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Ole E. Barndorff-Nielsen, José Manuel Corcuera, Mark Podolskij and Jeannette H.C. Woerner

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Bipower variation for Gaussian processes with stationary increments*

Ole E. Barndorff-Nielsen
University of Aarhus†

José Manuel Corcuera
University of Barcelona‡

Mark Podolskij
University of Aarhus and CREATES§

Jeannette H.C. Woerner
University of Göttingen¶

Abstract

Convergence in probability and central limit laws of bipower variation for Gaussian processes with stationary increments and for integrals with respect to such processes are derived. The main tools of the proofs are some recent powerful techniques of Wiener/Itô/Malliavin calculus for establishing limit laws, due to Nualart, Peccati and others.

Keywords: Bipower Variation, Central Limit Theorem; Chaos Expansion; Gaussian Processes; Multiple Wiener-Itô Integrals.

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†Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK–8000 Aarhus C, Denmark. e-mail: oebn@imf.au.dk

‡Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain. e-mail: jmcorcuera@ub.edu.

§CREATES, School of Economics and Management, University of Aarhus, Building 1322, DK-8000 Aarhus C, Denmark. e-mail: mpodolskij@creates.au.dk.

¶Institut für Mathematische Stochastik, Maschmühlenweg 8–10, 37073 Göttingen, Germany. e-mail: woerner@math.uni-goettingen.de.

1 Introduction

The theory of bipower, and more generally multipower, variation has developed out of problems in mathematical finance; for motivation and some first results and applications see [5], [6], [7], [8], [9], [10], [26], [27], [28], [29]. It is natural, therefore, that initially the focus was on Brownian semimartingales, for which a rather complete and comprehensive theory is now available, cf. [3] (also [4] and [18]). Extensions of the theory to Lévy processes and Itô semimartingales have been obtained, particularly by Jacod in [15] (cf. also [8]), and applications to finance of such extensions are discussed in [17] and [25].

A further avenue of generalisation is to stochastic integrals with respect to Gaussian processes having stationary increments. This was begun in [2], [12] which treated the power variation case, providing in particular a feasible central limit theorem for inference on the integrands in question[1]. The techniques used there, as well as in the present paper which considers the bipower case, come from very powerful recent results developed in the context of Wiener/Itô/Malliavin calculus, especially by Nualart, Peccati and coauthors, see [21], [22] and [23] (cf. also [19]). (In fact, we believe that there are no other tools available that would allow derivation of the conclusions in the present paper.)

The structure of the paper is as follows. Section 2 lists a number of background results needed for the proofs, given in the Appendix, of the main results, which are presented in Sections 3 and 4. Those Sections discuss limit laws of bipower variation for Gaussian processes with stationary increments and for integrals with respect to such processes, respectively. Section 5 concludes.

2 Background

In this section we review the basic concepts of the Wiener chaos expansion. In particular, we present a multiplication formula (Proposition 1) and a multivariate central limit theorem for a sequence of random variables which admit a chaos representation (Theorem 2). The latter is based on the theory for multiple stochastic integrals developed in [21], [23] and [14].

Consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a Gaussian subspace $\mathcal{H}_1$ of $L^2(\Omega, \mathcal{F}, P)$ whose elements are zero-mean Gaussian random variables. Let $\mathbb{H}$ be a separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and norm $||\cdot||_{\mathbb{H}}$. We will assume that there is an isometry

$$\begin{aligned} W &: \mathbb{H} \to \mathcal{H}_1 \\ h &\mapsto W(h) \end{aligned}$$

in the sense that

$$E[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_{\mathbb{H}}.$$

It is easy to see that this map has to be linear.

[1] As discussed in [2], an important early forerunner of that paper is a paper by Guyon and Leon [13] which derived quadratic variation limit results for stationary Gaussian processes.

For any $m \geq 2$, we denote by $\mathcal{H}_m$ the m-th Wiener chaos, that is, the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $H_m(X)$, where $X \in \mathcal{H}_1$, $E[X^2] = 1$, and H_m is the m-th Hermite polynomial, i.e. $H_0(x) = 1$ and $H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m}(e^{-\frac{x^2}{2}})$.

Suppose that $\mathbb{H}$ is infinite-dimensional and let $\{e_i, i \geq 1\}$ be an orthonormal basis of H. Denote by Λ the set of all sequences $a = (a_1, a_2, ...)$, $a_i \in \mathbb{N}$, such that all the terms, except a finite number of them, vanish. For $a \in \Lambda$ we set $a! = \Pi_{i=1}^{\infty} a_i!$ and $|a| = \sum_{i=1}^{\infty} a_i$. For any multindex $a \in \Lambda$ we define

$$\Phi_a = \frac{1}{\sqrt{a!}} \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)).$$

The family of random variables $\{\Phi_a, a \in \Lambda\}$ is an orthonormal system. In fact

$$E\left[\Pi_{i=1}^{\infty} H_{a_i}(W(e_i)) \Pi_{i=1}^{\infty} H_{b_i}(W(e_i))\right] = \delta_{ab} a! \ ,$$

where δ_{ab} denotes the Kronecker symbol. Moreover, $\{\Phi_a |\ a \in \Lambda,\ |a| = m\}$ is a complete orthonormal system in $\mathcal{H}_m$.

Let $a \in \Lambda$ with $|a| = m$. The mapping

$$\begin{array}{rcl} I_m : \mathbb{H}^{\odot m} & \to & \mathcal{H}_m \\ \widetilde{\otimes}_{i=1}^{\infty} e_i^{\otimes a_i} & \mapsto & \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)), \end{array}$$

between the symmetric tensor product $\mathbb{H}^{\odot m}$, equipped with the norm $\sqrt{m!}\, \|\cdot\|_{\mathbb{H}^{\otimes m}}$, and the m-th chaos $\mathcal{H}_m$ is a linear isometry. Here $\widetilde{\otimes}$ denotes the symmetrization of the tensor product $\otimes$ and I_0 is the identity in $\mathbb{R}$.

For any $h = h_1 \otimes \cdots \otimes h_m$ and $g = g_1 \otimes \cdots \otimes g_m \in \mathbb{H}^{\otimes m}$, we define the p-th contraction of h and g, denoted by $h \otimes_p g$, as the element of $\mathbb{H}^{\otimes 2(m-p)}$ given by

$$h \otimes_p g = \langle h_1, g_1 \rangle_{\mathbb{H}} \cdots \langle h_p, g_p \rangle_{\mathbb{H}} h_{p+1} \otimes \cdots \otimes h_m \otimes g_{p+1} \otimes \cdots \otimes g_m.$$

This definition can be extended by linearity to any element of $\mathbb{H}^{\otimes m}$. $h \otimes_p g$ does not necessarily belong to $\mathbb{H}^{\odot(2m-p)}$, even if h and g belong to $\mathbb{H}^{\odot m}$. We denote by $h \widetilde{\otimes}_p g$ the symmetrization of $h \otimes_p g$.

Proposition 1. *For any $h \in \mathbb{H}^{\otimes p}$ and $g \in \mathbb{H}^{\otimes q}$, we have*

$$I_p(h) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(h \widetilde{\otimes}_r g). \tag{2.1}$$

Proof: First, note that

$$I_1(e_i) = W(e_i).$$

Let $a \in \Lambda$ with $|a| = p$ and $q = 1$. Due to linearity of I_p it suffices to consider the case $h = \widetilde{\otimes}_{i=1}^{\infty} e_i^{\otimes a_i}$, $g = e_j$. It holds that

$$I_p(\widetilde{\otimes}_{i=1}^{\infty} e_i^{\otimes a_i}) I_1(e_j) = \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)) W(e_j).$$

Assume that j is an index such that $a_j = 0$. Then

$$\widetilde{\otimes}_{i=1}^{\infty} e_i^{\otimes a_i} \widetilde{\otimes}_1 e_j = 0$$

and

$$\Pi_{i=1}^{\infty} H_{a_i}(W(e_i))W(e_j) = I_{p+1}(\widetilde{\otimes}_{i=1}^{\infty}\, e_i^{\otimes a_i}\, \widetilde{\otimes}\, e_j),$$

so we have that

$$I_p(\widetilde{\otimes}_{i=1}^{\infty}\, e_i^{\otimes a_i}) I_1(e_j) = I_{p+1}(\widetilde{\otimes}_{i=1}^{\infty}\, e_i^{\otimes a_i}\, \widetilde{\otimes}\, e_j) + p I_{p-1}(\widetilde{\otimes}_{i=1}^{\infty} e_i^{\otimes a_i}\, \widetilde{\otimes}_1\, e_j).$$

Assume now that $a_j \neq 0$. Then we obtain the identity

$$\widetilde{\otimes}_{i=1}^{\infty}\, e_i^{\otimes a_i}\, \widetilde{\otimes}_1\, e_j = \frac{a_j}{p}\, \widetilde{\otimes}_{i=1}^{\infty}\, e_i^{\otimes a_i'}$$

with $a_i' = a_i$ if $i \neq j$ and $a_j' = a_j - 1$. Furthermore,

$$\begin{aligned} \Pi_{i=1}^{\infty} H_{a_i}(W(e_i))W(e_j) &= \Pi_{i=1, i\neq j}^{\infty} H_{a_i}(W(e_i)(H_{a_j+1}(W(e_j) + a_j H_{a_j-1}(W(e_j))) \\ &= I_{p+1}(\widetilde{\otimes}_{i=1}^{\infty}\, e_i^{\otimes a_i}\, \widetilde{\otimes}\, e_j) + p I_{p-1}(\widetilde{\otimes}_{i=1}^{\infty}\, e_i^{\otimes a_i}\, \widetilde{\otimes}_1\, e_j), \end{aligned}$$

since the Hermite polynomials verify

$$H_{n+1}(x) = x H_n(x) - n H_{n-1}(x).$$

Hence, the relationship (2.1) is true for $q = 1$. The general formula follows by induction through the lines of the proof of Proposition 1.1.3 in [20]. □

Remark 1. Note that if we take $h = e_i^{\otimes p}$, $g = e_i^{\otimes q}$ we obtain the well-known identity

$$H_p(W(e_i)) H_q(W(e_i)) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} H_{p+q-2r}(W(e_i)).$$

Now, let $\mathcal{G}$ be the σ-field generated by the random variables $\{W(h)|\ h \in \mathbb{H}\}$. Any square integrable random variable $F \in L^2(\Omega, \mathcal{G}, P)$ has a unique chaos decomposition

$$F = \sum_{m=0}^{\infty} I_m(h_m)\ ,$$

where $h_m \in \mathbb{H}^{\odot m}$ (see [20] for more details).

Finally, we present a multivariate central limit theorem for sequences of functionals $F_n \in L^2(\Omega, \mathcal{G}, P)$.

Theorem 2. *Consider a sequence of d-dimensional random vectors $F_n = (F_n^1, F_n^2, \ldots, F_n^d)$, such that $F_n^k \in L^2(\Omega, \mathcal{G}, P)$ and*

$$F_n^k = \sum_{m=0}^{\infty} I_m(h_{m,n}^k)\ ,$$

where $h_{m,n}^k \in \mathbb{H}^{\odot m}$. Assume that the following conditions hold:

(i) For any $k = 1, \ldots, d$ we have

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{m=N+1}^{\infty} m! ||h_{m,n}^k||^2_{\mathbb{H}^{\otimes m}} = 0.$$

(ii) For $k, l = 1, \ldots, d$ we have

$$m! \lim_{n\to\infty} ||h^k_{m,n}||^2_{\mathbb{H}^{\otimes m}} = \Sigma^m_{kk}.$$
$$m! \lim_{n\to\infty} \langle h^k_{m,n}, h^l_{m,n}\rangle_{\mathbb{H}^{\otimes m}} = \Sigma^m_{kl}, \qquad k \neq l \ ,$$

and $\sum_{m=1}^{\infty} \Sigma^m = \Sigma \in \mathbb{R}^{d\times d}$.

(iii) For any $m \geq 1$, $k = 1, \ldots, d$ and $r = 1, \ldots, m-1$

$$\lim_{n\to\infty} ||h^k_{m,n} \otimes_r h^k_{m,n}||^2_{\mathbb{H}^{\otimes 2(m-r)}} = 0.$$

Then we have

$$F_n - h_{0,n} \xrightarrow{\mathcal{D}} N_d(0, \Sigma) \ , \tag{2.2}$$

as n tends to infinity, and for any natural number N and $k = 1, \ldots, d$

$$\lim_{n\to\infty} E\bigg(\sum_{m=1}^{N} I_m(h^k_{m,n})\bigg)^4 = 3\bigg(\sum_{m=1}^{N} \Sigma^m_{kk}\bigg)^2. \tag{2.3}$$

Proof: Under the conditions (ii) and (iii) the weak convergence (2.2) of the vector

$$\big(I_{m_1}(h^1_{m_1,n}), I_{m_2}(h^2_{m_2,n}), \ldots, I_{m_d}(h^d_{m_d,n})\big) \ ,$$

is shown in [23] (moreover, these authors prove that (2.2) implies (2.3)). Under the additional condition (i) this result can be extended to general multivariate sequences F_n with squared integrable components (see [2]). □

Example 3. Consider a sequence of stationary, normalized, centered Gaussian random variables $(X_i)_{i\geq 1}$. We want to study the asymptotic behavior of the sequence

$$Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} H(X_i) \ ,$$

where H is a real-valued function of Hermite index $R \geq 1$, i.e.

$$H(x) = \sum_{m=R}^{\infty} c_m H_m(x)$$

with $c_R \neq 0$ (in particular, this implies that $E[H(X_i)] = 0$). Assume that $E[H^2(X_i)] = \sum_{m=R}^{\infty} m!\ c_m^2 < \infty$. We can take $\mathcal{H}_1$ =span$\{X_i, i \geq 1\}$, and $\mathbb{H} \equiv \mathcal{H}_1$. The inner product on $\mathbb{H}$ is then induced by the covariance function $\rho(k) = \mathrm{cov}(X_1, X_{1+k})$ of the sequence $(X_i)_{i\geq 1}$ (note that $\rho(0) = 1$). We obtain the following representation

$$\begin{aligned} Y_n &= \sum_{m=R}^{\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_m H_m(X_i) \\ &= \sum_{m=R}^{\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_m I_m(X_i^{\otimes m}) \\ &= \sum_{m=R}^{\infty} I_m\bigg(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_m X_i^{\otimes m}\bigg). \end{aligned}$$

Set

$$h_{m,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_m X_i^{\otimes m}.$$

Assume that

$$\sum_{j=1}^{\infty} |\rho(j)|^R < \infty. \tag{2.4}$$

It holds that

$$\sum_{m=R}^{\infty} m! ||h_{m,n}||^2_{\mathbb{H}^{\otimes m}} = \sum_{m=R}^{\infty} \frac{m! c_m^2}{n} \sum_{i,j=1}^{n} \rho^m(i-j)$$

$$= \sum_{m=R}^{\infty} m! c_m^2 \left(1 + 2 \sum_{j=1}^{n-1} \rho^m(j) \left(1 - \frac{j}{n} \right) \right) \to \sum_{m=R}^{\infty} m! c_m^2 \left(1 + 2 \sum_{j=1}^{\infty} \rho^m(j) \right) =: \sigma^2.$$

Note the identity

$$h_{m,n} \otimes_r h_{m,n} = \frac{c_m^2}{n} \sum_{i,j=1}^{n} \rho^r(i-j) X_i^{\otimes(m-r)} \otimes X_j^{\otimes(m-r)}.$$

This implies

$$\begin{aligned} &||h_{m,n} \otimes_r h_{m,n}||^2_{\mathbb{H}^{\otimes 2(m-r)}} \\ &\quad = \frac{c_m^4}{n^2} \sum_{i,j,k,l=1}^{n} \rho^r(i-j)\rho^r(k-l)\rho^{m-r}(i-k)\rho^{m-r}(j-l) \\ &\quad = \frac{c_m^4}{n} \sum_{i,j,k=0}^{n-1} \rho^r(i)\rho^r(j-k)\rho^{m-r}(j)\rho^{m-r}(i-k) \left(1 - \frac{i \vee j \vee k}{n} \right), \end{aligned}$$

where the last term converges to 0 under assumption (2.4) (see [12] for a detailed proof). Thus, under assumption (2.4), conditions (i)–(iii) of Theorem 2 are fulfilled, and we deduce that

$$Y_n \xrightarrow{\mathcal{D}} N(0, \sigma^2).$$

3 Asymptotic theory for bipower variation of Gaussian processes with stationary increments

We consider a Gaussian process $(G_t)_{t \ge 0}$, defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, P)$, with centered and stationary increments. The variance function R of the increments of G is defined as

$$R(t) = E[|G_{s+t} - G_s|^2], \qquad t \ge 0. \tag{3.1}$$

In this section we study the asymptotic behaviour of the *bipower variation processes*

$$V(G; p, q)_t^n = \frac{1}{n \tau_n^{p+q}} \sum_{i=1}^{[nt]} |\Delta_i^n G|^p |\Delta_{i+1}^n G|^q, \qquad p, q \ge 0, \tag{3.2}$$

where $\Delta_i^n G = G_{\frac{i}{n}} - G_{\frac{i-1}{n}}$ and $\tau_n^2 = R(\frac{1}{n}) = E[|\Delta_i^n G|^2]$, using the multiplication formula (2.1) and the central limit theorem discussed in the previous section. For this purpose we introduce the representation

$$|x|^p = \sum_{m=0}^{\infty} a_{p,m} H_m(x) \ , \tag{3.3}$$

where the H_m are Hermite polynomials as defined in Section 2.

In order to give a statement about the asymptotic behaviour of the bipower variation process $V(G;p,q)_t^n$ we require the following assumptions on the variance function R defined in (3.1), which were introduced by Guyon and Leon in [13]:

(A1) *$R(t) = t^{\beta} L_0(t)$ for some $\beta \in (0,2)$ and some positive slowly varying (at 0) function L_0, which is continuous on $(0,\infty)$.*

(A2) *$R''(t) = t^{\beta-2} L_2(t)$ for some slowly varying function L_2, which is continuous on $(0,\infty)$.*

(A3) *There exists $b \in (0,1)$ with*

$$K = \limsup_{x \to 0} \sup_{y \in [x, x^b]} \left| \frac{L_2(y)}{L_0(x)} \right| < \infty.$$

Recall that a function $L : (0,\infty) \to \mathbb{R}$ is called slowly varying at 0 when the identity

$$\lim_{x \searrow 0} \frac{L(tx)}{L(x)} = 1 \tag{3.4}$$

holds for any fixed $t > 0$. Provided L is continuous on $(0,\infty)$, we have

$$|L(x)| \le C x^{-\alpha} \ , \qquad x \in (0,T] \tag{3.5}$$

for any $\alpha > 0$ and any $T > 0$ (where the constant $C > 0$ depends on α and T).

Finally, we introduce the correlation function of the increments of G, i.e.

$$r_n(j) = \mathrm{Cov}\Big(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\Big) \ , \qquad j \ge 0. \tag{3.6}$$

By the triangular identity, and due to the stationarity of the increments of G, we know that $r_n(0) = 1$ and

$$r_n(j) = \frac{R(\frac{j+1}{n}) + R(\frac{j-1}{n}) - 2R(\frac{j}{n})}{2R(\frac{1}{n})} \ , \qquad j \ge 1. \tag{3.7}$$

We start with the weak law of large numbers for the sequence $V(G;p,q)_t^n$. Throughout this paper we write $Y^n \xrightarrow{ucp} Y$ when $\sup_{t \in [0,T]} |Y_t^n - Y_t| \xrightarrow{P} 0$ for any $T > 0$.

Theorem 4. *Assume that conditions **(A1)-(A3)** are satisfied. Then we have*

$$\frac{V(G;p,q)_t^n}{\rho_{p,q}^{(n)}} \xrightarrow{ucp} t \ , \tag{3.8}$$

where the quantity $\rho_{p,q}^{(n)}$ *is given by*

$$\rho_{p,q}^{(n)} = \sum_{m=0}^{\infty} a_{p,m} a_{q,m} m! \; r_n^m(1). \tag{3.9}$$

Proof: see Appendix.

Remark 2. Notice that by orthogonality of Hermite polynomials the identity

$$\rho_{p,q}^{(n)} = E\Big[\Big|\frac{\Delta_i^n G}{\tau_n}\Big|^p \Big|\frac{\Delta_{i+1}^n G}{\tau_n}\Big|^q\Big]$$

holds. Moreover, since the function L_0 is slowly varying at 0, assumption **(A1)**, (3.9) and (3.7) (and the dominated convergence theorem) imply that

$$\rho_{p,q} = \lim_{n\to\infty} \rho_{p,q}^{(n)} = \sum_{m=0}^{\infty} a_{p,m} a_{q,m} m! \; (2^{\beta-1} - 1)^m = E[|B_i^{\beta/2} - B_{i-1}^{\beta/2}|^p |B_{i+1}^{\beta/2} - B_i^{\beta/2}|^q] \; , \tag{3.10}$$

where $B^{\beta/2}$ is the fractional Brownian motion with Hurst parameter $\beta/2$. Consequently, Theorem 4 yields the uniform convergence

$$V(G;p,q)_t^n \xrightarrow{ucp} \rho_{p,q} t.$$

Next, we present the weak limit of the properly normalized sequence $V(G;p,q)_t^n$. Notice that the central limit theorem for bipower variation is valid under the same assumptions that are required to show the corresponding result for the power variation case (see [2]).

Theorem 5. *Assume that conditions **(A1)**–**(A3)** hold and* $0 < \beta < \frac{3}{2}$. *Then we obtain the weak convergence (in the space* $\mathcal{D}([0,T])^2$ *equipped with the Skorohod topology)*

$$\Big(G_t, \sqrt{n}\Big(\frac{V(G;p,q)_t^n}{\rho_{p,q}^{(n)}} - t\Big)\Big) \Longrightarrow \Big(G_t, \frac{\sigma_{p,q}}{\rho_{p,q}} W_t\Big) \; , \tag{3.11}$$

where W *is a Brownian motion that is defined on an extension of the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *and is independent of* $\mathcal{F}$, *and* $\sigma_{p,q}^2$ *is given by*

$$\sigma_{p,q}^2 = \lim_{n\to\infty} n \operatorname{Var}\Big(V(B^{\beta/2};p,q)_1^n\Big) \; , \tag{3.12}$$

where $B^{\beta/2}$ *is the fractional Brownian motion with Hurst parameter* $\beta/2$.

Proof: see Appendix.

Remark 3. In Theorem 5 the constant $\rho_{p,q}^{(n)}$ can not be replaced by its limit $\rho_{p,q}$ defined in (3.10). This is due to the fact that the bias $\sqrt{n}(\rho_{p,q}^{(n)} - \rho_{p,q})$ can, in general, converge to infinity.

Remark 4. The finiteness of $\sigma_{p,q}^2$ (for $0 < \beta < \frac{3}{2}$) and its exact representation is shown in (6.11) in the Appendix. Note that due to the assumption **(A1)** the behaviour of the function R near 0 is similar to that of the fractional Brownian motion with Hurst parameter $\beta/2$. This is reflected in the formula (3.12).

The proof of Theorem 5 relies on the methods developed in the previous section. In the first step we apply the multiplication formula (2.1) to obtain the chaos decomposition of the sequence $\sqrt{n}\big(\frac{V(G;p,q)^n_t}{\rho^{(n)}_{p,q}} - t\big)$. Then we show the convergence of finite dimensional distributions of the sequence given in (3.11). Finally, we prove the tightness condition.

Notice that the weak convergence in (3.11) is equivalent to the stable convergence (in $\mathcal{D}([0,T])^2$)

$$\sqrt{n}\Big(\frac{V(G;p,q)^n_t}{\rho^{(n)}_{p,q}} - t\Big) \stackrel{\mathcal{F}^G - st}{\longrightarrow} \frac{\sigma_{p,q}}{\rho_{p,q}} W_t \ , \tag{3.13}$$

where $\mathcal{F}^G$ denotes the σ-algebra generated by the process G (see [1], [16] or [24] for more details on stable convergence). The latter result is crucial for proving a functional central limit theorem for the bipower variation of integral processes which is presented in the next section.

4 Extensions to integral processes

In this section we extend the limit theorems of the previous section to integral processes

$$Z_t = \int_0^t u_s dG_s \tag{4.1}$$

defined on the same probability space as G, where the stochastic integral is the pathswise Riemann-Stieltjes integral. Assumption **(A1)** implies that G has finite r-variation for any $r > 2/\beta$ and hence by [30] the integral in (4.1) is well-defined for any stochastic process u of finite q-variation with $q < 1/(1-(\beta/2))$.

First we state the law of large numbers for the integral process which is valid under the same assumptions as in the power variation case.

Theorem 6. *Assume the conditions **(A1)**–**(A3)**. Suppose that $u = \{u_t, t \in [0,T]\}$ is a stochastic process with finite r-variation, where $r < \frac{1}{1-(\beta/2)}$. Set*

$$Z_t = \int_0^t u_s dG_s.$$

Then for $p, q > 0$ we obtain

$$V(Z;p,q)^n_t \stackrel{ucp}{\longrightarrow} \rho_{p,q} \int_0^t |u_s|^{p+q} ds,$$

as $n \to \infty$.

Proof: see Appendix.

Remark 5. Note that integrals with respect to fractional Brownian motion $Z_t = \int_0^t u_s dB_s^{\beta/2}$ are a special case of this setting leading to the same limit.

Next we provide the weak limit theorem of the properly normalized bipower variation.

Theorem 7. *Assume the conditions **(A1)–(A3)** and suppose that* $u = \{u_t, t \in [0,T]\}$ *is a stochastic process with finite* r*-variation, where* $r < \frac{1}{1-(\beta/2)}$*, and which is Hölder continuous of the order* a *with* $a > \max(1/(2(p \wedge 1)), 1/(2(q \wedge 1)))$*. Then we obtain for* $Z_t = \int_0^t u_s dG_s$ *and* $p, q > 0$

$$\left(G_t, \sqrt{n}\left(\frac{V(Z;p,q)_t^n}{\rho_{p,q}^{(n)}} - \int_0^t |u_s|^{p+q} ds \right)\right) \Longrightarrow \left(G_t, \frac{\sigma_{p,q}}{\rho_{p,q}} \int_0^t |u_s|^{p+q} dW_s \right)$$

as $n \to \infty$*, where the convergence is in* $\mathcal{D}([0,T])^2$ *and* W *is a Brownian motion defined on an extension of the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *and is independent of* $\mathcal{F}$*.*

Proof: see Appendix.

Combining Theorem 7 and 6 we can derive a feasible central limit theorem for the bipower variation.

Corollary 1. *Under the assumption of Theorem 7 it holds that*

$$\frac{\sqrt{n}\left(\frac{V(Z;p,q)_t^n}{\rho_{p,q}^{(n)}} - \int_0^t |u_s|^{p+q} ds \right)}{\sqrt{\frac{V(Z;2p,2q)_t^n}{\rho_{2p,2q}} \frac{\sigma_{p,q}^2}{\rho_{p,q}^2}}} \xrightarrow{\mathcal{D}} N(0,1).$$

5 Conclusion

In this paper we derived convergence in probability and stable central limit theorems for bipower variation of Gaussian processes with stationary increments and for associated integral processes. The corresponding asymptotic theory for multipower variation can be obtained similarly in a straightforward manner. Extensions of the results presented here to spatial and tempo-spatial settings would be of interest, as would simulation and empirical studies of how well the limit laws work in applications.

6 Appendix

In the following we denote all constants which do not depend on n by C.

Let $\mathcal{H}_1$ be the first Wiener chaos associated with the triangular array $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nt]}$, i.e. the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nt]}$. Notice that $\mathcal{H}_1$ can be seen as a separable Hilbert space with a scalar product induced by the covariance function of the process $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nt]}$. This means we can apply the theory of Section 2 with the canonical Hilbert space $\mathbb{H} = \mathcal{H}_1$. Denote by $\mathcal{H}_m$ the mth Wiener chaos associated with the triangular array $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nt]}$ and by I_m the corresponding linear isometry between the symmetric tensor product $\mathcal{H}_1^{\odot m}$ (equipped with the norm $\sqrt{m!}\,\|\cdot\|_{\mathcal{H}_1^{\otimes m}}$) and the mth Wiener chaos.

First, we present the chaos decomposition for the sequence $V(G;p,q)_t^n - \rho_{p,q}^{(n)} t$.

Lemma 1. *For any $t > 0$, we obtain the decomposition*

$$V(G;p,q)^n_t - \rho^{(n)}_{p,q} t = \sum_{m=2}^{\infty} I_m\Big(\frac{1}{n}\sum_{i=1}^{[nt]} f_i^m\Big) + O(n^{-1}) \ , \tag{6.1}$$

where the kernels $f_i^m \in \mathcal{H}_1^{\odot m}$ are given by

$$f_i^m = \sum_{h=0}^{m} s^{(n)}_{h,m} \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes h} \widetilde{\otimes} \Big(\frac{\Delta_{i+1}^n G}{\tau_n}\Big)^{\otimes m-h}. \tag{6.2}$$

(for simplicity we suppress the dependency of f_i^m on n) with

$$s^{(n)}_{h,m} = \sum_{l=0}^{\infty} a_{p,l+h} a_{q,l+m-h} l! \binom{l+h}{l}\binom{l+m-h}{l} r_n^l(1). \tag{6.3}$$

Proof of Lemma 1: Using the multiplication formula (2.1) and the linearity of the mapping I_m we obtain the representation

$$\begin{aligned} V(G;p,q)^n_t &= \frac{1}{n}\sum_{i=1}^{[nt]} \sum_{m_1,m_2=0}^{\infty} a_{p,m_1} a_{q,m_2} \Bigg[\sum_{l=0}^{m_1\wedge m_2} l! \binom{m_1}{l}\binom{m_2}{l} \\ &\qquad \times I_{m_1+m_2-2l}\Big(\Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes m_1} \otimes_l \Big(\frac{\Delta_{i+1}^n G}{\tau_n}\Big)^{\otimes m_2}\Big)\Bigg] \\ &= \frac{1}{n}\sum_{i=1}^{[nt]} \sum_{m_1,m_2=0}^{\infty} a_{p,m_1} a_{q,m_2} \Bigg[\sum_{l=0}^{m_1\wedge m_2} l! \binom{m_1}{l}\binom{m_2}{l} r_n^l(1) \\ &\qquad \times I_{m_1+m_2-2l}\Big(\Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes m_1-l} \otimes \Big(\frac{\Delta_{i+1}^n G}{\tau_n}\Big)^{\otimes m_2-l}\Big)\Bigg] \\ &= \sum_{m=0}^{\infty} I_m\Big(\frac{1}{n}\sum_{i=1}^{[nt]} f_i^m\Big). \end{aligned}$$

Notice that $a_{p,2m+1} = 0$ for all $m \ge 0$ and $p \ge 0$, because the H_{2m+1} are odd functions. This implies the identity

$$V(G;p,q)^n_t - \rho^{(n)}_{p,q} t = \sum_{m=2}^{\infty} I_m\Big(\frac{1}{n}\sum_{i=1}^{[nt]} f_i^m\Big) + O(n^{-1}) \ ,$$

which completes the proof of Lemma 1. □

Next, we present a lemma which has been shown in [2].

Lemma 2. *Suppose that conditions **(A1)–(A3)** hold. Let $\epsilon > 0$ with $\epsilon < 2-\beta$. Define the sequence $r(j)$ by*

$$r(j) = (j-1)^{\beta+\epsilon-2}, \qquad j \ge 2 \ , \tag{6.4}$$

and $r(0) = r(1) = 1$. Then we obtain the following assertions:

(i) It holds that

$$\frac{1}{n}\sum_{j=1}^{n} r^2(j) \to 0.$$

If, moreover, $\beta + \epsilon - 2 < -\frac{1}{2}$ *it holds that*

$$\sum_{j=1}^{\infty} r^2(j) < \infty.$$

(ii) For any $0 < \epsilon < 2 - \beta$ *from* (6.4) *there exists a natural number* $n_0(\epsilon)$ *such that*

$$|r_n(j)| \le Cr(j) \ , \qquad j \ge 0$$

for all $n \ge n_0(\epsilon)$.

(iii) Set $\rho(0) = 1$ *and* $\rho(j) = \frac{1}{2}\Big((j-1)^\beta - 2j^\beta + (j+1)^\beta\Big)$ *for* $j \ge 1$. *Then it holds that*

$$r_n(j) \to \rho(j)$$

for any $j \ge 0$.

Finally, we show the following result.

Lemma 3. *It holds that*

$$|s_{h,m}^{(n)}| \le \sum_{l=0}^{\infty} |a_{p,l+h}||a_{q,l+m-h}| l! \binom{l+h}{l}\binom{l+m-h}{l} |r_n^l(1)| \tag{6.5}$$

$$\le C\frac{\binom{m}{h}}{m!}\left(\frac{1}{|r_n(1)|(1-|r_n(1)|)}\right)^m , \tag{6.6}$$

where the constant C *does not depend on* n, m *and* h.

Proof of Lemma 3: First, notice the identity

$$\mathrm{Var}\left(\Big|\frac{\Delta_i^n G}{\tau_n}\Big|^p\right) = \sum_{l=2}^{\infty} a_{p,l}^2 l! < \infty.$$

From this we deduce that $a_{p,l}^2 \le \frac{C}{l!}$ (for any fixed $p \ge 0$). Now, recall that $|r_n(1)| < 1$ since r_n is a correlation function of a process with stationary increments. Consequently, we obtain the inequality

$$\sum_{l=0}^{\infty} |a_{p,l+h}||a_{q,l+m-h}| l! \binom{l+h}{l}\binom{l+m-h}{l} |r_n^l(1)|$$

$$\le \frac{C}{h!(m-h)!}\sum_{l=0}^{\infty}(l+1)\cdots(l+m)|r_n(1)|^l$$

$$= C\frac{\binom{m}{h}}{m!}\Big(\frac{1}{|r_n(1)|(1-|r_n(1)|)}\Big)^m.$$

Hence, we deduce the assertion of Lemma 3. □

Proof of Theorem 4: We will first show the pointwise convergence $V(G;p,q)^n_t - \rho^{(n)}_{p,q} t \xrightarrow{P} 0$. Using the expansion (6.1) and the stationarity of the increments of G we obtain the identity.

$$\mathrm{Var}\Big(V(G;p,q)^n_t - \rho^{(n)}_{p,q} t\Big) = \sum_{m=2}^{\infty} m!\bigg(\frac{[nt]}{n^2}||f_1^m||^2_{\mathcal{H}_1^{\otimes m}} + \frac{2}{n^2}\sum_{k=1}^{[nt]-1}([nt]-k)\langle f_1^m, f_{1+k}^m\rangle_{\mathcal{H}_1^{\otimes m}}\bigg). \tag{6.7}$$

Now, by (6.2), Lemma 2 (ii) and Lemma 3 we have the inequalities

$$||f_1^m||^2_{\mathcal{H}_1^{\otimes m}} \le \Big(\sum_{h=0}^{m} |s^{(n)}_{h,m}|\Big)^2 \le C\frac{4^m}{(m!)^2(|r_n(1)|(1-|r_n(1)|))^{2m}}\ , \tag{6.8}$$

$$\begin{aligned} |\langle f_1^m, f_{1+k}^m\rangle_{\mathcal{H}_1^{\otimes m}}| &\le C^m\Big(\sum_{h=0}^{m} |s^{(n)}_{h,m}|\Big)^2 r^m(k-1) \\ &\le C^m \frac{4^m r^m(k-1)}{(m!)^2(|r_n(1)|(1-|r_n(1)|))^{2m}}. \end{aligned} \tag{6.9}$$

for some $C>0$. Consequently, we deduce that

$$\begin{aligned} \mathrm{Var}\Big(V(G;p,q)^n_t - \rho^{(n)}_{p,q} t\Big) \le C\Bigg[&\frac{[nt]}{n^2}\exp\Big(\frac{4}{(|r_n(1)|(1-|r_n(1)|))^2}\Big) \\ &+ \frac{2t\exp\Big(\frac{4C}{(|r_n(1)|(1-|r_n(1)|))^2}\Big)}{n}\sum_{k=1}^{[nt]-1} r^2(k-1)\Bigg], \end{aligned}$$

since $r^m(k)$ is decreasing in m. This implies $\mathrm{Var}\Big(V(G;p,q)^n_t - \rho^{(n)}_{p,q} t\Big) \to 0$ (because $r_n(1) \to 2^{\beta-1}-1$) by Lemma 2 (i), and we obtain the pointwise convergence

$$\frac{V(G;p,q)^n_t}{\rho^{(n)}_{p,q}} \xrightarrow{P} t.$$

The ucp convergence follows immediately, because $\frac{V(G;p,q)^n_t}{\rho^{(n)}_{p,q}}$ is increasing in t and the limit process $g(t)=t$ is continuous. □

Before we proceed with the proof of Theorem 5 let us show the following lemma.

Lemma 4. *Under the assumptions **(A1)**–**(A3)** and* $0<\beta<\frac{3}{2}$ *we have*

$$\lim_{n\to\infty} n\,\mathrm{Var}\Big(V(G;p,q)^n_t - \rho^{(n)}_{p,q} t\Big) = \sigma^2_{p,q} t\ , \tag{6.10}$$

where $\sigma^2_{p,q}$ *is defined in* (3.12). *Moreover, we obtain the identity*

$$\sigma^2_{p,q} = \sum_{m=2}^{\infty} m!\Big(\sum_{l=0}^{m} c_l(m)\rho^l(1) + 2\sum_{(l_1,l_2,l_3)\in J_m} c_{l_1,l_2,l_3}(m)\sum_{k=1}^{\infty}\rho^{l_1}(k-1)\rho^{l_2}(k+1)\rho^{l_3}(k)\Big)\ , \tag{6.11}$$

where $J_m = \{(l_1, l_2, l_3) \in \mathbb{N}^3 |\ l_i \geq 0,\ l_1 + l_2 + l_3 = m\}$, $\rho(j)$ is the correlation function of the increments of the fractional Brownian motion $B^{\beta/2}$ defined in Lemma 2 (iii),

$$c_l(m) = \sum_{j=0}^{l} \sum_{h=j}^{m-l+j} s_h^m s_{h-2j+l}^m \binom{m}{h-2j+l}^{-1} \binom{h}{j} \binom{m-h}{l-j} ,$$

$$c_{l_1,l_2,l_3}(m) = \sum_{h=l_2}^{l_2+l_3} s_h^m s_{h-l_2+l_1}^m \binom{m}{h-l_2+l_1}^{-1} \binom{h}{l_2} \binom{m-h}{l_1} ,$$

and the quantity s_h^m is given by

$$s_h^m = \lim_{n \longrightarrow \infty} s_{h,m}^{(n)} = \sum_{l=0}^{\infty} a_{p,l+h} a_{q,l+m-h} l! \binom{l+h}{l} \binom{l+m-h}{l} \rho^l(1).$$

Proof of Lemma 4: We assume w.l.o.g. that $t = 1$. First, we prove the identities

$$||f_1^m||^2_{\mathcal{H}_1^{\otimes m}} = \sum_{l=0}^{m} c_l^{(n)}(m) r_n^l(1) , \tag{6.12}$$

$$\langle f_1^m, f_{1+k}^m \rangle_{\mathcal{H}_1^{\otimes m}} = \sum_{(l_1,l_2,l_3) \in J_m} c_{l_1,l_2,l_3}^{(n)}(m) r_n^{l_1}(k-1) r_n^{l_2}(k+1) r_n^{l_3}(k) , \tag{6.13}$$

where $c_l^{(n)}(m)$ (resp. $c_{l_1,l_2,l_3}^{(n)}(m)$) are defined exactly as $c_l(m)$ (resp. $c_{l_1,l_2,l_3}(m)$), but s_h^m is replaced by $s_{h,m}^{(n)}$.

Notice that

$$f_i^m = \sum_{h=0}^{m} s_{h,m}^{(n)} \binom{m}{h}^{-1} \sum_{t_j \in \{i,i+1\}, \sharp\{t_j = i\} = h} \frac{\Delta_{t_1}^n G}{\tau_n} \otimes \cdots \otimes \frac{\Delta_{t_m}^n G}{\tau_n}.$$

Next, we obtain

$$\begin{aligned} &||f_1^m||^2_{\mathcal{H}_1^{\otimes m}} \\ &= \sum_{h,h'=0}^{m} s_{h,m}^{(n)} s_{h',m}^{(n)} \binom{m}{h}^{-1} \binom{m}{h'}^{-1} \sum \left\langle \frac{\Delta_{t_1}^n G}{\tau_n}, \frac{\Delta_{t_1'}^n G}{\tau_n} \right\rangle_{\mathcal{H}_1} \cdots \left\langle \frac{\Delta_{t_m}^n G}{\tau_n}, \frac{\Delta_{t_m'}^n G}{\tau_n} \right\rangle_{\mathcal{H}_1} , \end{aligned}$$

where the second sum is over all indexes t_j, t_j' with $t_j, t_j' \in \{1, 2\}$, $\sharp\{t_j = 1\} = h$ and $\sharp\{t_j' = 1\} = h'$. Since $\langle \frac{\Delta_{t_1}^n G}{\tau_n}, \frac{\Delta_{t_1'}^n G}{\tau_n} \rangle_{\mathcal{H}_1} = r_n(1)$ or $r_n(0) = 1$, we get the representation

$$||f_1^m||^2_{\mathcal{H}_1^{\otimes m}} = \sum_{l=0}^{m} c_l^{(n)}(m) r_n^l(1) ,$$

where we have to compute $c_l^{(n)}(m)$. For this purpose we need to count all possible h, h' and all t_d, t_d' with $t_d, t_d' \in \{1, 2\}$, $\sharp\{t_d = 1\} = h$, $\sharp\{t_d' = 1\} = h'$ such that

$$\begin{aligned} &\sharp\{t_d = 1, t_d' = 2\} = j, && \sharp\{t_d = 1, t_d' = 1\} = h - j, \\ &\sharp\{t_d = 2, t_d' = 1\} = l - j, && \sharp\{t_d = 2, t_d' = 2\} = m - h + j - l, \end{aligned}$$

where $0 \le j \le l$. For this condition to be satisfied we require that $j \le h \le m-l+j$ and $h' = h - 2j + l$. Moreover, for fixed j and h there are

$$\binom{m}{h}\binom{h}{j}\binom{m-h}{l-j}$$

of the above-mentioned combinations. By summing over all possible j and h we obtain the identity (6.12).

The second identity can be deduced in a similar way. To compute $c^{(n)}_{l_1,l_2,l_3}(m)$ (for $(l_1,l_2,l_3) \in J_m$) we need to look at all possible h, h' and all t_d, t'_d with $t_d \in \{1,2\}$, $t'_d \in \{k+1, k+2\}$, $\sharp\{t_d = 1\} = h$, $\sharp\{t'_d = k+1\} = h'$ such that

$$\begin{aligned} &\sharp\{t_d = 2, t'_d = k+1\} = l_1, && \sharp\{t_d = 1, t'_d = k+2\} = l_2, \\ &\sharp\{t_d = 1, t'_d = k+1\} = h - l_2, && \sharp\{t_d = 2, t'_d = k+2\} = m - h - l_1. \end{aligned}$$

These conditions imply that $l_2 \le h \le m - l_1 = l_2 + l_3$ and $h' = h - l_2 + l_1$. Moreover, for fixed h there are

$$\binom{m}{h}\binom{h}{j}\binom{m-h}{l-j}$$

of the above-mentioned combinations. Hence, we obtain (6.13).

Now, recall the identity (6.7) (for $t = 1$):

$$n \operatorname{Var}\Big(V(G;p,q)^n_1 - \rho^{(n)}_{p,q}\Big) = \sum_{m=2}^{\infty} m!\Big(||f^m_1||^2_{\mathcal{H}_1^{\otimes m}} + \frac{2}{n}\sum_{k=1}^{n-1}(n-k)\langle f^m_1, f^m_{1+k}\rangle_{\mathcal{H}_1^{\otimes m}}\Big).$$

As in the proof of Theorem 4 we obtain the approximation

$$n \operatorname{Var}\Big(V(G;p,q)^n_1 - \rho^{(n)}_{p,q}\Big) \le C\Big(1 + 2\sum_{k=1}^{n-1} r^2(k-1)\Big),$$

where the function r is defined in Lemma 2 (i). When $0 < \beta < \frac{3}{2}$ the constant $\epsilon > 0$ (in the definition of the function r) can be chosen such that $\beta + \epsilon - 2 < -\frac{1}{2}$. In this case we have

$$\sum_{k=1}^{\infty} r^2(k-1) < \infty.$$

Finally, recall the convergence

$$r_n(j) \to \rho(j)$$

for any $j \ge 0$ (see Lemma 2 (iii)). By the dominated convergence theorem we deduce that

$$\begin{aligned} &n \operatorname{Var}\Big(V(G;p,q)^n_1 - \rho^{(n)}_{p,q}\Big) \\ &\quad \to \sum_{m=2}^{\infty} m!\Big(\sum_{l=0}^{m} c_l(m)\rho^l(1) + 2\sum_{(l_1,l_2,l_3)\in J_m} c_{l_1,l_2,l_3}(m)\sum_{k=1}^{\infty}\rho^{l_1}(k-1)\rho^{l_2}(k+1)\rho^{l_3}(k)\Big). \end{aligned}$$

On the other hand the right-hand side of the above convergence equals $\sigma^2_{p,q} = \lim_{n\to\infty} n \operatorname{Var}\big(V(B^{\beta/2};p,q)^n_1\big)$, because ρ is the correlation function of the increments of the fractional Brownian motion $B^{\beta/2}$. This proves (6.11). □

Proof of Theorem 5: We divide the proof of Theorem 5 into two steps. In the first step we prove the convergence of finite dimensional distribution of the sequence $\big(G_t, \sqrt{n}\big(\frac{V(G;p,q)^n_t}{\rho^{(n)}_{p,q}} - t\big)\big)$. Then we prove the tightness of this sequence.

Step 1: Define the vector $Y_n = (Y^1_n, \ldots, Y^d_n)^T$ by

$$Y^k_n = \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \left(\Big|\frac{\Delta^n_i G}{\tau_n}\Big|^p \Big|\frac{\Delta^n_{i+1} G}{\tau_n}\Big|^q - \rho^{(n)}_{p,q} \right) , \tag{6.14}$$

where $(a_k, b_k]$, $k = 1, \ldots, d$, are disjoint intervals contained in $[0,T]$. Clearly, it suffices to prove that

$$\big(G_{b_k} - G_{a_k}, Y^k_n\big)_{1\le k\le d} \xrightarrow{\mathcal{D}} \Big(G_{b_k} - G_{a_k}, \sigma_{p,q}(W_{b_k} - W_{a_k})\Big)_{1\le k\le d} ,$$

where $\sigma_{p,q}$ is given by (3.12) (because $\rho^{(n)}_{p,q} \to \rho_{p,q}$, where $\rho_{p,q}$ is given in (3.10)).

By Lemma 1 we obtain the representation

$$Y^k_n = \sum_{m=2}^{\infty} I_m\Big(\frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f^m_i\Big) + O(n^{-1/2}).$$

Since

$$E[(G_{b_k} - G_{a_k})Y^l_n] = 0$$

for any $1 \le k, l \le d$, it is sufficient to check the following conditions.

(i) For any $m \ge 1$ and $k = 1, \ldots, d$, the limit

$$\lim_{n\to\infty} m! \, \Big|\Big|\frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f^m_i\Big|\Big|^2_{\mathcal{H}_1^{\otimes m}} = \sigma^2_{p,q}(m,k)$$

exists and

$$\sum_{m=2}^{\infty} m! \, \sup_n \Big|\Big|\frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f^m_i\Big|\Big|^2_{\mathcal{H}_1^{\otimes m}} < \infty ,$$

(ii) For any $m \ge 1$ and $k \ne h$,

$$\lim_{n\to\infty} \Big\langle \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f^m_i, \frac{1}{\sqrt{n}} \sum_{i=[na_h]+1}^{[nb_h]} f^m_i \Big\rangle_{\mathcal{H}_1^{\otimes m}} = 0 ,$$

(iii) For any $m \ge 1$, $k = 1, \ldots, d$ and $1 \le p \le m-1$, we have that

$$\lim_{n\to\infty} \Big|\Big|\Big(\frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f^m_i\Big) \otimes_p \Big(\frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} f^m_i\Big)\Big|\Big|_{\mathcal{H}_1^{\otimes 2(m-p)}} = 0.$$

Under conditions (i)-(iii) we then obtain (by Theorem 2) the central limit theorem

$$Y_n \xrightarrow{\mathcal{D}} N_d\Big(0, \sigma^2_{p,q}\ \mathrm{diag}(b_1 - a_1, \ldots, b_d - a_d)\Big)\ , \tag{6.15}$$

where $\sigma^2_{p,q}$ is given by (3.12). Since the increments of the process G are stationary we will prove part (i) and (iii) only for $k = 1$, $a_1 = 0$ and $b_1 = 1$.

(i) By the same methods as presented in Lemma 4 we obtain

$$\sigma^2_{p,q}(m,1) = m!\Big(\sum_{l=0}^{m} c_l(m)\rho^l(1) + 2\sum_{(l_1,l_2,l_3)\in J_m} c_{l_1,l_2,l_3}(m)\sum_{k=1}^{\infty}\rho^{l_1}(k-1)\rho^{l_2}(k+1)\rho^{l_3}(k)\Big)\ ,$$

$$\sum_{m=2}^{\infty} m!\ \sup_n \Big|\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} f_i^m\Big|\Big|^2_{\mathcal{H}_1^{\otimes m}} < \infty$$

and $\sum_{m=2}^{\infty}\sigma^2_{p,q}(m,1)$=$\sigma^2_{p,q}$.

(ii) For any $1 \le k, h \le d$ with $b_k \le a_h$ we have

$$\Big\langle \frac{1}{\sqrt{n}}\sum_{i=[na_k]+1}^{[nb_k]} f_i^m, \frac{1}{\sqrt{n}}\sum_{i=[na_h]+1}^{[nb_h]} f_i^m\Big\rangle_{\mathcal{H}_1^{\otimes m}} = \frac{m!}{n}\sum_{j=[na_k]+1}^{[nb_k]}\sum_{i=[na_h]+1}^{[nb_h]}\langle f_1^m, f_{1+i-j}^m\rangle_{\mathcal{H}_1^{\otimes m}}.$$

Assume w.l.o.g. that $a_k = 0$, $b_k = a_h = 1$ and $b_h = 2$ (the case $b_k < a_h$ is much easier). By part (ii) of Lemma 2 with $0 < \epsilon < \frac{3}{2} - \beta$ in the definition of r (see (6.4)) we obtain the approximation (by (6.9))

$$\Big|\Big\langle \frac{1}{\sqrt{n}}\sum_{i=[na_k]+1}^{[nb_k]} f_i^m, \frac{1}{\sqrt{n}}\sum_{i=[na_h]+1}^{[nb_h]} f_i^m\Big\rangle_{\mathcal{H}_1^{\otimes m}}\Big| \le C_m\Big(\frac{1}{n}\sum_{j=1}^{n} j r^m(j) + \sum_{j=1}^{n-1} r^m(n+j)\Big)$$

for some constant $C_m > 0$. It follows that $r^m(j) \le (j-1)^{-1-\delta}$ for some $\delta > 0$ and for all $m, j \ge 2$. Hence, we obtain

$$\Big\langle \frac{1}{\sqrt{n}}\sum_{i=[na_k]+1}^{[nb_k]} f_i^m, \frac{1}{\sqrt{n}}\sum_{i=[na_h]+1}^{[nb_h]} f_i^m\Big\rangle_{\mathcal{H}_1^{\otimes m}} \to 0$$

as $n \to \infty$.

(iii) Fix $1 \le p \le m-1$. We obtain the identity

$$\begin{aligned} & f_i^m \otimes_p f_j^m \\ &\quad = \sum_{h,h'=0}^{m} s^{(n)}_{h,m} s^{(n)}_{h',m}\binom{m}{h}^{-1}\binom{m}{h'}^{-1}\sum\Big\langle\frac{\Delta^n_{t_1}G}{\tau_n}, \frac{\Delta^n_{t'_1}G}{\tau_n}\Big\rangle_{\mathcal{H}_1}\cdots\Big\langle\frac{\Delta^n_{t_p}G}{\tau_n}, \frac{\Delta^n_{t'_p}G}{\tau_n}\Big\rangle_{\mathcal{H}_1} \\ &\qquad \times \frac{\Delta^n_{t_{p+1}}G}{\tau_n}\otimes\cdots\otimes\frac{\Delta^n_{t_m}G}{\tau_n}\otimes\frac{\Delta^n_{t'_{p+1}}G}{\tau_n}\otimes\cdots\otimes\frac{\Delta^n_{t'_m}G}{\tau_n}\ , \end{aligned}$$

where the second sum is running over all indexes t_k, t'_k with $t_k \in \{i, i+1\}$, $t'_k \in \{j, j+1\}$, $\sharp\{t_k = i\} = h$ and $\sharp\{t'_k = j\} = h'$. Now, by Lemma 2 (ii) (with $0 < \epsilon < \frac{3}{2} - \beta$ in the definition of r) and Lemma 3 we deduce the inequality

$$\begin{aligned}
&\left\| \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i^m \right) \widetilde{\otimes}_p \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i^m \right) \right\|_{\mathcal{H}_1^{\otimes 2(m-p)}} \\
&\leq \frac{C_m}{n^2} \sum_{1 \leq j,l,h,k \leq n} r^p(|j-l|-1) r^p(|h-k|-1) \\
&\quad \times \left| \left\langle \left(\frac{\Delta_j^n G}{\tau_n} \right)^{\otimes(m-p)} \widetilde{\otimes} \left(\frac{\Delta_l^n G}{\tau_n} \right)^{\otimes(m-p)}, \left(\frac{\Delta_h^n G}{\tau_n} \right)^{\otimes(m-p)} \widetilde{\otimes} \left(\frac{\Delta_k^n G}{\tau_n} \right)^{\otimes(m-p)} \right\rangle_{\mathcal{H}_1^{\otimes 2(m-p)}} \right|
\end{aligned}$$

where "$\sim$" denotes the symmetrization and $r(-1) := 1$. Applying again Lemma 2 (ii) and Lemma 3 we see that it suffices to prove

$$\begin{aligned}
&n^{-2} \sum_{1 \leq j,l,h,k \leq n} r^p(|j-l|-1) r^p(|h-k|-1) \\
&\qquad \times r^{\alpha}(|j-h|-1) r^{m-p-\alpha}(|l-h|-1) r^{m-p-\alpha}(|j-k|-1) r^{\alpha}(|l-k|-1) \\
&\qquad \to 0,
\end{aligned}$$

where $0 \leq \alpha \leq m-p$. The latter term is smaller than

$$n^{-1} \sum_{-1 \leq j,l,k \leq n-2} r^p(|j-l|) r^p(k) r^{\alpha}(j) r^{m-p-\alpha}(l) r^{m-p-\alpha}(|j-k|) r^{\alpha}(|l-k|).$$

Without any loss of generality we can assume that $p = m-p = 1$ and $\alpha = 0$ or $\alpha = 1$. For $\alpha = 0$ and any $0 < \varepsilon < 1$ we get

$$\begin{aligned}
n^{-1} \sum_{-1 \leq j \leq n-2} \Big(\sum_{-1 \leq l \leq n-2} r(|j-l|) r(l) \Big)^2 &\leq n^{-1} \sum_{-1 \leq j \leq [n\varepsilon]} \Big(\sum_{-1 \leq l \leq n-2} r(|j-l|) r(l) \Big)^2 \\
&\quad + 2n^{-1} \sum_{[n\varepsilon] < j \leq n-2} \Big(\sum_{-1 \leq l \leq [n\varepsilon/2]} r(|j-l|) r(l) \Big)^2 \\
&\quad + 2n^{-1} \sum_{[n\varepsilon] < j \leq n-2} \Big(\sum_{[n\varepsilon/2] < l \leq n-1} r(|j-l|) r(l) \Big)^2 \\
&\leq 2\varepsilon \Big(\sum_{-1 \leq l < n-2} r^2(l) \Big)^2 + 6 \sum_{-1 \leq l < n-2} r^2(l) \sum_{[n\varepsilon/2] < l < \infty} r^2(l)
\end{aligned}$$

which converges to $2\varepsilon \left(\sum_{-1 \leq l < \infty} r^2(l) \right)^2$ as $n \to \infty$ by Lemma 2 (i). The desired result follows by letting ε tend to zero. $\square$

Step 2: It suffices to show the tightness of the sequence $\sqrt{n}\big(V(G;p,q)_t^n - \rho_{p,q}^{(n)} t\big)$. Set

$$Z_t^n = \sqrt{n}\Big(V(G;p,q)_t^n - \rho_{p,q}^{(n)} t\Big) = \sum_{m=2}^{\infty} I_m \Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} f_i^m \Big)$$

and

$$Z_t^{n,N} = \sum_{m=2}^{N} I_m\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} f_i^m\Big).$$

In Step 1 we have proved that conditions (i)-(iii) of Theorem 2 are satisfied. Then by (2.3) and the Cauchy-Schwarz inequality we obtain the approximation

$$\begin{aligned} P\Big(&|Z_t^{n,N} - Z_{t_1}^{n,N}| \geq \lambda, |Z_{t_2}^{n,N} - Z_t^{n,N}| \geq \lambda\Big) \\ &\leq \frac{E^{1/2}[|Z_t^{n,N} - Z_{t_1}^{n,N}|^4]E^{1/2}[|Z_{t_2}^{n,N} - Z_t^{n,N}|^4]}{\lambda^4} \\ &\leq C\frac{\sigma_{p,q}^4([nt]-[nt_1])([nt_2]-[nt])}{\lambda^4} \leq C\frac{\sigma_{p,q}^4(t_2-t_1)^2}{\lambda^4} \end{aligned}$$

for any $t_1 \leq t \leq t_2$ and $\lambda > 0$. On the other hand condition (i) of Theorem 2 also implies that

$$\lim_{N\to\infty}\limsup_{n\to\infty} E[|Z_t^n - Z_t^{n,N}|^2] = 0.$$

Using this we conclude that

$$P\Big(|Z_t^n - Z_{t_1}^n| \geq \lambda, |Z_{t_2}^n - Z_t^n| \geq \lambda\Big) \leq C\frac{\sigma_{p,q}^4(t_2-t_1)^2}{\lambda^4}$$

for any $t_1 \leq t \leq t_2$ and $\lambda > 0$, from which we deduce the tightness of the sequence Z_t^n by Theorem 15.6 in [11]. This completes the proof of Theorem 5. □

Proof of Theorem 6: We will show that

$$n^{-1}\tau_n^{-(p+q)}\sum_{i=1}^{[nt]-1}|\Delta_i^n Z|^p|\Delta_{i+1}^n Z|^q \xrightarrow{P} \rho_{p,q}\int_0^t |u_s|^{p+q}ds,$$

where we can follow similar ideas as in [12] and [2].

First we look at the case $p, q \leq 1$, then we obtain, for any $m \geq n$

$$\begin{aligned} m^{-1}&\tau_m^{-(p+q)}\sum_{j=1}^{[mt]}|\Delta_i^m Z|^p|\Delta_{i+1}^m Z|^q - \rho_{p,q}\int_0^t |u_s|^{p+q}ds \\ &= m^{-1}\tau_m^{-(p+q)}\sum_{j=1}^{[mt]}|\Delta_j^m Z|^p\left(|\Delta_{j+1}^m Z|^q - |u_{\frac{j}{m}}\Delta_{j+1}^m G|^q\right) \\ &\quad + m^{-1}\tau_m^{-(p+q)}\sum_{j=1}^{[mt]}|u_{\frac{j}{m}}\Delta_{j+1}^m G|^q\left(|\Delta_j^m Z|^p - |u_{\frac{j-1}{m}}\Delta_j^m G|^p\right) \\ &\quad + m^{-1}\tau_m^{-(p+q)}\Bigg(\sum_{j=1}^{[mt]}|u_{\frac{j}{m}}\Delta_{j+1}^m G|^q|u_{\frac{j-1}{m}}\Delta_j^m G|^p \\ &\qquad\qquad - \sum_{i=1}^{[nt]}\sum_{j\in I_n(i)}|u_{\frac{i}{n}}\Delta_{j+1}^m G|^q|u_{\frac{i-1}{n}}\Delta_j^m G|^p\Bigg) \end{aligned}$$

$$
\begin{aligned}
&+ \left(m^{-1}\tau_m^{-(p+q)} \sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q \sum_{j\in I_n(i)} |\Delta_{j+1}^m G|^q |\Delta_j^m G|^p \right. \\
&\qquad \left. - \frac{\rho_{p,q}}{n} \sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q \right) \\
&+ \rho_{p,q} \left(n^{-1} \sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q - \int_0^t |u_s|^{p+q} ds \right) \\
&= \mathrm{A}_t^{(m)} + \mathrm{B}_t^{(m)} + \mathrm{C}_t^{(n,m)} + \mathrm{D}_t^{(n,m)} + \mathrm{E}_t^{(n)},
\end{aligned}
$$

where $I_n(i) = \{j : \frac{j}{m} \in (\frac{i-1}{n}, \frac{i}{n}]\},\ 1 \le i \le [nt]$.

$\mathrm{A}_t^{(m)}$ converges in probability to zero, uniformly in t, as m tends to infinity. We denote by $||.||_\infty$ the supremum norm on $[0,T]$. Using the Hölder inequality we obtain

$$
\begin{aligned}
\left\|\mathrm{A}^{(m)}\right\|_\infty &\le m^{-1}\tau_m^{-(p+q)} \sum_{j=1}^{[mT]} \left|\Delta_j^m Z\right|^p \left| |\Delta_{j+1}^m Z|^q - |u_{\frac{j}{m}} \Delta_{j+1}^m G|^q \right| \\
&\le \left(m^{-1}\tau_m^{-2p} \sum_{j=1}^{[mT]} \left|\Delta_j^m Z\right|^{2p} \right)^{1/2} \\
&\quad \times \left(m^{-1}\tau_m^{-2q} \sum_{j=1}^{[mT]} \left(|\Delta_{j+1}^m Z|^q - |u_{\frac{j}{m}} \Delta_{j+1}^m Z|^q \right)^2 \right)^{1/2} \\
&\le \left(m^{-1}\tau_m^{-2p} \sum_{j=1}^{[mT]} \left|\Delta_j^m Z\right|^{2p} \right)^{1/2} \\
&\quad \times \left(m^{-1}\tau_m^{-2q} \sum_{j=1}^{[mT]} |\Delta_{j+1}^m Z - u_{\frac{j}{m}} \Delta_{j+1}^m G|^{2q} \right)^{1/2}.
\end{aligned}
$$

The first term tends to $\mu_{2p} \int_0^T |u_s|^{2p} ds$ in probability as follows by Theorem 2 of [2], and the second term can be shown to tend to zero as in the proof of that theorem.

The convergence of $\mathrm{B}_t^{(m)}$ to zero as $m \to \infty$ can be verified analogously.

For any fixed n, $\mathrm{D}_t^{(n,m)}$ converges in probability to zero, uniformly in t, as m tends to infinity. In fact,

$$
\begin{aligned}
\left\|\mathrm{D}^{(n,m)}\right\|_\infty \le &\sum_{i=1}^{[nT]} \left|u_{\frac{i-1}{n}}\right|^p \left|u_{\frac{i}{n}}\right|^q \\
&\times \left| m^{-1}\tau_m^{-(p+q)} \sum_{j\in I_n(i)} |\Delta_j^m G|^p |\Delta_{j+1}^m G|^q - \frac{\rho_{p,q}}{n} \right| \xrightarrow{P} 0
\end{aligned}
$$

since we know by Theorem 4 that

$$
m^{-1}\tau_m^{-(p+q)} \sum_{j=1}^{[mt]} |\Delta_j^m G|^p |\Delta_{j+1}^m G|^q \xrightarrow{ucp} \rho_{p,q} t.
$$

For the term $\mathrm{C}_t^{(n,m)}$ we obtain

$$\begin{aligned}
\left\|\mathrm{C}^{(n,m)}\right\|_\infty &\le m^{-1}\tau_m^{-(p+q)} \sum_{i=1}^{[nT]} \sum_{j\in I_n(i)} \left| |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q - |u_{\frac{j-1}{m}}|^p |u_{\frac{j}{m}}|^q \right| \left|\Delta_j^m G\right|^p \left|\Delta_{j+1}^m G\right|^q \\
&\quad + \left\||u|^{p+q}\right\|_\infty \sup_{0\le t\le T} m^{-1}\tau_m^{-(p+q)} \sum_{\frac{m}{n}([nt]-1)\le j\le \frac{m}{n}[nt]} \left|\Delta_j^m G\right|^p \left|\Delta_{j+1}^m G\right|^q \\
&\le m^{-1}\tau_m^{-(p+q)} \sum_{i=1}^{[nT]} \||u|^p\|_\infty \sup_{s\in\mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)} \left(\left|u_{\frac{i}{n}}\right|^q - |u_s|^q \right) \\
&\quad \times \sum_{j\in I_n(i)} \left|\Delta_j^m G\right|^p \left|\Delta_{j+1}^m G\right|^q \\
&\quad + m^{-1}\tau_m^{-(p+q)} \sum_{i=1}^{[nT]} \||u|^q\|_\infty \sup_{s\in\mathcal{I}_n(i+1)\cup\mathcal{I}_n(i)} \left(\left|u_{\frac{i-1}{n}}\right|^p - |u_s|^p \right) \\
&\quad \times \sum_{j\in I_n(i)} \left|\Delta_j^m G\right|^p \left|\Delta_{j+1}^m G\right|^q \\
&\quad + \left\||u|^{p+q}\right\|_\infty \sup_{0\le t\le T} m^{-1}\tau_m^{-(p+q)} \sum_{\frac{m}{n}([nt]-1)\le j\le \frac{m}{n}[nt]} \left|\Delta_j^m G\right|^p \left|\Delta_{j+1}^m G\right|^q ,
\end{aligned}$$

where we denote $\mathcal{I}_n(i) := (\frac{i-1}{n}, \frac{i}{n}]$, $1\le i\le [nt]$. As m tends to infinity, we find as above by Theorem 4, that this converges in probability to

$$\begin{aligned}
F_n = \frac{\rho_{p,q}}{n} \Bigg(& \||u|^p\|_\infty \sum_{i=1}^{[nT]} \sup_{s\in\mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)} \left| |u_{\frac{i}{n}}|^q - |u_s|^q \right| \\
& + \||u|^q\|_\infty \sum_{i=1}^{[nT]} \sup_{s\in\mathcal{I}_n(i+1)\cup\mathcal{I}_n(i)} \left| |u_{\frac{i-1}{n}}|^p - |u_s|^p \right| + \left\||u|^{p+q}\right\|_\infty \Bigg).
\end{aligned}$$

By the same argument as in [12], F_n tends to zero almost surely as n tends to infinity.

Finally we have to show that $\lim_{n\to\infty} \left\|\mathrm{E}^{(n)}\right\|_\infty = 0$.

$$\begin{aligned}
\left\|\mathrm{E}^{(n)}\right\|_\infty &\le \frac{\rho_{p,q}}{n} \sum_{i=1}^{[nT]} \sup_{s\in\mathcal{I}_n(i)} \left| |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q - |u_s|^{p+q} \right| + \rho_{p,q} \frac{\left\||u|^{p+q}\right\|_\infty}{n} \\
&\le \rho_{p,q} \Bigg(\||u|^p\|_\infty \, n^{-1} \sum_{i=1}^{[nT]} \sup_{s\in\mathcal{I}_n(i)} ||u_{\frac{i}{n}}|^q - |u_s|^q| \\
&\quad + \||u|^q\|_\infty \, n^{-1} \sum_{i=1}^{[nT]} \sup_{s\in\mathcal{I}_n(i)} ||u_{\frac{i-1}{n}}|^p - |u_s|^p| \\
&\quad + n^{-1} \sum_{i=1}^{[nT]} \sup_{s\in\mathcal{I}_n(i)} ||u_{\frac{i-1}{n}}|^p - |u_s|^p| \times ||u_{\frac{i}{n}}|^q - |u_s|^q| + \frac{\|u\|_\infty}{n} \Bigg),
\end{aligned}$$

where the terms tend to zero by the same arguments as for F_n.

Next we look at the other cases of p and q. Without loss of generality we may assume $p \le q$ and $q > 1$.

In the following we will use

$$\Big(\sum_i |a_i|^q |b_i + c_i|^p\Big)^{1/q} \le \Big(\sum_i |a_i|^q |b_i|^p\Big)^{1/q} + \Big(\sum_i |a_i|^q |c_i|^p\Big)^{1/q},$$

which follows by some straight forward application of Minkowski's inequality together with the triangular inequality.

Hence we can show that

$$\Big(m^{-1}\tau_m^{-(p+q)} \sum_{j=1}^{[mt]} |\Delta_j^m Z|^p |\Delta_{j+1}^m Z|^q\Big)^{1/q} \le I + II + III + IV + V + VI + VII$$

with

$$I = \Big(m^{-1}\tau_m^{-(p+q)} \sum_{j=1}^{[mt]} |\Delta_j^m Z|^p \left|\Delta_{j+1}^m Z - u_{\frac{j}{m}} \Delta_{j+1}^m G\right|^q\Big)^{1/q}$$

$$II = \Big(m^{-1}\tau_m^{-(p+q)} \sum_{j=1}^{[mt]} |u_{\frac{j}{m}} \Delta_{j+1}^m Z|^q \left|\Delta_j^m Z - u_{\frac{j-1}{m}} \Delta_j^m G\right|^p\Big)^{1/q}$$

$$III = \Big(m^{-1}\tau_m^{-(p+q)} \sum_{\frac{m}{n}([nt]) \le j \le \frac{m}{n}[nt]} |u_{\frac{j}{m}} \Delta_{j+1}^m G|^q |u_{\frac{j-1}{m}} \Delta_j^m G|^p\Big)^{1/q}$$

$$IV = \Big(m^{-1}\tau_m^{-(p+q)} \sum_{i=1}^{[nt]} \sum_{j \in I_n(i)} |u_{\frac{j}{m}} \Delta_{j+1}^m G|^q |(u_{\frac{j-1}{m}} - u_{\frac{i-1}{n}}) \Delta_j^m G|^p\Big)^{1/q}$$

$$V = \Big(m^{-1}\tau_m^{-(p+q)} \sum_{i=1}^{[nt]} \sum_{j \in I_n(i)} |(u_{\frac{j}{m}} - u_{\frac{i}{n}}) \Delta_{j+1}^m G|^q |u_{\frac{i-1}{n}} \Delta_j^m G|^p\Big)^{1/q}$$

$$VI = \Big(m^{-1}\tau_m^{-(p+q)} \sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q \sum_{j \in I_n(i)} |\Delta_{j+1}^m G|^q |\Delta_j^m G|^p - \frac{\rho_{p,q}}{n} \sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q\Big)^{1/q}$$

$$VIII = \Big(\frac{\rho_{p,q}}{n} \sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q\Big)^{1/q}$$

On account of this we obtain

$$\left|\Big(m^{-1}\tau_m^{-(p+q)} \sum_{j=1}^{[mt]} |\Delta_j^m Z|^p |\Delta_{j+1}^m Z|^q\Big)^{1/q} - \Big(\rho_{p,q} \int_0^t |u_s|^{p+q} ds\Big)^{1/q}\right|$$
$$\le |I| + |II| + |III| + |IV| + |V| + |VI| + \left|VII - \Big(\rho_{p,q} \int_0^t |u_s|^{p+q} ds\Big)^{1/q}\right|$$
$$\xrightarrow{P} 0$$

by the same arguments as for the case $p, q \le 1$. This completes the proof. □

Proof of Theorem 7: We obtain, for any $m \geq n$

$$
\begin{aligned}
&\left(\rho_{p,q}^{(m)}\right)^{-1} m^{-1/2}\tau_m^{-(p+q)} \sum_{j=1}^{[mt]} |\Delta_j^m Z|^p |\Delta_{j+1}^m Z|^q - m^{1/2}\int_0^t |u_s|^{p+q} ds \\
&= \left(\rho_{p,q}^{(m)}\right)^{-1} m^{-1/2}\tau_m^{-(p+q)} \sum_{j=1}^{[mt]} |\Delta_j^m Z|^p \left(|\Delta_{j+1}^m Z|^q - |u_{\frac{j}{m}}\Delta_{j+1}^m G|^q \right) \\
&\quad + \left(\rho_{p,q}^{(m)}\right)^{-1} m^{-1/2}\tau_m^{-(p+q)} \sum_{j=1}^{[mt]} |u_{\frac{j}{m}}\Delta_{j+1}^m G|^q \left(|\Delta_j^m Z|^p - |u_{\frac{j-1}{m}}\Delta_j^m G|^p \right) \\
&\quad + \left(\rho_{p,q}^{(m)}\right)^{-1} m^{-1/2}\tau_m^{-(p+q)} \sum_{j=1}^{[mt]} |u_{\frac{j}{m}}\Delta_{j+1}^m G|^q |u_{\frac{j-1}{m}}\Delta_j^m G|^p - m^{-1/2}\sum_{j=1}^{[mt]} |u_{\frac{j}{m}}|^q |u_{\frac{j-1}{m}}|^p \\
&\quad - \left(\rho_{p,q}^{(m)}\right)^{-1} m^{-1/2}\tau_m^{-(p+q)} \sum_{i=1}^{[nt]} \sum_{j\in I_n(i)} |u_{\frac{i}{n}}\Delta_{j+1}^m G|^q |u_{\frac{i-1}{n}}\Delta_j^m G|^p + \frac{m^{1/2}}{n}\sum_{i=1}^{[nt]} |u_{\frac{i}{n}}|^q |u_{\frac{i-1}{n}}|^p \\
&\quad + \left(\rho_{p,q}^{(m)}\right)^{-1} m^{-1/2}\tau_m^{-(p+q)} \sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q \sum_{j\in I_n(i)} |\Delta_{j+1}^m G|^q |\Delta_j^m G|^p \\
&\qquad\qquad - \frac{m^{1/2}}{n}\sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q \\
&\quad + m^{-1/2}\sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p |u_{\frac{i}{n}}|^q - m^{1/2}\int_0^t |u_s|^{p+q} ds \\
&= \mathrm{A}_t^{(m)} + \mathrm{B}_t^{(m)} + \mathrm{C}_t^{(n,m)} + \mathrm{D}_t^{(n,m)} + \mathrm{E}_t^{(n)}.
\end{aligned}
$$

By a combination of the arguments of Theorem 6 and Theorem 7 in [2] together with the result of Theorem 5 it can be shown that $||A^{(m)}||_\infty$, $||B^{(m)}||_\infty$, $||C^{(m,n)}||_\infty$, $||E^{(n)}||_\infty \to 0$ and $D_t^{(m,n)} \stackrel{\mathcal{F}^G-st}{\longrightarrow} \frac{\sigma_{p,q}}{\rho_{p,q}}\int_0^t |u_s|^{p+q} dW_s$ as $n, m \to \infty$. □

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