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Abstract

We consider dynamic proportional reinsurance strategies and derive the optimal strategies in a diffusion setup and a classical risk model. Optimal is meant in the sense of minimizing the ruin probability. Two basic examples are discussed.

Key words: optimal control, stochastic control, ruin probability, diffusion, Hamilton-Jacoby-Bellman equation, proportional reinsurance, subexponential distribution

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1. Introduction

Assume an insurance company models its surplus via a Markov process (X_t^1). For the premium calculation the company uses the expected value principle with safety loading $\eta > 0$, which means that $E[X_1^1]$ is η times the expected outflow in a unit interval. The risk left to the first-line insurer often is too large. Thus reinsurance is needed, see also [13]. Usually, reinsurance is also needed by legal restrictions.

Assume the company has the possibility of proportional reinsurance. The reinsurer also uses the expected value principle with safety loading θ. We have to assume that $\theta \geq \eta$. Otherwise, the insurer could reinsure the whole portfolio and make a riskless profit, because the reinsurance premium would be smaller than the premium income for the portfolio.

The company has now to choose the proportion b of the portfolio they are willing to take over. We will here consider two types of risk models: a classical Cramér-

Lundberg model and a diffusion approximation to the surplus process. The Cramér-Lundberg model is used in the literature as an approximation to reality if the number of individual contracts is large. Many characteristics of the risk process cannot be calculated in closed form. One therefore often uses a diffusion approximation to the risk model which, hopefully, help to take almost optimal decisions. For the theory of diffusion approximations see for instance [10], [6], [7], [14] or [12].

Waters [15] considered a general model where $(X_k - X_{k-1} : k \in \mathbb{N})$ was iid distributed. The most important special case is the Cramér-Lundberg model. The idea was to minimize the ruin probability, i.e. the probability that the surplus ever becomes negative. As an approximation to an optimal reinsurance treaty the adjustment coefficient was maximized. That is, it was assumed that there is a b such that there are constants $R(b)$, $\underline{C}$ and $\overline{C}$ such that

$$\underline{C} e^{-R(b)x} \le \psi_b(x) \le \overline{C}\, e^{-R(b)x}$$

and b was chosen such that $R(b)$ was maximized. This gives in some sense the asymptotically optimal strategy with respect to the ruin probability. In order that the problem has a non-trivial solution the assumption $\theta > \eta$ had to be made.

Højgaard and Taksar [9] considered a diffusion approximation and maximized the "expected future surplus" in the sense that

$$E\Big[\int_0^\tau X_s e^{-\delta s}\, ds\Big]$$

for some discounting factor $\delta > 0$. Here, τ is the time of ruin. In their model they assumed $\theta = \eta$ and at each time t the proportion b_t could be chosen.

In this paper we will also allow a proportion b_t that continuously can be changed. It will turn out to be of feedback form $b_t = b(X_{t-}^b)$ where (X_t^b) is the surplus process corresponding to the strategy $b = (b_t)$. Our aim is to minimize the ruin probability. The problem has a trivial solution $b_t = 0$ in the case $\theta = \eta$ because then ruin can never occur. We restrict b_t to be smaller than one. A proportional reinsurance with $b_t > 1$ would not be realistic. We will first consider the diffusion case (Section 2) and then the Cramér-Lundberg case (Section 3).

For the rest of this paper we work on a complete probability space $(\Omega, \mathcal{F}, P)$ on which the process (X_t) is defined. The information at time t is given by the complete filtration generated by (X_t). A *reinsurance strategy* is a previsible process (b_t) with values in $[0,1]$. The time of ruin is

$$\tau = \tau_b = \inf\{t \geq 0 : X_t^b < 0\}\,.$$

Here, (X_t^b) is the surplus under reinsurance strategy (b_t). The survival probability is

$$\delta_b(x) = P[\tau_b = \infty \mid X_0^b = x]\,.$$

Our aim is to find the optimal value

$$\delta(x) = \sup_{(b_t)} \delta_b(x)$$

and, hopefully, an optimal strategy (b_t^*) such that $\delta(x) = \delta_{b^*}(x)$.

At the moment we do not know whether an optimal strategy exists. It could happen that for any strategy (b_t) there exists a strategy $\tilde{b}_t$ such that $\delta_{\tilde{b}}(x) > \delta_b(x)$. But we know that it is possible to find a strategy which is near to be optimal. More specifically, for each $\varepsilon > 0$ we can find a strategy (b_t) such that $\delta(x) - \delta_b(x) < \varepsilon$. Assuming that $\delta(x)$ is a "nice" function we will find the Hamilton-Jacoby-Bellman equation. This equation which will be the starting point for solving the problem. Finding a candidate $f(x)$ for $\delta(x)$ and a candidate (b_t) for (b_t^*) we will then be able to verify that in fact $f(x) = \delta_b(x) = \delta(x)$.

The survival probability without reinsurance is then $\delta_1(x)$, giving $\delta(x) \geq \delta_1(x)$. Moreover, we have that $\delta(x)$ is strictly increasing, so $\delta'(x) > 0$ provided the derivative exists. Indeed, if $x_1 > x_0$ and (b_t) is a strategy for x_0 we can apply the strategy for initial capital x_1 until $\tau(x_0)$. In the diffusion case $X_{\tau(x_0)}(x_1) = x_1 - x_0$. If we let $b_t = 1$ for $t > \tau(x_0)$ there is a positive probability for survival. In the Cramér-Lundberg case there is a positive probability that $X_{\tau(x_0)}(x_1) \geq 0$. This yields also a positive probability for survival for initial capital x_1 but ruin for initial capital x_0. Thus $\delta(x_0) < \delta(x_1)$.

2. The diffusion case

Consider a Cramér-Lundberg model with claim arrival intensity λ and expected claim size μ. The premium income in $(0,t]$ is then $(1+\eta)\lambda\mu t$. A reinsurance contract with proportion b will cost $(1+\theta)\lambda b\mu t$. This gives a net premium $(b(1+\theta)-(\theta-\eta))\lambda\mu$ in a unit interval. in a unit time interval the expected value of the aggregate claims is $b\lambda\mu$ and its variance is $b^2\lambda\mu_2$. Here μ_2 is the second moment of the claim size distribution. The diffusion approximation considered in [6] is therefore

$$(b\theta - (\theta - \eta))\lambda\mu t + b\sqrt{\lambda\mu_2}W_t$$

where (W_t) is a standard Brownian motion. For simplicity we divide by $\lambda\mu$ and consider a model of the form

$$X_t^b = x + \int_0^t (b_s\theta - (\theta - \eta))\, ds + \sigma \int_0^t b_s \, dW_s \,. \tag{1}$$

For an introduction to stochastic integrals and stochastic analysis see for instance [5] or [11].

As ruin time we can consider here $\tau_b = \inf\{t \geq 0 : X_t^b = 0\}$ which coincides almost surely with the ruin time defined above. Note that $\delta(0) = 0$. Consider the following strategy

$$b_t = \begin{cases} b, & \text{if } 0 \leq t \leq h \wedge \tau, \\ \tilde{b}_{t-h}(X_h), & \text{if } t > h \text{ and } \tau > h. \end{cases}$$

Here b is an arbitrary constant and $\tilde{b}_t(x)$ is strategy such that $\delta_{\tilde{b}}(x) > \delta(x) - \varepsilon$ where ε is arbitrary but fixed. Then by the Markov property

$$\delta(x) \geq \delta_b(x) = E[\delta_{\tilde{b}}(X_h); \tau > h] = E[\delta_{\tilde{b}}(X_{h\wedge\tau})] \geq E[\delta(X_{h\wedge\tau})] - \varepsilon$$

and because ε is arbitrary

$$\delta(x) \geq E[\delta(X_{h\wedge\tau})] .$$

Let us now assume that $\delta(x)$ is twice continuously differentiable. By Itô's formula (see [5] or [11]),

$$\delta(X_{h\wedge\tau}) = \delta(x) + \int_0^{h\wedge\tau} \{(b\theta - (\theta - \eta))\delta'(X_s) + \frac{\sigma^2 b^2}{2}\delta''(X_s)\}\, ds + \int_0^{h\wedge\tau} b\sigma\delta'(X_s)\, dW_s .$$

If the stochastic integral is a martingale we obtain the equation

$$0 \geq E\Big[\int_0^{h\wedge\tau} \{(b\theta - (\theta - \eta))\delta'(X_s) + \frac{\sigma^2 b^2}{2}\delta''(X_s)\}\, ds\Big] .$$

Dividing the latter equation by h and letting $h \to 0$ we obtain, provided limit and expectation can be interchanged,

$$(b\theta - (\theta - \eta))\delta'(x) + \frac{\sigma^2 b^2}{2}\delta''(x) \leq 0 .$$

This equation must hold for all $b \in [0, 1]$. On the other side, for h small and b "optimal" $\delta_b(x)$ should be close to $\delta(x)$, so intuitively equality should be derived. This yields the Hamilton-Jacoby-Bellman equation

$$\sup_{b\in[0,1]} (b\theta - (\theta - \eta))\delta'(x) + \frac{\sigma^2 b^2}{2}\delta''(x) = 0 . \tag{2}$$

If $b^*(x)$ is the value at which the maximum above is attained we can conjecture that $(b^*(X_s^{b^*}))$ is an optimal strategy.

Equation (2) admits the following solution.

Lemma 1. *The function*

$$f(x) = 1 - e^{-\kappa x} \tag{3}$$

solves (2) where

$$\kappa = \begin{cases} \frac{\theta^2}{2\sigma^2(\theta-\eta)}, & \text{if } \eta < \theta < 2\eta, \\ \frac{2\eta}{\sigma^2}, & \text{if } \theta \geq 2\eta. \end{cases}$$

The value $b^*(x)$ that maximizes the left hand side of (2) is constant and

$$b^* = 2(1 - \eta/\theta) \wedge 1\,. \tag{4}$$

Proof. Of course, one can directly verify that $f(x)$ solves (2). But in order to illustrate the method we will solve (2) analytically. The equation is quadratic in b. If $f''(x) > 0$ then the minimum is attained at $-\theta/\sigma^2$. The maximum in $[0, 1]$ is therefore attained at $b = 1$. This yields the solution $f'(x) = C_1 e^{-2\eta x/\sigma^2} > 0$ and therefore $f''(x) < 0$ which is a contradiction. $f''(x) = 0$ implies that $b = 1$ and therefore $f'(x) = 0$. But our solution should satisfy $f'(x) > 0$. Thus $f''(x) < 0$ and the supremum is attained at

$$b = -\frac{\theta f'(x)}{\sigma^2 f''(x)}\,.$$

If b is larger than one then $b^*(x) = 1$ and the solution to (2) is $f'(x) = C_1 e^{-2\eta x/\sigma^2} > 0$. If $b \in (0, 1)$ we obtain

$$-\frac{\theta^2 f'(x)^2}{2\sigma^2 f''(x)} - (\theta - \theta) f'(x)\,.$$

Because the solution we are looking for satisfies $f'(x) > 0$ we can divide by $f'(x)$ and obtain $f'(x) = C_2 e^{-\theta^2 x/(2\sigma^2(\theta-\eta))}$. Note that $\theta^2/(2\sigma^2(\theta - \eta)) \geq 2\eta/\sigma^2$ and equality holds if and only if $\theta = 2\eta$. In the latter case we find $b = 1$, so the two solutions coincide. Assume there is a point x_0 where the solution $f(x)$ changes from one of the solutions $f_1(x)$ to another of the solutions $f_2(x)$. Because we want $f(x)$ to be twice continuously differentiable we get the two equation $f_1'(x_0) = f_2'(x_0)$ and $f_1''(x_0) = f_2''(x_0)$. These two equations can never be fulfilled simultaneously, except if $\theta = 2\eta$, thus the desired solution is one of the functions obtained above. From

$f(0) = 0$ and $\lim_{x\to\infty} f(x) = 1$ the possible solutions are (3) with one of the two possible values of κ given above. The maximum in (2) is attained for $b = \theta/(\sigma^2\kappa)$ which is smaller than one if and only if $\theta < 2\eta$. This proves the lemma. $\square$

We have now found a candidate $f(x)$ for $\delta(x)$ and a candidate $b^*(X_{t-})$ for the optimal control. We now have to verify that $f(x)$ is indeed the right solution. Note that $\delta_{b^*}(x) = f(x)$.

Theorem 1. *We have $\delta(x) = f(x)$ where $f(x)$ is given by (3) and b^* given by (4) is an optimal reinsurance strategy.*

Proof. Because $f(x) = \delta_{b*}(x)$ it remains to show that $f(x) \geq \delta(x)$. Let (b_t) be an arbitrary reinsurance strategy. It follows readily that either $\tau < \infty$ or $X_{\tau\wedge t} \to \infty$. Then by Itô's formula

$$\begin{aligned} f(X_{\tau\wedge t}) &= f(x) + \int_0^{\tau\wedge t} (b_s\theta - (\theta-\eta))f'(X_s) + \frac{b_s^2\sigma^2}{2} f''(X_s)\, ds + \int_0^{\tau\wedge t} b_s \sigma f'(X_s)\, dW_s \\ &\leq f(x) + \int_0^{\tau\wedge t} b_s\sigma f'(X_s)\, dW_s \end{aligned}$$

because $f(x)$ solves (2). Because $f'(x)$ is bounded by κ the stochastic integral is a martingale. Taking expectations yields

$$E[f(X_{\tau\wedge t})] \leq f(x)\,.$$

Letting $t \to \infty$ the left hand side converges to $\delta_b(x)$. Thus $\delta(x) \leq f(x)$. $\square$

3. The Cramér-Lundberg case

Let us now turn to a Cramér-Lundberg model. Here the number of claims N_t in $(0,t]$ is a Poisson process with rate λ and the claim sizes (Y_i) is a sequence of positive iid random variables independent of (N_t). Let $G(x) = P[Y_i \leq x]$, $E[Y_i] = \mu$ and assume that $G(x)$ is continuous. Let (T_i) be the occurrence time of the i-th claim. Then

$$X_t = x + \int_0^t (b_s(1+\theta) - (\theta-\eta))\lambda\mu\, ds - \sum_{i=1}^{N_t} b_{T_i} Y_i$$

is the surplus process under reinsurance policy (b_t). Note that in order that $\delta_b(x) < 1$ we need a strictly positive premium income, which shows that we can restrict to strategies with $b_t \in (\underline{b}, 1]$ where $\underline{b} = (\theta-\eta)/(1+\theta)$.

Let us first find the corresponding Hamilton-Jacoby-Bellman equation. Consider the following strategy

$$b_t = \begin{cases} b, & \text{if } 0 \le t \le h \wedge T_1, \\ \tilde{b}_{t-h\wedge T_1}(X_{h\wedge T_1}), & \text{if } t > h \wedge T_1 \text{ and } T_1 \wedge h < \tau. \end{cases}$$

Here $h > 0$ and $\tilde{b}_t(x)$ is a strategy such that $\delta_{\tilde{b}}(x) > \delta(x) - \varepsilon$ for some arbitrary but fixed $\varepsilon > 0$. Let $c_b = (b(1+\theta) - (\theta-\eta))\lambda\mu$. By the law of total probability we find

$$\begin{aligned}\delta(x) \ge \delta_b(x) &= e^{-\lambda h}\delta_b(x + c_b h) + \int_0^h \int_0^{(x+c_b t)/b} \delta_b(x + c_b t - by)\, dG(y)\, \lambda e^{-\lambda t}\, dt \\ &\ge e^{-\lambda h}\delta(x + c_b h) + \int_0^h \int_0^{(x+c_b t)/b} \delta(x + c_b t - by)\, dG(y)\, \lambda e^{-\lambda t}\, dt - \varepsilon\,.\end{aligned}$$

Because ε was arbitrary we can let $\varepsilon = 0$. Rearranging the terms and dividing by h yields

$$\begin{aligned}&\frac{\delta(x + c_b h) - \delta(x)}{h} e^{-\lambda h} - \frac{1 - e^{-\lambda h}}{h}\delta(x) \\ &\quad + \frac{1}{h}\int_0^h \int_0^{(x+c_b t)/b} \delta(x + c_b t - by)\, dG(y)\, \lambda e^{-\lambda t}\, dt \le 0\,.\end{aligned}$$

Assume that $\delta(x)$ is differentiable. Letting $h \to 0$ yields

$$(b(1+\theta) - (\theta-\eta))\lambda\mu\delta'(x) + \lambda \int_0^x \delta(x - by)\, dG(y) - \lambda\delta(x) \le 0\,.$$

This must hold for every b. For h small and b optimal the corresponding survival probability is almost optimal, thus equality in the above calculations should be obtained. This yields the Hamilton-Jacobi-Bellman equation

$$\sup_{b \in (\underline{b},1]} (b(1+\theta) - (\theta-\eta))\mu\delta'(x) + \int_0^{x/b} \delta(x - by)\, dG(y) - \delta(x) = 0\,. \tag{5}$$

To solve the above equation seems hard. Trying to find the $b(x)$ that maximizes the left hand side of (5) yields an integral equation for $\delta(x)$. We therefore try another approach, similar to the approach in [8].

Assume that $b(x)$ maximizes the left hand side of (5). Because

$$\int_0^{x/b} \delta(x-by)\,dG(y) < \int_0^{x/b} \delta(x)\,dG(y) = \delta(x)G(x/b)$$

and $\delta'(x) > 0$ it follows that $b(x)(1+\theta) - (\theta - \eta) > 0$. Solving for $\delta'(x)$ we find

$$\delta'(x) = \frac{\delta(x) - \int_0^{x/b(x)} \delta(x - b(x)y)\,dG(y)}{(b(x)(1+\theta) - (\theta-\eta))\mu}\,.$$

Observing that $\delta_b(x) = 1 - \int_x^\infty \delta_b'(y)\,dy$ we would maximize $\delta_b(x)$ if we could minimize $\delta_b'(y)$ for all y simultaneously. One therefore conjectures that $\delta(x)$ satisfies

$$\delta'(x) = \inf_{b \in (\underline{b},1]} \frac{\delta(x) - \int_0^{x/b} \delta(x-by)\,dG(y)}{(b(1+\theta) - (\theta-\eta))\mu}\,. \tag{6}$$

The latter equation determines solutions only up to a multiplicative constant. Let us therefore look for a solution $f(x)$ with $f(0) = \delta_1(0) = \eta/(1+\eta)$, where $\delta_1(x)$ is the survival probability for the strategy $b_t = 1$. The candidate for our solution will then be $f(x)/f(\infty)$.

Lemma 2. *There exists a unique solution $f(x)$ to (6) with $f(0) = f_0$ for each $f_0 > 0$. Moreover, $f'(x) > 0$, $f'(x)$ is continuous and $f(\infty) < \infty$.*

Proof. We can assume that $f_0 = \delta_1(0) = \eta/(1+\eta)$. Note that

$$f(x) - \int_0^{x/b} f(x-yb)\,dG(y) = f_0(1 - G(x/b)) + \int_0^x (1 - G((x-z)/b))f'(z)\,dz\,.$$

This means, for x small enough we must have $f'(x) > 0$. But then it follows from (6) that $f'(x) > 0$ for all $x > 0$.

Define the operator $\mathcal{V}$ on the space of differentiable functions via

$$(\mathcal{V}f)'(x) = \inf_{b \in (\underline{b},1]} \frac{f(x) - \int_0^{x/b} f(x-by)\,dG(y)}{(b(1+\theta) - (\theta-\eta))\mu}$$

and $\mathcal{V}f(0) = f_0$. The solution we are looking for fulfils $\mathcal{V}f = f$. Because

$$\frac{f_0(1 - G(x/b)) + \int_0^x (1 - G((x-z)/b)) f'(z)\, dz}{(b(1+\theta) - (\theta - \eta))\mu}$$

is both continuous in x and b it follows that $(\mathcal{V}f)'(x)$ is continuous and that the optimal $b(x)$ is bounded away from $\underline{b}$ on bounded intervals. We therefore can restrict to functions f with $f'(x)$ positive and continuous.

Let $f_1(x)$ and $f_2(x)$ be two functions and let $b_i(x)$ be the point where the infimum is taken. Then

$$\begin{aligned}
(\mathcal{V}f_1(x) - \mathcal{V}f_2(x))' &= \frac{f_0(1 - G(x/b_1)) + \int_0^x (1 - G((x-z)/b_1)) f_1'(z)\, dz}{(b_1(1+\theta) - (\theta - \eta))\mu} \\
&\qquad - \frac{f_0(1 - G(x/b_2)) + \int_0^x (1 - G((x-z)/b_2)) f_2'(z)\, dz}{(b_2(1+\theta) - (\theta - \eta))\mu} \\
&\le \frac{f_0(1 - G(x/b_2)) + \int_0^x (1 - G((x-z)/b_2)) f_1'(z)\, dz}{(b_2(1+\theta) - (\theta - \eta))\mu} \\
&\qquad - \frac{f_0(1 - G(x/b_2)) + \int_0^x (1 - G((x-z)/b_2)) f_2'(z)\, dz}{(b_2(1+\theta) - (\theta - \eta))\mu} \\
&= \frac{\int_0^x (1 - G((x-z)/b_2))(f_1'(z) - f_2'(z))\, dz}{(b_2(1+\theta) - (\theta - \eta))\mu} \\
&\le \frac{\int_0^x |f_1'(z) - f_2'(z)|\, dz}{(b_2(1+\theta) - (\theta - \eta))\mu}\,.
\end{aligned}$$

Interchanging the functions gives

$$|(\mathcal{V}f_1(x) - \mathcal{V}f_2(x))'| \le \frac{\int_0^x |f_1'(z) - f_2'(z)|\, dz}{((b_2 \wedge b_1)(1+\theta) - (\theta - \eta))\mu}\,. \tag{7}$$

Assume now $f_i(x) = \mathcal{V}f_i(x)$. Fix $x_0 \ge 0$ and let $x_1 = \inf\{((b_2(x) \wedge b_1(x))(1+\theta) - (\theta - \eta))\mu : 0 \le x \le x_0\}$. Then for $x < x_1$ we must have $f_1'(x) - f_2'(x) = 0$. This gives for $x \ge x_1$

$$|f_1'(x) - f_2'(x)| \le \frac{\int_{x_1}^x |f_1'(z) - f_2'(z)|\, dz}{((b_2 \wedge b_1)(1+\theta) - (\theta - \eta))\mu}$$

and $f_1'(x) = f_2'(x)$ for $x < 2x_1$ follows. Proceeding in the same way shows $f_1'(x) = f_2'(x)$ for all $x \le x_0$. Thus $f_1(x) = f_2(x)$ on $[0, x_0]$. Because x_0 was arbitrary any solution must be unique.

Let $f_0(x) = \delta_1(x)$ and define $f_n(x) = \mathcal{V} f_{n-1}(x)$. Because

$$f_0'(x) = \frac{f_0(x) - \int_0^x f(x-y)\,dG(y)}{(1+\eta)\mu}$$

it follows that $f_1'(x) \le f_0'(x)$. The considerations above show that

$$f_{n+1}'(x) - f_n'(x) \le \frac{\int_0^x (f_n'(z) - f_{n-1}'(z))\,dz}{(b_n(1+\theta) - (\theta-\eta))\mu} \le 0$$

and by induction $f_n'(x)$ is a decreasing sequence. Because $f_n'(x) > 0$ we have that $f_n'(x)$ converges to a function $f(x)$. It follows that $\mathcal{V} f(x) = f(x)$. Because $f_0(x) \le 1$ for all x we have $f(x) \le 1$ for all x. □

We now have to verify that $f(x)/f(\infty)$ indeed coincides with $\delta(x)$.

Theorem 2. *The function $\delta(x)$ is determined by $\delta(x) = f(x)/f(\infty)$ where $f(x)$ is the unique solution to (6). The strategy $(b^*(X_{t-}))$ is optimal, where $b^*(x)$ is a argument that minimizes the right hand side of (6).*

Proof. Let (b_t) be an arbitrary strategy. Then the process

$$f(X_{t\wedge\tau}) - \lambda \int_0^{t\wedge\tau} \Big[(b_s(1+\theta)-(\theta-\eta))\mu f'(X_s) + \int_0^{X_s/b_s} f(X_s - b_s y)\,dG(y) - f(X_s)\Big]\,ds$$

is a martingale, see for instance [2] or [1]. This gives

$$\begin{aligned} f(x) &= E[f(X_{t\wedge\tau})] - \lambda E\Big[\int_0^{t\wedge\tau} (b_s(1+\theta)-(\theta-\eta))\mu \\ &\qquad \times \Big(f'(X_s) - \frac{f(X_s) - \int_0^{X_s/b_s} f(X_s-b_s y)\,dG(y)}{(b_s(1+\theta)-(\theta-\eta))\mu}\Big)\,ds\Big] \\ &\ge E[f(X_{t\wedge\tau})] \end{aligned}$$

because $f(x)$ fulfils equation (6). Note that $f(x) \ge E[f(X_{t\wedge\tau})]$ also holds if $b_s(1+\theta) - (\theta-\eta) = 0$ for some s. Letting $t \to \infty$ gives $f(x) \ge f(\infty)P[\tau = \infty]$ or $f(x)/f(\infty) \ge \delta(x)$ because the strategy was arbitrary. Redoing the calculation with the strategy $(b^*(X_t))$ yields $f(x) = E[f(X_{t\wedge\tau})]$ and letting $t \to \infty$ gives $f(x) = f(\infty)P[\tau = \infty]$. Thus $f(x)/f(\infty) \le \delta(x)$ which ends the proof. □

For x small the optimal strategy $b^*(x)$ will just be $b^*(x) = 1$. This follows readily for $x = 0$. For x small we have that

$$\int_0^{x/b} \delta(x - by)\, dG(y) = \int_0^x (1 - G((x - z)/b))\delta'(z)\, dz$$

will only vary slowly with $b \in (\underline{b}, 1]$. Therefore the minimum in (6) will be determined by the maximum of the denominator in (6), i.e. at $b = 1$.

4. Two examples

In this section we consider two numerical examples. This is in order to illustrate how the optimal strategies and the survival function $\delta(x)$ can be calculated on a computer. The two examples come from the two main classes of claim size distributions: exponentially decreasing tail and subexponential distributions. From these examples it can be seen that $f_n(x) = \mathcal{V} f_{n-1}(x)$ converges very quickly. The quantity of interest, $b_n(x)$, the values of b where the minimum in the operator $\mathcal{V} f_{n-1}(x)$ is attained, will converge much slower. For x large, the function $\delta(x)$ only increases slowly, which means that the value to be minimized will not vary strongly with b near the optimal point. This makes it harder to find the optimal value. However, for large initial capital a choice of $b(x)$ near the optimal value will lead to an almost optimal strategy. This will be good enough for practical applications.

In the examples below we choose distributions such that $\mu = 1$. The other parameters are $\lambda = 1$, $\eta = 0.5$ and $\theta = 0.7$.

4.1. Exponentially distributed claim sizes

Let $G(x) = 1 - e^{-x}$. Then $\delta_1(x) = 1 - e^{-\eta x/(1+\eta)}/(1 + \eta) = 1 - 2e^{-x/3}/3$. This is an ideal choice for $f_0(x)$, see the proof of Lemma 2. In many cases the function $\delta_1(x)$ is not known explicitly. As an alternative to calculating first $\delta_1(x)$ one can start with

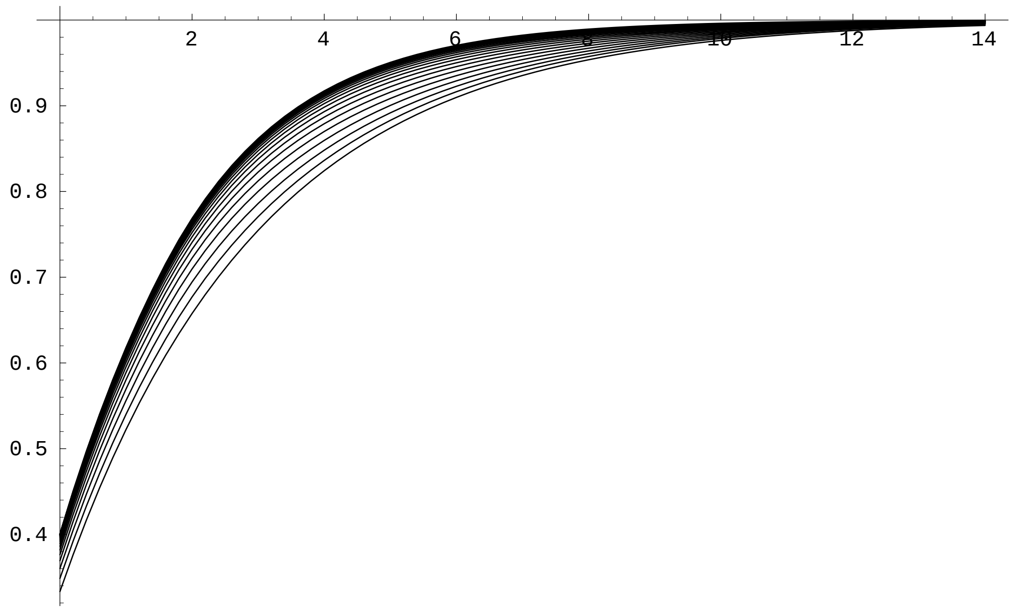

Figure 1: *The first 31 approximations to $\delta(x)$ for Exp(1) distributed claim sizes*

an approximation to $\delta_1(x)$, as for example the Cramér-Lundberg approximation, see [11]. As the derivation of (7) suggests, the operator $\mathcal{V}$ ultimately behaves like a contraction. Thus $f_n(x)$ also converges for arbitrary $f_0(x)$.

Figure 1 gives the first 31 approximations to $\delta(x)$. The lowest curve is $f_0(x) = \delta_1(x)$. We see that convergence $\delta(x)$ takes place very fast. The normalization of the functions $\tilde{f}_n(x)$ in Figure 1 is done in such a way that $\tilde{f}_n(x) = f_n(x)/f_n(20)$. Note that $1 - f_0(20) = 8.5 \cdot 10^{-4}$, so the error made by the normalization is negligible. For determining $\delta(x)$ a lower number of iterations had been enough. However, $b^*_{20}(x)$ determined after 20 iterations would not coincide for x near 14 with $b^*(x)$.

Figure 2 gives $b^*_{30}(x)$, the value of b based on $f_{30}(x)$. The graph suggests that the optimal strategy $b^*(x)$ is of the form $b(x) = \mathbb{I}_{x<m} + b_0 \mathbb{I}_{x \geq m}$. The function $\delta_b(x)$ can be calculated for such a strategy and will be of the form $A - Be^{-Rx}$, where the parameters A, B, R will be different for $x < m$ and $x \geq m$. But it is not trivial to check, whether $\delta_b(x)$ solves (6). Because this section is meant to illustrate the numerical evaluation of $b^*(x)$ we do not consider here the question whether the

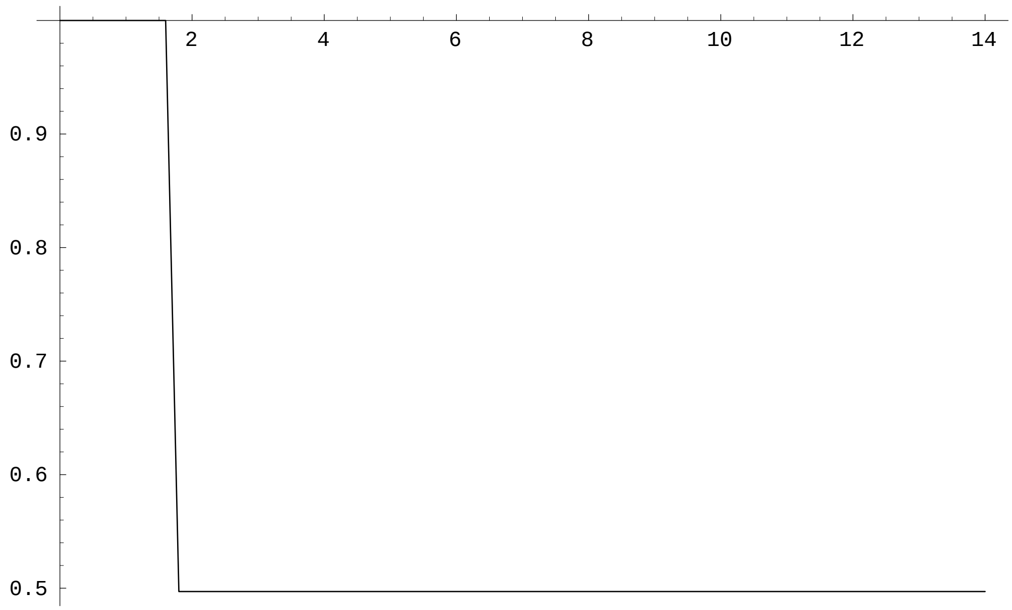

Figure 2: *The optimal strategy for Exp(1) distributed claim sizes*

conjecture $\delta(x) = \delta_b(x)$ is true or not.

Using the approach of Waters [15], the optimal choice of b in the sense that the adjustment coefficient $R(b)$ is maximized is

$$b_R = \max\Big\{\Big(1 - \frac{\eta}{\theta}\Big)\Big(1 + \frac{1}{\sqrt{1+\theta}}\Big), 1\Big\}$$

in the case of exponentially distributed claims. For our parameters this yields $b_R = 0.5048$. Because maximizing the adjustment coefficient yields the asymptotically best strategy, we expect $b^*(x)$ to tend to b_R. Figure 2 seems to verify this conjecture. In practice, this means that for x "large", we choose the strategy $b(x) = b_R$, and it is therefore not necessary to calculate $b^*(x)$ for "large" x.

4.2. Pareto distributed claim sizes

For $G(x) = 1 - (1+x)^{-2}$ no explicit expression for $\delta_1(x)$ is available. Of course, $\delta_1(x)$ could be calculated to any accuracy one likes, see for instance [3]. But this is

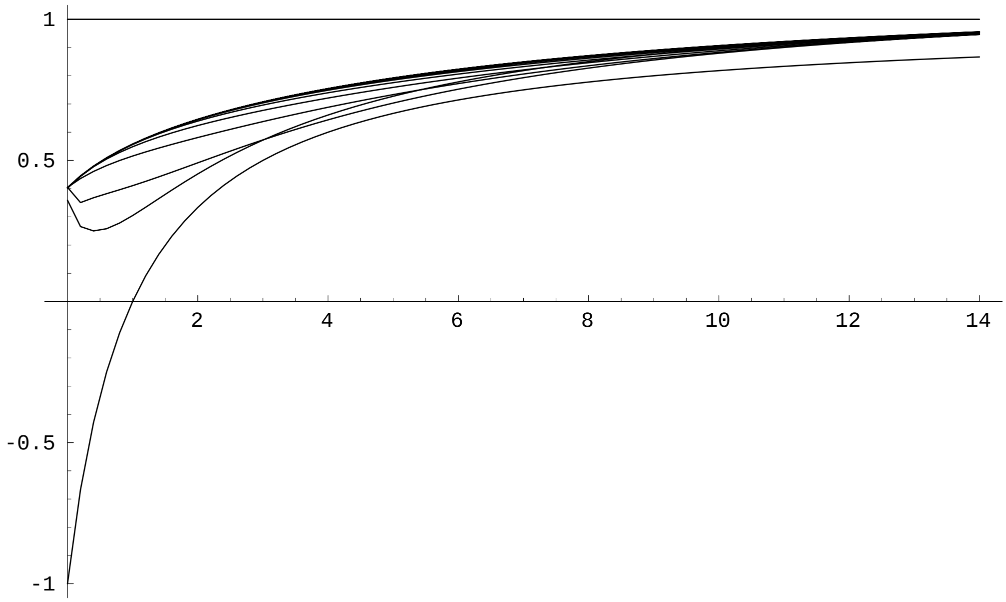

Figure 3: *The first 13 approximations to $\delta(x)$ for Par(2) distributed claim sizes*

not necessary. From [4] it is known that

$$\lim_{x\to\infty} \frac{1-\delta_1(x)}{\int_x^\infty (1-G(y))\,dy} = \frac{1}{\eta\mu}\,. \tag{8}$$

We therefore choose the initial function

$$f_0(x) = 1 - \frac{1}{\eta\mu}\int_x^\infty (1-G(y))\,dy\,.$$

Figure 3 shows the first 13 approximations to $\delta(x)$. 12 iterations were chosen because after 15, 20, 30 or 40 iterations Figures 3 and 4 will look the same. The lowest curve here is $f_0(x)$. We see that it is not even a problem that $f_0(x)$ is negative for small x. For the calculation we have chosen $f_n(0) = \eta/(1+\eta)$. As for the exponentially distributed claim sizes, the functions $\tilde{f}_n(x)$ in Figure 3 are normalized such that $\tilde{f}_n(x) = f_n(x)/f_n(20)$. Here $1 - f_0(20) = 0.095$. This accuracy is good enough because we mainly are interested in the optimal strategy $b^*(x)$.

The optimal strategy $b^*(x)$ given in Figure 4 looks differently to the strategy in the case of exponentially distributed claims. Note that $f_n(x)$ and $b_n(x)$ converge much faster than in the exponential case. This is do to the fact, that the convergence

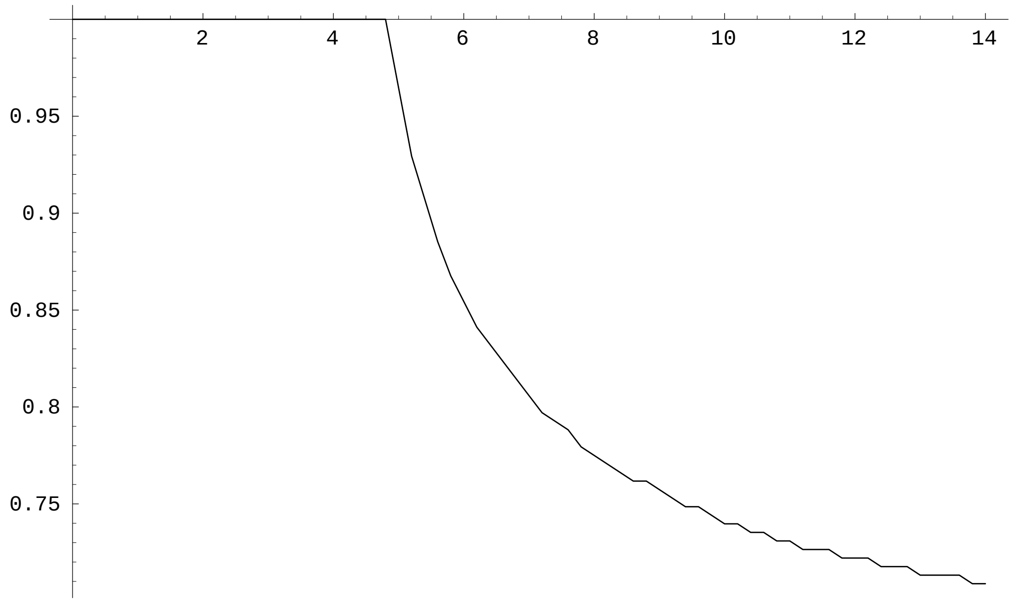

Figure 4: *The optimal strategy for Par(2) distributed claim sizes*

to zero of $1 - \delta(x)$ is behaving as x^{-1} and not as e^{-Rx} as in the exponential case. Thus $f_n'(x)$ is larger, and differences between different values of b on the right hand side of (6) will be larger. Thus $b_n(x)$ will be expected to converge faster to $b^*(x)$. It should be noted that the irregularity of the graph at the right end is do to the discretization used.

It is an open question whether $b^*(x)$ converges to some value as $x \to \infty$. Let us consider the following heuristic argument. For x large and b fixed we have, using the approximation (8),

$$1 - \delta_b(x) \approx \frac{1}{b\theta - (\theta - \eta)} \frac{b}{1 + x/b} = \frac{1}{(\theta - (\theta - \eta)/b)(1 + x/b)}.$$

Thus the ruin probability is minimized if $(\theta - (\theta - \eta)/b)(1 + x/b)$ is maximized. This gives

$$b = \frac{1(\theta - \eta)x}{\theta x - \theta - \eta}.$$

Letting $x \to \infty$ this indicates that the asymptotic value of $b^*(x)$ as $x \to \infty$ should be $b_a = 2(\theta - \eta)/\theta \wedge 1$. Note that this is the value (4) obtained from the diffusion approximation. In our example we obtain $b_a = 4/7 \approx 0.5714$. Suppose that $b^*(x)$

really converges to b_a. As indicated by Figure 4 at $x = 14$ the value $b^*(14)$ is far away from b_a and would only converge slowly to b_a. Thus the calculations would have to be done also for large initial capital. This, of course yields some numerical problems. In principle, however, with modern computers such a calculation could be done.

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