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LINKAGE AND CODES ON COMPLETE INTERSECTIONS

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ABSTRACT. This note is meant to be an introduction to cohomological methods and their use in the theory of error-correcting codes.

For an evaluating code on a complete intersection, the dimension is determined by the Koszul complex and a lower bound for the minimal distance is obtained through linkage.

CONTENTS

1. NOTATION AND COHOMOLOGY OF PROJECTIVE SPACE

Let k be an algebraically closed field and let $\mathbb{P}^l = \mathbb{P}^l(k), (l > 1)$ be the projective space over k. The structure sheaf $\mathcal{O}_{\mathbb{P}^l}$ of $\mathbb{P}^l$ is denoted by $\mathcal{O}$, the universal linebundle $\mathcal{O}_{\mathbb{P}^l}(1)$ by $\mathcal{O}(1)$ and tensorpowers of it by $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^l}(d)$. Correspondingly $\mathrm{H}^i(\mathcal{O}(d))$ denotes the cohomology group $\mathrm{H}^i(\mathbb{P}^l, \mathcal{O}(d))$ and $\mathrm{h}^i(\mathcal{O}(d)) = \dim_k(\mathrm{H}^i(\mathcal{O}(d)))$ it's dimension.

Remark 1.1. The following are well-known results on the cohomology of projective space.

i) $\mathrm{H}^0(\mathcal{O}(d)) \approx k[x_0, \ldots, x_l]_{(d)}$, consequently

ii) $\mathrm{h}^0(\mathcal{O}(d)) = \begin{cases} 0 & \text{if} \quad d < 0, \\ \binom{l+d}{l} & \text{if} \quad d \geq 0 \end{cases}$

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iii) $\mathrm{h}^i(\mathcal{O}(d)) = 0 \quad \text{if} \quad 0 < i < l$

iv) (Serre duality.) $\mathrm{H}^i(\mathcal{F}(d)) = \mathrm{H}^{l-i}(\mathcal{F}^\vee(-d-l-1))$ for all coherent sheaves $\mathcal{F}$, in particular

v) $\mathrm{h}^l(\mathcal{O}(d)) = \mathrm{h}(\mathcal{O}(-d-l-1))$.

Later k will be the finite field $\mathbb{F}_q$with q elements.

2. The ideal of a finite set of points in projective space

Let $X = \{P_1, \dots, P_n\}$ be a finite set of distinct points in $\mathbb{P}^l$. Let $S = k[x_0, \dots, x_l]$ be the homogenous coordinate ring of $\mathbb{P}^l$ and let $I \subset S$ be the homogenous ideal of the points $P_1, \dots, P_n$ and let $\mathcal{I} \subset \mathcal{O}$ be the corresponding sheaf of ideals.

Lemma 2.1. *The ring S/I is Cohen-Macaulay and I has a graded free resolution*

$$F_\bullet : \quad 0 \to F_{l-1} \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} I \to 0 \tag{1}$$

correspondingly in terms of sheaves

$$\mathcal{F}_\bullet : \quad 0 \to \mathcal{F}_{l-1} \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_1} \mathcal{F}_0 \xrightarrow{d_0} \mathcal{I} \to 0 \tag{2}$$

Proof. Each point P_i corresponds to a primideal I_{P_i} of height l in the homogenous coordinate ring S. Let I be the homogenous ideal of $P_1, \dots, P_s$, i.e. $I = I_{P_1} \cap \dots \cap I_{P_s}$, accordingly I has height l. Consider any ideal $\tilde{J} \subset S$, the quotient of I by an ideal J with $I \subset J \subset S$. Assume that $\mathrm{ht}(\tilde{J}) = r$ and $\tilde{J}$ is generated by r elements. Then there are two possibilities, namely $r = 0$ or $r = 1$, as $\dim(k[x_0, \dots, x_l]/I) = 1$. If $r = 0$, then $J = I$ and the associated prime ideals of S/I are precisely $I_{P_1}, \dots, I_{P_s}$, all of which has height l and $\tilde{J}$ is unmixed. If $r = 1$, then $\mathrm{ht}\, J = l + 1$, J is a maximal ideal in S and $\tilde{J}$ is trivially unmixed. The unmixedness theorem therefore holds in S/I, and we conclude that this ring is Cohen-Macaulay [M1, Theorem 32]. By the theorem of Auslander Buchsbaum [AB], we finally have that the homological dimension of S/I is l. □

Remark 2.2. Following [L], each F_i in (1) is of the form

$$F_i = \bigoplus (S(-d_{i,j}))^{\alpha_{i,j}}$$

and in terms of sheaves

$$\mathcal{F}_i = \bigoplus (\mathcal{O}(-d_{i,j}))^{\alpha_{i,j}}$$

where $d_{i,j} > 0$ are the degrees of the generators of $\operatorname{Im} d_i$ and $\alpha_{i,j}$ the number of generators in each degree. These numbers are accordingly positive integers.

Example 2.3. (Koszul complex) Let $X \subset \mathbb{P}^l$ be a complete intersection defined by a regular sequence of homogenous polynomials $f_1, \dots, f_l$ of degrees $d_1, \dots, d_l$. Let $S = \mathbb{F}_q[x_0, \dots, x_l]$ and let $E = S(-d_1) \oplus \cdots \oplus S(-d_l)$ be the free graded S-module and consider the S-linear homomorphism of degree 0:

$$s : E \to S$$

$$(g_1, \dots, g_l) \mapsto \sum_{i=1}^{l} g_i f_i$$

with image I, the ideal of X. Then the Koszul complex

$$\Lambda^j E \xrightarrow{d_j} \Lambda^{j-1} E,$$

provides a free resolution of I

$$0 \longrightarrow \Lambda^l E \xrightarrow{d_l} \dots \longrightarrow \Lambda_1 E = E \xrightarrow{s} I \longrightarrow 0 \tag{3}$$

Lemma 2.4. *Let X be a finite set of points in $\mathbb{P}^l$, let $\mathcal{I}$ be the defining sheaf of ideals. Let $\mathrm{H}_X(d)$ denote the Hilbert-function of X. Then for all d, we have*

i) $\mathrm{H}_X(d) = \mathrm{h}^0(\mathcal{O}(d)) - \mathrm{h}^0(\mathcal{I}(d)) = |X| - \mathrm{h}^1(\mathcal{I}(d))$

ii) $\mathrm{h}^i(\mathcal{I}(d)) = 0 \quad \textit{if} \quad 1 < i < l$

iii) $\mathrm{h}^l(\mathcal{I}(d)) = \begin{cases} 0 & \textit{if} \quad -l-1 < d, \\ \binom{-d-1}{l} & \textit{if} \quad -l-1 \geq d \end{cases}$

Proof. As X is assumed finite and therefore of dimension zero, we have that $\mathrm{H}^i(\mathcal{O}_X(d)) = 0$ for $i > 0$. Consider the short exact sequence

$$0 \to \mathcal{I}(d) \to \mathcal{O}(d) \to \mathcal{O}_X(d) \to 0.$$

To prove i) consider the start of the associated long exact cohomology sequence:

$$0 \to \mathrm{H}^0(\mathcal{I}(d)) \to \mathrm{H}^0(\mathcal{O}(d)) \to \mathrm{H}^0(\mathcal{O}_X(d)) \to \mathrm{H}^1(\mathcal{I}(d)) \to \mathrm{H}^1(\mathcal{O}(d)) \to$$

as the last term is zero by Remark 1.1 and $\mathrm{H}^0(\mathcal{O}_X(d))$ is the number of points in X, we obtain i) by taking the alternating sum dimensions.

As for ii) and iii) take $i \geq 2$ and consider the following part of the long exact sequence

$$\to \mathrm{H}^{i-1}(\mathcal{O}_X(d)) \to \mathrm{H}^i(\mathcal{I}(d)) \to \mathrm{H}^i(\mathcal{O}(d)) \to \mathrm{H}^i(\mathcal{O}_X(d)) \to$$

As the two extreme terms are zero, we have that

$$\mathrm{h}^i(\mathcal{I}(d)) = \mathrm{h}^i(\mathcal{O}(d)) \tag{4}$$

which in turn is zero by Remark 1.1 for $1 < i < l$. Finally we have from (4) and by Serre duality (Remark 1.1), that

$$\mathrm{h}^l(\mathcal{I}(d)) = \mathrm{h}^l(\mathcal{O}(d)) = \mathrm{h}^0(\mathcal{O}(-d-l-1))$$

and iii) follows fromRemark 1.1. □

In case X is a complete intersection intersection of l hypersurfaces the Koszul complex of Example 2.3 gives a resolution of I, taking alternating sums of dimensions in degree d in (3), we calculate the dimension in terms of the degrees of the defining equations.

Lemma 2.5. *Let $X \subset \mathbb{P}^l$ be a complete intersection defined by a regular sequence of homogenous polynomials $f_1, \dots, f_l$ of degrees $d_1, \dots, d_l$. Then*

$$\mathrm{H}_X(d) = \sum_{j=1}^{l} (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq n} \binom{d - \sum_{k=1}^{j} d_{i_k} + l}{l} + \binom{d+l}{l} \tag{5}$$

Later it will be of importance that a small number of points impose independent conditions on forms of sufficiently high degree.

Lemma 2.6. *Let V be a finite set in $\mathbb{P}^l$, then for $d \geq |V| - 1$:*

$$\mathrm{H}_V(d) = |V|.$$

Proof. For any m points in $\mathbb{P}^l$ with $m < d$, there exist hypersurfaces of degree d through the first $m-1$ points, but not through the last point. □

Lemma 2.7. *Let X be a finite set of points in $\mathbb{P}^l$, let $\mathcal{I}$ be the defining sheaf of ideals and let $\mathcal{F}.$ be a locally free resolution of $\mathcal{I}$, such that we have the exact sequence:*

$$0 \to \mathcal{F}_{l-1} \to \mathcal{F}_{l-2} \to \cdots \to \mathcal{F}_0 \to \mathcal{I} \to 0.$$

Let $\mathcal{N}_i = \mathrm{Im}(\mathcal{F}_i \mapsto \mathcal{F}_{i-1}) \subset \mathcal{F}_{i-1}$ for $i = 1, \dots, l-1$ and let $\mathcal{N}_0 = \mathcal{I}$. In particular, we have the short exact sequence

$$0 \to \mathcal{F}_{l-1} \to \mathcal{F}_{l-2} \to \mathcal{N}_{l-2} \to 0$$

For all d, we have

$$\mathrm{h}^0(\mathcal{I}(d)) = \sum_{i=0}^{l-1} (-1)^i \, \mathrm{h}^0(\mathcal{F}_i(d)). \tag{6}$$

$$\mathrm{h}^1(\mathcal{I}(d)) = \mathrm{h}^l(\mathcal{F}_{l-1}(d)) - \mathrm{h}^l(\mathcal{F}_{l-2}(d)) + \mathrm{h}^l(\mathcal{N}_{l-2}(d)) \tag{7}$$

Proof. The exact sequence

$$0 \to \mathcal{F}_{l-1}(d) \to \mathcal{F}_{l-2}(d) \to \cdots \to \mathcal{F}_0(d) \to \mathcal{I}(d) \to 0$$

breaks up into a number of short exact sequences, starting with

$$0 \to \mathcal{N}_1(d) \to \mathcal{F}_0(d) \to \mathcal{N}_0(d) = \mathcal{I}(d) \to 0 \tag{8}$$

the general being

$$0 \to \mathcal{N}_{j+1}(d) \to \mathcal{F}_j(d) \to \mathcal{N}_j(d) \to 0 \tag{9}$$

the penultimate being

$$0 \to \mathcal{N}_{l-1}(d) \to \mathcal{F}_{l-2}(d) \to \mathcal{N}_{l-2}(d) \to 0 \tag{10}$$

and finishing with

$$0 \to 0 \to \mathcal{F}_{l-1}(d) \to \mathcal{N}_{l-1}(d) \to 0. \tag{11}$$

Claim: $\mathrm{h}^i(\mathcal{N}_j(d))$=0 for $1 \leq i \leq j$

Proof of claim: Downward induction on j.

The induction starts with $j = l-1$ and we have

$$\mathrm{h}^i(\mathcal{N}_{l-1}(d)) = \mathrm{h}^i(\mathcal{F}_{l-1}(d)) = 0$$

for $1 \leq i \leq l-1$ by Remark 1.1.

For the induction step, let $1 \leq i \leq l-1$ and consider a part the long exact cohomology sequence, starting with 0 by Remark 1.1.

$$0 = \mathrm{H}^i(\mathcal{F}_j(d)) \to \mathrm{H}^i(\mathcal{N}_j(d)) \to \mathrm{H}^{i+1}(\mathcal{N}_{j+1}(d))$$

By induction assumption $\mathrm{H}^{i+1}(\mathcal{N}_{j+1}(d)) = 0$ for $i+1 \leq j+1$, hence $\mathrm{H}^i(\mathcal{N}_j(d)) = 0$ for $i \leq j$.

End proof of claim.

From the general short exact sequence above and the (proved) claim, we have the short exact sequence

$$0 = \mathrm{H}^0(\mathcal{N}_{j+1}(d)) \to \mathrm{H}^0(\mathcal{F}_j(d)) \to \mathrm{H}^0(\mathcal{N}_j(d)) \to \mathrm{H}^1(\mathcal{N}_{j+1}(d)) = 0$$

from which the statement (6) follows by taking alternating sums of dimensions.

To prove statement (7) consider the short exact sequence (8) and part of the associated long exact cohomology sequence:

$$\to \mathrm{H}^1(\mathcal{F}_0(d)) \to \mathrm{H}^1(\mathcal{I}(d)) \to \mathrm{H}^2(\mathcal{N}_1(d)) \to \mathrm{H}^2(\mathcal{F}_0(d)) \to$$

The two extreme terms are zero by Remark 1.1, hence

$$\mathrm{h}^2(\mathcal{N}_1(d)) = \mathrm{h}^1(\mathcal{N}_0(d)) = \mathrm{h}^1(\mathcal{I}(d)).$$

Inductively, using the same argument on (9) , we obtain:

$$\mathrm{h}^{l-1}(\mathcal{N}_{l-2}(d)) = \cdots = \mathrm{h}^2(\mathcal{N}_1(d)) = \mathrm{h}^1(\mathcal{N}_0(d)) = \mathrm{h}^1(\mathcal{I}(d)). \tag{12}$$

Now consider (10) and the top part of the associated long exact cohomology sequence:

$$\begin{array}{ccccccc} & \longrightarrow & \mathrm{H}^{l-1}(\mathcal{F}_{l-2}(d)) & \longrightarrow & \mathrm{H}^{l-1}(\mathcal{N}_{l-2}(d)) & \longrightarrow & \\ \mathrm{H}^l(\mathcal{N}_{l-1}(d)) & \longrightarrow & \mathrm{H}^l(\mathcal{F}_{l-2}(d)) & \longrightarrow & \mathrm{H}^l(\mathcal{N}_{l-2}(d)) & \longrightarrow & 0 \end{array}$$

By Remark 1.1 the first term is zero, taking alternating sum of dimensions and combining with (12) proves statement (7). □

3. Linkage

The idea of linkage is basically, that two closed subschemes $V_1, V_2 \subset \mathbb{P}^l$ are said to be linked if their union $V_1 \cup V_2$ is a complete intersection. More precisely

i) V_1, V_2 are equidimensional without embedded components and without common irreducible components.

*i*i) $V_1 \cup V_2$ is a complete intersection X in $\mathbb{P}^l$

In [PS, Proposition 2.5] there is a useful description of a projective resolution of the ideal $\mathcal{I}_{V_2}$ of V_2 linked to V_1, which we recollect. Let V_1, V_2 be linked and assume that V_1 is locally Cohen-Macaulay and let X be the complete intersection linking V_1 and V_2. Let $\mathcal{F}_\bullet$ be a projective resolution of $\mathcal{O}_X$, for example the Koszul resolution, and let $\mathcal{F}_{1\bullet}$ be a projective resolution of $\mathcal{O}_{V_1}$, i.e. we have two exact sequences:

$$\longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_{i-1} \longrightarrow \dots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{I}_X \longrightarrow 0$$

$$\longrightarrow \mathcal{F}_{1i} \longrightarrow \mathcal{F}_{1i-1} \longrightarrow \dots \longrightarrow \mathcal{F}_{10} \longrightarrow \mathcal{I}_{V_1} \longrightarrow 0$$

where $\mathcal{I}_{V_1}$ is the ideal of V_1 and $\mathcal{I}_X$ is the ideal of X. As $V_1 \subset X$, there is a canonical morphism $\alpha : \mathcal{O}_X \to \mathcal{O}_{V_1}$, which induces a morphism of complexes:

$$\begin{array}{cccccccccccc} \mathcal{F}_\bullet : & \longrightarrow & \mathcal{F}_i & \longrightarrow & \mathcal{F}_{i-1} & \longrightarrow & \dots & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_\bullet} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{F}_{1\bullet} : & \longrightarrow & \mathcal{F}_{1i} & \longrightarrow & \mathcal{F}_{1i-1} & \longrightarrow & \dots & \longrightarrow & \mathcal{F}_{10} & \longrightarrow & 0 \end{array}$$

Let e be the sum af the degrees of the equations of X and let N be the maximal length of the two complexes. Then the morphism of complexes above gives rise to a morphism of complexes:

$$\begin{array}{ccccccccc}
\mathcal{F}_1{}^{\vee}(-e)_\bullet : & 0 & \longrightarrow & \mathcal{F}_1{}^{\vee}_0(-e) & \longrightarrow & \dots & \longrightarrow & \mathcal{F}_1{}^{\vee}_N(-e) & \longrightarrow 0 \\
{\scriptstyle \alpha^{\vee}(-e)_\bullet}\downarrow & & & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{F}^{\vee}(-e)_\bullet : & 0 & \longrightarrow & \mathcal{F}^{\vee}_0(-e) & \longrightarrow & \dots & \longrightarrow & \mathcal{F}^{\vee}_N(-e) & \longrightarrow 0
\end{array}$$

The mapping cone of this morphism $\alpha^{\vee}(-e)_\bullet$ of complexes gives a resolution of the ideal $\mathcal{I}_{V_2}$, i.e. there is an exact sequence:

$$\begin{array}{c}
0 \\
\downarrow \\
\mathcal{F}_1{}^{\vee}_0(-e) \\
\downarrow \\
\mathcal{F}_1{}^{\vee}_1(-e) \oplus \mathcal{F}^{\vee}_0(-e) \\
\downarrow \\
\dots \\
\downarrow \\
\mathcal{F}_1{}^{\vee}_i(-e) \oplus \mathcal{F}^{\vee}_{i-1}(-e) \\
\downarrow \\
\mathcal{F}_1{}^{\vee}_{i+1}(-e) \oplus \mathcal{F}^{\vee}_i(-e) \\
\downarrow \\
\dots \\
\downarrow \\
\mathcal{F}_{N-1}{}^{\vee}_i(-e) \oplus \mathcal{F}^{\vee}_{N-2}(-e) \\
\downarrow \\
\mathcal{I}_{V_2} \\
\downarrow \\
0
\end{array} \tag{13}$$

4. Evaluation codes

From now on $k = \overline{\mathbb{F}}_q$, where $\mathbb{F}_q$ is the finite field with q elements.

The error-correcting codes we consider are codes where the code-words are the values of rational functions at a finite set of k-rational points. Such codes have been treated in [La], [RT1], [RT2] and [H].

Definition 4.1. Let $X = \{P_1, \dots, P_n\} \subset \mathbb{P}^l$ be a finite set of n points. Let $I \subset \mathbb{F}_q[x_0, \dots, x_l]$ be the ideal of X. Let $\mathbb{F}_q[x_0, \dots, x_l]_{(d)}$ be the homogenous polynomials of degree d and let $f_0 \in \mathbb{F}_q[x_0, \dots, x_l]_{(d)}$, such that $f_0(P) \neq 0$ for all $P \in X$.

Then for each $d > 0$ we denote by $C_d \subset {\mathbb{F}_q}^n$ the image

$$C_d = \Phi(\mathbb{F}_q[x_0, \dots, x_l]_{(d)}) \subset {\mathbb{F}_q}^n$$

of the evaluating morphism

$$\Phi : \mathbb{F}_q[x_0, \dots, x_l]_{(d)} \to {\mathbb{F}_q}^n \tag{14}$$

$$f \mapsto \left(\frac{f(P_1)}{f_0(P_1)}, \dots, \frac{f(P_n)}{f_0(P_1)} \right) \tag{15}$$

It is clear from the definition of the Hilbert-function, Lemma 2.4 and Lemma 2.5, that we have the following result

Proposition 4.2. *The dimension of the code C_d is determined by the Hilbert function*

$$\dim C_d = \mathrm{H}_X(d)$$

In particular if $X \subset \mathbb{P}^l$ is a complete intersection defined by a regular sequence of homogenous polynomials $f_1, \dots, f_l$ of degrees $d_1, \dots, d_l$. Then

$$\dim C_d = \sum_{j=1}^{l} (-1)^j \sum_{1 \le i_1 < \dots < i_j \le n} \binom{d - \sum_{k=1}^{j} d_{i_k} + l}{l} + \binom{d + l}{l}$$

Example 4.3. (Reed-Muller codes) Let $X = \mathbb{A}(\mathbb{F}_q) \subset \mathbb{P}^l$ be the $\mathbb{F}_q$-rational points in $\mathbb{P}^l$ with $x_0 \neq 0$. Then X is a complete intersection with defining ideal

$$I = (x_1^q - {x_0}^{q-1} x_1, \dots, x_l^q - {x_0}^{q-1} x_l)$$

From the proposition above, we have

$$\dim C_d = \sum_{j=0}^{l} (-1)^j \binom{l}{j} \binom{d - jq + l}{l}$$

4.1. **Minimal distance and linkage.** A lower bound for the minimal distance for evaluating codes on *complete intersections* are obtained with the help on linkage and the fact that a small number of points impose independent conditions on forms of sufficiently high degree. In [C] similar techniques are used to prove a generalisation of the Cayley-Bacharach theorem.

Theorem 4.4. *Let $l = 2$ and let $X \subset \mathbb{P}^l$ be a complete intersection of two hypersurfaces of degrees d_1, d_2. Let $d > d_i - 3$ for $i = 1, 2$ and let $d \leq d_1 + d_2 - 3$, then the evaluation code C_d has minimal distance $\geq d_1 + d_2 - d - 1$*

Proof. Let $V_1 \subset X$ be any subset of X with $|V_1| > n - (d_1 + d_2 - d - 3)$ and consider the canonical projection

$$C_d \to C(V_1)_d \to 0, \tag{16}$$

where $C(V_1)_d$ is the evaluation code on the points in V_1. We shall prove that the projection in (16) is injective.

Let V_2 be the the set residual to V_1 in X, such that V_1 and V_2 are linked by X.

We have resolutions for the three sheaves of ideals $\mathcal{I}_X(d), \mathcal{I}_{V_1}(d), \mathcal{I}_{V_2}(d)$, namely the Koszul resolution , see Example 2.3:

$$0 \to \mathcal{O}(-(d_1 + d_2)) \to \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \to \mathcal{I}_X \to 0, \tag{17}$$

a locally free resolution of $\mathcal{I}_{V_2}$ as in (2):

$$0 \to \mathcal{F}_{11} \to \mathcal{F}_{10} \to \mathcal{I}_{V_2} \to 0 \tag{18}$$

and the corresponding twisted mapping cone resolution (13) of $\mathcal{I}_{V_1}(d)$:

$$\begin{array}{c} 0 \\ \downarrow \\ \mathcal{F}_{10}^{\vee}(d - (d_1 + d_2)) \\ \downarrow \\ \mathcal{F}_{11}^{\vee}(d - (d_1 + d_2)) \oplus \mathcal{O}(d - d_1) \oplus \mathcal{O}(d - d_2) \\ \downarrow \\ \mathcal{I}_{V_1}(d) \\ \downarrow \\ 0 \end{array} \tag{19}$$

Taking the long exact cohomology sequence og (17) gives:

$$\begin{aligned}\mathrm{h}^1(\mathcal{I}_X(d)) &= \mathrm{h}^2(\mathcal{O}(d-(d_1+d_2))) \\ &= \mathrm{h}^0(\mathcal{O}(d_1+d_2-d-3))\end{aligned} \tag{20}$$

by Remark 1.1 as $d > d_i - 3$.

As $|V_2| - 1 < d_1 + d_2 - d - 3$ the points in V_2 impose according to Lemma 2.6 independent conditions on forms of degree $d_1 + d_2 - d - 3$, i.e.

$$\begin{aligned}\mathrm{h}^0(\mathcal{O}(d_1+d_2-d-3)) - \mathrm{h}^0(\mathcal{I}_{V_2}(d_1+d_2-d-3)) \\ = \mathrm{H}_{V_2}(d_1+d_2-d-3) \\ = |V_2|\end{aligned} \tag{21}$$

Combining (20) and (21) gives together with (6) applied to the resolution (18) that

$$\begin{aligned}\mathrm{h}^1(\mathcal{I}_X(d)) - |V_2| &= \mathrm{h}^0(\mathcal{I}_{V_2}(d_1+d_2-d-3)) \\ &= \mathrm{h}^0(\mathcal{F}_{10}(d_1+d_2-d-3)) \\ &\quad - \mathrm{h}^0(\mathcal{F}_{11}(d_1+d_2-d-3))\end{aligned} \tag{22}$$

On the other hand taking the long exact cohomology sequence of (19) gives

$$\begin{array}{c} \downarrow \\ \mathrm{H}^1(\mathcal{F}_{11}^{\vee}(d-(d_1+d_2)) \oplus \mathcal{O}(d-d_1) \oplus \mathcal{O}(d-d_2)) \\ \downarrow \\ \mathrm{H}^1(\mathcal{I}_{V_1}(d)) \\ \downarrow \\ \mathrm{H}^2(\mathcal{F}_{10}^{\vee}(d-(d_1+d_2)) \oplus \mathcal{O}(d-d_1) \oplus \mathcal{O}(d-d_2)) \\ \downarrow \\ \mathrm{H}^2(\mathcal{F}_{11}^{\vee}(d-(d_1+d_2)) \oplus \mathcal{O}(d-d_1) \oplus \mathcal{O}(d-d_2)) \\ \downarrow \\ \mathrm{H}^2(\mathcal{I}_{V_1}(d)) \\ \downarrow \end{array} \tag{23}$$

The top term is zero by Remark 1.1 and the bottom term is zero by Remark 2.4. Remark 1.1 gives $\mathrm{h}^2(\mathcal{O}(d-d_i)) = \mathrm{h}^0(\mathcal{O}(d_i-d-3)) = 0$

and

$$\begin{aligned} \mathrm{h}^1(\mathcal{I}_{V_1}(d)) &= \mathrm{h}^2(\mathcal{F}_{10}^{\vee}(d-(d_1+d_2)) \\ &\quad - \mathrm{h}^2(\mathcal{F}_{11}^{\vee}(d-(d_1+d_2)) \\ &= \mathrm{h}^0(\mathcal{F}_{10}(d_1+d_2-d-3)) \\ &\quad - \mathrm{h}^0(\mathcal{F}_{11}(d_1+d_2-d-3)) \end{aligned} \tag{24}$$

Finally (22) and (24) gives that

$$\mathrm{h}^1(\mathcal{I}_X(d)) - |V_2| = \mathrm{h}^1(\mathcal{I}_{V_1}(d)) \tag{25}$$

which by Lemma 2.4 implies that

$$\dim C_d = \mathrm{H}_X(d) = \mathrm{H}_{V_1}(d) = \dim C(V_1)_d \tag{26}$$

and therefore the projection in (16) has to be injective. □

Example 4.5. (Reed-Muller codes continued) Let $X = \mathbb{A}(\mathbb{F}_q) \subset \mathbb{P}^2$ be the $\mathbb{F}_q$- rational points in $\mathbb{P}^2$ with $x_0 \neq 0$. Then X is a complete intersection with defining ideal

$$I = (x_2^q - {x_0}^{q-1}x_2, \dots, x_l^q - {x_0}^{q-1}x_l)$$

$q-3 < q \leq 2q-3$, From the theorem above, we have that the minimal distance of C_d is $\geq 2q-d-1$ for $q-3 < d \leq 2q-3$,

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