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Kolmogorov and the Turbulence

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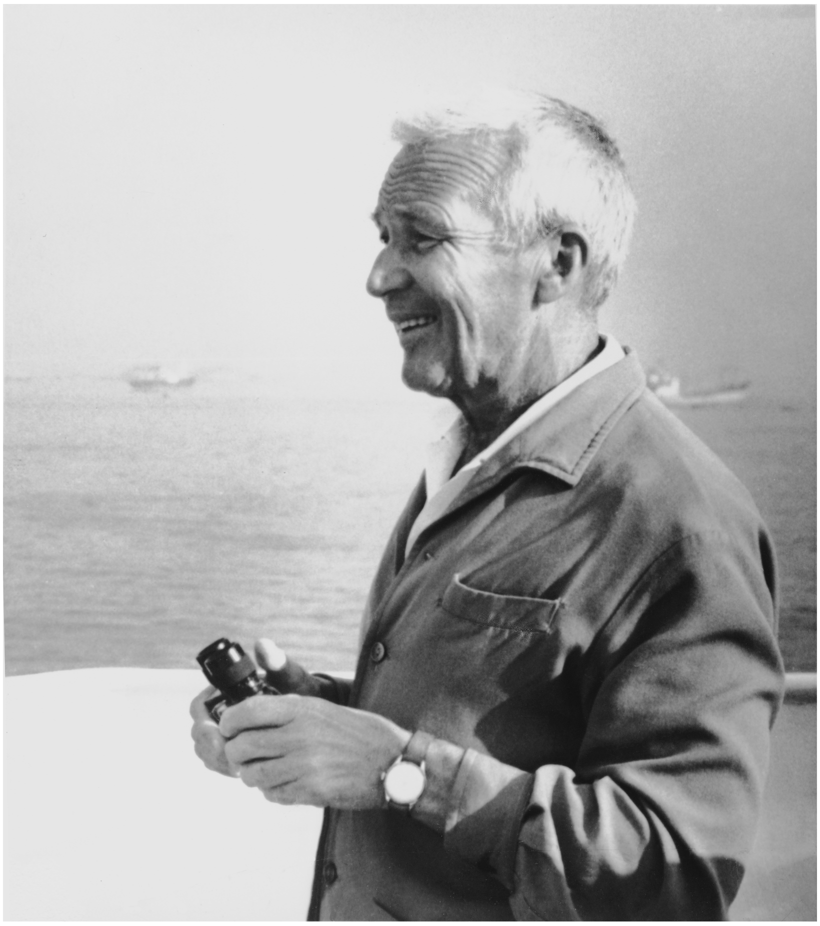

Kolmogorov on board the *Dimitry Mendelyeev*

Kolmogorov and the Turbulence

A. N. Shiryaev

May 20, 1999

1 Introduction

In August, 1990, at the Second International Congress of the Bernoulli Society in Uppsala (Sweden), I made a short report entitled "Everything about Kolmogorov was unusual ... ". I said: "*Andrei Nikolaevich Kolmogorov was one of a selected group of people, who made you feel that you had met an unusual, great and extraordinary person. That was the feeling of having met a wonder.*

Everything about Kolmogorov was unusual: his entire life, his school and university years, his pioneering discoveries in many areas of mathematics and in such disciplines as meteorology, hydrodynamics, history, linguistics, ... pedagogy ... His interests were unusually diverse including music, architecture, poetry and travelling. His erudition was unusual: he had a qualified opinion about everything".

This paper was prepared for the workshop "Turbulence and Finance" organized by MaPhySto (5-7 May, 1999, Aarhus) with a view to "discuss the striking similarities as well as the differences between key empirical features observed in the financial markets and the turbulence of fluids".

In the last few years the connection between these two disciplines has been stressed by several authors. In October 1995 O. E. Barndorff-Nielsen gave at Columbia University a talk "Probability and Statistics: Self-decomposability, *Finance* and *Turbulence*" where he discussed "key features of empirical data from finance and turbulence" which "are widely recognized as being essential for understanding and modelling within these two, quite different, subject areas", {10}.

In unison, with these words the journal "Nature" (Vol. 38, June,

1996) used on its cover the term

FINANCIAL TURBULENCE.

publishing in this issue the article "Turbulent Cascades in Foreign Exchange Markets" by five authors S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner and Y. Dodge. The article shows the similarity in the statistical behaviour of financial data and turbulent data. In the abstract to the article, the authors note that: "*The availability of high frequency data for financial markets has made it possible to study market dynamics on timescales shorter than a day. For foreign exchange (FX) rates, V. A. Muller et. al. (J. Banking Fin., 1990, 14, 1189-1208) have shown that there is a net flow of information from long to short timescales: the behaviour of long-term traders (who watch the markets only from time to time) influences the behaviour of short-time traders. Motivated by this hierarchical feature, we have studied FX market dynamics in greater detail and will show here an analogy between the dynamic and the hydrodynamic turbulence. Specifically, the relationship between the probability density of FX price changes* (ΔX) *and the time delay* (Δt) *(Fig. 1a) is much the same as the relationship between the probability density of the velocity differences* (ΔV) *of two points in a turbulent flow and their spatial separation* Δr *(Fig. 1b).*

Guided by this similarity, we claim that there is an information cascade in FX market dynamics that corresponds to the energy cascade in the hydrodynamic turbulence ... The analogy gives a conceptual framework for understanding the short-term dynamics of speculative markets". At the end of their article the authors conclude optimistically: " ... *We have reason to believe that the qualitative picture of turbulence that has been developed during the past 70 years, will help our understanding of the apparently remote field of financial markets*". (See also Chapters III and IV in {11}.)

In the present report, I am not going to compare the statistical data of the turbulence and of the finance. I would like to give a review of Kolmogorov's different periods of the turbulence study as well as of his main conceptions and the results achieved in this field, that influenced the later development of hydrodynamics.

There are *three* periods when Kolmogorov studied the turbulence.

The *first* period lasted from the late 30-s to the early 40-s. In that period he published his classical works (see also {1}):

[1] The local structure of turbulence in an incompressible fluid at very high Reynolds numbers. Dokl. Acad. Nauk USSR, 30 (1941), p. 299-303.

[2] The logarithmically normal distribution of the size of particles under the fragmentation. Dokl. Acad. Nauk USSR, 31 (1941), p. 99-101.

[3] The decay of isotropic turbulence in an incompressible viscous fluid. Dokl. Acad. Nauk USSR 31 (1941), p. 538-541.

[4] Energy dissipation in locally isotropic turbulence. Dokl. Acad. Nauk USSR, 32 (1941), p. 19-21.

On January 26, 1942, Kolmogorov presented his works on the turbulence at the Joint Meeting arranged in Kazan by the Department of Physics and Mathematics of the USSR Academy of Sciences.

[5] Equations of turbulent motion of an incompressible fluid. Izv. Acad. Nauk USSR, ser. Fiz. 6(1942), p. 56-58.

P.A. Kapitza and L.D. Landau took part in the discussion. Landau pointed out that "*A.N. Kolmogorov was the first to give the right conception of the local structure of the turbulent flow*".

Kolmogorov explained his interest in the turbulence as follows: "I took an interest in the study of turbulent flows of liquids and gases in the late thirties. It was clear to me from the very beginning that the main mathematical instrument in this study must be the theory of random functions of several variables (random fields) which had only then originated. Moreover, it soon became clear to me that there was no chance of developing a purely closed mathematical theory. Due to the lack of such a theory, it was necessary to use some hypotheses based on the results of the treatment of the experimental data. Therefore, it was important to find talented collaborators who were able to combine theoretical studies with the analysis of the experimental results. In this respect, I was quite successful". (Kolmogorov mentioned his students A. M. Obukhov, M. D. Millionshchikov, A. S. Monin and A. M. Yaglom; see {1})

The "**Law of two-thirds**" is the pearl of the first investigations by A.N. Kolmogorov.This is a universal law of the turbulence nature, supported by the experiments made for the fluids with high Reynolds numbers (see {1}).

The *second* period of Kolmogorovs's investigation of the turbulence started in the early 60-s. It was mainly related to his participation (with a group of his followers A. N. Obukhov, M. D. Millionschikov, A. M. Yaglom) in the two International Meetings on the Mechanics of Turbulence, arranged in Marseilles by the IUTAM (The International Union of Theoretical and Applied Mechanics) and the IUGG (The International Union of Geodesy and Geophysics).

The main ideas of Kolmogorov's report were presented in his two articles (see also {1}):

[6] Les Précesions sur la structure locale de la turbulence dans un flux visqueux aux nombres de Reynolds elevés. En Méchanique de la Turbulence. Coll. Int. du CNRS à Marseille, p. 447-458, Paris, CNRS, (1962).

[7] A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds numbers. J. Fluid Mech, 13 (1962), 82-85.

It can be seen from the titles of these articles that they are concerned with the refinement of the results obtained in the early 40-s. The verifications were mostly due to L.D. Landau's remarks made on the report [5] in 1942. Landau emphasized the fact that "the velocity rotor in the turbulent flow exists only in the restricted part of the space". These articles are closely connected with the report by A.M. Obukhov {3} that included new ideas about the development of the locally isotropic turbulence theory (see the last section of the article [5] and §6.4 in the book {3}).

Finally, the *third* (not widely known) period of Kolmogorov's study of the turbulence is related to his participation in the expeditions on board the "Dmitry Mendeleev" research ship (the 2^{nd} and the 5^{th} voyages).

The first expedition lasted from June 23 to September 18, 1969. Its duration was 87 days. The route was as follows:

> Kaliningrad → Reykjavik (the capital of Iceland) → Rio-de Janeiro (with a call at Konakri (Guinea)) → Dakar (Senegal) → Gibraltar → Kaliningrad.

The second expedition lasted from January 20 to May 12, 1971 (26 132 sea miles were covered). In fact, that was a round-the-world journey: by train from Moscow to Kaliningrad and a long voyage aboard the "Dmitry Mendeleev" ship. The route was as follows:

> Kaliningrad → The Kiel Canal → Pas de Calais → The Bay of Biscay → The Sargasso Sea → across The Gulf Stream to Cape Canaveral (The State of Florida) → The port of Kingston (Jamaica) → The Panama Canal → The Galapagos Islands (a territory possessed by Ecuador where Ch. Darwin elaborated his theory of the Origins of the Species) → The port of Honolulu (The Hawaii) → the passage to the South along the 159^o meridian of the western longitude → the coral Atoll of Fanning (Brit.) → the crossing of the Equator → the bay of Avarua on the island of Rapotong (the central island in the Cook Archipelago protected by New Zealand) → the passage to the West → the port of Suva (the Fiji Islands) → the port of Vila (Esratos Island), Touman Island, Malecula Island (the New Hebrides) → the port of Yokohama (Japan) → Vladivostok. Then Andrei Nikolaevich came back to Moscow by train. The picture on the front-page of this paper was taken at the Moscow train station.

In both voyages A. N. Kolmogorov was the Assistant Director in research. Prof. A. Monin, the Director of the expedition in 1971, wrote (see {5}): "Andrei Nikolaevich was responsible for the geophysical oceanic investigations. Five teams were engaged in that work, equipped with various devices. Some of the devices were new and did not function properly. Kolmogorov spared no effort and time in checking the measurement accuracy and the calibration of the devices as well as in revealing the interferences that distorted the readings."

It is a pity that a great mathematician had to waste time on solving insignificant problems. But he rejected any attempts to release him from his duties and would check himself the quality of the measurements.

I have known before that such an attitude was natural for Andrei Nikolaevich. He expressed it best in his last interview with a documentary-film maker A.N. Marutyan: “In fact, when the mathematician solves, for example, a hydrodynamics problem (I myself was dealing with the hydrodynamics of the ocean), it means that a hydrodynamics problem is solved by mathematical means. The mathematicians always want that their mathematics should be pure, that is, strict and provable, wherever possible. However, the most interesting and realistic problems could not usually be solved in that manner. Therefore, it is very important that the mathematician should be able to find the approximative (not necessarily strict but effective) ways of solving such problems. At any rate, I've always done it by this means... If turbulence is an object of my studies, I am dealing with the **turbulence**. I rate highly those mathematicians who, as a matter of fact, cannot be called “pure” mathematicians. They just solve applied problems by strict methods, if possible, or by making “hypotheses”.

– So, you are in favor of the flexibility of thinking?

– And in favor of the direct participation, where possible, in the experiments together with the physicists”.

2 Locally Isotropic Turbulence. The Law of 2/3.

The articles [1]–[5] were published during the *first* period of Kolmogorov's study of the turbulence in the early 40-s. These articles were preceded by the following two papers (see also {1}):

[8] Curves in a Hilbert space that are invariant under the one-parameter group of motions. Dokl. Acad. Nauk USSR, 26 (1940), p. 6-9.

[9] Wienersche Spiralen und einige andere interessante Kurven in Hilbertschen Raum. Dokl. Acad. Nauk USSR, 26 (1940), p. 115-118.

These articles are closely connected both with the turbulence and the general theory of random processes as well as with the stochastic finance.

We will now give some definitions and facts concerning homogeneous random processes and fields, in order to describe the mathematical notions of these papers.

Let $S = \{s\}$ be a homogeneous space of points s with a transitive group $G = \{g\}$ of transformations mapping the space S into itself $(S \to gS)$.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and let $X = X(s)$, $s \in S$ be a random field on this probability space, that is, $X(s) = X(s, \omega)$, $\omega \in \Omega$, $s \in S$ is a family of (complex-valued) random variables.

The random field $X = X(s)$, $s \in S$ is called (wide-sense) *homogeneous* if

$$\begin{aligned}
&\mathsf{E}\,|X(s)|^2 < \infty,\\
&\mathsf{E}X(s) = \mathsf{E}X(gs),\\
&\mathsf{E}X(s)\overline{X(t)} = \mathsf{E}X(gs)\overline{X(gt)}
\end{aligned}$$

for all $s, t \in S$ and $g \in G$.

The following special case is of particular importance: $S = \mathbb{R}^k$ and G is the group of parallel shifts. The homogeneous random field is often defined as a field of this type.

If G is a group of isometric transformations on $S = \mathbb{R}^k$ (generated by the parallel shifts, the rotations and the reflections), then $X = X(s)$, $s \in \mathbb{R}^k$ is called the *homogeneous isotropic random field.*

The special case of the homogeneous field is a (wide-sense) stationary process $X(s)$, $s \in \mathbb{R}$ where $\mathsf{E}\,|X(s)|^2 < \infty$, $\mathsf{E}X(s) = m(= const)$ and $\mathsf{E}X(s)\overline{X(t)}$ depends only on the difference $t - s$.

Assume $m = 0$ and denote by

$$R(t) = \mathsf{E}X(s+t)\overline{X(s)} \tag{1}$$

the *correlation* function of the process X.

Due to the Bochner-Khinchin theorem (for the mean-square continuous processes X), this function admits the *spectral representation*

$$R(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, F(d\lambda) \tag{2}$$

Here, $F = F(A)$ is a finite measure on the Borel sets $A \in \mathcal{B}(\mathbb{R})$. This measure is called the *spectral measure*, and the function

$$F(\lambda) = \int_{-\infty}^{\lambda} F(d\nu) \quad (= F(-\infty, \lambda]) \tag{3}$$

is called the *spectral function.* If $F'(\lambda)$ exists, then $f(\lambda) = F'(\lambda)$ is called the *spectral density* or the *energy spectrum*, or simply the spectrum. The magnitude $f(\lambda)\,d\lambda$ can be described as follows: it is a contribution to the "energy" of the harmonics whose frequencies are within the interval $(\lambda, \lambda + d\lambda)$.

It is important to note that (2) implies

$$\mathsf{E}[X(t) - X(s)]^2 = 2 \int_{-\infty}^{\infty} \left(1 - e^{i\lambda(t-s)}\right) F(d\lambda) \tag{4}$$

If X is a *real-valued* process, then $R(t) = R(-t)$ and

$$R(t) = \int_{-\infty}^{\infty} \cos(\lambda t)\, F(d\lambda), \tag{5}$$

$$\int_{-\infty}^{\infty} \sin(\lambda t)\, F(d\lambda) = 0. \tag{6}$$

Thus, the spectral function is symmetric with respect to the point $\lambda = 0$.

Set

$$G(\lambda) = F(\lambda) - F(-\lambda) \quad (= \int_{|\nu| \le \lambda} F(d\nu)).$$

Then we obtain

$$\begin{aligned} G(\lambda) &= 2F(\lambda) - R(0), \\ G(\lambda) &= \frac{2}{\pi} \int_0^{\infty} \frac{\sin(\lambda t)}{t} R(t)\, dt \end{aligned}$$

and

$$R(t) = \int_0^{\infty} \cos(\lambda t)\, G(d\lambda), \tag{7}$$

$$\mathsf{E}\,|X(t) - X(s)|^2 = 2 \int_0^{\infty} (1 - \cos(\lambda(t-s))\, G(d\lambda). \tag{8}$$

Essentially, the same is true for the *homogeneous one-dimensional random real-valued fields* $X(s)$, $s \in \mathbb{R}^k$ (with the group G of parallel shifts). Here, the spectral representation (2) is reformulated in the following way: for the mean-square continuous complex-valued random field $X = X(s)$, $s \in \mathbb{R}^k$, the covariance function (or the correlation function in the case of centered random fields) $R(t)$ admits the representation

$$R(t) = \int\limits_{\mathbb{R}^k} e^{i(t,\lambda)} F(d\lambda) \tag{9}$$

where $F = F(d\lambda)$ is a finite (uniquely determined) non-negative measure on $\mathcal{B}\left(\mathbb{R}^k\right)$.

For the homogeneous *isotropic* fields, the covariance function $R(t)$, $t = (t_1, \ldots, t_k)$ is defined as a function of $\|t\| = \sqrt{t_1^2 + \ldots + t_k^2}$: $R(t) = R(\|t\|)$. In this case, the integration over $\mathbb{R}^k$ in (9) is replaced by the integration over $\mathbb{R}_+$. To be more precise, we have

$$R(u) = 2^{\frac{k-2}{2}} \Gamma\left(\frac{k}{2}\right) \int\limits_{-\infty}^{\infty} \frac{I_{\frac{k-2}{2}}(\lambda u)}{(\lambda u)^{\frac{k-2}{2}}} Q(d\lambda) \tag{10}$$

where $I_\nu(x)$ is the Bessel function of index ν, Q is a non-negative random measure on $\mathcal{B}(\mathbb{R}_+)$ such that $Q(\mathbb{R}_+) = G(\mathbb{R}^k) = R(0)$.

Similar results are true for the homogeneous isotropic *vector fields* (defined on $\mathbb{R}^k$ with values in $\mathbb{R}^l$) (see, for example, {6}, {7}).

It is remarkable that for the (wide-sense) stationary mean-square continuous random processes $X = X(t)$, $t \in \mathbb{R}$, the spectral representation is valid both for the correlation function $R(t)$ and for the process X. The following result is due to Karhunen, Kolmogorov and Cramér: there exists a complex-valued random measure $Z = Z(A)$ (or $Z = Z(A;\omega)$) with orthogonal values (that is, $\mathsf{E} Z(A)\overline{Z(B)} = 0$ if $A \cap B = \emptyset$), such that

$$X(t) = \int\limits_{-\infty}^{\infty} e^{i\lambda t} Z(d\lambda). \tag{11}$$

Moreover, $\mathsf{E}\left|Z(A)\right|^2 = F(A)$.

The similar representation is valid for the homogeneous (both one-dimensional and vector) random fields on $\mathbb{R}^k$ with values in $\mathbb{R}^l$.

The role of the homogeneous random fields in the turbulence was for the first time emphasized by Taylor in $\{8\}$. This paper deals with the statistical theory of the turbulence. It was in this paper that Taylor introduced the important notion of the *isotropic* turbulence.

A.N. Kolmogorov pointed out in [1] that "Taylor's isotropy hypothesis has a good experimental support for the turbulence generated by the flow passing through the grating. In most other cases of practical interest, this hypothesis can be considered as a very rough approximation even for small areas ... and for extremely high Reynolds numbers".

It was mentioned above that the first of Kolmogorov's work on the turbulence was preceded by the papers [8], [9], in which he made the first step towards introducing the notion of *locally* homogeneous random fields. This notion (together with the notion of *local isotropy*) became the main mathematical means in analyzing the turbulent phenomena (especially in the case of high Reynolds numbers).

In [8], A.N. Kolmogorov considered a random field on $\mathbb{R}^1$ with values in $\mathbb{R}^1$. Such fields $Y(s)$, $s \in \mathbb{R}$, are the random processes with (wide-sense) stationary or homogeneous increments.

For such processes, the increments

$$\Delta_r Y(t) = Y(t) - Y(t-r)$$

(rather than the values of $Y(t)$) are supposed to be such that the function $\mathsf{E}\Delta_r Y(t)$ depends only on r ($\mathsf{E}\Delta_r Y(t) = m(r)$) and the function $\mathsf{E}\Delta_{r_1} Y(t+s)\Delta_{r_2} Y(s)$ does not depend on s for any s, t, r_1, r_2. This will be stressed by the notation

$$\mathsf{E}\Delta_{r_1} Y(t+s)\Delta_{r_2} Y(s) = D(t, r_1, r_2)$$

The function $D(t, r_1, r_2)$ is called the *structural* function of the process Y.

Due to the equality

$$(a-b)(c-d) = \frac{1}{2}\left[(a-d)^2 + (b-c)^2 - (a-c)^2 - (b-d)^2\right],$$

the structural function $D(t, r_1, r_2)$ can be represented as a function of one variable:

$$D(r) \equiv D(0, r, r) = \mathsf{E}\left|\Delta Y_r(t)\right|^2, \tag{12}$$

which is also called the structural function.

Kolmogorov obtained in [8] the following spectral representation for $D(r)$ and $D(t, r_1, r_2)$:

$$D(r) = 2 \int\limits_{\mathbb{R}\backslash\{0\}} (1 - \cos \lambda r)\Phi(d\lambda) + ar^2 \tag{13}$$

and

$$D(t, r_1, r_2) = \int\limits_{\mathbb{R}\backslash\{0\}} e^{i\lambda t} \left(1 - e^{-i\lambda r_1}\right) \left(1 - e^{i\lambda r_2}\right) \Phi(d\lambda) + ar^2. \tag{14}$$

Here, a is a constant, $a \geq m^2$, m satisfies the equality $m = m(r)$ $(= \mathsf{E}_{\Delta_r} Y(t))$ and Φ is a measure on $\mathbb{R}\backslash\{0\}$ such that

$$\int\limits_{\mathbb{R}\backslash\{0\}} \frac{\lambda^2 \Phi(d\lambda)}{1 + \lambda^2} < \infty. \tag{15}$$

Remark. *The comparison of formula* (13) *for* $a = 0$ *and formula* (8) *shows that they are very similar (this is not true for the covariance functions of* X *and* Y*).*

A.N. Kolmogorov obtained in [8] the following spectral representation for the process Y:

$$Y(t) = \int\limits_{-\infty}^{\infty} \left(e^{i\lambda t} - 1\right) Z(d\lambda) + ut + v \tag{16}$$

where u and v are random variables with finite second moments; $Z = Z(A)$, $A \in \mathcal{B}(\mathbb{R})$ is a random measure with orthogonal values and such that

$$\mathsf{E} \left|Z(A)\right|^2 = \Phi(A), \qquad A \in B(\mathbb{R}).$$

If the process Y is (wide-sense) stationary, then the spectral decomposition (11) can be derived from (16). Similarly to the passage from the stationary processes (which are automatically isotropic in the one-dimensional case) to the homogeneous fields, one can pass from the processes with the stationary increments to the locally homogeneous and locally isotropic $\mathbb{R}^l$-valued vector fields

$$\overline{Y} = \overline{Y}(\overline{t}) = (Y_1(\overline{t}), \dots, Y_l(\overline{t})).$$

Let

$$\left\|D_{ij}(\overline{t},\overline{r}_1,\overline{r}_2)\right\| = \left\|\mathsf{E}\Delta_{r_1}Y_i(\overline{t}+\overline{s})\Delta_{r_2}Y_j(\overline{s})\right\|$$

and

$$D_{ij}(\overline{r}) = D_{ij}(0,\overline{r},\overline{r}).$$

For the locally homogeneous and locally isotropic fields, one has

$$D_{ij}(\overline{r}) = [D_{ll}(r) - D_{kk}(r)]\,\frac{r_i r_j}{r^2} + D_{kk}(r)\delta_{ij}$$

where $r = \|\overline{r}\|$ and $D_{ll}(r)$, $D_{kk}(r)$ are the longitudinal and transverse structural functions:

$$D_{ll}(r) = \mathsf{E}\left|Y_l(\overline{t}+\overline{r}) - Y_l(\overline{t})\right|^2, \quad D_{kk}(r) = \mathsf{E}\left|Y_k(\overline{t}+\overline{r}) - Y_k(\overline{t})\right|^2.$$

Here, $Y_l(\overline{t})$ is a projection of the vector $\overline{Y}(\overline{t})$ on the direction $\overline{r}$ and $Y_k(\overline{t})$ is a projection of the same vector on the direction orthogonal to $\overline{r}$.

Remark. *The functions $D_{ij}(\overline{r})$ are simpler than the functions $D_{ij}(\overline{t},\overline{r_1},\overline{r_2})$. Therefore, it is important to find out whether the latter functions can be expressed by the former ones (that is, through $D_{ll}(r)$ and $D_{kk}(r)$). This can be done under the condition $D_{ij}(\overline{t},\overline{r}_1,\overline{r}_2) \to 0$, $|\overline{t}| \to \infty$ (see {6}). In particular, this can be done in all the applications to the turbulence of the locally homogeneous and locally isotropic random vector fields (for details, see {6}, p. 315).*

We will now consider the paper [9], which is closely connected with the paper [8]. A.N. Kolmogorov investigates in [9] the structure of the continuous Gaussian processes $X(t)$, $t \geq 0$ with stationary increments and with the self-similarity property, that is, for any $a > 0$, there exists $b > 0$ such that

$$Law\,(X(at);t \geq 0) = Law\,(bX(t);t \geq 0)\,.$$

It turns out that such processes with the zero mean have a special correlation function:

$$\mathsf{E}X(t)X(s) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right) \tag{17}$$

where $0 < H < 1$.

Kolmogorov called such Gaussian processes the "Wiener Spirals". Later they were named the Fractional Brownian Motion.

Note that

$$\mathsf{E}\|X(t) - X(s)\|^2 = |t - s|^{2H}. \tag{18}$$

The parameter H was called the Hurst parameter.

Let us now turn to the main results of the papers [1]–[4], which contain the "law of two-thirds".

Let $\overline{u}(\overline{x}) = (u_1(x_1, x_2, x_3), u_2(x_1, x_2, x_3), u_3(x_1, x_2, x_3))$ be a field of velocities of the turbulent flow at the point $x = (x_1, x_2, x_3)$.

Kolmogorov assumes that this field is locally homogeneous and is locally isotropic. So, unlike Taylor, he introduces the idea of *locality*. This reduces the analysis to the study of the structure of *increments*

$$\overline{u}(\overline{x} + \overline{r}) - \overline{u}(\overline{r}).$$

Further, Kolmogorov introduces "*the first hypothesis*" of similarity. Using this hypothesis, he deduces that the longitudinal structural function $D_{ll}(r)$ has the form r^m for a wide range of values r (compare with (18)).

In order to find the value m, Kolmogorov introduces "*the second hypothesis*" of similarity, which, together with the first one, yields $m = 2/3$. Therefore,

$$D_{ll}(r) \sim r^{2/3}. \tag{19}$$

Kolmogorov formulates this result more precisely in the following way: *if the dissipation rate ε of the kinematic energy is constant, then for the turbulent movements with a very high Reynolds number ($Re = \frac{Lv}{\nu}$ where L and v are length and velocity scales for the whole movement, the so-called typical scales, and ν is kinematic viscosity) and in the "inertia" interval of scales*

$$\lambda \ll r \ll L$$

where $\lambda = \varepsilon^{1/4}\nu^{-3/4}$ ("Kolmogorov's interior scale"), one has the following approximation:

$$D_{ll}(r) \approx C(\varepsilon r)^{2/3} \tag{20}$$

where C is a constant. Moreover, $D_{kk}(r) \approx \frac{3}{4} D_{ll}(r)$.

Along with the above "*correlational formulation*", the fundamental law of the small-scale turbulence (20) permits the *spectral formulation*. For $D_{ll}(r)$, one can obtain the spectral representation similar to (13) (with the spectral measure $\Phi_{ll}(dk) = E_{ll}(k)dk$). It turns out that

$$E_{ll}(k) \sim k^{-5/3} \tag{21}$$

for a wide range of frequencies k.

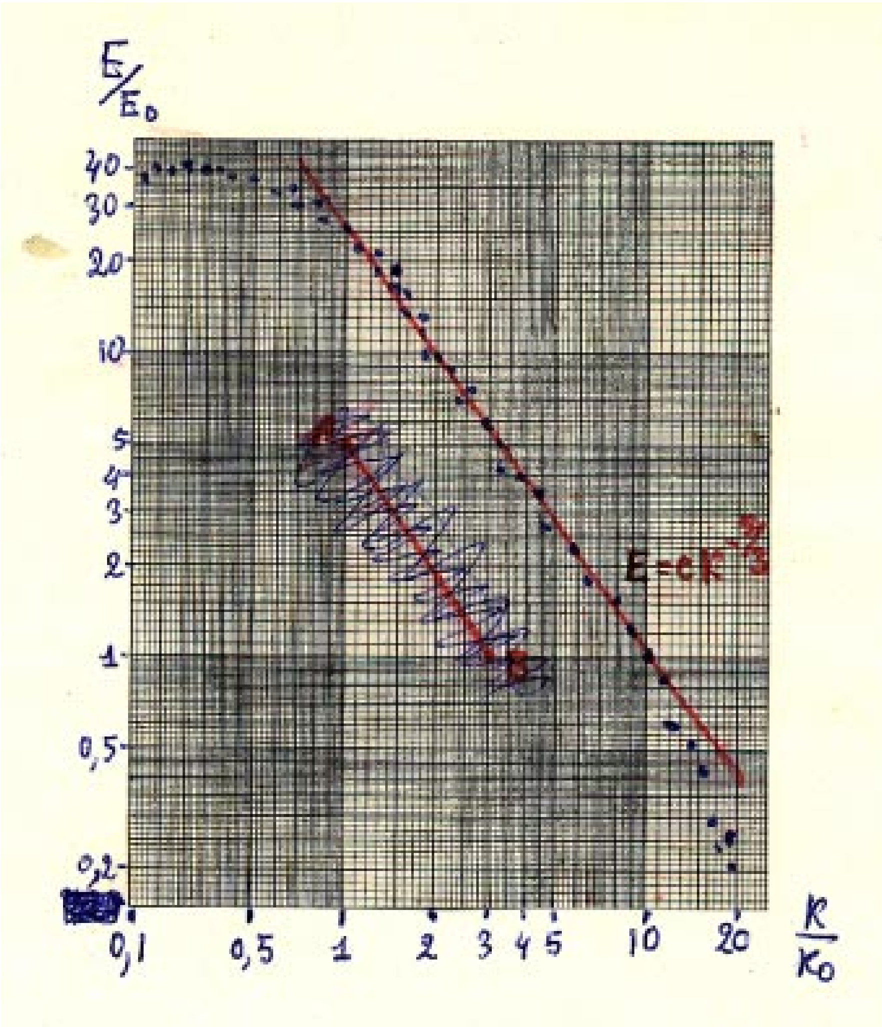

Figure 1: To the law "$E_{ll}(k) \sim k^{-5/3}$" (Kolmogorov's drawing)

In connection with (20) and (21), note that due to the properties of the Fourier transform for $f(\lambda) = |\lambda|^{-\alpha}$ with $1 < \alpha < 3$, one has

$$\int_{-\infty}^{\infty} (1 - e^{i\lambda\tau}) f(\lambda) d\lambda = \tau^{\alpha-1} \int_{-\infty}^{\infty} (1 - e^{i\lambda}) |\lambda|^{-\alpha} d\lambda,$$

and therefore, $\tau^{\alpha-1} = \tau^{5/3-1} = \tau^{2/3}$ for $\alpha = 5/3$.

At the end of the analysis of the main "turbulent" results published in Kolmogorov's papers in the 40-s, I would like to stress that Kolmogorov made these studies, in fact, as a physicist employing clear and natural physical assumptions.

Kolmogorov always used the experimental data in order to justify and verify his hypotheses.

As mentioned above, the *second* period of Kolmogorov's study of the turbulence was in the 60-s. The results of those studies were published in the papers [6] and [7].

These articles as well as the previous ones are notable for their physical style. Here, neither mathematical proofs nor complicated analytical calculations are present, but *three* new hypotheses of similarity are proposed instead of the *two* previous hypotheses.

It is interesting to note that the two previous hypotheses of similarity are related to the velocity increments while two of the new hypotheses are formulated in terms of velocity increments ratios.

These two new hypotheses were supplemented by the third hypothesis postulating the logarithmic normality of the energy dissipation rate ε and indicating the form of the logarithmic variance of the averaged dissipation rate $\overline{\varepsilon}_r$ (for details, see formulas (1) and (2) in [6]).

It turns out that these three hypotheses make it possible to account for L.D. Landau's remark that the variation of the energy dissipation

$$\varepsilon = \frac{\nu}{2} \sum_{\alpha,\beta} \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right)^2$$

should infinitely increase if the ratio $L : l$ (between the "exterior" scale L and "interior" scale l) increases.

3 Kolmogorov's Oceanographic Expeditions.

In the first expedition on board the "Dmitry Mendeleev" ship in 1969 and in the second expedition in 1971, A.N. Kolmogorov took an active part in performing the most of the experiments, in carrying out the programs of the measurements and in making the statistical analysis of the collected data.

It should be noted that difficulties are usually encountered in recording the oceanic turbulent fluctuations of flow rate, temperature, electric conductivity, sound velocity, refraction factor and other hydrodynamic parameters. This requires high-sensitivity and low-inertia devices. Serious problems arose in making the experiments during the voyage since the records of natural fluctuations were distorted due to the vibration of the towed devices, caused by the rocking of the ship and due to electric noises in a high-frequency range.

Another problem was that the frequency ranges of the turbulent fluctuations and those of the surface and internal waves largely overlapped. Thus, it was necessary to filter out mechanical and electric noises as well as the fluctuations created by the waves.

Kolmogorov wrote in his report: " My duty, as the Research Director of the hydro-physical investigations in the expedition (1971), was the coordination of the works performed by:

1. the hydrology team;
2. the turbulence team;
3. the small-scale turbulence team;
4. the team of acoustic methods;
5. the team employing the devices to measure the vertical flow of the moisture."

During the voyage in 1971, Kolmogorov wrote some articles (unpublished), for instance, "*The notes about the mathematical treatment of the observation results", "On the techniques employed to obtain integral spectra*".

In the first of these papers, one of the sections is called "The interpolation in depth and the calculation of gradients". Here, Kolmogorov writes that "these methods were used to treat the data of the conventional hydrologic station and to test the thermal sonde. The same methods were useful in restoring the shape of the isotherms from the data obtained. In all the cases, the scale of the observations is much larger than the "inner scale" of the turbulence. Therefore, it is necessary to account for the fact that the variation of the measured magnitudes will be irregular in depth and hence, that the linear interpolation will be useless. However, the linear interpolation could be considered, at least, as “harmless” since it gives intermediate values for the intermediate horizons. Here we do not lose useful information if we change the values obtained with the irregular intervals of the order of 1 m. for the linearly interpolated values on the scale with a 1 m. step.

The quadratic interpolation is also “harmless” in the measurements made with a conventional hydrologic station when non-standard horizons of the direct observations are close to the corresponding standard horizons. For example, if we restore the temperature at the 500 m. horizon from the temperatures at the 407 m., 492 m., 601 m. horizons, we could make the linear interpolation for the 492 m., 601 m. horizons and the extrapolation for the 407 m., 492 m. horizons. The quadratic interpolation gives the excessive average of these two results. By applying the quadratic interpolation, we can gain a lot in accuracy in the “good” case of smooth temperature curves and lose absolutely nothing in the “bad” cases.

The situation is different when the interpolation is made at the center of the interval between the horizons of the direct observations. We have examples of real data where the quadratic interpolation gives much worse results than the linear interpolation for rather typical temperature curves.

One should be even more careful when the interpolation is made by using the polynomials with the degree higher than two. It is notable, however, that the temperature curves are very smooth at great depths. The following exampleis not a single one:

50	14.54				
		1.24			
100	13.30		.54		
		.70		.18	
150	12.60		.36		−.02
		.34		.20	
200	12.26		.16		−.00
		.18		.20	
250	12.08		−.04		
		.22			
300	11.86				

Here, the third differences are almost constant and the fourth ones are almost equal to zero. In this case, we recommend that the interpolation should be made with the polynomials of the third degree.

If we restore the temperature at the 150 m. horizon by the linear interpolation between the 100 m. horizon and 200 m. horizon, we will get the value of 12.78.

The quadratic interpolation between the 50 m., 100 m., 200 m. horizons will give the value of 12.54.

The interpolation by the polynomial of the 3^{rd} degree along the 50 m., 100 m., 200 m., 300 m. horizons gives a precise result (with the accuracy up to 0.01) for the 150 m. horizon.

It would be of interest to find the reasons for getting "smooth" and "unsmooth" curves. But we recommend that the polynomials of the third degree should not be used in standard programs of interpolation.

The gradient must be calculated from the linear or the quadratic approximation of the curve. If the magnitude is measured with a small step in depth and with a noticeable error of each measurement, then it is reasonable that the linear or the quadratic approximation of the curve should be determined by the excessive number of points, using the least squares method".

We will now consider in greater detail the mathematical aspects of the second article "On the techniques employed to obtain integral spectra".

The integral spectrum is the following function of the frequency k:

$$J(k) = \int_k^\infty E(k')\, dk'. \tag{22}$$

Here, $E = E(k)$ is the spectral density. By setting

$$H_{k_0}(k) = \begin{cases} 0 & \text{if} \quad k < k_0 \\ 1 & \text{if} \quad k \geq k_0, \end{cases} \tag{23}$$

we get

$$J(k_0) = \int_0^\infty |H_{k_0}(k')|^2 E(k')\, dk'. \tag{24}$$

The equations like (24) naturally arise in connection with the linear transformations of the stationary random processes $X = X(t)$, $t \in \mathbb{R}$

having the following spectral representation (compare with (11)):

$$X(t) = \int_{-\infty}^{\infty} e^{itk}\, Z(dk) \tag{25}$$

where $\mathsf{E}|Z(dk)|^2 = E(k)\,dk$.

Indeed, let $Y(t) = \mathcal{L}\{X\}(t)$ be a linear transformation

$$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du \tag{26}$$

determined by the *weight* function $h = h(u)$ with the following Fourier transform:

$$H(k) = \int_{-\infty}^{\infty} e^{-iku}\, h(u)\, du. \tag{27}$$

For the physically realizable systems, one has $h(u) = 0$ if $u < 0$.

It follows from (25)–(27) that

$$Y(t) = \int_{-\infty}^{\infty} e^{itk}\, H(k)\, Z(dk) \tag{28}$$

and, obviously,

$$\mathsf{E}|Y(t)|^2 = \int_{-\infty}^{\infty} |H(k)|^2 E(k)\, dk. \tag{29}$$

Comparing (24) and (29), we conclude that it would be possible to estimate the integral spectrum $J(k_0)$ properly (using estimates for $\mathsf{E}|Y(t)|^2$ $(= \mathsf{E}|Y(0)|^2)$) if the frequency characteristic (transfer function) $H(k) = H(k_0, k)$, of a physically realizable filter is "close" to the function $H_{k_0}(k)$ from (23) which corresponds to the non-physically realizable filter. Let us denote by $\hat{J}(k_0)$ an estimate of the integral in (29) using some empirical estimator (of type of the arithmetic mean) for $\mathsf{E}|Y(t)|^2$ $(= \mathsf{E}|Y(0)|^2)$.

Kolmogorov points out that the appropriate selection of functions $H(k) = H(k_0, k)$ should be made with consideration for, at least, a rough idea of how the spectrum $E(k)$ or the integral spectrum $J(k)$ behaves.

A characteristic feature of the turbulence is that the following approximation is rather good for large intervals of the frequency variation.

$$J(k) = C_\beta \cdot k^\beta.$$

Thus, for the locally isotropic turbulence in the "inertial" range of the frequencies, one has

$$J(k) = C \cdot k^{-2/3}$$

since $E(k) \approx k^{-5/3}$.

In practice, it is very easy to apply the physically realizable filters for which the transfer function $H(k_0, k)$ has the following form:

$$H(k_0, k) = H\Big(\frac{k}{k_0}\Big)$$

where $H = H(\xi)$ is a function of $\xi = k/k_0$.

The spectral density $E(k)$ is usually several orders of magnitude greater for the low frequencies than for the high ones. Therefore, it is necessary that the filter characteristic $H(\xi)$ should sharply fall at low frequencies.

The filters $H_1(\xi)$ of the DISA company were used in the expedition in 1971. Kolmogorov compiled the following table for these filters:

ξ	0,01	0,02	0,05	0,1	0,2	0,5	1	2	5
$H_1^2(\xi)$	0,0001	0,0004	0,0025	0,012	0,044	0,21	0,56	0,86	1,00

It is easily seen from this table that the asymptotics $H_1^2(\xi) \approx \xi^2$ is rather good for $\xi \downarrow 0$. If these two filters $H_1(\xi)$ are placed in series, we obtain the filter $H_2(\xi)$ for which $H_2(\xi) = H_1^2(\xi)$. Then we have

ξ	$H_2^2(\xi) = H_1^4(\xi)$
0,01	0,00000001
0,02	0,00000016
0,05	0,0000062
0,1	0,00014
0,2	0,0019
0,5	0,044
1	0,31
2	0,74
5	1,00

Figure 2 illustrates the behaviour of $H_1^2(\xi)$ and $H_2^2(\xi)$.

The curve $H_2^2(\xi)$ is offset to the right with respect to $H_1^2(\xi)$. Therefore, if $H_1(k_0, k) = H_1\Big(\frac{k}{k_0}\Big)$ and $H_2(k_0, k) = H_2\Big(\frac{k}{k_0}\Big)$, then for a wide

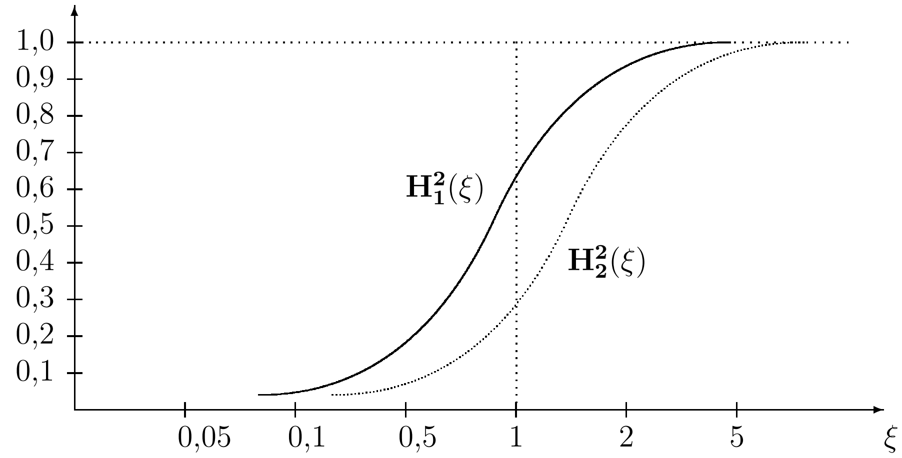

Figure 2: Behaviour of $H_1^2(\xi)$ and $H_2^2(\xi)$

range of frequencies k,

$$H_1(k_0, k) \approx H_2(1,35k_0, k).$$

In other words, when using the characteristic H_2, we deal with the new "nominal" frequency $k_0' \approx 1,35k_0$ instead of k_0 (nominal frequency for H_1). This should be remembered when one compares the results of employing different filters.

Furthermore, Kolmogorov gives a better method to make corrections for the filter properties.

Set

$$\Delta(\xi) = \begin{cases} H^2(\xi), & \xi < 1 \\ H^2(\xi) - 1, & \xi \geq 1 \end{cases}$$

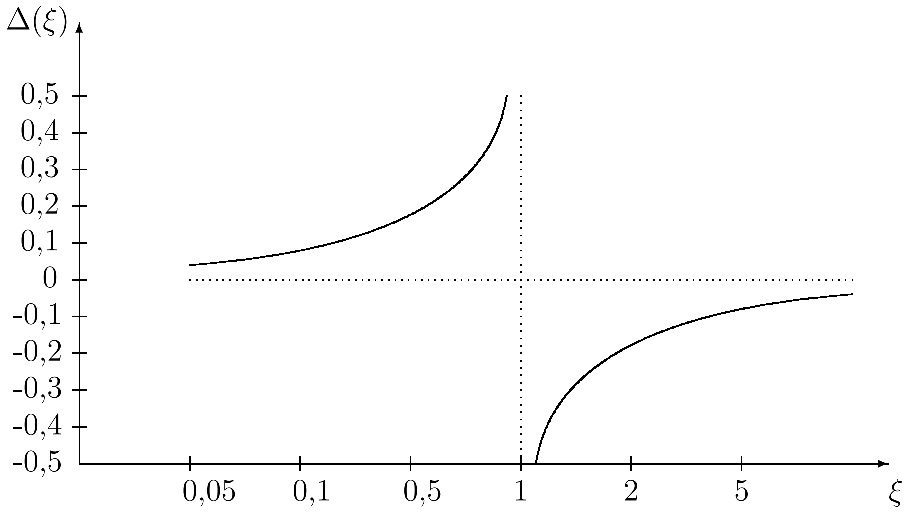

Figure 3: Graph of $\Delta(\xi)$

Then

$$\hat{J}(k_0) - J(k_0) = \int_0^\infty \Delta\Big(\frac{k'}{k_0}\Big)\, E(k')\, dk'.$$

The filters under consideration cut the low-frequency components of the spectrum and pass high frequencies with little distortion. The factor $\Delta\Big(\frac{k'}{k_0}\Big)$ significantly differs from zero in the limited range of the frequencies k' close to the frequency k_0. If $J(k) \approx C_\beta \cdot k^\beta$ ($\beta < 0$) in this frequency range, then $E(k) \approx \beta \cdot C_\beta \cdot k^{\beta-1}$. Set

$$M_\beta = \int_0^\infty \Delta(\xi)\, \xi^{\beta-1}\, d\xi.$$

Then

$$\int_0^\infty \Delta\Big(\frac{k'}{k_0}\Big)\, E(k')\, dk' = \int_0^\infty \Delta(\xi)\, E(k_0\xi)\, k_0\, d\xi \approx$$

$$\approx \int_0^\infty \beta\, C_\beta\, \Delta(\xi)\, (k_o\xi)^{\beta-1} k_0\, d\xi = \beta\, C_\beta\, k_0^\beta\, M_\beta$$

and, therefore,

$$\hat{J}(k_0) - J(k_0) \approx \beta\, C_\beta\, k_0^\beta\, M_\beta.$$

On the other hand, $J(k_0) \approx C_\beta \cdot k_0^\beta$. Consequently,

$$\hat{J}(k_0) \approx k_0^\beta\, C_\beta\, (1 + \beta M_\beta).$$

Since $J(k_0) \approx C_\beta \cdot k_0^\beta$, we deduce that

$$\hat{J}(k_0) \approx J(k_0')$$

if

$$k_0' = k_0\, (1 + \beta M_\beta)^{1/\beta}.$$

In other words, by getting the empirical value of $\hat{J}(k_0)$, we obtain the value of the integral spectrum at the "shifted" point k_0' rather than at k_0.

Let

$$\Delta^{(1)}(\xi), \quad M_\beta^{(1)}$$

and

$$\Delta^{(2)}(\xi), \quad M_\beta^{(2)}$$

denote the variables $\Delta(\xi)$ and M_β for the filters $H_1(\xi)$ and $H_2(\xi)$, respectively. Then we find the following values for $C_\beta^{(i)} = (1 + \beta M_\beta^{(i)})^{1/\beta}$:

β	-2	-1,5	-1	-0,5	0
$C_\beta^{(1)}$	0	0,52	0,62	0,72	0,83

and

β	-4	-3	-2	-1	0
$C_\beta^{(2)}$	0	0,75	1,00	1,20	1,36

As for the evaluation of β, we have

$$\ln J(k_0) \approx \ln C_\beta + \beta \ln k_0$$

assuming that $J(k_0) \approx C_\beta \cdot k_0^\beta$. Therefore, β is evaluated by the slope at the point k_0.

In conclusion, I would like to express my hope that I have illustrated some of Kolmogorov's ideas in the statistical theory of turbulence and also his views and methods of working with real statistical data in practical situations.

P.-S. Laplace used to address the mathematicians with the words: *"Read Euler, read Euler, he is our common teacher."* We may rightfully refer these words to Kolmogorov too.

References.

{1} *Kolmogorov A.N.* Selected works of A.N. Kolmogorov. Vol. I, *Mathematics and Mechanics.* Edited by Tikhomirov V.M. Dordrecht, Kluwer, 552 p., 1991.

{2} *Kolmogorov A.N.* Selected works of A.N. Kolmogorov. Vol. II, *Probability Theory and Mathematical Statistics.* Edited by Shiryaev A.N. Dordrecht, Kluwer, 598 p., 1992.

{3} *Frisch U.* Turbulence. *The legacy of A.N. Kolmogorov.* Cambridge Univ. Press, 1995.

{4} *Yaglom A.M.* A.N. Kolmogorov as a fluid mechanician and founder of a school in turbulence research. *Ann. Rev. Fluid Mech.* **26**, p. 1–22. Palo Alto, CA, 1994.

{5} *Monin A.S.* Kolmogorov's two voyages on "Dmitry Mendeleev". Manuscript, 1998.

{6} *Yaglom A.M.* Certain types of random fields in n-dimensional space similar to stationary stochastic processes. *Theory of Probability and its Applications* **2** (1957), p. 292–338.

{7} *Gikhman I.I., Skorokhod A.V., Yadrenko M.I.* A manual on probability theory and mathematical statistics. Moscow, 1985.

{8} *Taylor G.I.* Statistical theory of turbulence, I–IV, Proc. R. Soc., London, Ser. A **151** (1935), p. 421–478.

{9} *Mandelbrot B.B., van Ness J.W.* Fractional Brownian motions, fractional noises and applications. *SIAM Review* **10** (1968), p. 422-437.

{10} *Barndorff-Nielsen O.E.* Probability and Statistics: Self-decomposability, Finance and Turbulence. In *Probability towards 2000* (New York, 1995), 47–57, Lecture Notes in Statist., 128, Springer, New York, 1998.

{11} *Shiryaev A.N.* Essentials of Stochastic Finance. *Facts, Models, Theory.* World Scientific, 1999, Singapore, 834 p.