UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

ANALYTICITY ESTIMATES FOR THE NAVIER-STOKES EQUATIONS

by I. Herbst and E. Skibsted

Preprint Series No.: 6

July 2009
2009/07/27

Ny Munkegade 118, Bldg. 1530
DK-8000 Aarhus C, Denmark

http://www.imf.au.dk
institut@imf.au.dk

ANALYTICITY ESTIMATES FOR THE NAVIER-STOKES EQUATIONS

I. HERBST
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VA 22904 U.S.A.
IWH@VIRGINIA.EDU

E. SKIBSTED
INSTITUT FOR MATEMATISKE FAG
AARHUS UNIVERSITET
8000 AARHUS C, DENMARK
SKIBSTED@IMF.AU.DK

Abstract. We study spatial analyticity properties of solutions of the Navier-Stokes equation and obtain new growth rate estimates for the analyticity radius. We also study stability properties of strong global solutions of the Navier-Stokes equation with data in $H^r,\ r \geq 1/2$ and prove a stability result for the analyticity radius.

Contents

1. Introduction

We look at the following system of equations in the variables $(t, x) \in [0, \infty[\times \mathbb{R}^3$. The unknown u specifies at each argument a velocity $u = u(t, x) \in \mathbb{R}^3$. The unknown p specifies at each argument a pressure $p = p(t, x) \in \mathbb{R}$.

$$\begin{cases} \frac{\partial}{\partial t} u + (u \cdot \nabla)u + \nabla p = \triangle u \\ \nabla \cdot u = 0 \\ u(t=0) = u_0 \end{cases} . \tag{1.1}$$

We eliminate the pressure in the standard way using the Leray projection P. It is an orthogonal projection on $L^2(\mathbb{R}^3, \mathbb{R}^3)$ which is fibered in Fourier space, i.e.

Date: July 27, 2009.
2000 *Mathematics Subject Classification.* Primary 76D05.

$\widehat{(Pf)}(\xi) = P(\xi)\hat{f}(\xi)$, and it is given according to the following recipe: $P(\xi) = I - |\hat{\xi}\rangle\langle\hat{\xi}|$, $\hat{\xi} := \xi/|\xi|$.

$$\begin{cases} \left(\frac{\partial}{\partial t}u + P(u\cdot\nabla)u - \triangle u\right)(t,\cdot) = 0 \\ u(t) := u(t,\cdot) \in \operatorname{Ran}P \end{cases} . \tag{1.2}$$

We introduce a notion of strong global solution to (1.2) in terms of $A := \sqrt{-\triangle}$. In the following the Sobolev space H^r is the Hilbert space with norm $\|f\|_{H^r} = \|\langle A\rangle^r f\|_{L^2}$ where $\langle x\rangle = \sqrt{1+|x|^2}$.

Definition 1.1. Let $r \geq 1/2$. The set $\mathcal{G}_r$ is the set of $u \in C([0,\infty[,(H^r)^3)$ satisfying:
(1) $u(t) \in P(H^r)^3$ for all $t \geq 0$,
(2) The expression $A^{5/4}u(t)$ defines an element in $C(]0,\infty[,(L^2)^3)$ and

$$\lim_{t\to 0} t^{3/8}\|A^{5/4}u(t)\|_{(L^2)^3} = 0,$$

(3) $u \in C^1(]0,\infty[,\mathcal{S}'(\mathbb{R}^3))$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t}u = -A^2u - P(u\cdot\nabla)u;\ t>0.$$

Here the differentiability in t is meant in the weak* topology and the equation in (3) is meant in the sense of distributions. We refer to any $u \in \mathcal{G}_r$ as a *strong global solution* to the problem (1.2).

1.1. **Discussion of uniform real analyticity.** Fix $r \in \mathbb{R}$ and $f \in H^r$. We say that f is *uniformly H^r real analytic* if there exists $a > 0$ such that the function $\mathbb{R}^3 \ni \eta \to \mathrm{e}^{\mathrm{i}\eta\cdot p}f = f(\cdot+\eta) \in H^r$, $p := -\mathrm{i}\nabla$, extends to an analytic function $\tilde{f}$ on $\{|\operatorname{Im}\eta| < a\}$ and that

$$\sup_{|\operatorname{Im}\eta|<a} \|\tilde{f}(\eta)\|_{H^r} < \infty. \tag{1.3}$$

We define correspondingly the *analyticity radius* of f as

$$\operatorname{rad}(f) = \sup\{a > 0 |\text{ the property (1.3) holds}\}. \tag{1.4}$$

If f is not uniformly H^r real analytic we put $\operatorname{rad}(f) = 0$.

We note that the notions of uniform real analyticity and corresponding analyticity radius are independent of r, and that in fact

$$\operatorname{rad}(f) = \sup\{a \geq 0 |\, \mathrm{e}^{aA}f \in H^r\} = \sup\{a \geq 0 |\, \mathrm{e}^{aA}f \in L^2\}. \tag{1.5}$$

Moreover if $\operatorname{rad}(f) > 0$ then by the Sobolev embedding theorem, $H^s \subseteq L^\infty$ for $s > 3/2$, the function $\mathbb{R}^3 \ni x \to f(x) \in \mathbb{C}$ extends to the analytic function $\breve{f}$ on $\{|\operatorname{Im}\eta| < \operatorname{rad}(f)\}$ given by $\breve{f}(\eta) = \tilde{f}(\eta)(0)$. Conversely suppose a given function $\mathbb{R}^3 \ni x \to f(x) \in \mathbb{C}$ extends to an analytic function $\breve{f}$ on $\{|\operatorname{Im}\eta| < b\}$ and that $\tilde{f}(\eta) := \breve{f}(\cdot+\eta)$ obeys (1.3) for all $a < b$ then $\tilde{f}$ is an analytic H^r-valued function and $\operatorname{rad}(f) \geq b$.

If $f \in \dot{H}^r$ for some $r \in]-\infty, 3/2[$ (see Section 2 for the definition of homogeneous Sobolev spaces) one can introduce similar notions of uniform real analyticity and corresponding analyticity radius (by using (1.3) with $H^r \to \dot{H}^r$ and (1.4), respectively). If $f \in \dot{H}^r$ has a positive analyticity radius then $f = f_1 + f_2$ where f_1 has an entire analytic continuation and f_2 is uniformly H^s real analytic for any s. Note however that the concept of uniform $\dot{H}^r$ analyticity is dependent on r.

In any case $a > 0$ will be a lower bound of the analyticity radius of $f \in H^r$ or $f \in \dot{H}^r$ if $e^{aA} f \in H^r$ or $e^{aA} f \in \dot{H}^r$, respectively.

For $f = (f_1, f_2, f_3) \in (H^r)^3$ we define $\mathrm{rad}(f) = \min_j(\mathrm{rad}(f_j))$.

1.2. **Results on real analyticity of solutions.** It is a basic fact that for all $u \in \mathcal{G}_r$, $r \geq 1/2$,

$$\mathrm{rad}(u(t)) > 0 \text{ for all } t > 0. \tag{1.6}$$

In fact for any such u there exists $\lambda > 0$ such that

$$\mathrm{rad}(u(t)) \geq \lambda\sqrt{t} \text{ for all } t > 0; \tag{1.7}$$

we note that (1.7) is a consequence of Corollary 7.5, Lemma 7.13 and Proposition 4.1.

The main subject of this paper is the study of lower bounds of the quantity to the left in (1.6) (and the analogous question for solutions taking values in homogeneous Sobolev spaces). The study of analyticity of solutions of the Navier-Stokes equations originated with Foias and Temam in [FT] where they studied analyticity of periodic solutions in space and time (see also [FMRT]). Later Grujič and Kukavica [GK] studied space analyticity of the Navier-Stokes equations in R^3. Since then many authors have proven analyticity results. We mention the book by Lemarié-Rieusset [Le] where some results and references can be found.

There are two regimes to study, the small and the large time regimes. One of our main results on large time analyticity bounds is the following (a combination of Theorem 6.1 ii and Lemma 7.13):

Theorem 1.2. *Suppose $r \geq 1/2$ and $u \in \mathcal{G}_r$. Suppose that for some $\sigma \geq 0$ the following bound holds*

$$\|u(t)\|_{(L^2)^3} = O(t^{-\sigma/2}) \textit{ for } t \to \infty. \tag{1.8}$$

Let $0 \leq \tilde{\epsilon} < \epsilon \leq 1$ be given. Then there exist constants $t_0 > 1$ and $C > 0$ such that

$$\|\exp\Big(\sqrt{(1-\epsilon)(2\sigma+1)}\sqrt{t\ln t}A\Big)u(t)\|_{(L^2)^3} \leq Ct^{1/4-\tilde{\epsilon}(2\sigma+1)/4} \textit{ for all } t \geq t_0. \tag{1.9}$$

In particular

$$\liminf_{t\to\infty} \tfrac{\mathrm{rad}(u(t))}{\sqrt{t\ln t}} \geq \sqrt{2\sigma+1}. \tag{1.10}$$

Remarks. 1) For any $u \in \mathcal{G}_r$ the quantity $\|u(t)\|_{(L^2)^3} = o(t^0)$ as $t \to \infty$. In particular (1.8) is valid for $\sigma = 0$. Under some further conditions (partly generic, involving the condition $\int u_{0i}u_{0j}dx \neq c\delta_{ij}, u_0 = u(0)$) it is shown in [Sc1] that for some $C > 0$, $C^{-1}\langle t\rangle^{-5/4} \leq \|u(t)\|_{(L^2)^3} \leq C\langle t\rangle^{-5/4}$. References to many further works on the L^2 decay rate can be found in [Sc1] .

2) Our method of proof works more generally, in particular for the class of compressible flows given by taking $P = I$ in (1.2). In this setting we construct an example for which (1.8)–(1.10) hold with $\sigma = 5/2$ and for which *all* of the bounds (1.8)–(1.10) with $\sigma = 5/2$ are *optimal.* If P is the Leray projection in (1.2) we do not know if the bounds (1.9) and (1.10) are optimal under the decay condition (1.8).

As for the small time regime, we have less complete knowledge. It is a basic fact that for all $u \in \mathcal{G}_r$, $r \geq 1/2$,

$$\lim_{t\to 0} \tfrac{\mathrm{rad}(u(t))}{\sqrt{t}} = \infty. \tag{1.11}$$

This result is valid for any strong solution to (1.2) defined on an interval $I =]0,T]$ only (i.e. $I =]0,\infty[$ is not needed here). This class, $\mathcal{S}_{r,I}$, is introduced in Definition 7.1 in a similar way as the set of strong global solutions to (1.2) is introduced in Definition 1.1. For any $u_0 \in (H^r)^3$, $r \geq 1/2$, there exists a unique strong solution $u \in \mathcal{S}_{r,I}$ with $u(0) = u_0$ to the equations (1.2) provided that $T = |I|$ is small enough (alternatively, a unique small time solution to the initial value problem (1.1)).

One of our main results on small time analyticity bounds is the following (cf. Corollary 3.6):

Theorem 1.3. *Suppose $u_0 \in (H^r)^3$ for some $r \in]1/2, 3/2[$. Let u be the unique small time ($T = |I|$ small) strong solution $u \in \mathcal{S}_{r,I}$ with $u(0) = u_0$. Let $\epsilon \in]0, 2r-1]$. Then there exist constants $t_0 = t_0\big(\epsilon, r, \|\langle A\rangle^r u_0\|\big) \in]0,T]$ and $C = C\big(\epsilon, r, \|\langle A\rangle^r u_0\|\big) > 0$ such that*

$$\|\mathrm{e}^{\sqrt{2r-1-\epsilon}\ \sqrt{t|\ln t|}A} u(t)\|_{(H^r)^3} \leq C t^{1/4+\epsilon/4-r/2} \text{ for all } t \in]0, t_0]. \tag{1.12}$$

In particular

$$\liminf_{t\to 0} \frac{\mathrm{rad}(u(t))}{\sqrt{t|\ln t|}} \geq \sqrt{2r-1}. \tag{1.13}$$

Remark. There are several natural questions connected with these results: Are the bounds (1.12) and (1.13) optimal for $r \in]1/2, 3/2[$? Are there better bounds than those deducible from Theorem 1.3 if $r > 3/2$? Can (1.11) be improved for $r = 1/2$? (But in this connection see the discussion of an example in Subsection 3.4.)

1.3. **Results on stability of the analyticity radius.**

Definition 1.4. For $r \geq 1/2$ we denote by

$$\mathcal{I}_r = \{u_0 \in P(H^r)^3 |\, \exists u \in \mathcal{G}_r :\ u(0) = u_0\}, \tag{1.14}$$

and we endow $\mathcal{I}_r$ with the topology from the space $P(H^r)^3$.

Our main result on stability of the region of analyticity of global solutions to (1.2) is the following (from Theorem 7.12 and Lemma 7.13).

Theorem 1.5. *For all $r \geq 1/2$ the set $\mathcal{I}_r$ is open in $P(H^r)^3$. Given $u_0 \in \mathcal{I}_{1/2}$, if $\lambda > 0$ is given so that the corresponding solution $u(t)$ satisfies*

$$A^{1/2}\mathrm{e}^{\lambda\sqrt{\cdot}A} u(\cdot) \in C([0,\infty[, L^2), \tag{1.15}$$

then there is a $\delta_0 > 0$ so that if $\delta \leq \delta_0$ and $v_0 \in P(H^{1/2})^3$ with $\|A^{1/2}(v_0 - u_0)\| \leq \delta$ the solution v with initial data v_0 is in $\mathcal{G}_{1/2}$ and satisfies

$$\|A^{1/2}\mathrm{e}^{\lambda\sqrt{t}A}(v(t) - u(t))\| \leq K_1\delta, \tag{1.16a}$$

$$t^{3/8}\|A^{5/4}\mathrm{e}^{\lambda\sqrt{t}A}(v(t) - u(t))\| \leq K_2\delta. \tag{1.16b}$$

If $\|v_0 - u_0\|_{H^{1/2}} \leq \delta$ it follows in addition that

$$\langle t\rangle^{-1/4}\|\mathrm{e}^{\lambda\sqrt{t}A}(v(t) - u(t))\| \leq K_3\delta. \tag{1.16c}$$

In (1.16a)–(1.16c) the constants $K_1, K_2, K_3 > 0$ depend on λ, u, and δ_0 but not on δ, and all bounds are uniform in $t > 0$.

We note that the fact that $\mathcal{I}_r$ is open is a known result. References will be given in Subsection 7.3. We also note that indeed for any $u_0 \in \mathcal{I}_{1/2}$ the condition (1.15) holds for some $\lambda > 0$, cf. (1.7). We apply Theorem 1.5 (and some other results of this paper) to establish a new stability result for the L^2 norm. This result is presented in Subsection 7.4.

Remark. There is a natural question connected with Theorem 1.5: Is the analyticity radius lower semicontinuous in the $H^{1/2}$ topology? More precisely one may conjecture that for any fixed $u_0 \in \mathcal{I}_{1/2}$ and $t > 0$

$$\liminf \operatorname{rad}(v(t)) \geq \operatorname{rad}(u(t)) \text{ in the limit } \|v_0 - u_0\|_{H^{1/2}} \to 0? \tag{1.17}$$

For partial results in this direction see Proposition 7.8 and Corollary 7.10.

We shall use the standard notation $\langle \lambda \rangle := (1 + |\lambda|^2)^{1/2}$ for any real λ. For any given interval J and Hilbert space $\mathcal{H}$ the notation $BC(J, \mathcal{H})$ refers to the set of all bounded continuous functions $v : J \to \mathcal{H}$.

2. Integral equation

We look at the following (generalized) system of equations in the variables $(t, x) \in [0, \infty[\times\mathbb{R}^3$. The unknown u specifies at each argument a velocity $u = u(t, x) \in \mathbb{R}^3$. The quantity M is a fixed real 3×3–matrix. Corresponding to (1.1) $M = I$. The quantity P is an orthogonal projection on $L^2(\mathbb{R}^3, \mathbb{R}^3)$ which is fibered in Fourier space, i.e. $\widehat{(Pf)}(\xi) = P(\xi)\hat{f}(\xi)$. Corresponding to (1.1) P is the Leray projection given by $P(\xi) = I - |\hat{\xi}\rangle\langle\hat{\xi}|$, $\hat{\xi} := \xi/|\xi|$, but for the most part we will not assume this.

$$\begin{cases} \left(\frac{\partial}{\partial t}u + (Mu \cdot \nabla)u - \triangle u\right)(t, \cdot) \in \operatorname{Ran}\left(I - P\right) \text{ for } t > 0 \\ u(t) := u(t, \cdot) \in \operatorname{Ran}P \text{ for } t \geq 0 \\ u(0) = u_0 \end{cases} . \tag{2.1}$$

Similarly the operator $A := \sqrt{-\triangle}$ on $L^2(\mathbb{R}^3, \mathbb{R}^3)$ is fibered in Fourier space as $\widehat{(Af)}(\xi) = |\xi|\hat{f}(\xi)$. Upon multiplying the first equation by P and integrating we obtain (formally)

$$u(t) = \mathrm{e}^{-tA^2}u_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} P(Mu(s) \cdot \nabla)u(s)\ \mathrm{d}s. \tag{2.2}$$

Conversely, notice (formally) that a solution to (2.2) with $u_0 \in \operatorname{Ran}P$ obeys (2.1). In the bulk of this paper we shall study (2.2) without imposing the condition $u_0 \in \operatorname{Ran}P$. See though Section 7 for an exception. In fact in Subsection 7.1 we shall study (under some conditions) the relationship between (2.1) and (2.2). For the bulk of this paper this relationship is minor although traces are used already in Sections 5 and 6. The reader might prefer to read the present section and Subsection 7.1 before proceeding to Section 3.

In this section we consider the Cauchy problem in the form (2.2) using norms based essentially on Sobolev spaces. Although this material is well known (see [FK1], [FK2], and for example [KP], [Le], [Pl]) we give a self-contained account so that we can use the specific results and methods in our analysis of the spatial analyticity of solutions of (2.2) in the sections following.

Part of our motivation for studying equations more general than (1.2) is that such a study emphasizes what we actually use in our analysis. In particular, besides the case where $M = I$ and P is the Leray projection, we will consider the vector Burgers' equation, as an example, where $M = I$ and $P = I$ (see Subsections 3.4 and 6.2). The latter equation has been studied in [KL], [JS], [Ga], and elsewhere.

We define for any $r \in]-\infty, 3/2[$ the homogeneous Sobolev space $\dot{H}^r$ to be the set of $f \in \mathcal{S}'$ such that the Fourier transform $\hat{f}$ is a measurable function and $|\xi|^r \hat{f}(\xi) \in L^2(\mathbb{R}^3_\xi)$. The corresponding norm is $\|f\|_{\dot{H}^r} = \|A^r f\|$ where here and henceforth $\|\cdot\|$ refers to the L^2-norm. For simplicity we shall use the same notation for vectors $f \in (L^2)^3 = L^2 \oplus L^2 \oplus L^2$, viz. $\|f\| = \sqrt{\|f_1\|^2 + \|f_2\|^2 + \|f_3\|^2}$ for $f = (f_1, f_2, f_3) \in (L^2)^3$.

Let I be an interval of the form $I =]0, T]$ (if T is finite) or $I =]0, T[$ (if $T = \infty$). The closure $I \cup \{0\}$ will be denoted $\bar{I}$. Let $\zeta : I \to \mathbb{R}$ and $\theta : I \to [0, \infty[$ be given continuous functions. Let $s_1, s_2 \in [0, 3/2[$ be given. We shall consider the class of functions $I \ni t \to v(t) \in (\dot{H}^{s_2})^3$ for which the expression $\mathrm{e}^{-\zeta(t)} t^{s_1} \mathrm{e}^{\theta(t) A} A^{s_2} v(t)$ defines an element in $BC(I, (L^2)^3)$. The set of such functions, denoted by $\mathcal{B}_{\zeta,\theta,I,s_1,s_2}$, is a Banach space with the norm

$$\|v\|_{\zeta,\theta,I,s_1,s_2} := \sup_{t\in I} \mathrm{e}^{-\zeta(t)} t^{s_1} \|\mathrm{e}^{\theta(t)A} A^{s_2} v(t)\|. \tag{2.3}$$

In this section we discuss the case $\zeta = 0$ and $\theta = 0$ only which (upon choosing s_1 and s_2 suitably) corresponds to part of the pioneering work [FK1, FK2]. Consequently we omit throughout this section the subscripts ζ and θ in the above notation.

We recall the following class of Sobolev bounds (cf. [RS, (IX.19)]):

Lemma 2.1. *For all $r \in]0, 3/2[$, $f \in \dot{H}^r$ and all $g \in \dot{H}^{3/2-r}$ the product $fg \in L^2$, and there exists a constant $C = C(r) > 0$ such that*

$$\|fg\| \le C \|A^r f\| \|A^{3/2-r} g\|. \tag{2.4}$$

Due to Lemma 2.1 we can estimate

$$\|A^{5/4} \mathrm{e}^{-(t-s)A^2} P(Mu(s)\cdot\nabla) v(s)\| \le C_1 (t-s)^{-5/8} s^{-3/4} \|u\|_{I,3/8,5/4} \|v\|_{I,3/8,5/4}. \tag{2.5}$$

Here we used the spectral theorem to bound $\|A^{5/4}\mathrm{e}^{-(t-s)A^2}\|_{\mathcal{B}(L^2)} \le C(t-s)^{-5/8}$ and the boundedness of P and $A^{1/4}\partial_j A^{-5/4}$.

Motivated by (2.5) let us write the integral equation (2.2) as $X = Y + B(X,X)$ on the space $\mathcal{B} = \mathcal{B}_{I,s_1,s_2}$ with $s_1 = 3/8$ and $s_2 = 5/4$. Abbreviating $|v| = \|v\|_{I,s_1,s_2}$ we obtain that for all $u, v \in \mathcal{B}$

$$|B(u,v)| \le \gamma |u|\,|v|;\ \gamma = C_1 \sup_{t\in I} t^{3/8} \int_0^t (t-s)^{-5/8} s^{-3/4}\,\mathrm{d}s. \tag{2.6}$$

Notice that γ does not depend on I since C_1 is the constant coming from (2.5) and

$$\gamma = C_1 \int_0^1 (1-s)^{-5/8} s^{-3/4}\,\mathrm{d}s. \tag{2.7}$$

2.1. **Abstract scheme.** We shall study (motivated by (2.6)) the equation

$$X = Y + B(X,X), \tag{2.8}$$

where B is a continuous bilinear operator $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$ on a given Banach space $\mathcal{B}$. Let $\gamma \ge 0$ denote a corresponding bounding constant,

$$|B(u,v)| \le \gamma |u|\,|v|. \tag{2.9}$$

The elementary fixed point theorem applies if there exists $R \ge 0$ such that

$$|Y| \le R \text{ and } \kappa := 4\gamma R < 1. \tag{2.10}$$

In fact letting $B_{\widetilde{R}} = \{X \in \mathcal{B} \,|\, |X| \le \widetilde{R}\}$ for $\widetilde{R} \ge 0$ the conditions (2.10) assure that B_{2R} is mapped into itself by the map $X \to F(X) := Y + B(X,X)$ and that κ is a corresponding contraction constant. (This version of the fixed point theorem is implicitly used in [Pl], see [Pl, Lemma 1].) In particular under the condition (2.10) there exists a unique solution to (2.8) in B_{2R}. Letting $X_0 = Y$ and $X_n = Y + B(X_{n-1}, X_{n-1})$ for $n \in \mathbb{N}$ this solution can be represented as

$$X = \mathcal{B} - \lim_{n\to\infty} X_n. \tag{2.11}$$

2.2. **Local solvability in $\dot H^r$ and H^r, $r \in [1/2, 3/2[$.** We return to the integral equation (2.2) written as $X = Y + B(X,X)$ in the space $\mathcal{B} = \mathcal{B}_{I,s_1,s_2}$ with $s_1 = 3/8$ and $s_2 = 5/4$. We need to examine the first term $Y(t) = \mathrm{e}^{-tA^2}u_0$ for some "data" $u_0 \in (\mathcal{S}')^3$. More precisely we need to study the condition $Y \in \mathcal{B}$. Clearly due to (2.6) the requirement (2.10) is met if $|Y|$ is sufficiently small.

Let us first examine the special case $Y = \mathrm{e}^{-(\cdot)A^2}u_0 \in \mathcal{B}_\infty$ where $\mathcal{B}_\infty = \mathcal{B}_{]0,\infty[,s_1,s_2}$. This requirement is equivalent to finiteness of the expression

$$\sup_{q\in\mathbb{Z}} 2^{q/2}\Big(\int_{2^q\le|\xi|<2^{q+1}} |\hat u_0|^2 \,\mathrm{d}\xi\Big)^{1/2}, \tag{2.12}$$

cf. [Pl, Lemma 8]. Notice that finiteness of (2.12) and (2.3) are equivalent to finiteness of these expressions for each of the three components of u_0 and v, respectively. For notational convenience we shall in the following discussion slightly abuse notation by treating u_0 as a scalar-valued function rather than an $\mathbb{R}^3$-valued function and similarly for the elements in $\mathcal{B}_\infty$. The (finite) expression (2.12) is the norm of u_0 in the Besov space $\dot B^{1/2}_{2,\infty}$ which indeed consists of all $u_0 \in \mathcal{S}'$ with $\hat u_0$ a measurable function and (2.12) finite. In fact the norms (2.3) (with $\zeta = 0$, $\theta = 0$, $s_1 = 3/8$ and $s_2 = 5/4$) and (2.12) are equivalent on the subspace of $\mathcal{B}_\infty$ consisting of functions $t \to \mathrm{e}^{-tA^2}u_0$ where $u_0 \in \dot B^{1/2}_{2,\infty}$, henceforth for brevity denoted by $\mathrm{e}^{-(\cdot)A^2}\dot B^{1/2}_{2,\infty}$. Introducing $\dot B^{1/2,0}_{2,\infty}$ as the set of $u_0 \in \dot B^{1/2}_{2,\infty}$ with

$$2^{q/2}\Big(\int_{2^q\le|\xi|<2^{q+1}} |\hat u_0|^2 \,\mathrm{d}\xi\Big)^{1/2} \to 0 \text{ for } q \to +\infty,$$

obviously

$$\dot H^{1/2} \subseteq \dot B^{1/2,0}_{2,\infty} \subseteq \dot B^{1/2}_{2,\infty} \subseteq \cap_{1>\epsilon>0}\big(\dot H^{1/2-\epsilon} + \dot H^{1/2+\epsilon}\big). \tag{2.13}$$

Let $\mathcal{B}^0_\infty$ be the subspace of $\mathcal{B}_\infty$ consisting of functions $v \in \mathcal{B}_\infty$ obeying

$$t^{3/8}\|A^{5/4}v(t)\| \to 0 \text{ for } t \to 0. \tag{2.14}$$

We have the following identification of subspaces in $\mathcal{B}_\infty$ (which is easily proven)

$$\mathrm{e}^{-(\cdot)A^2}\dot B^{1/2,0}_{2,\infty} = \mathcal{B}^0_\infty \cap \mathrm{e}^{-(\cdot)A^2}\dot B^{1/2}_{2,\infty}. \tag{2.15}$$

Now returning to a general interval I we introduce the subspace of $\mathcal{B}$, denoted by $\mathcal{B}^0$, consisting of (vector-valued) functions v obeying (2.14).

Proposition 2.2. *Suppose $u_0 \in (\dot H^{1/2})^3$. Then for any $T = |I| > 0$ small enough (so that the conditions* (2.10) *hold for some $R > 0$) the integral equation* (2.8) *has a unique solution in the ball $B_{2R} \subset \mathcal{B}$ where $\mathcal{B} = \mathcal{B}_{I,s_1,s_2}$ with $s_1 = 3/8$, $s_2 = 5/4$ and $Y(t) = \mathrm{e}^{-tA^2}u_0$. This solution $X \in \mathcal{B}^0 \cap BC(\bar I, (\dot H^{1/2})^3)$ with $X(0) = u_0$. If in addition $u_0 \in (L^2)^3$ then $X \in BC(\bar I, (H^{1/2})^3)$.*

Proof. By combining (2.6) and (2.15) we conclude that the requirements (2.10) are met in the space $\mathcal{B}$ provided that the three components of u_0 belong to $\dot{B}^{1/2,0}_{2,\infty}$ and that the parameter T is taken small enough (to ensure that $|Y|$ is small). Whence there exists a unique solution $X \in B_{2R}$. We notice that X is also the unique solution to the fixed point problem in the ball $B^0_{2R} := \mathcal{B}^0 \cap B_{2R}$ (if the components of u_0 belong to $\dot{B}^{1/2,0}_{2,\infty}$ and $T > 0$ is small).

Using the first inclusion of (2.13) we obtain in particular a unique small time solution with "data" $u_0 \in (\dot{H}^{1/2})^3$ in B^0_{2R}. By an estimate very similar to (2.5) we obtain the bound

$$\|B(u,v)(t)\|_{(\dot{H}^{1/2})^3} \leq \eta |u|\,|v| \text{ for } t \in I. \tag{2.16}$$

Using (2.16) we see that in fact $B : \mathcal{B}^0 \times \mathcal{B}^0 \to BC(\bar{I}, (\dot{H}^{1/2})^3)$ and that the functions in the range of this map vanish at $t = 0$. Consequently, the constructed fixed point $X \in \mathcal{B}^0$ for $u_0 \in (\dot{H}^{1/2})^3$ belongs to the space $BC(\bar{I}, (\dot{H}^{1/2})^3)$ and the data $X(0) = u_0$ is attained continuously.

Moreover we have the bound

$$\|B(u,v)(t)\| \leq Ct^{1/4}|u|\,|v| \text{ for } t \in I. \tag{2.17}$$

So if in addition $u_0 \in (L^2)^3$ then $X \in BC(\bar{I}, (L^2)^3)$ with the data $X(0) = u_0$ attained continuously.

□

Remark 2.3. In [FK1, FK2] spaces $\mathcal{B}$ and $\mathcal{B}^0$ similar to ours are used for treating (1.1). The powers differ from ours: $t^{3/8} \to t^{1/8}$ and $A^{5/4} \to A^{3/4}$. These spaces are not suitable for the generalized problem (2.1).

If the data $u_0 \in (\dot{H}^r)^3$ for $r \in]1/2, 3/2[$ there is a similar result (in fact, as the reader will see, proved very similarly).

We let $\tilde{r} = \max(r, 5/4)$ and introduce $\mathcal{B} = \mathcal{B}_{I,s_1,s_2}$ with $s_1 = \tilde{r}/2 - r/4 - 1/8$ and $s_2 = \tilde{r}$.

We use Lemma 2.1 to bound

$$\|A^{2s_2-5/2}P(Mu(s)\cdot\nabla)v(s)\| \leq C\|A^{s_2}u(s)\|\,\|A^{s_2}v(s)\|, \tag{2.18}$$

and we obtain the following analogue of (2.5) by splitting $A^{s_2} = A^{5/2-s_2}A^{2s_2-5/2}$:

$$\|A^{s_2}\mathrm{e}^{-(t-s)A^2}P(Mu(s)\cdot\nabla)v(s)\| \leq C_1(t-s)^{s_2/2-5/4}s^{-2s_1}|u|\,|v|. \tag{2.19}$$

Consequently we infer that

$$|B(u,v)| \leq \gamma|u|\,|v|;\ \gamma = C_1 \sup_{t\in I} t^{s_1}\int_0^t (t-s)^{s_2/2-5/4}s^{-2s_1}\,\mathrm{d}s = C_2T^{r/4-1/8}. \tag{2.20}$$

The space $\mathcal{B}^0$ is now defined to be the space of functions $v \in \mathcal{B}$ obeying

$$t^{s_1}\|A^{s_2}v(t)\| \to 0 \text{ for } t \to 0, \tag{2.21}$$

cf. (2.14).

As for $Y := \mathrm{e}^{-(\cdot)A^2}u_0$ indeed $Y \in \mathcal{B}^0$ with

$$|Y| \leq C \sup_{t\in I} t^{r/4-1/8}\|u_0\|_{\dot{H}^r} = CT^{r/4-1/8}\|u_0\|_{\dot{H}^r}. \tag{2.22}$$

Here and henceforth we slightly abuse notation by abbreviating $(\dot{H}^r)^3$ as $\dot{H}^r$. Similarly from this point on we shall for convenience frequently abbreviate $(H^r)^3$ as H^r

and $(L^2)^3$ as L^2, respectively. (Hopefully the interpretation will be obvious in every concrete context.)

Proposition 2.4. *Suppose $u_0 \in \dot{H}^r$ with $r \in]1/2, 3/2[$. Let $\tilde{r} = \max(r, 5/4)$, $s_1 = \tilde{r}/2 - r/4 - 1/8$ and $s_2 = \tilde{r}$. For any $T = |I| > 0$ small enough (so that the conditions* (2.10) *hold for some $R > 0$) the integral equation* (2.8) *has a unique solution in the ball $B_{2R} \subset \mathcal{B}$ where $\mathcal{B} = \mathcal{B}_{I,s_1,s_2}$ and $Y(t) = \mathrm{e}^{-tA^2}u_0$. This solution $X \in \mathcal{B}^0 \cap BC(\bar{I}, \dot{H}^r)$ with $X(0) = u_0$. If in addition $u_0 \in L^2$ then $X \in BC(\bar{I}, H^r)$ (possibly we need at this point to take $T > 0$ smaller if $r \in]5/4, 3/2[$).*

Proof. Due to (2.20) and the fact that $Y \in \mathcal{B}^0$ indeed there exists a unique solution $X \in B_{2R}$ for $T > 0$ small enough, and we notice that X is also the unique solution to the fixed point problem in the ball $B^0_{2R} := \mathcal{B}^0 \cap B_{2R}$.

With the convention alluded to above the analogue of (2.16) reads

$$\|B(u,v)(t)\|_{\dot{H}^r} \leq \eta |u|\,|v| \text{ for } t \in I, \tag{2.23}$$

and we infer (as before) that $B : \mathcal{B}^0 \times \mathcal{B}^0 \to BC(\bar{I}, \dot{H}^r)$ and that the functions in the range of this map vanish at $t = 0$. (Note incidently that the constant η of (2.23) can be chosen independently of T but not as a vanishing power of T as in (2.20) and (2.22).) Whence indeed $X \in \mathcal{B}^0 \cap BC(\bar{I}, \dot{H}^r)$ with $X(0) = u_0$.

Moreover for $r \in]1/2, 5/4]$ we have the bound

$$\|B(u,v)(t)\| \leq Ct^{r/2}|u|\,|v| \text{ for } t \in I. \tag{2.24}$$

So if $r \in]1/2, 5/4]$ and in addition $u_0 \in L^2$ then $X \in BC(\bar{I}, L^2)$ with the data $X(0) = u_0$ attained continuously. Whence $X \in BC(\bar{I}, H^r)$ for $u_0 \in H^r$. In fact this holds for any $r \in]1/2, 3/2[$ (possibly by taking a smaller interval I if $r \in]5/4, 3/2[$). In the case $r \in]5/4, 3/2[$ we can obtain the result from the case $r = 5/4$ by invoking the embedding $H^r \subseteq H^{5/4}$ and using the representation (2.11) of the solutions with data in $\dot{H}^r$ and $\dot{H}^{5/4}$, respectively. We deduce that for data in $\dot{H}^r \cap \dot{H}^{5/4}$ the two constructed solutions, say $X_1 \in \mathcal{B}_1$ and $X_2 \in \mathcal{B}_2$, coincide on their common interval of definition $I_1 \cap I_2$.

□

2.3. **Local solvability in H^r, $r \in [5/4, \infty[$.** In this subsection we shall study local solutions in H^r for $r \in [5/4, \infty[$. The method of proof will be similar to that of Subsection 2.2. In particular our constructions will be based on the following modification of (2.3) (for simplicity we shall use the same notation).

$$\|v\|_{\zeta,\theta,I,s_1,s_2} := \sup_{t \in I} \mathrm{e}^{-\zeta(t)} t^{s_1} \|\mathrm{e}^{\theta(t)A}\langle A\rangle^{s_2} v(t)\|. \tag{2.25}$$

Again we consider here the case $\zeta = 0$ and $\theta = 0$ only. Let $\tilde{r} \in [5/4, 3/2[$ be arbitrarily given such that $r \geq \tilde{r}$. Then the parameters s_1 and s_2 in (2.25) are chosen as follows:

$$s_1 = \tilde{r}/4 - 1/8 \text{ and } s_2 = r, \tag{2.26}$$

and $\mathcal{B} = \mathcal{B}_{I,s_1,s_2}$ is the class of functions $I \ni t \to v(t) \in H^{s_2}$ for which the expression $t^{s_1}\langle A\rangle^{s_2} v(t)$ defines an element in $BC(I, L^2)$; $I =]0, T]$.

To bound $B : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ we let

$$\bar{r} = 5/2 - \tilde{r}, \tag{2.27}$$

and split $\langle A\rangle^{s_2} = \langle A\rangle^{\bar r}\langle A\rangle^{r-\bar r}$. Using again Lemma 2.1 we then obtain

$$\|\langle A\rangle^{s_2}\mathrm{e}^{-(t-s)A^2}P(Mu(s)\cdot\nabla)v(s)\| \le C_1\langle T\rangle^{\bar r/2}(t-s)^{-\bar r/2}s^{-2s_1}|u|\,|v|. \tag{2.28}$$

Consequently we infer that

$$|B(u,v)| \le \gamma|u|\,|v|;\ \gamma = C_1\langle T\rangle^{\bar r/2}\sup_{t\in I} t^{s_1}\int_0^t (t-s)^{-\bar r/2}s^{-2s_1}\,\mathrm{d}s = C_2\langle T\rangle^{\bar r/2}T^{s_1}. \tag{2.29}$$

The space $\mathcal{B}^0$ is the subclass of $v\in\mathcal{B}$ obeying

$$t^{s_1}\|\langle A\rangle^{s_2}v(t)\| \to 0 \text{ for } t\to 0. \tag{2.30}$$

Now suppose $u_0\in H^r$. Then clearly $Y := \mathrm{e}^{-(\cdot)A^2}u_0 \in \mathcal{B}^0$ with

$$|Y| \le C\sup_{t\in I} t^{s_1}\|u_0\|_{H^r} = CT^{s_1}\|u_0\|_{H^r}. \tag{2.31}$$

Next letting (as before) $B_{2R} := \{X\in\mathcal{B}|\,|X|\le 2|Y|\}$ and $B^0_{2R} := \mathcal{B}^0\cap B_{2R}$ we conclude from (2.29) and the fact that $Y\in\mathcal{B}^0$ that indeed the contraction condition (2.10) for the map $X\to Y+B(X,X)$ restricted to either B_{2R} or to B^0_{2R} is valid for any $T>0$ small enough. Consequently the common fixed point $X\in\mathcal{B}^0$. Moreover

$$\|B(u,v)(t)\|_{H^r} \le \eta\langle T\rangle^{\bar r/2}|u|\,|v| \text{ for } t\in I, \tag{2.32}$$

and we infer (as before) that $B:\mathcal{B}^0\times\mathcal{B}^0\to BC(\bar I, H^r)$ and that the functions in the range of this map vanish at $t=0$. We conclude:

Proposition 2.5. *Suppose $u_0\in H^r$ with $r\in[5/4,\infty[$. Let $\tilde r\in[5/4,3/2[$ be arbitrarily given such that $r\ge\tilde r$. Let s_1 and s_2 be given as in* (2.26). *For any $T=|I|>0$ small enough (so that the conditions* (2.10) *hold for some $R>0$) the integral equation* (2.8) *has a unique solution in the ball $B_{2R}\subset\mathcal{B}$ where $\mathcal{B}=\mathcal{B}_{I,s_1,s_2}$ and $Y(t)=\mathrm{e}^{-tA^2}u_0$. This solution $X\in\mathcal{B}^0\cap BC(\bar I,H^r)$ with $X(0)=u_0$.*

Remarks 2.6. 1) Clearly we may take $\tilde r = r$ in Proposition 2.5 if $r\in[5/4,3/2[$. In that case the (small) T-dependence of the bounds (2.22) and (2.20) coincides with that of (2.31) and (2.29), respectively.
2) Although there are different spaces involved in Propositions 2.2–2.5 the constructed solutions coincide on any common interval of definition, cf. the last argument of the proof of Proposition 2.4.

2.4. **Global solvability in $\dot H^{1/2}$ and $H^{1/2}$ for small data.** If the data in $\dot H^{1/2}$ is small we can improve on the conclusion of Proposition 2.2 to obtain a global solution. The proof is similar.

Proposition 2.7. *Suppose $u_0\in\dot H^{1/2}$. Then the integral equation* (2.8) *has a unique solution in the ball $B_{2R}\subset\mathcal{B}$ where $\mathcal{B}=\mathcal{B}_{I,s_1,s_2}$ with $I=]0,\infty[$, $s_1=3/8$, $s_2=5/4$ and $Y(t)=\mathrm{e}^{-tA^2}u_0$ provided that $\|u_0\|_{\dot H^{1/2}}$ is sufficiently small (so that for some $R>0$ and with γ given by* (2.7) *the conditions* (2.10) *hold). This solution $X\in\mathcal{B}^0\cap BC([0,\infty[,\dot H^{1/2})$ with $X(0)=u_0$. If in addition $u_0\in L^2$ then $X\in C([0,\infty[,H^{1/2})$.*

3. Analyticity bounds for small times

In this section we shall study analyticity properties of the short-time solutions of Propositions 2.2–2.5. This will be done by using more general spaces with norms given by (2.3) or (2.25) and modifying the proofs of Section 2.

3.1. **Local analyticity bounds in $\dot H^r$ and H^r , $r \in [1/2, 3/2[$.** In this subsection we specify the functions ζ and θ in the norm (2.3) in terms of a parameter $\lambda \geq 0$ as

$$\zeta = \lambda^2/4 + (\tilde r - r)\ln\langle\lambda\rangle \text{ and } \theta(t) = \lambda\sqrt{t}; \tag{3.1}$$

here $\tilde r$ is given as in Proposition 2.4 (also for $r = 1/2$). Let s_1 and s_2 be given as in Proposition 2.4, and let again $I =]0, T]$.

For applications in Subsection 3.3 we shall be concerned below with bounding various quantities independently of the parameter $\lambda \geq 0$ (rather than just proving Theorem 3.2 stated below). We shall use the following elementary bound (which follows from the spectral theorem).

Lemma 3.1. *For any $\alpha \geq 0$ there exists a constant $C \geq 0$ such that for all $f \in L^2$*

$$\sup_{\lambda,t\geq 0} \langle\lambda\rangle^{-\alpha} \mathrm{e}^{-\lambda^2/4} \|\big(\sqrt{t}A\big)^\alpha \mathrm{e}^{\lambda\sqrt{t}A}\mathrm{e}^{-tA^2} f\| \leq C\|f\|. \tag{3.2}$$

Due to (3.2) we can estimate the following norm of $Y = \mathrm{e}^{-(\cdot)A^2}u_0$, $u_0 \in \dot H^r$.

$$|Y| = \|Y\|_{\zeta,\theta,I,s_1,s_2} \leq C_1 T^{r/4-1/8}\|A^r u_0\|, \tag{3.3}$$

where the constant C_1 is independent of $\lambda \geq 0$.

Let $\mathcal{B}^0 = \mathcal{B}^0_{\zeta,\theta,I,s_1,s_2}$ be the space of functions $v \in \mathcal{B} := \mathcal{B}_{\zeta,\theta,I,s_1,s_2}$ obeying

$$t^{s_1}\|\mathrm{e}^{\lambda\sqrt{t}A}A^{s_2}v(t)\| \to 0 \text{ for } t \to 0, \tag{3.4}$$

Obviously it follows from (3.3) that $Y \in \mathcal{B}^0$ if $r > 1/2$. However this is also true for $r = 1/2$ which follows from the same bound and a simple approximation argument (using for instance that $H^{5/4}$ is dense in $\dot H^{1/2}$). Whence indeed

$$Y = \mathrm{e}^{-(\cdot)A^2}u_0 \in \mathcal{B}^0 \text{ for } u_0 \in \dot H^r,\ r \in [1/2, 3/2[. \tag{3.5}$$

We have the following generalization of Propositions 2.2 and 2.4 (abbreviating as before the norm on $\mathcal{B}$ as $|\cdot|$). Notice that the solutions of Theorem 3.2 coincide with those of Propositions 2.2 and 2.4 for $T > 0$ small enough, cf. Remark 2.6 2.

Theorem 3.2. *Let $\lambda \geq 0$ and $u_0 \in \dot H^r$ with $r \in [1/2, 3/2[$ be given. Let $\tilde r = \max(r, 5/4)$, $s_1 = \tilde r/2 - r/4 - 1/8$ and $s_2 = \tilde r$. For any $T = |I| > 0$ small enough (so that the conditions* (2.10) *hold for some $R > 0$) the integral equation* (2.8) *has a unique solution in the ball $B_{2R} \subset \mathcal{B}$ where $\mathcal{B} = \mathcal{B}_{\zeta,\theta,I,s_1,s_2}$ and $Y(t) = \mathrm{e}^{-tA^2}u_0$. This solution $X \in \mathcal{B}^0 \subseteq \mathcal{B}$, and it obeys that $\mathrm{e}^{\lambda\sqrt{(\cdot)}A}X \in BC(\bar I, \dot H^r)$ with $X(0) = u_0$. If in addition $u_0 \in L^2$ then $\mathrm{e}^{\lambda\sqrt{(\cdot)}A}X \in BC(\bar I, H^r)$ (possibly we need at this point to take $T > 0$ smaller if $r \in]5/4, 3/2[$).*

Proof. First we show that $B : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$. Using the triangle inequality in Fourier space we obtain the following analogue of (2.18):

$$\|\mathrm{e}^{\lambda\sqrt{s}A}A^{2s_2-5/2}P(Mu(s)\cdot\nabla)v(s)\| \leq C\|\mathrm{e}^{\lambda\sqrt{s}A}A^{s_2}u(s)\|\,\|\mathrm{e}^{\lambda\sqrt{s}A}A^{s_2}v(s)\|, \tag{3.6}$$

and consequently using that $\sqrt{t} \leq \sqrt{t-s} + \sqrt{s}$ and Lemma 3.1 we obtain, cf. (2.19),

$$\|\mathrm{e}^{\lambda\sqrt{t}A}A^{s_2}\mathrm{e}^{-(t-s)A^2}P(Mu(s)\cdot\nabla)v(s)\| \leq C_2\langle\lambda\rangle^{5/2-s_2}\mathrm{e}^{\lambda^2/4}\mathrm{e}^{2\zeta}(t-s)^{s_2/2-5/4}s^{-2s_1}|u|\,|v|. \tag{3.7}$$

We conclude, cf. (2.20), that indeed $B(u, v) \in \mathcal{B}$ with

$$|B(u, v)| \leq \gamma|u|\,|v|;\ \gamma = C_3\langle\lambda\rangle^{5/2-r}\mathrm{e}^{\lambda^2/2}T^{r/4-1/8}. \tag{3.8}$$

Obviously the same arguments show that $B : \mathcal{B}^0 \times \mathcal{B}^0 \to \mathcal{B}^0$. In conjunction with (3.5) we conclude that the integral equation (2.8) has a unique solution X in the ball B_{2R} provided that first $R > 0$ and then $T > 0$ are taken small enough, and that this $X \in \mathcal{B}^0$.

For the remaining statements of Theorem 3.2 we similarly mimic the proof of Proposition 2.4. For completeness of presentation let us state the analogues of (2.23) and (2.24)

$$\|\mathrm{e}^{\lambda\sqrt{t}A}B(u,v)(t)\|_{\dot H^r} \le C\langle\lambda\rangle^{5/2-r}\mathrm{e}^{3\lambda^2/4}|u|\,|v| \text{ for } t \in I, \tag{3.9}$$

and for $r \in [1/2, 5/4]$

$$\|\mathrm{e}^{\lambda\sqrt{t}A}B(u,v)(t)\| \le \widetilde{C}\langle\lambda\rangle^{5/2-2r}\mathrm{e}^{3\lambda^2/4}t^{r/2}|u|\,|v| \text{ for } t \in I. \tag{3.10}$$

□

3.2. **Local analyticity bounds in H^r, $r \in [5/4, \infty[$.** In this subsection we assume $r \in [5/4, \infty[$ and specify the functions ζ and θ in the norm (2.25) in terms of a parameter $\lambda \ge 0$ as

$$\zeta = \lambda^2/4 \text{ and } \theta(t) = \lambda\sqrt{t}. \tag{3.11}$$

Let s_1 and s_2 be given as in Proposition 2.5, and let again $I =]0, T]$.

Due to Lemma 3.1 we can estimate the following norm of $Y = \mathrm{e}^{-(\cdot)A^2}u_0$, $u_0 \in H^r$.

$$|Y| = \|Y\|_{\zeta,\theta,I,s_1,s_2} \le C_1 T^{s_1}\|\langle A\rangle^r u_0\|, \tag{3.12}$$

where the constant C_1 is independent of $\lambda \ge 0$.

Let $\mathcal{B}^0 = \mathcal{B}^0_{\zeta,\theta,I,s_1,s_2}$ be the space of functions $v \in \mathcal{B} := \mathcal{B}_{\zeta,\theta,I,s_1,s_2}$ obeying

$$t^{s_1}\|\mathrm{e}^{\lambda\sqrt{t}A}\langle A\rangle^{s_2}v(t)\| \to 0 \text{ for } t \to 0, \tag{3.13}$$

It follows from (3.12) that $Y = \mathrm{e}^{-(\cdot)A^2}u_0 \in \mathcal{B}^0$.

We have the following generalization of Proposition 2.5:

Theorem 3.3. *Let $\lambda \ge 0$ and $u_0 \in H^r$ with $r \in [5/4, \infty[$ be given. Let $\tilde r \in [5/4, 3/2[$ be arbitrarily given such that $r \ge \tilde r$. Let s_1 and s_2 be given as in* (2.26)*. For any $T = |I| > 0$ small enough (so that the conditions* (2.10) *hold for some $R > 0$) the integral equation* (2.8) *has a unique solution in the ball $B_{2R} \subset \mathcal{B}$ where $\mathcal{B} = \mathcal{B}_{\zeta,\theta,I,s_1,s_2}$ and $Y(t) = \mathrm{e}^{-tA^2}u_0$. This solution $X \in \mathcal{B}^0 \subseteq \mathcal{B}$, and it obeys that $\mathrm{e}^{\lambda\sqrt{(\cdot)}A}X \in BC(\bar I, H^r)$ with $X(0) = u_0$.*

Proof. Mimicking the proof of Proposition 2.5 we obtain

$$\|\mathrm{e}^{\lambda\sqrt{t}A}\langle A\rangle^{s_2}\mathrm{e}^{-(t-s)A^2}P(Mu(s)\cdot\nabla)v(s)\| \le C_2\langle\lambda\rangle^{\bar r}\mathrm{e}^{\lambda^2/4}\mathrm{e}^{2\zeta}\langle t\rangle^{\bar r/2}(t-s)^{-\bar r/2}s^{-2s_1}|u|\,|v|, \tag{3.14}$$

where $\bar r$ is given by (2.27). Consequently we infer that

$$|B(u,v)| \le \gamma|u|\,|v|; \tag{3.15}$$

$$\gamma = C_2\langle\lambda\rangle^{\bar r}\mathrm{e}^{\lambda^2/2}\langle T\rangle^{\bar r/2}\sup_{t\in I}t^{s_1}\int_0^t (t-s)^{-\bar r/2}s^{-2s_1}\,\mathrm{d}s = C_3\langle\lambda\rangle^{\bar r}\mathrm{e}^{\lambda^2/2}\langle T\rangle^{\bar r/2}T^{s_1}.$$

We conclude from (3.12) and (3.15) that indeed the contraction condition (2.10) for the map $X \to Y + B(X,X)$ on B_{2R} (or on $B^0_{2R} := \mathcal{B}^0 \cap B_{2R}$) is valid for any $T > 0$ small enough. Moreover

$$\|\mathrm{e}^{\lambda\sqrt{t}A}B(u,v)(t)\|_{H^r} \le C\langle\lambda\rangle^{\bar r}\mathrm{e}^{3\lambda^2/4}\langle T\rangle^{\bar r/2}|u|\,|v| \text{ for } t \in I, \tag{3.16}$$

and we infer (as before) that $\mathrm{e}^{\lambda\sqrt{(\cdot)}A}B : \mathcal{B}^0 \times \mathcal{B}^0 \to BC(\bar{I}, H^r)$ and that the functions in the range of this map vanish at $t = 0$. $\square$

3.3. **Improved local bounds of analyticity radii for** $r > 1/2$**.** In this subsection we shall modify the constructions of Subsections 3.1 and 3.2 in that the functions in (3.1) and (3.11) now will be taken with an additional time-dependence. Explicitly we define ζ and θ by (3.1) (for the setting of Subsection 3.1) and (3.11) (for the setting of Subsection 3.2) but now in terms of λ taken to have the following explicit time-dependence

$$\lambda = \lambda_0\sqrt{t/T}; \tag{3.17}$$

here $\lambda_0 \geq 0$ is an auxiliary parameter (which in the end will play the role of the previous parameter λ) and $T > 0$ is the right end point of the interval I (as in Subsection 3.1). The bounds (3.3), (3.6), (3.12) and (3.14) remain true where $\lambda = \lambda(s)$ in (3.6) and similar interpretations are needed in (3.14).

As for (3.8) the bounding constant has the form, cf. (3.7),

$$C_2 \sup_{t\in I} \mathrm{e}^{-\zeta(t)} t^{s_1} \int_0^t \langle\lambda(t-s)\rangle^{5/2-s_2} \mathrm{e}^{\lambda^2(t-s)/4} \mathrm{e}^{2\zeta(s)} (t-s)^{s_2/2-5/4} s^{-2s_1}\, \mathrm{d}s. \tag{3.18}$$

Since $\langle\lambda(t_1)\rangle^\alpha \leq \langle\lambda(t_2)\rangle^\alpha$ if $0 \leq t_1 \leq t_2 \leq T$ and $\alpha \geq 0$, and

$$\mathrm{e}^{-\lambda^2(t)/4}\mathrm{e}^{\lambda^2(t-s)/4}\mathrm{e}^{\lambda^2(s)/2} = \mathrm{e}^{\lambda^2(s)/4} \leq \mathrm{e}^{\lambda_0^2/4}, \tag{3.19}$$

we obtain

$$|B(u,v)| \leq \gamma|u|\,|v| \text{ with } \gamma = C_3\langle\lambda_0\rangle^{5/2-r}\mathrm{e}^{\lambda_0^2/4}T^{r/4-1/8}. \tag{3.20a}$$

(Notice that the cancellation (3.19) accounts for the "improvement" $\mathrm{e}^{\lambda^2/2} \to \mathrm{e}^{\lambda_0^2/4}$ compared to (3.8).)

As for (3.9) and (3.10) we obtain similar "improvements"

$$\|\mathrm{e}^{\theta(t)A}B(u,v)(t)\|_{\dot{H}^r} \leq C\langle\lambda_0\rangle^{5/2-r}\mathrm{e}^{\lambda_0^2/2}|u|\,|v| \text{ for } t \in I, \tag{3.20b}$$

and for $r \in [1/2, 5/4]$

$$\|\mathrm{e}^{\theta(t)A}B(u,v)(t)\| \leq \widetilde{C}\langle\lambda_0\rangle^{5/2-2r}\mathrm{e}^{\lambda_0^2/2}t^{r/2}|u|\,|v| \text{ for } t \in I. \tag{3.20c}$$

Arguing similarly for the setting of Subsection 3.2 we obtain in this case

$$|B(u,v)| \leq \gamma|u|\,|v| \text{ with } \gamma = C_3\langle\lambda_0\rangle^{\bar{r}}\mathrm{e}^{\lambda_0^2/4}\langle T\rangle^{\bar{r}/2}T^{s_1}, \tag{3.21a}$$

and

$$\|\mathrm{e}^{\theta(t)A}B(u,v)(t)\|_{H^r} \leq C\langle\lambda_0\rangle^{\bar{r}}\mathrm{e}^{\lambda_0^2/2}\langle T\rangle^{\bar{r}/2}|u|\,|v| \text{ for } t \in I. \tag{3.21b}$$

Let us now investigate the conditions (2.10) with $R = |Y|$: Due to (3.3) and (3.20a) it suffices to have

$$C_4\langle\lambda_0\rangle^{5/2-r}\mathrm{e}^{\lambda_0^2/4}T^{r/2-1/4} < 1;\; C_4 = 4C_1C_3\|A^r u_0\|. \tag{3.22}$$

Notice that the constants C_1 and C_3 from (3.3) and (3.20a), respectively, are independent of λ_0, T and u_0. Therefore also C_4 is independent of λ_0 and T.

Now, assuming $r \in]1/2, 3/2[$, we fix $\epsilon \in]0, 2r-1]$. Taking then

$$\lambda_0 = \sqrt{2r-1-\epsilon}\,\sqrt{|\ln T|} \tag{3.23}$$

indeed (3.22) is valid provided $T > 0$ is small enough, viz. $T \leq T_0 = T\big(\epsilon, r, \tilde{r}, \|A^r u_0\|\big)$.

Similarly in the setting of Subsection 3.2 (due to (3.12) and (3.21a)) the conditions (2.10) with $R = |Y|$ are valid if

$$C_4 \langle \lambda_0 \rangle^{\tilde{r}} \mathrm{e}^{\lambda_0^2/4} \langle T \rangle^{\tilde{r}/2} T^{2s_1} < 1; \; C_4 = 4C_1C_3 \|\langle A \rangle^r u_0\|. \tag{3.24}$$

Fix $\epsilon \in]0, 2\tilde{r} - 1]$. Taking then

$$\lambda_0 = \sqrt{2\tilde{r} - 1 - \epsilon} \; \sqrt{|\ln T|} \tag{3.25}$$

the bound (3.24) is valid provided $T \leq T_0 = T\big(\epsilon, r, \tilde{r}, \|\langle A \rangle^r u_0\|\big)$.

We have (almost) proved:

Theorem 3.4. i) *Suppose $u_0 \in \dot{H}^r$ for some $r \in]1/2, 3/2[$. Put $\tilde{r} = \max(r, 5/4)$, $s_1 = \tilde{r}/2 - r/4 - 1/8$ and $s_2 = \tilde{r}$, and let $\epsilon \in]0, 2r - 1]$. There exists $T_0 = T\big(\epsilon, r, \tilde{r}, \|A^r u_0\|\big) > 0$ such that for any $T \in]0, T_0]$ the integral equation* (2.8) *has a unique solution in the ball $B_{2|Y|} \subseteq \mathcal{B}$ where $\mathcal{B} = \mathcal{B}_{\zeta,\theta,I,s_1,s_2}$ has norm* (2.3) *with ζ and θ given by* (3.1), (3.17) *and* (3.23), *$I =]0, T]$ and $Y(t) = \mathrm{e}^{-tA^2} u_0$. This solution $X \in \mathcal{B}^0$, and it obeys that $\mathrm{e}^{\theta A} X \in BC(\bar{I}, \dot{H}^r)$ with $X(0) = u_0$. If in addition $u_0 \in L^2$ and $r \in]1/2, 5/4]$ then $\mathrm{e}^{\theta A} X \in BC(\bar{I}, H^r)$.*

Moreover there are bounds

$$\|\mathrm{e}^{\theta(t)A} X(t)\|_{\dot{H}^r} \leq C T^{-(r/2 - 1/4 - \epsilon/4)} \text{ for } t \in I, \tag{3.26a}$$

and assuming in addition $u_0 \in L^2$ and $r \in]1/2, 5/4]$

$$\|\mathrm{e}^{\theta(t)A} X(t)\| \leq \widetilde{C} T^{-(r/2 - 1/4 - \epsilon/4)} \text{ for } t \in I. \tag{3.26b}$$

The dependence of the constants C and $\widetilde{C}$ on u_0 is through the quantity $\|A^r u_0\|$ and through the quantities $\|A^r u_0\|$ and $\|u_0\|$, respectively.

ii) *Suppose $u_0 \in H^r$ for some $r \in [5/4, \infty[$. Let $\tilde{r} \in [5/4, 3/2[$ be given such that $r \geq \tilde{r}$ and let $\epsilon \in]0, 2\tilde{r} - 1]$. Put $s_1 = \tilde{r}/4 - 1/8$ and $s_2 = r$. There exists $T_0 = T\big(\epsilon, r, \tilde{r}, \|\langle A \rangle^r u_0\|\big) > 0$ such that for any $T \in]0, T_0]$ the integral equation* (2.8) *has a unique solution in the ball $B_{2|Y|} \subseteq \mathcal{B}$ where $\mathcal{B} = \mathcal{B}_{\zeta,\theta,I,s_1,s_2}$ has norm* (2.25) *with ζ and θ given by* (3.11), (3.17) *and* (3.25), *$I =]0, T]$ and $Y(t) = \mathrm{e}^{-tA^2} u_0$. This solution $X \in \mathcal{B}^0$, and it obeys that $\mathrm{e}^{\theta A} X \in BC(\bar{I}, H^r)$ with $X(0) = u_0$.*

Moreover

$$\|\mathrm{e}^{\theta(t)A} X(t)\|_{H^r} \leq C T^{-(\tilde{r}/2 - 1/4 - \epsilon/4)} \text{ for } t \in I. \tag{3.27}$$

The dependence of the constant C on u_0 is through the quantity $\|\langle A \rangle^r u_0\|$.

Proof. We use the arguments preceeding the theorem to get (unique) solutions.

As for i the bounds (3.26a) and (3.26b) follow from (3.20b) and (3.20c), respectively, and (3.3) and Lemma 3.1 (taking $\alpha = 0$ there). Notice that the contributions from the non-linear term $B(X, X)$ have better bounds.

As for ii the bound (3.27) follows from (3.12), (3.21b) and Lemma 3.1 (taking again $\alpha = 0$ there). Again the contribution from the non-linear term has a better bound. □

By choosing $t = T$ in (3.26a)–(3.27) we obtain:

Corollary 3.5. *Under the conditions of Theorem 3.4 i the solution X obeys*

$$\|\mathrm{e}^{\sqrt{2r - 1 - \epsilon} \; \sqrt{|\ln t|} \sqrt{t} A} X(t)\|_{\dot{H}^r} \leq C t^{-(r/2 - 1/4 - \epsilon/4)} \text{ for all } t \in]0, T_0], \tag{3.28a}$$

and assuming in addition $u_0 \in L^2$ *and* $r \in]1/2, 5/4]$

$$\|\mathrm{e}^{\sqrt{2r-1-\epsilon}\ \sqrt{|\ln t|}\sqrt{t}A} X(t)\| \leq \widetilde{C} t^{-(r/2-1/4-\epsilon/4)} \text{ for all } t \in]0, T_0]. \tag{3.28b}$$

Under the conditions of Theorem 3.4 ii the solution X *obeys*

$$\|\mathrm{e}^{\sqrt{2\tilde{r}-1-\epsilon}\ \sqrt{|\ln t|}\sqrt{t}A} X(t)\|_{H^r} \leq C t^{-(\tilde{r}/2-1/4-\epsilon/4)} \text{ for all } t \in]0, T_0]. \tag{3.29}$$

Clearly the dependence of the constant $\widetilde{C}$ in (3.28b) of $u_0 \in H^r$ can be taken through its norm $\|\langle A\rangle^r u_0\|$. Moreover, if $u_0 \in H^r$ for some $r \in [5/4, 3/2[$ we can choose $\tilde{r} = r$ in (3.29). Whence in particular we obtain from Corollary 3.5:

Corollary 3.6. *Suppose* $u_0 \in H^r$ *for some* $r \in]1/2, 3/2[$*. Let* X *be the solution to* (2.2) *with initial data* u_0 *as given in Proposition 2.4, and let* $\epsilon \in]0, 2r-1]$*. Then there exist constants* $C_0 = C\big(\epsilon, r, \|\langle A\rangle^r u_0\|\big) > 0$ *and* $T_0 = T\big(\epsilon, r, \|\langle A\rangle^r u_0\|\big) > 0$ *such that*

$$\|\mathrm{e}^{\sqrt{2r-1-\epsilon}\ \sqrt{t|\ln t|}A} X(t)\|_{H^r} \leq C_0 t^{-(r/2-1/4-\epsilon/4)} \text{ for all } t \in]0, T_0]. \tag{3.30}$$

In particular, using notation from Subsection 1.1, for this solution to (2.2)

$$\liminf_{t\to 0} \frac{\mathrm{rad}(X(t))}{\sqrt{t|\ln t|}} \geq \sqrt{2r-1}. \tag{3.31}$$

3.4. **Discussion.** We are only allowed to put $\tilde{r} = r$ in (3.29) if $r \in [5/4, 3/2[$ due to the restriction $\tilde{r} < 3/2$ of Theorem 3.4 ii.

One may conjecture that also in the case $r > 3/2$ the quantities

$$\|\mathrm{e}^{\sqrt{2r-1-\epsilon}\ \sqrt{t|\ln t|}A} X(t)\|_{H^r};\ \epsilon \in]0, 2r-1], \tag{3.32}$$

are all finite for t sufficiently small (given that $u_0 \in H^r$). However the proof for $r \in [5/4, 3/2[$ does not provide any indication.

On the other hand if u_0 is "much smoother", in fact analytic, there is indeed an improvement. Suppose there exists $\theta_0 > 0$ such that one of the following conditions holds

A) In addition to the assumptions of Theorem 3.4 i $\mathrm{e}^{\theta_0 A} u_0 \in \dot{H}^r$ (and possibly $\mathrm{e}^{\theta_0 A} u_0 \in L^2$).
B) In addition to the assumptions of Theorem 3.4 ii $\mathrm{e}^{\theta_0 A} u_0 \in H^r$.

Then we have the following version of Corollary 3.5: With A we can in (3.28a) replace $X(t)$ by $\mathrm{e}^{\theta_0 A} X(t)$ (and similarly in (3.28b)). With B we can in (3.29) replace $X(t)$ by $\mathrm{e}^{\theta_0 A} X(t)$.

The proof of these statements goes along the same line as the previous ones. Notice that we only have to check the previous proofs with $\theta \to \theta + \theta_0$ in various bounds. In particular this replacement is introduced in the construction of Banach spaces.

Another point to be discussed is the limit $r \to 1/2$. Clearly the expression $\sqrt{2r-1}$, related to (3.32), vanishes in this limit. One may ask if this kind of behaviour is expected. Clearly this question is related to whether Theorem 3.2 should be considered as being "optimal" for $r = 1/2$. There is a partial affirmative answer to the latter given by an example: We shall construct a specific (classical) solution to (2.1) for specific M and P such that

$$u \in BC([0,\infty[, H^r) \text{ and } u_0 \in H^r \cap \dot{B}^{1/2}_{2,\infty} \text{ for all } r < 1/2, \tag{3.33}$$

and for which

$$\lim_{t\to 0}\tfrac{\mathrm{rad}(u(t))}{\sqrt{t}}=\sqrt{6}. \tag{3.34}$$

This $u_0\notin \dot B^{1/2,0}_{2,\infty}$ and hence $u_0\notin \dot H^{1/2}$ (note the inclusions (2.13)).

Note, in comparison with (3.34), that by Theorem 3.2 for any $u_0\in H^{1/2}$ the corresponding solution obeys

$$\lim_{t\to 0}\tfrac{\mathrm{rad}(u(t))}{\sqrt{t}}=\infty. \tag{3.35}$$

Moreover the theory of Subsections 2.2 and 3.1 can be extended to the case of data $u_0\in \dot B^{1/2,0}_{2,\infty}$, cf. a discussion in the beginning of the proof of Proposition 2.2. In particular for $u_0\in L^2\cap \dot B^{1/2,0}_{2,\infty}$ (3.35) remains true for the corresponding solution. In fact the theory can be extended to the case $u_0\in \dot B^{1/2}_{2,\infty}$ provided that $\|u_0\|_{\dot B^{1/2}_{2,\infty}}$ is sufficiently small (so that (2.10) is fulfilled). This leads to the existence of a unique real-analytic global solution, cf. Subsection 2.4. However in that case we can only conclude weaker analyticity statements. To be specific, if $u_0\in L^2\cap \dot B^{1/2}_{2,\infty}$ and $\|u_0\|_{\dot B^{1/2}_{2,\infty}}$ is sufficiently small we can conclude that $\liminf_{t\to 0}\big(\mathrm{rad}(u(t))/\sqrt{t}\big)\geq\kappa$ for some $\kappa>0$ that depends on the (small) norm $\|u_0\|_{\dot B^{1/2}_{2,\infty}}$. We have not calculated this norm for the specific example given below, and consequently we do not know whether the example and therefore in particular (3.34) fit into this extended theory.

3.4.1. *An example.* Motivated by [WJZHJ] we use the Hopf-Cole transformation [Ev] and obtain a solution of the vector Burgers' equation: If $v=v(t,x)$ is a positive solution to the heat equation

$$\frac{\partial}{\partial t}v=\triangle v;\ t>0, \tag{3.36}$$

then $w:=-2\ln v$ fulfills

$$\frac{\partial}{\partial t}w+\tfrac12|\nabla w|^2=\triangle w. \tag{3.37}$$

Taking first order partial derivatives in (3.37) we get

$$\frac{\partial}{\partial t}\partial_j w+(\nabla w\cdot\nabla)\partial_j w=\triangle\partial_j w;\ j=1,2,3. \tag{3.38}$$

Next take $M=I$ (in fact M can be any invertible real 3×3–matrix), $P=I$ and $u=\nabla w$ (or more generally $u=M^{-1}\nabla w$) we obtain from (3.38)

$$\frac{\partial}{\partial t}u+(Mu\cdot\nabla)u-\triangle u=0 \text{ and } u=Pu. \tag{3.39}$$

In particular we have constructed a solution to (2.1) with $u_0=u(0,\cdot)$.

We choose

$$v(t,x)=1-(t+1)^{-3/2}\mathrm{exp}\big(-\tfrac{|x|^2}{4(t+1)}\big). \tag{3.40}$$

Clearly (3.36) holds. We compute

$$u_j(t,x)=\partial_j w(t,x)=-(t+1)^{-5/2}x_j\mathrm{exp}\big(-\tfrac{|x|^2}{4(t+1)}\big)\ /\ v(t,x), \tag{3.41}$$

from which we obtain

$$(u_0)_j(x)=\partial_j w_0(x)=-x_j\mathrm{exp}\big(-\tfrac{|x|^2}{4}\big)\ /\ \big(1-\mathrm{exp}\big(-\tfrac{|x|^2}{4}\big)\big). \tag{3.42}$$

Here the denominator vanishes like $|x|^2/4$ at $x=0$. Consequently the components of u_0 have a Coulomb singularity at $x=0$. In Fourier space this behaviour corresponds to a decay like $\xi_j|\xi|^{-3}$ at infinity, cf. (3.50) stated below. Whence indeed $u_0 \in L^2 \cap \big(\dot{B}^{1/2}_{2,\infty} \setminus \dot{B}^{1/2,0}_{2,\infty}\big)$.

As for the property (3.33) we notice the (continuous) embedding, cf. (2.13),

$$L^2 \cap \dot{B}^{1/2}_{2,\infty} \subseteq H^r \text{ for } r \in [0, 1/2[, \tag{3.43}$$

which by a scaling argument leads to the bound

$$\|f\|_{\dot{H}^r} \leq C_r \|f\|_{L^2}^{1-2r} \|f\|^{2r}_{\dot{B}^{1/2}_{2,\infty}} \text{ for } f \in L^2 \cap \dot{B}^{1/2}_{2,\infty} \text{ and } r \in [0, 1/2[. \tag{3.44}$$

We note the properties

$$u \in BC([0,\infty[, L^2), \tag{3.45}$$

$$u \in B([0,\infty[, \dot{B}^{1/2}_{2,\infty}). \tag{3.46}$$

Only (3.46), or equivalently

$$\sup_{t \geq 0} \|u(t,\cdot)\|_{\dot{B}^{1/2}_{2,\infty}} < \infty, \tag{3.47}$$

needs an elaboration.

To prove (3.47) we introduce for $\kappa \geq 1$ and $j = 1, 2, 3$ the functions

$$f_j(\kappa, y) = y_j \exp\big(-\tfrac{|y|^2}{4}\big) / \big(\kappa - \exp\big(-\tfrac{|y|^2}{4}\big)\big),$$
$$\tilde{f}_j(\kappa, y) = 4y_j / \big(\tilde{\kappa} + |y|^2\big);\ \tilde{\kappa} = 4(\kappa - 1),$$

and notice that

$$u_j(t,x) = -(t+1)^{-1/2} f_j((t+1)^{3/2}, x/\sqrt{t+1}).$$

We pick $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(s) = 1$ for $|s| < 1$. By a scaling argument (3.47) will follow from the bound

$$\sup_{\kappa \geq 1} \|\chi(|\cdot|) f_j(\kappa, \cdot)\|_{\dot{B}^{1/2}_{2,\infty}} < \infty. \tag{3.48}$$

The proof of (3.48) relies on a comparison argument. We notice that

$$\sup_{\kappa \geq 1} \|\chi(|\cdot|) \tilde{f}_j(\kappa, \cdot)\|_{\dot{B}^{1/2}_{2,\infty}} < \infty, \tag{3.49}$$

which may be seen as follows: First we notice the representation of the Fourier transform

$$(F\tilde{f}_j)(\kappa, \xi) = C\partial_{\xi_j} \big\{|\xi|^{-1} \int_0^\infty s^{-3/2} \exp\big(-(4s)^{-1}\big) \mathrm{e}^{-\tilde{\kappa}|\xi|^2 s} \mathrm{d}s\big\}. \tag{3.50}$$

By computing the derivative and then estimating the second exponential ≤ 1 we deduce the bound $|(F\tilde{f}_j)(\kappa, \xi)| \leq C|\xi|^{-2}$ uniformly in $\kappa \geq 1$. Using this estimate and the convoluton integral representation of the product we obtain that $|(F\{\chi\tilde{f}_j\})(\kappa, \xi)| \leq C\langle\xi\rangle^{-2}$ uniformly in $\kappa \geq 1$ from which (3.49) follows.

Due to (3.49) and (2.13) it suffices for (3.48) to show

$$\sup_{\kappa \geq 1} \|\chi(|\cdot|)\{f_j(\kappa, \cdot) - \tilde{f}_j(\kappa, \cdot)\}\|_{H^1} < \infty. \tag{3.51}$$

Clearly (3.51) follows from the uniform pointwise bounds

$$|\chi(|y|)\{f_j(\kappa, y) - \tilde{f}_j(\kappa, y)\}| \leq C|y| \text{ and } |\nabla\big(\chi(|y|)\{f_j(\kappa, y) - \tilde{f}_j(\kappa, y)\}\big)| \leq C,$$

which in turn follow from elementary Taylor expansion.

We conclude from (3.44)–(3.46) that indeed

$$u \in BC([0,\infty[, H^r) \text{ for } r \in [0,1/2[. \tag{3.52}$$

As for the property (3.34) we claim more generally that

$$\mathrm{rad}(u(t))^2 = 6(t+1)\ln(t+1) \text{ for all } t > 0. \tag{3.53}$$

To see this note that $v(t,iy)$ is zero on the surface $|y|^2 = 6(t+1)\ln(t+1)$ and if $|y|^2 < (1-\epsilon)6(t+1)\ln(t+1)$ then $|v(t,x+iy)| > 1-(t+1)^{-3\epsilon/2}$.

4. Analyticity bounds for all times

In this section we shall study analyticity properties of the global small data solutions of Proposition 2.7.

4.1. Global analyticity bounds in $\dot H^{1/2}$ and $H^{1/2}$ for small data. Let us begin this subsection by considering $r \in [1/2, 3/2[$ as in Subsection 3.1. The contraction condition (2.10) leads to the following combination of (3.3) and (3.8)

$$C_4\langle\lambda\rangle^{5/2-r}\mathrm{e}^{\lambda^2/2}T^{r/2-1/4} < 1;\ C_4 = 4C_1C_3\|A^r u_0\|. \tag{4.1}$$

The constants C_1 and C_3 from (3.3) and (3.8), respectively, are independent of λ, T and u_0 (but depend on r).

Obviously (4.1) cannot be fulfilled for $T=\infty$ unless $r=1/2$. On the other hand if $r=1/2$ and $\|A^r u_0\|$ is sufficiently small the condition is fulfilled for $T=\infty$ for $\lambda \geq 0$ smaller than some critical positive number. This observation leads to the following global analyticity result (using again Lemma 3.1, (3.9) and (3.10)):

Proposition 4.1. *Suppose $u_0 \in \dot H^{1/2}$ and that the constant $C_4 = 4C_1C_3\|A^{1/2}u_0\|$ in* (4.1) *(with $r=1/2$) obeys $C_4<1$. For $u_0 \neq 0$ define $\bar\lambda > 0$ as the solution to the equation*

$$4C_1C_3\|A^{1/2}u_0\|\langle\bar\lambda\rangle^2\mathrm{e}^{\bar\lambda^2/2} = 1.$$

If $u_0 = 0$ define $\bar\lambda = \infty$.

Then the solution X to the integral equation (2.8) *as constructed in Proposition 2.7 obeys the following bounds uniformly in $\lambda \in [0,\bar\lambda[$ and $t>0$*

$$\|A^{5/4}\mathrm{e}^{\lambda\sqrt{tA}}X(t)\| \leq 2C_1\|A^{1/2}u_0\|\langle\lambda\rangle^{3/4}\mathrm{e}^{\lambda^2/4}t^{-3/8}, \tag{4.2a}$$

$$\|A^{1/2}\mathrm{e}^{\lambda\sqrt{tA}}X(t)\| \leq C\Big(\mathrm{e}^{\lambda^2/4}\|A^{1/2}u_0\| + \langle\lambda\rangle^2\mathrm{e}^{3\lambda^2/4}\big(2C_1\|A^{1/2}u_0\|\big)^2\Big), \tag{4.2b}$$

and if in addition $u_0 \in L^2$,

$$\|\mathrm{e}^{\lambda\sqrt{tA}}X(t)\| \leq \widetilde C\Big(\mathrm{e}^{\lambda^2/4}\|u_0\| + \langle\lambda\rangle^{3/2}\mathrm{e}^{3\lambda^2/4}\big(2C_1\|A^{1/2}u_0\|\big)^2t^{1/4}\Big). \tag{4.2c}$$

4.2. Improved global analyticity bounds in $\dot H^{1/2}$ and $H^{1/2}$ for small data. In this subsection we improve on the results of Subsection 4.1 along the line of the method of Subsection 3.3. Whence we fix $u_0 \in \dot H^{1/2}$ and $T \in]0,\infty[$ and define the underlying Banach space $\mathcal{B}$ in terms of the finite interval $I =]0,T]$ and a time-dependent choice of λ (and (3.1) with $\tilde r - r = 3/4$, and the parameters $s_1 = 3/8$ and $s_2 = 5/4$). Explicitly we choose $\lambda = \lambda(t)$ as in (3.17) to be used in the expression (3.1) (with $\tilde r - r = 3/4$). As in the previous subsection we will need the parameter λ_0 of (3.17) to be smaller than a certain critical positive number dictated by the contraction condition (3.22) (with $r=1/2$). That is we need

$$C_4\langle\lambda_0\rangle^2\mathrm{e}^{\lambda_0^2/4} < 1;\ C_4 = 4C_1C_3\|A^{1/2}u_0\|; \tag{4.3}$$

here the constants C_1 and C_3 from (3.3) and (3.20a), respectively, are (again) independent of λ, T and u_0. (Notice that (4.3) "improves" (4.1) (for $r = 1/2$) in that the exponent $\lambda^2/2 \to \lambda_0^2/4$.) Mimicking the proofs of Theorem 3.4 and Corollary 3.5 we obtain under the condition (4.3) the following improvement of Proposition 4.1:

Theorem 4.2. *i) Suppose $u_0 \in \dot{H}^{1/2}$ and that the constant $C_4 = 4C_1C_3\|A^{1/2}u_0\|$ in (4.3) obeys $C_4 < 1$. For $u_0 \neq 0$ define $\bar{\lambda} > 0$ as the solution to the equation*

$$4C_1C_3\|A^{1/2}u_0\|\langle\bar{\lambda}\rangle^2 \mathrm{e}^{\bar{\lambda}^2/4} = 1. \tag{4.4}$$

If $u_0 = 0$ define $\bar{\lambda} = \infty$.

Then the solution X to the integral equation (2.8) as constructed in Proposition 2.7 obeys the following bounds uniformly in $\lambda_0 \in [0, \bar{\lambda}[$ and $t > 0$

$$\|A^{5/4}\mathrm{e}^{\lambda_0\sqrt{t}A}X(t)\| \leq 2C_1\|A^{1/2}u_0\|\langle\lambda_0\rangle^{3/4}\mathrm{e}^{\lambda_0^2/4}t^{-3/8}, \tag{4.5a}$$

$$\|A^{1/2}\mathrm{e}^{\lambda_0\sqrt{t}A}X(t)\| \leq C\Big(\mathrm{e}^{\lambda_0^2/4}\|A^{1/2}u_0\| + \langle\lambda_0\rangle^2\mathrm{e}^{\lambda_0^2/2}\big(2C_1\|A^{1/2}u_0\|\big)^2\Big). \tag{4.5b}$$

ii) Suppose in addition that $u_0 \in L^2$. Then

$$\|\mathrm{e}^{\lambda_0\sqrt{t}A}X(t)\| \leq \widetilde{C}\Big(\mathrm{e}^{\lambda_0^2/4}\|u_0\| + \langle\lambda_0\rangle^{3/2}\mathrm{e}^{\lambda_0^2/2}\big(2C_1\|A^{1/2}u_0\|\big)^2t^{1/4}\Big). \tag{4.5c}$$

We will use the following corollary in Subsection 7.3. We omit its proof.

Corollary 4.3. *Suppose $u_0 \in \dot{H}^{1/2}$ and that the constant $C_4 = 4C_1C_3\|A^{1/2}u_0\|$ in (4.3) obeys $C_4 < 1$. Suppose $\liminf_{t\to\infty}\|A^{1/2}X(t)\| = 0$ where X is the solution to the integral equation (2.8) as constructed in Proposition 2.7. Then for any $\lambda \geq 0$, as $t \to \infty$*

$$t^{3/8}\|A^{5/4}\mathrm{e}^{\lambda\sqrt{t}A}X(t)\| = o(1), \tag{4.6a}$$

$$\|A^{1/2}\mathrm{e}^{\lambda\sqrt{t}A}X(t)\| = o(1). \tag{4.6b}$$

Remark 4.4. If $u_0 \in H^{1/2}$ and $\bar{\lambda} > 0$ obeys (4.4) then clearly $\bar{\lambda}\sqrt{t}$ is a lower bound of $\mathrm{rad}(X(t))$. In particular in the sense of taking the limit $\|A^{1/2}u_0\| \to 0$ we obtain

$$\liminf \frac{\mathrm{rad}(X(t))}{\sqrt{-4\ln\|A^{1/2}u_0\|}\sqrt{t}} \geq 1 \text{ uniformly in } t > 0. \tag{4.7}$$

Clearly this statement is weak in the small time regime compared to (3.35). On the other hand, as demonstrated in Section 6, (4.7) is useful for obtaining further improved bounds of the analyticity radius in the large time regime.

5. Differential inequalities for small global solutions in $\dot{H}^{1/2}$ and $H^{1/2}$

In this section we continue our study of analyticity bounds of global small data solutions in $\dot{H}^{1/2}$ and $H^{1/2}$ initiated in the previous section. This study will be continued and completed in Section 6 where some optimal analyticity radius bounds in the large time regime will be presented. We impose throughout this section the conditions of Theorem 4.2 i. Notice that a simplified version of the bound (4.5b) takes the form

$$\|A^{1/2}\mathrm{e}^{\lambda_0\sqrt{t}A}X(t)\| \leq \breve{C}\langle\bar{\lambda}\rangle^{-2} \text{ for all } \lambda_0 \in [0, \bar{\lambda}[\text{ and } t > 0. \tag{5.1}$$

Our goal is twofold:

1) Under an additional decay condition of the quantity $\|A^{1/2}X(t)\|$ we shall improve on the right hand side of (5.1) in the large time regime. A similar improvement of (4.5c) will be established in terms of decay of the quantity $\|X(t)\|$.
2) Under an additional decay condition on the quantity $\|X(t)\|$ we shall show decay of the quantity $\|A^{1/2}X(t)\|$.

We think 1 has some independent interest, although more refined bounds will be presented in Section 6 (in particular presumably better bounds on analyticity radii than can be derived from the methods presented here). As for 2, our result will be used in Section 6.

The analysis is partly inspired by [FT], [Sc1] and [OT].

5.1. **Energy inequality.** Partly as a motivation we recall here a version of the energy inequality well-known for a class of solutions to (1.1). We state it for the function X of Theorem 4.2 i subject to the further conditions

$$u_0 = Pu_0 \in L^2 \text{ and } \nabla \cdot \big(MX(t)\big) = 0 \text{ for all } t > 0. \tag{5.2}$$

Notice that the second condition of (5.2) is fulfilled for the problem (1.1). Using Theorem 4.2, (2.1) and (5.2) we can derive

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|X(t)\|^2 = -2\|AX(t)\|^2 + 2\big\langle X(t), (MX(t)\cdot\nabla)X(t)\big\rangle = -2\|AX(t)\|^2 \text{ for all } t > 0. \tag{5.3}$$

We refer the reader to Subsection 7.1 for a discussion relevant for this derivation. (Actually a more general result than (5.3) is stated in Corollary 7.7 in Subsection 7.2.) We obtain by integrating (5.3)

$$\|X(t)\|^2 = \|u_0\|^2 - 2\int_0^t \|AX(s)\|^2\,\mathrm{d}s \text{ for all } t > 0. \tag{5.4}$$

In particular the *energy inequality* $\|X(t)\|^2 \leq \|u_0\|^2$ holds. Clearly this bound improves the bound $\|X(t)\| = O(t^{1/4})$ of (4.2c) at infinity.

We notice that for the problem (1.1) and under the above conditions it can be proven that the quantity $\|X(t)\| = o(t^0)$ as $t \to \infty$. Under some further (partly generic) conditions it is shown in [Sc1] that $\|X(t)\| = O(t^{-5/4})$ while $\|X(t)\| \neq o(t^{-5/4})$.

5.2. **Differential inequalities for exponentially weighted Sobolev norms.** Under the conditions of Theorem 4.2 i introduce for $r \geq 0$ and $\lambda_0 \in]0, \bar\lambda[$ the quantities

$$J_r(t) = \|A^r X(t)\|^2 \text{ and } G_r(t) = \|A^r \mathrm{e}^{\lambda_0\sqrt{t}A} X(t)\|^2 \text{ for } t > 0. \tag{5.5}$$

We are mainly interested in these quantities for $r = 1/2$ and $r = 0$. Any consideration for $r \in [0, 1/2[$ will involve the additional requirement $u_0 \in L^2$. Under the additional conditions (5.2) we have, due to the previous subsection, the a priori bound $J_0(t) = O(t^{-\sigma})$ with $\sigma = 0$ for $t \to \infty$ however in the following we do not assume (5.2).

Lemma 5.1. *For all $\kappa > 0$ and $\lambda_0 \in]0, \bar\lambda[$ there exists $K = K(\kappa, \lambda_0) > 0$ such that*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} G_{1/2}(t) \leq -\kappa t^{-1} G_{1/2}(t) + K t^{-1} J_{1/2}(t) \textit{ for all } t > 0. \tag{5.6}$$

Proof. We compute

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}G_{1/2}(t) &= -2G_{3/2}(t) + \lambda_0 t^{-1/2}G_1(t) + R(t); \\ R(t) &= 2\langle A^{1/2}\mathrm{e}^{\lambda_0\sqrt{t}A}X(t), A^{1/2}\mathrm{e}^{\lambda_0\sqrt{t}A}P(MX(t)\cdot\nabla)X(t)\rangle. \end{aligned} \tag{5.7}$$

By using the Cauchy-Schwarz inequality and Lemma 2.1 we are led to the bounds

$$R(t) \le 2G_1(t)^{1/2}CG_1(t)^{1/2}G_{3/2}(t)^{1/2} \le G_{3/2}(t) + C^2G_1(t)^2 \text{ for all } t > 0. \tag{5.8}$$

Now, pick any $\lambda_1 \in]\lambda_0, \bar\lambda[$. We can estimate the second term on the right hand side of (5.8) by first using the Cauchy-Schwarz inequality and (5.1) (with $\lambda_0 \to \lambda_1$) to obtain

$$\begin{aligned} G_1(t)^2 \le G_{3/2}(t)G_{1/2}(t) \le \tfrac{\sup_{x\ge 0} x^2\mathrm{e}^{-2x}}{(\lambda_1-\lambda_0)^2 t} G_{1/2,\lambda_1}(t)G_{1/2}(t) \le \widetilde{C}t^{-1}G_{1/2}(t); \\ \widetilde{C} = C(\lambda_0)\breve{C}\langle\bar\lambda\rangle^{-2}. \end{aligned} \tag{5.9}$$

Clearly (5.7)–(5.9) lead to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}G_{1/2}(t) \le -G_{3/2}(t) + \lambda_0 t^{-1/2}G_1(t) + C^2\widetilde{C}t^{-1}G_{1/2}(t). \tag{5.10}$$

Next we insert $0 = \kappa t^{-1}G_{1/2}(t) - Kt^{-1}J_{1/2}(t) - \kappa t^{-1}G_{1/2}(t) + Kt^{-1}J_{1/2}(t)$ on the right hand side of (5.10). We need to examine the condition

$$-G_{3/2}(t) + \lambda_0 t^{-1/2}G_1(t) + (\kappa + C^2\widetilde{C})t^{-1}G_{1/2}(t) - Kt^{-1}J_{1/2}(t) \le 0. \tag{5.11}$$

By the spectral theorem the bound (5.11) will follow from

$$-x^3\mathrm{e}^{2\lambda_0 x} + \lambda_0 x^2\mathrm{e}^{2\lambda_0 x} + (\kappa + C^2\widetilde{C})x\mathrm{e}^{2\lambda_0 x} \le Kx \text{ for all } x \ge 0. \tag{5.12}$$

The estimate (5.12) is obviously fulfilled for some $K = K(\kappa, \lambda_0) > 0$. ☐

Corollary 5.2. *i) For all $\kappa > 0$ and $\lambda_0 \in]0, \bar\lambda[$ there exists $K = K(\kappa, \lambda_0) > 0$ such that*

$$G_{1/2}(t) \le Kt^{-\kappa}\int_0^t s^{\kappa-1}J_{1/2}(s)\,\mathrm{d}s \text{ for all } t > 0. \tag{5.13}$$

ii) Suppose that for some $\sigma > -1/2$ the bound $J_{1/2}(t) = O(t^{-\sigma-1/2})$ for $t \to \infty$ holds. Then $J_r(t) = O(t^{-\sigma-r})$ and $G_r(t) = O(t^{-\sigma-r})$ for all $r \ge 1/2$.

iii) Suppose the conditions of Theorem 4.2 ii and that for some $\sigma > -1/2$ the bound $J_0(t) = O(t^{-\sigma})$ holds. Then $J_r(t) = O(t^{-\sigma-r})$ and $G_r(t) = O(t^{-\sigma-r})$ for all $r \ge 0$.

iv) Suppose the conditions of Theorem 4.2 ii and $J_0(t) = O(1)$. Then $G_0(t) = o(1)$.

Proof. As for i notice that t^κ is an integrating factor for (5.6).

For ii we choose $\kappa > 1/2 + \sigma$ in the bound (5.13) yielding the bound $G_{1/2}(t) = O(t^{-\sigma-1/2})$. Whence also $G_r(t) = O(t^{-\sigma-r})$ for all $r \ge 1/2$ (here we used the quantity $G_{1/2,\lambda_1}(t)$ of (5.9)). In particular $J_r(t) = O(t^{-\sigma-r})$ for all $r \ge 1/2$.

For iii we first prove that $G_{1/2}(t) = O(t^{-\sigma-1/2})$. By the Cauchy-Schwarz inequality

$$J_{1/2}(t) \le \sup_{x\ge 0} x^{1/2}\mathrm{e}^{-\lambda_0\sqrt{t}x}\, J_0(t)^{1/2}G_{1/2}(t)^{1/2} \le \tfrac{C}{\lambda_0\sqrt{t}}KJ_0(t) + G_{1/2}(t)/K. \tag{5.14}$$

In combination with (5.6) this estimate leads to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}G_{1/2}(t) \le -(\kappa-1)t^{-1}G_{1/2}(t) + C(\kappa,\lambda_0)t^{-3/2}J_0(t);\ C(\kappa,\lambda_0) = K^2\tfrac{C}{\lambda_0}. \tag{5.15}$$

By choosing $\kappa > 2+\sigma$ in (5.15) we deduce the following analogue of (5.13) where $\widetilde{K} := C(\kappa,\lambda_0)$ and $\tilde{\kappa} := \kappa - 1$

$$G_{1/2}(t) \le \widetilde{K} t^{-\tilde{\kappa}} \int_0^t s^{\tilde{\kappa}-3/2} J_0(s)\,\mathrm{d}s \text{ for all } t>0. \tag{5.16}$$

It follows from (5.16) that indeed $G_{1/2}(t) = O(t^{-\sigma-1/2})$.

To complete the proof of iii it suffices to show that $G_0(t) = O(t^{-\sigma})$. Split

$$G_0(t) = \|1_{[0,1[}(\sqrt{t}A)\mathrm{e}^{\lambda_0\sqrt{t}A}X(t)\|^2 + \|1_{[1,\infty[}(\sqrt{t}A)\mathrm{e}^{\lambda_0\sqrt{t}A}X(t)\|^2.$$

The first term bounded by $\mathrm{e}^{2\lambda_0}J_0(t) = O(t^{-\sigma})$. The second term is bounded by

$$\|\big(\sqrt{t}A\big)^{1/2}\mathrm{e}^{\lambda_0\sqrt{t}A}X(t)\|^2 = t^{1/2}G_{1/2}(t) = t^{1/2}O(t^{-\sigma-1/2}) = O(t^{-\sigma}).$$

To prove iv we use the integral equation (2.2) in the form $X = Y + B(X,X)$ where $Y(t) = \mathrm{e}^{-tA^2}u_0$. The decay of the first term is clear. For the nonlinear term we split the integral from 0 to t into an integral from 0 to T and another from T to t. Using the bound on $G_{5/4}$ from iii in the second integral we obtain a term which is $O(T^{-1/4})$ while the first integral is given by $\mathrm{e}^{-(t-T)A^2}g(T)$ for some $g(T) \in L^2$ and is thus $o(1)$. $\square$

Remark. In studying the types of inequalities proved in Corollary 5.2 we were motivated by [OT]. However the main theorem in that paper is stated incorrectly and the proof given there is also incorrect.

For completeness of presentation we end this section by giving another proof of Corollary 5.2 iii. Although the proof goes along similar lines it is somewhat more direct.

Theorem 5.3. *Suppose that $u_0 \in H^{1/2}$. For all $\kappa > 0$ and $\lambda_0 \in]0,\bar{\lambda}[$ there exists $K = K(\kappa,\lambda_0) > 0$ such that*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}G_0(t) \le -\kappa t^{-1}G_0(t) + Kt^{-1}J_0(t) \textit{ for all } t>0. \tag{5.17}$$

Whence

$$G_0(t) \le Kt^{-\kappa}\int_0^t s^{\kappa-1}J_0(s)\,\mathrm{d}s \textit{ for all } t>0. \tag{5.18}$$

In particular if for some $\sigma > -1/2$ the bound $J_0(t) = O(t^{-\sigma})$ holds, then $G_0(t) = O(t^{-\sigma})$ and whence, more generally, $J_r(t) = O(t^{-\sigma-r})$ and $G_r(t) = O(t^{-\sigma-r})$ for all $r \ge 0$.

Proof. We compute

$$\tfrac{\mathrm{d}}{\mathrm{d}t}G_0(t) = -2G_1(t) + \lambda_0 t^{-1/2}G_{1/2}(t) + R(t), \tag{5.19}$$

where

$$R(t) \le 2G_0(t)^{1/2}CG_1(t)^{1/2}G_{3/2}(t)^{1/2} \le G_1(t) + C^2G_0(t)G_{3/2}(t). \tag{5.20}$$

In particular, due to the estimate $G_{3/2}(t) = G_{3/2,\lambda_0}(t) \le C(\lambda_1-\lambda_0)t^{-1}G_{1/2,\lambda_1}(t)$, $\lambda_1 \in]\lambda_0,\bar{\lambda}[$, and (5.1) applied to $G_{1/2,\lambda_1}(t)$,

$$R(t) \le G_1(t) + \widetilde{C}t^{-1}G_0(t). \tag{5.21}$$

To obtain (5.17) we insert $0 = \kappa t^{-1}G_0(t) - Kt^{-1}J_0(t) - \kappa t^{-1}G_0(t) + Kt^{-1}J_0(t)$ on the right hand side of (5.19), and due to (5.21) we need only to examine the condition

$$-G_1(t) + \lambda_0 t^{-1/2}G_{1/2}(t) + \widetilde{C}t^{-1}G_0(t) + \kappa t^{-1}G_0(t) - Kt^{-1}J_0(t) \le 0. \tag{5.22}$$

By the spectral theorem the bound (5.22) will follow from

$$-x^2\mathrm{e}^{2\lambda_0 x} + \lambda_0 x\mathrm{e}^{2\lambda_0 x} + (\kappa + \widetilde{C})\mathrm{e}^{2\lambda_0 x} \le K \text{ for all } x \ge 0, \tag{5.23}$$

which in turn obviously is valid for some $K = K(\kappa, \lambda_0)$. Whence we have shown (5.17).

The remaining statements are immediate consequences of (5.17), cf. the proof of Corollary 5.2. □

6. Optimal rate of growth of analyticity radii

We shall combine Subsections 4.2 and 5.2 to obtain improved analyticity radius bounds of global solutions with small data in $\dot{H}^{1/2}$ or $H^{1/2}$ in the large time regime. An example shows that our bounds are optimal.

6.1. **Optimizing bounds of analyticity radii for large times.** Suppose the conditions of Theorem 4.2 i and that for some $\sigma > -1/2$ the bound $\|A^{1/2}X(t)\| = O(t^{-(2\sigma+1)/4})$ for $t \to \infty$ holds. We shall then apply Theorem 4.2 to $u_0 \to X(T)$ and $X \to u_T$, where $u_T(\tau) := X(\tau + T)$; here $T > 0$ is an auxiliary variable that in the end will be large (proportional to the time $t = \tau + T$). Notice that $X(T) \in \dot{H}^{1/2}$ (since we have assumed that $u_0 \in \dot{H}^{1/2}$), and that u_T is the unique small solution to the integral equation (2.8) with data $X(T)$, cf. Proposition 2.7. The fact that here indeed u_T is a solution to (2.8) requires an argument not given here. (We refer the reader to Subsection 7.1 for a thorough discussion of related issues in a different setting.) Since $T^{-(2\sigma+1)/4} \to 0$ for $T \to \infty$ we obtain for the critical value, $\bar{\lambda} = \bar{\lambda}(T)$ of (4.4), that $\bar{\lambda} \to \infty$ for $T \to \infty$. In fact

$$\liminf_{T\to\infty} \bar{\lambda}/\sqrt{\ln(T)} \ge (2\sigma + 1)^{1/2}.$$

Consequently for any $\epsilon_0 \in]0, 1[$, $\lambda_0 := \sqrt{(2\sigma + 1)(1 - \epsilon_0)\ln(T)}$ is a legitimate choice in Theorem 4.2 with $u_0 \to X(T)$ provided T is large enough. We shall use this observation to prove the following main result.

Theorem 6.1. *Suppose the conditions of Theorem 4.2 i, i.e. $u_0 \in \dot{H}^{1/2}$ and that the constant $4C_1C_3\|A^{1/2}u_0\| < 1$. Let X denote the corresponding solution to the integral equation* (2.8). *We have:*

i) Suppose that for some $\sigma > -1/2$ the following bound holds

$$\|A^{1/2}X(t)\| = O(t^{-(2\sigma+1)/4}) \text{ for } t \to \infty. \tag{6.1}$$

Let $0 \le \tilde{\epsilon} < \epsilon \le 1$ be given. Then there exist constants $t_0 > 1$ and $C > 0$ such that

$$\|A^{1/2}\exp\Big(\sqrt{(1-\epsilon)(2\sigma+1)}\sqrt{t\ln t}A\Big)X(t)\| \le Ct^{-\tilde{\epsilon}(2\sigma+1)/4} \text{ for all } t \ge t_0. \tag{6.2}$$

ii) Suppose $u_0 \in L^2$, and that for some $\sigma > -1/2$ the following bound holds

$$\|X(t)\| = O(t^{-\sigma/2}) \text{ for } t \to \infty. \tag{6.3}$$

Then (6.1) *holds (and therefore in particular the conclusion of i).*

Let $0 \le \tilde\epsilon < \epsilon \le 1$ be given. Then there exist constants $t_0 > 1$ and $C > 0$ such that

$$\|\exp\Big(\sqrt{(1-\epsilon)(2\sigma+1)}\sqrt{t\ln t}A\Big)X(t)\| \le Ct^{1/4-\tilde\epsilon(2\sigma+1)/4} \text{ for all } t \ge t_0. \tag{6.4}$$

In particular

$$\liminf_{t\to\infty} \tfrac{\mathrm{rad}(X(t))}{\sqrt{t\ln t}} \ge \sqrt{2\sigma+1}. \tag{6.5}$$

Proof. We prove first i. So fix $0 \le \tilde\epsilon < \epsilon \le 1$. To make contact to the discussion at the beginning of this subsection let us then choose $\epsilon_0 \in]\tilde\epsilon, \epsilon[$. We introduce in addition to the variable T a "new time" τ and a parameter n by the relations

$$t = \tau + T = (n+1)T;\ T \ge T_0. \tag{6.6}$$

We will let the parameters n and T_0 be chosen large. First we fix n by the condition

$$n(1-\epsilon_0) > (n+1)(1-\epsilon). \tag{6.7}$$

As noted at the beginning of this subsection we are allowed to choose

$$\lambda_0 = \sqrt{(2\sigma+1)(1-\epsilon_0)\ln(T)} \tag{6.8}$$

in Theorem 4.2 with $u_0 \to X(T)$ provided $T \ge T_0$ for some large $T_0 > 0$. We do that, and estimate using (6.6) and (6.7)

$$\begin{aligned}\lambda_0\sqrt{\tau} &= \sqrt{(2\sigma+1)(1-\epsilon_0)\ln(t/(n+1))}\sqrt{tn/(n+1)}\\ &\ge \sqrt{(2\sigma+1)(1-\epsilon)\ln(t)}\sqrt{t};\ \text{for all}\ \ t \ge t_0 := (n+1)T_0 \text{ for a } T_0 > 0.\end{aligned} \tag{6.9}$$

Here $T_0 > 0$ possibly needs to be chosen larger than before. Now fix such a T_0. Whence also t_0 is fixed, and with this value of t_0 indeed the left hand side of (6.2) is finite for all $t \ge t_0$. The bound (6.2) follows then from (4.5b). We have proved i.

As for ii, the first statement is a consequence of Corollary 5.2 iii. The second statement follows from the proof of i and (4.5c). ☐

6.2. **Example.** For the example presented in Subsection 3.4 we have the conditions of Theorem 6.1 ii fulfilled after a translation in time of the given solution u; i.e. by replacing $u \to u_{\widetilde T}$ for a sufficiently large $\widetilde T > 0$. This is with $\sigma = 5/2$. The estimate (6.5) is *sharp* by (3.53). Similarly the more precise bounds (6.4) are sharp. More precisely the power of t on the right of (6.4) cannot be improved for any $\epsilon \in]0,1[$ since indeed the estimate is false with $\tilde\epsilon = \epsilon$. This follows readily from an examination of the analytic extension of (3.41). Likewise (6.2) is sharp in the same sense. This can be seen by using the optimality of (6.4) discussed above and an argument similar to the one presented at the end of the proof of Corollary 5.2 iii.

7. Global solutions for arbitrary data

In this section we shall discuss strong solutions and in particular strong global solutions without assuming the $\dot H^{1/2}$-smallness condition of Subsection 2.4.

7.1. **Strong solutions.** Dealing with global large data solutions we need first to define a notion of global solutions without referring directly to the fixed point equation (2.8) (since the fixed point condition (2.10) now may fail). The definition needs to be based directly on (2.1). So we introduce (the equations written slightly differently):

$$\begin{cases} \big(\tfrac{\partial}{\partial t}u + P(Mu\cdot\nabla)u - \triangle u\big)(t,\cdot) = 0 \text{ for } t\in I \\ u(t) := u(t,\cdot) \in \mathrm{Ran}P \text{ for } t \in \bar{I} \end{cases}. \tag{7.1}$$

As in Subsection 2.1 I is an interval of the form $]0,T]$ or of the form $I=]0,\infty[$. We shall introduce a notion of strong solution to (7.1). The solutions with $I=]0,\infty[$ will be called strong global solutions. For that purpose we need the spaces appearing in Proposition 2.2. For simplicity we shall restrict our discussion to the H^r setting (leaving out the $\dot{H}^r$ setting with $r\in[1/2,3/2[$).

So let $\mathcal{B}=\mathcal{B}_{I,3/8,5/4}$ and $\mathcal{B}^0\subseteq\mathcal{B}$ be the spaces as specified in the beginning of Section 2 (constructed in terms of an arbitrarily given interval I).

Definition 7.1. Let $r\geq 1/2$. For $I=]0,T]$ we say that $u\in C(\bar{I},H^r)$ is a *strong solution* to the problem (7.1) if the following conditions hold:

(1) $u(t)\in PH^r$ for all $t\in\bar{I}$,
(2) $u\in\mathcal{B}^0$,
(3) $u\in C^1(I,\mathcal{S}'(\mathbb{R}^3))$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t}u = -A^2u - P(Mu\cdot\nabla)u;\ t\in I. \tag{7.2}$$

Here the differentiability in t is meant in the weak* topology and in (7.2) is meant in the sense of distributions. The class of such functions is denoted by $\mathcal{S}_{r,I}$. For $I=]0,\infty[$ we define $\mathcal{G}_r$ to be the subset of $C(\bar{I},H^r)$ consisting of u's such that $1_{\tilde{I}}u\in\mathcal{S}_{r,\tilde{I}}$ for all intervals of the form $\tilde{I}=]0,\tilde{T}]$, and we refer to any $u\in\mathcal{G}_r$ as a *strong global solution* to the problem (7.1) with $I=]0,\infty[$.

Remarks 7.2. 1) Obviously the condition 2 is redundant if $r\geq 5/4$.

2) For any strong solution u on I the first term on the right hand side of (7.2) is an element of $C(I,H^{r-2})$ while the second term is an element of $C(I,L^2)$, cf. (2.6). Consequently $u\in C^1(I,H^{\min(r-2,0)})$.
3) For any $u\in\mathcal{S}_{r,I}$

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{e}^{-(t-s)A^2}u(s)\big) = -\mathrm{e}^{-(t-s)A^2}P(Mu(s)\cdot\nabla)u(s) \text{ for all } 0<s<t\in I,$$

and consequently (by integration) the integral equation (2.2) with $u_0=u(0)$ holds for all $t\in I$. In fact it follows that $X=u$ is a solution to (2.8) in $\mathcal{B}^0$ (with $Y(t)=\mathrm{e}^{-tA^2}u_0$). Due to the uniqueness statement of Proposition 2.2 it follows that u coincides with the function X of Proposition 2.2 on a sufficiently small interval $\tilde{I}=]0,\tilde{T}]$. As a consequence similarly if $r\in]1/2,3/2[$ or $r\in[5/4,\infty[$, u coincides on a sufficiently small interval with the function X of Propositions 2.4 or 2.5, respectively.
4) Conversely, the solutions X of Propositions 2.2 and 2.7 with data $u_0=Pu_0\in H^{1/2}$ are indeed solutions in the sense of Definitions 7.1 (with $r=1/2$ and on the same interval I). Similarly it is readily verified that the solution X of Propositions 2.4 or 2.5 with data $u_0=Pu_0\in H^r$ for $r\in]1/2,3/2[$ or $r\in[5/4,\infty[$, respectively, is a solution in the sense of Definition 7.1 (with the same r and I).

5) The class $\mathcal{S}_{r,I}$ is right translation invariant, i.e. if $u \in \mathcal{S}_{r,I}$ and $t_0 \in]0,T[$ (where T is the right end point of I) then $u_{t_0}(\cdot) := u(\cdot + t_0) \in \mathcal{S}_{r,I_0}$; $I_0 :=]0,\infty[\cap \big(I - \{t_0\}\big)$. In particular $\mathcal{G}_r$ is right translation invariant ($u \in \mathcal{G}_r \Rightarrow u_{t_0} \in \mathcal{G}_r$ for any $t_0 > 0$).
6) With the modification of Definitions 7.1 given by omitting 1, the previous discussion, 1–5, is still appropriate (possibly slightly modified). Notice that we did not impose the condition 1 (viewed as a condition on the data) in the bulk of the paper.

Strong solutions to the same initial value problem are unique:

Proposition 7.3. *Suppose $u_1 \in \mathcal{S}_{r_1,I_1}$ and $u_2 \in \mathcal{S}_{r_2,I_2}$ obey $u_1(0) = u_2(0) = u_0$ for some $u_0 \in PH^{r_1} \cap PH^{r_2}$. Then $u_1 = u_2$ on $I_1 \cap I_2$.*

Proof. This is a standard argument for ODE's. We can assume that $r_1 = r_2$ and $I := I_1 = I_2$. Suppose $u_1 \neq u_2$ on I. Then let

$$t_0 = \inf\{t \in I | u_1(t) \neq u_2(t)\}.$$

Clearly $t_0 \in \bar{I}$, and by continuity $t_0 < T$ and $u_1(t_0) = u_2(t_0)$. Due to Remark 7.2 5 we can assume that $t_0 = 0$. Due to Remark 7.2 3 it follows that $u_1(t) = u_2(t)$ for all $t \in \tilde{I} =]0, \tilde{T}]$ for some sufficiently small $\tilde{T} > 0$. This is a contradiction.

□

7.2. **Sobolev and analyticity bounds for bounded intervals.** In this subsection we show that strong solutions are smooth, in fact real analytic, in the x-variable.

Proposition 7.4. *Let $r \geq 1/2$ and $0 < T_0 < T < \infty$ be given. Let $u \in \mathcal{S}_{r,I}$ where $I =]0,T]$, and denote by $|u|$ the norm $|u| = \|u\|_{\mathcal{B}}$. There exist $\delta = \delta(T_0, |u|) > 0$ and $C = C(T_0, T, |u|, \sup_{t\in I} \|u(t)\|_{H^{1/2}}) > 0$ such that*

$$\|e^{\delta A} u(t)\|_{H^{1/2}} \leq C \text{ for all } t \in [T_0, T]. \tag{7.3}$$

Proof. For all $u \in \mathcal{B}$ and $\widetilde{T} \in]0,T[$

$$\sup_{t\in[\tilde{T},T]} \|A^{5/4} u(t)\|_{L^2} \leq \tilde{T}^{-3/8} |u|. \tag{7.4}$$

For any given $u \in \mathcal{S}_{r,I}$ we shall obtain an analyticity bound for the restriction of u to $]t_0 - \epsilon, t_0]$ for $t_0 \in [T_0, T]$ and for suitable $\epsilon \in]0, T_0[$. For that we shall apply the procedure of the proof of Theorem 3.2 to the strong solution $u_{t_0,\epsilon} := u(\cdot + t_0 - \epsilon)$ on the interval $I_\epsilon =]0, \epsilon]$. The application will be with $\lambda = 1$ and $r = 1/2$ in the definitions of ζ and θ (given in (3.1)) and for $\epsilon > 0$ small, and the underlying Banach space will be $\tilde{\mathcal{B}} = \mathcal{B}_{\zeta,\theta,I_\epsilon,3/8,5/4}$. Indeed for $\epsilon > 0$ taken small enough the conditions (2.10) hold for some $R > 0$ that can be taken independent of T and $t_0 \in [T_0, T]$, cf. (3.8). Notice here that by Lemma 3.1 (with $\alpha = 0$ and $f = A^{5/4} u(t_0 - \epsilon)$) and (7.4)

$$|e^{-(\cdot)A^2} u(t_0 - \epsilon)|_{\tilde{\mathcal{B}}} \leq \tilde{C}_1 \epsilon^{3/8} \|A^{5/4} u(t_0 - \epsilon)\|_{L^2} \leq \tilde{C}_1 \epsilon^{3/8} (T_0 - \epsilon)^{-3/8} |u|. \tag{7.5}$$

So we can choose R in (2.10) to be equal to the constant on the right hand side of (7.5), and indeed the conditions (2.10) are fulfilled for all sufficiently small $\epsilon > 0$, cf. (3.8). Fix any such $\epsilon > 0$ and let $\delta = \sqrt{\epsilon}$. Then we invoke (3.9) and (3.10) with $u = v = u_{t_0,\epsilon}$ and with time $t = \epsilon$ as well as Lemma 3.1 (with $\alpha = 0$, $f = \langle A \rangle^{1/2} u(t_0 - \epsilon)$ and also applied for $t = \epsilon$).

□

In combination with Theorems 3.2 and 3.3 we obtain:

Corollary 7.5. *Let $r \geq 1/2$, $0 < T < \infty$ and $u \in \mathcal{S}_{r,I}$ be given; $I :=]0, T]$. Let $u_0 = u(0)$. There exist $\delta = \delta(u, r) > 0$ and $C = C(u, T, r) > 0$ such that*

$$\|\mathrm{e}^{\min(\sqrt{t},\delta)A} u(t)\|_{H^{1/2}} \leq C \textit{ for all } t \in I. \tag{7.6}$$

In particular, for all $\bar{r} \geq 1/2$ and with $\tilde{C} = C \max_{x \geq 0} x^{\bar{r}-1/2} \mathrm{e}^{-x}$

$$\|A^{\bar{r}} u(t)\|_{L^2} \leq \tilde{C} \min(\sqrt{t}, \delta)^{-(\bar{r}-1/2)} \textit{ for all } t \in I. \tag{7.7}$$

If $r > 1/2$ the dependence of δ and C on u can be chosen to be through $|u|$ and $\|u_0\|_{H^r}$ and through $|u|$, $\sup_{t \in I} \|u(t)\|_{H^{1/2}}$ and $\|u_0\|_{H^r}$, respectively.

For all $k \in \mathbb{N} \cup \{0\}$ and for all $\bar{r} \geq 1/2$

$$u \in C^k(I, H^{\bar{r}}). \tag{7.8}$$

Writing $u(t)(x) = u(t, x)$,

$$u \in C^\infty(I \times \mathbb{R}^3). \tag{7.9}$$

Proof. We apply Theorems 3.2 and 3.3 with $\lambda = 1$. There exist $T_0 \in]0, T[$ and $C > 0$ such that

$$\|\mathrm{e}^{\sqrt{t}A} u(t)\|_{H^{1/2}} \leq C \text{ for all } t \in]0, T_0]. \tag{7.10}$$

These constants are for $r = 1/2$ chosen in agreement with an approximation property of u_0. This is not the case for $r > 1/2$ where the bounds (3.3) and (3.12) can be used directly to get the appropriate smallness in terms of the quantities $\|u_0\|_{H^r}$ and r. We shall use Proposition 7.4 with the T_0 from (7.10). Whence for $r > 1/2$ we apply Proposition 7.4 with T_0 chosen as a function of the quantities $\|u_0\|_{H^r}$ and r. For $r = 1/2$ we apply Proposition 7.4 with T_0 depending on u through u_0.

As for (7.8) with $k = 0$ we apply (7.7) in combination with Propositions 2.5 and 7.3 (notice that we can assume that $\bar{r} > 5/4$ from the very definition of $\mathcal{S}_{r,I}$). The statement (7.8) with arbitrary $k \geq 1$ follows inductively by repeated differentiation of (7.2).

The statement (7.9) follows from (7.8) and the Sobolev embedding theorem.

□

Remark 7.6. For $r > 1/2$ the Sobolev bounds (7.7) can be improved in the short time regime due to Theorems 3.2 and 3.3: For any given $\bar{r} \geq r$ the quantity has a bound of the form $Ct^{-(\bar{r}-r)/2}$ for small $t > 0$.

The energy inequality was studied under certain conditions in Subsection 5.1. We can now prove it more generally:

Corollary 7.7. *Let $r \geq 1/2$, an interval $I =]0, T]$ and $u \in \mathcal{S}_{r,I}$ be given. Let $u_0 = u(0)$. Suppose in addition the condition*

$$\nabla \cdot \big(M u(t)\big) = 0 \textit{ for all } t \in I. \tag{7.11}$$

Then

$$\|u(t)\|^2 = \|u_0\|^2 - 2 \int_0^t \|A u(s)\|^2 \,\mathrm{d}s \textit{ for all } t \in I. \tag{7.12}$$

In particular $\|u(t)\| \leq \|u_0\|$ for all $t \in I$.

Proof. Due to (7.7) (applied for the first identity with $\bar{r} = 3/2$ in combination with Remark 7.2 2) and (7.11) the computation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = -2\|Au(t)\|^2 + 2\big\langle u(t), (Mu(t)\cdot\nabla)u(t)\big\rangle = -2\|Au(t)\|^2 \text{ for all } t \in I, \tag{7.13}$$

is legitimate. By integration of (7.13) we obtain (7.12). Note incidentally that $\|Au(s)\|^2 = O(s^{-1/2})$, due to (7.7), yielding an independent proof of the convergence of the integral in (7.12). □

7.3. **Global analyticity stability.** We shall study the set of data for which we have global solutions. There are several works (for example [PRST],[GIP1], [GIP2], [ADT], [FO], [Zh]) which study the stability of solutions to the Navier-Stokes equations. Perhaps the first result is [PRST] but there are many further results for different spaces. In particular the fact that $\mathcal{I}_r$ defined below is an open set in our setting is a known result ([GIP1], [GIP2], [ADT]). Although we give the openness result we concentrate particularly on the stability of the region of analyticity and corresponding estimates.

We shall prove two stability results. The first is for bounded intervals only, however it is used in the proof of our second (global) stability result (and besides it has some independent interest, see for example Corollary 7.10):

Proposition 7.8. *Let I be an interval of the form $I =]0,T]$, and let $\theta : \bar{I} \to [0,\infty[$ be a continuous function obeying the following estimate for some $\lambda \geq 0$:*

$$\theta(s+t) \leq \lambda\sqrt{s} + \theta(t) \text{ for } s, t, s+t \in \bar{I}. \tag{7.14}$$

Suppose $u \in \mathcal{S}_{1/2,I}$ obeys

$$A^{1/2}\mathrm{e}^{\theta(\cdot)A}u(\cdot) \in C(\bar{I}, L^2). \tag{7.15}$$

Let $u_0 = u(0)$. There exists $\delta_0 > 0$ such that:

i) If $\delta \leq \delta_0, v_0 \in PH^{1/2}$ and $\|A^{1/2}\mathrm{e}^{\theta(0)A}(v_0 - u_0)\| \leq \delta$ it follows that there exists $v \in \mathcal{S}_{1/2,I}$ with $v(0) = v_0$ obeying

$$\|A^{1/2}\mathrm{e}^{\theta(t)A}(v(t) - u(t))\| \leq K_1\delta, \tag{7.16a}$$

$$t^{3/8}\|A^{5/4}\mathrm{e}^{\theta(t)A}(v(t) - u(t))\| \leq K_2\delta. \tag{7.16b}$$

ii) If $\delta \leq \delta_0, v_0 \in PH^{1/2}$ and $\|\mathrm{e}^{\theta(0)A}(v_0 - u_0)\|_{H^{1/2}} \leq \delta$ it follows in addition that

$$\|\mathrm{e}^{\theta(t)A}(v(t) - u(t))\| \leq K_3\delta. \tag{7.16c}$$

In (7.16a)–(7.16c) *the constants $K_1, K_2, K_3 > 0$ depend on θ, u, T and δ_0 but not on δ, and all bounds are uniform in $t \in I$.*

In the proof we will use norms of the form

$$|w|_{s_0,t_0} := \sup_{0<s\leq\min(s_0,T-t_0)} s^{3/8}\|A^{5/4}\mathrm{e}^{\theta(s+t_0)A}w(s)\|;\ s_0 > 0,\ t_0 \in [0,T[. \tag{7.17}$$

Mimicking Subsection 3.1 thus with $\zeta(s) = 1$ and $\theta(s) \to \theta(s+t_0)$ we find

$$|B(w_1,w_2)|_{s_0,t_0} \leq \gamma_\lambda|w_1|_{s_0,t_0}\cdot|w_2|_{s_0,t_0}, \tag{7.18a}$$

$$\|A^{1/2}\mathrm{e}^{\theta(s+t_0)A}B(w_1,w_2)(s)\| \leq \gamma_\lambda|w_1|_{s_0,t_0}\cdot|w_2|_{s_0,t_0}, \tag{7.18b}$$

$$\|\mathrm{e}^{\theta(s+t_0)A}B(w_1,w_2)(s)\| \leq s^{1/4}\gamma_\lambda|w_1|_{s_0,t_0}\cdot|w_2|_{s_0,t_0}. \tag{7.18c}$$

Here $\gamma_\lambda = dc_\lambda$ where d is independent of λ, s_0, t_0 and T and

$$c_\lambda := \sup_{x\geq 0} \langle x\rangle^{5/4} \mathrm{e}^{\lambda x}\mathrm{e}^{-x^2}.$$

We will need the following lemma:

Lemma 7.9. *Suppose θ and $u \in \mathcal{S}_{1/2,I}$ are given as in Proposition 7.8. Let $\epsilon \in]0,(2\gamma_\lambda)^{-1}[$ be given. Then there is an $s_0 \in]0,1[, s_0 = s_0(\epsilon,\theta,u)$, so that*

$$\forall t_0 \in [0,T[\ \forall s \in]0, \min(s_0, T-t_0)] : \tag{7.19}$$
$$2s^{3/8}\|A^{5/4}\mathrm{e}^{\theta(s+t_0)A}\mathrm{e}^{-sA^2}u(t_0)\| \leq \epsilon \text{ and } s^{3/8}\|A^{5/4}\mathrm{e}^{\theta(s+t_0)A}u(s+t_0)\| \leq \epsilon.$$

Proof. Using $\theta(s+t_0) \leq \lambda\sqrt{s} + \theta(t_0)$ and the spectral theorem we have for any $N > 0$

$$\begin{aligned}
&s^{3/8}\|A^{5/4}\mathrm{e}^{\theta(s+t_0)A}\mathrm{e}^{-sA^2}u(t_0)\| \\
&\quad\leq \|(\sqrt{s}A)^{3/4}\mathrm{e}^{\lambda\sqrt{s}A}\mathrm{e}^{-sA^2}\| \cdot \|1_{[N,\infty[}(A)A^{1/2}\mathrm{e}^{\theta(t_0)A}u(t_0)\| \\
&\qquad + s^{3/8}\|\mathrm{e}^{\lambda\sqrt{s}A}\mathrm{e}^{-sA^2}\| \cdot \|A^{3/4}1_{[0,N]}(A)A^{1/2}\mathrm{e}^{\theta(t_0)A}u(t_0)\| \\
&\quad\leq c_\lambda\|1_{[N,\infty[}(A)A^{1/2}\mathrm{e}^{\theta(t_0)A}u(t_0)\| + c_\lambda s^{3/8}N^{3/4}\|A^{1/2}\mathrm{e}^{\theta(t_0)A}u(t_0)\|. \qquad (7.20)
\end{aligned}$$

Since the map $\bar{I} \ni t_0 \to A^{1/2}\mathrm{e}^{\theta(t_0)A}u(t_0)$ is continuous, it maps into a compact set on which $1_{[N,\infty[}(A) \to 0$ uniformly as $N \to \infty$. We then fix N so that the first term of (7.20) is less than $\epsilon/4$ for all $t_0 \in \bar{I}$. Once N is fixed we can choose $s_0 \in]0,1[$ so that the second term in (7.20) is less than $\epsilon/4$ for $s \in [0, s_0]$. We have proved the first estimate of (7.19), $|\mathrm{e}^{-(\cdot)A^2}u(t_0)|_{s_0,t_0} \leq \epsilon/2$.

To show the second estimate of (7.19) we go back to the integral equation (2.2) and use $u(t_0)$ as initial data following the scheme of Subsection 2.1 (with $R = \epsilon/2$). We use the first estimate in combination with (7.18a). By uniqueness the constructed fixed point $w = u_{t_0}$ where $u_{t_0}(s) = u(s+t_0)$. □

Proof of Proposition 7.8. We now choose $\epsilon = (3\gamma_\lambda)^{-1}$ and s_0 in accordance with Lemma 7.9. We can assume that $m_0 := T/s_0 \in \mathbb{N}$. We build the solution v in the interval I by constructing it in a series of intervals $[(m-1)s_0, ms_0], m = 1, 2, \ldots, m_0$. We assume inductively we have constructed $v(t)$ in the interval $0 \leq t \leq ms_0$ (with $m \leq m_0 - 1$) and that we have the estimate

$$\|A^{1/2}\mathrm{e}^{\theta(t)A}(v(t) - u(t))\| \leq (2c_\lambda)^m\delta \tag{7.21}$$

in this interval (this is true for $m = 0$). Let $t_0 = ms_0$ and $u_{t_0}(s) = u(s+t_0)$. Consider the map

$$F(w)(s) := \mathrm{e}^{-sA^2}(v(t_0) - u(t_0)) + B(w, u_{t_0})(s) + B(u_{t_0}, w)(s) + B(w,w)(s). \tag{7.22}$$

We have

$$|F(w)|_{s_0,t_0} \leq c_\lambda(2c_\lambda)^m\delta + 2\gamma_\lambda|w|_{s_0,t_0} \cdot |u_{t_0}|_{s_0,t_0} + \gamma_\lambda|w|^2_{s_0,t_0}, \tag{7.23a}$$
$$|F(w_1) - F(w_2)|_{s_0,t_0} \leq \gamma_\lambda(2|u_{t_0}|_{s_0,t_0} + |w_1|_{s_0,t_0} + |w_2|_{s_0,t_0}) \cdot |w_1 - w_2|_{s_0,t_0}. \tag{7.23b}$$

Then a simple computation shows $F : B_{2R} \to B_{2R}$ is a strict contraction if $R = c_\lambda(2c_\lambda)^m\delta$ and $\delta \leq \delta_0$ where $\delta_0 > 0$ is chosen small enough. If the fixed point is denoted by w, we define $v(t) = u(t) + w(t - t_0)$ for $t \in [ms_0, (m+1)s_0]$. The bound (7.21) with $m \to m+1$ in the interval $[ms_0, (m+1)s_0]$ follows from $w = F(w)$ and the estimate (7.18b). This completes the induction and gives (7.21) with

$m = m_0$ for $t \in I$. We have constructed a solution v obeying (7.16a). From the very construction we have partly shown (7.16b), however our bounds are somewhat poor at ms_0, $m = 1, \dots, m_0 - 1$ (assuming here $m_0 \geq 2$). In order to show (7.34b) near ms_0, $m = 1, \dots, m_0 - 1$, we can repeat the above procedure in the intervals $[(m-1/2)s_0, (m+1/2)s_0]$. The consistency of our definitions in overlapping intervals follows from uniqueness. For (7.16c) we use (7.18c) to show inductively

$$\|\mathrm{e}^{\theta(t)A}(v(t) - u(t))\| \leq (2c_\lambda)^m \delta \text{ for } 0 \leq t \leq ms_0, \tag{7.24}$$

cf. (7.21). □

We will use Proposition 7.8 to shed some light on (1.17), the conjectured lower semicontinuity of the analyticity radius of $u \in \mathcal{S}_{1/2,I}$. To motivate the construction in the following corollary, it should be noted that by definition of $\mathrm{rad}(u(t))$, if $t > 0$ and

$$\liminf_{s \uparrow t} \mathrm{rad}(u(s)) \geq \mathrm{rad}(u(t)) \tag{7.25}$$

then

$$\forall\, \alpha < \mathrm{rad}(u(t))\, \exists\, a < t \,\text{so that}\, \|\mathrm{e}^{\alpha A} u(s)\| < \infty \,\forall s \in]a, t] \tag{7.26}$$

but the uniform bound

$$\forall\, \alpha < \mathrm{rad}(u(t))\, \exists\, a < t \,\text{so that}\, \sup_{s \in]a,t]} \|\mathrm{e}^{\alpha A} u(s)\| < \infty \tag{7.27}$$

does not readily follow from the definitions.

Corollary 7.10. *Fix $u \in \mathcal{S}_{1/2,I}$ and let $u_0 = u(0)$. For $t \in I$ and any $v \in \mathcal{S}_{1/2,I}$ let $v_0 = v(0)$ and define*

$$r_t(v_0) := \sup\bigl\{\alpha \geq 0 \big| \sup_{s \in]a,t]} \|\mathrm{e}^{\alpha A} v(s)\| < \infty \text{ for some}\, a < t\bigr\} \tag{7.28}$$

Then $r_t(\cdot)$ is lower semicontinous at u_0 as a function of the initial data v_0 in the $H^{1/2}$ topology. More precisely, for $t \in I$

$$\liminf_{\|v_0 - u_0\|_{H^{1/2}} \to 0} r_t(v_0) \geq r_t(u_0). \tag{7.29}$$

If the analyticity radius of u satisfies (7.25) *and* (7.27) *then $r_t(u_0) = \mathrm{rad}(u(t))$ and the analyticity radius at t is lower semicontinous as a function of the initial data at u_0. More precisely,* (1.17) *is valid.*

Proof. Fix $t \in I$. Without loss we can assume $r_t(u_0) > 0$ and $I =]0, t]$. Choose $0 < \alpha < r_t(u_0)$. Then there exists $0 < a < t$ so that $\sup_{s \in]a,t]} \|\mathrm{e}^{\alpha A} u(s)\| < \infty$. Define $\theta : [0, t] \to [0, \infty[$:

$$\theta(s) = \begin{cases} 0, & \text{if } s \in [0, a]; \\ (\frac{s-a}{t-a})\alpha, & \text{if } s \in [a, t]. \end{cases} \tag{7.30}$$

Note that $\theta(\tau + s) \leq \lambda\sqrt{\tau} + \theta(s)$ with $\lambda = \alpha(t-a)^{-1/2}$ so that Proposition 7.8 applies. It follows that if $\|v_0 - u_0\|_{H^{1/2}}$ is small enough, $\sup_{\tau \in]a,t]} \|\mathrm{e}^{\theta(\tau)A} v(\tau)\| < \infty$. Thus by definition, for these $v_0, r_t(v_0) \geq \alpha$. This gives (7.29). As for the last statement of the corollary, following through the definitions it is easy to see that $r_t(u_0) = \mathrm{rad}(u(t))$ under the stated conditions. The definition of $r_t(v_0)$ also implies $\mathrm{rad}(v(t)) \geq r_t(v_0)$ for any $v \in \mathcal{S}_{1/2,I}$ and thus (7.29) gives the stated result. □

We now continue with our discussion of global stability.

Definition 7.11. For $r \geq 1/2$ we denote by

$$\mathcal{I}_r = \{u_0 \in PH^r \,|\, \exists u \in \mathcal{G}_r : \, u(0) = u_0\}, \tag{7.31}$$

and we endow $\mathcal{I}_r$ with the topology from the space PH^r.

Our result on global stability is as follows:

Theorem 7.12. *Suppose $u_0 \in \mathcal{I}_{1/2}$ and that the corresponding strong global solution u obeys*

$$\liminf_{t\to\infty} \|A^{1/2}u(t)\| = 0. \tag{7.32}$$

Suppose in addition that $\lambda > 0$ is given so that

$$A^{1/2}\mathrm{e}^{\lambda\sqrt{\cdot A}}u(\cdot) \in C([0,\infty[, L^2). \tag{7.33}$$

There exists $\delta_0 > 0$ such that:

i) If $\delta \leq \delta_0, v_0 \in PH^{1/2}$ and $\|A^{1/2}(v_0 - u_0)\| \leq \delta$ it follows that $v_0 \in \mathcal{I}_{1/2}$ and that for all $t > 0$ the corresponding strong global solution v satisfies

$$\|A^{1/2}\mathrm{e}^{\lambda\sqrt{t}A}(v(t) - u(t))\| \leq K_1\delta, \tag{7.34a}$$

$$t^{3/8}\|A^{5/4}\mathrm{e}^{\lambda\sqrt{t}A}(v(t) - u(t))\| \leq K_2\delta. \tag{7.34b}$$

ii) If in addition $u_0 \in PH^r$ with $r > 1/2$, then $u_0 \in \mathcal{I}_r$ and u_0 is an interior point of $\mathcal{I}_r$.

iii) If $\delta \leq \delta_0, v_0 \in PH^{1/2}$ and $\|v_0 - u_0\|_{H^{1/2}} \leq \delta$ it follows in addition that

$$\langle t\rangle^{-1/4}\|\mathrm{e}^{\lambda\sqrt{t}A}(v(t) - u(t))\| \leq K_3\delta. \tag{7.34c}$$

In (7.34a)–(7.34c) *the constants $K_1, K_2, K_3 > 0$ depend on λ, u, and δ_0 but not on δ, and all bounds are uniform in $t > 0$.*

Proof. Choose $T_0 > 1$ and large enough so that with $w(s) = u_{T_0}(s) = u(s + T_0)$

$$|w|_{\infty,T_0} := \sup_{s>0} s^{3/8}\|A^{5/4}\mathrm{e}^{\lambda\sqrt{s+T_0}A}w(s)\| \leq (3\gamma_\lambda)^{-1}. \tag{7.35}$$

This is possible by Corollary 4.3. On the one hand we apply Proposition 7.8 with $T = 2T_0$ and $\theta(t) = \lambda\sqrt{t}$ to construct a solution v in the interval $[0, T]$. We now construct v in the interval $[T_0, \infty[$ using the bound (7.35). This is done in a similar way as in the proof of Proposition 7.8 using the map F defined in (7.22) with the replacement $t_0 \to T_0$ and using the Banach space with norm $|\cdot|_{\infty,T_0}$ defined in (7.35). The contraction mapping argument then gives a fixed point w with $|w|_{\infty,T_0} \leq K\delta$. Finally we extend v to $[T_0, \infty[$ by setting $v(t) = u(t) + w(t - T_0)$. The estimates (7.34a)–(7.34c) follow easily. The statement ii follows from i in combination with Propositions 2.4 and 2.5 and Corollary 7.5.

□

Lemma 7.13. *Let $r \geq 1/2$. Suppose a given $u \in \mathcal{G}_r$ obeys* (7.11) *with $I =]0, \infty[$. Then the condition* (7.32) *holds.*

Proof. Let $u_0 = u(0)$, and let $n \in \mathbb{N}$ be given. Pick $\delta > 0$ such that $\delta\|u_0\| < n^{-2}$, and pick $\tilde{n} \geq n$ such that $\delta^2 2\tilde{n} > \|u_0\|^2$. Then, due to (7.12) with $t = 2\tilde{n}$, for some $t_n \in]\tilde{n}, 2\tilde{n}]$ we have $\|Au(t_n)\| < \delta$. For this time t_n

$$\|A^{1/2}u(t_n)\|^2 \leq \|Au(t_n)\|\,\|u(t_n)\| \leq \delta\|u_0\| < n^{-2},$$

so $\|A^{1/2}u(t_n)\| < 1/n$. Whence $\lim_{n\to\infty} \|A^{1/2}u(t_n)\| = 0$ for some sequence $t_n \to \infty$.

□

Using Theorem 7.12 and Lemma 7.13 we obtain for the system (1.1):

Corollary 7.14. *Let $M = I$ and P be given as the Leray projection. Then for all $r \geq 1/2$ the set $\mathcal{I}_r$ is open in PH^r.*

7.4. **L^2 stability.** Our final result on the stability of the L^2 norm is motivated by various previous works on L^2 decay properties, in particular [Sc1, Sc2, Wi]. In the following main result note the asymmetry between the solutions u and v reflected in the dependence of the constant K in (7.37) on u.

Proposition 7.15. *Suppose $u, v \in \mathcal{G}_{1/2}$ where $M = I$ and P is the Leray projection. Let $u_0(t) = \mathrm{e}^{-tA^2}u(0)$ and $v_0(t) = \mathrm{e}^{-tA^2}v(0)$. We suppose*

$$\|u_0(t)\| + \|v_0(t)\| \leq L\langle t\rangle^{-\sigma/2} \tag{7.36}$$

where $\sigma \geq 0$. Let $z(t) = v(t) - u(t) - w_0(t)$ where $w_0(t) = v_0(t) - u_0(t)$.

There exists $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$ and $\|v(0) - u(0)\|_{H^{1/2}} \leq \delta$ we have for any $\epsilon \in [0, 1]$ such that $(1 - \epsilon/2)\sigma \neq 1$,

$$\|z(t)\| \leq K\delta^\epsilon \langle t\rangle^{-\min((1-\epsilon/2)\sigma+1/4,\, 5/4)}. \tag{7.37}$$

Here K depends on L, δ_0, ϵ, and u.

Remarks 7.16. 1) The condition $\|u_0(t)\| = O(t^{-\sigma/2})$ of (7.36) is equivalent to the condition $\|1_{[0,r]}(A)u_0(0)\| = O(r^\sigma)$. This is one route to familiar sufficient conditions in terms of the L^p norms of $u_0(0)$ and of $xu_0(0)$. For these conditions and additional inequalities see [BJ].

2) The condition $\|u_0(t)\| = O(t^{-\sigma/2})$ of (7.36) implies the following decay of the solution $u \in \mathcal{G}_{1/2}$,

$$\|u(t)\| \leq C\langle t\rangle^{-\min(\sigma/2,\ 5/4)}. \tag{7.38}$$

The inequality (7.38) is proved in [Wi]. It follows from an argument similar to but simpler than an argument used in the proof of Proposition 7.15 to follow. Thus we omit the proof.

3) The positive parameter δ_0 of Proposition 7.15 can be determined as follows: Choose $u_0 = u(0)$ in Theorem 7.12 and $\lambda > 0$ in agreement with (7.33). Then according to Theorem 7.12 there exists $\delta_0 > 0$ so that (7.34c) holds. This δ_0 applies in Proposition 7.15 (in fact we shall only need (7.34c) with $\lambda = 0$).

4) The condition $(1 - \epsilon/2)\sigma \neq 1$ is introduced for simplicity to avoid logarithms in (7.37). If in addition to the hypotheses of Proposition 7.15 (excluding the requirement on ϵ) we require $\|w_0(t)\| \leq \delta\langle t\rangle^{-\sigma/2}$, then by repeating the proof of Proposition 7.15 the estimate (7.37) can be improved to

$$\|z(t)\| \leq \begin{cases} K\delta\langle t\rangle^{-\min(\sigma+1/4,\, 5/4)}, & \text{if } \sigma \neq 1 \\ K\delta\langle t\rangle^{-5/4}\ln(t+2), & \text{if } \sigma = 1 \end{cases}. \tag{7.39}$$

We will need the following lemma:

Lemma 7.17. *Assume the hypotheses of Proposition 7.15 and in addition the bound $\|u(t)\| \leq L\langle t\rangle^{-\sigma/2}$. Then for any $\epsilon \in [0, 1]$, and $0 < \delta \leq 1$,*

$$\|\nabla w_0(t)\|_\infty \leq C\delta^\epsilon t^{-1}\langle t\rangle^{-(1/4+(1-\epsilon)\sigma/2)}, \tag{7.40a}$$

$$\|\nabla u(t)\|_\infty \leq Ct^{-1}\langle t\rangle^{-(1/4+\sigma/2)}, \tag{7.40b}$$

$$\|w_0(t)\| \leq C\delta^\epsilon\langle t\rangle^{-(1-\epsilon)\sigma/2}. \tag{7.40c}$$

Proof. In the following the definition of C may change from line to line. We have

$$\|\nabla w_0(t)\|_\infty = \|\nabla \mathrm{e}^{-tA^2} w_0(0)\|_\infty = \|K_0(t) * w_0(0)\|_\infty \le \delta \|K_0(t)\| \tag{7.41}$$

where $K_0(t)$ is a (constant times) the inverse Fourier transform of $\xi \mathrm{e}^{-t|\xi|^2}$. We easily calculate $\|K_0(t)\| = Ct^{-5/4}$ and thus $\|\nabla w_0(t)\|_\infty \le C\delta t^{-5/4}$. We also have

$$\|\nabla w_0(t)\|_\infty = \|\nabla \mathrm{e}^{-tA^2/2} w_0(t/2)\|_\infty = \|K_0(t/2) * w_0(t/2)\|_\infty \le 2LCt^{-5/4} \cdot t^{-\sigma/2}.$$

Thus interpolating these two results we find for large time

$$\|\nabla w_0(t)\|_\infty \le C\delta^\epsilon t^{-5/4-(1-\epsilon)\sigma/2}. \tag{7.42}$$

For small and intermediate times

$$\|\nabla w_0(t)\|_\infty = \|\nabla A^{-1/2} \mathrm{e}^{-tA^2} A^{1/2} w_0(0)\|_\infty = \|K_1(t) * A^{1/2} w_0(0)\|_\infty \le \delta \|K_1(t)\| \tag{7.43}$$

where $K_1(t)$ is a (constant times) the inverse Fourier transform of $\xi|\xi|^{-1/2}\mathrm{e}^{-t|\xi|^2}$. We easily calculate $\|K_1(t)\| = Ct^{-1}$ and thus we have proved (7.40a).

The proof of (7.40b) goes along the same lines. We obtain (taking here $\lambda > 0$ sufficiently small)

$$\|\nabla u(t)\|_\infty = \|\nabla \mathrm{e}^{-\lambda\sqrt{t}A} \mathrm{e}^{\lambda\sqrt{t}A} u(t)\|_\infty = \|K_2(t) * \mathrm{e}^{\lambda\sqrt{t}A} u(t)\|_\infty \le \|K_2(t)\| \cdot G_0(t)^{1/2} \tag{7.44}$$

where here $K_2(t)$ is a (constant times) the inverse Fourier transform of $\xi \mathrm{e}^{-\lambda\sqrt{t}|\xi|}$ and $G_0(t) = \|\mathrm{e}^{\lambda\sqrt{t}A} u(t)\|^2$. We calculate $\|K_2(t)\| = Ct^{-5/4}$ and using Corollary 5.2 iii we deduce $G_0(t) \le Ct^{-\sigma}$. Thus we have the large time estimate

$$\|\nabla u(t)\|_\infty \le Ct^{-5/4-\sigma/2}. \tag{7.45}$$

To obtain the result for small and intermediate times we use (7.33) which gives

$$\begin{aligned}\|\nabla u(t)\|_\infty &= \|\nabla A^{-1/2} \mathrm{e}^{-\lambda\sqrt{t}A} A^{1/2} \mathrm{e}^{\lambda\sqrt{t}A} u(t)\|_\infty = \|K_3(t) * A^{1/2} \mathrm{e}^{\lambda\sqrt{t}A} u(t)\|_\infty \\ &\le C\|K_3(t)\|.\end{aligned} \tag{7.46}$$

We calculate $\|K_3(t)\| = Ct^{-1}$, completing the proof of (7.40b).

The proof of (7.40c) proceeds by interpolating the two bounds $\|w_0(t)\| \le \delta$ and $\|w_0(t)\| \le 2L\langle t\rangle^{-\sigma/2}$. □

Remark. We will not use the bounds in (7.40a) and (7.40b) for small t. We include these estimates for completeness.

Proof of Proposition 7.15. Note that from (7.34c) we have

$$\|z(t)\| \le C\delta\langle t\rangle^{1/4}, t \ge 0. \tag{7.47}$$

For large time we use Schonbek's technique [Sc1], [Sc2]. We differentiate $\|z(t)\|^2$ and find (after an integration by parts)

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 = -\|Az\|^2 - \langle z, z\cdot\nabla(u+w_0)\rangle - \langle z, u\cdot\nabla w_0 + w_0\cdot\nabla u\rangle - \langle z, w_0\cdot\nabla w_0\rangle. \tag{7.48}$$

We will need the estimate (7.38). Without loss of generality we can assume $\sigma \le 5/2$. In the following we shall also assume that $\delta_0 \le 1$ (in addition to the requirement of Remark 7.16 3); this is just to minimize notation.

We have (with some abbreviation)

$$\hat{z}(t,\xi) = -P(\xi)\int_0^t \mathrm{e}^{-(t-s)|\xi|^2} i\xi[\widehat{u\otimes w}(s,\xi) + \widehat{w\otimes u}(s,\xi) + \widehat{w\otimes w}(s,\xi)]ds \tag{7.49}$$

where $P(\xi)$ is the Leray projection in Fourier space. As a first step we get an initial bound for $\|z(t)\|$ of the form $\|z(t)\| \le C\delta^\epsilon$ for all $t > 0$. Define k so that $\sup_{t\ge 0}\|z(t)\| = k\delta^\epsilon$. In the last term of (7.49) we estimate

$$|\widehat{w \otimes w}(s,\xi)| \le \|w(s)\|^2 \le (\|u(s)\| + \|v(s)\|) \cdot \|w(s)\| \le (2\|u(0)\| + \delta)\|w(s)\|.$$

We note that by definition of k

$$\|w(s)\| \le \|w(0)\| + k\delta^\epsilon \le \delta + k\delta^\epsilon \le (1+k)\delta^\epsilon.$$

From (7.49) it thus follows that

$$|\hat{z}(t,\xi)| \le C|\xi|^{-1}(1+k)\delta^\epsilon. \tag{7.50}$$

Using (7.48) we obtain for $t \ge 1$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 \le &-\int_{|\xi|^2 t\ge a} |\xi|^2 |\hat{z}(t,\xi)|^2\,d\xi + \|z(t)\|^2(\|\nabla w_0(t)\|_\infty + \|\nabla u(t)\|_\infty) \\
&+ \|z(t)\|\big(\|u(t)\| \cdot \|\nabla w_0(t)\|_\infty + \|w_0(t)\| \cdot \|\nabla u(t)\|_\infty + \|w_0(t)\| \cdot \|\nabla w_0(t)\|_\infty\big) \\
\le &- (at^{-1} - Ct^{-5/4-\sigma/2})\|z(t)\|^2 \\
&+ \|z(t)\| C\delta^\epsilon t^{-5/4-(1-\epsilon/2)\sigma} + at^{-1}\int_{|\xi|^2 t\le a} |\hat{z}(t,\xi)|^2\,d\xi \\
\le &- (a/2t)\|z(t)\|^2 + C\delta^{2\epsilon} t^{-3/2-2(1-\epsilon/2)\sigma} + at^{-1}\int_{|\xi|^2 t\le a} |\hat{z}(t,\xi)|^2\,d\xi.
\end{aligned} \tag{7.51}$$

Here we have used (7.38), (7.40a)–(7.40c), and the "squaring inequality" for the term linear in $\|z(t)\|$. We have also taken a large. Inserting (7.50) and integrating we find for a large enough and $t \ge 1$

$$\|z(t)\| \le t^{-a/2}k\delta^\epsilon + Ct^{-1/4}(1+k)\delta^\epsilon.$$

We take $t_0 > 1$ large enough so that for $t \ge t_0$

$$\|z(t)\| \le k\delta^\epsilon/3 + (1+k)\delta^\epsilon/3.$$

If the supremum of $\|z(t)\|$ does not occur for $t \le t_0$ then

$$k\delta^\epsilon \le k\delta^\epsilon/3 + (1+k)\delta^\epsilon/3$$

so $k \le 1$. Otherwise $k\delta^\epsilon \le C\delta\langle t_0\rangle^{1/4}$ by (7.47) so that $k \le C\langle t_0\rangle^{1/4}$. Thus $\|z(t)\| \le C\delta^\epsilon$. We now make the inductive assumption that $\|z(t)\| \le C\delta^\epsilon\langle t\rangle^{-\mu/2}$. From (7.49) we obtain

$$\begin{aligned}
|\hat{z}(t,\xi)| &\le |\xi|\int_0^t (2\|u(s)\| \cdot \|w(s)\| + \|w(s)\|^2)\,ds \\
&\le C|\xi|\int_0^t \big(\|u(s)\| \cdot (\|w_0(s)\| + \|z(s)\|) + \|w_0(s)\|^2 + \|z(s)\|^2\big)\,ds \\
&\le C|\xi|\int_0^t \delta^\epsilon\big(\langle s\rangle^{-(1-\epsilon/2)\sigma} + \langle s\rangle^{-(\sigma+\mu)/2} + \langle s\rangle^{-\mu}\big)\,ds \\
&\le C|\xi|\int_0^t \delta^\epsilon\big(\langle s\rangle^{-(1-\epsilon/2)\sigma} + \langle s\rangle^{-\mu}\big)\,ds.
\end{aligned} \tag{7.52}$$

In the third inequality above we have used (7.40c). Assuming that $|\xi|^2 t \le a$ and integrating we find that if $\mu \ne 1$ and $\mu \le (1-\epsilon/2)\sigma$ then $|\hat{z}(t,\xi)| \le C\delta^\epsilon|\xi|^{\min(2\mu-1,1)}$ whereas if $\mu \ge (1-\epsilon/2)\sigma$ but $(1-\epsilon/2)\sigma < 1$ then $|\hat{z}(t,\xi)| \le C\delta^\epsilon|\xi|^{(2(1-\epsilon/2)\sigma-1)}$. If

$\mu \geq (1-\epsilon/2)\sigma > 1$ then $|\hat{z}(t,\xi)| \leq C\delta^{\epsilon}|\xi|$. Substituting in the differential inequality (7.51) and integrating we obtain (noting (7.47)) that in these three circumstances $\|z(t)\| \leq C\langle t\rangle^{-\mu'/2}$ where $\mu' = \min(2\mu+1/2, 5/2)$, $\mu' = 1/2+2(1-\epsilon/2)\sigma$ or $\mu' = 5/2$, respectively. Tracing through the iterations starting from $\mu = 0$ we find that at most three iterations gives the stated result. □

Acknowledgement. I.H. would like to thank the Mathematics Institute of Aarhus University and the Schrödinger Institute in Vienna for their support and hospitality while some of this research was completed.

References

[ADT] P. Auscher, S. Dubois, and P. Tchamitchian, *On the stability of global solutions of the Navier-Stokes equations in the space*, J. Math. Pures Appl. **83** (2004) 673–697.

[BJ] H.-O. Bae and B. Jin, *Temporal and spatial decays for the Navier-Stokes equations*, Proc. Roy. Soc. Edinburgh Sect. A **135** (2005), 441–477.

[Ev] L.C. Evans, *Partial differential equations*, Providence, Rhode Island, Am. Math. Soc. 1998.

[FO] J. Fan and T. Ozawa, *Asymptotic stability of the Navier-Stokes equations*, J. Evol. Equ. **8** (2008), 379–389.

[FMRT] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier-Stokes equations and turbulence* Encylopedia of Mathematics and its Applications, **83** Cambridge University Press, Cambridge 2001.

[FT] C. Foias, R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal. **87** (1989), 359–369.

[FK1] H. Fujita, T. Kato, *On the non-stationary Navier-Stokes system*, Rend. Sem. Math. Univ. Padova **32** (1962), 243–260.

[FK2] H. Fujita, T. Kato, *On the Navier-Stokes initial value problem I*, Arch. Rat. Mech. Anal. **16** (1964), 269–315.

[Ga] G.P. Galdi *An introduction to the Navier-Stokes initial-boundary value problem;* In *Fundamental directions in mathematical fluid mechanics* 1–70, Adv. Math. Fluid Mech., Birkhäuser, Basel 2000

[GIP1] I. Gallagher, D. Iftimie, and F. Planchon, *Non-explosion en temps grand et stabilité de solutions globales des équations de Navier-Stokes*, C.R. Math. Acad. Sci. Paris **334** (2002), no.4, 289–292.

[GIP2] I. Gallagher, D. Iftimie, and F. Planchon, *Asymptotics and stability for global solutions to the Navier-Stokes equations*, Ann. Inst. Fourier **53** (2003) 1387–1424.

[GK] Z. Grujič, I. Kukavica, *Space analyticity for the Navier-Stokes and related equations with initial data in L^p*, J. Funct. Anal. **152** (1998), 447–466.

[JS] K.T. Joseph and P.L. Sachdev, *Initial boundary value problems for scalar and vector Burgers equations* Stud. Appl. Math. **106** (2001), no. 4, 481–505.

[KP] T. Kato, G. Ponce *The Navier-Stokes equation with weak initial data*, Int. Math. Research Notices **10** (1994), 435–444.

[KL] A.A. Kiselev and O.A. Ladyzhenskaya, *On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid*, Izv. Akad. Nauk SSSR **21** (1957), 655–680.

[Le] P.G. Lemarié-Rieusset, *Recent development in the Navier-Stokes problem*, New York, Chapman & Hall/CRC 2002.

[OT] M. Oliver, E.S. Titi, *Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in $\mathbb{R}^n$*, J. Funct. Anal. **172** (2000), 1–18.

[Pl] F. Planchon, *Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$*, Ann. Inst. Henri Poincaré Section C **13** no.3 (1996), 319–336.

[PRST] G. Ponce, R. Racke, T.C. Sideris, E.S. Titi *Global Stability of Large Solutions to the 3D Navier-Stokes Equations*, Commun. Math. Phys. **159** (1994), 329–341.

[RS] M. Reed, B. Simon, *Methods of modern mathematical physics, v.2: Fourier analysis, self-adjointness*, New York, Academic Press 1975.

[Sc1] M.E. Schonbek *Lower Bounds of Rates of Decay for Solutions to the Navier-Stokes Equations*, J. American Mathematical Society **4** no. 3 (1991), 423–449.

[Sc2] M.E. Schonbek L^2 *decay for weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **88** (1985), 209–222.

[Wi] M. Wiegner, *Decay results for weak solutions of the Navier-Stokes equation in* R^n, J. London Math. Soc. **35** (1987) 303–313.

[WJZHJ] C. Wu, Z. Ji, Y. Zhang, J. Hao, and X. Jin, *Some new exact solutions for the two-dimensional Navier-Stokes equations*, Physics Letters A **371** (2007), 438–452.

[Zh] Y. Zhou, *Asymptotic stability of the Navier-Stokes equations in a marginal class*, Proc. Roy. Soc. Edinburgh **136** (2006) 1099–1109.