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ABSTRACT Using principal series Harish-Chandra modules, local cohomology with support in Schubert cells and twisting functors we construct certain modules parametrized by the Weyl group and a highest weight in the subcategory $\mathcal{O}$ of the category of representations of a complex semisimple Lie algebra. These are in a sense modules between a Verma module and its dual. We prove that the three different approaches lead to the same modules. Moreover, we demonstrate that they possess natural Jantzen type filtrations with corresponding sum formulae.

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and Weyl group W. In this paper we consider twisted Verma modules. These are in a sense representations between a Verma module and its dual. Fix a highest weight $\lambda \in \mathfrak{h}^*$. The twisted Verma modules $M^w(\lambda)$ corresponding to λ are parametrized by the Weyl group W. They have the same formal character as the Verma module $M(\lambda)$ (but in general not the same module structure). In the affine Kac-Moody setting these modules (turning out to be Wakimoto modules) have been studied by Feigin and Frenkel [6].

We give three rather different ways of constructing twisted Verma modules. First we obtain them as images of principal series Harish-Chandra modules (under the Bernstein-Gelfand-Joseph-Enright equivalence). In this setting Irving [7] applied wall crossing functors to describe principal series modules in a regular block inductively (Irving uses the term shuffled Verma module for a principal series module in a regular block). His inductive procedure inspired this work.

Let G be a complex semisimple algebraic group with Lie algebra $\mathfrak{g}$, B a Borel subgroup in G, $X = G/B^-$ the flag manifold of G, where B^- denotes the Borel subgroup opposite to B. Let w_0 denote the longest word and e the identity element in the Weyl group W of G. We let $C(w) = BwB^-/B^- \subseteq X$ denote the Schubert cell corresponding to $w \in W$. Notice that $\operatorname{codim} C(w) = \ell(w)$. It is known that the Verma module $M(\lambda)$ with integral highest weight λ can be realized as the top local cohomology group $\mathrm{H}^{\ell(w_0)}_{C(w_0)}(X, \mathcal{L}(w_0 \cdot \lambda))$ of the line bundle $\mathcal{L}(w_0 \cdot \lambda)$ with support in the point $C(w_0)$. The dual Verma module can be realized as the bottom local cohomology group $\mathrm{H}^0_{C(e)}(X, \mathcal{L}(\lambda))$ of the line bundle $\mathcal{L}(\lambda)$ with support in the big cell $C(e)$. Our second construction of twisted Verma modules (which was the starting point of this work) are the intermediate local cohomology groups of the line bundle $\mathcal{L}(w^{-1}\cdot\lambda)$ with support in an arbitrary Bruhat cell $C(w)$ — these are the modules in the global Grothendieck-Cousin complex [12]. The intermediate local cohomology groups $\mathrm{H}^{\ell(w)}_{C(w)}(X, \mathcal{L}(\lambda))$ are isomorphic to dual Verma modules for dominant weights λ. In this case the global Grothendieck-Cousin complex is the dual BGG-resolution. Let us be more precise about the link from local cohomology to principal series modules. Fix a regular antidominant integral weight λ. The principal series modules $M(x, y)$ in the block $\mathcal{O}_\lambda$ (under the Bernstein-Gelfand-Joseph-Enright equivalence) are parametrized by $(x, y) \in W \times W$. Let $C(w) = Bwx_0 \subseteq X$. Then our result says that

$$M(x, y) \cong \mathrm{H}^{\ell(x)}_{C(x)}(X, \mathcal{L}(y \cdot \lambda))$$

as $\mathfrak{g}$-modules. Our isomorphism is constructed using wall translation functors and gives a very explicit algorithm for obtaining the $\mathfrak{g}$-structure of the intermediate local cohomology groups (starting from a Verma module). Notice that the local cohomology approach only makes sense for integral weights.

Following Arkhipov we may for each $w \in W$ define twisting functors T_w of $\mathcal{O}$ (by tensoring with the “semiregular” $U(\mathfrak{g})$-bimodule S_w — see Section 6.1). When applied to a Verma module, T_w produces a twisted Verma module. Again it follows quite easily that the modules obtained in this way satisfy Irving's inductive procedure. This setup is probably the most powerful for studying twisted Verma modules and turns out to be the key for showing that the three approaches are isomorphic: the derived functor LT_w is a self-equivalence of the bounded derived category $D^b(\mathcal{O})$. This implies that twisted Verma modules only have constant $\mathfrak{g}$-endomorphisms (and therefore that they are indecomposable $\mathfrak{g}$-modules). This property allows us to deduce the required isomorphisms between the three approaches.

The twisting functors also give the required deformation theory for constructing Jantzen filtrations and proving sum formulas for twisted Verma modules (which turn out to be twisted versions of the original Jantzen sum formula). At the end of the paper we have used the sum formula to compute the structure of all twisted Verma modules in the B_2-case.

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We are grateful to S. Arkhipov for pointing out the paper [6] of Feigin and Frenkel and for explaining twisting functors to us during his stay in Aarhus, January 2001. We also thank M. Kashiwara and C. Stroppel for discussions that influenced this work.

1 Notation and preliminaries

Fix a *complex semisimple Lie algebra* $\mathfrak{g}$ with a *Cartan subalgebra* $\mathfrak{h}$. Let $R \subseteq \mathfrak{h}^*$ be the *root system* associated with $(\mathfrak{g}, \mathfrak{h})$ and $\mathbb{Z}R$ the *lattice of roots* in $\mathfrak{h}^*$. Fix a basis S of *simple roots* and let R^+ be the *positive roots* with respect to S. Let $\mathfrak{n}^+ = \sum_{\alpha>0} \mathfrak{g}_\alpha$, $\mathfrak{n}^- = \sum_{\alpha>0} \mathfrak{g}_{-\alpha}$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\sigma : \mathfrak{g} \to \mathfrak{g}$ the *Chevalley automorphism* of $\mathfrak{g}$. The positive roots give a *partial order* $\geq$ on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in \mathbb{N}R^+$. The *Weyl group* W of R acts naturally on $\mathfrak{h}^*$. It is generated by *simple reflections* (in the hyperplanes corresponding to the simple roots S). Let $\ell(w)$ denote the *length function* of an element $w \in W$. We let e and w_0 denote the *identity element* and the *unique element of maximal length* in W respectively. Fix a W-invariant positive definite symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ and let $\alpha^\vee$ denote the *dual root* of $\alpha \in R$ with respect to $(\cdot, \cdot)$. The *dot action* of W on $\mathfrak{h}^*$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $2\rho = \sum_{\alpha>0} \alpha$. A *fundamental region* for this action is the set C of *antidominant* weights translated by $-\rho$. Let $\mathrm{Stab}_W(\lambda) \subset W$ denote the *stabilizer subgroup* of $\lambda \in \mathfrak{h}^*$ with respect to the dot-action. A weight $\lambda \in \mathfrak{h}^*$ is called *regular* if $\mathrm{Stab}_W(\lambda) = \{e\}$ and *integral* if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for every $\alpha \in S$. We let C° denote the set of regular weights in C. The *enveloping algebra* associated with a complex Lie algebra L is denoted by $U(L)$.

1.1 The category $\mathcal{O}$

Let V be a (left) $U(\mathfrak{g})$-module. For $\lambda \in \mathfrak{h}^*$, we let $V_\lambda = \{m \in M \mid hm = \lambda(h)m, \text{for every h} \in \mathfrak{h}\} \subseteq \text{V}$ denote the *weight space* of V corresponding to λ. If $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, V is called $\mathfrak{h}$-*diagonalizable*.

A *highest weight vector* of weight λ in V is a non-zero vector in V_λ annihilated by $U(\mathfrak{n}^+)$. A *highest weight module* is a module generated by a highest weight vector. We let $\mathcal{O}$ (see [4]) denote the full subcategory of the category of left $U(\mathfrak{g})$-modules consisting of modules V such that

- V is finitely generated
- V is $\mathfrak{h}$-diagonalizable
- V is $U(\mathfrak{n}^+)$-finite

Any module in $\mathcal{O}$ has a finite filtration with highest weight modules as subquotients. A highest weight module of weight λ is a surjective image of the *Verma module* $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \in \mathcal{O}$. Simple $U(\mathfrak{g})$-modules are parametrized by their highest weight. We let $L(\lambda)$ denote the *simple module* corresponding to the highest weight $\lambda \in \mathfrak{h}^*$. Suppose that $M = \oplus_{\lambda \in \mathfrak{h}^*} M_\lambda \in \mathcal{O}$. Then the linear dual $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is not necessarily $\mathfrak{h}$-diagonalizable. This is remedied by putting $DM = U(\mathfrak{h})$-finite elements in M^*. In fact

$$DM = \oplus_{\lambda\in\mathfrak{h}^*} M^*_\lambda.$$

The $\mathfrak{g}$-module DM is an object of $\mathcal{O}$ after twisting its natural $\mathfrak{g}$-action with σ: $Xf(m) = f(\sigma(X)m)$, where $X \in \mathfrak{g}$ and $f \in DM$. Notice that D then becomes a *duality* of $\mathcal{O}$ fixing simple modules: $DL(\lambda) \cong L(\lambda)$ for $\lambda \in \mathfrak{h}^*$.

We let $\operatorname{ch} V = \sum_{\lambda\in\mathfrak{h}^*} \dim V_\lambda e^\lambda$ denote the *formal character* of an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module V with finite dimensional weight spaces.

Example 1.1. Let V be an $\mathfrak{h}$-diagonalizabe $\mathfrak{g}$-module with finite dimensional weight spaces. If $\operatorname{ch} V = \operatorname{ch} M$ for some $M \in \mathcal{O}$, then $V \in \mathcal{O}$.

Composition factors in Verma modules relate to the dot action by the fundamental result (Harish-Chandra) that $[M(\lambda) : L(\mu)] \neq 0$ implies that $\mu \in W \cdot \lambda$. The category $\mathcal{O}$ decomposes into blocks. We denote for $\lambda \in \mathfrak{h}^*$ by $\mathcal{O}_\lambda$ the *block* consisting of those $M \in \mathcal{O}$ whose composition factors have the form $L(w \cdot \lambda)$, where $w \in W$. Then $\mathcal{O} = \oplus_{\lambda\in C}\mathcal{O}_\lambda$. We let pr_μ denote the *projection* $\mathcal{O} \to \mathcal{O}_\mu$, where $\mu \in \mathfrak{h}^*$. To a pair of weights $\lambda, \mu \in C$, where $\mu - \lambda$ is integrable we have the *translation functor* $T^\mu_\lambda : \mathcal{O}_\lambda \to \mathcal{O}_\mu$. This functor is defined by

$$T^\mu_\lambda(M) = \operatorname{pr}_\mu(M \otimes E), M \in \mathcal{O}_\lambda,$$

where E is the simple finite dimensional $\mathfrak{g}$-module with extremal (integral) weight $\mu - \lambda$. The functors T^μ_λ and T^λ_μ are adjoint.

Definition 1.1. Let $\lambda \in C^\circ$. Pick $\mu \in C$ such that $\mu \in \lambda + \mathbb{Z}R$ and $\operatorname{Stab}_w(\mu) = \{1, s\}$. This defines the functor

$$\theta_s = T^\lambda_\mu \circ T^\mu_\lambda : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$$

called wall translation (translation through the s-wall). Different weights μ with the properties above define naturally isomorphic functors. The morphism $M \to \theta_s(M)$ corresponding to the identity $1 \in \operatorname{Hom}(T^\mu_\lambda(M), T^\mu_\lambda(M))$ under the adjunction isomorphism is called the *adjunction morphism*. We let $C_s(M)$ denote the cokernel of the adjunction morphism. Thus we have a short exact sequence

$$M \to \theta_s(M) \to C_s(M) \to 0.$$

Remark 1.1. The functor C_s is called the shuffling functor in [7].

Remark 1.2. On the level of derived categories the shuffling functor is a shadow of the functor

$$\tilde{C}_s : X \mapsto \operatorname{Con}(X \to \theta_s(X))$$

where $X \in D^b(\mathcal{O}_\lambda)$, Con refers to the mapping cone of a complex and θ_s is extended to complexes in the natural way. If X is concentrated in degree zero and the adjunction morphism for X is injective, then $\tilde{C}_s(X) \cong C_s(X)$. Using the fact that $\theta_s^2 \cong \theta_s + \theta_s$ one may prove that $\tilde{C}_s$ is a self-equivalence of $D^b(\mathcal{O}_\lambda)$. We owe this remark to R. Rouquier.

Remark 1.3. For ease of exposition we will restrict ourselves to only considering representations with integral weights until Section 5.

2 Formal properties of twisted Verma modules

In this section we formalize the properties of twisted Verma modules. We write down a set of properties that characterize twisted Verma modules in a block $\mathcal{O}_\lambda$ up to isomorphism.

2.1 Twisted Verma properties

A family of twisted Verma modules in $\mathcal{O}$ consists of twisted Verma modules in every block $\mathcal{O}_\lambda$, where $\lambda \in C$. A set of twisted Verma modules $M_\lambda(x, y)$ in $\mathcal{O}_\lambda$, where $x, y \in W$ is subject to the following properties (by abuse of notation we write $M(x, y)$ for $M_\lambda(x, y)$)

i. $M(e, e) \cong M(\lambda)$.

ii. $M(x, y) \cong M(xs, sy)$ if $xs > x$ and $sy > y$.

iii. If $\lambda \in C^\circ$ and $ys > y$, where s is a simple reflection, then the adjunction morphism on $M(x, y)$ is injective and fits the short exact sequence

$$0 \to M(x, y) \to \theta_s M(x, y) \to M(x, ys) \to 0$$

for every $x \in W$.

iv. If $\lambda \in C^\circ$ and $\mu \in C$, then

$$T_\lambda^\mu M_\lambda(x, y) \cong M_\mu(x, y).$$

v. If $\lambda \in C^\circ$ then

$$\theta_s M(x, y) \cong \theta_s M(x, ys)$$

if $ys > y$.

Lemma 2.1. *Let $M(x, y)$ be a set of twisted Verma modules in the block $\mathcal{O}_\lambda$, where $x, y \in W$. Then $M(x, e) \cong M(x \cdot \lambda)$ for every $x \in W$ and* $\operatorname{ch} M(x, y) = \operatorname{ch} M(xy \cdot \lambda)$ *for all $x, y \in W$.*

Proof. By Property iv) we may reduce to the case, where $\lambda \in C^0$. By Property i), $M(e,e) \cong M(\lambda)$. Now suppose by induction on $\ell(x)$ that $M(x,e) \cong M(x \cdot \lambda)$. Pick a simple reflection, such that $xs > x$. By Property ii) it follows that $M(xs,s) \cong M(x,e)$. Also by Property v) one gets that $\theta_s M(xs,e) \cong \theta_s M(xs,s)$. By Property iii) $M(xs,e)$ is identified with the kernel of a non-zero homomorphism $\theta_s M(x \cdot \lambda) \to M(x \cdot \lambda)$. This implies that $M(xs,e) \cong M(xs \cdot \lambda)$ ([9], 2.17). The fact that $\mathrm{ch}\, M(x,y) = \mathrm{ch}\, M(xy \cdot \lambda)$ follows from Property iii) and an easy induction on $\ell(y)$ (*cf. loc. cit.*). $\square$

Theorem 2.1. *A family of twisted Verma modules is unique up to isomorphism.*

Proof. By Property iv) it suffices to prove uniqueness of a set of twisted Verma modules in $\mathcal{O}_\lambda$, where $\lambda \in C^\circ$. Let $y = s_1 \dots s_r$ be a reduced decomposition of $y \in W$, then

$$M(x,y) \cong C_{s_r} \dots C_{s_1} M(x \cdot 0)$$

by Lemma 2.1 and Property iii). $\square$

Corollary 2.1. *Let $M(x,y)$ be a set of twisted Verma modules in a block $\mathcal{O}_\lambda$. Then $DM(x,y) \cong M(xw_0, w_0 y)$.*

Proof. We go through the properties for the modules $DM(xw_0, w_0 y)$. We may assume that $\lambda \in C^0$. Property ii) implies that $M(w_0,w_0) \cong M(e,e)$ and hence that $M(w_0,w_0) \cong M(\lambda)$ by Property i). But $M(\lambda) \cong DM(\lambda)$ as $M(\lambda)$ is simple. To verify Property ii), assume that $xs > x$ and $sy > y$. Then write $sw_0 = w_0 t$ for a suitable simple reflection t and therefore $DM(xsw_0, w_0 sy) \cong DM(xw_0 t, tw_0 y) \cong DM(xw_0, w_0 y)$. Suppose that $ys > y$, then Property iii) follows from applying Property v) and dualizing the short exact sequence

$$0 \to M(xw_0, w_0 ys) \to \theta_s M(xw_0, w_0 ys) \to M(xw_0, w_0 y) \to 0.$$

Properties iv) and v) are immediate using that translation commutes with duality D. By Theorem 2.1 we get that $DM(x,y) \cong M(xw_0, w_0 y)$. $\square$

Lemma 2.2. *Suppose that there exists a family of twisted Verma modules in $\mathcal{O}$ admitting only constant $\mathfrak{g}$-endomorphisms. A family of modules satisfying all properties of twisted Verma modules except that we only have a short exact sequence*

$$0 \to M(x,y) \to \theta_s M(x,y) \to M(x,ys) \to 0$$

in Property iii) (without any conditions on the morphisms involved) is a family of twisted Verma modules.

Proof. Let $M'(x,y)$ denote the modules in the family of twisted Verma modules with only constant $\mathfrak{g}$-endomorphisms and $M(x,y)$ the modules in the other family in a regular block $\mathcal{O}_\lambda$. We will prove that $M'(x,y) \cong M(x,y)$ by induction on $\ell(y)$. As

$$\mathrm{End}_{\mathfrak{g}}(M'_\mu(x,y)) = \mathrm{Hom}_{\mathfrak{g}}(T^\mu_\lambda M'_\lambda(x,y), T^\mu_\lambda M'_\lambda(x,y)),$$

we get $\mathrm{Hom}_{\mathfrak{g}}(M'_\lambda(x,y), \theta_s(M'_\lambda(x,y))) = \mathbb{C}$, so that the morphism $M(x,y) \to \theta_s M(x,y)$ in the relaxed Property iii) has to be a constant multiple of the adjunction morphism. By the proof of Lemma 2.1, $M_1(x,e) \cong M(x,e) \cong M(x\cdot\lambda)$ (the morphism $M(x,e) \to \theta_s M(x,e)$ in the relaxed Property iii) has to be a constant multiple of the adjunction morphism). Now suppose that $M_1(x,y) \cong M(x,y)$ and let s be a simple reflection with $ys > y$. Then

$$\mathrm{Hom}_{\mathfrak{g}}(M(x,y), \theta_s M(x,y)) \cong \mathrm{Hom}_{\mathfrak{g}}(M_1(x,y), \theta_s M_1(x,y)) \cong \mathbb{C}$$

and we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1(x,y) & \longrightarrow & \theta_s M_1(x,y) & \longrightarrow & M_1(x,ys) & \longrightarrow & 0 \\
 & & \Big\| & & \Big\| & & & & \\
0 & \longrightarrow & M(x,y) & \longrightarrow & \theta_s M(x,y) & \longrightarrow & M(x,ys) & \longrightarrow & 0
\end{array}$$

giving an isomorphism $M_1(x,ys) \cong M(x,ys)$. □

Remark 2.1. Using Remark 1.2 one may show that a twisted Verma module in a regular block only has constant $\mathfrak{g}$-endomorphisms. We need this result not only in regular blocks but in arbitrary blocks (or at least in (semiregular) blocks $\mathcal{O}_\lambda$, where λ is stabilized by at most one simple reflection). This is where the twisting functor approach is very useful. In the semiregular case C. Stroppel has proved that the principal series modules admit only constant $\mathfrak{g}$-endomorphisms using results of Joseph on completion functors.

3 Principal series Harish-Chandra modules

Here we recall basic properties of and results on Harish-Chandra modules following [7] and [10]. The goal is to prove that principal series Harish-Chandra modules when viewed in $\mathcal{O}$ through the categorical equivalence of Bernstein et. al., form a family of twisted Verma modules. Basically this has been done by Irving [7]. Here we reformulate his results in our setup.

3.1 Definition

Let M be a $\mathfrak{g}\times\mathfrak{g}$-module and view M as a $\mathfrak{g}$-module through the embedding $X \mapsto (X, -\sigma X)$. We let

$$F(M) = \{m \in M \mid \dim U(\mathfrak{g})m < \infty\}.$$

This is a $\mathfrak{g} \times \mathfrak{g}$-submodule of M. A $\mathfrak{g} \times \mathfrak{g}$-module M is called a Harish-Chandra module if $F(M) = M$. We let $\mathcal{H}$ denote the category of Harish-Chandra modules.

3.2 Constructions

Let M and N be $\mathfrak{g}$-modules. Then $\mathrm{Hom}_{\mathbb{C}}(M, N)$ and $(M \otimes_{\mathbb{C}} N)^*$ are $\mathfrak{g} \times \mathfrak{g}$-modules. We let

$$\begin{aligned}\mathcal{L}(M, N) &= F(\mathrm{Hom}_{\mathbb{C}}(M, N))\\ \mathcal{D}(M, N) &= F((M \otimes_{\mathbb{C}} N)^*)\end{aligned}$$

If $M \in \mathcal{O}$, then $\mathcal{D}(M, N) = \mathcal{L}(N, DM)$.

3.3 Principal series modules in $\mathcal{O}_\lambda$

Let $\lambda \in C$ and μ a dominant regular weight such that $\mu - \lambda \in \mathbb{Z}R$. Then

$$M \mapsto \mathcal{L}(M(\mu), M)$$

defines an equivalence of $\mathcal{O}_\lambda$ with a subcategory $\tilde{\mathcal{H}}$ of $\mathcal{H}$. This result is due to Bernstein-Gelfand, Joseph, Enright (see Chapter 6 in [10]). The principal series modules in $\tilde{\mathcal{H}}$ are

$$M(x, y) = \mathcal{D}(M(y \cdot \lambda), M(x^{-1} \cdot \mu))$$

where $x, y \in W$. Via the above equivalence these can be viewed as $\mathfrak{g}$-modules in $\mathcal{O}_\lambda$. To stress this we sometimes use the notation $M_\lambda(x, y)$.

3.4 Twisted Verma properties

In the following example and propositions we show that principal series modules satisfy the properties of twisted Verma modules.

Example 3.1. We have the following chain of isomorphisms (λ and μ as above)

$$\begin{aligned}M(x, e) &= \mathcal{D}(M(\lambda), M(x^{-1} \cdot \mu)) = \mathcal{L}(M(x^{-1} \cdot \mu), DM(\lambda))\\ &= \mathcal{L}(M(x^{-1} \cdot \mu), M(\lambda)) = \mathcal{L}(M(\mu), M(x \cdot \lambda))\end{aligned}$$

where the last equality follows from ([10], 7.23). This shows $M_\lambda(x, e) = M(x \cdot \lambda)$ and that Property i) holds for principal series modules.

Proposition 3.1. *Suppose that $\lambda, \bar{\lambda} \in C$. If $\lambda \in C^\circ$, then*

$$T_\lambda^{\bar{\lambda}} M_\lambda(x, y) \cong M_{\bar{\lambda}}(x, y)$$

for every $x, y \in W$.

Proof. This follows from the corresponding property

$$T_\lambda^{\bar\lambda} M(y\cdot\lambda) \cong M(y\cdot\bar\lambda)$$

for Verma modules and the fact that the translation functor $T_\lambda^{\bar\lambda}$ becomes left translation of Harish-Chandra modules under the equivalence $M \mapsto \mathcal{L}(M(\mu), M)$ (see [10], 6.33). $\square$

This verifies Property iv). The following proposition shows that Property iii) holds.

Proposition 3.2. *Let $\lambda \in C^\circ$. If $ys > y$, then there is an exact sequence*

$$0 \to M(x,y) \to \theta_s M(x,y) \to M(x,ys) \to 0,$$

in $\mathcal{O}_\lambda$, where the first homomorphism is the adjunction map.

Proof. This is ([7], Theorem 2.1). $\square$

The following proposition is Property ii) verbatim.

Proposition 3.3. *Let $\lambda \in C^\circ$. Suppose that $x < xs$ and $sy > y$. Then we have an isomorphism $M(x,y) \cong M(xs, sy)$ in $\mathcal{O}_\lambda$.*

Proof. This is ([7], Theorem 4.4). $\square$

By verifying the five properties of §2.1 we have proved that the principal series modules form a set of twisted Verma modules by Theorem 2.1.

4 Local cohomology

Let G be a complex semisimple algebraic group with Lie algebra $\mathfrak{g}$, $T \subseteq B \subseteq G$ a maximal torus and a Borel subgroup with Lie algebras $\mathfrak{h}$ and $\mathfrak{b}$ respectively. Let $X = G/B^-$ be the flag manifold of G, where B^- is the Borel subgroup opposite to B and let $C(w)$ denote the B-orbit BwB^-/B^- in X. Notice that $\operatorname{codim} C(w) = \ell(w)$. A representation M of B^- induces a G-equivariant vector bundle $\mathcal{L}(M)$ on X. We let $X(B^-) = X(T)$ denote the 1-dimensional representations of B^-. Notice that $X(T)$ can be identified with the integral weights in $\mathfrak{h}^*$. In general a G-linearized sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is naturally a sheaf of $\hat{G}$-modules (where $\hat{G}$ is the formal group of G) ([12], Lemma 11.1) or equivalently a sheaf of $\mathrm{Dist}(G) \cong U(\mathfrak{g})$-modules. The local cohomology group $\mathrm{H}^i_C(X, \mathcal{F})$ has a natural $U(\mathfrak{g})$-module structure for any locally closed subset $C \subseteq X$, where $i \geq 0$ ([12], Lemma 11.1).

For a B^--representation M and a locally closed subset $C \subseteq X$, we let $\mathrm{H}^i_C(M)$ denote the i-th local cohomology group of $\mathcal{L}(M)$ with support in

C with its natural $\mathfrak{g}$-action. By ([12], Lemma 12.8) the local cohomology groups are $\mathfrak{h}$-diagonalizable and

$$\operatorname{ch} \mathrm{H}^{\ell(w)}_{C(w)}(\lambda) = \operatorname{ch} M(w \cdot \lambda).$$

This implies by Example 1.1 that $\mathrm{H}^{\ell(w)}_{C(w)}(\lambda) \in \mathcal{O}$ (it belongs in fact to the block $\mathcal{O}_\lambda$).

4.1 Basic properties of local cohomology

Local cohomology exists only in one degree in the following sense.

Proposition 4.1. *Let V be a vector bundle on X and C an irreducible affinely embedded locally closed subset of X of codimension ℓ. Then*

$$\mathrm{H}^i_C(X, V) = 0 \quad \text{if } i \neq \ell.$$

Proof. On the level of sheaves $\mathcal{H}^i_C(V) = 0$ if $i \neq \ell$, since X is Cohen Macaulay and C irreducible of codimension ℓ. Now one uses the local to global spectral sequence

$$\mathrm{H}^p(X, \mathcal{H}^q_C(V)) \implies \mathrm{H}^{p+q}_C(X, V)$$

and the higher cohomology vanishing $\mathrm{H}^p(X, \mathcal{H}^q_C(V)) = 0, p > 0$, which follows from the assumption that C is affinely embedded, to deduce the result. □

Proposition 4.2. *Let V be a B^- representation, C a locally closed subset of X and E a finite dimensional $\mathfrak{g}$-representation. Then there is an isomorphism*

$$\mathrm{H}^i_C(V \otimes E) \cong \mathrm{H}^i_C(V) \otimes_{\mathbb{C}} E$$

of g-modules for $i \geq 0$.

Proof. We may lift E to a G-representation. On the level of G-sheaves we have an isomorphism $\mathcal{L}(V \otimes_{\mathbb{C}} E) \cong \mathcal{L}(V) \otimes_{\mathbb{C}} E$. This extends to an isomorphism of $\hat{G}$-sheaves giving the desired result. □

4.2 Principal series modules and local cohomology

We emphasize the following important lemma.

Lemma 4.1 (Kashiwara). *Let $\alpha \in S$ be a simple root, $w \in W$ and suppose that $\mu \in X(T)$ with $\langle \mu, \alpha^\vee \rangle \geq -1$ and that $ws_\alpha < w$. Then there is an isomorphism*

$$\mathrm{H}^{\ell(w)}_{C(w)}(\mu) \cong \mathrm{H}^{\ell(w)-1}_{C(ws_\alpha)}(s_\alpha \cdot \mu)$$

of $\mathfrak{g}$-modules.

Proof. This is Lemma 3.6.6 in [11]. □

Fix $\lambda \in C^\circ \cap X(T)$. We will prove that

$$\mathrm{H}^{\ell(x)}_{C(x)}(y \cdot \lambda)$$

satisfies the properties of twisted Verma modules, thereby showing the isomorphism

$$M(x,y) \cong \mathrm{H}^{\ell(x)}_{C(x)}(y \cdot \lambda)$$

between principal series modules in $\mathcal{O}_\lambda$ and local cohomology. Kashiwara's lemma is the key input for proving Property ii). In the above notation it states

Lemma 4.2 (Kashiwara'). *Let $\alpha \in S$ be a simple root and let $x, y \in W$, such that $x < xs_\alpha$ and $s_\alpha y > y$. Then there is an isomorphism*

$$\mathrm{H}^{\ell(x)}_{C(x)}(y \cdot \lambda) \cong \mathrm{H}^{\ell(x)+1}_{C(xs_\alpha)}(s_\alpha y \cdot \lambda)$$

of $\mathfrak{g}$-modules.

The above lemma is the content of Property ii) for local cohomology modules.

Proposition 4.3. *There is an isomorphism*

$$\mathrm{H}^0_{C(e)}(\lambda) \cong DM(\lambda)$$

of $\mathfrak{g}$-modules for any (integral) weight $\lambda \in X(T)$.

Proof. This is Proposition 3.6.2 in [11]. □

The above proposition shows that Property i) holds for local cohomology modules.

4.3 Translation and local cohomology

Proposition 4.4. *Let $0 \to K \to V \to L \to 0$ be an exact sequence of B^- modules. Then we get an exact sequence*

$$0 \to \mathrm{H}^i_{C(w)}(K) \to \mathrm{H}^i_{C(w)}(V) \to \mathrm{H}^i_{C(w)}(L) \to 0$$

of $\mathfrak{g}$-modules for every $i \geq 0$ and $w \in W$.

Proof. This follows from the long exact sequence and Proposition 4.1. □

Let $\Pi(\eta)$ denote the weights in the finite dimensional simple representation with extremal (integral) weight η. We have the following special case of a well known lemma due to Jantzen ([9], 2.9).

Lemma 4.3. *Let $y \in W$ and $\lambda, \mu \in C$, where $\lambda \in C^\circ$ and $\mu - \lambda$ is integral. Then*

$$W \cdot \mu \cap (y \cdot \lambda + \Pi(\mu - \lambda)) = \{y \cdot \mu\}.$$

If $\mathrm{Stab}_W(\mu) = \{1, s\}$, *then*

$$W \cdot \lambda \cap (y \cdot \mu + \Pi(\lambda - \mu)) = \{y \cdot \lambda, ys \cdot \lambda\}.$$

The following proposition shows that local cohomology modules satisfy Properties iii) and iv) of twisted Verma modules with the exception that one only has the short exact sequence in Property iii) (not knowing that the injection is the adjunction morphism). This unpleasant feature is resolved through Lemma 2.2 and the construction of twisted Verma modules using twisting functors (see Sections 6.4 and 6.7).

Proposition 4.5. *Suppose that $y \in W$ and $\lambda, \mu \in C$, where $\lambda \in C^\circ$. Then*

$$T_\lambda^\mu \mathrm{H}^i_{C(w)}(y \cdot \lambda) = \mathrm{H}^i_{C(w)}(y \cdot \mu).$$

If $\mathrm{Stab}_W(\mu) = \{1, s\}$ *and $ys \cdot \lambda > y \cdot \lambda$, then we have a short exact sequence*

$$0 \to \mathrm{H}^i_{C(w)}(y \cdot \lambda) \to T_\mu^\lambda \mathrm{H}^i_{C(w)}(y \cdot \mu) \to \mathrm{H}^i_{C(w)}(ys \cdot \lambda) \to 0$$

for every $i \geq 0$ and $w \in W$.

Proof. We use Proposition 4.2:

$$\begin{aligned} T_\lambda^\mu H^{\ell(w)}_{C(w)}(y \cdot \lambda) &= \mathrm{pr}_\mu (H^{\ell(w)}_{C(w)}(y \cdot \lambda) \otimes_{\mathbb{C}} E) \\ &= \mathrm{pr}_\mu \mathrm{H}^{\ell(w)}_{C(w)}(y \cdot \lambda \otimes E) \end{aligned}$$

where E is the finite dimensional simple module with extremal weight $\mu - \lambda$. Observe that $\mathrm{H}^i_{C(w)}(\eta) \in \mathcal{O}_\eta$ for arbitrary $i \geq 0, w \in W$ and $\eta \in X(T)$. Now take a B^--filtration $N = N_0 \supseteq N_1 \supseteq \dots$ of $N = y \cdot \lambda \otimes E$, such that $N_i/N_{i+1} = \mu_i$ and $i < j \implies \mu_i \not< \mu_j$. Then use Proposition 4.4 and Lemma 4.3 to get the desired result. □

Remark 4.1. Notice that we have proved the duality statement

$$D\mathrm{H}^{\ell(w)}_{C(w)}(X, \mathcal{L}(\lambda)) \cong \mathrm{H}^{\ell(ww_0)}_{C(ww_0)}(X, \mathcal{L}(w_0 \cdot \lambda))$$

of $\mathfrak{g}$-modules for arbitrary integral weights λ and Schubert cells $C(w)$. This follows from Corollary 2.1.

5 Reformulation of formal properties of twisted Verma modules

In this section we reformulate the properties in Section 2 describing a family of twisted Verma modules. This is partly because we want to introduce a new notation which is more natural in the setup in the following sections and partly because we want to generalize to the case of non-integral weights. Of course, the principal series modules considered in Section 3 also exist for non-integral weights (and in fact our definitions and results in Section 3 immediately generalize to this case, see [7]).

We fix an arbitrary weight $\lambda_0 \in \mathfrak{h}^*$ and set $\Lambda = \lambda_0 + \mathbb{Z}R \in \mathfrak{h}^*/\mathbb{Z}R$. Then $R(\lambda_0) = \{\alpha \in R \mid \langle \lambda_0, \alpha^\vee \rangle \in \mathbb{Z}\}$ is a root system with corresponding Weyl group $W(\lambda_0) = \{w \in W \mid w(\lambda_0) - \lambda \in \mathbb{Z}R\}$. A weight $\lambda \in \Lambda$ is called dominant (respectively antidominant) if $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$ (respectively ≤ 0) for all $\alpha \in R(\lambda_0) \cap R^+$.

We define $\mathcal{O}_\Lambda$ to be the subcategory of $\mathcal{O}$ consisting of those M whose weights all belong to Λ.

Definition 5.1. A family of twisted Verma modules in $\mathcal{O}_\Lambda$ is a collection of modules $(M^w(\lambda))$ parametrized by $\lambda \in \Lambda$ and $w \in W(\lambda_0)$ such that $M^w(\lambda) \in \mathcal{O}_\lambda$. It is required to have the following properties

i) $M^e(\lambda') = M(\lambda')$ for some regular antidominant weight $\lambda' \in \Lambda$.

ii) Let $w, y, s \in W(\lambda_0)$, where s is a simple reflection. If $ws > w$ and $w^{-1}y < sw^{-1}y$ then we have an isomorphism $M^w(y \cdot \lambda') \cong M^{ws}(y \cdot \lambda')$.

iii) Let $w, y, s \in W(\lambda_0)$, where s is a simple reflection. If $w^{-1}y > w^{-1}ys$ then we have a short exact sequence

$$0 \to M^w(y \cdot \lambda') \to \theta_s M^w(y \cdot \lambda') \to M^w(ys \cdot \lambda') \to 0.$$

iv) For every antidominant weight $\mu \in \Lambda$ we have $T_{\lambda'}^{\mu} M^w(\lambda) = M^w(\mu)$ for all $w \in W(\lambda_0), \lambda \in W(\lambda_0) \cdot \lambda'$.

In the next section we construct a family of twisted Verma modules and prove that all its modules have 1-dimensional endomorphism rings. Just as in Lemma 2.2 this shows that for any family of twisted Verma modules the first homomorphism in the exact sequence appearing in Property iii) is (up to a nonzero scalar) the adjunction morphism. As in Section 2 this leads to the following results

Theorem 5.1. *There is a unique family of twisted Verma modules in $\mathcal{O}_\Lambda$.*

Corollary 5.1. *If $(M^w(\lambda))_{\lambda \in \Lambda, w \in W(\lambda_0)}$ is a family of twisted Verma modules then $DM^w(\lambda) = M^{ww_0}(\lambda)$ for all $\lambda \in \Lambda, w \in W(\lambda_0)$.*

Remark 5.1. The correspondence between the above concept of a family of twisted Verma modules and the previously considered one is given by

$$M(x,y) = DM^x(xy \cdot \lambda').$$

It is straightforward to get the properties of $M(x,y)$ in 2.1 from the corresponding properties above.

6 Twisting functors

In this section we consider the twisting functors introduced by Arkhipov [2].

6.1 The semiregular modules

Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition of our semisimple complex Lie algebra $\mathfrak{g}$ as in Section 1. Recall that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is the Borel subalgebra corresponding to R^+. We shall write $U = U(\mathfrak{g})$, $N = \mathcal{U}(\mathfrak{n}^-)$ and $B = \mathcal{U}(\mathfrak{b})$.

The natural $\mathbb{Z}R$-grading on $\mathfrak{g}$ (where elements in $\mathfrak{h}$ have degree 0 and elements in $\mathfrak{g}_\alpha$ have degree α, $\alpha \in R$) gives rise to a grading on U,

$$U \cong \bigoplus_{\lambda \in \mathbb{Z}R} U_\lambda.$$

Let $\mathrm{ht} : \mathbb{Z}R \to \mathbb{Z}$ be the $\mathbb{Z}$-linear height function with $\mathrm{ht}(\alpha) = 1$ for all simple roots α. Then we get a $\mathbb{Z}$-grading $U \cong \oplus_{n\in\mathbb{Z}} U_n$, where

$$U_n = \bigoplus_{\mathrm{ht}(\lambda)=n} U_\lambda, \;\; n \in \mathbb{Z}.$$

Note that the subalgebra $N \subseteq U$ is negatively graded with $N_0^- = \mathbb{C}$.

For $w \in W$ we consider the subalgebra $\mathfrak{n}_w = \mathfrak{n}^- \cap w^{-1}(\mathfrak{n}^+)$ of $\mathfrak{n}^-$. The corresponding enveloping algebra $N_w = U(n_w)$ is then a (negatively) graded subalgebra of U with $(N_w)_0 = \mathbb{C}$. Note that $N_e = \mathbb{C}$ and $N_{w_0} = N$.

The (graded) dual of N_w is $N_w^* = \oplus_{n\geq 0}\mathrm{Hom}_{\mathbb{C}}((N_w)_n, \mathbb{C})$. This is a $\mathbb{Z}$-graded bimodule over N_w with $(N_w^*)_n = \mathrm{Hom}_{\mathbb{C}}((N_w)_{-n}, \mathbb{C}), n \in \mathbb{Z}$. The left action of N_w on N_w^* is given by $xf : n \mapsto f(nx), f \in N_w^*, x, n \in N_w$. The right action is defined similarly.

Then we define the corresponding semiregular module S_w by

$$S_w = U \otimes_{N_w} N_w^*.$$

Clearly, S_w is a left U-module and a right N_w-module. It is a non-trivial fact (see the theorem below) that S_w is in fact a U-bimodule. To state the precise result which gives this we first need a little more notation.

Let $e \in \mathfrak{n}^- \setminus \{0\}$. Then we set $U_{(e)} = U \otimes_{\mathbb{C}[e]} \mathbb{C}[e, e^{-1}]$. In particular, we shall consider the case where e is equal to the Chevalley generator $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$, α a simple root. Using this notation we can state

Theorem 6.1. *(Arkhipov [2])*

i) *For each $0 \neq e \in \mathfrak{n}^-$ we have that $U_{(e)}$ is an associative algebra which contains U as a subalgebra. We set $S_e = U_{(e)}/U$.*

ii) *For each simple root α with corresponding simple reflection $s \in W$ we have an isomorphism of left U-modules $S_s \simeq S_{e_\alpha}$.*

iii) *Let $w \in W$ and choose a filtration $\mathfrak{n}_w = F^0 \supset F^1 \supset \cdots \supset F^r \supset 0$ consisting of ideals $F^p \subset \mathfrak{n}^-$ of codimension p, $p = 0, 1, \cdots, r = l(w)$. If $e_p \in F^{p-1} \setminus F^p$ then we have an isomorphism of U-bimodules*

$$S_w \simeq S_{e_1} \otimes_U \cdots \otimes_U S_{e_r}.$$

iv) *(cf. Theorem 1.3 in [13]) For each $w \in W$ we have an isomorphism of right U-modules $S_w \simeq N_w^* \otimes_{N_w} U$.*

6.2 The twisting functors on $\mathcal{O}$

Let $\phi_w \in \mathrm{Aut}(\mathfrak{g})$ denote an automorphism corresponding to $w \in W$. If M is a $\mathfrak{g}$-module we can conjugate the action of $\mathfrak{g}$ on M by ϕ_w. The module obtained in this way we shall denote $\phi_w(M)$. Note that if $\lambda \in \mathfrak{h}^*$ then we have $\phi_w(M)_\lambda = M_{w(\lambda)}$.

Following Arkhipov [2] we define now a twisting functor T_w on the category of $\mathfrak{g}$-modules by

$$T_w M = \phi_w(S_w \otimes_U M).$$

Remark 6.1. i) It is clear from the definition that T_w is a right exact functor for all $w \in W$.

ii) Theorem 6.1 iii) shows that we have $T_{ws} = T_w \circ T_s$ whenever s is a simple reflection for which $ws > w$.

We shall now consider the composite of the twisting functor with induction from the subalgebra B.

Let E be a left B-module and set $T_w^B E = T_w(U \otimes_B E)$. Using Theorem 6.1 iv) and the fact that $U = N \otimes B$ we se that we may identify $T_w^B E$ with $\phi_w(N_w^* \otimes_{N_w} N \otimes E)$ (as vector spaces and as $\mathfrak{h}$-modules). Here and elsewhere $\otimes$ without a subscript denotes tensor product over $\mathbb{C}$.

Proposition 6.1. *Let $w \in W$.*

i) *The functor T_w^B is exact.*

ii) $\mathrm{ch}\, T_w M(\lambda) = \mathrm{ch}\, T_w^B \lambda = \mathrm{ch}\, M(w \cdot \lambda)$ *for all $\lambda \in \mathfrak{h}^*$.*

Proof. i) follows from the above by observing that N is free over N_w. Also ii) follows from the above identification via Theorem 6.1 i) and an easy induction on w. □

As an immediate consequence of Proposition 6.1 ii) we get

Corollary 6.1. *The functor T_w restricts to a functor from $\mathcal{O}$ to $\mathcal{O}$ and it preserves $\mathcal{O}_\lambda$ for all $\lambda \in \mathfrak{h}^*$.*

6.3 The tensor identity

The following tensor identity will be important in the following

Proposition 6.2. *Let $w \in W$ and suppose M and V are U-modules with V finite dimensional. Then we have a natural isomorphism $T_w(M \otimes V) \simeq (T_w M) \otimes V$.*

Likewise, if E is a B-module then $T_w^B(E \otimes V) \simeq (T_w^B E) \otimes V$.

Proof. Note that by Remark 6.1 ii) we may reduce to the case where $w = s$ for some simple reflection s. Since $\phi_s(V) \simeq V$ what we need to prove is $S_s \otimes_U (M \otimes V) \simeq (S_s \otimes_U M) \otimes V$. But this is clear from Theorem 6.1 i).

The last statement follows from the first by noting that the tensor identity for induction ensures that we have an isomorphism $U \otimes_B (E \otimes V) \simeq (U \otimes_B E) \otimes V$. □

Corollary 6.2. *Let $w \in W$ and let $\lambda, \mu \in C$. Then T_w commutes with T_μ^λ.*

Proof. Let $M \in \mathcal{O}$ and write $M = \bigoplus_\lambda \mathrm{pr}_\lambda M$. Since T_w preserves $\mathcal{O}_\lambda$ (Corollary 6.1) it follows that $T_w(\mathrm{pr}_\lambda M) = \mathrm{pr}_\lambda(T_w(M))$. Combining this with Proposition 6.2 we get the statement. □

6.4 Twisted Verma modules

Let $\lambda \in \mathfrak{h}^*$ and $w \in W$. Then we define the twisted Verma module $M^w(\lambda)$ by

$$M^w(\lambda) = T_w M(w^{-1} \cdot \lambda) = T_w^B(w^{-1} \cdot \lambda).$$

Theorem 6.2. *Let $\Lambda = \lambda_0 + \mathbb{Z}R \in \mathfrak{h}^*/\mathbb{Z}R$. Then $(M^w(\lambda))_{\lambda \in \Lambda, w \in W(\lambda_0)}$ is a family of twisted Verma modules (in the sense of Definition 5.1).*

Proof. Note that $M^w(\lambda) \in \mathcal{O}_\lambda$ by Corollary 6.1. We shall verify properties i), iii) and iv) in Definition 5.1 leaving ii) for later.

The above definition gives $M^e(\lambda) = M(\lambda)$ for all λ. So property i) is certainly satisfied. Via Corollary 6.2 we see that property iii) is a consequence of the corresponding fact for ordinary Verma modules.

Let now the notation and assumptions be as in Definition 5.1 iv). The well known effect of the wall crossing functor θ_s on ordinary Verma modules gives the short exact sequence

$$0 \to M(w^{-1}y \cdot \lambda') \to \theta_s M(w^{-1}y \cdot \lambda') \to M(w^{-1}ys \cdot \lambda') \to 0.$$

Applying the twisting functor T_w to this sequence we get (using Corollary 6.2 on the middle term)

$$0 \to M^w(y \cdot \lambda') \to \theta_s M^w(y \cdot \lambda') \to M^w(ys \cdot \lambda') \to 0.$$

The exactness of this sequence comes from Proposition 6.1 i). □

6.5 The $\mathfrak{sl}_2$-case

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with the usual basis $\{f, h, e\}$. Then $\mathfrak{n}^- = \mathbb{C}f, \mathfrak{h} = \mathbb{C}h$ and $\mathfrak{b} = \mathfrak{h} + \mathbb{C}e$. For $\lambda \in \mathfrak{h}^* = \mathbb{C}$ the Verma module $M(\lambda)$ is simple unless $\lambda \in \mathbb{N}$. On the other hand, when $\lambda \in \mathbb{N}$ we have an exact sequence

$$0 \to M(-\lambda - 2) \to M(\lambda) \to L(\lambda) \to 0.$$

In this case there is only one non-trivial element s in W. It is easy to see that $M^s(\lambda) = DM(\lambda)$. Hence $M^s(\lambda)$ is simple for $\lambda \notin \mathbb{N}$ and for $\lambda \in \mathbb{N}$ we have an exact sequence

$$0 \to L(\lambda) \to M^s(\lambda) \to M^s(-\lambda - 2) \to 0.$$

Combining the above two sequences we get the following four term exact sequence (still assuming $\lambda \in \mathbb{N}$)

$$0 \to M(-\lambda - 2) \to M(\lambda) \to M^s(\lambda) \to M^s(-\lambda - 2) \to 0.$$

Note that $M^s(-\lambda - 2) = L(-\lambda - 2) = M(-\lambda - 2)$.

6.6 Twist and induction

Returning to the general case we pick $\lambda \in \mathfrak{h}^*$ and fix a simple root α. We denote by $\mathfrak{p}_\alpha$ the minimal parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ corresponding to α. Then the $\mathfrak{p}_\alpha$-Verma module with highest weight λ is

$$M_\alpha(\lambda) = U(\mathfrak{p}_\alpha) \otimes_B \lambda.$$

In analogy with the above we can also define an s_α-twisted Verma module for $\mathfrak{p}_\alpha$ with highest weight λ, namely

$$M_\alpha^{s_\alpha}(\lambda) = \phi_{s_\alpha}(U(\mathfrak{p}_\alpha) \otimes_{N_{s_\alpha}} N_{s_\alpha}^* \otimes_{U(\mathfrak{p}_\alpha)} M_\alpha(s_\alpha \cdot \lambda))$$

Note that $N_{s_\alpha} = U(\mathfrak{g}_{-\alpha}) = \mathbb{C}[e_{-\alpha}]$.

Considered as a module for the Levi subalgebra in $\mathfrak{p}_\alpha$ we have that $M_\alpha^{s_\alpha}(\lambda)$ is dual to $M_\alpha(\lambda)$. In particular we get therefore as in the $\mathfrak{sl}_2$-case:

Lemma 6.1. *i) If* $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N}$ *then we have an isomorphism of* $\mathfrak{p}_\alpha$*-modules* $M_\alpha(\lambda) \simeq M_\alpha^{s_\alpha}(\lambda)$.

ii) If $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{N}$ *then we have an exact sequence of* $\mathfrak{p}_\alpha$*-modules*

$$0 \to M_\alpha(s_\alpha \cdot \lambda) \to M_\alpha(\lambda) \to M_\alpha^{s_\alpha}(\lambda) \to M_\alpha^{s_\alpha}(s_\alpha \cdot \lambda) \to 0.$$

The two extreme terms in this sequence are isomorphic.

In the above notation we clearly have

$$M(\lambda) = U \otimes_{U(\mathfrak{p}_\alpha)} M_\alpha(\lambda)$$

for all $\lambda \in \mathfrak{h}^*$. We claim that we have a similar transitivity result for twisted Verma modules, namely

$$M^{s_\alpha}(\lambda) = U \otimes_{U(\mathfrak{p}_\alpha)} M_\alpha^{s_\alpha}(\lambda).$$

This follows directly from the definitions: $M^{s_\alpha}(\lambda) = \phi_{s_\alpha}(S_{s_\alpha} \otimes_U M(s_\alpha \cdot \lambda)) = \phi_{s_\alpha}(U \otimes_{N_{s_\alpha}} N_{s_\alpha}^* \otimes_U U \otimes_B s_\alpha \cdot \lambda)) = \phi_{s_\alpha}(U \otimes_{U(\mathfrak{p}_\alpha)} U(\mathfrak{p}_\alpha) \otimes_{N_{s_\alpha}} N_{s_\alpha}^* \otimes_B s_\alpha \cdot \lambda) = U \otimes_{U(\mathfrak{p}_\alpha)} \phi_{s_\alpha}(U(\mathfrak{p}_\alpha) \otimes_{N_{s_\alpha}} N_{s_\alpha}^* \otimes_B s_\alpha \cdot \lambda) = U \otimes_{U(\mathfrak{p}_\alpha)} M_\alpha^{s_\alpha}(\lambda)$.

When we combine this with the results Lemma 6.1 we get

Lemma 6.2. *i) If* $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N}$ *then we have an isomorphism of* $\mathfrak{g}$*-modules* $M(\lambda) \simeq M^{s_\alpha}(\lambda)$.

ii) If $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{N}$ *then we have an exact sequence of* $\mathfrak{g}$*-modules*

$$0 \to M(s_\alpha \cdot \lambda) \to M(\lambda) \to M^{s_\alpha}(\lambda) \to M^{s_\alpha}(s_\alpha \cdot \lambda) \to 0.$$

Finally, let $w \in W$ and suppose $ws_\alpha > w$. Then $\beta = w(\alpha) \in \mathbb{R}^+$. By Remark 6.1 ii) we have $T_{ws_\alpha} = T_w \circ T_{s_\alpha}$ and hence by applying T_w the results in the above lemma we find (replacing λ by $w^{-1} \cdot \lambda$)

Proposition 6.3. *i) If* $\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N}$ *then* $M^w(\lambda) \simeq M^{ws_\alpha}(\lambda)$.

ii) If $\langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{N}$ *then we have an exact sequence of* $\mathfrak{g}$*-modules*

$$0 \to M^w(s_\beta \cdot \lambda) \to M^w(\lambda) \to M^{ws_\alpha}(\lambda) \to M^{ws_\alpha}(s_\beta \cdot \lambda) \to 0.$$

The fact that T_w preserves the exactness of the sequence in Lemma 6.2 is a consequence of Proposition 6.1 i).

Remark 6.2. i) Let the notation and assumptions be as in Definition 5.1 ii) and let α be the simple root with reflection s. Set $\beta = w(\alpha)$. Then the assumption $w^{-1}y < sw^{-1}y$ is equivalent to $w^{-1}y \cdot \lambda' < sw^{-1}y \cdot \lambda'$, i.e $\langle y(\lambda' + \rho), \beta^\vee \rangle = \langle w^{-1}y(\lambda' + \rho), \alpha^\vee \rangle < 0$. Hence i) in this proposition gives $M^w(y \cdot \lambda') \simeq M^{ws_\alpha}(y \cdot \lambda')$. We have thus completed the proof of Theorem 6.2.

ii) It is possible to derive the four term exact sequence in this proposition from the formal properties of a family of twisted Verma modules, see [7]. Here we have taken the opportunity to derive it directly from the $\mathfrak{sl}_2$-case. The sequence is sometimes called the Zelebenko-Duflo-Joseph sequence.

6.7 Endomorphisms

The twisting functor T_w on $\mathcal{O}$ extends to the functor LT_w on the derived category $\mathcal{D}^b(\mathcal{O})$. A key property of this functor is

Proposition 6.4. *(Arkhipov [3]) Let $w \in W$. The derived functor LT_w is an autoequivalence on $D^b(\mathcal{O})$.*

Corollary 6.3. *Let $\lambda \in \Lambda$ and $w \in W(\lambda_0)$. Then* $\mathrm{End}_{\mathcal{O}}(M^w(\lambda)) = \mathbb{C}$.

Proof. The result is well known for ordinary Verma modules. It then follows for twisted Verma modules by the above proposition. In fact, we get $\mathrm{End}_{\mathcal{O}}(M^w(\lambda)) = \mathrm{End}_{\mathcal{O}}(T_w M(w^{-1} \cdot \lambda)) = \mathrm{End}_{\mathcal{D}^b(\mathcal{O})}(LT_w M(w^{-1} \cdot \lambda)) = \mathrm{End}_{\mathcal{D}^b(\mathcal{O})}(M(w^{-1} \cdot \lambda)) = \mathrm{End}_{\mathcal{O}}(M(w^{-1} \cdot \lambda)) = \mathbb{C}$. $\square$

Remark 6.3. This corollary is essential for our arguments in Section 2 proving the uniqueness of a family of twisted Verma modules, see Lemma 2.2 . More precisely, we need for λ regular and s a simple reflection that $\mathrm{Hom}_{\mathfrak{g}}(M^w(\lambda), \theta_s M^w(\lambda)) = \mathbb{C}$. But this Hom-space equals

$$\mathrm{End}_{\mathfrak{g}}(T_\lambda^\mu M^w(\lambda)) = \mathrm{End}_{\mathfrak{g}}(M^w(\mu))$$

by property iv) in Definition 5.1. In other words we need the corollary for semi-regular weights. (In the Harish-Chandra module situation this is exactly the case handled by C. Stroppel, see Remark 2.1).

7 Filtrations and sum formulae

In this section we shall show that the twisted Verma modules considered in the previous sections have Jantzen type filtrations and we shall give the corresponding sum formulae.

7.1 Deformations

Let $A = \mathbb{C}[X]_{(X)}$ be the localization of the polynomial ring $\mathbb{C}[X]$ in the maximal ideal generated by X. We shall then consider the Lie algebra $\mathfrak{g}_A = \mathfrak{g} \otimes_{\mathbb{C}} A$ over A. Similarly, we set $\mathfrak{h}_A = \mathfrak{h} \otimes_{\mathbb{C}} A, \mathfrak{b}_A = \mathfrak{b} \otimes_{\mathbb{C}} A$, etc.

If $\lambda \in \mathfrak{h}_A^* = \mathrm{Hom}_A(\mathfrak{h}_A, A) \simeq \mathfrak{h}^* \otimes_{\mathbb{C}} A$ we have a Verma module over A $M_A(\lambda) = U_A \otimes_{B_A} \lambda$ where $U_A = U(\mathfrak{g}_A)$ and $B_A = U(\mathfrak{b}_A)$. Note that $M_A(\lambda)$

is free over A and that if $A \to \mathbb{C}$ is the specialization which takes X into 0 then we have $M_A(\lambda) \otimes_A \mathbb{C} \simeq M(\bar{\lambda})$. Here $\bar{\lambda} = \lambda \otimes 1 \in \mathfrak{h}_A^* \otimes_A \mathbb{C} = \mathfrak{h}^*$.

The twisting functors T_w, $w \in W$ may also be defined over A. We just extend scalars from $\mathbb{C}$ to A, i.e if M is a U_A-module we set $T_w M = \phi_w(S_w^A \otimes_{U_A} M)$ with $S_w^A = S_w \otimes_{\mathbb{C}} A$.

In particular, this allows us to define twisted Verma modules over A

$$M_A^w(\lambda) = T_w(M_A(w^{-1} \cdot \lambda)),$$

$\lambda \in \mathfrak{h}_A^*$, $w \in W$. These modules specialize to the twisted Verma modules $M^w(\bar{\lambda})$ considered in the previous section.

7.2 The $\mathfrak{sl}_2$-case revisited

Consider again $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then the Verma module $M_A(\lambda)$ for $\mathfrak{g}_A = \mathfrak{sl}_2(A)$ with highest weight $\lambda \in h_A^* = A$ has basis $\{v_0, v_1, \dots\}$ with v_0 a generator for $M_A(\lambda)_\lambda$ and $v_i = f^{(i)} v_0, i \geq 0$. Here $f^{(i)} = f^i/i! \in U_A$. The action of $\mathfrak{g}_A$ on $M_A(\lambda)$ is given by

$$hv_i = (\lambda - 2i)v_i, \;\; fv_i = (i+1)v_{i+1}, \;\; ev_i = (\lambda + 1 - i)v_{i-1}, i \geq 0.$$

for all $i \geq 0$ (we set $v_{-1} = 0$).

The twisted Verma module $M_A^s(\lambda)$ is dual to $M_A(\lambda)$. So if $\{v_0^*, v_1^*, \dots\}$ denotes the dual basis it is immediate to check that the linear map $\varphi_\lambda : M_A(\lambda) \to M_A^s(\lambda)$ given by

$$\varphi_\lambda(v_i) = \binom{\lambda}{i} v_i^*, i = 0, 1, \dots$$

is a $\mathfrak{g}_A$-homomorphism which generates $\mathrm{Hom}_{\mathfrak{g}_A}(M_A(\lambda), M_a^s(\lambda)) \cong A$.

Note that if $\lambda \notin \mathbb{N}$ then the elements $\binom{\lambda}{i} \in A$ are units in A for all i. Hence in this case φ_λ is an isomorphism. Define now $\psi_\lambda : M^s(\lambda) \to M(\lambda)$ to be the inverse of φ_λ when $\lambda \notin \mathbb{N}$ and

$$\psi_\lambda(v_i^*) = \begin{cases} 0, & \text{if } i \leq \lambda \\ (-1)^i \binom{i}{i-\lambda-1} v_i, & \text{if } i \geq \lambda + 1 \end{cases}$$

when $\lambda \in \mathbb{N}$.

It is easy to check that ψ_λ is a generator of $\mathrm{Hom}_{\mathfrak{g}_A}(M_A^s(\lambda), M_A(\lambda))$. When we pass to the specialization $A \to \mathbb{C}$ we have of course still that $\psi_{\bar{\lambda}}$ is an isomorphism when the specialization $\bar{\lambda}$ of λ is not in $\mathbb{N}$. If $\bar{\lambda} \in \mathbb{N}$ we get in analogy with the sequence involving $\varphi_{\bar{\lambda}}$ a four term exact sequence

$$0 \to M(\bar{\lambda})/M(-\bar{\lambda}-2) \to M^s(\bar{\lambda}) \to M(\bar{\lambda}) \to M(\bar{\lambda})/M(-\bar{\lambda}-2) \to 0$$

where the middle homomorphism is $\psi_{\bar{\lambda}}$.

Now we fix $\lambda \in \mathfrak{h}^*$ and look at the character $\lambda + X \in \mathfrak{h}_A^*$. Then we have that

$$M_A(\lambda + X) \overset{\varphi_\lambda}{\to} M_A^s(\lambda + X)$$

is an isomorphism if $\lambda \notin \mathbb{N}$ and fits into the exact sequence

$$0 \to M_A(\lambda + X) \overset{\varphi_\lambda}{\to} M_A^s(\lambda + X) \to M(-\lambda - 2) \to 0.$$

if $\lambda \in \mathbb{N}$.

In fact, the first claim is a special case of the situation dealt with above. To verify the second statement we assume $\lambda \in \mathbb{N}$. Then we see that $\binom{\lambda+X}{i}$ is a unit in A only when $0 \leq i \leq \lambda$ whereas $\binom{\lambda+X}{i} = u_i X$ for some unit $u_i \in A$ when $i > \lambda$. Moreover, we may identify $M_A(-\lambda-2+X)/XM_A(-\lambda-2+X)$ with $M(-\lambda-2)$ and we get a surjection $M_A^s(\lambda+X) \to M(-\lambda-2)$ by sending $v_i^* \mapsto (-1)^i v_{i-\lambda-1}$, $i > \lambda$, $v_i^* \mapsto 0, i \leq \lambda$. It is now easy to check that this leads to an exact sequence as claimed.

Similarly, just as over $\mathbb{C}$ we have for all λ a natural homomorphism $\psi_\lambda : M_A^s(\lambda + X) \to M_A(\lambda + X)$. When $\lambda \notin \mathbb{N}$ this is the inverse of φ_λ and for $\lambda \in \mathbb{N}$ we have the exact sequence

$$0 \to M_A^s(\lambda + X) \overset{\psi_\lambda}{\to} M(\lambda + X) \to M(\lambda)/M(-\lambda - 2) \to 0.$$

7.3 The general case

Consider now $\mathfrak{g}$ general. For each simple root α the results above transform easily into statements about $\mathfrak{p}_\alpha$-modules. So we may proceed exactly as in Section 6 to obtain the following results.

Proposition 7.1. *Let $\lambda \in \mathfrak{h}^*$ and consider $\lambda + X\rho \in \mathfrak{h}_A^*$. Suppose $w \in W$ and α is a simple root with $ws_\alpha > w$. Set $\beta = w(\alpha)$. Then* $\mathrm{Hom}_{\mathfrak{g}_A}(M_A^w(\lambda + X\rho), M_A^{ws_\alpha}(\lambda+X\rho)) \cong A \cong \mathrm{Hom}_{\mathfrak{g}_A}(M_A^{ws_\alpha}(\lambda+X\rho), M_A^w(\lambda+X\rho))$. *Moreover, if φ_λ^w and ψ_λ^w denote generators of these* Hom*-spaces. Then we have*

i) If $\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N}$ then φ_λ^w and ψ_λ^w are isomorphisms with $\varphi_\lambda^w = (\psi_\lambda^w)^{-1}$.

ii) If $\langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{N}$ then φ_λ^w and ψ_λ^w fit into the exact sequences

$$0 \to M_A^w(\lambda + X\rho) \overset{\varphi_\lambda^w}{\to} M_A^{ws_\alpha}(\lambda + X\rho) \to M^w(s_\beta \cdot \lambda) \to 0$$

and

$$0 \to M_A^{ws_\alpha}(\lambda + X\rho) \overset{\psi_\lambda^w}{\to} M_A^w(\lambda + X\rho) \to M^w(\lambda)/M^w(s_\beta \cdot \lambda) \to 0,$$

respectively.

Fix now $\lambda \in \mathfrak{h}^*$ and $w \in W$. Choose a reduced expression for w_0, $w_0 = s_1 s_2 \dots s_N$ such that $w = s_n s_{n-1} \dots s_1$. Let α_{i_j} denote the simple root corresponding to s_j. Set

$$\beta_j = \begin{cases} -w s_1 s_2 \dots s_{j-1}(\alpha_{i_j}), & \text{if } j \leq n \\ w s_1 s_2 \dots s_{j-1}(\alpha_{i_j}), & \text{if } j > n \end{cases}$$

Then $\{\beta_1, \beta_2, \dots, \beta_N\} = R^+$. If we set $R^+(w) = \{\beta \in R^+ \mid w^{-1}(\beta) \in R^-\}$, then $\{\beta_1, \beta_2, \dots, \beta_n\} = R^+(w)$. We shall also write $R^+(\lambda) = \{\beta \in R^+ \mid \langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{N}\}$.

Consider the composite $\Phi^w(\lambda)$

$$M_A^w(\lambda + X\rho) \to M_A^{ws_1}(\lambda + X\rho) \to M_A^{ws_1 s_2}(\lambda + X\rho) \to \dots \to M_A^{ww_0}(\lambda + X\rho)$$

where for each $j = 1, \dots, N$ the homomorphism $M_A^{ws_1 s_2 \dots s_{j-1}}(\lambda + X\rho) \stackrel{\varphi_j^w(\lambda)}{\to} M_A^{ws_1 s_2 \dots s_j}(\lambda + X\rho)$ is a generator of its Hom-space (see Proposition 7.1). Then we may define the Jantzen filtration of $M_A^w(\lambda + X\rho)$ by

$$M_A^w(\lambda + X\rho)^j = \{m \in M_A^w(\lambda + X\rho) \mid \Phi^w(\lambda)(m) \in X^j M_A^{ww_0}(\lambda + X\rho)\}.$$

Taking the images in $M^w(\lambda) = M_A^w(\lambda + X\rho)/X M_A^w(\lambda + X\rho)$ we obtain the Jantzen filtration $M^w(\lambda)^0 \supseteq M^w(\lambda)^1 \supseteq \dots$ of $M^w(\lambda)$.

These filtrations also filter the weight spaces of $M_A^w(\lambda + X\rho)$ and $M^w(\lambda)$. Note that for any $\mu \in \mathfrak{h}^*$ the weight space $M_A^w(\lambda)_{\mu + X\rho}$ is a finitely generated free A-module (of rank equal to $\dim_{\mathbb{C}} M^w(\lambda)_\mu = \dim_{\mathbb{C}} M(\lambda)_\mu$). Standard arguments (see e.g 5.1 in [9]) tell us that

$$\sum_{j \geq 1} \dim M^w(\lambda)^j_\mu = \nu_X(\det(\Phi^w(\lambda)_\mu)).$$

(Here and elsewhere the index μ on a homomorphism means the restriction of the homomorphism to the $\mu + X\rho$ weight space and ν_X is the X-adic valuation).

Clearly, the right hand side of this equation equals

$$\sum_{j=1}^n \nu_X(\det(\varphi_j^w(\lambda)_\mu)) = \sum_{j=1}^n \ell_X(\mathrm{Coker}(\varphi_j^w(\lambda)_\mu))$$

where ℓ_X denotes length of a module. Observe that by Proposition 7.1 i) we have that $\varphi_j^w(\lambda)$ is an isomorphism when $\beta_j \notin R^+(\lambda)$. By Proposition 7.1 ii) we have for $\beta_j \in R^+(\lambda)$

$$\ell_X(\mathrm{Coker}(\varphi_j^w(\lambda)_\mu)) = \begin{cases} \dim M(\lambda) - \dim M(s_{\beta_j} \cdot \lambda)_\mu, & \text{if } j \leq n \\ \dim M(s_{\beta_j} \cdot \lambda)_\mu, & \text{if } j > n. \end{cases}$$

Hence we have proved

Theorem 7.1. *Let λ, w be as above. Then $M^w(\lambda)$ has a Jantzen filtration*

$$M^w(\lambda) = M^w(\lambda)^0 \supseteq M^w(\lambda)^1 \supseteq \dots$$

such that $M^w(\lambda)/M^w(\lambda)^1 \cong \operatorname{Im}\Phi^w(\lambda) \subseteq M^{ww_0}(\lambda)$ and

$$\sum_{j\geq 1} \operatorname{ch} M^w(\lambda)^j = \sum_{\beta\in R^+(\lambda)\cap R^+(w)} (\operatorname{ch} M(\lambda) - \operatorname{ch} M(s_\beta\cdot\lambda)) + \sum_{\beta\in R^+(\lambda)\setminus R^+(w)} \operatorname{ch} M(s_\beta\cdot\lambda).$$

Remark 7.1. i) For $w = e$ we recover the usual Jantzen filtration and sum formula for the ordinary Verma module $M^e(\lambda) = M(\lambda)$ (Note that in this case $R^+(w) = \emptyset$).

ii) When reformulated using the notation from Sections 2–4 for twisted Verma modules the theorem reads as follows:

Let $x, y \in W$. Then $M(x,y) = M_\lambda(x,y)$ (with λ a regular integral and antidominant weight) has a Jantzen filtration

$$M(x,y) = M(x,y)^0 \supseteq M(x,y)^1 \supseteq \dots$$

such that $M(x,y)/M(x,y)^1 \cong \operatorname{Im}(M(x,y) \to M(xw_0, w_0y))$ and

$$\sum_{j\geq 1} \operatorname{ch} M(x,y)^j = \sum_{\beta\in R^+(xy)\setminus R^+(x)} (\operatorname{ch} M(xy\cdot\lambda) - \operatorname{ch} M(s_\beta xy\cdot\lambda)) + \sum_{\beta\in R^+(xy)\cap R^+(x)} \operatorname{ch} M(s_\beta xy\cdot\lambda).$$

7.4 The B_2-case

Example 7.1. Below we have listed all the twisted Verma modules with integral highest weights when the root system is B_2 together with their Jantzen filtrations. Since this is a multiplicity free case the sum formula in Theorem 7.1 completely determines the filtration. A simple module listed in the i-th row means that it occurs in the i-th layer of the filtration. In some cases a 0 occurs in the 0-th row. This means that the corresponding layer is 0.

Choose an integral regular antidominant weight λ and write $M^w(y)$ short for $M^w(y\cdot\lambda)$. Also write $L(x) = L(x\cdot\lambda)$. Let s (respectively t) be the simple reflections corresponding to the short (respectively long) simple root. Then $W = \{e, s, t, st, ts, sts, tst, w_0\}$.

Recall that $M^w(y) = DM^{ww_0}(y)$ for all $w, y \in W$. Therefore we have only listed half the twisted Verma modules. The others (and their Jantzen filtrations) are then obtained by dualizing. In the list below the twisted Verma modules are itemized according to their highest weight.

$\boxed{\lambda}$

$$M^w(e) = L(e)$$

$\boxed{s \cdot \lambda}$

$$M^{tst}(s) = M^{ts}(s) = M^t(s) = M^e(s) = \begin{array}{c} L(s) \\ L(e) \end{array}$$

$\boxed{t \cdot \lambda}$

$$M^{tst}(t) = M^{st}(t) = M^s(t) = M^e(t) = \begin{array}{c} L(t) \\ L(e) \end{array}$$

$\boxed{st \cdot \lambda}$

$$M^{ts}(st) = M^t(st) = M^e(st) = \begin{array}{c} L(st) \\ L(s) \;\; L(t) \\ L(e) \end{array}$$

$$M^s(st) = \begin{array}{c} L(t) \\ L(e) \;\; L(st) \\ L(s) \end{array}$$

$\boxed{ts \cdot \lambda}$

$$M^{st}(ts) = M^s(ts) = M^e(ts) = \begin{array}{c} L(ts) \\ L(s) \;\; L(t) \\ L(e) \end{array}$$

$$M^t(ts) = \begin{array}{c} L(s) \\ L(e) \;\; L(ts) \\ L(t) \end{array}$$

$\boxed{sts \cdot \lambda}$

$$M^t(sts) = M^e(sts) = \begin{array}{c} L(sts) \\ L(st)\;\; L(ts) \\ L(s)\;\; L(t) \\ L(e) \end{array}$$

$$M^s(sts) = \begin{array}{c} L(ts) \\ L(s)\;\; L(t)\;\; L(sts) \\ L(e)\;\; L(st) \end{array}$$

$$M^{st}(sts) = \begin{array}{c} 0 \\ L(e)\;\; L(ts) \\ L(s)\;\; L(t)\;\; L(sts) \\ L(st) \end{array}$$

$\boxed{tst \cdot \lambda}$

$$M^s(tst) = M^e(tst) = \begin{array}{c} L(tst) \\ L(st)\;\; L(ts) \\ L(s)\;\; L(t) \\ L(e) \end{array}$$

$$M^t(tst) = \begin{array}{c} L(st) \\ L(s)\;\; L(t)\;\; L(tst) \\ L(e)\;\; L(ts) \end{array}$$

$$M^{ts}(tst) = \begin{array}{c} 0 \\ L(e)\;\; L(st) \\ L(s)\;\; L(t)\;\; L(tst) \\ L(ts) \end{array}$$

$\boxed{w_0 \cdot \lambda}$

$$M^e(w_0) = \begin{array}{c} L(w_0) \\ L(sts) \;\; L(tst) \\ L(st) \;\; L(ts) \\ L(s) \;\; L(t) \\ L(e) \end{array}$$

$$M^s(w_0) = \begin{array}{c} L(tst) \\ L(st) \;\; L(ts) \;\; L(w_0) \\ L(s) \;\; L(t) \;\; L(sts) \\ L(e) \end{array}$$

$$M^t(w_0) = \begin{array}{c} L(sts) \\ L(st) \;\; L(ts) \;\; L(w_0) \\ L(s) \;\; L(t) \;\; L(tst) \\ L(e) \end{array}$$

$$M^{st}(w_0) = \begin{array}{c} 0 \\ L(t) \;\; L(tst) \\ L(e) \;\; L(st) \;\; L(ts) \;\; L(w_0) \\ L(s) \;\; L(sts) \end{array}$$

Remark 7.2. Recall that the Jantzen filtration of an ordinary Verma module is its unique Loewy series, see [8]. In particular, the radical series of the Verma modules can be read off from the above list and we have therefore a determination of all extensions between simple modules. Using this it is easy to see that there are twisted Verma modules which do not have simple heads. For instance, both $L(e)$ and $L(ts)$ are quotients of $M^{st}(sts)$. Likewise, both $L(e)$ and $L(sts)$ are submodules of $M^s(w_0)$ (this example of a non-rigid twisted Verma module was pointed out to us by C. Stroppel). It is also seen that $M^{st}(w_0)$ has non-simple head and socle.

The 0 occurring in the 0-th row for a module M in the list means that the composite $M \to DM$ (see 7.3) is zero. Nevertheless, the space $\mathrm{Hom}_{\mathfrak{g}}(M, DM)$ may be non-zero. For instance, one may check that

$$\mathrm{Hom}_{\mathfrak{g}}(M^{st}(w_0), DM^{st}(w_0))$$

is 2-dimensional.

Remark 7.3. Using that LT_w is an autoequivalence of the bounded derived category $D^b(\mathcal{O})$ one may prove that (similar to the proof of Corollary 6.3)

$$\mathrm{Hom}_{\mathfrak{g}}(M^w(\lambda), M^{ws}(\lambda)) = \mathbb{C}$$

where $w < ws$. Let φ be a generator of this Hom-space. It seems reasonable to expect that φ is well behaved with respect to the Jantzen filtration in the sense that

$$\varphi(M^w(\lambda)^j) \subseteq M^{ws}(\lambda)^{j+1}.$$

One may prove that $\mathrm{Soc} M^{w_0 s_\alpha}(\lambda) = L(s_\alpha \cdot \lambda)$, where α is a simple root and $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$. If φ respects the Jantzen filtration as above this leads to new and perhaps simpler proofs of non-vanishing Ext^1-groups between certain neighboring simple modules.

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