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1 Summary

The closed form of a rotational version of the famous Crofton formula is derived. In the simplest case where the sectioned object is a compact subset of $\mathbb{R}^d$ with a $(d-1)$-dimensional manifold of class C^2 as boundary, the rotational average of intrinsic volumes measured on sections passing through a fixed point can be expressed as an integral over the boundary involving hypergeometric functions. In the more general case of a compact subset of $\mathbb{R}^d$ of positive reach, the rotational average also involves hypergeometric functions.

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2 Introduction

Local stereology is a collection of sampling designs based on sections through a reference point of the structure under study, cf. [3]. The majority of the local stereological methods have been derived in the nineties, including methods of estimating number, length, surface area and volume. These methods have found numerous applications, in particular in the microscopic analysis of tissue samples, cf. [2] and the references therein.

Only very recently, a rotational integral formula has been derived for general intrinsic volumes, cf. [4]. This new formula opens up the possibility for developing local stereological methods of estimating curvature (for instance, integral of mean curvature). The formula shows how rotational averages of intrinsic volumes measured on sections are related to the geometry of the sectioned object $X \subset \mathbb{R}^d$. The rotational average considered is of the following form

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\mathrm{d}L_j^d, \tag{1}$$

$0 \le k \le j \le d$, where $\mathcal{L}_j^d$ is the set of $j-$dimensional *linear* subspaces in $\mathbb{R}^d$, V_k is the kth intrinsic volume and $\mathrm{d}L_j^d$ is the element of the rotation invariant measure on $\mathcal{L}_j^d$ with total measure

$$\int_{\mathcal{L}_j^d} \mathrm{d}L_j^d = c_{d,j}.$$

Here,

$$c_{d,j} = \frac{\sigma_d \sigma_{d-1} \cdots \sigma_{d-j+1}}{\sigma_j \sigma_{j-1} \cdots \sigma_1},$$

where $\sigma_k = 2\pi^{\frac{k}{2}}/\Gamma(\frac{k}{2})$ is the surface area of the unit sphere in $\mathbb{R}^k$.

In the simplest case where $X \subset \mathbb{R}^d$ is compact with a $(d-1)$-dimensional manifold of class C^2 as boundary ∂X, the rotational average (1) takes the following form, provided $0 \notin \partial X$,

$$\int_{\partial X} \sum_{\substack{I \subseteq \{1,\ldots,d-1\} \\ |I| = j-1-k}} w_{I,j}(x) \prod_{i \in I} \kappa_i(x)\, \mathcal{H}^{d-1}(\mathrm{d}x), \tag{2}$$

where $\mathcal{H}^k$ denotes the $k-$dimensional Hausdorff measure, $\kappa_i(x)$, $i = 1, \ldots, d-1$, are the principal curvatures at $x \in \partial X$ and $w_{I,j}$ is a real non-negative function defined on ∂X, cf. [4]. If X is a ball, the function $w_{I,j}$ is constant and the rotational average is therefore proportional to the $(d-j+k)$th intrinsic volume of X which has the following integral representation

$$V_{d-j+k}(X) = \frac{1}{\sigma_{j-k}} \int_{\partial X} \sum_{|I| = j-1-k} \prod_{i \in I} \kappa_i(x)\, \mathcal{H}^{d-1}(\mathrm{d}x),$$

cf. [7, Section 13.6] and [8, Section V.3].

In the present paper, we derive a closed form expression of the function $w_{I,j}$ involving hypergeometric functions. This expression allows us to study in detail how the rotational average depends on the local geometry of X in the non-spherical case.

The paper is organized as follows. In Section 2, we define the function $\omega_{I,j}$ and provide background knowledge on hypergeometric functions and other issues. In Section 3, the closed form expression of $\omega_{I,j}$ is derived while Section 4 contains a simplified expression of the rotational Crofton formula, under additional assumptions. The proof of one of the lemmas is deferred to an appendix.

3 Preliminaries

In [4], it was shown that the function $w_{I,j}$ satisfies the following equation

$$\sigma_{j-k} |x|^{d-j} w_{I,j}(x) = Q_j(x, n(x), A_I(x)), \tag{3}$$

where $n(x)$ is the outer unit normal to ∂X at x and $A_I(x)$ is the linear subspace spanned by the principal directions of curvature $a_i(x)$ with $i \notin I$. Furthermore, for any $x \in \mathbb{R}^d \setminus \{0\}$, $n \in S^{d-1}$ and $q-$dimensional linear subspace $A_q \subseteq \mathbb{R}^d$ perpendicular to n, Q_j is given by the following integral representation

$$Q_j(x, n, A_q) = \int_{\mathcal{L}^d_{j(1)}} \frac{\mathcal{G}(L_j, A_q)^2}{|p(n|L_j)|^{d-q}} \mathrm{d}L^d_{j(1)}, \tag{4}$$

where $\mathcal{L}^d_{j(1)}$ is the set of $j-$dimensional subspaces containing the line spanned by x, $p(\cdot|L_j)$ indicates orthogonal projection onto L_j and $\mathcal{G}(L_j, A_q)$ can be regarded as a generalized sinus of the angle between the subspaces L_j and A_q. A precise definition of $\mathcal{G}$ is provided at the end of this section. In the more general case of a compact subset $X \subset \mathbb{R}^d$ of positive reach, the rotational average (1) can also be expressed in terms of the functions $\omega_{I,j}$, cf. [4].

Note that in (2) and (3), we consider A_q with

$$q = d - 1 - (j - 1 - k) = d - j + k.$$

It follows that for such A_q we have $j + q \geq d$. If $j = 1$ and $x \perp n$, then the integrand in (4) is not defined; in this case we set $Q_1(x, n, n^\perp) = 0$. In all other cases, $n \not\perp L_j$ for $\mathrm{d}L^d_{j(1)}$-almost all L_j. Note that $Q_j(x, n, A_q)$ is finite whenever $x \not\perp n$ since $|p(n|L_j)| \geq |x \cdot n|/|x|$.

Only in the cases $q = 1$ and $q = d - 1$, [4] succeeded in finding closed form expressions for Q_j, involving hypergeometric functions. Recall that a hypergeometric function can be represented by a series of the following form

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}.$$

When $a = 0$ or $b = 0$, the hypergeometric function is identically equal to 1. The series converges absolutely for $|z| < 1$. In case $0 < b < c$, we can also represent the hypergeometric series by an integral

$$F(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 (1 - zy)^{-a} y^{b-1} (1-y)^{c-b-1} dy.$$

When $z = 1$, the extra assumption $c - a - b > 0$ is necessary. Transformations formulae for hypergeometric functions are often useful. In particular, we shall use the following formulae, cf. [1, (15.2.17) and (15.2.20)],

$$(c - a - 1)F(a, b; c; z) + aF(a + 1, b; c; z) = (c - 1)F(a, b; c - 1; z) \tag{5}$$
$$c(1 - z)F(a, b; c; z) + (c - b)zF(a, b; c + 1; z) = cF(a - 1, b; c; z). \tag{6}$$

For $q = d - 1$, it was shown in [4] that

$$Q_j(x, n, A_q) = c_{d-1,j-1} F(-1/2, (d-j)/2; (d-1)/2; \sin^2 \beta), \tag{7}$$

where $\beta = \angle(x, n)$. For $q = 1$, we must have $j = d - 1$. For $A_q = \mathrm{span}\{a\}$, $a \in S^{d-1}$, it was shown in [4] for x and n linearly independent that,

$$\begin{aligned} &Q_{d-1}(x, n, \mathrm{span}\{a\}) \\ &\quad = \frac{\pi^{(d-1)/2}}{2\Gamma((d+1)/2)} \sin^2 \alpha \Big[\sin^2 \theta F\Big(\frac{d-1}{2}, \frac{1}{2}; \frac{d+1}{2}; \sin^2 \beta\Big) \\ &\qquad + \cos^2 \theta F\Big(\frac{d-1}{2}, \frac{3}{2}; \frac{d+1}{2}; \sin^2 \beta\Big)\Big], \end{aligned} \tag{8}$$

where $\alpha = \angle(x,a)$, $\beta = \angle(x,n)$ and $\theta = \angle(m, p(a|x^\perp))$. Here, $m = \pi(n|x^\perp) := p(n|x^\perp)/|p(n|x^\perp)|$. Note that in the case where x is a multiple of a, θ is not well-defined. Then, (8) should be understood as

$$Q_{d-1}(x,n,\operatorname{span}\{a\}) = 0.$$

In the next section, we address the remaining cases where $1 < q < d-1$. Let us end this section by giving the precise definition of the function $\mathcal{G}$ which enters into Q_j. For this purpose, we let for $p \le d$ and $x_1,\dots,x_p \in \mathbb{R}^d$ $P(x_1,\dots,x_p)$ be the parallelotope spanned by $x_1,\dots,x_p$,

$$P(x_1,\dots,x_p) = \{\lambda_1 x_1 + \dots + \lambda_p x_p : 0 \le \lambda_i \le 1, i = 1,\dots,p\}.$$

We let

$$\nabla_p(x_1,\dots,x_p) = \mathcal{H}^p(P(x_1,\dots,x_p)).$$

Definition 1 (cf. [9] p. 532)**.** Let $L_p \in \mathcal{L}_p^d$ and $L_q \in \mathcal{L}_q^d$. Choose an orthonormal basis of $L_p \cap L_q$ and extend it to an orthonormal basis of L_p and an orthonormal basis of L_q. Then, $\mathcal{G}(L_p,L_q)$ is the d-dimensional volume of the parallelotope spanned by these vectors. □

It follows from Definition 1 that if $\dim(L_p + L_q) < d$ then

$$\mathcal{G}(L_p,L_q) = 0.$$

In the case $\dim(L_p + L_q) = d$ and either $p = 0$ or $q = 0$, $\mathcal{G}(L_p,L_q) = 1$. Finally, if $\dim(L_p + L_q) = d$ and $0 < p,q < d$, we can choose orthonormal bases for

$$\begin{aligned} L_p \cap L_q &: a_1,\dots,a_{p+q-d} \\ L_p \cap (L_p \cap L_q)^\perp &: b_1,\dots,b_{d-q} \\ L_q \cap (L_p \cap L_q)^\perp &: c_1,\dots,c_{d-p}. \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{G}(L_p,L_q) &= \nabla_d\left(a_1,\dots,a_{p+q-d},b_1,\dots,b_{d-q},c_1,\dots,c_{d-p}\right) \\ &= \nabla_{d-q}\left(p(b_1|L_q^\perp),\dots,p(b_{d-q}|L_q^\perp)\right) \\ &= \nabla_{d-p}\left(p(c_1|L_p^\perp),\dots,p(c_{d-p}|L_p^\perp)\right), \end{aligned}$$

cf. [3, Proposition 2.13 and 2.14].

4 Closed form of Q_j

We will now derive a closed form of $Q_j(x,n,A_q)$ valid for $1 < q < d-1$. Possible values for j are $j = d-q,\dots,d-1$. Note that we must have $d \ge 3$ and $j \ge 2$. We let $\alpha = \angle(x, A_q)$. The angle β and the unit vector m are defined as in the previous section. We shall assume that x lies in a general position with respect to n and A_q, so that $\alpha,\beta \in (0,\pi/2)$.

We first show the following lemma. We use here and in the following the notation $\mathcal{L}_r^s(M)$ for the set of r-dimensional linear subspaces contained in $M \in \mathcal{L}_s^d$.

Lemma 2. *Let $A_q \in \mathcal{L}_q^{d-1}(n^\perp)$, where $\mathcal{L}_q^{d-1}(n^\perp)$ is the set of q-subspaces contained in $n^\perp$. Let $L_j = L_{j-1} \oplus \operatorname{span}\{x\}$, where $L_{j-1} \in \mathcal{L}_{j-1}^{d-1}(x^\perp)$. Then,*

$$\mathcal{G}(L_j, A_q)^2 = \sin^2\alpha\, \mathcal{G}^{(x^\perp)}(L_{j-1}, p(A_q|x^\perp))^2 + \cos^2\alpha\, \mathcal{G}^{(x^\perp)}(L_{j-1}, A_q \cap x^\perp)^2, \tag{9}$$

where the upper index of $\mathcal{G}^{(x^\perp)}$ indicates that the function $\mathcal{G}$ is considered relatively in $x^\perp$. The second summand vanishes when $j+q=d$.

Proof. Consider first the case $j+q=d$. In this case,

$$\dim(L_{j-1} + A_q \cap x^\perp) < d-1$$

and the second summand of (9) vanishes because

$$\mathcal{G}^{(x^\perp)}(L_{j-1}, A_q \cap x^\perp) = 0.$$

In order to prove (9) in the case $j+q=d$, first notice that if $\dim(L_j + A_q) < d$, then left- and right-hand sides of (9) are both zero. If $\dim(L_j + A_q) = d$, we can proceed as follows. Let $\{a_1, \dots, a_q\}$ be an orthonormal basis of A_q such that $a_1 = \pi(x|A_q)$ and $a_i \perp x$, $i = 2, \dots, q$. Then we have

$$\begin{aligned}
\mathcal{G}(L_j, A_q) &= \nabla_q\big(p(a_1|L_j^\perp), p(a_2|L_j^\perp), \cdots, p(a_q|L_j^\perp)\big)\\
&= \nabla_q\big(p(p(a_1|x^\perp)|L_j^\perp), p(a_2|L_j^\perp), \cdots, p(a_q|L_j^\perp)\big)\\
&= |p(a_1|x^\perp)|\nabla_q\big(p(\pi(a_1|x^\perp)|L_j^\perp), p(a_2|L_j^\perp), \cdots, p(a_q|L_j^\perp)\big)\\
&= |p(a_1|x^\perp)|\nabla_q\big(p(\pi(a_1|x^\perp)|L_{j-1}^\perp), p(a_2|L_{j-1}^\perp), \cdots, p(a_q|L_{j-1}^\perp)\big)\\
&= |\sin\angle(x, A_q)|\mathcal{G}^{(x^\perp)}\big(L_{j-1}, p(A_q|x^\perp)\big).
\end{aligned}$$

Let now $j+q>d$ and choose an orthonormal basis $\{u_1, \dots, u_{j-1}\}$ of L_{j-1}. Given an index set $I \subseteq \{1, \dots, j-1\}$, we shall write L_I for the linear hull of $\{u_i \mid i \in I\}$. We have by [4, Lemma 1],

$$\mathcal{G}(L_j, A_q)^2 = \sum_{|I|=d-q} \mathcal{G}(L_I, A_q)^2 + \sum_{|I|=d-q-1} \mathcal{G}(L_I + \operatorname{span}\{x\}, A_q)^2.$$

By applying the identity [3, Proposition 5.1]

$$\mathcal{G}(L_I, A_q) = \cos\angle(x, A_q)\mathcal{G}^{(x^\perp)}(L_I, A_q \cap x^\perp)$$

to each summand in the first sum and by repeating the above procedure from the case $q+j=d$ to each summand of the second sum, we obtain

$$\begin{aligned}
&\mathcal{G}(L_j, A_q)^2\\
&\quad= \sum_{|I|=d-q} \cos^2\angle(x, A_q)\mathcal{G}^{(x^\perp)}(L_I, A_q \cap x^\perp)^2\\
&\qquad+ \sum_{|I|=d-q-1} \sin^2\angle(x, A_q)\mathcal{G}^{(x^\perp)}(L_I, p(A_q|x^\perp))^2\\
&\quad= \cos^2\angle(x, A_q)\mathcal{G}^{(x^\perp)}(L_{j-1}, A_q \cap x^\perp)^2 + \sin^2\angle(x, A_q)\mathcal{G}^{(x^\perp)}(L_{j-1}, p(A_{j-1}|x^\perp))^2.
\end{aligned}$$

□

Let $B_p \in \mathcal{L}_p^{d-1}(x^\perp)$. Define

$$I_{j-1}^{d-1}(m, B_p) = \int_{\mathcal{L}_{j-1}^{d-1}} f(\cos^2 \angle(m, L_{j-1}))\mathcal{G}^{(x^\perp)}(L_{j-1}, B_p)^2 \mathrm{d}L_{j-1}^{d-1},$$

where

$$f(z) = (\cos^2 \beta + z \sin^2 \beta)^{-\frac{d-q}{2}}.$$

Using Lemma 2, we have by definition of Q_j

$$Q_j(x, n, A_q) = \sin^2 \alpha I_{j-1}^{d-1}(m, p(A_q|x^\perp)) + \cos^2 \alpha I_{j-1}^{d-1}(m, A_q \cap x^\perp). \tag{10}$$

Note that the second term vanishes when $j + q = d$. In the next lemma, we give a useful expression for $I_{j-1}^{d-1}(m, B_p)$ in terms of an integral over a half-sphere that can be used for $B_p = p(A_q|x^\perp)$ and $B_p = A_q \cap x^\perp$.

Lemma 3. *Let $m \in S^{d-2}(x^\perp)$, the unit sphere in $x^\perp$, $B_p \in \mathcal{L}_p^{d-1}(x^\perp)$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ measurable. Then,*

$$I_{j-1}^{d-1}(m, B_p) = c_{d-3,j-2} \int_{S^{d-2}(x^\perp)\cap m^+} \frac{f(\cos^2 \angle(v, m))}{\tan^{j-2} \angle(v, m)} J(v)\mathcal{H}^{d-2}(\mathrm{d}v),$$

where $m^+ = \{x \in \mathbb{R}^d \mid x \cdot m > 0\}$,

$$J(v) = \cos^2 \angle(u, B_p)\big(k_{j-2,p-1}^{d-3} \sin^2 \angle(v, B_p \cap u^\perp) + k_{j-2,p-2}^{d-3} \cos^2 \angle(v, B_p \cap u^\perp)\big),$$

$u = \pi(m|v^\perp)$ and $k_{i,j}^d = \frac{i!j!}{(i+j-d)!d!}$ if $i + j \geq d$ and 0 otherwise. Note that the second term in $J(v)$ vanishes whenever $d = j + p$.

Proof. We apply the coarea formula for the mapping $g : L_{j-1} \mapsto \pi(m|L_{j-1})$ defined on $\mathcal{L}_{j-1}^{d-1} \cap \{L : m \not\perp L\}$ with Jacobian $J_{j-2}g(L_{j-1}) = \tan^{j-2} \angle(m, L_{j-1})$ (cf. [5, Lemma 4.2]). Using that

$$g^{-1}(v) = \{L_{j-2} \oplus \mathrm{span}\{v\} \mid L_{j-2} \in \mathcal{L}_{j-2}^{d-3}(x^\perp \cap v^\perp \cap m^\perp)\},$$

we get

$$\begin{aligned}
&I_{j-1}^{d-1}(m, B_p)\\
&\quad= \int_{\mathcal{L}_{j-1}^{d-1}} f(\cos^2 \angle(m, L_{j-1}))\mathcal{G}^{(x^\perp)}(L_{j-1}, B_p)^2 \mathrm{d}L_{j-1}^{d-1}\\
&\quad= \int_{S^{d-2}(x^\perp)\cap m^+} \int_{g^{-1}(v)} \frac{f(\cos^2 \angle(m, L_{j-1}))\mathcal{G}^{(x^\perp)}(L_{j-1}, B_p)^2}{J_{j-2}g(L_{j-1})} \mathrm{d}L_{j-2}^{d-3}\mathcal{H}^{d-2}(\mathrm{d}v)\\
&\quad= \int_{S^{d-2}(x^\perp)\cap m^+} \frac{f(\cos^2 \angle(m, v))}{\tan^{j-2} \angle(m, v)} \int_{g^{-1}(v)} \mathcal{G}^{(x^\perp)}(L_{j-1}, B_p)^2 \mathrm{d}L_{j-2}^{d-3}\mathcal{H}^{d-2}(\mathrm{d}v)\\
&\quad= \int_{S^{d-2}(x^\perp)\cap m^+} \frac{f(\cos^2 \angle(m, v))}{\tan^{j-2} \angle(m, v)} \int_{\mathcal{L}_{j-2}^{d-3}} \mathcal{G}^{(x^\perp)}(L_{j-2} \oplus \mathrm{span}\{v\}, B_p)^2 \mathrm{d}L_{j-2}^{d-3}\mathcal{H}^{d-2}(\mathrm{d}v).
\end{aligned}$$

It is enough to show that the inner integral is equal to $c_{d-3,j-2}$ times the Jacobian $J(v)$ in the lemma. Using [5, Lemma 4.1], we can apply the decomposition

$$\mathcal{G}^{(x^\perp)}(L_{j-1}, B_p)^2 = \cos^2 \angle(u, B_p)\mathcal{G}^{(x^\perp \cap u^\perp)}(L_{j-1}, B_p \cap u^\perp)^2.$$

Apply Lemma 2 to $\mathcal{G}^{(x^\perp \cap u^\perp)}(L_{j-1}, B_p \cap u^\perp)^2$, we get

$$\begin{aligned}
&\mathcal{G}^{(x^\perp \cap u^\perp)}(L_{j-1}, B_p \cap u^\perp)^2 \\
&\quad = \sin^2 \angle(v, B_p \cap u^\perp)\mathcal{G}^{(x^\perp \cap u^\perp \cap v^\perp)}(L_{j-2}, p(B_p \cap u^\perp | v^\perp))^2 \\
&\qquad + \cos^2 \angle(v, B_p \cap u^\perp)\mathcal{G}^{(x^\perp \cap u^\perp \cap v^\perp)}(L_{j-2}, B_p \cap u^\perp \cap v^\perp)^2.
\end{aligned}$$

Note that the second term vanishes when $d = p + j$. By integrating over $\mathcal{L}_{j-2}^{d-3}$ and using the identity

$$\int_{\mathcal{L}_i^d} \mathcal{G}(L_i, L_j)^2 \mathrm{d}L_i^d = k_{ij}^d c(d, i),$$

(cf. [5, Lemma 4.3]), we finally obtain the expression for $J(v)$. □

Using Lemma 3, it is possible to express I_{j-1}^{d-1} in terms of hypergeometric functions. The somewhat lengthy proof is deferred to the Appendix. The result is formulated in the lemma below.

Lemma 4. *Let the situation be as in Lemma 3. Then,*

$$\begin{aligned}
I_{j-1}^{d-1}(m, B_p) = &\frac{1}{p}\varsigma(j+1, p+1, d+2) \\
&\times \Bigg[\left(p - (d-1)\cos^2\theta\right) F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2\beta\right) \\
&\qquad + (d-1)\cos^2\theta F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right)\Bigg],
\end{aligned}$$

where $\varsigma(j, p, d) = k_{j-2,p-1}^{d-3} c_{d-3,j-2}$ *and* $\theta = \angle(m, B_p)$. □

We are now ready to formulate and prove the main result. It turns out that the result also holds for $q = 1$, $d = 1$, see below the proof of Theorem 5.

Theorem 5. *Let* $q = 1, \ldots, d-1$ *and* $j = d-q, \ldots, d-1$. *Furthermore, let* $x \in \mathbb{R}^d \setminus \{0\}$, $n \in S^{d-1}$ *and let* $A_q \in \mathcal{L}_q^{d-1}(n^\perp)$. *Let* $\alpha = \angle(x, A_q)$, $\beta = \angle(x, n)$ *and* $\cos\theta = \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}$. *Suppose that* $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$. *Then,*

$$\begin{aligned}
Q_j(x, n, A_q) = &\frac{(d-j)}{q}\varsigma(j+1, q+1, d+2) \\
&\times \Bigg\{ \sin^2\alpha \Bigg[\sin^2\theta F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2\beta\right) \\
&\qquad + \cos^2\theta F\left(\frac{d-q}{2}, \frac{d-j+2}{2}; \frac{d+1}{2}; \sin^2\beta\right)\Bigg] \\
&\quad + \frac{j+q-d}{d-j} F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2\beta\right)\Bigg\}.
\end{aligned}$$

Proof. We use the form of $Q_j(x,n,A_q)$ given in (10). In the first summand in (10) we have a factor of the form $I^{d-1}_{j-1}(m,B_p)$ with $p=\dim p(A_q|x^\perp)=q$. We need to determine $\angle(m,p(A_q|x^\perp))$. Since

$$m=\frac{n-p(n|x)}{|p(n|x^\perp)|}=\frac{1}{\sin\beta}\left(n-\frac{\cos\beta}{|x|}x\right),$$

we have

$$p\big(m|p(A_q|x^\perp)\big)=\frac{1}{\sin\beta}p\big(n|p(A_q|x^\perp)\big).$$

By using the decomposition $A_q=\operatorname{span}\{\pi(x|A_q)\}\oplus(A_q\cap x^\perp)$ and that $n\perp A_q$, we get

$$p\big(n|p(A_q|x^\perp)\big)=p\big(n|\pi(\pi(x|A_q)|x^\perp)\big),$$

where

$$\pi(\pi(x|A_q)|x^\perp)=\frac{\pi(x|A_q)-p(\pi(x|A_q)|x)}{|p(\pi(x|A_q)|x^\perp)|}=\frac{1}{\sin\alpha}\left(\pi(x|A_q)-\cos\alpha\frac{x}{|x|}\right).$$

Since $n\perp\pi(x|A_q)$, we obtain

$$\begin{aligned}\cos\theta=|p(m|p(A_q|x^\perp))|&=\frac{|\pi(\pi(x|A_q)|x^\perp)\cdot n|}{\sin\beta}\\&=\frac{1}{\sin\alpha\sin\beta}\left(\cos\alpha\frac{x\cdot n}{|x|}\right)=\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}.\end{aligned}$$

In the second summand we have a similar factor with $p=\dim(A_q\cap x^\perp)=q-1$ and $\theta=\frac{\pi}{2}$, i.e $\cos\theta=0$. Lemma 4 together with the identity

$$\varsigma(j+1,q,d+2)=\frac{j+q-d}{q}\varsigma(j+1,q+1,d+2)$$

implies

$$\begin{aligned}Q_j(x,n,A_q)=&\frac{\varsigma(j+1,q+1,d+2)}{q}\\&\times\Bigg\{\sin^2\alpha\bigg[(q-(d-1)\cos^2\theta)F\left(\frac{d-q}{2},\frac{d-j}{2};\frac{d+1}{2};\sin^2\beta\right)\\&\qquad+(d-1)\cos^2\theta F\left(\frac{d-q}{2},\frac{d-j}{2};\frac{d-1}{2};\sin^2\beta\right)\bigg]\\&\quad+\cos^2\alpha(j+q-d)F\left(\frac{d-q}{2},\frac{d-j}{2};\frac{d+1}{2};\sin^2\beta\right)\Bigg\},\end{aligned}\tag{11}$$

where $\cos\theta=\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}$. The result now follows by using (5). $\square$

Note that in case $q=1$ and $j=d-1$, the equation in Proposition 5 reduces to (8). When $q=d-1$, we have $\angle(x,A_q)=\frac{\pi}{2}-\angle(x,n)$; hence, $\cos\alpha=\sin\beta$ and $\cos\theta=1$. Then, by applying (6) and (11) and using the identity $\varsigma(j+1,d,d+2)=c_{d-1,j-1}$, we obtain (7).

5 The sum of Q_j

The resulting expression for the rotational average, obtained by combining (2), (3) and one of the expressions for Q_j, is quite evolved. In this section, we discuss simplified expression for the integrand of (2), under additional assumptions. First, we derive a result for the sum of Q_j.

Proposition 6. *Let the situation be as in Theorem 5. Then, for* $0 \le k < j < d$,

$$\sum_{\substack{I \subseteq \{1,\ldots,d-1\} \\ |I| = j-1-k}} Q_j(x, n, A_I(x)) = c_{d-1,j-1} \binom{j-1}{k} F\left(\frac{j-k-2}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right).$$

Proof. The sum of the Q_j-terms can be determined, using Theorem 5. Recall that

$$A_I(x) = \operatorname{span}\{a_i(x) | i \notin I\}.$$

If we let $\alpha_I = \angle(x, A_I(x))$ and $q = d - j + k$, we find

$$\begin{aligned}
\sum_{|I|=j-1-k} \cos^2 \alpha_I &= \sum_{|I|=j-1-k} |p(x|A_I(x))|^2 = \sum_{|I|=j-1-k} \sum_{i \notin I} (x \cdot a_i(x))^2 \\
&= \sum_{|I|=q} \sum_{i \in I} (x \cdot a_i(x))^2 = \sum_{i=1}^{d-1} \binom{d-2}{q-1} (x \cdot a_i(x))^2 \\
&= \binom{d-2}{q-1} \left| p(x| \operatorname{span}\{a_1(x), \ldots, a_{d-1}(x)\} \right|^2 = \binom{d-2}{q-1} |p(x|n^\perp)|^2 \\
&= \binom{d-2}{q-1} \sin^2\beta
\end{aligned}$$

and

$$\sum_{|I|=j-1-k} \sin^2 \alpha_I = \binom{d-1}{q} - \binom{d-2}{q-1} \sin^2 \beta = \binom{d-2}{q-1} \left(\frac{d-1}{q} - \sin^2\beta\right).$$

By using (5), (6) and the relation $\frac{d-1}{q}\binom{d-2}{q-1}\varsigma(j+1, q+1, d+2) = \binom{j-1}{d-q-1} c_{d-1,j-1}$, we arrive at the following formula

$$\begin{aligned}
&\sum_{|I|=j-1-k} Q(x, n, A_I(x)) \\
&= \frac{\varsigma(j+1, q+1, d+2)}{q} \binom{d-2}{q-1} \left[(j-1) \sin^2 \beta F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2\beta\right) \right. \\
&\qquad \left. + (d-1) \cos^2 \beta F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right) \right] \\
&= c_{d-1,j-1} \binom{j-1}{d-q-1} F\left(\frac{d-q-2}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right) \\
&= c_{d-1,j-1} \binom{j-1}{k} F\left(\frac{j-k-2}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right).
\end{aligned}$$

□

In case $q = 1$ and $j = d-1$, the expression above reduces to the one in [4], namely

$$\sum_{i=1}^{d-1} Q(x,n,a_i(x)) = c_{d-1,d-2} F\left(\frac{d-3}{2}, \frac{1}{2}; \frac{d-1}{2}; \sin^2\beta\right).$$

By using the series expansion of the hypergeometric function, the first order approximation of $\sum Q$ becomes

$$\begin{aligned}\sum_{|I|=j-1-k} Q(x,n,A_I(x)) &\approx \binom{j-1}{d-q-1}\left(1+\frac{(d-q-2)(d-j)}{2(d-1)}\sin^2\beta\right) c_{d-1,j-1} \\ &= \binom{j-1}{k}\left(1+\frac{(j-k-2)(d-j)}{2(d-1)}\sin^2\beta\right) c_{d-1,j-1}.\end{aligned}$$

We can use Proposition 6 to simplify the integrand of (2) for $x \in \partial X$ satisfying

$$\kappa_i(x) = \kappa(x), i = 1, \dots, d-1.$$

We find

$$\begin{aligned}\sum_{\substack{I \subseteq \{1,\dots,d-1\} \\ |I|=j-1-k}} w_{I,j}(x) \prod_{i \in I} \kappa_i(x) &= \frac{\binom{j-1}{k} c_{d-1,j-1}}{\sigma_{j-k}} \frac{1}{|x|^{d-j}} \kappa(x)^{j-1-k} \\ &\times F\left(\frac{j-k-2}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right).\end{aligned}$$

In particular, if B^d is the unit ball in $\mathbb{R}^d$, we get

$$\int_{\mathcal{L}_j^d} V_k(B^d \cap L_j) \mathrm{d}L_j^d = \binom{j-1}{k} c_{d-1,j-1} \frac{\sigma_d}{\sigma_{j-k}}.$$

This result can also be derived directly as follows

$$\begin{aligned}\int_{\mathcal{L}_j^d} V_k(B^d \cap L_j) \mathrm{d}L_j^d &= c_{d,j} V_k(B^j) \\ &= c_{d,j} \binom{j}{k} \frac{(j-k)}{j} \frac{\sigma_j}{\sigma_{j-k}} \\ &= \binom{j-1}{k} c_{d-1,j-1} \frac{\sigma_d}{\sigma_{j-k}}.\end{aligned}$$

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Appendix: Proof of Lemma 4

In the proof of Lemma 4, we will utilize the following integral equation

$$\begin{aligned}\int_0^\infty \left(\cos^2\beta + \frac{\sin^2\beta}{1+r^2}\right)^{-a} (r^2)^b (1+r^2)^{-c} \mathrm{d}r \\ = \frac{1}{2} B\left(b+\frac{1}{2}, c-\left(b+\frac{1}{2}\right)\right) F\left(a, b+\frac{1}{2}; c; \sin^2\beta\right),\end{aligned} \tag{12}$$

valid for $a,b,c\in\mathbb{R}_+$ whenever $0<b+\frac{1}{2}<c$ and $\beta\in\left(0,\frac{\pi}{2}\right)$. When $\beta=\frac{\pi}{2}$, the extra assumption $c-(a+b+\frac{1}{2})>0$ is necessary.

Let the situation be as in Lemma 3. If we let $\gamma=\angle(v,m)\in[0,\pi]$ be the angle between $\operatorname{span}(m)$ and $\operatorname{span}(v)$, we may write $m=p(m|v)+p(m|v^\perp)=v\cos\gamma+u\sin\gamma$, hence

$$|p(v|B_p\cap u^\perp)| = \frac{|p(m|B_p\cap u^\perp)|}{\cos\gamma}.$$

Consequently, by using Lemma 3,

$$\begin{aligned}\frac{I_{j-1}^{d-1}(m,B_p)}{\varsigma(j,p,d)} = \int_{S^{d-2}(x^\perp)\cap m^+} \frac{|p(u|B_p)|^2 f(\cos^2\gamma)}{\tan^{j-2}\gamma} \\ \times\left(1-\frac{d-j-1}{p-1}|p(m|B_p\cap u^\perp)|^2\cos^{-2}\gamma\right)\mathcal{H}^{d-2}(\mathrm{d}v),\end{aligned}$$

where $\varsigma(j,p,d)=k_{j-2,p-1}^{d-3}c_{d-3,j-2}$. We shall use the area formula with $\psi(v)=\pi(m|v^\perp)=u$ defined on $S^{d-2}(x^\perp\cap m^+)\setminus\operatorname{span}(m)$. Since ψ is bijective with Jacobian $J_{d-2}\psi(v)=\tan^{d-3}\angle(m,v^\perp)=\tan^{-(d-3)}\angle(m,v)$ and $\zeta=\angle(u,m)=\frac{\pi}{2}-\gamma$, the area formula implies that

$$\begin{aligned}\frac{I_{j-1}^{d-1}(m,B_p)}{\varsigma(j,p,d)} = \int_{S^{d-2}(x^\perp)\cap m^+} \frac{|p(u|B_p)|^2 f(\sin^2\zeta)}{\tan^{d-j-1}\zeta} \\ \times\left(1-\frac{d-j-1}{p-1}|p(m|B_p\cap u^\perp)|^2\sin^{-2}\zeta\right)\mathcal{H}^{d-2}(\mathrm{d}u),\end{aligned}$$

where $f(z)=(\cos^2\beta+z\sin^2\beta)^{-\frac{d-q}{2}}$. Hence,

$$I_{j-1}^{d-1}(m,B_p)=\varsigma(j,p,d)\left(K_1-\frac{d-j-1}{p-1}K_2\right), \tag{13}$$

where

$$K_1=\int_{S^{d-2}(x^\perp)\cap m^+}\frac{|p(u|B_p)|^2 f(\sin^2\zeta)}{\tan^{d-j-1}\zeta}\mathcal{H}^{d-2}(\mathrm{d}u) \tag{14}$$

and

$$K_2=\int_{S^{d-2}(x^\perp)\cap m^+}\frac{|p(m|B_p\cap u^\perp)|^2|p(u|B_p)|^2 f(\sin^2\zeta)\cos^{d-j-1}\zeta}{\sin^{d-j+1}\zeta}\mathcal{H}^{d-2}(\mathrm{d}u). \tag{15}$$

Using the coarea formula with $\varphi : (S^{d-2}(x^\perp) \cap m^+) \setminus \mathrm{span}(m) \to S^{d-3}(x^\perp \cap m^\perp)$ defined by $\varphi(u) = \pi(u|m^\perp) = u_0$ and with $J_{d-3}\varphi(u) = (\sin\angle(u,m))^{-(d-3)}$, we obtain (note: $m^+ = \{x \in \mathbb{R}^d | x \cdot m > 0\}$)

$$\begin{aligned}K_1 &= \int_{S^{d-3}(x^\perp\cap m^\perp)} \int_{\varphi^{-1}(u_0)} \frac{|p(u|B_p)|^2 f(\sin^2\angle(u,m))}{\tan^{d-j-1}\angle(u,m)} J^{-1}_{d-3}\varphi(u)\mathcal{H}^1(\mathrm{d}u)\mathcal{H}^{d-3}(\mathrm{d}u_0)\\ &= \int_{S^{d-3}(x^\perp\cap m^\perp)} \int_{\varphi^{-1}(u_0)} |p(u|B_p)|^2 f(\sin^2\angle(u,m))\\ &\qquad\qquad \times \cos^{d-j-1}\angle(u,m)\sin^{j-2}\angle(u,m)\mathcal{H}^1(\mathrm{d}u)\mathcal{H}^{d-3}(\mathrm{d}u_0).\end{aligned}$$

Define $\xi : \mathbb{R}_+ \to \varphi^{-1}(u_0)$ by $\xi(r) = \frac{u_0+rm}{|u_0+rm|} = u$ with $J_1\xi(r) = \frac{1}{1+r^2}$. The area formula implies

$$\begin{aligned}K_1 &= \int_{S^{d-3}(x^\perp\cap m^\perp)} \int_0^\infty |p(\xi(r)|B_p)|^2 f(\sin^2\angle(\xi(r),m))\\ &\qquad\qquad \times \cos^{d-j-1}\angle(\xi(r),m)\sin^{j-2}\angle(\xi(r),m)J_1\xi(r)\mathrm{d}r\mathcal{H}^{d-3}(\mathrm{d}u_0).\end{aligned}$$

We now use that $\sin^2\angle(\xi(r),m) = \frac{1}{1+r^2}$ and

$$|p(\xi(r)|B_p)|^2 = \frac{|p(u_0|B_p)|^2 + r^2|p(m|B_p)|^2 + 2rp(u_0|B_p)\cdot p(m|B_p)}{1+r^2},$$

which, in combination with the equality

$$\int_{S^{d-3}(x^\perp\cap m^\perp)} p(u_0|B_p)\cdot p(m|B_p)\mathcal{H}^{d-3}(du_0) = 0$$

and (12), lead us to the following expression

$$\begin{aligned}K_1 &= \int_{S^{d-3}(x^\perp\cap m^\perp)} \int_0^\infty \frac{(|p(u_0|B_p)|^2 + r^2|p(m|B_p)|^2) f(\frac{1}{1+r^2})(r^2)^{\frac{d-j-1}{2}}}{(1+r^2)^{\frac{d+1}{2}}}\mathrm{d}r\mathcal{H}^{d-3}(\mathrm{d}u_0)\\ &= \frac{1}{2}B\left(\frac{d-j}{2},\frac{j+1}{2}\right)F\left(\frac{d-q}{2},\frac{d-j}{2};\frac{d+1}{2};\sin^2\beta\right)H_1\\ &\quad + \frac{1}{2}B\left(\frac{d-j+2}{2},\frac{j-1}{2}\right)F\left(\frac{d-q}{2},\frac{d-j+2}{2};\frac{d+1}{2};\sin^2\beta\right)|p(m|B_p)|^2\sigma_{d-2},\end{aligned}$$

with $H_1 = \int_{S^{d-3}(x^\perp\cap m^\perp)} |p(u_0|B_p)|^2\mathcal{H}^{d-3}(\mathrm{d}u_0)$. The convergence criteria in (12) are satisfied since $1 < j < d$ and $0 < \beta < \frac{\pi}{2}$ by assumption. Note that the differences between K_1 and K_2 are the extra terms $\sin^2\zeta$ and

$$|p(m|B_p\cap u^\perp)|^2 = |p(m|B_p)|^2\frac{|p(u|B_p\cap m^\perp)|^2}{|p(u|B_p)|^2}.$$

Hence, K_2 can be rewritten as

$$K_2 = |p(m|B_p)|^2\int_{S^{d-2}(x^\perp)\cap m^+} \frac{f(\sin^2\zeta)|p(u|B_p\cap m^\perp)|^2\cos^{d-j-1}\zeta}{\sin^{d-j+1}\zeta}\mathcal{H}^{d-2}(\mathrm{d}u).$$

By applying the area formula for the mappings $\varphi : u \mapsto \pi(u|m^\perp)$ and $\xi : r \mapsto \frac{u_0+rm}{|u_0+rm|}$, the integral above becomes

$$\int_{S^{d-3}(x^\perp\cap m^\perp)}\int_{\varphi^{-1}(u_0)} f(\sin^2\zeta)|p(u|B_p\cap m^\perp)|^2\cos^{d-j-1}\zeta\sin^{j-4}\zeta\mathcal{H}^1(\mathrm{d}u)\mathcal{H}^{d-3}(\mathrm{d}u_0)$$
$$= \int_{S^{d-3}(x^\perp\cap m^\perp)}|p(u_0|B_p\cap m^\perp)|^2\int_0^\infty \frac{f\left(\frac{1}{1+r^2}\right)(r^2)^{\frac{d-j-1}{2}}}{(1+r^2)^{\frac{d-1}{2}}}\mathrm{d}r\mathcal{H}^{d-3}(\mathrm{d}u_0),$$

where we used $|p(\xi(r)|B_p\cap m^\perp)|^2 = \frac{|p(u_0|B_p\cap m^\perp)|^2}{1+r^2}$ for the last equality. Using (12), we obtain

$$K_2 = \frac{|p(m|B_p)|^2}{2}B\left(\frac{d-j}{2},\frac{j-1}{2}\right)F\left(\frac{d-q}{2},\frac{d-j}{2};\frac{d-1}{2};\sin^2\beta\right)H_2$$

with $H_2 = \int_{S^{d-3}(x^\perp\cap m^\perp)}\left|p(u_0|B_p\cap m^\perp)\right|^2\mathcal{H}^{d-3}(\mathrm{d}u_0)$. It remains to calculate the two integrals H_1 and H_2.

Define $\psi : S^{d-3}(x^\perp\cap m^\perp)\to S^{p-1}(B_p)$ by $\psi(u_0) = \pi(u_0|B_p) = u_1$ with

$$J_{p-1}\psi(u_0) = \frac{(\sin^2\theta+\cos^2\theta\cos^2\delta(u_1))^{\frac{1}{2}}}{|p(u_0|B_p)|^{p-1}}.$$

Here, $u_1 = \pi(u_0|B_p)$, $m_1 = \pi(m|B_p)$, $\delta = \delta(u_1) = \angle(u_1, m_1)$ and $\theta = \angle(m, B_p)$. The area formula gives us

$$H_1 = \int_{S^{p-1}(B_p)}\int_{\psi^{-1}(u_1)}|p(u_0|B_p)|^{p+1}\mathcal{H}^{d-p-2}(\mathrm{d}u_0)\frac{\mathcal{H}^{p-1}(\mathrm{d}u_1)}{(\sin^2\theta+\cos^2\theta\cos^2\delta)^{\frac{1}{2}}}.$$

Define $\zeta : m^\perp\cap B_p^\perp\cap x^\perp\to\psi^{-1}(u_1)$ by

$$\zeta(\omega) = \frac{(\sin\theta)u_1+(\cos\theta\cos\delta)m_2+\omega}{|(\sin\theta)u_1+(\cos\theta\cos\delta)m_2+\omega|},\qquad \omega\in m^\perp\cap B_p^\perp\cap x^\perp,$$

where $m_2 = \pi(m|B_p^\perp)$. The Jacobian of ζ is

$$J\zeta(\omega) = \frac{(\sin^2\theta+\cos^2\theta\cos^2\delta)^{\frac{1}{2}}}{(\sin^2\theta+\cos^2\theta\cos^2\delta+|\omega|^2)^{\frac{d-p-1}{2}}}.$$

Thus, by using the fact that

$$|p(\zeta(\omega)|B_p)| = \frac{\sin\theta}{|(\sin\theta)u_1+(\cos\theta\cos\delta)m_2+\omega|},$$

the area formula implies

$$\int_{\psi^{-1}(u_1)}|p(u_0|B_p)|^{p+1}\mathcal{H}^{d-p-2}(\mathrm{d}u_0)$$
$$= \int_{m^\perp\cap B_p^\perp\cap x^\perp}\frac{\sin^{p+1}\theta}{(\sin^2\theta+\cos^2\theta\cos^2\delta+|\omega|^2)^{\frac{p+1}{2}}}J\zeta(\omega)\mathcal{H}^{d-p-2}(\mathrm{d}\omega)$$
$$= (\sin^2\theta+\cos^2\theta\cos^2\delta)^{\frac{1}{2}}\int_{m^\perp\cap B_p^\perp\cap x^\perp}\frac{\sin^{p+1}\theta}{(\sin^2\theta+\cos^2\theta\cos^2\delta+|\omega|^2)^{\frac{d}{2}}}\mathcal{H}^{d-p-2}(\mathrm{d}\omega).$$

Hence,

$$\begin{aligned}
H_1 &= \int_{S^{p-1}(B_p)} \int_{m^\perp \cap B_p^\perp \cap x^\perp} \frac{\sin^{p+1}\theta}{(\sin^2\theta + \cos^2\theta\cos^2\delta + |\omega|^2)^{\frac{d}{2}}} \mathcal{H}^{d-p-2}(\mathrm{d}\omega)\mathcal{H}^{p-1}(\mathrm{d}u_1) \\
&= \sigma_{d-p-2} \int_{S^{p-1}(B_p)} \int_0^\infty \frac{\sin^{p+1}\theta}{(\sin^2\theta + \cos^2\theta\cos^2\delta + r^2)^{\frac{d}{2}}} r^{d-p-3}\mathrm{d}r\mathcal{H}^{p-1}(\mathrm{d}u_1) \\
&= \sigma_{d-p-2} b_{d-p-3,d} \sin^{p+1}\theta \int_{S^{p-1}(B_p)} \frac{1}{(\sin^2\theta + \cos^2\theta\cos^2\delta)^{\frac{p+2}{2}}} \mathcal{H}^{p-1}(\mathrm{d}u_1),
\end{aligned}$$

where $\sigma_m = \mathcal{H}^{m-1}(S^{m-1}) = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$ and

$$b_{m,n} = \int_0^\infty \frac{t^m}{(1+t^2)^{\frac{n}{2}}} dt = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n-m-1}{2}\right).$$

The last integral can be evaluated after substitution with $t = \sin^2\delta(u_1)$

$$H_1 = p\ \omega_{d-2} \sin^{p+1}\theta F\left(\frac{p+2}{2}, \frac{p-1}{2}; \frac{p}{2}; \cos^2\theta\right) = \omega_{d-2}(p - \cos^2\theta), \tag{16}$$

where $\omega_d = \pi^{\frac{d}{2}}/\Gamma(1+\frac{d}{2})$ is the volume of the unit ball in $\mathbb{R}^d$ and the second equality follows from the relation

$$F\left(\frac{p+2}{2}, \frac{p-1}{2}; \frac{p}{2}; z\right) = \left(1 - \frac{1}{p}z\right)(1-z)^{-\frac{p+1}{2}},$$

whenever $z \neq 1$. The computation of H_2 can be carried out similarly. Define $\psi : S^{d-3}(x^\perp \cap m^\perp) \to S^{p-2}(B_p \cap m^\perp)$ by $\psi(u_0) = \pi(u_0|B_p \cap m^\perp) = u_1$ with $J\psi(u_0) = |p(u_0|B_p \cap m^\perp)|^{-(p-2)}$. The coarea formula implies

$$\begin{aligned}
H_2 &= \int_{S^{p-2}(B_p\cap m^\perp)} \int_{\psi^{-1}(u_1)} |p(u_0|B_p \cap m^\perp)|^2 J\psi(u_0)^{-1} \mathcal{H}^{d-p-1}(\mathrm{d}u_0)\mathcal{H}^{p-2}(\mathrm{d}u_1) \\
&= \int_{S^{p-2}(B_p\cap m^\perp)} \int_{\psi^{-1}(u_1)} |p(u_0|B_p \cap m^\perp)|^p \mathcal{H}^{d-p-1}(\mathrm{d}u_0)\mathcal{H}^{p-2}(\mathrm{d}u_1).
\end{aligned}$$

The inner integral can be calculated using the area formula with $\zeta(\omega) = \frac{u_1+\omega}{|u_1+\omega|}$ defined on $(B_p \cap m^\perp)^\perp \cap m^\perp$ with $J_{d-p-1}\zeta(\omega) = \left(\frac{1}{1+|\omega|^2}\right)^{\frac{d-p}{2}}$. Using the equality $|p(\zeta(\omega)|B_p \cap m^\perp)|^2 = \frac{1}{1+|\omega|^2}$, we obtain

$$\begin{aligned}
&\int_{\psi^{-1}(u_1)} |p(u_0|B_p \cap m^\perp)|^p \mathcal{H}^{d-p-1}(\mathrm{d}u_0) \\
&\quad = \int_{(B_p\cap m^\perp)^\perp \cap m^\perp \cap x^\perp} \left(\frac{1}{1+|\omega|^2}\right)^{\frac{p}{2}} J_{d-p-1}\zeta(\omega)\mathcal{H}^{d-p-1}(\mathrm{d}\omega) \\
&\quad = \int_{(B_p\cap m^\perp)^\perp \cap m^\perp \cap x^\perp} \left(\frac{1}{1+|\omega|^2}\right)^{\frac{d}{2}} \mathcal{H}^{d-p-1}(\mathrm{d}\omega) \\
&\quad = \sigma_{d-p-1} \int_0^\infty \left(\frac{1}{1+r^2}\right)^{\frac{d}{2}} r^{d-p-2}\mathrm{d}r \\
&\quad = \sigma_{d-p-1} b_{d-p-2,d}.
\end{aligned}$$

Hence,

$$H_2 = \sigma_{p-1}\sigma_{d-p-1}b_{d-p-2,d} = \omega_{d-2}(p-1). \tag{17}$$

By inserting (16) into (14) and (17) into (15) we get

$$\begin{aligned}K_1 = {} & \frac{\omega_{d-2}(p-\cos^2\theta)}{2} B\left(\frac{d-j}{2}, \frac{j+1}{2}\right) F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2\beta\right) \\ & + \frac{\sigma_{d-2}\cos^2\theta}{2} B\left(\frac{d-j+2}{2}, \frac{j-1}{2}\right) F\left(\frac{d-q}{2}, \frac{d-j+2}{2}; \frac{d+1}{2}; \sin^2\beta\right),\end{aligned}$$

and

$$K_2 = \frac{\omega_{d-2}(p-1)\cos^2\theta}{2} B\left(\frac{d-j}{2}, \frac{j-1}{2}\right) F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right),$$

which, in combination with (13), implies

$$\begin{aligned}& I_{j-1}^{d-1}(m, B_p) \\ & \quad = \frac{1}{2}\varsigma(j,p,d)\omega_{d-2} B\left(\frac{d-j}{2}, \frac{j-1}{2}\right) \\ & \qquad \times \Bigg[(p-\cos^2\theta) F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2\beta\right) \\ & \qquad\quad + (d-2)\cos^2\theta \frac{d-j}{d-1} F\left(\frac{d-q}{2}, \frac{d-j+2}{2}; \frac{d+1}{2}; \sin^2\beta\right) \\ & \qquad\quad - \frac{(d-j-1)}{p-1}(p-1)\cos^2\theta \frac{j-1}{d-1} F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right)\Bigg].\end{aligned}$$

By using (5) the expression above can be rewritten

$$\begin{aligned}I_{j-1}^{d-1}(m, B_p) = {} & \frac{1}{2}\varsigma(j,p,d)\omega_{d-2} B\left(\frac{d-j}{2}, \frac{j-1}{2}\right) \\ & \times \Bigg[\left(\frac{(j-1)p}{d-1} - \frac{j-1+(d-2)(j-1)}{d-1}\cos^2\theta\right) \\ & \quad \times F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2\beta\right) \\ & \quad + (j-1)\cos^2\theta F\left(\frac{d-q}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right)\Bigg].\end{aligned}$$

Use

$$\frac{1}{p}\varsigma(j+1, p+1, d+2) = \frac{(j-1)\varsigma(j,p,d)\omega_{d-2}B(\frac{d-j}{2}, \frac{j-1}{2})}{2(d-1)}$$

and the proof is complete.

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