Semiclassical Analysis

Optimal Weyl law with an application

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PhD dissertation

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December 2019

Abstract

This thesis concerns optimal semiclassical analysis in two different settings. The first setting concerns an optimal semiclassical bound for the trace norm of certain commutators. The second setting is a more classic semiclassical question and it concerns an optimal Weyl law. Both settings concerns optimal semiclassical analysis but different methods are used in each setting.

The commutators considered is a non-magnetic Schrödinger operator commuted with either a positions operator or a momentum operator. For these commutators an optimal bound on the trace norm in terms of a semiclassical parameter is proven. Commutators of this type are not usual objects to consider in semiclassical analysis. But the bounds on the trace norm of the commutators correspond to a mean-field version of bounds introduced as an assumption by N. Benedikter, M. Porta and B. Schlein in a study of the evolution of a fermionic system.

What is presented here in this thesis on the Weyl law is work in progress. It is well established that a Weyl law is valid for self-adjoint differential operators with smooth coefficients under certain assumptions. The question considered in this thesis is what happens if the coefficients are not smooth. Can an optimal Weyl law still be proven?

In the thesis an optimal Weyl law is proven/reproven where the coefficients are once differentiable and the first derivative is Hölder continuous. In order to prove this Weyl law a class of rough symbols is defined. For this class a full symbolic and functional calculus is proven. Moreover a microlocal approximation of the propagator is constructed, which is not a Fourier integral operator!

The thesis also has a section on the possible future directions to go with the work on the Weyl law. There are a substantial number of interesting questions that could be pursued.

Resumé

Denne afhandling omhandler optimal semiklassisk analyse i to forskellige situationer. Den første situation omhandler en optimal semiklassisk begrænsning på spornormen af bestemte kommutatorer. Den anden situation er mere klassisk semiklassisk analyse, da det omhandler en optimal Weyl lov. Begge situation omhandler optimale semiklassiske resultater, men der bruges forskellige metoder i de to situationer.

Kommutatorerne, der betragtes i afhandlingen, er en ikke magnetisk Schrödinger operator kommuteret med enten en positions operator eller en impuls operator. For disse kommutatorer bevises der en optimal semiklassisk begrænsning på spornormen. Kommutatorer af denne type bliver normalt ikke betragtet i semiklassisk analyse, men begrænsningen på spornormen af disse kommutatorer svarer til en middel-felts approksimation af nogle begrænsninger introduceret som en antagelse af N. Benedikter, M. Porta and B. Schlein i et studie af evolutionen af et fermionisk system.

Det der præsenteres i afhandlingen om Weyl loven er i gangværende arbejde. Det er velkendt at der gælder en optimal Weyl lov for selvadjungerede differential operatorer med glatte koefficienter under visse antagelser. Det spørgsmål, der undersøges i afhandlingen er: Hvad sker der hvis koefficienterne ikke er glatte? Er det så stadig muligt at bevise en optimal Weyl lov?

I afhandlingen bevises/genbevises en optimal Weyl lov, hvor koefficienterne er en gang differentiable med Hölder kontinuerte første afledte. For at bevise denne Weyl lov introduceres en klasse af grove symboler. For denne klasse af symboler bevises der fuld symbol og funktional kalkyle. Yderligere konstrueres der også en mikrolokal approksimation til tidsudviklingen, der ikke er en Fourier integral operator!

Afhandlingen indeholder også en sektion om mulige retninger man kunne arbejde videre med i forbindelse med Weyl loven. Der er et betydeligt antal interessante spørgsmål man kunne forfølge.

Contents

Preface

"Mathematics is not a deductive science – that's a cliché. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork."
—Paul R. Halmos *I want to be a Mathematician*

This thesis marks the end of my studies as a PhD student at the Department of Mathematics, Aarhus University. The studies were supervised by Professor Søren Fournais and funded by the Sapere Aude project: Semiclassical Quantum Mechanics (DFF–4181-00221) from the Danish Council for Independent Research held by Søren Fournais.

The thesis consists of some introductory chapters and two papers. The papers are at first glance very different but they are both concerned with optimal semiclassical analysis. The main difference is the methods used in each paper. The two papers are:

- **Paper I:** An optimal semiclassical bound on certain commutators.
- **Paper II:** Optimal Weyl asymptotics for operators with irregular coefficients.

Paper I has been uploaded to arXiv with identification arXiv:1912.08467 and is presented in the same form in the thesis as the one on arXiv. This paper is co-authored with my supervisor. While we discussed all results and cooperated on the calculations I have done most of the typesetting for the paper with useful comments from my supervisor. Parts of paper I is advances on the result contained in my progress rapport for my qualifying exam.

Paper II is work in progress and is a self-contained review of what we so far have been able to prove/understand concerning optimal Weyl laws without full regularity. The paper is slightly rough around the edges since it really is work in progress. Most results have been discussed with my supervisor and some details have been work out in cooperation.

The structure of the thesis is such that the first chapter is an introduction to semiclassical analysis and the Weyl law. The second chapter is an introduction to Paper I followed by the paper itself. The third chapter is an introduction to Paper II and a discussion on in which directions we hope to be able to continue this research, after this follows Paper II. Each chapter has its own bibliography which is the last section of the chapter.

Acknowledgements

I would like to first and foremost thank my supervisor, Søren Fournais, for always having his door open and answering my sometimes not so intelligent questions. The discussions and answers have been very illuminating for me. Especial your intuition and eye for technicals details have been an indispensable help.

During the spring semester of 2019, I was so fortunate to part of the program "Spectral Methods in Mathematical Physics" at the Institute Mittag-Leffler in Stockholm and I thank them for hosting me. It was a truly inspiring semester program filled with great seminars and discussions. The institute itself and the surroundings was an incredible place to be allowed to work in and I am grateful for the experience.

Throughout my studies at Aarhus University I have shared an office with a lot of nice people and a special thanks goes to them for making my time in Aarhus so enjoyable. Especially Helene and Peter whom I spend the last couple of years sharing an office and both successes and frustrations with. Besides my office mates I would also like to thank my "Stamhold: Hold 2" including Jacob for making the start and the first five years of my University studies so enjoyable.

I would also like to thank Lars "daleif" Madsen for helping me with setting up the thesis and other LaTeX related issues throughout the years.

Lastly but not least I would like to thank my family and friends for their support thoughtout the time i have been studying at the university. You have all be invaluable to me.

Søren Mikkelsen
Aarhus, December 2019.

Chapter 1

Introduction to semiclassical analysis and Weyl's law

Semiclassical analysis can be viewed as the study of a system as a parameter tends to zero. In the context of quantum mechanics and classical mechanics, semiclassical analysis is the mathematically rigorous investigation of the Bohr correspondence principle i.e. that classical mechanics is the limit as $\hbar$ tends to zero of quantum mechanics. From a mathematical point of view letting a parameter tend to zero is not an unnatural case to consider but in physics the parameter $\hbar$ is Planck's (reduced) constant which is a fixed number up to the units in which one is working. But how can it be allowed to take a physical constant to zero. Here it should be noted that Planck's reduced constant is a small number $\hbar = 1.054571817 \times 10^{-34}$ J $\cdot$ s (joules times seconds). This number is negligibly small in a world where quantities are measured in joules and seconds (classical physics). But in the world of subatomic physics this is no longer a small scale (quantum physics). Hence letting $\hbar$ tend to zero can be interpreted as moving from microscopic to macroscopic scale which is the essence of the Bohr correspondence principle.

Besides a physical interpretation of semiclassical analysis it is also widely used within the theory of partial differential equations where the parameter often will appear by a scaling of the original partial differential equation. Moreover it also appears in a connection between geometry and analysis.

1.1 Weyl's law

In 1910 the physicist H. Lorentz came to Göttingen to give a seminar. In his lecture he proposed the mathematical problem of counting the eigenvalues of the Laplace operator on a domain (the standing waves or pure tones) and conjectured an asymptotic solution formula. Famously, D. Hilbert commented that this problem seemed too difficult to be solved in his lifetime. It was therefore surprising that D. Hilbert's own student H. Weyl succeeded in proving the expected formula already the next year [27]! Notes from the lectures by H. Lorentz were published in [19]. Actually this formula was also conjectured independently in 1910 by the physicist A. Sommerfeld in [26]. Both had based their conjecture on the book "The Theory of Sound" (1887) by Lord Rayleigh.

The operator considered by H. Weyl was the Dirichlet Laplacian on a bounded domain Ω in $\mathbb{R}^d$, where he proved the formula:

$$\operatorname{Tr}(\mathbf{1}_{(-\infty,\lambda]}(-\Delta_{D,\Omega})) = \frac{1}{(2\pi)^d}\omega_d \operatorname{Vol}(\Omega)\lambda^{\frac{d}{2}}(1+o(1)) \quad \text{as } \lambda \to \infty,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. During the following years H. Weyl published several papers [28–31] on the asymptotic distribution of eigenvalues and in [30] he conjectured the two term asymptotic formula

$$\begin{aligned} &\operatorname{Tr}(\mathbf{1}_{(-\infty,\lambda]}(-\Delta_{D,\Omega})) \\ &\quad = \frac{1}{(2\pi)^d}\omega_d \operatorname{Vol}(\Omega)\lambda^{\frac{d}{2}} - \frac{1}{4}\frac{\omega_{d-1}}{(2\pi)^{d-1}} \operatorname{Vol}'(\partial\Omega)\lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}}), \end{aligned} \tag{1.1}$$

as $\lambda \to \infty$, where $\operatorname{Vol}'(\partial\Omega)$ is the surface area of Ω. The formula is also formulated for Neumann boundary conditions where the minus in front of the second term should be a plus.[1]

H. Weyl's original proof was based on a technique which is now called Dirichlet-Neumann bracketing. This technique is simple, elegant and very robust. However, it cannot be used to obtain precise formulae including lower order correction terms. The variational methods were further developed but did not lead to sharp error bounds but generalisations instead.

The next advance was the application of Tauberian methods in this context first due to T. Carleman [4].[2] The idea of this method is to consider a function $F(A,t)$ of the operator A and an auxiliary parameter t. Then, under the right assumptions,

$$\operatorname{Tr}[F(A,t)] = \int_{\mathbb{R}} F(s,t)\, d\operatorname{Tr}[E_A(s)], \tag{1.2}$$

where $E_A(s)$ is the spectral projection. Now one tries to construct $\operatorname{Tr}[F(A,t)]$ using theory from partial differential equations and then try to recover $\operatorname{Tr}[E_A(s)]$. One of the very essential steps in this method is the choice of the function F. Some of the examples are $F(A,t) = e^{-tA}$, $F(A,t) = (t-A)^{-1}$ or $F(A,t) = A^t$. For each of these functions there is a rich theory in PDE but they are difficult in the Tauberian part (the recovering of $\operatorname{Tr}[E_A(s)]$) and have not yielded sharp error bounds so far (to the author's knowledge).

In the context of the Tauberian method V. Avakumović [2] and B. Levitan [18] used the function $F(A,t) = e^{itA}$. They were able to recover estimates of the order $\mathcal{O}(\lambda^{\frac{d-1}{2}})$ for closed manifolds and away from the boundary.

The next step was due to L. Hörmander who in [11] approximated the operator e^{itA} by Fourier integral operators and thereby extended the results of V. Avakumović and B. Levitan. The results of L. Hörmander was further extended by J. Duistermaat and V. Guillemin in [7], who recovered the error $o(\lambda^{\frac{d-1}{2}})$ for elliptic differential

[1] H. Weyl only considered the cases $d = 2, 3$.

[2] The Tauberian method or theorems originates from analytic number theory and is named after a Hungarian born, Austrian mathematician Alfred Tauber. The name Tauberian theorem was first introduced by Hardy and Littlewood in 1913.

operators on a compact smooth manifold without a boundary. For the proof they needed to assume that the measure of all periodic geodesics is 0.

The obstacle to get results with sharp remainder for smooth manifolds with boundary was the construction of the approximation to e^{itA} close to the boundary. This problem was partially circumvented by R. Seeley in [22]. It was first in 1980 that V. Ivrii proved the conjecture under two assumptions. The first assumption is that the boundary $\partial\Omega$ is smooth and the second is that the measure of all periodic billiards is zero. In the same year R. Melrose in [20] proved an analogous result for compact manifolds with smooth boundary but under more assumptions. One of these assumptions is that the boundary is everywhere strictly geodesically concave.

This short survey is not giving the full history of the progress on the Weyl conjecture. For a more detailed history about this problem see the surveys [1, 5, 15] or see the introductions in the mentioned papers.

We should in this section also mention that the development did not end with the proof of V. Ivrii in 1980. After this V. Ivrii and others tried to relax the assumptions on the boundary [3, 13, 14]. Others have not just considered the counting function but also the sum of the negative eigenvalues, where R. Frank and S. Larson this year (2019) obtained an optimal result on two terms asymptotic for the sum of negative eigenvalues for the Dirichlet Laplacian in a Lipschitz domain [8].

1.2 Generalisations of Weyl's law

The result obtained for the Laplacian on a manifold with or without boundary was during the period also extended to higher order differential operators. The higher order differential operators were assumed to be elliptic and defined on a manifold with or without boundary. A special case is the Schrödinger operator. They also started to consider the question of operators acting in $L^2(\mathbb{R}^d)$ and not just on a compact manifold. The question asked here was essentially the same: How many eigenvalues are there less than or equal to a certain number λ?

We will focus on the case of operators acting in $L^2(\mathbb{R}^d)$. Different types of problems were considered simultaneously but with similar methods. As an example we have the two following problems for Schrödinger operators:

- Find the spectral asymptotics as $\lambda \to \infty$ for $\mathrm{Tr}[\mathbf{1}_{(-\infty,\lambda]}(H)]$, the number of eigenvalues less than or equal to λ, where $H = -\Delta + V$ and $V(x) \to \infty$ as $|x| \to \infty$.
- Find the spectral asymptotics as $\hbar \to 0$ for $\mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(H_\hbar)]$, the number of eigenvalues less than or equal to 0, where $H = -\hbar^2\Delta + V$.

These two problems are quite different but the first can be deduced from the second under the right assumptions. The classical problem was the one receiving the most attention at the start but some also did work on the semiclassical problem. We will not give a full survey of the development for the semiclassical problem but instead fast forward to the results due to B. Helffer and D. Robert, which first appeared in [9] for the counting function. The results can also be found in the monograph [21]. We will consider more general operators than just the Schrödinger operator.

As we saw in the previous section optimal results relied on microlocal analysis and the same is true in the semiclassical setting. The class of operators they consider they call $\hbar$-admissible operators. We can think of these operators as operators of the form

$$A(\hbar) = \sum_{j\geq 0} \hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a_j),$$

where $\operatorname{Op}_\hbar^{\mathrm{w}}(a_j)$ is a Weyl-quantised $\hbar$-pseudo-differential operator of the smooth symbol a_j and the sum should be understood as a formal sum. The rigorous definition for these types of operators will be recalled later. The prime example for this type of operators is the semiclassical Schrödinger operator $H_\hbar = -\hbar^2\Delta + V$ which is given just by the principal symbol $a_0(x,p) = p^2 + V(x)$ for V which is smooth and not ill behaved. An optimal Weyl law for these operators is an expression of the form

$$\operatorname{Tr}[\mathbf{1}_{(-\infty,\lambda_0]}(A(\hbar))] = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,\lambda_0]}(a_0(x,p))\,dxdp + \mathcal{O}(\hbar^{1-d}),$$

for a number $\lambda_0 < \lambda$ such $a_0^{-1}((-\infty,\lambda])$ is compact and non-critical for $a_0(x,p)$. A number λ_0 is a non-critical value when

$$|\nabla a_0(x,p)| \geq c > 0 \quad \text{for all } (x,p) \in a_0^{-1}(\{\lambda_0\}).$$

This non-critical condition is essential for the proofs to be valid. It should be remarked that the form of the leading term of the Weyl conjecture is of the same form as the above phase space integral. Moreover in order for the proof to be valid we need extra assumptions on the symbols than stated here.

In the previous section we stated some of the techniques entering the proof of the Weyl conjecture. As it turns out these ideas also enter in the proof of the above formula. Their proof has roughly three main steps. First they use a functional calculus for $\hbar$-admissible operators to localise the problem in energy space such that all remaining values are non-critical. Then a Tauberian argument is used to smoothen the spectral measure and introduce a propagator. The third step is to approximate the propagator by a Fourier integral operator and do a stationary phase argument. Hence the steps used in this approach are very similar to the steps described in the previous section.

These results were further extended by B. Helffer and D. Robert in [10], where they consider functions of the form $g_s(t) = (t)_-^s$, where $(t)_- = (|t|-t)/2$ and $s \geq 0$, of the operators. Here they get more terms in the asymptotic expansions. In the case where the subprincipal synbol is zero $(a_1(x,p) = 0)$ the expansion becomes

$$\operatorname{Tr}[g_s(A(\hbar)-\lambda)] = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(a_0(x,p)-\lambda)\,dxdp + \mathcal{O}(\hbar^{1+s-d}),$$

for $0 \leq s \leq 1$ and a number $\lambda_0 < \lambda$ such $a_0^{-1}((-\infty,\lambda])$ is compact and non-critical for $a_0(x,p)$.

Similar results to these have also been proven by V. Ivrii [12] but with a slightly different method. In Ivrii's approach he considers the Schwarz kernel of the counting function directly and proves his results on the level of kernels. In his work there is a key observation: The construction of the propagator as a Fourier integral operator is not required to get full asymptotics for these types of problems. Usually the Fourier integral operator of the propagator is constructed for times in a small interval around zero. But by applying suitable scaling arguments and hyperbolic energy estimates he showed that the traces under consideration is negligible in the region $\hbar^\delta \leq |t| \leq T_0$ for $0 < \delta < 1$. For $|t| \leq \hbar^\delta$ he showed that a "rough" approximation to the propagator is sufficient to get the full asymptotics. This observation will be discussed further in Chapter 3.

The above generalisations have all been for global operators. There have also been work where the problem is localised by a multiplication operator with a smooth compactly supported function. Results of this kind have been obtain by V. Ivrii and can be found in [12, 16]. In the case of the Schrödinger operator A. V. Sobolev obtained local result in [23]. In the results obtained by A. V. Sobolev one does not need to assume the potential to be smooth in the whole space but only in a sufficiently large neighbourhood of the localisation.

The semicalssical results have so far all been with a non-critical condition at an energy λ. For a Schrödinger operator there is an approach due to V. Ivrii (see [12, 16]) where it is possible to prove optimal error terms with out a non-critical condition. This approach is also described by A. V. Sobolev in [24], where he extends the results mentioned above to operators which only locally are assumed to be Schrödinger operators. The approach is called multiscale analysis and the one suggested by V. Ivrii can be viewed as a discrete approach. There is also a continuous version suggested by J. P. Solovej and W. L. Spitzer in [25].

There are also works by L. Zielinski where he does not assume the non-critical condition [36, 38–40]. Here L. Zielinski rewrites the error in terms of volumes and is able to get a sharp bound with an assumption on the size of phase space volume.

Weyl laws have also be extended to symbols not taking values in $\mathbb{C}$ but in a Hilbert space. The case of the Hilbert space being a space of matricies can be fund in the monograph by M. Dimassi and J. Sjöstrand [6] and monographs [12, 16] by V. Ivrii.

1.3 Non-smooth theory

Almost all the previous results have assumed smoothness of the symbols. There were results in Section 1.1, where the boundary was not smooth. But what about the case of general differential operators with non-smooth coefficients? Can an optimal Weyl law still be proven?

These types of results were first proven by L. Zielinski in [32–35], where he obtained sharp estimates without assuming smoothness. We only mention L. Zielinski here,as he was the first to obtain sharp estimates, but others also considered the case of non smooth coefficients. L. Zielinski considered globally elliptic differential operators with non-smooth coefficients on compact manifolds with or without boundaries and on the whole space ($\mathbb{R}^d$). In these papers L. Zielinski considered the number of eigenvalues less than a number λ and the asymptotics as λ tends to infinity. As microlocal analysis requires smoothness of the symbols this type of theory can not a priori be applied. Instead the first step is to regularise the coefficients such that they become smooth. This regularisation was based on ideas that appeared in [17] by H. Kumano-go and M. Nagase. Then he could use techniques from microlocal analysis on the regularised operator and by comparing the spectral asymptotics for the two operators he could obtain the result. He proved the sharp estimate under the assumption that the coefficients are differentiable and the first derivative is Lipschitz continuous.

The results of L. Zielinski were generalised by V. Ivrii in the semiclassical setting in [13]. Here the coefficients are assumed to be differentiable and with a Hölder continuous first derivative. Moreover in the paper he uses the semiclassical result to prove the classical analogue. In [3] V. Ivrii together with M. Bronstein reduced the assumptions further by assuming the first derivative to have modulus of continuity $\mathcal{O}(|\log(x-y)|^{-1})$. In this paper they considered not only one regularisation of the original operator A but two framing operators $A_\varepsilon^\pm$ such

$$A_\varepsilon^- \leq A \leq A_\varepsilon^+,$$

in the sense of quadratic forms. The parameter ε is connected to the semiclassical parameter $\hbar$ in a way which depends on the regularity of the coefficients. One of the advantages of constructing two approximating operators is that by the min-max theorem we have

$$\mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(A_\varepsilon^+)] \leq \mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(A)] \leq \mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(A_\varepsilon^-)].$$

This implies that they can reduce the proof to showing a Weyl law for the approximating operators and compare the phase space integrals. They prove this Weyl law by microlocal techniques. We will later return to some of the ideas entering this approach.

In [37] L. Zielinski used the semiclassical setting to generalise the methods used in [32–35] to obtain sharp estimates for globally elliptic differential operators with non-smooth coefficients, which he assumed to be differentiable and with a Hölder continuously bounded first derivative.

For the non-smooth theory there are also works without a non-critical condition. The ones known to the author are the works by V. Ivrii and L. Zielinski.

1.4 Applications of Weyl's law

Weyl's law is not just a purely mathematical pursuit. It has a strong and deep connection to physical problems. As seen in the very start of this introduction the problem was first introduced by two physicists. Some of the connection to physics is in the description of vibrations of a string or membrane, heat radiation of a body in thermal equilibrium or the Schrödinger equation. We will not here go into further details about these connections here. In stead we refer to [1, 5], which is still not a complete list.

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Chapter 2

An optimal semiclassical bound on certain commutators

In the first paper we consider semiclassical bounds on the trace norm of the commutators

$$[\mathbf{1}_{(-\infty,0]}(H_\hbar), x_j], \quad [\mathbf{1}_{(-\infty,0]}(H_\hbar), Q_j] \quad \text{and} \quad [\mathbf{1}_{(-\infty,0]}(H_\hbar), e^{itx}]. \tag{2.1}$$

Here $H_\hbar$ is a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ of the form $H_\hbar = -\hbar^2\Delta + V$, where Δ is the Laplacian on $\mathbb{R}^d$ and V is a real-valued function. The operator Q_j is given by $Q_j = -i\hbar\partial_{x_j}$ for j in $\{1, \dots, d\}$, x_j is the multiplication operator with x_j for j in $\{1, \dots, d\}$ and e^{itx} is the multiplication operator with the function e^{itx}. The bound on the trace norm of the commutators we prove is

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), x_j]\|_1 \leq C\hbar^{1-d} \quad \text{and} \quad \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), Q_j]\|_1 \leq C\hbar^{1-d}, \tag{2.2}$$

where $\|\cdot\|_1$ denotes the trace norm. The bound on the trace norm of the last commutator in (2.1) follows as a corollary. The exact statement of our main theorem and our assumptions will follow shortly.

First of all why is this an interesting question? In order give some answers we need to set up some notation and terminology.

Definition 2.0.1. Let $\Omega \subseteq \mathbb{R}^d_x \times \mathbb{R}^d_p$ be open, ρ be in $[0, 1]$ and m a tempered weight function on $\mathbb{R}^d_x \times \mathbb{R}^d_p$. We call a function a a symbol with weights (m, ρ) if a is in $C^\infty(\Omega)$ and satisfies that

$$|\partial_x^\alpha \partial_p^\beta a(x,p)| \leq C_\alpha m(x,p)(1 + |(x,p)|)^{-\rho(|\alpha|+|\beta|)}, \tag{2.3}$$

for all (x, p) in Ω and α, β in $\mathbb{N}^d_0$. The space of these functions is denoted $\Gamma^m_\rho(\Omega)$. This space is a Fréchet space with the natural family of semi-norms.

The tempered weight function is also called an order function. This is the case in the monographs [5] and [11]. We have chosen to call them temperate weights to align with the terminology in the monographs [6] and [9].

With this definition we can define the Weyl-quantised $\hbar$ pseudo-differential operator ($\hbar$-ΨDO) we consider.

Definition 2.0.2. Let m be a tempered weight function. For a symbol a in the set $\Gamma^m_\rho\left(\mathbb{R}^d_x \times \mathbb{R}^d_p\right)$ we associate the operator $\mathrm{Op}^{\mathrm{w}}_\hbar(a)$ defined by

$$\mathrm{Op}^{\mathrm{w}}_\hbar(a)\psi(x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} a\left(\tfrac{x+y}{2}, p\right) \psi(y) \, dy \, dp, \tag{2.4}$$

for ψ in $\mathcal{S}(\mathbb{R}^d)$ (the Schwartz space). The integral in (2.4) shall be interpered as an oscillating integral.

For the properties of this type of operators we refer to one of the monographs [5, 6, 9, 11].

We can now give some answers to why this question is interesting. If we consider two $\hbar$-pseudo-differential operators $A(\hbar) = \mathrm{Op}_\hbar^{\mathrm{w}}(a)$ and $B(\hbar) = \mathrm{Op}_\hbar^{\mathrm{w}}(b)$ with symbols satisfying that $|\partial_x^\alpha \partial_p^\beta a(x,p)|$ and $|\partial_x^\alpha \partial_p^\beta b(x,p)|$ are elements of $L^1(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ for all multi indicies α and β in $\mathbb{N}_0^d$. Then by Theorem II-32 and Theorem II-49 in [9] we have the bound

$$\|[A(\hbar), B(\hbar)]\|_1 \leq C\hbar^{1-d}.$$

This bound implies that the bounds in (2.2) are optimal in terms of the semiclassical parameter $\hbar$ even though the operator $\mathbf{1}_{(-\infty,0]}(H_\hbar)$ is not a $\hbar$-ΨDO. Hence as a purely semiclassical question is interesting to see if optimal errors can be achieved in this setting.

The bounds in (2.2) correspond to a mean-field version of bounds assumed in a series of papers by N. Benedikter, M. Porta and B. Schlein et. al. [1–4]. In these papers they consider time evolutions of fermionic systems in certain regimes. They prove that if the initial state is a Slater determinant and the one-particle reduced density matrix of this state satisfies a semiclassical commutator bound as in (2.2), then the evolved state remains close to a Slater determinant. We will not here go into further details of the result, as it requires another formalism and machinery than we otherwise will use but refer to the papers [1–4] for further details. Bounds of this type have also been assumed in [7], which appeared this year (2019). So the semiclassical bounds of this type is not just interesting from a purely semiclassical view it also has applications in other areas.

2.1 Main theorem

Before we state the main theorem we will state the assumptions on V for which we can prove the theorem.

Assumption 2.1.1. Let $V : \mathbb{R}^d \to \mathbb{R}$ be a function for which there exists an open set $\Omega_V \subset \mathbb{R}^d$ and $\varepsilon > 0$ such that

1) V is in $C^\infty(\Omega_V)$.

2) There exists an open bounded set Ω_ε such that $\overline{\Omega}_\varepsilon \subset \Omega_V$ such that $V \geq \varepsilon$ for all $x \in \Omega_\varepsilon^c$.

3) $V\mathbf{1}_{\Omega_V^c}$ is an element of $L^1_{loc}(\mathbb{R}^d)$.

The assumption of smoothness in the set Ω_V is needed in order to use the theory of pseudo-differential operators. The second assumption is needed to ensure that we have noncontinuous spectrum in $(-\infty, 0]$ and enable us to localise the operator. The last assumption is just to ensure that we can define the operator $H_\hbar$ by a Friedrichs extension of the associated form. We can now state our main theorem:

Theorem 2.1.2. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$, where V satisfies Assumption 2.1.1 and let $Q_j = -i\hbar\partial_{x_j}$ for $j \in \{1, \dots, d\}$. Then the following bounds hold*

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), x_j]\|_1 \leq C\hbar^{1-d} \quad \textit{and} \quad \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), Q_j]\|_1 \leq C\hbar^{1-d}, \tag{2.5}$$

where C is a positive constant.

As a corollary to the above theorem we obtain:

Corollary 2.1.3. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$, where V satisfies Assumption 2.1.1. Then the following bound holds*

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), e^{i\langle t,x\rangle}]\|_1 \leq C|t|\hbar^{1-d}, \tag{2.6}$$

for all t in $\mathbb{R}^d$, where $\langle t, x\rangle$ is the Euclidean inner product and C is a positive constant.

The corollary follows by an application of the Duhamel formula/principle and the the first bound in Theorem 2.1.2. One thing we emphasise is that in our proof we found a way to circumvent the use of a smoothing procedure. This step will be briefly explained in the next section.

2.2 Ideas entering the proof of the main theorem

We will in the following use the notation from Assumption 2.1.1. Moreover we will only discuss the first commutator in Theorem 2.1.2 as the ideas used to prove the two bounds are the same.

The first step in the proof of Theorem 2.1.2 is to localise the problem. This is done by choosing a χ in $C_0^\infty(\Omega_V)$ such that $\chi(x) = 1$ for all x in $\overline{\Omega}_\varepsilon$. With this localisation we have

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), x_j]\|_1 = \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), \chi x_j]\|_1 + \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), (1-\chi)x_j]\|_1. \tag{2.7}$$

For the last term in the right hand side of (2.7) we use an Agmon type estimate and a corollary to the Cwikel-Lieb-Rosenbljum bound from [8] to prove it is negligible. What remains is the localised part of the commutator.

For the localised part we prove the bound in two steps. First we prove the bound under a non-critical assumption, where we are localised to a ball. Then this result is used to obtain the general local result by the multiscale argument.

Before we state the localised version with a non-critical condition we need some notation and to recall an assumption:

Assumption 2.2.1. Let $\mathcal{H}$ be an operator acting in $L^2(\mathbb{R}^d)$ such that

1) $\mathcal{H}$ is selfadjoint and lower semibounded.

2) There exists an open set $\Omega \subset \mathbb{R}^d$ and a realvalued function V_{loc} in $C_0^\infty(\mathbb{R}^d)$ such that $C_0^\infty(\Omega) \subset \mathcal{D}(\mathcal{H})$ and
$$\mathcal{H}u = H_\hbar^{loc} u$$
for all u in $C_0^\infty(\Omega)$, where $H_\hbar^{loc} = -\hbar^2\Delta + V_{loc}$.

This assumption is taken directly from [10] and we will also be applying a result from that paper. The first localised version we prove is:

Theorem 2.2.2. *Suppose the operator $\mathcal{H}$ acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$ obeys Assumption 2.2.1 with $\Omega = B(0, 4R)$ for $R > 0$ and*

$$|V(x)| + |\nabla V(x)|^2 + \hbar \geq c, \tag{2.8}$$

for all x in $B(0, 2R)$, where $c > 0$. For φ in $C_0^\infty(B(0, R/2))$ it holds that

$$\|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \varphi]\|_1 \leq C\hbar^{1-d} \quad \text{and} \quad \|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \varphi Q_j]\|_1 \leq C\hbar^{1-d}, \tag{2.9}$$

for all sufficiently small $\hbar$ and $j \in \{1, \dots, d\}$, where $Q_j = -i\hbar\partial_{x_j}$. The constant C only depends on the dimension, the numbers $\|\partial_x^\alpha V\|_\infty$ and $\|\partial_x^\alpha \varphi\|_\infty$ for all α in $\mathbb{N}_0^d$, and the numbers R and c in (2.8).

The new assumption on V in equation (2.8) ensures that 0 is a non-critical value of the operator. In order to prove the bounds here it is convenient to rewrite the commutator as

$$[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \varphi] = \mathbf{1}_{(-\infty,0]}(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H}) - \mathbf{1}_{(0,\infty)}(\mathcal{H})\varphi\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \tag{2.10}$$

where the bound is proven for each of the two terms in (2.10). In order to prove the bounds we localise in energy to a region close to zero where all values are non-critical and the rest. For the technical details, see the paper. This localisation gives us a term of the form
$$\mathbf{1}_{(-\eta,0]}(\mathcal{H})\varphi\mathbf{1}_{(0,\eta)}(\mathcal{H}),$$
where η is chosen such all values in $[-2\eta, 2\eta]$ are non-critical values. In order to estimate this term we make a $\hbar$ dependent dyadic decomposition of the interval $(-\eta, 0]$ by introducing the functions

$$\chi_{n,\hbar}(t) = \begin{cases} \mathbf{1}_{(\hbar,0]}(t) & n = 0 \\ \mathbf{1}_{(-4^n\hbar,-4^{n-1}\hbar]}(t) & n \in \mathbb{N}. \end{cases}$$

By letting $\widetilde{\chi}_{n,\hbar}(t) = \chi_{n,\hbar}(-t)$ we also have a decomposition of $(0, \eta)$. With these function we get

$$\|\mathbf{1}_{(-\eta,0]}(\mathcal{H})\varphi\mathbf{1}_{(0,\varepsilon]}(\mathcal{H})\|_1 \leq \sum_{n=0}^{N(\hbar)}\sum_{m=0}^{N(\hbar)} \|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1. \tag{2.11}$$

Why is this a useful estimate? Under the assumptions in the theorem one can prove that
$$\mathrm{Tr}[\mathbf{1}_{[-4^n\hbar,-4^{n-2}\hbar]}(\mathcal{H})\varphi] \leq 4^n\hbar^{1-d}C,$$

by applying [10, Theorem 4.1], which is an optimal Weyl law for this type of operators. Hence by a Cauchy-Schwarz inequality we have

$$\|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \leq C4^{\frac{n+m}{2}}\hbar^{1-d}.$$

The issue is that these terms are not summable. To make the sum converge we split the sum according to which of the numbers m and n are largest. In the case $n \leq m$ we use the simple fact that $t + 2^{2n-1}\hbar$ is large on the support of $\widetilde{\chi}_{m,\hbar}$ and small on the support of $\chi_{n,\hbar}$. This trick gives the estimate

$$\|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \leq \frac{1}{4^{2(m-1)}\hbar^2}\|\chi_{n,\hbar}(\mathcal{H})[[\varphi,\mathcal{H}],\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1. \tag{2.12}$$

What we have gained from this is a prefactor which will make the sum convergent if we can estimate the double commutator. The semiclassical parameter $\hbar^{-2}$ in the prefactor is canceled by the $\hbar^2$ coming from the double commutator. By the assumptions on the operator $\mathcal{H}$ the double commutator can be calculated explicitly and then the term can be estimated.

The above trick is the one that enable us to avoid a smoothing procedure which is often used to get optimal error bounds in such problems.

To move from this local result to a local result without a non-critical condition we use a multiscale argument as stated in [10]. We will not give further details here on how this argument works but we refer to the paper. This local result without a non-critical condition now gives the estimate for the first term of the right hand side in (2.7).

2.3 Possible extensions of the result

A first question could be to incorporate a magnetic potential. The answer to this is most likely positive as the optimal Weyl law we used is actually proven for a magnetic Schrödinger operator. Moreover the double commutator should still be computable. For other types of operators one would need to ensure an optimal Weyl law as the one we have used and an Agmon type estimate, as the one we applied heavily used that our operator was a Schrödinger operator.

Then there is the work to generalise this result to the precise assumption in [1–4], where the setup is that of a many-body problem. This would probably require some additional techniques than the ones we apply. However we consider the present contribution a first important step.

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Paper I

An optimal semiclassical bound on certain commutators

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Abstract: We prove an optimal semiclassical bound on the trace norm of the following commutators $[\mathbf{1}_{(-\infty,0]}(H_\hbar), x]$, $[\mathbf{1}_{(-\infty,0]}(H_\hbar), -i\hbar\nabla]$ and $[\mathbf{1}_{(-\infty,0]}(H_\hbar), e^{itx}]$, where $H_\hbar$ is a Schrödinger operator with a semiclassical parameter $\hbar$, x is the position operator and $-i\hbar\nabla$ is the momentum operator. These bounds corresponds to a mean-field version of bounds introduced as an assumption by N. Benedikter, M. Porta and B. Schlein in a study of the mean-field evolution of a fermionic system.

I.1 Introduction and main result

We consider a Schrödinger operator $H_\hbar = -\hbar^2\Delta + V$ acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$. Here Δ is the Laplacian acting in $L^2(\mathbb{R}^d)$ and V is a real valued function. We will be interested in the following trace norms of commutators:

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), x_j]\|_1, \quad \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), Q_j]\|_1 \quad \text{and} \quad \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), e^{itx}]\|_1,$$

where $Q_j = -i\hbar\partial_{x_j}$ and x_j is the position operator for $j \in \{1, \dots, d\}$. Moreover $\mathbf{1}_A$ denotes the characteristic function of a set A and $\|\cdot\|_1$ denotes the trace norm. The main theorem will be the bound for the first two commutators and the bound on the last will follow as a corollary.

Let us specify the assumptions on the function V for which we study the operator $H_\hbar$.

Assumption I.1.1. Let $V : \mathbb{R}^d \to \mathbb{R}$ be a function for which there exists an open set $\Omega_V \subset \mathbb{R}^d$ and $\varepsilon > 0$ such that

1) V is in $C^\infty(\Omega_V)$.

2) There exists an open bounded set Ω_ε such that $\overline{\Omega}_\varepsilon \subset \Omega_V$ such that $V \geq \varepsilon$ for all $x \in \Omega_\varepsilon^c$.

3) $V\mathbf{1}_{\Omega_V^c}$ is an element of $L^1_{loc}(\mathbb{R}^d)$.

The assumption of smoothness in the set Ω_V is needed in order to use the theory of pseudo-differential operators. The second assumption is needed to ensure that we have non continuous spectrum in $(-\infty,0]$ and enable us to localise the operator. The last assumption is just to ensure that we can define the operator $H_\hbar$ by a Friedrichs extension of the associated form. We can now state our main theorem:

Theorem I.1.2. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$, where V satisfies Assumption I.1.1 and let $Q_j = -i\hbar\partial_{x_j}$ for $j \in \{1,\dots,d\}$ futhermore, let $\hbar_0$ be a strictly positive number. Then the following bounds hold*

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), x_j]\|_1 \leq C\hbar^{1-d} \quad \textit{and} \quad \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), Q_j]\|_1 \leq C\hbar^{1-d}, \tag{2.1}$$

for all $\hbar$ in $(0,\hbar_0]$, where C is a positive constant.

From Theorem I.1.2 we get the corollary:

Corollary I.1.3. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$, where V satisfies Assumption I.1.1 futhermore, let $\hbar_0$ be a strictly positive number. Then the following bound holds*

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), e^{i\langle t,x\rangle}]\|_1 \leq C|t|\hbar^{1-d}, \tag{2.2}$$

for all t in $\mathbb{R}^d$ and all $\hbar$ in $(0,\hbar_0]$, where $\langle t,x\rangle$ is the Euclidean inner product and C is a positive constant.

Theorem I.1.2 and Corollary I.1.3 are semiclassical in the sense that they are of most interest in the cases where the semiclassical parameter $\hbar$ is small. The upper bound $\hbar_0$ on the semiclassical parameter is needed in order to control the constants as we do not have uniformity for $\hbar$ tending to infinity.

The proofs of Theorem I.1.2 and Corollary I.1.3 are given in section I.4. The proof of Theorem I.1.2 is divided into three parts. First a local version of the theorem (see Theorem I.3.3) is proven with a non-critical assumption (2.4). This proof is based on local Weyl-asymptotics proven in the paper [15] and an $\hbar$ dependent dyadic decomposition which will be introduced in the proof. In the first part we will not be considering the operator $H_\hbar$ directly but an abstract operator $\mathcal{H}$ which satisfies Assumption I.2.1 below. The abstract version is needed for the later multiscale argument.

The second part is to remove the non-critical condition by a multiscale argument as in [15] (see also [9, 10]). The main idea is to make a partition of unity and on each partition scale the operator in such a way that a non-critical assumption is achieved and then use the theorem with the non-critical condition. The final step in this part is to remove the dependence of the partition by integration.

The third part is to first note that the theorem obtained in the second part gives the desired estimate in the classically allowed region $\{V < \varepsilon\}$ and then prove that the classically forbidden region $\{V > \varepsilon\}$ contributes less to the error term than the desired estimate. This is done by applying an Agmon type bound on the eigenfunctions of the operator $H_\hbar$.

Commutator bounds of the type considered in this paper were introduced as assumptions in a series of papers by N. Benedikter, M. Porta and B. Schlein et. al. [2–5] where they considered mean-field dynamics of fermions in different settings. The bounds considered here are a first step to verifying their assumption, since the bounds proven here correspond to a mean field version of the bounds they need. The assumption reappeared in the paper [11].

Already the mean-field version of the bounds, treated in this paper, is non-trivial as they are optimal in terms of the semiclassical parameter $\hbar$, which is easily seen by the calculus of pseudo-differential operators.

I.2 Preliminaries

Assumptions and notation

First we will describe the operators we are working with. Under Assumption I.1.1 we can define the operator $H_\hbar = -\hbar^2\Delta + V$ as the Friedrichs extension of the quadratic form given by

$$\mathfrak{h}[f,g] = \int_{\mathbb{R}^d} \hbar^2 \sum_{i=1}^d \partial_{x_i} f(x)\overline{\partial_{x_i} g(x)} + V(x)f(x)\overline{g(x)}\, dx, \qquad f,g \in \mathcal{D}(\mathfrak{h}),$$

where

$$\mathcal{D}(\mathfrak{h}) = \left\{ f \in L^2(\mathbb{R}^d) | \int_{\mathbb{R}^d} |p|^2 |\hat{f}(p)|^2 \, dp < \infty \text{ and } \int_{\mathbb{R}^d} |V(x)||f(x)|^2 \, dx < \infty \right\}.$$

In this set up the Friedrichs extension will be unique and self-adjoint see e.g. [13]. Moreover, we will also consider operators that satisfy the following assumption

Assumption I.2.1. Let $\mathcal{H}$ be an operator acting in $L^2(\mathbb{R}^d)$ such that

1) $\mathcal{H}$ is selfadjoint and lower semibounded.

2) There exists an open set $\Omega \subset \mathbb{R}^d$ and a realvalued function V_{loc} in $C_0^\infty(\mathbb{R}^d)$ such that $C_0^\infty(\Omega) \subset \mathcal{D}(\mathcal{H})$ and

$$\mathcal{H}u = H_\hbar^{loc} u$$

for all u in $C_0^\infty(\Omega)$, where $H_\hbar^{loc} = -\hbar^2\Delta + V_{loc}$.

The above assumption is exactly the same as in [15]. It is important to note that the assumptions made on the the operator $H_\hbar$ in Theorem I.1.2 imply that $H_\hbar$ satisfies Assumption I.2.1 for a suitable V_{loc}. When referring to this assumption further on we will omit the *loc* on the operator $H_\hbar^{loc}$ and the function V_{loc} when we only consider an operator satisfying the assumption.

The construction of the operator via a Friedrichs extension will also work for the local Schrödinger operator, where V_{loc} is $C_0^\infty(\mathbb{R})$. But in this case the operator can also be constructed as the closure of an $\hbar$-pseudo-differential operator ($\hbar$-ΨDO) defined on the Schwarz space. By an $\hbar$-ΨDO, $A = \operatorname{Op}_\hbar^{\mathrm{w}}(a)$ we mean the operator with Weyl symbol a, that is

$$\operatorname{Op}_\hbar^{\mathrm{w}}(a)\psi(x) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} a\left(\tfrac{x+y}{2},p\right)\psi(y)\, dy\, dp,$$

for $\psi \in \mathcal{S}(\mathbb{R}^d)$ (the Schwarz space). The symbol a is assumed to be in $C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_p)$ and to satisfy the condition

$$|\partial_x^\alpha \partial_p^\beta a(x,p)| \le C_{\alpha,\beta} m(x,p), \tag{2.3}$$

for all multi-index α and β and some tempered weight function m. The above integrals should be understod as oscillating integrals. We need this as the results on Weyl-asymptotics needed is based on ($\hbar$-ΨDOs). For more details see e.g. the monographs [7, 14, 16].

We call a number E in $\mathbb{R}$ a non-critical value for a symbol a if

$$(\nabla_x a(x,p), \nabla_p a(x,p)) \neq 0 \qquad \forall (x,p) \in a^{-1}(\{E\}).$$

In the case where $a(x,p) = p^2 + V(x)$ the non-critical condition can be expressed only in terms of the function V by assuming that

$$|\nabla_x V(x)|^2 + |E - V(x)| > 0, \qquad \forall (x,p) \in a^{-1}(\{E\}),$$

since it is immediate that

$$|\nabla_x a(x,p)|^2 + |\nabla_p a(x,p)|^2 = |\nabla_x V(x)|^2 + 4|E - V(x)|, \qquad \forall (x,p) \in a^{-1}(\{E\}).$$

Optimal Weyl-asymptotics

We are interested in optimal Weyl-asymptotics for an operator $\mathcal{H}$ acting in $L^2(\mathbb{R}^d)$ satisfying Assumption I.2.1. When we only have one operator we will not write the *loc* subscript on the operator. In the following we will denote the open ball with radius R by $B(0,R)$. For this kind of operators we have from [15, Theorem 4.1] the following theorem:

Theorem I.2.2. *Suppose the operator $\mathcal{H}$ acting in $L^2(\mathbb{R}^d)$ with $d \ge 2$ obeys Assumption I.2.1 with $\Omega = B(0,4R)$ for $R > 0$ and*

$$|V(x)| + |\nabla V(x)|^2 + \hbar \ge c, \tag{2.4}$$

for all x in $B(0,2R)$ furthermore, let $\hbar_0$ be a strictly positive number. For φ in $C_0^\infty(B(0,R/2))$ it holds that

$$\Big|\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(\mathcal{H})\varphi] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{\{p^2+V(x)\le 0\}}(x,p)\varphi(x)\, dx\, dp\Big| \le C\hbar^{1-d},$$

for C a positive constant and all $\hbar$ in $(0,\hbar_0]$. The constant C depends on the numbers R, $\hbar_0$ and c in (2.4) and on the bounds on the derivatives of V and φ.

One can note that in our "non-critical" assumption (2.4) in the above theorem there has appeared an $\hbar$. This assumption would either imply that $|V(x)|+|\nabla V(x)|^2 \geq c/2$ or $\hbar \geq c/2$. In the first case the assumption gives us our non-critical assumption. In the second both sides will be finite and the formula can be made true by an appropriate choice of constants.

Proposition I.2.3. *Suppose the operator $\mathcal{H}$ acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$ obeys Assumption I.2.1 with $\Omega = B(0, 4R)$ for $R > 0$. Moreover suppose there is an $\varepsilon > 0$ such that*

$$|V(x) - E| + |\nabla V(x)|^2 + \hbar \geq c, \tag{2.5}$$

for all x in $B(0, 2R)$ and all E in $[-2\varepsilon, 2\varepsilon]$ furthermore let $\hbar_0$ be a strictly positive number. For φ in $C_0^\infty(B(0, R/2))$ and two numbers a and b such that

$$-\varepsilon < a \leq b < \varepsilon,$$

it holds that

$$\mathrm{Tr}[\mathbf{1}_{[a,b]}(\mathcal{H})\varphi] \leq C_1|b-a|\hbar^{-d} + C_2\hbar^{1-d},$$

for two positive constants C_1 and C_2 and all $\hbar$ in $(0, \hbar_0]$. The constants C_1 and C_2 depend only on the numbers R, $\hbar_0$ and c in (2.5) and on the bounds on the derivatives of V and φ.

Remark I.2.4. We suppose we have an operator $\mathcal{H}$ acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$, which obeys Assumption I.2.1 with $\Omega = B(0, 4R)$ for $R > 0$. If it is assumed that there exists a $c > 0$ for which

$$|V(x)| + |\nabla V(x)|^2 + \hbar \geq c,$$

for all x in $B(0, 2R)$, then by continuity this would imply the existence of a $\tilde{c} > 0$ and an $\varepsilon > 0$ such that

$$|V(x) - E| + |\nabla V(x)|^2 + \hbar \geq c,$$

for all x in $B(0, 2R)$ and all E in $[-2\varepsilon, 2\varepsilon]$. That is we could generalise the assumptions in the proposition. But we have chosen this form of the proposition due to later applications.

Proof. We can rewrite the trace of interest as

$$\mathrm{Tr}[\mathbf{1}_{[a,b]}(\mathcal{H})\varphi] = \mathrm{Tr}[\mathbf{1}_{(-\infty,b]}(\mathcal{H})\varphi] - \mathrm{Tr}[\mathbf{1}_{(-\infty,a)}(\mathcal{H})\varphi]. \tag{2.6}$$

If we consider the trace $\mathrm{Tr}[\mathbf{1}_{(-\infty,b]}(\mathcal{H})\varphi]$ then we can rewrite this in the following way

$$\mathrm{Tr}[\mathbf{1}_{(-\infty,b]}(\mathcal{H})\varphi] = \mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(\mathcal{H} - b)\varphi].$$

The operator $\mathcal{H} - b$ satisfies Assumption I.2.1 with V replaced by $V - b$ and by assumption we have

$$|V(x) - b| + |\nabla V(x)|^2 + \hbar \geq c, \tag{2.7}$$

for all x in $B(0,2R)$. The b should be understood as $b\chi(x)$ where χ is $C_0^\infty(B(0,4R))$ and $\chi(x)=1$ for x in $B(0,3R)$. Hence we can omit the χ when we are localised to $B(0,2R)$. By Theorem I.2.2 we have the following identity

$$\begin{aligned}\operatorname{Tr}[\mathbf{1}_{(-\infty,b]}(\mathcal{H})\varphi] &= \operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(\mathcal{H}-b)\varphi]\\ &= \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}_{\{p^2+V(x)-b\leq 0\}}(x,p)\varphi(x)\,dxdp+\mathcal{O}(\hbar^{1-d})\\ &= \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}_{\{p^2+V(x)\leq b\}}(x,p)\varphi(x)\,dxdp+\mathcal{O}(\hbar^{1-d}),\end{aligned}\tag{2.8}$$

where the error term is independent of b. Analogously we get that

$$\operatorname{Tr}[\mathbf{1}_{(-\infty,a]}(\mathcal{H})\varphi] = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}_{\{p^2+V(x)\leq a\}}(x,p)\varphi(x)\,dxdp+\mathcal{O}(\hbar^{1-d}).\tag{2.9}$$

Since the two error terms are of the same order we can, when subtracting the two traces, add the two error terms and obtain a new error term of order $\hbar^{1-d}$. Hence we will consider the integral

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}_{\{p^2+V(x)\leq b\}}(x,p)\varphi(x)-\mathbf{1}_{\{p^2+V(x)\leq a\}}(x,p)\varphi(x)\,dxdp.\tag{2.10}$$

By assumption this integral is finite. In order to evaluate these integrals we note that by assumption we are in one of the following two cases

$$\hbar > \frac{c}{2}\tag{2.11}$$

or

$$|V(x)-E|+|\nabla V(x)|^2 \geq \frac{c}{2},\tag{2.12}$$

for all x in $B(0,2R)$ and all E in $[-2\varepsilon,2\varepsilon]$. In the first case (2.11) we can estimate the integrals by a constant and replace $\hbar^{-d}$ by $\hbar^{1-d}$ at the cost of $\frac{2}{c}$. For the second case (2.12) we have, by the Coarea formula, the equality

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}_{\{p^2+V(x)\leq b\}}(x,p)\varphi(x)-\mathbf{1}_{\{p^2+V(x)\leq a\}}(x,p)\varphi(x)\,dxdp\\ &\qquad\qquad= \int_a^b\int_{\{p^2+V(x)=E\}}\varphi(x)\frac{1}{|(\nabla_x V(x),\nabla_p p^2)|}\,dSdE,\end{aligned}\tag{2.13}$$

where S is the surface measure. By support properties of φ and (2.12) we have that

$$\sup_{E\in[-\varepsilon,\varepsilon]}\int_{\{p^2+V(x)=E\}}\varphi(x)\frac{1}{|(\nabla_x V(x),\nabla_p p^2)|}\,dS\leq C.\tag{2.14}$$

Using (2.14) we get

$$\int_a^b\int_{\{p^2+V(x)=E\}}\varphi(x)\frac{1}{|(\nabla_x V(x),\nabla_p p^2)|}\,dSdE\leq\int_a^b C dE\leq C|b-a|,\tag{2.15}$$

where C is the constant from (2.14), which is independent of a, b and $\hbar$. By combining (2.6), (2.8), (2.9), (2.13) and (2.15) we get

$$\operatorname{Tr}[\mathbf{1}_{[a,b]}(\mathcal{H})\varphi]\leq C_1|b-a|\hbar^{-d}+C_2\hbar^{1-d}.$$

Which is the desired estimate and this ends the proof. □

The previous proposition gives that we can get the right order in $\hbar$ of the trace if we consider sufficiently small intervals. This will be a crucial point in the proof of Theorem I.3.3.

Furthermore we will be needing a corollary to the Cwikel-Lieb-Rosenbljum (CLR) bound. This corollary is stated in [12, Chapter 4].

Corollary I.2.5. *Let $d \geq 1$, $\gamma > 0$, $\lambda > 0$ and $H = -\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $(V + \frac{\lambda}{2})_-$ in $L^{\frac{d}{2}+\gamma}(\mathbb{R}^d)$ and V_+ in $L^1_{loc}(\mathbb{R}^d)$. Then*

$$\operatorname{Tr}(\mathbf{1}_{(-\infty,-\lambda]}(H)) \leq \frac{2^\gamma}{\lambda^\gamma} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\gamma)}{\Gamma(\frac{d}{2}+\gamma)} \int_{\mathbb{R}^d} (V(x) + \tfrac{\lambda}{2})_-^{\frac{d}{2}+\gamma}\, dx,$$

where Γ is the gamma function.

We will use this corollary in the following way.

Remark I.2.6. Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ and suppose it satisfies Assumption I.2.1. We will later need an a priori estimate on the number $\operatorname{Tr}(\mathbf{1}_{(-\infty,\frac{\varepsilon}{4}]}(H_\hbar))$. To obtain this we will consider the operator $\widetilde{H}_\hbar = -\hbar^2\Delta + V - \frac{\varepsilon}{2}$. Clearly,

$$\operatorname{Tr}(\mathbf{1}_{(-\infty,-\frac{\varepsilon}{4}]}(H_\hbar - \tfrac{\varepsilon}{2})) = \operatorname{Tr}(\mathbf{1}_{(-\infty,-\frac{\varepsilon}{4\hbar^2}]}(-\Delta + \tfrac{V}{\hbar^2} - \tfrac{\varepsilon}{2\hbar^2})). \tag{2.16}$$

If we apply Corollary I.2.5 to the right hand side of (2.16) with $\gamma = 1$ and $\lambda = \frac{\varepsilon}{4\hbar^2}$ we get

$$\begin{aligned} \operatorname{Tr}(\mathbf{1}_{(-\infty,-\frac{\varepsilon}{4\hbar^2}]}(-\Delta + \tfrac{V}{\hbar^2} - \tfrac{\varepsilon}{2\hbar^2})) &\leq c_d \frac{\hbar^2}{\varepsilon} \int_{\mathbb{R}^d} (\tfrac{V(x)}{\hbar^2} - \tfrac{3\varepsilon}{8\hbar^2})_-^{\frac{d}{2}+1}\, dx \\ &= \frac{c_d}{\varepsilon\hbar^d} \int_{\mathbb{R}^d} (V(x) - \tfrac{3\varepsilon}{8})_-^{\frac{d}{2}+1}\, dx. \end{aligned} \tag{2.17}$$

The last integral in (2.17) is finite by Assumption I.2.1 since the support of $(V(x) - \frac{3\varepsilon}{8})_-$ is compact and the function is continuous. Combining (2.16) with (2.17) we get the bound

$$\operatorname{Tr}(\mathbf{1}_{(-\infty,\frac{\varepsilon}{4}]}(H_\hbar)) \leq \frac{C}{\hbar^d}. \tag{2.18}$$

where we have absorbed the integral and ε into the constant.

Trace norm estimates of operators

In this subsection we will list some results on trace norms and estimates of trace norms for operators. First recall that for an operator A the trace norm is

$$\|A\|_1 = \operatorname{Tr}\left([AA^*]^{\frac{1}{2}}\right)$$

and the Hilbert-Schmidt norm is

$$\|A\|_2 = \sqrt{\operatorname{Tr}(AA^*)}$$

Moreover we will use the convention that $\|A\|$ is the operator norm of A. The following lemma is a modification of [15, Lemma 3.9]. The proofs are completely analogous.

Lemma I.2.7. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with V in $C_0^\infty(\mathbb{R}^d)$. Let f be in $C_0^\infty(\mathbb{R})$ and φ in $C_0^\infty(\mathbb{R}^d)$. We let $r \in \{0,1\}$, $\hbar_0 > 0$ and $Q_j = -i\hbar\partial_{x_j}$ for $j \in \{1,\dots,d\}$. Then*

$$\|\varphi Q_j^r f(H_\hbar)\|_1 \le C\hbar^{-d},$$

for all $\hbar$ in $(0,\hbar_0]$. If ψ is a bounded function from $C^\infty(\mathbb{R}^d)$ and $c > 0$ such that

$$\operatorname{dist}[\operatorname{supp}(\varphi), \operatorname{supp}(\psi)] \ge c. \tag{2.19}$$

Then for any N in $\mathbb{N}_0$

$$\|\varphi Q_j^r f(H_\hbar)\psi\|_1 \le C_N\hbar^N,$$

for all $\hbar$ in $(0,\hbar_0]$. Both constants C and C_N depend on the dimension, the functions φ and ψ, the numbers $\hbar_0$, $\|\partial^\alpha V\|_\infty$ for α in $\mathbb{N}_0^d$, $\|\partial^j f\|_\infty$ for j in $\mathbb{N}_0$, c in (2.19) and $\sup(\operatorname{supp}(f))$.

The following theorem is an extension of Theorem 3.12 from the paper [15] as an extra operator has been added. It is less general in the sense that we only consider compactly supported, smooth functions applied to the operator, whereas in the paper more general functions are considered. Again we omit the easy modifications of the proof in [15].

Theorem I.2.8. *Let $\mathcal{H}$ satisfy Assumption I.2.1 with $\Omega = B(0,4R)$ for an $R > 0$. Let f be in $C_0^\infty(\mathbb{R})$ and let $r \in \{0,1\}$, $\hbar_0 > 0$ and $Q_j = -i\hbar\partial_{x_j}$ for $j \in \{1,\dots,d\}$. If φ is in $C_0^\infty(B(0,3R))$ then for any $N \ge 0$*

$$\|\varphi Q_j^r[f(\mathcal{H}) - f(H_\hbar)]\|_1 \le C_N\hbar^N$$

and

$$\|\varphi Q_j^r f(\mathcal{H})\|_1 \le C\hbar^{-d}$$

for all $\hbar$ in $(0,\hbar_0]$, where the constants C_N and C only depend on the dimension and the numbers $\hbar_0$, $\|\partial^j f\|_\infty$ for j in $\mathbb{N}_0$ and $\|\partial^\alpha V\|_\infty$, $\|\partial^\alpha\varphi\|_\infty$ for α in $\mathbb{N}_0^d$.

I.3 Local case

In this section we will present the first step in the proof of Theorem I.1.2 where we prove a local version of the theorem under a non-critical condition. It should be noted that we are not trying to get optimal constants in the following.

Auxiliary bounds

Before we proceed we will consider a simple case where the function applied to the operator is a smooth function with compact support. Moreover we will prove a bound on a Hilbert-Schmidt norm which will prove to be useful.

The first auxiliary result is a simple case of Theorem I.3.3, where we consider the same commutators as in the theorem but we apply a smooth, compactly supported function to our operator instead of the characteristic function.

Lemma I.3.1. *Suppose the operator $\mathcal{H}$ obeys Assumption I.2.1 with $\Omega = B(0,4R)$ for $R > 0$ and let f be in $C_0^\infty(\mathbb{R})$ and $\hbar_0 > 0$. For φ in $C_0^\infty(B(0,3R))$ and $Q_j = -i\hbar\partial_{x_j}$ for $j \in \{1,\dots,d\}$ it holds that*

$$\|[f(\mathcal{H}),\varphi]\|_1 \leq C\hbar^{1-d} \qquad \text{and} \qquad \|[f(\mathcal{H}),\varphi Q_j]\|_1 \leq C\hbar^{1-d},$$

for all $\hbar$ in $(0,\hbar_0]$ and a positive constant C, where C only depend on the dimension, the function φ, the numbers $\hbar_0$, $\|\partial^\alpha V\|_\infty$ for α in $\mathbb{N}_0^d$, $\|\partial^j f\|_\infty$ for j in $\mathbb{N}_0$ and $\sup(\operatorname{supp}(f))$.

Proof. We start by proving the first commutator bound. By Theorem I.2.8 we note that for any $N \geq 0$

$$\|[f(\mathcal{H}),\varphi]\|_1 \leq \|[f(H_\hbar),\varphi]\|_1 + C_N\hbar^N, \tag{2.20}$$

hence we need only prove the bound for the trace norm of $[f(H_\hbar),\varphi]$. Let $g \in C_0^\infty(\mathbb{R})$ such that $g(t)f(t) = f(t)$ and $0 \leq g(t) \leq 1$ for all t in $\mathbb{R}$. Then we have that

$$\begin{aligned}
[f(H_\hbar),\varphi] &= f(H_\hbar)\varphi - \varphi f(H_\hbar) \\
&= g(H_\hbar)f(H_\hbar)\varphi - \varphi g(H_\hbar)f(H_\hbar) + g(H_\hbar)\varphi f(H_\hbar) - g(H_\hbar)\varphi f(H_\hbar) \\
&= g(H_\hbar)[f(H_\hbar),\varphi] + [g(H_\hbar),\varphi]f(H_\hbar).
\end{aligned}$$

These equalities implie that

$$\|[f(H_\hbar),\varphi]\|_1 \leq \|g(H_\hbar)[f(H_\hbar),\varphi]\|_1 + \|[g(H_\hbar),\varphi]f(H_\hbar)\|_1. \tag{2.21}$$

We start by considering the first trace norm $\|g(H_\hbar)[f(H_\hbar),\varphi]\|_1$ and the second can be treated by an analogous argument. Let $\widetilde{\varphi}$ be in $C_0^\infty(\mathbb{R}^d)$ such that $\widetilde{\varphi}\varphi = \varphi$ and $0 \leq \widetilde{\varphi} \leq 1$. Then we have that

$$\begin{aligned}
\|g(H_\hbar)[f(H_\hbar),\varphi]\|_1 &\leq \|g(H_\hbar)\widetilde{\varphi}[f(H_\hbar),\varphi]\|_1 + \|g(H_\hbar)(1-\widetilde{\varphi})f(H_\hbar)\varphi\|_1 \\
&\leq \|g(H_\hbar)\widetilde{\varphi}\|_1\|[f(H_\hbar),\varphi]\| + \|(1-\widetilde{\varphi})f(H_\hbar)\varphi\|_1 \\
&\leq C\hbar^{-d}\|[f(H_\hbar),\varphi]\| + C_N\hbar^N,
\end{aligned} \tag{2.22}$$

for all $N \geq 0$, where we have used Lemma I.2.7 in the last inequality. That

$$\|[f(H_\hbar),\varphi]\| \leq C\hbar, \tag{2.23}$$

is a consequence of the functional calculus for $\hbar$-ΨDOs presented in [14]. It also follows fairly easily from an argument using the Helffer-Sjöstrand formula [6] and the resolvent identities. The estimate on the second term in (2.21) is similar and will be left to the reader. This estimate concludes the proof. □

The next lemma is very similar to the above lemma.

Lemma I.3.2. *Suppose the operator $\mathcal{H}$ obeys Assumption I.2.1 with $\Omega = B(0,4R)$ for $R > 0$ and let f be in $C_0^\infty(\mathbb{R})$ and $\hbar_0 > 0$. For φ in $C_0^\infty(B(0,3R))$ it holds that*

$$\|[\mathcal{H},\varphi]f(\mathcal{H})\|_2 \leq C\hbar^{1-\frac{d}{2}},$$

for all $\hbar$ in $(0,\hbar_0]$ for a positive constant C, where C only depends on the dimension, the function φ, the numbers $\hbar_0$, $\|\partial^\alpha V\|_\infty$ for α in $\mathbb{N}_0^d$, $\|\partial^j f\|_\infty$ for j in $\mathbb{N}_0$ and $\sup(\operatorname{supp}(f))$.

Proof. Let φ_1 be in $C_0^\infty(B(0,3R))$ such that $\varphi_1\varphi = \varphi$ and $0 \leq \varphi_1 \leq 1$. Then by Assumption I.2.1 the commutator $[\mathcal{H},\varphi]$ is local in the sense that

$$[\mathcal{H},\varphi] = [H_\hbar,\varphi]\varphi_1,$$

where $H_\hbar$ is the operator from Assumption I.2.1 i.e. $H_\hbar = -\hbar^2\Delta + V$, where V is in $C_0^\infty(\mathbb{R}^d)$. Therefore there exists a $\lambda_0 \geq 0$ such that $-\lambda_0$ is in the resolvent set of $H_\hbar$ and the operator $H_\hbar + \lambda_0$ is positive (e.g. $\lambda_0 = 1 + \|V\|_\infty$) We then have that

$$\begin{aligned} \|[\mathcal{H},\varphi]f(\mathcal{H})\|_2 =&\|[H_\hbar,\varphi]\varphi_1 R_{H_\hbar}(-\lambda_0)(H_\hbar+\lambda_0)f(\mathcal{H})\|_2 \\ \leq&\|[H_\hbar,\varphi]R_{H_\hbar}(-\lambda_0)\varphi_1(H_\hbar+\lambda_0)f(\mathcal{H})\|_2 \\ &+\|[H_\hbar,\varphi][\varphi_1,R_{H_\hbar}(-\lambda_0)](H_\hbar+\lambda_0)f(\mathcal{H})\|_2, \end{aligned} \tag{2.24}$$

where $R_{H_\hbar}(z) = (H_\hbar - z)^{-1}$. If we now consider each of the terms separately we can for the first term note that by Assumption I.2.1 and Theorem I.2.8 we have

$$\begin{aligned} \|[H_\hbar,\varphi]R_{H_\hbar}(-\lambda_0)\varphi_1(H_\hbar+\lambda_0)f(\mathcal{H})\|_2 &\leq \|[H_\hbar,\varphi]R_{H_\hbar}(-\lambda_0)\|\|\varphi_1(\mathcal{H}+\lambda_0)f(\mathcal{H})\|_2 \\ &\leq c\hbar^{-\frac{d}{2}}\|[H_\hbar,\varphi]R_{H_\hbar}(-\lambda_0)\| \\ &\leq C\hbar^{1-\frac{d}{2}}, \end{aligned} \tag{2.25}$$

where we have used the bound

$$\|[H_\hbar,\varphi]R_{H_\hbar}(-\lambda_0)\| \leq \hbar\sum_{j=1}^d \|(2\varphi_{x_j}Q_j - i\hbar\varphi_{x_jx_j})R_{H_\hbar}(-\lambda_0)\| \leq c\hbar, \tag{2.26}$$

where we have calculated the commutator explicitly. The bound in (2.26) is valid since $\mathcal{D}(H_\hbar) \subset \mathcal{D}(Q_j)$ for all $j \in \{1,\dots,d\}$. Moreover in (2.25) we have used the following estimate

$$\begin{aligned} \|\varphi_1(\mathcal{H}+\lambda_0)f(\mathcal{H})\|_2^2 &= \mathrm{Tr}[\varphi_1(\mathcal{H}+\lambda_0)f(\mathcal{H})^2(\mathcal{H}+\lambda_0)\varphi_1] \\ &\leq \|\varphi_1(\mathcal{H}+\lambda_0)f(\mathcal{H})^2(\mathcal{H}+\lambda_0)\varphi_1\|_1 \leq C\hbar^{-d}, \end{aligned}$$

by Theorem I.2.8. For the other term on the right hand side of (2.24) we note that

$$\|[H_\hbar,\varphi][\varphi_1,R_{H_\hbar}(-\lambda_0)](H_\hbar+\lambda_0)f(\mathcal{H})\|_2 = \|[H_\hbar,\varphi]R_{H_\hbar}(-\lambda_0)[H_\hbar,\varphi_1]f(\mathcal{H})\|_2 \tag{2.27}$$

Let φ_2 be in $C_0^\infty(B(0,3R))$ such that $\varphi_2\varphi_1 = \varphi_1$ and $0 \leq \varphi_2 \leq 1$ and note that by Theorem I.2.8

$$\begin{aligned} &\|[H_\hbar,\varphi]R_{H_\hbar}(-\lambda_0)[H_\hbar,\varphi_1]f(\mathcal{H})\|_2 \\ &\quad= \|[H_\hbar,\varphi]R_{H_\hbar}(-\lambda_0)[H_\hbar,\varphi_1]\varphi_2 f(\mathcal{H})\|_2 \\ &\quad\leq \|[H_\hbar,\varphi]R_{H_\hbar}(-\lambda_0)^{\frac{1}{2}}\|\|R_{H_\hbar}(-\lambda_0)^{\frac{1}{2}}[H_\hbar,\varphi_1]\|\|\varphi_2 f(\mathcal{H})\|_2 \\ &\quad\leq C\hbar^{2-\frac{d}{2}}, \end{aligned} \tag{2.28}$$

where we have used that the commutators $[H_\hbar,\varphi]$ and $[H_\hbar,\varphi_1]$ can be calculated explicitly and that their domains contains the form domain of $H_\hbar$. Combining estimates (2.24), (2.25) and (2.28) we get the desired bound. □

Local case with a non-critical condition

We will now state and prove the local version of the main theorem (Theorem I.1.2) with a non-critical condition. It should be noted that we are only dealing with open balls as the domain in Assumption I.2.1 since when we extend the result we will use them to cover a general open set.

Theorem I.3.3. *Suppose the operator $\mathcal{H}$ acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$ obeys Assumption I.2.1 with $\Omega = B(0,4R)$ for $R > 0$ and*

$$|V(x)| + |\nabla V(x)|^2 + \hbar \geq c, \tag{2.29}$$

for all x in $B(0,2R)$, where $c > 0$. Furthermore, let $\hbar_0$ be a strictly positive number. For φ in $C_0^\infty(B(0,R/2))$ it holds that

$$\|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}),\varphi]\|_1 \leq C\hbar^{1-d} \quad \text{and} \quad \|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}),\varphi Q_j]\|_1 \leq C\hbar^{1-d}, \tag{2.30}$$

for all $\hbar$ in $(0,\hbar_0]$ and $j \in \{1,\dots,d\}$, where $Q_j = -i\hbar\partial_{x_j}$. The constant C only depends on the dimension, the numbers $\|\partial_x^\alpha V\|_\infty$ and $\|\partial_x^\alpha \varphi\|_\infty$ for all α in $\mathbb{N}_0^d$, and the numbers R and c in (2.29).

Proof. We start by proving the first bound in (2.30). We notice that

$$[\mathbf{1}_{(-\infty,0]}(\mathcal{H}),\varphi] = \mathbf{1}_{(-\infty,0]}(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H}) - \mathbf{1}_{(0,\infty)}(\mathcal{H})\varphi\mathbf{1}_{(-\infty,0]}(\mathcal{H}). \tag{2.31}$$

We will consider each of the terms in (2.31) separately and they can be handled with analogous arguments. So we only consider the term $\mathbf{1}_{(-\infty,0]}(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H})$. By (2.29) and continuity, there exists an $\varepsilon > 0$ such that for all E in $[-2\varepsilon, 2\varepsilon]$ we have

$$|E - V(x)| + |\nabla V(x)|^2 + \hbar \geq \frac{c}{2},$$

for all x in $B(0,2R)$. Without loss of generality we can assume $\varepsilon \leq 1$. Let g_1 and g_0 be two functions such that

- $g_1(\mathcal{H}) + g_0(\mathcal{H}) = \mathbf{1}_{(-\infty,0]}(\mathcal{H})$.
- $\operatorname{supp}(g_0) \subset [-\varepsilon, 0]$ and $g_0(t) = 1$ for $t \in [-\varepsilon/2, 0]$.
- $g_1 \in C_0^\infty(\mathbb{R})$.

That g_1 can assumed to be compactly supported is due to the fact that the spectrum of $\mathcal{H}$ is bounded from below. With these functions we get that

$$\begin{aligned} \mathbf{1}_{(-\infty,0]}(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H}) &= g_1(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H}) + g_0(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H}) \\ &= [g_1(\mathcal{H}),\varphi]\mathbf{1}_{(0,\infty)}(\mathcal{H}) + g_0(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H}). \end{aligned} \tag{2.32}$$

For the first term we note that by Lemma I.3.1 we have the estimate:

$$\|[g_1(\mathcal{H}),\varphi]\mathbf{1}_{(0,\infty)}(\mathcal{H})\|_1 \leq \|[g_1(\mathcal{H}),\varphi]\|_1 \leq C\hbar^{1-d}. \tag{2.33}$$

In order to estimate the term $g_0(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H})$ we let f be in $C_0^\infty(\mathbb{R})$ such that $f(t)=1$ on $[-\varepsilon,0]$ and $\operatorname{supp}(f)\subset[-2\varepsilon,\varepsilon]$. Then we have

$$\begin{aligned} g_0(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H}) &= g_0(\mathcal{H})f(\mathcal{H})\varphi\mathbf{1}_{(\varepsilon,\infty)}(\mathcal{H}) + g_0(\mathcal{H})\varphi\mathbf{1}_{(0,\varepsilon]}(\mathcal{H}) \\ &= g_0(\mathcal{H})[f(\mathcal{H}),\varphi]\mathbf{1}_{(\varepsilon,\infty)}(\mathcal{H}) + g_0(\mathcal{H})\varphi\mathbf{1}_{(0,\varepsilon]}(\mathcal{H}). \end{aligned}$$

Again from Lemma I.3.1 we have the estimate:

$$\|g_0(\mathcal{H})[f(\mathcal{H}),\varphi]\mathbf{1}_{(\varepsilon,\infty)}(\mathcal{H})\|_1 \leq \|[f(\mathcal{H}),\varphi]\|_1 \leq C\hbar^{1-d}. \tag{2.34}$$

What remains is to get an estimate of the trace norm of the term $g_0(\mathcal{H})\varphi\mathbf{1}_{(0,\varepsilon]}(\mathcal{H})$. In order to estimate this term we define the following $\hbar$ dependent dyadic decomposition:

$$\chi_{n,\hbar}(t) = \begin{cases} \mathbf{1}_{(\hbar,0]}(t) & n=0 \\ \mathbf{1}_{(-4^n\hbar,-4^{n-1}\hbar]}(t) & n\in\mathbb{N}_0. \end{cases}$$

moreover we let $\widetilde{\chi}_{n,\hbar}(t)=\chi_{n,\hbar}(-t)$. Then there exist $N(\hbar)$ in $\mathbb{N}_0$ such that

$$g_0(\mathcal{H}) = \sum_{n=0}^{N(\hbar)} g_0(\mathcal{H})\chi_{n,\hbar}(\mathcal{H}) \quad \text{and} \quad \mathbf{1}_{(0,\varepsilon]}(\mathcal{H}) = \sum_{m=0}^{N(\hbar)} \mathbf{1}_{(0,\varepsilon]}(\mathcal{H}).\widetilde{\chi}_{m,\hbar}(\mathcal{H}).$$

With these equalities we get the following inequality:

$$\begin{aligned} &\|g_1(\mathcal{H})\varphi\mathbf{1}_{(0,\varepsilon]}(\mathcal{H})\|_1 \leq \sum_{n=0}^{N(\hbar)}\sum_{m=0}^{N(\hbar)} \|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \\ &= \sum_{n=1}^{N(\hbar)}\sum_{m\geq n}^{N(\hbar)} \|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 + \sum_{m=1}^{N(\hbar)}\sum_{n>m}^{N(\hbar)} \|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \\ &\quad + \sum_{n=1}^{N(\hbar)} \|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{0,\hbar}(\mathcal{H})\|_1 + \sum_{m=1}^{N(\hbar)} \|\chi_{0,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 + \|\chi_{0,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{0,\hbar}(\mathcal{H})\|_1. \end{aligned} \tag{2.35}$$

We will start by considering a term from the first double sum. Hence we assume that $m\geq n>0$. The support of $\chi_{n,\hbar}(\mathcal{H})$ is $[-4^n\hbar,-4^{n-1}\hbar]=[-2^{2n}\hbar,-2^{2(n-1)}\hbar]$, which contains the point $-2^{2n-1}\hbar$, and similarly the support of $\widetilde{\chi}_{m,\hbar}(\mathcal{H})$ is $[4^{m-1}\hbar,4^m\hbar]=[2^{2(m-1)}\hbar,2^{2m}\hbar]$. We note that we can make the following estimate, using the spectral theorem.

$$\begin{aligned} &\|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \\ &\quad= \|\chi_{n,\hbar}(\mathcal{H})\varphi(\mathcal{H}+2^{2n-1}\hbar)\widetilde{\chi}_{m,\hbar}(\mathcal{H})(\mathcal{H}+2^{2n-1}\hbar)^{-1}\|_1 \\ &\quad\leq \|\chi_{n,\hbar}(\mathcal{H})\varphi(\mathcal{H}+2^{2n-1}\hbar)\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1(2^{2(m-1)}\hbar+2^{2n-1}\hbar)^{-1} \\ &\quad\leq (2^{2(m-1)}\hbar+2^{2n-1}\hbar)^{-1}\big\{\|\chi_{n,\hbar}(\mathcal{H})(\mathcal{H}+2^{2n-1}\hbar)\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \\ &\qquad + \|\chi_{n,\hbar}(\mathcal{H})[\varphi,\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1\big\} \\ &\quad\leq \frac{2^{2n-1}}{2^{2(m-1)}+2^{2n-1}}\|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \\ &\qquad + (2^{2(m-1)}\hbar+2^{2n-1}\hbar)^{-1}\|\chi_{n,\hbar}(\mathcal{H})[\varphi,\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1, \end{aligned}$$

With
$$a := \frac{2^{2n-1}}{2^{2(m-1)} + 2^{2n-1}}.$$
The above calculation implies that
$$(1-a)\|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \leq (2^{2(m-1)}\hbar + 2^{2n-1}\hbar)^{-1}\|\chi_{n,\hbar}(\mathcal{H})[\varphi,\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1.$$
This implies the following estimate
$$\begin{aligned}\|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \leq &(1-a)^{-1}(2^{2(m-1)}\hbar + 2^{2n-1}\hbar)^{-1}\|\chi_{n,\hbar}(\mathcal{H})[\varphi,\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \\ \leq &\frac{1}{2^{2(m-1)}\hbar}\|\chi_{n,\hbar}(\mathcal{H})[\varphi,\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1\end{aligned} \tag{2.36}$$

Due to the double sum in (2.35) we need to repeat the argument. By an analogous argument the following estimate holds
$$\|\chi_{n,\hbar}(\mathcal{H})[\varphi,\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \leq \frac{1}{2^{2(m-1)}\hbar}\|\chi_{n,\hbar}(\mathcal{H})[[\varphi,\mathcal{H}],\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1. \tag{2.37}$$
By combining (2.36) and (2.37) we get that
$$\|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \leq \frac{1}{4^{2(m-1)}\hbar^2}\|\chi_{n,\hbar}(\mathcal{H})[[\varphi,\mathcal{H}],\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1. \tag{2.38}$$
We will now prove that
$$\|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \leq C\frac{4^{\frac{m+n}{2}}\hbar^{3-d}}{4^{2(m-1)}\hbar^2} = \frac{16C}{4^{\frac{3}{4}m-\frac{1}{2}n}}\hbar^{1-d}, \tag{2.39}$$
for $m \geq n \geq 1$ is true. By Assumption I.2.1 we have that
$$[[\varphi,\mathcal{H}],\mathcal{H}] = [[\varphi,H_\hbar],H_\hbar], \tag{2.40}$$
since we have assumed that the operator $\mathcal{H}$ acts on $C_0^\infty(B(0,4R))$ as the operator $H_\hbar$. By a calculation we note that
$$[[\varphi,H_\hbar],H_\hbar] = \hbar^2\sum_{j=1}^{d}\sum_{l=1}^{d}\big[-2(Q_l\varphi_{x_jx_l}Q_j + Q_j\varphi_{x_jx_l}Q_l) + 2\varphi_{x_j}V_{x_j} + \hbar^2\varphi_{x_jx_jx_lx_l}\big], \tag{2.41}$$
where $Q_j = -i\hbar\partial_{x_j}$ and $\varphi_{x_j}(x) = (\partial_{x_j}\varphi)(x)$. With this form of the double commutator we have
$$\|R_{H_\hbar}(i)[[\varphi,\mathcal{H}],\mathcal{H}]R_{H_\hbar}(i)\| \leq c\hbar^2, \tag{2.42}$$
where $R_{H_\hbar}(i) = (H_\hbar - i)^{-1}$ is the resolvent at the point i, since $\mathcal{D}(H_\hbar) \subset \mathcal{D}(Q_j)$ for all $j \in \{1,\dots,d\}$. In order to estimate the right hand side in (2.37) let ψ be in $C_0^\infty(\mathbb{R}^d)$ such that $\psi(x) = 1$ for all x in $\operatorname{supp}(\varphi)$ and $\operatorname{supp}(\psi) \subset B(0,R/2)$. As the double commutator is local, which follows from (2.40) and (2.41), we have
$$\|\chi_{n,\hbar}(\mathcal{H})[[\varphi,\mathcal{H}],\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 = \|\chi_{n,\hbar}(\mathcal{H})\psi[[\varphi,\mathcal{H}],\mathcal{H}]\psi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1. \tag{2.43}$$

By inserting two resolvents, applying a Cauchy-Schwarz inequality and the estimate (2.42), we have

$$\begin{aligned}
&\|\chi_{n,\hbar}(\mathcal{H})\psi[[\varphi,\mathcal{H}],\mathcal{H}]\psi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \\
&\qquad= \|\chi_{n,\hbar}(\mathcal{H})\psi(H_\hbar-i)R_{H_\hbar}(i)[[\varphi,\mathcal{H}],\mathcal{H}]R_{H_\hbar}(i)(H_\hbar-i)\psi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \\
&\qquad\leq c\hbar^2\|\chi_{n,\hbar}(\mathcal{H})\psi(H_\hbar-i)\|_2\|(H_\hbar-i)\psi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_2.
\end{aligned} \tag{2.44}$$

If we consider the first of the two Hilbert-Schmidt norms we have

$$\|\chi_{n,\hbar}(\mathcal{H})\psi(H_\hbar-i)\|_2 \leq \|\chi_{n,\hbar}(\mathcal{H})(H_\hbar-i)\psi\|_2 + \|\chi_{n,\hbar}(\mathcal{H})[\psi,H_\hbar]\|_2. \tag{2.45}$$

By Assumption I.2.1 and Proposition I.2.3 we have

$$\|\chi_{n,\hbar}(\mathcal{H})(H_\hbar-i)\psi\|_2 \leq 2\|\chi_{n,\hbar}(\mathcal{H})\psi\|_2 = 2\sqrt{\mathrm{Tr}[\psi\chi_{n,\hbar}(\mathcal{H})^2\psi]} \leq 2\sqrt{C4^n\hbar^{1-d}}. \tag{2.46}$$

For the second term in (2.45) let f be in $C_0^\infty(\mathbb{R})$ such that $f(t)=1$ for $t\in[-\frac{3}{2}\varepsilon,\frac{3}{2}\varepsilon]$ and $f(t)=0$ for $|t|\geq 2\varepsilon$. Then we have the bound

$$\|\chi_{n,\hbar}(\mathcal{H})[\psi,H_\hbar]\|_2 = \|\chi_{n,\hbar}(\mathcal{H})f(\mathcal{H})[\psi,H_\hbar]\|_2 \leq \|f(\mathcal{H})[\psi,H_\hbar]\|_2 \leq c\hbar^{1-\frac{d}{2}}.$$

by Lemma I.3.2. Combining this estimate with (2.45) and (2.46) we get

$$\|\chi_{n,\hbar}(\mathcal{H})\psi(H_\hbar-i)\|_2 \leq \sqrt{\widetilde{C}4^n\hbar^{1-d}}. \tag{2.47}$$

By analogous estimates we also get

$$\|(H_\hbar-i)\psi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_2 \leq \sqrt{\widetilde{C}4^m\hbar^{1-d}}. \tag{2.48}$$

Now by combing(2.47) and (2.48) with (2.43) and (2.44) we get

$$\|\chi_{n,\hbar}(\mathcal{H})[[\varphi,\mathcal{H}],\mathcal{H}]\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \leq C4^{\frac{n+m}{2}}\hbar^{1-d}. \tag{2.49}$$

By (2.38) and (2.49) we have the estimate (2.39). Using (2.39) we can now estimate the double sum

$$\sum_{n=1}^{N(\hbar)}\sum_{m\geq n}^{N(\hbar)}\|\chi_{n,\hbar}(\mathcal{H})\varphi\widetilde{\chi}_{m,\hbar}(\mathcal{H})\|_1 \leq \sum_{n=1}^{\infty}\sum_{m\geq n}^{\infty}\frac{C}{4^{\frac{3}{4}m-\frac{1}{2}n}}\hbar^{1-d} \leq \widetilde{C}\hbar^{1-d}.$$

The remaining terms in (2.35) can be estimated in a similar way. The second double sum is estimated by the same argument but with the roles of m and n interchanged. To estimate the two single sums we only need to introduce one commutator to make the sum converge and then use the same arguments as for the double sum. For the last term we use a Cauchy-Schwarz inequality. Adding all our estimates up we have the bound

$$\|g_1(\mathcal{H})\varphi\mathbf{1}_{(0,\varepsilon]}(\mathcal{H})\|_1 \leq C\hbar^{1-d}. \tag{2.50}$$

By combining (2.50) with (2.33) and (2.34) we get the estimate

$$\|\mathbf{1}_{(-\infty,0]}(\mathcal{H})\varphi\mathbf{1}_{(0,\infty)}(\mathcal{H})\|_1 \leq C\hbar^{1-d}. \tag{2.51}$$

Since the trace norm satisfies the equality $\|A\|_1 = \|A^*\|_1$ we also have the bound,

$$\|\mathbf{1}_{(0,\infty)}(\mathcal{H})\varphi\mathbf{1}_{(-\infty,0]}(\mathcal{H})\|_1 \leq C\hbar^{1-d}. \tag{2.52}$$

By combining (2.51) and (2.52) with (2.31) we obtain the desired bound for the first part of (2.30).

For the second estimate in (2.30) we essentially repeat the argument. The main difference occurs when the double commutator $[[\varphi,\mathcal{H}],\mathcal{H}]$ is calculated. In this case, one has to calculate the commutator $[[\varphi Q_i,\mathcal{H}],\mathcal{H}]$. This can be done and one obtains the result

$$\begin{aligned}
[[\varphi Q_i,&\mathcal{H}],\mathcal{H}] = [[\varphi Q_i, H_\hbar], H_\hbar] \\
=&\hbar^2 \sum_{j=1}^{d} 2\varphi_{x_j} V_{x_i} Q_j + 2\varphi_{x_j} V_{x_j} Q_i - 2i\hbar\varphi_{x_j} V_{x_j x_i} - i\hbar\varphi_{x_j x_j} V_{x_i} \\
&+ \hbar^2 \sum_{k=1}^{d} \Big\{ 2(\varphi V_{x_i})_{x_k} Q_k - i\hbar(\varphi V_{x_i})_{x_k x_k} + \sum_{j=1}^{d} \big[-4Q_k \varphi_{x_j x_k} Q_i Q_j \\
&- 4i\hbar\varphi_{x_j x_k x_k} Q_i Q_j + 2i\hbar\varphi_{x_j x_k} Q_i Q_j + 2\hbar\varphi_{x_j x_j x_k} Q_i Q_k + \hbar^2 \varphi_{x_j x_j x_k x_k} Q_i \big] \Big\},
\end{aligned}$$

where we have used Assumption I.2.1. From this form we can note that again we have a bound of the type

$$\|R_{H_\hbar}(i)[[\varphi Q_i,\mathcal{H}],\mathcal{H}]R_{H_\hbar}(i)\| \leq c\hbar^2,$$

since $\mathcal{D}(H_\hbar) \subset \mathcal{D}(Q_j Q_i)$ for all $j, i \in \{1,\dots,d\}$. From here the proof proceeds as above just with some extra terms to consider. We omit the details. ☐

Local case without non-critical condition

In this subsection we will apply the multiscale techniques of [15] (see also [9, 10]). Using this approach will allow us to remove the non-critical assumption on the potential. Before we state and prove our theorem we will need a lemma and a remark.

Lemma I.3.4. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let f be a function in $C^1(\bar{\Omega})$ such that $f > 0$ on $\bar{\Omega}$ and assume that there exists ρ in $(0,1)$ such that*

$$|\nabla_x f(x)| \leq \rho, \tag{2.53}$$

for all x in Ω.

Then

i) There exists a sequence $\{x_k\}_{k=0}^{\infty}$ in Ω such that the open balls $B(x_k, f(x_k))$ form a covering of Ω. Furthermore, there exists a constant N_ρ, depending only on the constant ρ, such that the intersection of more than N_ρ balls is empty.

ii) *One can choose a sequence* $\{\varphi_k\}_{k=0}^{\infty}$ *such that* $\varphi_k \in C_0^\infty(B(x_k, f(x_k)))$ *for all* k *in* $\mathbb{N}_0$. *Moreover, for all multiindices* α *and all* k *in* $\mathbb{N}_0$

$$|\partial_x^\alpha \varphi_k(x)| \leq C_\alpha f(x_k)^{-|\alpha|},$$

and

$$\sum_{k=1}^{\infty} \varphi_k(x) = 1,$$

for all x *in* Ω.

This lemma is taken from [15] where it is Lemma 5.4. The proof is analogous to the proof of [8, Theorem 1.4.10].

Remark I.3.5. A crucial step in the following proof is scaling of our operator. Let D_f and T_z, for $f > 0$ and $z \in \mathbb{R}^d$, be the unitary dilation and translation operators defined by

$$(D_f u)(x) = f^{\frac{d}{2}} u(fx),$$

and

$$(T_z u)(x) = u(x + z),$$

for u in $L^2(\mathbb{R}^d)$. We let f be a positive number and suppose $\mathcal{H}$ satisfies Assumption I.2.1 with Ω being the open ball $B(z, f)$. We will consider the operator

$$\widetilde{\mathcal{H}} = f^{-2}(T_z U_f)\mathcal{H}(T_z U_f)^*.$$

The operator $\widetilde{\mathcal{H}}$ is selfadjoint and lower semibounded since $\mathcal{H}$ is assumed to be selfadjoint and lower semibounded which is the first part of Assumption I.2.1. The last part of the assumption will be fulfilled with the set $B(0,1)$, the function $\widetilde{V}_f(x) = f^{-2}V(fx + z)$ and a scaled $\hbar$ which we will call h. To see this note that for $\varphi \in C_0^\infty(B(0,1))$ it holds that $(T_z U_f)^*\varphi$ is an element of $C_0^\infty(B(z,f))$ since

$$(T_z U_f)^*\varphi(x) = f^{-\frac{d}{2}}\varphi\big(\tfrac{x-z}{f}\big).$$

Hence we have that, using Assumption I.2.1 for $\mathcal{H}$

$$\widetilde{\mathcal{H}}\varphi = -\left(\frac{\hbar}{f^2}\right)^2 \Delta\varphi(x) + f^{-2}V(fx+z)\varphi(x), \tag{2.54}$$

This calculation shows that our operator $\widetilde{\mathcal{H}}$ satisfies Assumption I.2.1 with $\Omega = B(0,1)$, $V_{loc} = \widetilde{V}_f$ and the new "Planck's constant" $h = \frac{\hbar}{f^2}$.

We are now ready to remove the non-critical assumption.

Theorem I.3.6. *Suppose the operator* $\mathcal{H}$ *acting in* $L^2(\mathbb{R}^d)$ *with* $d \geq 2$ *obeys Assumption I.2.1 with an open set* $\Omega \subset \mathbb{R}^d$ *and let* $\hbar_0$ *be a strictly positive number. For* ψ *in* $C_0^\infty(\Omega)$ *it holds that*

$$\|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \psi]\|_1 \leq C\hbar^{1-d} \quad \text{and} \quad \|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \psi Q_j]\|_1 \leq C\hbar^{1-d}, \tag{2.55}$$

for all $\hbar$ *in* $(0, \hbar_0]$, *where* C *is a positive constant.*

Proof. First note that by assumption ψ is in $C_0^\infty(\Omega)$. Hence there exists $\varepsilon > 0$ such that

$$\mathrm{dist}(\mathrm{supp}(\psi), \partial\Omega) > \varepsilon.$$

We define the function f by

$$f(x) = A^{-1}\left[V(x)^2 + |\nabla_x V(x)|^4 + \hbar^2\right]^{\frac{1}{4}} \qquad A > 0, \tag{2.56}$$

where we have to choose a sufficiently large A. It can be noted that f is a positive function due to $\hbar$ being a fixed positive number. We will need to choose A such that

$$f(x) \le \frac{\varepsilon}{9} \quad \text{and} \quad |\nabla_x f(x)| \le \rho < \frac{1}{8}. \tag{2.57}$$

Since V is smooth with compact support A can be chosen such that (2.57) is satisfied. The construction of f allows us to choose A such that the bounds are valid for all $\hbar$ in $(0, \hbar_0]$. Hence A will be independent of $\hbar$, for $\hbar$ in the interval $(0, \hbar_0]$. Moreover, we observe that this construction gives the estimates

$$|V(x)| \le Af(x)^2, \quad \text{and} \quad |\partial_{x_i} V(x)| \le Af(x). \tag{2.58}$$

This observation will prove useful for controlling bounds on some derivatives.

By Lemma I.3.4 with the set Ω and our function f there exists a sequence $\{x_k\}_{k=0}^\infty$ in Ω such that $\Omega \subset \bigcup_{k=0}^\infty B(x_k, f(x_k))$ and there exists a constant $N_{\frac{1}{8}}$ in $\mathbb{N}$ such that

$$\bigcap_{k\in\mathcal{I}} B(x_k, f(x_k)) = \emptyset,$$

for all $\mathcal{I} \subset \mathbb{N}$ such that $\#\mathcal{I} > N_{\frac{1}{8}}$. Moreover, there exists a sequence $\{\varphi_k\}_{k=0}^\infty$ such that $\varphi_k \in C_0^\infty(B(x_k, f(x_k)))$,

$$|\partial_x^\alpha \varphi_k| \le C_\alpha f(x_k)^{-|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^d,$$

and

$$\sum_{k=1}^\infty \varphi_k(x) = 1 \qquad \forall x \in \Omega.$$

Since $\mathrm{supp}(\psi) \subset \Omega$ the union $\bigcup_{k=0}^\infty B(x_k, f(x_k))$ forms an open cover of $\mathrm{supp}(\psi)$ by assumption the support is compact hence there exists $\mathcal{I} \subset \mathbb{N}$ such that $\#\mathcal{I} < \infty$ and

$$\Omega \subset \bigcup_{k\in\mathcal{I}} B(x_k, f(x_k)).$$

We can assume that each ball has a nontrivial intersection with Ω. Since at most $N_{\frac{1}{8}}$ balls intersect nontrivially we can without loss of generality assume that

$$\sum_{k\in\mathcal{I}} \varphi_k(x) = 1 \qquad \forall x \in \mathrm{supp}(\psi).$$

From this we get the following estimate:

$$\|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \psi]\|_1 \le \sum_{k\in\mathcal{I}} \|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \varphi_k\psi]\|_1. \tag{2.59}$$

We will consider each term separately. We can note that the function $\varphi_k\psi$ is smooth and supported in the ball $B(x_k, f(x_k))$. The idea is now to make a unitary conjugation of our commutator such that a non-critical assumption is obtained.

Let T_{x_k} be the unitary translation with x_k and let $U_{f(x_k)}$ be the unitary scaling operator with $f(x_k)$. We will use the notation from Remark I.3.5 and let

$$\widetilde{\varphi_k\psi}(x) = \varphi_k\psi(f(x_k)x + x_k).$$

Since the trace norm is invariant under unitary conjugation we have that

$$\begin{aligned}
&\|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \varphi_k\psi]\|_1 \\
&\qquad = f(x_k)^2\|f(x_k)^{-2}(T_{x_k}U_{f(x_k)})[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \varphi_k\psi](T_{x_k}U_{f(x_k)})^*\|_1 \\
&\qquad = f(x_k)^2\|[\mathbf{1}_{(-\infty,0]}(\widetilde{\mathcal{H}}), \widetilde{\varphi_k\psi}]\|_1.
\end{aligned}$$

By Remark I.3.5, $\widetilde{\mathcal{H}}$ satisfies Assumption I.2.1 with $h = \hbar f(x_k)^{-2}$, $\widetilde{V}_f$ and $B(0,8)$, since by construction we have that $B(x_k, 8f(x_k)) \subset \Omega$.

For all x in $B(x_k, 8f(x_k))$ we have that

$$\begin{aligned}
f(x) &= f(x) - f(x_k) + f(x_k) \\
&\geq -\max_{c\in[0,1]}|\nabla_x f(cx + (1-c)x_k)||x - x_k| + f(x_k) \\
&\geq (1 - 8\rho)f(x_k).
\end{aligned} \tag{2.60}$$

Analogously we can note that

$$f(x) \leq (1 + 8\rho)f(x_k), \tag{2.61}$$

for all x in $B(x_k, 8f(x_k))$. We note that the numbers $1 \pm 8\rho$ are independent of k. The aim is to use Theorem I.3.3. To see that the non-critical assumption (I.3.6) is satisfied we note that

$$\begin{aligned}
&|\widetilde{V}_f(x)| + h + |\nabla_x\widetilde{V}_f(x)|^2 \\
&\qquad = f(x_k)^{-2}\left(|V(f(x_k)x + x_k)| + \hbar + |(\nabla_x V)(f(x_k)x + x_k)|^2\right) \\
&\qquad = f(x_k)^{-2}\left(\sqrt{|V(f(x_k)x + x_k)|^2} + \sqrt{\hbar^2} + \sqrt{|(\nabla_x V)(f(x_k)x + x_k)|^4}\right) \\
&\qquad \geq f(x_k)^{-2}\left(|V(f(x_k)x + x_k)|^2 + \hbar^2 + |(\nabla_x V)(f(x_k)x + x_k)|^4\right)^{\frac{1}{2}} \\
&\qquad = f(x_k)^{-2}A^2 f(f(x_k)x + x_k)^2 \\
&\qquad \geq cA^2 > 0.
\end{aligned}$$

Here we used (2.60) and (2.61) to get the cancelation. Therefore the assumption (I.3.6) is valid for the operator $\widetilde{\mathcal{H}}$. In order to ensure uniformity of the error terms from Theorem I.3.3 we need the derivatives of $\widetilde{V}_f$ and $\widetilde{\varphi_k\psi}$ to be bounded uniformly in k. We note that

$$|\partial_x^\alpha\widetilde{V}_f| = |f(x_k)^{|\alpha|-2}(\partial_x^\alpha V)(f(x_k)x + x_k)| \leq C_\alpha,$$

where we in the cases of $\alpha = 0$ and $|\alpha| = 1$ use the estimates from equation (2.58). For $\widetilde{\varphi_k\psi}$ we note that

$$\begin{aligned}|\partial_x^\alpha \widetilde{\varphi_k\psi}| &= |f(x_k)^{|\alpha|} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (\partial_x^\beta \varphi_k)(f(x_k)x + x_k)(\partial_x^{\alpha-\beta}\psi)(f(x_k)x + x_k)| \\ &\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} f(x_k)^{|\alpha-\beta|} |(\partial_x^{\alpha-\beta}\psi)(f(x_k)x + x_k)| \leq \widetilde{C}_\alpha.\end{aligned}$$

Lastly we need to verify that the new semiclassical parameter is bounded. By the choice of A we have

$$h_k = \frac{\hbar}{f(x_k)^2} \leq A^2,$$

where we have used the definition of the function f (2.56). Hence we are in a situation where we can use Theorem I.3.3 which implies that

$$\begin{aligned}\|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}), \varphi_k\psi]\|_1 &= f(x_k)^2 \|[\mathbf{1}_{(-\infty,0]}(\widetilde{\mathcal{H}}), \widetilde{\varphi_k\psi}]\|_1 \\ &\leq f(x_k)^2 c \left(\frac{\hbar}{f(x_k)^2}\right)^{1-d} \\ &\leq C\hbar^{1-d} \int_{B(x_k, f(x_k))} f(x)^d \, dx,\end{aligned} \tag{2.62}$$

with C independent of k in $\mathcal{I}$ and where we also have used (2.60) and (2.61) in the last estimate. Since f is a bounded function and at most $N_{\frac{1}{8}}$ of the balls $B(x_k, f(x_k))$ can intersect non-empty we get the estimate

$$\sum_{k\in\mathcal{I}} \int_{B(x_k, f(x_k))} f(x)^d \, dx \leq C(N_{\frac{1}{8}})\mathrm{Vol}(\Omega). \tag{2.63}$$

By combining (2.62) and (2.63) with (2.59) we get the estimate

$$\|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}_\hbar), \psi]\|_1 \leq \sum_{k\in\mathcal{I}} \|[\mathbf{1}_{(-\infty,0]}(\mathcal{H}_\hbar), \varphi_k\psi]\|_1 \leq C\hbar^{1-d},$$

where C depends on the set Ω, the number $N_{\frac{1}{8}}$, the derivatives of ψ and the potential V. We now need to prove the second bound in (2.55). The proof of this bound is completely analogous. Notice that when the unitary conjugation is made one should multiply by $f(x_k)^3 f(x_k)^{-3}$ instead of $f(x_k)^2 f(x_k)^{-2}$ due to the extra derivative. This ends the proof. $\square$

I.4 of Theorem I.1.2 and Corollary I.1.3

In this section we will use the results obtained in the previous sections to prove Theorem I.1.2 and then use this theorem to prove Corollary I.1.3. First the proof of Theorem I.1.2:

Proof (Proof of Theorem I.1.2). Recall that we are in the setting with $H_\hbar = -\hbar^2\Delta + V$ being a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with $d \geq 2$, where V satisfies Assumption I.1.1 and $\hbar$ is bounded by a strictly positive number $\hbar_0$. We will here prove the following bounds

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), x_i]\|_1 \leq C\hbar^{1-d} \quad \text{and} \quad \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), Q_j]\|_1 \leq C\hbar^{1-d}, \tag{2.64}$$

where $Q_j = -i\hbar\partial_{x_i}$ and $j \in \{1, \dots, d\}$.

Without loss of generality we can assume that V attains negative values. If not, then $H_\hbar$ would be a positive operator with purely positive spectrum which implies both commutators would be zero and hence satisfy the estimate.

By assumption we have the open set Ω_V for which $V \in C^\infty(\Omega_V)$ and the bounded set Ω_ε satisfying that $\overline{\Omega}_\varepsilon \subset \Omega_V$. Hence we can find an open set U satisfying that it is bounded and

$$\Omega_\varepsilon \subset\subset U \subset\subset \Omega_V,$$

where $\subset\subset$ means compactly imbedded. We let χ be in $C_0^\infty(U)$ such that $0 \leq \chi \leq 1$ and $\chi(x) = 1$ for all x in $\overline{\Omega}_\varepsilon$. Moreover we let $\widetilde{\chi}$ be in $C_0^\infty(\Omega_V)$ such that $0 \leq \widetilde{\chi} \leq 1$ and $\widetilde{\chi}(x) = 1$ for all x in $\overline{U}$. With these sets and functions we have that our operator $H_\hbar$ satisfies Assumption I.2.1 with $\Omega = U$ and $V_{loc} = V\widetilde{\chi}$. With this setup we are ready to prove the bounds in (2.64).

We will now consider the first commutator in (2.64) and note that

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), x_i]\|_1 \leq \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), \chi x_i]\|_1 + \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), (1-\chi)x_i]\|_1. \tag{2.65}$$

For the first term in (2.65) we are in a situation where we can use Theorem I.3.6 since χx_i is in $C_0^\infty(U)$ and $H_\hbar$ satisfies Assumption I.2.1 with $\Omega = U$. Then the theorem gives us the bound:

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), \chi x_i]\|_1 \leq C\hbar^{1-d}. \tag{2.66}$$

For the other term we note that

$$\begin{aligned}
\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), (1-\chi)x_i]\|_1 &\leq \|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i\|_1 + \|(1-\chi)x_i\mathbf{1}_{(-\infty,0]}(H_\hbar)\|_1 \\
&= 2\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i\|_1.
\end{aligned}$$

By a Cauchy-Schwarz inequality we have that

$$\begin{aligned}
\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i\|_1 &\leq \|\mathbf{1}_{(-\infty,0]}(H_\hbar)\|_2\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i\|_2 \\
&= \mathrm{Tr}(\mathbf{1}_{(-\infty,0]}(H_\hbar))^{\frac{1}{2}}\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i\|_2.
\end{aligned} \tag{2.67}$$

The first term squared can be estimated by a constant times $\hbar^{-\frac{d}{2}}$ by Remark I.2.6. For the second term we calculate the trace in a basis of eigenfunctions for $H_\hbar$.

$$\begin{aligned}
\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i\|_2^2 &= \mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i^2(1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)] \\
&= \sum_{\lambda_n \leq \frac{\varepsilon}{4}} \langle \mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i^2(1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n, \psi_n\rangle \\
&= \sum_{\lambda_n \leq 0} \|(1-\chi)x_i\psi_n\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned} \tag{2.68}$$

In order to estimate the $L^2(\mathbb{R}^d)$-norm, we let $d(x) = \operatorname{dist}(x, \Omega_\varepsilon)$. For all x in the support of $1-\chi$ we have that $d(x) > 0$ since Ω_ε is a proper subset of the support of χ. We can note that V is an element of $L^1_{loc}(\mathbb{R}^d)$ hence Lemma I.1.1 gives the existence of a constant C only depending on V such that for all eigenvectors ψ_n with eigenvalue less than $\frac{\varepsilon}{4}$ we have the estimate

$$\|e^{\delta d\hbar^{-1}}\psi_n\|_{L^2(\mathbb{R}^d)} \leq C,$$

where $\delta = \frac{\sqrt{\varepsilon}}{8}$. With these observations we can note that for all norms in the last sum of (2.68) we have for all N in $\mathbb{N}_0$ the bound

$$\begin{aligned}\|(1-\chi)x_i\psi_n\|^2_{L^2(\mathbb{R}^d)} &\leq \|(1-\chi)x_i e^{-\delta\varphi\hbar^{-1}}\|^2_\infty \|e^{\delta\varphi\hbar^{-1}}\psi_n\|^2_{L^2(\mathbb{R}^d)} \\ &\leq C\|(1-\chi)x_i(\frac{\hbar}{\delta\varphi})^N(\frac{\delta\varphi}{\hbar})^N e^{-\delta\varphi\hbar^{-1}}\|^2_\infty \\ &\leq C_N\hbar^{2N},\end{aligned} \tag{2.69}$$

where the constant depends on the choice of the set U, $\delta(\varepsilon)$ and the power N. If we now combine this estimate with (2.68) we get

$$\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i\|_2^2 \leq C\hbar^{2N-d}, \tag{2.70}$$

where we have used Remark I.2.6 to estimate the number of terms in the sum in (2.68). Combining (2.70) with (2.67) we get

$$\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)x_i\|_1 \leq C_N\hbar^{N-d}.$$

Now by combining this bound with (2.66) we get the desired bound in (2.64).

For the second bound in (2.64) we take the same χ as above and note that

$$\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), Q_i]\|_1 \leq \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), \chi Q_i]\|_1 + \|[\mathbf{1}_{(-\infty,0]}(H_\hbar), (1-\chi)Q_i]\|_1.$$

The first term can as above be estimated by applying Theorem I.3.6. The second term will be proven to be small as before. We note that

$$\begin{aligned}\|[\mathbf{1}_{(-\infty,0]}(H_\hbar), (1-\chi)Q_i]\|_1 &\leq \|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)Q_i\|_1 + \|(1-\chi)Q_i\mathbf{1}_{(-\infty,0]}(H_\hbar)\|_1 \\ &\leq 2\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)Q_i\|_1 + \hbar\|\mathbf{1}_{(-\infty,0]}(H_\hbar)\partial_{x_i}\chi\|_1.\end{aligned}$$

The second term is on the same form as the left hand side of (2.67) and hence can be treated as above. For the first term we have that

$$\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)Q_i\|_1 \leq \|\mathbf{1}_{(-\infty,0]}(H_\hbar)\|_2\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)Q_i\|_2.$$

The first term can be controlled by Remark I.2.6. For the second term we have that

$$\begin{aligned}\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)Q_i\|_2 =& \|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)Q_i^2(1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\|_1^{\frac{1}{2}} \\ \leq& \|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)(H_\hbar+c)(1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\|_1^{\frac{1}{2}},\end{aligned}$$

where

$$c = 1 - \inf_{x \in \Omega_\varepsilon}(V(x)). \tag{2.71}$$

If we now calculate the trace norm by choosing a basis of eigenfunctions of $H_\hbar$ we get that

$$\begin{aligned}
&\|\mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)(H_\hbar + c)(1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\|_1 \\
&\qquad = \sum_{\lambda_n \leq \frac{\varepsilon}{4}} \langle \mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)(H_\hbar + c)(1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n, \psi_n\rangle.
\end{aligned}$$

If we consider just one of the terms we have by the IMS formula that

$$\begin{aligned}
&\langle \mathbf{1}_{(-\infty,0]}(H_\hbar)(1-\chi)(H_\hbar + c)(1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n, \psi_n\rangle \\
&\quad = c\langle (1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n, (1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n\rangle \\
&\qquad + \langle H_\hbar(1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n, (1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n\rangle \\
&\quad = c\langle (1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n, (1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n\rangle \\
&\qquad + \langle (1-\chi)H_\hbar\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n, (1-\chi)\mathbf{1}_{(-\infty,0]}(H_\hbar)\psi_n\rangle \\
&\qquad + \hbar^2 \int_{\mathbb{R}^d} |\nabla_x \chi|^2 |\psi_n|^2 \, dx \\
&\quad \leq (c + \lambda_n)\|(1-\chi)\psi_n\|^2_{L^2(\mathbb{R}^d)} + \hbar^2 \||\nabla_x\chi|\psi_n\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

We can note that the number $c + \lambda_n$ is less than or equal to $c + \frac{\varepsilon}{2}$ for the possible values of λ_n. For the two norms we can use the same trick as in (2.69) and thereby show that they are small in $\hbar$. This completes the proof. □

Now the proof of the corollary:

Proof (of Corollary I.1.3). We start by observing that the operator

$$[\mathbf{1}_{(-\infty,0]}(H_\hbar), x],$$

is a trace class operator by Theorem I.1.2, where the commutator is interpreted as the sum of the commutators with each entry in the vector x. Moreover we note that

$$\begin{aligned}
[\mathbf{1}_{(-\infty,0]}(H_\hbar), e^{i\langle t,x\rangle}] &= \mathbf{1}_{(-\infty,0]}(H_\hbar)e^{i\langle t,x\rangle} - e^{i\langle t,x\rangle}\mathbf{1}_{(-\infty,0]}(H_\hbar) \\
&= e^{i\langle t,x\rangle}\Big(e^{-i\langle t,x\rangle}\mathbf{1}_{(-\infty,0]}(H_\hbar)e^{i\langle t,x\rangle} - \mathbf{1}_{(-\infty,0]}(H_\hbar)\Big).
\end{aligned} \tag{2.72}$$

We define the function $f : \mathbb{R} \to \mathcal{B}(L^2(\mathbb{R}^d))$, where $\mathcal{B}(L^2(\mathbb{R}^d))$ are the bounded operators on $L^2(\mathbb{R}^d)$, by

$$f(s) = e^{-i\langle t,x\rangle s}\mathbf{1}_{(-\infty,0]}(H_\hbar)e^{i\langle t,x\rangle s}.$$

For this function we note that

$$e^{i\langle t,x\rangle}(f(1) - f(0)) = [\mathbf{1}_{(-\infty,0]}(H_\hbar), e^{i\langle t,x\rangle}].$$

By (2.72) we have that

$$\begin{aligned}\frac{d}{ds}f(s) &= -i\langle t,x\rangle e^{-i\langle t,x\rangle s}\mathbf{1}_{(-\infty,0]}(H_\hbar)e^{i\langle t,x\rangle s} + ie^{-\langle t,x\rangle s}\mathbf{1}_{(-\infty,0]}(H_\hbar)\langle t,x\rangle e^{i\langle t,x\rangle s} \\ &= ie^{-i\langle t,x\rangle s}[\mathbf{1}_{(-\infty,0]}(H_\hbar),\langle t,x\rangle]e^{i\langle t,x\rangle s}.\end{aligned}$$

With this we note by the fundamental theorem of calculus that

$$\begin{aligned}\|[\mathbf{1}_{(-\infty,0]}(H_\hbar),e^{i\langle t,x\rangle}]\|_1 &= \|\int_0^1 e^{i\langle t,x\rangle(1-s)}[\mathbf{1}_{(-\infty,0]}(H_\hbar),\langle t,x\rangle]e^{i\langle t,x\rangle s}\,ds\|_1 \\ &\leq \|[\mathbf{1}_{(-\infty,0]}(H_\hbar),\langle t,x\rangle]\|_1 \leq \sum_{j=1}^{d}|t_j|\|[\mathbf{1}_{(-\infty,0]}(H_\hbar),x_j]\|_1.\end{aligned}$$

With this bound the desired result follows from Theorem I.1.2. □

Appendix: Agmon type estimates

In this appendix we will prove an Agmon type estimate, that is exponential decay of eigenfunctions for a Schrödinger operator. Such results were proven by S. Agmon see [1].

Lemma I.1.1. *Let $H_\hbar = -\hbar^2\Delta + V$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$, where V is in $L^1_{loc}(\mathbb{R}^d)$ and suppose that there exist an $\varepsilon > 0$ and a open bounded sets U such that*

$$V(x) \geq \varepsilon \quad \text{when } x \in U^c.$$

Let $d(x) = \operatorname{dist}(x,\Omega_\varepsilon)$ and ψ be a normalised solution to the equation

$$H_\hbar\psi = E\psi,$$

with $E < \varepsilon/4$. Then there exists a $C > 0$ depending on V and ε such that

$$\|e^{\delta\hbar^{-1}d}\psi\|_{L^2(\mathbb{R}^d)} \leq C,$$

for $\delta = \frac{\sqrt{\varepsilon}}{8}$.

Proof. We start by defining the set Ω_ε by

$$\Omega_\varepsilon = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x,U) < 1\}.$$

For convenience and without loss of generality we assume that $0 \in U$, which implies that $d(x) \leq |x|$ for all x in $\mathbb{R}^d$. For $\gamma \in (0,1]$ we define the function φ_γ by

$$\varphi_\gamma(x) = \frac{d(x)}{1+\gamma|x|^2}.$$

Then φ_γ is a bounded function for all γ's by construction. Moreover we can note that $d(x)$ is almost everywhere differentiable with the norm of the gradient bounded by 1 since it is Lipschitz continuous with Lipschitz constant 1. Hence φ_γ is almost everywhere differentiable. We will prove the bound on the 2-norm is uniform in the parameter γ for the functions φ_γ and let γ tend to zero.

In order to prove the desired bound we need a partition of unity. We let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ for all x in Ω_ε^c and $\mathrm{Supp}(\chi) \subset U^c$. For this function we note that

$$\begin{aligned}\|e^{\delta\varphi_\gamma\hbar^{-1}}\psi\|_{L^2(\mathbb{R}^d)} \leq& \|e^{\delta\varphi_\gamma\hbar^{-1}}(1-\chi)\psi\|_{L^2(\mathbb{R}^d)} + \|e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\|_{L^2(\mathbb{R}^d)} \\ \leq& 1 + \|e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

where we have used that $1-\chi$ is supported in Ω_ε and $\varphi_\gamma(x) = 0$ for $x \in \Omega_\varepsilon$. Since φ_γ is a bounded function the left hand side in the above inequality is well defined. What remains is to estimate the last term in the above inequality.

To this end we note that since ψ is an eigenfunction with eigenvalue E we have that

$$\begin{aligned}(\tfrac{\varepsilon}{2} - E)\|e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\|^2_{L^2(\mathbb{R}^d)} &= (\tfrac{\varepsilon}{2} - E)\int_{\mathbb{R}^d} e^{2\delta\varphi_\gamma\hbar^{-1}}\chi^2|\psi|^2\,dx \\ &= \langle e^{2\delta\varphi_\gamma\hbar^{-1}}\chi^2\psi, (\tfrac{\varepsilon}{2} - H)\psi\rangle.\end{aligned}$$

Note that the above expression is real, hence we can take the real part of the right hand side without changing it. If we do this and use the IMS-formula we get that

$$\begin{aligned}\mathrm{Re}(\langle e^{2\delta\varphi_\gamma\hbar^{-1}}\chi^2\psi, (\tfrac{\varepsilon}{2} - H)\psi\rangle) =& \mathrm{Re}(\langle e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi, (\tfrac{\varepsilon}{2} - H)e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\rangle) \\ &+ \hbar^2\int_{\mathbb{R}^d}|\nabla e^{\delta\varphi_\gamma\hbar^{-1}}\chi|^2|\psi|^2\,dx.\end{aligned}$$

Note that the above gradient is well defined almost everywhere due to our previous observations. Since $e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi \in \mathcal{Q}(H)$ and is supported in U^c we have that

$$\mathrm{Re}(\langle e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi, (\tfrac{\varepsilon}{2} - H)e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\rangle) \leq 0,$$

since $(\frac{\varepsilon}{2} - H)$ is a negative operator when restricted to U^c. From this we obtain the inequality

$$(\tfrac{\varepsilon}{2} - E)\|e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\|^2_{L^2(\mathbb{R}^d)} \leq \hbar^2\int_{\mathbb{R}^d}|\nabla e^{\delta\varphi_\gamma\hbar^{-1}}\chi|^2|\psi|^2\,dx.$$

We note that

$$|\nabla e^{\delta\varphi_\gamma\hbar^{-1}}\chi|^2 \leq 4|\nabla e^{\delta\varphi_\gamma\hbar^{-1}}|^2\chi^2 + 4e^{2\delta\varphi_\gamma\hbar^{-1}}|\nabla\chi|^2, \tag{73}$$

where the gradients are defined almost everywhere with respect to the Lebesgue measure. The first term in (73) is almost everywhere given by

$$4|\nabla e^{\delta\varphi_\gamma\hbar^{-1}}|^2\chi^2 = 4\frac{\delta^2}{\hbar^2}|\nabla\varphi_\gamma|^2 e^{2\delta\varphi_\gamma\hbar^{-1}}\chi^2.$$

We note that for x in Ω_ε $|\nabla\varphi_\gamma(x)| = 0$, and for almost all x in Ω_ε^c

$$|\nabla\varphi_\gamma(x)| \leq \frac{|\nabla d(x)|}{1+\gamma|x|^2} + 2\frac{d(x)\gamma|x|}{(1+\gamma|x|^2)^2} \leq 1 + 2\frac{\gamma|x|^2}{(1+\gamma|x|^2)^2} \leq 2.$$

Hence for all x in $\mathbb{R}^d$ we have,

$$|\nabla\varphi_\gamma(x)| \leq 2.$$

With these estimates we get that

$$\begin{aligned}(\tfrac{\varepsilon}{2} - E)&\|e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\|^2_{L^2(\mathbb{R}^d)} \\ &\leq 8\delta^2 \int_{\mathbb{R}^d} e^{2\delta\varphi_\gamma\hbar^{-1}}\chi^2|\psi|^2 \, dx + 4\int_{\mathbb{R}^d} e^{2\delta\varphi_\gamma\hbar^{-1}}|\nabla\chi|^2|\psi|^2 \, dx \\ &= 8\delta^2\|e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\|^2_{L^2(\mathbb{R}^d)} + 4\int_{\mathbb{R}^d} e^{2\delta\varphi_\gamma\hbar^{-1}}|\nabla\chi|^2|\psi|^2 \, dx.\end{aligned}$$

This implies that

$$(\tfrac{\varepsilon}{2} - E - 8\delta^2)\|e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\|^2_{L^2(\mathbb{R}^d)} \leq 4\int_{\mathbb{R}^d} e^{2\delta\varphi_\gamma\hbar^{-1}}|\nabla\chi|^2|\psi|^2 \, dx.$$

With our choice of $\delta = \frac{\sqrt{\varepsilon}}{8}$ we have that

$$(\tfrac{\varepsilon}{2} - E - 8\delta^2) \geq \frac{\varepsilon}{2} - \frac{\varepsilon}{4} - 8\frac{\varepsilon}{64} = \frac{\varepsilon}{8},$$

which implies that

$$\|e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\|^2_{L^2(\mathbb{R}^d)} \leq \frac{32}{\varepsilon}\int_{\mathbb{R}^d} e^{2\delta\varphi_\gamma\hbar^{-1}}|\nabla\chi|^2|\psi|^2 \, dx.$$

We note that $|\nabla\chi|^2$ is supported on the set $\Omega_\varepsilon \setminus U$ and hence uniformly bounded by a constant which depends on the sets. Hence we get that

$$\begin{aligned}\int_{\mathbb{R}^d} e^{2\delta\varphi_\gamma\hbar^{-1}}|\nabla\chi|^2|\psi|^2 \, dx &\leq C\int_{\Omega_\varepsilon\setminus U} e^{2\delta\varphi_\gamma\hbar^{-1}}|\psi|^2 \, dx \\ &\leq C\int_{\Omega_\varepsilon\setminus U} |\psi|^2 \, dx \leq C,\end{aligned}$$

where we have used that $e^{2\delta\varphi_\gamma\hbar^{-1}} = 1$ for all x in Ω_ε. This implies that there exists a constant $C > 0$ which only depends on the potential V such that

$$\|e^{\delta\varphi_\gamma\hbar^{-1}}\chi\psi\|^2_{L^2(\mathbb{R}^d)} \leq C.$$

This estimate implies that we have the following uniform bound in γ

$$\|e^{\delta\varphi_\gamma\hbar^{-1}}\psi\|_{L^2(\mathbb{R}^d)} \leq 1 + C.$$

By monotone convergence we can take γ to zero and we obtain the desired result:

$$\|e^{\delta\varphi\hbar^{-1}}\psi\|_{L^2(\mathbb{R}^d)} \leq C,$$

with a constant only depending on the potential V. □

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Chapter 3

Weyl asymptotics with irregular coefficients

This paper draft is concerned proving Weyl laws for operators with irregular coefficients, where irregular means non-smooth and we consider operators acting in $L^2(\mathbb{R}^d)$. The presentation in the paper is self contained but it is work in progress. Some of the results in the paper is already know from the works of V. Ivrii and L. Zielinski. In the cases of already known results we have here given different proofs of these statements.

3.1 The main theorem

The prime operators in this paper are differential operators of the form

$$A(\hbar) = \sum_{|\alpha|,|\beta|\leq m} (\hbar D)^\alpha a_{\alpha\beta}(x)(\hbar D)^\beta, \tag{3.1}$$

which is of order $2m$, and defined via the associated quadratic form. Here we have used the notation

$$(\hbar D)^\alpha = \prod_{j=1}^{d} (-i\hbar\partial_{x_j})^{\alpha_j},$$

for all multi indices α in $\mathbb{N}_0^d$, where $\mathbb{N}_0$ is the natural numbers including 0. We remark that the semiclassical magnetic Schrödinger operator:

$$H_\hbar = (-i\hbar\nabla_x + A)^2 + V,$$

where A is a vector potential and V a potential, can be written in the form (3.1). In the case where the coefficients $a_{\alpha\beta}$ and the potentials are smooth there are condition implying an optimal Weyl law see e.g. [8]. In this paper we prove the following Weyl law:

Theorem 3.1.1. *Let $A(\hbar)$ be a differential operator of order $2m$ with the form*

$$A(\hbar) = \sum_{|\alpha|,|\beta|\leq m} (\hbar D)^\alpha a_{\alpha\beta}(x)(\hbar D)^\beta,$$

where the coefficients $a_{\alpha\beta}(x)$ are in $C^{1,\mu}(\mathbb{R}^d)$ for μ in $(0,1]$ and real. We suppose the following conditions on the coefficients are satisfied.

(*i*) *There is a* $\gamma_0 > 0$ *such that* $\min_{x\in\mathbb{R}^d}(a_{\alpha\beta}(x)) > -\gamma_0$ *for all* α *and* β.

(*ii*) *There is a* $\gamma_1 > \gamma_0$ *and* $C_1, M > 0$ *such that*

$$a_{\alpha\beta}(x) + \gamma_1 \leq C_1(a_{\alpha\beta}(y) + \gamma_1)(1 + |x-y|)^M,$$

for all x, y *in* $\mathbb{R}^d$.

(*iii*) *For all* j *in* $\{1,\dots,d\}$ *there is a* $c_j > 0$ *such that*

$$|\partial_{x_j} a_{\alpha\beta}(x)| \leq c_j(a_{\alpha\beta}(x) + \gamma_1).$$

Suppose there exists a constant C_2 *such that*

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)p^{\alpha+\beta} \geq C_2|p|^{2m}, \tag{3.2}$$

for all (x,p) *in* $\mathbb{R}^d_x \times \mathbb{R}^d_p$. *Moreover we suppose there is* $c > 0$ *such that*

$$|\nabla_p a_0(x,p)| \geq c \quad \text{for all } (x,p) \in a_0^{-1}(\{0\}), \tag{3.3}$$

where

$$a_0(x,p) = \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)p^{\alpha+\beta}.$$

Lastly we suppose there is a $\nu > 0$ *such that the set* $a_0^{-1}((-\infty,\nu])$ *is compact.*

Then we have

$$|\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(A(\hbar))] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,0]}(a_0(x,p))\,dxdp| \leq C\hbar^{1-d},$$

for all sufficiently small $\hbar$.

Where the set $C^{1,\mu}(\mathbb{R}^d)$ is defined by

$$\begin{aligned} &C^{1,\mu}(\mathbb{R}^d) \\ &\quad = \{f \in C^1(\mathbb{R}^d) \,|\, |\partial_{x_j} f(x) - \partial_{x_j} f(y)| \leq C|x-y|^\mu \;\forall x,y \in \mathbb{R}^d \text{ and } j \in \{1,\dots,d\}\}. \end{aligned}$$

The list of assumptions in the theorem is not short. But why do we need all these assumptions?

That we need some regularity of the coefficients is expected and that is why the coefficients are supposed to be in $C^{1,\mu}(\mathbb{R}^d)$ for a $\mu > 0$. The assumptions in (*i*), (*ii*) and (*iii*), can be seen assumptions on the behaviour of the coefficients for large values of x. Since in the case of the coefficients having compact support they are all verified. This regularity is need as we will use functional calculus of rough pseudo-differential operators.

Without assumption global ellipticity (3.2) we could easily be in a situation, where we there operator only had essential spectrum. This assumptions is also used to verify properties of the approximating operators.

The non-critical assumption (3.3) is essential for our proof to be valid. Compared to the usual non-critical assumption, where the whole gradient is assumed not to vanish, then this assumption is more strict as it imply the usual non-critical condition. But as we will see in the very last section in this chapter it is possible to use multiscale arguments with such a non-critical condition.

I strongly believe that these assumptions is not optimal. The assumptions that the coefficients are real on should be able to relax to the assumption that the coefficients may be complex but should satisfies that $a_{\alpha\beta}(x) = \overline{a_{\alpha\beta}}(x)$ for all α and β in $\mathbb{N}_0^d$. To allow the coefficients to be complex would require a slight change of the assumptions in (i), (ii) and (iii). These assumptions could possible also be changed slightly on their own, but I am at the moment unsure if this change would generalise them. As stated in the beginning this is really work in progress.

Results of this type was first obtained by L. Zielinski in [11–14] but with higher regularity. V. Ivrii generalised the result by L. Zielinski in [6] to coefficients which is differentiable and with a Hölder continuous first derivative. This was further generalised by M. Bronstein and V. Ivrii in [1], where they reduced the assumptions further by assuming the first derivative to have modulus continuity $\mathcal{O}(|\log(x-y)|^{-1})$. In all these papers they considered operators defined on a compact smooth manifold with and without a boundary.

Firstly we will briefly describe the main ideas entering a proof of this type of Weyl law in general. Usually an optimal Weyl law is proven by means of microlocal analysis and this is not possible to use when the coefficients are not smooth. Instead we construct two framing operators $A_\varepsilon^{\pm}(\hbar)$ such that

$$A_\varepsilon^{-}(\hbar) \leq A(\hbar) \leq A_\varepsilon^{+}(\hbar),$$

in the sense of quadratic forms. By the min-max theorem we have

$$\mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(A_\varepsilon^{+})] \leq \mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(A)] \leq \mathrm{Tr}[\mathbf{1}_{(-\infty,0]}(A_\varepsilon^{-})].$$

Hence what we need to do is to prove an optimal Weyl law for the framing operators such the difference between the phase space integrals is of the right order to.

These framing operators are what we call rough pseudo differential operators and we will in the next section state the definition and discuss some of their properties.

3.2 Rough symbols and pseudo differential operators

In this paper we consider a different types of symbols inspired by the framing procedure. This procedure can be seen in Section II.3 in the paper. We have chosen to call the rough symbols which aligns with the terminology in [7]. We define the rough symbols by:

Definition 3.2.1 (Rough symbol). Let $\Omega \subseteq \mathbb{R}_x^d \times \mathbb{R}_p^d \times \mathbb{R}_y^d$ be open, ρ be in $[0,1]$, $\varepsilon > 0$, τ be in $\mathbb{Z}$ and m a tempered weight function on $\mathbb{R}_x^d \times \mathbb{R}_p^d \times \mathbb{R}_y^d$. We call a function a_ε a rough symbol of regularity τ with weights (m, ρ, ε) if a_ε is in $C^\infty(\Omega)$

and satisfies that

$$
\begin{aligned}
&|\partial_x^\alpha \partial_p^\beta \partial_y^\gamma a_\varepsilon(x,p,y)| \\
&\leq \begin{cases} C_{\alpha\beta\gamma} m(x,p,y)(1+|(x,p,y)|)^{-\rho(|\alpha|+|\beta|+|\gamma|)} & \text{if } |\alpha|+|\gamma| \leq \tau \\ C_{\alpha\beta\gamma}\varepsilon^{\tau-|\alpha|-|\gamma|} m(x,p,y)(1+|(x,p,y)|)^{-\rho(|\alpha|+|\beta|+|\gamma|)} & \text{if } |\alpha|+|\gamma| > \tau, \end{cases}
\end{aligned} \tag{3.4}
$$

for all (x,p,y) in Ω and α, β, γ in $\mathbb{N}_0^d$, where the constants $C_{\alpha\beta\gamma}$'s do not depend on ε. The space of these functions is denoted $\Gamma_{\rho,\varepsilon}^{m,\tau}(\Omega)$. The space can be turned into a Fréchet space with semi norms associated to the estimates in (3.4).

This definition of the symbols is almost the same as the one in [8]. The new thing is the regularity parameter which can be interpreted as the measure of how smooth the "original" symbol was and the parameter ε. For further remarks on the dependence on the parameter ε see the paper. To these classes of symbols we can define associated operators and this is done in the paper.

To consider rough symbols is not new. In [7, Section 2.3 and 4.6] V. Ivrii considers a similar class of symbols and associated operators. We remark that generally in the monographs [5, 7] the symbols are not assumed to be smooth but to have a sufficient number of derivatives. This is due to the fact that in reality we never take an infinite number of derivatives but only a finite number when working with pseudo-differential operators.

In the monographs [2] and [19] they also consider rough symbols but here all variables become rough. These classes are introduced as these types of symbols naturally appear in a proof of the sharp Gårdinger inequality.

With this definition we can almost analogous to the definitions in [8], define $\hbar$-ε-admissible symbols, pseudo-differential operators with symbols from the above class and $\hbar$-ε-admissible operators. In order for the operators not to diverge in all norms as $\hbar$ tends to zero we need to assume there exists a $\delta \in (0,1)$ such that $\varepsilon \geq \hbar^{1-\delta}$. This assumption is no real restriction as we want to choose $\varepsilon = \hbar^{1-\delta}$ for a suitable δ. Moreover under the assumption $\varepsilon \geq \hbar^{1-\delta}$ it is also fairly easy to prove that the operators with positive regularity are well-defined operators from $\mathcal{S}(\mathbb{R}^d)$ into itself. The proof of this is complete analogous to the proof of the same statement for the operators considered in [8]. In the case of negative regularity the operator does not diverge in operator norm as $\hbar$ tends to zero if the operator comes with $\hbar$ raised to at least the absolute value of the regularity.

One of the main differences is we need to specify the regularity of these objects. An other significant difference concerns the error terms in asymptotic expansions of these new operators. To give an example of this we recall the definition of a $\hbar$-admissible operator from [8].

Definition 3.2.2. We call an operator $A(\hbar)$ from $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), L^2(\mathbb{R}^d))$ $\hbar$-admissible with weight m if the map

$$
A : (0,\hbar_0] \to \mathcal{L}(\mathcal{S}(\mathbb{R}^d), L^2(\mathbb{R}^d)),
$$

is smooth. There exists a sequence a_j in $\Gamma_0^m(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ and a sequence $R_N(\hbar)$ in $\mathcal{L}(L^2(\mathbb{R}^d))$ such that for $N \geq N_0$, N_0 sufficient large,

$$A(\hbar) = \sum_{j=0}^{N} \hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a_j) + \hbar^{N+1} R_N(\hbar),$$

and

$$\sup_{\hbar \in (0,\hbar_0]} \|R_N(\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} < \infty.$$

here $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), L^2(\mathbb{R}^d))$ is the linear operators from $\mathcal{S}(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$. If we compare with the definition of $\hbar$-ε-admissible operators which is

Definition 3.2.3. We call an operator $A_\varepsilon(\hbar)$ from $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), L^2(\mathbb{R}^d))$ $\hbar$-ε-admissible of regularity $\tau \geq 0$ with weight m if the map

$$A_\varepsilon : (0, \hbar_0] \to \mathcal{L}(\mathcal{S}(\mathbb{R}^d), L^2(\mathbb{R}^d)),$$

is smooth. There exists a sequence $a_{\varepsilon,j}$ in $\Gamma_{0,\varepsilon}^{m,\tau_j}(\mathbb{R}_x^d \times \mathbb{R}_y^d \times \mathbb{R}_p^d)$, where $\tau_0 = \tau$ and $\tau_{j+1} = \tau_j - 1$ and a sequence R_N in $\mathcal{L}(L^2(\mathbb{R}^d))$ such that for $N \geq N_0$, N_0 sufficient large,

$$A_\varepsilon(\hbar) = \sum_{j=0}^{N} \hbar^j \operatorname{Op}_\hbar(a_{\varepsilon,j}) + \hbar^{N+1} R_N(\varepsilon, \hbar),$$

and

$$\hbar^{N+1} \|R_N(\varepsilon, \hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \hbar^{\kappa(N)} C_N,$$

for a strictly positive increasing function κ.

At first glance the two definition semens the same bot in the later we need to introduce an increasing function κ in the estimate of the norm of the error terms. The necessity of this function is due to the error term which comes from application of the stationary phase theorem. If we just recall Quadratic stationary phase theorem:

Theorem 3.2.4. *Let B be a invertible, symmetric real $d \times d$ matrix and $(u, v) \to a(u, v; \hbar)$ be a function in $C^\infty(\mathbb{R}_u^d \times \mathbb{R}_v^n)$ for all $\hbar$ in $(0, \hbar_0]$. We suppose $v \to a(u, v; \hbar)$ has compact support for all u in $\mathbb{R}_u^d$ and $\hbar$ in $(0, \hbar_0]$. Moreover we let*

$$I(u; a, B, \hbar) = \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle Bv, v\rangle} a(u, v; \hbar)\, dv.$$

Then for each N in $\mathbb{N}$ we have

$$\begin{aligned} I(u;&a, B, \hbar) \\ &= (2\pi\hbar)^{\frac{n}{2}} \frac{e^{i\frac{\pi}{4}\operatorname{sgn}(B)}}{|\det(B)|^{\frac{1}{2}}} \sum_{j=0}^{N} \frac{\hbar^j}{j!} \Big(\frac{\langle B^{-1} D_v, D_v\rangle}{2i} \Big)^j a(u, v; \hbar)\Big|_{v=0} + \hbar^{N+1} R_{N+1}(u; \hbar), \end{aligned}$$

where $\mathrm{sgn}(B)$ *is the difference between the number of positive and negative eigenvalues of* B*. Moreover there exists a constant* c_n *only depending on the dimension such the error term* R_{N+1} *satisfies the bound*

$$|R_{N+1}(u;\hbar)| \leq c_n \| \frac{\langle B^{-1}D_v, D_v\rangle^{N+1}}{(N+1)!} a(u,v;\hbar)\|_{H^{[\frac{n}{2}]+1}(\mathbb{R}^n_v)}, \tag{3.5}$$

where $[\frac{n}{2}]$ *is the integer part of* $\frac{n}{2}$*, and the norm is the Sobolev norm.*

This version is from [8]. For the estimate on the error term in (3.5) it is evident what the challenge is. When we have to evaluate the Sobolev norm we need to take up to additional $[\frac{n}{2}]+1$ derivatives which does not come with a $\hbar$. Hence if these derivatives are in the rough variables we need to be able to compensate up to $\varepsilon^{-[\frac{n}{2}]-1}$. But this number is fixed so for application we just need to take a sufficient number of terms in the expansion. Hence it does not give rise to a problem just a detail to be aware of.

After defining these symbol classes and operators we prove that we still have a full symbolic calculus for the rough operators. We prove a Calderon-Vaillancourt type theorem and give criteria for the operators to be Hilbert-Schmidt and trace class. Finally we also prove that a full functional calculus is still valid for this type of operators under assumptions similar to the assumptions in [8]. To my knowledge the construction of the functional calculus has not been consider by others prior to this work.

3.3 The approximation of the propagator

In order to prove a Weyl law we will need some sort of approximation of a propagator. After the work of L. Hörmander in [4] this approximation, also called parametric, have usually been constructed as a Fourier integral operator. For the actual construction in the semiclassical setting see [2, 8]. The challenge in this approach is that the construction is not explicit. The phase function is the solution to the Hamilton-Jacobi equation associated to the principal symbol. Hence it becomes hard to see how the roughness of the principal symbol affects the Fourier integral operator.

Instead we do a microlocal approximation by an operator for which we directly construct the integral kernel. The kernel has the following form

$$\begin{aligned} &K_{U_N}(x,y,t,\varepsilon,\hbar) \\ &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} \sum_{j=0}^{N} (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\,dp, \end{aligned}$$

where the u_j's are compactly support in x and p so the integral do exists as a proper integral. This construction is inspired by the construed used by L. Zielinski in [15]. This construction is completely explicit and is made recursively by choosing a u_0 and then find u_1 and so fort. The construction is only valid as an approximation for short times of the order $\hbar^{1-\frac{\delta}{2}}$, where the δ is the one from the assumption on ε.

We need an approximation which is valid for a $\hbar$ independent time T_0 in order to be able to prove a Weyl law. What we do is to prove the following theorem.

Theorem 3.3.1. *Let $A_\varepsilon(\hbar)$ be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ which satisfies Assumption II.7.1, has a bounded principal symbol and there exists a δ in $(0,1)$ such that $\varepsilon \geq \hbar^{1-\delta}$. Suppose there exists a number $\eta > 0$ such $a_{\varepsilon,0}^{-1}([-2\eta, 2\eta])$ is compact and a constant $c > 0$ such*

$$|\nabla_p a_{\varepsilon,0}(x,p)| \geq c \qquad \text{for all } (x,p) \in a_{\varepsilon,0}^{-1}([-2\eta, 2\eta]),$$

where $a_{\varepsilon,0}$ is the principal symbol of $A_\varepsilon(\hbar)$. Let f be in $C_0^\infty((-\eta,\eta))$ and θ be in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such that $\operatorname{supp}(\theta) \subset a_{\varepsilon,0}^{-1}((-\eta,\eta))$. There exists a constant $T_0 > 0$ such that if χ is in $C_0^\infty((\frac{1}{2}\hbar^{1-\gamma}, T_0))$ for a γ in $(0,\delta]$, then for every N in $\mathbb{N}$, we have

$$|\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) \mathcal{F}_\hbar^{-1}[\chi](s - A_\varepsilon(\hbar)) \, \mathrm{Op}_\hbar^{\mathrm{w}}(\theta)]| \leq C_N \hbar^N$$

uniformly for s in $(-\eta,\eta)$.

Assumption II.7.1 is the assumption that ensures selfadjointness of the operator the exact assumption is in the paper and $\mathcal{F}_\hbar^{-1}[\chi]$ is the inverse to the semiclassical Fourier transform. This is a version of [2, Proposition 12..4], where we have modified the proof to fit in the framework of rough pseudo differential operators. The result in the theorem is of the same nature as the theorems in [7, Section 2.3].[1] Moreover there are similar result in the papers [16–18] by L. Zielinski proved by another method then in our paper and [7].

This theorem shows that the construction of the propagator in the $\hbar$ dependent interval is sufficient as the traces we consider is negligible for times in $(\frac{1}{2}\hbar^{1-\gamma}, T_0)$ under a non-critical condition, where the times are hidden the support of χ.

I was slightly surprised by the fact that the theorem is true. The way I think about it heuristically is that the theorem gives the bound

$$|\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta) e^{it\hbar^{-1}A_\varepsilon(\hbar)} \, \mathrm{Op}_\hbar^{\mathrm{w}}(\theta)]| \leq C_N \hbar^N,$$

for t in $(\frac{1}{2}\hbar^{1-\gamma}, T_0)$ under a non-critical assumption. The operator $\mathrm{Op}_\hbar^{\mathrm{w}}(\theta)$ can be viewed as localisations since θ has compact support. Then the term $e^{it\hbar^{-1}A_\varepsilon(\hbar)} \, \mathrm{Op}_\hbar^{\mathrm{w}}(\theta)$ is the evolution of a particle initially located in the support of θ under the operator $A_\varepsilon(\hbar)$. The non-critical condition says in terms of classical mechanics that the particle is moving. Hence if the support of θ is sufficiently small we should have moved out of the support. So when we compare against the non evolved θ we should have obtained that the support of the two operators are disjoint. Hence the trace becomes small. This heuristic idea is rather hard to see directly in the proof given in the paper.

3.4 The Weyl law

After establishing the symbolic and function calculus for the for certain rough pseudo-differential operators and constructing an approximation to the propagator we are in the paper able to prove the following Weyl law for rough pseudo-differential operators.

[1] V. Ivrii was the first to prove these types of theorems.

Theorem 3.4.1 (Weyl law). *Let $A_\varepsilon(\hbar)$ be a strongly $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ which satisfies Assumption II.7.1 and there exists a δ in $(0,1)$ such that $\varepsilon \geq \hbar^{1-\delta}$. Suppose there exists a $\eta > 0$ such $a_{\varepsilon,0}^{-1}((-\infty,\eta])$ is compact, where $a_{\varepsilon,0}$ is the principal symbol of $A_\varepsilon(\hbar)$. Moreover we suppose*

$$|\nabla_p a_{\varepsilon,0}(x,p)| \geq c \quad \text{for all } (x,p) \in a_{\varepsilon,0}^{-1}(\{0\}). \tag{3.6}$$

Then we have

$$|\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(A_\varepsilon(\hbar))] - \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,0]}(a_{\varepsilon,0}(x,p))\,dxdp| \leq C\hbar^{1-d},$$

for all sufficiently small $\hbar$.

The proof of this theorem is analogous to the proof of the corresponding theorem in the non-rough case in [8]. What is remarkable in this theorem is that we do not need to assume the operator to be a differential operator.

After this is proven we are able to prove the main theorem state in the beginning of this chapter (Theorem 3.1.1). The proof is to verify that the assumptions in the theorems ensures that the framing operators satisfies the assumption of the just stated Weyl law (Theorem 3.4.1). This is the last proof in the main part of the paper draft. For sake of completeness we have added an appendix on multivariate differentiation and Taylor's formula.

There is one remarkable observation to do concerning the results obtained here. If now the starting operator $A(\hbar)$ had been a $\hbar$-admissible operator which satisfies the assumptions in [8] for an optimal Weyl law. Then one might be interested in perturbing this operator by a non smooth potential V and consider the operator

$$A(\hbar) + V. \tag{3.7}$$

Then this operator would no longer be a $\hbar$-admissible operator. But it will be possible to frame this operator by two rough pseudo-differential operators which can be chosen to be $\hbar$-ε-admissible under the right assumptions on V. Then we could get an optimal Weyl law for these operators and compare phase-space integrals to see if an optimal Weyl law can be achieved for the operator (3.7). These kinds of results have not, to my knowledge, been obtained before.

3.5 Future work

As said in the beginning of this chapter, this is work in progress and what is presented in the thesis is where we are at the moment. In this section we will describe where we want to go from here.

A first thing we would like to do, is to extend the Weyl law to other functions than $\mathbf{1}_{(-\infty,0]}(t)$. In particular we are interested in functions of the form

$$g_s(t) = (t)_-^s = \Big(\frac{|t|-t}{2}\Big)^s, \tag{3.8}$$

for s in $(0,1]$. We should be able to use the techniques from [3] to obtain results of the form

$$|\operatorname{Tr}[g_s(A_\varepsilon(\hbar))] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(a_{\varepsilon,0}(x,p))\,dxdp| \leq C\hbar^{1+s-d},$$

in the case where the subprincipal symbol is zero ($a_{\varepsilon,1}(x,p) = 0$). In the case where the subprincipal symbol is not zero we expect an extra phase space integral. From this results we would like to extract similar results for the irregular operators. But we do suspect that more regularity of the coefficients are needed to obtain optimal results for $s > 0$ than the case $s = 0$, which requires the coefficients to be in $C^{1,\mu}(\mathbb{R}^d)$ for a $\mu > 0$. The case $s = 1$ is of particular interest for the Schrödinger operator.

If we in the following let $H_\hbar$ be a Schrödinger operator given by

$$H_\hbar = (-i\hbar\nabla_x + A)^2 + V,$$

where A is a vector potential and V a potential. Then we would also like to investigate if the methods used here can give local results for the Schrödinger operator without full regularity and with a non-critical condition, that is results of the form

$$|\operatorname{Tr}[\varphi g_s(H_\hbar)\varphi] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s((p+A(x))^2+V(x))\varphi(x)^2\,dxdp| \leq C\hbar^{1+s-d}, \tag{3.9}$$

where φ is a function from $C_0^\infty(\mathbb{R}^d)$. We suspect that in the case where $s = 0$ will require that the potentials is in $C^{1,\mu}(\mathbb{R}^d)$ for a $\mu > 0$ and we suspect the restrictions to be more strict in the case of a $s > 0$.

One of the main obstacles at the moment for our approach to work is that if we have the relation

$$H_\hbar^- \leq H_\hbar \leq H_\hbar^+$$

in the sense of quadratic forms. Then we can not use the min-max theorem to get the relation

$$\operatorname{Tr}[\varphi \mathbf{1}_{(-\infty,0]}(H_\hbar^+)\varphi] \leq \operatorname{Tr}[\varphi \mathbf{1}_{(-\infty,0]}(H_\hbar)\varphi] \leq \operatorname{Tr}[\varphi \mathbf{1}_{(-\infty,0]}(H_\hbar^-)\varphi]. \tag{3.10}$$

Moreover I do believe that a relation like the one in (3.10) is not valid in general. But it might be possible to compare the trace with small errors in the semiclassical parameter. But without this relation we need some other method to compare the traces. This other method is unclear to us at the moment of handing in this thesis.

An other interesting thing to investigate if some of the results also can be extended to also cover Pauli operators, but we will just leave this as a small remark.

Of cause it could also be interesting to see to what extent we can prove optimal Weyl laws without a non-critical condition. As mentioned this have already been studied before by V. Ivrii with collaborators see [1, 6, 7] and L. Zielinski see [17, 18].

In the case we are able to prove a statement as in (3.9) we will be able to remove the non-critical condition under some assumptions. Unfortunately one of these assumptions is on the dimension. This will be sketched in the following subsection. The method we will use is a multiscale argument as in Paper I but with some other choices of auxiliary function(s).

Before this argument is sketched and we end this introductory chapter we would like to make a small comment. One of the generalisations we do hope is possible to do is to generalise the result from [9, 10] by A. V. Sobolev to cases where the potentials does not have to be assumed smooth. But much work is needed before this will be possible.

Multiscale argument sketch

In this subsection we assume $H_\hbar$ to be a Schrödinger operator given by

$$H_\hbar = -\hbar^2 \Delta + V,$$

where we assume V to be at least one time differentiable. Assume that if

$$|V(x)| + \hbar^{\frac{2}{3}} \geq c > 0 \qquad \text{for all } x \in B(0, 4R), \tag{3.11}$$

for a $R > 0$ then we have for all ψ in $C_0^\infty(B(0, \frac{R}{2}))$ the estimate

$$| \operatorname{Tr}[g_s(H_\hbar)\psi] - \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_s(p^2 + V(x))\psi(x)\, dxdp| \leq C\hbar^{1+s-d}, \tag{3.12}$$

for all $\hbar$ in $(0, \hbar_0]$, where $\hbar_0$ is some positive number. Now assume we have a potential V which is at least once differentiable and a function φ from $C_0^\infty(\mathbb{R}^d)$. Denote the support of φ by Ω. Then we would like to study the trace

$$\operatorname{Tr}[g_s(H_\hbar)\varphi].$$

To do so we define the functions

$$l(x) = A^{-1}\sqrt{|V(x)|^2 + \hbar^{\frac{4}{3}}} \quad \text{and} \quad f(x) = \sqrt{l(x)} = A^{-\frac{1}{2}}(|V(x)|^2 + \hbar^{\frac{4}{3}})^{\frac{1}{4}},$$

for $A > 0$. We need to choose the number A such that

$$|\nabla_x l(x)| \leq \rho < \frac{1}{8}. \tag{3.13}$$

This choice can be made uniformly for $\hbar$ in $(0, \hbar_0]$. Then there exists a sequence $\{x_n\}_\mathbb{N} \subset \Omega$ such that

$$\Omega \subset \bigcup_{n\in\mathbb{N}} B(x_n, l(x_n)),$$

where at most N_ρ balls can intersect non-empty. Moreover there also exists φ_n in $C_0^\infty(B(x_n, l(x_n)))$ such that

$$\sum_{n\in\mathbb{N}} \varphi_n(x) = 1 \qquad \text{for all } x \in \Omega.$$

The existence of the sequence of points and sequence of function is ensured by Lemma I.3.4 from Paper I. Since Ω is compact we can find a finite subset $\mathcal{I}$ of $\mathbb{N}$ such that

$$\Omega \subset \bigcup_{n\in\mathcal{I}} B(x_n, l(x_n)).$$

By doing a possible finite extension of $\mathcal{I}$ we also have

$$\sum_{n\in\mathcal{I}} \varphi_n(x) = 1 \qquad \text{for all } x \in \Omega.$$

We will use the following notation

$$l_n = l(x_n), \qquad f_n = f(x_n) \quad \text{and} \quad h_n = \frac{\hbar}{l_n f_n}.$$

We have that h_n is uniformly bounded since

$$l(x)f(x) = A^{-\frac{3}{2}}(|V(x)|^2 + \hbar^{\frac{4}{3}})^{\frac{3}{4}} \geq A^{-\frac{3}{2}}\hbar,$$

for all x. We define the two unitary operators U_l and T_z by

$$U_l f(x) = l^{\frac{d}{2}} f(lx) \quad \text{and} \quad T_z f(x) = f(x+z) \qquad \text{for } f \in L^2(\mathbb{R}^d).$$

Moreover we set

$$\begin{aligned} \tilde{H}_{n,h_n} &= f_n^{-2}(T_{x_n}U_{l_n})H_\hbar(T_{x_n}U_{l_n})^* \\ &= -h_n^2\Delta + f_n^{-2}V(l_n x + x_n). \end{aligned} \tag{3.14}$$

What we would like is the function $\tilde{V}(x) = f_n^{-2}V(l_n x + x_n)$ to satisfy (3.11) for all x in $B(0,8)$ with h_n instead of $\hbar$. To see this note that by (3.13) we have

$$(1-8\rho)l_n \leq l(x) \leq (1+8\rho)l_n \qquad \text{for all } x \in B(x_n, 8l_n). \tag{3.15}$$

Hence for x in $B(0,8)$ we have

$$\begin{aligned} |\tilde{V}(x)| + h_n^{\frac{2}{3}} &= f_n^{-2}|V(l_n x + x_n)| + (\tfrac{\hbar}{f_n l_n})^{\frac{2}{3}} = l_n^{-1}(|V(l_n x + x_n)| + \hbar^{\frac{2}{3}}) \\ &\geq l_n^{-1} A l(l_n x + x_n) \geq (1-8\rho)A. \end{aligned} \tag{3.16}$$

That is $\tilde{V}(x)$ to satisfy (3.11) for all x in $B(0,8)$ with h_n instead of $\hbar$. Hence by (3.12) with $R=2$ we have for n in $\mathcal{I}$

$$\begin{aligned} &|\operatorname{Tr}[g_s(H_\hbar)\varphi_n\varphi] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(p^2 + V(x))\varphi_n\varphi(x)\,dxdp| \\ &= f_n^{2s}|\operatorname{Tr}[g_s(\tilde{H}_{n,h_n})(T_{x_n}U_{l_n})\varphi_n\varphi(T_{x_n}U_{l_n})^*] \\ &\quad - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f_n l_n)^d g_s(p^2 + f_n^{-2}V(l_n x + x_n))\varphi_n\varphi(l_n x + x_n)\,dxdp| \\ &\leq C h_n^{1+s-d} f_n^{2s}, \end{aligned} \tag{3.17}$$

where we have used that $(T_{x_n}U_{l_n})\varphi_n\varphi(T_{x_n}U_{l_n})^*$ acts as the multiplication operator $\varphi_n\varphi(l_n x + x_n)$ on functions supported in $B(0,1)$. If we try and sum the obtained error terms over n we get

$$\begin{aligned} \sum_{n\in\mathcal{I}} C h_n^{1+s-d} f_n^{2s} &= \sum_{n\in\mathcal{I}} \tilde{C}\hbar^{1+s-d}\int_{B(x_n,l_n)} l_n^{-d} f_n^{2s}(l_n f_n)^{d-1-s}\,dx \\ &= \sum_{n\in\mathcal{I}} \tilde{C}\hbar^{1+s-d}\int_{B(x_n,l_n)} l_n^{s-d} l_n^{\frac{3d-3-3s}{2}}\,dx \\ &\leq \sum_{n\in\mathcal{I}} \hat{C}\hbar^{1+s-d}\int_{B(x_n,l_n)} l(x)^{\frac{d-3-s}{2}}\,dx, \end{aligned} \tag{3.18}$$

where we have used the definition of f_n and (3.15). What we see from this is that in order to combine (3.17) and (3.18) into a estimate of the type

$$| \operatorname{Tr}[g_s(H_\hbar)\varphi] - \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_s(p^2 + V(x))\varphi(x)\, dxdp| \leq C\hbar^{1+s-d}. \tag{3.19}$$

we need to assume $d \geq 3 + s$ hence for $s = 1$ we need $d = 4$ for this argument to work. We have here followed the approach to multiscale analysis presented in [10] but arguments like this can also be found in [5, 7].

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Paper II

Optimal Weyl asymptotics for operators with irregular coefficients

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Abstract: This paper is a status/review on ongoing work concerning Weyl laws without full regularity. Some of the results are already known and reproved here. Others are to the authors knowledge new results. The main result is a Weyl for elliptic differential operators of order $2m$ where the coefficients are differentiable with a Hölder continious derivative. In order to establish this result a class of rough symbols is defined. For the associated rough operators we prove a symbolic and functional calculus. The paper also contains a microlocal construction of the propagator which is not a Fourier integral operator.

II.1 Introduction

In 1911 H. Weyl proved the first Weyl law in [22] and in [23] he conjectured the formula

$$\begin{aligned}\operatorname{Tr}(\mathbf{1}_{(-\infty,\lambda]}(-\Delta_{D,\Omega})) \\ = \frac{1}{(2\pi)^d}\omega_d \operatorname{Vol}(\Omega)\lambda^{\frac{d}{2}} - \frac{1}{4}\frac{\omega_{d-1}}{(2\pi)^{d-1}}\operatorname{Vol}'(\partial\Omega)\lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}}),\end{aligned} \tag{3.1}$$

as $\lambda \to \infty$, where $-\Delta_{D,\Omega}$ is the positive Laplacian on an open bounded domain Ω with Dirichlet boundary conditions, ω_d is the volume of the unit ball in $\mathbb{R}^d$, $\operatorname{Vol}(\Omega)$ is the volume of Ω and $\operatorname{Vol}'(\partial\Omega)$ is the surface area of Ω. This conjectured formula was first proven in 1980 by V. Ivrii in [9] under some extra assumptions on the set Ω. In the yeas between the conjecture was stated and the proof by V. Ivrii, a substantial number of mathematicians worked on the problem. We will not give a full review of the development here but we refer to the surveys [1, 3, 14] as our main interest is in a problem arising from the work on the Weyl conjecture.

This type of formulas was during the period also considered for other types of operators and slightly different settings. We will in this paper be working in the semiclassical setting and study differential operators of the form

$$A(\hbar) = \sum_{|\alpha|,|\beta|\leq m} (\hbar D)^\alpha a_{\alpha\beta}(x)(\hbar D)^\beta,$$

acting in $L^2(\mathbb{R}^d)$, where we have introduced a semiclassical parameter $\hbar$ and used the notation

$$(\hbar D)^\alpha = \prod_{j=1}^{d}(-i\hbar\partial_{x_j})^{\alpha_j},$$

for $\alpha \in \mathbb{N}_0^d$. In the case of smooth coefficients it was first proven in [6] by B. Helffer and D. Robert that a formula of the type

$$\mathrm{Tr}[\mathbf{1}_{(-\infty,\lambda_0]}(A(\hbar))] = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}_{(-\infty,\lambda_0]}(a_0(x,p))\,dxdp + \mathcal{O}(\hbar^{1-d}),$$

where

$$a_0(x,p) = \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)p^{\alpha+\beta},$$

is true for a number $\lambda_0 < \lambda$ such $a_0^{-1}((-\infty,\lambda])$ is compact and non-critical for $a_0(x,p)$. A number λ_0 is a non-critical value when

$$|\nabla a_0(x,p)| \geq c > 0 \quad \text{for all } (x,p) \in a_0^{-1}(\{\lambda_0\}).$$

Actually the proved such a formula for a larger class of pseudo-differential operators. The classical analog was proven by L. Hömander in [7], where the operator was defined on a smooth compact manifold without a boundary.

If we consider the assumptions above. Then this immediate raises two questions:

- What happens if the coefficients are not smooth? Can a Weyl law still be proven with optimal errors?
- What happens if a non-critical condition is not assumed? Can a Weyl law still be proven with optimal errors?

We are not the first to ask these questions. Answers to both have been provided in different cases. We will in this paper focus on the first question. But for the second question it is possible for Schrödinger operators to prove optimal Weyl laws with out a non critical condition by a multiscale argument see [2, 10, 15, 16], this approach is also described in [20]. This multiscale argument can be seen as a discreet approach and a continuous version have been proved and used in [21]. The essence of this approach is to localise and the locally introduce a non-critical condition by unitary conjugation. Then by an optimal Weyl law with a non-critical condition one obtain the right asymptotics locally. The last step is to average out the localisations. V. Ivrii has also considered multiscale analysis for higher order differential operators but to treat these cases extra assumptions on the Hessian of the principal symbol is needed see [12, 15]. There is also an other approach by L. Zielinski see [30, 31], where he proves optimal Weyl laws without a non-critical condition but with an extra assumption on a specific phase space volume.

If we consider the the first question. Then the first results with an optimal Weyl law was proven in the papers [24–27] by L. Zielinski. In these papers L. Zielinski obtained an optimal Weyl law under the assumption that the coefficients are differentiable with Lipschitz continuous first derivative. L. Zielinski did not in those papers consider the semiclassical setting. These results was generalised by V. Ivrii in the semiclassical setting in [11]. Here the coefficients is assumed to be differentiable and with a Hölder continuous first derivative. This was further generalised by M. Bronstein and V. Ivrii in [2], where they reduced the assumptions further by assuming the first derivative to have modulus continuity $\mathcal{O}(|\log(x-y)|^{-1})$. All these papers considered differential operators defined on a compact manifold.

We should mention that both V. Ivrii and L. Zielinski has considered both questions simultaneously.

We will in this paper consider differential operators acting in $L^2(\mathbb{R}^d)$. The main theorem we will prove in this paper is a Weyl law for differential operators of order $2m$ of the form

$$A(\hbar) = \sum_{|\alpha|,|\beta|\leq m} (\hbar D)^{\alpha} a_{\alpha\beta}(x)(\hbar D)^{\beta},$$

where the coefficients $a_{\alpha\beta}(x)$ are once differentiable with Hölder continuous derivatives. We will also need some other conditions on the operator. The exact statement of this Weyl law is in Theorem II.10.1.

In [29] L. Zielinski also considered operators acting in $L^2(\mathbb{R}^d)$ but he has to assume the first derivative to also be bounded, which we do not have to assume, hence our theorem generalises this assumption.

The structure of the paper is such that the first section is a preliminary section which fixes some notation. Then in the next section we construct two framing operators to approximate the operator of interest. Inspired by these framing operators we define a class of rough pseudo-differential operators. For this class of operators we verify symbolic and functional calculus. Then we construct an approximation to the time evolution. After this is done we are ready to prove a Weyl law for the rough pseudo-differential operators and use this to prove the main theorem.

II.2 Preliminaries

This preliminary section we mainly set up notation and some definitions. We will in this paper use the notation

$$\lambda(x) = (1+|x|^2)^{\frac{1}{2}}, \tag{3.2}$$

for x in $\mathbb{R}^d$ and not the usual bracket notation. Moreover for more vectors x, y, w from $\mathbb{R}^d$ we will use the convention

$$\lambda(x,y,w) = (1+|x|^2+|y|^2+|w|^2)^{\frac{1}{2}}, \tag{3.3}$$

and similar in the case of 2 or more vectors. We will denote the negative part of a number by $(t)_-$ for t in $\mathbb{R}$ defined by

$$(t)_- = \frac{|t|-t}{2}, \tag{3.4}$$

which is a positive number. We will denote the Schwartz space by $\mathcal{S}(\mathbb{R}^d)$ that is

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \sup_{x\in\mathbb{R}^d} |x^\beta D^\alpha f(x)| < \infty \ \forall \alpha,\beta \in \mathbb{N}_0^d\}, \tag{3.5}$$

where we use the convention

$$\mathbb{N} = \{1,2,3,\dots\} \quad \text{and} \quad \mathbb{N}_0 = \{0\} \cup \mathbb{N}_0.$$

When working with the Fourier transform we will use the following version for $\hbar > 0$

$$\mathcal{F}_\hbar[\varphi](p) := \int_{\mathbb{R}^d} e^{-i\hbar^{-1}\langle x,p\rangle} \varphi(x)\, dx,$$

and with inverse given by

$$\mathcal{F}_\hbar^{-1}[\psi](x) := \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x,p\rangle} \psi(p)\, dp,$$

where φ and ψ are elements of $\mathcal{S}(\mathbb{R}^d)$.

We will by $\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$ denote the linear bounded operators from the space $\mathcal{B}_1$ into $\mathcal{B}_1$ and $\mathcal{L}(\mathcal{B}_1)$ denotes the linear bounded operators from the space $\mathcal{B}_1$ into itself. For an operator A acting in a Hilbert space we will denote the spectrum of A by

$$\operatorname{spec}(A).$$

Finally we use the following definition of a non-critical value for an differential operator:

Definition II.2.1. For a differential operator of the form

$$A(\hbar) = \sum_{|\alpha|,|\beta|\leq m} (\hbar D)^\alpha a_{\alpha\beta}(x) (\hbar D)^\beta,$$

where the operator is defined via the associated quadratic form. We call a number E in $\mathbb{R}$ non-critical if there exists $c > 0$ such that

$$|\nabla_p a_0(x,p)| \geq c \quad \text{for all } (x,p) \in a_0^{-1}(\{E\}),$$

where

$$a_0(x,p) = \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x) p^{\alpha+\beta}.$$

This condition is not the usual non-critical condition as we only assume the gradient in p to be non vanishing. This assumption is called ξ-microhyperbolicty in [10, 15] and is also called a non-critical condition in [28].

II.3 Approximation of operators

In this section we will construct our approximating (framing) operators. The construction is similar to the one used in [2, 10, 12, 13] and [28]. The most important part in this construction is Proposition II.3.2, which also can be found in [2, 15]. Before we state it we need a definition. The definition is

Definition II.3.1. For k in $\mathbb{N}_0$ and μ in $(0,1]$ we denote by $C^{k,\mu}(\mathbb{R}^d)$ the subspace of $C^k(\mathbb{R}^d)$ defined by

$$C^{k,\mu}(\mathbb{R}^d) = \big\{ f \in C^k(\mathbb{R}^d) \,\big|\, |\partial_x^\alpha f(x) - \partial_x^\alpha f(y)| \leq C|x-y|^\mu \; \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| = k \big\}.$$

We can now state the proposition.

Proposition II.3.2. *Let f be in $C^{k,\mu}(\mathbb{R}^d)$ for a μ in $(0,1]$. Then for every $\varepsilon > 0$ there exists a function f_ε in $C^\infty(\mathbb{R}^d)$ such that*

$$\begin{aligned} |\partial_x^\alpha f_\varepsilon(x) - \partial_x^\alpha f(x)| \leq & C_\alpha \varepsilon^{k+\mu-|\alpha|} \qquad |\alpha| \leq k, \\ |\partial_x^\alpha f_\varepsilon(x)| \leq & C_\alpha \varepsilon^{k+\mu-|\alpha|} \qquad |\alpha| \geq k+1, \end{aligned} \tag{3.6}$$

where the constants is independent of ε.

The function f_ε is a smoothing (mollification) of f. Usually this is done by convolution with a compactly supported smooth function. However here we will use a Schwartz function in the convolution in order to ensure the stated error terms. The convolution with a compactly supported smooth function will in most cases "only" give an error of order ε except if k is equal to 1.

Proof. We start by letting $\mathcal{F}_1[\omega]$ be in $C_0^\infty(B(0,1))$ with $\mathcal{F}_1[\omega](p) = 1$ for all p in $B(0,\frac{1}{2})$, $0 \leq \mathcal{F}_1[\omega] \leq 1$ and $\mathcal{F}_1[\omega](p) = \mathcal{F}_1[\omega](-p)$. Then

$$\omega(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,p\rangle} \mathcal{F}_1[\omega](p)\, dp = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \cos(\langle x,p\rangle) \mathcal{F}_1[\omega](p)\, dp,$$

is a real Schwartz function. Hence for all n in $\mathbb{N}_0$ there exists a c_n such that

$$|\omega(y)| \leq \frac{c_n}{(1+|y|)^n}. \tag{3.7}$$

Moreover we can note that for all α in $\mathbb{N}_0$ with $|\alpha| > 0$ we have

$$\int_{\mathbb{R}^d} y^\alpha \omega(y)\, dy = (-D_p)^\alpha \mathcal{F}_1[\omega](p)\big|_{p=0} = 0. \tag{3.8}$$

We now let $\omega_\varepsilon(x) = \varepsilon^{-d}\omega(\varepsilon^{-1}x)$ and define

$$f_\varepsilon(x) = f * \omega_\epsilon(x) = \int_{\mathbb{R}^d} f(x-\varepsilon y)\omega(y)\, dy.$$

For α in $\mathbb{N}_0^d$ with $|\alpha| \leq k-1$ we let

$$R_\alpha(x,y,\varepsilon) = \partial_x^\alpha f(x-\varepsilon y) - \sum_{\beta : |\alpha+\beta| \leq k} \frac{(-\varepsilon y)^\beta}{\beta!} \partial_x^{\alpha+\beta} f(x),$$

that is R_α is the reminder term of the Taylor expansion of $\partial_x^\alpha f(x-\varepsilon y)$ around x up to order k. By Taylor expanding $\partial_x^\alpha f(x-\varepsilon y)$ with exact reminder estimate we get

$$\begin{aligned}
&|R_\alpha(x,y,\varepsilon)| \\
&\quad = \Big| \sum_{\beta:|\alpha+\beta|=k} k\frac{(-\varepsilon y)^\beta}{\beta!} \int_0^1 (1-s)^{k-1} \partial_x^{\beta+\alpha} f(x-\varepsilon s y)\, ds - \frac{(-\varepsilon y)^\beta}{\beta!} \partial_x^{\beta+\alpha} f(x) \Big| \\
&\quad \leq \sum_{\beta:|\alpha+\beta|=k} k\frac{|\varepsilon y|^\beta}{\beta!} \int_0^1 (1-s)^{k-1} |\partial_x^{\beta+\alpha} f(x-\varepsilon s y) - \partial_x^{\beta+\alpha} f(x)|\, ds \\
&\quad \leq c_f \varepsilon^{k+\mu-|\alpha|} (1+|y|)^{k+1},
\end{aligned}$$

where we in the last inequality have used the uniform Hölder continuity of the k'th derivative. Note that the constant c_f only depends on the function thought the constant in the Hölder continuity of the k'th derivative. If we now use this estimate, (3.7) and (3.8) we have

$$\begin{aligned}
|\partial_x^\alpha f_\varepsilon(x) - \partial_x^\alpha f(x)| &= \Big| \int_{\mathbb{R}^d} (\partial_x^\alpha f(x-\varepsilon y) - \partial_x^\alpha f(x)) \omega(y)\, dy \Big| \\
&= \Big| \int_{\mathbb{R}^d} (R_\alpha(x,y,\varepsilon) - \sum_{\beta:1\leq|\alpha+\beta|\leq k} \frac{(-\varepsilon y)^\beta}{\beta!} \partial_x^{\alpha+\beta} f(x)) \omega(y)\, dy \Big| \\
&\leq \int_{\mathbb{R}^d} |R_\alpha(x,y,\varepsilon)\omega(y)|\, dy \\
&\leq c\varepsilon^{k+\mu-|\alpha|} \int_{\mathbb{R}^d} \frac{1}{(1+|y|)^{d+1}}\, dy \\
&\leq C_\alpha \varepsilon^{k+\mu-|\alpha|}.
\end{aligned}$$

This gives the first statement in (3.6) in the case $|\alpha| \leq k-1$ for $|\alpha| = k$ we have by the uniform Hölder continuity

$$\begin{aligned}
|\partial_x^\alpha f_\varepsilon(x) - \partial_x^\alpha f(x)| &= \Big| \int_{\mathbb{R}^d} (\partial_x^\alpha f(x-\varepsilon y) - \partial_x^\alpha f(x)) \omega(y)\, dy \Big| \\
&\leq \int_{\mathbb{R}^d} |\partial_x^\alpha f(x-\varepsilon y) - \partial_x^\alpha f(x)| \omega(y)|\, dy \\
&\leq C\varepsilon^\mu \int_{\mathbb{R}^d} |y|^\mu \omega(y)\, dy \leq C_\alpha \varepsilon^\mu.
\end{aligned}$$

For the second statement in (3.6) we let $|\alpha| \geq k+1$ and take a $\beta < \alpha$ in $\mathbb{N}_0^d$ such that $|\alpha - \beta| = k$ then

$$\begin{aligned}
|\partial_x^\alpha f_\varepsilon(x)| &= \varepsilon^{-|\beta|} \Big| \int_{\mathbb{R}^d} (\partial_x^{\alpha-\beta} f(x-\varepsilon y) - \partial_x^{\alpha-\beta} f(x)) \partial_y^\beta \omega(y)\, dy \Big| \\
&\leq c\varepsilon^{\mu-|\beta|} \int_{\mathbb{R}^d} |\partial_y^\beta \omega(y)|\, dy \leq C_\alpha \varepsilon^{k+\mu-|\alpha|}.
\end{aligned}$$

this yields the second statement of (3.6) and hence concludes the proof. □

We will in the following give a situation where the framing operators can be found. We will consider is a differential operator of the form

$$A(\hbar) = \sum_{|\alpha|,|\beta|\leq m} (\hbar D)^\alpha a_{\alpha\beta}(x)(\hbar D)^\beta,$$

where the operator is defined via the associated quadratic form and the order is $2m$. In order to find the framing operators we need to assume the operator is globally elliptic. The type of framing operators used here is the same form as the framing operators used in [28].

Proposition II.3.3. *Let $A(\hbar)$ be a differential operator of order $2m$ of the form*

$$A(\hbar) = \sum_{|\alpha|,|\beta|\leq m} (\hbar D)^\alpha a_{\alpha\beta}(x)(\hbar D)^\beta,$$

where the coefficients $a_{\alpha\beta}(x)$ are in $C^{k,\mu}(\mathbb{R}^d)$. Suppose there exists a constant C such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)p^{\alpha+\beta} \geq C|p|^{2m}, \tag{3.9}$$

for all (x,p) in $\mathbb{R}^d_x \times \mathbb{R}^d_p$. Moreover we suppose $A(\hbar)$ is self-adjoint and lower semi-bounded and let $A_\varepsilon(\hbar)$ be the operator obtainded by replacing the coefficients of $A(\hbar)$ by $a^\varepsilon_{\alpha\beta}(x)$ which is the smoothed function of $a_{\alpha\beta}(x)$ according to Proposition II.3.2. Then there exists a set of framing operators $A^-_\varepsilon(\hbar)$ and $A^+_\varepsilon(\hbar)$ of the form

$$A^\pm_\varepsilon(\hbar) = A_\varepsilon(\hbar) \pm C_1\varepsilon^{k+\mu}(I - \hbar^2\Delta)^m,$$

where these operators are globally elliptic for all sufficiently small ε and satisfy the inequalities

$$A^-_\varepsilon(\hbar) \leq A(\hbar) \leq A^+_\varepsilon(\hbar),$$

in the sense of quadratic forms. Moreover if 0 is a non-critical value of $A(\hbar)$ in the sense of Definition II.2.1 then $A^-_\varepsilon(\hbar)$ and $A^+_\varepsilon(\hbar)$ can be chosen such 0 is also non-critical for these operators for all sufficiently small ε.

Remark II.3.4. If we consider the magnetic Schrödinger operator

$$H = (-i\hbar\nabla + A)^2 + V, \tag{3.10}$$

where $A(x)$ is a magnetic vector potential and V is the electric potential. Then will H, under suitable assumptions on A of V, satisfy the assumptions from the previous proposition. In the case of Schrödinger operator without a magnetic potential the framing operators $H^\pm_\varepsilon$ can be chosen as

$$H^\pm_\varepsilon = -\hbar^2\Delta + V_\varepsilon \pm c\varepsilon^{k+\mu}.$$

Proof. We start by considering the operator $A_\varepsilon(\hbar)$ of the form

$$A_\varepsilon(\hbar) = \sum_{|\alpha|,|\beta|\le m} (\hbar D)^\alpha a^\varepsilon_{\alpha\beta}(x)(\hbar D)^\beta$$

where we have replaced the coefficients of $A(\hbar)$ with smooth functions made according to Proposition II.3.2. For φ in $\mathcal{Q}(A(\hbar)) \cap \mathcal{Q}(A_\varepsilon(\hbar))$ we have by a Cauchy-Schwarz inequality

$$\begin{aligned}|A(\hbar)[\varphi,\varphi] - A_\varepsilon(\hbar)[\varphi,\varphi]| \le &\sum_{|\alpha|,|\beta|\le m} |\langle (a_{\alpha\beta} - a^\varepsilon_{\alpha\beta})(\hbar D)^\beta \varphi, (\hbar D)^\alpha \varphi\rangle| \\ \le &\sum_{|\alpha|,|\beta|\le m} \frac{1}{2\varepsilon^{k+\mu}} \|(a_{\alpha\beta} - a^\varepsilon_{\alpha\beta})(\hbar D)^\beta\varphi\|^2_{L^2(\mathbb{R}^d)} + \frac{\varepsilon^{k+\mu}}{2}\|(\hbar D)^\alpha \varphi\|^2_{L^2(\mathbb{R}^d)} \\ \le & c\varepsilon^{k+\mu} \sum_{|\alpha|\le m} \langle (\hbar D)^{2\alpha}\varphi,\varphi\rangle.\end{aligned} \tag{3.11}$$

We recognise the last bound in (3.11) as the quadratic form associated to $(I-\hbar^2\Delta)^m$. Hence for sufficiently choice of constant we can choose the framing operators by taking

$$A_\varepsilon(\hbar) \pm c\varepsilon^{k+\mu}(I-\hbar^2\Delta)^m,$$

where the operator are defined in the sense of quadratic forms. We first consider the ellipticity of the framing operators. Here we have

$$\begin{aligned}&\sum_{|\alpha|=|\beta|=m} a^\varepsilon_{\alpha\beta}(x)p^{\alpha+\beta} \pm c\varepsilon^{k+\mu}|p|^{2m} \\ &\qquad = \sum_{|\alpha|=|\beta|=m} (a^\varepsilon_{\alpha\beta}(x) - a_{\alpha\beta}(x))p^{\alpha+\beta} + \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)p^{\alpha+\beta} \pm \varepsilon^{k+\mu}|p|^{2m} \\ &\qquad \ge -\tilde{c}\varepsilon^{k+\mu}|p|^{2m} + C|p|^{2m} \ge \tilde{C}|p|^{2m},\end{aligned} \tag{3.12}$$

for sufficiently small ε and all (x,p) in $\mathbb{R}^d_x \times \mathbb{R}^d_p$. Hence if we choose C_1 such (3.11) and (3.12) both are satisfied and take

$$A^\pm_\varepsilon(\hbar) = A_\varepsilon(\hbar) \pm C_1\varepsilon^{k+\mu}(I-\hbar^2\Delta)^m,$$

then both operators are uniform elliptic and by (3.11) these operator satisfy the the inequalities

$$A^-_\varepsilon(\hbar) \le A(\hbar) \le A^+_\varepsilon(\hbar),$$

in the sense of quadratic forms.

For the last part we assume 0 is a non-critical value for the operator $A(\hbar)$ that is there exist a $c>0$ such that

$$|\nabla_p a_0(x,p)| \ge c \quad \text{for all } (x,p) \in a_0^{-1}(\{0\}),$$

where

$$a_0(x,p) = \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)p^{\alpha+\beta}.$$

In order to prove that 0 is a non-critical value for the framing operators we need to find an expression for $a_{\varepsilon,0}^{-1}(\{0\})$ for the framing operators, where we have omitted the + and − in the notation. By the ellipticity we can in the following calculation with out loss of generality assume p belongs to a bounded set. We have

$$\begin{aligned} &a_{\varepsilon,0}(x,p) \\ &\quad= \sum_{|\alpha|,|\beta|\leq m} (a^{\varepsilon}_{\alpha\beta}(x) - a_{\alpha\beta}(x))p^{\alpha+\beta} \pm C_1\varepsilon^{k+\mu}(1+p^2)^m + \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)p^{\alpha+\beta}. \end{aligned} \tag{3.13}$$

Since we can assume p to be in a compact set we have that

$$\Big| \sum_{|\alpha|,|\beta|\leq m} (a^{\varepsilon}_{\alpha\beta}(x) - a_{\alpha\beta}(x))p^{\alpha+\beta} \pm C_1\varepsilon^{k+\mu}(1+p^2)^m \Big| \leq C\varepsilon^{k+\mu}.$$

This combined with (3.13) implies the inclusion

$$\{(x,p)\in\mathbb{R}^{2d} \,|\, a_{\varepsilon,0}(x,p)=0\} \subseteq \{(x,p)\in\mathbb{R}^{2d} \,|\, |a_0(x,p)| \leq C\varepsilon^{k+\mu}\}.$$

Hence for a sufficiently small ε we have the inclusion

$$\{(x,p)\in\mathbb{R}^{2d} \,|\, a_{\varepsilon,0}(x,p)=0\} \subseteq \{(x,p)\in\mathbb{R}^{2d} \,|\, |\nabla_p a_0(x,p)| \geq \frac{c}{2}\}. \tag{3.14}$$

by continuity. For a point (x,p) in $\{(x,p)\in\mathbb{R}^{2d} \,|\, a_{\varepsilon,0}(x,p)=0\}$ we have

$$\begin{aligned} &\nabla_p a_{\varepsilon,0}(x,p) \\ &\quad= \sum_{|\alpha|,|\beta|\leq m} (a^{\varepsilon}_{\alpha\beta}(x) - a_{\alpha\beta}(x))\nabla_p p^{\alpha+\beta} \pm C_1\varepsilon^{k+\mu}\nabla_p(1+p^2)^m + \nabla_p a_0(x,p). \end{aligned} \tag{3.15}$$

Again since we can assume p to be contained in a compact set we have

$$\Big| \sum_{|\alpha|,|\beta|\leq m} (a^{\varepsilon}_{\alpha\beta}(x) - a_{\alpha\beta}(x))\nabla_p p^{\alpha+\beta} \pm C_1\varepsilon^{k+\mu}\nabla_p(1+p^2)^m \Big| \leq C\varepsilon^{k+\mu}.$$

Combining this with (3.14) and (3.15) we get

$$|\nabla_p a_{\varepsilon,0}(x,p)| \geq |\nabla_p a_0(x,p)| - C\varepsilon^{k+\mu} \geq \frac{c}{2} - C\varepsilon^{k+\mu} \geq \frac{c}{4}. \tag{3.16}$$

where the last inequality is for ε sufficiently small. This inequality proves 0 is also a non-critical value of the framing operators. □

The framing operators constructed in the previous proposition are operators with smooth coefficients. But when we take derivatives of these coefficients we start to get negative powers of ε from some point. Hence the classic theory of pseudo-differential operators can not a priori be applied.

We will in the following sections see that in fact it is possible to verify most of the results from classic theory of pseudo-differential operators. After this has been developed we will return to these framing operators.

Essential there is not a unique way to construct these framing operators but a large number of different choices.

II.4 Definitions and quantisations of rough pseudo-differential operators

In this section we will inspired by the approximation results in the previous section define a class of pseudo-differential operators with rough symbols and state and prove some of the properties of these operators relating to quantisation. The definitions and proof are very similar to the definitions in the monograph [17]. Before we define our rough symbols we recall the definition of a tempered weight function for the sake of completeness.

Definition II.4.1. A tempered weight function on $\mathbb{R}^D$ is a continuous function

$$m : \mathbb{R}^D \to [0, \infty[,$$

for which there exists positive constants C_0, N_0 such that for all points x_1 in $\mathbb{R}^D$ the estimate

$$m(x) \leq C_0 m(x_1)(1 + |x_1 - x|)^{N_0},$$

holds for all points x in $\mathbb{R}^D$.

For our purpose here we will consider the cases where $D = 2d$ or $D = 3d$. These types of functions is in the literature sometimes called order functions this is the case in the monographs [5, 32]. But we have chosen the name tempered weights to align with the terminology in the monographs [8, 17]. We can now define the symbols we will be working with.

Definition II.4.2 (Rough symbol). Let $\Omega \subseteq \mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y$ be open, ρ be in $[0, 1]$, $\varepsilon > 0$, τ be in $\mathbb{Z}$ and m a tempered weight function on $\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y$. We call a function a_ε a rough symbol of regularity τ with weights (m, ρ, ε) if a_ε is in $C^\infty(\Omega)$ and satisfies that

$$\begin{aligned} &|\partial_x^\alpha \partial_p^\beta \partial_y^\gamma a_\varepsilon(x, p, y)| \\ &\quad \leq \begin{cases} C_{\alpha\beta\gamma} m(x, p, y)(1 + |(x, p, y)|)^{-\rho(|\alpha|+|\beta|+|\gamma|)} & \text{if } |\alpha| + |\gamma| \leq \tau \\ C_{\alpha\beta\gamma} \varepsilon^{\tau-|\alpha|-|\gamma|} m(x, p, y)(1 + |(x, p, y)|)^{-\rho(|\alpha|+|\beta|+|\gamma|)} & \text{if } |\alpha| + |\gamma| > \tau, \end{cases} \end{aligned} \tag{3.17}$$

for all (x, p, y) in Ω and α, β, γ in $\mathbb{N}_0^d$, where the constants $C_{\alpha\beta\gamma}$'s do not depend on ε. The space of these functions is denoted $\Gamma^{m,\tau}_{\rho,\varepsilon}(\Omega)$. The space can be turned into a Fréchet space with semi norms associated to the estimates in (3.17).

Remark II.4.3. It is important to note that the semi norms on $\Gamma^{m,\tau}_{\rho,\varepsilon}(\Omega)$ should be chosen weighted such that the norms associated to a set of numbers α, β, γ will be bounded by the constant $C_{\alpha\beta\gamma}$ and hence independent of ε.

If ε is equal to 1, then are these symbols the same as the symbols defined in Robert [17] (Definition II-10). We will always assume $\varepsilon \leq 1$ as we are interested in the cases of very small ε.

We will later call a function $a_\varepsilon(x, p)$ or $b_\varepsilon(p, y)$ a rough symbol if it satisfies the above definition in the two variables x and p or p and y. This more general definition is made in order to define the different forms of quantisation and the interpolation between them.

If we say a symbol of regularity τ with tempered weight m we implicit assume that $\rho = 0$.

This type of rough symbols is contained in the class of rough symbols consider in [15, Section 2.3 and 4.6].

The following remark will be crucial.

Remark II.4.4. We will later assume that a rough symbol is a tempered weight. When this is done we will implicit assume that the constants from the definition of a tempered weight is independent of ε. This is an important assumption since we need the estimates we make to be uniform for $\hbar$ in $(0, \hbar_0]$ with $\hbar_0 > 0$ sufficiently small and then for a choice of δ in $(0, 1)$ we need the estimates to be uniform for ε in $[\hbar^{1-\delta}, 1]$.

Essentially the constants will be uniform for both $\hbar$ in $(0, \hbar_0]$ and ε in $(0, 1]$, but if $\varepsilon \leq \hbar$ then the estimates will diverge in the semiclassical parameter. Hence we will assume the lower bound on ε and when this bound is assumed we will hide ε in $\hbar$. The assumption that $\varepsilon \geq \hbar^{1-\delta}$ is in [10, 15] called a microlocal uncertainty principal. In [10, 15] there is two parameter instead of just one. This other parameter can to my knowledge be used to scale in the p-variable.

As we are interested in asymptotic expansions in the semiclassical parameter we will define $\hbar$-ε-admissible symbols, which is the symbols depending on the semiclassical parameter $\hbar$ for which we can make an expansion in $\hbar$.

Definition II.4.5. With the notation from Definition II.4.2. We call a symbol $a_\varepsilon(\hbar)$ $\hbar$-ε-admissible of regularity τ with weights (m, ρ, ε) in Ω, if for fixed ε and a $\hbar_0 > 0$ the map that takes $\hbar$ into $a_\varepsilon(\hbar)$ is smooth from $(0, \hbar_0]$ into $\Gamma^{m,\tau}_{\rho,\varepsilon}(\Omega)$ such that there exists a N_0 in $\mathbb{N}_0$ such for all $N \geq N_0$ we have

$$a_\varepsilon(x, p, y; \hbar) = a_{\varepsilon,0}(x, p, y) + \hbar a_{\varepsilon,1}(x, p, y) + \cdots + \hbar^N a_{\varepsilon,N}(x, p, y) + \hbar^{N+1} r_N(x, p, y; \hbar),$$

where $a_{\varepsilon,j}$ is in $\Gamma^{m,\tau_j}_{\rho,\varepsilon,-2j}(\Omega)$ with the notation $\tau_j = \tau - j$ and r_N is a symbol satisfying the bounds

$$\begin{aligned} &\hbar^{N+1} |\partial_x^\alpha \partial_p^\beta \partial_y^\gamma r_N(x, p, y; \hbar)| \\ &\quad \leq C_{\alpha\beta\gamma} \hbar^{\kappa_1(N)} \varepsilon^{-|\alpha|-|\gamma|} m(x, y, p)(1 + |(x, y, p)|)^{-\rho(\kappa_2(N)+|\alpha|+|\beta|+|\gamma|)}, \end{aligned}$$

where κ_1 is a positive strictly increasing function and κ_2 is non-decreasing function. For k in $\mathbb{Z}$ $\Gamma^{m,\tau}_{\rho,\varepsilon,k}(\Omega)$ is the space of rough symbols of regularity τ with weights $(m(1 + |(x, y, p)|)^{k\rho}, \rho, \varepsilon)$.

Remark II.4.6. We will also use the terminology $\hbar$-ε-admissible for symbols in two variables, where the definition is the same just in two variables. This definition is slightly different to the "usual" definition of an $\hbar$-admissible symbol [17, Definition II-11]. One difference is in the error term. Here is the fist sign of error terms getting small in the semiclassical parameter but not as fast as in the non-rough case. The functions κ_1 and κ_2 will in most cases be dependent on the tempered weight function through the constants in the definition of a tempered weight, the regularity τ and the dimension d. It should be noted that the function κ_2 might be constant negative.

We will now define the pseudo-differential operators associated to the rough symbols. We will call them rough pseudo-differential operators.

Definition II.4.7. Let m be a tempered weight function on $\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y$, ρ in $[0,1]$, $\varepsilon > 0$ and τ in $\mathbb{Z}$. For a rough symbol a_ε in $\Gamma^{m,\tau}_{\rho,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y)$ we associate the operator $\mathrm{Op}_\hbar(a_\varepsilon)$ defined by

$$\mathrm{Op}_\hbar(a_\varepsilon)\psi(x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} a_\varepsilon(x,p,y)\psi(y)\, dy\, dp,$$

for ψ in $\mathcal{S}(\mathbb{R}^d)$.

Remark II.4.8. With the notation from Definition II.4.7. We remark that the integral in the definition of $\mathrm{Op}_\hbar(a_\varepsilon)\psi(x)$ shall be considered as an oscillating integral. By applying the techniques for oscillating integrals it can be proven that $\mathrm{Op}_\hbar(a_\varepsilon)$ is a continuous linear operator from $\mathcal{S}(\mathbb{R}^d)$ into itself. The proof of this is analogous to the proof in [17] in the non-rough case. Hence by duality it is can also be defined as an operator from $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.

Definition II.4.9. We call an operator $A_\varepsilon(\hbar)$ from $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), L^2(\mathbb{R}^d))$ $\hbar$-ε-admissible of regularity $\tau \geq 0$ with tempered weight m if for fixed ε and a $\hbar_0 > 0$ the map

$$A_\varepsilon : (0, \hbar_0] \to \mathcal{L}(\mathcal{S}(\mathbb{R}^d), L^2(\mathbb{R}^d))$$

is smooth. There exists a sequence $a_{\varepsilon,j}$ in $\Gamma^{m,\tau_j}_{0,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y)$, where $\tau_0 = \tau$ and $\tau_{j+1} = \tau_j - 1$ and a sequence R_N in $\mathcal{L}(L^2(\mathbb{R}^d))$ such that for $N \geq N_0$, N_0 sufficient large,

$$A_\varepsilon(\hbar) = \sum_{j=0}^{N} \hbar^j \,\mathrm{Op}_\hbar(a_{\varepsilon,j}) + \hbar^{N+1} R_N(\varepsilon,\hbar), \tag{3.18}$$

and

$$\hbar^{N+1} \|R_N(\varepsilon,\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \hbar^{\kappa(N)} C_N,$$

for a strictly positive increasing function κ.

Remark II.4.10. By the results in Theorem II.6.1 we have if the tempered weight function m is in $L^\infty(\mathbb{R}^d)$ then for a $\hbar$-ε-admissible symbol $a_\varepsilon(\hbar)$ of regularity $\tau \geq 0$ with tempered weight m the operator $A_\varepsilon(\hbar) = \mathrm{Op}_\hbar(a_\varepsilon(\hbar))$ is a $\hbar$-ε-admissible operator of regularity τ.

Remark II.4.11. When we have an operator $A_\varepsilon(\hbar)$ with an expansion

$$A_\varepsilon(\hbar) = \sum_{j\geq 0} \hbar^j \,\mathrm{Op}_\hbar(a_{\varepsilon,j}),$$

where the sum is understood as a formal sum and in the sense that for all N sufficiently large there exists R_N in $\mathcal{L}(L^2(\mathbb{R}^d))$ such that the operator is of the same form as in (3.18). Then we call the symbol $a_{\varepsilon,0}$ the principal symbol and the symbol $a_{\varepsilon,1}$ the subprincipal symbol.

Definition II.4.12. Let $A_\varepsilon(\hbar)$ be a $\hbar$-ε-admissible of regularity τ with tempered weight m. For any t in $[0,1]$ we call all $\hbar$-ε-admissible symbols $b_\varepsilon(\hbar)$ in $\Gamma^{m,\tau}_{0,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p)$ such,

$$A_\varepsilon(\hbar)\psi(x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} b_\varepsilon((1-t)x+ty,p;\hbar)\psi(y)\,dy\,dp,$$

for all $\psi \in \mathcal{S}(\mathbb{R}^d)$ and all $\hbar \in]0,\hbar_0]$, where $\hbar_0$ is a strictly positive number, rough t-ε-symbols of regularity τ associated to $A_\varepsilon(\hbar)$.

Notation II.4.13. In general for a symbol $b_\varepsilon(\hbar)$ in $\Gamma^{m,\tau}_{\rho,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p)$ and ψ in $\mathcal{S}(\mathbb{R}^d)$ we will use the notation

$$\mathrm{Op}_{\hbar,\mathrm{t}}(b_\varepsilon)\psi(x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} b_\varepsilon((1-t)x+ty,p;\hbar)\psi(y)\,dy\,dp$$

We have the special case of Weyl quantisation when $t=\frac{1}{2}$, which is the one we will work the most with. In this case we write

$$\mathrm{Op}_{\hbar,\frac{1}{2}}(b_\varepsilon) = \mathrm{Op}_\hbar^{\mathrm{w}}(b_\varepsilon).$$

For some application we will need stronger assumptions than $\hbar$-ε-admissibility of our operators. The operators satisfying these stronger assumptions will be called strongly $\hbar$-ε-admissible operators with some regularity. As an example we could consider a symbol $a_\varepsilon(x,p)$ in $\Gamma^{m,\tau}_{\rho,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p)$. For this symbol we could then consider $\tilde{a}_\varepsilon(x,p,y) = a_\varepsilon(tx+(1-t)y,p)$ and ask if this symbol is in $\Gamma^{\tilde{m},\tau}_{\rho,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y)$, where $\tilde{m}(x,p,y) = m(tx+(1-t)y,p)$. The answer will not in general be positive. Hence in general we can not ensure decay in the variables (x,p,y) when viewing a function of (x,p) as a function of (x,p,y). With this in mind we define a new class of symbols and strongly $\hbar$-ε-admissible operators.

Definition II.4.14. A symbol a_ε belongs to the class $\tilde{\Gamma}^{m,\tau}_{\rho,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y)$ if a_ε is in $\Gamma^{m,\tau}_{0,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y)$ and there exists a positive ν such that

$$a_\varepsilon \in \Gamma^{m,\tau}_{\rho,\varepsilon}(\Omega_\nu),$$

where $\Omega_\nu = \{(x,p,y) \in \mathbb{R}^{3d} \,|\, |x-y| < \nu\}$.

Definition II.4.15. We call the family of operators $A_\varepsilon(\hbar) = \mathrm{Op}_\hbar(a_\varepsilon(\hbar))$ strongly $\hbar$-ε-admissible of regularity τ if $a_\varepsilon(\hbar)$ is an $\hbar$-ε-admissible symbol of regularity τ with respect to the weights $(m,0,\varepsilon)$ on $\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y$ and the weights (m,ρ,ε) on $\Omega_\nu = \{(x,p,y) \in \mathbb{R}^{3d} \,|\, |x-y| < \nu\}$ for a positive ν.

Remark II.4.16. It should be noted that a strongly $\hbar$-ε-admissible operator is also $\hbar$-ε-admissible but as a consequence of the definition the error term of a strongly $\hbar$-ε-admissible operator will be a pseudo-differential operator and not just a bounded operator as for the $\hbar$-ε-admissible operators.

In what follows we will investigate the connection between strongly $\hbar$-ε-admissible operators and operators defined by t-quantisation. Before we proceed with this we will just recall Quadratic stationary phase asymptotics.

Theorem II.4.17. *Let B be a invertible, symmetric real $d \times d$ matrix and $(u,v) \to a(u,v;\hbar)$ be a function in $C^\infty(\mathbb{R}^d_u \times \mathbb{R}^n_v)$ for all $\hbar$ in $(0,\hbar_0]$. We suppose $v \to a(u,v;\hbar)$ has compact support for all u in $\mathbb{R}^d_u$ and $\hbar$ in $(0,\hbar_0]$. Moreover we let*

$$I(u;a,B,\hbar) = \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle Bv,v\rangle} a(u,v;\hbar)\,dv.$$

Then for each N in $\mathbb{N}$ we have

$$\begin{aligned} I(u;&a,B,\hbar) \\ &= (2\pi\hbar)^{\frac{n}{2}} \frac{e^{i\frac{\pi}{4}\operatorname{sgn}(B)}}{|\det(B)|^{\frac{1}{2}}} \sum_{j=0}^{N} \frac{\hbar^j}{j!}\Big(\frac{\langle B^{-1}D_v, D_v\rangle}{2i}\Big)^j a(u,v;\hbar)\Big|_{v=0} + \hbar^{N+1} R_{N+1}(u;\hbar), \end{aligned}$$

where $\operatorname{sgn}(B)$ is the difference between the number of positive and negative eigenvalues of B. Moreover there exists a constant c_n only depending on the dimension such the error term R_{N+1} satisfies the bound

$$|R_{N+1}(u;\hbar)| \leq c_n \Big\| \frac{\langle B^{-1}D_v, D_v\rangle^{N+1}}{(N+1)!} a(u,v;\hbar)\Big\|_{H^{[\frac{n}{2}]+1}(\mathbb{R}^n_v)},$$

where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$ and $\|\cdot\|_{H^{[\frac{n}{2}]+1}(\mathbb{R}^n_v)}$ is the Sobolev norm.

A proof of the theorem can be found in e.g. [17] or [32]. It should be noted that we will apply this theorem where the function a is a rough symbol with some regularity. Hence we need to be aware of the number of derivatives we are taking in the rough variables and especially we will need to be aware when considering the error terms as the estimate involves a number of extra differentiation on the symbol.

We will now prove a connection between operators with symbols in the class $\tilde{\Gamma}^{m,\tau}_{\rho,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y)$ and t-quantised operators.

Theorem II.4.18. *Let a_ε be a symbol in $\tilde{\Gamma}^{m,\tau}_{\rho,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p \times \mathbb{R}^d_y)$ of regularity $\tau \geq 0$ with weights (m,ρ,ε) and*

$$A_\varepsilon(\hbar)\psi(x) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} a_\varepsilon(x,p,y)\psi(y)\,dy\,dp.$$

We suppose there is a δ in $(0,1)$ such $\varepsilon \geq \hbar^{1-\delta}$. Then for every t in $[0,1]$ we can associate a unique t-ε-symbol b_t of regularity τ with weights $(\tilde{m},\rho,\varepsilon)$, where $\tilde{m}(x,p) = m(x,x,p)$. The t-ε-symbol b_t is defined by the oscillating integral

$$b_t(x,p,\hbar) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle u,q\rangle} a_\varepsilon(x+tu,p+q,x-(1-t)u)\,dq\,du$$

and symbol b_t *has the following asymptotic expansion*

$$b_t(x,p;\hbar) = \sum_{j=0}^{N} \hbar^j a_{\varepsilon,j}(x,p) + \hbar^{N+1} r_{\varepsilon,N+1}(x,p;\hbar),$$

where

$$a_{\varepsilon,j}(x,p) = \frac{(-i)^j}{j!} \langle D_u, D_p \rangle^j a_\varepsilon(x+tu,p,x-(1-t)u)\Big|_{u=0},$$

and the error term satisfies that

$$\hbar^{N+1}|\partial_x^\alpha \partial_p^\beta r_{N+1}(x,p,\hbar)| \le C_{d,N,\alpha,\beta}\hbar^{N+1}\varepsilon^{-(\tau-N-2-d-|\alpha|)_-} m(x,p,x)\lambda(x,p)^{\rho N_0},$$

for all α *and* β *in* $\mathbb{N}_0^d$*. In particular we have that*

$$\begin{aligned} a_{\varepsilon,0}(x,p) &= a_\varepsilon(x,p,x) \\ a_{\varepsilon,1}(x,p) &= (1-t)(\nabla_y D_p a_\varepsilon)(x,p,x) - t(\nabla_x D_p a_\varepsilon)(x,p,x). \end{aligned}$$

Remark II.4.19. It can be noted that in order for the error term not to explode, when the semiclassical parameter tends to zero, one needs to take N such that

$$\tau - 1 - d + \delta(N+2+d) \ge 0.$$

If the symbol is a polynomial in one of the variables or both then the asymptotic expansion will be exact and a finite sum. This is in particular the case when "ordinary" differential operators are considered.

Proof. We start with the case where a_ε is a Schwartz function. Then the operator A_ε has the kernel

$$K_{a_\varepsilon,\hbar}(x,y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,q\rangle} a_\varepsilon(x,q,y)\,dq.$$

If now the operator $A_\varepsilon(\hbar)$ had an associated t-ε-symbol $b_t(x,p,\hbar)$ then we would have

$$K_{a_\varepsilon,\hbar}(x,y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,q\rangle} b_t((1-t)x+ty,q,\hbar)\,dq.$$

By the change of coordinates given by $x = \tilde{x} + tu$ and $y = \tilde{x} - (1-t)u$ in the above expressions we have

$$K_{a_\varepsilon,\hbar}(x+tu,x-(1-t)u) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle u,q\rangle} b_t(x,q,\hbar)\,dq.$$

This identity we recognise as the inverse Fourier transform of the t-ε-symbol. Hence the associated t-ε-symbol is given by the integral

$$\begin{aligned} b_t(x,p,\hbar) &= \int_{\mathbb{R}^d} e^{-i\hbar^{-1}\langle u,p\rangle} K_{a_\varepsilon,\hbar}(x+tu,x-(1-t)u)\,du \\ &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle u,q-p\rangle} a_\varepsilon(x+tu,q,x-(1-t)u)\,dq\,du \\ &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle u,q\rangle} a_\varepsilon(x+tu,p+q,x-(1-t)u)\,dq\,du. \end{aligned}$$

Which defines a unique symbol. In order to pass to general symbols which is not Schwarts functions we need to use oscillating integral techniques. That is we replace a_ε by $g_\sigma a_\varepsilon$ and let $\sigma \to \infty$, where g is a Schwartz function which is 1 in a neighbourhood of the origin and

$$g_\sigma(x,p,y) = g(\tfrac{x}{\sigma}, \tfrac{p}{\sigma}, \tfrac{y}{\sigma}).$$

By this we get the existence of the t-ε-symbol as an oscillating integral and hence also the uniqueness.

We now turn to the asymptotic expansion of the t-ε-symbol b_t. Here we will apply quadratic stationary phase approximation and in order to do this we will need a localisation. We introduce a smooth cut-off function χ such that $\operatorname{supp}(\chi) \subset [-2,2]$ and $\chi(t) = 1$ for t in $[-1,1]$ and let

$$\chi_4(u,q) = \chi\Big(\frac{|u|^2 + |q|^2}{\frac{1}{4}\lambda(x,p)^{2\rho}}\Big).$$

With this localisation we split the symbol a_ε in the two parts $a_\varepsilon^{(1)} = \chi_4 a_\varepsilon$ and $a_\varepsilon^{(2)} = (1-\chi_4)a_\varepsilon$ and $b_t^{(j)}$ is the part of the t-ε-symbol corresponding to $a_\varepsilon^{(j)}$.

We start with studing the term $b_t^{(1)}$. Here we use quadratic stationary phase asymptotic (Theorem II.4.17). We will use the theorem with the block matrix B given by

$$B = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix},$$

where $|\det(B)| = 1$, $\operatorname{sgn}(B) = 0$ and $B^{-1} = B$. Thus we have for N in $\mathbb{N}$

$$b_t^{(1)}(x,p,\hbar) = \sum_{j=0}^{N} \hbar^j a_{\varepsilon,j}(x,p) + \hbar^{N+1} r_{N+1}(x,p,\hbar),$$

where

$$a_{\varepsilon,j}(x,p) = \frac{(-i)^j}{j!}\langle D_u, D_q\rangle^j a_\varepsilon(x+tu, p+q, x-(1-t)u)\Big|_{\substack{u=0\\q=0}},$$

and

$$|r_{N+1}(x,p,\hbar)| \leq \Big\| \frac{\langle D_u, D_q\rangle^{N+1}}{(N+1)!}\chi_4 a_\varepsilon(x+tu,p+q,x-(1-t)u)\Big\|_{H^{d+1}(\mathbb{R}^d_u\times\mathbb{R}^d_q)}.$$

In order to control this error term we note that on the support of $a_\varepsilon^{(1)}$ we have

$$|u|^2 + |q|^2 \leq \frac{1}{2}\lambda(x,p)^{2\rho}.$$

From this inequality one can deduce

$$\frac{1}{2}\lambda(x,p) \leq \lambda(x+tu,p+q,x-(1-t)u) \leq 4\lambda(x,p).$$

Moreover since m is a tempered weight function there exists N_0 in $\mathbb{N}_0$ and a positive constant C such that

$$m(x+tu,p+q,x-(1-t)u) \leq C m(x,p,x)\lambda(x,p)^{N_0\rho}.$$

With this we get the following bounds on the error term

$$\begin{aligned}
&\hbar^{N+1}|r_{N+1}(x,p,\hbar)| \\
&\qquad \leq C_{d,N}\hbar^{N+1} \sup_{\substack{(u,q)\in\mathbb{R}^{2d} \\ |\alpha|=|\beta|=N+1 \\ |\gamma|+|\delta|\leq d+1}} |\partial_u^{\gamma+\alpha}\partial_q^{\delta+\beta}\chi_4(u,q)a_\varepsilon(x+tu,p+q,x-(1-t)u)| \\
&\qquad \leq C_{d,N}\hbar^{N+1}\varepsilon^{-(\tau-N-1-d)_-}m(x,x,p)\lambda(x,p)^{\rho N_0}.
\end{aligned}$$

For α and β in $\mathbb{N}_0^d$ an analogous argument yields the bound.

$$\hbar^{N+1}|\partial_x^\alpha\partial_p^\beta r_{N+1}(x,p,\hbar)| \leq C_{d,N,\alpha,\beta}\hbar^{N+1}\varepsilon^{-(\tau-N-2-d-|\alpha|)_-}m(x,p,x)\lambda(x,p)^{\rho N_0}.$$

Which is the desired estimates on the first part of the error term. If we consider the $a_{\varepsilon,j}$'s then by our assumptions on a_ε we have

$$a_{\varepsilon,j}(x,p) \in \Gamma^{\tilde{m},\tau_j}_{\rho,\varepsilon,-2j}\left(\mathbb{R}_x^d\times\mathbb{R}_p^d\right),$$

where the class is defined in Definition II.4.5. What remains is to estimate the part of the error term arising from $b_t^{(2)}$.

On the support of $a_\varepsilon^{(2)}$ we have $|u|^2+|q|^2 \geq \frac{1}{4}\lambda(x,p)^{2\rho}$ this implies the following operator

$$L = \frac{-i\hbar}{|u|^2+|q|^2}\sum_{j=1}^d [q_j\partial_{u_j}+u_j\partial_{q_j}],$$

is well-defined when acting on $a_\varepsilon^{(2)}$. The real tronsposed of L is

$$L^t = i\hbar\sum_{j=1}^d \frac{q_j\partial_{u_j}+u_j\partial_{q_j}}{|u|^2+|q|^2} - \frac{2u_jq_j}{(|u|^2+|q|^2)^2}.$$

By induction we get for k in $\mathbb{N}$

$$(L^t)^k = \frac{(i\hbar)^k}{(|u|^2+|q|^2)^{\frac{k}{2}}}\sum_{|\alpha|+|\beta|\leq k} f_{\alpha,\beta}(u,v)\partial_u^\alpha\partial_q^\beta,$$

where $f_{\alpha,\beta}(u,v)$ are uniformly bounded on the support of $a_\varepsilon^{(2)}$. We note that

$$L(e^{i\hbar^{-1}\langle u,q\rangle}) = e^{i\hbar^{-1}\langle u,q\rangle},$$

which implies

$$\begin{aligned}
&b_t^{(2)}(x,p,\hbar) \\
&\quad = \left(\frac{\hbar}{i}\right)^k \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle u,q\rangle}(L^t)^k a_\varepsilon^{(2)}(x+tu,p+q,x-(1-t)u)\,dq\,du.
\end{aligned}$$

By our assumptions on the symbol a_ε and the definition of a tempered weight, there exist a N_0 in $\mathbb{N}$ and a positive constant C such that

$$|\partial_u^\alpha\partial_q^\beta a_\varepsilon^{(2)}(x+tu,p+q,x-(1-t)u)| \leq \varepsilon^{-(\tau-|\alpha|)_-}Cm(x,p,x)(1+|u|+|q|)^{N_0}$$

for all α and β in $\mathbb{N}_0^d$. Now for $k \geq 2d+2+N_0$ we have

$$
\begin{aligned}
&|b_t^{(2)}(x,p,\hbar)| \\
&\leq \hbar^{k-d} C_k \sum_{|\alpha|+|\beta|\leq k} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mathbf{1}_{\{|u|^2+|q|^2\geq \frac{1}{4}\lambda(x,p)^{2\rho}\}}}{(|u|^2+|q|^2)^{\frac{k}{2}}} \\
&\qquad\qquad \times |\partial_u^\alpha \partial_q^\beta a_\varepsilon^{(2)}(x+tu,p+q,x-(1-t)u)|\,dq\,du \\
&\leq C\hbar^{k-d}\varepsilon^{-(\tau-k)_-} m(x,p,x)\lambda(x,p)^{-\rho(k+1-2d-N_0)}.
\end{aligned}
$$

By analogous arguments we get the estimate

$$
|\partial_x^\alpha \partial_p^\beta b_t^{(2)}(x,p,\hbar)| \leq C\hbar^{k-d}\varepsilon^{-(\tau-k-|\alpha|)_-} m(x,p,x)\lambda(x,p)^{-\rho(k+1-2d-N_0)}.
$$

Hence by choosing k sufficiently large we get a better estimate for $|b_t^{(2)}(x,p,\hbar)|$ and $|\partial_x^\alpha \partial_p^\beta b_t^{(2)}(x,p,\hbar)|$ for α and β in $\mathbb{N}_0^d$, than for the error term from Quadratic stationary phase asymptotic. This yields the desired estimate. □

From this Theorem we immediate obtain the following Corollary, which will prove useful in later sections.

Corollary II.4.20. *Let t_1 be in $[0,1]$ and b_{t_1} be a t_1-ε-symbol of regularity $\tau \geq 0$ with weights (m,ρ,ε) and suppose $\varepsilon \geq \hbar^{1-\delta}$ for a δ in $(0,1)$. Let $A_\varepsilon(\hbar)$ be the associated operator acting on a Schwarzt function by the formula*

$$
A_\varepsilon(\hbar)\psi(x) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} b_{t_1}((1-t_1)x+t_1 y,p)\psi(y)\,dy\,dp.
$$

Then for every t_2 in $[0,1]$ we can associate an admissible t_2-ε-symbol given by the expansion

$$
b_{t_2}(\hbar) = \sum_{j=0}^{N} \hbar^j b_{t_2,j} + \hbar^{N+1} r_{\varepsilon,N+1}(x,p;\hbar),
$$

where

$$
b_{t_2,j}(x,p) = \frac{(t_2-t_1)^j}{j!}(\nabla_x D_p)^j b_{t_1}(x,p),
$$

and the error term satisfies that

$$
\hbar^{N+1}|\partial_x^\alpha\partial_p^\beta r_{N+1}(x,p,\hbar)| \leq C_{d,N,\alpha,\beta}\hbar^{N+1}\varepsilon^{-(\tau-N-2-d-|\alpha|)_-} m(x,p)\lambda(x,p)^{\rho N_0},
$$

for all α and β in $\mathbb{N}_0^d$, the number N_0 is the number connected to the tempered weight m.

This corollary can also be proven directly by considering the kernel as an oscillating integral and the integrant as a function in the variable t_1. To obtain the corollary do a Taylor expansion in t_1 at the point t_2, then do partial integration a number of times and then one would recover the result.

II.5 Composition of rough pseudo-differential operators

With the rough pseudo-differential operators defined and the ability to interpolate between the different quantisations our next aim is results concerning composition of rough pseudo-differential operators. The theorems and proofs in this section is almost equivalent to the ones in [17]. The first result on composition of operators is the following theorem.

Theorem II.5.1. *Let $A_\varepsilon(\hbar)$ and $B_\varepsilon(\hbar)$ be two t-quantised operators given by*

$$A_\varepsilon(\hbar)\psi(x) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-z,p\rangle} a_\varepsilon((1-t)x+tz,p)\psi(z)\,dz\,dp$$

and

$$B_\varepsilon(\hbar)\psi(z) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle z-y,q\rangle} b_\varepsilon((1-t)z+ty,q)\psi(y)\,dy\,dq.$$

Where a_ε and b_ε be two rough symbols of regularity $\tau_1,\tau_2 \geq 0$ with weights (m_1,ρ,ε) and (m_2,ρ,ε) respectively. We suppose there exists a number $\delta>0$ such that $\varepsilon \geq \hbar^{1-\delta}$. Then the operator $C_\varepsilon(\hbar) = A_\varepsilon(\hbar)\circ B_\varepsilon(\hbar)$ is strongly $\hbar$-ε-admissible and $C_\varepsilon(\hbar) = \mathrm{Op}_{\hbar,\mathrm{t}}(c_\varepsilon)$, where c_ε is a rough admissible symbol of regularity $\tau=\min(\tau_1,\tau_2)$ with weights $(m_1m_2,\rho,\varepsilon)$. The symbol c_ε satisfies the following: For every $N\geq N_\delta$ we have

$$c_\varepsilon(\hbar) = \sum_{j=0}^{N}\hbar^j c_{\varepsilon,j} + \hbar^{N+1} r_{\varepsilon,N+1}(a_\varepsilon,b_\varepsilon;\hbar)$$

with

$$c_{\varepsilon,j}(x,p) = \frac{(i\sigma(D_u,D_\mu;D_v,D_\nu))^j}{j!}[\tilde{a}_\varepsilon(x,p;u,v,\mu,\nu)\tilde{b}_\varepsilon(x,p;u,v,\mu,\nu)]\Big|_{\substack{u=v=0\\ \mu=\nu=0}},$$

where

$$\begin{aligned}
\sigma(u,\mu;v,\nu) &= \langle v,\mu\rangle - \langle u,\nu\rangle\\
\tilde{a}_\varepsilon(x,p;u,v,\mu,\nu) &= a_\varepsilon(x+tv+t(1-t)u,\nu+(1-t)\mu+p)\\
\tilde{b}_\varepsilon(x,p;u,v,\mu,\nu) &= b_\varepsilon(x+(1-t)v-t(1-t)u,\nu-t\mu+p),
\end{aligned}$$

Moreover the error term $r_{\varepsilon,N+1}(a_\varepsilon,b_\varepsilon;\hbar)$ satisfies that for every multi indices α,β in $\mathbb{N}_0^d$ there exists a constant $C(N,\alpha,\beta)$ independent of a_ε and b_ε and a natural number M such that:

$$\begin{aligned}
&\hbar^{N+1}|\partial_x^\alpha\partial_p^\beta r_{\varepsilon,N+1}(a_\varepsilon,b_\varepsilon;x,p,\hbar)|\\
&\quad\leq C\varepsilon^{-|\alpha|}\hbar^{\delta(\tau-N-2d-2)_-+\tau-2d-1}\mathcal{G}^{\alpha,\beta}_{M,\tau}(a_\varepsilon,m_1,b_\varepsilon,m_2)m_1(x,\xi)m_2(x,\xi)\\
&\quad\times\lambda(x,\xi)^{-\rho(\tilde{N}(M)+|\alpha|+|\beta|)},
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{G}^{\alpha,\beta}_{M,\tau}(a_\varepsilon,m_1,b_\varepsilon,m_2)\\
&\qquad= \sup_{\substack{|\gamma_1+\gamma_2|+|\eta_1+\eta_2|\leq M\\ (x,\xi)\in\mathbb{R}^{2d}}} \varepsilon^{(\tau-M)_-+|\alpha|}\frac{|\partial_x^\alpha\partial_\xi^\beta(\partial_x^{\gamma_1}\partial_\xi^{\eta_1}a_\varepsilon(x,\xi)\partial_x^{\gamma_2}\partial_\xi^{\eta_2}b_\varepsilon(x,\xi))|}{m_1(x,\xi)m_2(x,\xi)}\\
&\qquad\quad\times\lambda(x,\xi)^{\rho(|\gamma_1+\gamma_2|+|\eta_1+\eta_2|)}.
\end{aligned}$$

The function $\tilde{N}(M)$ is also depending on the weights m_1, m_2 and the dimension d.

Remark II.5.2. The number N_δ is explicit and it is the smallest number such that

$$\delta(N_\delta + 2d + 2 - \tau) + \tau > 2d + 1.$$

This restriction is made in order to ensure that the error term is estimated by the semiclassical parameter raised to a positive power. If one compares this result to the classic result of composition of t-quantised operators. Then there are some similarities and differences. The similarities are in the form of the symbol for the composition and how it is proven. The main differences is that for this new class there is a minimum of terms in the expansion of the symbol for the composition in order to obtain an error that does not diverge as $\hbar \to 0$.

The form of the c's we obtain in the theorem is sometimes written as

$$\begin{aligned} c_\varepsilon(x,\xi;\hbar) = & e^{i\hbar\sigma(D_u,D_\mu;D_v,D_\nu)}[a_\varepsilon(x+tv+t(1-t)u, \nu+(1-t)\mu+\xi) \\ & \times b_\varepsilon(x+(1-t)v-t(1-t)u, \nu - t\mu+\xi)]\Big|_{\substack{u=v=0 \\ \mu=\nu=0}}. \end{aligned}$$

Proof. The first step is to notice that the kernes of $A_\varepsilon(\hbar)$ and $B_\varepsilon(\hbar)$ are

$$K_{A_\varepsilon(\hbar)}(x,z) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-z,p\rangle}, a_\varepsilon((1-t)x+tz,p)\, dp$$

and

$$K_{B_\varepsilon(\hbar)}(z,y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle z-y,q\rangle} b_\varepsilon((1-t)z+ty,q)\, dq,$$

where the integrals is oscillating integrals. Hence the kernel of the operator $C_\varepsilon(\hbar)$ is given by

$$\begin{aligned} K_{C_\varepsilon(\hbar)}(x,y) = \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} & e^{i\hbar^{-1}(\langle x-z,p\rangle + \langle z-y,q\rangle)} a_\varepsilon((1-t)x+tz,p) \\ & \times b_\varepsilon((1-t)z+ty,q)\, dq\, dp\, dz. \end{aligned}$$

As in the proof of Theorem II.4.18 we now have that a t-ε-symbol for the operator $C_\varepsilon(\hbar)$ is given by the following expression in order to correspond to the kernel above.

$$\begin{aligned} c_\varepsilon(x,\xi;\hbar) = & \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}(\langle x+tu-z,p\rangle + \langle z-(x-(1-t)u),q\rangle - \langle u,\xi\rangle)} \\ & a_\varepsilon((1-t)(x+tu)+tz,p) b_\varepsilon((1-t)z+t(x-(1-t)u),q)\, dq\, dp\, dz\, du. \end{aligned}$$

It is important to note that the above integrals all should be understood as oscillating integrals and one can verify that all phase functions satisfies the assumption for the integrals to be well-defined. In order to apply Quadratic stationary phase asymptotic we need to make a change of variables to the variables (μ,ν,v,u) given by

$$\begin{aligned} u &= u & \nu &= tp + (1-t)q - \xi \\ v &= z - x & \mu &= p - q. \end{aligned}$$

The old coordinates can be recovered by the equations

$$\begin{aligned} u &= u & q &= \nu - t\mu + \xi \\ z &= v + x & p &= \nu + (1-t)\mu + \xi. \end{aligned}$$

The determinant of the Jacobmatrix for this change of variable is 1. Hence when we make the change of variables we get

$$\begin{aligned} &c_\varepsilon(x,\xi;\hbar) = \\ &\frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\hbar^{-1}(\langle v,\mu\rangle - \langle u,\nu\rangle)} a_\varepsilon(x+tv+t(1-t)u, \nu+(1-t)\mu+\xi) \\ &\qquad\qquad \times b_\varepsilon(x+(1-t)v - t(1-t)u, \nu - t\mu+\xi)\, d\mu\, d\nu\, dv\, du. \end{aligned}$$

With this change of variables we have transformed the phase function into the map $\sigma(u,\mu;v,\nu) = \langle v,\mu\rangle - \langle u,\nu\rangle$ which corresponds to the Quadratic form on $\mathbb{R}^{4d}$ given by the matrix B defined by

$$B = \begin{pmatrix} 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & 0 & -I_n & 0 \end{pmatrix}.$$

We have that $|\det(B)| = 1$, $\operatorname{sgn}(B) = 0$ and $B^{-1} = B$. In order to apply quadratic stationary phase we have to first make a partition of unity. We let χ be in $C_0^\infty(\mathbb{R})$ such that $\chi(t) = 1$ for $t \in [-1,1]$ and $\chi(t) = 0$ for $|t| \geq 2$. With this function we define

$$\chi_{\frac{1}{16}}(x,\xi;v,\mu,u,\nu) = \chi\Big(\frac{|v|^2+|\mu|^2+|u|^2+|\nu|^2}{\frac{1}{16}\lambda(x,\xi)^{2\rho}}\Big),$$

and we let

$$\begin{aligned} d_1(x,\xi;v,\mu,u,\nu) =& \chi_{\frac{1}{16}}(x,\xi;v,\mu,u,\nu) a_\varepsilon(x+tv+t(1-t)u, \nu+(1-t)\mu+\xi) \\ &\times b_\varepsilon(x+(1-t)v - t(1-t)u, \nu - t\mu+\xi) \\ d_2(x,\xi;v,\mu,u,\nu) =& (1-\chi_{\frac{1}{16}}(x,\xi;v,\mu,u,\nu)) a_\varepsilon(x+tv+t(1-t)u, \nu+(1-t)\mu+\xi) \\ &\times b_\varepsilon(x+(1-t)v - t(1-t)u, \nu - t\mu+\xi). \end{aligned}$$

We now split the expression for $c_\varepsilon(x,\xi;\hbar)$ up in two parts $c_\varepsilon^1(x,\xi;\hbar)$ and $c_\varepsilon^2(x,\xi;\hbar)$ corresponding to the integral over d_1 and d_2 respectively.

We start by considering consider the part arising from the integral of the function $d_2(x,\xi;v,\mu,u,\nu)$. On the support of d_2 we have

$$|v|^2+|\mu|^2+|u|^2+|\nu|^2 \geq \frac{1}{16}\lambda(x,\xi)^{2\rho}.$$

Hence we can construct a linear first order differential operator L which acts on $e^{-i\hbar^{-1}(\langle v,\mu\rangle - \langle u,\nu\rangle)}$ as the identity and then use partial integration. The operator L is given by

$$L = i\hbar \frac{\sum_{j=1}^d [\mu_j\partial_{v_j} + v_j\partial_{\mu_j} - u_j\partial_{\nu_j} - \nu_j\partial_{u_j}]}{|v|^2+|\mu|^2+|u|^2+|\nu|^2}.$$

The real transposed of the operator L is given by

$$L^t = -i\hbar \sum_{j=1}^{d} \Big[\frac{\mu_j \partial_{v_j} + v_j \partial_{\mu_j} - u_j \partial_{\nu_j} - \nu_j \partial_{u_j}}{|v|^2 + |\mu|^2 + |u|^2 + |\nu|^2} - \frac{\mu_j v_j - u_j \nu_j}{(|v|^2 + |\mu|^2 + |u|^2 + |\nu|^2)^2} \Big].$$

By induction we get for M in $\mathbb{N}$

$$(L^t)^M = \frac{(-i\hbar)^M}{(|v|^2 + |\mu|^2 + |u|^2 + |\nu|^2)^{\frac{M}{2}}} \sum_{|\alpha|+|\beta|+|\gamma|+|\delta| \leq M} f^M_{\alpha\beta\gamma\delta}(v, \mu, u, \nu) \partial_v^\alpha \partial_\mu^\beta \partial_u^\gamma \partial_\nu^\delta,$$

where the functions $f^M_{\alpha\beta\gamma\delta}(v, \mu, u, \nu)$ are smooth uniformly bounded functions defined on the support of d_2. We now have

$$\begin{aligned} &c^2_\varepsilon(x, \xi; \hbar) \\ &\quad = \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\hbar^{-1}(\langle v,\mu\rangle - \langle u,\nu\rangle)} (L^t)^M d_2(x, \xi; v, \mu, u, \nu)\, d\mu\, d\nu\, dv\, du. \end{aligned}$$

We will shortly impose conditions on the number M for the integral to be convergent. If we consider the absolute value of the integrant we have

$$\begin{aligned} &|(L^t)^M d_2(x, \xi; v, \mu, u, \nu)| \\ &\leq \frac{\hbar^M}{(|v|^2 + |\mu|^2 + |u|^2 + |\nu|^2)^{\frac{M}{2}}} \sum_{|\alpha|+|\beta|+|\gamma|+|\eta| \leq M} C^M_{\alpha\beta\gamma\eta} |\partial_v^\alpha \partial_\mu^\beta \partial_u^\gamma \partial_\nu^\delta d_2(x, \xi; v, \mu, u, \nu)| \end{aligned}$$

The function d_2 is a product of three different functions $(1 - \chi_{\frac{1}{4}})$, $\tilde{a}_\varepsilon$ and $\tilde{b}_\varepsilon$ hence we need Leibniz's formula in order to estimate the derivatives of d_2. Firstly we note that all derivatives of $(1 - \chi_{\frac{1}{4}})$ are uniformly bounded. Hence in estimating the derivatives of d_2 what is important is the derivatives of $\tilde{a}_\varepsilon$ and $\tilde{b}_\varepsilon$. In the following we will use the notation and estimates

$$\begin{aligned} \tilde{m}_1(x, \xi; v, \mu, u, \nu) &= m_1(x + tv + t(1-t)u, \nu + (1-t)\mu + \xi) \\ &\leq C m_1(x, \xi)(1 + |u| + |v| + |\nu| + |\mu|)^{N_0}, \end{aligned}$$

and

$$\begin{aligned} \tilde{m}_2(x, \xi; v, \mu, u, \nu) &= m_2(x + (1-t)v - t(1-t)u, \nu - t\mu + \xi) \\ &\leq C m_2(x, \xi)(1 + |u| + |v| + |\nu| + |\mu|)^{N_0}, \end{aligned}$$

where the existence of the constants is ensured by the definition of tempered weights. If we just consider the sum in the above expression we get by applying Leibniz's formula a number of times

$$
\begin{aligned}
&\sum_{|\alpha|+|\beta|+|\gamma|+|\eta|\leq M} C^M_{\alpha\beta\gamma\eta}|\partial_v^\alpha\partial_\mu^\beta\partial_u^\gamma\partial_\nu^\eta d_2(x,\xi;v,\mu,u,\nu)| \\
&\leq \sum_{|\alpha|+|\beta|+|\gamma|+|\eta|\leq M}\ \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \gamma_1+\gamma_2=\gamma}}\ \sum_{\substack{\beta_1+\beta_2=\beta\\ \eta_1+\eta_2=\eta}} C^M_{\alpha\beta\gamma\eta}|\partial_v^{\alpha_1}\partial_\mu^{\beta_1}\partial_u^{\gamma_1}\partial_\nu^{\eta_1}\tilde{a}_\varepsilon(x,\xi;v,\mu,u,\nu)| \\
&\qquad\qquad \times|\partial_v^{\alpha_2}\partial_\mu^{\beta_2}\partial_u^{\gamma_2}\partial_\nu^{\eta_2}\tilde{b}_\varepsilon(x,\xi;v,\mu,u,\nu)| \\
&\leq \sum_{|\alpha|+|\beta|\leq M}\ \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} C^M_{\alpha\beta}|(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\tilde{a}_\varepsilon(x,\xi;v,\mu,u,\nu)||(\partial_x^{\alpha_2}\partial_\xi^{\beta_2}\tilde{b}_\varepsilon(x,\xi;v,\mu,u,\nu)| \\
&\leq \varepsilon^{-(\tau-M)_-}\sum_{|\alpha|+|\beta|\leq M}\ \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} C^M_{\alpha\beta}\varepsilon^{(\tau-M)_-}\frac{\tilde{m}_1(x,\xi;v,\mu,u,\nu)}{\tilde{m}_1(x,\xi;v,\mu,u,\nu)}\frac{\tilde{m}_2(x,\xi;v,\mu,u,\nu)}{\tilde{m}_2(x,\xi;v,\mu,u,\nu)} \\
&\qquad\qquad \times|(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\tilde{a}_\varepsilon)(x,\xi;v,\mu,u,\nu)(\partial_x^{\alpha_2}\partial_\xi^{\beta_2}\tilde{b}_\varepsilon)(x,\xi;v,\mu,u,\nu)| \\
&\leq C_M\varepsilon^{-(\tau-M)_-}m_1(x,\xi)m_2(x,\xi)\mathcal{G}_{M,\tau}(a_\varepsilon,m_1,b_\varepsilon,m_2)(1+|u|+|v|+|\nu|+|\mu|)^{2N_0},
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{G}_{M,\tau}(a_\varepsilon,m_1,b_\varepsilon,m_2) \\
&\quad=\sup_{\substack{|\alpha_1+\alpha_2|+|\beta_1+\beta_2|\leq M\\ (x,\xi)\in\mathbb{R}^{2d}}}\varepsilon^{(\tau-M)_-}\frac{|\partial_x^{\alpha_1}\partial_\xi^{\beta_1}a_\varepsilon(x,\xi)\partial_x^{\alpha_2}\partial_\xi^{\beta_2}b_\varepsilon(x,\xi)|}{m_1(x,\xi)m_2(x,\xi)}\lambda(x,\xi)^{\rho(|\alpha_1+\alpha_2|+|\beta_1+\beta_2|)}.
\end{aligned}
$$

The number $\mathcal{G}_{M,\tau}(a_\varepsilon,m_1,b_\varepsilon,m_2)$ is by assumption finite for all M in $\mathbb{N}$ and independent of ε because of the factor $\varepsilon^{(\tau-M)_-}$ since $\tau=\min(\tau_1,\tau_2)$. The inclusion of $\lambda(x,\xi)$ in $\mathcal{G}_{M,r}(a_\varepsilon,m_1,b_\varepsilon,m_2)$ is technically not required here but it will be useful for later estimates. If we use this estimate in the expression of $c_\varepsilon^2(x,\xi;\hbar)$ we have

$$
\begin{aligned}
|c_\varepsilon^2(x,\xi;\hbar)| \leq&\frac{1}{(2\pi\hbar)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|(L^t)^M d_2(x,\xi;v,\mu,u,\nu)|\,d\mu\,d\nu\,dv\,du \\
\leq&\frac{C\hbar^M\varepsilon^{-(\tau-M)_-}}{(2\pi\hbar)^{2d}}m_1(x,\xi)m_2(x,\xi)\mathcal{G}_{M,\tau}(a_\varepsilon,m_1,b_\varepsilon,m_2) \\
&\times\int_\Omega\frac{(1+|u|+|v|+|\nu|+|\mu|)^{2N_0}}{(|v|^2+|\mu|^2+|u|^2+|\nu|^2)^{\frac{M}{2}}}\,d\mu\,d\nu\,dv\,du,
\end{aligned}
$$

where

$$
\Omega=\{(v,\mu,u,\nu)\in\mathbb{R}^{4d}\,|\,|v|^2+|\mu|^2+|u|^2+|\nu|^2\geq\frac{1}{16}\lambda(x,\xi)^{2\rho}\}.
$$

We observe that the integral is convergent if $M>2N_0+4d$. But in order to make the constant arising from the integral independent of N_0 we choose $M>4N_0+4d$ then we have

$$
\begin{aligned}
&\int_\Omega\frac{(1+|u|+|v|+|\nu|+|\mu|)^{2N_0}}{(|v|^2+|\mu|^2+|u|^2+|\nu|^2)^{\frac{N}{2}}}\,d\mu\,d\nu\,dv\,du \\
&\quad\leq\int_\Omega\frac{(|u|+|v|+|\nu|+|\mu|)^{4N_0}}{(|v|^2+|\mu|^2+|u|^2+|\nu|^2)^{\frac{N}{2}}}\,d\mu\,d\nu\,dv\,du\leq c\lambda(x,\xi)^{-\rho(M-4N_0-4d)}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&|c_\varepsilon^2(x,\xi;\hbar)| \\
&\quad \leq C_M \hbar^{M-2d} \varepsilon^{-(r-M)_-} m_1(x,\xi) m_2(x,\xi) \mathcal{G}_{M,\tau}(a_\varepsilon, m_1, b_\varepsilon, m_2) \lambda(x,\xi)^{-\rho(M-4N_0-4d)}.
\end{aligned}
$$

By our assumptions on ε we have for $M \geq \tau$

$$
\hbar^{M-2d} \varepsilon^{-(\tau-M)_-} \leq \hbar^{\tau+\delta(M-\tau)-2d}.
$$

Hence for every N in $\mathbb{N}$ there exists an M in $\mathbb{N}$ and a constant only depending on N such that

$$
|c_\varepsilon^2(x,\xi;\hbar)| \leq C_N \hbar^N m_1(x,\xi) m_2(x,\xi) \mathcal{G}_{M,r}(a_\varepsilon, m_1, b_\varepsilon, m_2) \lambda(x,\xi)^{-\rho \tilde{N}(M)}.
$$

By analogous arguments this extents too the following: For every N in $\mathbb{N}$ and α,β in $\mathbb{N}_0^d$ there exists an M in $\mathbb{N}$ and a constant $C(N,\alpha,\beta)$ only depending N, α and β such that

$$
\begin{aligned}
&|\partial_x^\alpha \partial_\xi^\beta c_\varepsilon^2(x,\xi;\hbar)| \\
&\quad \leq C(N,\alpha,\beta) \hbar^N \varepsilon^{-|\alpha|} m_1(x,\xi) m_2(x,\xi) \mathcal{G}_{M,r}^{\alpha,\beta}(a_\varepsilon, m_1, b_\varepsilon, m_2) \lambda(x,\xi)^{-\rho \tilde{N}(M)}.
\end{aligned}
\tag{3.19}
$$

This was first part of the error term. The second part will come from the remainder when applying Quadratic stationary phase asymptotics.

We now turn to the part of the integral where we integrate $d_1(x,\xi;v,\mu,u,\nu)$. For this integral we can now apply Quadratic stationary phase asymptotics hence Theorem II.4.17 gives for N in $\mathbb{N}$ the expansion

$$
\begin{aligned}
c_\varepsilon^1(x,\xi;\hbar) &= \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\hbar^{-1}(\langle v,\mu\rangle - \langle u,\nu\rangle)} d_1(x,\xi;v,\mu,u,\nu) \, d\mu \, d\nu \, dv \, du \\
&= \sum_{j=0}^N \frac{(i\sigma(D_u,D_\mu;D_v,D_\nu))^j}{j!} [\tilde{a}_\varepsilon(x,p;u,v,\mu,\nu) \tilde{b}_\varepsilon(x,p;u,v,\mu,\nu)] \Big|_{\substack{u=v=0\\ \mu=\nu=0}} \\
&\quad + \hbar^{N+1} \tilde{r}_{\varepsilon,N+1}(a_\varepsilon, b_\varepsilon;\hbar)
\end{aligned}
$$

where we in the sum have used that the localising part of d_1 is constant 1 in a neighbourhood of 0, hence if a derivative have been applied to it and we evaluate at 0 we get 0. Moreover the reminder $\tilde{r}_{\varepsilon,N+1}$ satisfies that

$$
|\tilde{r}_{\varepsilon,N+1}(a_\varepsilon, b_\varepsilon;\hbar)| \leq c_d \Big\| \tfrac{(\sigma(D_u,D_\mu;D_v,D_\nu))^{N+1}}{(N+1)!} d_1(x,\xi;v,\mu,u,\nu) \Big\|_{H^{2d+1}(\mathbb{R}_u^d \times \mathbb{R}_\mu^d \times \mathbb{R}_v^d \times \mathbb{R}_\nu^d)}.
$$

We can note that we have the desired form for the terms contributing to the new symbol. What remains is to estimate the error term from the Quadratic stationary phase asymptotic and the contribution from the integral of d_2. For the error from the Quadratic stationary phase asymptotic we start by noticing

$$
\begin{aligned}
&\Big\| \tfrac{(\sigma(D_u,D_\mu;D_v,D_\nu))^{N+1}}{(N+1)!} d_1(x,\xi;v,\mu,u,\nu) \Big\|_{H^{2d+1}(\mathbb{R}_u^d \times \mathbb{R}_\mu^d \times \mathbb{R}_v^d \times \mathbb{R}_\nu^d)} \\
&\leq C_{N,d} \lambda(x,\xi)^{2d\rho} \sup_{\substack{|\eta| \leq 2d+1 \\ \{|v|^2+|\mu|^2+|u|^2+|\nu|^2 \leq \frac{1}{8}\lambda(x,\xi)^{2\rho}\}}} |\partial_{u\mu v\nu}^\eta (\sigma(D_u,D_\mu;D_v,D_\nu))^{N+1} d_1(x,\xi;v,\mu,u,\nu)|
\end{aligned}
\tag{3.20}
$$

If we only consider the expression $\sigma(D_u, D_\mu; D_v, D_\nu))^{N+1} d_1(x,\xi; v, \mu, u, \nu)$ we have

$$\begin{aligned}\sigma(D_u, D_\mu; D_v, D_\nu))^{N+1} & d_1(x,\xi; v, \mu, u, \nu) \\ &= (\langle D_v, D_\mu\rangle - \langle D_u, D_\nu\rangle)^{N+1} d_1(x,\xi; v, \mu, u, \nu) \\ &= \sum_{|\alpha|+|\beta|=N+1} J_{\alpha\beta}\partial_v^\alpha \partial_\mu^\alpha \partial_u^\beta \partial_\nu^\beta d_1(x,\xi; v, \mu, u, \nu),\end{aligned}$$

where the $J_{\alpha\beta}$'s are constants which may be negative. In particular one should note that from the above expression we see that in the rough variables u and v we can at most get $N+1$ derivatives. This is the important part in the above calculation. How the derivatives exactly are is not important for the next estimate. We now have, as above, for $M = N+1+|\eta|$

$$\begin{aligned}&|\partial_{u\mu v\nu}^\eta (\sigma(D_u, D_\mu; D_v, D_\nu))^{N+1} d_1(x,\xi; v, \mu, u, \nu)| \\ &\quad \leq C_{N,d}\varepsilon^{-(\tau-M)_-} \mathcal{G}_{M,r}(a_\varepsilon, m_1, b_\varepsilon, m_2) m_1(x+tv+t(1-t)u, \nu+(1-t)\mu+\xi) \\ &\qquad \times m_2(x+(1-t)v - t(1-t)u, \nu - t\mu + \xi) \\ &\qquad \times \sum_{\substack{2(N+1) \\ \leq j+k+l \leq \\ 2(N+1)+|\eta|}} \lambda(x,\xi)^{-l\rho}\lambda(x+tv+t(1-t)u, \nu+(1-t)\mu+\xi)^{-j\rho} \\ &\qquad\qquad\qquad \times \lambda(x+(1-t)v - t(1-t)u, \nu - t\mu+\xi)^{-k\rho},\end{aligned} \tag{3.21}$$

where again

$$\begin{aligned}&\mathcal{G}_{M,\tau}(a_\varepsilon, m_1, b_\varepsilon, m_2) \\ &\quad = \sup_{\substack{|\alpha_1+\alpha_2|+|\beta_1+\beta_2|\leq M \\ (x,\xi)\in\mathbb{R}^{2d}}} \varepsilon^{(\tau-M)_-} \frac{|\partial_x^{\alpha_1}\partial_\xi^{\beta_1} a_\varepsilon(x,\xi)\partial_x^{\alpha_2}\partial_\xi^{\beta_2} b_\varepsilon(x,\xi)|}{m_1(x,\xi)m_2(x,\xi)} \lambda(x,\xi)^{\rho(|\alpha_1+\alpha_2|+|\beta_1+\beta_2|)}.\end{aligned}$$

The different powers j, k and l in (3.21) corresponds to j derivatives on $\tilde{a}_\varepsilon$, k derivatives on $\tilde{b}_\varepsilon$ and l derivatives on $\chi_{\frac{1}{16}}$. Now since we, on the support of d_1, have the estimate $|v|^2 + |\mu|^2 + |u|^2 + |\nu|^2 \leq \frac{1}{8}\lambda(x,\xi)^{2\rho}$ we get the following two estimates

$$\begin{aligned}\lambda(x+tv+t(1-t)u, \nu+(1-t)\mu+\xi)^{-1} &\leq 2\lambda(x,\xi)^{-1}, \\ \lambda(x+(1-t)v - t(1-t)u, \nu - t\mu+\xi)^{-1} &\leq 2\lambda(x,\xi)^{-1}.\end{aligned} \tag{3.22}$$

By the properties of tempered weight functions there exists $C_1 > 0$ and N_0 in $\mathbb{N}_0$ such that

$$\begin{aligned}m_1(x+tv+t(1-t)u, \nu+(1-t)\mu+\xi) &\leq C_1\lambda(x,\xi)^{N_0\rho}, \\ m_2(x+(1-t)v - t(1-t)u, \nu - t\mu+\xi) &\leq C_1\lambda(x,\xi)^{N_0\rho}.\end{aligned} \tag{3.23}$$

Now by combining (3.20), (3.21), (3.22) and (3.23) and with $M = N+2d+2$ we have

$$
\begin{aligned}
&|\tilde{r}_{\varepsilon,N+1}(a_\varepsilon, b_\varepsilon; \hbar)| \\
&\qquad \leq C_{d,N}\varepsilon^{-(\tau-N-2d-2)_-}\mathcal{G}_{M,\tau}(a_\varepsilon, m_1, b_\varepsilon, m_2) \\
&\qquad\qquad\qquad\qquad \times m_1(x,\xi)m_2(x,\xi)\lambda(x,\xi)^{-\rho 2(N+1-N_0-d)} \\
&\qquad \leq C_{d,N}\hbar^{(\delta-1)(\tau-N-2d-2)_-}\mathcal{G}_{M,\tau}(a_\varepsilon, m_1, b_\varepsilon, m_2) \\
&\qquad\qquad\qquad\qquad\quad \times m_1(x,\xi)m_2(x,\xi)\lambda(x,\xi)^{-\rho 2(N+1-N_0-d)}.
\end{aligned}
$$

Recall that the error term comes with $\hbar^{N+1}$. Hence if we expand to at least a number N_δ such that $\delta(N_\delta + 2d + 2 - \tau) + \tau > 2d + 1$ we will have an estimate with $\hbar$ raised to a positive power. For α and β in $\mathbb{N}_0^d$ we can by an analogous argument find a positive constant $C = C(\alpha, \beta, d, N)$ such that

$$
\begin{aligned}
&|\partial_x^\alpha \partial_\xi^\beta \tilde{r}_{\varepsilon,N+1}(a_\varepsilon, b_\varepsilon; \hbar)| \\
&\qquad \leq C\varepsilon^{-|\alpha|}\hbar^{(\delta-1)(\tau-N-2d-2)_-}\mathcal{G}^{\alpha,\beta}_{M,r}(a_\varepsilon, m_1, b_\varepsilon, m_2) \\
&\qquad \times m_1(x,\xi)m_2(x,\xi)\lambda(x,\xi)^{-\rho 2(N+1+|\alpha|+|\beta|-N_0-d)}
\end{aligned}
\tag{3.24}
$$

Now we can combine the estimates on the two different parts (3.19) and (3.24) of the error term and then we arrive at the estimate

$$
\begin{aligned}
\hbar^{N+1}|\partial_x^\alpha \partial_\xi^\beta r_{\varepsilon,N+1}(a_\varepsilon, b_\varepsilon; \hbar)| &= \hbar^{N+1}|\partial_x^\alpha \partial_\xi^\beta \tilde{r}_{\varepsilon,N+1}(a_\varepsilon, b_\varepsilon; \hbar) + \partial_x^\alpha \partial_\xi^\beta c^2_\varepsilon(x,\xi;\hbar)| \\
&\leq C\varepsilon^{-|\alpha|}\hbar^{\delta(\tau-N-2d-2)_- + \tau - 2d-1}\mathcal{G}^{\alpha,\beta}_{M,\tau}(a_\varepsilon, m_1, b_\varepsilon, m_2)m_1(x,\xi)m_2(x,\xi) \\
&\quad \times \lambda(x,\xi)^{-\rho(\tilde{N}(M)+|\alpha|+|\beta|)},
\end{aligned}
$$

for $N \geq N_\delta$, where we have used that the contribution to the error from $c^2_\varepsilon(x,\xi;\hbar)$ can be arbitrary small. Hence the main error term is the part from the Quadratic stationary phase theorem. This ends the proof. ☐

Remark II.5.3 (Particular cases of Theorem II.5.1). We will see the 3 most important cases for this presentation of the composition for t-quantised operators. We suppose the assumptions of Theorem II.5.1 is satisfied.

$\boldsymbol{t = 0}$**:** In this case the amplitude will be independent of u hence we have

$$
c_\varepsilon(x,p;\hbar) = e^{i\hbar\langle D_y, D_q,\rangle}[a_\varepsilon(x,q)b_\varepsilon(y,p)]\Big|_{\substack{y=x \\ p=q}}.
$$

This gives the formula

$$
c_{\varepsilon,j}(x,p) = \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial_p^\alpha a_\varepsilon(x,p) D_x^\alpha b_\varepsilon(x,p).
$$

***t* = 1:** This case is similar to the one above, except a change of signs. The composition formula is given by

$$c_\varepsilon(x,p;\hbar) = e^{-i\hbar\langle D_y,D_q,\rangle}[a_\varepsilon(y,p)b_\varepsilon(x,q)]\Big|_{\substack{y=x\\p=q}}.$$

This gives the formula

$$c_{\varepsilon,j}(x,p) = (-1)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} D_x^\alpha a_\varepsilon(x,p)\partial_p^\alpha b_\varepsilon(x,p).$$

***t* = $\frac{1}{2}$ (Weyl-quatisation):** In order to obtain a not to complicated formula for the c_j's we will need an extra change of variables in the proof. Recall that before applying stationary phase we had the following expression for c_ε

$$\begin{aligned} c_\varepsilon(x,\xi;\hbar) =& \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\hbar^{-1}(\langle v,\mu\rangle - \langle u,\nu\rangle)} \\ & a_\varepsilon(x+\frac{v}{2}+\frac{u}{4},\nu+\frac{\mu}{2}+\xi)b_\varepsilon(x+\frac{v}{2}-\frac{u}{4},\nu-\frac{\mu}{2}+\xi)\,d\mu\,d\nu\,dv\,du, \end{aligned}$$

in the case $t=\frac{1}{2}$. If we do the change of variables

$$\begin{aligned} \frac{1}{2}v + \frac{1}{4}u &= w & \qquad \frac{\mu}{2}+\nu &= \eta \\ \frac{1}{2}v - \frac{1}{4}u &= r & \qquad \nu - \frac{\mu}{2} &= \tau \end{aligned}$$

we note that the determinant for this change of variables is 2^{2d} and the function σ satisfies that

$$\sigma(u,\mu;v,\nu) = 2\sigma(w,\rho;r,\tau).$$

Hence we obtain

$$\begin{aligned} c_\varepsilon(x,\xi;\hbar) =& \frac{2^{2d}}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\hbar^{-1}2\sigma(w,\rho;r,\tau)} \\ & a_\varepsilon(x+w,\rho+\xi)b_\varepsilon(x+r,\tau+\xi)\,d\rho\,d\tau\,dw\,dr. \end{aligned}$$

As in the proof of Theorem II.5.1 we can obtain

$$c_\varepsilon(x,p;\hbar) = e^{i\frac{\hbar}{2}\sigma(D_x,D_p;D_y,D_q)}[a_\varepsilon(x,p)b_\varepsilon(y,q)]\Big|_{\substack{y=x\\p=q}}$$

with

$$c_{\varepsilon,j}(x,p) = \Big(\frac{i}{2}\Big)^j \frac{1}{j!}[\sigma(D_x,D_p;D_y,D_q)]^j a_\varepsilon(x,p)b_\varepsilon(y,q)\Big|_{\substack{y=x\\p=q}}.$$

The last equation can be rewritten by some algebra to the classic formula

$$c_{\varepsilon,j}(x,p) = \sum_{|\alpha|+|\beta|=j} \frac{1}{\alpha!\beta!}\Big(\frac{1}{2}\Big)^{|\alpha|}\Big(-\frac{1}{2}\Big)^{|\beta|}(\partial_p^\alpha D_x^\beta a_\varepsilon)(\partial_p^\beta D_x^\alpha b_\varepsilon)(x,p).$$

The above formula could also have been derived from the expression in Theorem II.5.1 but this is a slightly harder calculation.

In all three cases we can note that the symbols for the compositions of operators is the same as in the non-rough case.

We now have composition of operators given by a single symbol. The next result generalises the previous to composition of strongly $\hbar$-ε-admissible operators. Moreover it verifies that the strongly $\hbar$-ε-admissible operators form an algebra. More precisely we have.

Theorem II.5.4. *Let $A_\varepsilon(\hbar)$ and $B_\varepsilon(\hbar)$ be two strongly $\hbar$-ε-admissible operators of regularity $\tau_a \geq 0$ and $\tau_b \geq 0$. with weights (m_1, ρ, ε) and (m_2, ρ, ε) respectively and of the form*

$$A_\varepsilon(\hbar) = \mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon) \quad \text{and} \quad B_\varepsilon(\hbar) = \mathrm{Op}_\hbar^{\mathrm{w}}(b_\varepsilon)$$

We suppose $\varepsilon \geq \hbar^{1-\delta}$ for a δ in $(0,1)$ and let $\tau = \min(\tau_a, \tau_b)$. Then is $C_\varepsilon(\hbar) = A_\varepsilon(\hbar) \circ B_\varepsilon(\hbar)$ a strongly $\hbar$-ε-admissible operators of regularity $\tau \geq 0$ with weights $(m_1 m_2, \rho, \varepsilon)$. The symbol $c_\varepsilon(x,p;\hbar)$ of $C_\varepsilon(\hbar)$ has for $N \geq N_\delta$ the expansion

$$c_\varepsilon(x,p;\hbar) = \sum_{j=0}^{N} \hbar^j c_{\varepsilon,j}(x,p) + \hbar^{N+1} \zeta_\varepsilon(a_\varepsilon(\hbar), b_\varepsilon(\hbar); \hbar),$$

where

$$c_{\varepsilon,j}(x,p) = \sum_{|\alpha|+|\beta|+k+l=j} \frac{1}{\alpha!\beta!} \Big(\frac{1}{2}\Big)^{|\alpha|} \Big(-\frac{1}{2}\Big)^{|\beta|} (\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,l})(x,p).$$

The symbols $a_{\varepsilon,k}$ and $b_{\varepsilon,l}$ are from the expansion of a_ε and b_ε respectively. Let

$$a_\varepsilon(x,p) = \sum_{k=0}^{N} \hbar^j a_{\varepsilon,j}(x,p) + \hbar^{N+1} r_{\varepsilon,N+1}(a_\varepsilon, x, p; \hbar)$$

and equvalint for $b_\varepsilon(x,p)$. Then for every multi indices α, β there exists a constant $C(\alpha,\beta,N)$ independent of a_ε and b_ε and an integer M such that

$$\begin{aligned}
&\hbar^{N+1} |\partial_x^\alpha \partial_p^\beta \zeta_\varepsilon(a_\varepsilon(\hbar), b_\varepsilon(\hbar); x, p; \hbar)| \\
&\quad \leq C(\alpha,\beta,N) \hbar^{\delta(\tau-N-2d-2)_- + \tau - 2d-1} \varepsilon^{-|\alpha|} m_1(x,p) m_2(x,p) \lambda(x,p)^{-\rho(\tilde{N}(M)+|\alpha|+|\beta|)} \\
&\quad\quad \times \Big[\sum_{j=0}^{N} \{ \mathcal{G}_{M,\tau}^{\alpha,\beta}(a_{\varepsilon,j}, m_1, r_{\varepsilon,N+1}(b_\varepsilon(\hbar)), m_2) + \mathcal{G}_{M,\tau}^{\alpha,\beta}(r_{\varepsilon,N+1}(a_\varepsilon(\hbar)), m_1, b_{\varepsilon,j}, m_2) \} \\
&\quad\quad + \sum_{N \leq j+k \leq 2N} \mathcal{G}_{M,\tau}^{\alpha,\beta}(a_{\varepsilon,j}, m_1, b_{\varepsilon,k}, m_2) + \mathcal{G}_{M,\tau}^{\alpha,\beta}(r_{\varepsilon,N+1}(a_\varepsilon(\hbar)), m_1, r_{\varepsilon,N+1}(b_\varepsilon(\hbar)), m_2) \Big],
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}_{M,\tau}^{\alpha,\beta}&(a_\varepsilon, m_1, b_\varepsilon, m_2) \\
&= \sup_{\substack{|\gamma_1+\gamma_2|+|\eta_1+\eta_2| \leq M \\ (x,\xi)\in\mathbb{R}^{2d}}} \varepsilon^{(\tau-M)_- + |\alpha|} \frac{|\partial_x^\alpha \partial_\xi^\beta (\partial_x^{\gamma_1} \partial_\xi^{\eta_1} a_\varepsilon(x,\xi) \partial_x^{\gamma_2} \partial_\xi^{\eta_2} b_\varepsilon(x,\xi))|}{m_1(x,\xi) m_2(x,\xi)} \\
&\quad \times \lambda(x,\xi)^{\rho(|\gamma_1+\gamma_2|+|\eta_1+\eta_2|)}.
\end{aligned}$$

The function $\tilde{N}(M)$ is also depending on the weights m_1, m_2 and the dimension d.

The proof of this theorem is an application of Theorem II.5.1 a number of times and recalling that the error operator of a strongly $\hbar$-ε-admissible operator of some regularity is a quantised pseudo-differential operator.

II.6 Rough pseudo-differential operators acting on $L^2(\mathbb{R}^d)$

So far we have only considered operators acting on $\mathcal{S}(\mathbb{R}^d)$ or $\mathcal{S}'(\mathbb{R}^d)$. Hence they can be viewed as unbounded operators acting in $L^2(\mathbb{R}^d)$ with domain $\mathcal{S}(\mathbb{R}^d)$. The question is then when is this a bounded operator? The first theorem of this section gives a criteria for when the operator can be extended to a bounded operator. This theorem is a Calderon-Vaillancourt type theorem and the proof uses the Calderon-Vaillancourt Theorem for the non-rough pseudo-differential operators. We will not recall this theorem but refer to [5, 17, 32].

Theorem II.6.1. *Let a_ε be in $\Gamma^{m,\tau}_{0,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p)$, where m is a bounded tempered weight function, $\tau \geq 0$ and there exists a δ in $(0,1)$ such that $\varepsilon \geq \hbar^{1-\delta}$. Then there exists a constant C_d and an integer k_d only depending on the dimension such that*

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\psi\|_{L^2} \leq C_d \sup_{\substack{|\alpha|,|\beta|\leq k_d \\ (x,p)\in\mathbb{R}^{2d}}} \hbar^{(1-\delta)|\alpha|}|\partial_x^\alpha \partial_p^\beta a_\varepsilon(x,p)|\|\psi\|_{L^2(\mathbb{R}^d)},$$

for all ψ in $\mathcal{S}(\mathbb{R}^d)$. Especially can $\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)$ be extended to a bounded operator on $L^2(\mathbb{R}^d)$.

That the above result is in-fact true is some kind of miracle. But as we shall see in the proof most of the work nedded to prove the theorem is actually to prove the classical Calderon-Vaillancourt theorem for non-rough symbols. Which we know is valid.

Proof. We start by writing the L^2-norm of interest and make a change of variables

$$\begin{aligned}
&\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\psi\|^2_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi\hbar)^{2d}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} a_\varepsilon(\tfrac{x+y}{2},p)\psi(y)\,dydp\Big|^2\,dx \\
&= \int_{\mathbb{R}^d} \frac{\hbar^{(1-\delta)d}}{(2\pi)^{2d}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle \tilde{x}-\tilde{y},\tilde{p}\rangle} a_\varepsilon(\hbar^{1-\delta}\tfrac{\tilde{x}+\tilde{y}}{2},\hbar^\delta\tilde{p})\psi(\hbar^{1-\delta}\tilde{y})\,d\tilde{y}d\tilde{p}\Big|^2\,d\tilde{x} \\
&= \int_{\mathbb{R}^d} \frac{\hbar^{(1-\delta)d}}{(2\pi)^{2d}} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle \tilde{x}-\tilde{y},\tilde{p}\rangle} \tilde{a}_\varepsilon(\tfrac{\tilde{x}+\tilde{y}}{2},\tilde{p})\tilde{\psi}(\tilde{y})\,d\tilde{y}d\tilde{p}\Big|^2\,d\tilde{x},
\end{aligned} \tag{3.25}$$

where we have used the change of variables

$$\tilde{x} = \hbar^{-1+\delta}x \qquad \tilde{y} = \hbar^{-1+\delta}y \qquad \tilde{p} = \hbar^{-\delta}p$$

and the following definition of functions

$$\tilde{a}_\varepsilon(\tilde{x},\tilde{p}) \coloneqq a_\varepsilon(x,p) = a_\varepsilon(\hbar^{1-\delta}\tilde{x},\hbar^\delta\tilde{p}) \qquad \tilde{\psi}(\tilde{x}) \coloneqq \psi(x) = \psi(\hbar^{1-\delta}\tilde{x}).$$

We can from this change of variables note that we are now calculating the L^2-norm of the function $\mathrm{Op}_1^{\mathrm{w}}(\tilde{a}_\varepsilon)\tilde{\psi}$ times $\hbar^{(1-\delta)d}$. If we consider the new symbol $\tilde{a}_\varepsilon$, then it is still a symbol and in the new coordinates and

$$|\partial_{\tilde{x}}^\alpha\partial_{\tilde{p}}^\beta \tilde{a}_\varepsilon(\tilde{x},\tilde{p})| = |\partial_{\tilde{x}}^\alpha\partial_{\tilde{p}}^\beta a_\varepsilon(\hbar^{1-\delta}\tilde{x},\hbar^\delta\tilde{p})| \leq \hbar^{(1-\delta)|\alpha|+\delta|\beta|}\varepsilon^{-|\alpha|}m(x,p) \leq m(x,p),$$

for all α and β in $\mathbb{N}_0^d$. Hence in the new coordinates the symbol is not rough. Now from the classical Calderon-Vaillancourt theorem we get existence of a constant C_d and an integer k_d only depending on the dimension such that

$$\|\mathrm{Op}_1^{\mathrm{w}}(\tilde{a}_\varepsilon)\tilde{\psi}\|_{L^2(\mathbb{R}^d)} \leq C_d \sup_{\substack{|\alpha|,|\beta|\leq k_d \\ (\tilde{x},\tilde{p})\in\mathbb{R}^{2d}}} |\partial_{\tilde{x}}^\alpha \partial_{\tilde{p}}^\beta \tilde{a}_\varepsilon(\tilde{x},\tilde{p})| \|\tilde{\psi}\|_{L^2(\mathbb{R}^d)},$$

Now by combining this with (3.25) we get

$$\begin{aligned} \|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\psi\|_{L^2(\mathbb{R}^d)}^2 &= \hbar^{(1-\delta)d}\|\mathrm{Op}_1^{\mathrm{w}}(\tilde{a}_\varepsilon)\tilde{\psi}\|_{L^2(\mathbb{R}^d)} \\ &\leq \hbar^{(1-\delta)d} C_d \sup_{\substack{|\alpha|,|\beta|\leq k_d \\ (\tilde{x},\tilde{p})\in\mathbb{R}^{2d}}} |\partial_{\tilde{x}}^\alpha \partial_{\tilde{p}}^\beta \tilde{a}_\varepsilon(\tilde{x},\tilde{p})| \|\tilde{\psi}\|_{L^2(\mathbb{R}^d)} \\ &\leq C_d \sup_{\substack{|\alpha|,|\beta|\leq k_d \\ (x,p)\in\mathbb{R}^{2d}}} \hbar^{(1-\delta)|\alpha|} |\partial_x^\alpha \partial_p^\beta a_\varepsilon(x,p)| \|\psi\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

This is the desired estimate and this concludes the proof. $\square$

We have now established criteria for which a rough pseudo-differential operator is bounded. But we also need some criteria for which they are Hilbert-Schmidt and trace class. First we consider the Hilbert-Schmidt case.

Proposition II.6.2. *Let a_ε be in $\Gamma_{0,\varepsilon}^{m,\tau}(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ with $\tau \geq 0$ and suppose a_ε is an element of $L^2(\mathbb{R}_x^d \times \mathbb{R}_p^d)$. Then is*

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\|_{\mathrm{HS}}^2 = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |a_\varepsilon(x,p)|^2 \, dxdp.$$

Proof. By assumption the object

$$\mathcal{F}_\hbar^{-1}[a_\varepsilon(\tfrac{x+y}{2},\cdot)](x-y) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} a_\varepsilon(\tfrac{x+y}{2},p)\,dp$$

exists as an oscillating integral and is an element of $L^2(\mathbb{R}_x^d \times \mathbb{R}_y^d)$. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}_x^d \times \mathbb{R}_y^d)$ then by Parsival's formula

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\|_{\mathrm{HS}}^2 = \sum_{n=1}^\infty \|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\varphi_n\|_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathcal{F}_\hbar^{-1}[a_\varepsilon(\tfrac{x+y}{2},\cdot)](x-y)|^2\,dydx$$

With the change of variables

$$w = \frac{x+y}{2} \qquad\qquad z = x-y,$$

which have a determinant with absolute value 1, and the Plancherel theorem we have

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\|_{\mathrm{HS}}^2 = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |a_\varepsilon(x,p)|^2 \, dxdp.$$

This is the desired equality and this ends the proof. $\square$

The above Proposition is a complete characteristic of all rough pseudo differential operators which are Hilbert-Schmidt operators. In the case of rough pseudo differential operators we are only able to give a sufficient condition for the operator to be trace class.

Before we continue we will just recall/prove the following Lemma

Lemma II.6.3. *Let b_ε be a rough function of regularity $\tau \geq 0$ in $C_0^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_p)$. Then the estimate*

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(b_\varepsilon)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \frac{1}{(\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |b_\varepsilon(x,p)|\, dxdp$$

holds.

Proof. Let φ and ψ be two functions from $C_0^\infty(\mathbb{R}^d)$. Then we have

$$\langle \mathrm{Op}_\hbar^{\mathrm{w}}(b_\varepsilon)\varphi,\psi\rangle = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} b_\varepsilon(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\, dydpdx.$$

With the change of variables

$$w = \frac{x+y}{2} \qquad\qquad z = x-y,$$

which have a determinant with absolute value 1, we have

$$|\langle \mathrm{Op}_\hbar^{\mathrm{w}}(b_\varepsilon)\varphi,\psi\rangle| \leq \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b_\varepsilon(w,p)\int_{\mathbb{R}^d}\varphi(w-\frac{z}{2})\overline{\psi}(w+\frac{z}{2})\, dzdpdw.$$

Now by changing z into $2z$ and apply a Cauchy-Schwarz inequality in z we have

$$|\langle \mathrm{Op}_\hbar^{\mathrm{w}}(b_\varepsilon)\varphi,\psi\rangle| \leq \frac{1}{(\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b_\varepsilon(w,p)\, dpdw \int_{\mathbb{R}^d}|\varphi(z)|^2\, dz\int_{\mathbb{R}^d}|\psi(z)|^2\, dz.$$

This inequality implies the desired estimate and this ends the proof. ☐

We can now give a criteria for the rough pseudo differential operators to be trace class. The criteria will be sufficient but not necessary. Hence it does not provide a full characteristic for the set of rough pseudo differential operators which are trace class.

Theorem II.6.4. *There exists a constant $C(d)$ only depending on the dimension such*

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\|_{\mathrm{Tr}} \leq \frac{C(d)}{\hbar^d}\sum_{|\alpha|+|\beta|\leq 2d+2}\hbar^{|\beta|}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\partial_x^\alpha\partial_p^\beta a_\varepsilon(x,p)]\, dxdp.$$

for every a_ε in $\Gamma^{m,\tau}_{0,\varepsilon}(\mathbb{R}^d_x\times\mathbb{R}^d_p)$ with $\tau\geq 0$.

Remark II.6.5. The above estimate does not a priori imply that the trace norm of a rough pseudo differential operator is of the order $\hbar^{-d}$ as there might appear extra factors of ε^{-1} from the integrands. These extra factors will be determined by the regularity of the symbol.

If one had used a semiclassical Harmonic oscillator in the proof the estimate would have been

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\|_{\mathrm{Tr}} \leq \frac{C(d)}{\hbar^{2d+1}} \sum_{|\alpha|+|\beta|\leq 2d+2} \hbar^{|\alpha|+|\beta|} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\partial_x^\alpha \partial_p^\beta a_\varepsilon(x,p)]\,dxdp.$$

which in terms of the semiclassical parameter $\hbar$ is a "worse" estimate even though we do not get factors of ε^{-1} from the derivatives.

In the case where the rough symbol a_ε is a Schwarz function it is possible to obtain a bound of the type

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\|_{\mathrm{Tr}} \leq \frac{C(d,a_\varepsilon)}{\hbar^{d+\kappa}}$$

for any $\kappa > 0$, where the constant now also depends on the symbol a_ε. Hence this indicates that the right order of the semiclassical parameter should still be $\hbar^{-d}$ for the trace norm of the rough pseudo differential operators.

In the following section we will use estimates of the type

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\|_{\mathrm{Tr}} \leq \frac{C(d,a_\varepsilon)}{\hbar^{2d+1}} \tag{3.26}$$

under the assumption that the symbol and all derivatives are integrable. Estimates is the worst we can possible get from the theorem but they are sufficient for our applications.

Proof. In this proof we let $|(x,p)|_\infty = \max_j(|x_j|,|p_j|)$. We start by letting χ be in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such $\chi(x,p) = 1$ on the set $\{(x,p) \in \mathbb{R}^{2d} \,|\, |(x,p)|_\infty \leq \frac{2}{3}\}$ and with support contained in $\{(x,p) \in \mathbb{R}^{2d} \,|\, |(x,p)|_\infty \leq 1\}$. With this function we let

$$\chi_{\gamma,\eta}(x,p) = \frac{\chi(x-\gamma,p-\eta)}{\sum_{\gamma',\eta'\in\mathbb{N}_0^d} \chi(x-\gamma',p-\eta')}.$$

for γ,η in $\mathbb{N}_0^d$. This is a partition of unity for $\mathbb{R}_x^d \times \mathbb{R}_p^d$ and hence we have

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\|_{\mathrm{Tr}} \leq \sum_{\gamma,\eta\in\mathbb{N}_0^d} \|\mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{\gamma,\eta}a_\varepsilon)\|_{\mathrm{Tr}}. \tag{3.27}$$

We start by considering one of the terms in the sum, hence let a γ and η be given. We define the unitary operators T_γ and U_η by

$$(T_\gamma f)(x) = f(x-\gamma) \quad \text{and} \quad (U_\eta f)(x) = e^{-i\hbar^{-1}\langle\eta,x\rangle} f(x).$$

With these operators we have

$$(T_\gamma U_\eta)\,\mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{\gamma,\eta}a_\varepsilon)(T_\gamma U_\eta)^* = \mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{0,0}\tilde{a}_\varepsilon),$$

where $\tilde{a}_\varepsilon(x,p) = a_\varepsilon(x+\gamma,p+\eta)$. Sine the trace norm is invariant under unitary conjugation we have

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{\gamma,\eta}a_\varepsilon)\|_{\mathrm{Tr}} = \|\mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{0,0}\tilde{a}_\varepsilon)\|_{\mathrm{Tr}}.$$

We now let $H_d = -\Delta + x^2$ (the harmonic oscillator). This operator is positive, self-adjoint and has pure point spectrum. The eigenvalues are given by

$$E_j = (2j_1 + 1) + (2j_2 + 1) + \cdots + (2j_d + 1),$$

for j in $\mathbb{N}_0^d$. For this operator we have

$$\|H_d^{-d-1}\|_{\mathrm{Tr}} = \sum_{j \in \mathbb{N}_0^d} \frac{1}{E_j^{d+1}} = \tilde{C}(d) < \infty.$$

The above number only depend on the dimension. We now have

$$\begin{aligned} \|\mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{0,0}\tilde{a}_\varepsilon)\|_{\mathrm{Tr}} &= \|H_d^{-d-1} H_d^{d+1} \mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{0,0}\tilde{a}_\varepsilon)\|_{\mathrm{Tr}} \\ &\leq \tilde{C}(d) \|H_d^{d+1} \mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{0,0}\tilde{a}_\varepsilon)\|_{\mathcal{L}(L^2(\mathbb{R}^d))}. \end{aligned} \tag{3.28}$$

The operator H_d^{d+1} is a pseudo differential operator with a symbol $b(x,p)$ which is a polynomial in x and p of degree $2d+2$. If we choose to consider the Weyl-quantisation of H_d^{d+1} and apply the result on composition of pseudo-differential operators we get that $H_d^{d+1} \mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{0,0}\tilde{a}_\varepsilon) = \mathrm{Op}^{\mathrm{w}}(c_\varepsilon)$ is a rough pseudo-differential operator. The symbol $c_\varepsilon(x, \hbar p)$ of this operator satisfy the bound

$$\begin{aligned} |c_\varepsilon(x,\hbar p)| &\leq \sum_{|\alpha|+|\beta| \leq 2d+2} c_{\alpha,\beta} \hbar^\beta |\partial_x^\alpha \partial_p^\beta \chi_{0,0}(x,\hbar p) \tilde{a}_\varepsilon(x,\hbar p)| \\ &\leq c \sum_{|\alpha|+|\beta| \leq 2d+2} \hbar^{|\beta|} \mathbf{1}_{\mathrm{supp}(\chi_{0,0})}(x,p) |(\partial_x^\alpha \partial_p^\beta \tilde{a}_\varepsilon)(x,\hbar p)|, \end{aligned}$$

where we have used the correspondence between a semiclassical pseudo-differential operator and a non semiclassical pseudo-differential operator and that the support of $\chi_{0,0}$ is contained in $\{(x,p) \in \mathbb{R}^{2d} \,|\, |(x,p)|_\infty \leq 1\}$. The constant c is dependent on the multi indices α and β (as it is the maximum of the $c_\alpha\beta$'s) and the estimates on the derivatives of $\chi_{0,0}$. All of these numbers is only dependent on the dimension. Lemma II.6.3 now imply

$$\begin{aligned} &\|H_d^{d+1} \mathrm{Op}_\hbar^{\mathrm{w}}(\chi_{0,0}\tilde{a}_\varepsilon)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \\ &\quad \leq \frac{c}{(\pi\hbar)^d} \sum_{|\alpha|+|\beta| \leq 2d+2} \hbar^{|\beta|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathrm{supp}(\chi_{0,0})}(x,p) |(\partial_x^\alpha \partial_p^\beta \tilde{a}_\varepsilon)(x,p)| \, dxdp \\ &\quad = \frac{\tilde{c}}{\hbar^d} \sum_{|\alpha|+|\beta| \leq 2d+2} \hbar^{|\beta|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathrm{supp}(\chi_{\gamma,\eta})}(x,p) |(\partial_x^\alpha \partial_p^\beta a_\varepsilon)(x,p)| \, dxdp. \end{aligned}$$

For the functions $\chi_{\gamma,\eta}$ at most 2^{2d} of them has not disjoint support. Hence combing the above estimate with (3.27) and (3.28) we get

$$\begin{aligned} &\| \mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)\|_{\mathrm{Tr}} \\ &\quad \leq \frac{\tilde{C}(d)\tilde{c}}{\hbar^d} \sum_{\gamma,\eta \in \mathbb{N}_0^d} \sum_{|\alpha|+|\beta| \leq 2d+2} \hbar^{|\beta|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathrm{supp}(\chi_{\gamma,\eta})}(x,p) |(\partial_x^\alpha \partial_p^\beta a_\varepsilon)(x,p)| \, dxdp \\ &\quad \leq \frac{\tilde{C}(d)\tilde{c}2^{2d}}{\hbar^d} \sum_{\gamma,\eta \in \mathbb{N}_0^d} \sum_{|\alpha|+|\beta| \leq 2d+2} \hbar^{|\beta|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\partial_x^\alpha \partial_p^\beta a_\varepsilon)(x,p)| \, dxdp. \end{aligned}$$

By letting $C(d) = \tilde{C}(d)\tilde{c}2^{2d}$ we have the desired estimate and this ends the proof. □

The previous theorem gives us a sufficient condition for the rough pseudo differential operators to be trace class. The next theorem gives the form of the trace for the rough pseudo differential operators.

Theorem II.6.6. *Let a_ε be in $\Gamma^{m,\tau}_{0,\varepsilon}(\mathbb{R}^d_x \times \mathbb{R}^d_p)$ with $\tau \geq 0$ and suppose $\partial_x^\alpha \partial_p^\beta a_\varepsilon(x,p)$ is an element of $L^1(\mathbb{R}^d_x \times \mathbb{R}^d_p)$ for all $|\alpha| + |\beta| \leq 2d+2$. Then is $\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)$ trace class and*

$$\operatorname{Tr}(\mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a_\varepsilon(x,p)\, dxdp.$$

The proof of this theorem is analogous to the proof of [17, Theorem II-53] and will not be given here.

II.7 Self-adjointness and functional calculus for rough pseudo-differential operator

In this section we will establish a functional calculus for rough pseudo-differential operators. The construction is similar to the one made in [17] except we need to be more aware when taking derivatives in x.

First we will give criteria for the operator to be semi-lower bounded and self adjoint.

Assumption II.7.1. Let $A_\varepsilon(\hbar)$ be a $\hbar$-ε-admissible operator of regularity τ and suppose that

(H$_1$) $A_\varepsilon(\hbar)$ is symmetric on $\mathcal{S}(\mathbb{R}^n)$ for all $\hbar$ in $]0,\hbar_0]$.

(H$_2$) The principal symbol $a_{\varepsilon,0}$ satisfies that

$$\min_{(x,p)\in\mathbb{R}^{2n}} a_{\varepsilon,0}(x,p) = \gamma_0 > -\infty.$$

(H$_3$) Let $\gamma_1 < \gamma_0$ and $\gamma_1 \leq 0$. Then $a_{\varepsilon,0} - \gamma_1$ is a tempered weight function with constants independent of ε and

$$a_{\varepsilon,j} \in \Gamma^{a_{\varepsilon,0}-\gamma_1,\tau-j}_{0,\varepsilon}\left(\mathbb{R}^d_x \times \mathbb{R}^d_p\right),$$

for all j in $\mathbb{N}_0$.

Remark II.7.2. The assumption in (H$_3$) that $a_{\varepsilon,0} - \gamma_1$ is a tempered weight function with constants independent of ε is crucial. If this is not satisfied then all constants will start to be dependent on ε, which is not desirable. Written out the assumption is that there should exist $C_0 > 0$ and N_0 in $\mathbb{N}$ such that

$$a_{\varepsilon,0}(x,p) - \gamma_1 \leq C_0(a_{\varepsilon,0}(x_0,p_0) - \gamma_1)\Big(1 + \sqrt{|x-x_0|^2 + |p-p_0|^2}\Big)^{N_0} \tag{3.29}$$

for all (x,p) and (x_0,p_0) in $\mathbb{R}^d_x \times \mathbb{R}^d_p$ and all ε in $(0,1]$.

Theorem II.7.3. *Let $A_\varepsilon(\hbar)$, for $\hbar$ in $(0,\hbar_0]$, be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ with tempered weight m and symbol*

$$a_\varepsilon(\hbar) = \sum_{j\geq 0} \hbar^j a_{\varepsilon,j}.$$

Suppose that $A_\varepsilon(h)$ satisfies Assumption II.7.1. Then there exists $\hbar_1$ in $]0,\hbar_0]$ such that for all $\hbar$ in $]0,\hbar_1]$ $A_\varepsilon(\hbar)$ is essential self-adjoint and lower semi-bounded.

Proof. We let $t < \gamma_0$, where γ_0 is the number from Assumption II.7.1. For this t we define the symbol

$$b_{\varepsilon,t}(x,p) = \frac{1}{a_{\varepsilon,0}(x,p) - t}.$$

By assumption we have that $b_{\varepsilon,t} \in \Gamma_{0,\varepsilon}^{(a_{\varepsilon,0}-\gamma_1)^{-1},\tau}(\mathbb{R}_x^d \times \mathbb{R}_p^d)$. For N sufficiently large we get by the formula for composition of symbols and the Calderon-Vaillancourt theorem that

$$\begin{aligned}(A_\varepsilon(\hbar) - t)\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,t}) = & \sum_{j=0}^{N} \hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,j})\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,t}) \\ & + \hbar^{N+1} R_N(\hbar)\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,t}) \\ = & I + \hbar S_N(\varepsilon,\hbar),\end{aligned}$$

where the operator S_N satisfies that $\sup_{\hbar\in(0,\hbar_0]}\|S_N(\varepsilon,\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} < \infty$. In the above calculation have we chosen N sufficiently large such that the operator has the form written above. We note that if $\hbar$ is chosen such that $\hbar\|S_N(\varepsilon,\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} < 1$ then the operator $I + \hbar S_N(\varepsilon,\hbar)$ will be invertible. We have by the Calderon-Vaillancourt theorem that $\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,t})$ is a bounded operator. This implies that the expression $\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,t})(I+\hbar S_N(\varepsilon,\hbar))^{-1}$ is a well defined bounded operator. Hence we have that the operator $(A_\varepsilon(\hbar)-t)$ maps its domain surjective onto all of $L^2(\mathbb{R}^d)$. By [19, Proposition 3.11] this implies that $A_\varepsilon(\hbar)$ is essential self-adjoint.

Since we have for all $t < \gamma_0$ that $(A_\varepsilon(\hbar) - t)$ maps its domain surjective onto all of $L^2(\mathbb{R}^d)$ they are all in the resolvent set and hence the operator has to be lower semi-bounded. ☐

If the rough symbol is positive, does this imply that the associated operator than positive? In general this is not true but it is close to be true as the next theorems shows. which is a version of the sharp Gårdinger inequality for rough pseudo-differential operators.

Theorem II.7.4. *Let a_ε be a bounded rough symbol of regularity $\tau \geq 0$ which satisfies*

$$a_\varepsilon(x,p) \geq 0 \quad \text{for all } (x,p) \in \mathbb{R}_x^d \times \mathbb{R}_p^d,$$

and suppose there exists $\delta > 0$ such that $\varepsilon > \hbar^{1-\delta}$. Then there exists a $C_0 > 0$ and $\hbar_0 > 0$ such

$$\langle \operatorname{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)g, g\rangle \geq -\hbar^\delta C\|g\|_{L^2(\mathbb{R}^d)}$$

for all g in $L^2(\mathbb{R}^d)$.

Proof. We will really on the "usual" semiclassical sharp Gårdinger inequality with the semiclassical parameter $h = \hbar\varepsilon^{-1}$. We have by a change of variables

$$
\begin{aligned}
\langle \mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)g, g\rangle =& \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} a_\varepsilon(\tfrac{x+y}{2},p) g(y)\overline{g(x)}\, dydpdx \\
=& \frac{\varepsilon^{2d}}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\varepsilon\hbar^{-1}\langle \tilde{x}-\tilde{y},p\rangle} a_\varepsilon(\varepsilon\tfrac{\tilde{x}+\tilde{y}}{2},p) g(\varepsilon\tilde{y})\overline{g(\varepsilon\tilde{x})}\, d\tilde{y}dpd\tilde{x} \\
=& \varepsilon^d \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ih^{-1}\langle \tilde{x}-\tilde{y},p\rangle} \tilde{a}_\varepsilon(\tfrac{\tilde{x}+\tilde{y}}{2},p) \tilde{g}(\tilde{y})\overline{\tilde{g}(\tilde{x})}\, d\tilde{y}dpd\tilde{x} \\
=& \varepsilon^d \langle \mathrm{Op}_h^{\mathrm{w}}(\tilde{a}_\varepsilon)\tilde{g}, \tilde{g}\rangle,
\end{aligned}
\tag{3.30}
$$

where the change of variables was

$$
\tilde{x} = \varepsilon^{-1}x \qquad\qquad \tilde{y} = \varepsilon^{-1}y,
$$

and the new functions are defined by

$$
\tilde{a}_\varepsilon(\tfrac{\tilde{x}+\tilde{y}}{2},p) \coloneqq a_\varepsilon(\tfrac{x+y}{2},p) = a_\varepsilon(\varepsilon\tfrac{\tilde{x}+\tilde{y}}{2},p) \qquad\qquad \tilde{g}(\tilde{x}) \coloneqq g(x) = g(\varepsilon x).
$$

The change of variables have given us a new operator to consider $\mathrm{Op}_h^{\mathrm{w}}(\tilde{a}_\varepsilon)$ with the new semiclassical parameter h. The new symbol satisfies for all α and β in $\mathbb{N}_0^d$ the estimate

$$
|\partial_{\tilde{x}}^\alpha \partial_p^\beta \tilde{a}_\varepsilon(\tilde{x},p)| = |\partial_{\tilde{x}}^\alpha \partial_p^\beta a_\varepsilon(\varepsilon\tilde{x},p)| \le \varepsilon^{|\alpha|}\varepsilon^{-|\alpha|}C \le C,
$$

by our assumptions on the symbol a_ε. Hence in the new coordinates and with the new semiclassical parameter h the shapr Gårdinger inequality imply there exists a constant $C_0 > 0$ and $h_0 > 0$ such for $h \in (0, h_0]$ we have

$$
\langle \mathrm{Op}_h^{\mathrm{w}}(\tilde{a}_\varepsilon)\tilde{g}, \tilde{g}\rangle \ge -C_0 h \|\tilde{g}\|_{L^2(\mathbb{R}^d)}.
$$

Now combing this with (3.30) we arrive at

$$
\langle \mathrm{Op}_\hbar^{\mathrm{w}}(a_\varepsilon)g, g\rangle \ge -C_0\varepsilon^d h \|\tilde{g}\|_{L^2(\mathbb{R}^d)} \ge -C_0\hbar^\delta \|g\|_{L^2(\mathbb{R}^d)}.
$$

This is the desired estimate and this ends the proof. □

The next theorem shows that the resolvent of an operator which satisfies Assumption II.7.1 is a rough pseudo-differential operator.

Theorem II.7.5. *Let $A(\hbar)$, for $\hbar$ in $(0, \hbar_0]$, be a $\hbar$-ε-admissible operator of regularity $\tau \ge 1$ with tempered weight m and symbol*

$$
a_\varepsilon(\hbar) = \sum_{j\ge 0} \hbar^j a_{\varepsilon,j}.
$$

Suppose that $A_\varepsilon(h)$ satisfies Assumption II.7.1 with the numbers γ_0 and γ_1. For z in $\mathbb{C}\setminus[\gamma_1,\infty)$ we define the sequence of symbols

$$\begin{aligned} b_{\varepsilon,z,0} &= (a_{\varepsilon,0}-z)^{-1} \\ b_{\varepsilon,z,j+1} &= -b_{\varepsilon,z,0}\cdot \sum_{\substack{l+|\alpha|+|\beta|+k=j+1\\ 0\le l\le j}} \frac{1}{\alpha!\beta!}\frac{1}{2^{|\alpha|}}\frac{1}{(-2)^{|\beta|}}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,l}), \end{aligned} \tag{3.31}$$

for $j\ge 1$. Moreover we define

$$B_{\varepsilon,z,M}(\hbar)=\sum_{j=0}^{M}\hbar^j b_{\varepsilon,z,j}.$$

Then for N in $\mathbb{N}_0$ we have that

$$(A_\varepsilon(h)-z)\operatorname{Op}_\hbar^{\mathrm{w}} B_{\varepsilon,z,N}=I+h^{N+1}\Delta_{z,N+1}(h), \tag{3.32}$$

with

$$\hbar^{N+1}\|\Delta_{z,N+1}(h)\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\le C\hbar^{\kappa(N)}\left(\frac{|z|}{\operatorname{dist}(z,[\gamma_1,\infty))}\right)^{q(N)}, \tag{3.33}$$

where κ is a positive strictly increasing function and $q(N)$ is a positive integer depending on N. In particular we have for all z in $\mathbb{C}\setminus[\gamma_1,\infty)$ and all $\hbar$ in $(0,\hbar_1]$ ($\hbar_1$ sufficient small and independent of z), that $(A_\varepsilon(h)-z)^{-1}$ is a $\hbar$-ε-admissible operator with respect to the tempered weight $(a_{\varepsilon,0}-\gamma_1)^{-1}$ and of regularity τ with symbol:

$$B_z(h)=\sum_{j\ge 0}h^j b_{\varepsilon,z,j}. \tag{3.34}$$

Before we prove the actual theorem we will need some lemma's with the same setting.

Lemma II.7.6. *Let the setting be as in Theorem II.7.5. For every j in $\mathbb{N}$ we have*

$$b_{\varepsilon,z,j}=\sum_{k=1}^{2j-1}d_{\varepsilon,j,k}b_{\varepsilon,z,0}^{k+1}, \tag{3.35}$$

where $d_{\varepsilon,j,k}$ are universal polynomials in $\partial_p^\alpha\partial_x^\beta a_{\varepsilon,l}$ for $|\alpha|+|\beta|+l\le j$ and $d_{\varepsilon,j,k}\in\Gamma_{0,\varepsilon}^{(a_0-\gamma_1)^k,\tau-j}$ for all k, $1\le k\le 2j-1$. In particular we have that

$$b_{\varepsilon,z,1}=-a_{\varepsilon,1}b_{\varepsilon,z,0}^2.$$

In order to prove this Lemma we will be needing the following Lemma:

Lemma II.7.7. *Let the setting be as in Lemma II.7.6. For any j and k in $\mathbb{N}$ we let $d_{\varepsilon,j,k}b_{\varepsilon,z,0}^{k+1}$ be one of the elements in the expansion of $b_{\varepsilon,z,j}$. Then for all multiindices α and β it holds that*

$$\partial_p^\beta\partial_x^\alpha d_{\varepsilon,j,k}b_{\varepsilon,z,0}^{k+1}=\sum_{n=0}^{|\alpha|+|\beta|}\tilde{d}_{\varepsilon,j,k,n,\alpha,\beta}b_{\varepsilon,z,0}^{k+1+n},$$

where $\tilde{d}_{\varepsilon,j,k,n,\alpha,\beta}$ are polynomials in $\partial_x^{\alpha'}\partial_p^{\beta'}a_{\varepsilon,k}$ with $|\alpha'|+|\beta'|+k\le j+|\alpha|+|\beta|$ of degree at most $k+n$ and they are of regularity at least $\tau-j-|\alpha|$. They are determined only by α, β and $d_{\varepsilon,j,k}$.

Proof. The proof is an application of Theorem II.1.2 (Faà di Bruno formula) and the Corollary II.1.3 to the formula. For our α, β we have by the Leibniz's formula (Theorem II.1.1) that:

$$\begin{aligned}
\partial_p^\beta \partial_x^\alpha d_{\varepsilon,j,k} b_{\varepsilon,z,0}^{k+1} =& \partial_p^\beta \{(\partial_x^\alpha d_{\varepsilon,j,k}) b_{\varepsilon,z,0}^{k+1} + \sum_{\substack{|\gamma|=1 \\ \gamma\leq\alpha}}^{|\alpha|} \binom{\alpha}{\gamma} \partial_x^{\alpha-\gamma} d_{\varepsilon,j,k} \partial_x^\gamma b_{\varepsilon,z,0}^{k+1}\} \\
=& (\partial_p^\beta \partial_x^\alpha d_{\varepsilon,j,k}) b_{\varepsilon,z,0}^{k+1} + \sum_{\substack{|\zeta|=1 \\ \zeta\leq\beta}}^{|\beta|} \binom{\beta}{\zeta} \partial_p^{\beta-\zeta} \partial_x^\alpha d_{\varepsilon,j,k} \partial_p^\zeta b_{\varepsilon,z,0}^{k+1} \\
&+ \sum_{\substack{|\gamma|=1 \\ \gamma\leq\alpha}}^{|\alpha|} \binom{\alpha}{\gamma} (\partial_p^\beta \partial_x^{\alpha-\gamma} d_{\varepsilon,j,k}) \partial_x^\gamma b_{\varepsilon,z,0}^{k+1} \\
&+ \sum_{\substack{|\zeta|=1 \\ \zeta\leq\beta}}^{|\beta|} \sum_{\substack{|\gamma|=1 \\ \gamma\leq\alpha}}^{|\alpha|} \binom{\beta}{\zeta} \binom{\alpha}{\gamma} (\partial_p^{\beta-\zeta} \partial_x^{\alpha-\gamma} d_{\varepsilon,j,k}) \partial_p^\zeta \partial_x^\gamma b_{\varepsilon,z,0}^{k+1}.
\end{aligned}$$

We will here consider each of the three sums separately for the first we get by the Faà di Bruno formula (Theorem II.1.2)

$$\begin{aligned}
&\sum_{\substack{|\zeta|=1 \\ \zeta\leq\beta}}^{|\beta|} \binom{\beta}{\zeta} \partial_p^{\beta-\zeta} \partial_x^\alpha d_{\varepsilon,j,k} \partial_p^\zeta b_{\varepsilon,z,0}^{k+1} \\
&= \sum_{\substack{|\zeta|=1 \\ \zeta\leq\beta}}^{|\beta|} \binom{\beta}{\zeta} \partial_p^{\beta-\zeta} \partial_x^\alpha d_{\varepsilon,j,k} \sum_{n=1}^{|\zeta|} (-1)^n \frac{(k+n)!}{k!} b_{\varepsilon,z,0}^{k+1+n} \sum_{\substack{\zeta_1+\cdots+\zeta_n=\zeta \\ |\zeta_i|>0}} c_{\zeta_1,\dots,\zeta_n} \partial_p^{\zeta_1} a_0 \cdots \partial_p^{\zeta_n} a_0 \\
&= \sum_{n_\beta=1}^{|\beta|} \Big\{ \sum_{\substack{|\zeta|\geq n_\beta \\ \zeta\leq\beta}}^{|\beta|} c_{k,n_\beta,\beta,\zeta} \partial_p^{\beta-\zeta} \partial_x^\alpha d_{\varepsilon,j,k} \sum_{\substack{\zeta_1+\cdots+\zeta_{n_\beta}=\zeta \\ |\zeta_i|>0}} c_{\zeta_1,\dots,\zeta_n} \partial_p^{\zeta_1} a_0 \cdots \partial_p^{\zeta_{n_\beta}} a_0 \Big\} b_{\varepsilon,z,0}^{k+1+n_\beta} \\
&= \sum_{n_\beta=1}^{|\beta|} \tilde{d}_{\varepsilon,j,k,\alpha,\beta,n_\beta} b_{\varepsilon,z,0}^{k+1+n_\beta}.
\end{aligned}$$

This calculation implies that we have a polynomial structure where the polynomials $\tilde{d}_{\varepsilon,j,k,\alpha,\beta,n_\beta}$ are polynomials in $\partial_x^{\alpha'} \partial_p^{\beta'} a_{\varepsilon,k}$ with $|\alpha'| + |\beta'| + k \leq j + |\alpha| + |\beta|$ and of regularity $\tau - j - |\alpha|$.

For the second sum we again use Faà di Bruno formula (Theorem II.1.2) and get

$$
\begin{aligned}
&\sum_{\substack{|\gamma|=1\\ \gamma\le\alpha}}^{|\alpha|}\binom{\alpha}{\gamma}(\partial_p^{\beta}\partial_x^{\alpha-\gamma}d_{\varepsilon,j,k})\partial_x^{\gamma}b_{\varepsilon,z,0}^{k+1}\\
&=\sum_{\substack{|\gamma|=1\\ \gamma\le\alpha}}^{|\alpha|}\binom{\alpha}{\gamma}\partial_p^{\beta}\partial_x^{\alpha-\gamma}d_{\varepsilon,j,k}\sum_{n=1}^{|\gamma|}(-1)^n\frac{(k+n)!}{k!}b_{\varepsilon,z,0}^{k+1+n}\sum_{\substack{\gamma_1+\cdots+\gamma_n=\gamma\\ |\gamma_i|>0}}c_{\gamma_1,\dots,\gamma_n}\partial_x^{\gamma_1}a_0\cdots\partial_x^{\gamma_n}a_0\\
&=\sum_{n_\alpha=1}^{|\alpha|}\Big\{\sum_{\substack{|\gamma|\ge n_\alpha\\ \gamma\le\alpha}}^{|\alpha|}c_{k,n_\alpha,\alpha,\gamma}\partial_p^{\beta}\partial_x^{\alpha-\gamma}d_{\varepsilon,j,k}\sum_{\substack{\gamma_1+\cdots+\gamma_{n_\alpha}=\gamma\\ |\gamma_i|>0}}c_{\gamma_1,\dots,\gamma_n}\partial_x^{\gamma_1}a_0\cdots\partial_x^{\gamma_{n_\alpha}}a_0\Big\}b_{\varepsilon,z,0}^{k+1+n_\alpha}\\
&=\sum_{n_\alpha=1}^{|\alpha|}\tilde d_{\varepsilon,j,k,\alpha,\beta,n_\alpha}b_{\varepsilon,z,0}^{k+1+n_\alpha}.
\end{aligned}
$$

This calculation implies that we have a polynomial structure where the polynomials $\tilde d_{\varepsilon,j,k,\alpha,\beta,n_\alpha}$ are polynomials in $\partial_x^{\alpha'}\partial_p^{\beta'}a_{\varepsilon,k}$ with $|\alpha'|+|\beta'|+k\le j+|\alpha|+|\beta|$ and they are of at least regularity $\tau-j-|\alpha|$.

For the last sum we need a slightly modified version of the Faà di Bruno formula which is Corollary II.1.3. If we use this we get that

$$
\begin{aligned}
&\sum_{\substack{|\zeta|=1\\ \zeta\le\beta}}^{|\beta|}\sum_{\substack{|\gamma|=1\\ \gamma\le\alpha}}^{|\alpha|}\binom{\beta}{\zeta}\binom{\alpha}{\gamma}(\partial_p^{\beta-\zeta}\partial_x^{\alpha-\gamma}d_{\varepsilon,j,k})\partial_p^{\zeta}\partial_x^{\gamma}b_{\varepsilon,z,0}^{k+1}\\
&=\sum_{\substack{|\zeta|=1\\ \zeta\le\beta}}^{|\beta|}\sum_{\substack{|\gamma|=1\\ \gamma\le\alpha}}^{|\alpha|}\binom{\beta}{\zeta}\binom{\alpha}{\gamma}(\partial_p^{\beta-\zeta}\partial_x^{\alpha-\gamma}d_{\varepsilon,j,k})\sum_{n=1}^{|\zeta|+|\gamma|}c_nb_{\varepsilon,z,0}^{k+1+n}\\
&\qquad\times\sum_{\mathcal{I}_n(\gamma,\zeta)}c_{\gamma_1\cdots\gamma_k}^{\zeta_1\cdots\zeta_k}\partial_p^{\zeta_1}\partial_x^{\gamma_1}a_{\varepsilon,0}\cdots\partial_p^{\zeta_n}\partial_x^{\gamma_n}a_{\varepsilon,0}\\
&=\sum_{n=1}^{|\alpha|+|\beta|}\Big\{\sum_{\substack{|\zeta|+|\gamma|\ge n\\ \zeta\le\beta,\ \gamma\le\alpha}}^{|\beta|+|\alpha|}\sum_{\mathcal{I}_n(\gamma,\zeta)}c_{k,n,\alpha,\beta,\gamma,\zeta}(\partial_p^{\beta-\zeta}\partial_x^{\alpha-\gamma}d_{\varepsilon,j,k})\\
&\qquad\times\partial_p^{\zeta_1}\partial_x^{\gamma_1}a_{\varepsilon,0}\cdots\partial_p^{\zeta_n}\partial_x^{\gamma_n}a_{\varepsilon,0}\Big\}b_{\varepsilon,z,0}^{k+1+n}\\
&=\sum_{n=1}^{|\alpha|+|\beta|}\tilde d_{\varepsilon,j,k,n,\alpha,\beta}b_{\varepsilon,z,0}^{k+1+n},
\end{aligned}
$$

where $\tilde d_{\varepsilon,j,k,n,\alpha,\beta}$ are polynomials in $\partial_x^{\alpha'}\partial_p^{\beta'}a_{\varepsilon,k}$ with $|\alpha'|+|\beta'|+k\le j+|\alpha|+|\beta|$ of degree at most $k+n$ and they are of regularity at least $\tau-j-|\alpha|$. If we combine all of the above calculations we get the desired result:

$$\partial_p^\beta \partial_x^\alpha d_{\varepsilon,j,k} b_{\varepsilon,z,0}^{k+1} = \sum_{n=0}^{|\alpha|+|\beta|} \tilde{d}_{\varepsilon,j,k,n,\alpha,\beta} b_{\varepsilon,z,0}^{k+1+n},$$

where $\tilde{d}_{\varepsilon,j,k,n,\alpha,\beta}$ are polynomials in $\partial_x^{\alpha'} \partial_p^{\beta'} a_{\varepsilon,k}$ with $|\alpha'| + |\beta'| + k \leq j + |\alpha| + |\beta|$ of degree at most $k + n$ and they are of regularity at least $\tau - j - |\alpha|$. The form of the polynomials is entirely determined by the multi-indices α, β and the polynomial $d_{\varepsilon,j,k}$. □

Proof (Proof of Lemma II.7.6). The proof will be induction in the parameter j. We start by considering the case $j = 1$, where we by definition of $b_{\varepsilon,z,1}$ have

$$\begin{aligned} b_{\varepsilon,z,1} &= -b_{\varepsilon,z,0} \cdot \sum_{|\alpha|+|\beta|+k=1} \frac{1}{\alpha!\beta!} \frac{1}{2^{|\alpha|}} \frac{1}{(-2)^{|\beta|}} (\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,0}) \\ &= -b_{\varepsilon,z,0}\big(a_{\varepsilon,1} b_{\varepsilon,z,0} + \sum_{n=1}^{d} \frac{-i}{2} \partial_{p_n} a_{\varepsilon,0} \partial_{x_n} b_{\varepsilon,z,0} + \frac{i}{2} \partial_{x_n} a_{\varepsilon,0} \partial_{p_n} b_{\varepsilon,z,0}\big) \\ &= -b_{\varepsilon,z,0}\big(a_{\varepsilon,1} b_{\varepsilon,z,0} - \frac{i}{2} \sum_{n=1}^{d} \partial_{p_n} a_{\varepsilon,0} \partial_{x_n} a_{\varepsilon,0} b_{\varepsilon,z,0}^2 + \partial_{x_n} a_{\varepsilon,0} \partial_{p_n} a_{\varepsilon,0} b_{\varepsilon,z,0}^2\big) \\ &= -a_{\varepsilon,1} b_{\varepsilon,z,0}^2. \end{aligned}$$

This calculation verifies the form of $b_{\varepsilon,z,1}$ stated in the lemma and that it has the form given by (3.35) with $d_{\varepsilon,1,1} = -a_{\varepsilon,1}$ which is in the symbol class $\Gamma_{0,\varepsilon}^{(a_0-\gamma_1),\tau-1}$ by assumption.

Assume the lemma to be correct for $b_{\varepsilon,z,j}$ and consider $b_{\varepsilon,z,j+1}$. By the definition of $b_{\varepsilon,z,j+1}$ and our assumption we have

$$\begin{aligned} b_{\varepsilon,z,j+1} =& - b_{\varepsilon,z,0} \cdot \sum_{\substack{l+|\alpha|+|\beta|+k=j+1 \\ 0 \leq l \leq j}} \frac{1}{\alpha!\beta!} \frac{1}{2^{|\alpha|}} \frac{1}{(-2)^{|\beta|}} (\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,l}) \\ =& - b_{\varepsilon,z,0}\Big\{ \sum_{|\alpha|+|\beta|+k=j+1} c_{\alpha,\beta} (\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,0}) \\ &+ \sum_{l=1}^{j} \sum_{l+|\alpha|+|\beta|+k=j+1} c_{\alpha,\beta} (\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,l}) \Big\} \\ =& - b_{\varepsilon,z,0}\Big\{ \sum_{|\alpha|+|\beta|+k=j+1} c_{\alpha,\beta} (\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,0}) \\ &+ \sum_{l=1}^{j} \sum_{m=1}^{2l-1} \sum_{l+|\alpha|+|\beta|+k=j+1} c_{\alpha,\beta} (\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha d_{\varepsilon,l,m} b_{\varepsilon,z,0}^{m+1}) \Big\}. \end{aligned}$$

We will consider each of the sums separately. To calculate the first sum we get by applying Corollary II.1.3

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|+k=j+1} c_{\alpha,\beta}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,0})\\
&\quad= c_{\alpha,\beta} a_{\varepsilon,j+1} b_{\varepsilon,z,0} + \sum_{\substack{|\alpha|+|\beta|+k=j+1\\|\alpha|+|\beta|\geq 1}} \Big\{ c_{\alpha,\beta}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k}) \sum_{n=1}^{|\alpha|+|\beta|} c_n b_{\varepsilon,z,0}^{n+1}\\
&\qquad \times \sum_{\mathcal{I}_n(\alpha,\beta)} c_{\alpha_1\cdots\alpha_n}^{\beta_1\cdots\beta_n} \partial_p^{\beta_1} D_x^{\alpha_1} a_{\varepsilon,0}\cdots \partial_p^{\beta_n} D_x^{\alpha_n} a_{\varepsilon,0}\Big\}\\
&\quad= c_{\alpha,\beta} a_{\varepsilon,j+1} b_{\varepsilon,z,0} + \sum_{n=1}^{j+1}\Big\{ \sum_{\substack{|\alpha|+|\beta|+k=j+1\\|\alpha|+|\beta|\geq n}} \sum_{\mathcal{I}_n(\alpha,\beta)} c_{\alpha,\beta,n}\\
&\qquad \times (\partial_p^\alpha D_x^\beta a_{\varepsilon,k}) \partial_p^{\beta_1} D_x^{\alpha_1} a_{\varepsilon,0}\cdots \partial_p^{\beta_n} D_x^{\alpha_n} a_{\varepsilon,0}\Big\} b_{\varepsilon,z,0}^{n+1}\\
&\quad= \sum_{n=0}^{j+1} \tilde{d}_{\varepsilon,j,n,\alpha,\beta} b_{\varepsilon,z,0}^{n+1},
\end{aligned}$$

where $\tilde{d}_{\varepsilon,j,n,\alpha,\beta}$ are polynomials in $\partial_x^{\alpha'}\partial_p^{\beta'} a_{\varepsilon,k}$ with $|\alpha'|+|\beta'|+k\leq j+1$ of degree $n+1$ and of regularity at least $\tau-j-1$. The index set $\mathcal{I}_n(\alpha,\beta)$ is defined by

$$\begin{aligned}
\mathcal{I}_n(\alpha,\beta) = \{&(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)\in\mathbb{N}_0^{2nd}\\
&\mid \sum_{l=1}^n \alpha_l=\alpha,\, \sum_{l=1}^n \beta_l=\beta,\, \max(|\alpha_l|,|\beta_l|)\geq 1\,\forall l\}.
\end{aligned}$$

The form of the polynomials is determined by the multi indices α and β. Moreover we have that the polynomials $\tilde{d}_{\varepsilon,j,n,\alpha,\beta}$ are elements of $\Gamma_{0,\varepsilon}^{(a_0-\gamma_1)^n,\tau-j-1}$.

If we now consider the triple sum and apply Lemma II.7.7 we get

$$\begin{aligned}
&\sum_{l=1}^{j}\sum_{m=1}^{2l-1}\sum_{l+|\alpha|+|\beta|+k=j+1} c_{\alpha,\beta}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha d_{\varepsilon,l,m} b_{\varepsilon,z,0}^{m+1})\\
&= \sum_{l=1}^{j}\sum_{m=1}^{2l-1}\sum_{l+|\alpha|+|\beta|+k=j+1} c_{\alpha,\beta}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(-i)^{|\alpha|}\sum_{n=0}^{|\alpha|+|\beta|}\tilde{d}_{\varepsilon,l,m,n,\alpha,\beta} b_{\varepsilon,z,0}^{m+1+n}\\
&= \sum_{m=1}^{2j-1}\sum_{l=\lceil\frac{m-1}{2}\rceil+1}^{j}\sum_{l+|\alpha|+|\beta|+k=j+1}\sum_{n=0}^{|\alpha|+|\beta|} c_{\alpha,\beta}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(-i)^{|\alpha|}\tilde{d}_{\varepsilon,l,m,n,\alpha,\beta} b_{\varepsilon,z,0}^{m+1+n}\\
&= \sum_{m=1}^{2j-1}\sum_{l=\lceil\frac{m-1}{2}\rceil+1}^{j}\sum_{n=0}^{j+1-l}\sum_{|\alpha|+|\beta|=n} c_{\alpha,\beta}(\partial_p^\alpha D_x^\beta a_{\varepsilon,j+1-l-n})(-i)^{|\alpha|}\tilde{d}_{\varepsilon,l,m,n,\alpha,\beta} b_{\varepsilon,z,0}^{m+1+n},
\end{aligned}$$

where the $\tilde{d}_{\varepsilon,l,m,n\alpha,\beta}$ are the polynomials from Lemma II.7.7. The way we have expressed the sums ensures k always is the fixed value $j+1-l-n$. From Lemma II.7.7 we have that $\tilde{d}_{\varepsilon,l,m,n,\alpha,\beta}$ are polynomials in $\partial_x^{\alpha'}\partial_p^{\beta'} a_{\varepsilon,m}$ with $|\alpha'|+|\beta'|+m\leq l+n\leq j+1$ of degree $n+m$ and with regularity at least $\tau-l-|\alpha|$. Hence the factors $c_{\alpha,\beta}(\partial_p^\alpha D_x^\beta a_{\varepsilon,j+1-l-n})(-i)^{|\alpha|}\tilde{d}_{\varepsilon,l,m,n,\alpha,\beta}$ will be polynomials in $\partial_x^{\alpha'}\partial_p^{\beta'}a_{\varepsilon,m}$ with $|\alpha'|+|\beta'|+m \leq j+1$ of degree $n+m+1$. The regularity of the terms will be at least

$$\tau - l - |\alpha| - (j+1-l-n) - |\beta| = \tau - (j+1),$$

where most terms will have more regularity. By rewriting and renaming some of the terms we get the following equality

$$\sum_{l=1}^{j}\sum_{m=1}^{2l-1}\sum_{l+|\alpha|+|\beta|+k=j+1} c_{\alpha,\beta}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha d_{\varepsilon,l,m} b_{\varepsilon,z,0}^{m+1}) = \sum_{n=1}^{2j} \tilde{d}_{\varepsilon,j,n,\alpha,\beta} b_{\varepsilon,z,0}^{n+1},$$

where $\tilde{d}_{\varepsilon,j,n,\alpha,\beta}$ again are polynomials in $\partial_x^{\alpha'}\partial_p^{\beta'}a_{\varepsilon,k}$ with $|\alpha'|+|\beta'|+k \leq j+1$ of degree $n+1$ of regularity at least $\tau-(j+1)$. By combing these calculation we arrive at the expression

$$\begin{aligned} b_{\varepsilon,z,j+1} &= - b_{\varepsilon,z,0}\cdot \sum_{\substack{l+|\alpha|+|\beta|+k=j+1\\ 0\leq l\leq j}} \frac{1}{\alpha!\beta!}\frac{1}{2^{|\alpha|}}\frac{1}{(-2)^{|\beta|}}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,l}) \\ &= - b_{\varepsilon,z,0}\Big\{\sum_{n=0}^{j+1}\tilde{d}_{\varepsilon,j,n,\alpha,\beta}b_{\varepsilon,z,0}^{n+1} + \sum_{n=1}^{2j}\tilde{d}_{\varepsilon,j,n,\alpha,\beta}b_{\varepsilon,z,0}^{n+1}\Big\} \\ &= \sum_{k=1}^{2j+1} d_{\varepsilon,j+1,k}b_{\varepsilon,z,0}^{k+1}, \end{aligned}$$

where the polynomials $d_{\varepsilon,j+1,k}$ are universal polynomials in $\partial_x^{\alpha'}\partial_p^{\beta'}a_{\varepsilon,k}$ with $|\alpha'|+|\beta'|+k \leq j+1$ of degree k and with regularity at least $\tau-j-1$. Hence they are elements of $\Gamma_{0,\varepsilon}^{(a_0-\gamma_1)^k,\tau-(j+1)}$. This ends the proof. ☐

Lemma II.7.8. *Let the setting be as in Theorem II.7.5. For every j in $\mathbb{N}_0$ and α, β in $\mathbb{N}_0^d$ there exists a number $c_{j,\alpha,\beta} > 0$ such that*

$$|\partial_p^\beta\partial_x^\alpha b_{\varepsilon,z,j}| \leq c_{j,\alpha,\beta}\varepsilon^{-(\tau-j-|\alpha|)_-}(a_{\varepsilon,0}-\gamma_1)^{-1}\left(\frac{|z-\gamma_1|}{\operatorname{dist}(z,[\gamma_1,\infty))}\right)^{2j+|\alpha|+|\beta|},$$

for all $z \in \mathbb{C}\setminus[\gamma_1,\infty)$ and all $(x,p)\in\mathbb{R}_x^d\times\mathbb{R}_p^d$.

Proof. We start by considering the fraction $\frac{|a_{\varepsilon,0}-\gamma_1|}{|a_{\varepsilon,0}-z|}$ and we will divide it into two cases according to the real part of z. If $\operatorname{Re}(z) < \gamma_1$ then

$$\frac{|a_{\varepsilon,0}-\gamma_1|}{|a_{\varepsilon,0}-z|} \leq 1 \leq \frac{|z-\gamma_1|}{\operatorname{dist}(z,[\gamma_1,\infty))}.$$

If instead $\operatorname{Re}(z)\geq\gamma_1$ and $|\operatorname{Im} z| > 0$ we have by the law of sines that

$$\frac{|a_{\varepsilon,0}-z|}{\sin(\phi_1)} = \frac{|a_{\varepsilon,0}-\gamma_1|}{\sin(\phi_2)} \geq |a_{\varepsilon,0}-\gamma_1|,$$

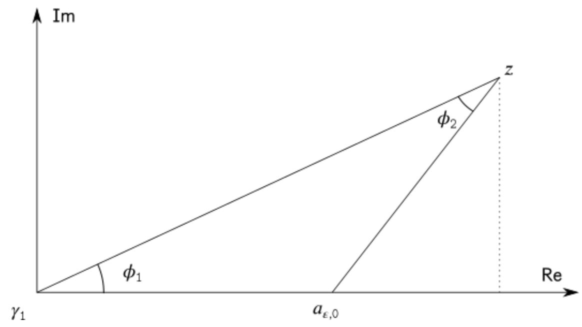

Figure 31: One case of $\mathrm{Re}(z) \geq \gamma_1$ and $|\mathrm{Im}\, z| > 0$.

where we have used that $0 < \sin(\phi_2) \leq 1$. If we apply this inequality and the law of sines again we arrive at the following expression

$$\frac{|a_{\varepsilon,0} - \gamma_1|}{|a_{\varepsilon,0} - z|} \leq \frac{1}{\sin(\phi_1)} = \frac{|z - \gamma_1|}{|\mathrm{Im}(z)|}.$$

Combining these two cases we get the estimate

$$\frac{|a_{\varepsilon,0} - \gamma_1|}{|a_{\varepsilon,0} - z|} \leq \frac{|z - \gamma_1|}{\mathrm{dist}(z, [\gamma_1, \infty))} \qquad \text{For all } z \in \mathbb{C} \setminus [\gamma_1, \infty). \tag{3.36}$$

If we now consider a given $b_{\varepsilon,z,j}$ and α, β in $\mathbb{N}_0^d$. Lemma II.7.6 and Lemma II.7.7 gives us

$$\partial_p^\beta \partial_x^\alpha b_{\varepsilon,z,j} = \sum_{k=1}^{2j-1} \partial_p^\beta \partial_x^\alpha (d_{\varepsilon,j,k} b_{\varepsilon,z,0}^{k+1}) = \sum_{k=1}^{2j-1} \sum_{n=0}^{|\alpha|+|\beta|} \tilde{d}_{\varepsilon,j,k,n,\alpha,\beta} b_{\varepsilon,z,0}^{k+1+n},$$

with $\tilde{d}_{\varepsilon,j,k,\alpha,\beta}$ in $\Gamma_{0,\varepsilon}^{(a_0-\gamma_1)^{k+n},\tau-j-|\alpha|}$. By taking absolute value and applying (3.36) we get that

$$\begin{aligned}
|\partial_p^\beta \partial_x^\alpha b_{\varepsilon,z,j}| \leq & \sum_{k=1}^{2j-1} \sum_{n=0}^{|\alpha|+|\beta|} |\tilde{d}_{\varepsilon,j,k,n,\alpha,\beta} b_{\varepsilon,z,0}^{k+1+n}| \\
\leq & |b_{\varepsilon,z,0}| \sum_{k=1}^{2j-1} \sum_{n=0}^{|\alpha|+|\beta|} \varepsilon^{-(\tau-j-|\alpha|)_-} c_{j,k,\alpha,\beta} \left(\frac{|a_{\varepsilon,0} - \gamma_1|}{|a_{\varepsilon,0} - z|} \right)^{k+n} \\
\leq & c_{j,\alpha,\beta} \varepsilon^{-(\tau-j-|\alpha|)_-} (a_{\varepsilon,0} - \gamma_1)^{-1} \left(\frac{|z - \gamma_1|}{\mathrm{dist}(z, [\gamma_1, \infty))} \right)^{2j+|\alpha|+|\beta|},
\end{aligned}$$

where we have use

$$|b_{\varepsilon,z,0}| = \frac{(a_{\varepsilon,0} - \gamma_1)}{|a_{\varepsilon,0} - z|(a_{\varepsilon,0} - \gamma_1)} \leq \frac{1}{(a_{\varepsilon,0} - \gamma_1)} \left(\frac{|z - \gamma_1|}{\mathrm{dist}(z, [\gamma_1, \infty))} \right).$$

We have now obtained the desired estimate and this ends the proof. □

Proof (Proof of Theorem II.7.5). We have by Lemma II.7.6 that the symbols $b_{\varepsilon,z,j}$ are in the class $\Gamma_{0,\varepsilon}^{(a_0-\gamma_1)^{-1},\tau-j}$ for every j in $\mathbb{N}_0$, where $b_{\varepsilon,z,j}$ is defined (3.31). Hence we have that

$$B_{\varepsilon,z,N}(\hbar) = \sum_{j=0}^{N} \hbar^j b_{\varepsilon,z,j}.$$

is a well defined symbol for every N in $\mathbb{N}_0$. Moreover as $(a_0-\gamma_1)^{-1}$ is a bounded function we have by Theorem II.6.1 that $\operatorname{Op}_\hbar^{\mathrm{w}}(B_{\varepsilon,z,N}(\hbar))$ is a bounded operator. Now for N sufficiently large we have by assumption

$$A_\varepsilon(\hbar)-z=\operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,0}-z)+\sum_{k=1}^{N}\hbar^k\operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,k})+\hbar^{N+1}R_N(\varepsilon,\hbar),$$

where the error term satisfies

$$\hbar^{N+1}\|R_N(\varepsilon,\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq\hbar^{\kappa(N)}C_N$$

for a positive strictly increasing function κ. If we consider the composition of $A_\varepsilon(\hbar)$ and $\operatorname{Op}_\hbar^{\mathrm{w}}(B_{\varepsilon,z,N}(\hbar))$ we get

$$\begin{aligned}A_\varepsilon(\hbar)\operatorname{Op}_\hbar^{\mathrm{w}}(B_{\varepsilon,z,N}(\hbar))=&\sum_{k=0}^{N}\hbar^k\operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,k})\sum_{j=0}^{N}\hbar^j\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j})\\&+\sum_{j=0}^{N}\hbar^{N+1+j}R_N(\varepsilon,\hbar)\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j}).\end{aligned}\tag{3.37}$$

If we consider the first part then this corresponds to a composition of two strongly $\hbar$-ε-admissible operators. As we want to apply Theorem II.5.4 we need to ensure N satisfies the inequality

$$\delta(N+2d+2-\tau)+\tau>2d+1.$$

As this is the condition that ensures a positive power in front of the error term. If we assume N satisfies the inequality. Then by Theorem II.5.4 we have

$$\begin{aligned}&\sum_{k=0}^{N}\hbar^k\operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,k})\sum_{j=0}^{N}\hbar^j\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j})\\&\qquad=\sum_{l=0}^{N}\hbar^l\operatorname{Op}_\hbar^{\mathrm{w}}(c_{\varepsilon,l})+\hbar^{N+1}\operatorname{Op}_\hbar^{\mathrm{w}}(\zeta_\varepsilon(a_\varepsilon(\hbar),B_\varepsilon,z,N(\hbar);\hbar)),\end{aligned}$$

where

$$c_{\varepsilon,l}(x,p)=\sum_{|\alpha|+|\beta|+k+j=l}\frac{1}{\alpha!\beta!}\Big(\frac{1}{2}\Big)^{|\alpha|}\Big(-\frac{1}{2}\Big)^{|\beta|}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,j})(x,p).$$

The error term satisfies for every multi indices α, β in $\mathbb{N}_0^d$, that there exists a constant $C(\alpha,\beta,N)$ independent of a_ε and $B_{\varepsilon,z,N}$ and an integer M such that

$$\begin{aligned}\hbar^{N+1}|\partial_x^\alpha\partial_p^\beta\zeta_\varepsilon(a_\varepsilon(\hbar),&b_\varepsilon(\hbar);x,p;\hbar)|\leq C(\alpha,\beta,N)\hbar^{\delta(\tau-N-2d-2)_-+\tau-2d-1}\varepsilon^{-|\alpha|}\\&\times\sum_{j+k\leq 2N}\mathcal{G}^{\alpha,\beta}_{M,\tau-j-k}(a_{\varepsilon,k},(a_{\varepsilon,0}-\gamma_1),b_{\varepsilon,z,j},(a_{\varepsilon,0}-\gamma_1)^{-1}),\end{aligned}$$

where $\mathcal{G}^{\alpha,\beta}_{M,\tau-j-k}$ are as defined in Theorem II.5.4. By Lemma II.7.8 we have for all $j+k \leq 2N$

$$
\begin{aligned}
&\mathcal{G}^{\alpha,\beta}_{M,\tau-j-k}(a_{\varepsilon,k},(a_{\varepsilon,0}-\gamma_1),b_{\varepsilon,z,j},(a_{\varepsilon,0}-\gamma_1)^{-1}) \\
&\quad= \sup_{\substack{|\gamma_1+\gamma_2|+|\eta_1+\eta_2|\leq M \\ (x,\xi)\in\mathbb{R}^{2d}}} \varepsilon^{(\tau-j-k-M)_-+|\alpha|}|\partial_x^\alpha\partial_\xi^\beta(\partial_x^{\gamma_1}\partial_\xi^{\eta_1}a_{\varepsilon,k}(x,\xi)\partial_x^{\gamma_2}\partial_\xi^{\eta_2}b_{\varepsilon,z,j}(x,\xi))| \\
&\quad\leq C_{\alpha,\beta,M} \sup_{\substack{|\gamma_1+\gamma_2|+|\eta_1+\eta_2|\leq M \\ (x,\xi)\in\mathbb{R}^{2d}}} \varepsilon^{(\tau-j-k-M)_- -(\tau-k-\gamma_1)_- -(\tau-j-\gamma_2)_-} \\
&\qquad\qquad\qquad \times\left(\frac{|z-\gamma_1|}{\operatorname{dist}(z,[\gamma_1,\infty))}\right)^{2j+M+|\alpha|+|\beta|} \\
&\quad\leq C_{\alpha,\beta,M}\left(\frac{|z-\gamma_1|}{\operatorname{dist}(z,[\gamma_1,\infty))}\right)^{2j+M+|\alpha|+|\beta|}.
\end{aligned}
$$

Now by Theorem II.6.1 (Calderon-Vaillancourt Theorem) there exists a number M_d such that

$$
\hbar^{N+1}\|\operatorname{Op}_\hbar^{\mathrm{w}}(\zeta_\varepsilon(a_\varepsilon(\hbar),B_\varepsilon,z,N(\hbar);\hbar))\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\hbar^{\kappa(N)}\left(\frac{|z-\gamma_1|}{\operatorname{dist}(z,[\gamma_1,\infty))}\right)^{M_d}.
$$

If we now consider the symbols $c_{\varepsilon,l}(x,p)$ for $0 \leq l \leq N$. For $l=0$ we have

$$
c_{\varepsilon,0}(x,p) = (a_{\varepsilon,0}(x,p)-z)b_{\varepsilon,z,0}(x,p) = 1.
$$

By definition of $b_{\varepsilon,z,0}(x,p)$. Now for $1 \leq l \leq N$ we have

$$
\begin{aligned}
c_{\varepsilon,l} &= \sum_{|\alpha|+|\beta|+k+j=l} \frac{1}{\alpha!\beta!}\Big(\frac{1}{2}\Big)^{|\alpha|}\Big(-\frac{1}{2}\Big)^{|\beta|}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,j}) \\
&= \sum_{\substack{|\alpha|+|\beta|+k+j=l \\ 0\leq j\leq l-1}} \frac{1}{\alpha!\beta!}\Big(\frac{1}{2}\Big)^{|\alpha|}\Big(-\frac{1}{2}\Big)^{|\beta|}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,j}) + (a_{\varepsilon,0}-z)b_{\varepsilon,z,l} \\
&= \sum_{\substack{|\alpha|+|\beta|+k+j=l \\ 0\leq j\leq l-1}} \frac{1}{\alpha!\beta!}\Big(\frac{1}{2}\Big)^{|\alpha|}\Big(-\frac{1}{2}\Big)^{|\beta|}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,j}) \\
&\quad - \sum_{\substack{|\alpha|+|\beta|+k+j=l \\ 0\leq j\leq l-1}} \frac{1}{\alpha!\beta!}\frac{1}{2^{|\alpha|}}\frac{1}{(-2)^{|\beta|}}(\partial_p^\alpha D_x^\beta a_{\varepsilon,k})(\partial_p^\beta D_x^\alpha b_{\varepsilon,z,j}) \\
&=0,
\end{aligned}
$$

by definition of $b_{\varepsilon,z,l}$. These two equalities implies

$$
\sum_{k,j=0}^{N} \hbar^{k+j}\operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,k})\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j}) = I + \hbar^{N+1}\operatorname{Op}_\hbar^{\mathrm{w}}(\zeta_\varepsilon(a_\varepsilon(\hbar),B_\varepsilon,z,N(\hbar);\hbar)). \tag{3.38}
$$

This was the first part of equation (3.37). If we now consider the second part of (3.37):

$$
\sum_{j=0}^{N}\hbar^{N+1+j}R_N(\varepsilon,\hbar)\operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j}).
$$

By Theorem II.6.1 and Lemma II.7.8 there exist constants M_d and C such that

$$\hbar^j \|\mathrm{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j})\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C \left(\frac{|z-\gamma_1|}{\operatorname{dist}(z,[\gamma_1,\infty))} \right)^{2j+M_d}$$

for all j in $\{0,\dots,N\}$. Hence by assumption we have

$$\sum_{j=0}^N \hbar^{N+1+j} \|R_N(\varepsilon,\hbar)\,\mathrm{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j})\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C\hbar^{\kappa(N)} \left(\frac{|z-\gamma_1|}{\operatorname{dist}(z,[\gamma_1,\infty))} \right)^{q(N)}$$

Now by combining this with (3.37) and (3.38) we get

$$(A_\varepsilon(h)-z)\,\mathrm{Op}_\hbar^{\mathrm{w}} B_{\varepsilon,z,N} = I + h^{N+1}\Delta_{z,N+1}(h)$$

with

$$\hbar^{N+1}\|\Delta_{z,N+1}(h)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C\hbar^{\kappa(N)} \left(\frac{|z|}{\operatorname{dist}(z,[\gamma_1,\infty))} \right)^{q(N)},$$

where κ is a positive strictly increasing function and $q(N)$ is a positive integer depending on N. Which is the desired form and this ends the proof. $\square$

We are now almost ready to construct/prove a functional calculus for operators satisfying Assumption II.7.1. First we need to settle some terminology and recall a theorem.

Definition II.7.9. For a smooth function $f:\mathbb{R}\to\mathbb{R}$ we define the almost analytical extension $\tilde{f}:\mathbb{C}\to\mathbb{C}$ of f by

$$\tilde{f}(x+iy) = \left(\sum_{r=0}^n f^{(r)}(x)\frac{(iy)^r}{r!} \right)\sigma(x,y),$$

where $n\ge 1$ and

$$\sigma(x,y) = \omega\left(\tfrac{y}{\lambda(x)}\right),$$

for some smooth function ω, defined on $\mathbb{R}$ such that $\omega(t)=1$ for $|t|\le 1$ and $\omega(t)=0$ for $|t|\ge 2$. Moreover we will use the notation

$$\begin{aligned}\bar{\partial}\tilde{f}(x+iy) &:= \frac{1}{2}\left(\frac{\partial \tilde{f}}{\partial x} + i\frac{\partial \tilde{f}}{\partial y} \right) \\ &= \frac{1}{2}\left(\sum_{r=0}^n f^{(r)}(x)\frac{(iy)^r}{r!} \right)(\sigma_x(x,y)+i\sigma_y(x,y)) + \frac{1}{2}f^{(n+1)}(x)\frac{(iy)^n}{n!}\sigma(x,y),\end{aligned}$$

where σ_x and σ_y are the partial derivatives of σ with respect to x and y respectively.

Remark II.7.10. The above choice is one way to define an almost analytic extension and it is not unique. Once an n has been fixed the extension has the property that

$$|\bar{\partial}\tilde{f}(x+iy)| = \mathcal{O}(|y|^n)$$

as $y \to 0$. Hence when making calculation a choice has to be made concerning how fast $|\bar{\partial}\tilde{f}|$ vanishes when approaching the real axis. If f is a $C_0^\infty(\mathbb{R})$ function one can find an almost analytic extension $\tilde{f}$ in $C_0^\infty(\mathbb{C})$ such $f(x) = \tilde{f}(x)$ for x in $\mathbb{R}$ and

$$|\bar{\partial}\tilde{f}(x+iy)| \leq C_N |y|^N, \quad \text{for all } N \geq 0.$$

without chancing the extension. This type of extension could be based on a Fourier transform hence it may not work for a general smooth function. For details see [5, Chapter 8].

The type of functions for which we can construct a functional calculus is introduced in the next definition:

Definition II.7.11. For ρ in $\mathbb{R}$ we define the set S^ρ to be the set of smooth functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f^{(r)}(x)| := \Big|\frac{d^r f}{dx^r}(x)\Big| \leq c_r \lambda(x)^{\rho - r}$$

for some $c_r < \infty$, all x in $\mathbb{R}$ and all integers $r \geq 0$, where $\lambda(x) = (1 + |x|^2)^{1/2}$. Moreover we define $\mathcal{A}$ by

$$\mathcal{A} := \bigcup_{\rho < 0} S^\rho.$$

We can now recall the form of the spectral theorem which we will use:

Theorem II.7.12 (The Helffer-Sjöstrand formula). *Let H be a self-adjoint operator acting on a Hilbert space $\mathscr{H}$ and f a function from $\mathcal{A}$. Moreover let $\tilde{f}$ be an almost analytic extension of f with n terms. Then the bounded operator $f(H)$ is given by the equation*

$$f(H) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(z - H)^{-1}\, L(dz),$$

where $L(dz) = dxdy$ is the Lebesgue measure on $\mathbb{C}$. The formula holds for all numbers $n \geq 1$.

We are now ready to state and prove the functional calculus for a certain class of rough pseudo-differential operators.

Theorem II.7.13. *Let $A_\varepsilon(\hbar)$, for $\hbar$ in $(0, \hbar_0]$, be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ and with symbol*

$$a_\varepsilon(\hbar) = \sum_{j \geq 0} \hbar^j a_{\varepsilon,j}.$$

Suppose that $A_\varepsilon(h)$ satisfies Assumption II.7.1. Then for any function f from $\mathcal{A}$, $f(A_\varepsilon(h))$ is a $\hbar$-ε-admissible operator of regularity τ with respect to a constant tempered weight function. $f(A_\varepsilon(h))$ has the symbol

$$a_\varepsilon^f(\hbar) = \sum_{j \geq 0} \hbar^j a_{\varepsilon,j}^f,$$

where

$$\begin{aligned} a^f_{\varepsilon,0} &= f(a_{\varepsilon,0}), \\ a^f_{\varepsilon,j} &= \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} d_{\varepsilon,j,k} f^{(k)}(a_{\varepsilon,0}) \qquad \textit{for } j \geq 1, \end{aligned} \tag{3.39}$$

the symbols $d_{\varepsilon,j,k}$ *are the polynomials from Lemma II.7.6. Especially we have*

$$a^f_{\varepsilon,1} = a_{\varepsilon,1} f^{(1)}(a_{\varepsilon,0}).$$

The proof is an application of Theorem II.7.12 and the fact that the resolvent is a $\hbar$-ε-admissible operator as well.

Proof. By Theorem II.7.3 the operator $A_\varepsilon(\hbar)$ is essentially self-adjoint for sufficiently small $\hbar$. Hence Theorem II.7.12 gives us

$$f(A_\varepsilon(\hbar)) = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(z - A_\varepsilon(\hbar))^{-1} \, L(dz),$$

where $\tilde{f}$ is an almost analytic extension of f. For the almost analytic extension of f we will need a suffiently large number of terms which we assume to have chosen from the start. Theorem II.7.5 gives that the resolvent is a $\hbar$-ε-admissible operator and the explicit form of it as well. Hence

$$\begin{aligned} f(A_\varepsilon(\hbar)) =& \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z) \sum_{j=0}^{M} \hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j}) \, L(dz) \\ &- \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z) h^{N+1} (z - A_\varepsilon(\hbar))^{-1} \Delta_{z,N+1}(h) \, L(dz), \end{aligned}$$

where the symbols $b_{\varepsilon,z,j}$ and the operator $\Delta_{z,N+1}(h)$ are as defined in Theorem II.7.5. If we start by considering the error term we have by Theorem II.7.5 the estimate

$$\begin{aligned} \|(z - A_\varepsilon(\hbar))^{-1} \Delta_{z,N+1}(h)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq& C\hbar^{\kappa(N)} \frac{1}{|\mathrm{Im}(z)|} \left(\frac{|z|}{\mathrm{dist}(z, [\gamma_1, \infty))} \right)^{q(N)} \\ \leq& C\hbar^{\kappa(N)} \frac{|z|^{q(N)}}{|\mathrm{Im}(z)|^{q(N)+1}}, \end{aligned}$$

for N sufficiently large and where $q(N)$ is some integer dependent of the number N. We have

$$|\bar{\partial}\tilde{f}(z)| \leq c_1 \sum_{r=0}^{n} |\tilde{f}^{(r)}(\mathrm{Re}(z))| \lambda(\mathrm{Re}(z))^{r-1} \mathbf{1}_U(z) + c_2 |\tilde{f}^{(n+1)}(\mathrm{Re}(z))| |\operatorname{Im}(z)|^n \mathbf{1}_V(z),$$

where

$$U = \{z \in \mathbb{C} \,|\, \lambda(\mathrm{Re}(z)) < |\mathrm{Im}(z)| < 2\lambda(\mathrm{Re}(z))\},$$

and

$$V = \{z \in \mathbb{C} \,|\, 0 < |\mathrm{Im}(z)| < 2\lambda(\mathrm{Re}(z))\}.$$

This estimate follows directly from the definition of $\tilde{f}$. By combining these estimates and the definition of the class of functions $\mathcal{A}$ we have

$$\Big\| \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z) h^{N+1}(z-A_\varepsilon(\hbar))^{-1}\Delta_{z,N+1}(h)\, L(dz) \Big\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\hbar^{\kappa(N)}.$$

What remains to prove the following equality

$$\sum_{j=0}^{M} \hbar^j \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z) \operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j})\, L(dz) = \sum_{j=0}^{M} \hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a^f_{\varepsilon,j}), \tag{3.40}$$

where the symbols $a^f_{\varepsilon,j}$ are as defined in the theorem. We will only consider one of the terms as the rest is treated analogously. Hence we need to establish the equality

$$\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z) \operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j})\, L(dz) = \operatorname{Op}_\hbar^{\mathrm{w}}(a^f_{\varepsilon,j}).$$

As both operators are bounded we need only establish the equality weekly for a dense subset of $L^2(\mathbb{R}^d)$. Hence let φ and ψ be two functions from $C_0^\infty(\mathbb{R}^d)$ and a j be given. We have

$$\langle \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z) \operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j})\, L(dz)\varphi, \psi\rangle = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z) \langle \operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j})\varphi, \psi\rangle\, L(dz), \tag{3.41}$$

where we have

$$\begin{aligned} &\langle \operatorname{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j})\varphi, \psi\rangle \\ &\quad = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\, dydpdx \\ &\quad = \lim_{\sigma\to\infty} \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} g_\sigma(x,y,p) b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\, dydpdx, \end{aligned} \tag{3.42}$$

where the function g is a positiv Schwartz function bounded by 1 and identical 1 in a neighbourhood of 0. I the above we have set $g_\sigma(x,y,p) = g(\frac{x}{\sigma}, \frac{y}{\sigma}, \frac{p}{\sigma})$. The next step in the proof is to apply dominated convergence to move the limit outside the integral over z.

We let χ be in $C_0^\infty(\mathbb{R}^d)$ such that $\chi(p) = 1$ for $|p| \leq 1$ and $\chi(p) = 0$ for $|p| \geq 2$. With this function we have

$$\begin{aligned} &\frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} g_\sigma(x,y,p) b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\, dydpdx \\ &\quad = \frac{1}{(2\pi\hbar)^d}\Big[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} g_\sigma(x,y,p)\chi(p) b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\, dydpdx \\ &\qquad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} g_\sigma(x,y,p)(1-\chi(p)) b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\, dydpdx\Big]. \end{aligned} \tag{3.43}$$

By Lemma II.7.8 we have

$$\begin{aligned} &\Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} g_\sigma(x,y,p)\chi(p) b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\, dydpdx \Big| \\ &\qquad \leq C_j \varepsilon^{-(\tau-j)_-} \Big(\frac{|z-\gamma_1|}{|\mathrm{Im}(z)|}\Big)^{2j}, \end{aligned} \tag{3.44}$$

where the γ_1 is the number from Assumption II.7.1. The factor $\varepsilon^{-(\tau-j)_-}$ is not an issue as the operator we consider has $\hbar^j$ in front. We have just omitted to write this factor. This bound is clearly independent of σ. We now need to bound the term with $1-\chi(p)$. Here we use that on the support of $1-\chi(p)$ we have $|p|>1$. Hence the operator

$$M = \frac{(-i\hbar)^{2d}}{|p|^{2d}}(\sum_{k=1}^{d}\partial_{y_k}^2)^d = \frac{(-i\hbar)^{2d}}{|p|^{2d}}\sum_{|\alpha|=d}\partial_y^{2\alpha},$$

is well defined when acting on functions supported in $\mathrm{supp}(1-\chi)$. We have

$$Me^{i\hbar^{-1}\langle x-y,p\rangle} = e^{i\hbar^{-1}\langle x-y,p\rangle}.$$

We now have

$$\begin{aligned}
&|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}g_\sigma(x,y,p)(1-\chi(p))b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\,dydpdx| \\
&=|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(Me^{i\hbar^{-1}\langle x-y,p\rangle})g_\sigma(x,y,p)(1-\chi(p))b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\,dydpdx| \\
&=|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}(1-\chi(p))M^t(g_\sigma(x,y,p)b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y))\overline{\psi}(x)\,dydpdx|.
\end{aligned}$$

If we consider the the expression $M^t g_\sigma(x,y,p)b_{\varepsilon,z,j}(\frac{x+y}{2},p)\varphi(y)$ we have by Leibniz's formula

$$\begin{aligned}
&|M^t g_\sigma(x,y,p)b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)| \\
&\quad= \frac{\hbar^{2d}}{|p|^{2d}}\Big|\sum_{|\alpha|=d}\partial_y^{2\alpha}g_\sigma(x,y,p)b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\Big| \\
&\quad= \frac{\hbar^{2d}}{|p|^{2d}}\Big|\sum_{|\alpha|=d}\sum_{\beta\leq 2\alpha}\partial_y^{2\alpha-\beta}(g_\sigma(x,y,p)\varphi(y))\partial_y^{\beta}b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\Big| \\
&\quad\leq C_j\frac{\hbar^{2d}}{|p|^{2d}}\mathbf{1}_{\mathrm{supp}(\varphi)}(y)\sum_{|\alpha|=d}\sum_{\beta\leq 2\alpha}\varepsilon^{-(\tau-j-|\beta|)_-}\left(\frac{|z-\gamma_1|}{|\mathrm{Im}(z)|}\right)^{2j+|\beta|} \\
&\quad\leq C\frac{\hbar^{2d}}{|p|^{2d}}\varepsilon^{-(\tau-j-2d)_-}\mathbf{1}_{\mathrm{supp}(\varphi)}(y)\left(1+\frac{|z-\gamma_1|}{|\mathrm{Im}(z)|}\right)^{2j+2d},
\end{aligned}$$

where we again have used Lemma II.7.8. This imply the estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}g_\sigma(x,y,p)(1-\chi(p))b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\,dydpdx\Big| \\
&\qquad\leq C_j\varepsilon^{-(\tau-j)_-}\left(1+\frac{|z-\gamma_1|}{|\mathrm{Im}(z)|}\right)^{2j+2d}.
\end{aligned}$$

If we combine this estimate with (3.43) and (3.44) we have

$$\begin{aligned}
&\Big|\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}g_\sigma(x,y,p)b_{\varepsilon,z,j}(\tfrac{x+y}{2},p)\varphi(y)\overline{\psi}(x)\,dydpdx\Big| \\
&\qquad\leq C\varepsilon^{-(\tau-j)_-}\left(1+\frac{|z-\gamma_1|}{|\mathrm{Im}(z)|}\right)^{2j+2d}.
\end{aligned}$$

As above we have

$$\Big| \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) \left(1 + \frac{|z-\gamma_1|}{|\mathrm{Im}(z)|} \right)^{2j+2d} L(dz) \Big| < \infty.$$

Hence we can apply dominated convergence and by an analogous argument we can also apply Fubini's Theorem. This gives

$$\begin{aligned} &\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) \langle \mathrm{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j}) \varphi, \psi \rangle \, L(dz) \\ &\quad = \lim_{\sigma \to \infty} \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p \rangle} g_\sigma(x,y,p) \\ &\qquad\qquad \times \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) b_{\varepsilon,z,j}(\tfrac{x+y}{2},p) \, L(dz) \varphi(y) \overline{\psi}(x) \, dydpdx \end{aligned} \tag{3.45}$$

If we only consider the integral over z then we have by a Cauchy formula and Lemma II.7.6 that

$$\begin{aligned} &\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) b_{\varepsilon,z,j}(\tfrac{x+y}{2},p) \, L(dz) \\ &\quad = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) \sum_{k=1}^{2j-1} d_{\varepsilon,j,k}(\tfrac{x+y}{2},p) b_{\varepsilon,z,0}^{k+1}(\tfrac{x+y}{2},p) \, L(dz) \\ &\quad = \sum_{k=1}^{2j-1} d_{\varepsilon,j,k}(\tfrac{x+y}{2},p) \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) \Big(\frac{1}{a_{\varepsilon,0}(\frac{x+y}{2},p) - z} \Big)^{k+1} L(dz) \\ &\quad = \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} d_{\varepsilon,j,k}(\tfrac{x+y}{2},p) f^{(k)}(a_{\varepsilon,0}(\tfrac{x+y}{2},p)) = a_{\varepsilon,j}^f(\tfrac{x+y}{2},p), \end{aligned}$$

which is the desired form of $a_{\varepsilon,j}^f$ given in (3.39). Now combing this with (3.41), (3.42) and (3.45) we arrive at

$$\langle \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) \, \mathrm{Op}_\hbar^{\mathrm{w}}(b_{\varepsilon,z,j}) \, L(dz) \varphi, \psi \rangle = \langle \mathrm{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,j}^f) \varphi, \psi \rangle.$$

The remaning j' can be treated analogously and hence we obtain the equality (3.40). This ends the proof. □

With the functional calculus we can now prove some useful theorems and lemmas. One of them is an asymptotic expansion of certain traces. But before we do this we have the following theorem.

Theorem II.7.14. *Let $A_\varepsilon(\hbar)$, for $\hbar$ in $(0,\hbar_0]$, be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ and symbol*

$$a_\varepsilon(\hbar) = \sum_{j \geq 0} \hbar^j a_{\varepsilon,j}.$$

Suppose that $A_\varepsilon(\hbar)$ satisfies Assumption II.7.1. Let $E_1 < E_2$ be two real numbers and suppose there exists an $\eta > 0$ such $a_{\varepsilon,0}^{-1}([E_1-\eta, E_2+\eta])$ is compact. Then we have

$$\mathrm{spec}(A_\varepsilon(\hbar)) \cap [E_1.E_2] \subseteq \mathrm{spec}_{pp}(A_\varepsilon(\hbar)), \tag{3.46}$$

for $\hbar$ sufficiently small, where $\mathrm{spec}_{pp}(A_\varepsilon(\hbar))$ is the pure point spectrum of $A_\varepsilon(\hbar)$.

Proof. Let f and g be in $C_0^\infty((E_1-\eta,E_2+\eta))$ such $g(t)=1$ for $t\in[E_1,E_2]$ and $f(t)=1$ for t in $\operatorname{supp}(g)$. By Theorem II.7.13 we have

$$f(A_\varepsilon(\hbar)) = A_{\varepsilon,f,N}(\hbar) + \hbar^{N+1}R_{N+1,f}(\varepsilon,\hbar), \tag{3.47}$$

where the terms $A_{\varepsilon,f,N}(\hbar)$ consists of the first N terms in the expansion in $\hbar$ of $f(A_\varepsilon(\hbar))$. We get by (3.47) and the definition of g and f that

$$g(A_\varepsilon(\hbar))(I-\hbar^{N+1}R_{N+1,f}(\varepsilon,\hbar)) = g(A_\varepsilon(\hbar))A_{\varepsilon,f,N}(\hbar).$$

Hence for $\hbar$ sufficiently small we have

$$g(A_\varepsilon(\hbar)) = g(A_\varepsilon(\hbar))A_{\varepsilon,f,N}(\hbar)(I-\hbar^{N+1}R_{N+1,f}(\varepsilon,\hbar))^{-1},$$

thereby we have the inequality

$$\|g(A_\varepsilon(\hbar))\|_{\mathrm{Tr}} \leq c\|g\|_\infty \|A_{\varepsilon,f,N}(\hbar)\|_{\mathrm{Tr}} \leq C\hbar^{-2d-1}, \tag{3.48}$$

where we have used Theorem II.6.4. Since $\mathbf{1}_{[E_1,E_2]}(t)\leq g(t)$ we have that $\mathbf{1}_{[E_1,E_2]}(A_\varepsilon(\hbar))$ is a trace class operator by (3.48). This implies the inclusion

$$\operatorname{spec}(A_\varepsilon(\hbar))\cap[E_1.E_2] \subseteq \operatorname{spec}_{pp}(A_\varepsilon(\hbar)),$$

for $\hbar$ sufficiently small. This ends the proof. □

Theorem II.7.15. *Let $A_\varepsilon(\hbar)$, for $\hbar$ in $(0,\hbar_0]$, be a $\hbar$-ε-admissible operator of regularity $\tau\geq 1$ and symbol*

$$a_\varepsilon(\hbar) = \sum_{j\geq 0}\hbar^j a_{\varepsilon,j}.$$

Suppose that $A_\varepsilon(\hbar)$ satisfies Assumption II.7.1. Let $E_1<E_2$ be two real numbers and suppose there exists an $\eta>0$ such $a_{\varepsilon,0}^{-1}([E_1-\eta,E_2+\eta])$ is compact. Then for every f in $C_0^\infty((E_1,E_2))$ and any N_0 in $\mathbb{N}$ there exists an N in $\mathbb{N}$ such that

$$|\operatorname{Tr}[f(A_\varepsilon(\hbar))] - \frac{1}{(2\pi\hbar)^d}\sum_{j=0}^N \hbar^j T_j(f,A_\varepsilon(\hbar))| \leq C\hbar^{N_0+1-d}.$$

for all sufficiently small $\hbar$, where the T_j's is given by

$$T_j(f,A_\varepsilon(\hbar)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{k=1}^{2j-1}\frac{(-1)^k}{k!}d_{\varepsilon,j,k}f^{(k)}(a_{\varepsilon,0})\,dxdp,$$

where $d_{\varepsilon,j,k}$ are the polynomials from Lemma II.7.6. In particular we have

$$T_0(f,A_\varepsilon(\hbar)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(a_{\varepsilon,0})\,dxdp$$

and

$$T_1(f,A_\varepsilon(\hbar)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a_{\varepsilon,1}f^{(1)}(a_{\varepsilon,0})\,dxdp.$$

The proof is an application of Theorem II.7.13 which gives the form of the operator $f(A_\varepsilon(\hbar))$ combined with the trace formula from Theorem II.6.6 and we use some of the same ideas as in the proof of Theorem II.7.14.

Proof. Let f in $C_0^\infty((E_1, E_2))$ be given and fix a g in $C_0^\infty((E_1-\eta, E_2+\eta))$ such that $g(t) = 1$ for $t \in [E_1, E_2]$. By Theorem II.7.13 we have

$$f(A_\varepsilon(\hbar)) = A_{\varepsilon,f,N}(\hbar) + \hbar^{N+1} R_{N+1,f}(\varepsilon, \hbar),$$

and

$$g(A_\varepsilon(\hbar)) = A_{\varepsilon,g,N}(\hbar) + \hbar^{N+1} R_{N+1,g}(\varepsilon, \hbar),$$

where the terms $A_{\varepsilon,f,N}(\hbar)$ and $A_{\varepsilon,g,N}(\hbar)$ consist of the first N terms in the expansion in $\hbar$ of $f(A_\varepsilon(\hbar))$ and $g(A_\varepsilon(\hbar))$ respectively. Since $f(A_\varepsilon(\hbar))g(A_\varepsilon(\hbar)) = f(A_\varepsilon(\hbar))$ we have

$$\begin{aligned} &f(A_\varepsilon(\hbar)) \\ &\quad = (A_{\varepsilon,f,N}(\hbar) + \hbar^{N+1} R_{N+1,f}(\varepsilon, \hbar))(A_{\varepsilon,g,N}(\hbar) + \hbar^{N+1} R_{N+1,g}(\varepsilon, \hbar)) \\ &\quad = A_{\varepsilon,f,N}(\hbar)A_{\varepsilon,g,N}(\hbar) + \hbar^{N+1}[A_{\varepsilon,f,N}(\hbar)R_{N+1,g}(\varepsilon, \hbar) + R_{N+1,f}(\varepsilon, \hbar)A_{\varepsilon,g,N}(\hbar)]. \end{aligned}$$

By Theorem II.6.4 we have that

$$\|f(A_\varepsilon(\hbar)) - A_{\varepsilon,f,N}(\hbar)A_{\varepsilon,g,N}(\hbar)\|_{\mathrm{Tr}} \leq C\hbar^{\kappa(N)-2d-1} \tag{3.49}$$

as $\hbar \to 0$. Hence taking N sufficiently large we can consider the composition of the operators $A_{\varepsilon,f,N}(\hbar)A_{\varepsilon,g,N}(\hbar)$ instead of $f(A_\varepsilon(\hbar))$. By the choice of g and Theorem II.5.4 (composition of operators) we have

$$\begin{aligned} A_{\varepsilon,f,N}(\hbar)A_{\varepsilon,g,N}(\hbar) &= \sum_{j=0}^{N} \hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,j}^f) + \hbar^{N+1} R_{N,f,g}(\varepsilon\hbar, a_\varepsilon) \\ &= A_{\varepsilon,f,N}(\hbar) + \hbar^{N+1} R_{N,f,g}(\varepsilon\hbar, a_\varepsilon). \end{aligned} \tag{3.50}$$

Hence we have Theorem II.6.4 that

$$\|A_{\varepsilon,f,N}(\hbar) - A_{\varepsilon,f,N}(\hbar)A_{\varepsilon,g,N}(\hbar)\|_{\mathrm{Tr}} \leq C\hbar^{\kappa(N)-2d-1}, \tag{3.51}$$

where we have used that the error term in (3.50) is a $\hbar$-pseudo-differential operator, which follows from Theorem II.5.4. Theorem II.6.6 now gives

$$\begin{aligned} \mathrm{Tr}[A_{\varepsilon,f,N}(\hbar)] &= \sum_{j=0}^{N} \hbar^j \,\mathrm{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,j}^f)] \\ &= \frac{1}{(2\pi\hbar)^d} \sum_{j=0}^{N} \hbar^j \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} d_{\varepsilon,j,k} f^{(k)}(a_{\varepsilon,0})\, dxdp. \end{aligned} \tag{3.52}$$

By combining (3.49) and (3.51) implies that

$$|\mathrm{Tr}[f(A_\varepsilon(\hbar))] - \mathrm{Tr}[A_{\varepsilon,f,N}(\hbar)]| \leq C\hbar^{\kappa(N)-2d-1}. \tag{3.53}$$

Hence by choosing N sufficiently large and combining (3.52) and (3.53) we get the desired estimate. $\square$

The next Lemmas will be use-full in the proof of the Weyl law. Both of these Lemmas are proven by applying the functional calculus and the results on compositions of operators.

Lemma II.7.16. *Let $A_\varepsilon(\hbar)$, for $\hbar$ in $(0,\hbar_0]$, be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ and symbol*

$$a_\varepsilon(\hbar) = \sum_{j\geq 0} \hbar^j a_{\varepsilon,j}.$$

Suppose that $A_\varepsilon(h)$ satisfies Assumption II.7.1. Let $E_1 < E_2$ be two real numbers and suppose there exists an $\eta > 0$ such $a_{\varepsilon,0}^{-1}([E_1-\eta, E_2+\eta])$ is compact. Let f be in $C_0^\infty((E_1,E_2))$ and suppose θ is in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such $\operatorname{supp}(\theta) \subset a_{\varepsilon,0}^{-1}([E_1-\eta, E_2+\eta])$ and $\theta(x,p) = 1$ for all (x,p) in $\operatorname{supp}(f(a_{0,\varepsilon}))$. Then we have the bound

$$\|(1-\mathrm{Op}_\hbar^{\mathrm{w}}(\theta))f(A_\varepsilon(\hbar))\|_{\mathrm{Tr}} \leq C_N \hbar^N$$

for every N in $\mathbb{N}_0$

Proof. We choose g in $C_0^\infty((E_1,E_2))$ such $g(t)f(t) = f(t)$. We now have

$$\|(1-\mathrm{Op}_\hbar^{\mathrm{w}}(\theta))f(A_\varepsilon(\hbar))\|_{\mathrm{Tr}} \leq \|(1-\mathrm{Op}_\hbar^{\mathrm{w}}(\theta))f(A_\varepsilon(\hbar))\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\|g(A_\varepsilon(\hbar))\|_{\mathrm{Tr}}$$

As in the proof of Theorem II.7.15 and combined with Theorem II.6.4 we get

$$\|g(A_\varepsilon(\hbar))\|_{\mathrm{Tr}} \leq C_d \hbar^{-2d-1}.$$

Form Theorem II.7.13 $f(A_\varepsilon(\hbar))$ is $\hbar$-ε-admissible operator with symbols

$$a_\varepsilon^f(\hbar) = \sum_{j\geq 0} \hbar^j a_{\varepsilon,j}^f,$$

where

$$a_{\varepsilon,j}^f = \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} d_{\varepsilon,j,k} f^{(k)}(a_{\varepsilon,0}),$$

the symbols $d_{\varepsilon,j,k}$ are the polynomials from Lemma II.7.6. The support of these functions is disjoint from the support of symbol for the operator $(1-\mathrm{Op}_\hbar^{\mathrm{w}}(\theta))$. Hence by Theorem II.5.4 we get the desired estimate. ☐

Lemma II.7.17. *Let $A_\varepsilon(\hbar)$, for $\hbar$ in $(0,\hbar_0]$, be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ and symbol*

$$a_\varepsilon(\hbar) = \sum_{j\geq 0} \hbar^j a_{\varepsilon,j}.$$

Suppose that $A_\varepsilon(h)$ satisfies Assumption II.7.1. Let $E_1 < E_2$ be two real numbers and suppose there exists an $\eta > 0$ such $a_{\varepsilon,0}^{-1}([E_1-\eta, E_2+\eta])$ is compact. Suppose θ is in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such $\operatorname{supp}(\theta) \subset a_{\varepsilon,0}^{-1}((E_1,E_2))$.

Then for every f in $C_0^\infty([E_1-\eta, E_2+\eta])$ such $f(t) = 1$ for all t in $[E_1-\frac{\eta}{2}, E_2+\frac{\eta}{2}]$ the bound

$$\|\mathrm{Op}_\hbar^{\mathrm{w}}(\theta)(1-f(A_\varepsilon(\hbar)))\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_N \hbar^N,$$

is true for every N in $\mathbb{N}_0$

Proof. Theorem II.7.13 gives us that $f(A_\varepsilon(\hbar))$ is $\hbar$-ε-admissible operator with symbols

$$a_\varepsilon^f(\hbar) = \sum_{j\geq 0} \hbar^j a_{\varepsilon,j}^f,$$

where

$$a_{\varepsilon,j}^f = \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} d_{\varepsilon,j,k} f^{(k)}(a_{\varepsilon,0}),$$

the symbols $d_{\varepsilon,j,k}$ are the polynomials from Lemma II.7.6. Hence we have that the principal symbol of $(1 - f(A_\varepsilon(\hbar)))$ is $1 - f(a_{\varepsilon,0})$. By assumption we then have that the support of θ and the support of every symbol in $f(A_\varepsilon(\hbar))$ are disjoint. Hence Theorem II.5.4 implies the desired estimate. $\square$

II.8 Microlocal approximation and properties of propagator

In this section we will study the solution to the operator valued Cauchy problem:

$$\begin{cases} \hbar\partial_t U(t,\hbar) - iU(t,\hbar)A_\varepsilon(\hbar) = 0 & t \neq 0 \\ U(0,\hbar) = \theta(x,D) & t = 0, \end{cases}$$

where A_ε is self-adjoint and the symbol θ is in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$. In particular we will only consider the case where $A_\varepsilon(\hbar)$ is a strongly $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ which satisfies Assumption II.7.1. Hence for sufficiently small $\hbar$ the operator $A_\varepsilon(\hbar)$ is self-adjoint by Theorem II.7.3. It is well-known that the solution to the operator valued Cauchy problem is the micro localised propagator $\theta(x,D)e^{i\hbar^{-1}tA_\varepsilon(\hbar)}$.

We are interested in the propagators as they turn up in a smoothing of functions applied to the operators we consider. In this smoothing procedure we need to know the behaviour of the propagator for t in a small interval around zero. Usually this is done by constructing a Fourier integral operator (FIO) as an approximation to propagator. For our set up the FIO approximation is not desirable as we can not control the number of derivatives in the space variables and hence we can not be certain about how the operator behave. Instead we will use a microlocal approximation for times in $[-\hbar^{1-\frac{\delta}{2}}, \hbar^{1-\frac{\delta}{2}}]$.

The construction of the approximation is recursive and inspired by the construction in the works of L. Zielinski. If the construction is compered to the approximation in the works of V. Ivrii, one can note that Ivrii's construction is successive. Hence the constructions are not the same.

Our objective is to construct the approximation $U_N(t,\hbar)$ such that

$$\|\hbar\partial_t U_N(t,\hbar) - iU_N(t,\hbar)A_\varepsilon\|_{\mathcal{L}(L^2(\mathbb{R}^d))},$$

is small and the trace of the operator has the "right" asymptotic behaviour. The approximation we will construct as a polynomial in t and $\frac{1}{\hbar}$ with operator coefficients.

These operators will have kernels given by some rough functions. The kernel of the approximation will have the following form

$$(x,y) \to \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} e^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^{N} (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\, dp,$$

where N is chosen such that the error are of a desired order, and the u_j's are compactly supported rough functions in x and p. It can be noted that this kernel is much like a kernel of a rough pseudo-differential operator.

If we consider the expression we want to show is small and assume A_ε having quantisation $\mathrm{Op}_{\hbar,1}(a_\varepsilon)$ and use the results on how to compose two operators given by kernels. Then the kernel of $\hbar\partial_t U_N(t,\hbar) - iU_N(t,\hbar)A_\varepsilon$ is given by

$$\begin{aligned}(x,y) \to \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} \{\hbar\partial_t[e^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^{N} (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)] \\ - ie^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^{N} (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\tilde{a}_\varepsilon(y,p)\}\, dp.\end{aligned}$$

If we now make a Taylor expansion in the space variable y centred at the point x of $\tilde{a}_\varepsilon(y,p)$ we get

$$\begin{aligned}\tilde{a}_\varepsilon(y,p) &= \sum_{|\alpha|\leq N} \frac{(y-x)^\alpha}{\alpha!} \partial_x^\alpha \tilde{a}_\varepsilon(x,p) \\ &+ \sum_{|\alpha|=N+1} (N+1)\frac{(y-x)^\alpha}{\alpha!} \int_0^1 (1-s)^N \partial_x^\alpha \tilde{a}_\varepsilon(x+s(y-x),p)\, ds.\end{aligned}$$

When inserting the Taylor expansion without the error term in the kernel we get

$$\begin{aligned}(x,y) \to \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} \{\hbar\partial_t[e^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^{N} t^j u_j(x,p,\hbar,\varepsilon)] \\ - ie^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^{N} t^j u_j(x,p,\hbar,\varepsilon) \sum_{|\alpha|\leq N} \frac{(y-x)^\alpha}{\alpha!} \partial_x^\alpha \tilde{a}_\varepsilon(x,p)\}\, dp.\end{aligned}$$

By partial integration in p and applying the identity

$$(y-x)^\alpha e^{i\hbar^{-1}\langle x-y,p\rangle} = (i\hbar)^{|\alpha|} \partial_p^\alpha e^{i\hbar^{-1}\langle x-y,p\rangle},$$

the kernel becomes

$$\begin{aligned}(x,y) \to \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} \{\hbar\partial_t[e^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^{N} t^j u_j(x,p,\hbar,\varepsilon)] \\ -i \sum_{|\alpha|\leq N} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \partial_p^\alpha [e^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^{N} t^j u_j(x,p,\hbar,\varepsilon) \partial_x^\alpha \tilde{a}_\varepsilon(x,p)]\}\, dp.\end{aligned}$$

We note that we act we the operator

$$\hbar\partial_t - i\mathcal{P}_N : C^\infty(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_p) \to C^\infty(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_p),$$

where

$$\mathcal{P}_N b(x,p) = \sum_{|\alpha|\leq N} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \partial_p^\alpha\{b(x,p)\partial_x^\alpha \tilde{a}_\varepsilon(x,p)\},$$

on the expression

$$e^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^N (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon).$$

Hence in order to obtain the desired estimates we need to understand what the action of $\hbar\partial_t - i\mathcal{P}_N$ is.

The above discussion is the heuristic behind the next theorem, where we construct the approximation explicitly.

Theorem II.8.1. *Let $A_\varepsilon(\hbar)$ be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ with tempered weight m which is self-adjoint for all $\hbar$ in $(0,\hbar_0]$, for $\hbar_0 > 0$ and with $\varepsilon \geq \hbar^{1-\delta}$ for a $\delta > 0$. Let $\theta(x,p)$ be a function in $C_0^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_p)$. Then for all $N_0 \in \mathbb{N}_0$ there exist an operator $U_N(t,\varepsilon,\hbar)$ with integral kernel*

$$\begin{aligned} &K_{U_N}(x,y,t,\varepsilon,\hbar) \\ &\quad = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} \sum_{j=0}^N (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\, dp, \end{aligned}$$

such that $K_{U_N}(x,y,0,\varepsilon,\hbar)$ is the kernel of the operator $\mathrm{Op}_{\hbar,0}(\theta) = \theta(x,\hbar D)$. The terms in the sum satisfies $u_0(x,p,\hbar,\varepsilon) = \theta(x,p)$,

$$u_j(x,p,\hbar,\varepsilon) \in C_0^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_p)$$

and they satisfies the bounds

$$|\partial_x^\beta \partial_p^\alpha u_j(x,p,\hbar,\varepsilon)| \leq \begin{cases} C_{\alpha\beta} & j=0 \\ C_{\alpha\beta}\hbar\varepsilon^{-|\beta|} & j=1 \\ C_{\alpha\beta}\hbar^{1+\delta(j-2)}\varepsilon^{-|\beta|} & j\geq 2 \end{cases}$$

for all α and β in $\mathbb{N}_0^d$ in the case $\tau = 1$. For $\tau \geq 2$ the u_j's satisfies the bounds

$$|\partial_x^\beta \partial_p^\alpha u_j(x,p,\hbar,\varepsilon)| \leq \begin{cases} C_{\alpha\beta} & j=0 \\ C_{\alpha\beta}\hbar\varepsilon^{-|\beta|} & j=1,2 \\ C_{\alpha\beta}\hbar^{2+\delta(j-3)}\varepsilon^{-|\beta|} & j\geq 3 \end{cases}$$

for all α and β in $\mathbb{N}_0^d$. Moreover U_N satisfies the following bound:

$$\|\hbar\partial_t U_N(t,\hbar) - iU_N(t,\hbar)A_\varepsilon\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\hbar^{N_0}$$

for $|t| \leq \hbar^{1-\frac{\delta}{2}}$.

Remark II.8.2. If the operator satisfies Assumption II.7.1, then by Theorem II.7.3 the operator will be self-adjoint for all sufficiently small $\hbar$. Hence Assumption II.7.1 would be sufficient but not necessary for the above theorem to be true.

The number N is explicit dependent on N_0, d and δ as N need to greater than or equal to the number $2(\frac{N_0+d-1}{\delta}+1)$. This follows directly from the proof.

Proof. We start by fixing N such that N is the smallest integer greater than or equal to

$$2(\frac{N_0+d-1}{\delta}+1)$$

since this implies

$$1+\delta\left(\frac{N}{2}-1\right)-d\geq N_0.$$

By assumption we have for sufficiently large M in $\mathbb{N}$ the following form of $A_\varepsilon(\hbar)$

$$A_\varepsilon(\hbar)=\sum_{j=0}^{M}\hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,j})+\hbar^{M+1}R_M(\varepsilon,\hbar). \tag{3.54}$$

We can choose and fix M such the following estimate is true

$$\hbar^{M+1}\|R_M(\varepsilon,\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq C_M\hbar^{N_0}.$$

With this M we consider the sum in the expression of $A_\varepsilon(\hbar)$. By Corollary II.4.20 there exists a sequence $\{\tilde{a}_{\varepsilon,j}\}_{j\in\mathbb{N}_0}$ of symbols where $\tilde{a}_{\varepsilon,j}$ is of regularity $\tau-j$ and a $\tilde{M}$ such

$$\sum_{j=0}^{M}\hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,j})=\sum_{j=0}^{\tilde{M}}\hbar^j \operatorname{Op}_{\hbar,1}(\tilde{a}_{\varepsilon,j})+\hbar^{\tilde{M}+1}\tilde{R}_{\tilde{M}}(\varepsilon,\hbar), \tag{3.55}$$

where $a_{\varepsilon,0}=\tilde{a}_{\varepsilon,0}$ and

$$\hbar^{\tilde{M}+1}\|\tilde{R}_{\tilde{M}}(\varepsilon,\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq C_{\tilde{M}}\hbar^{N_0}.$$

We will for the reminder of the proof use the notation

$$\tilde{a}_\varepsilon(x,p)=\sum_{j=0}^{\tilde{M}}\hbar^j\tilde{a}_{\varepsilon,j}(x,p). \tag{3.56}$$

The function $\tilde{a}_\varepsilon(y,p)$ is a rough function of regularity τ. These choices and definitions will become important again at the end of the proof.

For our fixed N we define the operator $\hbar\partial_t-i\mathcal{P}_N : C^\infty(\mathbb{R}_t\times\mathbb{R}_x^d\times\mathbb{R}_p^d)\to C^\infty(\mathbb{R}_t\times\mathbb{R}_x^d\times\mathbb{R}_p^d)$, where

$$\mathcal{P}_N b(t,x,p)=\sum_{|\alpha|\leq N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}\partial_p^\alpha\{b(t,x,p)\partial_x^\alpha\tilde{a}_\varepsilon(x,p)\}$$

for a $b\in C^\infty(\mathbb{R}_t\times\mathbb{R}_x^d\times\mathbb{R}_p^d)$. First step is to observe how the operator $\hbar\partial_t-i\mathcal{P}_N$ acts on $e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\psi(x,p)$ for $\psi\in C_0^\infty(\mathbb{R}_x^d\times\mathbb{R}_p^d)$. We will in the following calculation omit the dependence of the variables x and p. By Leibniz's formula and the chain rule we get

$$
\begin{aligned}
(\hbar\partial_t - i\mathcal{P}_N)e^{it\hbar^{-1}a_{\varepsilon,0}}\psi &= \hbar\partial_t e^{it\hbar^{-1}a_{\varepsilon,0}}\psi - i\sum_{|\alpha|\leq N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}\partial_p^\alpha\{e^{it\hbar^{-1}a_{\varepsilon,0}}\psi\partial_x^\alpha\tilde{a}_\varepsilon\} \\
=& ia_{\varepsilon,0}e^{it\hbar^{-1}a_{\varepsilon,0}}\psi - i\sum_{|\alpha|\leq N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial_p^\beta e^{it\hbar^{-1}a_{\varepsilon,0}}\partial_p^{\alpha-\beta}\{\psi\partial_x^\alpha\tilde{a}_\varepsilon\} \\
=& ia_{\varepsilon,0}e^{it\hbar^{-1}a_{\varepsilon,0}}\psi - ie^{it\hbar^{-1}a_{\varepsilon,0}}\psi\tilde{a}_\varepsilon - i\sum_{|\alpha|=1}^{N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}e^{it\hbar^{-1}a_{\varepsilon,0}}\partial_p^\alpha\{\psi\partial_x^\alpha\tilde{a}_\varepsilon\} \\
&- i\sum_{|\alpha|=1}^{N}\sum_{\beta\leq\alpha}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}\binom{\alpha}{\beta}\partial_p^\beta e^{it\hbar^{-1}a_{\varepsilon,0}}\partial_p^{\alpha-\beta}\{\psi\partial_x^\alpha\tilde{a}_\varepsilon\} \\
=& ie^{it\hbar^{-1}a_\varepsilon}\psi(a_{\varepsilon,0}-\tilde{a}_\varepsilon) - i\sum_{|\alpha|=1}^{N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}e^{it\hbar^{-1}a_{\varepsilon,0}}\partial_p^\alpha\{\psi\partial_x^\alpha\tilde{a}_\varepsilon\} - i\sum_{|\alpha|=1}^{N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!} \\
&\times\sum_{k=1}^{|\alpha|}(it\hbar^{-1})^k\sum_{\substack{\beta_1+\cdots+\beta_k\leq\alpha\\|\beta_j|>0}}c_{\alpha,\beta_1\cdots\beta_k}e^{it\hbar^{-1}a_{\varepsilon,0}}\prod_{j=1}^{k}\partial_p^{\beta_j}a_{\varepsilon,0}\partial_p^{\alpha-(\beta_1+\cdots+\beta_k)}\{\psi\partial_x^\alpha\tilde{a}_\varepsilon\} \\
=& e^{it\hbar^{-1}a_{\varepsilon,0}}\Big[i\psi(a_{\varepsilon,0}-\tilde{a}_\varepsilon) - i\sum_{|\alpha|=1}^{N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}\partial_p^\alpha\{\psi\partial_x^\alpha\tilde{a}_\varepsilon\} - i\sum_{k=1}^{N}(it\hbar^{-1})^k \\
&\times\sum_{|\alpha|=k}^{N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}\sum_{\substack{\beta_1+\cdots+\beta_k\leq\alpha\\|\beta_j|>0}}c_{\alpha,\beta_1\cdots\beta_k}\prod_{j=1}^{k}\partial_p^{\beta_j}a_{\varepsilon,0}\partial_p^{\alpha-(\beta_1+\cdots+\beta_k)}\{\psi\partial_x^\alpha\tilde{a}_\varepsilon\}\Big] \\
=& ie^{it\hbar^{-1}a_{\varepsilon,0}}\sum_{k=0}^{N}(it\hbar^{-1})^k q_k(\psi,x,p,\hbar,\varepsilon)
\end{aligned}
\tag{3.57}
$$

From this we note after acting with $\hbar\partial_t - i\mathcal{P}_N$ on $e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\psi(x,p)$ we get $ie^{it\hbar^{-1}a_{\varepsilon,0}}$ times a polynomial in $it\hbar^{-1}$ with coefficients in $C_0^\infty(\mathbb{R}_x^d\times\mathbb{R}_p^d)$ depending on ψ, $\hbar$ and ε. If we consider the coefficients in the polynomial we have

$$
\begin{aligned}
&|q_0(\psi,x,p,\hbar,\varepsilon)| \\
&\quad= |\psi(a_{\varepsilon,0}-\tilde{a}_\varepsilon) + \sum_{|\alpha|=1}^{N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}\partial_p^\alpha\{\psi\partial_x^\alpha\tilde{a}_\varepsilon\}| \\
&\quad\leq c_1\hbar + \sum_{|\alpha|=1}^{N}\frac{\hbar^{|\alpha|}}{\alpha!}|\partial_p^\alpha\{\psi\partial_x^\alpha\tilde{a}_\varepsilon\}| \leq c_1\hbar + \sum_{|\alpha|=1}^{\tau}\frac{\hbar^{|\alpha|}}{\alpha!}c_\alpha + \sum_{|\alpha|=\tau+1}^{N}\frac{\hbar^{|\alpha|}}{\alpha!}c_\alpha\varepsilon^{\tau-|\alpha|} \\
&\quad\leq c_1\hbar + \sum_{|\alpha|=1}^{\tau}\frac{\hbar^{|\alpha|}}{\alpha!}c_\alpha + \sum_{|\alpha|=\tau+1}^{N}\frac{\hbar^{|\alpha|}}{\alpha!}c_\alpha\hbar^{(1-\delta)(\tau-|\alpha|)} \leq C\hbar
\end{aligned}
\tag{3.58}
$$

where C depends on the p-derivatives of ψ and $\partial_x^\alpha a_\varepsilon$ on the support of ψ for $|\alpha| \leq N$. For $1 \leq k \leq \tau$ we have

$$\begin{aligned} &|q_k(\psi, x, p, \hbar, \varepsilon)| \\ &\quad =| \sum_{|\alpha|=k}^{N} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \sum_{\substack{\beta_1+\cdots+\beta_k\leq\alpha \\ |\beta_j|>0}} c_{\alpha,\beta_1\cdots\beta_k} \prod_{j=1}^{k} \partial_p^{\beta_j} a_{\varepsilon,0} \partial_p^{\alpha-(\beta_1+\cdots+\beta_k)} \{\psi \partial_x^\alpha \tilde{a}_\varepsilon\}| \\ &\quad \leq \sum_{|\alpha|=k}^{\tau} c_\alpha \frac{\hbar^{|\alpha|}}{\alpha!} + \sum_{|\alpha|=\tau+1}^{N} c_\alpha \frac{\hbar^{|\alpha|}}{\alpha!} \varepsilon^{\tau-|\alpha|} \\ &\quad \leq \sum_{|\alpha|=k}^{\tau} c_\alpha \frac{\hbar^{|\alpha|}}{\alpha!} + \sum_{|\alpha|=\tau+1}^{N} c_\alpha \frac{\hbar^{|\alpha|}}{\alpha!} \hbar^{(1-\delta)(\tau-|\alpha|)} \leq C\hbar^k. \end{aligned} \tag{3.59}$$

where C depends on the p-derivatives of ψ and $\partial_x^\alpha a_\varepsilon$ on the support of ψ. For $\tau < k \leq N$ we have

$$\begin{aligned} &|q_k(\psi, x, p, \hbar, \varepsilon)| \\ &\quad =| \sum_{|\alpha|=k}^{N} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \sum_{\substack{\beta_1+\cdots+\beta_k\leq\alpha \\ |\beta_j|>0}} c_{\alpha,\beta_1\cdots\beta_k} \prod_{j=1}^{k} \partial_p^{\beta_j} a_{\varepsilon,0} \partial_p^{\alpha-(\beta_1+\cdots+\beta_k)} \{\psi \partial_x^\alpha \tilde{a}_\varepsilon\}| \\ &\quad \leq \sum_{|\alpha|=k}^{N} c_\alpha \frac{\hbar^{|\alpha|}}{\alpha!} \varepsilon^{\tau-|\alpha|} \leq \sum_{|\alpha|=k}^{N} c_\alpha \frac{\hbar^{|\alpha|}}{\alpha!} \hbar^{(1-\delta)(\tau-|\alpha|)} \leq C\hbar^{\tau+(k-\tau)\delta}. \end{aligned} \tag{3.60}$$

where C depends on the p-derivatives of ψ and $\partial_x^\alpha a_\varepsilon$ on the support of ψ. It is important the the the coefficients only depends on derivatives in p for the function we apply the operator to. One should also note that if ψ had $\hbar$ to some power multiplied to it. Then it should be multiplied to the new power obtained. In the reminder of the proof we will continue to denote the coefficients obtained by acting with $\hbar\partial_t - i\mathcal{P}_N$ by q_j and the exact form can be found in (3.57).

We are now ready to start constructing the kernel. We set $u_0(x, p, \hbar, \varepsilon) = \theta(x, p)$ which gives the first term. In order to find u_1 we act with the operator $\hbar\partial_t - i\mathcal{P}_N$ on $e^{it\hbar^{-1}a_{\varepsilon,0}} u_0(x, p, \hbar, \varepsilon)$ (where we in the reminder of the construction of the approximation will omit writing the dependence of the variables (x, p) in the exponential). By the calculation (3.57) we get

$$(\hbar\partial_t - i\mathcal{P}_N) e^{it\hbar^{-1}a_{\varepsilon,0}} u_0(x, p, \hbar, \varepsilon) = i e^{it\hbar^{-1}a_{\varepsilon,0}} \sum_{k=0}^{N} (it\hbar^{-1})^k q_k(u_0, x, p, \hbar, \varepsilon).$$

This would not lead to the desired estimate. So we now take

$$u_1(x, p, \hbar, \varepsilon) = -q_0(u_0, x, p, \hbar, \varepsilon).$$

We can note by the previous estimates (3.58) we have

$$|u_1(x, p, \hbar, \varepsilon)| = |q_0(u_0, x, p, \hbar, \varepsilon)| \leq \hbar C. \tag{3.61}$$

If we now use the operator $\hbar\partial_t - i\mathcal{P}_N$ on $e^{it\hbar^{-1}a_{\varepsilon,0}}(u_0(x,p,\hbar,\varepsilon) + it\hbar^{-1}u_1(x,p,\hbar,\varepsilon))$. Then according to (3.57) we get

$$\begin{aligned}
&(\hbar\partial_t - i\mathcal{P}_N)(e^{it\hbar^{-1}a_{\varepsilon,0}}(u_0(x,p,\hbar,\varepsilon) + it\hbar^{-1}u_1(x,p,\hbar,\varepsilon))) \\
&\quad= ie^{it\hbar^{-1}a_{\varepsilon,0}}\sum_{k=0}^{N}(it\hbar^{-1})^k q_k(u_0,x,p,\hbar,\varepsilon) + ie^{it\hbar^{-1}a_{\varepsilon,0}}u_1(x,p,\hbar,\varepsilon) \\
&\qquad + it\hbar^{-1}ie^{it\hbar^{-1}a_{\varepsilon,0}}\sum_{k=0}^{N}(it\hbar^{-1})^k q_k(u_1,x,p,\hbar,\varepsilon) \\
&\quad= ie^{it\hbar^{-1}a_{\varepsilon,0}}(\sum_{k=1}^{N}(it\hbar^{-1})^k q_k(u_0,x,p,\hbar,\varepsilon) + \sum_{k=0}^{N}(it\hbar^{-1})^{k+1} q_k(u_1,x,p,\hbar,\varepsilon)).
\end{aligned}$$

If we now take $u_2(x,p,\hbar,\varepsilon) = -\frac{1}{2}(q_1(u_0,x,p,\hbar,\varepsilon) + q_0(u_1,x,p,\hbar,\varepsilon))$ and act with the operator $\hbar\partial_t - i\mathcal{P}_N$ according to (3.57) we get

$$\begin{aligned}
&(\hbar\partial_t - i\mathcal{P}_N)e^{it\hbar^{-1}a_{\varepsilon,0}}\sum_{j=0}^{2}(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon) \\
&\quad= ie^{it\hbar^{-1}a_{\varepsilon,0}}\Big[\sum_{j=0}^{2}\sum_{k=0}^{N}(it\hbar^{-1})^{k+j} q_k(u_j,x,p,\hbar,\varepsilon) + \sum_{j=1}^{2} j(it\hbar^{-1})^{j-1}u_j(x,p,\hbar,\varepsilon)\Big] \\
&\quad= ie^{it\hbar^{-1}a_{\varepsilon,0}}\sum_{j=0}^{2}\sum_{k=2-j}^{N}(it\hbar^{-1})^{k+j} q_k(u_j,x,p,\hbar,\varepsilon).
\end{aligned}$$

We note that the "lowest" power of $it\hbar^{-1}$ is 2. Hence it is these terms which should be used to construct u_3. Moreover we note that by (3.58) and (3.59) we have

$$|u_2(x,p,\hbar,\varepsilon)| = \tfrac{1}{2}|q_1(u_0,x,p,\hbar,\varepsilon) + q_0(u_1,x,p,\hbar,\varepsilon)| \leq \tfrac{1}{2}C(\hbar + \hbar^2) \leq C\hbar, \tag{3.62}$$

and u_2 is a smooth compactly supported function in the variables x and p. Generally for $2 \leq j \leq N$ we have

$$u_j(x,p,\hbar,\varepsilon) = -\frac{1}{j}\sum_{k=0}^{j-1} q_{j-1-k}(u_k,x,p,\hbar,\varepsilon).$$

We now need estimates for these terms. In the case $\tau = 1$ the next step will be empty, but for $\tau \geq 2$ it is needed. For $\tau \geq 2$ we have

$$|u_3(x,p,\hbar,\varepsilon)| \leq \frac{1}{3}\sum_{k=0}^{2}|q_{2-k}(u_k,x,p,\hbar,\varepsilon)| \leq C\hbar^2,$$

where we have used (3.58), (3.59), (3.61) and (3.62). For the rest of the u_j's we split in the two cases $\tau = 1$ or $\tau \geq 2$. First the cases $\tau = 1$ for $2 \leq j \leq N$ the estimate is

$$|u_j(x,p,\hbar,\varepsilon)| \leq C\hbar^{1+\delta(j-2)}$$

Note that u_2 satisfies the above equation hence if we assume it okay for $j-1$ between 2 and $N-1$ we want to show the above estimate for j. We note that

$$
\begin{aligned}
|u_j(x,p,\hbar,\varepsilon)| \leq& \frac{1}{j}\sum_{k=0}^{j-1}|q_{j-1-k}(u_k,x,p,\hbar,\varepsilon)| \\
\leq& C(|q_{j-1}(u_0,x,p,\hbar,\varepsilon)| + |q_{j-2}(u_1,x,p,\hbar,\varepsilon)| \\
&+\sum_{k=2}^{j-2}|q_{j-1-k}(u_k,x,p,\hbar,\varepsilon)| + |q_0(u_{j-1},x,p,\hbar,\varepsilon)|) \\
\leq& C(\hbar^{j-1}+\hbar^{j-1}+\sum_{k=2}^{j-2}\hbar^{1+\delta(j-1-k-1)+1+\delta(k-2)}+\hbar^{2+\delta(j-3)}) \\
\leq& C(\hbar^{1+\delta(j-2)}+\hbar^{2+\delta(j-2)}+\hbar^{2+\delta(j-4)}+\hbar^{2+\delta(j-3)}) \leq C\hbar^{1+\delta(j-2)},
\end{aligned}
$$

where the last inequality only holds for $\delta \leq \frac{1}{2}$ and we have used (3.58), (3.60) and the induction assumption.

Now the case $\tau \geq 2$ which we will treat as $\tau = 2$, here the estimate is

$$
|u_j(x,p,\hbar,\varepsilon)| \leq C\hbar^{2+\delta(j-3)}
$$

for $3 \leq j \leq N$. To prove this bound is the same as in the case of $\tau = 1$. In order to prove the bound with the derivatives as stated in the theorem the above arguments are repeated with a number of derivatives on the u_j's otherwise it is analogous.

What remains is to prove this construction satisfies the bound

$$
\|\hbar\partial_t U_N(t,\hbar) - iU_N(t,\hbar)A_\varepsilon\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\hbar^{N_0}
$$

Here we only consider the case $\tau = 1$ as the cases $\tau \geq 2$ will have better estimates. Hence from the above estimates we have for k in $\{0,\dots,N\}$ and $|t| \leq \hbar^{1-\frac{\delta}{2}}$

$$
|(it\hbar^{-1})^k u_k(x,p,\hbar,\varepsilon)| \leq \begin{cases} C & k=0 \\ C\hbar^{1-\frac{\delta}{2}} & k=1 \\ C\hbar^{1+\delta(\frac{k}{2}-2)} & k\geq 2 \end{cases} \tag{3.63}
$$

The first step is to apply the operator $\hbar\partial_t - i\mathcal{P}_N$ on then "full" kernel and see what error this produces. By construction we have

$$
\begin{aligned}
(\hbar\partial_t - i\mathcal{P}_N)&e^{it\hbar^{-1}a_{\varepsilon,0}}\sum_{k=0}^{N}(it\hbar^{-1})^k u_k(x,p,\hbar,\varepsilon) \\
&= \sum_{j=0}^{N}\sum_{k=N-j}^{N}(it\hbar^{-1})^{k+j}q_k(u_j,x,p,\hbar,\varepsilon).
\end{aligned}
$$

If we start by considering j equal 0 and 1 we note that:

$$
\begin{aligned}
\Big|\sum_{j=0}^{1}\sum_{k=N-j}^{N}&(it\hbar^{-1})^{k+j}q_k(u_j,x,p,\hbar,\varepsilon)\Big|\\
&\leq C(\hbar^{-\frac{\delta}{2}N}\hbar^{1+\delta(N-1)}+\hbar^{-\frac{\delta}{2}N}\hbar^{2+\delta(N-2)}+\hbar^{-\frac{\delta}{2}(N+1)}\hbar^{2+\delta(N-1)})\\
&\leq C(\hbar^{1+\delta(\frac{N}{2}-1)}+\hbar^{2+\delta(\frac{N}{2}-2)}+\hbar^{2+\frac{\delta}{2}(N-3)})\\
&\leq \tilde{C}\hbar^{1+\delta(\frac{N}{2}-1)},
\end{aligned}
$$

where we have used (3.63). For the rest of the terms we have that

$$
\begin{aligned}
\Big|\sum_{j=2}^{N}\sum_{k=N-j}^{N}&(it\hbar^{-1})^{k+j}q_k(u_j,x,p,\hbar,\varepsilon)\Big|\\
&\leq C\sum_{j=2}^{N}\sum_{k=\max(N-j,1)}^{N}\hbar^{-\frac{\delta}{2}(k+j)}\hbar^{2+\delta(k-1)+\delta(j-2)}+\hbar^{-\frac{\delta}{2}N}\hbar^{2+\delta(N-2)}\\
&\leq C\sum_{j=2}^{N}\sum_{k=\max(N-j,1)}^{N}\hbar^{2+\delta(\frac{j+k}{2}-3)}+\hbar^{2+\delta(\frac{N}{2}-2)}\leq \tilde{C}\hbar^{1+\delta(\frac{N}{2}-1)},
\end{aligned}
$$

where we have used (3.63) and that in the double sum $k+j\geq N$ and $\delta<\frac{1}{2}$. When these estimates is combined we have

$$
|(\hbar\partial_t-i\mathcal{P}_N)e^{it\hbar^{-1}a_{\varepsilon,0}}\sum_{k=0}^{N}(it\hbar^{-1})^k u_k(x,p,\hbar,\varepsilon)|\leq C\hbar^{1+\delta(\frac{N}{2}-1)}. \tag{3.64}
$$

We now let $U_N(t,\hbar)$ be the operator with the integral kernel:

$$
\begin{aligned}
K_{U_N}&(x,y,t,\varepsilon,\hbar)\\
&=\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}e^{i\hbar^{-1}\langle x-y,p\rangle}e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{j=0}^{N}(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\,dp,
\end{aligned}
$$

which is well defined due to our previous estimates. In particular we have that it is a bounded operator by the Schur test. We now need to find an expression for

$$
\hbar\partial_t U_N(t,\hbar)-iU_N(t,\hbar)A_\varepsilon(\hbar).
$$

In the start of the proof we wrote the operator $A_\varepsilon(\hbar)$ in two different ways (3.54) and (3.55). If we combine these we have

$$
\begin{aligned}
A_\varepsilon(\hbar)=&\sum_{j=0}^{\tilde{M}}\hbar^j\operatorname{Op}_{\hbar,1}(\tilde{a}_{\varepsilon,j})+\hbar^{\tilde{M}+1}\tilde{R}_{\tilde{M}}(\varepsilon,\hbar)+\hbar^{M+1}R_M(\varepsilon,\hbar)\\
=&\operatorname{Op}_{\hbar,1}(\tilde{a}_\varepsilon)+\hbar^{\tilde{M}+1}\tilde{R}_{\tilde{M}}(\varepsilon,\hbar)+\hbar^{M+1}R_M(\varepsilon,\hbar),
\end{aligned}
$$

where the two reminder terms satisfies

$$
\|\hbar^{\tilde{M}+1}\tilde{R}_{\tilde{M}}(\varepsilon,\hbar)+\hbar^{M+1}R_M(\varepsilon,\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq C\hbar^{N_0}.
$$

If we use this form of $A_\varepsilon(\hbar)$ we have

$$\begin{aligned}\hbar\partial_t U_N(t,\hbar) - iU_N(t,\hbar)A_\varepsilon(\hbar) = \hbar\partial_t U_N(t,\hbar) - iU_N(t,\hbar)\operatorname{Op}_{\hbar,1}(\tilde{a}_\varepsilon)\\ - i\hbar^{\tilde{M}+1}U_N(t,\hbar)\tilde{R}_{\tilde{M}}(\varepsilon,\hbar) - i\hbar^{M+1}U_N(t,\hbar)R_M(\varepsilon,\hbar).\end{aligned}$$

If we consider the the operator norm of the two last terms we have

$$\|\hbar^{\tilde{M}+1}U_N(t,\hbar)\tilde{R}_{\tilde{M}}(\varepsilon,\hbar) + \hbar^{M+1}U_N(t,\hbar)R_M(\varepsilon,\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\hbar^{N_0} \tag{3.65}$$

as $U_N(t,\hbar)$ is a bounded operator. What remains is the expression

$$\hbar\partial_t U_N(t,\hbar) - iU_N(t,\hbar)\operatorname{Op}_{\hbar,1}(\tilde{a}_\varepsilon).$$

The rules for composition of kernels gives by a straight forward calculation that the kernel of the above expression is

$$\begin{aligned}K(x,y;\varepsilon,\hbar) := \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}(\hbar\partial_t - i\tilde{a}_\varepsilon(y,p))e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\\ \times\sum_{j=0}^N (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\,dp.\end{aligned}$$

The next step is to rewrite the above kernel. This is done by doing a Taylor expansion of $\tilde{a}_\varepsilon$ in the variable y centred at x. This gives

$$\begin{aligned}\tilde{a}_\varepsilon(y,p) &= \sum_{|\alpha|\leq N}\frac{(y-x)^\alpha}{\alpha!}\partial_x^\alpha\tilde{a}_\varepsilon(x,p)\\ &+ \sum_{|\alpha|=N+1}(N+1)\frac{(y-x)^\alpha}{\alpha!}\int_0^1(1-s)^N\partial_x^\alpha\tilde{a}_\varepsilon(x+s(y-x),p)\,ds.\end{aligned}$$

We replace $\tilde{a}_\varepsilon(y,p)$ by the above Taylor expansion in the kernel and start by considering the part of the kernel with the first sum. Here we have

$$\begin{aligned}&\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}[\hbar\partial_t e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{j=0}^N(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\\ &\qquad - i\sum_{|\alpha|\leq N}\frac{(y-x)^\alpha}{\alpha!}\partial_x^\alpha\tilde{a}_\varepsilon(x,p)e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{j=0}^N(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)]\,dp\\ &= \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}[\hbar\partial_t e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{j=0}^N(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\\ &\qquad - i\sum_{|\alpha|\leq N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}\partial_p^\alpha[\partial_x^\alpha\tilde{a}_\varepsilon(x,p)e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{j=0}^N(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)]]\,dp\\ &= \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}(\hbar\partial_t - i\mathcal{P}_N)[e^{it\hbar^{-1}a_\varepsilon(x,p)}\sum_{j=0}^N(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)]\,dp,\end{aligned}$$

where we have used the identity

$$(y-x)^\alpha e^{i\hbar^{-1}\langle x-y,p\rangle} = (i\hbar)^\alpha \partial_p^\alpha e^{i\hbar^{-1}\langle x-y,p\rangle},$$

and integration by parts. If we now consider the part of the kernel with the error term we have

$$\begin{aligned}
&\frac{-i}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} e^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^{N}(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon) \\
&\times \sum_{|\alpha|=N+1}(N+1)\frac{(y-x)^\alpha}{\alpha!}\int_0^1 (1-s)^N \partial_x^\alpha \tilde{a}_\varepsilon(x+s(y-x),p)\,dsdp \\
&= \frac{-i}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} \sum_{|\alpha|=N+1}(N+1)\frac{(-i\hbar)^\alpha}{\alpha!}\partial_p^\alpha[e^{-it\hbar^{-1}a_\varepsilon(x,p)}\sum_{j=0}^{N}(-it\hbar^{-1})^j \\
&\times u_j(x,p,\hbar,\varepsilon)\int_0^1 (1-s)^N \partial_x^\alpha \tilde{a}_\varepsilon(x+s(y-x),p)\,ds]dp,
\end{aligned}$$

where we again have used the above identity and partial integration. Combing the two expressions we get

$$\begin{aligned}
&K(x,y;\varepsilon,\hbar) \\
&= \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}(\hbar\partial_t - i\mathcal{P}_N)[e^{it\hbar^{-1}a_\varepsilon(x,p)}\sum_{j=0}^{N}(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)]\,dp \\
&\quad + \frac{-i}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}\sum_{|\alpha|=N+1}(N+1)\frac{(-i\hbar)^\alpha}{\alpha!}\partial_p^\alpha[e^{it\hbar^{-1}a_\varepsilon(x,p)}\sum_{j=0}^{N}(it\hbar^{-1})^j \\
&\quad \times u_j(x,p,\hbar,\varepsilon)\int_0^1 (1-s)^N \partial_x^\alpha \tilde{a}_\varepsilon(x+s(y-x),p)\,ds]dp
\end{aligned} \tag{3.66}$$

In order to estimate the operator norm we will divide the kernel into two parts. We do this by considering a part localised in y and the reminder. To localise in y we let ψ be a smooth function on $\mathbb{R}^d$ such $\psi(y)=1$ on the set $\{y\in\mathbb{R}^d \mid \operatorname{dist}[y,\operatorname{supp}_x(\theta)]\leq 1\}$ and supported in the set $\{y\in\mathbb{R}^d \mid \operatorname{dist}[y,\operatorname{supp}_x(\theta)]\leq 2\}$. With this function our kernel can be written as

$$K(x,y;\varepsilon,\hbar) = K(x,y;\varepsilon,\hbar)\psi(y) + K(x,y;\varepsilon,\hbar)(1-\psi(y)) \tag{3.67}$$

If we consider the part multiplied by $\psi(y)$ then this part has the form as in (3.66) but each term is multiplied by $\psi(y)$. By the estimate in (3.64) we have for the first part of $K(x,y;\varepsilon,\hbar)\psi(y)$ the following estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}(\hbar\partial_t - i\mathcal{P}_N)[e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{j=0}^{N}(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)]\psi(y)\,dp\Big| \\
&\leq \psi(x)\psi(y)C\hbar^{1+\delta(\frac{N}{2}-1)}.
\end{aligned} \tag{3.68}$$

For the second part of $K(x,y;\varepsilon,\hbar)\psi(y)$ we have by Leibniz's formula and Faà di Bruno formula (Theorem II.1.2) for each term in the sum over α

$$
\begin{aligned}
&\frac{(-i\hbar)^{|\alpha|}}{\alpha!}\partial_p^\alpha[e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} \\
&\qquad\qquad \times\sum_{j=0}^{N}(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\int_0^1(1-s)^N\partial_x^\alpha\tilde{a}_\varepsilon(x+s(y-x),p)\,ds]\psi(y) \\
&=\sum_{j=0}^{N}\frac{(-i\hbar)^{|\alpha|}}{\alpha!}e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{k=0}^{N+1}(it\hbar^{-1})^{k+j}\sum_{\substack{\beta_1+\cdots+\beta_k\leq\alpha\\|\beta_j|>0}}c_{\alpha,\beta_1\cdots\beta_k}\prod_{n=1}^{k}\partial_p^{\beta_n}a_{\varepsilon,0}(x,p) \\
&\quad\times\partial_p^{\alpha-(\beta_1+\cdots+\beta_k)}[u_j(x,p,\hbar,\varepsilon)\int_0^1(1-s)^N\partial_x^\alpha\tilde{a}_\varepsilon(x+s(y-x),p)\,ds]\psi(y).
\end{aligned}
$$

We note that for j equal 0 we have an estimate of the following form:

$$
\begin{aligned}
&\Big|\frac{(-i\hbar)^{|\alpha|}}{\alpha!}e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{k=0}^{N+1}(it\hbar^{-1})^{k}\sum_{\substack{\beta_1+\cdots+\beta_k\leq\alpha\\|\beta_j|>0}}c_{\alpha,\beta_1\cdots\beta_k}\prod_{n=1}^{k}\partial_p^{\beta_n}a_{\varepsilon,0}(x,p) \\
&\qquad\times\partial_p^{\alpha-(\beta_1+\cdots+\beta_k)}[u_0(x,p,\hbar,\varepsilon)\int_0^1(1-s)^N\partial_x^\alpha\tilde{a}_\varepsilon(x+s(y-x),p)\,ds]\psi(y)\Big| \\
&\qquad\leq C\sum_{k=0}^{N+1}\hbar^{N+1}\hbar^{-\frac{\delta}{2}k}\varepsilon^{-N}\psi(x)\psi(y)\leq C\hbar^{N+1}\hbar^{-\frac{\delta}{2}(N+1)}\hbar^{-N+\delta N}\psi(x)\psi(y) \\
&\qquad\leq C\hbar^{1+\frac{\delta}{2}(N-1)}\psi(x)\psi(y).
\end{aligned}
\tag{3.69}
$$

We note that for j equal 1 we have an error of the following form:

$$
\begin{aligned}
&\Big|\frac{(-i\hbar)^{|\alpha|}}{\alpha!}e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{k=0}^{N+1}(it\hbar^{-1})^{k+1}\sum_{\substack{\beta_1+\cdots+\beta_k\leq\alpha\\|\beta_j|>0}}c_{\alpha,\beta_1\cdots\beta_k}\prod_{n=1}^{k}\partial_p^{\beta_n}a_{\varepsilon,0}(x,p) \\
&\qquad\times\partial_p^{\alpha-(\beta_1+\cdots+\beta_k)}[u_1(x,p,\hbar,\varepsilon)\int_0^1(1-s)^N\partial_x^\alpha\tilde{a}_\varepsilon(x+s(y-x),p)\,ds]\psi(y)\Big| \\
&\qquad\leq C\sum_{k=0}^{N+1}\hbar^{N+1}\hbar^{-\frac{\delta}{2}(k+1)}\hbar\varepsilon^{-N}\psi(x)\psi(y)\leq C\hbar^{N+2}\hbar^{-\frac{\delta}{2}(N+2)}\hbar^{-N+\delta N}\psi(x)\psi(y) \\
&\qquad\leq C\hbar^{2+\frac{\delta}{2}(N-2)}\psi(x)\psi(y),
\end{aligned}
\tag{3.70}
$$

where we have used the estimate $|u_1(x,p,\hbar,\varepsilon)|\leq c\hbar$. We note that for j greater than or equal to 2, we have an error of the following form:

$$
\begin{aligned}
&\Big|\frac{(-i\hbar)^{|\alpha|}}{\alpha!} e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} \sum_{k=0}^{N+1} (it\hbar^{-1})^{k+j} \sum_{\substack{\beta_1+\cdots+\beta_k\leq\alpha \\ |\beta_j|>0}} c_{\alpha,\beta_1\cdots\beta_k} \prod_{n=1}^{k} \partial_p^{\beta_n} a_{\varepsilon,0}(x,p) \\
&\quad\times \partial_p^{\alpha-(\beta_1+\cdots+\beta_k)}[u_j(x,p,\hbar,\varepsilon) \int_0^1 (1-s)^N \partial_x^\alpha \tilde{a}_\varepsilon(x+s(y-x),p)\,ds]\psi(y)\Big| \\
&\quad\leq C\sum_{k=0}^{N+1} \hbar^{N+1}\hbar^{-\frac{\delta}{2}(k+j)}\hbar^{1+\delta(j-2)}\varepsilon^{-N}\psi(x)\psi(y) \\
&\quad\leq C\hbar^{N+1}\hbar^{-\frac{\delta}{2}(N+1+j)}\hbar^{1+\delta(j-2)}\hbar^{-N+\delta N}\psi(x)\psi(y) \leq C\hbar^{2+\frac{\delta}{2}(N+j-1)-2\delta}\psi(x)\psi(y) \\
&\quad\leq C\hbar^{1+\frac{\delta}{2}(N+j-1)}\psi(x)\psi(y),
\end{aligned}
\tag{3.71}
$$

where we have used the estimate $|u_j(x,p,\hbar,\varepsilon)| \leq c\hbar^{1+\delta(j-2)}$. Now by combining (3.71), (3.70) and (3.71) we arrive at

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} \sum_{|\alpha|=N+1} (N+1)\frac{(-i\hbar)^\alpha}{\alpha!}\partial_p^\alpha[e^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^{N}(it\hbar^{-1})^j \\
&\qquad\times u_j(x,p,\hbar,\varepsilon)\int_0^1 (1-s)^N \partial_x^\alpha \tilde{a}_\varepsilon(x+s(y-x),p)\,ds]\psi(y)\,dp\Big| \\
&\quad\leq C\hbar^{1+\frac{\delta}{2}(N-1)}\psi(x)\psi(y).
\end{aligned}
$$

Combining this estimate with (3.68) we have

$$
|K(x,y;\varepsilon,\hbar)\psi(y)| \leq C\hbar^{1+\frac{\delta}{2}(N-1)-d}\psi(x)\psi(y) \leq C\hbar^{N_0}\psi(x)\psi(y), \tag{3.72}
$$

where the last inequality is due to our choice of N made in the start of the proof. Now we turn to the term $K(x,y;\varepsilon,\hbar)(1-\psi(y))$. On the support of this kernel we have

$$
1 \leq |x-y|
$$

due to the definition of ψ. This imply we can divide by the difference between x and y where the kernel is supported. The idea is now to multiply the kernel with $\frac{|x-y|}{|x-y|}$ to an appropriate power η. We take η such

$$
\frac{m(x+s(y-x),p)}{|x-y|^{2\eta}} \leq \frac{C}{|x-y|^{d+1}} \quad \text{for } (x,p)\in\operatorname{supp}(\theta),
$$

where m is the tempered weight function associated to our operator. The existence of such a η is ensured by the definition of the tempered weight. By (3.66) the kernel $K(x,y;\varepsilon,\hbar)(1-\psi(y))$ is of the form

$$
\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}\varphi(x,y,p;\hbar,\varepsilon)(1-\psi(y))dp,
$$

where the exact form of φ is not important at the moment. Now for our choice of η we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}\varphi(x,y,p;\hbar,\varepsilon)(1-\psi(y))dp \\
&\quad = \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}\frac{|x-y|^{2\eta}}{|x-y|^{2\eta}}\varphi(x,y,p;\hbar,\varepsilon)(1-\psi(y))dp \\
&\quad = \int_{\mathbb{R}^d} (-i\hbar)^{2\eta}\sum_{|\gamma|=\eta} c_\gamma \partial_p^{2\gamma}(e^{i\hbar^{-1}\langle x-y,p\rangle})\frac{1}{|x-y|^{2\eta}}\varphi(x,y,p;\hbar,\varepsilon)(1-\psi(y))dp \\
&\quad = \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}\frac{1-\psi(y)}{|x-y|^{2\eta}}\sum_{|\gamma|=\eta} c_\gamma (i\hbar)^{2\eta}\partial_p^{2\gamma}\varphi(x,y,p;\hbar,\varepsilon)dp.
\end{aligned}$$

By analogous estimates to the estimate used above we have

$$\begin{aligned}
&\Big|\frac{1-\psi(y)}{|x-y|^{2\eta}}\sum_{|\gamma|=\eta}(i\hbar)^{2\eta}\partial_p^{2\gamma}\varphi(x,y,p;\hbar,\varepsilon)\Big| \\
&\qquad \leq C\hbar^{2\eta(1-\frac{\delta}{2})+1+\frac{\delta}{2}(N-1)}\frac{1-\psi(y)}{|x-y|^{d+1}}\mathbf{1}_{\mathrm{supp}(\theta)}(x,p),
\end{aligned}$$

where the term $\hbar^{-\eta\delta}$ is due to the exponentials $e^{it\hbar^{-1}a_\varepsilon(x,p)}$ in φ which gives $it\hbar^{-}1$ when we take a derivative with respect to p_j for all j in $\{1,\dots,d\}$ and that $|t|\leq \hbar^{1-\frac{\delta}{2}}$. The rest of the powers in $\hbar$ is found analogous to above. Hence we have

$$\begin{aligned}
|K(x,y;\varepsilon,\hbar)(1-\psi(y))| \leq& C\hbar^{2\eta(1-\frac{\delta}{2})+1+\frac{\delta}{2}(N-1)-d}\mathbf{1}_{\mathrm{supp}_x(\theta)}(x)\frac{1-\psi(y)}{|x-y|^{d+1}} \\
\leq& C\hbar^{N_0}\psi(x)\frac{1-\psi(y)}{|x-y|^{d+1}}.
\end{aligned}$$

By combining this with (3.67) and (3.72) we have

$$|K(x,y;\varepsilon,\hbar)| \leq C\hbar^{N_0}\Big[\psi(x)\psi(y)+\mathbf{1}_{\mathrm{supp}_x(\theta)}(x)\frac{1-\psi(y)}{|x-y|^{d+1}}\Big]. \tag{3.73}$$

We have by definition of ψ the estimates

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\psi(x)\psi(y)+\mathbf{1}_{\mathrm{supp}_x(\theta)}(x)\frac{1-\psi(y)}{|x-y|^{d+1}}|\,dy \leq c+\int_{|y|\geq 1}\frac{1}{|y|^{d+1}}\,dy \leq C_1 \\
&\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\psi(x)\psi(y)+\mathbf{1}_{\mathrm{supp}_x(\theta)}(x)\frac{1-\psi(y)}{|x-y|^{d+1}}|\,dx \leq C_2
\end{aligned}$$

These estimates combined with the Schur test, (3.65) and (3.73) gives

$$\|\hbar\partial_t U_N(t,\hbar) - iU_N(t,\hbar)A_\varepsilon\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\hbar^{N_0}$$

for $|t|\leq \hbar^{1-\frac{\delta}{2}}$. This is the desired estimate which ends the proof. □

In the previous proof we constructed a microlocal approximation for the propagator for short times dependent on $\hbar$. It would be preferable to not have this dependence of $\hbar$ in the time. In the following Lemma we prove that under a non-critical condition on the principal symbol a localised trace of the approximation becomes negligible.

Lemma II.8.3. *Let $A_\varepsilon(\hbar)$ be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ with tempered weight m which is self-adjoint for all $\hbar$ in $(0, \hbar_0]$ and with $\varepsilon \geq \hbar^{1-\delta}$ for a $\delta > 0$. Let $\theta(x,p)$ be a function in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$. Suppose*

$$|\nabla_p a_{\varepsilon,0}(x,p)| \geq c > 0 \quad \textit{for all } (x,p) \in \operatorname{supp}(\theta),$$

where $a_{\varepsilon,0}$ is the principal symbol of $A_\varepsilon(\hbar)$. Moreover let the operator $U_N(t,\hbar)$ be the one constructed in Theorem II.8.1 with the function θ. Then for $|t| \in [\frac{1}{2}\hbar^{1-\frac{\delta}{2}}, 1]$ and every N_0 in $\mathbb{N}_0$ it holds

$$|\operatorname{Tr}[U_N(t,\hbar)\operatorname{Op}_{\hbar,1}(\theta)]| \leq C\hbar^{N_0}$$

for a constant $C > 0$, which depends on the constant from the non-critical condition.

The essence of the proof is partial integration and cyclicality of the trace.

Proof. Recall that the kernel of $U_N(t,\hbar)$ is given by

$$\begin{aligned} &K_{U_N}(x,y,t,\varepsilon,\hbar) \\ &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} e^{it\hbar^{-1}a_\varepsilon(x,p)} \sum_{j=0}^N (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\,dp, \end{aligned}$$

where we have from Theorem II.8.1 the estimate

$$\sup_{x,p}\sup_{|t|\leq 1} |\partial_x^\alpha \partial_p^\beta \sum_{j=0}^N (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)| \leq C_{\alpha\beta}\hbar^{(\delta-1)N}\varepsilon^{-|\alpha|}.$$

This initial estimate is a priori not desirable. In order to make the notation less complicated we let B be an operator with integral kernel

$$K_B(x,y,t,\varepsilon,\hbar) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} b(x,p,t,\hbar,\varepsilon)\,dp,$$

where $b \in C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$, $\operatorname{supp}(b) \subset \operatorname{supp}(\theta)$ and satisfies

$$\sup_{x,p}\sup_{|t|\leq 1} |\partial_x^\alpha \partial_p^\beta b(x,p,t,\hbar,\varepsilon)| \leq C_{\alpha\beta}\hbar^{-l}\varepsilon^{-|\alpha|},$$

for some positive number l and all α and β in $\mathbb{N}_0^d$. Here we note that $U_N(t,\hbar)$ is an operator of this kind. Since we suppose $|\nabla_p a_{\varepsilon,0}| \geq c > 0$ on the support of $\theta(x,p)$ we have

$$\begin{aligned} &K_B(x,y,t,\varepsilon,\hbar) \\ &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} b(x,p,t,\hbar,\varepsilon)\,dp \\ &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} \frac{\sum_{j=1}^d (\partial_{p_j} a_{\varepsilon,0}(x,p))^2}{|\nabla_p a_{\varepsilon,0}(x,p)|^2} e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} b(x,p,t,\hbar,\varepsilon)\,dp \\ &= \frac{-i\hbar t^{-1}}{(2\pi\hbar)^d} \sum_{j=1}^d \int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} \frac{\partial_{p_j} a_{\varepsilon,0}(x,p)}{|\nabla_p a_{\varepsilon,0}(x,p)|^2} \partial_{p_j} e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} b(x,p,t,\hbar,\varepsilon)\,dp \\ &= \frac{i\hbar t^{-1}}{(2\pi\hbar)^d} \sum_{j=1}^d \int_{\mathbb{R}^d} e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} \partial_{p_j}[e^{i\hbar^{-1}\langle x-y,p\rangle} \frac{\partial_{p_j} a_{\varepsilon,0}(x,p)}{|\nabla_p a_{\varepsilon,0}(x,p)|^2} b(x,p,t,\hbar,\varepsilon)]\,dp. \end{aligned}$$

If we calculate the derivative we get that

$$
\begin{aligned}
&\partial_{p_j}[e^{i\hbar^{-1}\langle x-y,p\rangle}\frac{\partial_{p_j}a_{\varepsilon,0}(x,p)}{|\nabla_p a_{\varepsilon,0}(x,p)|^2}b(x,p,t,\hbar,\varepsilon)]\\
&= i\hbar^{-1}(x_j-y_j)e^{i\hbar^{-1}\langle x-y,p\rangle}\frac{\partial_{p_j}a_{\varepsilon,0}(x,p)}{|\nabla_p a_{\varepsilon,0}(x,p)|^2}b(x,p,t,\hbar,\varepsilon)\\
&\quad + e^{i\hbar^{-1}\langle x-y,p\rangle}\partial_{p_j}\Big[\frac{\partial_{p_j}a_{\varepsilon,0}(x,p)}{|\nabla_p a_{\varepsilon,0}(x,p)|^2}b(x,p,t,\hbar,\varepsilon)\Big].
\end{aligned}
$$

Combining these calculations we have that

$$
B = -t^{-1}\sum_{j=1}^{d}[x_j, B_{1,j}] + i\hbar t^{-1}\sum_{j=1}^{d} B_{2,j},
$$

where $B_{1,j}$ and $B_{2,j}$ are operators given by the kernels:

$$
\begin{aligned}
&K_{B_{1,j}}(x,y,t,\hbar,\varepsilon)\\
&\quad = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\frac{\partial_{p_j}a_{\varepsilon,0}(x,p)}{|\nabla_p a_{\varepsilon,0}(x,p)|^2}b(x,p,t,\hbar,\varepsilon)\,dp,
\end{aligned}
$$

and

$$
\begin{aligned}
&K_{B_{2,j}}(x,y,t,\hbar,\varepsilon)\\
&\quad = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\partial_{p_j}\Big[\frac{\partial_{p_j}a_{\varepsilon,0}(x,p)}{|\nabla_p a_{\varepsilon,0}(x,p)|^2}b(x,p,t,\hbar,\varepsilon)\Big]\,dp.
\end{aligned}
$$

If we now consider the trace and use the cyclisity of the trace we have

$$
\begin{aligned}
\operatorname{Tr}(B\operatorname{Op}_{\hbar,1}(\theta)) &= t^{-1}\sum_{j=1}^{d}\operatorname{Tr}([x_j,B_{1,j}]\operatorname{Op}_{\hbar,1}(\theta)) - i\hbar t^{-1}\sum_{j=1}^{d}\operatorname{Tr}(B_{2,j}\operatorname{Op}_{\hbar,1}(\theta))\\
&= t^{-1}\sum_{j=1}^{d}\operatorname{Tr}(B_{1,j}[x_j,\operatorname{Op}_{\hbar,1}(\theta)]) - i\hbar t^{-1}\sum_{j=1}^{d}\operatorname{Tr}(B_{2,j}\operatorname{Op}_{\hbar,1}(\theta))\\
&= \hbar t^{-1}\sum_{j=1}^{d}\operatorname{Tr}(B_{1,j}\operatorname{Op}_{\hbar,1}(\partial_{p_j}\theta)) - i\hbar t^{-1}\sum_{j=1}^{d}\operatorname{Tr}(B_{2,j}\operatorname{Op}_{\hbar,1}(\theta)).
\end{aligned}
$$

By our assumptions on t we note that we have gained $\hbar^{\frac{\delta}{2}}$ compared to our naive first estimate hence if do this procedure again on the operators $B_{1,j}$ and $B_{2,j}$ we will gain an additional factor of $\hbar^{\frac{\delta}{2}}$. By continuing this a sufficient amout of times we end up with the desired estimate. □

The previous Lemma showed that under a non-critical assumption on the principal symbol a localised trace of our approximation becomes negligible. But we would also need a result similar to this for the true propagator. Actually this can be proven in a setting for which we will need it, which is the content of the next Thoerem. An observation of this type was first made by V. Ivrii (see [11]). Here we will follow the proof of such a statement as made by M. Dimassi and J. Sjöstrand in [5]. The statement is:

Theorem II.8.4. *Let $A_\varepsilon(\hbar)$ be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ which satisfies Assumption II.7.1, has a bounded principal symbol and suppose there exists a δ in $(0,1)$ such that $\varepsilon \geq \hbar^{1-\delta}$. Furthermore, suppose there exists a number $\eta > 0$ such $a_{\varepsilon,0}^{-1}([-2\eta, 2\eta])$ is compact and a constant $c > 0$ such*

$$|\nabla_p a_{\varepsilon,0}(x,p)| \geq c \quad \textit{for all } (x,p) \in a_{\varepsilon,0}^{-1}([-2\eta, 2\eta]),$$

where $a_{\varepsilon,0}$ is the principal symbol of $A_\varepsilon(\hbar)$. Let f be in $C_0^\infty((-\eta,\eta))$ and θ be in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such that $\operatorname{supp}(\theta) \subset a_{\varepsilon,0}^{-1}((-\eta,\eta))$.

Then there exists a constant $T_0 > 0$ such that if χ is in $C_0^\infty((\frac{1}{2}\hbar^{1-\gamma}, T_0))$ for a γ in $(0,\delta]$, then for every N in $\mathbb{N}$, we have

$$|\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) \mathcal{F}_\hbar^{-1}[\chi](s - A_\varepsilon(\hbar)) \mathrm{Op}_\hbar^{\mathrm{w}}(\theta)]| \leq C_N \hbar^N$$

uniformly for s in $(-\eta,\eta)$.

Remark II.8.5. Theorems of this type for non-regular operators can be found in the works of V. Ivrii see [15] and L. Zielinski see [30, 32]. In both cases the proof of such theorems is different from the one we present here. The techniques used by both is based on propagation of singularities. The propagation of singularities is not directly present in the proof presented here but hidden in the techniques used.

In both [5] and [15] they assume the symbol to microhyperbolic in some direction. It might also be possible to extend the Theorem here to a general microhyperbolic assumption instead of the non-critical assumption. The challenge in this will be that for the proof to work under a general microhyperbolic assumption the symbol should be change such that microhyperbolic assumption similar to the non-critical assumption is achieved. This change might be problematic to do since it could mix the rough and non rough variables.

The localising operators $\mathrm{Op}_\hbar^{\mathrm{w}}(\theta)$ could be omitted if the first step of the proof is change to introducing them by applying Lemma II.7.16. We have chosen to state the theorem with them since when we will apply the theorem we have the localisations.

Proof. We start by remarking that it suffices to show the estimate with a function $\chi_\xi(t) = \chi(\frac{t}{\xi})$, where χ is in $C_0^\infty((\frac{1}{2},1))$ uniformly for ξ in $[\hbar^{1-\gamma}, T_0]$. Indeed assume such an estimate has been prove. We can split the interval $(\frac{1}{2}\hbar^{1-\gamma}, T_0)$ in $\frac{2T_0}{\hbar^{1-\gamma}}$ intervals of size $\frac{1}{2}\hbar^{1-\gamma}$ and make a partition of unity which members is supported in each of these intervals. Hence by linearity of the inverse Fourier transform and trace we would have

$$\begin{aligned}
&| \mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) \mathcal{F}_\hbar^{-1}[\chi](s - A_\varepsilon(\hbar)) \mathrm{Op}_\hbar^{\mathrm{w}}(\theta)]| \\
&\qquad \leq \sum_{j=1}^{M(\hbar)} |\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) \mathcal{F}_\hbar^{-1}[\chi_{\xi_j}](s - A_\varepsilon(\hbar)) \mathrm{Op}_\hbar^{\mathrm{w}}(\theta)]| \leq \tilde{C}_N \hbar^{N-1+\delta}.
\end{aligned}$$

Hence we will consider the trace

$$\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) \mathcal{F}_\hbar^{-1}[\chi_\xi](s - A_\varepsilon(\hbar)) \mathrm{Op}_\hbar^{\mathrm{w}}(\theta)],$$

with $\chi_\xi(t) = \chi(\frac{t}{\xi})$, where χ is in $C_0^\infty((\frac{1}{2},1))$ and ξ in $[\hbar^{1-\gamma}, T_0]$. For the rest of the proof we let a N in $\mathbb{N}$ be given as the error we want.

Without loss of generality we can assume $\theta = \sum_k \theta_k$, where the θ_k's satisfies that if $\text{supp}(\theta_k) \cap \text{supp}(\theta_l) \neq \emptyset$ then there exists j in $\{1,\dots,d\}$ such $|\partial_{p_j} a_{\varepsilon,0}(x,p)| > \tilde{c}$ on the set $\text{supp}(\theta_k) \cup \text{supp}(\theta_l)$. With this splitting of θ we have

$$\begin{aligned}&\text{Tr}[\text{Op}_\hbar^{\text{w}}(\theta) f(A_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi_\xi](s - A_\varepsilon(\hbar)) \,\text{Op}_\hbar^{\text{w}}(\theta)] \\ &\qquad = \sum_k \sum_l \text{Tr}[\text{Op}_\hbar^{\text{w}}(\theta_k) f(A_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi_\xi](s - A_\varepsilon(\hbar)) \,\text{Op}_\hbar^{\text{w}}(\theta_l)].\end{aligned}$$

By the cyclicity of the trace and the formulas for composition of pseudo-differential operators we observe if $\text{supp}(\theta_k) \cap \text{supp}(\theta_l) = \emptyset$ then the term is negligible. Hence what remains is the terms with $\text{supp}(\theta_k) \cap \text{supp}(\theta_l) \neq \emptyset$. All terms of the form are estimated with analogous techniques but some different indexes. Hence we will suppose we have a pair $\text{supp}(\theta_k) \cap \text{supp}(\theta_l) \neq \emptyset$ such $|\partial_{p_1} a_{\varepsilon,0}(x,p)| > \tilde{c}$ on the set $\text{supp}(\theta_k) \cup \text{supp}(\theta_l)$. This imply we either have $\partial_{p_1} a_{\varepsilon,0}(x,p) > \tilde{c}$ or $-\partial_{p_1} a_{\varepsilon,0}(x,p) > \tilde{c}$. We suppose we are in the first case. The other case is treated in the same manner but with a change of some signs.

To sum up we have reduced to the case where we consider

$$\text{Tr}[\text{Op}_\hbar^{\text{w}}(\theta_k) f(A_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi_\xi](s - A_\varepsilon(\hbar)) \,\text{Op}_\hbar^{\text{w}}(\theta_l)],$$

where $\partial_{p_1} a_{\varepsilon,0}(x,p) > \tilde{c}$ on the the set $\text{supp}(\theta_k) \cup \text{supp}(\theta_l)$. The next step is to change the principal symbol of our operator such it becomes global microhyperbolic in the direction $(\mathbf{0}; (1,0,\dots,0))$, where $\mathbf{0}$ is the d-dimensional vector with only zeros.

We let φ_2 be a function in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such $\varphi_2(x,p) = 1$ on a small neighbourhood of $\text{supp}(\theta_k) \cup \text{supp}(\theta_l)$ and have support contained in the set

$$\{(x,p) \in \mathbb{R}^{2d} \,|\, |\partial_{p_1} a_{\varepsilon,0}(x,p)| > \tfrac{\tilde{c}}{2}\}.$$

Moreover we let φ_1 be a function in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such $\varphi_1(x,p) = 1$ on $\text{supp}(\varphi_2)$ and such that

$$\text{supp}(\varphi_1) \subseteq \{(x,p) \in \mathbb{R}^{2d} \,|\, |\partial_{p_1} a_{\varepsilon,0}(x,p)| > \tfrac{\tilde{c}}{4}\}. \tag{3.74}$$

With these functions we define the symbol

$$\tilde{a}_{\varepsilon,0}(x,p) = a_{\varepsilon,0}(x,p)\varphi_1(x,p) + C(1 - \varphi_2(x,p)),$$

where the constant C is chosen such $\tilde{a}_{\varepsilon,0}(x,p) \geq 1 + \eta$ outside the support of $\varphi_2(x,p)$. We have

$$\partial_{p_1} \tilde{a}_{\varepsilon,0}(x,p) = (\partial_{p_1} a_{\varepsilon,0})(x,p)\varphi_1(x,p) + a_{\varepsilon,0}(x,p)(\partial_{p_1}\varphi_1)(x,p) - C\partial_{p_1}\varphi_2(x,p).$$

Hence there exist constants c_0 and c_1 such

$$\partial_{p_1} \tilde{a}_{\varepsilon,0}(x,p) \geq c_0 - c_1(\tilde{a}_{\varepsilon,0}(x,p))^2, \tag{3.75}$$

for all (x,p) in $\mathbb{R}^{2d}$. To see this we observe that on $\text{supp}(\theta_k) \cup \text{supp}(\theta_l)$ we have the inequality

$$\partial_{p_1} \tilde{a}_{\varepsilon,0}(x,p) \geq \tilde{c}.$$

By continuity there exists an open neighbourhood Ω of $\operatorname{supp}(\theta_k) \cup \operatorname{supp}(\theta_l)$ such $\partial_{p_1} \tilde{a}_{\varepsilon,0}(x,p) \geq \frac{\tilde{c}}{3}$ and $(1-\varphi_2) \neq 0$ on Ω^c. Hence outside Ω we get the the bound

$$\partial_{p_1} \tilde{a}_{\varepsilon,0}(x,p) \geq c_0 - c_1(\tilde{a}_{\varepsilon,0}(x,p))^2,$$

by choosing c_1 sufficiently large. This estimates is that our new symbol is global microhyperbolic in the direction $(\mathbf{0};(1,0,\dots,0))$.

Our assumptions on the operator $A_\varepsilon(\hbar)$ imply the form

$$A_\varepsilon(\hbar) = \sum_{j=0}^{N_0} \hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,j}) + \hbar^{N_0+1} R_{N_0}(\hbar,\varepsilon),$$

where N_0 is chosen such

$$\hbar^{N_0+1} \| R_{N_0}(\hbar,\varepsilon)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\hbar^{N+d}.$$

By $\tilde{A}_\varepsilon(\hbar)$ we denote the operator obtained by taking the N_0 first terms of $A_\varepsilon(\hbar)$ and exchanging the principal symbol $a_{\varepsilon,0}$ of $A_\varepsilon(\hbar)$ by $\tilde{a}_{\varepsilon,0}$. Note that the operator $\tilde{A}_\varepsilon(\hbar)$ still satisfies Assumption II.7.1 as the original symbols where assumed to be bounded. We have

$$A_\varepsilon(\hbar) - \tilde{A}_\varepsilon(\hbar) = \operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,0} - \tilde{a}_{\varepsilon,0}) + \hbar^{N_0+1} R_{N_0}(\hbar,\varepsilon),$$

and by construction is $a_{\varepsilon,0} - \tilde{a}_{\varepsilon,0}$ supported away from $\operatorname{supp}(\theta_k) \cup \operatorname{supp}(\theta_l)$. Hence if we apply the identity

$$(z - A_\varepsilon(\hbar))^{-1} - (z - \tilde{A}_\varepsilon(\hbar))^{-1} = (z - A_\varepsilon(\hbar))^{-1}(A_\varepsilon(\hbar) - \tilde{A}_\varepsilon(\hbar))(z - \tilde{A}_\varepsilon(\hbar))^{-1},$$

and use the formula for composition of operators we get for $N_1 \geq \tau$ the estimate

$$\begin{aligned} \|\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k)&((z - A_\varepsilon(\hbar))^{-1} - (z - \tilde{A}_\varepsilon(\hbar))^{-1}) \operatorname{Op}_\hbar^{\mathrm{w}}(\theta_l)\|_{\mathrm{Tr}} \\ &= C\hbar^{-d} \|\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k)[(z - A_\varepsilon(\hbar))^{-1}(A_\varepsilon(\hbar) - \tilde{A}_\varepsilon(\hbar))(z - \tilde{A}_\varepsilon(\hbar))^{-1}]\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \\ &\leq C_{N_1} \frac{\hbar^{(N_1-\tau)\delta+\tau-d}}{|\mathrm{Im}(z)|^{N+2}} + C_{N_0} \frac{\hbar^N}{|\mathrm{Im}(z)|^2}, \end{aligned} \tag{3.76}$$

for z in $\mathbb{C}$ with $|\operatorname{Im}(z)| > 0$. In order to use the above estimate we will use Theorem II.7.12 and hence we need to make an almost analytic extensions of the function f. Let $\tilde{f}$ be an almost analytic extension of f, such $\tilde{f}$ is in $C_0^\infty(\mathbb{C})$ and

$$\begin{aligned} \tilde{f}(x) &= f(x) \text{ for all } x \in \mathbb{R} \\ \bar{\partial}\tilde{f}(z) &\leq C_N |\mathrm{Im}(z)|^N \text{ for all } N \in \mathbb{N}. \end{aligned}$$

Such an extension exists according to Remark II.7.10. As $\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)$ is an analytic function in z we have by Theorem II.7.12 the identity

$$\begin{aligned} \operatorname{Tr}[&\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k) f(A_\varepsilon(\hbar)) \mathcal{F}_\hbar^{-1}[\chi_\xi](s - A_\varepsilon(\hbar)) \operatorname{Op}_\hbar^{\mathrm{w}}(\theta_l)] \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}(\tilde{f})(z) \mathcal{F}_\hbar^{-1}[\chi_\xi](s-z) \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k)(z - A_\varepsilon(\hbar))^{-1} \operatorname{Op}_\hbar^{\mathrm{w}}(\theta_l)]\, L(dz). \end{aligned} \tag{3.77}$$

This identity is also true for $\tilde{A}_\varepsilon(\hbar)$. On the support of $\tilde{f}$ we have

$$\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}} e^{it\hbar^{-1}(s-z)}\chi_\xi(t)\,dt \leq C\frac{\xi}{\hbar}.$$

Now by the properties of $\tilde{f}$, (3.76), (3.77) and the above estimate we have for $N_1 \geq \tau$

$$\begin{aligned}
&|\operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k)f(A_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi_\xi](s-A_\varepsilon(\hbar))\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_l)] \\
&\quad - \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k)f(\tilde{A}_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi_\xi](s-\tilde{A}_\varepsilon(\hbar))\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_l)]| \\
&\quad \leq \frac{1}{\pi}\int_{\mathbb{C}} |\bar{\partial}(\tilde{f})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z) \\
&\qquad\qquad \times \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k)((z-A_\varepsilon(\hbar))^{-1}-(z-\tilde{A}_\varepsilon(\hbar))^{-1})\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_l)]|\,L(dz) \\
&\quad \leq C_{N_1}\hbar^{(N_1-\tau)\delta+\tau-d-1}+C\hbar^N.
\end{aligned}$$

Hence by choosing N_1 sufficiently large we can change the principal symbol. Note that the constant C_{N_1} also depends on the symbols.

For the reminder of the proof we will omit the tilde on the operator and principal symbol but instead assume the principal symbol to be global micro-hyperbolic in the direction $(\mathbf{0};(1,0,\dots,0))$ ((3.75) without the tildes).

In order to estimate

$$\operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k)f(A_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi_\xi](s-A_\varepsilon(\hbar))\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_l)],$$

we will need an auxiliary function. Let ψ be in $C^\infty(\mathbb{R})$ such $\psi(t)=1$ for $t\leq 1$ and $\psi(t)=0$ for $t\geq 2$. Moreover let M be a sufficiently large constant which will be fixed later and put

$$\psi_{\mu_1}(z) = \psi\Big(\frac{\operatorname{Im}(z)}{\mu_1}\Big),$$

where $\mu_1 = \frac{M\hbar}{\xi}\log(\frac{1}{\hbar})$. With this function we have

$$|\bar{\partial}(\tilde{f}\psi_{\mu_1})| \leq \begin{cases} C_N|\operatorname{Im}(z)|^N, & \text{if } \operatorname{Im}(z)<0 \\ C_N\psi_{\mu_1}(z)|\operatorname{Im}(z)|^N + \mu_1^{-1}\mathbf{1}_{[1,2]}(\frac{\operatorname{Im}(z)}{\mu_1}), & \text{if } \operatorname{Im}(z)\geq 0, \end{cases} \tag{3.78}$$

for any N in $\mathbb{N}$. Again we can use Theorem II.7.12 for the operator $A_\varepsilon(\hbar)$ on the function $(\tilde{f}\psi_{\mu_1})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)$. This gives

$$\begin{aligned}
&(\tilde{f}\psi_{\mu_1})(A_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi_\xi](s-A_\varepsilon(\hbar)) \\
&\qquad = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}(\tilde{f}\psi_{\mu_1})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)(z-A_\varepsilon(\hbar))^{-1}\,L(dz),
\end{aligned}$$

where we have used that $\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)$ is an analytic function in z. Hence the trace we consider is

$$\begin{aligned}
&\operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k)f(A_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi_\xi](s-A_\varepsilon(\hbar))\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_l)] \\
&= -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}(\tilde{f}\psi_{\mu_1})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_k)(z-A_\varepsilon(\hbar))^{-1}\operatorname{Op}_\hbar^{\mathrm{w}}(\theta_l)]\,L(dz) \\
&= -\frac{1}{\pi}\int_{\operatorname{Im}(z)<0}\cdots\,L(dz) - \frac{1}{\pi}\int_{\operatorname{Im}(z)\geq 0}\cdots\,L(dz)
\end{aligned} \tag{3.79}$$

If we shortly investigate each of the integrals. Firstly we note the bound

$$|\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_k)(z - A_\varepsilon(\hbar))^{-1}\,\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_l)]| \leq \frac{C}{\hbar^d |\mathrm{Im}(z)|}.$$

If we consider the integral over the negative imaginary part we have

$$\begin{aligned}
&\Big|\frac{1}{\pi}\int_{\mathrm{Im}(z)<0} \bar{\partial}(\tilde{f}\psi_{\mu_1})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\,\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_k)(z - A_\varepsilon(\hbar))^{-1}\,\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_l)]\,L(dz)\Big| \\
&\qquad\leq \frac{C_{2N}\xi}{\pi\hbar^{d+1}}\int_{\mathrm{Im}(z)<0} \frac{\mathrm{Im}(z)^{2N}}{|\,\mathrm{Im}(z)|} e^{\frac{\xi\,\mathrm{Im}(z)}{2\hbar}}\,d\,\mathrm{Im}(z) \\
&\qquad\leq \frac{C_{2N}}{\pi\hbar^{d}}(\frac{\hbar}{\xi})^{2N-2} \leq \tilde{C}\hbar^{(2N-2)\gamma-d},
\end{aligned}$$

for any N in $\mathbb{N}$. We have in the above calculation used partial integration and the estimate

$$|\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)| \leq C\frac{\xi}{\hbar}e^{\frac{\xi\,\mathrm{Im}(z)}{2\hbar}}.$$

The above estimate imply that the contribution to the trace from the negative integral is negligible. If we split the integral over positive imaginary part up according to μ_1 we have by (3.78) the estimate

$$\begin{aligned}
&|\frac{1}{\pi}\int_{0\leq\mathrm{Im}(z)\leq\mu_1} \bar{\partial}(\tilde{f}\psi_{\mu_1})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\,\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_k)(z - A_\varepsilon(\hbar))^{-1}\,\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_l)]\,L(dz)| \\
&\quad\leq \frac{C_{2N}\xi}{\pi\hbar^{d+1}}\int_{0\leq\mathrm{Im}(z)\leq\mu_1} \psi_{\mu_1}(z)|\mathrm{Im}(z)|^N e^{\frac{\xi\,\mathrm{Im}(z)}{2\hbar}}\,d\,\mathrm{Im}(z) \\
&\quad\leq \tilde{C}\frac{\xi}{\hbar^{d+1}}\mu_1^{N+1} \leq \tilde{C}\frac{\xi}{\hbar^{d+1}}\frac{M\hbar}{\xi}\log(\frac{1}{\hbar})^{N+1} \leq \tilde{C}M\hbar^{(N+1)\gamma-d-1},
\end{aligned}$$

For any N in $\mathbb{N}$. Hence this terms also becomes negligible. What remains from (3.79) is the expression

$$-\frac{1}{\pi}\int_{\mathrm{Im}(z)>\mu_1} \bar{\partial}(\tilde{f}\psi_{\mu_1})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\,\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_k)(z - A_\varepsilon(\hbar))^{-1}\,\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_l)]\,L(dz). \tag{3.80}$$

In order to estimates this we will need to change all our operators. This is done by introducing an auxiliary variable in the symbols and make an almost analytic extension in this variable. Recall we have change the operator $A_\varepsilon(\hbar)$ such it is a sum of Weyl quantised pseudo differential operators. Hence in the following we let $q(x,p)$ be one of our symbols and we let $q_t(x,p) = q(x,(p_1+t,p_2,\dots,p_d))$. We now take t be complex and make an almost analytic extension $\tilde{q}_t$ of q_t in t according to Definition II.7.9 for $|\,\mathrm{Im}(t)| < 1$. The form of $\tilde{q}_t$ is

$$\tilde{q}_t(x,p) = \sum_{r=0}^{n}(\partial_{p_1}^r q)(x,(p_1+\mathrm{Re}(t),p_2,\dots,p_d))\frac{(i\,\mathrm{Im}(t))^r}{r!},$$

Recalling the identity

$$\mathrm{Op}_\hbar^{\mathrm{w}}(q_{\mathrm{Re}(t)}) = e^{-i\,\mathrm{Re}(t)\hbar^{-1}x_1}\,\mathrm{Op}_\hbar^{\mathrm{w}}(q)e^{i\,\mathrm{Re}(t)\hbar^{-1}x_1},$$

we have

$$\operatorname{Op}_\hbar^{\mathrm{w}}(\tilde{q}_t) = \sum_{r=0}^{n} \frac{(i\operatorname{Im}(t))^r}{r!} e^{-i\operatorname{Re}(t)\hbar^{-1}x_1} \operatorname{Op}_\hbar^{\mathrm{w}}(\partial_{p_1}^r q) e^{i\operatorname{Re}(t)\hbar^{-1}x_1}. \tag{3.81}$$

If we take derivatives with respect to $\operatorname{Re}(t)$ and $\operatorname{Im}(t)$ in operator sense we see

$$\begin{aligned}\frac{\partial}{\partial\operatorname{Re}(t)} \operatorname{Op}_\hbar^{\mathrm{w}}(\tilde{q}_t) &= -\frac{i}{\hbar}\sum_{r=0}^{n} \frac{(i\operatorname{Im}(t))^r}{r!} e^{-i\operatorname{Re}(t)\hbar^{-1}x_1} [x_1, \operatorname{Op}_\hbar^{\mathrm{w}}(\partial_{p_1}^r q)] e^{i\operatorname{Re}(t)\hbar^{-1}x_1} \\ &= \sum_{r=0}^{n} \frac{(i\operatorname{Im}(t))^r}{r!} e^{-i\operatorname{Re}(t)\hbar^{-1}x_1} \operatorname{Op}_\hbar^{\mathrm{w}}(\partial_{p_1}^{r+1} q) e^{i\operatorname{Re}(t)\hbar^{-1}x_1},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\operatorname{Im}(t)} \operatorname{Op}_\hbar^{\mathrm{w}}(\tilde{q}_t) &= i\sum_{r=1}^{n} \frac{(i\operatorname{Im}(t))^{r-1}}{(r-1)!} e^{-i\operatorname{Re}(t)\hbar^{-1}x_1} \operatorname{Op}_\hbar^{\mathrm{w}}(\partial_{p_1}^r q) e^{i\operatorname{Re}(t)\hbar^{-1}x_1} \\ &= \sum_{r=0}^{n-1} \frac{(i\operatorname{Im}(t))^r}{r!} e^{-i\operatorname{Re}(t)\hbar^{-1}x_1} \operatorname{Op}_\hbar^{\mathrm{w}}(\partial_{p_1}^{r+1} q) e^{i\operatorname{Re}(t)\hbar^{-1}x_1}.\end{aligned}$$

In the above calculation the unbounded operator x_1 appear, but for all the symbols we consider the commutator $[x_1, \operatorname{Op}_\hbar^{\mathrm{w}}(\partial_{p_1}^r q)]$ will be bounded. This calculation gives

$$\Big(\frac{\partial}{\partial\operatorname{Re}(t)} + i\frac{\partial}{\partial\operatorname{Im}(t)}\Big) \operatorname{Op}_\hbar^{\mathrm{w}}(\tilde{q}_t) = \frac{(i\operatorname{Im}(t))^n}{n!} e^{-i\operatorname{Re}(t)\hbar^{-1}x_1} \operatorname{Op}_\hbar^{\mathrm{w}}(\partial_{p_1}^{n+1} q) e^{i\operatorname{Re}(t)\hbar^{-1}x_1}$$

This imply

$$\begin{aligned}&\|\frac{\partial}{\partial\bar{t}} \operatorname{Op}_\hbar^{\mathrm{w}}(\tilde{\theta}_{j,t})\|_{\mathrm{Tr}} \leq C_n\hbar^{-d}|\operatorname{Im}(t)|^n \qquad \text{for } j = k, l \\ &\|\frac{\partial}{\partial\bar{t}}\tilde{A}_\varepsilon(\hbar;t)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_n|\operatorname{Im}(t)|^n,\end{aligned} \tag{3.82}$$

for any n in $\mathbb{N}_0$ by choosing an almost analytic expansion of this order. The operator $\tilde{A}_\varepsilon(\hbar;t)$ is the operator where we have made the above construction for each symbol in the expansion of the operator. Moreover we have by the construction of $\tilde{A}_\varepsilon(\hbar;t)$

$$\tilde{A}_\varepsilon(\hbar;t) = e^{-i\operatorname{Re}(t)\hbar^{-1}x_1} A_\varepsilon(\hbar) e^{i\operatorname{Re}(t)\hbar^{-1}x_1} + i\operatorname{Im}(t)B_\varepsilon(\hbar;t)$$

where $B_\varepsilon(\hbar;t)$ is a bounded operator this form is obtained from (3.81) with q replaced by the symbol of $A_\varepsilon(\hbar)$. This gives

$$z - \tilde{A}_\varepsilon(\hbar;t) = (z - U^*A_\varepsilon(\hbar)U)[I + (z - U^*A_\varepsilon(\hbar)U)^{-1} i\operatorname{Im}(t)B_\varepsilon(\hbar;t)],$$

where $U = e^{i\operatorname{Re}(t)\hbar^{-1}x_1}$. Hence if $|\operatorname{Im}(t)| \leq \frac{|\operatorname{Im}(z)|}{C_1}$ the operator $z - \tilde{A}_\varepsilon(\hbar;t)$ has an inverse where $C_1 \geq \|B_\varepsilon(\hbar;t)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} + 1$. This imply that the following function

$$\eta(t,z) = \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\tilde{\theta}_{k,t})(z - \tilde{A}_\varepsilon(\hbar;t))^{-1} \operatorname{Op}_\hbar^{\mathrm{w}}(\tilde{\theta}_{l,t})]$$

is well defined for $|\mathrm{Im}(t)| \leq \frac{|\mathrm{Im}(z)|}{C_1}$. The function have by construction the properties

$$\begin{aligned} |\eta(t,z)| &\leq \frac{c}{\hbar^d |\mathrm{Im}(z)|} \\ |\Big(\frac{\partial}{\partial \operatorname{Re}(t)} + i \frac{\partial}{\partial \operatorname{Im}(t)}\Big)\eta(t,z)| &\leq \frac{c_n |\mathrm{Im}(t)|^n}{\hbar^d |\mathrm{Im}(z)|^2}. \end{aligned}$$

for n in $\mathbb{N}_0$. But by cyclicity of the trace the function $\eta(t,z)$ is independent of $\mathrm{Re}(t)$. Hence we have

$$|\eta(\pm i \operatorname{Im}(t), z) - \eta(0,z)| \leq \frac{c_N |\mathrm{Im}(t)|^n}{\hbar^d |\mathrm{Im}(z)|^2}$$

by the fundamental theorem of calculus. The construction of η gives us that

$$\eta(0,z) = \mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta_k)(z - A_\varepsilon(\hbar))^{-1} \mathrm{Op}_\hbar^{\mathrm{w}}(\theta_l)].$$

Hence we can exchange the trace in (3.80) by $\eta(-i\frac{\mu_1}{C_1}, z)$ with an error of the order $\hbar^{\gamma n - d}$. This is due to our choice of $\mu_1 = \frac{M\hbar}{\xi}\log(\frac{1}{\hbar})$ in the start of the proof and that the integral is only over a compact region where $|\mathrm{Im}(z)| > \frac{\mu_1}{C_1}$ due to the definition of ψ_{μ_1}. It now remains to estimate the term

$$-\frac{1}{\pi}\int_{\mathrm{Im}(z)>\mu_1} \bar{\partial}(\tilde{f}\psi_{\mu_1})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\eta(-i\mu_2, z)\, L(dz), \tag{3.83}$$

where

$$\eta(-i\mu_2, z) = \mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\tilde{\theta}_{k,-i\mu_2})(z - \tilde{A}_\varepsilon(\hbar; -i\mu_2))^{-1} \mathrm{Op}_\hbar^{\mathrm{w}}(\tilde{\theta}_{l,-i\mu_2})],$$

and $\mu_2 = \frac{\mu_1}{C_1}$. From the construction of the almost analytic extension we have the following form of the principal symbol of $z - \tilde{A}_\varepsilon(\hbar; -i\mu_2)$

$$z - \tilde{a}_{\varepsilon,0}(x,p;-i\mu_2) = z - (a_{\varepsilon,0}(x,p) - i\mu_2(\partial_{p_1}a_{\varepsilon,0})(x,p) + \mathcal{O}(\mu_2^2)).$$

For $-\frac{c_0\mu_2}{4} < \mathrm{Im}(z) < 0$, where c_0 is the constant from the global micro-hyperbolicity (3.75), we have by the global micro-hyperbolicity for $|\mathrm{Re}(z)| < \eta$ and $\hbar$ sufficiently small

$$\mathrm{Im}(z - \tilde{a}_{\varepsilon,0}(x,p;-i\mu_2)) \geq c_0\mu_2 + \mathrm{Im}(z) - C\mu_2(\mathrm{Re}(z) - a_{\varepsilon,0}(x,p))^2.$$

To see this recall how the principal symbol was changed and that if $\mathrm{Re}(z) - a_{\varepsilon,0}(x,p)$ is zero or small then is $(\partial_{p_1}a_{\varepsilon,0})(x,p) > 2c_0$ hence we have to assume $\hbar$ sufficiently small. This implies there exists a C_2 such we have the inequality

$$\begin{aligned} \mathrm{Im}(z - \tilde{a}_{\varepsilon,0}(x,p;-i\mu_2) + C_2\mu_2\overline{(z - \tilde{a}_{\varepsilon,0}(x,p;-i\mu_2))}(z - \tilde{a}_{\varepsilon,0}(x,p;-i\mu_2)) \\ \geq \frac{c_0}{2}\mu_2 + \mathrm{Im}(z), \end{aligned}$$

Where we again assume $\hbar$ sufficiently small and that all terms from the product in the above equation which is not $(\mathrm{Re}(z) - a_{\varepsilon,0}(x,p))^2$ comes with at least one extra

μ_2. Now by Theorem II.7.4 we have for every g in $L^2(\mathbb{R}^d)$

$$\begin{aligned}
&\mathrm{Im}(\langle \mathrm{Op}_\hbar^{\mathrm{w}}(z-\tilde{a}_{\varepsilon,0}(-i\mu_2))g,g\rangle) + C_2\mu_2\|\mathrm{Op}_\hbar^{\mathrm{w}}(z-\tilde{a}_{\varepsilon,0}(-i\mu_2))g\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad\geq \langle \mathrm{Op}_\hbar^{\mathrm{w}}(\mathrm{Im}(z-\tilde{a}_{\varepsilon,0}(-i\mu_2)) + C_2\mu_2(\overline{z-\tilde{a}_{\varepsilon,0}(-i\mu_2)})(z-\tilde{a}_{\varepsilon,0}(-i\mu_2)))g,g\rangle \\
&\qquad\qquad\qquad\qquad - c\mu_2\hbar^\delta\|g\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad\geq (\frac{c_0\mu_2}{2} + \mathrm{Im}(z))\|g\|_{L^2(\mathbb{R}^d)}^2 - \tilde{c}(\hbar^\delta + \mu_2\hbar^\delta)\|g\|_{L^2(\mathbb{R}^d)}^2 \geq \frac{c_0\mu_2}{6}\|g\|_{L^2(\mathbb{R}^d)}^2,
\end{aligned}$$

for $\hbar$ sufficiently small. Now by a Hölder inequality we have

$$\begin{aligned}
&\frac{c_0\mu_2}{6}\|g\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad\leq |\langle \mathrm{Op}_\hbar^{\mathrm{w}}(z-\tilde{a}_{\varepsilon,0}(-i\mu_2))g,g\rangle| + C_2\mu_2\|\mathrm{Op}_\hbar^{\mathrm{w}}(z-\tilde{a}_{\varepsilon,0}(-i\mu_2))g\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad\leq \frac{c_0\mu_2}{12}\|g\|_{L^2(\mathbb{R}^d)}^2 + (\frac{6}{2c_0\mu_2} + C_2\mu_2)\|\mathrm{Op}_\hbar^{\mathrm{w}}(z-\tilde{a}_{\varepsilon,0}(-i\mu_2))g\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned}$$

This shows that there exists a constant C such

$$\frac{c_0\mu_2}{C}\|g\|_{L^2(\mathbb{R}^d)} \leq \|\mathrm{Op}_\hbar^{\mathrm{w}}(z-\tilde{a}_{\varepsilon,0}(-i\mu_2))g\|_{L^2(\mathbb{R}^d)},$$

for all g in $L^2(\mathbb{R}^d)$. Since $\mathrm{Op}_\hbar^{\mathrm{w}}(z-\tilde{a}_{\varepsilon,0}(-i\mu_2))$ is the principal part of $\tilde{A}_\varepsilon(\hbar;-i\mu_2)$ and the rest comes with an extra $\hbar$ in front as we have assumed regularity $\tau \geq 1$ the above estimate imply

$$\frac{c_0\mu_2}{2C}\|g\|_{L^2(\mathbb{R}^d)} \leq \|z-\tilde{A}_\varepsilon(\hbar;-i\mu_2)g\|_{L^2(\mathbb{R}^d)},$$

for $\hbar$ sufficiently small. We can do the above argument again for $\mathrm{Im}(z) \geq 0$ and obtain the same result. The estimate implies that the set $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > -\frac{c_0\mu_2}{4}\}$ is in the regularity set of $\tilde{A}_\varepsilon(\hbar;-i\mu_2)$. Since $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > -\frac{c_0\mu_2}{4}\}$ is connect we have that this is a subset of the resolvent set if just one point of the set is in the resolvent set. For a z in $\mathbb{C}$ with positive imaginary part and $|z| \geq 2\|\tilde{A}_\varepsilon(\hbar;-i\mu_2)\|$ we have existence of $(z-\tilde{A}_\varepsilon(\hbar;-i\mu_2))^{-1}$ as a Neumann series. Hence we can conclude that $(z-\tilde{A}_\varepsilon(\hbar;-i\mu_2))^{-1}$ extends to a holomorphic function for z in $\mathbb{C}$ such $\mathrm{Im}(z) \geq -\frac{c_0\mu_2}{4C_1}$. This implies

$$\begin{aligned}
0 &= -\frac{1}{\pi}\int_{\mathbb{C}} (\tilde{f}\psi_{\mu_1}\psi_{-\frac{c_0\mu_2}{4C_1}})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\bar{\partial}\eta(-i\mu_2,z)\,L(dz) \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}(\tilde{f}\psi_{\mu_1}\psi_{-\frac{c_0\mu_2}{4C_1}})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\eta(-i\mu_2,z)\,L(dz) \\
&= \frac{1}{\pi}\int_{\mathrm{Im}(z)\geq 0} \bar{\partial}(\tilde{f}\psi_{\mu_1})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\eta(-i\mu_2,z)\,L(dz) \\
&\quad + \frac{1}{\pi}\int_{\mathrm{Im}(z)<0} \bar{\partial}(\tilde{f}\psi_{\mu_1}\psi_{-\frac{c_0\mu_2}{4C_1}})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\eta(-i\mu_2,z)\,L(dz),
\end{aligned}$$

where we have used that $\psi_{-\frac{c_0\mu_2}{4C_1}}(z) = 1$ for all z in $\mathbb{C}$ with $\mathrm{Im}(z) \geq 0$. This equality gives us the following rewriting of (3.83)

$$\begin{aligned}
-\frac{1}{\pi}\int_{\mathrm{Im}(z)>\mu_1} &\bar{\partial}(\tilde{f}\psi_{\mu_1})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\eta(-i\mu_2,z)\,L(dz) \\
&= \frac{1}{\pi}\int_{\mathrm{Im}(z)<0} \bar{\partial}(\tilde{f}\psi_{\mu_1}\psi_{-\frac{c_0\mu_2}{4C_1}})(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\eta(-i\mu_2,z)\,L(dz) + \mathcal{O}(\hbar^{N_0}),
\end{aligned} \tag{3.84}$$

for any N_0 in $\mathbb{N}_0$. We have

$$\bar{\partial}(\tilde{f}\psi_{\mu_1}\psi_{-\frac{c_0\mu_2}{4C_1}})(z) = \bar{\partial}(\tilde{f})(z)(\psi_{\mu_1}\psi_{-\frac{c_0\mu_2}{4C_1}})(z) + \tilde{f}\psi_{\mu_1}(z)\bar{\partial}\psi_{-\frac{c_0\mu_2}{4C_1}}(z),$$

for $\mathrm{Im}(z) < 0$, where we have used that $\psi_{\mu_1}(z) = 1$ for $\mathrm{Im}(z) \leq 1$. The part of the integral on the right hand side of (3.84) with the derivative on $\tilde{f}$ will be small due to the same argumentation as previously in the proof. What remains is the part where the derivative is on $\psi_{-\frac{c_0\mu_2}{4C_1}}$. For this part we have

$$\begin{aligned}
\frac{1}{\pi}\Big|\int_{\mathrm{Im}(z)<0} &\tilde{f}(z)\bar{\partial}\psi_{-\frac{c_0\mu_2}{4C_1}}(z)\mathcal{F}_\hbar^{-1}[\chi_\xi](s-z)\eta(-i\mu_2,z)\,L(dz)\Big| \\
&\leq \frac{C}{\hbar^d\left(\frac{M\hbar}{C_1\xi}\log(\frac{1}{\hbar})\right)^2}\int_{\substack{-\eta<\mathrm{Re}(z)<\eta \\ -\frac{M\hbar c_0}{2C_1^2\xi}\log(\frac{1}{\hbar})<\mathrm{Im}(z)<-\frac{M\hbar c_0}{4C_1^2\xi}\log(\frac{1}{\hbar})}} \frac{\xi}{\hbar}e^{\frac{\xi\,\mathrm{Im}(z)}{2\hbar}}\,L(dz) \\
&= \frac{C}{\hbar^d\left(\frac{M\hbar}{C_1\xi}\log(\frac{1}{\hbar})\right)^2)}e^{-\frac{c_0M}{2C_1^2}\log(\frac{1}{\hbar})} = \frac{\tilde{C}\xi^2}{\hbar^{d+2}M^2\log(\frac{1}{\hbar})^2}\hbar^{\frac{c_o}{2C_1^2}M}.
\end{aligned}$$

Hence by choosing M sufficiently large we can make the above expression smaller than $\hbar^N$ for any N in $\mathbb{N}_0$. This concludes the proof. ☐

This proposition actually imply a stronger version of it self, where the assumption of boundedness is not needed.

Corollary II.8.6. *Let $A_\varepsilon(\hbar)$ be a strongly $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ which satisfies Assumption II.7.1 and there exists a δ in $(0,1)$ such that $\varepsilon \geq \hbar^{1-\delta}$. Suppose there exists a number $\eta > 0$ such $a_{\varepsilon,0}^{-1}([-2\eta,2\eta])$ is compact and a constant $c > 0$ such*

$$|\nabla_p a_{\varepsilon,0}(x,p)| \geq c \quad \textit{for all } (x,p) \in a_{\varepsilon,0}^{-1}([-2\eta,2\eta]),$$

where $a_{\varepsilon,0}$ is the principal symbol of $A_\varepsilon(\hbar)$. Let f be in $C_0^\infty((-\eta,\eta))$ and θ be in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such that $\mathrm{supp}(\theta) \subset a_{\varepsilon,0}^{-1}((-\eta,\eta))$. There exists a constant $T_0 > 0$ such that if χ is in $C_0^\infty((\frac{1}{2}\hbar^{1-\gamma},T_0))$ for a γ in $(0,\delta]$, then for every N in $\mathbb{N}$, we have

$$|\mathrm{Tr}[\mathrm{Op}_\hbar^{\mathrm{w}}(\theta)f(A_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi](s-A_\varepsilon(\hbar))\,\mathrm{Op}_\hbar^{\mathrm{w}}(\theta)]| \leq C_N\hbar^N,$$

uniformly for s in $(-\eta,\eta)$.

Proof. The operator $A_\varepsilon(\hbar)$ satisfies the assumptions of Theorem II.7.13. This gives us the functional calculus for the pseudo differential operator for functions in the set $\mathcal{A}$ which contains all functions from $C_0^\infty(\mathbb{R})$. It can be remarked that the function $f(t)\mathcal{F}_\hbar^{-1}[\chi](s-t)$ is a $C_0^\infty(\mathbb{R})$ in t and both imaginary and real part is also in $C_0^\infty(\mathbb{R})$ just with real values. By $g(t)$ we denote either the real or imaginary part of $f(t)\mathcal{F}_\hbar^{-1}[\chi](s-t)$. Theorem II.7.13 gives

$$g(A_\varepsilon(h)) = \sum_{j\geq 0} \hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a_{\varepsilon,j}^g),$$

where

$$\begin{aligned} a_{\varepsilon,0}^g &= g(a_{\varepsilon,0}) \\ a_{\varepsilon,j}^g &= \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} d_{\varepsilon,j,k} g^{(k)}(a_{\varepsilon,0}) \qquad \text{for } j \geq 1, \end{aligned} \tag{3.85}$$

the symbols $d_{\varepsilon,j,k}$ are the polynomials from Lemma II.7.6. Now Let φ be in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such $\varphi(x,p) = 1$ on a neighbourhood of $\operatorname{supp}(f(a_{\varepsilon,0})\mathcal{F}_\hbar^{-1}[\chi](s-a_{\varepsilon,0}))$. Then if we define the operator $\tilde{A}_\varepsilon(\hbar)$ as the operator with symbol

$$\tilde{a}_\varepsilon(\hbar) = \sum_{j\geq 0} \hbar^j \varphi a_{\varepsilon,j}.$$

This operator satisfies the assumptions in Theorem II.8.4. Hence we have

$$|\operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) f(\tilde{A}_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\chi](s-\tilde{A}_\varepsilon(\hbar)) \operatorname{Op}_\hbar^{\mathrm{w}}(\theta)]| \leq C_N \hbar^N \tag{3.86}$$

But by construction we have

$$\|g(A_\varepsilon(h)) - g(\tilde{A}_\varepsilon(h))\| \leq C_n \hbar^n,$$

for every n in $\mathbb{N}_0$, where we have used the form of the symbols given in (3.85). Combining this with (3.86) we achieve the desired estimate. ☐

II.9 Weyl law for Rough pseudo-differential operators

In this section we will prove a Weyl law for rough pseudo differential operators and we will do it with the approach used in [17]. Hence we will first consider some asymptotic expansions of certain integrals.

Theorem II.9.1. *Let $A_\varepsilon(\hbar)$ be a $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ which satisfies Assumption II.7.1 and there exists a δ in $(0,1)$ such that $\varepsilon \geq \hbar^{1-\delta}$. Suppose there exists $\eta > 0$ such that $a_{0,\varepsilon}^{-1}([-2\eta, 2\eta])$ is compact, where $a_{\varepsilon,0}$ is the principal symbol of $A_\varepsilon(\hbar)$. Let χ be in $C_0^\infty((-T_0,T_0))$ and $\chi = 1$ in a neighbourhood of 0, where T_0 is the number from Corollary II.8.6. Then for every f in $C_0^\infty((-\eta,\eta))$ we have*

$$\int_{\mathbb{R}} \operatorname{Tr}[f(A_\varepsilon(\hbar))e^{it\hbar^{-1}A_\varepsilon(\hbar)}]e^{-its\hbar^{-1}}\chi(t)\,dt = (2\pi\hbar)^{1-d}\Big[\sum_{j=0}^{N_0} \hbar^j \xi_j(s) + \mathcal{O}(\hbar^N)\Big].$$

where the error term is uniform with respect to $s \in (-\eta, \eta)$ and the number N_0 depends on the desired error. The functions $\xi_j(s)$ are smooth functions in s and are given by

$$\xi_0(s) = f(s) \int_{\{a_{\varepsilon,0}=s\}} \frac{1}{|\nabla a_{\varepsilon,0}|} \, dS_s,$$

$$\xi_j(s) = \sum_{k=1}^{2j-1} \frac{1}{k!} f(s) \partial_s^k \int_{\{a_{\varepsilon,0}=s\}} \frac{d_{\varepsilon,j,k}}{|\nabla a_{\varepsilon,0}|} \, dS_s,$$

where the symbols $d_{\varepsilon,j,k}$ are the polynomials from Lemma II.7.6. In particular we have

$$\xi_1(s) = -f(s)\partial_s \int_{\{a_{\varepsilon,0}=s\}} \frac{a_{\varepsilon,1}}{|\nabla a_{\varepsilon,0}|} \, dS_s.$$

The proof of the theorem is split in two parts. First is the existence of the expansion proven by a stationary phase theorem. Next is the form of the coefficients found by application of the functional calculus developed earlier.

Proof. In order to be in a situation, where we can apply the stationary phase theorem we need to exchange the propagator with the approximation of it constructed in Section II.8. As the construction required auxiliary localisation we need to introduce these. Let θ be in $C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$ such $\operatorname{supp}(\theta) \subset a_{\varepsilon,0}^{-1}((-\eta,\eta))$ and $\theta(x,p) = 1$ for all (x,p) in $\operatorname{supp}(f(a_{\varepsilon,0}))$. Now by Lemma II.7.16 we have

$$\|(1 - \operatorname{Op}_\hbar^{\mathrm{w}}(\theta)) f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}\|_{\mathrm{Tr}} \leq \|(1 - \operatorname{Op}_\hbar^{\mathrm{w}}(\theta)) f(A_\varepsilon(\hbar))\|_{\mathrm{Tr}} \leq C_N \hbar^N, \tag{3.87}$$

for every N in $\mathbb{N}$. Hence we have

$$|\operatorname{Tr}[f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}] - \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}]| \leq C_N \hbar^N,$$

for any N in $\mathbb{N}$. This implies the estimate

$$\begin{aligned} &\int_{\mathbb{R}} \operatorname{Tr}[f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}] e^{-its\hbar} \chi(t) \, dt \\ &\qquad = \int_{\mathbb{R}} \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}] e^{-its\hbar} \chi(t) \, dt + \mathcal{O}(\hbar^N). \end{aligned} \tag{3.88}$$

In order to use the results of Section II.8 we need also to localise in time. To do this we let χ_2 be in $C_0^\infty(\mathbb{R})$ such $\chi_2(t) = 1$ for t in $[-\frac{1}{2}\hbar^{1-\frac{\delta}{2}}, \frac{1}{2}\hbar^{1-\frac{\delta}{2}}]$ and $\operatorname{supp}(\chi_2) \subset [-\hbar^{1-\frac{\delta}{2}}, \hbar^{1-\frac{\delta}{2}}]$. With this function we have

$$\begin{aligned} &\int_{\mathbb{R}} \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}] e^{-its\hbar} \chi(t) \, dt \\ &\qquad = \int_{\mathbb{R}} \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}] e^{-its\hbar} \chi_2(t)\chi(t) \, dt \\ &\qquad + \int_{\mathbb{R}} \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}] e^{-its\hbar} (1-\chi_2(t))\chi(t) \, dt. \end{aligned} \tag{3.89}$$

We will use the notation $\tilde{\chi}(t) = (1 - \chi_2(-t))\chi(-t)$ in the following. If we start with the second term which we will prove is negligible. Before we do so we need an extra localisation. This localisation can be introduced as the first hence if we use the estimate in (3.87) and cyclicity of the trace again we have

$$\operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta)f(A_\varepsilon(\hbar))e^{it\hbar^{-1}A_\varepsilon(\hbar)}] = \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta)f(A_\varepsilon(\hbar))e^{it\hbar^{-1}A_\varepsilon(\hbar)}\operatorname{Op}_\hbar^{\mathrm{w}}(\theta)] + C_N\hbar^N.$$

Now by Corollary II.8.6 we have

$$\begin{aligned}\int_{\mathbb{R}} &\operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta)f(A_\varepsilon(\hbar))e^{it\hbar^{-1}A_\varepsilon(\hbar)}\operatorname{Op}_\hbar^{\mathrm{w}}(\theta)]e^{-its\hbar}\tilde{\chi}(-t)\,dt \\ &= 2\pi\hbar\operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta)f(A_\varepsilon(\hbar))\mathcal{F}_\hbar^{-1}[\tilde{\chi}](s-A_\varepsilon(\hbar))\operatorname{Op}_\hbar^{\mathrm{w}}(\theta)] \\ &\leq \tilde{C}_N\hbar^N,\end{aligned} \tag{3.90}$$

uniformly in s in $[-\eta,\eta]$ and any N in $\mathbb{N}_0$. What remains in (3.89) is the first term. We need to change the positions of the operators $f(A_\varepsilon(\hbar))$ and $e^{it\hbar^{-1}A_\varepsilon(\hbar)}$ but they commute so we can just do it. Here we have to change the quantisation of the localisation. By Corollary II.4.20 we have for any N in $\mathbb{N}_0$

$$\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) = \operatorname{Op}_{\hbar,0}(\theta_0^N) + \hbar^{N+1}R_N(\hbar),$$

where R_N is a bounded operator uniformly in $\hbar$ since θ is a non-rough symbol. Moreover we have

$$\theta_0^N(x,p) = \sum_{j=0}^N \frac{\hbar^j}{j!}\big(-\frac{1}{2}\big)^j(\nabla_x D_p)^j\theta(x,p)$$

If we choose N sufficiently large (greater than or equal to 2) we can exchange $\operatorname{Op}_\hbar^{\mathrm{w}}(\theta)$ by $\operatorname{Op}_{\hbar,0}(\theta_0^N)$ plus a negligible error. We will in the following omit the N on θ_0^N. For the first term on the right hand side in (3.89) we have $|t| \leq \hbar^{1-\frac{\delta}{2}}$. Now by Theorem II.8.1 there exists $U_N(t,\varepsilon,\hbar)$ with integral kernel

$$\begin{aligned}&K_{U_N}(x,y,t,\varepsilon,\hbar) \\ &=\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle}e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)}\sum_{j=0}^N(it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\,dp,\end{aligned}$$

such that

$$\|\hbar\partial_t U_N(t,\varepsilon,\hbar) - iU_N(t,\varepsilon,\hbar)A_\varepsilon(\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\hbar^{N_0}, \tag{3.91}$$

for $|t| \leq \hbar^{1-\frac{\delta}{2}}$ and $U_N(0,\varepsilon,\hbar) = \operatorname{Op}_{\hbar,0}(\theta_0)$. We emphasise that the number N in the operator U_N is dependent on the error N_0. We now have

$$\begin{aligned}&|\operatorname{Tr}[\operatorname{Op}_{\hbar,0}(\theta_0)e^{it\hbar^{-1}A_\varepsilon(\hbar)}f(A_\varepsilon(\hbar))] - \operatorname{Tr}[U_N(t,\varepsilon,\hbar)f(A_\varepsilon(\hbar))]| \\ &= |\operatorname{Tr}[\int_0^t \partial_s(U_N(t-s,\varepsilon,\hbar)e^{is\hbar^{-1}A_\varepsilon(\hbar)}f(A_\varepsilon(\hbar))\,ds]| \\ &= |\operatorname{Tr}[\int_0^t(-(\partial_t U_N)(t-s,\varepsilon,\hbar) + i\hbar^{-1}U_N(t-s,\varepsilon,\hbar)A_\varepsilon(\hbar))e^{is\hbar^{-1}A_\varepsilon(\hbar)}f(A_\varepsilon(\hbar))\,ds]| \\ &\leq \hbar^{-1}\int_0^t \|\hbar\partial_s U_N(s,\varepsilon,\hbar) - iU_N(s,\varepsilon,\hbar)A_\varepsilon(\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \\ &\qquad\qquad \times \|e^{i(t-s)\hbar^{-1}A_\varepsilon(\hbar)}f(A_\varepsilon(\hbar))\|_{\mathrm{Tr}}\,ds \\ &\leq C_N\hbar^{N_0-d},\end{aligned}$$

where we have used (3.91). By combining this with (3.89) and (3.90) we have

$$\begin{aligned}
&\int_{\mathbb{R}} \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}] e^{-its\hbar} \chi_2(t)\chi(t)\, dt \\
&\qquad = \int_{\mathbb{R}} \operatorname{Tr}[U_N(t,\varepsilon,\hbar) f(A_\varepsilon(\hbar))] e^{-its\hbar} \chi_2(t)\chi(t)\, dt + \mathcal{O}(\hbar^N).
\end{aligned} \tag{3.92}$$

Before we proceed we will change the quantisation of $f(A_\varepsilon(\hbar))$. From Theorem II.7.13 we have

$$f(A_\varepsilon(\hbar)) = \sum_{j\geq 0} \hbar^j \operatorname{Op}_\hbar^{\mathrm{w}}(a^f_{\varepsilon,j}),$$

where

$$a^f_{\varepsilon,j} = \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} d_{\varepsilon,j,k} f^{(k)}(a_{\varepsilon,0}), \tag{3.93}$$

the symbols $d_{\varepsilon,j,k}$ are the polynomials from Lemma II.7.6. We choose a sufficiently large N and consider the first N terms of the operator $f(A_\varepsilon(\hbar))$. For each of these terms we can use Corollary II.4.20 and this yields

$$\operatorname{Op}_\hbar^{\mathrm{w}}(a^f_{\varepsilon,j}) = \operatorname{Op}_{\hbar,1}(a^{f,M}_{\varepsilon,j}) + \hbar^{M+1} R_M,$$

where $\hbar^{M+1}R_M$ is a bounded by $C_M\hbar^N$ in operator norm. The symbol $a^{f,M}_{\varepsilon,j}$ is given by

$$a^{f,M}_{\varepsilon,j} = \sum_{j=0}^{M} \frac{\hbar^j}{j!} \big(\frac{1}{2}\big)^j (\nabla_x D_p)^j a^f_{\varepsilon,j}.$$

If we choose N sufficiently large we can exchange $f(A_\varepsilon(\hbar))$ by

$$\operatorname{Op}_{\hbar,1}(\tilde{a}^{f,M}_\varepsilon) := \sum_{j=0}^{N} \hbar^j \operatorname{Op}_{\hbar,1}(a^{f,M}_{\varepsilon,j}),$$

plus a negligible error as $U_N(t,\varepsilon,\hbar)$ is trace class. We will omit the M when writing $\tilde{a}^{f,M}_\varepsilon$. Hence we have the equality

$$\begin{aligned}
&\int_{\mathbb{R}} \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}] e^{-its\hbar} \chi_2(t)\chi(t)\, dt \\
&\qquad = \int_{\mathbb{R}} \operatorname{Tr}[U_N(t,\varepsilon,\hbar) \operatorname{Op}_{\hbar,1}(\tilde{a}^{f,M}_\varepsilon)] e^{-its\hbar} \chi_2(t)\chi(t)\, dt + \mathcal{O}(\hbar^N).
\end{aligned}$$

As we have the non-critical assumption we have by Lemma II.8.3 that the trace in the above expression is negligible for $\frac{1}{2}\hbar^{1-\frac{\delta}{2}} \leq |t| \leq T_0$. Hence we can omit the $\hat{\chi}_2(t)$ in the expression and then we have

$$\begin{aligned}
&\int_{\mathbb{R}} \operatorname{Tr}[\operatorname{Op}_\hbar^{\mathrm{w}}(\theta) f(A_\varepsilon(\hbar)) e^{it\hbar^{-1}A_\varepsilon(\hbar)}] e^{-its\hbar} \chi_2(t)\chi(t)\, dt \\
&\qquad = \int_{\mathbb{R}} \operatorname{Tr}[U_N(t,\varepsilon,\hbar) \operatorname{Op}_{\hbar,1}(\tilde{a}^{f,M}_\varepsilon)] e^{-its\hbar} \chi(t)\, dt + \mathcal{O}(\hbar^N).
\end{aligned} \tag{3.94}$$

The two operators $U_N(t,\varepsilon,\hbar)$ and $\mathrm{Op}_{\hbar,1}(\tilde{a}_\varepsilon^{f,M})$ are both given by kernels and the composition of the operators has the kernel

$$
\begin{aligned}
&K_{U_N(t,\varepsilon,\hbar)\,\mathrm{Op}_{\hbar,1}(\tilde{a}_\varepsilon^{f,M})}(x,y) \\
&\quad= \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\hbar^{-1}\langle x-y,p\rangle} e^{it\hbar^{-1}a_{\varepsilon,0}(x,p)} \sum_{j=0}^{N} (it\hbar^{-1})^j u_j(x,p,\hbar,\varepsilon)\tilde{a}_\varepsilon^{f,M}(y,p)\,dp.
\end{aligned}
$$

We can now calculate the trace and we get

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathrm{Tr}[U_N(t,\varepsilon,\hbar)\,\mathrm{Op}_{\hbar,1}(\tilde{a}_\varepsilon^{f,M})]e^{-its\hbar}\chi(t)\,dt \\
&\quad= \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi(t)e^{it\hbar^{-1}(a_{\varepsilon,0}(x,p)-s)}u(x,p,t,\hbar,\varepsilon)\tilde{a}_\varepsilon^{f,M}(x,p)\,dxdpdt,
\end{aligned}
\tag{3.95}
$$

where

$$
u(x,p,\hbar,\varepsilon) = \sum_{j=0}^{N}(it\hbar^{-1})^j u_j(x,p,t,\hbar,\varepsilon).
$$

In order to evaluate the integral we will need the stationary phase theorem. We will use the theorem in t and one of the p coordinates after using a prober partition of unity according to p. By assumption we have that $|\nabla_p a_\varepsilon| > c$ on the support of θ. Hence we can make a partition Ω_j such that $\partial_{p_j} a_\varepsilon \neq 0$ on Ω_j and with loss of generality we can assume that Ω_j is connected. To this partition we choose a partition of the unit supported on each of the sets Ω_j. When we have localised to each of these sets the calculation will be identical with some indicies changed. Hence we assume that $\partial_{p_1} a_\varepsilon \neq 0$ on the entire support of the integrant. We will now make a change of variables in the integral in the following way:

$$
F : (x,p) \to (X,P) = (x_1,\dots,x_d, a_{\varepsilon,0}(x,p), p_2,\dots,p_d).
$$

This transformation has the following jacobian matrix

$$
DF = \begin{pmatrix} I_d & 0_{d\times d} \\ \nabla_x a_{\varepsilon,0}^t & \nabla_p a_{\varepsilon,0}^t \\ 0_{d-1\times d+1} & I_{d-1} \end{pmatrix},
$$

where I_d is the d-dimensional identity matrix, $\nabla_x a_\varepsilon^t$ and $\nabla_p a_\varepsilon^t$ are the transposed of the respective gradients and the zeros are corresponding matrices with only zeroes and the dimensions indicated in the subscript. We note that

$$
\det(DF) = \partial_{p_1} a_{\varepsilon,0},
$$

which is non zero by our assumptions. Hence the inverse map exists and we will denote it by F^{-1} and the inverse. For the inverse we denote the part that gives p as a function of (X,P) by F_2^{-1}. By a change of variables we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi(t)e^{it\hbar^{-1}(a_{\varepsilon,0}(x,p)-s)}u(x,p,t,\hbar,\varepsilon)\tilde{a}_\varepsilon^{f,M}(x,p)\,dxdpdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi(t)e^{it\hbar^{-1}(P_1-s)} \frac{u(X,F_2^{-1}(X,P),t,\hbar,\varepsilon)\tilde{a}_\varepsilon^{f,M}(X,F_2^{-1}(X,P))}{\partial_{p_1}a_{\varepsilon,0}(X,F_2^{-1}(X,P))}\,dXdPdt,
\end{aligned}
$$

where we have omitted the prefactor $(2\pi\hbar)^{-d}$. If we do the variable change $\tilde{P}_1 = P_1 - s$ we arrive at a situation we we can apply quadratic stationary phase. Hence by stationary phase in the variables $\tilde{P}_1$ and t, (3.88), (3.89), (3.92), (3.94) and (3.95) we get

$$\int_{\mathbb{R}} \mathrm{Tr}[f(A_\varepsilon(\hbar))e^{it\hbar^{-1}A_\varepsilon(\hbar)}]e^{-its\hbar}\chi(t)\,dt = (2\pi\hbar)^{1-d}\Big[\sum_{j=0}^{N_0}\hbar^j\xi_j(s) + \mathcal{O}(\hbar^N)\Big], \tag{3.96}$$

uniformly for s in $(-\eta,\eta)$. This ends the proof of the existence of the expansion.

From the above expression we have that $\xi_j(s)$ are smooth functions in s hence the above expression defines a distribution on $C_0^\infty((-\eta,\eta))$. So in order to find the expressions of the $\xi_j(s)$'s we consider the action of the distribution. We let φ be in $C_0^\infty((-\eta,\eta))$ and consider the expresion

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{Tr}[f(A_\varepsilon(\hbar))e^{it\hbar^{-1}A_\varepsilon(\hbar)}]e^{-its\hbar}\chi(t)\varphi(s)\,dtds. \tag{3.97}$$

Using that f is supported in the pure point spectrum of $A_\varepsilon(\hbar)$ we have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} &\mathrm{Tr}[f(A_\varepsilon(\hbar))e^{it\hbar^{-1}A_\varepsilon(\hbar)}]e^{-its\hbar}\chi(t)\varphi(s)\,dtds \\ &= \mathrm{Tr}[f(A_\varepsilon(\hbar))\int_{\mathbb{R}}\mathcal{F}_1[\chi](\tfrac{s}{\hbar})\varphi(A_\varepsilon(\hbar)-s)\,ds],\end{aligned} \tag{3.98}$$

where we have used Fubini's theorem. That f is supported in the pure point spectrum follows from Theorem II.7.14. If we consider the integral in the right hand side of (3.98) and let ψ be in $C_0^\infty((-2,2))$ such that $\psi(t)=1$ for $|t|\leq 1$ we have

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{F}_1[\chi](\tfrac{s}{\hbar})\varphi(A_\varepsilon(\hbar)-s)\,ds = &\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-its\hbar^{-1}}\chi(t)\psi(s)\varphi(A_\varepsilon(\hbar)-s)\,dsdt \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-its\hbar^{-1}}\chi(t)(1-\psi(s))\varphi(A_\varepsilon(\hbar)-s)\,dsdt.\end{aligned} \tag{3.99}$$

If we consider the last integral on the right hand side of (3.99). Then by the identity

$$\Big(\frac{i\hbar}{s}\Big)^n\partial_t^n e^{-its\hbar^{-1}} = e^{-its\hbar^{-1}},$$

partial integration, the spectral theorem and that the function $(1-\psi(s))$ is support on $|s|\geq 1$, we have that

$$\Big\|\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-its\hbar^{-1}}\chi(t)(1-\psi(s))\varphi(A_\varepsilon(\hbar)-s)\,dsdt\Big\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = C_N\hbar^N, \tag{3.100}$$

for any N in $\mathbb{N}$. Now for the first integral in the right hand side of (3.99) we have by Theorem II.4.17 (Quadratic stationary phase)

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}&e^{-its\hbar^{-1}}\chi(t)\psi(s)\varphi(A_\varepsilon(\hbar)-s)\,dsdt \\ &= 2\pi\hbar\sum_{j=0}^{N}\hbar^j\frac{(i)^j}{j!}\chi^{(j)}(0)\varphi^{(j)}(A_\varepsilon(\hbar)) + \hbar^{N+1}R_{N+1}(\hbar),\end{aligned} \tag{3.101}$$

where we have used that $\psi(0) = 1$ and $\psi^{(j)}(0) = 0$ for all $j \in \mathbb{N}$. Moreover we have from Theorem II.4.17 the estimate

$$|R_{N+1}(\hbar)| \leq c \sum_{l+k=2} \int_{\mathbb{R}} \int_{\mathbb{R}} |\chi^{(N+1+l)}(t) \partial_s^{N+1+k} \psi(s) \varphi(A_\varepsilon(\hbar) - s)| \, dt ds.$$

As the integrants are supported on a compact set the integral will be convergent and since φ is $C_0^\infty(\mathbb{R})$ we have by the spectral theorem

$$\|R_{N+1}(\hbar)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C. \tag{3.102}$$

If we now use that χ is 1 in a neighbourhood of 0 and combine (3.98)–(3.102) we have

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{Tr}[f(A_\varepsilon(\hbar)) e^{it\hbar^{-1} A_\varepsilon(\hbar)}] e^{-its\hbar} \chi(t) \varphi(s) \, dt ds \\ &\qquad = 2\pi\hbar \operatorname{Tr}[f(A_\varepsilon(\hbar)) \varphi(A_\varepsilon(\hbar))] + C_N \hbar^N. \end{aligned} \tag{3.103}$$

Since both f and φ are $C_0^\infty((-\eta, \eta))$ functions we have by Theorem II.7.15 the identity

$$\operatorname{Tr}[f(A_\varepsilon(\hbar)) \varphi(A_\varepsilon(\hbar))] = \frac{1}{(2\pi\hbar)^d} \sum_{j=0}^{N} \hbar^j T_j(f\varphi, A_\varepsilon(\hbar)) + \mathcal{O}(\hbar^{N+1-d}). \tag{3.104}$$

From Theorem II.7.15 we have the exact form of the terms $T_j(f\varphi, A_\varepsilon(\hbar))$, which is given by

$$T_j(f\varphi, A_\varepsilon(\hbar)) = \begin{cases} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f\varphi)(a_{\varepsilon,0}) \, dx dp & j = 0 \\ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a_{\varepsilon,1} (f\varphi)^{(1)}(a_{\varepsilon,0}) \, dx dp & j = 1 \\ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} d_{\varepsilon,j,k} (f\varphi)^{(k)}(a_{\varepsilon,0}) \, dx dp & j \geq 2, \end{cases}$$

where the symbols $d_{\varepsilon,j,k}$ are the polynomials from Lemma II.7.6. If we combine (3.96), (3.103) and (3.104) we get

$$\int_{\mathbb{R}} \xi_j(s) \varphi(s) \, ds = T_j(f\varphi, A_\varepsilon(\hbar)).$$

If we consider $T_0(f\varphi, A_\varepsilon(\hbar))$ we have

$$\begin{aligned} T_0(f\varphi, A_\varepsilon(\hbar)) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f\varphi)(a_{\varepsilon,0}) \, dx dp \\ &= \int_{\mathbb{R}} f(\omega) \varphi(\omega) \int_{\{a_{\varepsilon,0} = \omega\}} \frac{1}{|\nabla a_{\varepsilon,0}|} \, dS_\omega d\omega, \end{aligned}$$

where S_ω is the euclidian surface measure on the surface in $\mathbb{R}_x^d \times \mathbb{R}_p^d$ given by the equation $a_{\varepsilon,0}(x,p) = \omega$. If we now consider $T_j(f\check{\varphi}, A_\varepsilon(\hbar))$ we have

$$\begin{aligned} T_j(f\varphi, A_\varepsilon(\hbar)) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} d_{\varepsilon,j,k} (f\varphi)^{(k)}(a_{\varepsilon,0}) \, dx dp \\ &= \sum_{k=1}^{2j-1} \frac{(-1)^k}{k!} \int_{\mathbb{R}} (f\varphi)^{(k)}(\omega) \int_{\{a_{\varepsilon,0} = \omega\}} \frac{d_{\varepsilon,j,k}}{|\nabla a_{\varepsilon,0}|} \, dS_\omega d\omega \\ &= \sum_{k=1}^{2j-1} \frac{1}{k!} \int_{\mathbb{R}} (f\varphi)(\omega) \partial_\omega^k \int_{\{a_{\varepsilon,0} = \omega\}} \frac{d_{\varepsilon,j,k}}{|\nabla a_{\varepsilon,0}|} \, dS_\omega d\omega, \end{aligned} \tag{3.105}$$

where we in the last equality used partial integration. These equalities implies the stated form of the functions ξ_j. □

We will now fixing some notation which will be useful for the rest of this section. We let χ be a function in $C_0^\infty((-T_0, T_0))$, where the T_0 is a sufficiently small number. The number will be the number T_0 from Corollary II.8.6. We suppose χ is even and $\chi(t) = 1$ for $|t| \leq \frac{T_0}{2}$. We then set

$$\hat{\chi}_1(s) = \frac{1}{2\pi} \int_{\mathbb{R}} \chi(t) e^{-its} \, dt.$$

We assume $\hat{\chi}_1 \geq 0$ and that there is a $c > 0$ such that $\hat{\chi}_1(t) \geq c$ in a small interval around 0. These assumptions can be guaranteed by replacing χ by $\chi * \chi$. With this we set

$$\hat{\chi}_\hbar(s) = \frac{1}{\hbar} \hat{\chi}_1(\tfrac{s}{\hbar}) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \chi(t) e^{its\hbar^{-1}} \, dt.$$

Before we prove a Weyl law we recall a Tauberian theorem from [17, Theorem V–13].

Theorem II.9.2. *Let $\tau_1 < \tau_2$ and $\sigma_\hbar : \mathbb{R} \to \mathbb{R}$ be a family of increasing functions, where $\hbar$ is in $(0, 1]$. Suppose that*

(i) $\sigma_\hbar(\tau) = 0$ for every $\tau \leq \tau_1$.

(ii) $\sigma_\hbar(\tau)$ is constant for $\tau \geq \tau_2$.

(iii) $\sigma_\hbar(\tau) = \mathcal{O}(\hbar^{-n})$ as $\hbar \to 0$, $n \geq 1$ and uniformly with respect to τ in $\mathbb{R}$.

(iv) $\partial_\tau \sigma_\hbar * \hat{\chi}_\hbar(\tau) = \mathcal{O}(\hbar^{-n})$ as $\hbar \to 0$, with the same n as above and uniformly with respect to τ in $\mathbb{R}$.

where $\hat{\chi}_\hbar$ is defined as above. Then we have

$$\sigma_\hbar(\tau) = \sigma_\hbar * \hat{\chi}_\hbar(\tau) + \mathcal{O}(\hbar^{1-d}),$$

as $\hbar \to 0$ and uniformly with respect to τ in $\mathbb{R}$.

We can now formulate and prove a Weyl law for the rough pseudo differential operators.

Theorem II.9.3 (Weyl law). *Let $A_\varepsilon(\hbar)$ be a strongly $\hbar$-ε-admissible operator of regularity $\tau \geq 1$ which satisfies Assumption II.7.1 and there exists a δ in $(0, 1)$ such that $\varepsilon \geq \hbar^{1-\delta}$. Suppose there exists a $\eta > 0$ such $a_{\varepsilon,0}^{-1}((-\infty, \eta])$ is compact, where $a_{\varepsilon,0}$ is the principal symbol of $A_\varepsilon(\hbar)$. Moreover we suppose*

$$|\nabla_p a_{\varepsilon,0}(x, p)| \geq c \quad \text{for all } (x, p) \in a_{\varepsilon,0}^{-1}(\{0\}). \tag{3.106}$$

Then we have

$$|\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(A_\varepsilon(\hbar))] - \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,0]}(a_{\varepsilon,0}(x, p)) \, dx dp| \leq C\hbar^{1-d},$$

for all sufficiently small $\hbar$.

Proof. By the assumption in (3.106) there exists a $\nu > 0$ such

$$|\nabla_p a_{\varepsilon,0}(x,p)| \geq \frac{c}{2} \quad \text{for all } (x,p) \in a_{\varepsilon,0}^{-1}([-2\nu, 2\nu]).$$

More over by Theorem II.7.3 we have that the spectrum of $A_\varepsilon(\hbar)$ is bounded from below uniformly in $\hbar$ and let E denote a number with distance 1 to the bottom of the spectrums. We now take two functions f_1 and f_2 in $C_0^\infty(\mathbb{R})$ such

$$f_1(t) + f_2(t) = 1,$$

for every t in $[E, 0]$, $\text{supp}(f_2) \subset [-\frac{\nu}{4}, \frac{\nu}{4}]$, $f_2(t) = 1$ for t in $[-\frac{\nu}{8}, \frac{\nu}{8}]$ and $f_2(t) = f_2(-t)$ for all t. With these functions we have

$$\text{Tr}[\mathbf{1}_{(-\infty,0]}(A_\varepsilon(\hbar))] = \text{Tr}[f_1(A_\varepsilon(\hbar))] + \text{Tr}[f_2(A_\varepsilon(\hbar))\mathbf{1}_{(-\infty,0]}(A_\varepsilon(\hbar))]. \tag{3.107}$$

For the first term on the right hand in the above equality we have by Theorem II.7.15 that

$$\text{Tr}[f_1(A_\varepsilon(\hbar))] = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_1(a_{\varepsilon,0}(x,p))\, dxdp + \mathcal{O}(\hbar^{1-d}). \tag{3.108}$$

In order to calculate the second term on the right hand side in (3.107) we will study the function

$$\omega \to M(\omega;\hbar) = \text{Tr}[f_2(A_\varepsilon(\hbar))\mathbf{1}_{(-\infty,\omega]}(A_\varepsilon(\hbar))]. \tag{3.109}$$

We have that $M(\omega;\hbar)$ satisfies the three first conditions in Theorem II.9.2. We will use the notation

$$\mathcal{P} = \text{supp}(f_2) \cap \text{spec}(A_\varepsilon(\hbar)),$$

where $\text{spec}(A_\varepsilon(\hbar))$ is the spectrum of the operator $A_\varepsilon(\hbar)$. The function M can be written in the following form

$$M(\omega;\hbar) = \sum_{e_j \in \mathcal{P}} f_2(e_j)\mathbf{1}_{[e_j,\infty)}(\omega),$$

since f_2 is supported in the pure point spectrum of $A_\varepsilon(\hbar)$. This follows from Theorem II.7.14. Let $\hat{\chi}_\hbar$ be defined as above. Then we will consider the convolution

$$(M(\cdot;\hbar) * \hat{\chi}_\hbar)(\omega) = \int_{\mathbb{R}} M(s;\hbar)\hat{\chi}_\hbar(\omega - s)\, ds = \sum_{e_j \in \mathcal{P}} f_2(e_j) \int_{e_j}^{\infty} \hat{\chi}_\hbar(\omega - s)\, ds.$$

If we take a derivative with respect to ω we get

$$\begin{aligned}\partial_\omega(M(\cdot;\hbar) * \hat{\chi}_\hbar)(\omega) &= \sum_{e_j \in \mathcal{P}} f_2(e_j)\hat{\chi}_\hbar(\omega - e_j) \\ &= \frac{1}{2\pi\hbar}\int_{\mathbb{R}} \text{Tr}[f_2(A_\varepsilon(\hbar))e^{it\hbar^{-1}A_\varepsilon(\hbar)}]e^{-it\omega\hbar^{-1}}\chi(t)\, dt,\end{aligned}$$

by the definition of $\hat{\chi}_\hbar$. We get now by Theorem II.9.1 the identity

$$\partial_\omega(M(\cdot;\hbar) * \hat{\chi}_\hbar)(\omega) = \frac{1}{(2\pi\hbar)^d} f_2(\omega)\int_{\{a_{\varepsilon,0}=\omega\}} \frac{1}{|\nabla a_{\varepsilon,0}|}\, dS_\omega + \mathcal{O}(\hbar^{1-d}). \tag{3.110}$$

This verifies the fourth condition in Theorem II.9.2 for $M(\cdot;\hbar)$, hence the Theorem gives the identity

$$\begin{aligned}
&\operatorname{Tr}[f_2(A_\varepsilon(\hbar))\mathbf{1}_{(-\infty,0]}(A_\varepsilon(\hbar))] \\
&\quad= \frac{1}{(2\pi\hbar)^d}\int_{-\infty}^0 f_2(\omega)\int_{\{a_{\varepsilon,0}=\omega\}}\frac{1}{|\nabla a_{\varepsilon,0}|}\,dS_\omega d\omega + \mathcal{O}(\hbar^{1-d}) \\
&\quad= \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_2(a_{\varepsilon,0}(x,p))\mathbf{1}_{(-\infty,0]}(a_{\varepsilon,0}(x,p))\,dxdp + \mathcal{O}(\hbar^{1-d}).
\end{aligned} \tag{3.111}$$

By combining (3.107), (3.108) and (3.111) we get

$$\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(A_\varepsilon(\hbar))] = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}_{(-\infty,0]}(a_{\varepsilon,0}(x,p))\,dxdp + \mathcal{O}(\hbar^{1-d}).$$

This is the desired estimate and this ends the proof. □

II.10 Weyl law for differential operators with irregular coefficients

We now return to the differential operator of the form

$$A(\hbar) = \sum_{|\alpha|,|\beta|\le m}(\hbar D)^\alpha a_{\alpha\beta}(x)(\hbar D)^\beta,$$

where the operator is defined via the associated quadratic form and the order is $2m$. As we saw in Section II.3 we could find framing operators for this type of operators. The aim now is to use the theory that we have just developed to prove a Weyl law for these operators.

Theorem II.10.1. *Let $A(\hbar)$ be a differential operator of order $2m$ of the form*

$$A(\hbar) = \sum_{|\alpha|,|\beta|\le m}(\hbar D)^\alpha a_{\alpha\beta}(x)(\hbar D)^\beta,$$

where the coefficients $a_{\alpha\beta}(x)$ are in $C^{1,\mu}(\mathbb{R}^d)$ for μ in $(0,1]$ and real. We suppose the following conditions on the coeficients are satisfied

(i) *There is a $\gamma_0>0$ such that $\min_{x\in\mathbb{R}^d}(a_{\alpha\beta}(x)) > -\gamma_0$ for all α and β.*

(ii) *There is a $\gamma_1>\gamma_0$ and $C_1, M>0$ such that*

$$a_{\alpha\beta}(x)+\gamma_1 \le C_1(a_{\alpha\beta}(y)+\gamma_1)(1+|x-y|)^M,$$

for all x,y in $\mathbb{R}^d$.

(iii) *For all j in $\{1,\dots,d\}$ there is a $c_j>0$ such that*

$$|\partial_{x_j}a_{\alpha\beta}(x)| \le c_j(a_{\alpha\beta}(x)+\gamma_1).$$

Suppose there exists a constant C_2 such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)p^{\alpha+\beta} \geq C_2|p|^{2m}, \tag{3.112}$$

for all (x,p) in $\mathbb{R}^d_x \times \mathbb{R}^d_p$. Moreover we suppose there is $c>0$ such that

$$|\nabla_p a_0(x,p)| \geq c \quad \text{for all } (x,p) \in a_0^{-1}(\{0\}),$$

where

$$a_0(x,p) = \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)p^{\alpha+\beta}.$$

Lastly we suppose there is a $\nu>0$ such that the set $a_0^{-1}((-\infty,\nu])$ is compact.

Then we have

$$|\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(A(\hbar))] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,0]}(a_0(x,p))\,dxdp| \leq C\hbar^{1-d},$$

for all sufficiently small $\hbar$.

There are quite a number of assumptions in this theorem. We will shortly here discuss them. The assumptions in (*i*), (*ii*) and (*iii*), can be seen assumptions on the behaviour of the coefficients for large values of x. Since in the case of the coefficients having compact support they are all verified. This regularity is need as we will use functional calculus of rough pseudo-differential operators.

Without assumption global ellipticity (3.2) we could easily be in a situation, where we there operator only had essential spectrum. This assumptions is also used to verify properties of the approximating operators. The non-critical assumption (3.3) is essential for our proof to be valid.

Proof. By Proposition II.3.3 we can find two framing operators $A_\varepsilon^-(\hbar)$ and $A_\varepsilon^+(\hbar)$ of the form

$$A_\varepsilon^\pm(\hbar) = A_\varepsilon(\hbar) \pm C_1\varepsilon^{1+\mu}(I-\hbar^2\Delta)^m,$$

where $A_\varepsilon(\hbar)$ is the original operator with the coefficients replaced by $a^\varepsilon_{\alpha\beta}(x)$ which is the smoothed function of $a_{\alpha\beta}(x)$ according to Proposition II.3.2. The proposition also gives for a sufficiently small ε, 0 is a non-critical value for the framing operators and they are globally elliptic. With out loss of generality we can assume ε to be less than or equal to 1.

If we consider the new coefficients $a^\varepsilon_{\alpha\beta}(x)$ the by construction they are given by

$$a^\varepsilon_{\alpha\beta}(x) = \int_{\mathbb{R}^d} a_{\alpha\beta}(x-\varepsilon y)\omega(y)\,dy,$$

where $\omega(y)$ is a real Schwarz function integrating to 1. Hence there exists a sequence of numbers c_n for n in $\mathbb{N}_0$ such that

$$|\omega(y)| \leq \frac{c_n}{(1+|y|)^n}. \tag{3.113}$$

By taking a slightly larger γ than γ_1 in the assumptions we have that $a^\varepsilon_{\alpha\beta}(x)+\gamma$ is positive. Moreover by assumption we have for any x and z in $\mathbb{R}^d$

$$\begin{aligned} a^\varepsilon_{\alpha\beta}(x)+\gamma =& \int_{\mathbb{R}^d} (a_{\alpha\beta}(x-\varepsilon y)+\gamma)\omega(y)\,dy \\ \leq& C_1(a_{\alpha\beta}(z)+\gamma)\int_{\mathbb{R}^d}(1+|x-\varepsilon y-z|)^M\omega(y)\,dy \\ \leq& C_1(a_{\alpha\beta}(z)+\gamma)\int_{\mathbb{R}^d} 2^M[(1+|x-z|)^M+|y|^M]\omega(y)\,dy \\ \leq& C(a^\varepsilon_{\alpha\beta}(z)+a_{\alpha\beta}(z)-a^\varepsilon_{\alpha\beta}(z)+\gamma)(1+|x-z|)^M \\ \leq& \tilde{C}(a^\varepsilon_{\alpha\beta}(z)+\gamma)(1+|x-z|)^M, \end{aligned} \tag{3.114}$$

where we have used

$$|a_{\alpha\beta}(z)-a^\varepsilon_{\alpha\beta}(z)|\leq c\varepsilon^{1+\mu}, \tag{3.115}$$

by Proposition II.3.2 and we have used that $\varepsilon\leq 1$. This calculation verifies that the functions $a^\varepsilon_{\alpha\beta}(x)+\gamma$ are tempered weights and the numbers $\tilde{C}$ and M are both independent of ε in $(0,1]$. Moreover for j in $\{1,\dots,d\}$ we have

$$\begin{aligned} |\partial_{x_j}a^\varepsilon_{\alpha\beta}(x)| =& |\int_{\mathbb{R}^d}\partial_{x_j}a_{\alpha\beta}(x-\varepsilon y)\omega(y)\,dy| \leq \int_{\mathbb{R}^d}|\partial_{x_j}a_{\alpha\beta}(x-\varepsilon y)\omega(y)|\,dy \\ \leq& \int_{\mathbb{R}^d} c_j(a_{\alpha\beta}(x-\varepsilon y)+\gamma_1)|\omega(y)|\,dy \\ \leq& C(a_{\alpha\beta}(x)+\gamma_1)\int_{\mathbb{R}^d}(1+\varepsilon|y|)^M|\omega(y)|\,dy \\ \leq& C_j(a^\varepsilon_{\alpha\beta}(x)+\gamma), \end{aligned} \tag{3.116}$$

where we again have used (3.115). The calculation also shows that C_j is uniform for ε in $(0,1]$. For any η in $\mathbb{N}^d$ with $|\eta|\geq 2$ we have

$$|\partial^\eta_x a^\varepsilon_{\alpha\beta}(x)|\leq c\varepsilon^{1+\mu-|\eta|}\leq c_\alpha\varepsilon^{1-|\eta|}(a^\varepsilon_{\alpha\beta}(x)+\gamma), \tag{3.117}$$

by Proposition II.3.2 with a constant which is uniform for ε in $(0,1]$. All these estimates will prove useful later. If we consider the framing operators $A^-_\varepsilon(\hbar)$ and $A^+_\varepsilon(\hbar)$ they have the form

$$A^\pm_\varepsilon(\hbar) = \sum_{|\alpha|,|\beta|\leq m}(\hbar D)^\alpha a^\varepsilon_{\alpha\beta}(x)(\hbar D)^\beta \pm C_1\varepsilon^{1+\mu}(I-\hbar^2\Delta)^m, \tag{3.118}$$

as the coefficients are smooth we can represent the operators $A^\pm_\varepsilon(\hbar)$ as Weyl quantised rough pseudo differential operators,

$$A^\pm_\varepsilon(\hbar) = \sum_{j=1}^{2m}\hbar^j\,\mathrm{Op}^{\mathrm{w}}_\hbar(a^\pm_{\varepsilon,j}), \tag{3.119}$$

where the principal symbol is

$$a^\pm_{\varepsilon,0}(x,p) = \sum_{|\alpha|,|\beta|\leq m} a^\varepsilon_{\alpha\beta}(x)p^{\alpha+\beta}\pm C_1\varepsilon^{1+\mu}(1+p^2)^m. \tag{3.120}$$

The subprincipal symbol is the terms where 1 derivative have been used on one of the coefficients. The symbol $a_{\varepsilon,2}$ contains the cases where 2 derivatives have been used on the coefficients. This continues for all the symbols. By the estimates in (3.116) and (3.117) gives us that the framing operators satisfies Assumptions II.7.1 and has regularity $\tau = 1$. What remains in order to be able to apply Theorem II.9.3 is the existence of a $\tilde{\nu} > 0$ such that the preimage of $(-\infty, \tilde{\nu}]$ under $a^{\pm}_{\varepsilon,0}$ is compact. By the uniform ellipticity we have that the preimage is compact in p. Hence if we choose $\tilde{\nu} = \frac{\nu}{2}$ and note that as in the proof of Proposition II.3.3 we have the estimate

$$|\sum_{|\alpha|,|\beta|\leq m} (a^{\varepsilon}_{\alpha\beta}(x) - a_{\alpha\beta}(x))p^{\alpha+\beta} \pm C_1\varepsilon^{k+\mu}(1+p^2)^m| \leq C\varepsilon^{k+\mu}, \tag{3.121}$$

since we can assume p to be in a compact set. This implies the inclusion

$$\{(x,p) \in \mathbb{R}^{2d} \,|\, a^{\pm}_{\varepsilon,0}(x,p) \leq \frac{\nu}{2}\} \subseteq \{(x,p) \in \mathbb{R}^{2d} \,|\, |a_0(x,p)| \leq \frac{\nu}{2} + C\varepsilon^{k+\mu}\}.$$

Hence for a sufficiently small ε we have that $\{(x,p) \in \mathbb{R}^{2d} \,|\, a^{\pm}_{\varepsilon,0}(x,p) \leq \frac{\nu}{2}\}$ is compact due to our assumptions. Now by Theorem II.9.3 we get for sufficiently small $\hbar$ and $\varepsilon \geq \hbar^{1-\delta}$ for a positive $\delta \leq \frac{1}{2}$ that

$$|\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(A^{\pm}_{\varepsilon}(\hbar))] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,0]}(a^{\pm}_{\varepsilon,0}(x,p))\,dxdp| \leq C\hbar^{1-d}. \tag{3.122}$$

Here we choose $\delta = \frac{\mu}{1+\mu}$. Now if we consider the following difference between integrals

$$\begin{aligned} &|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,0]}(a^{\pm}_{\varepsilon,0}(x,p))\,dxdp - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,0]}(a_{\varepsilon,0}(x,p))\,dxdp| \\ &\quad \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{[-C\varepsilon^{1+\mu},C\varepsilon^{1+\mu}]}(a_{\varepsilon,0}(x,p))\,dxdp \leq \tilde{C}\varepsilon^{1+\mu}, \end{aligned} \tag{3.123}$$

for ε and hence $\hbar$ sufficiently small. Where we in the last inequality have used the non-critical condition. By combining (3.122) and (3.123) we get

$$|\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(A^{\pm}_{\varepsilon}(\hbar))] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,0]}(a^{(}_{\varepsilon,0}x,p))\,dxdp| \leq C\hbar^{1-d} + \tilde{C}\varepsilon^{1+\mu}\hbar^{-d}. \tag{3.124}$$

If we take $\varepsilon = \hbar^{1-\delta}$ we have that

$$\varepsilon^{1+\mu} = \hbar^{(1+\mu)(1-\delta)} = \hbar.$$

Hence (3.124) with this choice of δ and ε gives the estimate

$$|\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(A^{\pm}_{\varepsilon}(\hbar))] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{(-\infty,0]}(a_{\varepsilon,0}(x,p))\,dxdp| \leq C\hbar^{1-d}. \tag{3.125}$$

Now as the framing operators satisfied the relation

$$A^{-}_{\varepsilon}(\hbar) \leq A(\hbar) \leq A^{+}_{\varepsilon}(\hbar),$$

we get by the min-max-theorem the ralation

$$\operatorname{Tr}[A^{+}_{\varepsilon}(\hbar)] \leq \operatorname{Tr}[A(\hbar)] \leq \operatorname{Tr}[A^{-}_{\varepsilon}(\hbar)].$$

Combining this with (3.125) we get the estimate

$$|\operatorname{Tr}[\mathbf{1}_{(-\infty,0]}(A(\hbar))] - \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}_{(-\infty,0]}(a_{\varepsilon,0}(x,p))\,dxdp| \leq C\hbar^{1-d}. \tag{3.126}$$

Which is the desired estimate and this ends the proof. □

Remark II.10.2. Both Weyl laws have been for the function $\mathbf{1}_{(-\infty.0]}(t)$ applied to the operator. By translations and changing the assumptions slightly we could have taken any number E instead of 0 and we could have considered $\mathbf{1}_{[E_1,E_2]}(t)$ also under a slight change of assumptions.

Remark II.10.3. This is how far we got before i handed in my thesis. One observation to do, is that the method we use here to prove the Weyl law for the irregular differential operators actually also would work if in stead the operator had been a $\hbar$-admissible operator as in [17] perturbed by an irregular potential. Of cause this is only under the right conditions on the $\hbar$-admissible operator and the irregular potential. But to our knowledge this has note been covered before.

Appendix: Multivariate differentiation and Taylor's formula

In this appendix we will recall some results about multivariate differentiation and the multivariate Taylor's formula. We will start with Leibniz's formula:

Theorem II.1.1 (Leibniz's formula). *Let α be in $\mathbb{N}_0^d$. For any $C^{|\alpha|}(\mathbb{R}^d)$ functions f and g it holds*

$$\partial_x^\alpha f(x)g(x) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial_x^\beta f(x)\partial_x^{\alpha-\beta}g(x).$$

A proof of the forumla can eg. be found in [18]. The next result gives a multivariate chain rule for any number of derivatives:

Theorem II.1.2 (Faà di Bruno formula). *Let f be a function from $C^\infty(\mathbb{R})$ and g a function from $C^\infty(\mathbb{R}^d)$. Then for all multi indices α with $|\alpha|\geq 1$ the following formula holds:*

$$\partial_x^\alpha f(g(x)) = \sum_{k=1}^{|\alpha|} f^{(k)}(g(x)) \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ |\alpha_j|>0}} c_{\alpha_1\cdots\alpha_k}\partial_x^{\alpha_1}g(x)\cdots\partial_x^{\alpha_k}g(x),$$

where $f^{(k)}$ is the k'th derivative of f. The second sum should be understod as a sum over all ways to split the multi index α in k non-trivial parts. The numbers $c_{\alpha_1\cdots\alpha_k}$'s are combinatorial constants independent of the functions.

A proof of the Faà di Bruno formula can be found in [4], where they prove the formula in greater generality than stated here. It is also possible to find the constants from their proof, but for our purpose here the exact value of the constants are not important. The next Corollary is the Faà di Bruno formula in the case of a $\mathbb{R}^{2d}$ instead of just $\mathbb{R}^d$. But we need to controle the exact number of derivatives in the first d components hence it is stated separately.

Corollary II.1.3. *Let f be a function from $C^\infty(\mathbb{R})$ and a a function from $C^\infty(\mathbb{R}_x^d \times \mathbb{R}_p^d)$. Then for all multi indices α and β with $|\alpha|+|\beta| \geq 1$ the following formula holds:*

$$\partial_p^\beta \partial_x^\alpha f(a(x,p)) = \sum_{k=1}^{|\alpha|+|\beta|} f^{(k)}(a(x,p)) \sum_{\mathcal{I}_k(\alpha,\beta)} c_{\alpha_1\cdots\alpha_k}^{\beta_1\cdots\beta_k} \partial_p^{\beta_1}\partial_x^{\alpha_1} a(x,p) \cdots \partial_p^{\beta_k}\partial_x^{\alpha_k} a(x,p),$$

where the set $\mathcal{I}_k(\alpha,\beta)$ is defined by

$$\mathcal{I}_k(\alpha,\beta) = \{(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k) \in \mathbb{N}_0^{2kd} \\ \mid \sum_{l=1}^k \alpha_l = \alpha, \sum_{l=1}^k \beta_l = \beta, \max(|\alpha_l|,|\beta_l|) \geq 1 \, \forall l\}.$$

The second sum is a sum over all elements in the set $\mathcal{I}_k(\alpha,\beta)$, the constants $c_{\alpha_1\cdots\alpha_k}^{\beta_1\cdots\beta_k}$ are combinatorial constants independent of the functions and $f^{(k)}$ is the k'th derivative of the function f.

We will only give a short sketch of the proof of this corollary.

Proof (Sketch). We have by the Faà di Bruno formula (Theoram II.1.2) the identity

$$\partial_x^\alpha f(a(x,p)) = \sum_{k=1}^{|\alpha|} f^{(k)}(a(x,p)) \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ |\alpha_j|>0}} c_{\alpha_1\cdots\alpha_k} \partial_x^{\alpha_1} a(x,p) \cdots \partial_x^{\alpha_k} a(x,p) \tag{127}$$

By Leibniz's formula we have

$$\partial_p^\beta \partial_x^\alpha f(a(x,p)) \\ = \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \sum_{k=1}^{|\alpha|} \partial_p^\gamma f^{(k)}(a(x,p)) \sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha \\ |\alpha_j|>0}} c_{\alpha_1\cdots\alpha_k} \partial_p^{\beta-\gamma}[\partial_x^{\alpha_1} a(x,p) \cdots \partial_x^{\alpha_k} a(x,p)]. \tag{128}$$

In order to obtain the form stated in the corollary we need to use the Faà di Bruno formula on the terms

$$\partial_p^\gamma f^{(k)}(a(x,p)), \tag{129}$$

and we need to use Leibniz's formula (multiple times) on the terms

$$\partial_p^{\beta-\gamma}[\partial_x^{\alpha_1} a(x,p) \cdots \partial_x^{\alpha_k} a(x,p)]. \tag{130}$$

If this is done, then by using some algebra the stated form can be obtained. The particular form of the index set $\mathcal{I}_k(\alpha,\beta)$ also follows from this algebra. □

We end this appendix by recalling the multidimensional Taylor's formula just for sake of completeness as it is used multiple times.

Theorem II.1.4 (Taylor's formula). *Let f be in $C^k(\mathbb{R}^d)$; then for x and y in $\mathbb{R}^d$ one has*

$$f(x+y) = \sum_{|\alpha|<k} \frac{y^\alpha}{\alpha!} \partial_x^\alpha f(x) + \sum_{|\alpha|=k} k \frac{y^\alpha}{\alpha!} \int_0^1 (1-s)^{k-1} \partial_x^\alpha f(x+sy)\, ds.$$

A proof of the formula can e.g. be found in [18].

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