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Abstract

In this paper, we introduce Lévy driven Cox point processes (LCPs) as Cox point processes with driving intensity function Λ defined by a kernel smoothing of a Lévy basis (an independently scattered infinitely divisible random measure). We also consider log Lévy driven Cox point processes (LLCPs) with Λ equal to the exponential of such a kernel smoothing. Special cases are shot noise Cox processes, log Gaussian Cox processes and log shot noise Cox processes. We study the theoretical properties of Lévy based Cox processes, including moment properties described by nth order product densities, mixing properties, specification of inhomogeneity and spatio-temporal extensions.

1 Introduction

Cox point processes constitute one of the most important and versatile classes of point process models for clustered point patterns [10, 11, 38]. During the last decades several new classes of Cox point process models have appeared in the literature – e.g. shot noise Cox processes defined by means of generalized gamma measures [4], log Gaussian Cox processes [8, 29] and shot noise Cox processes [26]. These models share some common properties and differ in others, depending on how the driving intensity measure of the Cox process is constructed. One of the aims of this paper is to introduce a unified framework which is able to include all the different models mentioned above thus showing them in new light, investigate their relationships and define further natural extensions of those models.

The starting point for us will be the notion of a Lévy basis – an independently scattered infinitely divisible random measure. The terminology of a Lévy basis has been introduced in [1, 2]. Lévy bases include Poisson random measures, mixed Poisson random measures, Gaussian random measures as well as so-called $G-$measures [4]. Thus having in mind the construction of the shot noise Cox processes the second step in defining the driving intensity of the Cox process should be a kernel smoothing of the Lévy basis. By this we arrive at the definition of the Lévy driven Cox processes (LCPs) – i.e. Cox processes with the random driving intensity function defined by an integral of a weight function with respect to a Lévy basis. This construction has earlier been discussed by Robert L. Wolpert under the name of Lévy moving average processes [44], see also [45, 46]. It will be shown that LCPs are, under regularity conditions, shot noise Cox processes with additional random noise.

Furthermore, it is also possible to define the driving intensity as the exponential of a kernel smoothing of a Lévy basis (now allowing for non-positive weight functions and non-positive Lévy bases) thus arriving at the log Lévy driven Cox processes (LLCPs). It will be shown that LLCPs have, under regularity conditions, a driving field of the form $\Lambda = \Lambda_1 \cdot \Lambda_2$, where Λ_1 and Λ_2 are independent, Λ_1 is a log Gaussian field and Λ_2 is a log shot noise field. The latter process may describe clustered point patterns with randomly placed empty holes.

Shot noise Cox processes, log Gaussian Cox processes and log shot noise Cox processes will appear as natural building blocks in a modelling framework for Cox processes. Different types of combinations of the building blocks (corresponding to thinning and superposition) will be discussed in the present paper.

Having defined the framework the second aim is to study the theoretical properties of Lévy based Cox processes, including moment properties described by nth order product densities, mixing properties, specification of inhomogeneity and spatio-temporal extensions.

The present paper is organized as follows. In Section 2 we give a short overview of the theory of Lévy bases and integration with respect to such bases. In Section 3 we recall standard results about Cox processes. In Section 4 we introduce and study the Lévy driven Cox processes and in Section 5 the log Lévy driven Cox processes. Combinations of LCPs and LLCPs are discussed in Section 6, while inhomogeneous LCPs and LLCPs are considered in Section 7. We conclude with a discussion. In two appendices, integrability issues as well as further results for LCPs are addressed.

2 Lévy bases

This section provides a brief overview of the general theory of Lévy bases, in particular the theory of integration with respect to Lévy bases. For a more detailed exposition, see [2, 34] and references therein. Moreover, we discuss integrals of a kernel function with respect to a Lévy basis.

Let $(\mathcal{R}, \mathcal{A})$ be a measurable space. We will always suppose that $\mathcal{R}$ is a Borel subset of $\mathbb{R}^d$ and $\mathcal{A}$ is the δ-ring $\mathcal{B}_b(\mathcal{R})$ of bounded Borel subsets of $\mathcal{R}$.

Following [34], we consider a collection of real-valued random variables $L = \{L(A), A \in \mathcal{A}\}$ with the following properties

- for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{A}$, $L(A_1), \dots, L(A_n), \dots$ are independent random variables and $L(\cup_n A_n) = \sum_n L(A_n)$ a.s. provided $\cup_n A_n \in \mathcal{B}_b(\mathcal{R})$,
- for every A in $\mathcal{A}$, $L(A)$ is infinitely divisible.

If L has these properties, L will be called a *Lévy basis*, cf. [2]. If $L(A) \geq 0$ for all $A \in \mathcal{A}$, L is called a non-negative Lévy basis.

For a random variable X, let us denote the *cumulant function* $\log \mathsf{E}(\mathrm{e}^{ivX})$ by $C(v, X)$. When L is a Lévy basis, the cumulant function of $L(A)$ can by the Lévy-

Khintchine representation be written as

$$C(v, L(A)) = iva(A) - \frac{1}{2}v^2 b(A) + \int_{\mathbb{R}} (\mathrm{e}^{ivr} - 1 - ivr\mathbf{1}_{[-1,1]}(r))\, U(\mathrm{d}r, A), \qquad (1)$$

where a is a $\sigma-$additive set function on $\mathcal{A}$, b is a measure on $\mathcal{A}$, $U(\mathrm{d}r, A)$ is a measure on $\mathcal{A}$ for fixed $\mathrm{d}r$ and a Lévy measure on $\mathcal{B}(\mathbb{R})$ for each fixed $A \in \mathcal{A}$ (i.e. $U(\{0\}, A) = 0$ and $\int_{\mathbb{R}} (1 \wedge r^2)\, U(\mathrm{d}r, A) < \infty$, where $\wedge$ denotes minimum). The measure U is referred to as the *generalized Lévy measure* and L is said to have the *characteristic triplet* (a, b, U). If $b = 0$ then L is called a Lévy jump basis, if $U = 0$ then L is a Gaussian basis, see the examples below. A general Lévy basis L can always be written as a sum of a Gaussian basis and an independent Lévy jump basis.

Let $|a|$ denote the total variation measure generated by a and let μ denote the measure defined by

$$\mu(A) = |a|(A) + b(A) + \int_{\mathbb{R}} (1 \wedge r^2) U(\mathrm{d}r, A),$$

for $A \in \mathcal{A}$, and extended to a non-negative measure on $\sigma(\mathcal{A})$. We will call μ the *control measure*. In [34, Lemma 2.3] it has been shown that the generalized Lévy measure U factorizes as

$$U(\mathrm{d}r, \mathrm{d}\eta) = V(\mathrm{d}r, \eta)\mu(\mathrm{d}\eta), \qquad (2)$$

where $V(\mathrm{d}r, \eta)$ is a Lévy measure for fixed η. Moreover a and b are absolutely continuous with respect to μ, i.e.

$$a(\mathrm{d}\eta) = \tilde{a}(\eta)\mu(\mathrm{d}\eta), \qquad b(\mathrm{d}\eta) = \tilde{b}(\eta)\mu(\mathrm{d}\eta), \qquad (3)$$

and obviously $|\tilde{a}|, \tilde{b} \le 1$ μ a.s. .

Let $L'(\eta)$ be a random variable with the cumulant function

$$C(v, L'(\eta)) = iv\tilde{a}(\eta) - \frac{1}{2}v^2 \tilde{b}(\eta) + \int_{\mathbb{R}} (\mathrm{e}^{ivr} - 1 - ivr\mathbf{1}_{[-1,1]}(r))\, V(\mathrm{d}r, \eta). \qquad (4)$$

Then, we get the representation

$$C(v, L(\mathrm{d}\eta)) = C(v, L'(\eta))\, \mu(\mathrm{d}\eta). \qquad (5)$$

The random variables $L'(\eta)$ will play an important role in the following and will be called *spot variables*. For later use, note that if $\mathsf{E}(L'(\eta))$ and $\mathsf{Var}(L'(\eta))$ exists, then

$$\mathsf{E}(L'(\eta)) = \tilde{a}(\eta) + \int_{[-1,1]^C} rV(\mathrm{d}r, \eta),$$
$$\mathsf{Var}(L'(\eta)) = \tilde{b}(\eta) + \int_{\mathbb{R}} r^2 V(\mathrm{d}r, \eta).$$

The results given above have the consequence that it is no restriction if we for modelling purposes only consider Lévy bases with characteristic triplet (a, b, U) of the form

$$a(\mathrm{d}\eta) = \tilde{a}_\nu(\eta)\nu(\mathrm{d}\eta) \qquad (6)$$
$$b(\mathrm{d}\eta) = \tilde{b}_\nu(\eta)\nu(\mathrm{d}\eta) \qquad (7)$$
$$U(\mathrm{d}r, \mathrm{d}\eta) = V_\nu(\mathrm{d}r, \eta)\nu(\mathrm{d}\eta) \qquad (8)$$

where ν is a non-negative measure on $\sigma(\mathcal{A})$, $a_\nu : \mathcal{R} \to \mathbb{R}$ and $b_\nu : \mathcal{R} \to [0, \infty)$ are measurable functions and $V_\nu(\mathrm{d}r, \eta)$ is a Lévy measure for fixed η. The random variable satisfying (5) with μ replaced by ν will be denoted by $L'_\nu(\eta)$. For simplicity, we let $L'_\mu(\eta) = L'(\eta)$, $\tilde{a}_\mu = \tilde{a}$, $\tilde{b}_\mu = \tilde{b}$ and $V_\mu(\mathrm{d}r, \eta) = V(\mathrm{d}r, \eta)$. If $V_\nu(\cdot, \eta)$, $\tilde{a}_\nu(\eta)$ and $\tilde{b}_\nu(\eta)$ do not depend on η neither does the distribution of $L'_\nu(\eta)$ and the Lévy basis L is called *$\nu-$factorizable*. If moreover the measure ν is proportional to the Lebesgue measure, L is called *homogeneous* and all the finite dimensional distributions of L are translation invariant.

Let us now consider integration of a measurable function f on $\mathcal{R}$ with respect to a Lévy basis L.

Lemma 1 *Let f be a measurable function on $\mathcal{R}$ and L a Lévy basis on $\mathcal{R}$ with characteristic triplet (a, b, U). If the following conditions*

(i) $\int_\mathcal{R} |f(\eta)|\, |a|(\mathrm{d}\eta) < \infty$

(ii) $\int_\mathcal{R} f(\eta)^2\, b(\mathrm{d}\eta) < \infty$

(iii) $\int_\mathcal{R} \int_\mathbb{R} |f(\eta)r|\, V(\mathrm{d}r, \eta)\, \mu(\mathrm{d}\eta) < \infty$

are satisfied, then the function f is integrable with respect to L and $\int_\mathcal{R} f\, \mathrm{d}L$ is a well defined random variable with the cumulant function

$$C\left(v, \int_\mathcal{R} f\, \mathrm{d}L\right) = iv \int_\mathcal{R} f(\eta) a(\mathrm{d}\eta) - \frac{1}{2} v^2 \int_\mathcal{R} f(\eta)^2 b(\mathrm{d}\eta) + \int_\mathcal{R} \int_\mathbb{R} \left(e^{if(\eta)vr} - 1 - if(\eta)vr \mathbf{1}_{[-1,1]}(r)\right) V(\mathrm{d}r, \eta)\, \mu(\mathrm{d}\eta). \tag{9}$$

The proof of Lemma 1 is given in Appendix A. The conclusions of Lemma 1 hold under weaker assumptions, see [18, Proposition 5.6] or [34, Theorem 2.7]. The assumptions in Lemma 1 are simple to check and suffice for our purposes. The master thesis [18] also contains new selfcontained proofs of a number of other results concerning integration with respect to a Lévy basis.

Using equation (4) we can rewrite (9) as

$$C\left(v, \int_\mathcal{R} f\, \mathrm{d}L\right) = \int_\mathcal{R} C(vf(\eta), L'(\eta))\, \mu(\mathrm{d}\eta). \tag{10}$$

If the *kumulant function* (defined for a random variable X and $v \in \mathbb{R}$ by $K(v, X) = \log \mathsf{E}(e^{-vX})$ of the integral exists, then

$$K\left(v, \int_\mathcal{R} f\, \mathrm{d}L\right) = \int_\mathcal{R} K(vf(\eta), L'(\eta))\, \mu(\mathrm{d}\eta). \tag{11}$$

Example 1 (Gaussian Lévy basis). If L is a Gaussian Lévy basis with characteristic triplet $(a, b, 0)$, then $L(A)$ is $N(a(A), b(A))$ distributed for each set $A \in \mathcal{A}$. If (6) and (7) hold, we obtain $L'_\nu(\eta) \sim N(\tilde{a}_\nu(\eta), \tilde{b}_\nu(\eta))$. Furthermore,

$$C\left(v, \int_\mathcal{R} f\, \mathrm{d}L\right) = iv \int_\mathcal{R} f(\eta)\, a(\mathrm{d}\eta) - \frac{1}{2} v^2 \int_\mathcal{R} f(\eta)^2\, b(\mathrm{d}\eta).$$

It follows that

$$\int_{\mathcal{R}} f \, \mathrm{d}L \sim N\Big(\int_{\mathcal{R}} f(\eta)\, a(\mathrm{d}\eta), \int_{\mathcal{R}} f(\eta)^2\, b(\mathrm{d}\eta)\Big).$$

The basis is $\nu-$factorizable when $\tilde{a}_\nu$ and $\tilde{b}_\nu$ are constant. A concrete example of a Gaussian Lévy basis is obtained by attaching independent Gaussian random variables $\{X_i\}$ to a locally finite sequence $\{\eta_i\}$ of fixed points and let

$$L(A) = \sum_{\eta_i \in A} X_i, \quad A \in \mathcal{A}.$$

Another example of a Gaussian Lévy basis is the white noise process, cf. e.g. [23, Section 1.3]. $\square$

Example 2 (Poisson Lévy basis). The simplest Lévy jump basis is the Poisson basis for which $L(A) \sim Po(\nu(A))$, where ν is a non-negative measure on $\sigma(\mathcal{A})$. Clearly, L is a non-negative Lévy basis. This basis has characteristic triplet $(\nu, 0, \delta_1(\mathrm{d}r)\nu(\mathrm{d}\eta))$, where δ_c denotes the Dirac measure concentrated at c. Note that $\tilde{a}_\nu(\eta) \equiv 1$ and $V_\nu(\mathrm{d}r, \eta) = \delta_1(\mathrm{d}r)$. This basis is always $\nu-$factorizable. The random variable $L'_\nu(\eta)$ has a $Po(1)$ distribution. $\square$

Example 3 (generalized G-Lévy basis). A broad and versatile class of (non-negative) Lévy jump bases are the so-called generalized G–Lévy bases with characteristic triplet of the form $(a, 0, U)$ depending on a non-negative measure ν on $\sigma(\mathcal{A})$. The measures a and U satisfy (6) and (8) with

$$V_\nu(\mathrm{d}r, \eta) = \mathbf{1}_{\mathbb{R}_+}(r)\frac{r^{-\alpha-1}}{\Gamma(1-\alpha)}\,\mathrm{e}^{-\theta(\eta)r}\,\mathrm{d}r \quad \text{and} \quad \tilde{a}_\nu(\eta) = \int_0^1 \frac{r^{-\alpha}}{\Gamma(1-\alpha)}\mathrm{e}^{-\theta(\eta)r}\,\mathrm{d}r,$$

where $\alpha \in (-\infty, 1)$ and $\theta : \mathcal{R} \to (0, \infty)$ is a measurable function. Γ denotes the gamma function. The class includes two important special cases – the gamma Lévy basis for $\alpha = 0$ with $L'_\nu(\eta) \sim \Gamma(1, \theta(\eta))$, and the inverse Gaussian Lévy basis for $\alpha = \frac{1}{2}$ with $L'_\nu(\eta) \sim IG(\sqrt{2}, \sqrt{2\theta(\eta)})$. In case the function θ is constant $\theta(\eta) = \theta$ we get that $L(A) \sim G(\alpha, \nu(A), \theta)$, i.e. L is a G-measure as defined in [4, Section 2]. $\square$

The following theorem is a special case of the Lévy-Ito decomposition. This theorem will play a crucial role for the interpretation of some of the Lévy driven Cox processes to be considered in the subsequent sections.

Theorem 2 *Suppose that the Lévy basis L has no Gaussian part ($b = 0$) and its generalized Lévy measure U satisfies the following conditions*

- $U(\{(r, \eta)\}) = 0$ *for all* $(r, \eta) \in \mathbb{R} \times \mathcal{R}$ *(U is diffuse),*
- $\int_{[-1,1]\times A} |r|\, U(\mathrm{d}r, \mathrm{d}\eta) < \infty$ *for all* $A \in \mathcal{A}$.

Then

$$L(A) = a_0(A) + \int_{\mathbb{R}} rN(\mathrm{d}r, A), \quad A \in \mathcal{A}, \tag{12}$$

where

$$a_0(A) = a(A) - \int_{[-1,1]} rU(\mathrm{d}r, A), \quad A \in \mathcal{A},$$

and N is a Poisson measure on $\mathbb{R} \times \mathcal{R}$ with intensity measure U.

The conditions of Theorem 2 are satisfied for a Poisson Lévy basis and a generalized G-Lévy basis if ν is a diffuse locally finite measure on $\sigma(\mathcal{A})$.

3 Cox processes

Let $\mathcal{S}$ be a Borel subset of $\mathbb{R}^d$ and suppose that $\{\Lambda(\xi) : \xi \in \mathcal{S}\}$ is a non-negative random field which is almost surely integrable (with respect to the Lebesgue measure) on bounded Borel subsets of $\mathcal{S}$. A point process X on $\mathcal{S}$ is a *Cox process with the driving field* Λ, if conditionally on Λ, X is a Poisson process with intensity Λ ([9, 10, 28]). The driving measure Λ_M of the Cox process X is defined by

$$\Lambda_M(B) = \int_B \Lambda(\xi)\,\mathrm{d}\xi, \quad B \in \mathcal{B}_b(\mathcal{S}).$$

In the following, the intensity function of X will be denoted by $\rho(\xi)$ and, more generally, $\rho^{(n)}(\xi)$ is the nth order product density of X. It follows from the conditional structure of X that $\rho^{(n)}$ can be computed from Λ by

$$\rho^{(n)}(\xi_1, \dots, \xi_n) = \mathsf{E} \prod_{i=1}^{n} \Lambda(\xi_i), \qquad \xi_i \in \mathcal{S}. \tag{13}$$

(for a proof, using moment measures, see e.g. [10]). A useful characteristic of a point process is the pair correlation function

$$g(\xi_1, \xi_2) = \frac{\rho^{(2)}(\xi_1, \xi_2)}{\rho^{(1)}(\xi_1)\,\rho^{(1)}(\xi_2)}, \qquad \xi_1, \xi_2 \in \mathcal{S}.$$

Note that for a Cox process, the pair correlation function can be calculated as

$$g(\xi_1, \xi_2) = \frac{\mathsf{E}\,\Lambda(\xi_1, \xi_2)}{\mathsf{E}\,\Lambda(\xi_1)\,\mathsf{E}\,\Lambda(\xi_2)}.$$

It can be shown that a Cox process is overdispersed relative to the Poisson process, i.e.

$$\mathsf{Var}(X(B)) \geq \mathsf{E}\,X(B),$$

where $X(B)$ denotes the number of points from X falling in B.

Examples of Cox processes include shot noise Cox processes (SNCPs, see [4, 26, 45]) with driving field of the form

$$\Lambda(\xi) = \sum_{(r,\eta) \in \Phi} r\,k(\xi, \eta),$$

where k is a probability kernel ($k(\cdot, \eta)$ is a probability density) and Φ is the atoms of a Poisson measure on $\mathbb{R}_+ \times \mathcal{R}$, say. Concrete examples of probability kernels are the uniform kernel

$$k(\xi, \eta) = \frac{1}{\omega_d R^d} \mathbf{1}_{[0,R]}(\|\xi - \eta\|),$$

where $\omega_d = \pi^{d/2}/\Gamma(1 + d/2)$ is the volume of the unit ball in $\mathbb{R}^d$, and the Gaussian kernel

$$k(\xi, \eta) = \frac{1}{(2\pi\sigma^2)^{d/2}} \exp(-\|\xi - \eta\|^2/2\sigma^2), \tag{14}$$

where $\sigma^2 > 0$. Another important class of Cox processes are the log Gaussian Cox processes (LGCPs, see [29]) driven by the exponential of a Gaussian field Ψ

$$\Lambda(\xi) = \exp(\Psi(\xi)).$$

4 Lévy driven Cox processes (LCPs)

4.1 Definition

Let $\mathcal{S}$ be a Borel subset of $\mathbb{R}^d$. A point process X on $\mathcal{S}$ is called a *Lévy driven Cox process (LCP)* if X is a Cox process with a driving field of the form

$$\Lambda(\xi) = \int_{\mathcal{R}} k(\xi, \eta) L(\mathrm{d}\eta), \qquad \xi \in \mathcal{S}, \tag{15}$$

where L is a non-negative Lévy basis on $\mathcal{R}$. Furthermore, k is a non-negative function on $\mathcal{S} \times \mathcal{R}$ such that $k(\xi, \cdot)$ is integrable with respect to L for each $\xi \in \mathcal{S}$ and $k(\cdot, \eta)$ is integrable with respect to the Lebesgue measure on $\mathcal{S}$ for each $\eta \in \mathcal{R}$.

Note that it is always possible for each pair (k, L) to construct an associated pair $(\tilde{k}, \tilde{L})$ generating the same driving field Λ where now $\tilde{k}(\cdot, \eta)$ is a probability kernel. We may simply let

$$\begin{aligned} \tilde{k}(\xi, \eta) &= k(\xi, \eta)/\alpha(\eta), \\ \tilde{L}(\mathrm{d}\eta) &= \alpha(\eta) L(\mathrm{d}\eta) \end{aligned}$$

where

$$\alpha(\eta) = \int_{\mathcal{S}} k(\xi, \eta) \, \mathrm{d}\xi.$$

In the formulation and analysis of the models it is however convenient not always to restrict to probability kernels.

It is important to note that from the non-negativity of the Lévy basis L and [10, Theorem 6.1.VI], we get that L is equivalent to a random measure on $\mathcal{R}$. Thus, the measurability of Λ defined in (15) follows from measurability of k as a function of η and ξ and Tonelli's theorem. Therefore, Λ is a well-defined random field and (under the condition of local integrability - see below) the driving measure $\int_B \Lambda(\xi) \, \mathrm{d}\xi$, $B \in \mathcal{B}_b(\mathcal{S})$, is also a well-defined random measure determined by the finite-dimensional distributions of L (for further discussion of measurability issues, see [19]).

The function k and the Lévy basis L will be chosen such that Λ is almost surely locally integrable, i.e. $\int_B \Lambda(\xi)\,\mathrm{d}\xi < \infty$ with probability 1 for $B \in \mathcal{B}_b(\mathcal{S})$. A sufficient condition for the last property is that, cf. [28, Remark 5.1],

$$\int_B \mathsf{E}\,\Lambda(\xi)\,\mathrm{d}\xi < \infty, \qquad B \in \mathcal{B}_b(\mathcal{S}). \tag{16}$$

If L is factorizable, then (16) is satisfied if the following conditions hold

$$\int_1^\infty rV(\,\mathrm{d}r) < \infty,$$
$$\int_B \int_{\mathcal{R}} k(\xi,\eta)\mu(\,\mathrm{d}\eta)\,\mathrm{d}\xi < \infty, \quad B \in \mathcal{B}_b(\mathcal{S}).$$

4.2 The nth order product densities of an LCP

It is possible to derive a number of properties of LCPs, using the theory of Lévy bases presented in Section 2. Below, the nth order product densities are derived while the generating functional and void probabilities of an LCP are considered in Appendix B.

Proposition 3 *Suppose that*

$$\mathsf{E}\left(\int_{\mathcal{R}} k(\xi,\eta)L(\mathrm{d}\eta)\right)^n < \infty$$

and

$$\int_{\mathcal{R}}\int_{\mathbb{R}_+} (k(\xi,\eta)r)^n V(\mathrm{d}r,\eta)\mu(\mathrm{d}\eta) < \infty,$$

for all $\xi \in \mathcal{S}$. Then, the nth order product density of an LCP is given by

$$\rho^{(n)}(\xi_1,\ldots,\xi_n) = \frac{1}{2^n n!}\sum_{t\in T_n}\left(\prod_{j=1}^n t_j\right) B_n(\kappa_1(t),\ldots,\kappa_n(t)),$$

$\xi_1,\ldots,\xi_n \in \mathcal{S}$, where T_n denotes the set of all functions from $\{1,\ldots,n\}$ to $\{-1,1\}^n$, B_n is the nth complete Bell polynomial evaluated at

$$\kappa_j(t) = \int_{\mathcal{R}}\left(\sum_{i=1}^n t_i k(\xi_i,\eta)\right)^j \kappa_j(L'(\eta))\mu(\mathrm{d}\eta), \quad j = 1,\ldots,n,$$

and $\kappa_j(L'(\eta))$ is the jth cumulant moment of the spot variable $L'(\eta)$.

Proof. First we rewrite $\rho^{(n)}(\xi_1,\ldots,\xi_n) = \mathsf{E}\prod_{i=1}^n \Lambda(\xi_i)$, using the polarization formula ([13, p. 43])

$$\mathsf{E}\prod_{i=1}^n \Lambda(\xi_i) = \frac{1}{2^n n!}\sum_{t\in T_n}\left(\prod_{i=1}^n t_i\right)\mathsf{E}\left(\sum_{i=1}^n t_i\Lambda(\xi_i)\right)^n. \tag{17}$$

The terms

$$\mathsf{E}\Big(\sum_{i=1}^{n} t_i\Lambda(\xi_i)\Big)^n$$

can be computed by evaluating the nth complete Bell polynomial in the first n cumulants of $\sum_{i=1}^n t_i\Lambda(\xi_i) = \int_{\mathcal{R}} \sum_{i=1}^n t_i k(\xi_i,\eta)\, L(\mathrm{d}\eta)$. Thus, we have

$$\mathsf{E}\Big(\sum_{i=1}^{n} t_i\Lambda(\xi_i)\Big)^n = B_n(\kappa_1(t),\ldots,\kappa_n(t)),$$

where $\kappa_j(t)$ is the jth cumulant of

$$\int_{\mathcal{R}} \sum_{i=1}^{n} t_i k(\xi_i,\eta) L(\mathrm{d}\eta).$$

Under the assumptions of the proposition, $\kappa_j(t)$ can be calculated by differentiating (10) j times with $f(\eta) = \sum_{i=1}^n t_i k(\xi_i,\eta)$. We get

$$\kappa_j(t) = \int_{\mathcal{R}} \Big(\sum_{i=1}^{n} t_i k(\xi_i,\eta)\Big)^j \kappa_j(L'(\eta))\mu(\mathrm{d}\eta).$$

Note that

$$\mathsf{E}\Big(\int_{\mathcal{R}} \sum_{i=1}^{n} t_i k(\xi_i,\eta) L(\mathrm{d}\eta)\Big)^j < \infty$$

and

$$\int_{\mathcal{R}}\int_{\mathbb{R}_+} \Big(\sum_{i=1}^{n} t_i k(\xi_i,\eta)\Big)^j V(\mathrm{d}r,\eta)\mu(\mathrm{d}\eta) < \infty,$$

$j = 1,\ldots,n$, under the assumptions of the proposition. □

Corollary 4 *Suppose that $k(\xi,\cdot)$ satisfies the assumptions of Lemma 1 for each $\xi \in \mathcal{S}$. Then, the intensity function of the LCP exists and is given by*

$$\rho(\xi) = \int_{\mathcal{R}} k(\xi,\eta)\, \mathsf{E}(L'(\eta))\mu(\mathrm{d}\eta) \tag{18}$$

for all $\xi \in \mathcal{S}$. Furthermore, if

$$\mathsf{E}\left(\int_{\mathcal{R}} k(\xi,\eta) L(\mathrm{d}\eta)\right)^2 < \infty, \tag{19}$$

and

$$\int_{\mathcal{R}}\int_{\mathbb{R}} (k(\xi,\eta)r)^2\, V(\mathrm{d}r,\eta)\, \mu(\mathrm{d}\eta) < \infty, \tag{20}$$

for each $\xi \in \mathcal{S}$, the pair correlation function of the process exists and is given by

$$g(\xi,\zeta) = 1 + \frac{\int_{\mathcal{R}} k(\xi,\eta)k(\zeta,\eta)\, \mathsf{Var}(L'(\eta))\mu(\mathrm{d}\eta)}{\rho(\xi)\rho(\zeta)}, \tag{21}$$

for all $\xi,\zeta \in \mathcal{S}$.

Proof. The result follows from Proposition 3, using that the first and second complete Bell polynomials are given by $B_1(x) = x$, $B_2(x_1, x_2) = x_1^2 + x_2$. Also recall that $\kappa_1(L'(\eta)) = \mathsf{E}(L'(\eta))$ and $\kappa_2(L'(\eta)) = \mathsf{Var}(L'(\eta))$. $\square$

Corollary 5 (Stationary LCP) *Let $\mathcal{S} = \mathcal{R} = \mathbb{R}^d$ and assume that k is a homogeneous kernel in the sense that*

$$k(\xi, \eta) = k(\xi - \eta) \qquad \textit{for all} \quad \xi, \eta \in \mathbb{R}^d. \tag{22}$$

Let $\int k(\eta)\,\mathrm{d}\eta = \alpha$. Assume that L is a homogenous Lévy basis with control measure $\mu(\mathrm{d}\eta) = c\,\mathrm{d}\eta$ for some $c > 0$. Then, (18) and (21) take the following simplified form

$$\begin{aligned} \rho =& c\,\mathsf{E}\,L'\alpha \\ g(\xi, \zeta) =& 1 + \frac{\mathsf{Var}\,L'}{(\mathsf{E}\,L')^2}\,\frac{I_k(\zeta - \xi)}{c}, \end{aligned}$$

where I_k only depend on the kernel k

$$I_k(\zeta - \xi) = \int_{\mathbb{R}^d} \frac{k(\zeta - \xi + \eta)k(\eta)}{\alpha^2}\,\mathrm{d}\eta.$$

Note that the fraction $\frac{\mathsf{Var}\,L'}{(\mathsf{E}\,L')^2}$ is equal to $\frac{1}{\mathsf{E}\,L'}$, 1 and $\mathsf{E}\,L'$ for the Poisson, gamma and inverse Gaussian basis, respectively. The choice of the Lévy basis changes substantially the correlations in the LCP and the overall variability in the point pattern even when the corresponding LCPs are stationary and all other parameters of the model are the same. As an illustration, Figure 1 on the following page shows 3 stationary LCPs observed on a $[0, 100] \times [0, 200]$ window with $c = 0.003$, $\mathsf{E}\,L' = 2$ and a Gaussian kernel obtained as 10 times the kernel (14) with $\sigma = 4$. The spot variable L' is distributed as a $\mathsf{E}\,L'Po(1)-$variable, a $\Gamma(1, \mathsf{E}\,L')$-distributed variable and a $IG(1, 1/\,\mathsf{E}\,L')$-variable, respectively. From left to right, an increasing irregularity is clearly visible.

4.3 Mixing properties

The following proposition gives conditions for stationarity and mixing of an LCP. Mixing and ergodicity are important e.g. for establishing the consistency of model parameter estimates, including nonparametric estimates of the n-th order product density $\rho^{(n)}$ and the pair correlation function g. Mixing [10, Definition 10.3.I] implies ergodicity [10, p. 341]. The case of an LCP with G-Lévy basis has been treated in [4, Proposition 2.2].

Proposition 6 *Let $\mathcal{S} = \mathcal{R} = \mathbb{R}^d$ and assume that the Lévy basis L and the kernel k are homogeneous. Then, an LCP with driving field Λ of the form (15) is stationary and mixing.*

Proof. Note that a Cox process is stationary/mixing if and only if the driving field of the Cox process has the same property [10, Proposition 10.3.VII]. Using the assumptions of the proposition it is easily seen that $\{\Lambda(\xi + x) : \xi \in \mathbb{R}^d\}$ has the same distribution as $\{\Lambda(\xi) : \xi \in \mathbb{R}^d\}$ for all $x \in \mathbb{R}^d$.

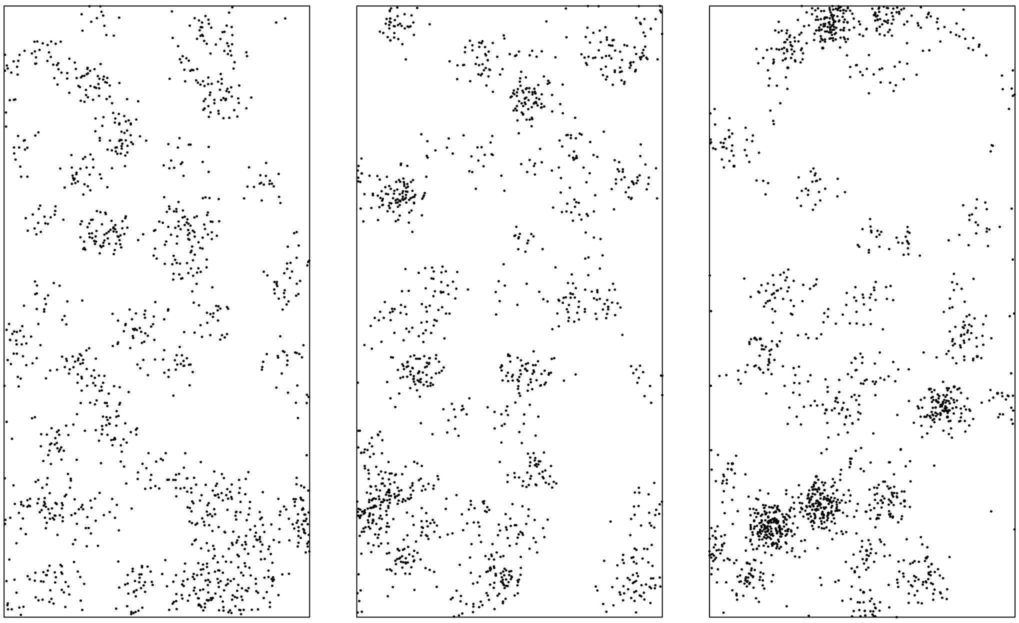

Figure 1: Examples of realizations of homogeneous LCPs with Poisson (left), gamma (middle) and inverse Gaussian (right) Lévy bases. For details, see the text.

According to [10, Proposition 10.3.VI(a)], Λ is mixing if and only if

$$L_\Lambda[h_1 + T_x h_2] \to L_\Lambda[h_1] L_\Lambda[h_2],$$

as $\|x\| \to \infty$. Here, h_1 and h_2 are arbitrary non-negative bounded functions on $\mathbb{R}^d$ of bounded support and L_Λ is the Laplace functional defined by

$$L_\Lambda[h] = \mathsf{E} \exp\left(- \int_{\mathbb{R}^d} h(\xi)\Lambda(\xi) \,\mathrm{d}\xi \right),$$

$T_x h(\xi) = h(\xi + x)$, $\xi, x \in \mathbb{R}^d$. We get

$$\begin{aligned}
&L_\Lambda[h_1 + T_x h_2] \\
&\quad = \mathsf{E} \exp\left(- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (h_1(\xi) + h_2(\xi + x)) k(\xi - \eta) L(\mathrm{d}\eta) \,\mathrm{d}\xi \right) \\
&\quad = \mathsf{E} \exp\left(- \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h_1(\xi) k(\xi - \eta) \,\mathrm{d}\xi + \int_{\mathbb{R}^d} h_2(\xi) k(\xi - \eta - x) \,\mathrm{d}\xi \right) L(\mathrm{d}\eta) \right) \\
&\quad = \mathsf{E} \left[\exp\left(- \int_{\mathbb{R}^d} \tilde{h}_1(\eta) L(\mathrm{d}\eta) \right) \cdot \exp\left(- \int_{\mathbb{R}^d} \tilde{h}_2(\eta + x) L(\mathrm{d}\eta) \right) \right],
\end{aligned}$$

where

$$\tilde{h}_i(\eta) = \int_{\mathbb{R}^d} h_i(\xi) k(\xi - \eta) \,\mathrm{d}\xi.$$

If k has bounded support, then we can find a $C > 0$ such that for $\|x\| > C$

$$\{\eta \in \mathbb{R}^d : \tilde{h}_1(\eta) > 0\} \cap \{\eta \in \mathbb{R}^d : \tilde{h}_2(\eta + x) > 0\} = \emptyset.$$

It follows that for $\|x\| > C$

$$\begin{aligned} L_\Lambda[h_1 + T_x h_2] &= \mathsf{E}\exp\left(-\int_{\mathbb{R}^d} \tilde{h}_1(\eta) L(\mathrm{d}\eta)\right) \cdot \mathsf{E}\exp\left(-\int_{\mathbb{R}^d} \tilde{h}_2(\eta + x) L(\mathrm{d}\eta)\right) \\ &= L_\Lambda[h_1] L_\Lambda[h_2], \end{aligned}$$

since L is independently scattered. If k does not have bounded support, we define a series of functions with bounded support

$$k_n(\xi - \eta) = k(\xi - \eta)\mathbf{1}_{[0,n)}(\|\xi - \eta\|), \quad n = 1, 2, \ldots$$

that converges monotonically from below to k. It follows that $\tilde{h}_{i,n}$ defined by

$$\tilde{h}_{i,n}(\eta) = \int_{\mathbb{R}^d} h_i(\xi) k_n(\xi - \eta)\,\mathrm{d}\xi$$

converges monotonically from below to $\tilde{h}_i(\eta)$ and for fixed n we can find C_n such that for $\|x\| > C_n$

$$\begin{aligned} &\mathsf{E}\left[\exp\left(-\int_{\mathbb{R}^d} \tilde{h}_{1,n}(\eta) L(\mathrm{d}\eta)\right) \cdot \exp\left(-\int_{\mathbb{R}^d} \tilde{h}_{2,n}(\eta + x) L(\mathrm{d}\eta)\right)\right] \\ &\quad = \mathsf{E}\exp\left(-\int_{\mathbb{R}^d} \tilde{h}_{1,n}(\eta) L(\mathrm{d}\eta)\right) \cdot \mathsf{E}\exp\left(-\int_{\mathbb{R}^d} \tilde{h}_{2,n}(\eta + x) L(\mathrm{d}\eta)\right) \end{aligned}$$

Using the reasoning just after [10, Proposition 10.3.VI], it follows that

$$L_\Lambda[h_1 + T_x h_2] \to L_\Lambda[h_1] L_\Lambda[h_2],$$

for the original functions h_1 and h_2. □

4.4 Examples of LCPs

4.4.1 Shot noise Cox processes (SNCPs) with random noise

Under the assumptions of Theorem 2, the driving field of an LCP takes the form

$$\Lambda(\xi) = \int_{\mathcal{R}} k(\xi, \eta) a_0(\mathrm{d}\eta) + \sum_{(r,\eta)\in\Phi} r k(\xi, \eta), \tag{23}$$

where Φ is the atoms of a Poisson measure on $\mathbb{R}_+ \times \mathcal{R}$ with intensity measure U. An LCP X with such a driving field is distributed as a superposition $X_1 \cup X_2$ where X_1 and X_2 are independent, X_1 is a Poisson point process with intensity function

$$\rho_1(\xi) = \int_{\mathcal{R}} k(\xi, \eta) a_0(\mathrm{d}\eta)$$

and X_2 is a shot noise Cox process as defined in [26] with driving field

$$\Lambda_2(\xi) = \sum_{(r,\eta)\in\Phi} r k(\xi, \eta).$$

An LCP with driving field Λ of the form (23) is therefore an SNCP with additional random noise. Simulation of the associated Lévy basis can be performed, using the algorithm introduced in [14], if L is factorizable, otherwise the algorithm developed in [46] may be used, see also [43]. A third option is the method used in [26]. An overview of available methods of simulating Lévy processes can be found in [37].

For $a_0 \equiv 0$, we get the familiar SNCPs. In [26], three specific examples of stationary SNCPs are considered. Using the notion of a Lévy basis, they are specified by $U(\mathrm{d}r, \mathrm{d}\eta) = V(\mathrm{d}r, \eta)\nu(\mathrm{d}\eta)$, where $\nu(\mathrm{d}\eta) \propto \mathrm{d}\eta$ and

- V is concentrated in a single point $c > 0$, i.e. $V(\mathrm{d}r) = \delta_c(\mathrm{d}r)$. If $c = 1$, the corresponding Lévy basis is Poisson. If $c \neq 1$, $L(A) \sim cPo(\nu(A))$. LCPs of this type are the well-known Matérn cluster process [24] and the Thomas process [39].
- $V((0,\infty)) < \infty$. In this case, Φ can be represented as a marked Poisson point process. Examples of LCPs with such a Lévy basis are the Neyman-Scott processes, cf. [32].
- $V(\mathrm{d}r) = \mathbf{1}_{\mathbb{R}_+}(r)\frac{r^{-\alpha-1}}{\Gamma(1-\alpha)}\,\mathrm{e}^{-\theta r}\,\mathrm{d}r$ corresponding to a G-Lévy basis. The resulting LCP is a so-called shot noise G Cox process [4].

In Figure 2, we show an example of a SNCP with a homogeneous Poisson process (a_0 is proportional to Lebesgue measure) as additional random noise. More precisely, the process $X = X_1 \cup X_2$ is defined on $[0, 200] \times [0, 100]$, X_1 is a Poisson process with intensity 0.01 and X_2 is an SNCP with Gaussian kernel (14) with $\sigma = 2$ and an intensity measure U of the form $U(\mathrm{d}r, \mathrm{d}\eta) = \delta_{25}(r) \cdot 0.0025\,\mathrm{d}\eta$. The process X_2 is thereby a Thomas process.

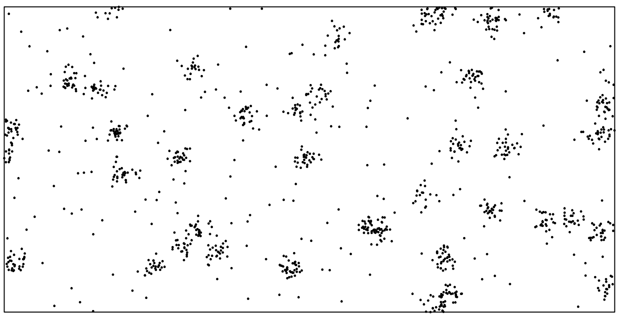

Figure 2: Example of a shot noise Cox process with extra noise. For details, see the text.

4.4.2 LCPs driven by smoothed discrete random fields

We suppose that $\{\eta_i\}$ is a locally finite sequence of fixed points and let

$$L(A) = \sum_{\eta_i \in A} X_i,$$

where $\{X_i\}$ is a sequence of independent and identically distributed non-negative random variables with infinitely divisible distribution. If, for instance, X_i is gamma or inverse Gaussian distributed, then L is a special case of a gamma or inverse Gaussian Lévy basis, respectively. The driving intensity of the associated LCP will take the form

$$\Lambda(\xi) = \sum_{\eta_i} k(\xi, \eta_i) X_i.$$

5 Log Lévy driven Cox processes (LLCPs)

5.1 Definition

Let $\mathcal{S}$ be a Borel subset of $\mathbb{R}^d$. A point process X on $\mathcal{S}$ is called a *log Lévy driven Cox process (LLCP)* if X is a Cox process with intensity field of the form

$$\Lambda(\xi) = \exp\left(\int_{\mathcal{R}} k(\xi, \eta) L(\mathrm{d}\eta)\right), \tag{24}$$

where L is a Lévy basis and k is a kernel such that $k(\xi, \cdot)$ is integrable with respect to L for each $\xi \in \mathcal{S}$, $k(\cdot, \eta)$ is integrable with respect to Lebesgue measure on $\mathcal{S}$ for each $\eta \in \mathcal{R}$ and Λ is almost surely locally integrable.

Since the driving intensity field of an LLCP is always non-negative because of the exponential function, we can generally use kernels and Lévy bases which also have negative values. Moreover, using the Lévy-Khintchine representation (1), we see that each Lévy basis L is equal to a sum of two independent parts – a Lévy jump part (let us denote it by L_J) and a Gaussian part (let us denote it by L_G). Thus we can represent the driving intensity of an LLCP as a product of two independent driving fields

$$\Lambda(\xi) = \exp\left(\int_{\mathcal{R}} k(\xi, \eta) L_J(\mathrm{d}\eta)\right) \exp\left(\int_{\mathcal{R}} k(\xi, \eta) L_G(\mathrm{d}\eta)\right) = \Lambda_J(\xi)\Lambda_G(\xi). \tag{25}$$

If $L_J \equiv 0$, Λ is the driving field of a log Gaussian Cox process (LCP) [8, 29]; if $L_G \equiv 0$, Λ is under regularity conditions the driving field of a log shot noise Cox process, see the examples in Section 5.4 below.

Note that we can simulate an LLCP with driving field (25) by first generating a log Gaussian Cox process X with driving field Λ_G and then thin X according to a realization of Λ_J. The roles of Λ_J and Λ_G may be interchanged. Simulated versions of Λ_J and Λ_G may also simply be multiplied and the result used directly as random intensity function.

Because of the exponential function in the definition of $\Lambda(\xi)$, stronger conditions on k and L are needed in order to ensure that Λ is almost surely locally integrable. A sufficient condition is that the kumulant transform $K(-k(\xi, \eta), L'(\eta))$ exists for all $\xi \in \mathcal{S}$ and $\eta \in \mathcal{R}$, and that

$$\int_B \exp\left(\int_{\mathcal{R}} K(-k(\xi, \eta), L'(\eta))\, \mu(\mathrm{d}\eta)\right) \mathrm{d}\xi < \infty, \quad \text{for all } B \in \mathcal{B}_b(\mathcal{S}). \tag{26}$$

This result follows from the definition of the kumulant function and from the key relation (11) for the kumulant transform. In particular, we use that

$$\mathsf{E}\,\Lambda(\xi) = \exp\left(K(1, -\int_{\mathcal{R}} k(\xi, \eta) L(\mathrm{d}\eta))\right) = \exp\left(\int_{\mathcal{R}} K(-k(\xi, \eta), L'(\eta))\, \mu(\mathrm{d}\eta)\right).$$

Note that

$$K(-k(\xi,\eta),L'(\eta)) = k(\xi,\eta)\tilde{a}(\eta) + \tfrac{1}{2}k(\xi,\eta)^2\tilde{b}(\eta) + \int_{\mathbb{R}} (e^{k(\xi,\eta)r} - 1 - k(\xi,\eta)r\mathbf{1}_{[-1,1]}(r))V(\mathrm{d}r,\eta).$$

If L is factorizable, then (26) is satisfied if *either* there exist $B > 0$, $C > 0$ and $D > 0$ such that

$$|k(\xi,\eta)| \le C \text{ for all } \xi \in \mathcal{S}, \eta \in \mathcal{R} \tag{27}$$

$$\int_{\mathcal{R}} |k(\xi,\eta)|^i \mu(\mathrm{d}\eta) < B \cdot D^i, \quad i = 1,2,\ldots, \quad \xi \in \mathcal{S} \tag{28}$$

$$\int_{\mathbb{R}} \left(e^{(C\vee D)|r|} - 1 - (C \vee D)|r|\mathbf{1}_{[-1,1]}(r)\right) V(\mathrm{d}r) < \infty, \tag{29}$$

or there exist $C > 0$ and $R > 0$ such that

$$|k(\xi,\eta)| \le C \text{ for all } \xi \in \mathcal{S}, \eta \in \mathcal{R} \tag{30}$$

$$k(\xi,\eta) = 0 \text{ for } \|\xi - \eta\| > R \tag{31}$$

$$\mu \text{ is locally finite} \tag{32}$$

$$\int_{\mathbb{R}} \left(e^{C|r|} - 1 - C|r|\mathbf{1}_{[-1,1]}(r)\right) V(\mathrm{d}r) < \infty. \tag{33}$$

Note that (27) and (28) are satisfied for the Gaussian kernel if μ is Lebesgue measure, while (30) and (31) hold for the uniform kernel. In the case of a purely Gaussian basis, (28) is only needed for $i = 2$ and conditions (29) and (33) are trivially fulfilled since $V \equiv 0$.

5.2 The nth order product densities of an LLCP

The nth order product densities of LLCPs are easily derived, using Lévy theory.

Proposition 7 *The nth order product density is given by*

$$\rho^{(n)}(\xi_1,\ldots,\xi_n) = \exp\left(\int_{\mathcal{R}} K(-\sum_{i=1}^n k(\xi_i,\eta), L'(\eta))\mu(\mathrm{d}\eta)\right), \tag{34}$$

$\xi_1,\ldots,\xi_n \in \mathcal{S}$, *provided the right-hand side exists.*

Proof. The formula follows directly from the definition of the kumulant function and from the key relation (11). We get

$$\begin{aligned}
\rho^{(n)}(\xi_1,\ldots,\xi_n) &= \mathsf{E}\prod_{i=1}^n \Lambda(\xi_i) = \mathsf{E}\exp\left(\sum_{i=1}^n \int_{\mathcal{R}} k(\xi_i,\eta)L(\mathrm{d}\eta)\right) \\
&= \exp\left(K(1, -\sum_{i=1}^n \int_{\mathcal{R}} k(\xi_i,\eta)L(\mathrm{d}\eta))\right) \\
&= \exp\left(\int_{\mathcal{R}} K(-\sum_{i=1}^n k(\xi_i,\eta), L'(\eta))\mu(\mathrm{d}\eta)\right).
\end{aligned}$$

□

Corollary 8 *The intensity function of an LLCP X is given by*

$$\rho(\xi) = \exp\left(\int_{\mathcal{R}} K(-k(\xi,\eta), L'(\eta))\,\mu(\mathrm{d}\eta)\right), \tag{35}$$

provided the right-hand side exists. When the second order product density exists, the pair correlation function of an LLCP takes the following form

$$\begin{aligned} g(\xi,\zeta) = &\exp\Bigg(\int_{\mathcal{R}}[K(-k(\xi,\eta)-k(\zeta,\eta), L'(\eta)) \\ &\qquad - K(-k(\xi,\eta), L'(\eta)) - K(-k(\zeta,\eta), L'(\eta))]\mu(\mathrm{d}\eta)\Bigg) \\ = &\exp\Bigg(\int_{\mathcal{R}} k(\xi,\eta)k(\zeta,\eta)b(\mathrm{d}\eta) \\ &\qquad + \int_{\mathcal{R}}\int_{\mathbb{R}}\left[e^{(k(\xi,\eta)+k(\zeta,\eta))r} - e^{k(\xi,\eta)r} - e^{k(\zeta,\eta)r} + 1\right] V(\mathrm{d}r,\eta)\,\mu(\mathrm{d}\eta)\Bigg). \end{aligned}$$

Corollary 9 (Stationary LLCP) *Let $\mathcal{S} = \mathcal{R} = \mathbb{R}^d$. Assume that k is a homogeneous kernel and L a homogeneous Lévy basis with $\mu(\mathrm{d}\eta) = c\,\mathrm{d}\eta$ for some $c > 0$. Then,*

$$\rho = \exp\left(c\int_{\mathbb{R}^d} K(-k(\eta), L')\,\mathrm{d}\eta\right)$$

and

$$\begin{aligned} g(\xi,\zeta) = &\exp\Bigg(\tilde{b}c\int_{\mathbb{R}^d} k(\xi-\zeta+\eta)k(\eta)\,\mathrm{d}\eta \\ &\qquad + c\int_{\mathbb{R}^d}\int_{\mathbb{R}}(e^{(k(\xi-\zeta+\eta)+k(\eta))r} - e^{k(\xi-\zeta+\eta)r} - e^{k(\eta)r} + 1)V(\mathrm{d}r)\,\mathrm{d}\eta\Bigg). \end{aligned}$$

5.3 Mixing properties

Proposition 10 *Let $\mathcal{S} = \mathcal{R} = \mathbb{R}^d$ and assume that the Lévy basis L is homogeneous and the kernel k is homogeneous in the sense of Proposition 6. Then, an LLCP with driving field of the form (24) is stationary and mixing.*

Proof. As in the proof of Proposition 6, we immediately get the stationarity. The method of proving mixing has to be modified compared to the one used in Proposition 6. First, rewrite

$$\begin{aligned} &L_\Lambda[h_1 + T_x h_2] \\ &\quad = \mathsf{E}\Bigg[\exp\left(-\int_{\mathbb{R}^d} h_1(\xi)\exp\left(\int_{\mathbb{R}^d} k(\xi-\eta)L(\mathrm{d}\eta)\right)\mathrm{d}\xi\right) \\ &\qquad\quad \cdot\exp\left(-\int_{\mathbb{R}^d} h_2(\xi+x)\exp\left(\int_{\mathbb{R}^d} k(\xi-\eta)L(\mathrm{d}\eta)\right)\mathrm{d}\xi\right)\Bigg] \\ &\quad = \mathsf{E}[A\cdot B_x], \end{aligned}$$

say. If k has bounded support, A and B_x will be independent if $\|x\|$ is large enough. If k does not have bounded support, we use a series of functions k_n with bounded support that converges to k. To be precise, let as in Proposition 6

$$k_n(u) = k(u)\mathbf{1}_{[0,n)}(\|u\|), \quad u \in \mathbb{R}^d,$$

$n = 1, 2, \dots$. We have $k_n \to k$ and $|k_n| \le |k|$. Now, let

$$A_n = \exp\left(-\int_{\mathbb{R}^d} h_1(\xi)\exp\left(\int_{\mathbb{R}^d} k_n(\xi-\eta)L(\mathrm{d}\eta)\right)\mathrm{d}\xi\right)$$

and

$$B_{x,n} = \exp\left(-\int_{\mathbb{R}^d} h_2(\xi+x)\exp\left(\int_{\mathbb{R}^d} k_n(\xi-\eta)L(\mathrm{d}\eta)\right)\mathrm{d}\xi\right)$$

Note that $0 \le A, A_n, B_x, B_{x,n} \le 1$. Now, consider the following inequality

$$\begin{aligned}
&|\,\mathsf{E}[A\cdot B_x] - \mathsf{E}\,A\cdot\mathsf{E}\,B_x| \\
&\quad\le |\,\mathsf{E}[A\cdot B_x] - \mathsf{E}[A\cdot B_{x,n}]| + |\,\mathsf{E}[A\cdot B_{x,n}] - \mathsf{E}[A_n\cdot B_{x,n}]| \\
&\qquad + |\,\mathsf{E}[A_n\cdot B_{x,n}] - \mathsf{E}\,A_n\cdot\mathsf{E}\,B_{x,n}| + |\,\mathsf{E}\,A_n\cdot\mathsf{E}\,B_{x,n} - \mathsf{E}\,A_n\cdot\mathsf{E}\,B_x| \\
&\qquad + |\,\mathsf{E}\,A_n\cdot\mathsf{E}\,B_x - \mathsf{E}\,A\cdot\mathsf{E}\,B_x|. \\
&\quad= \delta_{1xn} + \delta_{2xn} + \delta_{3xn} + \delta_{4xn} + \delta_{5xn},
\end{aligned}$$

say. Let us evaluate each of these five terms. Using that $0 \le A \le 1$ and that L is homogeneous, we get

$$\begin{aligned}
\delta_{1xn} &= |\,\mathsf{E}[A\cdot(B_x - B_{x,n})]| \\
&\le \mathsf{E}[A\cdot|B_x - B_{x,n}|] \\
&\le \mathsf{E}\,|B_x - B_{x,n}| \\
&= \mathsf{E}\,|B_0 - B_{0,n}|.
\end{aligned}$$

Now, since $k_n \to k$ and $|k_n| \le |k|$, where k is $L-$integrable, it follows that

$$\int_{\mathbb{R}^d} k_n(\xi-\eta)L(\mathrm{d}\eta) \to \int_{\mathbb{R}^d} k(\xi-\eta)L(\mathrm{d}\eta),$$

almost surely. We can therefore find n_1 (not dependent on x) such that for $n \ge n_1$, $\delta_{1xn} \le \epsilon$, say. Using the same type of arguments, we can find n_2, n_4, n_5 such that for $n \ge n_i$, $\delta_{ixn} \le \epsilon$, $i = 2, 4, 5$. Now choose a fixed $n \ge \max(n_1, n_2, n_4, n_5)$ and consider

$$\delta_{3xn} = |\,\mathsf{E}[A_n\cdot B_{x,n}] - \mathsf{E}\,A_n\cdot\mathsf{E}\,B_{x,n}|.$$

Using the previous results for bounded functions of bounded support, we finally find a constant $C > 0$ such that for x with $\|x\| > C$ we have $\delta_{3xn} \le \epsilon$. This completes the proof. □

5.4 Examples of LLCPs

5.4.1 Log shot noise Cox processes (LSNCPs)

Under the assumptions of Theorem 2, the driving field of an LLCP takes the form

$$\Lambda(\xi) = \exp\Bigg(d(\xi) + \sum_{(r,\eta)\in\Phi} rk(\xi,\eta)\Bigg), \tag{36}$$

where $d(\xi)$ is a deterministic function and Φ is the atoms of a Poisson measure on $\mathbb{R} \times \mathcal{R}$ with intensity measure U. Such a process is called a *log shot noise Cox process (LSNCP)*.

It is important to realize that SNCPs and LSNCPs are quite different model classes. An SNCP X with driving field of the form

$$\Lambda(\xi) = \sum_{(r,\eta)\in\Phi} rk(\xi,\eta)$$

is a superposition of independent Poisson processes $X_{(r,\eta)}$, $(r,\eta) \in \Phi$, where $X_{(r,\eta)}$ has intensity function $rk(\cdot,\eta)$. (The process $\{\eta : (r,\eta) \in \Phi\}$ is usually called the centre process (although it is not necessarily locally finite) while $X_{(r,\eta)}$ is called a cluster around η.) The presence of a particular cluster $X_{(r,\eta)}$ will not affect the presence of the other clusters.

In contrast to this, the driving field of an LSNCP takes the form

$$\Lambda(\xi) = \exp(d(\xi)) \prod_{(r,\eta)\in\Phi} \exp(rk(\xi,\eta)).$$

A cluster $X_{(r,\eta)}$ with negative, numerically large values of $rk(\cdot,\eta)$ will very likely contain 0 points and moreover, wipe out points from other clusters in the neighbourhood of η. In the resulting point pattern, empty holes may therefore be present. Examples of such point patterns are shown in Figure 3. Here, $\{\eta\}$ is a homogeneous Poisson process on $[0,100] \times [0,200]$ with intensity $c = 0.003$, $V(\mathrm{d}r) = \frac{1}{3}\delta_1(r) + \frac{2}{3}\delta_{-1}(r)$ and the kernel is (left) $k(\xi) = 1(|\xi| \le R)$ and (right) $k(\xi) = (1 - \frac{|\xi|^3}{R^3})1(|\xi| \le R)$, respectively, with $R = 10$.

5.4.2 Log Gaussian Cox processes (LGCPs)

In this subsection, we consider LLCPs with driving field of the form

$$\Lambda(\xi) = \exp\left(\int_{\mathcal{R}} k(\xi,\eta)L(\mathrm{d}\eta)\right), \tag{37}$$

where L is a Gaussian Lévy basis.

Clearly, the resulting process is an LGCP [8, 29]. If k and L are homogeneous, the process is stationary. In this case, the random intensity function $\Lambda(\xi)$ is well defined for all $\xi \in \mathbb{R}^d$ and almost surely integrable if

$$k(\xi) \le C, \xi \in \mathbb{R}^d, \text{ and } \int_{\mathbb{R}^d} k(\xi)^2 \,\mathrm{d}\xi < \infty. \tag{38}$$

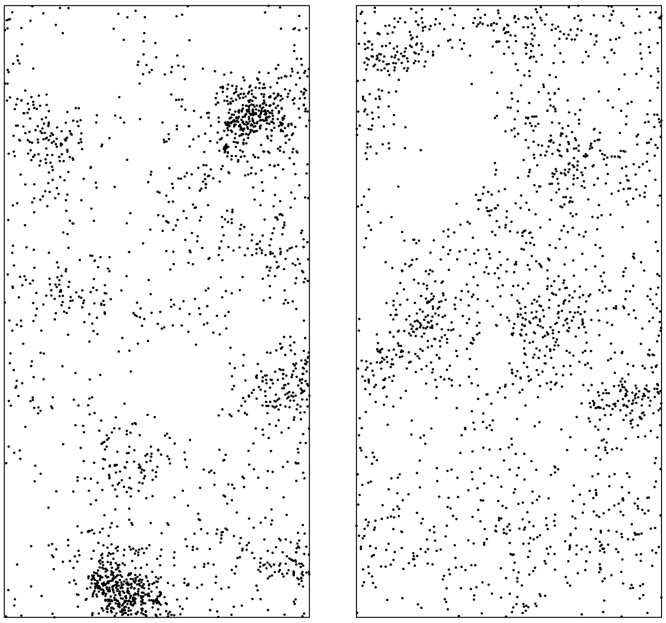

Figure 3: Examples of log shot noise Cox processes. Notice the circular empty holes in the point patterns. For details, see the text.

The covariance function of the Gaussian field

$$\Psi(\xi) = \int_{\mathbb{R}^d} k(\xi - \eta) L(\mathrm{d}\eta)$$

takes the form

$$\mathsf{Cov}(\Psi(\xi_1), \Psi(\xi_2)) = \int_{\mathbb{R}^d} k(\xi_1 - \xi_2 + \eta) k(\eta)\,\mathrm{d}\eta = c(\xi_1 - \xi_2),$$

say. Note that under (38) c is integrable. Under the mild additional assumption that the set of discontinuity points of k has Lebesgue measure 0, c is also continuous. In the proposition below, we show that any stationary LGCP with a continuous and integrable covariance function can indeed be obtained as a kernel smoothing (37) of a Gaussian Lévy basis. The proposition is a generalization of a result mentioned in [20].

Proposition 11 *Any stationary Gaussian random field with continuous and integrable covariance function can be generated by a kernel smoothing of a homogeneous Lévy basis.*

Proof. Let $\{\Psi(\xi) : \xi \in \mathbb{R}^d\}$ be an arbitrary stationary zero mean Gaussian field. Let $c(\xi_1, \xi_2) = c(\xi_1 - \xi_2)$ denote its covariance function which is a function of $\xi = \xi_1 - \xi_2$ due to the stationarity. Since c is continuous and positive definite, it follows from Bochner's Theorem that

$$c\,(\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot\eta} \tau\,(\mathrm{d}\eta)$$

for some non-negative measure τ. Since c is integrable and symmetric, τ has a symmetric density f, which can be found using the inverse Fourier-transform. $\sqrt{f}$ is continuous and a member of $L^2\left(\mathbb{R}^d\right)$. Note: For a symmetric function defined on $\mathbb{R}^d$ the Fourier transform and its inverse are the same up to multiplication/division with the constant $(2\pi)^{d/2}$.

By the convolution theorem for the Fourier(-Plancerel) transform we get

$$\widehat{\left(\hat{\sqrt{f}} * \hat{\sqrt{f}}\right)}^{-1} = \widehat{\hat{\sqrt{f}}}^{-1} \cdot \widehat{\hat{\sqrt{f}}}^{-1} = f,$$

thus

$$\hat{\sqrt{f}} * \hat{\sqrt{f}}\,(\xi) = c\,(\xi)\,.$$

Put $k = \hat{\sqrt{f}}$ and let L denote a homogeneous Lévy basis, with characteristic triplet $(0,1,0)$. Then, since the covariance function for $\int k dL$ is equal to $k * k$, our proof is complete. $\square$

In [29, Theorem 3], conditions for ergodicity is given in the special case of a stationary LGCP. Note that under (38) $c(\xi) \to 0$ for $\|\xi\| \to \infty$ and the conditions for ergodicity stated in [29, Theorem 3b] are satisfied.

6 Combinations of LCPs and LLCPs

The driving field of an LLCP has the form

$$\Lambda(\xi) = \exp\left(\int_{\mathcal{R}} k(\xi,\eta) L_J(\mathrm{d}\eta)\right) \exp\left(\int_{\mathcal{R}} k(\xi,\eta) L_G(\mathrm{d}\eta)\right) = \Lambda_J(\xi)\Lambda_G(\xi).$$

It seems natural to extend the model such that the kernels used in the jump part and the Gaussian part do not need to be the same. We thereby arrive at Cox processes with driving field of the form

$$\Lambda(\xi) = \exp\left(\int_{\mathcal{R}} k(\xi,\eta) L_J(\mathrm{d}\eta)\right) \Lambda_G(\xi). \tag{39}$$

with Λ_G an arbitrary log Gaussian random field.

If L_J satisfies the regularity conditions of Theorem 2, we get

$$\Lambda(\xi) = \exp\left(d(\xi) + \sum_{(r,\eta)\in\Phi} r k(\xi,\eta) + Y(\xi)\right),$$

where $d(\xi)$ is a deterministic function, Φ is the atoms of a Poisson measure with intensity measure U and Y is an independent Gaussian field.

A related model can be found in [35] for modelling the positions of offsprings in a long-leaf pine forest given the positions of the parents and information about the topography. The model is in [35] formulated conditional on the positions η of the parents.

There are, of course, other possibilities for combining shot noise components and log Gaussian components in the driving field than the one suggested above. For instance, if L_J is a non-negative Lévy jump basis, we may consider Cox processes driven by

$$\begin{aligned}\Lambda(\xi) &= \left(\int_{\mathcal{R}} k(\xi,\eta) L_J(\mathrm{d}\eta)\right)\Lambda_G(\xi) \\ &= \left(\int_{\mathcal{R}} k(\xi,\eta) a_0(\mathrm{d}\eta) + \sum_{(r,\eta)\in N} r k(\xi,\eta)\right)\Lambda_G(\xi),\end{aligned} \tag{40}$$

cf. [42]. In [11, pp. 92-100], a Cox process model of the type described in (40) has been considered but now with the Gaussian field replaced by a Boolean field. Such a model will be able to produce shot noise point patterns with empty holes generated by the Boolean field.

7 Inhomogeneous LCPs and LLCPs

In [31], it has recently been suggested to introduce inhomogeneity into a Cox process such that the resulting process becomes second-order intensity reweighted stationary. In this section, we describe four different ways of introducing inhomogeneity. Only one of them leads to second-order intensity reweighted stationary processes.

We concentrate on SNCPs with $a_0 \equiv 0$, cf. Section 4.4.1. The interpretation of the type of inhomogeneity introduced may be facilitated by using the cluster representation of a shot noise Cox process X. It is not needed that the process of cluster centres (mothers) is locally finite in order to use this interpretation.

Example 4 (Type 1). The kernel is assumed to be homogeneous $k(\xi,\eta) = k(\xi-\eta)$ while the Lévy basis satisfies $V(\mathrm{d}r,\eta) = V(\mathrm{d}r)$, $\nu(\mathrm{d}\eta) = cf(\eta)\,\mathrm{d}\eta$. If the function f is non-constant, mothers will be unevenly distributed (according to ν) but the distribution of the clusters will not depend on the location in the sense that the distribution of $X_{(r,\eta)} - \eta$ does not depend on η. □

Example 5 (Type 2). The kernel is assumed to be homogeneous $k(\xi,\eta) = k(\xi-\eta)$ while the Lévy basis satisfies $V(\mathrm{d}r,\eta) = V(\mathrm{d}(\frac{r}{f(\eta)}))$ and $\nu(\mathrm{d}\eta) = c\,\mathrm{d}\eta$. In this case, the mothers will be evenly distributed while the distribution of the clusters may be location dependent. A model with (k,V) replaced by $k(\xi,\eta) = k(\xi-\eta)f(\eta)$ and $V(\mathrm{d}r,\eta) = V(\mathrm{d}r)$ will result in the same type of LCP. □

Example 6 (Type 3). The kernel is inhomogeneous of the form $k(\xi,\eta) = k(\xi-\eta)f(\xi)$ while the Lévy basis is homogeneous $V(\mathrm{d}r,\eta) = V(\mathrm{d}r)$ and $\nu(\mathrm{d}\eta) = c\,\mathrm{d}\eta$. The resulting LCP will be a location dependent thinning of a stationary LCP. This option has been discussed in [31, 41] with the following log-linear specification of the function f

$$f(\xi) = \exp(z(\xi)\cdot\beta).$$

Here, $z(\xi)$ is a list of explanatory variables and β a parameter vector. Note that Type 2 and 3 inhomogeneity will typically have a similar appearance. The reason is that they can be regarded as only differing in the specification of the kernel as either of the form

$$k(\xi,\eta) = k(\xi-\eta)f(\eta)\ \text{(Type 2)}$$

or

$$k(\xi, \eta) = k(\xi - \eta) f(\xi) \text{ (Type 3)},$$

and $k(\xi - \eta)(f(\eta) - f(\xi))$ is only non-negligible if ξ and η are close enough so that $k(\xi - \eta)$ is non-negligible and at the same time there is an abrupt change in f between ξ and η. □

Example 7 (Type 4). Inhomogeneity may also be introduced into the process by a local scaling mechanism [15, 16]. Here, the kernel is inhomogeneous

$$k(\xi, \eta) = k\left(\frac{\xi - \eta}{f(\eta)}\right) \frac{1}{f(\eta)^d}.$$

while $V(\mathrm{d}r, \eta) = V(\mathrm{d}r)$ and $\nu(\mathrm{d}\eta) = c\,\mathrm{d}\eta / f(\eta)^d$. The inhomogeneity of the resulting point process can be explained by local scaling. □

In Figure 4, examples of inhomogeneous LCPs of Type 1, 2 and 4 are given on $\mathcal{S} = \mathcal{R} = [0, 100] \times [0, 200]$. Here, k is the Gaussian kernel (14) with $\sigma = 2$, $c = 1/200$ and V is concentrated in $r = 18$. The inhomogeneity function f is linear in all three cases, $f(x, y) = y/100$.

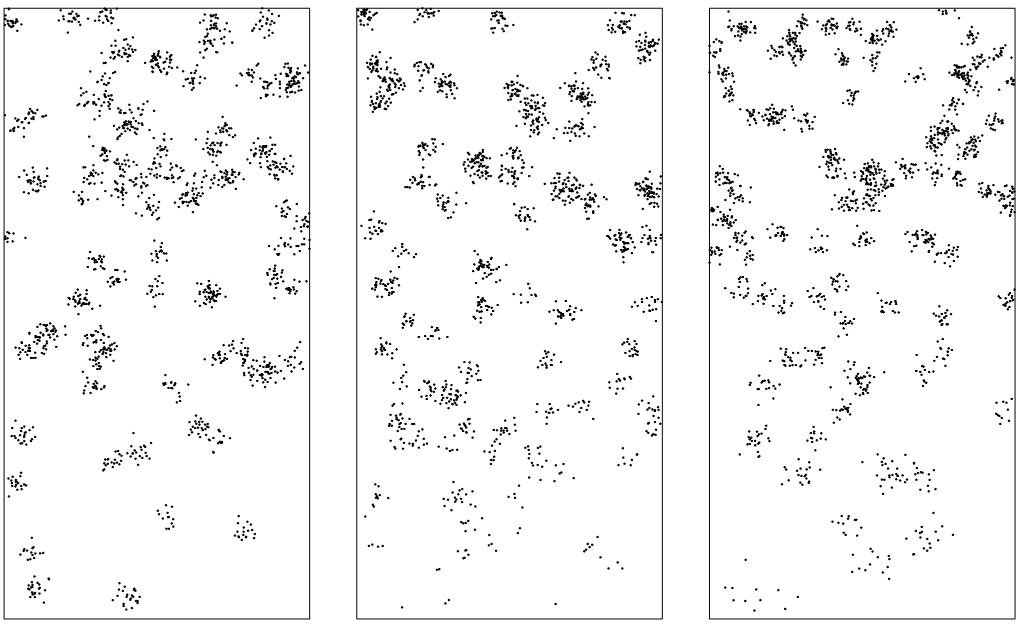

Figure 4: Examples of realizations of inhomogeneous LCPs. From left to right, Type 1, 2 and 4, respectively. For details, see the text.

Inhomogeneity may be introduced into an LSNCP by changing L or k as indicated in the examples above. Compared to LCPs, the effects are now multiplicative.

8 Discussion

During the last years, there has been some debate concerning which one of the two model classes (SNCP or LGCP) are most appropriate [27, 30, 36, 45]. The modelling framework described in the present paper provides the possibility for using models involving both SNCP and LGCP components and subsequently test whether it is appropriate to reduce the model to a pure SNCP model or a pure LGCP model. Below, we discuss a few additional issues.

8.1 Probability densities of LCPs and LLCPs

It is possible to derive an expression for the density of an LCP or an LLCP, using the methodology of Lévy bases. For instance, in the case of an LCP with $a_0 \equiv 0$, the density of X_B for $B \in \mathcal{B}_b(\mathcal{S})$ can be written as an expansion involving complete Bell polynomials evaluated at certain cumulants. The derivation of this result utilizes (17). Unfortunately, the expansion seems to be too complicated to be of practical use for inference. The same type of conclusion was reached when likelihood functions for G-shot-noise Cox processes were discussed in [4, Section 4.2.1]. Closed form expressions for densities of other types of Cox processes are available ([25]).

8.2 Spatio-temporal extensions

The LCPs and LLCPs extend easily to spatio-temporal Cox processes. The set $\mathcal{S}$ on which the process is living is now a Borel subset of $\mathbb{R}^d \times \mathbb{R}$ where the last copy of $\mathbb{R}$ indicates time. The dependency on the past at time t and position x may be modelled using an ambit set

$$A_t(x), \quad x \in \mathbb{R}^d, t \in \mathbb{R},$$

satisfying

$$\begin{aligned} (x,t) &\in A_t(x) \\ A_t(x) &\subseteq \mathbb{R}^d \times (-\infty, t] \end{aligned}$$

A spatio-temporal LCP is then defined by a driving intensity of the form

$$\Lambda(x,t) = \int_{A_t(x)} k((x,t),(y,s)) L(\mathrm{d}(y,s)),$$

where L is a non-negative Lévy basis on $\mathcal{R} \subseteq \mathbb{R}^d \times \mathbb{R}$ and k is a non-negative weight function. Likewise, a spatio-temporal LLCP has driving field of the form

$$\Lambda(x,t) = \exp\left(\int_{A_t(x)} k((x,t),(y,s)) L(\mathrm{d}(y,s)) \right),$$

where L and k do not need to be non-negative anymore. Using the tools of Lévy theory, it is possible to derive moment relations as shown in the present paper for the purely spatial case [33]. This approach to spatio-temporal modelling is expected to be very flexible and has earlier been used with success in growth modelling [22], see also [21]. It will be interesting to investigate how it performs compared to earlier methods described in [5, 6, 7, 12].

8.3 Statistical inference

Statistical inference for Cox processes has been studied earlier in a number of papers, including [3, 17, 27, 30, 31, 40]. It remains to investigate to what degree known procedures, based on summary statistics, likelihood or Bayesian reasoning, can be adjusted to deal with LCPs and LLCPs. For a stationary LCP with $\Lambda = \rho_1 + \Lambda_2$, it is easy to determine the summary statistics F, G and J in terms of the corresponding characteristics F_2, G_2 and J_2 of the shot noise component with intensity field Λ_2. Thus,

$$\begin{aligned} 1 - F(r) &= \exp(-\rho_1 |B(0,r)|)(1 - F_2(r)), \\ 1 - G(r) &= \exp(-\rho_1 |B(0,r)|) \left(\frac{\rho_1}{\rho_1 + \rho_2}(1 - F_2(r)) + \frac{\rho_2}{\rho_1 + \rho_2}(1 - G_2(r)) \right), \\ J(r) &= \frac{\rho_1}{\rho_1 + \rho_2} + \frac{\rho_2}{\rho_1 + \rho_2} J_2(r). \end{aligned}$$

However, in general simple expressions for G_2 and J_2 in terms of model parameters are not available. Likewise, it does not seem to be possible to derive general closed form expressions for F, G and J in the case of an LLCP.

In order to evaluate whether both a jump part and a Gaussian part is needed in an LLCP, we may consider a third order summary statistic, suggested in the paper [29] (for stationary point processes)

$$z(t) = \frac{1}{\pi^2 t^4} \int_{\|\xi\| \le t} \int_{\|\zeta\| \le t} \frac{\rho^{(3)}(\xi, \zeta)}{(\rho^{(1)})^3 \, g(\xi) \, g(\zeta) \, g(\xi - \zeta)} \, \mathrm{d}\xi \, \mathrm{d}\zeta, \qquad t > 0, \tag{41}$$

where the following abbreviated notation is used due to the stationarity

$$\begin{aligned} g(\xi_1, \xi_2) &= g(\xi_2 - \xi_1), \\ \rho^{(3)}(\xi_1, \xi_2, \xi_3) &= \rho^{(3)}(\xi_2 - \xi_1, \xi_3 - \xi_1). \end{aligned}$$

When computing the integrand in $z(t)$ for an LLCP we obtain

$$\begin{aligned} &\frac{\rho^{(3)}(\xi_1, \xi_2, \xi_3)}{(\rho^{(1)})^3 \, g(\xi_1, \xi_2) \, g(\xi_2, \xi_3) \, g(\xi_1, \xi_3)} \\ &\qquad = \frac{\mathsf{E}(\prod_{i=1}^3 \Lambda_J(\xi_i)) \, (\prod_{i=1}^3 \mathsf{E} \, \Lambda_J(\xi_i))}{\mathsf{E}(\Lambda_J(\xi_1)\Lambda_J(\xi_2)) \; \mathsf{E}(\Lambda_J(\xi_2)\Lambda_J(\xi_3)) \; \mathsf{E}(\Lambda_J(\xi_1)\Lambda_J(\xi_3))}, \end{aligned} \tag{42}$$

where $\Lambda_J(\xi) = \exp\left(\int_{\mathcal{R}} k(\xi, \eta) L_J(\mathrm{d}\eta)\right)$ is the part of the driving intensity originating from the pure jump part of the Lévy basis. Thus, this characteristic of X is not influenced by the Gaussian part of the model. In particular, $z \equiv 1$ for log Gaussian Cox processes. A non-parametric unbiased estimator of $z(t)$ has been derived in [29, Theorem 2].

Assessment of the full potential of the new modelling framework described in the present paper will also require more detailed studies of inhomogeneity and practical experience with concrete applications of the models.

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Appendix A – proof of Lemma 1

It suffices to check that the regularity conditions of [34, Theorem 2.7] are satisfied under the assumptions of Lemma 1. More specifically we need to check that

(a) $\int_{\mathcal{R}} |h(f(\eta),\eta)|\mu(\mathrm{d}\eta) < \infty$,

(b) $\int_{\mathcal{R}} f(\eta)^2\tilde{b}(\eta)\mu(\mathrm{d}\eta) < \infty$,

(c) $\int_{\mathcal{R}} \left(\int_{\mathbb{R}} \min\{1,(rf(\eta))^2\}V(\mathrm{d}r,\eta)\right)\mu(\mathrm{d}\eta) < \infty$,

where

$$h(u,\eta) = u\tilde{a}_\tau(\eta) + \int_{\mathbb{R}} (\tau(ru) - u\tau(r))V(\mathrm{d}r,\eta).$$

Here,

$$\tau(r) = r\mathbf{1}_{[-1,1]}(r) + \frac{r}{|r|}\mathbf{1}_{[-1,1]^C}(r)$$

and

$$\tilde{a}_\tau(\eta) = \tilde{a}(\eta) + \int_{[-1,1]^C} \frac{r}{|r|}V(\mathrm{d}r,\eta).$$

To proof (a), note that $|\tau(ru) \leq |ur|$. Therefore,

$$|h(f(\eta),\eta)| \leq |f(\eta)\tilde{a}_\tau(\eta)| + 2\int_{\mathbb{R}} |f(\eta)r|V(\mathrm{d}r,\eta).$$

Using (i) and (iii) of Lemma 1, it follows that

$$\int_{\mathcal{R}} |h(f(\eta),\eta)|\mu(\mathrm{d}\eta) \leq \int_{\mathcal{R}} |f(\eta)\tilde{a}(\eta)|\mu(\mathrm{d}\eta) + 3\int_{\mathcal{R}}\int_{\mathbb{R}} |f(\eta)r|V(\mathrm{d}r,\eta)\mu(\mathrm{d}\eta) < \infty.$$

Condition (b) is the same as (ii) and (c) follows from (iii) and

$$\min\{1,(rf(\eta))^2\} \leq |rf(\eta)|.$$

Appendix B – further results for LCPs

The distribution of a point process X on $\mathcal{S}$ can be characterized by the probability generating functional G_X. This functional is defined by

$$G_X(u) = \mathsf{E}\prod_{\xi\in X} u(\xi),$$

for functions $u : \mathcal{S} \to [0,1]$ with $\{\xi \in \mathcal{S} : u(\xi) < 1\}$ bounded. As proved e.g. in [10] the probability generating functional of a Cox process can be computed by

$$G_X(u) = \mathsf{E}\exp\left(-\int_{\mathcal{S}} (1-u(\xi))\Lambda(\xi)\,\mathrm{d}\xi\right). \tag{43}$$

Void probabilities can be calculated as

$$v(B) := P(X \cap B = \emptyset) = \mathsf{E}\exp\left(-\int_B \Lambda(\xi)\,\mathrm{d}\xi\right), \quad B \in \mathcal{B}_b(\mathcal{S}).$$

Proposition 12 *The probability generating functional of an LCP has the following form*

$$
\begin{aligned}
G_X(u) = \exp\Bigg(&- \int_{\mathbb{R}_+}\int_{\mathcal{R}} \left[1 - \exp\left(-\int_{\mathcal{S}} (1-u(\xi))k(\xi,\eta) r \,\mathrm{d}\xi\right)\right] U(\mathrm{d}r,\mathrm{d}\eta) \\
&- \int_{\mathcal{R}}\int_{\mathcal{S}} (1-u(\xi))k(\xi,\eta)\,\mathrm{d}\xi a_0(\mathrm{d}\eta)\Bigg),
\end{aligned}
$$

while the void probabilities are given by

$$
\begin{aligned}
v(B) = \exp\Bigg(&- \int_{\mathbb{R}_+}\int_{\mathcal{R}} \left[1 - \exp(-r\int_B k(\xi,\eta)\,\mathrm{d}\xi)\right] U(\mathrm{d}r,\mathrm{d}\eta) \\
&- \int_{\mathcal{R}}\int_B k(\xi,\eta)\,\mathrm{d}\xi a_0(\mathrm{d}\eta)\Bigg), \quad B \in \mathcal{B}_b(\mathcal{S}).
\end{aligned}
$$

Proof. Since $\Lambda(\xi)$ is almost surely locally integrable,

$$
\int_{\mathcal{S}} (1-u(\xi))\Lambda(\xi)\,\mathrm{d}\xi \le \int_{\mathcal{S}} \mathbf{1}_{\mathrm{supp}(1-u)}(\xi)\Lambda(\xi)\,\mathrm{d}\xi < \infty \tag{44}
$$

is a well-defined non-negative random variable and its kumulant transform exists. (In (44), the support of the function $1-u$ is denoted $\mathrm{supp}(1-u)$.) Using the key relation (11) for the kumulant function, we get

$$
\begin{aligned}
\log\left(G_X(u)\right) = {} & \log\left(\mathsf{E}\exp(-\int_{\mathcal{S}}(1-u(\xi))\Lambda(\xi)\,\mathrm{d}\xi)\right) \\
&= K\left(1, \int_{\mathcal{S}}(1-u(\xi))\Lambda(\xi)\,\mathrm{d}\xi\right) \\
&= K\left(1, \int_{\mathcal{S}}(1-u(\xi))\int_{\mathcal{R}} k(\xi,\eta)L(\mathrm{d}\eta)\,\mathrm{d}\xi\right) \\
&= K\left(1, \int_{\mathcal{R}}\left(\int_{\mathcal{S}}(1-u(\xi))k(\xi,\eta)\,\mathrm{d}\xi\right) L(\mathrm{d}\eta)\right) \\
&= \int_{\mathcal{R}} K\left(\int_{\mathcal{S}}(1-u(\xi))k(\xi,\eta)\,\mathrm{d}\xi,\, L'(\eta)\right)\mu(\mathrm{d}\eta) \\
&= -\int_{\mathcal{R}}\int_{\mathcal{S}}(1-u(\xi))k(\xi,\eta)\,\mathrm{d}\xi a_0(\mathrm{d}\eta) \\
&\quad + \int_{\mathcal{R}}\int_{\mathbb{R}_+}\left(\exp\left(-\int_{\mathcal{S}}(1-u(\xi))k(\xi,\eta) r\,\mathrm{d}\xi\right) - 1\right) V(\mathrm{d}r,\eta)\mu(\mathrm{d}\eta).
\end{aligned}
$$

The result for the void probabilities is obtained by choosing $u(\xi) = \mathbf{1}_{B^c}(\xi)$. □