DEPARTMENT OF OPERATIONS RESEARCH
UNIVERSITY OF AARHUS

Working Paper no. 2008 / 1 2008 / 05 / 15

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ISSN 1600-8987

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A note on Fenchel cuts for the single-node flow problem

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Abstract

The single-node flow problem, which is also known as the single-sink fixed-charge transportation problem, consists in finding a minimum cost flow from a number of nodes to a single sink. The flow cost comprise an amount proportional to the quantity shipped as well as a fixed charge. In this note, some structural properties of Fenchel cutting planes for this problem are described. Such cuts might then be applied for solving, e.g., fixed-charge transportation problems and more general fixed-charge network flow problems.

Key words: fixed-charge transportation problem, single-node flow problem, Lagrangian relaxation, cutting planes, Fenchel cuts

1 Introduction

Let $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$, denote the set or a subset of all feasible solutions to an integer program. For simplicity assume that the convex hull $\text{conv}(X)$ of X is bounded. Given a point y, it is well known from convex analysis that $y \notin \text{conv}(X)$ if and only if there exists a hyperplane $H = \{x \in \mathbb{R}^n : \pi x = \pi_0\}$ such that $\pi y > \pi_0$, whilst $\pi x \leq \pi_0$ holds for every $x \in \text{conv}(X)$.

Consider now a cutting plane approach that seeks to separate a solution $\hat{x} \in \hat{X} \supset \text{conv}(X)$ from the convex hull of X. Instead of considering polyhedral cuts, one might also try to solve the separation problem directly by finding any (not too week) valid inequality $\pi x \leq \pi_0$ that cuts off $\hat{x}$. The inequality's right hand side π_0 can obviously be replaced by

$$f(\pi) := \max\{\pi x : x \in X\}\,. \tag{1}$$

The resulting inequality

$$\pi x \leq f(\pi) \tag{2}$$

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is known as *Fenchel inequality* and as a *Fenchel cut* if $\pi\hat{x} > \pi_0$. Boyd (1994) first proposed these type of cuts in the framework of integer programming. Ralphs *et al.* (2003) use a principally equivalent type of cuts called *Farkas cuts* for the capacitated vehicle routing problem. Also the so-called (partial) convexification cuts introduced in Sherali *et al.* (2005) are very closely related to this concept.

The set $P := \{(\pi, \pi_0) \in \mathbb{R}^n : \pi x \leq \pi_0 \; \forall\, x \in \text{conv}(X)\}$ of all valid inequalities of $\text{conv}(X)$ is called the *polar* of the polyhedron $\text{conv}(X)$ (Nemhauser and Wolsey, 1988, Chap. I.4.5). Since $\pi_0 = f(\pi)$ must hold in any non-dominated valid inequality, $\text{conv}(X)$ is fully described by the set of Fenchel inequalities, that is

$$\text{conv}(X) = \left\{ x \in \mathbb{R}^n : \pi x \leq f(\pi) \; \forall\, \pi \in \mathbb{R}^n \right\}.$$

From $\pi_0 \geq \max_{x \in X} \pi x$ it furthermore follows that any inequality $\pi x \leq \pi_0$ valid for X is either dominated or equivalent to a Fenchel inequality. Fenchel inequalities should thus give sharp cutting planes; the separation problem is however computationally demanding.

In order to determine a Fenchel cutting plane for separating a given point $\hat{x}$ from $\text{conv}(X)$, the separation problem

$$\max_{\pi} \{v(\pi) : \pi \in \Pi\} \tag{3}$$

has to be solved, where

$$v(\pi) = \pi\hat{x} - f(\pi)\,. \tag{4}$$

The set Π just specifies some normalisation constraints on the coefficients π_j of the valid inequality, e.g, $\Pi = \{\pi : \|\pi\| = 1\}$ or $\Pi = \{\pi : -1 \leq \pi_j \leq 1 \; \forall\, j\}$. We then have $\hat{x} \notin \text{conv}(X)$ if and only if $v(\pi) > 0$ for some $\pi \in \Pi$ (Boyd, 1994). In the sequel, it is assumed that

$$\Pi = \left\{ \pi : \pi_j^{\min} \leq \pi_j \leq \pi_j^{\max} \; \forall\, j \right\}. \tag{5}$$

Generally $-\pi_j^{\min} = \pi_j^{\max} = 1$, but additional structural information might have been used to further restrict the coefficients π_j. The separation problem (3) is a non-differentiable optimisation problem (v is a is continuous piecewise linear concave function) and can be solved by any method suited for this purpose as, e.g., sub-gradient optimisation, column generation and Dantzig-Wolfe decomposition, resp., bundle methods, interior point decomposition methods, etc. To ease the computations it can be very useful to further restrict the inequality's coefficients.

2 Domain constraints on the inequality's coefficients

Additonal constraints on the inequality's coefficients can be helpful to speed up procedures for solving the separation problem.

Lemma 1. *Assume that* $0 \leq l_j \leq x_j \leq u_j$ *holds for all feasible integer solutions* $x \in X$. *If* $\hat{x}_j = u_j$ *then there exists an optimal solution* π^* *to the separation problem* (3) *with* Π *given by* (5) *such hat* $\pi_j^* = \pi_j^{\max}$.

Proof. Let $\pi_j < \pi_j^{\max}$ for given $\pi \in \Pi$. Set $\pi' = \pi + (\pi_j^{\max} - \pi_j)e_j$, where e_j denotes the j-th unit vector. Then

$$f(\pi') = \max\Big\{\pi' x : x \in X\Big\} = \max\Big\{\pi x + (\pi_j^{\max} - \pi_j)x_j : x \in X\Big\},$$

and from $\pi_j^{\max} - \pi_j > 0$ and $x_j \leq u_j$ we get

$$\begin{aligned} f(\pi') &\leq \max\{\pi x : x \in X\} + (\pi_j^{\max} - \pi_j)\max\{x_j : x \in X\} \\ &= f(\pi) + (\pi_j^{\max} - \pi_j)\max\{x_j : x \in X\} \\ &\leq f(\pi) + (\pi_j^{\max} - \pi_j)u_j \,. \end{aligned} \tag{6}$$

Furthermore

$$\begin{aligned} v(\pi) &= \pi\hat{x} - f(\pi) = \pi'\hat{x} - (\pi_j^{\max} - \pi_j)\hat{x}_j - f(\pi) \\ &= \pi'\hat{x} - \Big(f(\pi) + (\pi_j^{\max} - \pi_j)u_j\Big) \leq \pi'\hat{x} - f(\pi') = v(\pi')\,, \end{aligned}$$

where the last inequality above follows from (6). □

Lemma 2. *Assume that $l_j \leq x_j \leq u_j$ holds for all feasible integer solutions $x \in X$. If $\hat{x}_j = l_j$ then there exists an optimal solution π^* to the separation problem* (3) *with Π given by* (5) *such hat $\pi_j^* = \pi_j^{\min}$.*

Proof. Let $\pi_j > \pi_j^{\min}$ for given $\pi \in \Pi$. Set $\pi' = \pi + (\pi_j^{\min} - \pi_j)e_j$, where e_j denotes the j-th unit vector. Then

$$f(\pi') = \max\Big\{\pi' x : x \in X\Big\} = \max\Big\{\pi x - (\pi_j - \pi_j^{\min})x_j : x \in X\Big\}.$$

From $\pi_j - \pi_j^{\min} > 0$ and $x_j \geq l_j$ we get

$$\begin{aligned} f(\pi') &\leq \max\{\pi x : x \in X\} - (\pi_j - \pi_j^{\min})\min\{x_j : x \in X\} \\ &= f(\pi) - (\pi_j - \pi_j^{\min})\min\{x_j : x \in X\} \\ &\leq f(\pi) - (\pi_j - \pi_j^{\min})l_j \,. \end{aligned} \tag{7}$$

Furthermore

$$\begin{aligned} v(\pi) &= \pi\hat{x} - f(\pi) = \pi'\hat{x} + (\pi_j - \pi_j^{\min})\hat{x}_j - f(\pi) \\ &= \pi'\hat{x} - \Big(f(\pi) - (\pi_j - \pi_j^{\min})l_j\Big) \leq \pi'\hat{x} - f(\pi') = v(\pi')\,, \end{aligned}$$

where the last inequality above follows from (7). □

3 Fenchel cuts for the single-node flow problem

The set X of feasible solutions (x, y) to the single-node flow problem is the set of all flow vectors $x \in \mathbb{R}^n$ and binary vectors $y \in \{0,1\}^n$ meeting the constraints

$$\sum_{j=1}^{n} x_j = D \tag{8}$$

and

$$0 \le x_j \le b_j y_j \quad \text{for } j = 1, \ldots, n\,. \tag{9}$$

It is assumed that $0 < b_j \le D$ and $\sum_j b_j \ge D$. A valid Fenchel inequality for this set is then given by

$$\sum_{j=1}^{n} (\pi_j x_j - \lambda_j y_j) \le \pi_0\,, \tag{10}$$

where

$$\pi_0 = f(\pi, \lambda) := \max\Big\{\pi x - \lambda y : (x, y) \in X\Big\}\,. \tag{11}$$

Because of (8), any constant term might be added to the coefficients π_j of the flow variables x_j in order to obtain the same, equivalent inequality. Hence, $\pi_j \ge 0$ for $j = 1, \ldots, n$ (and also $\pi_0 > 0$) can be assumed without loss of generality. The following lemma can then be used to further restrict the range of the coefficients of the binary variables y_j.

Lemma 3. *For given $(\hat{x}, \hat{y})$ there always exists an optimal solution (π^*, λ^*) to the separation problem*

$$\max_{\lambda,\pi}\Big\{v(\pi, \lambda) : 0 \le \pi_j \le 1\,,\ -1 \le \lambda_j \le 1 \textit{ for } j = 1, \ldots, n\Big\}\,, \tag{12}$$

where

$$v(\pi, \lambda) = \pi\hat{x} - \lambda\hat{y} - f(\pi, \lambda)\,, \tag{13}$$

such that $\lambda_j^* \ge 0$.

Proof. If $\lambda_k < 0$, then it is optimal to set $y_k = 1$ in (11). Hence, in this case

$$f(\pi, \lambda) = -\lambda_k + \max\Big\{\sum_{j=1}^{n} \pi_j x_j - \sum_{j \ne k} \lambda_j y_j : (x, y) \in X\,,\ y_k = 1\Big\}$$

Also, if $\lambda_k = 0$ it remains optimal to set $y_k = 1$ in (11). Thus, if $\lambda^0 = \lambda - e_k \lambda_k$ results from λ just by setting λ_k to zero, we get

$$f(\pi, \lambda) = -\lambda_k + f(\pi, \lambda^0)$$

and

$$\begin{aligned}
v(\pi, \lambda) &= \pi\hat{x} - \lambda\hat{y} - f(\pi, \lambda) \\
&= \pi\hat{x} - \lambda\hat{y} + \lambda_k - f(\pi, \lambda^0) \\
&= \pi\hat{x} - \lambda^0\hat{y} - \lambda_k\hat{y}_k + \lambda_k - f(\pi, \lambda^0) \\
&= v(\pi, \lambda^0) + \lambda_k(1 - \hat{y}_k) \\
&\le v(\pi, \lambda^0)\,,
\end{aligned}$$

where the last inequality follows from $\lambda_k < 0$ and $\hat{y}_k \le 1$. ☐

In summary, in the Fenchel inequality (10) it can w.l.o.g. be assumed that

$$0 \le \pi_j \le 1 \quad \text{and} \quad 0 \le \lambda_j \le 1$$

for $j = 1, \ldots, n$. Furthermore, if the point $(\hat{x}, \hat{y})$ should be separated by a Fenchel inequality (10), we get from Lemma 1 and 2

$$\begin{aligned} \hat{x_j} = 0 \Rightarrow \pi_j = 0 \quad &\text{and} \quad \hat{x_j} = b_j \Rightarrow \pi_j = 1\,, \\ \hat{y_j} = 0 \Rightarrow \lambda_j = 1 \quad &\text{and} \quad \hat{y_j} = 1 \Rightarrow \lambda_j = 0\,. \end{aligned}$$

4 Solving the separation problem

Let $\{(x^t, y^t) : t \in T\}$ denote the set of all extreme points of $\operatorname{conv}(X)$, where X is the set of all feasible solutions to the single-node flow problem. The separation problem (12) can then be rewritten as the linear program

$$\max \sum_{j=1}^{n} \Big(\pi_j \hat{x}_j - \lambda_j \hat{y}_j\Big) - \pi_0 \tag{14}$$

$$\text{s.t.:} \sum_{j=1}^{n} \Big(\pi_j x_j^t - \lambda_j y_j^t\Big) - \pi_0 \le 0 \quad \forall\, t \in T\,, \tag{15}$$

$$\pi_j \le 1 \quad \text{for } j = 1, \ldots, n\,, \tag{16}$$

$$\lambda_j \le 1 \quad \text{for } j = 1, \ldots, n\,, \tag{17}$$

$$\lambda,\, \pi \ge 0\,,\ \pi_0 \in \mathbb{R}\,. \tag{18}$$

In the sequel, a subgradient optimisation procedure for approximately solving the separation problem (14)–(18) is described. Subgradient methods can take advantage from a (sharp) upper bound on the maximal function value. Sect. 4.1 shows how such upper bounds can be obtained. Sect. 4.2 then summarises a subgradient algorithm for maximising $v(\pi, \lambda)$ and Sect. 4.3 illustrates the method's behaviour on small numerical example.

4.1 Upper bounds on the separation problem's objective value

Let α_t, $t \in T$, as well as η_j and ν_j for $j = 1, \ldots, n$ be dual multipliers of the constraints (15), (16) and (17). The dual of the separation problem (14)–(18) then reads as

$$\min \sum_{j=1}^{n} \Big(\eta_j + \nu_j\Big) \tag{19}$$

$$\text{s.t.:} \sum_{t \in T} \alpha_t x_j^t + \eta_j \ge \hat{x}_j \quad \text{for } j = 1, \ldots, n\,, \tag{20}$$

$$\sum_{t \in T} \alpha_t y_j^t - \nu_j \le \hat{y}_j \quad \text{for } j = 1, \ldots, n\,, \tag{21}$$

$$\sum_{t \in T} \alpha_t = 1\,, \tag{22}$$

$$\alpha,\, \eta,\, \nu \geq 0\,, \tag{23}$$

which is equivalent to the programming problem

$$\min \sum_{j=1}^{n} \Big(\eta_j + \nu_j\Big) \tag{24}$$

$$\text{s.t.: } x + \eta \geq \hat{x}\,, \tag{25}$$

$$y - \nu \leq \hat{y}\,, \tag{26}$$

$$(x, y) \in \text{conv}(X)\,, \tag{27}$$

$$\eta,\, \nu \geq 0\,. \tag{28}$$

Without inclusion of the normalisation and additional constraints imposed on the inequality's coefficient π and λ, constraints (20) and (21) as well as (25) and (26) would be equality constraints. Moreover, we would have $\eta = \nu = 0$ and the programming problem (19)–(23) and (24)–(28), resp., would just ask if the given point $(\hat{x}, \hat{y})$ is contained in the convex hull of X or not. Since $\eta_j = \max\{0, \hat{x}_j - x_j\}$ and $\nu_j = \max\{0, y_j - \hat{y}_j\}$ must hold in any optimal solution to (24)–(28), this program may for short also be stated as

$$\min\Big\{\sum_{j=1}^{n}\Big(\max\{0, \hat{x}_j - x_j\} + \max\{0, y_j - \hat{y}_j\}\ \Big) : (x, y) \in \text{conv}(X)\Big\}. \tag{29}$$

This shows that

$$U(x, y) := \sum_{j=1}^{n}\Big(\max\{0, \hat{x}_j - x_j\} + \max\{0, y_j - \hat{y}_j\}\Big) \tag{30}$$

provides an upper bound on $\max_{\pi,\lambda} v(\pi, \lambda)$ for any solution $(x, y) \in \text{conv}(X)$.

4.2 Subgradient procedure

The linear program (19)–(23) can be solved exactly by optimisation means of column generation. Alternatively, subgradient can be applied for approximately maximising the piecewise linear and concave function $v(\pi, \lambda)$ in (13). At the point $(\pi, \lambda) = (\overline{\pi}, \overline{\lambda})$, the function $f(\pi, \lambda)$ in (11) is given by

$$f(\overline{\pi}, \overline{\lambda}) = \overline{\pi}\overline{x} - \overline{\lambda}\overline{y}$$

and a subgradient of the function $v(\pi, \lambda) = \pi\hat{x} - \lambda\hat{y} - f(\pi, \lambda)$ at this point thus reads as

$$\Big((\hat{x} - \overline{x})\,,\ (\overline{y} - \hat{y})\Big)\,.$$

The components $\hat{x} - \overline{x}$ and $\overline{y} - \hat{y}$ of this subgradient are however of totally different dimensions. Whilst $\hat{y}_j$ is a dimensionless number between zero and one, $\hat{x}_j$ denotes a quantity. This is avoided by replacing variables y_j with $z_j = b_j y_j$ and restating the set X of feasible solutions to the single-node flow problem as

$$X = \Big\{(x, z) : \sum_{j=1}^{n} x_j = D\,,\ 0 \leq x_j \leq z_j \text{ and } z_j \in \{0, b_j\}\ \forall\, j\Big\}\,. \tag{31}$$

The Fenchel inequality then reads as

$$\sum_{j=1}^{n}(\pi_j x_j - \lambda_j z_j) \leq f(\pi, \lambda)\,, \tag{32}$$

where

$$f(\pi, \lambda) = \max\Big\{\pi_j x_j - \lambda_j z_j : (x, z) \in X\Big\} \tag{33}$$

and

$$0 \leq \pi_j \leq 1 \quad \text{as well as} \quad 0 \leq \lambda_j \leq 1 \quad \forall\, j\,.$$

In the separation problem (14)–(18), we then just have to replace y_j by z_j, and a subgradient of

$$v(\pi, \lambda) = \pi\hat{x} - \lambda\hat{z} - f(\pi, \lambda)$$

at the point $(\overline{\pi}, \overline{\lambda})$ is now given by

$$\Big((\hat{x} - \overline{x})\,,\ (\overline{z} - \hat{z})\Big)\,,$$

if $(\overline{x}, \overline{z})$ is such that

$$f(\overline{\pi}, \overline{\lambda}) = \overline{\pi}\overline{x} - \overline{\lambda}\overline{z}\,.$$

During subgradient optimisation, we might further use the restrictions

$$\begin{array}{rcl} \hat{x}_j = 0 & \Rightarrow & \pi_j = 0\,, \\ \hat{x}_j = b_j & \Rightarrow & \pi_j = 1\,, \\ \hat{z}_j = 0 & \Rightarrow & \lambda_j = 1\,, \\ \hat{z}_j = b_j & \Rightarrow & \lambda_j = 0\,. \end{array} \tag{34}$$

Using the step size parameter σ, $0 < \sigma \leq 2$, and a suitable direction (h_x, h_z), the step size θ in the subgradient step

$$(\pi', \lambda') = (\overline{\pi}, \overline{\lambda}) + \theta(h_x, h_z)\,,$$

is then usually computed according to

$$\theta = \sigma \frac{U - v(\overline{\pi}, \overline{\lambda})}{\Big\|(h_x, h_z)\Big\|^2}\,,$$

where U is an upper bound on $\max_{\pi,\lambda} v(\pi, \lambda)$. According to (30) such upper bounds are readily available from

$$U(x, z) = \sum_{j=1}^{n}\Big(\max\{0, \hat{x}_j - x_j\} + \max\{0, z_j - \hat{z}_j\}\Big) \tag{35}$$

and any solution $(x, z) \in \operatorname{conv}(X)$. In case that the point $(\hat{x}, \hat{z})$ to be separated meets all constraints except the requirements, the solution $x = \hat{x}$ and $z = z(\hat{x})$, where

$$z_j(\hat{x}) := \begin{cases} b_j & \text{if } \hat{x}_j > 0 \\ 0 & \text{if } \hat{x}_j = 0 \end{cases}$$

is feasible for the single-node flow problem and thus generates the upper bound

$$U(\hat{x}, z(\hat{x})) = \sum_{j:\hat{x}_j>0} (b_j - z_j(\hat{x}))\,.$$

Moreover, in the course of a subgradient algorithm for maximising the function $v(\pi, \lambda)$, the subproblem

$$\begin{aligned} f(\pi^k, \lambda^k) = \max \ & \sum_{j=1}^{n} (\pi_j^k x_j - \lambda_j^k z_j) \\ \text{s.t.: } & \sum_{j=1}^{n} x_j = D\,, \\ & 0 \le x_j \le z_j \text{ for } j = 1, \ldots, n\,, \\ & z_j \in \{0, b_j\} \text{ for } j = 1, \ldots, n \end{aligned} \tag{36}$$

has repeatedly to be solved with (π^k, λ^k) being the current iterate. Let then $(x^\ell, z^\ell) \in X$ for $\ell = 1, \ldots, k$ denote the solutions obtained to the subproblem in iterations 1 to k. The average solution

$$(\overline{x}^k, \overline{z}^k) = \frac{1}{k} \sum_{\ell=1}^{k} (x^\ell, z^\ell)$$

is from $\mathrm{conv}(X)$ and a corresponding upper bound $U(\overline{x}^k, \overline{z}^k)$ can be obtained from (35).

A subgradient procedure for solving the separation problem (14)–(18) that makes use of these upper bounds may then be summarised as follows.

Step 1: Set $k = 1$, $L = 0$, $U = U(\hat{x},\, z(\hat{x}))$, $\sigma = 2$, $(\overline{x}^k, \overline{z}^k) = (0, 0)$. For $j = 1, \ldots, n$ set

$$\pi_j^k = \begin{cases} 0 & \text{, if } \hat{x}_j < b_j \\ 1 & \text{, if } \hat{x}_j = b_j \end{cases} \qquad \text{and} \qquad \lambda_j^k = \begin{cases} 1 & \text{, if } \hat{z}_j = 0 \\ 0 & \text{, if } \hat{z}_j > 0 \end{cases}$$

Step 2: Solve the subproblem (36) and let $(x^k,\, z^k)$ be the solution. Set

$$g^k = (\hat{x} - x^k\,,\ z^k - \hat{z}) \quad \text{and} \quad v(\pi^k, \lambda^k) = \sum_{j=1}^{n} \Big(\pi_j^k (\hat{x}_j - x_j^k) + \lambda_j^k (z_j^k - \hat{z}_j) \Big)\,.$$

If $v(\pi^k, \lambda^k) > L$, set $L := v(\pi^k, \lambda^k)$.

Step 3: Choose a suitable direction, e.g., $h^k = g^k$.

Step 4: Set $(\overline{x}^k, \overline{z}^k) = \frac{(k-1)}{k} (\overline{x}^{k-1}, \overline{z}^{k-1}) + \frac{1}{k} (x^k,\, z^k)$ and compute the upper bound

$$U^k = \min\Big\{ U(x^k, z^k)\,,\ U(\overline{x}^k, \overline{z}^k) \Big\}\,.$$

If $U > U^k$, set $U = U^k$. If $(U - L)/\max\{1, L\} < \epsilon$, where $\epsilon > 0$ is a given small tolerance value, then terminate the procedure; otherwise continue with Step 5.

Step 5: Perform the subgradient step by setting

$$\theta = \sigma \frac{U - v(\pi^k, \lambda^k)}{\left\|h^k\right\|^2} \quad \text{and} \quad \left(\pi', \lambda'\right) = \left(\pi^k, \lambda^k\right) + \theta h^k .$$

Then make the the following adjustments for $j = 1, \ldots, n$ in order to take the additional restrictions (34) on the coefficients π and λ into account:

$$\pi_j^{k+1} := \begin{cases} 0 & \text{, if } \hat{x}_j = 0 \\ 1 & \text{, if } \hat{x}_j = b_j \\ \min\{\max\{0, \pi'_j\}, 1\} & \text{, if } 0 < \hat{x}_j < b_j \end{cases}$$

and

$$\lambda_j^{k+1} := \begin{cases} 0 & \text{, if } \hat{z}_j = b_j \\ 1 & \text{, if } \hat{z}_j = 0 \\ \min\{\max\{0, \lambda'_j\}, 1\} & \text{, if } 0 < \hat{z}_j < b_j . \end{cases}$$

Update the step size parameter σ and set $k := k + 1$. If a maximum number of iterations is reached or σ falls below a given threshold value, stop. Otherwise return to Step 2.

Within a standard implementation of the subgradient procedure, the step size parameter σ is usually halved if the lower bound L did not improve after a given number of, e.g, 5 or 10 subsequent iterations. In Step 3, it might be favourable to select a different direction than the current subgradient g^k. Baker and Sheasby (1999), e.g., propose to use an exponentially smoothed average of the generated subgradients. That is, the direction h^k is computed from

$$h^k = h^{k-1} + \gamma(g^k - h^{k-1}) ,$$

where $0 < \gamma < 1$ and $h^0 = g^1$. Camerini *et al.* (1975) propose instead a subgradient deflection method and compute the direction h^k from

$$h^k = g^k - \delta_k h^{k-1} ,$$

where $\delta_k \geq 0$ is the deflection parameter given by

$$\delta_k = \begin{cases} 0, \text{ if } (g^k)^T h^{k-1} \geq 0 , \\ \tau_k (g^k)^T h^{k-1} / \|h^{k-1}\|^2, \text{ otherwise.} \end{cases}$$

with $1 \leq \tau_k < 2$.

4.3 Results of a small computational experiment

The proposed subgradient procedure were tested on a smaller instance of the fixed-charge transportation problem. Each time the LP relaxation was solved, it was tried

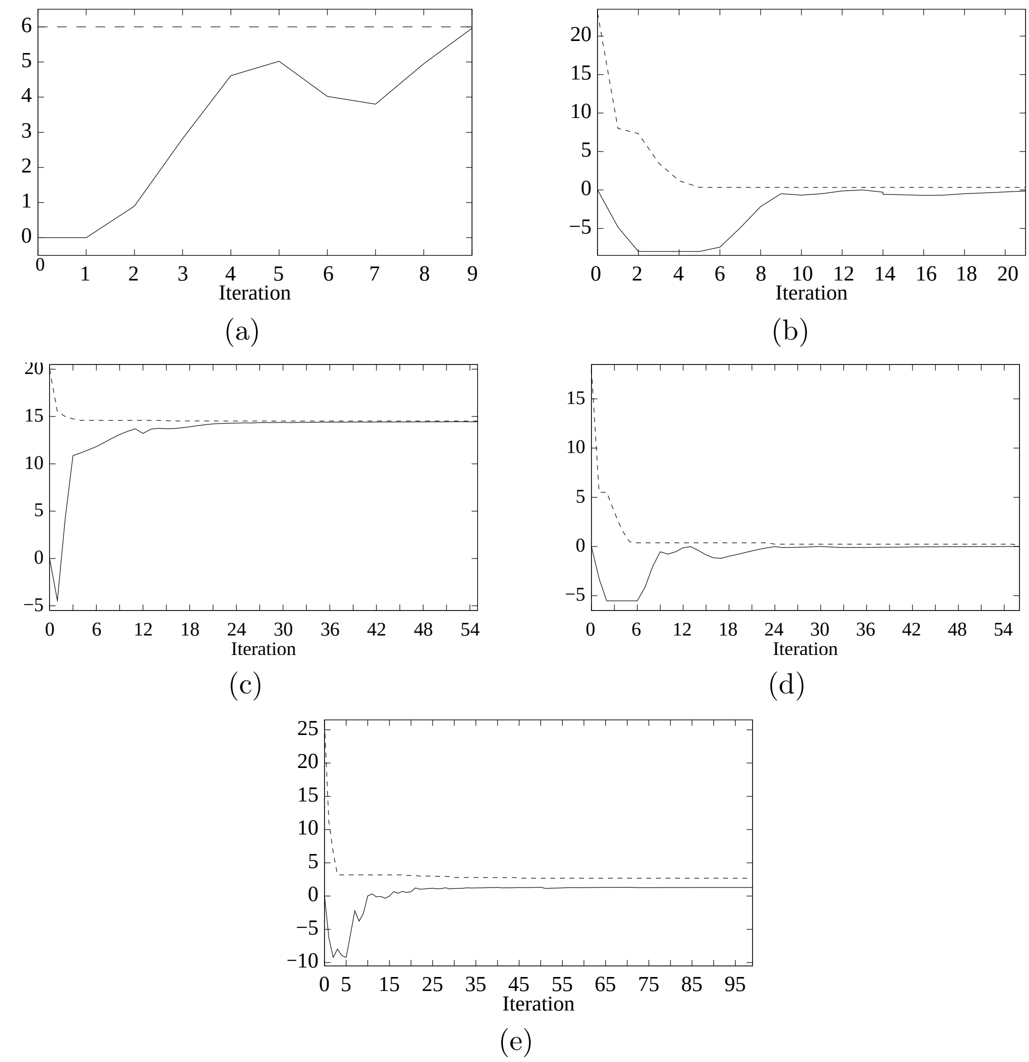

Figure 1. Convergence of the subgradient procedure (U: dashed, $v(\pi^k, \lambda^k)$: solid)

to generate a Fenchel cutting plane for each of the single-node flow structures that can be obtained from variable upper bounds on the flows and each single demand and supply constraint. At most one hundred subgradient steps were executed. The step size parameter σ was initially set to two and halved if after five subsequent iterations the lower bound L did not improve. In each step, the current subgradient was used as search direction, since neither exponentially smoothed nor deflected subgradients performed significantly better. Fig. 1 shows a few typical examples of the subgradient procedure's convergence behaviour. The figures plot the current best upper bound U on $\max_{\pi,\lambda} v(\pi, \lambda)$ and the current function value $v(\pi^k, \lambda^k)$ versus the current iteration k. In the cases of figures 1(a)–1(c), a Fenchel inequality is violated, while no such inequality exists in case of figure 1(d) and 1(e). Generally, convergence is slower if no Fenchel inequality is violated and thus $\max_{\pi,\lambda} v(\pi, \lambda) = 0$ results as, e.g, in Fig. 1(e) were even after executing the maximal number of iterations, $(\hat{x}, \hat{z}) \in X$ could not be proved. Nevertheless did the upper bound $U(\overline{x}^k, \overline{z}^k)$ that is based on the average $(\overline{x}^k, \overline{z}^k)$ of the subproblem solutions often allow to prematurely stop the subgradient procedure, even in the more difficult case of $(\hat{x}, \hat{z}) \in X$ as, e.g, in Fig. 1(d). Faster convergence was usually achieved in case of $(\hat{x}, \hat{z}) \notin X$, that is

$\max_{\pi,\lambda} v(\pi, \lambda) > 0$. At times, just a few subgradient steps were in this case required to solve the separation problem as, e.g, in figures 1(a) and 1(b). There are, of course, also a number of cases where a substantial number of subgradient steps had to be carried out in order to determine the final Fenchel inequality as, e.g, in Fig. 1(c). On average, however, the upper bound $U(\overline{x}^k, \overline{z}^k)$ contributed to a significant decrease in the number of subgradient steps that need to be performed for solving the separation problem. Despite its simplicity, it seems that the average of the subproblem solutions is a quite reasonable estimate of the primal one, that is the solution to the primal separation problem (29).

5 Summary and conclusions

This paper discussed Fenchel cutting planes derived from the single-node flow polytope. We first showed some properties regarding the sign of the inequality's coefficients and also derived domain constraints that can be exploited by methods for solving the separation problem. We then further discussed the separation problem and a subgradient method for its resolution. Although convergence of the subgradient method can greatly be improved by means of upper bounds derived from average solutions to the subproblem, the effort required for determining the cutting planes should usually still be too large compared to the gain in the strength of the linear relaxation. For performing exact separation of the binary knapsack problem, Avella *et al.* (2007) and Kaparis and Letchford (2007) proposed to first fix variables that take on an integer solution value in the current LP solution, then to solve the separation problem over the reduced polytope and to apply thereafter a sequential lifting of the obtained inequality. This way, the separation problem is usually reduced to such an extend that an exact decomposition method can be applied for solving it. Compared to a subgradient method, this has the advantage that the separation problem is solved exactly, which should result in better cutting planes. A similar approach might probably also be taken in case of an exact separation of valid inequalities for the single-node flow polytope. In this case, however, it is desirable to lift not only binary variables but also the continuous flow variables. Since these two types of variables are so closely related, they should be lifted simultaneously, which considerably complicates the lifting problem. For the case of flow cover inequalities and some other polyhedral inequalities, Gu *et al.* (1999, 2000) were able to make the lifting sequence independent by deriving superadditive lifting functions. Probably a similar approach can be taken here in order to obtain at least an approximate sequence independent lifting that can be carried out efficiently and still gives an inequality that is strong enough.

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