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A rotational integral formula for intrinsic volumes

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Summary

A rotational version of the famous Crofton formula is derived. The motivation for deriving the formula comes from local stereology, a new branch of stereology based on sections through fixed reference points. The formula shows how rotational averages of intrinsic volumes measured on sections passing through fixed points are related to the geometry of the sectioned object. In particular it is shown how certain weighting factors, appearing in the rotational integral formula, can be expressed in terms of hypergeometric functions. Close connections to geometric tomography will be pointed out. Applications to stereological particle analysis are discussed.

Keywords. Geometric measure theory, integral geometry, rotational integral, Grassmann manifold, intrinsic volume, set with positive reach, stereology, unit normal bundle.

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1 Introduction

Classical stereology makes it possible to obtain information about quantitative properties of a spatial structure from randomly positioned and orientated sections through the structure. The same stereological methods apply for arbitrarily positioned and orientated sections if the spatial structure is translation and rotation invariant. Up-to-date monographs on stereology are Baddeley and Jensen [3] and Beneš and Rataj [4].

Prompted by advances in microscopic sampling and measurement techniques, a new branch of stereology, *local stereology*, has been developed during the last decades, cf. [5, 11, 17, 18, 22, 27, 29]. The microscopic techniques involve optical sectioning by means of which virtual sections can be generated through a reference point of the structure. A typical example is optical sectioning of a biological cell through its nucleus. A technical advantage of such sectioning of biological material is that the boundary of a central section is much more clearly seen than the boundary of a periferal section. More importantly, central sections carry more information about the structure than arbitrary sections. The main field of application of local stereology is in quantitative analysis of cell populations. The local methods do not require specific assumptions of the shape of the cells which is a great advantage in practice. Local stereology is by now recognized as being very powerful in biomedicine, especially in neuroscience and cancer grading. Recent important examples of applications are [1, 12, 13].

In local stereology, geometric identities involving sections through a fixed point are used. A geometric identity has the following general form

$$\int \alpha(X \cap L) \mathrm{d}L = \beta(X),$$

where α and β are geometrical quantities (volume, surface area or, more generally, intrinsic volumes), X is the spatial object of interest, L is the probe (line, plane, grid of parallel lines, linear subspace, affine subspace) and $\mathrm{d}L$ is 'uniform integration' over positions of L (integration with respect to a measure invariant under a certain group action). In local stereology, we focus on geometric identities for $j-$dimensional planes L_j in $\mathbb{R}^d$ passing through O (L_j is a $j-$dimensional linear subspace in $\mathbb{R}^d$, called a $j-$subspace in the following). The mathematical foundation of local stereology has been developed in [15]. It should be noted that local stereology is closely related to geometric tomography, especially to central concepts of the dual Brunn-Minkowski theory, as pointed out in [10], see also [8]. In geometric tomography, $\alpha(X \cap L)$ is called a section function for particular choices of α.

A number of geometric identities have been developed in local stereology, including a generalized Blaschke-Petkantschin formula [16], a slice formula [19], a geometric identity for surface area [14] and a vertical section formula [4]. Affine versions of the vertical section formula and the generalized Blaschke-Petkantschin formula appeared already in Baddeley [2] and Zähle [31], respectively. A review of these geometric identities has recently been given in [21].

To the best of our knowledge a geometric identity involving rotational averages of general intrinsic volumes is not yet available. In the present paper, we derive such a geometric identity. Recall that for a subset X of $\mathbb{R}^d$, satisfying certain regularity, we can define $d+1$ intrinsic volumes $V_k(X)$, $k = 0, \ldots, d$. For d=2 and 3, the intrinsic volumes have the following interpretations, cf. e.g. [3],

$$\begin{aligned}
d = 2: \qquad & V_2(X) = A(X) \text{ area} \\
& 2V_1(X) = L(X) \text{ boundary length} \\
& V_0(X) = \chi(X) \text{ Euler-Poincaré characteristic} \\
d = 3: \qquad & V_3(X) = V(X) \text{ volume} \\
& 2V_2(X) = S(X) \text{ surface area} \\
& \pi V_1(X) = M(X) \text{ integral of mean curvature} \\
& V_0(X) = \chi(X) \text{ Euler-Poincaré characteristic}
\end{aligned}$$

The formula to be derived in the present paper shows how the rotational average of intrinsic volumes relates to principal curvatures and their corresponding principal directions of the original spatial structure. The formula can be regarded as a rotational version of the classical Crofton formula, relating integrals of intrinsic volumes defined on $j-$dimensional affine subspaces to intrinsic volumes of the original set X,

$$\int_{\mathcal{F}_j^d} V_k(X \cap F_j) \mathrm{d}F_j^d = c_{d,j,k} V_{d-j+k}(X), \tag{1}$$

$j = 0, 1, \ldots, d$, $k = 0, 1, \ldots, j$. Here, $\mathcal{F}_j^d$ is the set of $j-$dimensional affine subspaces in $\mathbb{R}^d$ and $F_j = x + L_j$, L_j $j-$subspace, $x \in L_j^\perp$. Furthermore, $\mathrm{d}F_j^d = \mathrm{d}x^{d-j}\mathrm{d}L_j^d$ is the element of the motion invariant measure on $j-$dimensional affine subspaces, where $\mathrm{d}L_j^d$ is the element of the rotation invariant measure on $\mathcal{L}_j^d$, the set of $j-$subspaces, and $\mathrm{d}x^{d-j}$ is the element of the Lebesgue measure in $L_j^\perp$. Finally, $c_{d,j,k}$ is a known constant.

The formula to be derived in the present paper focuses instead on the rotational average

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j) \mathrm{d}L_j^d. \tag{2}$$

An early version of the formula has already been presented in [14] but here the rotational average of intrinsic volumes is related to curvatures on sections. The formula derived in the present paper states more clearly how the rotational average of intrinsic volumes is related to geometric properties of the original spatial structure.

The derived identity will allow us to relate averages of measurements of intrinsic volumes in section planes passing through a fixed point to quantitative properties of the set under study. Knowledge of rotational averages of sectional curvatures are of particular importance in relation to the study of cell populations. Change in curvature properties may be associated with deficiencies of the cell population as discussed in [9]. The latter paper did however not relate curvatures measured in a section plane to properties of the original set.

The proof of the formula uses the representation of curvature measures as integral currents carried on the unit normal bundle of the set (see [30]). The main technical tool is Federer's coarea formula for currents and the form of the curvature defining current for the flat section of a body which has already been used in [23]. We use the framework of compact sets with positive reach introduced by Federer in 1959 [6] in connection with curvature measures; this set class extends the family of convex bodies, and a restriction to convex bodies would only save almost no effort. An extension to finite unions of sets with positive reach (encompassing polyconvex sets) is mentioned as well.

The paper is organized as follows. In Section 2, basic concepts from geometric measure theory are shortly summarized. The rotational integral formula for intrinsic volumes is presented in Section 3. In Section 4, we show how certain weight factors appearing in the rotational integral formula can be expressed in terms of hypergeometric functions. Modified sectional intrinsic volumes with a more clear relation to geometric properties of the original set are introduced in Section 5. Applications to stereological particle analysis is shortly discussed in Section 6. Section 7 is devoted to a geometric measure theoretic proof of the main theorem. In Section 8, extensions of the main theorem are discussed. The paper has been written such that a reading of Sections 2 to 6 does not require specialist knowledge in geometric measure theory. The presented results are illustrated by 6 simple examples.

2 Preliminaries

For the background of multilinear algebra and geometric measure theory, we refer to Federer's book [7]. We shall also use the notation of [7] throughout the paper, unless otherwise stated. In particular, $\mathcal{H}^k$ denotes the $k-$dimensional Hausdorff measure in $\mathbb{R}^d$.

We will consider compact sets $X \subseteq \mathbb{R}^d$ of positive reach. To explain this notion, consider the parallel set of amount $s \geq 0$ defined as $X_s := X + sB(0,1)$ where $B(0,1)$ is the closed unit ball in $\mathbb{R}^d$. Following [6], the supremum of all $s \geq 0$, such that for any $y \in X_s$, there exists a unique point in X nearest to y, is called reach X. The normal cone $\mathrm{Nor}(X,x)$

$$\mathrm{Nor}(X,x) = \{w \in \mathbb{R}^d : v \cdot w \leq 0 \text{ for } v \in \mathrm{Tan}(X,x)\}$$

is the dual cone to the tangent cone $\mathrm{Tan}(X,x)$ of X at x (which is always a convex cone if $\mathrm{reach}\, X > 0$). For an illustration, see Figure 1. The unit normal bundle of X is given by

$$\mathrm{nor}\, X = \{(x,n) : x \in \partial X, n \in \mathrm{Nor}(X,x) \cap S^{d-1}\},$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$.

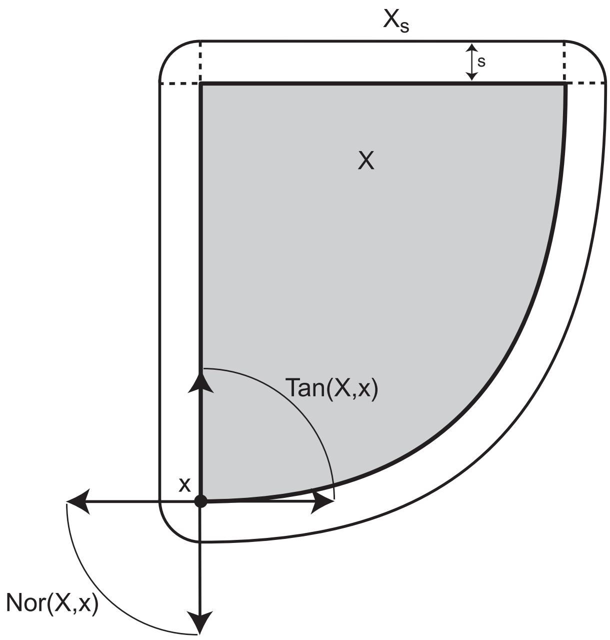

Figure 1: Illustration of the parallel set X_s of X, the tangent cone $\mathrm{Tan}(X,x)$ and the normal cone $\mathrm{Nor}(X,x)$.

Using the parallel sets it is possible for $\mathcal{H}^{d-1}$-almost all points $(x,n) \in \mathrm{nor}\,(X)$ to define (generalized) principal curvatures $\kappa_i(x,n) \in [-\mathrm{reach}\, X, \infty]$ and corresponding principal directions $a_i(x,n)$ at (x,n), $i=1,\dots,d-1$, see [24] and [30]. We assume that the principal directions are ordered in such a way that

$$a_1(x,n),\dots,a_{d-1}(x,n),n$$

form a positively oriented orthonormal basis of $\mathbb{R}^d$.

To each compact set X of positive reach we can associate $d+1$ intrinsic volumes $V_k(X)$, $k=0,1,\dots,d$. The intrinsic volume $V_d(X)$ is the volume (Lebesgue measure) of X, $V_{d-1}(X)$ one half of the surface area (provided that X is d-dimensional in the sense that the normal cone $\mathrm{Nor}(X,x)$ does not contain a line for almost all boundary points x), and $V_0(X)$ is the Euler-Poincaré characteristic of X, see [6, Theorem 5.19]. For $k=0,1,\dots,d-1$, it can be shown that the kth intrinsic volume has the following integral representation, cf. [30],

$$V_k(X)=\frac{1}{\sigma_{d-k}}\int_{\mathrm{nor}\,X}\sum_{|I|=d-1-k}\frac{\prod_{i\in I}\kappa_i(x,n)}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2(x,n)}}\mathcal{H}^{d-1}(\mathrm{d}(x,n)), \tag{3}$$

where $\sigma_k=2\pi^{k/2}/\Gamma(k/2)=\mathcal{H}^{k-1}(S^{k-1})$ is the surface area of the unit sphere in $\mathbb{R}^k$, I is a subset of $\{1,\dots,d-1\}$ and $|I|$ denotes the number of its elements. Since the principal curvatures may be infinite, we set $\frac{\infty}{\sqrt{1+\infty^2}}=1$ and $\frac{1}{\sqrt{1+\infty^2}}=0$. In the special case where ∂X is a $(d-1)-$dimensional manifold of class C^2, the principal curvatures $\kappa_i(x,n)=\kappa_i(x)$ are functions of $x\in\partial X$ only and (3) reduces to

$$V_k(X)=\frac{1}{\sigma_{d-k}}\int_{\partial X}\sum_{|I|=d-1-k}\prod_{i\in I}\kappa_i(x)\,\mathcal{H}^{d-1}(\mathrm{d}x),$$

cf. [28, Section V.3] and [26, Section 13.6].

Let $\mathcal{L}_j^d$ be the Grassmann manifold of j-dimensional linear subspaces of $\mathbb{R}^d$, $0\le j\le d$. The elements of $\mathcal{L}_j^d$ will usually be denoted by L_j, $L_j^\perp$ stands for the orthogonal complement of L_j which is a $(d-j)$-dimensional subspace of $\mathbb{R}^d$. Note that $\mathcal{L}_j^d$ can be regarded as a $j(d-j)$-dimensional smooth compact submanifold of a Euclidean space (see [7, §3.2.28]) and, hence, we can equip it with the Hausdorff measure $\mathcal{H}^{j(d-j)}$. We shall use the shortened notation here

$$\mathrm{d}L_j^d=\mathcal{H}^{j(d-j)}(\mathrm{d}L_j).$$

The total mass of the measure is

$$\int_{\mathcal{L}_j^d}\mathrm{d}L_j^d=c_{d,j},$$

where

$$c_{d,j}=\frac{\sigma_d\sigma_{d-1}\cdots\sigma_{d-j+1}}{\sigma_j\sigma_{j-1}\cdots\sigma_1}.$$

The resulting measure on $\mathcal{L}_j^d$ is the unique, up to multiplication with a positive constant, rotation invariant measure. If $0\le q<j$ and $L_q\in\mathcal{L}_q^d$ is fixed, then $\mathcal{L}_{j(q)}^d$ denotes the set of j-subspaces containing the fixed subspace L_q (note that $\mathcal{L}_{j(q)}^d$ is isomorphic to $\mathcal{L}_{j-q}^{d-q}$). The measure described by the integration $\mathrm{d}L_{j(j-1)}^d\,\mathrm{d}L_{j-1}^d$ is

clearly rotation invariant on $\mathcal{L}_j^d$ and, after computation of its total mass, we get the relation

$$\mathrm{d}L_j^d = \frac{\sigma_1}{\sigma_j}\,\mathrm{d}L_{j(j-1)}^d\, dL_{j-1}^d. \tag{4}$$

Note that a subspace $L_j \in \mathcal{L}_{j(j-1)}^d$ can be written as

$$L_j = L_{j-1} \oplus \mathrm{Lin}\,\{z\},$$

where $\oplus$ indicates orthogonal sum and $\mathrm{Lin}\,\{z\}$ is the linear space spanned by $z \in L_{j-1}^{\perp} \cap S^{d-1}$. The space $\mathcal{L}_{j(j-1)}^d$ is thereby isomorphic to the unit sphere S^{d-j} in $L_{j-1}^{\perp}$ modulo change of sign and we can write

$$\mathrm{d}L_{j(j-1)}^d = \tfrac{1}{2}\mathcal{H}^{d-j}(\mathrm{d}z).$$

In the main text of the paper, we will use the following result, valid for $u \in \mathbb{R}^d \setminus L_q$ and a measurable non-negative function g, cf. [15, Proposition 3.9],

$$\int_{\mathcal{L}_{j(q)}^d} g\left(\frac{|p(u|L_j^{\perp})|^2}{|p(u|L_q^{\perp})|^2}\right)\frac{\mathrm{d}L_{j(q)}^d}{c_{d-q,j-q}} = \frac{1}{B((d-j)/2,(j-q)/2)}\int_0^1 g(y)y^{\frac{d-j}{2}-1}(1-y)^{\frac{j-q}{2}-1}\mathrm{d}y, \tag{5}$$

$0 \le q < j < d$. Here, $p(\cdot|L_k)$ denotes the orthogonal projection onto L_k.

The Grassmann manifold $\mathcal{L}_j^d$ can be embedded into the linear space $\bigwedge_j \mathbb{R}^d$ of j-vectors in $\mathbb{R}^d$ as the submanifold of simple unit j-vectors modulo change of sign (cf. [7]). The space $\bigwedge_j \mathbb{R}^d$ is equipped with the scalar product which can be defined on simple j-vectors as

$$(u_1 \wedge \cdots \wedge u_j)\cdot(v_1 \wedge \cdots \wedge v_j) = \det\big(u_i \cdot v_l\big)_{i,l=1}^{j}.$$

Given two linear subspaces L_p, L_q with sum of dimensions $p+q \ge d$, we define $\mathcal{G}(L_p, L_q)$ as the determinant of the orthogonal projection of $(L_p \cap L_q)_{L_p}^{\perp}$ (the orthogonal complement of $L_p \cap L_q$ in L_p) onto $L_q^{\perp}$, cf. [15, p. 47]. We have $0 \le \mathcal{G}(L_p, L_q) \le 1$. Note that if $p+q=d$ then $\mathcal{G}(L_p, L_q) = |L_p \cdot L_q^{\perp}|$, with the scalar product introduced above. In the main part of the paper we shall often use that $\mathcal{G}(L_{d-1}, L_q) = |p(n|L_q)|$, where n is a unit normal of L_{d-1}. For $d=3$, $\mathcal{G}(L_p, L_q)$ is simply $|\sin\alpha|$ where α is the angle between L_p and L_q.

The following result concerning the $\mathcal{G}$ functions turns out to be useful.

Lemma 1. *Let L_p, L_q be subspaces of dimensions p, q, respectively, $p+q \ge d$, and let $\{v_1, \ldots, v_q\}$ be an orthonormal basis of L_q. Then*

$$\mathcal{G}(L_p, L_q)^2 = \sum_{\substack{I \subseteq \{1,\ldots,q\}\\ |I| = d-p}} \mathcal{G}(L_p, \mathrm{Lin}\,\{v_i : i \in I\})^2.$$

Proof. We shall identify the subspaces $\operatorname{Lin}\{v_i : i \in I\}$ with the simple unit $|I|$-vectors $\bigwedge_{i\in I} v_i$. In order to show the result of the lemma we use that for any index subset I with cardinality $d-p$,

$$\mathcal{G}\Big(L_p, \bigwedge_{i\in I} v_i\Big) = \mathcal{G}(L_p, L_q)\mathcal{G}\Big(L_p \cap L_q, \bigwedge_{i\in I} v_i\Big)$$

(see [15, Proposition 5.1]), where the last function $\mathcal{G}$ has to be understood as defined relatively in the $q-$subspace L_q. If $\dim(L_p \cap L_q) > p+q-d$ then $\mathcal{G}(L_p, L_q) = 0$ and the equality is obviously true. We shall suppose in the sequel that $\dim(L_p \cap L_q) = p+q-d$. It is enough to show that

$$\sum_{\substack{I\subseteq\{1,\ldots,q\}\\ |I|=d-p}} \mathcal{G}\Big(L_p \cap L_q, \bigwedge_{i\in I} v_i\Big)^2 = 1. \tag{6}$$

We may represent the orthogonal complement of $L_p \cap L_q$ in L_q as a unit $(d-p)$-vector in L_q and $\mathcal{G}(L_p \cap L_q, \bigwedge_{i\in I} v_i)$ is its scalar product with $\bigwedge_{i\in I} v_i$ in $\bigwedge_{d-p} L_q$. Since $\{\wedge_{i\in I} v_i : |I| = d-p\}$ forms an orthonormal basis of $\bigwedge_{d-p} L_q$, (6) follows. $\square$

In the new rotational formula to be derived in this paper, hypergeometric functions play an important role. A hypergeometric function can be represented by a series of the following form

$$\begin{aligned} F(\alpha, \beta; \gamma; z) &= \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1}(\alpha+i)\prod_{i=0}^{k-1}(\beta+i)}{\prod_{i=0}^{k-1}(\gamma+i)} \frac{z^k}{k!} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)} \frac{z^k}{k!}. \end{aligned} \tag{7}$$

The coefficient of z^k is for $k=0$ equal to 1. We shall always assume that $\alpha+\beta-\gamma < 0$ which ensures that the series is convergent for $|z| < 1$. In case $0 < \beta < \gamma$, we can also represent the hypergeometric series by an integral

$$F(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 (1-zy)^{-\alpha} y^{\beta-1}(1-y)^{\gamma-\beta-1} \mathrm{d}y. \tag{8}$$

3 The main Theorem

The particular cases relating to rotational averages of sectional Lebesgue measure can easily be derived. In the simplest case where $k = j = 1$, we get

$$\begin{aligned} \int_{\mathcal{L}_1^d} V_1(X \cap L_1)\mathrm{d}L_1^d &= \int_{\mathcal{L}_1^d} \int_{X\cap L_1} \mathrm{d}x^1 \mathrm{d}L_1^d \\ &= \int_X |x|^{-(d-1)} \mathrm{d}x^d, \end{aligned}$$

where we at the last equality sign have used polar decomposition in $\mathbb{R}^d$:

$$\mathrm{d}x^d = |x|^{d-1}\mathrm{d}x^1\mathrm{d}L_1^d.$$

More generally, the rotational average can for $k = j$, where $j = 1, 2, \ldots, d$, be expressed as follows

$$\int_{\mathcal{L}_j^d} V_j(X \cap L_j) \mathrm{d}L_j^d = c_{d-1,j-1} \int_X |x|^{-(d-j)} \mathrm{d}x^d. \tag{9}$$

The proof of this geometric identity can be based on the Blaschke-Petkantschin formula [16, 31].

Example 1. *For $d = 3$ and $j = 2$, we get, cf. (9),*

$$\int_{\mathcal{L}_2^3} A(X \cap L_2) \mathrm{d}L_2^3 = \beta(X),$$

where

$$\beta(X) = \pi \int_X |x|^{-1} \mathrm{d}x^3.$$

□

In order to solve the more difficult remaining case $k < j$, we consider a compact set $X \subseteq \mathbb{R}^d$ with positive reach. Given $O \neq x \in \mathbb{R}^d$, $n \in S^{d-1}$ and $A_q \subseteq \mathbb{R}^d$ a q-subspace perpendicular to n, we define

$$Q_j(x, n, A_q) = \int_{\mathcal{L}_{j(1)}^d} \frac{\mathcal{G}(L_j, A_q)^2}{|p(n|L_j)|^{d-q}} \, dL_{j(1)}^d,$$

where the integral runs over all j-subspaces containing the line through O spanned by x. Note that $Q_j(x, n, A_q)$ is finite whenever $n \not\perp x$ since $|p(n|L_j)| \geq |x \cdot n| \,/\, |x|$.

For a subset I of $\{1, \ldots, d-1\}$ and a point $(x, n) \in \operatorname{nor} X$ with principal directions $a_i(x, n)$, we shall use the notation $A_I = A_I(x, n)$ for the $(d-1-|I|)$-subspace spanned by all the vectors $a_i(x, n)$ with $i \notin I$.

Theorem. *Assume that $O \notin \partial X$ and that for almost all $L_j \in \mathcal{L}_j^d$,*

$$(x, n) \in \operatorname{nor} X,\ x \in L_j \implies n \not\perp L_j. \tag{10}$$

Then for any $0 \leq k < j$, $1 \leq j \leq d$,

$$\begin{aligned} &\int_{\mathcal{L}_j^d} V_k(X \cap L_j) \, \mathrm{d}L_j^d \\ &\quad = \frac{1}{\sigma_{j-k}} \int_{\operatorname{nor} X} \frac{1}{|x|^{d-j}} \\ &\quad \times \sum_{|I|=j-1-k} Q_j(x, n, A_I) \frac{\prod_{i \in I} \kappa_i(x, n)}{\prod_{i=1}^{d-1} \sqrt{1 + \kappa_i^2(x, n)}} \, \mathcal{H}^{d-1}(\mathrm{d}(x, n)), \end{aligned} \tag{11}$$

provided that the integral on the right-hand side exists.

It is worthwhile to compare the rotational version (11) of the Crofton formula with the classical Crofton formula (1). The right-hand side of the classical Crofton formula is, up to a known constant, $V_{d-j+k}(X)$. The right-hand side of the rotational version of the Crofton formula has an integral representation similar to that of $V_{d-j+k}(X)$ but with the additional terms $1/|x|^{d-j}$ and $Q_j(x,n,A_I)$. For $j=d$, these terms are identically equal to 1 and (11) reduces to the well-known integral form (3) for intrinsic volumes. In the next section, we will show for $j<d$ that $Q_j(x,n,A_I)$ can be expressed in terms of hypergeometric functions. If X is a ball, then $1/|x|^{d-j}$ and $Q_j(x,n,A_I)$ are constant and the right-hand side of (11) is proportional to $V_{d-j+k}(X)$.

Corollary 1. *Let the situation be as in Theorem. Assume furthermore that ∂X is a $(d-1)-$dimensional manifold of class C^2. Then,*

$$\int_{\mathcal{L}_j^d} V_k(X\cap L_j)\,\mathrm{d}L_j^d$$
$$= \frac{1}{\sigma_{j-k}}\int_{\partial X}\frac{1}{|x|^{d-j}}\sum_{|I|=j-1-k} Q_j(x,n(x),A_I)\prod_{i\in I}\kappa_i(x)\,\mathcal{H}^{d-1}(\mathrm{d}x), \tag{12}$$

where $n(x)$ is the unique outer unit normal to ∂X at x.

Proposition 1 below shows that the regularity condition (10) is mild; in particular, the second statement of Proposition 1 implies that (10) can be violated only for exceptional choices of the origin.

Proposition 1. *Assume that $O\notin\partial X$. Then, the regularity condition* (10) *holds whenever X is convex. Furthermore, if X is a compact set with positive reach, then*

$$\mathcal{H}^d\{z\in\mathbb{R}^d:\, z+X \text{ does not satisfy (10)}\}=0. \tag{13}$$

Proof. To verify (13), it is enough to show that $(\mathcal{H}^d\times\mathcal{H}^{j(d-j)})(N)=0$, where

$$N=\{(z,L_j)\in\mathbb{R}^d\times\mathcal{L}_j^d:\,\exists(x,n)\in\operatorname{nor}X, z+x\in L_j, n\perp L_j\}.$$

The image of N under the projection $\Pi:(z,L_j)\mapsto(p(z|L_j^\perp),L_j)$ is the subset of j-flats in $\mathbb{R}^d$ "locally colliding with X" which is known to have finite r-dimensional measure with $r=d-1+j(d-1-j)$ (see [25]). Hence, the invariant $(d-j+j(d-j))$-dimensional measure of $\Pi(N)$, and, consequently, also the $(d+j(d-j))$-dimensional measure of N, is zero. $\square$

Sufficient conditions for the boundedness of the integral in Theorem are given in the following proposition.

Proposition 2. *The integral in Theorem converges if X is convex or if $j-k\le 2$, in particular, always in $\mathbb{R}^3$.*

Proof. If X is convex then all principal curvatures are finite and nonnegative, hence the integrated function is nonnegative. One easily sees that the integral on the left hand side is bounded, hence the right hand side is bounded as well.

For the second assertion, note that since $n \perp A_I$, we have

$$\begin{aligned}\mathcal{G}(L_j, A_I) &= \mathcal{G}(L_j, n^\perp)\mathcal{G}(L_j \cap n^\perp, A_I)\\ &\le \mathcal{G}(L_j, n^\perp)\\ &= |p(n|L_j)|.\end{aligned}$$

Consequently, $Q_j(x,n,A_I) \le \int |p(n|L_j)|^{2+k-j}\, dL^d_{j(1)} =: c(j,k,d)$ which is clearly finite if $j-k \le 2$. Thus, we have

$$\begin{aligned}&\int_{\operatorname{nor} X} \left| \frac{1}{|x|^{d-j}} \sum_{|I|=j-1-k} Q_j(x,n,A_I) \frac{\prod_{i\in I} \kappa_i(x,n)}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2(x,n)}} \right| \mathcal{H}^{d-1}(\mathrm{d}(x,n))\\ &\quad\le \int_{\operatorname{nor} X} \frac{1}{|x|^{d-j}} \sum_{|I|=j-1-k} Q_j(x,n,A_I) \left| \frac{\prod_{i\in I} \kappa_i(x,n)}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2(x,n)}} \right| \mathcal{H}^{d-1}(\mathrm{d}(x,n))\\ &\quad\le \frac{1}{(\operatorname{dist}(O,\partial X))^{d-j}} \binom{d-1}{j-1-k} c(j,k,d)\mathcal{H}^{d-1}(\operatorname{nor} X) < \infty.\end{aligned}$$

The fact that $\mathcal{H}^{d-1}(\operatorname{nor} X) < \infty$ follows from the (locally) $(d-1)-$rectifiability of $\operatorname{nor} X$, cf. [30, p. 560]. □

4 Explicit forms of $Q_j(x,n,A_q)$

We will in this section evaluate the integral $Q_j(x,n,A_q)$ where $O \ne x \in \mathbb{R}^d$, $n \in S^{d-1}$ and A_q is a $q-$subspace perpendicular to n. The dimensions j,q satisfy $1 \le j,q \le d-1$ and $j+q \ge d$. We will first consider the case $q=d-1$, next $q=1$ and finally $1<q<d-1$, representing increasing degree of complexity.

4.1 The case $q = d-1$

Here, $A_q = n^\perp$ and $\mathcal{G}(L_j, n^\perp) = |p(n|L_j)|$. It follows that

$$\begin{aligned}Q_j(x,n,A_q) &= \int_{\mathcal{L}^d_{j(1)}} \frac{\mathcal{G}(L_j,A_q)^2}{|p(n|L_j)|^{d-q}} dL^d_{j(1)}\\ &= \int_{\mathcal{L}^d_{j(1)}} |p(n|L_j)| dL^d_{j(1)}\\ &= c_{d-1,j-1} F\big(-1/2, (d-j)/2; (d-1)/2; |p(n|L_1^\perp)|^2\big),\end{aligned}$$

where we have used (5) and (8) at the last equality sign. Since L_1 is the line spanned by x,

$$|p(n|L_1^\perp)|^2 = \sin^2\beta, \tag{14}$$

where $\beta = \angle(x,n)$. Using the series expansion of the hypergeometric function, a first-order approximation of Q_j becomes

$$Q_j(x,n,A_q) \approx c_{d-1,j-1}\left(1 - \frac{d-j}{2(d-1)}\sin^2\beta\right).$$

In the particular case where $\beta = 0$ we have

$$Q_j(x, n, A_q) = c_{d-1,j-1}.$$

Example 2. *For $d = 3$ and $j = 2$, we find, using (8),*

$$\begin{aligned}
&F\left(-\frac{1}{2}, \frac{d-j}{2}; \frac{d-1}{2}; |p(n|L_1^\perp)|^2\right) \\
&\quad = \frac{2}{\pi}\int_0^{\pi/2} (1 - |p(n|L_1^\perp)|^2 \sin^2\phi)^{1/2} \mathrm{d}\phi \\
&\quad = \frac{2}{\pi} E(|p(n|L_1^\perp)|, \pi/2),
\end{aligned}$$

where E is the elliptic integral of the second kind. For $X \subseteq \mathbb{R}^3$ such that ∂X is a 2-dimensional manifold of class C^2, we find, cf. (12),

$$\int_{\mathcal{L}_2^3} L(X \cap L_2)\mathrm{d}L_2^3 = \beta(X),$$

where

$$\beta(X) = 2\int_{\partial X} |x|^{-1} E(|p(n(x)|L_1^\perp)|, \pi/2)\mathcal{H}^2(\mathrm{d}x)$$

and $n(x)$ is the unique outer unit normal to ∂X at x. □

4.2 The case $q = 1$

Since $j + q \geq d$, we have $j = d - 1$ or $j = d$. Since the case $j = d$ is trivial, we concentrate on $j = d - 1$. We will assume that $d \geq 3$ because the planar case $d = 2$ has been treated in the previous subsection.

As shown in the proposition below, $Q_j(x, n, A_q)$ becomes a linear combination of hypergeometric functions.

Proposition 3. *Let $q = 1$ and $j = d-1$. Let A_q be spanned by a and let $\alpha = \angle(x, a)$, $\beta = \angle(x, n)$ and $\theta = \angle(m, p(a|x^\perp))$, where $m = \pi(n|x^\perp) := p(n|x^\perp) \ / \ |p(n|x^\perp)|$. Then,*

$$\begin{aligned}
Q_{d-1}(x, n, a) = \frac{\pi^{(d-1)/2}}{2\Gamma((d+1)/2)} \sin^2\alpha \Bigg[&\sin^2\theta F\left(\frac{d-1}{2}, \frac{1}{2}; \frac{d+1}{2}; \sin^2\beta\right) \\
&+ \cos^2\theta F\left(\frac{d-1}{2}, \frac{3}{2}; \frac{d+1}{2}; \sin^2\beta\right)\Bigg].
\end{aligned} \tag{15}$$

Furthermore,

$$\sum_{i=1}^{d-1} Q_{d-1}(x, n, a_i(x, n)) = c_{d-1,d-2} F\left(\frac{d-3}{2}, \frac{1}{2}; \frac{d-1}{2}; \sin^2\beta\right). \tag{16}$$

Proof. Let L_{d-1} be spanned by x and $L_{d-2} \subset x^\perp$. Then it can be shown that

$$\mathcal{G}(L_{d-1}, a) = \sin\alpha\, \mathcal{G}^{x^\perp}(L_{d-2}, p(a|x^\perp)),$$

where the upper index $x^\perp$ of $\mathcal{G}$ indicates that the function $\mathcal{G}$ is here considered relatively in $x^\perp$. Furthermore,

$$|p(n|L_{d-1})|^2 = \cos^2\beta + \sin^2\beta\cos^2\angle(m, L_{d-2}).$$

It follows that

$$\begin{aligned}
&Q_{d-1}(x,n,a)\\
&\quad= \int_{\mathcal{L}^d_{d-1(1)}} \frac{\mathcal{G}(L_{d-1},a)^2}{|p(n|L_{d-1})|^{d-1}} \mathrm{d}L^d_{d-1(1)}\\
&\quad= \sin^2\alpha \int_{\mathcal{L}^{d-1}_{d-2}} \mathcal{G}^{x^\perp}(L_{d-2}, p(a|x^\perp))^2[1-\sin^2\beta\sin^2\angle(m, L_{d-2})]^{-\frac{d-1}{2}} \mathrm{d}L^{d-1}_{d-2},
\end{aligned}$$

where $\mathcal{L}^{d-1}_{d-2}$ is the set of $(d-2)-$subspaces of $x^\perp$. Each such subspace L_{d-2} can be identified with its unit normals $v, -v \in S^{d-2} \subset x^\perp$. Using this identification, we get

$$Q_{d-1}(x,n,a) = \frac{\sin^2\alpha}{2}\int_{S^{d-2}} (v\cdot\pi(a|x^\perp))^2[1-\sin^2\beta(m\cdot v)^2]^{-\frac{d-1}{2}} \mathrm{d}v^{d-2}.$$

Using the coarea formula on the mapping $\varphi : v \to (m\cdot v)^2$ with Jacobian

$$J\varphi(v; S^{d-2}) = 2|m\cdot v|\sqrt{1-(m\cdot v)^2},$$

cf. [15, Proposition 2.11], we finally get after some manipulation

$$\begin{aligned}
&Q_{d-1}(x,n,a)\\
&\quad= \frac{\sin^2\alpha}{2}\int_0^1\int_{S^{d-2}\cap\varphi^{-1}(y)} (v\cdot\pi(a|x^\perp))^2[1-(\sin^2\beta)y]^{-\frac{d-1}{2}} \frac{1}{2\sqrt{y}\sqrt{1-y}} \mathrm{d}v^{d-3}\mathrm{d}y^1\\
&\quad= \frac{\pi^{\frac{d-1}{2}}}{2\Gamma(\frac{d+1}{2})}\sin^2\alpha\Big[\sin^2\theta F\left(\frac{d-1}{2},\frac{1}{2};\frac{d+1}{2};\sin^2\beta\right)\\
&\qquad\qquad + \cos^2\theta F\left(\frac{d-1}{2},\frac{3}{2};\frac{d+1}{2};\sin^2\beta\right)\Big],
\end{aligned}$$

where $\theta = \angle(m, p(a|x^\perp))$ satisfies

$$\cos\theta = \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

A direct way of proving (16) is the following

$$
\begin{aligned}
\sum_{i=1}^{d-1} Q_{d-1}(x,n,a_i) &= \int_{\mathcal{L}^d_{d-1(1)}} \frac{\sum_{i=1}^{d-1}\mathcal{G}(L_{d-1},a_i)^2}{|p(n|L_{d-1})|^{d-1}} dL^d_{d-1(1)} \\
&= \int_{\mathcal{L}^d_{d-1(1)}} \frac{|p(n|L_{d-1})|^2}{|p(n|L_{d-1})|^{d-1}} dL^d_{d-1(1)} \\
&= \int_{\mathcal{L}^d_{d-1(1)}} |p(n|L_{d-1})|^{3-d} dL^d_{d-1(1)} \\
&= c_{d-1,d-2} F\left(\frac{d-3}{2}, \frac{1}{2}; \frac{d-1}{2}; \sin^2\beta\right),
\end{aligned}
$$

where we at the last equality sign have used (5) and (8). □

Example 3. *For $d=3$ and $j=2$, we find that*

$$
\sum_{i=1}^{d-1} Q_{d-1}(x,n,a_i(x,n)) = c_{2,1} = \pi
$$

does not depend on x and n. It follows for $X \subseteq \mathbb{R}^3$ with ∂X a 2-dimensional manifold of class C^2 that, cf. (12),

$$
\int_{\mathcal{L}^3_2} \chi(X \cap L_2) \mathrm{d}L^3_2 = \beta(X),
$$

where

$$
\beta(X) = \frac{1}{2}\int_{\partial X} |x|^{-1} \sum_{i=1}^{2} \omega_i(x)\kappa_i(x)\mathcal{H}^2(\mathrm{d}x)
$$

and $\omega_i(x) = Q_2(x, n(x), a_{2-i+1}(x))/\pi$, $i=1,2$, sum to 1. □

4.3 The case $1 < q < d-1$

This case is more complicated than the two previous cases. We conjecture that $Q_j(x,n,A_q)$ can be written as a linear combination of four hypergeometric functions. The details will be worked out in a future paper. Note that the previous cases cover all cases of immediate practical interest ($d=3$).

5 Modifications for applications

In the previous sections, we have derived new geometric identities of the form

$$
\int_{\mathcal{L}^d_j} V_k(X \cap L_j)\mathrm{d}L^d_j = \beta(X),
$$

showing how the rotational averages of the sectional intrinsic volumes depend on the principal curvatures and their principal directions of the set X. The 'opposite' problem of finding functions α defined on $X \cap L_j$ with rotational average equal to the intrinsic volumes of X is also of interest for applications, see Section 6 below. So in this section we will study the problem of finding α such that

$$\int_{\mathcal{L}_j^d} \alpha(X \cap L_j)\mathrm{d}L_j^d = V_{d-j+k}(X),$$

$0 \leq j \leq d$, $0 \leq k \leq j$. It turns out that the cases $k = j$ and $k = j - 1$ can be solved but otherwise the problem is largely open.

5.1 The case $k = j$

For $k = j$, V_{d-j+k} is Lebesgue measure and the Blaschke-Petkantchin formula implies that, cf. e.g. [15, Proposition 4.5],

$$\int_{\mathcal{L}_j^d} \tilde{V}_{d,j}(X \cap L_j)\mathrm{d}L_j^d = V_d(X),$$

$1 \leq j \leq d$, where

$$\tilde{V}_{d,j}(X \cap L_j) = \frac{1}{c_{d-1,j-1}} \int_{X \cap L_j} |x|^{d-j}\mathrm{d}x^j. \tag{17}$$

In geometric tomography, $\tilde{V}_{d,j}$ is a special case of a dual volume, cf. [8, (A.63)]. Dual volumes have a number of interesting properties, cf. [10, Section 4]. In particular, they satisfy a generalization of the dual Kubota integral recursion (see [8, Theorem A.7.2]). Also, (17) can be expressed as a section function as defined in [8, Section 7.2].

Example 4. *For $d = 3$ and $j = 2$, we find*

$$\int_{\mathcal{L}_2^3} \alpha(X \cap L_2)\mathrm{d}L_2^3 = V(X),$$

where

$$\alpha(X \cap L_2) = \frac{1}{\pi} \int_{X \cap L_2} |x|\mathrm{d}x^2.$$

□

5.2 The case $k = j - 1$

For $k = j - 1$, results in [15, Proposition 5.4 and Section 5.6] can be used to show the following proposition.

Proposition 4. *Let X be a subset of $\mathbb{R}^d$ with ∂X a $(d-1)-$dimensional manifold of class C^1 with finite surface area. Assume that $O \notin \partial X$ and that*

$$\mathcal{H}^{d-1}(\{x \in \partial X : n(x) \perp x\}) = 0.$$

Then, for $1 < j < d$,

$$\int_{\mathcal{L}_j^d} \tilde{V}_{d,j-1}(X \cap L_j)\, dL_j^d = V_{d-1}(X),$$

where

$$2c_{d-1,j-1}\tilde{V}_{d,j-1}(X\cap L_j) \\ = \int_{\partial X\cap L_j} |x|^{d-j} F\left(-\frac{1}{2}, -\frac{d-j}{2}; \frac{j-1}{2}; |p(n_{L_j}(x)|L_1^{\perp})|^2\right)\mathcal{H}^{j-1}(\mathrm{d}x),$$

$n_{L_j}(x)\in L_j$ is the unit normal to $\partial X\cap L_j$ at $x\in\partial X\cap L_j$ and $L_1=\operatorname{Lin}\{x\}$.

Proof. Using [15, p. 142–144], we find that

$$\frac{1}{2}\int_{\mathcal{L}_1^d}\int_{\mathcal{L}_{j(1)}^d} \tilde{m}_j^{(d)}(\partial X, L_j; L_1)\frac{\mathrm{d}L_{j(1)}^d}{c_{d-1,j-1}}\frac{\mathrm{d}L_1^d}{c_{d,1}} = V_{d-1}(X), \tag{18}$$

where

$$\tilde{m}_j^{(d)}(\partial X, L_j; L_1) = \frac{\pi^{d/2}}{\Gamma(d/2)}\sum_{x\in\partial X\cap L_1}|x|^{d-1}|p(n_{L_j}(x)|L_1)|^{-1} \\ \times F\left(-\frac{1}{2}, -\frac{d-j}{2}; \frac{j-1}{2}; |p(n_{L_j}(x)|L_1^{\perp})|^2\right).$$

Interchanging the order of integration in (18) and applying [15, Proposition 5.4], we obtain the result. □

Note that $\tilde{V}_{d,j-1}(X\cap L_j)$ can be determined from information in L_j alone.

Example 5. *Let $d=3$ and $j=2$. Furthermore, let $|p(n_{L_2}(x)|L_1^{\perp})| = \sin\gamma(x)$. Then,*

$$F\left(-\frac{1}{2}, -\frac{d-j}{2}; \frac{j-1}{2}; \sin^2\gamma(x)\right) = \cos\gamma(x)+\gamma(x)\sin\gamma(x),$$

cf. [15, Example 5.10]. It follows that

$$\int_{\mathcal{L}_2^3}\alpha(X\cap L_2)\mathrm{d}L_2^3 = S(X),$$

where

$$\alpha(X\cap L_2) = \frac{1}{\pi}\int_{\partial X\cap L_2}|x|(\cos\gamma(x)+\gamma(x)\sin\gamma(x))\mathcal{H}^1(\mathrm{d}x).$$

□

5.3 The case $k<j-1$

In order to make some progress in the case $k<j-1$, let us consider the following generalized intrinsic volumes

$$\tilde{V}_{i,k}^d(X) = \frac{1}{\sigma_{d-k}}\int_{\operatorname{nor}X}|x|^{i-k}\sum_{|I|=d-1-k}\frac{\prod_{i\in I}\kappa_i(x,n)}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i(x,n)^2}}\mathcal{H}^{d-1}(\mathrm{d}(x,n)),$$

$0 \le k \le d-1$, $i \ge k$. Note that $\tilde{V}^d_{k,k}(X) = V_k(X)$. It follows from the proof of the main Theorem that for $0 \le k < j$, $1 \le j \le d$,

$$\int_{\mathcal{L}^d_j} \tilde{V}^j_{d-j+k,k}(X \cap L_j)\,\mathrm{d}L^d_j$$
$$= \frac{1}{\sigma_{j-k}} \int_{\operatorname{nor} X} \sum_{|I|=j-1-k} Q_j(x,n,A_I) \frac{\prod_{i\in I} \kappa_i(x,n)}{\prod_{i=1}^{d-1} \sqrt{1+\kappa_i^2(x,n)}} \mathcal{H}^{d-1}(\mathrm{d}(x,n)). \quad (19)$$

Comparing with the integral representation (3) of $V_{d-j+k}(X)$, the right-hand side of (19) will be proportional to $V_{d-j+k}(X)$ if $Q_j(x,n,A_I)$ is constant. This is, of course, not the case in general. But the sum of the Q_js are constant in the case of practical interest discussed in the example below.

Example 6. *Let $d=3$ and $j=2$. Suppose, for simplicity, that ∂X is a 2-dimensional manifold of class C^2. In $\mathbb{R}^3$, it still remains to find α such that*

$$\int_{\mathcal{L}^3_2} \alpha(X \cap L_2)\mathrm{d}L^3_2 = M(X),$$

where

$$M(X) = \int_{\partial X} \frac{1}{2}[\kappa_1(x)+\kappa_2(x)]\mathcal{H}^2(\mathrm{d}x)$$

is the integral of mean curvature of X, cf. the list of intrinsic volumes in $\mathbb{R}^3$ given in the Introduction. Using (19), we can obtain the following related geometric identity

$$\int_{\mathcal{L}^3_2} \alpha(X \cap L_2)\mathrm{d}L^3_2 = \tilde{M}(X), \quad (20)$$

where

$$\tilde{M}(X) = \int_{\partial X} \sum_{i=1}^{2} \omega_i(x)\kappa_i(x)\mathcal{H}^2(\mathrm{d}x),$$

and the weights $\omega_i(x)$ are defined in Example 3 and sum to 1. Furthermore, the α in (20) is given by

$$\alpha(X \cap L_2) = \frac{1}{\pi} \int_{\partial X \cap L_2} |x|\kappa_{L_2}(x)\mathcal{H}^1(\mathrm{d}x),$$

where $\kappa_{L_2}(x)$ is the curvature of $\partial X \cap L_2$ at $x \in L_2$. □

6 Applications to stereological particle analysis

In this section we will briefly discuss how the derived geometric identities can be used in the stereological analysis of particle populations. The particles are regarded as a realization of a marked point process $\Psi = \{[x_i;\Xi_i]\}$ where the x_is are points in $\mathbb{R}^d$ and the marks Ξ_i are compact subsets of $\mathbb{R}^d$ of positive reach. The ith particle of the process is represented by $X_i = x_i + \Xi_i$. In this framework, x_i is called the nucleus of the ith particle and Ξ_i the 'primary' or 'centred' particle.

Under assumptions of stationarity and isotropy of the particle process, it can be shown for any nonnegative measurable function h that

$$E\sum_i h(x_i,\Xi_i) = \lambda \int_{\mathbb{R}^d}\int_{\mathcal{P}} h(x,K)P_m(\mathrm{d}K)\mathrm{d}x^d, \tag{21}$$

where λ is the particle intensity and P_m is a probability distribution on the set $\mathcal{P}$ of compact sets in $\mathbb{R}^d$ of positive reach. The distribution P_m is called the particle distribution. We let Ξ_0 be a random compact set of positive reach with distribution P_m.

In relation to such a particle population, a geometric identity

$$\int_{\mathcal{L}_j^d} \alpha(X\cap L_j)\mathrm{d}L_j^d = \beta(X) \tag{22}$$

can be used to express the mean value of a specified measurement α on sectioned particles in terms of a certain $\beta-$content of the original particles (Examples 1–3). A geometric identity may also give the measurement α to be determined on sectioned particles in order to estimate the mean particle $\beta-$content for a specified β (Examples 4–6).

To be more specific, note that for the generic particle Ξ_0, we get from (22) that

$$E\beta(\Xi_0) = E\int_{\mathcal{L}_j^d}\alpha(\Xi_0\cap L_j)\mathrm{d}L_j^d = \int_{\mathcal{L}_j^d} E\alpha(\Xi_0\cap L_j)\mathrm{d}L_j^d.$$

Since the distribution of Ξ_0 is invariant under rotations, $E\alpha(\Xi_0\cap L_j)$ does not depend on L_j and it follows that for an arbitrary but fixed $j-$subspace L_{j0}

$$\frac{1}{c_{d,j}}\alpha(\Xi_0\cap L_{j0})$$

is an unbiased estimator of $E\beta(\Xi_0)$, i.e. the mean value of $\alpha(\Xi_0\cap L_{j0})/c_{d,j}$ with respect to the distribution P_m of Ξ_0 is equal to $E\beta(\Xi_0)$. In practice, a sample of particle $\{x_i+\Xi_i : x_i\in W\}$ is collected in a sampling window and a central section

$$(x_i+\Xi_i)\cap(x_i+L_{j0})$$

is determined through each particle. The resulting estimator of $E\beta(\Xi_0)$ based on this sample becomes

$$\frac{1}{c_{d,j}}\sum_{\{i:x_i\in W\}}\alpha(\Xi_i\cap L_{j0})/N_W, \tag{23}$$

where N_W is the number of sampled particles. Using (21), it can be shown that this estimator is ratio-unbiased for $E\beta(\Xi_0)$, i.e. the ratio of the mean values of the numerator and denominator is equal to $E\beta(\Xi_0)$.

7 Proof of main Theorem

Let $L_{j-1} \in \mathcal{L}^d_{j-1}$ be fixed and write $S^{d-j} = S^{d-j}(L^\perp_{j-1})$. Let further L^z_j be the linear space spanned by L_{d-1} and z whenever z is a vector which does not lie in L_{j-1}. We introduce the following mappings.

$$\begin{aligned} f:\ \operatorname{nor} X \setminus \{(x,n):\ n \perp L^x_j\} &\to \mathbb{R}^d \times S^{d-1}, \\ (x,n) &\mapsto (x, \pi(n|L^x_j)), \\ g:\ \operatorname{nor} X \setminus (L_{j-1} \times S^{d-1}) &\to S^{d-j}, \\ (x,n) &\mapsto \pi(x|L^\perp_{j-1}), \end{aligned}$$

where $\pi(\cdot|L) = p(\cdot|L)/|p(\cdot|L)|$ denotes the spherical projection onto the unit sphere in a subspace L.

Lemma 2. *The differentials of the mappings f, g are*

$$Dg(x,n)(u,v) = \frac{p(u|(L^x_j)^\perp)}{|p(x|L^\perp_{j-1})|}, \tag{24}$$

$$\begin{aligned} Df(x,n)(u,v) = \Bigg(u, \frac{p(v|L^x_j \cap n^\perp)}{|p(n|L^x_j)|} &+ \frac{\big(n \cdot p(u|(L^x_j)^\perp)\big) p\big(p(x|L^\perp_{j-1})\big|L^x_j \cap n^\perp\big)}{|p(x|L^\perp_{j-1})|^2\, |p(n|L^x_j)|} \\ &+ \frac{\big(n \cdot p(x|L^\perp_{j-1})\big) p(u|(L^x_j)^\perp)}{|p(x|L^\perp_{j-1})|^2\, |p(n|L^x_j)|} \Bigg), \end{aligned} \tag{25}$$

$(u,v) \in \mathbb{R}^d \times \mathbb{R}^d$.

Proof. The formulae are obtained by a routine calculation, using the representation

$$p(n|L^x_j) = p(n|L_{j-1}) + \big(n \cdot \pi(x|L^\perp_{j-1})\big)\pi(x|L^\perp_{j-1}).$$

Note that the differential of the spherical projection $\pi_L :\ n \mapsto \pi(n|L)$ is

$$D\pi_L(n)v = \frac{p(v|L \cap n^\perp)}{|p(n|L)|}.$$

□

The idea of the following procedure is as follows. Given any linear subspace $L \in \mathcal{L}^d_j$ of $\mathbb{R}^d$ which does not 'osculate' with X (i.e., there is no pair $(x,n) \in \operatorname{nor} X$ with $n \perp L$), then $X \cap L$ has positive reach and its unit normal bundle (relative to L) is

$$\operatorname{nor}^{(j)}(X \cap L) = \{(x, \pi(n|L)) :\ (x,n) \in \operatorname{nor} X\}.$$

This fact follows from [6, Theorem 4.10]. Note also that if X and L do osculate than $X \cap L$ need not have positive reach; therefore, such cases have to be avoided by assumptions.

At first, we shall show a technical lemma stating that for $\mathcal{H}^{d-j}$-almost all $z \in S^{d-j}$, $\mathcal{H}^{j-1}$-almost all points in $f(g^{-1}\{z\})$ have a unique pre-image under f. This will enable us later to use the area formula for f without multiplicities. (For an analogous result for the translative formula, see [32].)

Let $f^{(z)}$ denote the restriction of f to $g^{-1}\{z\}$.

Lemma 3. *For $\mathcal{H}^{d-j}$-almost all $z \in S^{d-j}$ we have*

$$\mathcal{H}^{j-1}\left(\{(x,v) \in f(g^{-1}\{z\}) : \operatorname{card} f^{-1}\{(x,v)\} > 1\}\right) = 0.$$

Proof. Let N denote the set of all $(x,n) \in \operatorname{nor} X$, $n \not\perp L_j^x$, such that there exists another unit vector $n' \neq n$, $n' \not\perp L_j^x$, with $(x,n') \in \operatorname{nor} X$ and $f(x,n) = f(x,n')$. We have to show that

$$\int_{S^{d-j}} \mathcal{H}^{j-1}\big(f(N \cap g^{-1}\{z\})\big)\, \mathcal{H}^{d-j}(dz) = 0. \tag{26}$$

Using the area and co-area formulae, the last integral can be bounded from above by

$$\begin{aligned}\int_{S^{d-j}} \int_{N\cap g^{-1}\{z\}} & J_{j-1} f^{(z)}\, d\mathcal{H}^{j-1}\, \mathcal{H}^{d-j}(dz) \\ &= \int_N J_{d-j} g(x,n) J_{j-1} f^{(z)}(x,n)\, \mathcal{H}^{d-1}(d(x,n)),\end{aligned}$$

where $J_{j-1}f^{(z)}$, $J_{d-j}g(x,n)$ is the $(j-1)$-dimensional Jacobian of $f^{(z)}$ at $z = g(x,n)$, $(d-j)$-dimensional Jacobian of g at (x,n), respectively. We shall show that almost everywhere on N, at least one of the Jacobians is zero . To end this, note that

$$\ker Dg(x,n) = \operatorname{Tan}(g^{-1}\{z\},(x,n)) = \operatorname{Tan}(\operatorname{nor} X,(x,n)) \cap (L_j^x \times \mathbb{R}^d).$$

If $\dim\ker Dg(x,n) > j-1$ then $J_{d-j}g(x,n) = 0$. Assume thus that $\dim\ker Dg(x,n) \leq j-1$. Due to the definition of N, if $(x,n) \in N$ then there exists a nonzero vector

$$\xi := (o, \pi_{n^\perp}(n'-n)) \in \operatorname{Tan}(g^{-1}\{z\},(x,n))$$

such that $Df^{(z)}(x,n)\xi = 0$, hence $J_{j-1}f^{(z)}(x,n) = 0$. $\square$

In what follows we shall use the representation of curvature measures (intrinsic volumes) of a set X with positive reach by means of the associated normal cycle N_X due to Zähle [30]. N_X is a $(d-1)$-dimensional current on $\mathbb{R}^{2d}$

$$N_X = (\mathcal{H}^{d-1} \llcorner \operatorname{nor} X) \wedge a_X,$$

i.e., the $(d-1)$-dimensional Hausdorff measure restricted to $\operatorname{nor} X$, multiplied with a unit $(d-1)$-vectorfield orienting $\operatorname{nor} X$; this can be given in the following form:

$$a_X(x,n) = \bigwedge_{i=1}^{d-1} \left(\frac{1}{\sqrt{1+\kappa_i(x,n)^2}} a_i(x,n), \frac{\kappa_i(x,n)}{\sqrt{1+\kappa_i(x,n)^2}} a_i(x,n) \right) \tag{27}$$

(recall the convention $\frac{\infty}{\sqrt{1+\infty^2}} = 1$, $\frac{1}{\sqrt{1+\infty^2}} = 0$). The current N_X acts on $(d-1)$-forms ϕ on $\mathbb{R}^{2d}$ as

$$N_X(\phi) = \int_{\operatorname{nor} X} \langle a_X(x,n), \phi(x,n) \rangle\, \mathcal{H}^{d-1}(d(x,n)).$$

The Lipschitz-Killing curvature form φ_k on $\mathbb{R}^{2d}$ of order $k = 0, \dots, d-1$ is defined as

$$\langle (u_0^1, u_1^1) \wedge \cdots \wedge (u_0^{d-1}, u_1^{d-1}), \varphi_k(x,n) \rangle = \frac{1}{\sigma_{d-k}} \sum_{\substack{\varepsilon_i = 0,1 \\ \varepsilon_1 + \cdots + \varepsilon_{d-1} = d-1-k}} \langle u_{\varepsilon_1}^1 \wedge \cdots \wedge u_{\varepsilon_{d-1}}^{d-1} \wedge n, \Omega_d \rangle,$$

where Ω_d is the volume d-form in $\mathbb{R}^d$. The kth intrinsic volume of X can then be expressed as

$$V_k^d(X) = N_X(\varphi_k).$$

When considering a section of X with a j-subspace L^j, the upper index (j) will always indicate that the corresponding notion is considered in the j-dimensional space L_j, and not in the whole $\mathbb{R}^d$. (E.g., $N^{(j)}_{X \cap L_j}$ is a $(j-1)$-dimensional current in $L_j \times L_j$.)

For the definition of the slice $\langle N_X, g, z \rangle$ of the current N_X with the Lipschitz mapping g at a point z see [7, §4.2.1 and §4.3.13]. We need to fix an orientation of the unit sphere S^{d-j} in $L_{j-1}^\perp$ which is the image of g. To do this, we fix a unit simple $(j-1)$-vector ω_{j-1} orienting L_{j-1} and a unit simple $(d-j+1)$-vector ω_{d-j+1} orienting $L_{j-1}^\perp$, so that $\langle \omega_{j-1} \wedge \omega_{d-j+1}, \Omega_d \rangle = 1$. Let Ω_{j-1}, Ω_{d-j+1} be the dual multi-covectors to ω_{j-1}, ω_{d-j+1}. If $z \in S^{d-j}$ we choose $\omega_{d-j}(z) = \omega_{d+j+1} \llcorner dz$ as the unit simple $(d-j)$-vectorfield orienting S^{d-j}. Its dual form will be denoted $\Omega_{d-j}(z) = z \lrcorner \Omega_{d-j+1}$ and we denote also by $\Omega_j(z) = \omega_{d-1} \wedge dz$ a volume form in L_j^z. Note that

$$\Omega_j(z) \wedge \Omega_{d-j}(z) = \Omega_d.$$

Further, $f^\# \phi$ denotes the push-forward of a differential form ϕ by a Lischitz mapping f, whereas $f_\# T$ is the dual pull-back of a current T.

Lemma 4. *Assume that*

$$\mathcal{H}^{j-1}\left(\{(x,n) \in \operatorname{nor} X : x \in L_{j-1}\}\right) = 0 \tag{28}$$

and that for $\mathcal{H}^{d-j}$-almost all $z \in S^{d-j}$, (10) holds with $L_j = L_j^z$. Then

$$N^{(j)}_{X \cap L_j^z} = f_\#^{(z)} \langle N_X, g, z \rangle + f_\#^{(-z)} \langle N_X, g, -z \rangle$$

for $\mathcal{H}^{d-j}$-almost all $z \in S^{d-j}$.

Proof. First, we apply [7, §4.3.8,13] to get the expression of the section current

$$\langle N_X, g, z \rangle = \left(\mathcal{H}^{j-1} \llcorner g^{-1}\{z\} \right) \wedge \zeta$$

for $\mathcal{H}^{j-1}$-almost all $z \in S^{d-j}$, with the unit vector field

$$\zeta(x,n) = \frac{a_X(x,n) \llcorner \left(\bigwedge^{d-j} Dg(x,n) \right) \Omega_{d-j}(g(x,n))}{J_{j-1} g(x,n)}$$

associated with $g^{-1}\{z\}$. Further, we apply [7, §4.1.30] (area theorem for currents) together with Lemma 3 and obtain

$$f_\#^{(z)} \langle N_X, g, z \rangle = \left(\mathcal{H}^{j-1} \llcorner f(g^{-1}\{z\}) \right) \wedge \eta$$

with unit vector field

$$\eta_z(x,v) = \frac{\left(\bigwedge_{j-1} Df(f^{-1}(x,v))\right)\zeta}{J_{j-1}f(f^{-1}(x,v))}.$$

(In fact, $f^{(z)}$ cannot be extended to a locally Lipschitz mapping over the whole space $L_j^z \times S^{j-1}$, nevertheless, due to (10) and since the unit normal bundle is closed, we can find a compact set containing $g^{-1}\{z\}$ to which $f^{(z)}$ can be extended as a Lipschitz function, verifying so the assumption of [7, §4.1.3].)

Conditions (10) and (28) assure that for $\mathcal{H}^{d-j}$-almost all $z \in S^{d-j}$, $\operatorname{nor}^{(j)}(X \cap L_j^z)$ agrees with the disjoint union of $f(g^{-1}\{z\})$ and $f(g^{-1}\{-z\})$ up to a set of $\mathcal{H}^{j-1}$-measure zero. It is thus sufficient to verify that the vector fields η and $a_{X\cap L_j^z}$ coincide almost everywhere, for $\mathcal{H}^{d-j}$-almost all $z \in S^{d-j}$. Since both are unit tangent vector fields associated with the same set, if suffices to show that they have the same orientation. To check the orientation, it is sufficient to verify that for almost all $(x,v) \in \operatorname{nor}^{(j)}(X \cap L_j)$, $\langle \eta(x,v), \varphi_p^{(j)}(v)\rangle > 0$ if $d-1-p$ is the number of infinite principal curvatures of $X \cap L_j$ at (x,v). We have

$$\begin{aligned}\langle \eta, \varphi_p^{(j)}\rangle &= \alpha\langle a_X \llcorner g^\# \Omega_{d-j}, f^\# \varphi_p^{(j)}\rangle\\ &= \alpha\langle a_X, g^\#\Omega_{d-j} \wedge f^\#\varphi_p^{(j)}\rangle,\end{aligned}$$

with a positive factor α. The last expression will be calculated later and the form (33) shows that it is positive at points where exactly $d-1-p$ principal curvatures are infinite. □

For the application of Lemma 4, the following result will be needed.

Lemma 5. *If $O \notin \partial X$ then* (28) *is fulfilled for almost all $L_{j-1} \in \mathcal{L}_{j-1}^d$.*

Proof. Note that (28) can be written equivalently as

$$\mathcal{H}^{j-1}(\operatorname{nor} X \cap (L_{j-1} \times \mathbb{R}^d)) = 0 \quad \text{for almost all } L_{j-1}. \tag{29}$$

If $j = 1$, the assertion follows from the assumption $O \notin \partial X$. Let us procede by induction on j. Assume that $j > 1$ and (29) is true for $j-1$. We shall show that

$$\mathcal{H}^{j-1}(\operatorname{nor} X \cap (L_{j-1(j-2)} \times \mathbb{R}^d)) = 0$$

for almost all $L_{j-1(j-2)}$ and almost all L_{j-2}, which is equivalent to (29). For L_{j-2} fixed, consider the locally Lipschitz mapping

$$\phi : \operatorname{nor} X \setminus (L_{j-2} \times \mathbb{R}^d) \to\ S^{d-j+1}(L_{j-2}^\perp)$$

qiven by $\phi(x,n) = \pi(x|L_{j-2}^\perp)$. Applying [7, §3.2.22(2)] to ϕ, we get that $\phi^{-1}\{z\}$ is locally $(\mathcal{H}^{j-2}, j-2)$ rectifiable, hence $\mathcal{H}^{j-1}(\phi^{-1}\{z\}) = 0$, for $\mathcal{H}^{d-j+1}$-almost all $z \in S^{d-j+1}(L_{j-2}^\perp)$. Since

$$\operatorname{nor} X \cap (L_{j-1(j-2)} \times \mathbb{R}^d) = \phi^{-1}(L_{j-1(j-2)} \cap L_{j-2}^\perp) \cup (\operatorname{nor} X \cap (L_{j-2} \times \mathbb{R}^d))$$

and $L_{j-1(j-2)} \cap L_{j-2}^\perp$ has only two points, the assertion follows. □

Proof of Theorem. Using the desintegration (4) we can write

$$\begin{aligned}
\int_{\mathcal{L}_j^d} & V_k(X \cap L_j)\, dL_j \\
&= \frac{2}{\sigma_j} \int_{\mathcal{L}_{j-1}^d} \int_{\mathcal{L}_{j(j-1)}^d} V_k(X \cap L_j)\, dL_{j(j-1)}\, dL_{j-1} \\
&= \frac{1}{\sigma_j} \int_{\mathcal{L}_{j-1}^d} \mathcal{I}(L_{j-1}) dL_{j-1}, \qquad (30)
\end{aligned}$$

and

$$\mathcal{I}(L_{j-1}) := \int_{S^{d-j}} V_k(X \cap L_j^z)\, \mathcal{H}^{d-j}(dz),$$

S^{d-j} being the unit sphere in $L_{j-1}^\perp$ (recall that L_j^z denotes the subspace spanned by L_{j-1} and z). The subspace L_{j-1} will be fixed in the following. Our next aim is to evaluate the integral $\mathcal{I}(L_{j-1})$. Let us remark that we do not know at this moment whether the integral exists since $V_k(X \cap L_j^z)$ can change sign.

Assume that (28) holds and that

$$\text{(10) holds for } \mathcal{H}^{d-j} - \text{almost all } z \in S^{d-j}. \qquad (31)$$

Using Lemma 4 and [7, §4.3.13], we get

$$\begin{aligned}
\mathcal{I}(L_{j-1}) &= 2 \int_{S^{d-j}} \left(f_\#^{(z)} \langle N_X, g, z \rangle \right) (\varphi_k^{(j)})\, \mathcal{H}^{d-j}(dz) \\
&= 2 \int_{S^{d-j}} (\langle N_X, g, z \rangle)\, (f^\# \varphi_k^{(j)})\, \mathcal{H}^{d-j}(dz) \\
&= 2 \left(N_X \llcorner g^\# \Omega_{d-j} \right) (f^\# \varphi_k^{(j)}) \\
&= 2 \int_{\operatorname{nor} X} \langle a_X, g^\# \Omega_{d-j} \wedge f^\# \varphi_k^{(j)} \rangle\, d\mathcal{H}^{d-1}.
\end{aligned}$$

We can represent the $(d-1)$-vector a_X in the form

$$a_X = (u_1, v_1) \wedge \cdots \wedge (u_{d-1}, v_{d-1})$$

with $u_{d-j+1}, \ldots, u_{d-1} \in L_j^z$, since

$$\dim \left(\operatorname{Tan}(\operatorname{nor} X, (x,n)) \cap (L_j^x \times \mathbb{R}^d) \right) \geq (d-1) + (j+d) - 2d = j - 1.$$

Then $(u_i, v_i) \in \ker Dg(x,n)$, $i = d-j+1, \ldots, d-1$, and, hence,

$$\begin{aligned}
\mathcal{I}(L_{j-1}) = 2 \int_{\operatorname{nor} X} & \left\langle \bigwedge_{i=1}^{d-j} Dg(x,n)(u_i, v_i), \Omega_{d-j}(z) \right\rangle \\
& \times \left\langle \bigwedge_{i=d-j+1}^{d-1} Df(x,n)(u_i, v_i), \varphi_k^{(j)}(\pi(n|L_j^z)) \right\rangle \mathcal{H}^{d-1}(d(x,n)).
\end{aligned}$$

Using the definition of $\varphi_k^{(j)}$ and (25), we obtain

$$\begin{aligned}
&\Big\langle \bigwedge_{i=d-j+1}^{d-1} Df(x,n)(u_i,v_i), \varphi_k^{(j)}(\pi_{L_j^z} n)\Big\rangle \\
&\quad = \sum_{|I|=k} (\operatorname{sgn} I) \Big\langle \bigwedge_{i\in I} u_i \wedge \bigwedge_{i\in I^C} \frac{p(v_i|L_j^x \cap n^\perp)}{|p(n|L_j^x)|} \wedge \pi(n|L_j), \Omega_j \Big\rangle \\
&\quad = \frac{1}{\sigma_{j-k}} \frac{1}{|p(n|L_j)|^{j-k}} \sum_{|I|=k} (\operatorname{sgn} I) \Big\langle \bigwedge_{i\in I} p(u_i|L_j^x \cap n^\perp) \\
&\qquad\qquad \wedge \bigwedge_{i\in I^C} p(v_i|L_j^x \cap n^\perp) \wedge p(n|L_j), \Omega_j \Big\rangle,
\end{aligned}$$

with summation over all index sets $I \subseteq \{1,\dots,d-1\}$ of given cardinality $|I|$, where $\operatorname{sgn} I$ is the sign of the permutation which maps the numbers $1,\dots,|I|$ in an increasing order onto I, and $|I|+1,\dots,d-1$ in an increasing order to $I^C := \{1,\dots,d-1\} \setminus I$. On the other hand, (24) yields

$$\Big\langle \bigwedge_{i=1}^{d-j} Dg(x,n)(u_i,v_i), \Omega_{d-j}(z)\Big\rangle = \frac{1}{|p(x|L_{j-1}^\perp)|^{d-j}} \Big\langle \bigwedge_{i=1}^{d-j} p(u_i|(L_j^x)^\perp), \Omega_{d-j}(z)\Big\rangle.$$

Thus we get

$$\mathcal{I}(L_{j-1}) = \frac{2}{\sigma_{j-k}} \int_{\operatorname{nor} X} \frac{1}{|p(x|L_{j-1}^\perp)|^{d-j} |p(n|L_j^x)|^{j-k}} \tau(x,n)\, \mathcal{H}^{d-1}(d(x,n)) \tag{32}$$

with

$$\begin{aligned}
\tau(x,n) = \sum_{|I|=k} (\operatorname{sgn} I) \Big\langle & \bigwedge_{i\in I} p(u_i|L_j^x \cap n^\perp) \wedge \bigwedge_{i\in I^C} p(v_i|L_j^x \cap n^\perp) \wedge p(n|L_j) \\
& \wedge \bigwedge_{i=j}^{d-1} p(u_i|(L_j^x)^\perp), \Omega_d \Big\rangle.
\end{aligned}$$

Using the fact that $u_i \in L_j$ for $i \le j-1$, we can write

$$\begin{aligned}
\tau(x,n) &= \sum_{\substack{I\subseteq J\\ |I|=k, |J|=j-1}} (\operatorname{sgn} I)(\operatorname{sgn} J) \Big\langle \bigwedge_{i\in I} p(u_i|L_j^x \cap n^\perp) \\
&\qquad \wedge \bigwedge_{i\in J\setminus I} p(v_i|L_j^x \cap n^\perp) \wedge p(n|L_j) \wedge \bigwedge_{i\in J^C} p(u_i|(L_j^x)^\perp), \Omega_d \Big\rangle \\
&= \sum_{\substack{I\subseteq J\\ |I|=k, |J|=j-1}} (\operatorname{sgn} I)(\operatorname{sgn} J) \Big\langle \bigwedge_{i\in I} p(p_0(u_i,v_i)u_i|L_j^x \cap n^\perp) \\
&\qquad \wedge \bigwedge_{i\in J\setminus I} p(p_1(u_i,v_i)|L_j^x \cap n^\perp) \wedge p(n|L_j) \\
&\qquad \wedge \bigwedge_{i\in J^C} p(p_0(u_i,v_i)|(L_j^x)^\perp), \Omega_d \Big\rangle;
\end{aligned}$$

here $p_0(u,v) - u$ and $p_1(u,v) = v$ are the orthogonal projections and the summations is taken over all index subsets $I \subseteq J \subseteq \{1,\dots,d-1\}$ of given cardinalities. The last expression is the value of a $(d-1)$-form applied to a_X, hence it does not depend on the particular representation of the $(d-1)$-vector a_X. Using the representation (27), we get

$$\begin{aligned}
\tau(x,n) &= \sum_{\substack{I\subseteq J\\ |I|=k, |J|=j-1}} (\operatorname{sgn} J) \frac{\prod_{i\in J\setminus I} \kappa_i}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2}} \\
&\qquad \times \left\langle \bigwedge_{i\in J} p(a_i|L_j^x \cap n^\perp) \wedge p(n|L_j) \wedge \bigwedge_{i\in J^C} p(a_i|(L_j^x)^\perp), \Omega_d \right\rangle \\
&= \sum_{|I'|=j-1-k} \frac{\prod_{i\in I'} \kappa_i}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2}} \sum_{\substack{J\supseteq I'\\ |J|=j-1}} \left| \bigwedge_{i\in J} p(a_i|L_j^x \cap n^\perp) \wedge p(n|L_j) \right| \\
&\qquad \times \left| \bigwedge_{i\in J^C} p(a_i|(L_j^x)^\perp) \right| \\
&= \sum_{|I'|=j-1-k} \frac{\prod_{i\in I'} \kappa_i}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2}} \sum_{\substack{J\supseteq I'\\ |J|=j-1}} \left| \bigwedge_{i\in J} p(a_i|L_j^x) \wedge p(n|L_j) \right| \\
&\qquad \times \left| \bigwedge_{i\in J^C} p(a_i|(L_j^x)^\perp) \right| \\
&= \sum_{|I'|=j-1-k} \frac{\prod_{i\in I'} \kappa_i}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2}} \sum_{\substack{J\supseteq I'\\ |J|=j-1}} \mathcal{G}\Big(L_j^x, \bigwedge_{i\in J^c} a_i\Big)^2.
\end{aligned}$$

Applying Lemma 1 to the subspaces L_j^x and A_I, we obtain

$$\tau(x,n) = \sum_{|I|=j-1-k} \frac{\prod_{i\in I} \kappa_i}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2}} \mathcal{G}\left(L_j^x, A_I\right)^2. \tag{33}$$

Revoking (30) and (32), we arrive at

$$\begin{aligned}
&\int_{\mathcal{L}_j^d} V_k(X\cap L_j)\, dL_j \\
&\qquad = \frac{2}{\sigma_j \sigma_{j-k}} \int_{\operatorname{nor} X} \sum_{|I|=j-1-k} \frac{\prod_{i\in I} \kappa_i}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2}} \widetilde{Q}(x,n,A_I)\, \mathcal{H}^{d-1}(d(x,n)),
\end{aligned}$$

with

$$\widetilde{Q}(x,n,A_I) = \int_{\mathcal{L}_{j-1}^d} \frac{1}{|p(x|L_{j-1}^\perp)|^{d-j} |p(n|L_j^x)|^{j-k}} \mathcal{G}(L_j^x, A_I)^2\, dL_{j-1}.$$

Finaly, we apply the coarea formula for the projection h of the subspace L_{j-1} into the orthogonal complement of x, see Lemma 6:

$$\begin{aligned}
\widetilde{Q}(x,n,A_I) &= \int_{\mathcal{L}^d_{j-1}} J_{(j-1)(d-j)}h(L_{j-1})\frac{1}{|x|^{d-j}}\frac{1}{|p(n|L^x_j)|^{j-k}}\mathcal{G}(L^x_j,A_I)^2\,dL_{j-1} \\
&= \int_{\mathcal{L}^{d-1}_{j-1}(x^\perp)} \mathcal{H}^{j-1}(h^{-1}(L_{j-1}))\frac{1}{|x|^{d-j}}\frac{1}{|p(n|L^x_j)|^{j-k}}\mathcal{G}(L^x_j,A_I)^2\,dL_{j-1} \\
&= \frac{\sigma_j}{2}\frac{1}{|x|^{d-j}}\int_{\mathcal{L}^d_{j(j-1)}} \frac{\mathcal{G}(L^x_j,A_I)^2}{|p(n|L^x_j)|^{j-k}}\,dL_{j(j-1)} \\
&= \frac{\sigma_j}{2}Q(x,n,A_I)
\end{aligned}$$

(we have used the fact that $h^{-1}(L_{j-1})$ is isomorphic to the space $\mathcal{L}^j_{j-1}$ of $(j-1)$-subspaces of L^x_j), and the assertion follows. □

Lemma 6. *Given a nonzero vector $x\in\mathbb{R}^d$ and $1\le q\le d-1$, consider the mapping*

$$\begin{aligned}
h:\ \mathcal{L}^d_q\setminus\mathcal{L}^d_{q(1)} &\to \mathcal{L}^{d-1}_q(x^\perp), \\
L_q &\mapsto p(L_q|x^\perp).
\end{aligned}$$

Then the Jacobian of h is given by

$$J_{q(d-1-q)}h(L_q)=\frac{|x|^{d-1-q}}{|p(x|L_q^\perp)|^{d-1-q}}.$$

Proof. Choose an orthonormal basis $\{u_1,\dots,u_d\}$ of $\mathbb{R}^d$ such that $u_1=\pi(x|L_q)$, $u_d=\pi(x|L_q^\perp)$ and L_q is spanned by $u_1,\dots,u_q$. Considering $\mathcal{L}^d_q$ as a submanifold of $\bigwedge_q\mathbb{R}^d$, the q-vectors

$$\xi^r_i=u_1\wedge\cdots\wedge u_{i-1}\wedge u_r\wedge u_{i+1}\cdots\wedge u_q,\quad 1\le i\le j,\ j+1\le r\le d,$$

form an orthonormal basis of the tangent space $\mathrm{Tan}(\mathcal{L}^d_q,L_q)$. Then, denoting $v_1=\pi(u_1|x^\perp)$,

$$h(L_q)=v_1\wedge u_2\wedge\cdots\wedge u_q,$$

and if the q-vectors ζ^r_i are defined as ξ^r_i with u_1 replaced by v_1, then

$$\zeta^r_i:\quad 1\le i\le q,\ q+1\le r\le d-1,$$

form an orthonormal basis of $\mathrm{Tan}(\mathcal{L}^{d-1}_q,h(L_q))$. We can evaluate the differential $Dh(L_q)$ at these basis vectors:

$$\begin{aligned}
&Dh(L_q)(\xi^r_1)=\frac{1}{\sin\angle(x,u_1)}\zeta^r_1, && j+1\le r\le d-1,\\
&Dh(L_q)(\xi^d_i)=0, && 1\le i\le q,\\
&Dh(L_q)(\xi^r_i)=\zeta^r_i, && 2\le i\le q, q+1\le r\le d-1.
\end{aligned}$$

Consequently,

$$J_{q(d-1-q)}h(L_q)=\frac{1}{|\sin\angle(x,u_1)|^{d-1-q}}=\frac{|x|^{d-1-q}}{|p(x|L_q^\perp)|^{d-1-q}}.$$

□

8 Extensions of the main theorem

A very slight modification of the proof of Theorem yields a local variant of (11) for curvature measures: Let the assumptions of Theorem be fulfilled and let, moreover, h be a nonnegative measurable function on $\mathbb{R}^d$. Then

$$\begin{aligned}
&\int_{\mathcal{L}_j^d}\int_{L_j} h(x) C_k(X\cap L_j, dx)\, dL_j^d \\
&\quad = \frac{1}{\sigma_{j-k}}\int_{\operatorname{nor} X} h(x)\frac{1}{|x|^{d-j}} \\
&\qquad \times \sum_{|I|=j-1-k} Q_j(x,n,A_I)\frac{\prod_{i\in I}\kappa_i(x,n)}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2(x,n)}}\,\mathcal{H}^{d-1}(\mathrm{d}(x,n)).
\end{aligned}$$

(Recall that the curvature measure $C_k(X,A)$ of X at A is defined by a formula analogous to (3), with the indicator function $1_A(x)$ added to the integral, see e.g. [30].)

Furthermore, we can use the additivity of curvature measures to generalise Theorem to finite unions of sets with positive reach. A set $X\subseteq\mathbb{R}^d$ is called a $\mathcal{U}_{PR}$ set if it can be represented as a locally finite union $X=\bigcup_{i=1}^{\infty} X_i$ for some $m\in\mathbb{N}$ such that for any index set $I\subseteq\{1,\dots,m\}$, the intersection $\bigcap_{i\in I} X_i$ has positive reach (provided that it is nonempty). Note that, in particular, sets from the extended convex ring are $\mathcal{U}_{PR}$ sets. Using the index function

$$i_X(x,n) := \mathbf{1}_X(x)\Big(1-\lim_{r\to 0_+}\lim_{s\to 0_+}\chi\big(X\cap B(x+((r+s)n,r))\big)\Big),$$

$x\in\mathbb{R}^d$, $n\in S^{d-1}$ ($B(y,t)$ denotes the closed ball of centre y and radius t and χ stands for the Euler-Poincaré characteristic), we can define the unit normal bundle of X as the support of i_X and the normal cycle of X as

$$N_X = (\mathcal{H}^{d-1}\llcorner \operatorname{nor} X)\wedge i_X a_X,$$

where a_X is a unit simple $(d-1)$ vector field orienting $\operatorname{nor} X$ in the same way as in the case of sets with positive reach. Applying N_X to the Lipschitz-Killing curvature forms, we obtain additive extensions of curvature measures for $\mathcal{U}_{PR}$-sets (see [24]).

Corollary 2. *Let X be a compact $\mathcal{U}_{PR}$ set with an $\mathcal{U}_{PR}$ representation $X=\bigcup_{i=1}^m X_i$ such that for any $I\subseteq\{1,\dots,m\}$, $O\notin\partial\bigcap_{i\in I}X_i$ and $\bigcap_{i\in I}X_i$ fulfills* (10)*. Then for any $0\le k<j$,*

$$\begin{aligned}
&\int_{\mathcal{L}_j^d} V_k^d(X\cap L_j)\, dL_j^d = \frac{1}{\sigma_{j-k}} \\
&\quad \times\int_{\operatorname{nor} X} i_X(x,n)\frac{1}{|x|^{d-j}}\sum_{|I|=j-1-k} Q(x,n,A_I)\frac{\prod_{i\in I}\kappa_i(x,n)}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2(x,n)}}\,\mathcal{H}^{d-1}(d(x,n)),
\end{aligned}$$

provided that the integral on the right hand side has sense.

Remark. It follows from Propositions 1 and 2 that the assumptions of Corollary 2 are fulfilled and the integral converges whenever X is a compact set from the convex ring with $O\notin\partial X$.

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