some fixed c with $0 < c < 1$ we have $\sqrt{8} \le f(n) < (1-c)g(n)$ and $g(n)^2 \le e^n$ for all $n \in \mathbb{N}$. Define the sets*

$$E_{f,g} := \{[0; a_1, a_2, a_3, \dots] \in \mathfrak{F} : \forall n \in \mathbb{N} \quad f(n) \le \|a_n\| \le g(n)\}.$$

Then, $\dim_H E_{f,g} = 1$.

5 Proofs of Main Results

Proof of Theorem 1.3. From $E \subseteq E'_L$ for $L \in \mathbb{N}_{\geq 3}$ and the first part of Lemma 4.1, we obtain

$$\dim_H E \leq 1.$$

Let f and g be as in the second part of Lemma 4.3, then $E_{f,g} \subseteq E$ and the aforementioned lemma implies

$$1 \leq \dim_H E_{f,g} \leq \dim_H E. \qquad \square$$

Remark.. With the notation of Theorem 1.3, we also get that the set

$$E_f = \{z = [0; a_1, a_2, \dots] : \forall n \in \mathbb{N} \quad f(n) \leq |a_n|\}$$

has Hausdorff dimension 1, for $E_{f,g} \subseteq E_f \subseteq E$.

Proof of Corollary 1.4. The monotonicity of the limit is clear, because $L \mapsto \dim_H E_L$ is non-increasing. Let $\varepsilon > 0$. By Lemma 4.3, we can take $L = L(\varepsilon) \in \mathbb{N}$ and $M = M(L, \varepsilon) \in \mathbb{N}$ be such that

$$1 - \varepsilon \leq \dim_H E_L^M \leq \dim_H E_L \leq 1 + \varepsilon.$$

Since an irrational complex number belongs to $\mathbf{Bad}_{\mathbb{C}}$ if and only if its HCF is bounded, $E_L^M \subseteq E_L \cap \mathbf{Bad}_{\mathbb{C}} \subseteq E_L$ and the result follows. $\square$

6 Proofs of Preliminary Lemmas

6.1 Proof of Lemma 4.1

Proof of Lemma 4.1. i. Take $L \geq \sqrt{8}$. Note that for sufficiently large $M > L$ we have

$$\#\{b \in \mathbb{Z}[i] : L \leq |b| \leq M\} \geq M^2.$$

Let n be a positive integer and $\mathbf{a} \in \mathbb{Z}[i]^n$ be such that

$$\forall j \in \{1, \dots, n\} \quad L \leq |a_j| \leq M.$$

Set $\gamma_3 = \gamma_1/\gamma_2$ with $0 < \gamma_1 < \gamma_2$ the constants implied in (2). For any $\beta > 0$ and any large M we have

$$\begin{aligned}\sum_{L \leq |b| \leq M} |\mathcal{C}_{n+1}(\mathbf{a}b)|^\beta &\geq \frac{\gamma_1^\beta}{(M+2)^{2\beta}} \sum_{L \leq |b| \leq M} \frac{1}{|q_n|^{2\beta}} \geq \frac{\gamma_1^\beta M^2}{\gamma_2^\beta (M+2)^{2\beta}} \frac{\gamma_2^\beta}{|q_n|^{2\beta}} \\ &\geq \frac{M^{2-2\beta}\gamma_3^\beta}{\left(1 + \frac{2}{M}\right)^{2\beta}} |\mathcal{C}_n(\mathbf{a})|^{2\beta}.\end{aligned}$$

The coefficient of $|\mathcal{C}_n(\mathbf{a})|^{2\beta}$ is at least 1 if and only if

$$(2 - 2\beta)\log M + \beta \log \gamma_3 - 2\beta \log\left(1 + \frac{2}{M}\right) \geq 0,$$

which, after rearranging the terms, is seen to be equivalent to

$$\beta \leq \frac{2\log M}{2\log M + 2\log\left(1+\frac{2}{M}\right) - \log\gamma_3} < 1.$$

Since the quotient tends to 1 as $M \to \infty$, we can take $\beta = \beta(M) < 1$ as close to 1 as we please by considering a large M. Hence, in view of the First Generalized Jarník Lemma, we conclude that $\dim_H E_L^M \geq 1 + \mathcal{O}((\log M)^{-1})$. The implied constant depends on L.

ii. For any $L \geq \sqrt{8}$ and any $\varepsilon > 0$

$$\sum_{|b|\geq L} |\mathcal{C}_{n+1}(\mathbf{a}b)|^{1+\varepsilon} \leq \frac{\gamma_2^{1+\varepsilon}}{|q_n|^{2+2\varepsilon}} \sum_{|b|\geq L} \frac{1}{\left|b+\frac{q_{n-1}}{q_n}\right|^{2+2\varepsilon}} \ll \frac{\gamma_2^{1+\varepsilon}}{|q_n|^{2+2\varepsilon}} \sum_{L\leq|b|} \frac{1}{|b|^{2+2\varepsilon}},.$$

with γ_1, γ_2 as before; hence,

$$\sum_{|b|\geq L} |\mathcal{C}_{n+1}(\mathbf{a}b)|^{1+\varepsilon} \ll \frac{1}{|q_n|^{2+2\varepsilon}} \int_L^\infty \frac{\mathrm{d}r}{r^{1+2\varepsilon}} = \frac{1}{|q_n|^{2+2\varepsilon}} \frac{1}{2\varepsilon L^{2\varepsilon}} \ll_\varepsilon \frac{1}{L^{2\varepsilon}} |\mathcal{C}_n(\mathbf{a})|^{1+\varepsilon}.$$

Since $L^{-2\varepsilon} \to 0$ as $L \to \infty$, the Second Jarník Generalized Lemma yields that for any large integer L

$$\dim_H E_L \leq 1 + \varepsilon. \qquad \square$$

6.2 Proof of Lemma 4.3

We start with an observation which is stems from the triangle inequality.

Proposition 6.1. *Let $\alpha, \beta \in \mathfrak{F}$ such that $|a_1(\alpha)| \geq \sqrt{8}$ and $|a_1(\beta)| \geq \sqrt{8}$. Then, there exists an absolute constant $c_1 > 0$ such that for any $a, b \in \mathbb{Z}[i]$, $a \neq b$,*

$$|(a+\alpha) - (b+\beta)| > c_1.$$

Proof of Lemma 4.3. Let $\mathbf{a} \in \mathbb{Z}[i]^n$ be such that

$$\forall j \in \{1, \ldots, n\} \quad f(j) \leq \|a_j\| \leq g(j).$$

Let $b, c \in \mathbb{Z}[i]$ satisfy $f(n+1) \leq \|b\|, \|c\| \leq g(n+1)$. Let $((p_n)_{n\geq 0}, (q_n)_{n\geq 0})$ be the $\mathcal{Q}$-pair of $\mathbf{a}b$, and p'_{n+1}, q'_{n+1} the last terms of the $\mathcal{Q}$-pair of $\mathbf{a}c$. Any $\beta, \gamma \in \mathfrak{F}$ with $f(n+2) \leq \|a_1(\beta)\|, \|a_1(\gamma)\| \leq g(n+2)$ satisfy

$$\begin{aligned}
d\left(\mathcal{C}_{n+1}(\mathbf{a}r) \cap E_{\sqrt{8}}, \mathcal{C}_{n+1}(\mathbf{a}s) \cap E_{\sqrt{8}}\right) &\geq \left|\frac{\beta p_n + p_{n+1}}{\beta q_n + q_{n+1}} - \frac{\gamma p_n + p'_{n+1}}{\gamma q_n + q'_{n+1}}\right| \\
&= \frac{1}{|q_n|^2} \frac{|c+\gamma-(b+\beta)|}{\left|\left(b+z+\frac{q_{n-1}}{q_n}\right)\left(c+w+\frac{q_{n-1}}{q_n}\right)\right|} \\
&\gg \frac{1}{|q_n|^2} \frac{1}{(g(n)-2)^2} \gg \frac{1}{g(n)^2} \left|\mathcal{C}_{n+1}(\mathbf{a}) \cap E_{\sqrt{8}}\right|.
\end{aligned}$$

As in the proof of Lemma 4.1, we have

$$\sum_{f(n+1)\le\|b\|\le g(n+1)} |\mathcal{C}_{n+1}(\mathbf{a}b)\cap E_{\sqrt{8}}|^{\beta} \ge g(n)^{2-2\beta}\frac{\gamma_3^{\beta}\left(1-\frac{f(n)}{g(n)}\right)}{\left(1+\frac{2}{g(n)}\right)^{2\beta}}|\mathcal{C}_n(\mathbf{a})\cap E_{\sqrt{8}}|^{\beta}$$

for any $\mathbf{a}\in\mathbb{Z}[i]^n$ with $f(j)\le\|a_j\|\le g(j)$, $1\le j\le n$. The coefficient of $|\mathcal{C}_n(\mathbf{a})|^{\beta}$ is at least 1 if and only if

$$\frac{2\log g(n)+\log\left(1-\frac{f(n)}{g(n)}\right)}{2\log g(n)-\log\gamma_3+\log\left(1+\frac{2}{g(n)}\right)}\ge\beta.$$

Because of $f\le(1-c)g$, the left-hand side tends to 1 as $n\to\infty$ and, by the First Generalized Jarník Lemma, $\dim_H E_{f,g}\ge 1$. The inequality $\dim_H E_{f,g}\le 1$ follows from the second part of Lemma 4.1 and Lemma 4.2. □

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On a theorem of Bengoechea on equivalent numbers

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Abstract

Let x, y be real irrational numbers and suppose their regular continued fraction expansion is $x = [a_0; a_1, a_2, \ldots]$, $y = [b_0; b_1, b_2, \ldots]$. A classical result of Serret tells us that there exists $\gamma \in \mathrm{GL}(2, \mathbb{Z})$ such that $\gamma x = y$ (acting through Möbius transformations) if and only if for some $s, t \in \mathbb{N}$ we have $a_{n+t} = b_{n+s}$ for all natural numbers n. Recently, P. Bengoechea gave an estimate for t in terms of γ. In this paper, we show a theorem equivalent to Bencoechea's result using a geometric continued fraction perspective.

1 Introduction

For any sequence of integers $(a_n)_{n\geq 0}$ with $a_n \geq 1$ for $n \geq 1$ we adopt the usual notation

$$[a_0; a_1, a_2, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

and similarly when the sequence is finite. We refer to the previous expression as a regular continued fraction. Continued fractions are a classical topic in elementary number theory, for they possess important properties which allow to obtain non-trivial conclusions from the number they represent. For example, a real number is irrational if and only if its continued fraction is infinite. While infinite continued fractions expansions are unique, every non-zero rational number has exactly two expansions one ending in 1 and the other one ending in an integer at least 2.

Consider the group $\mathrm{GL}(2, \mathbb{Z})$ acting on $\mathbb{R} \cup \{\infty\}$ via Möbius transformations; that is

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \quad \forall x \in \mathbb{R} \quad \gamma x := \frac{ax + b}{cx + d}.$$

A famous theorem of J.A. Serret characterizes the orbits of $\mathrm{GL}(2, \mathbb{Z})/\mathbb{R}$ in terms of their continued fraction expansion.

Theorem 1.1 (J.A. Serret, 1866)**.** *V Let x, y be two real numbers. Then, x, y belong to the same orbit of* $\mathrm{GL}(2,\mathbb{Z})/\mathbb{R}$ *if and only if $x, y \in \mathbb{Q}$ or if their continued fraction expansion satisfy*

$$x = [a_0; a_1, a_2, \ldots, a_{s-1}, a_s, a_{s+1}, \ldots], \quad y = [b_0; b_1, b_2, \ldots, b_{t-1}, a_s, a_{s+1}, \ldots].$$

In a recent paper, P. Bengoechea gave an estimate for s in terms of γ. Namely, she showed the following theorem.

Theorem 1.2 (P. Bengoechea, 2015)**.** *Let $\gamma \in \mathrm{GL}(2,\mathbb{Z})$ and let r be the length of the continued fraction of $\gamma^{-1}[\infty]$. Then, for each $x \in \mathbb{R}$ there is a natural number $s \leq r+3$ such that*

$$x = [a_0; a_1, a_2, \ldots], \quad \gamma x = [b_0; b_1, \ldots, b_{t-1}, a_s, a_{s+1}, a_{s+2}, \ldots].$$

In her proof, P. Bengoechea's studied carefully a sequence of matrices in $\mathrm{GL}(2,\mathbb{Z})$ that naturally arises from the continued fraction expansion of x. Relying on different tools, namely the geometry of continued fractions, we show a result equivalent to Theorem 1.2.

Theorem 1.3. *Let $\mathrm{GL}(2,\mathbb{Z})$ act on $\mathbb{R}$ through Möbius transformations and let L be the length of the continued fraction expansion of $\gamma(0) \in \mathbb{Q}$. Then, for every $x \in \mathbb{R} \setminus \mathbb{Q}$ there exists $s \in \mathbb{N}$ such that $s \leq L+3$ and*

$$x = [a_0; a_1, a_2, a_3, \ldots], \quad \gamma x = [b_0, b_1, \ldots, b_n, a_s, a_{s+1}, a_{s+2}, \ldots].$$

2 Geometry of Continued Fractions

In 1895, Felix Klein suggested a geometric approach to the study of continued fractions. For a detailed account of the geometry of continued fractions, we refer the reader to citekarpenkov.

For any pair of integers m, n denote its greatest common divisor by $\gcd(m,n) \geq 1$.

Definition 2.1. *The **integer length** of two points $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{Z}^2$, $1d(\mathbf{x},\mathbf{y})$, is*

$$1\ell(\mathbf{x},\mathbf{y}) \coloneqq \gcd(x_1 - y_1, x_2 - y_2).$$

*Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^2$ be three different points. The **angle** $\theta = \angle\mathbf{xzy}$ is the ordered triple $(\mathbf{x}, \mathbf{z}, \mathbf{y})$ and its **integer sine** is*

$$1\sin(\angle\mathbf{xzy}) = \frac{|\det(\mathbf{x}-\mathbf{z}, \mathbf{y}-\mathbf{z})|}{1\ell(\mathbf{x},\mathbf{z})1\ell(\mathbf{z},\mathbf{y})}.$$

For any linearly independent $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ with $\mathbf{x} \in \mathbb{Q}^2$ write $L_x^+ := \{tx : t \geq 0\}$, $L_y^+ := \{tx : t \geq 0\}$. The **sail** of L_x^+ and L_y^+, $S = S(\mathbf{x}, \mathbf{y})$, is the boundary of the set

$$\overline{\mathrm{co}}\{(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\} : \exists r_x, r_y \geq 0 \; \exists t \in [0,1] \; (a,b) = r_x x + t(r_y y - r_x x)\}. \tag{1}$$

In general, S is an infinite broken line and hence is homeomorphic to $\mathbb{R}$. If we assume that $\mathbf{x} \in \mathbb{Q}^2$, then S and L_x^+ eventually coincide. Moreover, since S and $\mathbb{R}$ are homeomorphic, we can call V_0 the unique extreme point of the set in (1) lying on L_x^+, and successively denote by V_n the vertices of S.

Definition 2.2. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ be linearly independent with $\mathbf{x} \in \mathbb{Q}$. The **LLS' sequence** of $\mathbf{x}, \mathbf{y}$ is the sequence of integers $(a_n)_{n \geq 0}$ given by*

$$\forall n \in \mathbb{N} \quad a_{2n} = 1\ell(V_n, V_{n+1}), \quad a_{2n+1} = 1\sin(\angle V_n V_{n+1} V_{n+2}).$$

Remark. The LLS and LLS' sequences coincide up to a shift and multiplication by -1 of the indices.

The following result is a restatement of Theorems 3.1. and 3.4. in [Ka13].

Theorem 2.1 (F. Klein, 1895)**.** *Let $\alpha > 1$ be an irrational number. The sequence of vertices of the angle $\theta = \angle((1,0),(0,0),(1,\alpha))$, $(V_n)_{n \geq 0}$ satisfies*

$$\forall n \in \mathbb{N}_0 \quad V_n = (q_{2n-2}, p_{2n-2}).$$

If $\alpha = [a_0; a_1, a_2, \ldots]$, the LLS'-sequence of θ is $(a_n)_{n \geq 0}$. The sequence of vertices $\theta' = \angle((1,0),(0,0),(1,\alpha))$, $(W_n)_{n \geq 0}$ satisfies

$$\forall n \in \mathbb{N}_0 \quad W_n = (q_{2n-1}, p_{2n-1}).$$

If $\alpha = [a_0; a_1, a_2, \ldots]$, the LLS'-sequence of θ is $(a_n)_{n \geq 0}$. A similar result holds for $0 < \alpha < 1$, in this case $V_n = (q_{2n}, p_{2n})$, $W_n(q_{2n+1}, p_{2n+1})$. If $\alpha = [0; a_1, a_2, \ldots]$, then the LLS' sequence of θ is $(a_n)_{n \geq 1}$.

3 Two Elementary Lemmas

For the terminology regarding Farey sequences, see [NMZ91].

Lemma 3.1. *Let r, s, u, v be integers such that $rv - su = 1$ and $v, s \geq 1$. Then, $\frac{r}{s}$ and $\frac{u}{v}$ admit a regular continued fraction expansion of length n and m, respectively, such that $|n - m| \leq 1$.*

Proof. Recall that that for any $A, B, C, D \in \mathbb{Z}$ satisfying $B, D \geq 1$ and $AD - BC = 1$ the numbers $\frac{A}{B}$ and $\frac{C}{D}$ are adjacent in the the Farey sequence of order $\max\{B, D\}$ (citenimozu, p.301). Thus, the numbers adjacent to the regular continued fraction $\frac{p_k}{q_k} = [a_0; a_1, \ldots, a_k]$, $a_k \geq 2$, in the Farey sequence of order q_k are

$$\frac{p_{k-1}}{q_{k-1}} = [a_0; a_1, \ldots, a_{k-1}], \quad \frac{(a_k - 1)p_{k-1} + p_{k-2}}{(a_k - 1)q_{k-1} + q_{k-2}} = [a_0; a_1, \ldots, a_{k-1}, a_k - 1, 2].$$

The lemma follows after applying the previous observation to the the $\frac{r}{s}$ or $\frac{u}{v}$ depending on which of the inequalities $s > v$ or $v > s$ hold. □

Now we recall a well known continued fraction identity.

Lemma 3.2. *Let α be a positive real number with continued fraction $\alpha = [a_0; a_1, a_2, \ldots] > 0$, then*

$$-\alpha = \begin{cases} [-a_0 - 1; 1, a_1 - 1, a_2, a_3, \ldots] & \text{if } a_1 \neq 1, \\ [-a_0 - 1; 1, a_2 + 1, a_3, \ldots] & \text{if } a_1 = 1. \end{cases}$$

4 Proof of the Main Theorem

Let us first note that for any $n \in \mathbb{Z}$ we have that

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & c + nd \\ c & d \end{pmatrix}.$$

Hence, since translations by integers affect only the first term in the continued fraction expansion, we will not lose any generality by assuming that $b < 0 < d$ and $ac \leq 0$. Moreover, if the determinant is 1, then we must have $ac = 0$ and $\max\{|a|, |c|\} = 1$. In any case, direct verification with Lemma 3.2 give the result. Moreover, using the transformation $x \mapsto -x$ and Lemma 2, the result for matrices with determinant -1 follows from the one for matrices with positive determinant.

Proof. The preceding discussion tells us that we do not lose generality by assuming that $\det \gamma = 1$ and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad b < 0 < d, \quad c < 0 < a. \tag{2}$$

Note that by the invariance of the LLS sequence under the action of $\mathrm{GL}(2, \mathbb{Z})$, we show that the tails of the sails $\tilde{\gamma}[L_x]$ and S_y eventually coincide.

Write $L_\alpha = \{t(1, \alpha) : t \in \mathbb{R}\}$ for any $\alpha \in \mathbb{R}$. Let x be an irrational number and $y = \gamma x$, so $\overline{\gamma}[L_x] = L_y$ where

$$\overline{\gamma} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Consider $L_1 = L_{\frac{b}{d}}$ and $L_2 := L_{\frac{a}{c}}$. We will also call S_x the sail of the angle $(0,1)$, $(0,0)$, $(1, x)$.

Case I: $0 < x, y$. Our assumptions imply that the angle formed by (d,b), $(0,0)$ and $(1,y)$ lies entirely in the semi-plane $\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 1\}$. As a consequence, we must have that $(1,0) \in \tilde{\gamma}[S_x]$ for otherwise there should be $\mathbf{a} = (a_1,a_2), \mathbf{b} = (b_1,b_2) \in \mathbb{N} \times \mathbb{Z}$ and $0 < t < 1$ such that

$$\|t\mathbf{a} + (1-t)\mathbf{b}\| < \|(1,0)\| = 1.$$

But $a_1, b_1 \geq 1$ makes the previous inequality impossible. As a consequence, the tail of the broken line $\gamma[S_x]$ starting from $(1,0)$ is S_y. The broken starting segment of $\tilde{\gamma}[S_x]$ from (d,b) to $(1,0)$ is the reflection on the x axis of $S_{\frac{b}{d}}$. Therefore, since its LLS sequence gives the continued fraction expansion of $\frac{b}{d}$, we conclude that $[a_{L+1}; a_{L+2}, a_{L+3}, \ldots]$ is a tail of the expansion of y.

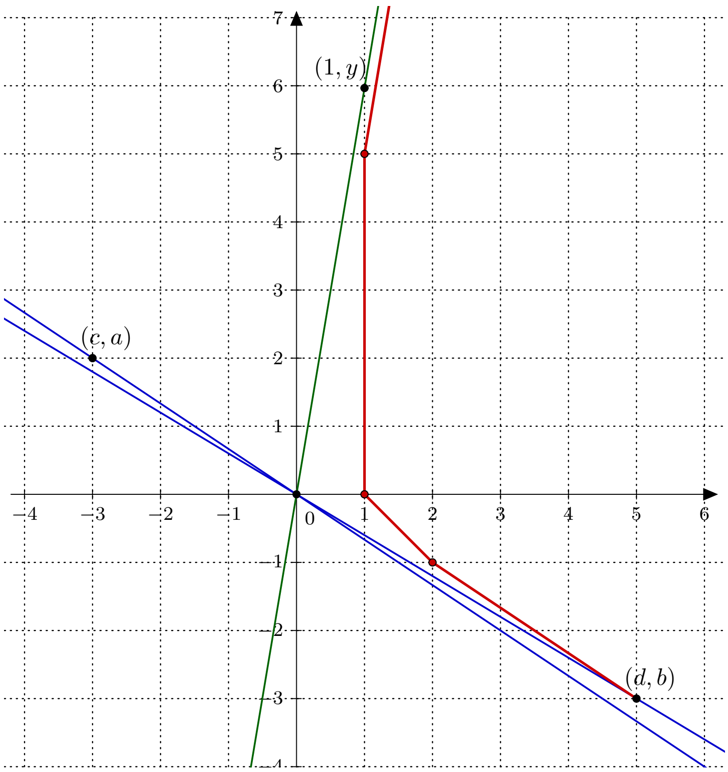

Figure 1: Case I. $0 < x, y$.

Case II: $y < 0 < x$. Consider the integer points (Q,P) and (Q',P') with

$$Q = \max\{q \in \mathbb{Z}_{\leq 0} : \exists p \in \mathbb{N} \quad (q,p) \in \tilde{\gamma}[S_x]\},$$
$$Q' = \min\{q \in \mathbb{N} : \exists p \in \mathbb{Z} \quad (q,p) \in \tilde{\gamma}[S_x]\},$$

and let P, P' be the corresponding integers. If $Q = 0$, then $P = 1$ (i.e. $(0,1) \in \tilde{\gamma}[S_x]$) and the tail of S_x starting from $(0,1)$ coincides with the S'_{-y}. Assume that $Q > 0$ and let $0 < \alpha^* < 1$ be such that $(0, \alpha^*)$ is the intersection of $\tilde{\gamma}[S_x]$ with the vertical axis. Let L' be the infinite broken line obtained from the intersection of $\tilde{\gamma}[S_x]$ and the second quadrant and replacing the segment (P,Q), $(0,\alpha^*)$ with (P,Q), $(0,1)$. We obtain the sail of the angle $(-1,y),(0,0),(0,1)$, which gives the continued fraction of $-y$ and, hence, of y. We do the same for (P',Q'), and we conclude that $[x_{L+2}; x_{L+3}, x_{L+4}, \dots]$ is a tail of y.

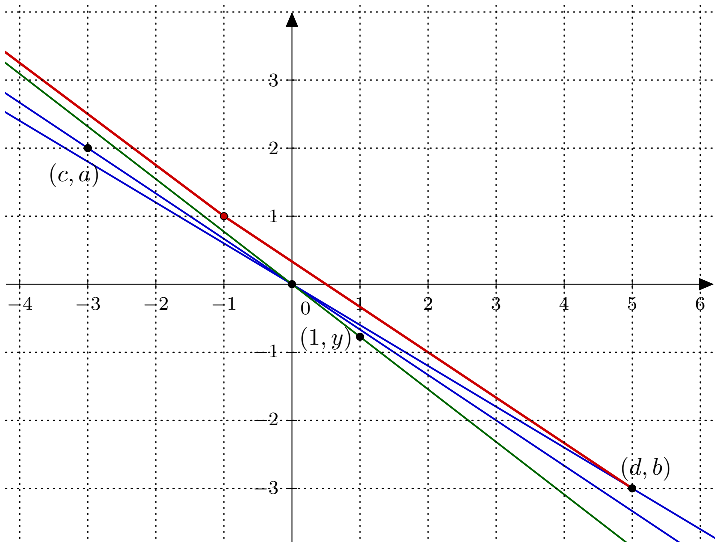

Figure 2: Case II. $y < 0 < x$.

Case III: $x, y < 0$. We start by noting that for any $m \in \mathbb{Z}$ we have that

$$\gamma\tau_m = \begin{pmatrix} a & am+b \\ c & cm+d \end{pmatrix} = \begin{pmatrix} a & b_m \\ c & d_m \end{pmatrix} \quad y = \gamma\tau_m(x-m).$$

Hence, take $m \in \mathbb{N}$ large such that $c, d' := d_m < 0 < a, b := b_m$ and note that $x - m < x < 0$. An argument similar to the one in the previous case gives the result.

In general, let $\gamma \in \mathrm{GL}(2,\mathbb{Z})$ have $\det\gamma = 1$. By considering compositions $\tau_{-n}\gamma$ for $n \in \mathbb{N}$ we can take γ into the previous treated cases. Moreover, since translations only affect the first term of the continued fraction, the conclusions on the tails hold.

To conclude the result for this case, we must estimate the length of the continued fraction of b'/d'. This follows, however, directly from Lemma 3.1 and $ad - bc = a(d+mc) - (b+ma)c = 1$. $\square$

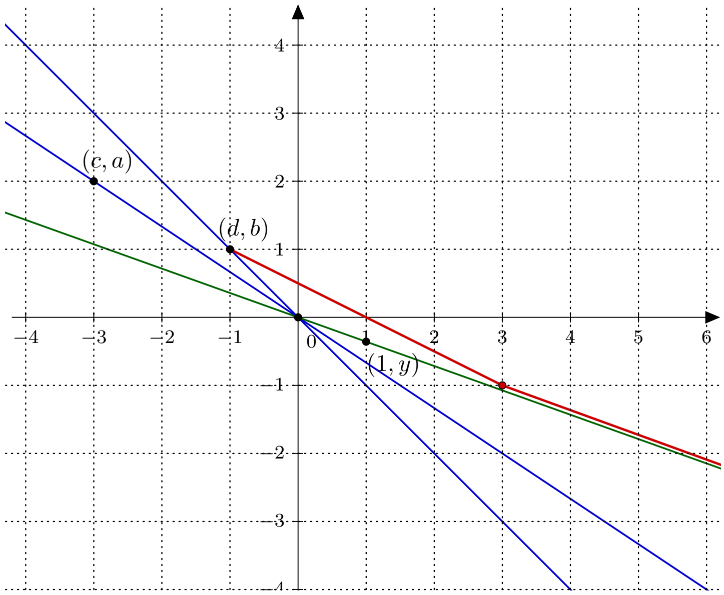

Figure 3: Case III. $y, x < 0$.

5 Further Comments

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