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ON A VARIANT OF WILSON'S FUNCTIONAL EQUATION ON GROUPS

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On a variant of Wilson's functional equation on groups

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Abstract

We study the solutions $f, g : G \to \mathbb{C}$ of the functional equation

$$f(xy) + f(y^{-1}x) = 2f(x)g(y), \quad x, y \in G,$$

where G is a group. We prove that if G is a connected Lie group (or more generally is generated by its squares), and $f \neq 0$, then g satisfies d'Alembert's equation, and f is either proportional to g, or it satisfies Kannappan's condition.

Mathematical Subject Classification 2000: 39B32
Key words: Functional equation, Wilson.

I Introduction

We will study the functional equation

$$f(xy) + f(y^{-1}x) = 2f(x)g(y), \quad x, y \in G, \tag{I.1}$$

where G is a group and $f, g : G \to \mathbb{C}$ are functions on G that we want information about. The functional equation (I.1) is very similar in appearance to Wilson's functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G, \tag{I.2}$$

but it differs from it on the second term, where the new equation (I.1) has $f(y^{-1}x)$, while the old one (I.2) has $f(xy^{-1})$. The two versions of course agree if G is abelian.

Ng [9] proved in the special case of $g = 1$ that each solution of

$$f(xy) + f(y^{-1}x) = 2f(x), \quad x, y \in G, \tag{I.3}$$

also satisfies Jensen's functional equation

$$f(xy) + f(xy^{-1}) = 2f(x), \quad x, y \in G. \tag{I.4}$$

On certain groups (I.4) possesses more solutions than (I.3) (e.g. the Heisenberg group, see [12, Example 5.1]). So the functional equations (I.1) and (I.2) need not have the same solutions. On the other hand, the functional equation

$$g(xy) + g(y^{-1}x) = 2g(x)g(y), \quad x, y \in G, \tag{I.5}$$

and d'Alembert's functional equation

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \quad x, y \in G, \tag{I.6}$$

have the same set of solutions on any group G (Proposition B.1).

Inspired by these results the purpose of the present paper is to study the solutions of (I.1) and their relations to the solutions of (I.2). We are in particular interested in conditions on G ensuring that each solution of (I.1) satisfies (I.2) as well, like in the cases of Jensen's and d'Alembert's functional equations above. We prove that so is the case, if G is generated by its squares, but also that it is not true in general (Example IX.2). Another reason for the present study is that we want to illuminate the relations between the variant (I.1) of Wilson's functional equation and d'Alembert's functional equation.

Let us briefly describe one of the main results, because it indicates that the equation (I.1) often is simpler to study than its counterpart (I.2): Let G be a connected topological group, and let $\{f, g\}$ be a continuous solution of (I.1) on G such that $f \neq 0$. Then g satisfies d'Alembert's functional equation, and there are two possibilities: Either f is proportional to g in which case (I.1) becomes d'Alembert's functional equation, or both f and g satisfy Kannappan condition, so that the form of the solutions is the same as in the abelian case (Theorem VI.3).

For complex-valued solutions our results encompass those of [9] on large classes of groups as for example the class of connected Lie groups.

We illustrate the theory by a number of examples in Sections VIII and IX.

From the aesthetical point of view it might be added that (I.1) has a more homogeneous structure than (I.2): The left hand side of (I.1) is a sum of representations (the right regular representation R and the left regular representation L), while the left hand side of (I.2) is the sum of a representation and an anti-homomorphism (R and the map $y \mapsto R(y^{-1})$ respectively).

II Notation

G denotes a group with neutral element e and center $Z(G)$. $\langle \text{squares} \rangle$ is the subgroup of G generated by the squares of elements of G. The commutator subgroup $[G, G]$ is the subgroup of G generated by the commutators $[x, y] = xyx^{-1}y^{-1}$, $x, y \in G$.

We let L and R denote respectively the left and right regular representation of G on functions on G, i.e. $[L(y)F](x) = F(y^{-1}x)$ and $[R(y)F](x) = F(xy)$ for $x, y \in G$ and $F : G \to \mathbb{C}$. L and R commute (i.e. $L(x)R(y) = R(y)L(x)$ for all $x, y \in G$) as is well known and also easy to check.

For any complex-valued function F on G we introduce certain new functions on G by

$$\begin{aligned} \check{F}(x) &= F(x^{-1}), \quad x \in G, \\ F_e &= (F + \check{F})/2, \quad F_o = (F - \check{F})/2, \\ m_F(x) &= 2F(x)^2 - F(x^2), \quad x \in G. \end{aligned}$$

F is said to be even, resp. odd, if $\check{F} = F$, resp. $\check{F} = -F$.

Kannappan's condition on $F : G \to \mathbb{C}$ is that $F(xyz) = F(xzy)$ for all $x, y, z \in G$. As is well known, Kannappan's condition is equivalent to F being a function on the abelian group $G/[G, G]$. By a classical solution of (I.1) or (I.2) we mean a solution $\{f, g\}$ where both f and g satisfy Kannappan's condition.

$\mathbb{C}^*$ denotes the multiplicative group of all non-zero complex numbers.

III Jensen's functional equation

In [9] Ng studied (I.1) in the special case of $g = 1$, where (I.1) reduces to the version of (I.3) of Jensen's functional equation. To treat a general g we must proceed in a different way than the one of [9], because the fact that $g = 1$ plays a crucial role in the computations in [9].

Our next result, Theorem III.1, is a corollary of [9, eq. (2.14)], but we present it nevertheless here, partly because it is short and simple, partly because its proof generalizes to the functional equation (I.1).

Theorem III.1. *The solutions $f : G \to \mathbb{C}$ of* (I.3) *are the functions of the the form $f = a + \alpha$, where $a : G \to \mathbb{C}$ is additive and α is a complex constant.*

Proof. Let $f : G \to \mathbb{C}$ be a solution of (I.3). The identity (I.3) says that $[L(y) + R(y)]f = 2f$, so that f for any $y \in G$ is an eigenfunction for the operator $L(y)+R(y)$ corresponding to the eigenvalue 2. Applying the operator twice to f we find that

$$\begin{aligned} 4f = [L(y) + R(y)]^2 f &= [L(y^2) + R(y^2) + 2L(y)R(y)]f \\ &= 2f + 2L(y)R(y)f, \end{aligned}$$

which reduces to $f = L(y)R(y)f$, i.e. to $f(x) = f(y^{-1}xy)$. In other words, f is invariant under inner automorphisms. It is then a solution of Jensen functional equation $f(xy) + f(xy^{-1}) = 2f(x)$. But any solution of Jensen's functional equation which is invariant under inner automorphisms, has the desired form (see [1, Lemma 1]).

The other direction of the proof is trivial to verify. $\square$

Remark III.2. It should be mentioned that the range space of f in [9] is a general abelian group H, whereas we in the present paper only work with $H = \mathbb{C}$. Our proof above of Theorem III.1 works if f takes its values in an abelian group $(H, +)$ with the property that $[h \in H,\, 2h = 0] \Rightarrow h = 0$.

IV Properties of the solutions on any group

In this section we derive basic properties of the solutions of (I.1), and we find necessary and sufficient conditions for a solution of (I.1) to satisfy (I.2).

In the case of an abelian group it is well known that the g in a solution $\{f, g\}$ of Wilson's functional equation is a solution of d'Alembert's functional equation. Recalling that m_F for any complex-valued function F on G is defined by $m_F(x) = 2F(x)^2 - F(x^2)$, $x \in G$, we have

Lemma IV.1. *Let G be any group. If $g : G \to \mathbb{C}$ is a non-zero solution of d'Alembert's functional equation* (I.6)*, then $g(e) = 1$, g is invariant under inner automorphisms, $g = \check{g}$ and $m_g = 1$.*

Proof. [11, Lemma V.1]. $\square$

As an elementary example we mention that the solution $g(x) = \cos x$ of (I.1) on $G = \mathbb{R}$ has $m_g(x) = 2(\cos x)^2 - \cos(2x) = 1$, i.e. $m_g \equiv 1$. More generally we find

Lemma IV.2. *Let the pair $f, g : G \to \mathbb{C}$ be a solution of the variant* (I.1) *of Wilson's functional equation such that $f \neq 0$.*

(a) $g(e) = 1$, g is invariant under inner automorphisms, and $g = \check{g}$.

(b) m_g is a homomorphism of G into the multiplicative group $\{\pm 1\}$.

(c) $f(y^{-1}xy) = m_g(y)f(x)$ for all $x, y \in G$.

(d) $f = m_g f$ and $g = m_g g$.

Proof. (a) To get the first statement put $y = e$ in (I.1). The identity (I.1) may be rewritten as $[L(y) + R(y)]f = 2g(y)f$ for all $y \in G$. Applying $L(z) + R(z)$ to this we get (starting with the right hand side) that

$$\begin{aligned} 4g(y)g(z)f &= [L(z) + R(z)][L(y) + R(y)]f \\ &= [L(zy) + R(zy)]f + [L(z)R(y) + R(z)L(y)]f \\ &= 2g(zy)f + [L(z)R(y) + R(z)L(y)]f, \end{aligned}$$

so

$$2g(zy)f = 4g(y)g(z)f - [L(z)R(y) + R(z)L(y)]f, \quad y, z \in G. \qquad \text{(IV.1)}$$

The value of the right hand side of (IV.1) does not change if z and y are interchanged, because L and R commute. f being non-zero, we infer that g is invariant under inner automorphisms.

Let $x_0 \in G$ be arbitrary. Assume first that either $f(x_0) \neq 0$ or $f(e) \neq 0$. Then the pair $\{f, g\}$ is a solution of (I.1) on the subgroup $\langle x_0 \rangle$ of G generated by x_0, and f is not identically 0 on $\langle x_0 \rangle$. The group $\langle x_0 \rangle$ is abelian, so the pair $\{f, g\}$ is a solution of Wilson's functional equation (I.2) on $\langle x_0 \rangle$, from which it follows that $g(x_0) = g(x_0^{-1})$.

We may thus assume that $f(e) = f(x_0) = 0$. Putting $x = e$ in (I.1) we see that f is odd. We get for any $x \in G$ that

$$\begin{aligned}
&2f(x)[g(x_0) - g(x_0^{-1})] = f(xx_0) + f(x_0^{-1}x) - f(xx_0^{-1}) - f(x_0x) \\
&= f(xx_0) + f(x_0^{-1}x) + f(x_0x^{-1}) + f(x^{-1}x_0^{-1}) \\
&= f(x_0x^{-1}) + f(xx_0) + f(x_0^{-1}x) + f(x^{-1}x_0^{-1}) \\
&= 2f(x_0)g(x^{-1}) + 2f(x_0^{-1})g(x) = 0 - 2f(x_0)g(x) = 0 - 0 = 0,
\end{aligned}$$

which implies that $g(x_0) = g(x_0^{-1})$, because $f \neq 0$.

(b) Putting $z = y$ in (IV.1) we find that

$$[2g(y)^2 - g(y^2)]f(x) = f(y^{-1}xy), \quad x, y \in G. \qquad \text{(IV.2)}$$

Defining $[i(y)F](x) := F(y^{-1}xy)$ for any complex-valued function F on G we have by (IV.2) that $i(y)f = m_g(y)f$ for all $y \in G$. Noting that $i(y_1y_2) = i(y_1)i(y_2)$ we get from the assumption $f \neq 0$ that $m_g : G \to \mathbb{C}$ is multiplicative. Since m_g is not identically 0 [this would by (IV.2) force f to be 0] it follows that $m_g(G) \subseteq \mathbb{C}^*$ and so that $m_g : G \to \mathbb{C}^*$ is a homomorphism. From $g = \check{g}$ we infer that $m_g = \check{m}_g$, so that $m_g(x)^2 = m_g(x)m_g(x^{-1}) = m_g(e) = 1$, which implies that $m_g(G) \subseteq \{\pm 1\}$.

(c) is just a reformulation of the identity (IV.2).

(d) The first result comes about when we in the formula $f(y^{-1}xy) = m_g(y)f(x)$ from Lemma IV.2(c) put $x = y$. For any $x, y \in G$ we get that

$$\begin{aligned}
&2f(x)g(y) = f(xy) + f(y^{-1}x) = f(y^{-1}yxy) + f(y^{-1}xy^{-1}y) \\
&= m_g(y)[f(yx) + f(xy^{-1})] = m_g(y)[f((y^{-1})^{-1}x) + f(xy^{-1})] \\
&= m_g(y)[f(xy^{-1}) + f((y^{-1})^{-1}x)] = 2m_g(y)f(x)g(y).
\end{aligned}$$

Since $f \neq 0$ we get that $g(y) = m_g(y)g(y)$. □

Theorem IV.3. *Let the pair $f, g : G \to \mathbb{C}$ be a solution of the variant* (I.1) *of Wilson's functional equation such that $f \neq 0$. Then the following 4 statements are equivalent:*

(a) $\{f,g\}$ *is a solution of Wilson's functional equation* (I.2)*.*

(b) f *is invariant under inner automorphisms.*

(c) g *satisfies d'Alembert's functional equation* (I.6)*.*

(d) $m_g = 1$*.*

Proof. (a) $\Leftrightarrow$ (b): Assuming (a) we subtract (I.2) from (I.1) and get that $f(y^{-1}x) = f(xy^{-1})$, which shows (b). The converse statement is trivial.

(a) $\Rightarrow$ (c): By the assumption $\{f,g\}$ is a solution of Wilson's functional equation, so g is a solution of d'Alembert's long functional equation

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \quad x, y \in G, \tag{IV.3}$$

([3, Lemma 1]). Since g is invariant under inner automorphisms (Lemma IV.2(a)) it is also a solution of d'Alembert's (short) functional equation.

(c) $\Rightarrow$ (d): $m_g = 1$ by Lemma IV.1.

(d) $\Rightarrow$ (b): This is immediate from Lemma IV.2(c). $\square$

Let the pair $f, g : G \to \mathbb{C}$ be a solution of the variant (I.1) of Wilson's functional equation such that $f \neq 0$. Then f and g may not satisfy Wilson's and d'Alembert's functional equations on all of G, but according to Lemma IV.4 they do satisfy signed versions of them on G, and on the subgroup $\ker(m_g) = \{x \in G \mid m_g(x) = 1\}$ of G they satisfy Wilson's and d'Alembert's functional equation. Outside the subgroup both functions f and g vanish identically (follows from Lemma IV.2(d)).

Lemma IV.4. *Let the pair* $f, g : G \to \mathbb{C}$ *be a solution of the variant* (I.1) *of Wilson's functional equation such that* $f \neq 0$*.*

(a) $f(xy) + m_g(y)f(xy^{-1}) = 2f(x)g(y)$ *for all* $x, y \in G$*.*

(b) $g(xy) + m_g(y)g(xy^{-1}) = 2g(x)g(y)$ *for all* $x, y \in G$*.*

Proof. (a) is an immediate consequence of Lemma IV.2(c).

(b) The identity (IV.1) says that

$$2g(zy)f(x) = 4g(y)g(z)f(x) - f(z^{-1}xy) - f(y^{-1}xz), \quad x, y, z \in G.$$

Applying Lemma IV.2(c) to it we find for any $x, y, z \in G$ that

$$\begin{aligned} 2g(zy)f(x) &= 4g(y)g(z)f(x) - m_g(y)f(yz^{-1}x) - m_g(y)f(xzy^{-1}) \\ &= 4g(y)g(z)f(x) - m_g(y)[f(yz^{-1}x) + f(xzy^{-1})] \\ &= 4g(y)g(z)f(x) - 2m_g(y)f(x)g(zy^{-1}). \end{aligned}$$

Since $f \neq 0$ we may divide through by $2f$ to get $g(zy) + m_g(y)g(zy^{-1}) = 2g(y)g(z)$, which up to a change of notation is the same as the result of (b). $\square$

We next consider the even and odd parts of the solutions.

Lemma IV.5. *Let the pair $f, g : G \to \mathbb{C}$ be a solution of the variant* (I.1) *of Wilson's functional equation.*

(a) f is odd if and only if $f(e) = 0$.

(b) Both f_e and f_o satisfy (I.1) *with the same g as for f.*

(c) $f_e = f(e)g$ and $f_e(xy) = f_e(yx)$ for all $x, y \in G$.

(d) The odd part f_o of f satisfies

$$\frac{f_o(xy) + f_o(yx)}{2} = f_o(x)g(y) + f_o(y)g(x) \text{ for all } x, y \in G. \tag{IV.4}$$

Proof. (a) follows immediately when you put $x = e$ in (I.1).

(b) For the odd part f_o of f we get

$$\begin{aligned} f_o(xy) + f_o(y^{-1}x) &= \frac{1}{2}[f(xy) - f(y^{-1}x^{-1})] + \frac{1}{2}[f(y^{-1}x) - f(x^{-1}y)] \\ &= \frac{1}{2}[f(xy) + f(y^{-1}x)] - \frac{1}{2}[f(x^{-1}y) + f(y^{-1}x^{-1})] \\ &= f(x)g(y) - f(x^{-1})g(y) = [f(x) - f(x^{-1})]g(y) = 2f_o(x)g(y). \end{aligned}$$

The computation for f_e proceeds along the same lines.

(c) Putting $x = e$ in (I.1) we get that $f_e = f(e)g$. The last statement is immediate from Lemma IV.2(a).

(d) Using that f_o satisfies (I.1) we reformulate the right hand side of (IV.4) as follows:

$$\begin{aligned} f_o(x)g(y) + f_o(y)g(x) &= \frac{f_o(xy) + f_o(y^{-1}x)}{2} + \frac{f_o(yx) + f_o(x^{-1}y)}{2} \\ &= \frac{f_o(xy) + f_o(yx)}{2} + \frac{f_o(y^{-1}x) + f_o(x^{-1}y)}{2} \\ &= \frac{f_o(xy) + f_o(yx)}{2} + \frac{f_o(y^{-1}x) - f_o(y^{-1}x)}{2} = \frac{f_o(xy) + f_o(yx)}{2}, \end{aligned}$$

which is the desired result. □

V Properties of m_g

In this section we find sufficient conditions for the function m_g to be identically 1 and study what happens if $m_g(x_0) = -1$ for some $x_0 \in G$.

Proposition V.1. *If the pair $f, g : G \to \mathbb{C}$ is a solution of* (I.1) *such that $f \neq 0$, then $m_g \equiv 1$ on the subgroups $Z(G)$ and $\langle squares \rangle$.*

Proof. By assumption there exists an $x_0 \in G$ such that $f(x_0) \neq 0$. For any $z \in Z(G)$ we get by Lemma IV.2(c) that $f(x_0) = f(z^{-1}x_0z) = m_g(z)f(x_0)$, implying that $m_g(z) = 1$. The last statement follows immediately from Lemma IV.2(b). $\square$

Theorem V.2. *Let the pair $f, g : G \to \mathbb{C}$ be a solution of* (I.1) *such that $f \neq 0$. Then $m_g \equiv 1$ if one of the following conditions holds:*

(a) G is abelian.

(b) G is generated by its squares, i.e. $G = \langle squares \rangle$.

(c) G is a finite group of odd order.

(d) G is a connected Lie group.

(e) G is a connected topological group and g is continuous.

(f) $f(e) \neq 0$.

(g) g is a solution of d'Alembert's functional equation.

Proof. (a) and (b) are immediate from Proposition V.1.
(c) In Appendix A we observe that a finite group of odd order is generated by its squares, so we may refer to (b).
(d) In Lemma A.3 we observe that a connected Lie group is generated by its squares, so we may refer to (b).
(e) m_g is continuous and takes only the values ± 1. G being connected m_g is constant, so $m_g(x) = m_g(e) = 1$.
(f) Putting $x = e$ in Lemma IV.2(c) we get that $m_g(y) = 1$ for all $y \in G$.
(g) is part of Lemma IV.1. $\square$

It might be pointed out that any solution $\{f, g\}$ of (I.1) is a solution of (I.2), unless possibly when f is odd. This is a consequence of Theorem V.2(f), combined with (a) of Lemma IV.5.

We will next examine what the consequences are if m_g is not identically 1. This phenomenon actually happens, for instance on the Heisenberg group with integer entries (See Example IX.2). It causes among other things f to be odd as the following Proposition V.3 shows. The proposition is also useful in computations in concrete examples.

Proposition V.3 (Properties of m_g). *Let the pair $f, g : G \to \mathbb{C}$ be a solution of the variant* (I.1) *of Wilson's functional equation such that $f \neq 0$.*

(a) Let $x_0 \in G$. Then $m_g(x_0) = -1 \Rightarrow f(x_0) = g(x_0) = 0$ and $g(x_0^2) = 1$.

(b) Assume that f and g are not proportional. If $x_0 \in G$, then $f(x_0) = g(x_0) = 0 \Rightarrow m_g(x_0) = -1$. If $m_g \equiv 1$, then f and g have no common zeros.

(c) Assume that there exists an $x_0 \in G$ such that $m_g(x_0) = -1$. Then $f \equiv 0$ on the centralizer of x_0 in G. In particular $f \equiv 0$ on $Z(G)$, so f is odd.

Proof. (a) The first implication follows immediately from Lemma IV.2(d) and the computation $-1 = m_g(x_0) = 2g(x_0)^2 - g(x_0^2) = 0 - g(x_0^2) = -g(x_0^2)$.

(b) Assume that $f(x_0) = g(x_0) = 0$ and that f and g are not proportional. $f_e = f(e)g$ (Lemma IV.5(c)) is proportional to g, so our assumption implies that f_o is not proportional to g. In particular $f_o \neq 0$. Furthermore f_o is a solution of (I.1) by Lemma IV.5(b). Finally $f_o(x_0) = f(x_0) - f_e(x_0) = 0 - f(e)g(x_0) = 0$, so we may replace f by f_o. Doing so we get for any $x \in G$ from the functional equation (I.1) that

$$f(x_0x) + f(x^{-1}x_0) = 2f(x_0)g(x) = 0, \tag{V.1}$$

$$f(xx_0) + f(x_0^{-1}x) = 2f(x)g(x_0) = 0. \tag{V.2}$$

Now, using (V.1), (V.2) and that f is odd in that order we find for any $x \in G$ that

$$\begin{aligned} f(x_0x) &= -f(x^{-1}x_0) = f(x_0^{-1}x^{-1}) = -f(xx_0) \\ &= -f(x_0^{-1}x_0xx_0) = -m_g(x_0)f(x_0x). \end{aligned}$$

Since $x \in G$ is arbitrary and $f \neq 0$ we see that $m_g(x_0) = -1$.

The last statement of (b) is an immediate consequence of the first statement.

(c) If z is in the centralizer of x_0 then $f(z) = f(x_0^{-1}x_0z) = f(x_0^{-1}zx_0) = m_g(x_0)f(z) = -f(z)$, so $f(z) = 0$. That f is odd comes from Lemma IV.5(a). ☐

(b) is illustrated by the fact that Sine and Cosine have no common zeros on $G = \mathbb{R}$.

VI Explicite solution formulas when $m_g = 1$

As we shall see below (Proposition VI.2) f and g satisfy Kannappan's condition unless f is even. Kannappan's condition reduces the considerations to the abelian case. We recall the result for Wilson's functional equation:

Theorem VI.1. *Let $\{f, g\}$ be a solution of* (I.1) *or* (I.2) *such that $f \neq 0$ and such that f satisfies Kannappan's condition. Then so does g.*

Furthermore there exists a homomorphism $\chi : G \to \mathbb{C}^*$ such that $g = (\chi + \check{\chi})/2$. This fixes the homomorphism χ except for interchange of χ and $\check{\chi}$.

If $\chi \neq \check{\chi}$, then $f = \alpha(\chi + \check{\chi})/2 + \beta(\chi - \check{\chi})/2$ for some constants $\alpha, \beta \in \mathbb{C}$, and if $\chi = \check{\chi}$, then $f = (\alpha + a)\chi$, where $\alpha \in \mathbb{C}$ and $a : G \to \mathbb{C}$ is additive.

Conversely, any pair $\{f, g\}$ of functions of the forms described above solves (I.1) *and* (I.2).

Proof. See, e.g., [8, Lemma 4.2] or [10, Theorem III.4]. $\square$

The next results focus on Kannappan's condition for solutions of (I.1). Let $\{f, g\}$ be a solution of (I.1) such that $f \neq 0$. A necessary condition for f to satisfy Kannappan's condition is that $m_g \equiv 1$. Indeed, if f satisfies Kannappan's condition then f is invariant under inner automorphisms and so $m_g \equiv 1$ by Theorem IV.3.

Proposition VI.2. *Let the pair $f, g : G \to \mathbb{C}$ be a solution of the variant* (I.1) *of Wilson's functional equation, such that $m_g = 1$.*

(a) If g satisfies Kannappan's condition, then f also satisfies Kannappan's condition.

(b) If $f_o \neq 0$, then both f and g satisfy Kannappan's condition.

Proof. By Theorem IV.3 we get that f is invariant under inner automorphisms. Hence so are f_e and f_o. From Lemma IV.5(d) we then infer that $f_o(xy) = f_o(x)g(y) + f_o(y)g(x)$ for all $x, y \in G$. It is known that all solutions $\{f_o, g\}$ with $f_o \neq 0$ of this functional equation satisfy Kannappan's condition (See [2] or [5]).

(a) Since $f_e = f(e)g$ by Lemma IV.5(c) we see that f_e satisfies Kannappan's condition. By the above so does f_o and hence also $f = f_e + f_o$.

(b) From the remarks at the beginning of the proof we get since $f_o \neq 0$ that g satisfies Kannappan's condition. We are done by point (a). $\square$

By help of Proposition VI.2 we now describe the complete solution of (I.1) under the assumption that $m_g = 1$. This encompasses for example all connected Lie groups.

Theorem VI.3. *Let the pair $f, g : G \to \mathbb{C}$ be a solution of* (I.1), *such that $f \neq 0$ and $m_g = 1$. Then either*

(1) $\{f, g\} = \{cg, g\}$, where $c \in \mathbb{C}^*$ and g is a non-zero solution of d'Alembert's functional equation, that does not satisfy Kannappan's condition, or

(2) There exists a homomorphism $\chi : G \to \mathbb{C}^*$ such that $g = (\chi + \check{\chi})/2$. This fixes the homomorphism χ except for interchange of χ and $\check{\chi}$.

   (a) If $\chi \neq \check{\chi}$, then $f = \alpha(\chi + \check{\chi})/2 + \beta(\chi - \check{\chi})/2$ for some constants $\alpha, \beta \in \mathbb{C}$.

   (b) If $\chi = \check{\chi}$, then $f = (\alpha + a)\chi$, where $\alpha \in \mathbb{C}$ and $a : G \to \mathbb{C}$ is additive.

Conversely, any pair $\{f, g\}$ of functions of the forms described above satisfies (I.1).

To take another example, let G be any group and consider the variant

$$f(xy) + f(y^{-1}x) = 2f(x), \quad x, y \in G, \tag{VI.1}$$

of Jensen's functional equation. Here $g = 1$, so $m_g = 1$. Now, (1) of Theorem VI.3 does not apply, and in (2) we have $\chi = \check{\chi} = 1$, so that $f = \alpha + a$, where $\alpha \in \mathbb{C}$ and $a : G \to \mathbb{C}$ is additive.

VII On the zeros of a solution

In Proposition V.3(b) and (c) we have encountered some of the properties of the zeros of f and g. In this section we list one more. It is a generalization of the fact that the zeros of Sine is a subgroup of $(\mathbb{R}, +)$.

Proposition VII.1. *Let* $\{f, g\}$ *be a solution of* (I.1) *such that* f *is odd and* $m_g = 1$. *Then* $\{x \in G \,|\, f(x) = 0\}$ *is a normal subgroup of* G.

Proof. Let $N_f = \{x \in G \,|\, f(x) = 0\}$. Clearly $e \in N_f$ (Lemma IV.5(a)), and $x \in N_f \Rightarrow x^{-1} \in N_f$, because f is odd.

Let $x_1, x_2 \in N_f$. From the functional equation (I.1) we get that

$$\begin{aligned} &f(x_1x_2) + f(x_2^{-1}x_1) = 0, \text{ and} \\ &f(x_2x_1) + f(x_1^{-1}x_2) = 0, \end{aligned}$$

where the last identity, due to f being odd, is equivalent to

$$f(x_1^{-1}x_2^{-1}) + f(x_2^{-1}x_1) = 0.$$

Comparing this with the first identity we get that

$$f(x_1x_2) = f(x_1^{-1}x_2^{-1}) = -f(x_2x_1) = -m_g(x_1)f(x_1x_2) = -f(x_1x_2),$$

which implies that $f(x_1x_2) = 0$.

That N_f is normal is seen by a simple computation: If $x \in N_f$ and $y \in G$, then $f(yxy^{-1}) = m_g(y)f(x) = m_g(y) \cdot 0 = 0$. $\square$

VIII Two general examples

First two examples covering two general classes of groups.

Example VIII.1. Let G be a connected nilpotent Lie group. The Heisenberg group is an example of such a group.

Let the pair $\{f, g\}$ be a solution of (I.1) on G such that $f \neq 0$. Then $m_g = 1$ by Theorem V.2(d), from which it follows that g satisfies d'Alembert's functional equation (Theorem IV.3). According to [4, Corollary 2.8] the function g then satisfies Kannappan's condition. From Proposition VI.2(a) we see that so does f.

Conclusion: If the pair $\{f, g\}$ is a solution of (I.1) on G such that $f \neq 0$, then both f and g satisfy Kannappan's condition, and so they are given by the formulas of Theorem VI.1.

Example VIII.2. Let G be a group for which $G = [G, G]$, for example a connected semisimple Lie group ([14, Corollary 3.18.10]). The group $SL(2, \mathbb{R})$ is an example of such a group.

Then $g = 1$ is the only classical solution of d'Alembert's functional equation on G. Indeed, if g satisfies Kannappan's condition then by Kannappan's original result $g = (\chi + \check{\chi})/2$ for some homomorphism $\chi : G \to \mathbb{C}^*$. But $G = [G, G]$, so any homomorphism of G into $\mathbb{C}^*$ is identically 1.

Let $\{f, g\}$ be a solution of (I.1) such that $f_o \neq 0$. G is generated by its squares (Corollary A.1), so that $m_g = 1$ by Theorem V.2(b). Theorem IV.3 tells us that g is a solution of d'Alembert's functional equation. Noting that g satisfies Kannappan's condition (Proposition VI.2(b)) we get from the remarks above that $g = 1$. f_o is a solution of (I.1) (Proposition IV.5(b)), which with $g = 1$ reduces to the modified Jensen functional equation (I.3). But then f_o is additive (Theorem III.1) which implies that $f_o = 0$, because $G = [G, G]$. Thus the case of $f_o \neq 0$ does not occur, so f is even. Then $f = f_e = f(e)g$ (Lemma IV.5(c)) which by Theorem IV.3(c) means that we deal with d'Alembert's functional equation.

Conclusion: The solutions $\{f, g\}$ of (I.1) with $f \neq 0$ are the function pairs of the form $\{f, g\} = \{cg, g\}$, where $c \in \mathbb{C}^*$ and g is a non-zero solution of d'Alembert's functional equation.

IX Specific examples

We continue by discussing the functional equation (I.1) on specific groups.

Example IX.1. $G = SL(2, \mathbb{R})$ is a connected semisimple Lie group. By Example VIII.2 the only solutions $\{f, g\}$ of (I.1) with $f \neq 0$ are those of the form $\{f, g\} = \{cg, g\}$, where $c \in \mathbb{C}^*$ and g is a non-zero solution of d'Alembert's functional equation. Also $g = 1$ is the only classical non-zero solution of d'Alembert's functional equation.

Straightforward computations show that the function $g(x) = \frac{1}{2}\,\mathrm{tr}(x)$, $x \in SL(2, \mathbb{R})$, is a solution of d'Alembert's functional equation. This g does not satisfy Kannappan's condition, because $g \neq 1$, a fact that of course also can be seen directly:

$$g(\begin{Bmatrix} 1 & 0 \\ 1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 1 \\ -1 & 0 \end{Bmatrix}) = \frac{1}{2}, \text{ but}$$

$$g(\begin{Bmatrix} 1 & 0 \\ 1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 1 \\ -1 & 0 \end{Bmatrix} \begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}) = 1.$$

The example shows that the condition $f_o \neq 0$ in Proposition VI.2(b) cannot be left out.

On any group G the solutions of d'Alembert's functional equation are constant on each of the conjugacy class $C_x = \{yxy^{-1} \mid y \in G\}$, because they are invariant under inner automorphisms (Lemma IV.1). An analysis of the set of conjugacy classes shows that the only continuous non-zero solutions g of d'Alembert's functional equation on $SL(2,\mathbb{R})$ are the two functions met above, i.e. $g = 1$ and $g(x) = \frac{1}{2}\operatorname{tr}(x)$, $x \in SL(2,\mathbb{R})$. We skip the details.

Example IX.2 (The Heisenberg group). The Heisenberg group

$$H_3(\mathbb{R}) = \{(x,y,z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{R}\}$$

is covered by Example VIII.1, so all solutions of (I.1) on it are classical. It might be added that the solutions of (I.2) on H_3 are written down in [13].

As a contrast, on the Heisenberg group with integer entries $G = H_3(\mathbb{Z})$ there exist solutions of (I.1) that are not solutions of (I.2) as well. Here is an example:

Let us for $(k,m,n) \in H_3(\mathbb{Z})$ define

$$\begin{aligned} f(k,m,n) &:= \frac{i^k - i^{-k}}{2} 1_{2\mathbb{Z}}(m)(-1)^n, \\ g(k,m,n) &:= \frac{i^k + i^{-k}}{2} 1_{2\mathbb{Z}}(m)(-1)^n. \end{aligned} \tag{IX.1}$$

Elementary calculations show that $\{f,g\}$ is a solution of (I.1).

A small computation based on the definition of m_g reveals that

$$m_g(k,m,n) = (-1)^m \text{ for } k,m,n \in \mathbb{Z}, \tag{IX.2}$$

so that $m_g \neq 1$. Hence $\{f,g\}$ is not a solution of Wilson's functional equation.

The above also provides an example of a solution $\{f,g\}$ of (I.1) such that neither f nor g satisfies Kannappan's condition. Indeed, as noted just before Proposition VI.2, $m_g \equiv 1$ is a necessary condition for f to satisfy Kannappan's condition, and $m_g \not\equiv 1$ here. For the claim about g we note that $g((1,1,0)(1,0,0)(0,1,0)) = g(2,2,2) = -1$, while $g((1,1,0)(0,1,0)(1,0,0)) = g(2,2,1) = 1$.

Example IX.3 (The quaternion group). The quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is neither abelian, nor is it generated by its squares. We shall see that the set of solutions of (I.1) nevertheless is a proper subset of the set of solutions of (I.2).

Let $\{f,g\}$ be a solution of (I.1) such that $f \neq 0$. We will show that $m_g \equiv 1$.

Assume first that $f_o \neq 0$. We go on by noting that $f_o(\pm 1) = 0$ because $x = x^{-1}$ for $x = \pm 1$. Since $f_o \neq 0$ there is a $y \in \{\pm i, \pm j, \pm k\}$ such that $f_o(y) \neq 0$. Now $\{f_o, g\}$ is also a solution of (I.1) (Lemma IV.5(b)), so

$$g(x) = \frac{1}{2f_o(y)}(f_o(yx) + f_o(x^{-1}y)) \text{ for all } x \in Q_8.$$

Taking $x = -1$ here we get, since $y^{-1} = -y$ for the y's in question, that

$$g(-1) = \frac{1}{2f_o(y)}(f_o(-y) + f_o(-y)) = \frac{1}{2f_o(y)}(-f_o(y) - f_o(y)) = -1.$$

If $m_g(i) = -1$, then $g(i) = 0$ by Proposition V.3(a), and so $m_g(i) = 2g(i)^2 - g(i^2) = 2 \cdot 0^2 - g(-1) = 0 - (-1) = 1$. Also $m_g(-i) = m_g(i^{-1}) = m_g(i) = 1$. Similarly we find that $m_g(\pm j) = m_g(\pm k) = 1$. Finally $m_g(\pm 1) = 1$, because $m_g = 1$ on $Z(G)$ (Proposition V.1). Thus $m_g \equiv 1$.

If $f_o = 0$ so that $f \neq 0$ is even, then f is proportional to g, so that the equation (I.1) becomes d'Alembert's equation. And then $m_g \equiv 1$ by Lemma IV.1. So $m_g \equiv 1$ in any case.

We conclude by Theorem IV.3 that on Q_8 all solutions of (I.1) satisfy Wilson's functional equation (I.2). We next describe these solutions and show that they form a proper subset of the solutions of (I.2).

We see from Theorem VI.3 that there apart from the classical solutions only may occur solutions of the form $\{f, g\} = \{cg, g\}$, where $c \in \mathbb{C}^*$ and g is a non-zero solution of d'Alembert's functional equation, that does not satisfy Kannappan's condition. As shown in [13, Example 7.4] there is exactly one such g, viz. $g = g_0$, where $g_0(\pm 1) = \pm 1$, $g_0(x) = 0$ for $x \in Q_8 \setminus \{\pm 1\}$.

In [13, Example 11.3] the solutions $\{f, g_0\}$ of Wilson's functional equation are described. There are solutions for which f is odd. Actually the dimension of the space of these odd solutions is 3. None of the pairs $\{f, g_0\}$ with f odd is a solution of (I.1), unless $f = 0$. So on Q_8 the set of solutions of (I.1) is a proper subset of the set of solutions of (I.2).

Example IX.4. Here we consider a group G such that $G/\langle\text{squares}\rangle$ is cyclic. This is a natural extension of the requirement $G = \langle\text{squares}\rangle$ that we have met earlier, for example in Theorem V.2. It is interesting, because [9, Theorem 2.2] implies that a solution of the variant (I.3) of Jensen's functional equation is a solution of Jensen's functional equation (I.4) as well, if $G/\langle\text{squares}\rangle$ is cyclic. However, as we now shall see, this result does not carry over to (I.1), when $g \neq 1$.

Let $G = S_3 = \{e, \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2\}$ be the group of permutations of 3 objects, where $\tau_1 = \{1, 2\}$, $\tau_2 = \{1, 3\}$, $\tau_3 = \{2, 3\}$, $\sigma_1 = \{1, 2, 3\}$ and $\sigma_2 = \{1, 3, 2\}$. Then $S_3/\langle\text{squares}\rangle$ is a cyclic group of order 2, generated by $\tau_1\langle\text{squares}\rangle$.

By inspection you prove that

$$f(e) = f(\tau_1) = f(\tau_2) = f(\tau_3) = 0, \quad f(\sigma_1) = -f(\sigma_2) = \exp\{2\pi i/3\},$$
$$g(e) = 1, \quad g(\tau_1) = g(\tau_2) = g(\tau_3) = 0, \quad g(\sigma_1) = g(\sigma_2) = -\frac{1}{2},$$

defines a solution $\{f, g\}$ of (I.1). The corresponding function m_g is

$$m_g(e) = 1, \quad m_g(\tau_1) = m_g(\tau_2) = m_g(\tau_3) = -1, \quad m_g(\sigma_1) = m_g(\sigma_2) = 1.$$

Now, m_g is not identically 1, so $\{f, g\}$ is not a solution of (I.2).

This example is also a counter-example to Theorem V.2(c), if we there replace the condition 'of odd order' by 'of even order'.

A On the subgroup generated by the squares

In this appendix we study the condition $G = \langle\text{squares}\rangle$. The subgroup $\langle\text{squares}\rangle$ contains the commutator group $[G, G]$ (indeed, $[x, y] = x^2(x^{-1}y)^2y^{-2}$ for any $x, y \in G$), so $\langle\text{squares}\rangle$ is a normal subgroup of G, and the quotient group $G/\langle\text{squares}\rangle$ is abelian.

Corollary A.1. *If $[G, G] = G$, then G is generated by its squares. In particular, any simple, non-abelian group is generated by its squares.*

Example A.2. The group

$$G = \langle a_1, a_2, a_3, a_4 \mid$$
$$[a_2^{-1}, a_1^{-1}] = a_2,\ [a_3^{-1}, a_2^{-1}] = a_3,\ [a_4^{-1}, a_3^{-1}] = a_3,\ [a_1^{-1}, a_4^{-1}] = a_1\rangle$$

that was considered by Higman in [7], has $G = [G, G]$, so it is generated by its squares.

The next lemma reveals that many important topological groups have the algebraic property of being generated by their squares. $\mathbb{Z}$ is an example of a group, which is not generated by its squares.

Lemma A.3. *Any connected Lie group is generated by its squares.*

Proof. Let the Lie group be G and its Lie algebra $\mathfrak{g}$. It is very well known that the image $U = \exp(\mathfrak{g})$ of the exponential map is a neighborhood of e in G. U consists of squares, because $\exp(sX)\exp(tX) = \exp((s + t)X)$ for all $s, t \in \mathbb{R}$ and $X \in \mathfrak{g}$, so that in particular $\exp(X/2)\exp(X/2) = \exp(X)$ for all $X \in \mathfrak{g}$. Since G is connected we have that $G = \bigcup_{n=1}^{\infty} V^n$ for any neighborhood V of $e \in G$ (see [6, Theorem 7.4]). Taking $V = U$ we see that G is generated by U and hence by the squares of G. $\square$

Infinite-dimensional examples also exist: Let $(\mathfrak{X}, \|\cdot\|)$ be a real or complex Banach space. Let $\mathcal{B}(\mathfrak{X})$ be the set of all bounded linear operators on $\mathfrak{X}$, and let $GL(\mathfrak{X}) = \{A \in \mathcal{B}(\mathfrak{X}) | A \text{ is invertible}\}$. Then $GL(\mathfrak{X})$ is a topological group with the identity operator I as its neutral element.

Lemma A.4. *The identity component of $GL(\mathfrak{X})$ is generated by its squares.*

Proof. Like in the proof of Lemma A.3 it suffices to produce a neighborhood U of I, such that U consists of squares from the identity component. As U we may take $\{A \in \mathcal{B}(\mathfrak{X}) \,|\, \|A - I\| < 1\}$. Indeed, if $\|A - I\| < 1$ then $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (A - I)^n$ is a square root of A in the identity component. □

The condition $G = \langle\text{squares}\rangle$ is not found in the standard textbooks on algebra, so let us give an equivalent characterization of it.

Lemma A.5. *$G = \langle\text{squares}\rangle$ if and only if G has no subgroup of index 2.*

Proof. We will show the equivalent statement that $G \neq \langle\text{squares}\rangle$ if and only if G has a subgroup of index 2.

Let us first assume that $G \neq \langle\text{squares}\rangle$. Then $G/\langle\text{squares}\rangle$ is a non-trivial abelian group. Let $x_0 \in G \setminus \langle\text{squares}\rangle$. Viewing G as a discrete group we get from the theory of locally compact abelian groups that there exists a homomorphism $\chi : G/\langle\text{squares}\rangle \to \mathbb{T}$ such that $\chi(x_0\langle\text{squares}\rangle) \neq 1$ ([6, Theorem 22.17]). Consider $m = \chi \circ \pi$, where $\pi : G \to G/\langle\text{squares}\rangle$ is the canonical projection. $m : G \to \mathbb{T}$ is a homomorphism, and $m(x_0) \neq 1$. Since m is identically 1 on squares, the values of m are only ± 1. Since $m(x_0) \neq 1$ we see that $m(x_0) = -1$. Now $H = \ker m$ is a subgroup of G of index 2.

Let us conversely assume that G has a subgroup H of index 2. The index being 2, H is normal. Again, since the index is 2, we see that any square belongs to H, so $\langle\text{squares}\rangle \subseteq H$. But then $\langle\text{squares}\rangle \neq G$, because H is a proper subgroup of G. □

In particular any finite group of odd order is generated by its squares.

To take an example to the contrary, we note that the symmetric group S_n is not generated by its squares, because it has a subgroup of index 2, viz. the alternating group A_n.

B Symmetry of the right hand side

The Proposition below applies to various functional equations. It is surely known and easy to prove, but I don't know of any explicite reference. It shows for example that we get nothing new by replacing the left hand side of d'Alembert's functional equation or of the quadratic functional equation by the versions corresponding to the variant (I.1) of Wilson's functional equation.

Proposition B.1. *Let G be any group, and let H be an abelian group. Let $g : G \to H$, and let $B : G \times G \to H$ be symmetric. Then*

$$g(xy) + g(y^{-1}x) = B(x, y), \quad \forall x, y \in G,$$
$$\Updownarrow$$
$$g(xy) + g(xy^{-1}) = B(x, y), \quad \forall x, y \in G.$$

If one of the two identities holds, then g is even and invariant under inner automorphisms.

Proof. Let us assume that the first identity holds, i.e. that $g(xy)+g(y^{-1}x) = B(x, y)$, for all $x, y \in G$. Putting $y = e$ we get that $2g(x) = B(x, e)$. Using that we get putting $x = e$ in the first identity that $g(y)+g(y^{-1}) = B(e, y) = B(y, e) = 2g(y)$, proving the claim $g = \check{g}$.

Now, we get for any $x, y \in G$ that

$$g(xy) + g(y^{-1}x) = B(x, y) = B(y, x) = g(yx) + g(x^{-1}y)$$
$$= g(yx) + \check{g}(x^{-1}y) = g(yx) + g(y^{-1}x),$$

which implies the second claim, i.e. that $g(xy) = g(yx)$ for all $x, y \in G$. The second claim ensures that the second identity holds.

The proof of the implication in the other direction proceeds along the same lines as the one just given, so we leave it out. □

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