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The Dynamics of Stochastic Processes

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Contents

Preface

This dissertation constitutes the result of my PhD studies at the Department of Mathematical Science, Aarhus University. These studies have been carried out from February 1, 2006 to January 31, 2010 under the supervision of Jan Pedersen (Aarhus University).

Main problems

The present dissertation focuses primarily on the dynamics (i.e. the evolution over time) of different kinds of stochastic processes. In particular the semimartingale property will be important to us, but also path properties such as p-variation, continuity and integrability of seminorms will be considered. The dynamics of solutions to ordinary stochastic differential equations, as in e.g. Protter [8], are always semimartingales and hence most of their probabilistic properties, as e.g. path properties, are well understood. However, for more complicated models such as stochastic fractional differential equations (see [2, 1]), stochastic partial differential equations (see [3, 9]), stochastic delay equations (see [5]) or stochastic Volterra equations (see [6, 7]), the solution is in general not a semimartingale and it is only in special cases that the dynamics of such processes is known. Moreover, many phenomenons, e.g. in finance and turbulence, are well described by stationary or stationary increment processes; an important subclass herein is moving averages. Both in theory and applications it is crucial to know the dynamics of such processes; but this remains an open problem except in simple cases, see e.g. Barndorff-Nielsen and Schmiegel [4]. In addition to the above problems we will also be interested in properties of stationary solutions to the Langevin equation driven by a stationary increment process, and a development of an applicable martingale theory for processes indexed $\mathbb{R}$.

About the Dissertation

The dissertation consists of the following eight manuscripts:

Manuscript A: Representation of Gaussian semimartingales with application to the covariance function. *Stochastics: An International Journal of Probability and Stochastic Processes*, (2009), 21 pages. In Press.

Manuscript B: Spectral representation of Gaussian semimartingales. *Journal of Theoretical Probability 22*(4), (2009), 811–826.

Manuscript C: Gaussian moving averages and semimartingales. *Electronic Journal of Probability 13*, no. 39, (2008), 1140–1165.

Manuscript D: Lévy driven moving averages and semimartingales (with J. Pedersen). *Stochastic Processes and their Applications 119*(9), (2009), 2970–2991.

Manuscript E: Path and semimartingale properties of chaos processes (with S.-E. Graversen). *Stochastic Processes and their Applications*, (2009), 19 pages. doi: 10.1016/j.spa.2009.12.001.

Manuscript F: Integrability of seminorms, (2009), 18 pages. Submitted.

Manuscript G: Martingale-type processes indexed by $\mathbb{R}$ (with S.-E. Graversen and J. Pedersen), (2009), 24 pages. Submitted.

Manuscript H: Quasi Ornstein-Uhlenbeck processes (with O. E. Barndorff-Nielsen), (2009), 25 pages. Submitted.

Manuscripts A–C are written during the first two years of the PhD program, where after I obtained the masters degree. Manuscripts D–H are written during the last two years of the PhD program. In addition to the above manuscripts the dissertation consists of a summary chapter, which sets the stage for the manuscripts and provides an overview of some of the results obtained in them.

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Andreas Basse-O'Connor

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Introduction

The purpose of the present chapter is to (1) introduce some of the problems addressed in the dissertation, (2) describe some of the main results obtained, and (3) briefly relate the dissertation to the literature. Section 1 introduces our basic setting. In Section 2 we are concerned with Gaussian semimartingales and we will primarily focus on representations, the covariance function, the spectral measure and expansions of filtrations. It summarizes results from Manuscripts A, C and E. Section 3 is mainly about the semimartingale property of moving averages. Our focus is primarily on Gaussian, infinitely divisible and chaos processes and we will study the semimartingale property in three different filtrations. This part relies on results from Manuscripts C–E. We conclude this section with a brief review on the results obtained in Manuscript B on the spectral representation of Gaussian semimartingales. The results in Manuscript E rely on an integrability result for seminorms obtained in Manuscript F, which generalizes, in a natural way, a result by X. Fernique [22]. Manuscripts G and H have a slightly different focus, although they are still concerned with the dynamics of stochastic processes. Indeed, in Manuscript G we study martingale-type processes indexed by the real numbers; see Section 5 below. Finally, we study stationary solutions to the Langevin equation driven by a stationary increments process in Manuscript H; see Section 6. Throughout this chapter $(\Omega, \mathcal{F}, \mathrm{P})$ will be a complete probability space on which all random variables are defined.

1 Fundamental classes of stochastic processes

In this section we will introduce some classes of stochastic processes studied in the dissertation. We will start by introducing semimartingales and then proceed with some properties of stationary processes. We conclude the section with some properties of two natural generalizations of Gaussian processes; namely, infinitely divisible processes and chaos processes.

1.1 Semimartingales

By a filtration $\mathscr{F} = (\mathcal{F}_t)_{t\geq 0}$ we mean an increasing family of sub σ-algebras of $\mathcal{F}$ satisfying the usual conditions of completeness and right-continuity. Given a process $X = (X_t)_{t\geq 0}$ we let $\mathscr{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ denote the least filtration to which X is adapted. Similarly, for a process $X = (X_t)_{t\in\mathbb{R}}$ indexed by $\mathbb{R}$, we let $\mathscr{F}^{X,\infty} = (\mathcal{F}_t^{X,\infty})_{t\geq 0}$ denote the least filtration to which $(X_t)_{t\geq 0}$ is adapted and that satisfies $\sigma(X_s : s \in (-\infty, 0]) \subseteq \mathcal{F}_0^{X,\infty}$. A stochastic process $M = (M_t)_{t\geq 0}$ is called a local martingale with respect to a filtration $\mathscr{F}$ if there exists an increasing sequence of $\mathscr{F}$-stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n \uparrow \infty$ a.s. and for all $n \geq 1$, the stopped process $M^{\tau_n} = (M_{t\wedge\tau_n})_{t\geq 0}$ is a martingale with respect to $\mathscr{F}$. A function $f\colon \mathbb{R}_+ \to \mathbb{R}$ is said to be of bounded variation if $\mathrm{V}(f)_t < \infty$ for all $t > 0$, where

$$\mathrm{V}(f)_t = \sup_{\pi} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|, \tag{1.1}$$

and the sup is taken over all finite subdivisions $\pi = \{t_0, \dots, t_n\}$ where $n \geq 1$ and $0 = t_0 < \cdots < t_n = t$.

Given a filtration $\mathscr{F}$, a processes $X = (X_t)_{t\geq 0}$ is said to be a semimartingale with respect to $\mathscr{F}$, if it has a decomposition as

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0, \tag{1.2}$$

where $(A_t)_{t\geq 0}$ is a càdlàg $\mathscr{F}$-adapted process of bounded variation starting at 0, $(M_t)_{t\geq 0}$ is a càdlàg $\mathscr{F}$-local martingale starting at 0, and X_0 is $\mathcal{F}_0$-measurable. (Càdlàg means right-continuous with left-hand limits). We will use the notation $\mathcal{SM}(\mathscr{F})$ to denote the space of all $\mathscr{F}$-semimartingales. Moreover, X is called a special semimartingale if there exists a decomposition (1.2) with A predictable; in this case the decomposition with A predictable is unique and it is called the canonical decomposition of X. For each $p \geq 1$, let H^p denote the space of all special semimartingales $X = X_0 + A + M$ for which $\mathrm{E}[\mathrm{V}(A)_t^p + [M]_t^{p/2}] < \infty$ for all $t < \infty$. Let $\mathscr{G} = (\mathcal{G})_{t\geq 0}$ and $\mathscr{F} = (\mathcal{F}_t)_{t\geq 0}$ be two filtrations such that $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t \geq 0$, then by a theorem of Stricker [46], all semimartingales with respect to $\mathscr{F}$ are also semimartingales with respect to $\mathscr{G}$ provided they are $\mathscr{G}$-adapted. We refer to [16], [25] and [36] for surveys of semimartingale theory.

In what follows we will recall some results from stochastic integration theory. For a fixed filtration $\mathscr{F} = (\mathcal{F}_t)_{t\geq 0}$, $\mathscr{P}$ will denote the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$, i.e., $\mathscr{P}$ is the σ-algebra generated by $(s,t] \times A$ where $0 \leq s < t$ and $A \in \mathcal{F}_s$, and $\{0\} \times A$ where $A \in \mathcal{F}_0$. We will say that $H = (H_t)_{t\geq 0}$ is a simple predictable process if for some $n \geq 1$, H is of the form

$$H_t = Y_0 1_{\{0\}}(t) + \sum_{i=1}^{n} Y_i 1_{(t_i, t_{i+1}]}(t), \qquad t \geq 0, \tag{1.3}$$

where $0 \leq t_1 < \cdots < t_{n+1} < \infty$ and for all $i = 0, \dots, n$, Y_i is a bounded $\mathcal{F}_{t_i}$-measurable random variable. Let $\mathrm{s}\mathscr{P}$ be the space of all simple predictable processes $H = (H_t)_{t\geq 0}$ equipped with the sup norm $\|H\| = \sup_{(s,\omega)\in\mathbb{R}_+\times\Omega} |H_s(\omega)|$. For each càdlàg process Z and $H \in \mathrm{s}\mathscr{P}$ of the form (1.3) define for all $t \geq 0$,

$$\int_0^t H_s \,\mathrm{d}Z_s = \sum_{i=1}^{n} Y_{i-1}(Z_{t_i \wedge t} - Z_{t_{i-1}\wedge t}). \tag{1.4}$$

The following theorem, shown independently by Bichteler [7, 8] and Dellacherie [15], states that semimartingales is the largest class of processes for which the stochastic integral depends continuously on the integrand.

Theorem 1.1 (Bichteler-Dellacherie)**.** *Let $Z = (Z_t)_{t\geq 0}$ be a càdlàg and $\mathscr{F}$-adapted process. Then for all $t > 0$ the map*

$$I_Z : (H \in \mathrm{s}\mathscr{P}) \mapsto \left(\int_0^t H_s \,\mathrm{d}Z_s \in L^0 \right), \tag{1.5}$$

is continuous if and only if Z is a semimartingale with respect to $\mathscr{F}$.

As usual, L^0 is equipped with the topology corresponding to convergence in probability. In the case where Z is a semimartingale, we can immediately extend the stochastic integral $\int_0^t H_s \, \mathrm{d}Z_s$, by continuity, to all bounded predictable processes H. Indeed, let $\mathrm{b}\mathscr{P}$ denote the space of all bounded predictable processes equipped with sup norm. Then, I_Z is a bounded linear operator on $\mathrm{s}\mathscr{P}$ and hence it extends uniquely, by continuity, to a bounded operator on the closure of $\mathrm{s}\mathscr{P}$, which is $\mathrm{b}\mathscr{P}$. Moreover, this extension, also to be denoted $\int_0^\cdot H_s \, \mathrm{d}Z_s$, is a semimartingale. In fact, we can extend the stochastic integral in a reasonable way to a much larger class of predictable processes H such that the integral process still is a semimartingale; see e.g. [44].

In mathematical finance a discounted price process is an $\mathscr{F}$-adapted càdlàg process X, and a simple strategy π is a pair (x, H) where $x \in \mathbb{R}$ and $H \in \mathrm{s}\mathscr{P}$. For all simple strategies $\pi = (x, H)$ define the discounted capital process V^π as

$$V_t^\pi = x + \int_0^t H_s \, \mathrm{d}X_s, \qquad t \geq 0. \tag{1.6}$$

Thus the Bichteler-Dellacherie Theorem shows that the capital process V^π depends continuously on the strategy π if and only if X is a semimartingale with respect to $\mathscr{F}$. This is just one of the reasons why semimartingales is one of the basic model classes in continuous time mathematical finance.

1.2 Stationary and related processes

In this subsection we will recall some results about stationary and stationary increment processes. Recall that a process $X = (X_t)_{t \in \mathbb{R}}$ is said to be stationary if for all $s \in \mathbb{R}$, $(X_t)_{t \in \mathbb{R}}$ has the same finite dimensional distributions as $(X_{s+t})_{t \in \mathbb{R}}$. Moreover, $X = (X_t)_{t \in \mathbb{R}}$ is said to have stationary increments if for all $s \in \mathbb{R}$, $(X_t - X_0)_{t \in \mathbb{R}}$ has the same finite dimensional distributions as $(X_{t+s} - X_s)_{t \in \mathbb{R}}$. Two important examples are Lévy processes and the fractional Brownian motion (fBm). We shall say that a process $X = (X_t)_{t \in \mathbb{R}}$ is a *moving average* if it has a representation of the form

$$X_t = \int_{\mathbb{R}} [\phi(t-s) - \psi(-s)] \, \mathrm{d}Z_s, \qquad t \in \mathbb{R}, \tag{1.7}$$

where $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are two real-valued functions and $Z = (Z_t)_{t \in \mathbb{R}}$ is a suitable process with stationary increments to be specified later on. For all reasonable triplets (ϕ, ψ, Z), the corresponding moving average X has stationary increments and in fact, moving averages are a very important subclass of stationary increment processes. Furthermore, if $\psi \equiv 0$ then X is stationary and if $\psi = \phi$ then $X_0 = 0$. When $\psi \equiv 0$ and ϕ is zero on $(-\infty, 0)$ then X given by (1.7) is called a *backward moving average*. Theorem 1.2 below shows that all second-order stationary processes with absolutely continuous spectral measure satisfying an integrability condition is a backward moving average. In this dissertation we will primarily focus on moving averages where Z is a Lévy process or of the form $\mathrm{d}Z_t = \sigma_t \, \mathrm{d}B_t$, where σ is a stationary process and B is a Brownian motion. Moving averages of the latter type are closely related to ambit processes, induced in Barndorff-Nielsen and Schmiegel [5, 3], and are used e.g. in modeling of turbulence. When Z is a Lévy process, X is infinitely divisible (to be defined in the next subsection) and when Z is a Brownian motion, X is Gaussian. Two examples of moving averages are the Ornstein-Uhlenbeck type process, which corresponds to $\psi = 0$, $\phi = e^{-\beta t} 1_{\mathbb{R}_+}(t)$ and Z a Lévy process with $\mathrm{E}[\log^+ |Z_1|] < \infty$, and the fBm, which corresponds to $\phi(t) = \psi(t) = t^{H-1/2} 1_{\mathbb{R}_+}(t)$ for some $H \in (0,1)$ and Z a Brownian motion.

A square-integrable process X is said to be second-order stationary if its covariance function $\mathrm{Cov}(X_t, X_u)$ only depends on $t-u$ and its mean-value function $\mathrm{E}[X_t]$ does not depend on t. Let $X = (X_t)_{t\in\mathbb{R}}$ be an L^2-continuous second-order stationary process, and let F_X denote its spectral measure, i.e., F_X is the unique finite and symmetric measure on $\mathbb{R}$ satisfying

$$\mathrm{Cov}(X_t, X_u) = \int_{\mathbb{R}} e^{i(t-u)x}\, F_X(\mathrm{d}x), \qquad t, u \in \mathbb{R}. \tag{1.8}$$

Moreover, let F'_X denote the density of the absolutely continuous part of F_X. For each $t \in \mathbb{R}$ let

$$\mathcal{X}_t = \overline{\mathrm{span}}\{X_s : s \le t\}, \quad \mathcal{X}_{-\infty} = \cap_{t\in\mathbb{R}} \mathcal{X}_t, \quad \mathcal{X}_\infty = \overline{\mathrm{span}}\{X_s : s \in \mathbb{R}\}, \tag{1.9}$$

($\overline{\mathrm{span}}$ denotes the L^2-closure of the linear span). Following Karhunen [30], X is called *deterministic* if $\mathcal{X}_{-\infty} = \mathcal{X}_\infty$ and *purely non-deterministic* if $\mathcal{X}_{-\infty} = \{0\}$. (Note that deterministic does *not* mean that X is non-random). The next theorem, which is given in [30, Satz 5–6] (cf. also [18]), provides a decomposition of a stationary process as a sum of a deterministic process and a purely non-deterministic process; in fact, the purely non-deterministic process is decomposed as a backward moving average.

Theorem 1.2 (Karhunen)**.** *Let X be an L^2-continuous second-order stationary process with spectral measure F_X. If*

$$\int_{\mathbb{R}} \frac{|\log F'_X(x)|}{1+x^2}\,\mathrm{d}x < \infty \tag{1.10}$$

then there exists a unique decomposition of X as

$$X_t = \int_{-\infty}^t \phi(t-s)\,\mathrm{d}Z_s + V_t, \qquad t \in \mathbb{R}, \tag{1.11}$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is a Lebesgue square-integrable deterministic function, and Z is a process with second-order stationary and orthogonal increments satisfying $\mathrm{E}[|Z_u - Z_s|^2] = |u-s|$ *for all $u, t \in \mathbb{R}$, and for all $t \in \mathbb{R}$, $\mathcal{X}_t = \overline{\mathrm{span}}\{Z_s - Z_u : -\infty < u < s \le t\}$, and V is a deterministic second-order stationary process.*

Moreover, if F_X is absolutely continuous and satisfies (1.10) *then $V \equiv 0$ and hence X is a backward moving average. Finally, the integral in* (1.10) *is infinite if and only if X is deterministic.*

The integral of ϕ with respect to Z in (1.11) is defined in L^2-sense; see e.g. [18]. Note also that if X is Gaussian then the process Z in (1.11) is a standard Brownian motion. Note finally that also stationary increment processes have a spectral measure; see e.g. Section 7 in Manuscript C for the precise definition.

1.3 Infinitely divisible processes

In this subsection we will recall some properties and characteristics of infinitely divisible processes. A probability measure μ on $\mathbb{R}^n$ is called infinitely divisible (ID) if for all $k \ge 1$ there exists a probability measure μ_k on $\mathbb{R}^n$ such that $\mu = \mu_k^{*k}$, where μ_k^{*k} is the k-fold convolution of μ_k; see e.g. [43]. Similarly, an $\mathbb{R}^n$-valued random vector X is called ID if its law, P_X, is an ID probability measure on $\mathbb{R}^n$. Key examples of ID distributions are Gaussian, α-stable, gamma and Poisson. Let T denote a non-empty set. Then a process

$X = (X_t)_{t\in T}$ is called an ID process if for all $n \geq 1$ and $t_1, \ldots, t_n \in T$, $(X_{t_1}, \ldots, X_{t_n})$ is an $\mathbb{R}^n$-valued ID random vector. A key example of an ID process is a Lévy process. Recall that $(Z_t)_{t\geq 0}/(Z_t)_{t\in\mathbb{R}}$ is said to be a Lévy process indexed by $\mathbb{R}_+/\mathbb{R}$ if it has independent, stationary increments, càdlàg sample paths and $Z_0 = 0$.

Let S be a non-empty set equipped with a δ-ring $\mathcal{S}$, i.e., $\mathcal{S}$ is a family of subsets of S which is closed under union, countable intersection and set difference. We will say that $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ is an ID independently scattered random measure (random measure, for short) if for all pairwise disjoint set $(A_n)_{n\geq 1} \subseteq \mathcal{S}$ we have that $\{\Lambda(A_n) : n \geq 1\}$ are independent ID random variables, and if $\cup_{n=1}^\infty A_n \in \mathcal{S}$, we have that

$$\Lambda\left(\cup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \Lambda(A_n) \qquad \text{a.s.} \tag{1.12}$$

As usual assume also that $\mathcal{S}$ is σ-finite, that is, there exists $(S_n)_{n\geq 1} \subseteq \mathcal{S}$ such that $\cup_{n\geq 1} S_n = S$. We will equip S with the σ-algebra $\sigma(\mathcal{S})$.

Given a random measure Λ, there exists a positive measurable function $\sigma^2 \colon S \to \mathbb{R}_+$, a measurable function $a \colon S \to \mathbb{R}$, a measurable parametrization of Lévy measures on $\mathbb{R}$ $\nu(dx, s)$, and a σ-finite measure m on S such that for all $A \in \mathcal{S}$,

$$E[e^{iy\Lambda(A)}] = \exp\left(\int_A K(y,s)\, m(\mathrm{d}s)\right), \qquad y \in \mathbb{R}, \tag{1.13}$$

where for $y \in \mathbb{R}$, $s \in S$ and $\tau(x) = x1_{\{|x|\leq 1\}} + \mathrm{sign}(x)1_{\{|x|>1\}}$,

$$K(y,s) = iya(s) - \sigma^2(s)y^2/2 + \int_{\mathbb{R}} \left(e^{iyx} - 1 - iy\tau(x)\right) \nu(\mathrm{d}x, s), \tag{1.14}$$

see Proposition 2.4 in Rajput and Rosiński [37]. Furthermore, m is called the control measure of Λ. A measurable function $f \colon S \to \mathbb{R}$ is said to be Λ-integrable if there exists a sequence $(f_n)_{n\geq 1}$ of simple functions such that $\lim_n f_n = f$ m-a.s. and for all $A \in \mathcal{S}$ the limit $\lim_n \int_A f_n(s)\, \Lambda(\mathrm{d}s)$ exists in probability. In this case the stochastic integral of f with respect to Λ is defined as

$$\int_S f(s)\, \Lambda(\mathrm{d}s) = \lim_{n\to\infty} \int_S f_n(s)\, \Lambda(\mathrm{d}s) \qquad \text{in probability.} \tag{1.15}$$

We refer to Theorem 2.7 in [37] for necessary and sufficient conditions on f, a, σ^2, ν and m for existence of the stochastic integral $\int_S f(s)\, \Lambda(\mathrm{d}s)$. Note that integration with respect to a centered Gaussian random measures is particularly simple since in this case the integral can be constructed through an L^2-isometry. Let Z be a Lévy process indexed by $\mathbb{R}_+$ or $\mathbb{R}$ then there exists a unique random measure on respectively $(\mathbb{R}_+, \mathcal{B}_b(\mathbb{R}_+))$ or $(\mathbb{R}, \mathcal{B}_b(\mathbb{R}))$ which for $a < b$ is given by $\Lambda((a, b]) = Z_b - Z_a$. (For $A \subseteq \mathbb{R}$, $\mathcal{B}_b(A)$ denotes the δ-ring of bounded Borel subsets of A).

Let S and T be separable and complete metric spaces, e.g. $[0, 1]$ or $\mathbb{R}$, and assume that S is uncountable. Let $(X_t)_{t\in T}$ be an ID process separable in probability. Then, under minimal conditions, Theorem 4.11 in [37] ensures that there exists a random measure Λ on S, and Λ-integrable functions $f(t, \cdot) \colon S \to \mathbb{R}$ for all $t \in T$ such that

$$(X_t)_{t\in T} \stackrel{\mathscr{D}}{=} \left(\int_S f(t,s)\, \Lambda(\mathrm{d}s)\right)_{t\in T}, \tag{1.16}$$

where $\stackrel{\mathscr{D}}{=}$ denotes equality in finite dimensional distributions. Such a representation is called a spectral representation of X, and when X is α-stable for some $\alpha \in (0, 2]$, we may choose the random measure Λ to be α-stable as well.

We conclude this subsection with the following two results concerning spectral representations: Assume that $X = (X_t)_{t\in T}$ has spectral representation (1.16) with $\sigma^2 \equiv 0$. If the sample paths of X belong to a closed vector space V of $\mathbb{R}^T$ a.s. then by Rosiński [41] it follows that $t \mapsto f_t(s)$ belongs to V for m-a.a. $s \in S$. That is, X inherits all the properties of the functions $t \mapsto f_t(s)$. Using zero-one laws Rosiński [40] shows that for an α-stable process with $\alpha \in (0,2)$ the sample paths of X belong to V if and only if $t \mapsto f_t(s)$ belongs to V for m-a.a. $s \in S$.

1.4 Chaos processes

We refer to Manuscripts E–F for the general definition of chaos process and some of their properties, and to Janson [27] for a nice introduction to different aspects of the Gaussian case. Let us here briefly recall the definition of a Gaussian chaos process. To do so, let $\mathcal{G}$ be a vector space of Gaussian random variables and for all $d \geq 1$ let $\overline{\Pi}^d_{\mathcal{G}}$ be the L^2-closure of the random variables of the form

$$p(Z_1, \dots, Z_n) \tag{1.17}$$

where for $n \geq 1$, $p\colon \mathbb{R}^n \to \mathbb{R}$ is polynomial of degree at most d and $Z_1, \dots, Z_n \in \mathcal{G}$. A stochastic process $(X_t)_{t\in T}$ is called a Gaussian chaos process of order d if for all $t \in T$, $X_t \in \overline{\Pi}^d_{\mathcal{G}}$. When $\mathcal{G} = \{\int_0^1 h(s)\,\mathrm{d}B_s : h \in L^2([0,1])\}$ a result by Itô [24] shows that $\overline{\Pi}^d_{\mathcal{G}}$ is exactly the space of multiple Wiener-Itô integrals with respect to B, that is, the random variables of the form

$$\sum_{k=0}^{d} \int_{[0,1]^k} f_k(s_1, \dots, s_k)\,\mathrm{d}B_{s_1} \cdots \mathrm{d}B_{s_k}, \tag{1.18}$$

where $f_k \in L^2([0,1]^k)$ for all $k = 0, \dots, d$.

2 The semimartingale property

In this chapter we will discuss the semimartingale and related properties of Gaussian and related processes. Jain and Monrad [26] show that the bounded variation and martingale components of a Gaussian quasimartingale both are Gaussian processes. Relying on a classical result by Fernique [22], Stricker [45] extends this result to cover all Gaussian semimartingales $X = (X_t)_{t\geq 0}$ and obtains moreover that $X \in H^p$ for all $p \geq 1$. This shows, in particular, that a Gaussian semimartingale is special. The key idea is to approximate the bounded variation component similarly to K. M. Rao's [38] proof of the Doob-Meyer decomposition. Emery [20] shows that the covariance function Γ_X of a Gaussian semimartingale X is of bounded variation; moreover, he obtains a characterization of the semimartingale property of X in terms of integrals with respect to the Lebesgue-Stieltjes measure on $\mathbb{R}^2_+$ induced by Γ_X; see the Introduction in Manuscript A for further details. For a stationary Gaussian process $X = (X_t)_{t\in\mathbb{R}}$, Jeulin and Yor [29, Proposition 19] have characterized the spectral measure of X for $(X_t)_{t\geq 0}$ to be an $\mathscr{F}^{X,\infty}$-semimartingale.

In Theorem 4.5 in Manuscript A we extend a representation result due to Stricker [45], from Gaussian processes of the form $X_t = B_t + \int_0^t Z_s\,\mathrm{d}s$ where B is a Brownian motion, to general Gaussian semimartingales. This result shows that the bounded variation component A of a Gaussian semimartingale $X = X_0 + M + A$ can represented as

$$A_t = \int_0^t \left(\int_0^r \Psi_r(s)\,\mathrm{d}M_s \right) \mu(\mathrm{d}r) + \int_0^t Y_r\,\mu(\mathrm{d}r) \tag{2.1}$$

where $\Psi\colon \mathbb{R}_+^2 \to \mathbb{R}$ is a deterministic kernel, μ is a Radon measure on $\mathbb{R}_+$, and Y is a Gaussian process which is independent by M. So, in particular, A is decomposed into an $\mathscr{F}^M$-adapted component and a component which is independent of M. This result relies on a result by Jeulin [28], which shows that if a Gaussian process Y is of bounded variation then almost surely it is absolutely continuous with respect to the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto \mathrm{E}[\mathrm{V}(Y)_t]$. Furthermore, in Theorem 5.2, Manuscript A, we use decomposition (2.1) to characterize the covariance function of Gaussian semimartingales. This is an alternative to the result obtained by Emery [20] mentioned above. In Manuscript A our characterization is then used to study properties of the Lebesgue-Stieltjes measure on $\mathbb{R}_+^2$ induced by the covariance function of a Gaussian semimartingale.

Next we will consider some extensions to chaos processes. The following results are given in Manuscript D. Extending results by Jain and Monrad [26] and Jeulin [28] from the Gaussian case, we show in Theorem 3.1, Manuscript D, that if a Gaussian chaos process X is of bounded variation then it is necessarily absolutely continuous with respect to μ_X, which is the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto \mathrm{E}[\mathrm{V}(X)_t]$. Using this result we show in Proposition 3.5, Manuscript D, that if a chaos process X is of bounded p-variation for some $p \geq 1$ then it has almost surely continuous paths if and only if it is continuous in probability. Thereafter, extending a result by Stricker [45], mentioned above, we show in Theorem 4.1, Manuscript D, that if a chaos process is a semimartingale then both its martingale and bounded variation components are chaos processes and $X \in H^p$ for all $p \geq 1$; recall the definition of H^p from Subsection 1.1. Likewise, in Proposition 4.2, Manuscript D, the canonical decomposition of Dirichlet processes is characterized as well.

Let us return to the Gaussian case. Consider a Gaussian process $X = (X_t)_{t\in\mathbb{R}}$ which is either stationary or has stationary increments and $X_0 = 0$. In both cases the distribution of X is uniquely determined by its spectral measure; see Section 1.2. In Manuscript C, Theorems 6.4 and 7.1, we characterize the $\mathscr{F}^{X,\infty}$-semimartingale property of $(X_t)_{t\geq 0}$ in terms of its spectral measure. (Recall the definition of $\mathscr{F}^{X,\infty}$ from Section 1.1). Theorem 6.4 gives an alternative to Jeulin and Yor [29, Proposition 19]. To state Theorem 7.1 let $(X_t)_{t\in\mathbb{R}}$ be a Gaussian process with stationary increments, $X_0 = 0$ and spectral measure $F_X = F_X^s + F_X' \,\mathrm{d}\lambda$, where F_X^s and $F_X' \,\mathrm{d}\lambda$ are respectively the singular and absolute continuous component of F_X. Then Theorem 7.1, Manuscript C, says that $(X_t)_{t\geq 0}$ is an $\mathscr{F}^{X,\infty}$-semimartingale if and only if F_X^s is a finite measure and there exist $\alpha \in \mathbb{R}$ and $h \in L^2(\lambda)$ which is zero on $(-\infty, 0)$ if $\alpha \neq 0$ such that

$$f = |\alpha + \hat{h}|^2. \tag{2.2}$$

When X is a fBm of index $H \in (0,1)$ we have $F_X^s = 0$ and $f(s) = c_H|s|^{1-2H}$. In this case it is easily seen that f is of the form (2.2) if and only if $H = 1/2$, which then gives a different proof of the well-known fact that the fBm is an $\mathscr{F}^{X,\infty}$-semimartingale if and only if $H = 1/2$. Theorem 7.1 relies heavily on complex function theory, in particular a decomposition result for Hardy functions due to Beurling [6]; see Chapter 2, Manuscript C, for a brief survey on Hardy functions. These functions will also be crucial for some of the results discussed in Subsection 3.2.

As above assume that $X = (X_t)_{t\in\mathbb{R}}$ is a Gaussian process which is either stationary or has stationary increments and $X_0 = 0$. It is of interest to consider the relationship between being a semimartingale with respect to $\mathscr{F}^{X,\infty}$ or $\mathscr{F}^X$. Since $\mathcal{F}_t^X \subseteq \mathcal{F}_t^{X,\infty}$ for all $t \geq 0$ it follows by Stricker's Theorem, see Subsection 1.1, that it is a weaker property to be an $\mathscr{F}^X$-semimartingale than being an $\mathscr{F}^{X,\infty}$-semimartingale. Let $X = (X_t)_{t\geq 0}$

is a Gaussian $\mathscr{F}^X$-semimartingale with canonical decomposition $X = X_0 + M + A$. Theorem 4.8(i–iii), Manuscript A, studies the canonical decomposition of X, and in particular Theorem 4.8(iii) gives the following expansion of filtration result: $(X_t)_{t\geq 0}$ is an $\mathscr{F}^{X,\infty}$-semimartingale if and only if $t \mapsto \mathrm{E}[\mathrm{V}(A)_t]$ is Lipschitz continuous on $\mathbb{R}_+$.

3 The semimartingale property of moving averages

Let us first warm up with some preliminary observations concerning the semimartingale property and then afterwards, in Subsection 3.2, we go into a deeper study of this and related properties.

3.1 Preliminary observations

Let $Z = (Z_t)_{t\geq 0}$ be an $\mathscr{F}$-semimartingale and H be a predictable process. Recall from Subsection 1.1 that if the integral

$$X_t = \int_0^t H_s \,\mathrm{d}Z_s, \qquad t \geq 0, \tag{3.1}$$

exists, then X is a semimartingale with respect to $\mathscr{F}$. But what happens when the integrand H depends on t also? The simplest case is when instead of H_s we integrate $\phi(t-s)$ where $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ is a deterministic function, and then X is the stochastic convolution between ϕ and Z, given by

$$X_t = (\phi * Z)_t = \int_0^t \phi(t-s)\,\mathrm{d}Z_s, \qquad t \geq 0. \tag{3.2}$$

By use of a stochastic Fubini result, see e.g. [36, Chapter IV, Theorem 65], one can show that X is an $\mathscr{F}$-semimartingale if ϕ is absolutely continuous with a locally square-integrable density, i.e., there exists a function $\phi' \in L^2_{\mathrm{loc}}(\mathbb{R}_+,\lambda)$ (where λ denotes the Lebesgue measure) and $\alpha \in \mathbb{R}$ such that

$$\phi(t) = \alpha + \int_0^t \phi'(s)\,\mathrm{d}s, \qquad t \geq 0. \tag{3.3}$$

If for some $\beta > 0$ we let $\phi(t) = e^{-\beta t}$ for $t \geq 0$ then ϕ is of the above type and in this case the stochastic convolution $X = \phi * Z$ is the Ornstein-Uhlenbeck process driven by Z which starts at 0. On the other hand, if $\phi = 1_{[0,1]}$ then $X_t = Z_t - Z_{(t-1)\vee 0}$ and if e.g. Z is a Brownian motion then X is not a semimartingale in any filtration. Thus, there are simple examples of ϕ's for which the stochastic convolution $X = \phi * Z$ is a semimartingale, and for which it is not.

3.2 Characterization of the semimartingale property

In this subsection we will discuss the semimartingale property of stochastic convolutions of the form (3.2) and of moving averages of the form

$$X_t = \int_{-\infty}^t [\phi(t-s) - \psi(-s)]\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}. \tag{3.4}$$

For a moving average $(X_t)_{t\geq 0}$ of the form (3.4) there are at least three filtrations in which it is natural to consider the semimartingale property; namely $\mathscr{F}^X$, $\mathscr{F}^{X,\infty}$ and $\mathscr{F}^{Z,\infty}$ (recall their definitions from Section 1.1). Note that these filtrations satisfy

$$\mathcal{F}_t^X \subseteq \mathcal{F}_t^{X,\infty} \subseteq \mathcal{F}_t^{Z,\infty}, \qquad t \geq 0, \tag{3.5}$$

and hence by Stricker's Theorem we have

$$\mathcal{SM}(\mathscr{F}^{Z,\infty}) \subseteq \mathcal{SM}(\mathscr{F}^{X,\infty}) \subseteq \mathcal{SM}(\mathscr{F}^{X}), \tag{3.6}$$

i.e., it is easiest to be a semimartingale in $\mathscr{F}^X$ and hardest in $\mathscr{F}^{Z,\infty}$. In the below table we gather some results characterizing the semimartingale property of X; vertical is the filtration and horizontal is the driving process Z.

filtration\\Z	Brownian motion	Lévy process	Chaos process
$\mathscr{F}^{Z,\infty}$	Knight [31], $(*)$	Manuscript D	Manuscript E
$\mathscr{F}^{X,\infty}$	Jeulin and Yor [29], Manuscript A	—	—
$\mathscr{F}^{X}$	Manuscript B	—	—

$(*)$: Cherny [14], Cheridito [12], Manuscript B.

Let us discuss some of the results mentioned in this table. The first necessary and sufficient conditions on (ϕ,ψ) for $X=(X_t)_{t\geq 0}$ to be a semimartingale are due to Frank B. Knight. To discuss these and related results let us, as long as not mentioned otherwise, assume that Z is a Brownian motion and that X is a moving average of the form (3.4). Note that since ϕ and ψ are deterministic, X is then a Gaussian process. By using the result of Stricker [45] on Gaussian semimartingales discussed in Section 2, it is shown by Knight [31] that X is an $\mathscr{F}^{Z,\infty}$-semimartingale if and only if ϕ is absolutely continuous with a square-integrable density. Cheridito [12] extends this, using Novikov's condition, by showing that if X is an $\mathscr{F}^{Z,\infty}$-semimartingale then it is locally equivalent to a Brownian motion, i.e., for each finite $T>0$ the law of $(c_0X_t)_{t\in[0,T]}$ is equivalent to the Wiener measure on $C([0,T];\mathbb{R})$, where $c_0=1/\mathrm{E}[X_1^2]^{1/2}$. The first characterization of the $\mathscr{F}^{X,\infty}$-semimartingale property of X is due to Jeulin and Yor [29]. They used Knight's result together with a result on Hardy functions to obtain necessary and sufficient conditions in terms of $|\hat{\phi}|^2$ for X to be a semimartingale with respect to $\mathscr{F}^{X,\infty}$. ($\hat{\phi}$ denotes the Fourier transform of ϕ).

Let S^1 be the unit circle in the complex field $\mathbb{C}$, i.e., $S^1=\{z\in\mathbb{C}:|z|=1\}$, λ be the Lebesgue measure on $\mathbb{R}$ and assume for simplicity that $\psi=\phi$. Theorem 3.2, Manuscript C, shows that $X=(X_t)_{t\geq 0}$ is an $\mathscr{F}^{X,\infty}$-semimartingale if and only if ϕ can be decomposed as

$$\phi(t)=\beta+\alpha\tilde{f}(t)+\int_0^t \widehat{f\hat{h}}(s)\,ds, \qquad \lambda\text{-a.a. } t\in\mathbb{R}, \tag{3.7}$$

where $\alpha,\beta\in\mathbb{R}$, $f\colon\mathbb{R}\to S^1$ is a measurable function such that $\overline{f}=f(-\cdot)$, and $h\in L^2_{\mathbb{R}}(\lambda)$ is 0 on $\mathbb{R}_+$ when $\alpha\neq 0$. Furthermore, $\tilde{f}\colon\mathbb{R}\to\mathbb{R}$ is the $\sim$-transform of f, given by

$$\tilde{f}(t)=\lim_{a\to\infty}\int_{-a}^{a}\frac{e^{its}-1_{[-1,1]}(s)}{is}f(s)\,\mathrm{d}s \tag{3.8}$$

where the limit is in λ-measure. This new transform, which shares some properties with the Fourier transform, is studied in detail in Manuscript C. Key ingredients in the proof of (3.7) are again decompositions of Hardy functions and the result by Knight. Furthermore, our result generalizes Knight's result in the sense that we may choose $f\equiv 1$ if and only if

X is an $\mathscr{F}^{Z,\infty}$-semimartingale. Thus, we have given necessary and sufficient conditions for an $\mathscr{F}^{X,\infty}$-semimartingale to be an $\mathscr{F}^{Z,\infty}$-semimartingale. In this context recall also Theorem 4.8(iii), Manuscript A, which characterizes when an $\mathscr{F}^{X}$-semimartingale is an $\mathscr{F}^{X,\infty}$-semimartingale; see the end of Section 2 of the present chapter.

Let us mention that the above result gives a constructive way to obtain decompositions of the Brownian motion and the following is an example of this: Let $(X_t)_{t\geq 0}$ be a stationary Ornstein-Uhlenbeck process driven by a Brownian motion B, in short $\mathrm{d}X_t = -X_t\,\mathrm{d}t + \mathrm{d}B_t$. Then $Y = (Y_t)_{t\geq 0}$, given by

$$Y_t = B_t - 2\int_0^t X_s\,\mathrm{d}s, \qquad t \geq 0, \tag{3.9}$$

is a Brownian motion in its natural filtration; see the end of Section 3, Manuscript C. Moreover, in Proposition 6.3, Manuscript C, we provide necessary and sufficient conditions for the $\mathscr{F}^{X}$-Markov property of a moving average X; this result relies on a characterization of the Ornstein-Uhlenbeck process due to Doob [17]. In particular, if X is an $\mathscr{F}^{X}$-Markov process then Proposition 6.3 shows that X is an $\mathscr{F}^{X,\infty}$-semimartingale and gives necessary and sufficient conditions for X to be an $\mathscr{F}^{Z,\infty}$-semimartingale.

All results mentioned above are concerned with the case where the driven process Z is a Brownian motion. Next we will consider more general processes; we will start with the case where Z is a Lévy process. Consider a stochastic convolution $X = (X_t)_{t\geq 0}$ of the form

$$X_t = \int_0^t \phi(t-s)\,\mathrm{d}Z_s, \qquad t \geq 0, \tag{3.10}$$

where $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ is a deterministic function and $Z = (Z_t)_{t\geq 0}$ is a Lévy process. The main result in Manuscript D, Theorem 3.1, provides necessary and sufficient conditions on ϕ for X to be an $\mathscr{F}^{Z}$-semimartingale. As an example, when Z is an α-stable Lévy process and $\alpha \in (1,2]$, these conditions show that X is an $\mathscr{F}^{Z}$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally α-integral density, i.e., $\int_0^t |\phi'(s)|^\alpha\,\mathrm{d}s < \infty$ for all $t > 0$. The proof of Theorem 3.1 relies mainly on various results by J. Rosiński and co-authors, in particular the moment estimates in [35] and the integrability of seminorm result in [42]. As a special case, Theorem 3.1 provides necessary and sufficient conditions for X to be of bounded variation, hereby generalizing results by Doob [18] and Ibragimov [23] from the Gaussian case. Manuscript D is concluded with a study of moving averages X of the form (3.4) where $Z = (Z_t)_{t\in\mathbb{R}}$ is a Lévy process. A complete characterization of the semimartingale property is in this case still missing. However, some further extensions were obtained joint with Jan Rosiński, under a visit at the University of Tennessee, USA, in April, 2009. This work is still in progress and not included in the dissertation.

Manuscript E, Section 5, uses the results on the semimartingale property of chaos processes, mentioned in Section 2, to extend the result of Knight [31] from the case where Z is a Brownian motion to where it is a chaos process. We consider both moving averages of the form (3.4) and stochastic convolutions of the form (3.10). For example, assume $B = (B_t)_{t\geq 0}$ is a Brownian motion, $\sigma = (\sigma_t)_{t\geq 0}$ is a Gaussian chaos process which is, say, left-continuous in probability and $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ a deterministic function such that

$$X_t = \int_0^t \phi(t-s)\sigma_s\,\mathrm{d}B_s, \qquad t \geq 0, \tag{3.11}$$

is well-defined. Then Corollary 5.4, Manuscript E, shows that X is an $\mathscr{F}^{Z}$-semimartingale if and only if ϕ is absolutely continuous with a locally square-integrable density.

The spectral decomposition of, say, centered Gaussian processes, generalizes moving averages and stochastic convolutions driven by the Brownian motion in a natural way to non-stationary processes. Recall from Subsection 1.3 that each centered Gaussian process X, continuous in probability, has a spectral decomposition in distribution of the form (1.16). In Manuscript B the semimartingale property is characterized in terms of spectral representation and hereby we answer a question raised by Knight [31], about extending his result on moving averages to non-stationary Gaussian processes; see Manuscript B, Theorem 4.6. The semimartingale property is studied both in the natural filtration of X, i.e. in $\mathscr{F}^X$, (see Theorem 4.1, Manuscript B) and in the filtration spanned by the background measure Λ (see Theorem 4.6, Manuscript B). Our set-up includes, in particular, processes of the form

$$(*)\quad X_t = \int_0^t f(t,s)\,\mathrm{d}B_s \quad \text{and} \quad (**)\quad X_t = \int_{[0,t]\times\mathbb{R}^d} f(t-s,x)\,\Lambda(\mathrm{d}s,\mathrm{d}x), \tag{3.12}$$

that occur as solutions to fractional and partial stochastic differential equations; see [2, 1, 4, 48]. (In (3.12) B is a Brownian motion and Λ is a centered Gaussian random measure on $\mathbb{R}_+\times\mathbb{R}^d$). For example, if X of the form (3.12)($*$), then by Theorem 4.6, Manuscript B, X is an $\mathscr{F}^B$-semimartingale if and only if f is of the form

$$f(t,s) = g(s) + \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r), \qquad t,s\in\mathbb{R}_+, \tag{3.13}$$

where $g\in L^2_{\mathrm{loc}}(\mathbb{R}_+,\lambda)$, μ is a Radon measure on $\mathbb{R}_+$ and $\Psi\colon\mathbb{R}^2_+\to\mathbb{R}$ is a measurable function satisfying $\|\Psi_r(\cdot)\|_{L^2(\lambda)}=1$ for all $r\geq 0$. In the case where $f(t,s)=\phi(t-s)$, a minor extension of Knight's result shows that X is an $\mathscr{F}^B$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density; that is, g is constant, μ is the Lebesgue measure and $\Psi_r(s)$ depends only on $r-s$.

4 Integrability of seminorms

When studying the dynamic of stochastic processes it is often important to have have integrability and moment estimates of functionals of the process of interest. For example all the classical continuity results for Gaussian processes due to Dudley [19], Fernique [21] and Talagrand [47] rely on very precise moment estimates for functionals of the process. However, our main motivation is due to the fact that such integrability results are a crucial tool when studying the semimartingale property as in Section 2. The following classical result by Fernique [22] covers the Gaussian case: Let T be a countable set, $X=(X_t)_{t\in T}$ be a Gaussian process and $N\colon\mathbb{R}^T\to[0,\infty]$ be a measurable seminorm on $\mathbb{R}^T$ such that $N(X)<\infty$ a.s. Then there exists an $\epsilon>0$ such that $\mathrm{E}[e^{\epsilon N(X)^2}]<\infty$. A key example of N is $N(f)=\sup_{t\in T}|f(t)|$ for all $f\in\mathbb{R}^T$. We refer to Manuscript E, Section 1.2, for a survey of results providing integrability of seminorms. A general definition of chaos processes is introduced in Manuscript E which includes infinitely divisible processes (see Section 1.3), Gaussian chaos processes (see Section 1.4) and linear processes. Manuscript E partly unifies and partly extends known results, and in particular Theorem 2.7, Manuscript E, shows that chaos processes provide a setting in which the above result of Fernique extends in a natural way; further, this theorem gives explicit constants for equivalence of L^p-norms. For example, if $X=(X_t)_{t\in T}$ is a symmetric normal inverse Gaussian process and N is a seminorm on $\mathbb{R}^T$ such that $N(X)<\infty$ a.s. then Theorem 2.7, Manuscript E, shows that $\mathrm{E}[N(X)^p]<\infty$ for all $p\in(0,1)$. Moreover, Proposition 2.9, Manuscript E, provides a simple proof of a result on Gaussian chaos processes due to Borell [9]. Our proofs rely strongly on results from probability in Banach spaces, in particular those on hypercontractivity properties mainly due to Borell [10, 11] and Krakowiak and Szulga [32, 33].

5 Martingale-type processes indexed by $\mathbb{R}$

The theory of martingales $M = (M_t)_{t \geq 0}$ indexed by $\mathbb{R}_+$ is very well developed. However, stationary processes are always indexed by $\mathbb{R}$. Hence the question raises: What is the right definition of martingales indexed by $\mathbb{R}$ and what are their properties? We shall say that a process $M = (M_t)_{t \in \mathbb{R}}$ is a *martingale* if $\mathrm{E}[M_t | \mathcal{F}_s] = M_s$ for all $s, t \in \mathbb{R}$ with $s \leq t$ and an *increment martingale* if for all $s \in \mathbb{R}$, $(M_{t+s} - M_s)_{t \geq 0}$ is a martingale (in the usual sense). Observe that increment martingales are a more general type of processes than martingales and that e.g. a Brownian motion $B = (B_t)_{t \in \mathbb{R}}$ indexed by $\mathbb{R}$ is not a martingale but only an increment martingale. The object of Manuscript G is to study the basic properties of increment martingales such as their relationship to martingales, their behavior at $-\infty$ but also their Doob-Meyer decompositions. Two such results are the following in which $M = (M_t)_{t \in \mathbb{R}}$ is an increment martingale. By Proposition 3.9, Manuscript G, we have that $M_{-\infty}$ exists a.s. and $M - M_{-\infty}$ is a martingale if and only if $\{M_0 - M_t : t \in (-\infty, 0]\}$ is uniformly integrable. Next assume that M is square-integrable, then we have by Theorem 3.14, Manuscript G that there exists a predictable and increasing process $\langle M \rangle = (\langle M \rangle_t)_{t \in \mathbb{R}}$ such that $\lim_{t \to -\infty} \langle M \rangle_t = 0$ a.s. and $M^2 - \langle M \rangle$ is a martingale if and only if $M_{-\infty}$ exists a.s. and $M - M_{-\infty}$ is a square-integrable martingale.

6 Quasi Ornstein-Uhlenbeck processes

Let $\lambda > 0$ be a positive real number and $N = (N_t)_{t \in \mathbb{R}}$ be a measurable process with stationary increments. In Manuscript H we study stationary solutions $X = (X_t)_{t \in \mathbb{R}}$ to the Langevin equation

$$\mathrm{d}X_t = -\lambda X_t \,\mathrm{d}t + \mathrm{d}N_t. \tag{6.1}$$

Such solutions are called quasi Ornstein-Uhlenbeck (QOU) processes. The existing literature has mainly focused on the classical case where N has independent increments (see [39]) or where it is a fBm (see [13]). However, Maejima and Yamamoto [34] study the case where N is a linear fractional α-stable motion. In Theorem 2.4, Manuscript H, we show that if N has finite first-moments then there exists a unique in law QOU process $X = (X_t)_{t \in \mathbb{R}}$, and this is given by

$$X_t = N_t - \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} N_s \,\mathrm{d}s, \qquad t \in \mathbb{R}. \tag{6.2}$$

By this result we show, in particular, existences of the linear fractional α-stable motion for all $\alpha \in (1, 2]$ and $H \in (0, 1)$, which on page 4 in Maejima and Yamamoto [34] is conjectured not to exist. Let X be a QOU process and R_X be its autocovariance function, that is,

$$\mathrm{R}_X(t) = \mathrm{Cov}(X_t, X_0), \qquad t \in \mathbb{R}. \tag{6.3}$$

In Manuscript H, the asymptotic behavior of the autocovariance function is studied for the limits 0 and ∞. Assuming that $N_0 = 0$ a.s. and with $\mathrm{V}_N(t) = \mathrm{Var}(N_t)$ denoting the variance function of N, we show in particular that under minor conditions on N, we have for $t \to \infty$ that

$$\mathrm{R}_X(t) \sim \left(\frac{1}{2\lambda^2}\right) \mathrm{V}_N''(t). \tag{6.4}$$

Manuscript H is concluded with a specialization to the case where N is a moving average. To be able to handle this case we show and apply a stochastic Fubini result in Manuscript H (Theorem 3.1), which generalizes earlier results from literature and Manuscript A (Lemma 3.2), Manuscript B (Lemma 3.4(ii)) and Manuscript D (Lemma 4.9).

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PAPER A

Representation of Gaussian semimartingales with application to the covariance function

Andreas Basse-O'Connor

Abstract

The present paper is concerned with various aspects of Gaussian semimartingales. Firstly, generalizing a result of (Stricker, 1983, Semimartingales gaussiennes—application au problème de l'innovation. *Z. Wahrsch. Verw. Gebiete 64*(3)), we provide a convenient representation of Gaussian semimartingales $X = X_0+M+A$ as an $\mathcal{F}^M$-semimartingale plus a process of bounded variation which is independent of M. Secondly, we study stationary Gaussian semimartingales and their canonical decomposition. Thirdly, we give a new characterisation of the covariance function of Gaussian semimartingales which enable us to characterize the class of martingales and the processes of bounded variation among the Gaussian semimartingales. We conclude with two applications of the results.

Keywords: semimartingales; Gaussian processes; covariance functions; stationary processes

AMS Subject Classification: 60G15; 60G10; 60G48

1 Introduction

Recently, there has been renewed interest in some of the fundamental properties of Gaussian processes, such as the semimartingale property and the existence of quadratic variation; see e.g. Barndorff-Nielsen and Schmiegel [1].

Knight [13], Jeulin and Yor [12], Cherny [6], Cheridito [5] and Basse [2] studied the semimartingale property of a certain class of Gaussian processes with stationary increments (or of a deterministic transformation of such processes). In Basse and Pedersen [3] some of these results are extended in to a class of infinitely divisible processes. Jain and Monrad [10] studied, among other topics, certain properties of Gaussian process of bounded variation. A good review of the literature about Gaussian semimartingales can be found in Liptser and Shiryayev [14].

Stricker [19, Théorème 2] showed the following. Let $(X_t)_{t\geq 0}$ be a Gaussian semimartingale with canonical decomposition $X_t = W_t + \int_0^t Z_s \,\mathrm{d}s$, where $(W_t)_{t\geq 0}$ is a Brownian motion. Then there exists a Gaussian process $(Y_t)_{t\geq 0}$ which is independent of $(W_t)_{t\geq 0}$ and a deterministic function $(r,s) \mapsto \Psi_r(s)$ such that

$$X_t = W_t + \int_0^t \left(\int_0^r \Psi_r(s)\,\mathrm{d}W_s \right) \mathrm{d}r + \int_0^t Y_r \,\mathrm{d}r. \tag{1.1}$$

One of the purposes of the present paper is to generalize this result. Indeed, we show that a Gaussian process $(X_t)_{t\geq 0}$ is a semimartingale if and only if it can be decomposed as

$$X_t = X_0 + M_t + \int_0^t \left(\int_0^r \Psi_r(s)\,\mathrm{d}M_s \right) \mu(\mathrm{d}r) + \int_0^t Y_r \,\mu(\mathrm{d}r), \tag{1.2}$$

where $(M_t)_{t\geq 0}$ is a Gaussian martingale, $(Y_t)_{t\geq 0}$ is a Gaussian process which is independent of $(M_t)_{t\geq 0}$, μ is a Radon measure on $\mathbb{R}_+$ and $(r,s) \mapsto \Psi_r(s)$ is a deterministic function. As a part of this we study Gaussian processes of bounded variation.

A second purpose of the paper is to study the canonical decomposition of stationary Gaussian semimartingales. Let $(X_t)_{t\in\mathbb{R}}$ be a stationary Gaussian process such that $(X_t)_{t\geq 0}$ is a semimartingale. We study the canonical decomposition of $(X_t)_{t\geq 0}$ and give a necessary and sufficient condition for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, where $\mathcal{F}_t^{X,\infty} := \sigma(X_s : s \in (-\infty, t])$ for $t \geq 0$.

In the last section of the paper we study the the covariance structure of Gaussian semimartingales. Let $(X_t)_{t\in\mathbb{R}}$ be a stationary Gaussian process. Then, Proposition 19 in Jeulin and Yor [12] gives a necessary and sufficient condition on the spectral measure of $(X_t)_{t\in\mathbb{R}}$ for $(X_t)_{t\in\mathbb{R}}$ to be a semimartingale. Emery [8] showed that a Gaussian process $(X_t)_{t\geq 0}$ is a semimartingale if and only if the mean-value function and the covariance function Γ of $(X_t)_{t\geq 0}$ are of bounded variation and there exists an right-continuous increasing function F such that for each $0 \leq s < t$ and each elementary function $u \mapsto f_s(u)$ with $f_s(u) = 0$ for $u > s$ we have

$$\frac{\left| \int_s^t \int_0^s f_s(v)\,\Gamma(\mathrm{d}u,\mathrm{d}v) \right|}{\sqrt{\int_0^s \int_0^s f_s(u) f_s(v)\,\Gamma(\mathrm{d}u,\mathrm{d}v)}} \leq F(t) - F(s). \tag{1.3}$$

However, based on the decomposition (1.2) we provide a new alternative characterisation of the covariance function (see Theorem 5.2). Some applications will be given as well. For example, we study the fractional Brownian motion.

The paper is organised as follows. Section 2 contains some preliminary results. We show that Gaussianity is preserved under various operations on a Gaussian semimartingale. Moreover, a suitable version of Fubini's Theorem is provided. Section 3 contains some representation results for Gaussian semimartingales. First, extending a result of Jeulin [11], we characterize Gaussian process of bounded variation. Afterwards the decomposition (1.2) is provided. In section 4 the covariance function of Gaussian semimartingales is considered. We conclude with a few examples.

1.1 Notation

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space. By a filtration we mean an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras satisfying the usual conditions of right-continuity and completeness. If $(X_t)_{t\geq 0}$ is a stochastic process we denote by $(\mathcal{F}_t^X)_{t\geq 0}$ the least filtration to which $(X_t)_{t\geq 0}$ is adapted.

A separable subspace $\mathbb{G}$ of $L^2(\mathrm{P})$ which contains all constants, is called a Gaussian space if $(X_1, \dots, X_n)$ follows a multivariate Gaussian distribution whenever $n \geq 1$ and $X_1, \dots, X_n \in \mathbb{G}$. Let $\mathbb{G}$ denote a Gaussian space and $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Then we say that $\mathbb{G}$ is $(\mathcal{F}_t)_{t\geq 0}$-stable if $X \in \mathbb{G}$ implies $\mathrm{E}[X|\mathcal{F}_t] \in \mathbb{G}$ for all $t \geq 0$. A typical example is $\mathbb{G} := \overline{\mathrm{span}}\{X_t : t \geq 0\}$ for a càdlàg Gaussian process $(X_t)_{t\geq 0}$ ($\overline{\mathrm{span}}$ denotes the $L^2(\mathrm{P})$-closure of the linear span) and $(\mathcal{F}_t)_{t\geq 0} = (\mathcal{F}_t^X)_{t\geq 0}$.

We say that a stochastic process $(X_t)_{t\geq 0}$ has stationary increments if for all $n \geq 1$, $0 \leq t_0 < \cdots < t_n$ and $0 < t$ we have

$$(X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}) \stackrel{\mathscr{D}}{=} (X_{t_1+t} - X_{t_0+t}, \dots, X_{t_n+t} - X_{t_{n-1}+t}), \tag{1.4}$$

where $\stackrel{\mathscr{D}}{=}$ denotes equality in distribution.

Let μ be a σ-finite measure on $\mathbb{R}$ and $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a function. Then f is said to be absolutely continuous w.r.t. μ if f is of bounded variation and the total variation measure of f is absolutely continuous w.r.t. μ. A stochastic process $(X_t)_{t\geq 0}$ starting at 0 is said to be absolutely continuous w.r.t. μ if almost all sample paths of $(X_t)_{t\geq 0}$ are absolutely continuous w.r.t. μ. Moreover for a locally μ-integrable function f we define $\int_a^b f \,\mathrm{d}\mu := \int_{(a,b]} f \,\mathrm{d}\mu$ for all $0 \leq a < b$.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Recall that an $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process $(X_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale, if there exists a decomposition of $(X_t)_{t\geq 0}$ as

$$X_t = X_0 + M_t + A_t, \tag{1.5}$$

where $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-local martingale starting at 0 and $(A_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-adapted process of finite variation starting at 0. We say that $(X_t)_{t\geq 0}$ is a semimartingale if it is an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale. Moreover $(X_t)_{t\geq 0}$ is called a special $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if it is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale such that $(A_t)_{t\geq 0}$ in (1.5) can be chosen $(\mathcal{F}_t)_{t\geq 0}$-predictable. In this case the representation (1.5) with $(A_t)_{t\geq 0}$ $(\mathcal{F}_t)_{t\geq 0}$-predictable is unique and is called the canonical decomposition of $(X_t)_{t\geq 0}$. From Liptser and Shiryayev [14, Chapter 4, Section 9, Theorem 1] it follows that if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale then it is also an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale.

If $(A_t)_{t\geq 0}$ is a right-continuous Gaussian process of bounded variation then $(A_t)_{t\geq 0}$ is of integrable variation (see Stricker [19, Proposition 4 and 5]) and we let μ_A denote the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto \mathrm{E}[V_t(A)]$. For every Gaussian martingale $(M_t)_{t\geq 0}$ let μ_M denote the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto \mathrm{E}[M_t^2]$.

2 Preliminary results

In the following μ denotes a Radon measure on $\mathbb{R}_+$ and $(E, \mathcal{E}, \nu)$ is a σ-finite measure space.

Lemma 2.1. *Let $\Psi_t \in L^2(\nu)$ for $t \geq 0$ and define $S := \overline{\operatorname{span}}\{\Psi_t : t \geq 0\}$. Assume S is a separable subset of $L^2(\nu)$ and $t \mapsto \int \Psi_t(s)g(s)\,\nu(\mathrm{d}s)$ is measurable for $g \in S$. Then, there exists a measurable mapping $\mathbb{R} \times E \ni (t,s) \mapsto \tilde{\Psi}_t(s) \in \mathbb{R}$ such that $\tilde{\Psi}_t = \Psi_t$ ν-a.s. for $t \geq 0$.*

Proof. Since S is a separable normed space, the Borel σ-algebra on S induced by the norm-topology equals the σ-algebra induced by the mappings $S \ni f \mapsto \int fg\,\mathrm{d}\nu \in \mathbb{R}$ for $g \in S$. Therefore $t \mapsto \Psi_t$ is Bochner measurable, and thus a uniform limit of elements of the form $\Psi_t^n(s) = \sum_{k\geq 1} f_k^n(s) 1_{A_k^n}(t)$ where $f_k^n \in L^2(\nu)$ for $n, k \geq 1$ and $(A_k^n)_{k\geq 1}$ are disjoint $\mathcal{B}(\mathbb{R}_+)$-measurable sets for $n \geq 1$. Reducing if necessary to a subsequence we may assume that

$$\sup_{t\in\mathbb{R}_+} \|\Psi_t^n - \Psi_t\|_{L^2(\nu)} \leq 2^{-n}, \qquad n \geq 1. \tag{2.1}$$

Let $B := \{(t,s) \in \mathbb{R}_+ \times E : \limsup_{n\to\infty} |\Psi_t^n(s)| < \infty\}$ and define

$$\tilde{\Psi}_t(s) := \limsup_{k\to\infty} \Psi_t^n(s) 1_B((t,s)), \qquad (t,s) \in \mathbb{R}_+ \times E. \tag{2.2}$$

Then $(t,s) \mapsto \tilde{\Psi}_t(s)$ is measurable. Moreover by (2.1) it follows that $\tilde{\Psi}_t = \Psi_t$ ν-a.s. for $t \in \mathbb{R}_+$, which completes the proof. $\square$

Let $L^{2,1}(\nu, \mu)$ denote the space of all measurable mappings $\mathbb{R}_+ \times E \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ satisfying $\Psi_t \in L^2(\nu)$ for $t \geq 0$ and

$$\int_0^t \|\Psi_r\|_{L^2(\nu)}\,\mu(\mathrm{d}r) < \infty, \qquad t > 0. \tag{2.3}$$

Furthermore $\mathcal{BV}(\nu)$ denotes the space of all measurable mappings $\mathbb{R}_+ \times E \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ for which $\Psi_t \in L^2(\nu)$ for all $t \geq 0$ and there exists a right-continuous increasing function f such that $\|\Psi_t - \Psi_u\|_{L^2(\nu)} \leq f(t) - f(u)$ for $0 \leq u \leq t$.

Lemma 2.2. *Let $\Psi \in L^{2,1}(\nu,\mu)$. Then $r \mapsto \Psi_r(s)$ is locally μ-integrable for ν-a.a. $s \in E$ and by setting $\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) = 0$ if $r \mapsto \Psi_r(s)$ is not locally μ-integrable we have*

$$(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) \in \mathcal{BV}(\nu). \tag{2.4}$$

If in addition S is a closed subspace of $L^2(\nu)$ such that $\Psi_r \in S$ for all $r \in [0,t]$, then

$$s \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) \in S. \tag{2.5}$$

Proof. Let $t \geq 0$ be given. Tonelli's Theorem and Cauchy-Schwarz' inequality imply

$$\int \Big(\int_0^t |\Psi_r(s)|\,\mu(\mathrm{d}r) \Big)^2 \nu(\mathrm{d}s) \tag{2.6}$$
$$= \int_0^t \int_0^t \Big(\int |\Psi_r(s)\Psi_v(s)|\,\nu(\mathrm{d}s) \Big) \mu(\mathrm{d}r)\,\mu(dv) \leq \Big(\int_0^t \|\Psi_r\|_{L^2(\nu)}\,\mu(\mathrm{d}r) \Big)^2 < \infty. \tag{2.7}$$

This shows that $r \mapsto \Psi_r(s)$ is locally μ-integrable for ν-a.a. $s \in E$. By setting

$$\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) = 0 \qquad \text{if } r \mapsto \Psi_r(s) \text{ is not locally } \mu\text{-integrable,} \tag{2.8}$$

we have that $(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r)$ is measurable and $s \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) \in L^2(\nu)$. Calculations as in (2.6) show that

$$\Big\| \int_0^t \Psi_r\,\mu(\mathrm{d}r) - \int_0^u \Psi_r\,\mu(\mathrm{d}r) \Big\|_{L^2(\nu)} \le \int_u^t \|\Psi_r\|_{L^2(\nu)}\,\mu(\mathrm{d}r) \tag{2.9}$$

$$= \int_0^t \|\Psi_r\|_{L^2(\nu)}\,\mu(\mathrm{d}r) - \int_0^u \|\Psi_r\|_{L^2(\nu)}\,\mu(\mathrm{d}r), \tag{2.10}$$

which yields (2.4). To show (2.5) fix $t \ge 0$. By the Projection Theorem it is enough to show

$$\Big\langle \int_0^t \Psi_r\,\mu(\mathrm{d}r), g \Big\rangle_{L^2(\nu)} = 0 \qquad \text{for } g \in S^\perp. \tag{2.11}$$

Fix $g \in S^\perp$. Tonelli's Theorem and Cauchy-Schwarz' inequality shows that

$$\iint_0^t |\Psi_r(s) g(s)|\,\mu(\mathrm{d}r)\,\nu(\mathrm{d}s) \le \|g\|_{L^2(\nu)} \int_0^t \|\Psi_r\|_{L^2(\nu)}\,\mu(\mathrm{d}r) < \infty. \tag{2.12}$$

Thus Fubini's Theorem shows that

$$\Big\langle \int_0^t \Psi_r\,\mu(\mathrm{d}r), g \Big\rangle_{L^2(\nu)} = \int_0^t \langle \Psi_r, g\rangle_{L^2(\nu)}\,\mu(\mathrm{d}r) = 0, \tag{2.13}$$

which completes the proof. □

For $\Psi \in L^{2,1}(\nu,\mu)$ we always define $(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r)$ as in the above lemma.

Lemma 2.3. *For every $\Psi \in \mathcal{BV}(\nu)$ there exists a measurable mapping $(t,s) \mapsto \tilde{\Psi}_t(s)$ such that $t \mapsto \tilde{\Psi}_t(s)$ is right-continuous and of bounded variation for $s \in E$ and $\Psi_t = \tilde{\Psi}_t$ ν-a.s. for $t \ge 0$.*

Proof. Define $\mathcal{D} := \{i2^{-n} : n \ge 1,\ i \ge 0\}$. We first show that $(A_t)_{t\in\mathcal{D}}$ has finite upcrossing over each finite interval P-a.s. by showing that $(\Psi_t)_{t\in\mathcal{D}\cap[0,N]}$ is of bounded variation ν-a.s. for all $N \ge 1$. Fix $N \ge 1$. We have

$$\int \sup_{n\ge1} \sum_{i=1}^{N2^n} |\Psi_{i2^{-n}} - \Psi_{(i-1)2^{-n}}|\,\mathrm{d}\nu = \int \liminf_{n\to\infty} \sum_{i=1}^{N2^n} |\Psi_{i2^{-n}} - \Psi_{(i-1)2^{-n}}|\,\mathrm{d}\nu \tag{2.14}$$

$$\le \liminf_{n\to\infty} \sum_{i=1}^{N2^n} \int |\Psi_{i2^{-n}} - \Psi_{(i-1)2^{-n}}|\,\mathrm{d}\nu \tag{2.15}$$

$$\le \liminf_{n\to\infty} \sum_{i=1}^{N2^n} \|\Psi_{i2^{-n}} - \Psi_{(i-1)2^{-n}}\|_{L^2(\nu)} < \infty, \tag{2.16}$$

where the last inequality follows since $\Psi \in \mathcal{BV}(\nu)$. Since $(\Psi_t)_{t\in\mathcal{D}}$ has finite upcrossing over each finite interval ν-a.s.

$$\tilde{\Psi}_t := \lim_{u\downarrow t,\ u\in\mathcal{D}} \Psi_u, \qquad t \ge 0, \tag{2.17}$$

is a well-defined càdlàg process. Moreover, since $\Psi \in \mathcal{BV}(\nu)$, $t \mapsto \Psi_t \in L^2(\nu)$ is right-continuous. This implies that $\tilde{\Psi}_t = \Psi_t$ ν-a.s. for $t \ge 0$ and so $\tilde{\Psi} \in \mathcal{BV}(\nu)$. Thus it follows from calculations as above that $(\tilde{\Psi}_t)_{t\ge0}$ is of integrable variation. This completes the proof. □

3 General properties of Gaussian semimartingales

Our next result shows that the Gaussian property is preserved under various operations on a Gaussian semimartingale.

Lemma 3.1. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{G}$ denote an $(\mathcal{F}_t)_{t\geq 0}$-stable Gaussian space. We have the following.*

(i) Let $(X_t)_{t\geq 0} \subseteq \mathbb{G}$ be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale. Then $(X_t)_{t\geq 0}$ is a special $(\mathcal{F}_t)_{t\geq 0}$-semimartingale. Let $X_t = M_t + A_t + X_0$ be the $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$. Then, $(A_t)_{t\geq 0}, (M_t)_{t\geq 0} \subseteq \mathbb{G}$ and $(M_t)_{t\geq 0}$ is a (true) $(\mathcal{F}_t)_{t\geq 0}$-martingale which is independent of X_0.

(ii) Let $(M_t)_{t\geq 0} \subseteq \mathbb{G}$ be a Gaussian martingale starting at 0. Then

$$\{\int_0^t f(s)\,\mathrm{d}M_s : f \in L^2(\mu_M)\} = \overline{\operatorname{span}}\{M_u : u \leq t\}, \qquad t \geq 0. \tag{3.1}$$

In particular if $Y \in \mathbb{G}$ is an $\mathcal{F}_t^M$-measurable random variable with mean zero then there exists an $f \in L^2(\mu_M)$ such that

$$Y = \int_0^t f(s)\,\mathrm{d}M_s. \tag{3.2}$$

Proof. (i) follows by Stricker [19, Proposition 4 and 5].
(ii): Fix $t \geq 0$. To show the inclusion '$\subseteq$' let $f \in L^2(\mu_M)$ be given. Since $\int_0^t f(s)\,\mathrm{d}M_s$ is the $L^2(\mathrm{P})$-limit of $\int_0^t f_n(s)\,\mathrm{d}M_s$ where the f_n's are step functions such that $f_n \to f$ in $L^2(\mu_M)$, it follows that

$$\int_0^t f(s)\,\mathrm{d}M_s \in \overline{\operatorname{span}}\{M_u : u \leq t\}. \tag{3.3}$$

Since $M_u = \int_0^t 1_{(0,u]}(s)\,\mathrm{d}M_s$ for $u \in [0,t]$ and the left-hand side of (3.1) is closed the '$\supseteq$' inclusion follows and thus we have shown (3.1). Now assume that $Y \in \mathbb{G}$ is an $\mathcal{F}_t^M$-measurable random variable with mean zero. Let $(a_n)_{n\geq 1}$ be a dense subset of $[0,t]$ containing t. By Lévy's Theorem it follows that

$$\mathrm{E}[Y|M_{a_1},\ldots,M_{a_n}] \to \mathrm{E}[Y|\mathcal{F}_t^M] = Y \qquad \text{in } L^2(\mathrm{P}). \tag{3.4}$$

Since $(Y, M_{a_1},\ldots,M_{a_n})$ is simultaneously Gaussian for every $n \geq 1$ the left-hand side of (3.4) belongs to the linear span of $\{M_{a_i} : 1 \leq i \leq n\}$. This shows that $Y \in \overline{\operatorname{span}}\{M_u : u \leq t\}$, which by (3.1) completes the proof of (ii). □

Let $(M_t)_{t\geq 0}$ denote a càdlàg Gaussian martingale and $(t,s) \mapsto \Psi_t(s)$ be a measurable mapping satisfying $\Psi_t \in L^2(\mu_M)$ for $t \geq 0$. Then we may and do choose $(\int \Psi_t(s)\,\mathrm{d}M_s)_{t\geq 0}$ jointly measurable in (t,ω). To see this note that $S := \overline{\operatorname{span}}\{M_t : t \geq 0\}$ is a separable subspace of $L^2(\mathrm{P})$. Moreover Lemma 3.1 (ii) shows that each element in S is on the form $\int f(s)\,\mathrm{d}M_s$ for such $f \in L^2(\mu_M)$. Thus for $\int f(s)\,\mathrm{d}M_s \in S$ we have

$$\mathrm{E}\Big[\int \Psi_t(s)\,\mathrm{d}M_s \int f(s)\,\mathrm{d}M_s\Big] = \int \Psi_t(s) f(s)\,\mu_M(\mathrm{d}s), \tag{3.5}$$

which shows that $t \mapsto \mathrm{E}[\int \Psi_t(s)\,\mathrm{d}M_s \int f(s)\,\mathrm{d}M_s]$ is measurable. Hence by Lemma 2.1 there exists a measurable modification of $(\int \Psi_t(s)\,\mathrm{d}M_s)_{t\geq 0}$.

Lemma 3.2 (Stochastic Fubini result)**.** *Let μ be a σ-finite measure on $\mathbb{R}_+$, $(M_t)_{t\geq 0}$ be a càdlàg Gaussian martingale and $\Psi \in L^{2,1}(\mu_M, \mu)$. Then $t \mapsto \int \Psi_t(s)\,\mathrm{d}M_s$ is locally μ-integrable* P*-a.s. and*

$$\int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}M_s\Big)\mu(\mathrm{d}r) = \int \Big(\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r)\Big) dM_s, \qquad t \geq 0. \tag{3.6}$$

Proof. We have

$$\mathrm{E}\Big[\int_0^t \Big|\int \Psi_r(s)\,\mathrm{d}M_s\Big|\mu(\mathrm{d}r)\Big] \leq \int_0^t \|\Psi_r\|_{L^2(\mu_M)}\,\mu(\mathrm{d}t) < \infty, \tag{3.7}$$

which shows that $r \mapsto \int \Psi_r(s)\,\mathrm{d}M_s$ is locally μ-integrable P-a.s. Thus both sides of (3.6) are well-defined. The right-hand side belongs to $\overline{\mathrm{span}}\{M_t : t \geq 0\}$ and so does the left-hand side by Lemma 2.2. From Lemma 3.1 (ii) it follows that all elements in $\overline{\mathrm{span}}\{M_t : t \geq 0\}$ are on the form $\int g(s)\,\mathrm{d}M_s$ for a $g \in L^2(\mu_M)$. Fix $\int g(s)\,\mathrm{d}M_s \in \overline{\mathrm{span}}\{M_t : t \geq 0\}$. We have

$$\mathrm{E}\Big[\int g(s)\,\mathrm{d}M_s \int \Big(\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r)\Big) dM_s\Big] = \int g(s) \int_0^t \Psi_r(s)\,\mu(\mathrm{d}t)\,\mu_M(\mathrm{d}s). \tag{3.8}$$

Moreover, it follows from Fubini's Theorem that

$$\mathrm{E}\Big[\int g(s)\,\mathrm{d}M_s \int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}M_s\Big)\mu(\mathrm{d}r)\Big] = \int_0^t \mathrm{E}\Big[\int g(s)\,\mathrm{d}M_s \int \Psi_r(s)\,\mathrm{d}M_s\Big]\,\mu(\mathrm{d}r) \tag{3.9}$$

$$= \int_0^t \int g(s)\Psi_r(s)\,\mu_M(\mathrm{d}s)\,\mu(\mathrm{d}r) = \int \int_0^t g(s)\Psi_t(s)\,\mu(\mathrm{d}r)\,\mu_M(\mathrm{d}s). \tag{3.10}$$

Hence, the left- and the right-hand side of (3.6) have the same inner product with all elements of $\overline{\mathrm{span}}\{M_t : t \geq 0\}$ which means that they are equal. This completes the proof. □

4 Representation of Gaussian semimartingales

Proposition 4.1. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{G}$ be a Gaussian space. Moreover let $(A_t)_{t\geq 0} \subseteq \mathbb{G}$ be $(\mathcal{F}_t)_{t\geq 0}$-adapted, right-continuous and of bounded variation. Then there exists an $(\mathcal{F}_t)_{t\geq 0}$-optional process $(Y_t)_{t\geq 0} \subseteq \mathbb{G}$ such that $\|Y_t\|_{L^2(\mathrm{P})} \leq 3$ for $t \geq 0$ and*

$$A_t = \int_0^t Y_s\,\mu_A(\mathrm{d}s), \qquad t \geq 0. \tag{4.1}$$

If $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)$-predictable then $(Y_t)_{t\geq 0}$ can be chosen $(\mathcal{F}_t)_{t\geq 0}$-predictable and if $(A_t)_{t\geq 0}$ is a centered process we have $\|Y_r\|_{L^2(\mathrm{P})} = \sqrt{\pi/2}$ for $r \geq 0$.

Proof. It follows from Jeulin [11, Proposition 2] that $(A_t)_{t\geq 0}$ is absolutely continuous w.r.t. μ_A. By Jacod and Shiryaev [9, Proposition 3.13] there exists an $(\mathcal{F}_t)_{t\geq 0}$-optional process $(Z_t)_{t\geq 0}$ such that $A_t = \int_0^t Z_s\,\mu_A(\mathrm{d}s)$ for $t \in \mathbb{R}_+$. Define

$$Z_s^n := \sum_{i=1}^{n2^n} \frac{A_{i2^{-n}} - A_{(i-1)2^{-n}}}{\mu_A(((i-1)2^{-n}, i2^{-n}])} 1_{((i-1)2^{-n}, i2^{-n}]}(s), \qquad s \geq 0, \tag{4.2}$$

where $0/0 := 0$. By reducing to probability measures we get from Dellacherie and Meyer [7, page 50] that for almost all $\omega \in \Omega$, $Z^n_\cdot(\omega)$ converges to $Z_\cdot(\omega)$ μ_A-a.s. Thus, Tonelli's Theorem shows that there exists a measurable μ_A-null set N such that for $t \notin N$, we have Z^n_t converges to Z_t P-a.s. For $t \geq 0$ define $Y_t := Z_t 1_{N^c}(t)$. Then $(Y_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-optional, $(Y_t)_{t\geq 0} \subseteq \mathbb{G}$ and $(Y_t)_{t\geq 0}$ satisfies (4.1). For all Gaussian random variables X we have $\|X\|_{L^2(\mathrm{P})} \leq 3\|X\|_{L^1(\mathrm{P})}$. Now it follows

$$\mu_A((0,t]) = \mathrm{E}[\int_0^t |Y_s|\,\mu_A(\mathrm{d}s)] \geq 1/3 \int_0^t \|Y_s\|_{L^2(\mathrm{P})}\,\mu_A(\mathrm{d}s), \tag{4.3}$$

by which we conclude that $\|Y_t\|_{L^2(\mathrm{P})} \leq 3$ for μ_A-a.a. $t \geq 0$.

If $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable Jacod and Shiryaev [9, Proposition 3.13] shows that the above $(Z_t)_{t\geq 0}$ can be chosen $(\mathcal{F}_t)_{t\geq 0}$-predictable and therefore $(Y_t)_{t\geq 0}$ will be $(\mathcal{F}_t)_{t\geq 0}$-predictable as well. ☐

The above result characterizes Gaussian processes of bounded variation. Indeed it follows from Proposition 4.1 and Lemma 2.2 that $(A_t)_{t\geq 0}$ is a Gaussian process which is right-continuous and of bounded variation if and only if

$$A_t = \int_0^t Y_r\,\mu(\mathrm{d}r) \qquad t \geq 0, \tag{4.4}$$

for a Radon measure μ on $\mathbb{R}_+$ and a measurable Gaussian process $(Y_t)_{t\geq 0}$ which is bounded in $L^2(\mathrm{P})$.

Recall the definition of μ_M on page 18. Moreover, recall (e.g. from Rogers and Williams [17]) the definition of the dual predictable projection of non-adapted processes.

Proposition 4.2. *Let μ be Radon measure on $\mathbb{R}_+$, $(M_t)_{t\geq 0}$ be a càdlàg Gaussian martingale and $\Psi \in L^{2,1}(\mu_M,\mu)$. Define*

$$A_t := \int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}M_s\Big)\mu(\mathrm{d}r), \qquad t \geq 0. \tag{4.5}$$

Then the dual $(\mathcal{F}_t)_{t\geq 0}$-predictable projection of $(A_t)_{t\geq 0}$ is for $t \geq 0$ given by

$$A^p_t = \int_0^t \Big(\int_s^t \Psi_r(s)\,\mu(\mathrm{d}r)\Big)\mathrm{d}M_s = \int_0^t \Big(\int 1_{(0,r)}(s)\Psi_r(s)\,\mathrm{d}M_s\Big)\mu(\mathrm{d}r). \tag{4.6}$$

In particular $(A_t)_{t\geq 0}$ is $(\mathcal{F}^M_t)_{t\geq 0}$-predictable if and only if $\Psi_t(s) = 0$ for $\mu_M \otimes \mu$-a.a. (s,t) with $s \geq t$.

Proof. Since $\Psi \in L^{2,1}(\mu_M,\mu)$ Lemma 2.2 shows that $(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) \in \mathcal{BV}(\mu)$. Now Lemma 3.2 and Lemma 4.3 below shows that

$$A^{\mathrm{p}}_t = \int_0^t \Big(\int_s^t \Psi_r(s)\,\mu(\mathrm{d}r)\Big)\mathrm{d}M_s, \qquad t \geq 0. \tag{4.7}$$

The last identity in (4.6) follows from Lemma 3.2.

To conclude we note that $(A_t)_{t\geq 0}$ is $(\mathcal{F}^M_t)_{t\geq 0}$-predictable if and only if $A_t = A^{\mathrm{p}}_t$ for all $t \geq 0$. From (4.6) this is the case if and only if for $\mathrm{P} \otimes \mu$-a.a. (ω, r) we have

$$\int 1_{(0,r)}(s)\Psi_r(s)\,\mathrm{d}M_s(\omega) = \int \Psi_r(s)\,\mathrm{d}M_s(\omega). \tag{4.8}$$

which by the isometric property of the integral corresponds to $1_{(0,r)}(s)\Psi_r(s) = \Psi_r(s)$ for $\mu_M \otimes \mu$-a.a. (s,r). ☐

Lemma 4.3. *Let $(M_t)_{t\geq 0}$ be a càdlàg Gaussian martingale and let $\Psi \in \mathcal{BV}(\mu_M)$ satisfy that $t \mapsto \Psi_t(s)$ is càdlàg for $s \geq 0$. Then $s \mapsto \Psi_s(s)$ is locally μ_M-square integrable. Let furthermore $(A_t)_{t\geq 0}$ be a modification of $(\int \Psi_t(s)\,\mathrm{d}M_s)_{t\geq 0}$ which is right-continuous and of bounded variation. (Such a modification exists according to Lemma 2.3). Then the dual $(\mathcal{F}_t^M)_{t\geq 0}$-predictable projection of $(A_t)_{t\geq 0}$ exists and is given by*

$$A_t^p = \int_0^t \Big(\Psi_t(s) - \Psi_s(s)\Big)\mathrm{d}M_s, \qquad t \geq 0. \tag{4.9}$$

In particular, $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t^M)_{t\geq 0}$-predictable if and only if for $t \geq 0$ we have $\Psi_t(s) = 0$ for μ_M-a.a. $s \in [t, \infty)$.

Proof. Fix $t \geq 0$. General theory shows that for $t \geq 0$ we have

$$\frac{1}{h}\int_0^t \mathrm{E}[A_{u+h} - A_u | \mathcal{F}_u^M]\,\mathrm{d}u \to A_t^{\mathrm{p}} \text{ in the } \sigma(L^1, L^\infty)\text{-topology, as } h \downarrow 0, \tag{4.10}$$

see e.g. Dellacherie and Meyer [7, Theorem 21.1]. Thus from Gaussianity the convergence also takes place in the $\sigma(L^2, L^2)$-topology. We have

$$\frac{1}{h}\int_0^t \mathrm{E}[A_{u+h} - A_u | \mathcal{F}_u^M]\,\mathrm{d}u = \frac{1}{h}\int_0^t \Big(\int_0^u \big(\Psi_{u+h}(s) - \Psi_u(s)\big)\,\mathrm{d}M_s\Big)\mathrm{d}u \tag{4.11}$$

$$= \int_0^t \Big(\frac{1}{h}\int_s^t \big(\Psi_{u+h}(s) - \Psi_u(s)\big)\,\mathrm{d}u\Big)\mathrm{d}M_s, \tag{4.12}$$

where the second equality follows from Lemma 3.2 since $\Psi \in \mathcal{BV}(\mu_M) \subseteq L^{2,1}(\mu_M, \lambda)$ (λ denotes the Lebesgue measure on $\mathbb{R}$). Thus (4.10) implies that there exists an $f_t \in L^2(\mu_M)$ such that

$$1_{[0,t]}(s)\frac{1}{h}\int_s^t \big(\Psi_{u+h}(s) - \Psi_u(s)\big)\,\mathrm{d}u \xrightarrow[h\downarrow 0]{} f_t(s) \qquad \text{in the } \sigma(L^2, L^2) \tag{4.13}$$

and $A_t^{\mathrm{p}} = \int_0^t f_t(s)\,\mathrm{d}M_s$. Fix $s \in [0,t]$. The right-continuity of $t \mapsto \Psi_t(s)$ implies that

$$\frac{1}{h}\int_s^t \big(\Psi_{u+h}(s) - \Psi_u(s)\big)\,\mathrm{d}u \tag{4.14}$$

$$= \frac{1}{h}\int_t^{t+h} \Psi_u(s)\,\mathrm{d}u - \frac{1}{h}\int_s^{s+h} \Psi_u(s)\,\mathrm{d}u \to \Psi_t(s) - \Psi_s(s), \qquad \text{as } h \downarrow 0. \tag{4.15}$$

This shows $f_t(s) = \Psi_t(s) - \Psi_s(s)$ for μ_M-a.a. $s \in [0,t]$ and the proof of (4.9) is complete.

Since $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t^M)_{t\geq 0}$-predictable if and only if $A_t = A_t^{\mathrm{p}}$ for $t \geq 0$ the last part of the result is immediate. □

Remark 4.4. By writing $s \mapsto \Psi_s(s)$ as a telescoping sum of the functions $s \mapsto \Psi_t(s)$ it can also be seen directly that $s \mapsto \Psi_s(s)$ is locally μ_M-square integrable.

We are now ready to state and prove one of the main results of the paper which describes the bounded variation component of a Gaussian semimartingale and generalizes a result of Stricker [19].

Theorem 4.5. *$(X_t)_{t\geq 0}$ is a Gaussian semimartingale if and only if for $t \geq 0$ we have*

$$X_t = X + M_t + \Big(\int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}M_s\Big)\mu(\mathrm{d}r) + \int_0^t Y_r\,\mu(\mathrm{d}r)\Big), \tag{4.16}$$

where μ is a Radon measure, $(M_t)_{t\geq 0}$ is a Gaussian martingale starting at 0, Ψ is a measurable mapping such that $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(\mu_M)$ and $\Psi_t(s) = 0$ for $\mu_M \otimes \mu$-a.a. (s,t) with $s \geq t$, $(Y_t)_{t\geq 0}$ is a measurable process which is bounded in $L^2(\mathrm{P})$ and X is a random variable such that $\{Y_t, X : t \geq 0\}$ is Gaussian and independent of $(M_t)_{t\geq 0}$.

In this case, $(X_t)_{t\geq 0}$ is (in addition) an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale, where $\mathcal{F}_t := \mathcal{F}_t^M \vee \sigma(X, Y_s : s \geq 0)$ for $t \geq 0$ and (4.16) is the $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$.

Remark 4.6. We actually prove the following. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{G}$ be an $(\mathcal{F}_t)_{t\geq 0}$-stable Gaussian space. Assume that $(X_t)_{t\geq 0} \subseteq \mathbb{G}$ and that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale with $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition $X_t = X_0 + M_t + A_t$. Then $(X_t)_{t\geq 0}$ can be decomposed as in (4.16) with $\mu = \mu_A$, $(Y_t)_{t\geq 0}$ $(\mathcal{F}_t)_{t\geq 0}$-predictable and $(M_t)_{t\geq 0}, (Y_t)_{t\geq 0} \subseteq \mathbb{G}$.

Theorem 4.5 also shows the following.

Remark 4.7. A Gaussian semimartingale $(X_t)_{t\geq 0}$ with martingale component $(M_t)_{t\geq 0}$ can be decomposed as $X_t = Z_t + B_t$, where $(Z_t)_{t\geq 0}$ is a Gaussian $(\mathcal{F}_t^M)_{t\geq 0}$-semimartingale and $(B_t)_{t\geq 0}$ is a Gaussian $(\mathcal{F}_t^X)_{t\geq 0}$-predictable process independent of $(M_t)_{t\geq 0}$ which is right-continuous and of bounded variation. In particular $\mathcal{F}_t^X = \mathcal{F}_t^M \vee \mathcal{F}_t^B$.

Proof of Theorem 4.5. Only if: We prove the more general result stated in Remark 4.6. Thus let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{G}$ be an $(\mathcal{F}_t)_{t\geq 0}$-stable Gaussian space. Assume $(X_t)_{t\geq 0} \subseteq \mathbb{G}$ and that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale with $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition $X_t = X_0 + M_t + A_t$. It follows from Lemma 3.1 (i) that $(A_t)_{t\geq 0}, (M_t)_{t\geq 0} \subseteq \mathbb{G}$, and since $(A_t)_{t\geq 0}$ is of bounded variation, Proposition 4.1 shows that there exists an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(Z_t)_{t\geq 0} \subseteq \mathbb{G}$ such that $\|Z_r\|_{L^2(\mathrm{P})} \leq 3$ for $r \geq 0$ and

$$A_t = \int_0^t Z_s \, \mu_A(\mathrm{d}s), \qquad t \geq 0. \tag{4.17}$$

Let $({}^{\mathrm{p}}Z_t)_{t\geq 0}$ denote the $(\mathcal{F}_t^M)_{t\geq 0}$-predictable projection of $(Z_t)_{t\geq 0}$. The definition of $({}^{\mathrm{p}}Z_t)_{t\geq 0}$ shows that for $t \geq 0$ we have ${}^{\mathrm{p}}Z_t = \mathrm{E}[Z_t | \mathcal{F}_{t-}^M]$. From Gaussianity it follows that ${}^{\mathrm{p}}Z_t$ is the projection of Z_t on $\overline{\mathrm{span}}\{M_s : s < t\}$ and thus ${}^{\mathrm{p}}Z_t \in \mathbb{G}$ for $t \geq 0$. This means that $\|Z_s\|_{L^2(\mathrm{P})} \geq \|{}^{\mathrm{p}}Z_s\|_{L^2(\mathrm{P})}$ for $r \geq 0$. Define $Y_t := Z_t - {}^{\mathrm{p}}Z_t$ for $t \geq 0$. Then $(Y_t)_{t\geq 0} \subseteq \mathbb{G}$ is bounded in $L^2(\mathrm{P})$. We claim that

$$\mathrm{E}[Y_u M_t] = 0 \qquad \text{for } t, u \geq 0. \tag{4.18}$$

Since ${}^{\mathrm{p}}Z_u$ is the projection of Z_u on $\overline{\mathrm{span}}\{M_v : v < t\}$, (4.18) is obviously true for $0 \leq t < u$. Moreover since $(M_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale it remains to be shown $\mathrm{E}[Y_u M_u] = 0$ for $u \geq 0$. Fix $u \geq 0$. We have

$$\mathrm{E}[Y_u M_u] = \mathrm{E}[Y_u \mathrm{E}[M_u | \mathcal{F}_{u-}]] = \mathrm{E}[Y_u M_{u-}] = 0, \tag{4.19}$$

where the first equality follows since Y_u is $\mathcal{F}_{u-}$ measurable and the second equality follows since $(M_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale. This completes the proof of (4.18).

Since $(Y_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$ both are subsets of $\mathbb{G}$ and $(M_t)_{t\geq 0}$ is a centered processes, (4.18) implies that $(Y_t)_{t\geq 0}$ is independent of $(M_t)_{t\geq 0}$. It follows from Lemma 3.1 (ii) and Lemma 2.1 that ${}^{\mathrm{p}}Z_t = \mathrm{E}[{}^{\mathrm{p}}Z_t] + \int \Psi_t(s) \, \mathrm{d}M_s$ for $t \geq 0$ and some measurable mapping Ψ such that $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(\mu_M)$. Since $({}^{\mathrm{p}}Z_t)_{t\geq 0}$ is $(\mathcal{F}_t^M)_{t\geq 0}$-predictable, Proposition 4.2 shows $\Psi_t(s) = 0$ for $\mu_M \otimes \mu_A$-a.a. (s,t) with $s \geq t$. This completes the proof of (4.16), by using $\tilde{Y}_t := Y_t + \mathrm{E}[{}^{\mathrm{p}}Z_t]$ instead of $(Y_t)_{t\geq 0}$.

If: Assume conversely that (4.16) is satisfied. By Lemma 4.2

$$\int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}M_s \Big) \mu(\mathrm{d}r), \qquad t \geq 0, \tag{4.20}$$

is $(\mathcal{F}_t^M)_{t\geq 0}$-predictable. Hence, $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale ($\mathcal{F}_t := \mathcal{F}_t^M \vee \sigma(X, Y_s : s \geq 0)$) and the $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is (4.16). Since $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale, Stricker's Theorem (see Protter [15, Chapter 2, Theorem 4]) shows that $(X_t)_{t\geq 0}$ in particular is a semimartingale, that is an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale. This completes the proof. □

In the following we study the canonical decomposition of a Gaussian semimartingales. For a stochastic process $(X_t)_{t\in\mathbb{R}}$ we let $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$ denote the least filtration for which X_s is $\mathcal{F}_t^{X,\infty}$-measurable for $t \geq 0$ and $s \in (-\infty, t]$.

Theorem 4.8. *Let $(X_t)_{t\in\mathbb{R}}$ be Gaussian process which either is stationary or has stationary increments and satisfies $X_0 = 0$. Assume $(X_t)_{t\geq 0}$ is a semimartingale with canonical decomposition $X_t = X_0 + M_t + A_t$. Then we have*

(i) $(M_t)_{t\geq 0}$ is a Wiener process and hence μ_M equals the Lebesgue measure up to a scaling constant. Moreover μ_A is absolutely continuous with increasing density.

(ii) $(A_t)_{t\geq 0}$ has stationary increments if and only if $(M_t)_{t\geq 0}$ is independent of $(X_t)_{t\leq 0}$.

(iii) $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if μ_A has a bounded density.

Proof. The stationary increments of $(X_t)_{t\geq 0}$ imply that $(X_t)_{t\geq 0}$ has no fixed points of discontinuity. Since in addition $(X_t)_{t\geq 0}$ is a Gaussian semimartingale, it is continuous (see Stricker [19, Proposition 3]). By continuity of $(X_t)_{t\geq 0}$ it follows that $(A_t)_{t\geq 0}$ is continuous as well.

(i)**:** Since $(A_t)_{t\geq 0}$ is continuous we have $[M]_t = [X]_t$ for $t \geq 0$. (For a process $(Z_t)_{t\geq 0}$, $[Z]_t$ denotes the quadratic variation of $(Z_t)_{t\geq 0}$ on $[0, t]$.) By the stationary increments of $(X_t)_{t\geq 0}$, it follows that $[X]_t = Kt$ for all $t \geq 0$ and some constant $K \in \mathbb{R}_+$. Thus by Gaussianity it follows that $(M_t)_{t\geq 0}$ has stationary increments and therefore is a Wiener process with parameter K.

Let $v \geq 0$ be given and define $\mathcal{F}_t^{X,v} := \mathcal{F}_t^X \vee \sigma(X_s : s \in [-v, 0])$ for $t \geq 0$. In the following we shall use that for $0 \leq t_0 < t_1 < \cdots < t_n$ we have

$$(\mathrm{E}[X_{t_i+v} - X_{t_{i-1}+v} | \mathcal{F}^X_{t_{i-1}+v}])_{i=1}^n \overset{\mathscr{D}}{=} (\mathrm{E}[X_{t_i} - X_{t_{i-1}} | \mathcal{F}^{X,v}_{t_{i-1}}])_{i=1}^n. \tag{4.21}$$

In the case where $(X_t)_{t\in\mathbb{R}}$ has stationary increments and satisfies $X_0 = 0$ this is due to

$$\big(X_{t_i+v} - X_{t_{i-1}+v}, (X_s)_{s\in[0,t_{i-1}+v]}\big)_{i=1}^n \overset{\mathscr{D}}{=} \big(X_{t_i} - X_{t_{i-1}}, (X_s - X_{-v})_{s\in[-v,t_{i-1}]}\big)_{i=1}^n, \tag{4.22}$$

and $\sigma(X_s - X_{-v} : s \in [-v, t_{i-1}]) = \mathcal{F}^{X,v}_{t_{i-1}}$ for $i = 1, \ldots, n$. In the stationary case it follows since

$$\big(X_{t_i+v} - X_{t_{i-1}+v}, (X_s)_{s\in[0,t_{i-1}+v]}\big)_{i=1}^n \overset{\mathscr{D}}{=} \big(X_{t_i} - X_{t_{i-1}}, (X_s)_{s\in[-v,t_{i-1}]}\big)_{i=1}^n. \tag{4.23}$$

From (4.21) it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,v})_{t\geq 0}$-local quasimartingale and therefore also an $(\mathcal{F}_t^{X,v})_{t\geq 0}$-special semimartingale. Let $(A_t^v)_{t\geq 0}$ be the bounded variation component of $(X_t)_{t\geq 0}$ in the filtration $(\mathcal{F}_t^{X,v})_{t\geq 0}$. For $0 \leq u \leq t$ we have

$$A_t - A_u = \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} \mathrm{E}[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}^X_{(i-1)/2^n}] \tag{4.24}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} \mathrm{E}[A^v_{i/2^n} - A^v_{(i-1)/2^n} | \mathcal{F}^X_{(i-1)/2^n}] \qquad \text{in } L^2(\mathrm{P}), \tag{4.25}$$

which shows

$$\|A_t - A_u\|_{L^1(\mathrm{P})} \leq \lim_{n\to\infty} \sum_{i=[u2^n]+1}^{[t2^n]} \|A^v_{i/2^n} - A^v_{(i-1)/2^n}\|_{L^1(\mathrm{P})} = \mu_{A^v}((u,t]). \tag{4.26}$$

From (4.21) it follows that (the limits are in $L^2(\mathrm{P})$)

$$A^v_t - A^v_u = \lim_{n\to\infty} \sum_{[u2^n]+1}^{[t2^n]} \mathrm{E}[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}^{X,v}_{(i-1)/2^n}] \tag{4.27}$$

$$\overset{\mathscr{D}}{=} \lim_{n\to\infty} \sum_{[u2^n]+1}^{[t2^n]} \mathrm{E}[X_{i/2^n+v} - X_{(i-1)/2^n+v} | \mathcal{F}^X_{(i-1)/2^n+v}] = A_{t+v} - A_{u+v}, \tag{4.28}$$

and hence $\mu_{A^v}((u,t]) = \mu_A((u+v, t+v])$. Thus by (4.26) we conclude that

$$\mu_A((u,t]) \leq \mu_A((u+v,t+v]), \qquad 0 \leq u \leq t,\ 0 \leq v. \tag{4.29}$$

Define $f(t) := \mu_A((0,t])$ for $t \geq 0$ and let $T \geq 0$ be given. Choose a $t_0 \geq T$ such that f is differentiable at t_0. Moreover let $t, h \geq 0$ satisfy $t + h \leq T$. Then

$$\mu_A((t,t+h]) = f(t+h) - f(t) = \sum_{i=1}^{n} f(t+ih/n) - f(t+(i-1)h/n) \tag{4.30}$$

$$\leq \sum_{i=1}^{n} f(t_0 + h/n) - f(t_0) = h\frac{f(t_0+h/n) - f(t_0)}{h/n} \xrightarrow[n\to\infty]{} hf'(t_0), \tag{4.31}$$

which shows f is locally Lipschitz continuous and hence μ_A is absolutely continuous. From (4.29) it follows that μ_A has an increasing density.

(ii): Assume $(A_t)_{t\geq 0}$ has stationary increments. For $t \geq 0$ (4.21) shows

$$A_t \overset{\mathscr{D}}{=} A_{t+v} - A_v = \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} \mathrm{E}[X_{i/2^n+v} - X_{(i-1)/2^n+v} | \mathcal{F}^X_{(i-1)/2^n+v}] \tag{4.32}$$

$$\overset{\mathscr{D}}{=} \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} \mathrm{E}[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}^{X,v}_{(i-1)/2^n}] =: A^v_t \qquad \text{in } L^2(\mathrm{P}). \tag{4.33}$$

By the rules of successive conditioning it follows that $\mathrm{E}[A_t A^v_t] = \mathrm{E}[A^2_t]$. Since in addition $A_t \overset{\mathscr{D}}{=} A^v_t$ this shows that

$$\mathrm{E}[(A_t - A^v_t)^2] = \mathrm{E}[(A^v_t)^2] - \mathrm{E}[A^2_t] = 0, \tag{4.34}$$

and hereby

$$A_t = \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} \mathrm{E}[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}^{X,v}_{(i-1)/2^n}] \qquad \text{in } L^2(\mathrm{P}). \tag{4.35}$$

This yields

$$M_t = \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} X_{i/2^n} - X_{(i-1)/2^n} - \mathrm{E}[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}^{X,v}_{(i-1)/2^n}] \qquad \text{in } L^2(\mathrm{P}), \tag{4.36}$$

which implies that M_t is independent of X_u for $u \in [-v, 0]$. Since $v, t \in \mathbb{R}_+$ were arbitrarily chosen, we conclude that $(M_t)_{t\geq 0}$ is independent of $(X_t)_{t\leq 0}$.

Assume conversely that $(M_t)_{t\geq 0}$ is independent of $(X_t)_{t\leq 0}$ and hence $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale with $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-canonical decomposition given by $X_t = X_0 + A_t + M_t$. For $0 \leq u \leq t$ and $0 \leq v$ we have

$$A_{t+v} - A_{u+v} = \lim_{n\to\infty} \sum_{i=[u2^n]+1}^{[t2^n]} \mathrm{E}[X_{i/2^n+v} - X_{(i-1)/2^n+v} | \mathcal{F}^{X,\infty}_{(i-1)/2^n+v}] \qquad \text{in } L^2(\mathrm{P}) \tag{4.37}$$

from which we conclude that $(A_t)_{t\geq 0}$ has stationary increments.

(iii): Let $(X_t)_{t\geq 0}$ be an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale and let $(B_t)_{t\geq 0}$ denote the $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-bounded variation component of $(X_t)_{t\geq 0}$. By arguments as above it follows that $(B_t)_{t\geq 0}$ has stationary increments and hence $\mathrm{E}[|B_t - B_u|] \leq K(t-u)$ for $0 \leq u \leq t$ and some constant $K \in \mathbb{R}_+$. For $t \geq 0$ we have

$$A_t = \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} \mathrm{E}[B_{i/2^n} - B_{(i-1)/2^n} | \mathcal{F}^{X}_{(i-1)/2^n}] \qquad \text{in } L^2(\mathrm{P}), \tag{4.38}$$

and hence

$$\mathrm{E}[|A_t - A_u|] \leq \lim_{n\to\infty} \sum_{i=[u2^n]+1}^{[t2^n]} \mathrm{E}[|\mathrm{E}[B_{i/2^n} - B_{(i-1)/2^n} | \mathcal{F}^{X}_{(i-1)/2^n}]|] \leq K(t-u), \tag{4.39}$$

which shows μ_A has a bounded density.

Assume conversely that μ_A has a bounded density and let $K \in \mathbb{R}_+$ be a constant dominating the density. For $0 \leq u \leq t$ we have

$$\begin{aligned}
\mathrm{E}[|\mathrm{E}[X_t - X_u|\mathcal{F}^{X,\infty}_u]|] &= \lim_{n\to\infty} \mathrm{E}[|\mathrm{E}[X_t - X_u|\mathcal{F}^{X,n}_u]|] && (4.40)\\
&= \lim_{n\to\infty} \mathrm{E}[|\mathrm{E}[X_{t+n} - X_{u+n}|\mathcal{F}^{X}_{u+n}]|] = \lim_{n\to\infty} \mathrm{E}[|\mathrm{E}[A_{t+n} - A_{u+n}|\mathcal{F}^{X}_{u+n}]|] && (4.41)\\
&\leq \lim_{n\to\infty} \mu_A((u+n, t+n]) \leq K(t-u). && (4.42)
\end{aligned}$$

This shows $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-quasimartingale on bounded intervals and hence an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale. $\square$

Let $(X_t)_{t\geq 0}$ be a stationary Gaussian semimartingale with covariance function $\gamma(t) := \mathrm{Cov}(X_{u+t}, X_u) = \mathrm{E}[X_t X_0] - \mathrm{E}[X_0^2]$ for $t \geq 0$. Then γ is locally Lipschitz continuous and Lipschitz continuous if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale.

To show this, let $(A_t)_{t\geq 0}$ be the bounded variation component of $(X_t)_{t\geq 0}$. For $0 \leq u, t$ we have

$$|\gamma(t+u) - \gamma(u)| = |\mathrm{E}[(A_{t+u} - A_u)X_0]| \leq \|A_t - A_u\|_{L^2(\mathrm{P})} \|X_0\|_{L^2(\mathrm{P})}, \tag{4.43}$$

and the statement thus follows from Theorem 4.8.

We believe that among the stationary Gaussian processes $(X_t)_{t\geq 0}$, the class of $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingales is strictly larger than the class of $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingales. However, we haven't found an example of an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale which isn't an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. This is equivalent (according to Theorem 4.8 (iii)) to finding a stationary Gaussian semimartingale $(X_t)_{t\geq 0}$ for which μ_A has an unbounded density ($(A_t)_{t\geq 0}$ denotes the bounded variation component of $(X_t)_{t\geq 0}$).

5 The covariance function of Gaussian semimartingales

If $(X_t)_{t\geq 0}$ is a Gaussian process we let Γ_X denote the corresponding covariance function, i.e. $\Gamma_X(t,s) := \mathrm{E}[(X_t - \mathrm{E}[X_t])(X_s - \mathrm{E}[X_s])]$ for all $s, t \geq 0$. We need the following.

Condition 5.1. A function $G\colon \mathbb{R}_+^2 \to \mathbb{R}$ satisfies Condition 5.1, if G is symmetric, positive semi-definite and there exists a right-continuous increasing function f such that for all $0 \leq s \leq t$

$$\sqrt{G(t,t) + G(s,s) - 2G(s,t)} \leq f(t) - f(s). \tag{5.1}$$

Recall that G is positive semi-definite if

$$\sum_{i,j=1}^{n} a_i G(t_i, t_j) a_j \geq 0 \tag{5.2}$$

for all $n \geq 1$, $a_1, \dots, a_n \in \mathbb{R}$ and $t_1, \dots, t_n \in \mathbb{R}_+$.

Assume that G satisfies Condition 5.1 and denote by $(A_t)_{t\geq 0}$ a centered Gaussian process satisfying $\Gamma_A = G$. Then by Lemma 2.3 there exists a modification of $(A_t)_{t\geq 0}$ which is right-continuous and of bounded variation. Conversely, if $(A_t)_{t\geq 0}$ is a right-continuous Gaussian process of bounded variation, then Γ_A satisfies Condition 5.1 with $f(t) = \mathrm{E}[V_t(A)]$ for $t \geq 0$ since $(A_t)_{t\geq 0}$ is of integrable variation.

Thus G satisfies Condition 5.1 if and only if $G = \Gamma_A$ for some right-continuous Gaussian process $(A_t)_{t\geq 0}$ of bounded variation.

A measurable mapping $\mathbb{R}_+^2 \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ is said to be a Volterra type kernel if $\Psi_t(s) = 0$ for all $s > t$. (A Volterra kernel is often assumed to be an $L^2(\lambda)$-kernel see e.g. Baudoin and Nualart [4] and Smithies [18]. However, the latter assumption is not needed here.) Let $\mathbb{1}$ denote the Volterra type kernel given by $\mathbb{R}_+^2 \ni (t,s) \mapsto \mathbb{1}_t(s) = 1_{[0,t]}(s)$.

The next result is a new characterization of the covariance function of Gaussian semimartingales. The result is only formulated for centered Gaussian processes. This is no restriction since a Gaussian process $(X_t)_{t\geq 0}$ is a semimartingale if and only if $t \mapsto \mathrm{E}[X_t]$ is right-continuous and of bounded variation and $(X_t - \mathrm{E}[X_t])_{t\geq 0}$ is a semimartingale. To see this it is enough to show that the mean-value function of a Gaussian semimartingale is of bounded variation. Let $(X_t)_{t\geq 0}$ be a Gaussian semimartingale with bounded variation component $(A_t)_{t\geq 0}$. For $0 \leq u \leq t$ we have

$$|\mathrm{E}[X_t] - \mathrm{E}[X_u]| = |\mathrm{E}[A_t] - \mathrm{E}[A_u]| \leq \mathrm{E}[V_t(A)] - \mathrm{E}[V_u(A)], \tag{5.3}$$

by which we conclude that the mean-value function of $(X_t)_{t\geq 0}$ is of bounded variation.

Theorem 5.2. *Let $(X_t)_{t\geq 0}$ be a centered Gaussian process. Then the following conditions are equivalent:*

(i) *$(X_t)_{t\geq 0}$ is a semimartingale.*

(ii) *There exists a Radon measure μ on $\mathbb{R}_+$, a Volterra type kernel Φ such that $\Phi - \mathbb{1} \in \mathcal{BV}(\mu)$, and a function G satisfying Condition 5.1 such that*

$$\Gamma_X(t,u) = G(t,u) + \int \Phi_t(s)\Phi_u(s)\,\mu(\mathrm{d}s), \qquad u,t \geq 0. \tag{5.4}$$

(iii) *There exist Radon measures μ and ν on $\mathbb{R}_+$, a function G satisfying Condition 5.1 and a Volterra type kernel Ψ such that $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(\nu)$ and such that for $0 \leq u,t$ we have*

$$\Gamma_X(t,u) = G(t,u) + \nu((0,t\wedge u]) + \int_0^t \int_0^u \Psi_r(s)\,\nu(\mathrm{d}s)\,\mu(\mathrm{d}r) \tag{5.5}$$

$$+ \int_0^t \int_0^u \Psi_r(s)\,\mu(\mathrm{d}r)\nu(\mathrm{d}s) + \int_0^t \int_0^u \langle \Psi_r, \Psi_v\rangle_{L^2(\nu)}\,\mu(\mathrm{d}r)\,\mu(\mathrm{d}v). \tag{5.6}$$

Proof. We show (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).
Assume (iii) is satisfied. Equation (5.5) can be written as

$$\Gamma_X(t,u) = G(t,u) + \int \Big(\mathbb{1}_t(s) + \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r)\Big)\Big(\mathbb{1}_u(s) + \int_0^u \Psi_r(s)\,\mu(\mathrm{d}r)\Big)\,\nu(\mathrm{d}s). \tag{5.7}$$

By Lemma 2.2, $(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) \in \mathcal{BV}(\nu)$ which shows (ii).

Assume (ii) is satisfied. To show that $(X_t)_{t\geq 0}$ is a semimartingale it is enough to show that there exists a Gaussian semimartingale $(Z_t)_{t\geq 0}$ such that $(Z_t)_{t\geq 0}$ is distributed as $(X_t)_{t\geq 0}$. Indeed, assume that $(Z_t)_{t\geq 0}$ has been constructed. Then since $(Z_t)_{t\geq 0}$ is a càdlàg process, $(X_t)_{t\geq 0}$ is càdlàg through the rational numbers, and since $(X_t)_{t\geq 0}$ is right-continuous in $L^2(\mathrm{P})$, is it possible to choose a càdlàg modification of $(X_t)_{t\geq 0}$. For all $0 \leq s \leq t$ we have

$$\mathrm{E}[|\mathrm{E}[Z_t - Z_s|\mathcal{F}_s^Z]|] = \mathrm{E}[|\mathrm{E}[X_t - X_s|\mathcal{F}_s^X]|]. \tag{5.8}$$

Since a Gaussian process is a semimartingale if and only if it is quasimartingale on $[0,T]$ for all $T > 0$ according to Liptser and Shiryayev [14] [Chapter 4, Section 9, Corollary of Theorem 1] and [Chapter 2, Section 1, Theorem 4], equation (5.8) shows that $(X_t)_{t\geq 0}$ is a semimartingale.

To construct $(Z_t)_{t\geq 0}$, note that since G satisfies Condition 5.1 there exist two independent processes $(A_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$, with the properties that $(M_t)_{t\geq 0}$ is a càdlàg centered Gaussian martingale with $\mu_M = \mu$ for all $t \geq 0$ and $(A_t)_{t\geq 0}$ is a right-continuous centered Gaussian process of bounded variation such that $\Gamma_A = G$. Let $\Theta := \Phi - \mathbb{1}$ and

$$Z_t := M_t + \int \Theta_t(s)\,\mathrm{d}M_s + A_t. \tag{5.9}$$

Then $(Z_t)_{t\geq 0}$ is a well-defined centered Gaussian process. Since $\Theta \in \mathcal{BV}(\mu)$, Lemma 2.3 implies that $(\int \Theta_t(s)\,\mathrm{d}M_s)_{t\geq 0}$ can be chosen right-continuous and of bounded variation. Moreover, since Θ is a Volterra type kernel, $(\int \Theta_t(s)\,\mathrm{d}M_s)_{t\geq 0}$ is $(\mathcal{F}_t^M)_{t\geq 0}$-adapted. Hence, since $(A_t)_{t\geq 0}$ is independent of $(M_t)_{t\geq 0}$, $(Z_t)_{t\geq 0}$ is a semimartingale. Since $\Gamma_X = \Gamma_Z$, Gaussianity implies that $(X_t)_{t\geq 0}$ is distributed as $(Z_t)_{t\geq 0}$, which completes the proof of (i).

Assume finally (i) is satisfied i.e. that $(X_t)_{t\geq 0}$ is a semimartingale. Choose, according to Remark 4.6, $(M_t)_{t\geq 0}, (Y_t)_{t\geq 0}, \Psi$ and μ_A such that for $t \geq 0$ we have

$$X_t = M_t + \int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}M_s \Big) \mu_A(\mathrm{d}r) + \int_0^t Y_r\,\mu_A(\mathrm{d}r) + X_0. \tag{5.10}$$

Since $\big(\int_0^t Y_r\,\mu_A(\mathrm{d}r)\big)_{t\geq 0}$ is a Gaussian process of bounded variation, it follows that

$$G(t,u) := \mathrm{E}\Big[\Big(\int_0^t Y_r\,\mu_A(\mathrm{d}r) + X_0\Big)\Big(\int_0^u Y_r\,\mu_A(\mathrm{d}r) + X_0\Big)\Big], \qquad t,u \geq 0 \tag{5.11}$$

satisfies Condition 5.1. Since $\{(M_t)_{t\geq 0}, (Y_t)_{t\geq 0}, X_0\}$ are centered simultaneously Gaussian random variables and $(M_t)_{t\geq 0}$ is independent of $\{X_0, (Y_t)_{t\geq 0}\}$, it follows that (5.5) is satisfied. This completes the proof. $\square$

The following definitions are taken from Jain and Monrad [10]. Let $f\colon \mathbb{R}_+^2 \to \mathbb{R}$. For $0 \leq s_1 \leq s_2$ and $0 \leq t_1 \leq t_2$ define

$$\Delta f((s_1,t_1);(s_2,t_2)) := f(s_2,t_2) - f(s_1,t_2) - f(s_2,t_1) + f(s_1,t_1) \tag{5.12}$$

and

$$V_{s,t}(f) := \sup \sum_{i,j} \big|\Delta f((s_{i-1},t_{j-1});(s_i,t_j))\big| + \sum_j |f(0,t_j) - f(0,t_{j-1})| \tag{5.13}$$

$$+ \sum_i |f(s_i,0) - f(s_{i-1},0)| + |f(0,0)|, \tag{5.14}$$

where the sup is taken over all subdivisions $0 = s_0 < \cdots < s_p = s$ and $0 = t_0 < \cdots < t_q = t$ of $[0,s]\times[0,t]$. We say f is of bounded variation if $V_{s,t}(f) < \infty$ for all $s,t > 0$.

From the representation (5.5) it is easily seen that the covariance function of a Gaussian semimartingale is of bounded variation (a direct proof can be found e.g. in Liptser and Shiryayev [14]). Thus if $(X_t)_{t\geq 0}$ is a Gaussian semimartingale, Γ_X induces a Radon signed measure λ_{Γ_X} on $\mathbb{R}_+^2$ satisfying

$$\lambda_{\Gamma_X}((0,t]\times(0,s]) = \Gamma_X(t,s) - \Gamma_X(0,s) - \Gamma_X(t,0) + \Gamma_X(0,0), \qquad t,s \geq 0. \tag{5.15}$$

A function $f\colon \mathbb{R}_+^2 \to \mathbb{R}$ of bounded variation is said to be absolutely continuous if $(s,t) \mapsto V_{s,t}(f)$ is the restriction to $\mathbb{R}_+^2$ of the distribution function of a measure on $\mathbb{R}^2$ which is absolutely continuous w.r.t. λ_2 (the planar Lebesgue measure). This is equivalent to the existence of three locally integrable functions h_1, h_2 and g such that

$$f(s,t) = \int_0^s \int_0^t g(u,v)\,\mathrm{d}u\,\mathrm{d}v + \int_0^s h_1(u)\,\mathrm{d}u + \int_0^t h_2(v)\,\mathrm{d}v + f(0,0). \tag{5.16}$$

If μ is a Radon measure on $\mathbb{R}_+$ let $\mu\Delta\mu$ denote the measure on $\mathbb{R}_+^2$ for which $(\mu\Delta\mu)(A\times B) = \mu(A\cap B)$ for all $A,B \in \mathcal{B}(\mathbb{R})$. Let $\Delta := \{(x,y)\in\mathbb{R}^2 : x = y\}$ denote the diagonal of $\mathbb{R}_+^2$ and note that $\mu\Delta\mu$ is concentrated on Δ.

Corollary 5.3. *Let $(X_t)_{t\geq 0}$ be a continuous Gaussian semimartingale with martingale component $(M_t)_{t\geq 0}$. Then the restriction of λ_{Γ_X} to Δ equals $\mu_M\Delta\mu_M$.*

Proof. Let $X_t = X_0 + M_t + A_t$ be the canonical decomposition of $(X_t)_{t\geq 0}$ and let $(A_t)_{t\geq 0}$ be decomposed as in Remark 4.6. For $0 \leq u, t$ Fubini's Theorem shows

$$\Gamma_X(t,u) = \mathrm{Cov}\Big[M_t + \int_0^t \int \Psi_r(s)\,\mathrm{d}M_s\,\mu_A(\mathrm{d}r), M_u + \int_0^u \int \Psi_r(s)\,\mathrm{d}M_s\,\mu_A(\mathrm{d}r)\Big] \tag{5.17}$$

$$+ \mathrm{Cov}\Big[X_0 + \int_0^t Y_r\,\mu_A(\mathrm{d}r), X_0 + \int_0^u Y_r\,\mu_A(\mathrm{d}r)\Big] \tag{5.18}$$

$$= \mu_M((0,t\wedge u]) + \int_0^t \int_0^u \Psi_r(s)\,\mu_M(\mathrm{d}s)\,\mu_A(\mathrm{d}r) + \int_0^t \int_0^u \Psi_r(s)\,\mu_A(\mathrm{d}r)\,\mu_M(\mathrm{d}s) \tag{5.19}$$

$$+ \int_0^t \int_0^u \langle \Psi_r, \Psi_v\rangle_{L^2(\mu_M)}\,\mu_A(\mathrm{d}r)\,\mu_A(\mathrm{d}v) + \int_0^t \int_0^u \mathrm{E}[Y_rY_v]\,\mu_A(\mathrm{d}r)\,\mu_A(\mathrm{d}v) \tag{5.20}$$

$$+ \int_0^t \mathrm{E}[X_0Y_r]\,\mu_A(\mathrm{d}r) + \int_0^u \mathrm{E}[X_0Y_r]\,\mu_A(\mathrm{d}r) + \mathrm{E}[X_0^2]. \tag{5.21}$$

Since $(X_t)_{t\geq 0}$ is a continuous semimartingale, μ_M and μ_A are nonatomic measures on $\mathbb{R}$. Hence, (5.18) shows that there exists a nonatomic measure μ on $\mathbb{R}$ and a measurable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that for $0 \leq u, t$ we have

$$\lambda_{\Gamma_X}((0,t] \times (0,u]) = \mu_M\Delta\mu_M((0,u] \times (0,t]) + \int_{-\infty}^t \int_{-\infty}^u f\,\mathrm{d}\mu \otimes \mu. \tag{5.22}$$

Since μ is nonatomic it follows that $\mu \otimes \mu$ has no mass on Δ, which together with (5.22) completes the proof. □

Note that the distribution of a Gaussian martingale $(M_t)_{t\geq 0}$ is uniquely determined by μ_M. Moreover Corollary 5.3 shows that for a continuous Gaussian semimartingale $(X_t)_{t\geq 0}$ with martingale component $(M_t)_{t\geq 0}$ we have

$$\mu_M((0,t]) = \lambda_{\Gamma_X}((s_1,s_2) \in \mathbb{R}_+^2 : s_1 = s_2 \leq t), \qquad t \geq 0. \tag{5.23}$$

Thus it is easy to find the distribution of the martingale component $(M_t)_{t\geq 0}$ from Γ_X. The following result characterizes the Gaussian martingales and Gaussian processes of bounded variation among the Gaussian semimartingales.

Corollary 5.4. *Let $(X_t)_{t\geq 0}$ be a Gaussian semimartingale with canonical decomposition $X_t = X_0 + M_t + A_t$. Assume μ_A and μ_M are absolutely continuous. Then $\lambda_{\Gamma_X} - \mu_M\Delta\mu_M$ is absolutely continuous. In particular we have the following for all $T \geq 0$.*

(i) $(X_t)_{t\geq T}$ is a martingale if and only if

$$\frac{\partial^2\Gamma_X}{\partial u \partial t} = 0 \ \lambda_2\text{-a.s. on } [T,\infty)^2 \text{ and } \frac{\partial \Gamma_X}{\partial t}(0,t) = 0 \text{ for } \lambda\text{-a.a. } t \geq T. \tag{5.24}$$

(ii) $(X_t)_{t\geq T}$ is of bounded variation if and only if Γ_X is absolutely continuous on $[T,\infty)^2$.

Remark 5.5. We have μ_A and μ_M are absolutely continuous if and only if $(A_t)_{t\geq 0}$ and $([X]_t)_{t\geq 0}$ are absolutely continuous. This is in particular satisfied if $(X_t)_{t\geq 0}$ is stationary or has stationary increments and $X_0 = 0$ (see Theorem 4.8 (i)).

Proof. Calculations as in (5.18) show $(u,t) \mapsto \Gamma_X(u,t) - \mu_M((0,u\wedge t])$ is absolutely continuous.

(i): Let $(X_t)_{t\geq T}$ be a martingale. Then $\Gamma_X(u,t) = \mu_M((0, u\wedge t])$, which implies that (5.24) is satisfied. Assume conversely that (5.24) is satisfied. Since $(u,t)\mapsto \Gamma_X(u,t) - \mu_M((0,u\wedge t])$ is absolutely continuous and Γ_X satisfies (5.24), we have that $\Gamma_X(u,t) = \mu_M((0,u\wedge t]) + \mathrm{E}[X_0^2]$ for all $u,t\geq T$, which implies that $(X_t)_{t\geq T}$ is a martingale.

(ii): Assume that Γ_X is absolutely continuous on $[T,\infty)^2$. Since $(u,t)\mapsto \Gamma_X(u,t) - \mu_M((0,u\wedge t])$ is absolutely continuous we have that $\mu_M\wedge\mu_M$ is absolutely continuous. But $\mu_M\wedge\mu_M$ is concentrated on the diagonal of $\mathbb{R}_+^2$ and thereby singular to λ_2, which implies that $\mu_M = 0$. This shows that $(X_t)_{t\geq T}$ is of bounded variation. Assume conversely that $(X_t)_{t\geq T}$ is of bounded variation. Then a calculation as in (5.18) shows that Γ_X is absolutely continuous on $[T,\infty)^2$. □

The following two examples are applications of Corollary 5.4.

Example 5.6. The fractional Brownian Motion (fBm) with Hurst parameter $H\in(0,1)$ is a centered Gaussian processes $(X_t)_{t\geq 0}$ with covariance function

$$\Gamma_X(t,u) = \frac{1}{2}(t^{2H} + u^{2H} - |t-u|^{2H}). \tag{5.25}$$

Let $\epsilon>0$ be given. We prove that fBm is a semimartingale on $[0,\epsilon]$ only if $H=1/2$, i.e. $(X_t)_{t\in[0,\epsilon]}$ is a semimartingale only if it is a Brownian Motion. Let $H\in(0,1/2)\cup(1/2,1)$ and assume (for contradiction) that $(X_t)_{t\in[0,\epsilon]}$ is a semimartingale. Since $(X_t)_{t\geq 0}$ has stationary increments and satisfies $X_0=0$, it follows from Theorem 4.8 (i) (which also applies on bounded intervals) that μ_M and μ_A are absolutely continuous. Using (5.25) it follows that

$$\int_0^t\int_0^u \frac{\partial^2\Gamma_X}{\partial s\partial v}\,d\lambda_2 = \Gamma_X(t,u), \qquad t,u\geq 0, \tag{5.26}$$

which shows Γ_X is absolutely continuous. By Corollary 5.4 (ii) we conclude that $(X_t)_{t\in[0,\epsilon]}$ is of bounded variation on $[0,\epsilon]$ and hence of integrable variation. But this contradicts that

$$\|X_t - X_u\|_{L^1(\mathrm{P})} = \sqrt{2/\pi}\,|t-u|^H, \qquad t,u\geq 0, \tag{5.27}$$

and therefore $(X_t)_{t\in[0,\epsilon]}$ can not be a semimartingale. For $H=1/2$, we have $\frac{\partial^2\Gamma_X}{\partial s\partial v} = 0$ λ_2-a.s. and hence (5.26) doesn't hold. ◊

Example 5.7. Let $(W_t)_{t\geq 0}$ be a canonical Brownian Motion and define $(X_t)_{t\geq 0} := (W_{t+1} - W_t)_{t\geq 0}$. We show $(X_t)_{t\in[0,1+\epsilon]}$ is not a semimartingale for any $\epsilon>0$. We have

$$\Gamma_X(t,u) = (1-|t-u|)1_{[0,1]}(|t-u|), \qquad t,u\geq 0. \tag{5.28}$$

Assume that $(X_t)_{t\in[0,1+\epsilon]}$ is a semimartingale. Since

$$\frac{\partial^2\Gamma_X}{\partial u\partial t} = 0 \quad \lambda_2\text{-a.s.} \qquad \text{and} \qquad \Gamma_X(t,0) = 0 \text{ for all } t\geq 1, \tag{5.29}$$

and $(X_t)_{t\geq 0}$ is a stationary process, it follows from Corollary 5.4 (i) that $(X_t)_{t\in[1,1+\epsilon]}$ is a martingale. This contradicts that Γ_X does not depend only on $t\wedge u$ for $t,u\in[1,1+\epsilon]$.

Even though $(X_t)_{t\geq 0}$ is not a semimartingale on $\mathbb{R}_+$, we now show that on $[0,1]$ it is. By Yor [20], $(W_t + W_1)_{t\in[0,1]}$ is a semimartingale with canonical decomposition

$$\Big(W_t - \int_0^t \frac{W_1 - W_s}{1-s}\,\mathrm{d}s\Big) + \int_0^t \frac{W_1-W_s}{1-s}\,\mathrm{d}s + W_1. \tag{5.30}$$

Let

$$\mathcal{F}_t := \sigma(W_{s+1} - W_1 : s \in [0,t]) \vee \sigma(W_s : s \in [0,t]) \vee \sigma(W_1), \qquad t \geq 0. \tag{5.31}$$

Then (5.30) shows that $(X_t)_{t\in[0,1]}$ is a $(\mathcal{F}_t)_{t\in[0,1]}$-semimartingale with $(\mathcal{F}_t)_{t\in[0,1]}$-canonical decomposition given by

$$X_t = \Big[W_{t+1} - W_1 - W_t + \int_0^t \frac{W_1 - W_s}{1-s}\,\mathrm{d}s\Big] - \int_0^t \frac{W_1 - W_s}{1-s}\,\mathrm{d}s + X_0, \tag{5.32}$$

where the term in the first bracket is the martingale component. By forming the dual $(\mathcal{F}_t^X)_{t\in[0,1]}$-predictable projection on the bounded variation component of (5.32) it follows that the $(\mathcal{F}_t^X)_{t\in[0,1]}$-canonical decomposition of $(X_t)_{t\in[0,1]}$ is given by

$$X_t = \Big(W_{t+1} - W_1 - W_t + \int_0^t \frac{W_1 - \mathrm{E}[W_s|\mathcal{F}_s^X]}{1-s}\,\mathrm{d}s\Big) - \int_0^t \frac{W_1 - \mathrm{E}[W_s|\mathcal{F}_s^X]}{1-s}\,\mathrm{d}s + X_0. \tag{5.33}$$

Note that, even though $(X_t)_{t\geq 0}$ is not a semimartingale on $\mathbb{R}_+$ the quadratic variation of $(X_t)_{t\geq 0}$ does exist, and it is given by $[X]_t = 2t$ for all $t \geq 0$. $\Diamond$

It is known that the processes in Example 5.6 and 5.7 not are semimartingales (for the fBm case see Rogers [16]). However, the proofs presented here are new and indicate the usefulness of the results in this paper.

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PAPER B

Spectral representation of Gaussian semimartingales

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Abstract

The aim of the present paper is to characterize the spectral representation of Gaussian semimartingales. That is, we provide necessary and sufficient conditions on the kernel K for $X_t = \int K_t(s)\, dN_s$ to be a semimartingale. Here, N denotes an independently scattered Gaussian random measure on a general space S. We study the semimartingale property of X in three different filtrations. First the $\mathcal{F}^X$-semimartingale property is considered and afterwards the $\mathcal{F}^{X,\infty}$-semimartingale property is treated in the case where X is a moving average process and $\mathcal{F}_t^{X,\infty} = \sigma(X_s : s \in (-\infty, t])$. Finally we study a generalization of Gaussian Volterra processes. In particular we provide necessary and sufficient conditions on K for the Gaussian Volterra process $\int_{-\infty}^t K_t(s)\, dW_s$ to be an $\mathcal{F}^{W,\infty}$-semimartingale (W denotes a Wiener process). Hereby we generalize a result of Knight (*Foundations of the Prediction Process*, 1992) to the non-stationary case.

Keywords: semimartingales; Gaussian processes; Volterra processes; stationary processes; moving average processes

AMS Subject Classification: 60G15; 60G10; 60G48; 60G57

1 Introduction

Recently there has been major interest in Gaussian Volterra processes. That is, processes $(X_t)_{t\geq 0}$ given by

$$X_t = \int_{-\infty}^{t} K_t(s)\,\mathrm{d}W_s, \qquad t \geq 0, \tag{1.1}$$

where $(W_t)_{t\in\mathbb{R}}$ is a Wiener process with parameter space $\mathbb{R}$ and $s \mapsto K_t(s)$ is a square integrable function for $t \geq 0$. Knight [10, Theorem 6.5], Cherny [5], Cheridito [4] and Jeulin and Yor [9] studied Gaussian Volterra processes with K on the form $K_t(s) = k(t-s) + f(s)$ (such processes are called moving average processes). They characterized the set of K's for which $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale, where $\mathcal{F}_t^{W,\infty} := \sigma(W_s : s \in (-\infty, t])$. In the case where $K_t(s) = k(t-s)$ Jeulin and Yor [9, Proposition 19] gave a condition on the Fourier transform of k for $(X_t)_{t\geq 0}$ to an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale by using complex function theory (in particular Hardy theory).

A fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is an example of a Gaussian Volterra process (it is in fact a moving average process). In this case K is given by

$$K_t(s) = ((t-s)^+)^{H-1/2} - ((-s)^+)^{H-1/2}. \tag{1.2}$$

It is well-known (see Rogers [15]) that the fBm is a semimartingale if and only if $H = 1/2$, i.e. it is a Brownian motion. Inspired by the fBm there has been developed (using Malliavin calculus) an integral for some Gaussian Volterra processes which are not semimartingales, see Alòs et al. [1], Decreusefond [6] and Marquardt [12]. This integral lacks some of the usual properties of the semimartingale integral by the characterization of semimartingales as stochastic integrators (the Bichteler-Dellacherie Theorem), see Protter [13, Chapter 3, Theorem 43]. Hence it is important to characterize the set of K's for which $(X_t)_{t\geq 0}$ is a semimartingale.

According to Kuelbs [11] every centered Gaussian process $(X_t)_{t\geq 0}$, which is right-continuous in probability, has a spectral representation in distribution, i.e. $(X_t)_{t\geq 0}$ is distributed as $(\int K_t(s)\,\mathrm{d}N_s)_{t\geq 0}$, where N is an independently scattered centered Gaussian random measure and $(t,s) \mapsto K_t(s)$ is a deterministic function. The semimartingale property of Gaussian processes is determined by the distribution of the process. Hence, $(X_t)_{t\geq 0}$ is a semimartingale if and only if $(\int K_t(s)\,\mathrm{d}N_s)_{t\geq 0}$ has this property. The purpose of this paper is to characterize the spectral representation of Gaussian semimartingales, that is we characterize the family of kernels K for which

$$\Big(\int K_t(s)\,\mathrm{d}N_s\Big)_{t\geq 0} \tag{1.3}$$

is a semimartingale. Note that the processes on the form (1.3) constitute a generalization of the Gaussian Volterra processes. We study the semimartingale property with respect to the natural filtration and with respect to two larger filtrations. In particular we characterize the K's for which a Gaussian Volterra process $(X_t)_{t\geq 0}$ given by (1.1) is an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale or an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale (the latter condition is strongest). Hereby we generalize results of Cheridito [4], Knight [10, Theorem 6.5] and Cherny [5]. Our setting also covers Ambit processes with deterministic volatility, see Barndorff-Nielsen and Schmiegel [2]. Moreover, we characterize the functions k for which $(X_t)_{t\in\mathbb{R}} = (\int k(t-s)\,\mathrm{d}W_s)_{t\in\mathbb{R}}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

The paper is organised as follows. Section 2 contains notation and preliminary results about Gaussian random measures. Section 3 contains measure-theoretic and Gaussian results. In section 4 we characterize the spectral representation of Gaussian semimartingales.

2 Notation and random measures

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space. By a filtration we mean an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras satisfying the usual conditions of right-continuity and completeness. If $(X_t)_{t\geq 0}$ is a stochastic process we denote by $(\mathcal{F}_t^X)_{t\geq 0}$ the least filtration to which $(X_t)_{t\geq 0}$ is adapted. Let T equal $\mathbb{R}_+$ or $\mathbb{R}$. Then $(X_t)_{t\in T}$ is said to have stationary increments if for all $n \geq 1$, $t_0 < \cdots < t_n$ and $0 < t$ we have

$$(X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}) \overset{\mathscr{D}}{=} (X_{t_1+t} - X_{t_0+t}, \ldots, X_{t_n+t} - X_{t_{n-1}+t}), \tag{2.1}$$

where $\overset{\mathscr{D}}{=}$ denotes equality in distribution.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Recall that an $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process $(X_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale, if there exists a decomposition of $(X_t)_{t\geq 0}$ as

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0, \tag{2.2}$$

where $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-local martingale starting at 0 and $(A_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-adapted process of finite variation starting at 0. We say that $(X_t)_{t\geq 0}$ is a semimartingale if it is an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale. Moreover $(X_t)_{t\geq 0}$ is called a special $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if it is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale such that $(A_t)_{t\geq 0}$ in (2.2) can be chosen $(\mathcal{F}_t)_{t\geq 0}$-predictable. In this case the representation (2.2) with $(A_t)_{t\geq 0}$ $(\mathcal{F}_t)_{t\geq 0}$-predictable is unique and is called the canonical decomposition of $(X_t)_{t\geq 0}$. From Stricker's Theorem (see Protter [13, Chapter 2, Theorem 4]) it follows that if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale then it is also an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale.

For each function $f : \mathbb{R}_+ \to \mathbb{R}$ of bounded variation, $V_t(f)$ denotes the total variation of f on $[0, t]$ for $t \geq 0$. If $(A_t)_{t\geq 0}$ is a right-continuous Gaussian process of bounded variation then $(A_t)_{t\geq 0}$ is of integrable variation (see Stricker [16]) and we let μ_A denote the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto \mathrm{E}[V_t(A)]$. For every Gaussian martingale $(M_t)_{t\geq 0}$ let μ_M denote the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto \mathrm{E}[M_t^2]$.

A process $(W_t)_{t\in\mathbb{R}}$ is said to be a Wiener process if for all $n \geq 1$ and $t_0 < \cdots < t_n$,

$$W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}} \tag{2.3}$$

are independent, for $-\infty < s < t < \infty$ $W_t - W_s$ follows a centered Gaussian distribution with variance $t - s$, and $W_0 = 0$.

We now give a short survey of properties of independently scattered centered Gaussian random measures. Let S denote a non-empty set and $\mathcal{A}$ be a family of subsets of S. Then $\mathcal{A}$ is called a ring if for every pair of sets in $\mathcal{A}$ the union, intersection and set difference are also in $\mathcal{A}$. A ring $\mathcal{A}$ is called a δ-ring if $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ implies $\cap A_n \in \mathcal{A}$. If $\mathcal{A}$ is a δ-ring and there exists a sequence $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ satisfying $\cup A_n = S$ then $\mathcal{A}$ is said to be σ-finite. Throughout the paper let $\mathcal{A}$ denote a σ-finite δ-ring on a nonempty set S.

A family $N = \{N(A) : A \in \mathcal{A}\}$ of random variables is said to be an independently scattered centered Gaussian random measure if

1. For every sequence $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ of pairwise disjoint sets with $\cup A_n \in \mathcal{A}$, $\sum_{i=1}^n N(A_i)$ converges to $N(\cup_{i=1}^\infty A_i)$ in probability as n tends to infinity.

2. For all $n \geq 1$ and all disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$, $N(A_1), \ldots, N(A_n)$ are independent centered Gaussian random variables.

For a general treatment of independently scattered random measures, see Rajput and Rosiński [14]. Let N denote an independently scattered centered Gaussian random measure. It is readily seen that there is a σ-finite measure ν on $(S, \sigma(\mathcal{A}))$ such that $N(A)$ has a centered Gaussian distribution with variance $\nu(A)$ for all $A \in \mathcal{A}$. Following Rajput and Rosiński [14], ν is called the control measure of N. Throughout the paper N denotes a independently scattered centered Gaussian random measure with control measure ν. We shall assume in addition that $L^2(\nu)$ is separable.

Let $f = \sum_{i=1}^n \alpha_i 1_{A_i}$ be a simple function. That is, $n \geq 1$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathcal{A}$. Define $\int f(s)\,\mathrm{d}N_s := \sum_{i=1}^n \alpha_i N(A_i)$. By a standard argument the integral $\int f(s)\,\mathrm{d}N_s$ can be defined through the isometry

$$\|\int f(s)\,\mathrm{d}N_s\|_{L^2(\mathrm{P})} = \|f\|_{L^2(\nu)} \tag{2.4}$$

for all $f \in L^2(S, \sigma(\mathcal{A}), \nu)$.

If $S = \mathbb{R}_+$, N could be the independently scattered random measure induced by a Brownian motion. More generally, if $S = \mathbb{R}_+^d$, N could be the independently scattered random measure induced by a d-parameter Brownian sheet. In this case ν is the Lebesgue measure on $\mathbb{R}_+^d$ and we can choose $\mathcal{A}$ to be the bounded Borel sets of $\mathbb{R}_+^d$. Another example is when $S = \mathbb{R}$ and N is the independently scattered random measure induced by a Brownian motion $(W_t)_{t\in\mathbb{R}}$ with parameter space $\mathbb{R}$.

3 Preliminary results

In this section we collect some measure-theoretical and Gaussian results. We let $(E, \mathcal{E}, m)$ be a σ-finite measure space and μ be a Radon measure on $\mathbb{R}_+$. If H is a normed space and $A \subseteq H$, then $\overline{\mathrm{span}}A$ denotes the closure of the linear span of A. For each mapping $\mathbb{R}_+ \times E \ni (t, s) \mapsto \Psi_t(s) \in \mathbb{R}$ we denote by Ψ_t the mapping $s \mapsto \Psi_t(s)$ for $t \geq 0$. The following Lemma 3.1 – 3.2 are taken from Basse [3].

Lemma 3.1. *Let $\Psi_t \in L^2(\nu)$ for $t \geq 0$ and define $V := \overline{\mathrm{span}}\{\Psi_t : t \geq 0\}$. Assume V is a separable subset of $L^2(m)$ and $t \mapsto \int \Psi_t(s)g(s)\,m(\mathrm{d}s)$ is measurable for $g \in V$. Then there exists a measurable mapping $\mathbb{R}_+ \times E \ni (t, s) \mapsto \tilde{\Psi}_t(s) \in \mathbb{R}$ such that $\tilde{\Psi}_t = \Psi_t$ m-a.s. for $t \geq 0$.*

For a locally μ-integrable function f we define $\int_a^b f\,d\mu := \int_{(a,b]} f\,d\mu$ for $0 \leq a < b$. Let $\mathcal{BV}(m)$ denote the space of all measurable mappings $\mathbb{R}_+ \times S \ni (r, s) \mapsto \Psi_r(s) \in \mathbb{R}$ for which $\Psi_r \in L^2(m)$ for $r \geq 0$ and there exists a right-continuous increasing function f such that $\|\Psi_t - \Psi_u\|_{L^2(m)} \leq f(t) - f(u)$ for $0 \leq u \leq t$.

Lemma 3.2. *Let $(r,s) \mapsto \Psi_r(s)$ be a measurable mapping for which $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(m)$. Then $r \mapsto \Psi_r(s)$ is locally μ-integrable for m-a.a. $s \in E$ and by setting $\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) := 0$ for $t \geq 0$ if $r \mapsto \Psi_r(s)$ is not locally m-integrable we have*

$$(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) \in \mathcal{BV}(m). \tag{3.1}$$

If in addition V is a closed subspace of $L^2(m)$ such that $\Psi_r \in V$ for all $r \in [0,t]$ then

$$s \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) \in V. \tag{3.2}$$

For a measurable mapping $(r,s) \mapsto \Psi_r(s)$ for which $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(m)$ we always define the mapping $(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r)$ as in the above lemma.

Lemma 3.3. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $(Y_t)_{t\geq 0} \subseteq L^1(\mathrm{P})$ be a measurable process with locally μ-integrable sample paths. Define*

$$A_t := \int_0^t Y_r\,\mu(\mathrm{d}r), \qquad t \geq 0. \tag{3.3}$$

Then $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable if and only if Y_t is $\mathcal{F}_{t-}$-measurable for μ-a.a. $t \geq 0$.

Proof. Assume $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable. Then there exists an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(Z_t)_{t\geq 0}$ with locally μ-integrable sample paths such that $A_t = \int_0^t Z_r\,\mu(\mathrm{d}r)$ for $t \geq 0$, see Jacod and Shiryaev [8, Proposition 3.13]. Hence $Y_t = Z_t$ P-a.s. for μ-a.a. $t \geq 0$ and we conclude that Y_t is $\mathcal{F}_{t-}$-measurable for μ-a.a. $t \geq 0$.

Assume conversely that Y_t is $\mathcal{F}_{t-}$-measurable for μ-a.a. $t \geq 0$ and let $({}^{\mathrm{p}}Y_t)_{t\geq 0}$ denote the $(\mathcal{F}_t)_{t\geq 0}$-predictable projection of $(Y_t)_{t\geq 0}$. Since Y_t is $\mathcal{F}_{t-}$-measurable for μ-a.a. $t \geq 0$ it follows that ${}^{\mathrm{p}}Y_t = Y_t$ P-a.s. for μ-a.a. $t \geq 0$. Thus

$$A_t = \int_0^t {}^{\mathrm{p}}Y_s\,\mu(\mathrm{d}s), \qquad t \geq 0, \tag{3.4}$$

and it follows that $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable. This completes the proof. $\square$

Recall that N denotes an independently scattered centered Gaussian random measure with control measure ν. Let $\mathbb{R}_+ \times S \ni (r,s) \mapsto \Psi_r(s)$ be a measurable mapping for which $\Psi_r \in L^2(\nu)$ for $r \geq 0$. Then we may and do choose $(\int \Psi_t(s)\,\mathrm{d}N_s)_{t\geq 0}$ jointly measurable in (t,ω). To see this note that $V := \overline{\mathrm{span}}\{N(A) : A \in \mathcal{A}\}$ is a separable subspace of $L^2(\mathrm{P})$ and

$$V = \{\int f(s)\,\mathrm{d}N_s : f \in L^2(\nu)\}. \tag{3.5}$$

Hence for each element $\int f(s)\,\mathrm{d}N_s \in V$ we have

$$\mathrm{E}[\int \Psi_t(s)\,\mathrm{d}N_s \int f(s)\,\mathrm{d}N_s] = \int \Psi_t(s) f(s)\,\nu(\mathrm{d}s), \tag{3.6}$$

which shows $t \mapsto \mathrm{E}[\int \Psi_t(s)\,\mathrm{d}N_s \int f(s)\,\mathrm{d}N_s]$ is measurable. The existence of a measurable modification of $(\int \Psi_t(s)\,\mathrm{d}N_s)_{t\geq 0}$ now follows from Lemma 3.1.

Lemma 3.4. *We have the following.*

(i) *Let $(Y_t)_{t\geq 0}$ be a measurable process such that $(Y_t)_{t\geq 0} \subseteq \overline{\operatorname{span}}\{N(A) : A \in \mathcal{A}\}$. Then there exists a measurable mapping $\mathbb{R}_+ \times S \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ with $\Psi_t \in L^2(\nu)$ for $t \geq 0$ and such that $Y_t = \int \Psi_t(s)\,\mathrm{d}N_s$ for $t \geq 0$.*

(ii) *Let $(r,s) \mapsto \Psi_r(s)$ be a measurable mapping for which $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(\nu)$. Then $r \mapsto \int \Psi_r(s)\,\mathrm{d}N_s$ is locally μ-integrable* P*-a.s. and for $t \geq 0$ we have*

$$\int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}N_s\Big)\mu(\mathrm{d}r) = \int \Big(\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r)\Big)\,\mathrm{d}N_s. \tag{3.7}$$

(iii) *Let $K_t \in L^2(\nu)$ for $t \geq 0$ and $(X_t)_{t\geq 0}$ be a right-continuous process satisfying $X_t = \int K_t(s)\,\mathrm{d}N_s$ for $t \geq 0$. Then for $0 \leq u \leq t$ we have*

$$\mathrm{E}[X_t|\mathcal{F}_u^X] = \int \big(\mathcal{P}_u K_t\big)(s)\,\mathrm{d}N_s, \tag{3.8}$$

where $\mathcal{P}_u K_t$ denotes the $L^2(\nu)$-projection of K_t on $\overline{\operatorname{span}}\{K_v : v \in [0,u]\}$.

(iv) *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $(A_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-predictable centered Gaussian process which is right-continuous and of bounded variation. Then there exists an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(Y_t)_{t\geq 0} \subseteq \overline{\operatorname{span}}\{A_t : t \geq 0\}$ satisfying $\|Y_t\|_{L^2(\mathrm{P})} = 1$ for $t \geq 0$ and*

$$A_t = \int_0^t Y_r\,\mu(\mathrm{d}r), \qquad t \geq 0, \tag{3.9}$$

where $\mu := \sqrt{2/\pi}\,\mu_A$.

Proof. (i): For $t \geq 0$ there exists, by (3.5), a $\Phi_t \in L^2(\nu)$ such that $Y_t = \int \Phi_t(s)\,\mathrm{d}N_s$. Moreover for $f \in L^2(\nu)$, $t \mapsto \int \Phi_t(s) f(s)\,\nu(\mathrm{d}s)$ is measurable since

$$\mathrm{E}[Y_t \int f(s)\,\mathrm{d}N_s] = \int \Phi_t(s) f(s)\,\nu(\mathrm{d}s). \tag{3.10}$$

Hence it follows from Lemma 3.1 that there exists a Ψ as stated in (i).
(ii): Since for $t \geq 0$ we have

$$\mathrm{E}[\int_0^t |\int \Psi_r(s)\,\mathrm{d}N_s|\mu(\mathrm{d}r)] \leq \int_0^t \|\Psi_r\|_{L^2(\nu)}\,\mu(\mathrm{d}s) < \infty, \tag{3.11}$$

the mapping $r \mapsto \int \Psi_r(s)\,\mathrm{d}N_s$ is locally μ-integrable P-a.s. Thus both sides of (3.7) are well-defined. The right-hand side belongs to $\overline{\operatorname{span}}\{N(A) : A \in \mathcal{A}\}$ and so does the left-hand side by Lemma 3.2. Fix $Y = \int g(s)\,\mathrm{d}N_s$ in $\overline{\operatorname{span}}\{N(A) : A \in \mathcal{A}\}$. We have

$$\mathrm{E}[Y \int \Big(\int f(t,s)\,\mu(\mathrm{d}t)\Big)\mathrm{d}N_s] = \int g(s) \int f(t,s)\,\mu(\mathrm{d}t)\,\nu(\mathrm{d}s). \tag{3.12}$$

Moreover from Fubini's Theorem we have

$$\mathrm{E}[Y \int \Big(\int f(t,s)\,\mathrm{d}N_s\Big)\mu(\mathrm{d}t)] = \int \mathrm{E}[Y \int f(t,s)\,\mathrm{d}N_s]\,\mu(\mathrm{d}t) \tag{3.13}$$

$$= \int\int g(s) f(t,s)\,\nu(\mathrm{d}s)\,\mu(\mathrm{d}t) = \int\int g(s) f(t,s)\,\mu(\mathrm{d}t)\,\nu(\mathrm{d}t). \tag{3.14}$$

Hence, the left- and right-hand side of (3.7) have the same inner product with all elements of $\overline{\operatorname{span}}\{N(A) : A \in \mathcal{A}\}$, from which equality follows.

(iii): From Gaussianity it follows that $\mathrm{E}[X_t|\mathcal{F}_u^X]$ is the $L^2(\mathrm{P})$-projection of X_t on $\overline{\operatorname{span}}\{X_v : v \le u\}$ and therefore (3.5) shows

$$\mathrm{E}[X_t|\mathcal{F}_u^X] = \int f(s)\,\mathrm{d}N_s, \tag{3.15}$$

for some $f \in L^2(\nu)$. Since $L^2(\nu) \ni g \mapsto \int g(s)\,\mathrm{d}N_s \in L^2(\mathrm{P})$ is an isometry it is readily seen that $f = \mathcal{P}_u K_t$.
(iv) is an immediate consequence of Basse [3, Proposition 4.1]. □

4 Main results

In this section we characterize the spectral representation of Gaussian semimartingales $(X_t)_{t\ge 0}$. We study three different filtrations. First we consider the natural filtration of $(X_t)_{t\ge0}$. Then we assume $(X_t)_{t\in\mathbb{R}}$ is a moving average process and the filtration is $(\mathcal{F}_t^{X,\infty})_{t\ge0}$, where $(\mathcal{F}_t^{X,\infty})_{t\ge0}$ is the least filtration for which X_s is $\mathcal{F}_t^{X,\infty}$-measurable for $t \ge 0$ and $s \in (-\infty, t]$. Finally the filtration is generated by the background driving random measure N. Recall that ν is the control measure of N.

Theorem 4.1. *Let $\mathbb{R}_+ \ni t \mapsto K_t \in L^2(\nu)$ be a right-continuous mapping and $(X_t)_{t\ge0}$ be given by $X_t = \int K_t(s)\,\mathrm{d}N_s$ for $t \ge 0$. Then the following three conditions are equivalent:*

(i) $(X_t)_{t\ge0}$ is a semimartingale (in its natural filtration).

(ii) For $t \ge 0$ we have

$$K_t(s) = K_0(s) + H_t(s) + \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r), \qquad \nu\text{-a.a. } s \in S, \tag{4.1}$$

where $\mathbb{R}_+ \ni t \mapsto H_t \in L^2(\nu)$ is a right-continuous mapping satisfying $H_0 = 0$ and

$$\int \big(H_t(s) - H_u(s)\big) K_v(s)\,\nu(\mathrm{d}s) = 0, \qquad 0 \le v \le u \le t, \tag{4.2}$$

$\mathbb{R}_+ \times S \ni (r,s) \mapsto \Psi_r(s) \in \mathbb{R}$ is a measurable mapping such that $\|\Psi_r\|_{L^2(\nu)} = 1$ and $\Psi_r \in \overline{\operatorname{span}}\{K_v : v < r\}$ for $r \ge 0$, and μ is a Radon measure.

(iii) There exists a right-continuous increasing function $f\colon \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|\mathcal{P}_u K_t - K_u\|_{L^2(\nu)} \le f(t) - f(u), \qquad 0 \le u \le t, \tag{4.3}$$

where $\mathcal{P}_u K_t$ denotes the $L^2(\nu)$-projection of K_t on $\overline{\operatorname{span}}\{K_v : v \le u\}$.

The decomposition (4.1) is unique and if K is represented as in (4.1) then the canonical decomposition of $(X_t)_{t\ge0}$ is given by

$$X_t = X_0 + \int H_t(s)\,\mathrm{d}N_s + \int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}N_s\Big)\mu(\mathrm{d}r). \tag{4.4}$$

Proof of Theorem 4.1. (i) ⇒ (ii): Assume $(X_t)_{t\ge0}$ is a semimartingale. By Stricker [16, Théorème 1] $(X_t)_{t\ge0}$ is a special semimartingale with bounded variation component $(A_t)_{t\ge0} \subseteq \overline{\operatorname{span}}\{X_t : t \ge 0\}$. Hence by Lemma 3.4 (iv) there exists an $(\mathcal{F}_t^X)_{t\ge0}$-predictable process $(Z_t)_{t\ge0} \subseteq \overline{\operatorname{span}}\{X_t : t\ge 0\}$ with $\|Z_r\|_{L^2(\mathrm{P})} = 1$ such that $A_t = \int_0^t Z_r\,\mu(\mathrm{d}r)$ for

$t \geq 0$, where $\mu = \sqrt{2/\pi}\mu_A$. Moreover Lemma 3.4 (i) shows that there exists a measurable mapping $(r, s) \mapsto \Psi_r(s)$ satisfying $\Psi_r \in L^2(\nu)$ and $Z_r = \int \Psi_r(s)\,\mathrm{d}N_s$ for $r \geq 0$. Since Z_r is $\mathcal{F}^X_{r-}$-measurable, it follows from Gaussianity that $\Psi_r \in \overline{\mathrm{span}}\{K_v : v < r\}$ for $r \geq 0$. From Lemma 3.4 (ii) we have

$$A_t = \int \Big(\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) \Big) \mathrm{d}N_s, \qquad t \geq 0. \tag{4.5}$$

Due to the fact that $(M_t)_{t\geq 0} \subseteq \overline{\mathrm{span}}\{X_t : t \geq 0\}$, Lemma 3.4 (i) shows that for all $t \geq 0$, $M_t = \int H_t(s)\,\mathrm{d}N_s$ for some $H_t \in L^2(\nu)$. The mapping $t \mapsto H_t \in L^2(\nu)$ is right-continuous since $(M_t)_{t\geq 0}$ is right-continuous. Stricker [16, Théorème 1] shows that $(M_t)_{t\geq 0}$ is a true $(\mathcal{F}^X_t)_{t\geq 0}$-martingale and hence

$$0 = \mathrm{E}[(M_t - M_u)X_v] = \int \big(H_t(s) - H_u(s)\big) K_v(s)\,\nu(\mathrm{d}s), \qquad 0 \leq v \leq u \leq t. \tag{4.6}$$

This completes the proof of (4.1).
(ii) $\Rightarrow$ (i): Assume (4.1) is satisfied. We show that $(X_t)_{t\geq 0}$ is a semimartingale with canonical decomposition given by (4.4). For $t \geq 0$ define

$$M_t := \int H_t(s)\,\mathrm{d}N_s \qquad \text{and} \qquad A_t := \int \Big(\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r) \Big) \mathrm{d}N_s. \tag{4.7}$$

Note $X_t = X_0 + M_t + A_t$. Lemma 3.4 (ii) shows that

$$A_t = \int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}N_s \Big) \mu(\mathrm{d}r), \qquad t \geq 0, \tag{4.8}$$

which implies that $(A_t)_{t\geq 0}$ is right-continuous and of bounded variation. Let $r \geq 0$. Since $\Psi_r \in \overline{\mathrm{span}}\{K_v : v < r\}$, $\int \Psi_r(s)\,\mathrm{d}N_s$ is $\mathcal{F}^X_{r-}$-measurable and hence it follows from Lemma 3.3 that $(A_t)_{t\geq 0}$ is $(\mathcal{F}^X_t)_{t\geq 0}$-predictable.

The only thing left to show is that $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}^X_t)_{t\geq 0}$-martingale. Since $M_t = X_t - X_0 - A_t$, $(M_t)_{t\geq 0}$ is $(\mathcal{F}^X_t)_{t\geq 0}$-adapted. Equation (4.2) shows that $\mathrm{E}[(M_t - M_u)X_v] = 0$ for $0 \leq v \leq u \leq t$ and hence from Gaussianity it follows that $M_t - M_u$ is independent of X_v. The $(\mathcal{F}^X_t)_{t\geq 0}$-martingale property of $(M_t)_{t\geq 0}$ therefore follows by the $L^2(\mathrm{P})$ right-continuity of $(M_t)_{t\geq 0}$. Since $(\mathcal{F}^X_t)_{t\geq 0}$ satisfies the usual conditions we can choose a càdlàg modification of $(M_t)_{t\geq 0}$. Thus $(X_t)_{t\geq 0}$ is a semimartingale with canonical decomposition given by (4.4).
(i) $\Leftrightarrow$ (iii): From Stricker [16, Théorème 1] it follows that $(X_t)_{t\geq 0}$ is a semimartingale if and only if it is a quasimartingale on each bounded interval. That is, for $t \geq 0$ we have

$$\sup \sum_{i=1}^n \mathrm{E}[|\mathrm{E}[X_{t_i} - X_{t_{i-1}} | \mathcal{F}^X_{t_{i-1}}]|] < \infty, \tag{4.9}$$

where the sup is taken over all finite partitions $0 = t_0 < \cdots < t_n = t$ of $[0, t]$. This is equivalent to the existence of a right-continuous and increasing function f satisfying

$$\mathrm{E}[|\mathrm{E}[X_t - X_u | \mathcal{F}^X_u]|] \leq f(t) - f(u), \qquad 0 \leq u \leq t. \tag{4.10}$$

The function f can be chosen to be the left-hand side of (4.9). Moreover Lemma 3.4 (iii) shows that

$$\|\mathcal{P}_u K_t - K_u\|_{L^2(\nu)} = \|\mathrm{E}[X_t - X_u | \mathcal{F}^X_u]\|_{L^2(\mathrm{P})} = \sqrt{\frac{\pi}{2}}\,\mathrm{E}[|\mathrm{E}[X_t - X_u | \mathcal{F}^X_u]|], \tag{4.11}$$

which implies that (i) and (iii) are equivalent.

Decompose K as in (4.1). We show that this decomposition is unique. In the proof of "(ii) $\Rightarrow$ (i)"we showed that (4.4) is the canonical decomposition of $(X_t)_{t\geq 0}$ and since this is unique we have that $\mathbb{R}_+ \ni t \mapsto H_t \in L^2(\nu)$ is unique. Let $(A_t)_{t\geq 0}$ be the bounded variation component of the semimartingale $(X_t)_{t\geq 0}$. We have

$$\mathrm{E}[V_t(A)] = \mathrm{E}[\int_0^t |\int \Psi_r(s)\,\mathrm{d}N_s|\,\mu(\mathrm{d}r)] = \int_0^t \mathrm{E}[|\int \Psi_r(s)\,\mathrm{d}N_s|]\,\mu(\mathrm{d}r) \tag{4.12}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^t \|\Psi_r\|_{L^2(\nu)}\,\mu(\mathrm{d}r) = \sqrt{\frac{2}{\pi}}\,\mu((0,t]), \tag{4.13}$$

and hence μ is uniquely determined and it follows that $(t,s)\mapsto \Psi_t(s)$ is uniquely determined $\mu\otimes\nu$-a.s. This completes the proof. $\square$

The functions $t \mapsto H_t(s)$ can behave very differently for different H in the above theorem. An example of such an H is $H_t(s) = 1_{(0,t]}(s)$. In this case $t \mapsto H_t(s)$ is constant except at s where it has a jump of size one. But there are also examples of H for which $t \mapsto H_t(s)$ is continuous and nowhere differentiable (and hence of unbounded variation).

We now apply Theorem 4.1 on an example.

Example 4.2. Let $g, h \in C^1(\mathbb{R})$ be two strictly increasing functions such that $0 \leq g < h$ and $g(\infty) = \infty$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f > 0$. Define $K_t(s) = 1_{[g(t),h(t)]}(s)f(s)$ and let $(W_t)_{t\geq 0}$ be a Wiener process. We show that $(X_t)_{t\geq 0}$ given by

$$X_t = \int K_t(s)\,\mathrm{d}W_s = \int_{g(t)}^{h(t)} f(s)\,\mathrm{d}W_s, \qquad t \geq 0, \tag{4.14}$$

is not a semimartingale.

Choose $(a,b) \subseteq \mathbb{R}_+$ such that $h(0) \leq g(x) \leq h(a)$ for $x \in (a,b)$ and let $u, t \in (a,b)$ with $u \leq t$ be given. Moreover choose $c, d \geq 0$ satisfying $c \leq d \leq u$, $h(c) = g(u)$ and $h(\mathrm{d}) = g(t)$ and define $\psi := K_d - K_c = (1_{[g(u),g(t)]} - 1_{[g(c),g(\mathrm{d})]})f$. Let $\mathcal{P}_u$ respectively $\mathcal{P}_\psi$ denote the projection on $\overline{\mathrm{span}}\{K_v : v \in [0,u]\}$ respectively $\overline{\mathrm{span}}\{\psi\}$, where the closure is in $L^2(\lambda)$ (λ denotes the Lebesgue measure). We have that

$$\|\mathcal{P}_u K_t - K_u\|_{L^2(\lambda)} = \|\mathcal{P}_u f 1_{[g(u),g(t)]}\|_{L^2(\lambda)} \geq \|\mathcal{P}_\psi f 1_{[g(u),g(t)]}\|_{L^2(\lambda)}, \tag{4.15}$$

and by choosing $K_1, K_2 \in (0,\infty)$ such that $K_1 \leq f^2(s) \leq K_2$ for $s \in [0, g(t)]$, we get

$$|\mathcal{P}_\psi f 1_{[g(u),g(t)]}| = \Big|\frac{\langle \psi, f1_{[g(u),g(t)]}\rangle}{\langle \psi,\psi\rangle}\,\psi\Big| \geq \frac{K_1(g(t)-g(u))}{K_2(g(t)-g(u)+g(\mathrm{d})-g(c))}\,|\psi|. \tag{4.16}$$

Thus, by setting $\phi = g\circ h^{-1}\circ g$, it follows that

$$\|\mathcal{P}_u K_t - K_u\|_{L^2(\lambda)} \geq K_1 K_2^{-1}\frac{g(t)-g(u)}{g(t)-g(u)+g(\mathrm{d})-g(c)}\|\psi\|_{L^2(\lambda)} \tag{4.17}$$

$$\geq K_1^{3/2}K_2^{-1}\frac{g(t)-g(u)}{g(t)-g(u)+g(\mathrm{d})-g(c)}\sqrt{g(t)-g(u)+g(\mathrm{d})-g(c)} \tag{4.18}$$

$$= K_1^{3/2}K_2^{-1}\frac{g(t)-g(u)}{\sqrt{g(t)-g(u)+\phi(t)-\phi(u)}} \geq K\sqrt{t-u}, \tag{4.19}$$

for some $K > 0$. Hence we conclude, by Theorem 4.1, that $(X_t)_{t\geq 0}$ is not a semimartingale. $\lozenge$

Let $(W_t)_{t\in\mathbb{R}}$ be a given Wiener process and k and f be measurable functions satisfying $k(t-\cdot)-f(-\cdot)\in L^2(\lambda)$ for $t\in\mathbb{R}$ (λ denotes the Lebesgue measure on $\mathbb{R}$). Then $(X_t)_{t\in\mathbb{R}}$ is said to be a (W_t)-moving average process with parameter (k,f) if

$$X_t = \int_{\mathbb{R}} k(t-s)-f(-s)\,\mathrm{d}W_s, \qquad t\in\mathbb{R}. \tag{4.20}$$

For short we say $(X_t)_{t\in\mathbb{R}}$ is a (W_t)-moving average process. Note that we do not assume k and f are 0 on $(-\infty,0)$. It is readily seen that all (W_t)-moving average processes have stationary increments. By Doob [7, page 533] it follows that an $L^2(\mathrm{P})$-continuous, stationary and centered Gaussian process has absolutely continuous spectral measure if and only if it is a (W_t)-moving average process with parameter $(k,0)$, for some Wiener process $(W_t)_{t\in\mathbb{R}}$ and function k. Recall the definition of the filtration $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$ on page 42.

Lemma 4.3. *Let $(\mathcal{F}_t)_{t\geq0}$ be a filtration and $(X_t)_{t\in\mathbb{R}}$ be a (W_t)-moving average. If $(X_t)_{t\geq0}$ is an $(\mathcal{F}_t)_{t\geq0}$-semimartingale and either the martingale component or the bounded variation component of $(X_t)_{t\geq0}$ is a (W_t)-moving average process, then $(X_t)_{t\geq0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq0}$-semimartingale.*

Proof. Let $(X_t)_{t\in\mathbb{R}}$ be a process given by $X_t=\int k(t-s)-f(-s)\,\mathrm{d}W_s$ and assume $(X_t)_{t\geq0}$ is an $(\mathcal{F}_t)_{t\geq0}$-semimartingale where the martingale or the bounded variation component is a (W_t)-moving average. In either case the martingale component of $(X_t)_{t\geq0}$ is given by $M_t=\int h(t-s)-h(-s)\,\mathrm{d}W_s$ for $t\geq0$ for some measurable function h. For $t,v\in\mathbb{R}_+$ we have

$$\begin{aligned}
\mathrm{E}[M_tX_{-v}] &= \mathrm{E}[M_t(X_{-v}-X_0)] && (4.21)\\
&= \int \big(h(t-s)-h(-s)\big)\big(k(-v-s)-k(-s)\big)\,\mathrm{d}s && (4.22)\\
&= \int \big(h(t+v-s)-h(v-s)\big)\big(k(-s)-k(v-s)\big)\,\mathrm{d}s && (4.23)\\
&= \mathrm{E}[(M_{t+v}-M_v)(X_0-X_v)]=0, && (4.24)
\end{aligned}$$

and it follows from Gaussianity that $(M_t)_{t\geq0}$ is independent of $(X_t)_{t\leq0}$. This shows that $(X_t)_{t\geq0}$ is an $(\mathcal{F}_t\vee\mathcal{G})_{t\geq0}$-semimartingale, where $\mathcal{G}:=\sigma(X_s:s\in(-\infty,0))$, and hence in particular an $(\mathcal{F}_t^{X,\infty})_{t\geq0}$-semimartingale. □

Theorem 4.4. *Let $(X_t)_{t\in\mathbb{R}}$ be a (W_t)-moving average process with parameters $(k,0)$. Then $(X_t)_{t\geq0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq0}$-semimartingale if and only if*

$$k(t)=h(t)+\int_0^t \psi(r)\,\mathrm{d}r, \qquad \lambda\text{-a.a. } t\in\mathbb{R}, \tag{4.25}$$

where h and ψ are measurable functions satisfying $h(t-\cdot)-h(-\cdot)\in L^2(\lambda)$ for $t\geq0$,

$$\int \big(h(t-s)-h(u-s)\big)k(v-s)\,\mathrm{d}s=0, 0\leq v\leq u\leq t, \tag{4.26}$$

and

$$\psi(t-\cdot)\in\overline{\mathrm{span}}\{k(v-\cdot):v\in(-\infty,t]\}\subseteq L^2(\lambda), 0\leq t. \tag{4.27}$$

The above k and h are uniquely determined and the $(\mathcal{F}_t^{X,\infty})_{t\geq0}$-canonical decomposition of $(X_t)_{t\geq0}$ is given by

$$X_t = X_0+\int h(t-s)-h(-s)\,\mathrm{d}W_s+\int_0^t\Big(\int\psi(r-s)\,\mathrm{d}W_s\Big)\mathrm{d}r, \tag{4.28}$$

and the martingale and the bounded variation component of $(X_t)_{t\geq 0}$ *are* (W_t)*-moving average processes.*

For each function $g\colon \mathbb{R}\to\mathbb{R}$ and $u\in\mathbb{R}$, we let $\theta_u g$ denote the function $s\mapsto g(s-u)$.

Proof of Theorem 4.4. Let $K_t(s):=k(t-s)$ for $t,s\in\mathbb{R}$.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. By the stationary increments, $(X_t)_{t\geq 0}$ has no fixed points of discontinuity. Moreover since $(X_t)_{t\geq 0}$ is a Gaussian semimartingale it follows from Stricker [16, Proposition 3] that $(X_t)_{t\geq 0}$ is a continuous process. Let $X_t=X_0+M_t+A_t$ be the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$. For $u\in\mathbb{R}_+$, let $\mathcal{P}_u\colon L^2(\lambda)\to L^2(\lambda)$ denote the projection on $\overline{\operatorname{span}}\{K_v: v\in(-\infty,u]\}$ and note that $\mathcal{P}_{v+u}K_{t+u}=\theta_u\mathcal{P}_vK_t$ for $v\leq t$ and $0\leq u$. Standard theory shows that for $t\geq 0$ we have

$$A_t=\lim_{n\to\infty}\sum_{i=1}^{[t2^n]}\mathrm{E}[X_{i/2^n}-X_{(i-1)/2^n}|\mathcal{F}^{X,\infty}_{(i-1)/2^n}] \tag{4.29}$$

$$=\lim_{n\to\infty}\int\sum_{i=1}^{[t2^n]}\big(\mathcal{P}_{(i-1)/2^n}K_{i/2^n}(s)-K_{(i-1)/2^n}(s)\big)\,\mathrm{d}W_s \qquad \text{in } L^2(\mathrm{P}), \tag{4.30}$$

where the second equality follows from Lemma 3.4 (iii). Thus with

$$G_t:=\lim_{n\to\infty}\sum_{i=1}^{[t2^n]}\big(\mathcal{P}_{(i-1)/2^n}K_{i/2^n}-K_{(i-1)/2^n}\big) \qquad \text{in } L^2(\lambda), \tag{4.31}$$

we have $A_t=\int G_t(s)\,\mathrm{d}W_s$. For $t,u\in\mathbb{R}_+$ it follows that

$$G_{t+u}-G_u=\lim_{n\to\infty}\sum_{i=[u2^n]+1}^{[(t+u)2^n]}\mathcal{P}_{(i-1)/2^n}\big(K_{i/2^n}-K_{(i-1)/2^n}\big) \tag{4.32}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{[t2^n]}\mathcal{P}_{(i-1)/2^n+u}\big(K_{i/2^n+u}-K_{(i-1)/2^n+u}\big) \tag{4.33}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{[t2^n]}\theta_u\mathcal{P}_{(i-1)/2^n}\big(K_{i/2^n}-K_{(i-1)/2^n}\big)=\theta_uG_t \qquad \text{in } L^2(\lambda). \tag{4.34}$$

Which shows $(A_t)_{t\geq 0}$ has stationary increments and therefore μ_A equals the Lebesgue measure up to a scaling constant. Arguments as in the prove of '(i) $\Rightarrow$ (ii)' in Theorem 4.1 shows that

$$A_t=\int\Big(\int_0^t\Psi_r(s)\,\mathrm{d}r\Big)\mathrm{d}W_s, \qquad t\geq 0, \tag{4.35}$$

for some measurable mapping $(t,s)\mapsto\Psi_t(s)$ satisfying that $t\mapsto\|\Psi_t\|_{L^2(\lambda)}$ is constant and $\Psi_t\in\overline{\operatorname{span}}\{K_u: u\in(-\infty,t]\}$ for $t\geq 0$. Hence $G_t(s)=\int_0^t\Psi_r(s)\,\mathrm{d}r$ for λ-a.a. $s\in\mathbb{R}$ for $t\geq 0$. For $t,u\in\mathbb{R}_+$, (4.33) yields

$$\int_0^t\Psi_{r+u}(s)\,\mathrm{d}r=\int_u^{t+u}\Psi_r(s)\,\mathrm{d}r=\theta_u\int_0^t\Psi_r(s)\,\mathrm{d}r=\int_0^t\theta_u\Psi_r(s)\,\mathrm{d}r, \tag{4.36}$$

for λ-a.a. $s\in\mathbb{R}$, which implies that $\Psi_{r+u}=\theta_u\Psi_r$ λ-a.s. Thus there exists a $\psi\in L^2(\lambda)$ such that for $r\geq 0$, $\Psi_r(s)=\psi(r-s)$ for λ-a.a. $s\in\mathbb{R}$. By setting $h(t)=k(t)-\int_0^t\psi(r)\,\mathrm{d}r$

for $t \in \mathbb{R}$, it follows that $h(t-\cdot) - h(-\cdot) \in L^2(\lambda)$ and $M_t = \int h(t-s) - h(-s)\,\mathrm{d}W_s$ for $t \geq 0$. The $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale property of $(M_t)_{t\geq 0}$ shows that h satisfies (4.26). This completes the proof of the *only if* statement.

Assume conversely k is on the form (4.25). By approximating k with continuous functions with compact support it is readily seen that

$$\lim_{t\to 0} \int \big(k(t-s) - k(-s)\big)^2\,\mathrm{d}s = 0. \tag{4.37}$$

Since $(X_t)_{t\geq 0}$ is a stationary process, (4.37) shows that it is $L^2(\mathrm{P})$-continuous. For $t \geq 0$ define

$$M_t := \int h(t-s) - h(-s)\,\mathrm{d}W_s \qquad \text{and} \qquad A_t := \int_0^t \Big(\int \psi(r-s)\,\mathrm{d}W_s\Big)\mathrm{d}r. \tag{4.38}$$

By Lemma 3.4 (ii) we have that

$$A_t = \int \Big(\int_0^t \psi(r-s)\,\mathrm{d}r\Big)\mathrm{d}W_s, \qquad t \geq 0, \tag{4.39}$$

which shows $X_t = X_0 + M_t + A_t$ for $t \geq 0$. Since $\psi(r-\cdot) \in \overline{\mathrm{span}}\{K_v : v \in (-\infty, r]\}$ for $r \geq 0$ it follows that $\int \psi(r-s)\,\mathrm{d}W_s$ is $\mathcal{F}_r^{X,\infty}$-measurable for $r \geq 0$ and therefore $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-adapted and hence by continuity $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-predictable.

Equation (4.26) and the translation invariancy of the Lebesgue measure shows

$$\int \big(h(t-s) - h(u-s)\big)k(v-s)\,\mathrm{d}s = 0, \qquad -\infty < v \leq u \leq t. \tag{4.40}$$

This yields $\mathrm{E}[(M_t - M_u)X_v] = 0$ for $-\infty < v \leq u \leq t$ where $0 \leq u$ and it follows by Gaussianity that $M_t - M_u$ is independent of X_v. Since $M_t = X_t - X_0 - A_t$, $(M_t)_{t\geq 0}$ is continuous in $L^2(\mathrm{P})$. Moreover since $(M_t)_{t\geq 0}$ is a centered process we conclude that $(M_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale. Since $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$ satisfies the usual conditions, $(M_t)_{t\geq 0}$ has a càdlàg modification. Hence $(X_t)_{t\geq 0}$ is an semimartingale with canonical decomposition given by (4.28).

We finally show that h and k are uniquely determined. Thus assume (4.25) is satisfied for k, h and $\tilde{k}, \tilde{h}$. By the uniqueness of the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-decomposition of $(X_t)_{t\geq 0}$ is follows from (4.28) and Lemma 3.4 (ii) that

$$\int_0^t \psi(r-s)\,\mathrm{d}r = \int_0^t \tilde{\psi}(r-s)\,\mathrm{d}r, \qquad \lambda\text{-a.a. } s \in \mathbb{R},\ \text{all } t \geq 0, \tag{4.41}$$

which shows $\psi(r-s) = \tilde{\psi}(r-s)$ for λ-a.a. $r \geq 0$ and λ-a.a. $s \in \mathbb{R}$ and hence $\psi = \tilde{\psi}$ λ-a.s. Hereby it follows from (4.25) that $h = \tilde{h}$ λ-a.s. and the proof is complete. □

As a consequence of Lemma 4.3 and Theorem 4.4 we have the following.

Corollary 4.5. *Let $(X_t)_{t\in\mathbb{R}}$ be a (W_t)-moving average process with parameter $(k, 0)$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if there exists a filtration in which $(X_t)_{t\geq 0}$ is a semimartingale with a martingale component which is a (W_t)-moving average process.*

For a (W_t)-moving average process on the form

$$X_t = \int_{-\infty}^t k(t-s)\,\mathrm{d}W_s, \qquad t \in \mathbb{R}, \tag{4.42}$$

Knight [10, Theorem 6.5] proved that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if $k(t) = \alpha + \int_0^t g(s)\,\mathrm{d}s$ for λ-a.a. $t \geq 0$, where $\alpha \in \mathbb{R}$ and $g \in L^2(\lambda)$. After proving this result he wrote "an interesting project for further research might be to test the present methods in the non-stationary Gaussian case". The following result generalizes his theorem to the non-stationary Gaussian case, but uses a different approach.

Let $(C_t)_{t\geq 0}$ be a family of increasing $\sigma(\mathcal{A})$-measurable sets satisfying

$$\bigcap_{u\in(t,\infty)} C_u = C_t, \qquad t \geq 0. \tag{4.43}$$

Let $(\mathcal{F}_t^N)_{t\geq 0}$ be the smallest filtration satisfying $N(A)$ is $\mathcal{F}_t^N$-measurable for $A \in \mathcal{A}$ with $A \subseteq C_t$, and let $(X_t)_{t\geq 0}$ be given by $X_t = \int_{C_t} K_t(s)\,\mathrm{d}N_s$ for $t \geq 0$.

Theorem 4.6. *Let $(X_t)_{t\geq 0}$ and $(\mathcal{F}_t^N)_{t\geq 0}$ be given as above. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-semimartingale if and only if for $t \geq 0$ we have*

$$K_t(s) = g(s) + \int_0^t \Psi_r(s)\,\mu(\mathrm{d}r), \qquad \nu\text{-a.a. } s \in C_t, \tag{4.44}$$

where $g\colon S \to \mathbb{R}$ is square integrable w.r.t. ν on C_t for $t \geq 0$, μ is a Radon measure on $\mathbb{R}_+$ and $\mathbb{R}_+ \times S \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ is a measurable mapping satisfying $\|\Psi_r\|_{L^2(\nu)} = 1$ and $\Psi_r(s) = 0$ for ν-a.a. $s \notin \cup_{u<r} C_u$.

The decomposition (4.44) is unique and if K is represented as in (4.44), then the $(\mathcal{F}_t^N)_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is given by

$$X_t = X_0 + \int_{C_t\setminus C_0} g(s)\,\mathrm{d}N_s + \int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}N_s\Big)\mu(\mathrm{d}r). \tag{4.45}$$

Proof. Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-semimartingale with $(\mathcal{F}_t^N)_{t\geq 0}$-canonical decomposition $X_t = X_0 + M_t + A_t$. From Stricker [16, Proposition 4 and 5] it follows that $(M_t)_{t\geq 0} \subseteq \overline{\operatorname{span}}\{X_t : t \geq 0\}$. Thus for each $t \geq 0$ there exists an $H_t \in L^2(\nu)$ such that $M_t = \int_{C_t} H_t(s)\,\mathrm{d}N_s$. Let $0 \leq u \leq t$ be given. The $(\mathcal{F}_t^N)_{t\geq 0}$-martingale property of $(M_t)_{t\geq 0}$ implies

$$\begin{aligned} 0 &= \mathrm{E}\big[\big(\mathrm{E}[M_t - M_u|\mathcal{F}_u^N]\big)^2\big] && (4.46)\\ &= \mathrm{E}\Big[\Big(\int_{C_u} H_t(s) - H_u(s)\,\mathrm{d}N_s\Big)^2\Big] = \int_{C_u} \big(H_t(s) - H_u(s)\big)^2\,\nu(\mathrm{d}s), && (4.47) \end{aligned}$$

which shows $H_t(s) = H_u(s)$ for ν-a.a. $s \in C_u$. Thus there exists a measurable function $g\colon S \to \mathbb{R}$ which equals H_t ν-a.s. on C_t for $t \geq 0$. By Lemma 3.4 (iv) there exists a Radon measure μ and an $(\mathcal{F}_t^N)_{t\geq 0}$-predictable process $(Y_t)_{t\geq 0} \subseteq \overline{\operatorname{span}}\{A_t : t \geq 0\}$ satisfying $\|Y_r\|_{L^2(\mathrm{P})} = 1$ for $r \geq 0$ and

$$A_t = \int_0^t Y_r\,\mu_A(\mathrm{d}r), \qquad t \geq 0. \tag{4.48}$$

In particular Y_r is $\mathcal{F}_{r-}^N$ measurable for $r \geq 0$. Thus by Lemma 3.4 (i) there exists a measurable mapping $(r,s) \mapsto \Psi_r(s)$ satisfying $\Psi_r(s) = 0$ for ν-a.a. $s \notin \cup_{u<r} C_u$ and $Y_r = \int \Psi_r(s)\,\mathrm{d}N_s$. From Lemma 3.4 (ii) it follows that

$$X_t = \int_{C_t} \Big(g(s) + K_0(s)\Big)\,\mathrm{d}N_s + \int \Big(\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r)\Big)\mathrm{d}N_s, \tag{4.49}$$

which shows (4.44).

Assume conversely (4.44) is satisfied. We show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-semimartingale with canonical decomposition given by (4.45). From Lemma 3.4 (ii) it follows that

$$X_t = X_0 + \int_{C_t\setminus C_0} g(s)\,\mathrm{d}N_s + \int \Big(\int_0^t \Psi_r(s)\,\mu(\mathrm{d}r)\Big)\mathrm{d}N_s \tag{4.50}$$

$$= X_0 + \int_{C_t\setminus C_0} g(s)\,\mathrm{d}N_s + \int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}N_s\Big)\mu(\mathrm{d}r). \tag{4.51}$$

Since $(\int_{C_t\setminus C_0} g(s)\,\mathrm{d}N_s)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t^N)_{t\geq 0}$ it is enough to show that $\int_0^t \Big(\int \Psi_r(s)\,\mathrm{d}N_s\Big)\mu(\mathrm{d}r)$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-predictable process. But this follows from Lemma 3.3 since $\int \Psi_r(s)\,\mathrm{d}N_s$ is $\mathcal{F}_{r-}^N$-measurable for $r\geq 0$.

To conclude the proof assume that K is decomposed as in (4.44). By uniqueness of the martingale component of $(X_t)_{t\geq 0}$ it follows that g is determined uniquely ν-a.s. on $\cup_{t\geq 0} C_t$. Using once more that $\|\Psi_r\|_{L^2(\nu)} = 1$ for $r\geq 0$, we have that $\mu = (2/\pi)^{1/2}\mu_A$ where $(A_t)_{t\geq 0}$ is the bounded variation component of $(X_t)_{t\geq 0}$, and hence μ is uniquely determined and it follows from (4.44) that Ψ is uniquely determined up to $\mu\otimes\nu$-null sets. This completes the proof. □

Let the setting be as in Theorem 4.6 and assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-semimartingale. Then Theorem 4.6 in particular shows that $K_t = \tilde{K}_t$ ν-a.s. on C_t, where $(t,s)\mapsto \tilde{K}_t(s)$ is a measurable mapping satisfying that $t\mapsto \tilde{K}_t(s)$ is right-continuous and of bounded variation for $s\in S$.

If $(X_t)_{t\in\mathbb{R}}$ is given by

$$X_t = \int_{-\infty}^t k(t-s)\,\mathrm{d}W_s, \qquad t\in\mathbb{R}, \tag{4.52}$$

then $(X_t)_{t\geq 0}$ satisfies the following relations

$$(\mathcal{F}_t^{W,\infty})_{t\geq 0}\text{-semimartingale} \Rightarrow (\mathcal{F}_t^{X,\infty})_{t\geq 0}\text{-semimartingale} \Rightarrow (\mathcal{F}_t^{X})_{t\geq 0}\text{-semimartingale}. \tag{4.53}$$

Hence the assumptions on $(X_t)_{t\geq 0}$ are strongest in Theorem 4.6, weaker in Theorem 4.4 and weakest in Theorem 4.1.

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PAPER C

Gaussian moving averages and semimartingales

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Abstract

In the present paper we study moving averages driven by a Wiener process and with a deterministic kernel. Necessary and sufficient conditions on the kernel are provided for the moving average to be a semimartingale in its natural filtration. Our results are constructive - meaning that they provide a simple method to obtain kernels for which the moving average is a semimartingale or a Wiener process. Several examples are considered. In the last part of the paper we study general Gaussian processes with stationary increments. We provide necessary and sufficient conditions on spectral measure for the process to be a semimartingale.

Keywords: semimartingales; Gaussian processes; stationary processes; moving averages; stochastic convolutions; non-canonical representations

AMS Subject Classification: 60G15; 60G10; 60G48; 60G57

1 Introduction

In this paper we study moving averages, that is processes $(X_t)_{t\geq 0}$ on the form

$$X_t = \int (\phi(t-s) - \psi(-s))\,\mathrm{d}W_s, \qquad t \in \mathbb{R}, \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is a Wiener process and ϕ and ψ are two locally square integrable functions such that $s \mapsto \phi(t-s) - \psi(-s) \in L^2_{\mathbb{R}}(\lambda)$ for all $t \in \mathbb{R}$ (λ denotes the Lebesgue measure). We are concerned with the semimartingale property of $(X_t)_{t\geq 0}$ in the filtration $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$, where $\mathcal{F}^{X,\infty}_t := \sigma(X_s : s \in (-\infty, t])$ for all $t \geq 0$.

The class of moving averages includes many interesting processes. By Doob [10, page 533] the case $\psi = 0$ corresponds to the class of centered Gaussian $L^2(\mathrm{P})$-continuous stationary processes with absolutely continuous spectral measure. Moreover, (up to scaling constants) the fractional Brownian motion corresponds to $\phi(t) = \psi(t) = (t \vee 0)^{H-1/2}$, and the Ornstein-Uhlenbeck process to $\phi(t) = e^{-\beta t} 1_{\mathbb{R}_+}(t)$ and $\psi = 0$. It is readily seen that all moving averages are Gaussian with stationary increments. Note however that in general we do not assume that ϕ and ψ are 0 on $(-\infty, 0)$. In fact, Karhunen [16, Satz 5] shows that a centered Gaussian $L^2(\mathrm{P})$-continuous stationary process has the representation (1.1) with $\psi = 0$ and $\phi = 0$ on $(-\infty, 0)$ if and only if it has an absolutely continuous spectral measure and the spectral density f satisfies

$$\int \frac{\log(f(u))}{1+u^2}\,\mathrm{d}u > -\infty. \tag{1.2}$$

In the case where $\psi = 0$ and ϕ is 0 on $(-\infty, 0)$, it follows from Knight [17, Theorem 6.5] that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^{W,\infty}_t)_{t\geq 0}$-semimartingale if and only if

$$\phi(t) = \alpha + \int_0^t h(s)\,\mathrm{d}s, \qquad t \geq 0, \tag{1.3}$$

for some $\alpha \in \mathbb{R}$ and $h \in L^2_{\mathbb{R}}(\lambda)$. Related results, also concerning general ψ, are found in Cherny [7] and Cheridito [6]. Knight's result is extended to the case $X_t = \int_{-\infty}^t K_t(s)\,\mathrm{d}W_s$ in Basse [3, Theorem 4.6].

The results mentioned above are all concerned with the semimartingale property in the $(\mathcal{F}^{W,\infty}_t)_{t\geq 0}$-filtration. Much less is known when it comes to the $(\mathcal{F}^X_t)_{t\geq 0}$-filtration or the $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-filtration ($\mathcal{F}^X_t := \sigma(X_s : 0 \leq s \leq t)$). In particular no simple necessary and sufficient conditions, as in (1.3), are available for the semimartingale property in these filtrations. Let $(X_t)_{t\geq 0}$ be given by (1.1) and assume it is $(\mathcal{F}^{W,\infty}_t)_{t\geq 0}$-adapted; it is then easier for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale than an $(\mathcal{F}^{W,\infty}_t)_{t\geq 0}$-semimartingale and harder than being an $(\mathcal{F}^X_t)_{t\geq 0}$-semimartingale. It follows from Basse [2, Theorem 4.8, iii] that when ψ equals 0 or ϕ and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^X_t)_{t\geq 0}$-semimartingale with canonical decomposition $X_t = X_0 + M_t + A_t$, then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale as well if and only if $t \mapsto \mathrm{E}[\mathrm{Var}_{[0,t]}(A)]$ is Lipschitz continuous on $\mathbb{R}_+$ ($\mathrm{Var}_{[0,t]}(A)$ denotes the total variation of $s \mapsto A_s$ on $[0,t]$). In the case $\psi = 0$, Jeulin and Yor [15, Proposition 19] provides necessary and sufficient conditions on the Fourier transform of ϕ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale.

In the present paper we provide necessary and sufficient conditions on ϕ and ψ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale. The approach taken relies heavily on Fourier theory and Hardy functions as in Jeulin and Yor [15]. Our main result can be described as follows. Let S^1 denote the unit circle in the complex plane $\mathbb{C}$. For each measurable function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$, define $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(t) := \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\,\mathrm{d}s, \tag{1.4}$$

where the limit is in λ-measure. For simplicity let us assume $\psi = \phi$. We then show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if ϕ can be decomposed as

$$\phi(t) = \beta + \alpha\tilde{f}(t) + \int_0^t \widehat{f\hat{h}}(s)\,\mathrm{d}s, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{1.5}$$

where $\alpha, \beta \in \mathbb{R}$, $f\colon \mathbb{R} \to S^1$ such that $\overline{f} = f(-\cdot)$, and $h \in L^2_{\mathbb{R}}(\lambda)$ is 0 on $\mathbb{R}_+$ when $\alpha \neq 0$. In this case $(X_t)_{t\geq 0}$ is in fact a continuous $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, where the martingale component is a Wiener process and the bounded variation component is an absolutely continuous Gaussian process. Several applications of (1.5) are provided.

In the last part of the paper we are concerned with the spectral measure of $(X_t)_{t\geq 0}$, where $(X_t)_{t\geq 0}$ is either a stationary Gaussian semimartingale or a Gaussian semimartingale with stationary increments and $X_0 = 0$. In both cases we provide necessary and sufficient conditions on the spectral measure of $(X_t)_{t\geq 0}$ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

2 Notation and Hardy functions

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space. By a filtration we mean an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras satisfying the usual conditions of right-continuity and completeness. For a stochastic process $(X_t)_{t\geq 0}$ let $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$ denote the least filtration subject to $\sigma(X_s : s \in (-\infty, t]) \subseteq \mathcal{F}_t^{X,\infty}$ for all $t \geq 0$.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Recall that an $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process $(X_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if there exists a decomposition of $(X_t)_{t\geq 0}$ such that

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0, \tag{2.1}$$

where $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-local martingale which starts at 0 and $(A_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-adapted process of finite variation which starts at 0.

A process $(W_t)_{t\geq 0}$ is said to be a Wiener process if for all $n \geq 1$ and $t_0 < \cdots < t_n$

$$W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}} \tag{2.2}$$

are independent, for $-\infty < s < t < \infty$ $W_t - W_s$ follows a centered Gaussian distributed with variance $\sigma^2(t-s)$ for some $\sigma^2 > 0$, and $W_0 = 0$. If $\sigma^2 = 1$, $(W_t)_{t\geq 0}$ is said to be a standard Wiener process.

Let $f\colon \mathbb{R} \to \mathbb{R}$. Then (unless explicitly stated otherwise) all integrability matters of f are with respect to the Lebesgue measure λ on $\mathbb{R}$. If f is a locally integrable function and $a < b$, then $\int_b^a f(s)\,\mathrm{d}s$ should be interpreted as $-\int_a^b f(s)\,\mathrm{d}s = -\int 1_{[a,b]}(s) f(s)\,\mathrm{d}s$. For $t \in \mathbb{R}$ let $\tau_t f$ denote the function $s \mapsto f(t-s)$.

Remark 2.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally square integrable function satisfying $\tau_t f - \tau_0 f \in L^2_{\mathbb{R}}(\lambda)$ for all $t \in \mathbb{R}$. Then $t \mapsto \tau_t f - \tau_0 f$ is a continuous mapping from $\mathbb{R}$ into $L^2_{\mathbb{R}}(\lambda)$.

A similar result is obtained in Cheridito [6, Lemma 3.4]. However, a short proof is given as follows. By approximation with continuous functions with compact support it follows that $t \mapsto 1_{[a,b]}(\tau_t f - \tau_0 f)$ is continuous for all $a < b$. Moreover, since $\tau_t f - \tau_0 f = \lim_n 1_{[-n,n]}(\tau_t f - \tau_0 f)$ in $L^2_{\mathbb{R}}(\lambda)$, the Baire Characterization Theorem (or more precisely a generalization of it to functions with values in abstract spaces, see e.g. Reĭnov [21] or Stegall [23]) states that the set of continuity points C of $t \mapsto \tau_t f - \tau_0 f$ is dense in $\mathbb{R}$. Furthermore, since the Lebesgue measure is translation invariant we obtain $C = \mathbb{R}$ and it follows that $t \mapsto \tau_t f - \tau_0 f$ is continuous.

For measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfying $\int |f(t-s)g(s)| \,\mathrm{d}s < \infty$ for $t \in \mathbb{R}$, we let $f * g$ denote the convolution between f and g, that is $f * g$ is the mapping

$$t \mapsto \int f(t-s)g(s) \,\mathrm{d}s. \tag{2.3}$$

A locally square integrable function $f\colon \mathbb{R} \to \mathbb{R}$ is said to have orthogonal increments if $\tau_t f - \tau_0 f \in L^2_{\mathbb{R}}(\lambda)$ for all $t \in \mathbb{R}$ and for all $-\infty < t_0 < t_1 < t_2 < \infty$ we have that $\tau_{t_2} f - \tau_{t_1} f$ is orthogonal to $\tau_{t_1} f - \tau_{t_0} f$ in $L^2_{\mathbb{R}}(\lambda)$.

We now give a short survey of Fourier theory and Hardy functions. For a comprehensive survey see Dym and McKean [11]. The Hardy functions will become an important tool in the construction of the canonical decomposition of a moving average. Let $L^2_{\mathbb{R}}(\lambda)$ and $L^2_{\mathbb{C}}(\lambda)$ denote the spaces of real and complex valued square integrable functions from $\mathbb{R}$. For $f, g \in L^2_{\mathbb{C}}(\lambda)$ define their inner product as $\langle f, g\rangle_{L^2_{\mathbb{C}}(\lambda)} := \int f\overline{g} \,\mathrm{d}\lambda$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. For $f \in L^2_{\mathbb{C}}(\lambda)$ define the Fourier transform of f as

$$\hat{f}(t) := \lim_{a\downarrow -\infty,\ b\uparrow\infty} \int_a^b f(x)e^{ixt} \,\mathrm{d}x, \tag{2.4}$$

where the limit is in $L^2_{\mathbb{C}}(\lambda)$. The Plancherel identity shows that for all $f, g \in L^2_{\mathbb{C}}(\lambda)$ we have $\langle \hat{f}, \hat{g}\rangle_{L^2_{\mathbb{C}}(\lambda)} = 2\pi \langle f, g\rangle_{L^2_{\mathbb{C}}(\lambda)}$. Moreover, for $f \in L^2_{\mathbb{C}}(\lambda)$ we have that $\hat{\hat{f}} = 2\pi f(-\cdot)$. Thus, the mapping $f \mapsto \hat{f}$ is (up to the factor $\sqrt{2\pi}$) a linear isometry from $L^2_{\mathbb{C}}(\lambda)$ onto $L^2_{\mathbb{C}}(\lambda)$. Furthermore, if $f \in L^2_{\mathbb{C}}(\lambda)$, then f is real valued if and only if $\overline{\hat{f}} = \hat{f}(-\cdot)$.

Let $\mathbb{C}_+$ denote the open upper half plane of the complex plane $\mathbb{C}$, i.e. $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$. An analytic function $H\colon \mathbb{C}_+ \to \mathbb{C}$ is a Hardy function if

$$\sup_{b>0} \int |H(a+ib)|^2 \,\mathrm{d}a < \infty. \tag{2.5}$$

Let $\mathbb{H}^2_+$ denote the space of all Hardy functions. It can be shown that a function $H\colon \mathbb{C}_+ \to \mathbb{C}$ is a Hardy function if and only if there exists a function $h \in L^2_{\mathbb{C}}(\lambda)$ which is 0 on $(-\infty, 0)$ and satisfies

$$H(z) = \int e^{izt} h(t) \,\mathrm{d}t, \qquad z \in \mathbb{C}_+. \tag{2.6}$$

In this case $\lim_{b\downarrow 0} H(a+ib) = \hat{h}(a)$ for λ-a.a. $a \in \mathbb{R}$ and in $L^2_{\mathbb{C}}(\lambda)$.

Let $H \in \mathbb{H}^2_+$ with h given by (2.6). Then H is called an outer function if it is non-trivial and for all $a + ib \in \mathbb{C}_+$ we have

$$\log(|H(a+ib)|) = \frac{b}{\pi} \int \frac{\log(|\hat{h}(u)|)}{(u-a)^2 + b^2} \,\mathrm{d}u. \tag{2.7}$$

An analytic function $J\colon \mathbb{C}_+ \to \mathbb{C}$ is called an inner function if $|J| \leq 1$ on $\mathbb{C}_+$ and with $j(a) := \lim_{b\downarrow 0} J(a+ib)$ for λ-a.a. $a \in \mathbb{R}$ we have $|j| = 1$ λ-a.s. For $H \in \mathbb{H}^2_+$ (with h

given by (2.6)) it is possible to factor H as a product of an outer function H^o and an inner function J. If h is a real function, J can be chosen such that $\overline{J(z)} = J(-\overline{z})$ for all $z \in \mathbb{C}_+$.

3 Main results

By S^1 we shall denote the unit circle in the complex field $\mathbb{C}$, i.e. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. For each measurable function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$ we define $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(t) := \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\,\mathrm{d}s, \tag{3.1}$$

where the limit is in λ-measure. The limit exists since for $a \geq 1$ we have

$$\int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\,\mathrm{d}s = \int_{-1}^{1} \frac{e^{its} - 1}{is} f(s)\,\mathrm{d}s + \int_{-a}^{a} e^{its} 1_{[-1,1]^c}(s) f(s)(is)^{-1}\,\mathrm{d}s, \tag{3.2}$$

and the last term converges in $L^2_{\mathbb{R}}(\lambda)$ to the Fourier transform of

$$s \mapsto 1_{[-1,1]^c}(s) f(s)(is)^{-1}. \tag{3.3}$$

Moreover, $\tilde{f}$ takes real values since $\overline{f} = f(-\cdot)$. Note that $\tilde{f}(t)$ is defined by integrating $f(s)$ against the kernel $(e^{its} - 1_{[-1,1]}(s))/is$, whereas the Fourier transform $\hat{f}(t)$ occurs by integration of $f(s)$ against e^{its}.

For $u \leq t$ we have

$$\tilde{f}(t + \cdot) - \tilde{f}(u + \cdot) = \widehat{1_{[u,t]} f}, \qquad \lambda\text{-a.s.} \tag{3.4}$$

Using this it follows that $\tilde{f}$ has orthogonal increments. To see this let $t_0 < t_1 < t_2 < t_3$ be given. Then

$$\langle \tilde{f}(t_3 - \cdot) - \tilde{f}(t_2 - \cdot), \tilde{f}(t_1 - \cdot) - \tilde{f}(t_0 - \cdot)\rangle_{L^2_{\mathbb{C}}(\lambda)} \tag{3.5}$$

$$= 2\pi \langle \hat{1}_{[t_2,t_3]} f, \hat{1}_{[t_0,t_1]} f\rangle_{L^2_{\mathbb{C}}(\lambda)} = \langle \hat{1}_{[t_2,t_3]}, \hat{1}_{[t_0,t_1]}\rangle_{L^2_{\mathbb{C}}(\lambda)} = \langle 1_{[t_2,t_3]}, 1_{[t_0,t_1]}\rangle_{L^2_{\mathbb{C}}(\lambda)} = 0, \tag{3.6}$$

which shows the result.

In the following let $t \mapsto \operatorname{sign}(t)$ denote the signum function defined by $\operatorname{sign}(t) = -1_{(-\infty,0)}(t) + 1_{(0,\infty)}(t)$. Let us calculate $\tilde{f}$ in three simple cases.

Example 3.1. We have the following:

(i) if $f \equiv 1$ then $\tilde{f}(t) = \pi \operatorname{sign}(t)$,

(ii) if $f(t) = (t + i)(t - i)^{-1}$ then $\tilde{f}(t) = 4\pi(e^{-t} - 1/2)1_{\mathbb{R}_+}(t)$,

(iii) if $f(t) = i\operatorname{sign}(t)$ then $\tilde{f}(t) = -2(\gamma + \log|t|)$, where γ denotes Euler's constant.

(i) follows since $\int_0^x \frac{\sin(s)}{s}\,\mathrm{d}s \to \pi/2$ as $x \to \infty$. Let f be given as in (ii). Then for all $t \in \mathbb{R}$ we have

$$\int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\,\mathrm{d}s = 4\int_0^a \frac{\cos(ts) - 1_{[0,1]}(s)}{s^2 + 1}\,\mathrm{d}s + 2\int_0^a \frac{\sin(ts)}{s}\frac{s^2 - 1}{s^2 + 1}\,\mathrm{d}s, \tag{3.7}$$

which converges to

$$\begin{cases} 4\frac{\pi}{4}(2e^{-t}-1)+2\frac{\pi}{2}(2e^{-t}-1)=2\pi(2e^{-t}-1), & t>0,\\ 4\frac{\pi}{4}(2e^{-t}-1)-2\frac{\pi}{2}(2e^{-t}-1)=0, & t<0, \end{cases} \tag{3.8}$$

as $a\to\infty$. This shows (ii).

Finally let $f(t)=i\mathrm{sign}(t)$. For $t>0$ and $a\geq 1$,

$$\int_{-a}^{a}\frac{e^{its}-1_{[-1,1]}(s)}{is}f(s)\,\mathrm{d}s=\int_{-a}^{a}\frac{\cos(ts)-1_{[-1,1]}(s)}{is}f(s)\,\mathrm{d}s \tag{3.9}$$

$$=2\int_0^{at}\frac{\cos(s)-1_{[0,t]}(s)}{is}f(s/t)\,\mathrm{d}s=2\Big(\int_0^{at}\frac{\cos(s)-1_{[0,1]}(s)}{s}\,\mathrm{d}s-\log(t)\Big), \tag{3.10}$$

which shows (iii) since $\tilde{f}(-t)=\tilde{f}(t)$. ◊

Let $(W_t)_{t\geq 0}$ be a standard Wiener process and $\phi,\psi\colon\mathbb{R}\to\mathbb{R}$ be two locally square integrable functions such that $\phi(t-\cdot)-\psi(-\cdot)\in L^2_{\mathbb{R}}(\lambda)$ for all $t\in\mathbb{R}$. In the following we let $(X_t)_{t\geq 0}$ be given by

$$X_t=\int(\phi(t-s)-\psi(-s))\,\mathrm{d}W_s,\qquad t\in\mathbb{R}. \tag{3.11}$$

Now we are ready to characterize the class of $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingales.

Theorem 3.2. *$(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if the following two conditions* (a) *and* (b) *are satisfied:*

(a) ϕ can be decomposed as

$$\phi(t)=\beta+\alpha\tilde{f}(t)+\int_0^t\widehat{f\hat{h}}(s)\,\mathrm{d}s,\qquad \lambda\text{-a.a. } t\in\mathbb{R}, \tag{3.12}$$

where $\alpha,\beta\in\mathbb{R}$, $f\colon\mathbb{R}\to S^1$ is a measurable function such that $\overline{f}=f(-\cdot)$, and $h\in L^2_{\mathbb{R}}(\lambda)$ is 0 on $\mathbb{R}_+$ when $\alpha\neq 0$.

(b) Let $\xi:=f\widehat{(\widehat{\phi-\psi})}$. If $\alpha\neq 0$ then

$$\int_0^r\Bigg(\frac{|\xi(s)|}{\sqrt{\int_s^\infty\xi(u)^2\,\mathrm{d}u}}\Bigg)\mathrm{d}s<\infty,\qquad\forall\, r>0, \tag{3.13}$$

where $\frac{0}{0}:=0$.

In this case $(X_t)_{t\geq 0}$ is a continuous $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale where the martingale component is a Wiener process with parameter $\sigma^2=(2\pi\alpha)^2$ and the bounded variation component is an absolutely continuous Gaussian process. In the case $X_0=0$ we may choose α,β,h and f such that the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is given by $X_t=M_t+A_t$, where

$$M_t=\alpha\int\Big(\tilde{f}(t-s)-\tilde{f}(-s)\Big)\,\mathrm{d}W_s\quad\text{and}\quad A_t=\int_0^t\Big(\int\widehat{f\hat{h}}(s-u)\,\mathrm{d}W_u\Big)\,\mathrm{d}s. \tag{3.14}$$

Furthermore, when $\alpha\neq 0$ and $X_0=0$, the law of $(\frac{1}{2\pi\alpha}X_t)_{t\in[0,T]}$ is equivalent to the Wiener measure on $C([0,T])$ for all $T>0$. ◊

The proof is given in Section 5. Let us note the following:

Remark 3.3.

1. The case $X_0 = 0$ corresponds to $\psi = \phi$. In this case condition (b) is always satisfied since we then have $\xi = 0$.

2. When $f \equiv 1$, (a) and (b) reduce to the conditions that ϕ is absolutely continuous on $\mathbb{R}_+$ with square integrable density and ϕ and ψ are constant on $(-\infty, 0)$. Hence by Cherny [7, Theorem 3.1] an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if we may choose $f \equiv 1$.

3. The condition imposed on ξ in (b) is the condition for expansion of filtration in Chaleyat-Maurel and Jeulin [5, Theoreme I.1.1].

Corollary 3.4. *Assume $X_0 = 0$. Then $(X_t)_{t\geq 0}$ is a Wiener process if and only if $\phi = \beta + \alpha \tilde{f}$, for some measurable function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$ and $\alpha, \beta \in \mathbb{R}$.*

The corollary shows that the mapping $f \mapsto \tilde{f}$ (up to affine transformations) is onto the space of functions with orthogonal increments (recall the definition on page 54). Moreover, if $f, g\colon \mathbb{R} \to S^1$ are measurable functions satisfying $\overline{f} = f(-\cdot)$ and $\overline{g} = g(-\cdot)$ and $\tilde{f} = \tilde{g}$ λ-a.s. then (3.4) shows that for $u \leq t$ we have

$$\hat{1}_{[u,t]} f = \hat{1}_{[u,t]} g, \qquad \lambda\text{-a.s.} \tag{3.15}$$

which implies $f = g$ λ-a.s. Thus, we have shown:

Remark 3.5. The mapping $f \mapsto \tilde{f}$ is one to one and (up to affine transformations) onto the space of functions with orthogonal increments.

For each measurable function $f\colon \mathbb{R} \to S^1$ such that $\overline{f} = f(-\cdot)$ and for each $h \in L^2_{\mathbb{R}}(\lambda)$ we have

$$\int_0^t \widehat{f\hat{h}}(s)\, \mathrm{d}s = \langle 1_{[0,t]}, \widehat{f\hat{h}} \rangle_{L^2_{\mathbb{C}}(\lambda)} = \langle \hat{1}_{[0,t]}, (f\hat{h})(-\cdot) \rangle_{L^2_{\mathbb{C}}(\lambda)} \tag{3.16}$$

$$= \langle \hat{1}_{[0,t]} f, \hat{h}(-\cdot) \rangle_{L^2_{\mathbb{C}}(\lambda)} = \langle \widehat{\hat{1}_{[0,t]} f}, h \rangle_{L^2_{\mathbb{C}}(\lambda)} = \int \big(\tilde{f}(t+s) - \tilde{f}(s)\big) h(s)\, \mathrm{d}s, \tag{3.17}$$

which gives an alternative way of writing the last term in (3.12).

In some cases it is of interest that $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted. This situation is studied in the next result. We also study the case where $(X_t)_{t\geq 0}$ is a stationary process, which corresponds to $\psi = 0$.

Proposition 3.6. *We have*

(i) Assume $\psi = 0$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if ϕ satisfies (a) *of Theorem 3.2 and $t \mapsto \alpha + \int_0^t h(-s)\, \mathrm{d}s$ is square integrable on $\mathbb{R}_+$ when $\alpha \neq 0$.*

(ii) Assume ψ equals 0 or ϕ and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. Then $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted if and only if we may choose f and h of Theorem 3.2 (a) *such that $f(a) = \lim_{b\downarrow 0} J(-a+ib)$ for λ-a.a. $a \in \mathbb{R}$, for some inner function J, and h is 0 on $\mathbb{R}_+$. In this case there exists a constant $c \in \mathbb{R}$ such that*

$$\phi = \beta + \alpha \tilde{f} + (\tilde{f} - c) * g, \qquad \lambda\text{-a.s.} \tag{3.18}$$

where $g = h(-\cdot)$.

According to Beurling [4] (see also Dym and McKean [11, page 53]), $J\colon \mathbb{C}_+ \to \mathbb{C}$ is an inner function if and only if it can be factorized as:

$$J(z) = Ce^{i\alpha z} \exp\Big(\frac{1}{\pi i}\int \frac{1+sz}{s-z}\,F(\mathrm{d}s)\Big)\prod_{n\geq 1}\epsilon_n\frac{z_n - z}{\overline{z}_n - z}\,, \tag{3.19}$$

where $C \in S^1$, $\alpha \geq 0$, $(z_n)_{n\geq 1} \subseteq \mathbb{C}_+$ satisfies $\sum_{n\geq 1}\Im(z_n)/(|z_n|^2{+}1) < \infty$ and $\epsilon_n = z_n/\overline{z}_n$ or 1 according as $|z_n| \leq 1$ or not, and F is a nondecreasing bounded singular function. Thus, a measurable function $f\colon \mathbb{R} \to S^1$ with $\overline{f} = f(-\cdot)$ satisfies the condition in Proposition 3.6 (ii) if and only if

$$f(a) = \lim_{b\downarrow 0} J(-a+ib), \qquad \lambda\text{-a.a. } a \in \mathbb{R}, \tag{3.20}$$

for a function J given by (3.19). If $f\colon \mathbb{R} \to S^1$ is given by $f(t) = i\mathrm{sign}(t)$, then according to Example 3.1, $\tilde{f}(t) = -2(\gamma + \log|t|)$. Thus this f does not satisfy the condition in Proposition 3.6 (ii).

In the next example we illustrate how to obtain (ϕ,ψ) for which $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale or a Wiener process (in its natural filtration). The idea is simply to pick a function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$ and calculate $\tilde{f}$. Moreover, if one wants $(X_t)_{t\geq 0}$ to be $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted one has to make sure that f is given as in (3.20).

Example 3.7. *Let $(X_t)_{t\geq 0}$ be given by*

$$X_t = \int (\phi(t-s) - \phi(-s))\,\mathrm{d}W_s, \qquad t \in \mathbb{R}. \tag{3.21}$$

(i) If ϕ is given by $\phi(t) = (e^{-t} - 1/2)1_{\mathbb{R}_+}(t)$ or $\phi(t) = \log|t|$ for all $t \in \mathbb{R}$, then $(X_t)_{t\geq 0}$ is a Wiener process (in its natural filtration).

(ii) If ϕ is given by

$$\phi(t) = \log|t| + \int_0^t \log\Big|\frac{s-1}{s}\Big|\,\mathrm{d}s, \qquad t \in \mathbb{R}, \tag{3.22}$$

then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

(i) is a consequence of Corollary 3.4 and Example 3.1 (ii)-(iii). To show (ii) let $f(t) = i\mathrm{sign}(t)$ as in Example 3.1 (iii). According to Theorem 3.2 it is enough to show

$$\widehat{f\hat{h}}(t) = \log\Big|\frac{t-1}{t}\Big|, \qquad t \in \mathbb{R}, \tag{3.23}$$

for some $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $\mathbb{R}_+$. Let $h(t) = 1_{[-1,0]}(t)$. Due to the fact that $\hat{h}(t) = \frac{1-\cos(t)}{it} + \frac{\sin(t)}{t}$, we have

$$\int_{-a}^{a} e^{its}\hat{h}(s)f(s)\,\mathrm{d}s = 2\Big(\int_0^a \frac{\cos(ts) - (\cos(ts)\cos(s) + \sin(ts)\sin(s))}{s}\,\mathrm{d}s\Big) \tag{3.24}$$

$$= 2\int_0^a \frac{\cos(ts) - \cos((t-1)s)}{s}\,\mathrm{d}s \tag{3.25}$$

$$= 2\int_0^{ta} \frac{\cos(s) - \cos(s(t-1)/t)}{s}\,\mathrm{d}s \to 2\log\Big|\frac{t-1}{t}\Big| \tag{3.26}$$

as $a \to \infty$, for all $t \in \mathbb{R}\setminus\{0,1\}$. This shows that $h/2$ satisfies (3.23) and the proof of (ii) is complete. $\Diamond$

As a consequence of Example 3.7 (i) we have the following: Let $(X_t)_{t\geq 0}$ be the stationary Ornstein-Uhlenbeck process given by

$$X_t = X_0 - \int_0^t X_s \,\mathrm{d}s + W_t, \qquad t \geq 0, \tag{3.27}$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process and $X_0 \overset{\mathscr{D}}{=} N(0, 1/2)$ is independent of $(W_t)_{t\geq 0}$. Then $(B_t)_{t\geq 0}$, given by

$$B_t := W_t - 2\int_0^t X_s \,\mathrm{d}s, \qquad t \geq 0, \tag{3.28}$$

is a Wiener process (in its natural filtration). Representations of the Wiener process have been extensively studied by Lévy [18], Cramér [8], Hida [13] and many others. One famous example of such a representation is

$$B_t = W_t - \int_0^t \frac{1}{s} W_s \,\mathrm{d}s, \qquad t \geq 0, \tag{3.29}$$

see Jeulin and Yor [14].

Let $X_t = \int (\phi(t-s) - \phi(-s)) \,\mathrm{d}W_s$ for $t \in \mathbb{R}$. Then ϕ has to be continuous on $[0, \infty)$ (in particular bounded on compacts of $\mathbb{R}$) for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. This is not the case for the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale property. Indeed, Example 3.7 shows that if $\phi(t) = \log|t|$ then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale, but ϕ is unbounded on $[0, 1]$.

4 Functions with orthogonal increments

In the following we collect some properties of functions with orthogonal increments. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with orthogonal increments. For $t \in \mathbb{R}$ we have

$$\begin{aligned} &\|\tau_t f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)} && (4.1)\\ &\quad = \|\tau_t f - \tau_{t/2} f\|^2_{L^2_{\mathbb{R}}(\lambda)} + \|\tau_{t/2} f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)} = 2\|\tau_{t/2} f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)}. && (4.2) \end{aligned}$$

Moreover, since $t \mapsto \|\tau_t f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)}$ is continuous by Remark 2.1 (recall that f by definition is locally square integrable), equation (4.1) shows that $\|\tau_t f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)} = K|t|$, where $K := \|\tau_1 f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)}$. This implies that $\|\tau_t f - \tau_u f\|^2_{L^2_{\mathbb{R}}(\lambda)} = K|t-u|$ for $u, t \in \mathbb{R}$. For a step function $h = \sum_{j=1}^k a_j 1_{(t_{j-1}, t_j]}$ define the mapping

$$\int h(u) \,\mathrm{d}\tau_u f := \sum_{j=1}^k a_j (\tau_{t_j} f - \tau_{t_{j-1}} f). \tag{4.3}$$

Then $v \mapsto (\int h(u) \,\mathrm{d}\tau_u f)(v)$ is square integrable and

$$\sqrt{K} \|h\|_{L^2_{\mathbb{R}}(\lambda)} = \|\int h(u) \,\mathrm{d}\tau_u f\|_{L^2_{\mathbb{R}}(\lambda)}. \tag{4.4}$$

Hence, by standard arguments we can define $\int h(u) \,\mathrm{d}\tau_u f$ through the above isometry for all $h \in L^2_{\mathbb{R}}(\lambda)$ such that $h \mapsto \int h(u) \,\mathrm{d}\tau_u f$ is a linear isometry from $L^2_{\mathbb{R}}(\lambda)$ into $L^2_{\mathbb{R}}(\lambda)$.

Assume that $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a measurable function, and μ is a finite measure such that

$$\int\int g(u,v)^2 \,\mathrm{d}u\,\mu(\mathrm{d}v) < \infty. \tag{4.5}$$

Then $(v,s) \mapsto (\int g(u,v)\,\mathrm{d}\tau_u f)(s)$ can be chosen measurable and in this case we have

$$\int \Big(\int g(u,v)\,\mathrm{d}\tau_u f\Big)\mu(\mathrm{d}v) = \int \Big(\int g(u,v)\,\mu(\mathrm{d}v)\Big)\mathrm{d}\tau_u f. \tag{4.6}$$

Lemma 4.1. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by*

$$g(t) = \begin{cases} \alpha + \int_0^t h(v)\,\mathrm{d}v & t \geq 0 \\ 0 & t < 0, \end{cases} \tag{4.7}$$

where $\alpha \in \mathbb{R}$ and $h \in L^2_{\mathbb{R}}(\lambda)$. Then, $g(t-\cdot) - g(-\cdot) \in L^2_{\mathbb{R}}(\lambda)$ for all $t \in \mathbb{R}$.

Let f be a function with orthogonal increments.

(i) Let ϕ be a measurable function. Then there exists a constant $\beta \in \mathbb{R}$ such that

$$\phi(t) = \beta + \alpha f(t) + \int_0^\infty \big(f(t-v) - f(-v)\big)h(v)\,\mathrm{d}v, \quad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{4.8}$$

if and only if for all $t \in \mathbb{R}$ we have

$$\tau_t\phi - \tau_0\phi = \int (g(t-u) - g(-u))\,\mathrm{d}\tau_u f, \qquad \lambda\text{-a.s.} \tag{4.9}$$

(ii) Assume g is square integrable. Then there exists a $\beta \in \mathbb{R}$ such that λ-a.s.

$$\int g(-u)\,\mathrm{d}\tau_u f = \beta + \alpha f(-\cdot) + \int_0^\infty \big(f(-u-\cdot) - f(-u)\big)h(u)\,\mathrm{d}u. \tag{4.10}$$

Proof. From Jensen's inequality and Tonelli's Theorem it follows that

$$\int \Big(\int_{-s}^{t-s} h(u)\,\mathrm{d}u\Big)^2 \mathrm{d}s \leq t \int \Big(\int_{-s}^{t-s} h(u)^2\,\mathrm{d}u\Big)\,\mathrm{d}s = t^2 \int h(s)^2\,\mathrm{d}u < \infty, \tag{4.11}$$

which shows $g(t-\cdot) - g(-\cdot) \in L^2_{\mathbb{R}}(\lambda)$.

(i): We may and do assume that h is 0 on $(-\infty, 0)$. For $t, u \in \mathbb{R}$ we have

$$g(t-u) - g(-u) = \begin{cases} \alpha 1_{(0,t]}(u) + \int_{-u}^{t-u} h(v)\,\mathrm{d}v, & t \geq 0, \\ -\alpha 1_{(t,0]}(u) - \int_{t-u}^{-u} h(v)\,\mathrm{d}v, & t < 0, \end{cases} \tag{4.12}$$

which by (4.6) implies that for $t \in \mathbb{R}$ we have λ-a.s.

$$\int (g(t-u) - g(-u))\,\mathrm{d}\tau_u f = \alpha(\tau_t f - \tau_0 f) + \int (\tau_{t-v} f - \tau_{-v} f)\,h(v)\,\mathrm{d}v. \tag{4.13}$$

First assume (4.9) is satisfied. For $t \in \mathbb{R}$ it follows from (4.13) that

$$\tau_t\phi - \tau_0\phi = \alpha(\tau_t f - \tau_0 f) + \int (\tau_{t-v} f - \tau_{-v} f)\,h(v)\,\mathrm{d}v, \qquad \lambda\text{-a.s.} \tag{4.14}$$

Hence, by Tonelli's Theorem there exists a sequence $(s_n)_{n\geq 1}$ such that $s_n \to 0$ and such that

$$\phi(t-s_n) = \phi(-s_n) - \alpha f(s_n) + \alpha f(t-s_n) \tag{4.15}$$

$$+ \int (f(t-v-s_n) - f(-v-s_n))\, h(v)\, \mathrm{d}v, \qquad \forall n \geq 1,\ \lambda\text{-a.a. } t \in \mathbb{R}. \tag{4.16}$$

From Remark 2.1 it follows that $\phi(\cdot - s_n) - \phi(\cdot)$ and $f(\cdot - s_n) - f(\cdot)$ converge to 0 in $L^2_{\mathbb{R}}(\lambda)$ and

$$\int \big(f(t-v-s_n) - f(-v-s_n)\big) h(v)\, \mathrm{d}v \to \int [f(t-v) - f(-v)] h(v)\, \mathrm{d}v, \qquad t \in \mathbb{R}. \tag{4.17}$$

Thus we obtain (4.10) by letting n tend to infinity in (4.15).

Assume conversely (4.8) is satisfied. For $t \in \mathbb{R}$ we have

$$\tau_t\phi - \tau_0\phi = \alpha(\tau_t f - \tau_0 f) + \int (\tau_{t-v} f - \tau_{-v} f)\, h(v)\, \mathrm{d}v, \qquad \lambda\text{-a.s.} \tag{4.18}$$

and hence we obtain (4.9) from (4.13).

(ii): Assume in addition that $g \in L^2_{\mathbb{R}}(\lambda)$. By approximation we may assume h has compact support. Choose $T > 0$ such that h is 0 outside $(0,T)$. Since $g \in L^2_{\mathbb{R}}(\lambda)$, it follows that $\alpha = -\int_0^T h(s)\, \mathrm{d}s$ and therefore g is on the form

$$g(t) = -1_{[0,T]}(t) \int_t^T h(s)\, \mathrm{d}s, \qquad t \in \mathbb{R}. \tag{4.19}$$

From (4.6) it follows that

$$\int g(-u)\, \mathrm{d}\tau_u f = \int \left(\int -1_{(-u,T]}(s) 1_{[0,T]}(-u) h(s)\, \mathrm{d}s \right) \mathrm{d}\tau_u f \tag{4.20}$$

$$= \int \left(\int -1_{(-u,T]}(s) 1_{[0,T]}(-u) h(s)\, \mathrm{d}\tau_u f \right) \mathrm{d}s = \int_0^T -h(s) \left(\int_{-s}^0 \mathrm{d}\tau_u f \right) \mathrm{d}s \tag{4.21}$$

$$= \int_0^T -h(s)\, (\tau_0 f - \tau_{-s} f)\, \mathrm{d}s = \alpha \tau_0 f + \int_0^T h(s) \tau_{-s} f\, \mathrm{d}s. \tag{4.22}$$

Thus, if we let $\beta := \int_0^T h(s) f(-s)\, \mathrm{d}s$, then

$$\int g(-u)\, \mathrm{d}\tau_u f = \beta + \alpha f(-\cdot) + \int h(s)\, (f(-s-\cdot) - f(-s))\, \mathrm{d}s, \tag{4.23}$$

which completes the proof. □

Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function with orthogonal increments and let $(B_t)_{t\geq 0}$ be given by

$$B_t = \int (f(t-s) - f(-s))\, \mathrm{d}W_s, \qquad t \in \mathbb{R}. \tag{4.24}$$

Then it follows that $(B_t)_{t\geq 0}$ is a Wiener process and

$$\int q(s)\, \mathrm{d}B_s = \int \Big(\int q(u)\, \mathrm{d}\tau_u f \Big)(s)\, \mathrm{d}W_s, \qquad \forall\, q \in L^2_{\mathbb{R}}(\lambda). \tag{4.25}$$

This is obvious when q is a step function and hence by approximation it follows that (4.25) is true for all $q \in L^2_{\mathbb{R}}(\lambda)$.

Let $f\colon \mathbb{R} \to S^1$ denote a measurable function satisfying $\overline{f} = f(-\cdot)$. Then

$$\int q(u)\,\mathrm{d}\tau_u \tilde{f} = (\widehat{qf})(-\cdot), \qquad \forall\, q \in L^2_{\mathbb{R}}(\lambda). \tag{4.26}$$

To see this assume first q is a step function on the form $\sum_{j=1}^k a_j 1_{(t_{j-1},t_j]}$. Then

$$\Big(\int q(u)\,\mathrm{d}\tau_u \tilde{f}\Big)(s) = \sum_{j=1}^k a_j \left(\tilde{f}(t_j - s) - \tilde{f}(t_{j-1} - s)\right) \tag{4.27}$$

$$= \int \sum_{j=1}^k a_j \frac{e^{it_j u} - e^{it_{j-1}u}}{iu} f(u) e^{-isu}\,\mathrm{d}u = \int \widehat{q}(u) f(u) e^{-isu}\,\mathrm{d}u = (\widehat{qf})(-s), \tag{4.28}$$

which shows that (4.26) is valid for step functions and hence the result follows for general $q \in L^2_{\mathbb{R}}(\lambda)$ by approximation. Thus, if $(B_t)_{t\geq 0}$ is given by $B_t = \int (\tilde{f}(t-s) - \tilde{f}(-s))\,\mathrm{d}W_s$ for all $t \in \mathbb{R}$, then by combining (4.25) and (4.26) we have

$$\int q(s)\,\mathrm{d}B_s = \int (\widehat{qf})(-s)\,\mathrm{d}W_s, \qquad \forall\, q \in L^2_{\mathbb{R}}(\lambda). \tag{4.29}$$

Lemma 4.2. *Let $f\colon \mathbb{R} \to S^1$ be a measurable function such that $\overline{f} = f(-\cdot)$. Then $\tilde{f}$ is constant on $(-\infty, 0)$ if and only if there exists an inner function J such that*

$$f(a) = \lim_{b\downarrow 0} J(-a + ib), \qquad \lambda\text{-a.a. } a \in \mathbb{R}. \tag{4.30}$$

Proof. Assume $\tilde{f}$ is constant on $(-\infty, 0)$ and let $t \geq 0$ be given. We have $\widehat{\hat{1}_{[0,t]}\overline{f}}(-s) = 0$ for λ-a.a. $s \in (-\infty, 0)$ due to the fact that $\widehat{\hat{1}_{[0,t]}\overline{f}}(-s) = \tilde{f}(s) - \tilde{f}(-t+s)$ for λ-a.a. $s \in \mathbb{R}$ and hence $\hat{1}_{[0,t]}\overline{f} \in \mathbb{H}^2_+$. Moreover, since $\hat{1}_{[0,t]}\overline{f}$ has outer part $\hat{1}_{[0,t]}$ we conclude that $\overline{f}(a) = \lim_{b\downarrow 0} J(a+ib)$ for λ-a.a. $a \in \mathbb{R}$ and an inner function $J\colon \mathbb{C}_+ \to \mathbb{C}$.

Assume conversely (4.30) is satisfied and fix $t \geq 0$. Let $G \in \mathbb{H}^2_+$ be the Hardy function induced by $1_{[0,t]}$. Since J is an inner function, we obtain $GJ \in \mathbb{H}^2_+$ and thus

$$G(z)J(z) = \int e^{itz} \kappa(t)\,\mathrm{d}t, \qquad z \in \mathbb{C}_+, \tag{4.31}$$

for some $\kappa \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$. The remark just below (2.6) shows

$$\widehat{1_{[0,t]}}(a)\overline{f}(a) = \lim_{b\downarrow 0} G(a+ib)J(a+ib) = \hat{\kappa}(a), \qquad \lambda\text{-a.a. } a \in \mathbb{R}, \tag{4.32}$$

which implies

$$\tilde{f}(s) - \tilde{f}(-t+s) = \widehat{\hat{1}_{[0,t]}\overline{f}}(-s) = \hat{\hat{\kappa}}(-s) = 2\pi k(s), \tag{4.33}$$

for λ-a.a. $s \in \mathbb{R}$. Hence, we conclude that $\tilde{f}$ is constant on $(-\infty, 0)$ λ-a.s. $\square$

5 Proofs of main results

Let $(X_t)_{t\geq 0}$ denote a stationary Gaussian process. Following Doob [10], $(X_t)_{t\geq 0}$ is called deterministic if $\overline{\text{span}}\{X_t : t \in \mathbb{R}\}$ equals $\overline{\text{span}}\{X_t : t \leq 0\}$ and when this is not the case $(X_t)_{t\geq 0}$ is called regular. Let $\phi \in L^2_{\mathbb{R}}(\lambda)$ and let $(X_t)_{t\geq 0}$ be given by $X_t = \int \phi(t-s)\,\mathrm{d}W_s$ for all $t \in \mathbb{R}$. By the Plancherel identity $(X_t)_{t\geq 0}$ has spectral measure given by $(2\pi)^{-1}|\hat{\phi}|^2\,\mathrm{d}\lambda$. Thus according to Szegö's Alternative (see Dym and McKean [11, page 84]), $(X_t)_{t\geq 0}$ is regular if and only if

$$\int \frac{\log|\hat{\phi}|(u)}{1+u^2}\,\mathrm{d}u > -\infty. \tag{5.1}$$

In this case the remote past $\cap_{t<0}\sigma(X_s : s < t)$ is trivial and by Karhunen [16, Satz 5] (or Doob [10, Chapter XII, Theorem 5.3]) we have

$$X_t = \int_{-\infty}^{t} g(t-s)\,\mathrm{d}B_s,\ t \in \mathbb{R} \qquad \text{and} \qquad (\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}, \tag{5.2}$$

for some Wiener process $(B_t)_{t\geq 0}$ and some $g \in L^2_{\mathbb{R}}(\lambda)$. However, we need the following explicit construction of $(B_t)_{t\geq 0}$.

Lemma 5.1 (Main Lemma). *Let $\phi \in L^2_{\mathbb{R}}(\lambda)$ and $(X_t)_{t\geq 0}$ be given by $X_t = \int \phi(t-s)\,\mathrm{d}W_s$ for $t \in \mathbb{R}$, where $(W_t)_{t\geq 0}$ is a Wiener process.*

(i) If $(X_t)_{t\geq 0}$ is a regular process then there exist a measurable function $f\colon \mathbb{R} \to S^1$ with $\overline{f} = f(-\cdot)$, a function $g \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$ such that we have the following: First of all $(B_t)_{t\geq 0}$ defined by

$$B_t = \int \Big(\tilde{f}(t-s) - \tilde{f}(-s)\Big)\,\mathrm{d}W_s, \qquad t \in \mathbb{R}, \tag{5.3}$$

is a Wiener process. Moreover,

$$X_t = \int_{-\infty}^{t} g(t-s)\,\mathrm{d}B_s, \qquad t \in \mathbb{R}, \tag{5.4}$$

and finally $(\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$.

(ii) If ϕ is 0 on $(-\infty, 0)$ and $\phi \neq 0$, then $(X_t)_{t\geq 0}$ is regular and the above f is given by $f(a) = \lim_{b\downarrow 0} J(-a+ib)$ for λ-a.a. $a \in \mathbb{R}$, where J is an inner function.

Proof. (i): Due to the fact that $|\hat{\phi}|^2$ is a positive integrable function which satisfies (5.1), Dym and McKean [11, Chapter 2, Section 7, Exercise 4] shows there is an outer Hardy function $H^o \in \mathbb{H}^2_+$ such that $|\hat{\phi}|^2 = |\hat{h}^0|^2$ and $\overline{\hat{h}^o} = \hat{h}^o(-\cdot)$, where h^0 is given by (2.6). Additionally, H^o is given by

$$H^o(z) = \exp\Big(\frac{1}{\pi i}\int \frac{uz+1}{u-z}\frac{\log|\hat{\phi}|(u)}{u^2+1}\,\mathrm{d}u\Big), \qquad z \in \mathbb{C}_+. \tag{5.5}$$

Define $f\colon \mathbb{R} \to S^1$ by $\overline{f} = \hat{\phi}/\hat{h}^o$ and note that $\overline{f} = f(-\cdot)$. Let $(B_t)_{t\geq 0}$ be given by (5.3), then $(B_t)_{t\geq 0}$ is a Wiener process due to the fact that $\tilde{f}$ has orthogonal increments. Moreover, by definition of f we have $\widehat{\tau_t h^o} f = \widehat{\tau_t \phi}$, which shows that

$$\widehat{(\widehat{\tau_t h^o} f)} = 2\pi\tau_t\phi(-\cdot). \tag{5.6}$$

Thus if we let $g := (2\pi)^{-1}h^o$, then $g \in L^2_{\mathbb{R}}(\lambda)$ and (5.4) follows by (4.29) and (5.6). Furthermore, since H^o is an outer function we have $(\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$ according to page 95 in Dym and McKean [11].

(ii): Assume $\phi \in L^2_{\mathbb{R}}(\lambda)$ is 0 on $(-\infty, 0)$ and $\phi \neq 0$. By definition $(X_t)_{t\geq 0}$ is clearly regular. Let h^o, f and $(B_t)_{t\geq 0}$ be given as above (recall that $\overline{f} = f(-\cdot)$). It follows by Dym and McKean [11, page 37] that $J := H/H^o$ is an inner function and by definition of J, $f(-a) = \lim_{b\downarrow 0} J(a+ib)$ for λ-a.a. $a \in \mathbb{R}$, which completes the proof. □

The following lemma is related to Hardy and Littlewood [12, Theorem 24] and hence the proof is omitted.

Lemma 5.2. *Let κ be a locally integrable function and let $\Delta_t\kappa$ denote the function*

$$s \mapsto t^{-1}(\kappa(t+s) - \kappa(s)), \qquad t > 0. \tag{5.7}$$

Then $(\Delta_t\kappa)_{t>0}$ is bounded in $L^2_{\mathbb{R}}(\lambda)$ if and only if κ is absolutely continuous with square integrable density.

The following simple, but nevertheless useful, lemma is inspired by Masani [19] and Cheridito [6].

Lemma 5.3. *Let $(X_t)_{t\geq 0}$ denote a continuous and centered Gaussian process with stationary increments. Then there exists a continuous, stationary and centered Gaussian process $(Y_t)_{t\geq 0}$, satisfying*

$$Y_t = X_t - e^{-t}\int_{-\infty}^t e^s X_s \,\mathrm{d}s \quad \text{and} \quad X_t - X_0 = Y_t - Y_0 + \int_0^t Y_s \,\mathrm{d}s, \tag{5.8}$$

for all $t \in \mathbb{R}$, and $\mathcal{F}_t^{X,\infty} = \sigma(X_0) \vee \mathcal{F}_t^{Y,\infty}$ for all $t \geq 0$.

Furthermore, if $(X_t)_{t\geq 0}$ is given by (3.11),

$$\kappa(t) := \int_{-\infty}^0 e^u\big(\phi(t) - \phi(u+t)\big)\,\mathrm{d}u, \qquad t \in \mathbb{R}, \tag{5.9}$$

is a well-defined square integrable function and $(Y_t)_{t\geq 0}$ is given by $Y_t = \int \kappa(t-s)\,\mathrm{d}W_s$ for $t \in \mathbb{R}$.

The proof is simple and hence omitted.

Remark 5.4. A càdlàg Gaussian process $(X_t)_{t\geq 0}$ with stationary increments has P-a.s. continuous sample paths. Indeed, this follows from Adler [1, Theorem 3.6] since $\mathrm{P}(\Delta X_t = 0) = 1$ for all $t \geq 0$ by the stationary increments.

Proof of Theorem 3.2. If: Assume (a) and (b) are satisfied. We show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

(1): The case $\alpha \neq 0$. Let $(B_t)_{t\geq 0}$ denote the Wiener process given by

$$B_t := \int \Big(\tilde{f}(t-s) - \tilde{f}(-s)\Big)\,\mathrm{d}W_s, \qquad t \in \mathbb{R}, \tag{5.10}$$

and let $g\colon \mathbb{R} \to \mathbb{R}$ be given by

$$g(t) = \begin{cases} \alpha + \int_0^t h(-u)\,\mathrm{d}u & t \geq 0 \\ 0 & t < 0. \end{cases} \tag{5.11}$$

Since ϕ satisfies (3.12) it follows by (3.16), Lemma 4.1 and (4.25) that

$$X_t - X_0 = \int (\tau_t\phi(s) - \tau_0\phi(s))\,\mathrm{d}W_s = \int (g(t-s) - g(-s))\,\mathrm{d}B_s, \qquad t \in \mathbb{R}. \tag{5.12}$$

From Cherny [7, Theorem 3.1] it follows that $(X_t - X_0)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-semimartingale with martingale component $(\alpha B_t)_{t\geq 0}$. Let $k = (2\pi)^{-2}\xi \in L^2_{\mathbb{R}}(\lambda)$ (ξ is given in (b)). Since $\widehat{\hat{k}f} = \phi - \psi$ it follows by (4.29) that $X_0 = \int k(s)\,\mathrm{d}B_s$. Moreover, since k satisfies (3.13) it follows from Chaleyat-Maurel and Jeulin [5, Theoreme I.1.1] that $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t^B \vee \sigma(\int_0^\infty k(s)\,\mathrm{d}B_s))_{t\geq 0}$-semimartingale and since $\mathcal{F}_t^B \vee \sigma(\int_0^\infty k(s)\,\mathrm{d}B_s) \vee \sigma(B_u : u \leq 0) = \mathcal{F}_t^{B,\infty} \vee \sigma(X_0)$, $(B_t)_{t\geq 0}$ is also an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale. Thus we conclude that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale and hence also an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, since $\mathcal{F}_t^{X,\infty} \subseteq \mathcal{F}_t^{B,\infty} \vee \sigma(X_0)$ for all $t \geq 0$.

(2): The case $\alpha = 0$. Let us argue as in Cherny [7, page 8]. Since ϕ is absolutely continuous with square integrable density, Lemma 5.2 implies

$$\mathrm{E}[(X_t - X_u)^2] = \int \big(\phi(t-s) - \phi(u-s)\big)^2 \mathrm{d}s \leq K|t-u|^2, \qquad t, u \geq 0, \tag{5.13}$$

for some constant $K \in \mathbb{R}_+$. The Kolmogorov-Čentsov Theorem shows that $(X_t)_{t\geq 0}$ has a continuous modification and from (5.13) it follows that this modification is of integrable variation. Hence $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Only if: Assume conversely that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale and hence continuous, according to Remark 5.4.

(3): First assume (in addition) that $(X_t)_{t\geq 0}$ is of unbounded variation. Let κ and $(Y_t)_{t\geq 0}$ be given as in Lemma 5.3. Since

$$Y_t = X_t - e^{-t}\int_{-\infty}^t e^s X_s\,\mathrm{d}s, \quad t \geq 0, \quad \text{and} \quad (\mathcal{F}_t^{Y,\infty} \vee \sigma(X_0))_{t\geq 0} = (\mathcal{F}_t^{X,\infty})_{t\geq 0}, \tag{5.14}$$

we deduce that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale of unbounded variation. This implies that $\mathcal{F}_0^{Y,\infty} \neq \mathcal{F}_\infty^{Y,\infty}$ and we conclude that $(Y_t)_{t\geq 0}$ is regular. Now choose f and g according to Lemma 5.1 (with (ϕ, X) replaced by (κ, Y)) and let $(B_t)_{t\geq 0}$ be given as in the lemma such that

$$Y_t = \int_{-\infty}^t g(t-s)\,\mathrm{d}B_s, \quad t \in \mathbb{R}, \quad \text{and} \quad (\mathcal{F}_t^{Y,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}. \tag{5.15}$$

Since $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-semimartingale, Knight [17, Theorem 6.5] shows that

$$g(t) = \alpha + \int_0^t \zeta(u)\,\mathrm{d}u, \qquad t \geq 0, \tag{5.16}$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$ and some $\zeta \in L^2_{\mathbb{R}}(\lambda)$ and the $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-martingale component of $(Y_t)_{t\geq 0}$ is $(\alpha B_t)_{t\geq 0}$. Equation (5.14) shows that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale, and since $(\mathcal{F}_t^{Y,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$, $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale. Hence $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale. As in (1) we have $X_0 = \int k(s)\,\mathrm{d}B_s$ where $k := (2\pi)^{-2}\xi$. Since $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale and $\mathcal{F}_t^B \vee \sigma(\int_0^\infty k(s)\,\mathrm{d}B_s) \subseteq \mathcal{F}_t^{B,\infty} \vee \sigma(X_0)$, $(B_t)_{t\geq 0}$ is also a semimartingale with respect to $(\mathcal{F}_t^B \vee \sigma(\int_0^\infty k(s)\,\mathrm{d}B_s))_{t\geq 0}$. Thus according to Chaleyat-Maurel and Jeulin [5, Theoreme I.1.1] k satisfies (3.13) which shows condition (b). From this theorem it follows that the bounded variation component is an absolutely continuous Gaussian process and the martingale component is a Wiener process with parameter $\sigma^2 = (2\pi\alpha)^2$. Let $\eta := \zeta + g$ and let ρ be given by

$$\rho(t) = \alpha + \int_0^t \eta(u)\,\mathrm{d}u, \quad t \geq 0, \quad \text{and} \quad \rho(t) = 0, \qquad t < 0. \tag{5.17}$$

For all $t \in \mathbb{R}$ we have

$$X_t - X_0 = Y_t - Y_0 - \int_0^t Y_u\,\mathrm{d}u = Y_t - Y_0 - \int \left(\int_0^t g(u-s)\,\mathrm{d}u \right) \mathrm{d}B_s \tag{5.18}$$

$$= \int \left(g(t-s) - g(-s) + \int_{-s}^{t-s} g(u)\,\mathrm{d}u \right) \mathrm{d}B_s = \int (\rho(t-s) - \rho(-s))\,\mathrm{d}B_s, \tag{5.19}$$

where the second equality follows from Protter [20, Chapter IV, Theorem 65]. Thus from (4.25) we have

$$\tau_t\phi - \tau_0\phi = \int (\rho(t-u) - \rho(-u))\,\mathrm{d}\tau_u\tilde{f}, \qquad \lambda\text{-a.s. } \forall t \in \mathbb{R}, \tag{5.20}$$

which by Lemma 4.1 (i) implies

$$\phi(t) = \beta + \alpha\tilde{f}(t) + \int_0^\infty \left(\tilde{f}(t-v) - \tilde{f}(-v) \right) \eta(v)\,\mathrm{d}v, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{5.21}$$

for some $\beta \in \mathbb{R}$. We obtain (3.12) (with $h = \eta(-\cdot)$) by (3.16). This completes the proof of (a).

Let us study the canonical decomposition of $(X_t)_{t\geq 0}$ in the case $X_0 = 0$. For $t \geq 0$ we have

$$X_t - X_0 = \alpha B_t + \int \left(\int_{-s}^{t-s} \widehat{f\hat{h}}(u)\,\mathrm{d}u \right) \mathrm{d}W_s = \alpha B_t + \int_0^t \left(\int \widehat{f\hat{h}}(s-u)\,\mathrm{d}W_u \right) \mathrm{d}s, \tag{5.22}$$

and by (4.29) we have

$$\int \widehat{f\hat{h}}(s-u)\,\mathrm{d}W_u = \int h(u-s)\,\mathrm{d}B_u. \tag{5.23}$$

Recall that $(\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$. From (5.23) it follows that the last term of (5.22) is $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-adapted and hence the canonical $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-decomposition of $(X_t)_{t\geq 0}$ is given by (5.22). Furthermore, by combining (5.22) and (5.23), Cheridito [6, Proposition 3.7] shows that the law of $(\frac{1}{2\pi\alpha}X_t)_{t\in[0,T]}$ is equivalent to the Wiener measure on $C([0,T])$ for all $T > 0$, when $X_0 = 0$.

(4) : Assume $(X_t)_{t\geq 0}$ is of bounded variation and therefore of integrable variation (see Stricker [24]). By Lemma 5.2 we conclude that ϕ is absolutely continuous with square integrable density and hereby on the form (3.12) with $\alpha = 0$ and $f \equiv 1$. This completes the proof. □

Proof of Proposition 3.6. To prove (ii) assume that ψ equals 0 or ϕ and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Only if: Assume $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted. By studying $(X_t - X_0)_{t\geq 0}$ we may and do assume that $\psi = \phi$. Furthermore, it follows that ϕ is constant on $(-\infty, 0)$ since $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted. Let us first assume that $(X_t)_{t\geq 0}$ is of bounded variation.

By arguing as in (4) in the proof of Theorem 3.2 it follows that ϕ is on the form (3.12) where h is 0 on $\mathbb{R}_+$ and $f \equiv 1$ (these h and f satisfies the additional conditions in (ii)). Second assume $(X_t)_{t\geq 0}$ is of unbounded variation. Proceed as in (3) in the proof of Theorem 3.2. Since ϕ is constant on $(-\infty, 0)$ it follows by (5.9) that κ is 0 on $(-\infty, 0)$. Thus according to Lemma 5.1 (ii), f is given by $f(a) = \lim_{b\downarrow 0} J(-a + ib)$ for some inner function J and the proof of the *only if* part is complete.

If: According to Lemma 4.2, $\tilde{f}$ is constant on $(-\infty, 0)$ λ-a.s. and from (3.16) it follows that (recall that h is 0 on $\mathbb{R}_+$)

$$\int_0^t \widehat{f\hat{h}}(s)\,\mathrm{d}s = \int_{-\infty}^0 \Big(\tilde{f}(t+s) - \tilde{f}(s)\Big)\, h(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}. \tag{5.24}$$

This shows that ϕ is constant on $(-\infty, 0)$ λ-a.s. and hence $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted since ψ equals 0 or ϕ.

To prove (3.18) assume that ϕ is represented as in (3.12) with $f(a) = \lim_{b\downarrow 0} J(-a+b)$ for λ-a.a. $a \in \mathbb{R}$ for some inner function J and h is 0 on $\mathbb{R}_+$. Lemma 4.2 shows that there exists a constant $c \in \mathbb{R}$ such that $\tilde{f} = c$ λ-a.s. on $(-\infty, 0)$. Let $g := h(-\cdot)$. By (3.16) we have

$$\begin{aligned} \int_0^t \widehat{f\hat{h}}(s)\,\mathrm{d}s &= \int \Big(\tilde{f}(t-s) - \tilde{f}(-s)\Big)\, g(s)\,\mathrm{d}s \qquad &(5.25)\\ &= \int \Big(\tilde{f}(t-s) - c\Big)\, g(s)\,\mathrm{d}s = \Big((\tilde{f} - c) * g\Big)(t), \qquad &(5.26) \end{aligned}$$

where the third equality follows from the fact that g only differs from 0 on $\mathbb{R}_+$ and on this set $\tilde{f}(-\cdot)$ equals c. This shows (3.18).

To show (i) assume $\psi = 0$.

Only if: We may and do assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale of unbounded variation. We have to show that we can decompose ϕ as in (a) of Theorem 3.2 where $\alpha + \int_0^\cdot h(-s)\,\mathrm{d}s$ is square integrable on $\mathbb{R}_+$. However, this follows as in (3) in the proof of Theorem 3.2 (without referring to Lemma 5.3).

If: Assume (a) of Theorem 3.2 is satisfied with α, β, h and f and that g defined by

$$g(t) = \begin{cases} \alpha + \int_0^t h(-v)\,\mathrm{d}v & t \geq 0 \\ 0 & t < 0, \end{cases} \tag{5.27}$$

is square integrable. From Lemma 4.1 (ii) it follows that there exists a $\tilde{\beta} \in \mathbb{R}$ such that

$$\int g(-u)\,\mathrm{d}\tau_u \tilde{f} = \tilde{\beta} + \alpha \tilde{f}(-\cdot) + \int \big(\tilde{f}(-v-\cdot) - \tilde{f}(-v)\big) h(-v)\,\mathrm{d}v, \qquad \lambda\text{-a.s.} \tag{5.28}$$

which by (3.12) and (3.16) implies

$$\int g(-u)\,\mathrm{d}\tau_u \tilde{f} = \tilde{\beta} - \beta + \phi(-\cdot), \qquad \lambda\text{-a.s.} \tag{5.29}$$

The square integrability of ϕ shows $\tilde{\beta} = \beta$ and by (4.26) it follows that $\widehat{\hat{\phi} f} = (2\pi)^2 g(-\cdot)$. Since $g(-\cdot)$ is zero on $\mathbb{R}_+$ this shows that condition (b) in Theorem 3.2 is satisfied and hence it follows by Theorem 3.2 that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. $\square$

6 The spectral measure of stationary semimartingales

For $t \in \mathbb{R}$, let $X_t = \int_{-\infty}^t \phi(t-s)\,\mathrm{d}W_s$ where $\phi \in L^2_{\mathbb{R}}(\lambda)$. In this section we use Knight [17, Theorem 6.5] to give a condition on the Fourier transform of ϕ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. In the case where $(X_t)_{t\geq 0}$ is a Markov process we use this to provide a simple condition on $\hat{\phi}$ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. In the last part of this section we study a general stationary Gaussian process $(X_t)_{t\geq 0}$. As in Jeulin and Yor [15] we provide conditions on the spectral measure of $(X_t)_{t\geq 0}$ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Proposition 6.1. *Let $(X_t)_{t\geq 0}$ be given by $X_t = \int \phi(t-s)\,\mathrm{d}W_s$, where $\phi \in L^2_{\mathbb{R}}(\lambda)$ and $(W_t)_{t\geq 0}$ is a Wiener process. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if*

$$\hat{\phi}(t) = \frac{\alpha + \hat{h}(t)}{1-it}, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{6.1}$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$.

The result follows directly from Knight [17, Theorem 6.5], once we have shown the following technical result.

Lemma 6.2. *Let $\phi \in L^2_{\mathbb{R}}(\lambda)$. Then ϕ is on the form*

$$\phi(t) = \begin{cases} \alpha + \int_0^t h(s)\,\mathrm{d}s & t \geq 0 \\ 0 & t < 0, \end{cases} \tag{6.2}$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2_{\mathbb{R}}(\lambda)$ if and only if

$$\hat{\phi}(t) = \frac{c + \hat{k}(t)}{1-it}, \tag{6.3}$$

for some $c \in \mathbb{R}$ and some $k \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$.

Proof. Assume ϕ satisfies (6.2). By square integrability of ϕ we can find a sequence $(a_n)_{n\geq 1}$ converging to infinity such that $\phi(a_n)$ converges to 0. For all $n \geq 1$ we have

$$\begin{aligned}
\int_0^{a_n} \phi(s)e^{its}\,\mathrm{d}s &= \int_0^{a_n} \alpha e^{its}\mathrm{d}s + \int_0^{a_n} \left(\int_0^s h(u)\,\mathrm{d}u\right) e^{its}\,\mathrm{d}s && (6.4)\\
&= \frac{\alpha(e^{ia_n t}-1)}{it} + \int_0^{a_n} h(u)\left(\int_u^{a_n} e^{its}\mathrm{d}s\right)\mathrm{d}u && (6.5)\\
&= \frac{\alpha(e^{ia_n t}-1)}{it} + \int_0^{a_n} h(u)\left(\frac{e^{ia_n t}-e^{iut}}{it}\right)\mathrm{d}u && (6.6)\\
&= \frac{1}{it}\left(e^{ia_n t}\left(\alpha + \int_0^{a_n} h(u)\mathrm{d}u\right) - \alpha - \int_0^{a_n} h(u)e^{itu}\,\mathrm{d}u\right) && (6.7)\\
&= \frac{1}{it}\left(e^{ia_n t}\phi(a_n) - \alpha - \int_0^{a_n} h(u)e^{itu}\mathrm{d}u\right). && (6.8)
\end{aligned}$$

Hence by letting n tend to infinity it follows that $\hat{\phi}(t) = -(it)^{-1}(\alpha + \hat{h}(t))$ and we obtain (6.3).

Assume conversely that (6.3) is satisfied and let $e(t) := e^{-t}1_{\mathbb{R}_+}(t)$ for $t \in \mathbb{R}$. We have

$$\hat{\phi}(t) = \frac{c + \hat{k}}{1-it} = c\hat{e}(t) + \hat{k}(t)\hat{e}(t). \tag{6.9}$$

Note that $k * e$ is square integrable and $\widehat{k * e} = \hat{k}\hat{e}$. Thus from (6.9) it follows that $\phi = ce + k * e$ λ-a.s. This shows in particular that ϕ is 0 on $(-\infty, 0)$ and $k(t) - k * e(t) = ce(t) + k(t) - \phi(t) =: f(t)$, which implies that

$$h(t) - h(0) = f(t) - f(0) - \int_0^t f(s)\,\mathrm{d}s, \tag{6.10}$$

and hence

$$\phi(t) = \begin{cases} \phi(0) + \int_0^t (\phi(s) - k(s))\,\mathrm{d}s & t \geq 0 \\ 0 & t < 0. \end{cases} \tag{6.11}$$

This completes the proof of (6.2). □

Let $(X_t)_{t\geq 0}$ be given by $X_t = \int_{-\infty}^t \phi(t-s)\,\mathrm{d}W_s$ for some $\phi \in L^2_{\mathbb{R}}(\lambda)$. Below we characterize when $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process by means of two constants and an inner function. Moreover, we provide a simple condition on the inner function for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. Finally, this condition is used to construct a rather large class of ϕ's for which $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale but not an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. Cherny [7, Example 3.4] constructs a ϕ for which $(X_t)_{t\geq 0}$ given by (3.11) (with $\psi = \phi$) is an $(\mathcal{F}_t^X)_{t\geq 0}$-Wiener process but not an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Proposition 6.3. *Let $(X_t)_{t\geq 0}$ be given by $X_t = \int \phi(t-s)\,\mathrm{d}W_s$, for $t \in \mathbb{R}$, where $\phi \in L^2_{\mathbb{R}}(\lambda)$ is non-trivial and 0 on $(-\infty, 0)$.*

(i) $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process if and only if ϕ is given by

$$\hat{\phi}(t) = \frac{cj(t)}{\theta - it}, \qquad t \in \mathbb{R}, \tag{6.12}$$

where J is an inner function satisfying $\overline{J(z)} = J(-\overline{z})$, $j(a) = \lim_{b\downarrow 0} J(a+ib)$ and $c, \theta > 0$. In this case $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, and an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if $J - \alpha \in \mathbb{H}^2_+$ for some $\alpha \in \{-1, 1\}$.

(ii) In particular, let ϕ be given by (6.12), where J is a singular inner function, i.e. on the form

$$J(z) = \exp\Big(\frac{-1}{\pi i}\int \frac{sz+1}{s-z}\frac{1}{1+s^2}\,F(\mathrm{d}s)\Big), \qquad z \in \mathbb{C}_+, \tag{6.13}$$

where F is a singular measure which integrates $s \mapsto (1+s^2)^{-1}$, and assume F is symmetric, concentrated on $\mathbb{Z}$, $(F(\{k\}))_{k\in\mathbb{Z}}$ is bounded and $\sum_{k\in\mathbb{Z}} F(\{k\})^2 = \infty$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process, an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale and $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted, but not an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale.

Proof. Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process and let J denote the inner part of the Hardy function induced by ϕ. Note that $\overline{J(z)} = J(-\overline{z})$. Since $(X_t)_{t\geq 0}$ is an $L^2(\mathrm{P})$-continuous, centered Gaussian $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process it follows by Doob [9, Theorem 1.1] that $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process and hence

$$|\hat{\phi}(t)|^2 = \frac{c}{\theta + t^2}, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{6.14}$$

for some $\theta, c > 0$. This implies that the outer part of $\hat{\phi}$ is $z \mapsto c/(\theta - iz)$ and thus ϕ satisfies (6.12). Assume conversely that ϕ is given by (6.12). It is readily seen that ϕ is a real function which is 0 on $(-\infty, 0)$. Moreover, since $|\hat{\phi}|^2 = c^2/(\theta^2+t^2)$ it follows that $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process and hence an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process and an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. According to Proposition 6.1, $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if

$$\hat{\phi}(t) = \frac{\alpha + \hat{h}(t)}{\theta - it}, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{6.15}$$

for some $\alpha \in \mathbb{R}$ and $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$, which by (6.12) is equivalent to $J - \alpha/c = H/c$, where H is the Hardy function induced by h. This completes the proof of (i).

To prove (ii), note first that $\overline{J(z)} = J(-\overline{z})$ since F is symmetric. Moreover,

$$|J(a+ib)| = \exp\Big(\int \frac{-b}{\pi((s-a)^2+b^2)}\, F(\mathrm{d}s)\Big). \tag{6.16}$$

If $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded measurable function then $f \in L^2_{\mathbb{R}}(\lambda)$ if and only if $e^f - 1 \in L^2_{\mathbb{R}}(\lambda)$. We will use this on

$$f(a) := \int \frac{-b}{\pi((s-a)^2+b^2)}\, F(\mathrm{d}s), \qquad a \in \mathbb{R}. \tag{6.17}$$

The function f is bounded since $k \mapsto F(\{k\})$ is bounded. Moreover, $f \notin L^2_{\mathbb{R}}(\lambda)$ since

$$\int |f(a)|^2\, \mathrm{d}a = \Big(\frac{b}{\pi}\Big)^2 \int \Big(\sum_{j\in\mathbb{Z}} \frac{F(\{j\})}{(j-a)^2+b^2}\Big)^2 da \tag{6.18}$$

$$\geq \Big(\frac{b}{\pi}\Big)^2 \int \sum_{j\in\mathbb{Z}} \Big(\frac{F(\{j\})}{(j-a)^2+b^2}\Big)^2 \mathrm{d}a = \Big(\frac{b}{\pi}\Big)^2 \sum_{j\in\mathbb{Z}} \int \Big(\frac{F(\{j\})}{(j-a)^2+b^2}\Big)^2 \mathrm{d}a \tag{6.19}$$

$$= \Big(\frac{b}{\pi}\Big)^2 \sum_{j\in\mathbb{Z}} \int \Big(\frac{F(\{j\})}{a^2+b^2}\Big)^2 \mathrm{d}a = \Big(\frac{b}{\pi}\Big)^2 \int \Big(\frac{1}{a^2+b^2}\Big)^2 \mathrm{d}a \sum_{j\in\mathbb{Z}} [F(\{j\})]^2 = \infty, \tag{6.20}$$

where the first inequality follows from the fact that the terms in the sum are positive. It follows that $e^f - 1 \notin L^2_{\mathbb{R}}(\lambda)$. Let $\alpha \in \{-1, 1\}$. Then

$$|J(a+ib) - \alpha| \geq ||J(a+ib)| - 1| = e^{f(a)} - 1, \tag{6.21}$$

which shows that $J - \alpha \notin \mathbb{H}^2_+$ and hence $(X_t)_{t\geq 0}$ is not an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. $\square$

Let $(X_t)_{t\geq 0}$ denote an $L^2(\mathrm{P})$-continuous centered Gaussian process. Recall that the symmetric finite measure μ satisfying

$$\mathrm{E}[X_t X_u] = \int e^{i(t-u)s}\, \mu(\mathrm{d}s), \qquad \forall\, t, u \in \mathbb{R}, \tag{6.22}$$

is called the spectral measure of $(X_t)_{t\geq 0}$. The proof of the next result is quite similar to the proof of Jeulin and Yor [15, Proposition 19].

Proposition 6.4. *Let $(X_t)_{t\geq 0}$ be an $L^2(\mathrm{P})$-continuous stationary centered Gaussian process with spectral measure $\mu = \mu_s + f\,\mathrm{d}\lambda$ (μ_s is the singular part of μ). Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if $\int t^2\, \mu_s(\mathrm{d}t) < \infty$ and*

$$f(t) = \frac{|\alpha + \hat{h}(t)|^2}{1+t^2}, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{6.23}$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$ when $\alpha \neq 0$. Moreover, $(X_t)_{t\geq 0}$ is of bounded variation if and only if $\alpha = 0$.

Proposition 6.4 extends the well-known fact that an $L^2(\mathrm{P})$-continuous stationary Gaussian process is of bounded variation if and only if $\int t^2\,\mu(\mathrm{d}t) < \infty$.

Proof of Proposition 6.4. Only if: If $(X_t)_{t\geq 0}$ is of bounded variation then $\int t^2\,\mu(\mathrm{d}t) < \infty$ and therefore μ is on the stated form. Thus, we may and do assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale of unbounded variation. It follows that $(X_t)_{t\geq 0}$ is a regular process and hence it can be decomposed as (see e.g. Doob [10])

$$X_t = V_t + \int_{-\infty}^t \phi(t-s)\,\mathrm{d}W_s, \qquad t\in\mathbb{R}, \tag{6.24}$$

where $(W_t)_{t\geq 0}$ is a Wiener process which is independent of $(V_t)_{t\in\mathbb{R}}$ and $W_r - W_s$ is $\mathcal{F}_t^{X,\infty}$-measurable for $s\leq r\leq t$. The process $(V_t)_{t\in\mathbb{R}}$ is stationary Gaussian and V_t is $\mathcal{F}_{-\infty}^{X,\infty}$-measurable for all $t\in\mathbb{R}$, where

$$\mathcal{F}_{-\infty}^{X,\infty} := \bigcap_{t\in\mathbb{R}} \mathcal{F}_t^{X,\infty}. \tag{6.25}$$

Moreover, $(V_t)_{t\in\mathbb{R}}$ respectively $(X_t - V_t)_{t\in\mathbb{R}}$ has spectral measure μ_s respectively $f\,\mathrm{d}\lambda$. For $0\leq u\leq t$ we have

$$\begin{aligned}\mathrm{E}[|V_t - V_u|] &= \mathrm{E}[|\mathrm{E}[V_t - V_u|\mathcal{F}_u^{V,\infty}]|] = \mathrm{E}[|\mathrm{E}[X_t - X_u|\mathcal{F}_u^{V,\infty}]|] &(6.26)\\ &\leq \mathrm{E}[|\mathrm{E}[X_t - X_u|\mathcal{F}_u^{X,\infty}]|], &(6.27)\end{aligned}$$

which shows that $(V_t)_{t\geq 0}$ is of integrable variation and hence $\int t^2\,\mu_s(\mathrm{d}t) < \infty$. The fact that $(V_t)_{t\geq 0}$ is $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-adapted and of bounded variation implies that

$$\Big(\int_{-\infty}^t \phi(t-s)\,\mathrm{d}W_s\Big)_{t\geq 0} \tag{6.28}$$

is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale and therefore also an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. Thus, by Proposition 6.1 we conclude that

$$f(t) = |\hat{\phi}(t)|^2 = \frac{|\alpha + \hat{h}(t)|^2}{1+t^2}, \qquad \lambda\text{-a.a. } t\in\mathbb{R}, \tag{6.29}$$

for some $\alpha\in\mathbb{R}$ and some $h\in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty,0)$.

If: If $\int t^2\,\mu(\mathrm{d}t) < \infty$, then $(X_t)_{t\geq 0}$ is of bounded variation and hence an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. Thus, we may and do assume $\int t^2 f(t)\,\mathrm{d}t = \infty$. We show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale by constructing a process $(Z_t)_{t\geq 0}$ which equals $(X_t)_{t\geq 0}$ in distribution and such that $(Z_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. By Lemma 6.2 there exists a $\beta\in\mathbb{R}$ and a $g\in L^2_{\mathbb{R}}(\lambda)$ such that with $\phi(t) = \beta + \int_0^t g(s)\,\mathrm{d}s$ for $t\geq 0$ and $\phi(t) = 0$ for $t<0$, we have $|\hat{\phi}|^2 = f$. Define $(Z_t)_{t\geq 0}$ by

$$Z_t = V_t + \int_{-\infty}^t \phi(t-s)\,\mathrm{d}W_s, \qquad t\in\mathbb{R}, \tag{6.30}$$

where $(V_t)_{t\in\mathbb{R}}$ is a stationary Gaussian process with spectral measure μ_s and $(W_t)_{t\geq 0}$ is a Wiener process which is independent of $(V_t)_{t\in\mathbb{R}}$. The processes $(X_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are identical in distribution due to the fact that they are centered Gaussian processes with the same spectral measure and hence it is enough to show that $(Z_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. It is well-known that $(V_t)_{t\geq 0}$ is of bounded variation since $\int t^2\,\mu_s(\mathrm{d}t) < \infty$ and by Knight [17, Theorem 6.5] the second term on the right-hand side of (6.30) is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. Thus we conclude that $(Z_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. □

7 The spectral measure of semimartingales with stationary increments

Let $(X_t)_{t\geq 0}$ be an $L^2(\mathrm{P})$-continuous Gaussian process with stationary increments such that $X_0 = 0$. Then there exists a unique positive symmetric measure μ on $\mathbb{R}$ which integrates $t \mapsto (1+t^2)^{-1}$ and satisfies

$$\mathrm{E}[X_t X_u] = \int \frac{(e^{its}-1)(e^{-ius}-1)}{s^2}\,\mu(\mathrm{d}s), \qquad t,u\in\mathbb{R}. \tag{7.1}$$

This μ is called the spectral measure of $(X_t)_{t\geq 0}$. The spectral measure of the fractional Brownian motion (fBm) with Hurst parameter $H\in(0,1)$ is

$$\mu(\mathrm{d}s) = c_H |s|^{1-2H}\,\mathrm{d}s, \tag{7.2}$$

where $c_H \in \mathbb{R}$ is a constant (see e.g. Yaglom [25]). In particular the spectral measure of the Wiener process ($H = 1/2$) equals the Lebesgue measure up to a scaling constant.

Theorem 7.1. *Let $(X_t)_{t\geq 0}$ be an $L^2(\mathrm{P})$-continuous, centered Gaussian process with stationary increments such that $X_0 = 0$. Moreover, let $\mu = \mu_s + f\,\mathrm{d}\lambda$ be the spectral measure of $(X_t)_{t\geq 0}$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if μ_s is a finite measure and*

$$f = |\alpha + \hat{h}|^2, \qquad \lambda\text{-a.s.} \tag{7.3}$$

for some $\alpha\in\mathbb{R}$ and some $h\in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty,0)$ when $\alpha\neq 0$. Moreover, $(X_t)_{t\geq 0}$ is of bounded variation if and only if $\alpha = 0$.

Proof. Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. Let $(Y_t)_{t\geq 0}$ be the stationary centered Gaussian process given by Lemma 5.3, that is

$$X_t = Y_t - Y_0 + \int_0^t Y_s\,\mathrm{d}s, \qquad t\in\mathbb{R}, \tag{7.4}$$

and let ν denote the spectral measure of $(Y_t)_{t\geq 0}$, that is ν is a finite measure satisfying

$$\mathrm{E}[Y_t Y_u] = \int e^{i(t-u)a}\,\nu(da), \qquad t,u\in\mathbb{R}. \tag{7.5}$$

By using Fubini's Theorem it follows that

$$\mathrm{E}[X_t X_u] = \int \frac{\big(e^{its}-1\big)\big(e^{-ius}-1\big)}{s^2}(1+s^2)\,\nu(\mathrm{d}s), \qquad t,u\in\mathbb{R}. \tag{7.6}$$

Thus, by uniqueness of the spectral measure of $(X_t)_{t\geq 0}$ we obtain $\mu(\mathrm{d}s) = (1+s^2)\,\nu(\mathrm{d}s)$. Since $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale (7.4) implies that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale and hence Proposition 6.4 shows that the singular part ν_s of ν satisfies $\int t^2\,\nu(\mathrm{d}t) < \infty$ and the absolute continuous part is on the form

$$|\alpha + \hat{h}(s)|^2 (1+s^2)^{-1}\,\mathrm{d}s, \tag{7.7}$$

for some $\alpha\in\mathbb{R}$, and some $h\in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty,0)$ when $\alpha\neq 0$. This completes the *only if* part of the proof.

Conversely assume that μ_s is a finite measure and $f = |\alpha + \hat{h}|^2$ for an $\alpha \in \mathbb{R}$ and an $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$ when $\alpha \neq 0$. Let $(Y_t)_{t\geq 0}$ be a centered Gaussian process such that

$$\mathrm{E}[Y_t Y_u] = \int \frac{e^{i(t-u)a} f(a)}{1+a^2}\,\mathrm{d}a, \qquad t, u \in \mathbb{R}. \tag{7.8}$$

By Proposition 6.4 it follows that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale. Thus, $(Z_t)_{t\in\mathbb{R}}$ defined by

$$Z_t := Y_t - Y_0 + \int_0^t Y_s\,\mathrm{d}s, \qquad t \in \mathbb{R}, \tag{7.9}$$

is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale and therefore also an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. Moreover, by calculations as in (7.6) it follows that $(Z_t)_{t\in\mathbb{R}}$ is distributed as $(X_t)_{t\geq 0}$, which shows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. This completes the proof. □

Let $(X_t)_{t\geq 0}$ denote a fBm with Hurst parameter $H \in (0,1)$ (recall that the spectral measure of $(X_t)_{t\geq 0}$ is given by (7.2)). If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale then Theorem 7.1 shows that $c_H|s|^{1-2H} = |\alpha + \hat{h}(s)|$, for some $\alpha \in \mathbb{R}$ and some $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$ when $\alpha \neq 0$. This implies $H = 1/2$. It is well-known from Rogers [22] that the fBm is not a semimartingale (even in the filtration $(\mathcal{F}_t^X)_{t\geq 0}$) when $H \neq 1/2$. However, the proof presented is new and illustrates the usefulness of the theorem. As a consequence of the above theorem we also have:

Corollary 7.2. *Let $(X_t)_{t\geq 0}$ be a Gaussian process with stationary increments. Then $(X_t)_{t\geq 0}$ is of bounded variation if and only if $(X_t - X_0)_{t\in\mathbb{R}}$ has finite spectral measure.*

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PAPER D

Lévy driven moving averages and semimartingales

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Abstract

The aim of the present paper is to study the semimartingale property of continuous time moving averages driven by Lévy processes. We provide necessary and sufficient conditions on the kernel for the moving average to be a semimartingale in the natural filtration of the Lévy process, and when this is the case we also provide a useful representation. Assuming that the driving Lévy process is of unbounded variation, we show that the moving average is a semimartingale if and only if the kernel is absolutely continuous with a density satisfying an integrability condition.

Keywords: semimartingales; moving averages; Lévy processes; bounded variation; absolutely continuity; stable processes; fractional processes

AMS Subject Classification: 60G48; 60H05; 60G51; 60G17

1 Introduction

The present paper is concerned with the semimartingale property of moving averages (also known as stochastic convolutions) which are driven by Lévy processes. More precisely, let $(X_t)_{t\geq 0}$ be a moving average of the form

$$X_t = \int_0^t \phi(t-s)\,\mathrm{d}Z_s, \qquad t \geq 0, \tag{1.1}$$

where $(Z_t)_{t\geq 0}$ is a Lévy process and $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ is a deterministic function for which the integral exists. We are interested in the question whether $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale, where $(\mathcal{F}^Z)_{t\geq 0}$ denotes the natural filtration of $(Z_t)_{t\geq 0}$. In addition, two-sided moving averages (see (1.6)) are studied as well.

According to Doob [13, page 533], a stationary process is a moving average if and only if its spectral measure is absolutely continuous. Key examples of moving averages are the Ornstein-Uhlenbeck process, the fractional Brownian motion, and their generalizations, the Ornstein-Uhlenbeck type process (see [31]) and the linear fractional stable motion (see [37]). Moving averages occur naturally in many different contexts, e.g. in stochastic Volterra equations (see [26]), in stochastic delay equations (see [30]), and in turbulence (see [1]). Moreover, to capture the long-range dependence of log-returns in financial markets it is natural to consider the fractional Brownian motion instead of the Brownian motion in the Black-Scholes model (see Biagini et al. [6, Part III]), and to capture also heavy tails one is often led to more general moving averages.

It is often important that the process of interest is a semimartingale, and in particular the following two properties are crucial: Firstly, if $(X_t)_{t\geq 0}$ models an asset price which is locally bounded and satisfies the No Free Lunch with Vanishing Risk condition then $(X_t)_{t\geq 0}$ has to be an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale (see Delbaen and Schachermayer [11, Theorem 7.2]). Secondly, it is possible to define a "reasonable" stochastic integral $\int_0^t H_s\,\mathrm{d}X_s$ for all locally bounded $(\mathcal{F}^Z)_{t\geq 0}$-predictable processes $(H_t)_{t\geq 0}$ if and only if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale due to the Bichteler-Dellacherie Theorem (see Bichteler [7, Theorem 7.6]). In view of the numerous applications of moving averages it is thus natural to study the semimartingale property of these processes.

Let $(Z_t)_{t\geq 0}$ denote a general semimartingale, $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ be absolutely continuous with a bounded density and let $(X_t)_{t\geq 0}$ be given by (1.1). Then by a stochastic Fubini result it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale, see e.g. Protter [26, Theorem 3.3] or Reiß et al. [30, Theorem 5.2]. In the case where $(Z_t)_{t\in\mathbb{R}}$ is a two-sided Wiener process, $\phi \in L^2(\mathbb{R}_+, \lambda)$ (λ denotes the Lebesgue measure) and $(X_t)_{t\geq 0}$ is given by

$$X_t = \int_{-\infty}^t \phi(t-s)\,\mathrm{d}Z_s, \qquad t \geq 0, \tag{1.2}$$

Knight [19, Theorem 6.5] shows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if ϕ is absolutely continuous with a square integrable density ($\mathcal{F}_t^{Z,\infty} := \sigma(Z_s : -\infty < s \leq t)$). Related results can be found in Cherny [10], Cheridito [9] and Basse [3]. Moreover, results characterizing when $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale are given in Jeulin and Yor [17] and Basse [2].

The above presented results only provide sufficient conditions on ϕ or are only concerned with the Brownian case. In the present paper we study the case where $(Z_t)_{t\geq 0}$ is a Lévy process and we provide necessary and sufficient conditions on ϕ for $(X_t)_{t\geq 0}$, given by (1.1), to be an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. Assume $(Z_t)_{t\geq 0}$ is of unbounded variation and has characteristic triplet (γ, σ^2, ν). Our main result is the following:

$(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a density ϕ' satisfying

$$\int_0^t \int_{[-1,1]} \Big(|x\phi'(s)|^2 \wedge |x\phi'(s)|\Big)\, \nu(\mathrm{d}x)\,\mathrm{d}s < \infty, \qquad \forall t>0,\ \text{if } \sigma^2 = 0, \tag{1.3}$$

$$\int_0^t |\phi'(s)|^2\,\mathrm{d}s < \infty, \qquad \forall t>0,\ \text{if } \sigma^2 > 0. \tag{1.4}$$

In the case where $(Z_t)_{t\geq 0}$ is a symmetric α-stable Lévy process, (1.3) corresponds to $\phi' \in L^\alpha([0,t],\lambda)$ for all $t>0$ when $\alpha \in (1,2)$ and to $|\phi'|\log^+(|\phi'|) \in L^1([0,t],\lambda)$ for all $t>0$ when $\alpha = 1$.

Assume $(Z_t)_{t\geq 0}$ is of unbounded variation. If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale it can be decomposed as

$$X_t = \phi(0)Z_t + \int_0^t \Big(\int_0^u \phi'(u-s)\,\mathrm{d}Z_s\Big)\,\mathrm{d}u, \qquad t\geq 0. \tag{1.5}$$

As a corollary of (1.5) it follows that $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation if and only if it is absolutely continuous, which is also equivalent to ϕ is absolutely continuous on $\mathbb{R}_+$ with a density satisfying (1.3)–(1.4) and $\phi(0)=0$.

Finally we study two-sided moving averages, i.e. where $(X_t)_{t\geq 0}$ is given by

$$X_t = \int_{-\infty}^t (\phi(t-s) - \psi(-s))\,\mathrm{d}Z_s, \qquad t\geq 0, \tag{1.6}$$

$(Z_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process and $\phi,\psi\colon \mathbb{R}\to\mathbb{R}$ are deterministic functions for which the integral exists. Note that in this case $(X_t)_{t\geq 0}$ has stationary increments, and when $\psi = 0$ it is a stationary process. Several examples, including fractional Lévy processes and hence also the linear fractional stable motion, are given in Section 5.

The conditions on ϕ from the one-sided case translate into necessary conditions in the two-sided case. That is, if $(Z_t)_{t\in\mathbb{R}}$ is of unbounded variation and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale then ϕ is absolutely continuous on $\mathbb{R}_+$ with a density satisfying (1.3)–(1.4). Moreover, Knight [19, Theorem 6.5] is extended from the Gaussian case to the α-stable case with $\alpha \in (1,2]$.

The paper is organized as follows. In Section 2 we collect some preliminary results. The main results are presented in Section 3. All proofs are given in Section 4. The two-sided case is considered in Section 5.

2 Preliminaries

Throughout the paper $(\Omega,\mathcal{F},\mathrm{P})$ denotes a complete probability space. Let $(Z_t)_{t\geq 0}$ denote a Lévy process with characteristic triplet (γ,σ^2,ν), that is for $t\geq 0$, $\mathrm{E}[e^{i\theta Z_t}] = e^{t\kappa(\theta)}$ for all $\theta\in\mathbb{R}$, where

$$\kappa(\theta) = i\gamma\theta - \sigma^2\theta^2/2 + \int \big(e^{i\theta s} - 1 - i\theta s 1_{\{|s|\leq 1\}}\big)\,\nu(\mathrm{d}s), \qquad \theta\in\mathbb{R}. \tag{2.1}$$

For a general treatment of Lévy processes we refer to [38], [5] or [27]. Let $f\colon\mathbb{R}\to\mathbb{R}$ denote a measurable function. Following Rajput and Rosiński [28, page 460] we say that f is Z-integrable if there exists a sequence of simple functions $(f_n)_{n\geq 1}$ such that $f_n \to f$ λ-a.s. and $\lim_n \int_A f_n(s)\,\mathrm{d}Z_s$ exists in probability for all $A\in\mathcal{B}([0,t])$ and all

$t > 0$ (recall that λ denotes the Lebesgue measure). In this case we define $\int_0^t f(s)\,\mathrm{d}Z_s$ as the limit in probability of $\int_0^t f_n(s)\,\mathrm{d}Z_s$. By Rajput and Rosiński [28, Theorem 2.7], f is Z-integrable if and only if the following three conditions are satisfied for all $t > 0$:

$$\int_0^t f(s)^2\sigma^2\,\mathrm{d}s < \infty, \tag{2.2}$$

$$\int_0^t \int \Big(|xf(s)|^2 \wedge 1\Big)\,\nu(\mathrm{d}x)\,\mathrm{d}s < \infty, \tag{2.3}$$

$$\int_0^t \Big|f(s)\Big(\gamma + \int x(1_{\{|xf(s)|\le 1\}} - 1_{\{|x|\le 1\}})\,\nu(\mathrm{d}x)\Big)\Big|\,\mathrm{d}s < \infty. \tag{2.4}$$

In this case $\int_0^t f(s)\,\mathrm{d}Z_s$ is infinitely divisible with characteristic triplet $(\gamma_f, \sigma_f^2, \nu_f)$ given by

$$\gamma_f = \int_0^t f(s)\Big(\gamma + \int x(1_{\{|xf(s)|\le 1\}} - 1_{\{|x|\le 1\}})\,\nu(\mathrm{d}x)\Big)\,\mathrm{d}s, \tag{2.5}$$

$$\sigma_f^2 = \int_0^t f(s)^2\sigma^2\,\mathrm{d}s, \tag{2.6}$$

$$\nu_f(A) = (\nu \times \lambda)((x,s) \in \mathbb{R} \times [0,t] : xf(s) \in A \setminus \{0\}), \qquad A \in \mathcal{B}(\mathbb{R}). \tag{2.7}$$

If f is locally square integrable it is easily shown that (2.2)–(2.4) are satisfied and hence $\int_0^t f(s)\,\mathrm{d}Z_s$ is well-defined for all $t \geq 0$. Note also that (2.4) is satisfied if $(Z_t)_{t\geq 0}$ is symmetric. Recall that $(Z_t)_{t\geq 0}$ is a symmetric α-stable Lévy process with $\alpha \in (0,2]$ if $\gamma = \sigma^2 = 0$ and ν has density $s \mapsto c|s|^{-1-\alpha}$ for some $c > 0$ when $\alpha \in (0,2)$, and $\nu = 0$ and $\gamma = 0$ when $\alpha = 2$. In this case (2.2)–(2.4) reduce to $f \in L^\alpha([0,t],\lambda)$ for all $t > 0$.

A function $f\colon \mathbb{R}_+ \to \mathbb{R}$ is said to be of bounded variation if on each finite interval $[0,t]$ the total variation of f is finite, that is

$$\mathrm{Var}_t(f) := \sup \sum_{i=1}^n |f(t_i) - f(t_{i-1})| < \infty, \tag{2.8}$$

where the sup is taken over all partitions $0 = t_0 < \cdots < t_n = t$, $n \geq 1$ of $[0,t]$. Note that a Lévy process $(Z_t)_{t\geq 0}$ is of bounded variation if and only if $\int_{[-1,1]}|s|\,\nu(\mathrm{d}s) < \infty$ and $\sigma^2 = 0$ (see e.g. Sato [38, Theorem 21.9]). Let I denote an interval and $f\colon I \to \mathbb{R}$. Then f is said to be absolutely continuous if there exists a locally integrable function h such that

$$f(t) - f(u) = \int_u^t h(s)\,\mathrm{d}s, \qquad u,t \in I,\ u \leq t, \tag{2.9}$$

and in this case h is called the density of f. If $f\colon I \to \mathbb{R}$ and $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ are two measurable functions, then f is said to have locally g-moment if

$$\int_u^t g(|f(s)|)\,\mathrm{d}s < \infty, \qquad u,t \in I,\ u \leq t. \tag{2.10}$$

If (2.10) is satisfied with $g(x) = x^\alpha$ for some $\alpha > 0$ then f is said to have locally α-moment.

An increasing family of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ is called a filtration if it satisfies the usual conditions of right-continuity and completeness. For each process $(Y_t)_{t\geq 0}$ we let $(\mathcal{F}_t^Y)_{t\geq 0}$ denote its the natural filtration, i.e. $(\mathcal{F}_t^Y)_{t\geq 0}$ is the least filtration for which $(Y_t)_{t\geq 0}$ is

$(\mathcal{F}_t^Y)_{t\geq 0}$-adapted. Let $(\mathcal{F}_t)_{t\geq 0}$ denote a filtration. We say that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if it admits the following representation

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0, \tag{2.11}$$

where $(M_t)_{t\geq 0}$ is a càdlàg local $(\mathcal{F}_t)_{t\geq 0}$-martingale starting at 0 and $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted, càdlàg, of bounded variation and starting at 0, and X_0 is $\mathcal{F}_0$-measurable. (Recall that càdlàg means right-continuous with left-hand limits).

We need the following standard notation: For functions $f, g\colon \mathbb{R} \to (0,\infty)$ we write $f(x) \approx g(x)$ as $x \to \infty$ if f/g is bounded above and below on some interval (K,∞), where $K > 0$. Furthermore we write $f(x) = o(g(x))$ as $x \to \infty$ if $f(x)/g(x) \to 0$ as $x \to \infty$. A similar notation is used as $x \to 0$.

Assume ν has positive mass on $[-1,1]$. Similar to [23] we let $\xi\colon [0,\infty) \to [0,\infty)$ be given by

$$\xi(x) = \int_{[-1,1]} \Big(|sx|^2 \wedge |sx|\Big)\, \nu(\mathrm{d}s), \qquad x \geq 0. \tag{2.12}$$

Note that ξ is 0 at 0, continuous and increasing and satisfies:

(i) $\xi(x)/x \to \int_{[-1,1]} |s|\, \nu(\mathrm{d}s) \in (0,\infty]$ as $x \to \infty$,

(ii) If $\int_{[-1,1]} |s|^\alpha\, \nu(\mathrm{d}s) < \infty$ for $\alpha \in (1,2]$ then $\xi(x) = o(x^\alpha)$ as $x \to \infty$.

To show (i)–(ii) let

$$H(x) = x \int_{x^{-1}\leq |s| \leq 1} |s|\, \nu(\mathrm{d}s) \qquad \text{and} \qquad K(x) = x^2 \int_{|s|<x^{-1}} s^2\, \nu(\mathrm{d}s), \tag{2.13}$$

and note that $\xi(x) = H(x) + K(x)$ for $x > 1$. We have

$$\int_{x^{-1}\leq |s|\leq 1} |s|\, \nu(\mathrm{d}s) \leq \xi(x)x^{-1} \leq \int_{[-1,1]} |s|\, \nu(\mathrm{d}s), \qquad x > 1, \tag{2.14}$$

where the first inequality follows from $H \leq \xi$ and the second from (2.12) since $|xs|^2 \wedge |xs| \leq |xs|$. Hence by (2.14) and monotone convergence (i) follows. To show (ii) assume $\int_{[-1,1]} |s|^\alpha\, \nu(\mathrm{d}s) < \infty$ for some $\alpha \in (1,2]$. For all $\epsilon > 0$ we have

$$\limsup_{x\to\infty} H(x)x^{-\alpha} \leq \int_{[-\epsilon,\epsilon]} |s|^\alpha\, \nu(\mathrm{d}s), \tag{2.15}$$

and

$$K(x)x^{-\alpha} \leq \int_{|s|<x^{-1}} |s|^\alpha\, \nu(\mathrm{d}s), \tag{2.16}$$

which shows $\xi(x)x^{-\alpha} \to 0$ as $x \to \infty$ and completes the proof of (ii).

Assume ν is absolutely continuous in a neighborhood of zero with a density f satisfying $f(x) \approx |x|^{-\alpha-1}$ as $x \to 0$ for some $\alpha \in (0,2)$ (this is satisfied in the α-stable case). An easy calculation shows:

(1) $\xi(x) \approx x^\alpha$ as $x \to \infty$ if $\alpha \in (1,2)$,

(2) $\xi(x) \approx x\log(x)$ as $x \to \infty$ if $\alpha = 1$,

(3) $\int_{[-1,1]} |s|\, \nu(\mathrm{d}s) < \infty$ if $\alpha \in (0,1)$.

3 Main results

Let $(Z_t)_{t\geq 0}$ denote a nondeterministic Lévy process with characteristic triplet (γ, σ^2, ν) and $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ be a measurable function which is Z-integrable (see (2.2)–(2.4)). Throughout this section we let $(X_t)_{t\geq 0}$ be the moving average

$$X_t = \int_0^t \phi(t-s)\,\mathrm{d}Z_s, \qquad t \geq 0. \tag{3.1}$$

Theorem 3.1 below is the main result of the paper. It provides a complete characterization of when $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. Recall the definition of the function ξ in (2.12).

Theorem 3.1. *Assume $(Z_t)_{t\geq 0}$ is of unbounded variation. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a density ϕ' which is locally square integrable when $\sigma^2 > 0$ and has locally ξ-moment when $\sigma^2 = 0$ (that is, ϕ' satisfies* (1.3)–(1.4)*).*

Assume $(Z_t)_{t\geq 0}$ is of bounded variation. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale if and only if it is of bounded variation which is also equivalent to ϕ is of bounded variation.

In particular, if $\sigma^2 = 0$, $\int_{[-1,1]} |x|^\alpha\,\nu(\mathrm{d}x) < \infty$ for some $\alpha \in (1,2]$ and ϕ is absolutely continuous on $\mathbb{R}_+$ with a density having locally α-moment then it follows by (ii) on page 79 and the above theorem that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. In the case where $(X_t)_{t\geq 0}$ is a semimartingale the next proposition provides a useful representation of this process.

Proposition 3.2. *Assume $(Z_t)_{t\geq 0}$ is of unbounded variation and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. Then*

$$X_t = \phi(0) Z_t + \int_0^t \Big(\int_0^u \phi'(u-s)\,\mathrm{d}Z_s \Big)\,\mathrm{d}u, \qquad t \geq 0, \tag{3.2}$$

where ϕ' denotes the density of ϕ and $(\int_0^u \phi'(u-s)\,\mathrm{d}Z_s)_{u\geq 0}$ is chosen measurable.

Hence we obtain the following corollary.

Corollary 3.3. *Assume $(Z_t)_{t\geq 0}$ is of unbounded variation. Then the following four statements are equivalent:*

(a) $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation,

(b) $(X_t)_{t\geq 0}$ is absolutely continuous,

(c) $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale and $\phi(0) = 0$,

(d) ϕ is absolutely continuous with a density satisfying (1.3)–(1.4) *and $\phi(0) = 0$.*

In the symmetric α-stable case with $\alpha \in (1,2)$ the equivalence between (b) and (d) follows by Rosiński [32, Theorem 6.1]. [8] study, among other things, processes $(Y_t)_{t\geq 0}$ on the form $Y_t = \int_0^t f(t,s)\,\mathrm{d}Z_s$, where $(Z_t)_{t\geq 0}$ is a symmetric Lévy process and f is a deterministic function. Their Theorem 5.1 provides necessary and sufficient conditions on $f(t,s)$ for $(X_t)_{t\geq 0}$ to be absolutely continuous. In [24] and [20] necessary and sufficient conditions on ϕ are obtained for $(X_t)_{t\geq 0}$ to have locally bounded or continuous sample paths.

The next corollary follows by Theorem 3.1 and the estimates on ξ given in (1)–(3) on page 79.

Corollary 3.4. *Assume $\sigma^2 = 0$ and ν is absolutely continuous in a neighborhood of zero with a density f satisfying $f(x) \approx |x|^{-\alpha-1}$ as $x \to 0$ for some $\alpha \in (0,2)$ (this is satisfied in the α-stable case with $\alpha \in (0,2)$). Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale if and only if*

(i) ϕ is absolutely continuous with a density having locally α-moment when $\alpha \in (1,2)$,

(ii) ϕ is absolutely continuous with a density having locally $x\log^+(x)$-moment when $\alpha = 1$,

(iii) ϕ is of bounded variation when $\alpha \in (0,1)$.

Here $\log^+$ denotes the positive part of log, i.e. $\log^+(x) = \log(x)$ for $x \geq 1$ and 0 otherwise.

In the following let $(X_t)_{t\geq 0}$ be the Riemann-Liouville fractional integral given by

$$X_t = \int_0^t (t-s)^\tau \, \mathrm{d}Z_s, \qquad t \geq 0, \tag{3.3}$$

where τ is such that the integral exists. If $(Z_t)_{t\geq 0}$ is a Wiener process and $\tau > -1/2$, $(X_t)_{t\geq 0}$ is called a Lévy fractional Brownian motion (see Mandelbrot and Van Ness [22, page 424]). Assume $(Z_t)_{t\geq 0}$ has no Brownian component (i.e. $\sigma^2 = 0$). Using (2.2)–(2.4) it follows that for $(X_t)_{t\geq 0}$ to be well-defined one of the following (I)–(III) must be satisfied:

(I) $\tau > -1/2$,

(II) $\tau = -1/2$ and $\int_{[-1,1]} x^2 |\log|x|| \, \nu(\mathrm{d}x) < \infty$,

(III) $\tau < -1/2$ and $\int_{[-1,1]} |x|^{-1/\tau} \, \nu(\mathrm{d}x) < \infty$.

Condition (I) is also sufficient for $(X_t)_{t\geq 0}$ to be well-defined and when $(Z_t)_{t\geq 0}$ is symmetric, the conditions (I)–(III) are both necessary and sufficient for $(X_t)_{t\geq 0}$ to be well-defined. When $\tau = 0$, $(X_t)_{t\geq 0} = (Z_t)_{t\geq 0}$; thus let us assume $\tau \neq 0$. As a consequence of Theorem 3.1 we have the following.

Corollary 3.5. *Let $(X_t)_{t\geq 0}$ be given by* (3.3) *and assume $(Z_t)_{t\geq 0}$ has no Brownian component. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale if and only if one of the following* (1)–(3) *is satisfied:*

(1) $\tau > 1/2$,

(2) $\tau = 1/2$ and $\int_{[-1,1]} x^2 |\log|x|| \, \nu(\mathrm{d}x) < \infty$,

(3) $\tau \in (0,1/2)$ and $\int_{[-1,1]} |x|^{1/(1-\tau)} \, \nu(\mathrm{d}x) < \infty$.

Note that $1/(1-\tau) \in (1,2)$ when $\tau \in (0,1/2)$. Let us in particular consider

$$X_t = \int_0^t (t-s)^{H-1/\alpha} \, \mathrm{d}Z_s, \qquad t \geq 0, \tag{3.4}$$

where $(Z_t)_{t\geq 0}$ is a symmetric α-stable Lévy process with $\alpha \in (0,2]$ and $H > 0$ (note that $(X_t)_{t\geq 0}$ is well-defined). To avoid trivialities assume $H \neq 1/\alpha$. As a consequence of Corollary 3.5 ($\alpha \in (0,2)$) and Theorem 3.1 ($\alpha = 2$) it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale if and only if $H > 1$ when $\alpha \in [1,2]$ or $H > 1/\alpha$ when $\alpha \in (0,1)$.

4 Proofs

Throughout this section $(X_t)_{t\geq 0}$ is given by (3.1). We extend ϕ to a function from $\mathbb{R}$ into $\mathbb{R}$ by setting $\phi(s)=0$ for $s\in(-\infty,0)$. For any function $f\colon\mathbb{R}\to\mathbb{R}$, let $\Delta_t f$ denote the function $s\mapsto t(f(1/t+s)-f(s))$ for all $t>0$. We start by the following extension of Hardy and Littlewood [15, Theorem 24].

Lemma 4.1. *Let I be either $\mathbb{R}_+$ or $\mathbb{R}$, $f\colon I\to\mathbb{R}$ be locally integrable and $g\colon\mathbb{R}_+\to\mathbb{R}_+$ be an increasing convex function satisfying $g(x)/x\to\infty$ as $x\to\infty$ and let $(r_k)_{k\geq 1}$ be a sequence satisfying $r_k\to\infty$. Then f is absolutely continuous with a density having locally g-moment if and only if $(g(|\Delta_{r_k}f|))_{k\geq 1}$ is bounded in $L^1([a,b],\lambda)$ for all $a,b\in I$ with $a<b$. In this case $\{g(|\Delta_t f|):t>\epsilon\}$ is bounded in $L^1([a,b],\lambda)$ for all $a,b\in I$ with $a<b$ and all $\epsilon>0$.*

If $(Z_t)_{t\geq 0}$ is of unbounded variation the above lemma can be applied with ξ playing the role of g (ξ is given by (2.12)), since in this case ξ satisfies all the conditions imposed on g except ξ is not convex. But h, defined by $h(x)=x^2 1_{\{x\leq 1\}}+(2x-1)1_{\{x>1\}})$ for all $x\geq 0$, is convex and if we let

$$g(x)=\int_{[-1,1]} h(|xs|)\,\nu(\mathrm{d}s),\qquad x\geq 0, \tag{4.1}$$

then g satisfies all the conditions in the lemma and $g/2\leq\xi\leq g$. Thus, if $f\colon I\to\mathbb{R}$ is locally integrable then f is absolutely continuous with a density having locally ξ-moment if and only if $(\xi(|\Delta_{r_k}f|))_{k\geq 1}$ is bounded in $L^1([a,b],\lambda)$ for all $a,b\in I$ with $a<b$.

Proof. Note that g is continuous and $x\mapsto g(|x|)$ is a convex function from $\mathbb{R}$ into $\mathbb{R}$, since g is increasing and convex. Let $a,b\in I$ satisfying $a<b$ be given and assume $(g(|\Delta_{r_k}f|))_{k\geq 1}$ is bounded in $L^1([a,b],\lambda)$. Since $g(x)/x\to\infty$ as $x\to\infty$, $\{\Delta_{r_k}f:k\geq 1\}$ is uniformly integrable and hence weakly sequentially compact in $L^1([a,b],\lambda)$ (see e.g. Dunford and Schwartz [14, Chapter IV.8, Corollary 11]). Choose a subsequence $(n_k)_{k\geq 1}$ of $(r_k)_{k\geq 1}$ and an $h\in L^1([a,b],\lambda)$ such that $\Delta_{n_k}f\to h$ in the weak $L^1([a,b],\lambda)$-topology. For all $c,d\in[a,b]$ with $c<d$ we have

$$\int_c^d \Delta_{n_k}f\,\mathrm{d}\lambda\to\int_c^d h\,\mathrm{d}\lambda,\qquad \text{for } k\to\infty. \tag{4.2}$$

Moreover,

$$\int_c^d \Delta_{n_k}f\,\mathrm{d}\lambda = n_k\Big(\int_{c+1/n_k}^{d+1/n_k} f\,\mathrm{d}\lambda-\int_c^d f\,\mathrm{d}\lambda\Big) \tag{4.3}$$

$$= n_k\int_d^{d+1/n_k} f\,\mathrm{d}\lambda - n_k\int_c^{c+1/n_k} f\,\mathrm{d}\lambda\to f(d)-f(c),\qquad \text{for } k\to\infty, \tag{4.4}$$

for $\lambda\times\lambda$-a.a. $c<d$. Thus, we conclude that f is absolutely continuous with density h. Since $\Delta_{n_k}f\to h$ in the weak $L^1([a,b],\lambda)$-topology we may choose a sequence $(\kappa_n)_{n\geq 1}$ of convex combinations of $(\Delta_{n_k}f)_{k\geq 1}$ such that $\kappa_n\to h$ in $L^1([a,b],\lambda)$, see Rudin [36, Theorem 3.13]. By convexity and continuity of g we have

$$\int_a^b g(|h|)\,\mathrm{d}\lambda\leq\liminf_{n\to\infty}\int_a^b g(|\kappa_n|)\,\mathrm{d}\lambda\leq\sup_{k\geq 1}\int_a^b g(|\Delta_{n_k}f|)\,\mathrm{d}\lambda<\infty, \tag{4.5}$$

which shows that h has g-moment on $[a,b]$. This completes the proof of the *if*-part.

Assume conversely that f is absolutely continuous with a density, h, having locally g-moment. For all $t > \epsilon$, we have by Jensen's inequality that

$$\int_a^b g\Big(\Big|t\int_s^{s+1/t} h(u)\,\mathrm{d}u\Big|\Big)\,\mathrm{d}s \leq \int_a^b \Big(t\int_0^{1/t} g(|h(u+s)|)\,\mathrm{d}u\Big)\,\mathrm{d}s \tag{4.6}$$

$$= t\int_0^{1/t}\int_a^b g(|h(u+s)|)\,\mathrm{d}s\,\mathrm{d}u \leq \int_a^{b+1/\epsilon} g(|h(s)|)\,\mathrm{d}s < \infty, \tag{4.7}$$

which shows that $\{g(|\Delta_t f|) : t > \epsilon\}$ is bounded in $L^1([a,b],\lambda)$ and completes the proof. □

In following we are going to use two Lévy-Itô decompositions of $(Z_t)_{t\geq 0}$ (see e.g. Sato [38, Theorem 19.2]).

(a) Decompose $(Z_t)_{t\geq 0}$ as $Z_t = Z_t^1 + Z_t^2$, where $(Z_t^1)_{t\geq 0}$ and $(Z_t^2)_{t\geq 0}$ are two independent Lévy processes with characteristic triplets $(0,\sigma^2,\nu_1)$ respectively $(\gamma,0,\nu_2)$, where $\nu_1 = \nu|_{[-1,1]}$ and $\nu_2 = \nu|_{[-1,1]^c}$. $(Z_t^1)_{t\geq 0}$ and $(Z_t^2)_{t\geq 0}$ are $(\mathcal{F}^Z)_{t\geq 0}$-adapted. Moreover, when ϕ is locally bounded we let

$$X_t^1 = \int_0^t \phi(t-s)\,\mathrm{d}Z_s^1, \quad \text{and} \quad X_t^2 = \int_0^t \phi(t-s)\,\mathrm{d}Z_s^2, \qquad t \geq 0. \tag{4.8}$$

(b) Decompose $(Z_t)_{t\geq 0}$ as $Z_t = W_t + Y_t$, where $(W_t)_{t\geq 0}$ is a Wiener process with variance parameter σ^2 and $(Y_t)_{t\geq 0}$ is a Lévy process with characteristic triplet $(\gamma,0,\nu)$. $(W_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are independent and $(\mathcal{F}^Z)_{t\geq 0}$-adapted. Moreover, let

$$X_t^W = \int_0^t \phi(t-s)\,\mathrm{d}W_s, \quad \text{and} \quad X_t^Y = \int_0^t \phi(t-s)\,\mathrm{d}Y_s, \qquad t \geq 0. \tag{4.9}$$

If $\sigma^2 = 0$ and $(X_t)_{t\geq 0}$ is càdlàg it follows by Rosiński [33, Theorem 4] and a symmetrization argument that by modification on a set of Lebesgue measure 0, we may and do choose ϕ càdlàg.

The following lemma is closely related to Knight [19, Theorem 6.5].

Lemma 4.2. *We have the following:*

(i) $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale if ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density.

(ii) Assume $(Z_t)_{t\geq 0}$ is a Wiener process. Then ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale.

Proof. (i): Decompose $(Z_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ as in (a) above. Since both ϕ and $(Z_t^2)_{t\geq 0}$ are càdlàg and of bounded variation, $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation as well. Hence, it is enough to show $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. Since

$$X_t^1 = \int_0^t (\phi(t-s) - \phi(0))\,\mathrm{d}Z_s^1 + \phi(0)Z_t^1, \qquad t \geq 0, \tag{4.10}$$

we may and do assume $\phi(0) = 0$. Then, ϕ is absolutely continuous on $\mathbb{R}$ with locally square integrable density and hence for all $T > 0$, $\|\Delta_t \phi\|_{L^2([-T,T],\lambda)} \leq K$ for some constant $K > 0$ and all $t > 1/T$ by Lemma 4.1 with $g(x) = x^2$. By letting $c = \mathrm{E}[|Z_1^1|^2]$ we have (recall that ϕ is zero on $(-\infty, 0)$)

$$\mathrm{E}[(X_t^1 - X_u^1)^2] = c\|\phi(t-\cdot) - \phi(u-\cdot)\|^2_{L^2([0,t],\lambda)} \le cK^2(t-u)^2, \quad \forall\, 0 \le u \le t \le T, \tag{4.11}$$

which by the Kolmogorov-Čentsov Theorem (see Karatzas and Shreve [18, Chapter 2, Theorem 2.8]) shows that $(X_t^1)_{t\ge 0}$ has a continuous modification (also to be denoted $(X_t^1)_{t\ge 0}$). Moreover, for all $0 = t_0 < \cdots < t_n = T$ we have

$$\mathrm{E}\Big[\sum_{i=1}^n |X_{t_i}^1 - X_{t_{i-1}}^1|\Big] \le \sum_{i=1}^n \|X_{t_i}^1 - X_{t_{i-1}}^1\|_{L^2(\mathrm{P})} \le \sqrt{c}KT, \tag{4.12}$$

which shows that $(X_t^1)_{t\ge 0}$ is of integrable variation and hence an $(\mathcal{F}^Z)_{t\ge 0}$-semimartingale.

To show (ii) assume $(Z_t)_{t\ge 0}$ is a standard Wiener process and $(X_t)_{t\ge 0}$ is an $(\mathcal{F}^Z)_{t\ge 0}$-semimartingale. Since $(X_t)_{t\ge 0}$ is a Gaussian process, Stricker [39, Proposition 4+5] entails that $(X_t)_{t\ge 0}$ is an $(\mathcal{F}^Z)_{t\ge 0}$-quasimartingale on each compact interval $[0, N]$. For $0 \le u \le t$ we have

$$\mathrm{E}[|\mathrm{E}[X_t - X_u|\mathcal{F}_u^Z]|] = \mathrm{E}[|\int_0^u \big(\phi(t-s) - \phi(u-s)\big)\,\mathrm{d}Z_s|] \tag{4.13}$$

$$= \sqrt{\frac{2}{\pi}}\|\int_0^u \big(\phi(t-s) - \phi(u-s)\big)\,\mathrm{d}Z_s\|_{L^2(\mathrm{P})} \tag{4.14}$$

$$= \sqrt{\frac{2}{\pi}}\Big(\int_0^u \big(\phi(t-s) - \phi(u-s)\big)^2\,\mathrm{d}s\Big)^{1/2} \tag{4.15}$$

$$= \sqrt{\frac{2}{\pi}}\Big(\int_0^u \big(\phi(t-u+s) - \phi(s)\big)^2\,\mathrm{d}s\Big)^{1/2}, \tag{4.16}$$

where the second equality follows by Gaussianity, which implies that

$$\sum_{i=1}^{nN} \mathrm{E}[|\mathrm{E}[X_{i/n} - X_{(i-1)/n}|\mathcal{F}^Z_{(i-1)/n}]|] \ge \frac{Nn}{\sqrt{\pi 2}}\Big(\int_0^{N/2} \big(\phi(1/n+s) - \phi(s)\big)^2\,\mathrm{d}s\Big)^{1/2}. \tag{4.17}$$

Since $(X_t)_{t\ge 0}$ is an $(\mathcal{F}^Z)_{t\ge 0}$-quasimartingale on $[0, N]$, the left-hand side of (4.17) is bounded in n (see Dellacherie and Meyer [12, Chapter VI, Definition 38]), showing that $(\Delta_n\phi)_{n\ge 1}$ is bounded in $L^2([0, N/2], \lambda)$. By Lemma 4.1 with $g(x) = x^2$ this shows that ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density. □

Lemma 4.3. *If $(X_t)_{t\ge 0}$ is an $(\mathcal{F}^Z)_{t\ge 0}$-semimartingale then $(X_t^1)_{t\ge 0}$ is an $(\mathcal{F}_t^{Z^1})_{t\ge 0}$-semimartingale.*

Proof. Assume $(X_t)_{t\ge 0}$ is an $(\mathcal{F}^Z)_{t\ge 0}$-semimartingale, fix $T > 0$ and let

$$A := \{\Delta Z_t^2 = 0 \ \forall t \in [0, T]\}. \tag{4.18}$$

Note that $\mathrm{P}(A) > 0$ and $(Z_t^1)_{t\ge 0}$ is P-independent of A. Let Q^A denote the probability measure given by $\mathrm{Q}^A(B) := \mathrm{P}(B\cap A)/\mathrm{P}(A)$. $(X_t)_{t\ge 0}$ is an $(\mathcal{F}^Z)_{t\ge 0}$-semimartingale under Q^A, since Q^A is absolutely continuous with respect to P. Moreover, since $(Z_t)_{t\ge 0}$ and $(Z_t^1)_{t\ge 0}$ are Q^A-indistinguishable it follows that $(X_t^1)_{t\ge 0}$ is an $(\mathcal{F}_t^{Z^1})_{t\ge 0}$-semimartingale under Q^A and since A is independent of $(Z_t^1)_{t\ge 0}$ this is also true under P. □

In the next lemma we study the jump structure of $(X_t)_{t\ge 0}$.

Lemma 4.4. *Assume $\sigma^2 = 0$ and $(X_t)_{t\geq 0}$ is càdlàg. Then $(\Delta X_t 1_{\{\Delta Z_t \neq 0\}})_{t\geq 0}$ and $(\phi(0)\Delta Z_t)_{t\geq 0}$ are indistinguishable.*

Before proving the lemma we note the following:

Remark 4.5.

(a) Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ denote two independent càdlàg processes such that $\mathrm{P}(\Delta X_t = 0) = \mathrm{P}(\Delta Y_t = 0) = 1$ for all $t \geq 0$. Then as a consequence of Tonelli's Theorem we have $\mathrm{P}(\Delta X_t \Delta Y_t = 0,\ \forall t \geq 0) = 1$.

(b) If ν is concentrated on $[-1, 1]$ then the mapping $t \mapsto \int_0^t \phi(t-s)\,\mathrm{d}Z_s$ is continuous from $\mathbb{R}_+$ into $L^1(\mathrm{P})$. This follows by approximating ϕ with continuous functions.

Proof of Lemma 4.4. Since $X_t = \int_0^t (\phi(t-s)-1)\,\mathrm{d}Z_s + Z_t$ we may and do assume $\phi(0) \neq 0$. Recall from page 83 that ϕ is chosen càdlàg; moreover $\Delta\phi(0) = \phi(0)$.

First we show the lemma in the case where ν is a finite measure. Let τ_n denote the time of the nth jump of $(Z_t)_{t\geq 0}$ ($(\tau_{n+1} - \tau_n)_{n\geq 1}$ is thus an i.i.d. sequence of exponential distributions) and let $(\sigma_n)_{n\geq 1} \subseteq [0, \infty)$ denote the jump times of ϕ. Note that the event

$$B := \{\exists\,(j, k) \neq (j', k') : \tau_j + \sigma_k = \tau_{j'} + \sigma_{k'}\}, \tag{4.19}$$

has probability zero. Since $(Z_t)_{t\geq 0}$ only has finitely many jumps on each compact interval we may regard $(X_t)_{t\geq 0}$ as a pathwise Lebesgue-Stieltjes integral and hence it follows that

$$(\Delta X_t)_{t\geq 0} = \Big(\sum_{k\geq 1} \Delta Z_{t-\sigma_k} \Delta\phi(\sigma_k)\Big)_{t\geq 0}. \tag{4.20}$$

Let us show that on B^c the series $\sum_{k\geq 1} \Delta Z_{t-\sigma_k}\Delta\phi(\sigma_k)$ has at most one term which differs from zero for all $t \geq 0$. Indeed, to see this assume that $\Delta Z_{t-\sigma_k}\Delta\phi(\sigma_k)$ and $\Delta Z_{t-\sigma_{k'}}\Delta\phi(\sigma_{k'})$ both differ from zero, where $k \neq k'$. Then there exist $n, n' \geq 1$ such that $\tau_n = t - \sigma_k$ and $\tau_{n'} = t - \sigma_{k'}$ which implies $\tau_n + \sigma_k = \tau_{n'} + \sigma_{k'}$, and hence we have a contradiction. In particular, if $\Delta Z_t \neq 0$ then $\Delta Z_t \Delta\phi(0) \neq 0$ and thus $\Delta X_t = \Delta Z_t \Delta\phi(0) = \phi(0)\Delta Z_t$.

Now let $(Z_t)_{t\geq 0}$ be a general Lévy process for which $\sigma^2 = 0$. For each $n \geq 1$, decompose $(Z_t)_{t\geq 0}$ as $Z_t = Y_t^n + U_t^n$, where $(Y_t^n)_{t\geq 0}$ and $(U_t^n)_{t\geq 0}$ are two independent Lévy processes with characteristic triplets $(0, 0, \nu|_{[-1/n, 1/n]})$ respectively $(0, 0, \nu|_{[-1/n,1/n]^c})$. Moreover, set

$$X_t^{Y^n} = \int_0^t \phi(t-s)\,\mathrm{d}Y_s^n \quad \text{and} \quad X_t^{U^n} = \int_0^t \phi(t-s)\,\mathrm{d}U_s^n. \tag{4.21}$$

Since $(U_t^n)_{t\geq 0}$ has piecewise constant sample paths the second integral is a pathwise Lebesgue-Stieltjes integral. Hence $(X_t^{U^n})_{t\geq 0}$ is càdlàg and it follows that $(X_t^{Y^n})_{t\geq 0}$ is càdlàg as well. Set

$$C := \bigcap_{n\geq 1} \{\Delta X_t^{Y^n} \Delta U_t^n = 0,\ \forall t \geq 0\}, \tag{4.22}$$

$$D := \bigcap_{n\geq 1} \{\Delta X_t^{U^n} 1_{\{\Delta U_t^n \neq 0\}} = \phi(0)\Delta U_t^n,\ \forall t \geq 0\}. \tag{4.23}$$

From Remark 4.5 (b) it follows that $\mathrm{P}(\Delta X_t^{Y^n} = 0) = 1$ for all $t \geq 0$ which together with Remark 4.5 (a) shows that C has probability one. Moreover, from the first part of the proof it follows that D has probability one. When $\Delta Z_t \neq 0$, choose $n \geq 1$ such that $|\Delta Z_t| > 1/n$. Thus, $\Delta U_t^n \neq 0$, and hence $\Delta X_t^{Y^n} = 0$ on C, which shows $\Delta X_t = \Delta X_t^{U^n} = \phi(0)\Delta U_t^n = \phi(0)\Delta Z_t$ on $C \cap D$ and completes the proof. ☐

Lemma 4.6. *Assume $\sigma^2 = \gamma = 0$, ν is concentrated on $[-1,1]$ and $(X_t)_{t\geq 0}$ is a special $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. Then $(\phi(0)Z_t)_{t\geq 0}$ is the martingale component of $(X_t)_{t\geq 0}$.*

Proof. Let $X_t = M_t + A_t$ denote the canonical decomposition of $(X_t)_{t\geq 0}$. Since $(Z_t)_{t\geq 0}$ is a Lévy process, it is quasi-left-continuous (see Jacod and Shiryaev [16, Chapter II, Corollary 4.18]) and thus there exists a sequence of totally inaccessible stopping times $(\tau_n)_{n\geq 1}$ which exhausts the jumps of $(Z_t)_{t\geq 0}$. On the other hand, since $(A_t)_{t\geq 0}$ is predictable there exists a sequence of predictable times $(\sigma_n)_{n\geq 1}$ which exhausts the jumps of $(A_t)_{t\geq 0}$. From the martingale representation theorem for Lévy processes (see Jacod and Shiryaev [16, Chapter III, Theorem 4.34]) it follows that $(M_t)_{t\geq 0}$ is a purely discontinuous martingale which jumps only when $(Z_t)_{t\geq 0}$ does. Furthermore, since

$$\mathrm{P}(\exists\, n,k \geq 1 : \tau_n = \sigma_k < \infty) = 0, \tag{4.24}$$

Lemma 4.4 shows

$$\phi(0)\Delta Z_{\tau_n} = \Delta X_{\tau_n} = \Delta M_{\tau_n} + \Delta A_{\tau_n} = \Delta M_{\tau_n}, \quad \text{P-a.s. on } \{\tau_n < \infty\}\ \forall\, n \geq 1. \tag{4.25}$$

Hence $(\Delta M_t)_{t\geq 0}$ and $(\phi(0)\Delta Z_t)_{t\geq 0}$ are indistinguishable which implies that $(M_t)_{t\geq 0}$ and $(\phi(0)Z_t)_{t\geq 0}$ are indistinguishable since they both are purely discontinuous martingales (see Jacod and Shiryaev [16, Chapter I, Corollary 4.19]). This completes the proof. $\square$

The following lemma is concerned with the bounded variation case and it relies on an inequality by Marcus and Rosiński [23].

Lemma 4.7. *Assume $\gamma = \sigma^2 = 0$, ν is concentrated on $[-1,1]$ and $(Z_t)_{t\geq 0}$ is of unbounded variation. Then $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a density having locally ξ-moment and $\phi(0) = 0$.*

Recall the definition of $\Delta_t\phi$ on page 82 and of $\mathrm{Var}_t(f)$ in (2.8).

Proof. Let $N \geq 1$ be given. We start by showing the following (i) and (ii) under the assumptions stated in the lemma:

(i) If $(X_t)_{t\geq 0}$ is of bounded variation then $\mathrm{E}[\mathrm{Var}_N(X)] < \infty$ for all $N \geq 1$.

(ii) For all $N \geq 1$,

$$\begin{aligned} &\frac{N}{8}\sup_{n\geq 1}\Big\{\Big(\int_{-N/2}^{N/2}\xi(|\Delta_{2^n}\phi(s)|)\,\mathrm{d}s\Big) \wedge \Big(\int_{-N/2}^{N/2}\xi(|\Delta_{2^n}\phi(s)|)\,\mathrm{d}s\Big)^{1/2}\Big\} &(4.26)\\ &\qquad\leq \mathrm{E}[\mathrm{Var}_N^{\mathcal{D}}(X)] \leq 3N\sup_{n\geq 1}\Big\{\int_{-N}^{N}\xi(|\Delta_{2^n}\phi(s)|)\,\mathrm{d}s + 1\Big\}, &(4.27)\end{aligned}$$

where for each $f\colon \mathbb{R}_+ \to \mathbb{R}$ we let

$$\mathrm{Var}_N^{\mathcal{D}}(f) = \sup_{n\geq 1}\sum_{i=1}^{2^n N}|f(i/2^n) - f((i-1)/2^n)|. \tag{4.28}$$

To show (i) assume $(X_t)_{t\geq 0}$ is of bounded variation. By Rosiński [33, Theorem 4], $\phi(\cdot - s)$ is of bounded variation for λ-a.a. $s \in \mathbb{R}_+$; in particular there exists an $s \in \mathbb{R}_+$ such that $\phi(\cdot - s)$ is of bounded variation. Hence ϕ is of bounded variation. Let $T := [0,N] \cap \mathbb{Q}$,

$\underline{X}: \Omega \to \mathbb{R}^T$ denote the canonical random element induced by $(X_t)_{t\in T}$ and let μ be given by

$$\mu(A) = (\lambda \times \nu)\left((s,x) \in [0,t_0] \times \mathbb{R} : x\phi(\cdot - s) \in A \setminus \{\underline{0}\}\right), \qquad A \in \mathcal{B}(\mathbb{R}^T). \tag{4.29}$$

For all $t_1,\dots,t_n \in T$, $(X_{t_1},\dots,X_{t_n})$ is infinitely divisible with Lévy measure $\mu \circ p^{-1}_{t_1,\dots,t_n}$, where $p_{t_1,\dots,t_n}(f) = (f(t_1),\dots,f(t_n))$ for all $f \in \mathbb{R}^T$. For $f \in \mathbb{R}^T$ let $q(f)$ denote the total variation of f on T. Then $q: \mathbb{R}^T \to [0,\infty]$ is clearly a lower-semicontinuous pseudonorm on $\mathbb{R}^T$ (see Rosiński and Samorodnitsky [35, page 998]). Since ν has compact support and ϕ is of bounded variation there exists an $r_0 > 0$ such that $\mu(f \in \mathbb{R}^T : q(f) > r_0) = 0$ and hence by Lemma 2.2 in [35], $\mathrm{E}[e^{\epsilon q(\underline{X})}] < \infty$ for some $\epsilon > 0$. In particular $(X_t)_{t\geq 0}$ is of integrable variation on $[0,N]$.

(ii): From Marcus and Rosiński [23, Corollary 1.1] we have

$$1/4 \min(a_{i,n}, a_{i,n}^{1/2}) \leq \mathrm{E}[|2^n(X_{i/2^n} - X_{(i-1)/2^n})|] \leq 3\max(a_{i,n}, a_{i,n}^{1/2}), \tag{4.30}$$

where

$$a_{i,n} := \int_{-1/2^n}^{(i-1)/2^n} \xi(|\Delta_{2^n}\phi(s)|)\,\mathrm{d}s. \tag{4.31}$$

By monotone convergence we have

$$\mathrm{E}[\mathrm{Var}_N^{\mathcal{D}}(X)] = \sup_{n\geq 1} \frac{1}{2^n}\sum_{i=1}^{2^n N} \mathrm{E}[|2^n(X_{i/2^n} - X_{(i-1)/2^n})|], \tag{4.32}$$

and hence

$$\frac{N}{2}\sup_{n\geq 1}\inf_{2^n N/2 < i \leq 2^n N} \mathrm{E}[|2^n(X_{i/2^n} - X_{(i-1)/2^n})|] \leq \mathrm{E}[\mathrm{Var}_{0,N}^{\mathcal{D}}(X)] \tag{4.33}$$

$$\leq N \sup_{n\geq 1}\sup_{1\leq i\leq 2^n N} \mathrm{E}[|2^n(X_{i/2^n} - X_{(i-1)/2^n})|], \tag{4.34}$$

which by (4.30) shows (4.26).

Assume $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation and hence by (i) of integrable variation. From (ii) it follows that $(\xi(\Delta_{2^n}\phi))_{n\geq 1}$ is bounded in $L^1([-a,a],\lambda)$ for all $a > 0$. Conversely, if $(\xi(\Delta_{2^n}\phi))_{n\geq 1}$ is bounded in $L^1([-a,a],\lambda)$ for all $a > 0$, (ii) shows that $\mathrm{E}[\mathrm{Var}_N^{\mathcal{D}}(X)] < \infty$; in particular $\mathrm{Var}_N^{\mathcal{D}}(X) < \infty$ P-a.s. Since in addition $(X_t)_{t\geq 0}$ is right-continuous in probability by Remark 4.5 (b) it has a has a càdlàg modification (also to be denoted $(X_t)_{t\geq 0}$), which is of bounded variation since $\mathrm{Var}_N(X) = \mathrm{Var}_N^{\mathcal{D}}(X) < \infty$ P-a.s.

Finally, the discussion just below Lemma 4.1 completes the proof, since $(Z_t)_{t\geq 0}$ is of unbounded variation. □

We have the following consequence of the Bichteler-Dellacherie Theorem.

Lemma 4.8. *Let $(Y_t)_{t\geq 0}, (U_t)_{t\geq 0}, (\tilde{Y}_t)_{t\geq 0}$ and $(\tilde{U}_t)_{t\geq 0}$ denote four processes such that $(Y_t)_{t\geq 0}$ is $(\mathcal{F}_t^U)_{t\geq 0}$-adapted, $(\tilde{Y}_t)_{t\geq 0}$ is $(\mathcal{F}_t^{\tilde{U}})_{t\geq 0}$-adapted and $(Y_\cdot, U_\cdot) \stackrel{\mathscr{D}}{=} (\tilde{Y}_\cdot, \tilde{U}_\cdot)$. If $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^U)_{t\geq 0}$-semimartingale then $(\tilde{Y}_t)_{t\geq 0}$ has a modification which is an $(\mathcal{F}_t^{\tilde{U}})_{t\geq 0}$-semimartingale.*

Proof. Since $(Y_t)_{t\geq 0}$, by assumption, is càdlàg and $(Y_t)_{t\geq 0} \stackrel{\mathscr{D}}{=} (\tilde{Y}_t)_{t\geq 0}$ we may choose a càdlàg modification of $(\tilde{Y}_t)_{t\geq 0}$ (also to be denoted $(\tilde{Y}_t)_{t\geq 0}$). By the Bichteler-Dellacherie

Theorem (see Dellacherie and Meyer [12, Theorem 80]) we must show that for all $t > 0$ the set of random variables given by

$$\Big\{ \sum_{i=1}^{n} \tilde{H}_{t_{i-1}}(\tilde{Y}_{t_i} - \tilde{Y}_{t_{i-1}}) : n \geq 1,\, 0 \leq t_0 < \cdots < t_n \leq t,\, \tilde{H}_{t_i} \in \mathcal{F}^{\tilde{U}}_{t_i},\, |\tilde{H}_{t_i}| \leq 1 \Big\} \tag{4.35}$$

is bounded in $L^0(\mathrm{P})$. Since each $\tilde{H}_s \in \mathcal{F}^{\tilde{U}}_s$ satisfying $|\tilde{H}_s| \leq 1$ is given by

$$\tilde{H}_s = \lim_{n\to\infty} F_n((\tilde{U}_u)_{u \leq s+1/n}) \qquad \text{P-a.s.}, \tag{4.36}$$

for some $F_n \colon \mathbb{R}^{[0,s+1/n]} \to [-1,1]$ which is $\mathcal{B}(\mathbb{R})^{[0,s+1/n]}$-measurable, our assumptions imply that for each random variable in the above set there exist $H_{t_i} \in \mathcal{F}^U_{t_i}$ satisfying $|H_{t_i}| \leq 1$ for $i = 0, \dots, n-1$ such that

$$\sum_{i=1}^{n} \tilde{H}_{t_{i-1}}(\tilde{Y}_{t_i} - \tilde{Y}_{t_{i-1}}) \stackrel{\mathscr{D}}{=} \sum_{i=1}^{n} H_{t_{i-1}}(Y_{t_i} - Y_{t_{i-1}}). \tag{4.37}$$

Thus since $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}^U_t)_{t\geq 0}$-semimartingale, another application of the Bichteler-Dellacherie Theorem shows that the set given in (4.35) is bounded in $L^0(\mathrm{P})$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We prove the result in the following three steps (1)–(3). Recall (a) and (b) on page 83.

(1) Let $\sigma^2 > 0$.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. Let $\tilde{Z}_t = Y_t - W_t$ and $\tilde{X}_t = \int_0^t \phi(t-s)\,\mathrm{d}\tilde{Z}_s$. We have $\mathcal{F}^Z_t = \mathcal{F}^W_t \vee \mathcal{F}^Y_t = \mathcal{F}^{-W}_t \vee \mathcal{F}^Y_t = \mathcal{F}^{\tilde{Z}}_t$ and since $(X., Z.) \stackrel{\mathscr{D}}{=} (\tilde{X}., \tilde{Z}.)$, Lemma 4.8 shows that $(\tilde{X}_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. Therefore $(X^W_t)_{t\geq 0} := (X_t - \tilde{X}_t)/2)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale and thus an $(\mathcal{F}^W_t)_{t\geq 0}$-semimartingale, and by Lemma 4.2 (ii) we conclude that ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density.

On the other hand, if ϕ is absolutely continuous with a locally square integrable density it follows by Lemma 4.2 (i) that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale.

(2) Let $\sigma^2 = 0$ and $(Z_t)_{t\geq 0}$ be of unbounded variation.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. By Lemma 4.3 it follows that $(X^1_t)_{t\geq 0}$ is an $(\mathcal{F}^{Z^1}_t)_{t\geq 0}$-semimartingale. Let $T = \mathbb{Q} \cap [0,t]$, $q(f) = \sup_{s\in T}|f(s)|$ for all $f \in \mathbb{R}^T$ and μ be given by (4.29) with ν replaced by ν_1. Since ν_1 has compact support and ϕ is locally bounded (recall from page 83 that ϕ is chosen càdlàg) there exists an $r_0 > 0$ such that $\mu(f \in \mathbb{R}^T : q(f) \geq r_0) = 0$ and hence, according to Rosiński and Samorodnitsky [35, Lemma 2.2], $\mathrm{E}[\sup_{s\in[0,t]}|X^1_s|] < \infty$. This shows that $(X^1_t)_{t\geq 0}$ is a special $(\mathcal{F}^{Z^1}_t)_{t\geq 0}$-semimartingale. Let $X^1_t = M_t + A_t$ denote the canonical $(\mathcal{F}^{Z^1}_t)_{t\geq 0}$-decomposition of $(X^1_t)_{t\geq 0}$. Then Lemma 4.6 yields $(M_t)_{t\geq 0} = (\phi(0)Z^1_t)_{t\geq 0}$ and hence $(A_t)_{t\geq 0}$, given by

$$A_t = \int_0^t \psi(t-s)\,\mathrm{d}Z^1_s, \qquad t \geq 0, \tag{4.38}$$

where $\psi(t) = \phi(t) - \phi(0)$ for $t \geq 0$, is of bounded variation. Thus, by Lemma 4.7 we conclude that ψ, and hence also ϕ, is absolutely continuous on $\mathbb{R}_+$ with a density having locally ξ-moment.

Assume conversely that ϕ is absolutely continuous with a density having locally ξ-moment. Since ϕ and $(Z^2_t)_{t\geq 0}$ are càdlàg and of bounded variation it follows that $(X^2_t)_{t\geq 0}$

is càdlàg and of bounded variation as well. Let $(A_t)_{t\geq 0}$ be given by (4.38). By Lemma 4.7 it follows that $(A_t)_{t\geq 0}$ is càdlàg and of bounded and hence $(X^1_t)_{t\geq 0} = (\phi(0)Z^1_t + A_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale and we have shown that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale.

(3) Let $(Z_t)_{t\geq 0}$ be of bounded variation.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. By arguing as in (2) it follows that $(A_t)_{t\geq 0}$ given by (4.38) is of bounded variation. Hence Rosiński [33, Theorem 4] and a symmetrization argument shows that ψ, and hence also ϕ, is of bounded variation.

Assume conversely that ϕ is of bounded variation. Since $(Z_t)_{t\geq 0}$ is càdlàg and of bounded variation it follows that $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation and hence an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. ☐

To show Proposition 3.2 we need the following Fubini type result.

Lemma 4.9. *Let $T > 0$, μ denote a finite measure on $\mathbb{R}_+$ and let $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ be a measurable function such that either* (i) *or* (ii) *are satisfied, where*

(i) $\sigma^2 = 0$, $\xi(|f(t,\cdot)|) \in L^1([0,T],\lambda)$ for all $t \geq 0$ and $\xi(|f|) \in L^1(\mathbb{R}_+ \times [0,T], \mu \times \lambda)$.

(ii) $\sigma^2 > 0$, $f(t,\cdot) \in L^2([0,T],\lambda)$ for all $t \geq 0$, and $f \in L^2(\mathbb{R}_+ \times [0,T], \mu \times \lambda)$.

Then $(\int_0^T f(t,s)\,\mathrm{d}Z_s)_{t\geq 0}$ can be chosen measurable and in this case

$$\int \Big(\int_0^T f(t,s)\,\mathrm{d}Z_s \Big) \mu(\mathrm{d}t) = \int_0^T \Big(\int f(t,s)\,\mu(\mathrm{d}t) \Big) \mathrm{d}Z_s \qquad \text{P-}a.s. \tag{4.39}$$

Proof. Assume (i) is satisfied. To show (4.39) we may and do assume that $(Z_t)_{t\geq 0}$ has characteristic triplet $(0,0,\nu)$ where ν is concentrated on $[-1,1]$. Let g be given by (4.1). Since g is 0 at 0, symmetric, increasing, convex, $\lim_{x\to\infty} g(x) = \infty$ and $g(2x) \leq 4g(x)$ for all $x \geq 0$, g is a Young function satisfying the Δ_2-condition (see Rao and Ren [29, page 5+22]). Let $L^g([0,T],\lambda)$ denote the Orlicz space of measurable functions with finite g-moment on $[0,T]$ equipped with the norm

$$\|h\|_g = \inf\{c > 0 : \int_0^T g(c^{-1}h(s))\,\mathrm{d}s \leq 1\}. \tag{4.40}$$

According to Chapter 3.3, Theorem 10, and Chapter 3.5, Theorem 1, in [29], $L^g([0,T],\lambda)$ is a separable Banach space. Let $f_t := f(t,\cdot)$ for all $t \geq 0$. Since $\xi(|f_t|) \in L^1([0,T],\lambda)$ for all $t \geq 0$, it is easy to check that f_t satisfies (2.2)–(2.4) and hence $Y_t := \int_0^T f_t(s)\,\mathrm{d}Z_s$ is well-defined for all $t \geq 0$. We show that $(Y_t)_{t\geq 0}$ has a measurable modification. Since $L^g([0,T],\lambda)$ is separable and $t \mapsto \|f_t - h\|_g$ is measurable for all $h \in L^g([0,T],\lambda)$ it follows that $t \mapsto f_t$ is a measurable mapping from $\mathbb{R}_+$ into $L^g([0,T],\lambda)$. Furthermore, since $L^g([0,T],\lambda)$ is separable there exists $(h^n_k)_{n,k\geq 1} \subseteq L^g([0,T],\lambda)$ and disjoint measurable sets $(A^n_k)_{k\geq 1}$ for all $n \geq 1$ such that with

$$f^n_t(s) = \sum_{k\geq 1} h^n_k(s) 1_{A^n_k}(t), \tag{4.41}$$

we have $\|f_t - f^n_t\|_g \leq 2^{-n}$ for all $t \geq 0$. Set $Y^n_t = \sum_{k\geq 1} \int_0^T h^n_k(s)\,\mathrm{d}Z_s 1_{A^n_k}(t)$ for all $t \geq 0$ and $n \geq 1$. Then $(Y^n_t)_{t\geq 0}$ is a measurable process and by Marcus and Rosiński [23, Theorem 2.1] it follows that

$$\|Y^n_t - Y_t\|_{L^1(\mathrm{P})} \leq 3\|f^n_t - f_t\|_g \leq 3 \times 2^{-n}, \qquad \forall t \geq 0, \forall n \geq 1. \tag{4.42}$$

For all $t \geq 0$ and $\omega \in \Omega$ let $\tilde{Y}_t(\omega) = \lim_n Y_t^n(\omega)$ when the limit exists in $\mathbb{R}$ and zero otherwise. Then $(\tilde{Y}_t)_{t\geq 0}$ is measurable and for all $t \in \mathbb{R}$, $\tilde{Y}_t = Y_t$ P-a.s. by (4.42). Thus we have constructed a measurable modification of $(Y_t)_{t\geq 0}$.

Let us show that both sides of (4.39) are well-defined. Since $g/2 \leq \xi \leq g$ and $\xi(ax) \leq (a+1)^2\xi(x)$ for all $x, a > 0$, it follows by Jensen's inequality that

$$\int_0^T \xi\Big(\int |f(t,s)|\,\mu(\mathrm{d}t)\Big)\,\mathrm{d}s \leq \frac{2(\mu(\mathbb{R})+1)^2}{\mu(\mathbb{R})}\int_0^T \int \xi(|f(t,s)|)\,\mu(\mathrm{d}t)\,\mathrm{d}s < \infty. \tag{4.43}$$

Thus, the right-hand side of (4.39) is well-defined. The left-hand side is well-defined as well since

$$\mathrm{E}\Big[\int \Big|\int_0^T f(t,s)\,\mathrm{d}Z_s\Big|\mu(\mathrm{d}t)\Big] \tag{4.44}$$

$$\leq 3\int \Big(\int_0^T \xi(|f_t(s)|)\,\mathrm{d}s\Big) \vee \Big(\int_0^T \xi(|f_t(s)|)\,\mathrm{d}s\Big)^{1/2}\mu(\mathrm{d}t) < \infty, \tag{4.45}$$

where the first inequality follows by Marcus and Rosiński [23, Corollary 1.1]. Furthermore, (4.39) is obviously true for simple f on the form

$$f(t,s) = \sum_{i=1}^n \alpha_i 1_{(s_{i-1},s_i]}(t)1_{(t_{i-1},t_i]}(s). \tag{4.46}$$

If f is a given function satisfying (i) we can choose a sequence of simple $(f_n)_{n\geq 1}$ converging to f and satisfying $|f_n| \leq |f|$. We have

$$\int \Big(\int_0^T f_n(u,s)\,\mathrm{d}Z_u\Big)\mu(\mathrm{d}s) = \int_0^T \Big(\int f_n(u,s)\,\mu(\mathrm{d}s)\Big)\mathrm{d}Z_u, \tag{4.47}$$

and by estimates as above it follows that we can go to the limit in $L^1(\mathrm{P})$ in (4.47), which shows (4.39).

The case (ii) follows by a similar argument. In this case we have to work in $L^2([0,T],\lambda)$ instead of $L^g([0,T],\lambda)$. □

Proposition 3.2 is an immediate consequence of Theorem 3.1 and Lemma 4.9, since

$$\phi(t-s) = \phi(0) + \int_0^{t-s}\phi'(u)\,\mathrm{d}u = \phi(0) + \int_0^t 1_{\{s\leq u\}}\phi'(u-s)\,\mathrm{d}u, \qquad s \in [0,t]. \tag{4.48}$$

5 The two-sided case

Let $(X_t)_{t\geq 0}$ be given by

$$X_t = \int_{-\infty}^t (\phi(t-s) - \psi(-s))\,\mathrm{d}Z_s, \qquad t \geq 0, \tag{5.1}$$

where $(Z_t)_{t\in\mathbb{R}}$ is a (two-sided) nondeterministic Lévy process with characteristic triplet (γ, σ^2, ν) and $\phi, \psi\colon \mathbb{R} \to \mathbb{R}$ are measurable functions for which the integral exists (still in the sense of Rajput and Rosiński [28, page 460]). Also assume that ϕ and ψ are 0 on $(-\infty, 0)$ and let $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$ denote the least filtration for which $\sigma(Z_s : -\infty < s \leq t) \subseteq$

$\mathcal{F}_t^{Z,\infty}$ for all $t \geq 0$. From Rajput and Rosiński [28, Theorem 2.8] it follows that $(X_t)_{t\geq 0}$ is well-defined if and only if

$$X_t^1 = \int_0^t \phi(t-s)\,\mathrm{d}Z_s, \quad \text{and} \quad X_t^2 = \int_{-\infty}^0 (\phi(t-s) - \psi(-s))\,\mathrm{d}Z_s, \tag{5.2}$$

are well-defined. Similar to Lemma 4.8 we have the following.

Lemma 5.1. *Let $(Y_t)_{t\geq 0}, (U_t)_{t\in\mathbb{R}}, (\tilde{Y}_t)_{t\geq 0}$ and $(\tilde{U}_t)_{t\in\mathbb{R}}$ denote four processes such that $(Y_t)_{t\geq 0}$ is $(\mathcal{F}_t^{U,\infty})_{t\geq 0}$-adapted, $(\tilde{Y}_t)_{t\geq 0}$ is $(\mathcal{F}_t^{\tilde{U},\infty})_{t\geq 0}$-adapted and $(Y_\cdot, U_\cdot) \overset{\mathscr{D}}{=} (\tilde{Y}_\cdot, \tilde{U}_\cdot)$. If $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{U,\infty})_{t\geq 0}$-semimartingale then $(\tilde{Y}_t)_{t\geq 0}$ has a modification which is an $(\mathcal{F}_t^{\tilde{U},\infty})_{t\geq 0}$-semimartingale.*

Lemma 5.2. *Assume $(Z_t)_{t\in\mathbb{R}}$ is symmetric. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale and $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation.*

Proof. The *if*-part is trivial. To show the *only if*-part assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. Let $\tilde{X}_t = X_t^1 - X_t^2$ and let $\tilde{Z}_t = Z_t$ for $t \geq 0$ and $\tilde{Z}_t = -Z_t$ when $t < 0$. Since $(Z_t)_{t\in\mathbb{R}}$ is symmetric $(X_\cdot, Z_\cdot) \overset{\mathscr{D}}{=} (\tilde{X}_\cdot, \tilde{Z}_\cdot)$ and from Lemma 5.1 it follows that $(\tilde{X}_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{\tilde{Z},\infty})_{t\geq 0}$-semimartingale and hence an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale since $(\mathcal{F}_t^{\tilde{Z},\infty})_{t\geq 0} = (\mathcal{F}_t^{Z,\infty})_{t\geq 0}$. Thus, $(X_t^1)_{t\geq 0} = ((X_t + \tilde{X}_t)/2)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale and hence an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. Moreover, $(X_t^2)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale and hence càdlàg and of bounded variation since X_t^2 is $\mathcal{F}_0^{Z,\infty}$-measurable for all $t \geq 0$. $\square$

We have the following consequence of Lemma 5.2 and Theorem 3.1.

Proposition 5.3. *Let $(X_t)_{t\geq 0}$ be given by (5.1) and assume it is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale.*

If $(Z_t)_{t\in\mathbb{R}}$ is of unbounded variation then ϕ is absolutely continuous on $\mathbb{R}_+$ with a density ϕ' satisfying (1.3)–(1.4).

If $(Z_t)_{t\in\mathbb{R}}$ is of bounded variation then $(X_t)_{t\geq 0}$ is of bounded variation and ϕ is of bounded variation as well.

Proof. Let $\tilde{Z}_t = Z_t - Z_t'$ where $(Z_t')_{t\in\mathbb{R}}$ is an independent copy of $(Z_t)_{t\in\mathbb{R}}$ and let $(X_t')_{t\geq 0}$ be given by

$$X_t' = \int_{-\infty}^t (\phi(t-s) - \psi(-s))\,\mathrm{d}Z_s', \qquad t \geq 0. \tag{5.3}$$

By Lemma 5.1, $(X_t')_{t\geq 0}$ is an $(\mathcal{F}_t^{Z',\infty})_{t\geq 0}$-semimartingale, which by independence of filtrations shows that $(\tilde{X}_t)_{t\geq 0} := (X_t - X_t')_{t\geq 0}$ is a semimartingale in the $(\mathcal{F}_t^{Z,\infty} \vee \mathcal{F}_t^{Z',\infty})_{t\geq 0}$-filtration and hence in the $(\mathcal{F}_t^{\tilde{Z},\infty})_{t\geq 0}$-filtration. Since $(\tilde{Z}_t)_{t\in\mathbb{R}}$ is symmetric Lemma 5.2 shows that $(\tilde{X}_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^{\tilde{Z}})_{t\geq 0}$-semimartingale and since $(\tilde{Z}_t)_{t\geq 0}$ has characteristic triplet $(0, 2\sigma^2, \tilde{\nu})$ where $\tilde{\nu}(A) = \nu(A) + \nu(-A)$, the proposition follows by Theorem 3.1. $\square$

Let $(X_t)_{t\geq 0}$ denote a fractional Lévy motion, that is

$$X_t = \int_{-\infty}^t ((t-s)^\tau - (-s)_+^\tau)\,\mathrm{d}Z_s, \qquad t \geq 0, \tag{5.4}$$

where τ is such that the integral exists and $x_+ := x \vee 0$ for all $x \in \mathbb{R}$. In the following let us assume $(Z_t)_{t\in\mathbb{R}}$ has no Brownian component. Recall the definition of X_t^2 in (5.2). From Rajput and Rosiński [28, Theorem 2.8] it follows that it is necessary (and sufficient when $(Z_t)_{t\geq 0}$ is symmetric) that

$$\int_0^\infty \int \left(|x((t+s)^\tau - s^\tau)|^2 \wedge 1 \right) \nu(\mathrm{d}x)\, \mathrm{d}s < \infty \tag{5.5}$$

for X_t^2 to be well-defined. A simple calculation shows that (5.5) is satisfied if and only if

$$\tau < 1/2 \qquad \text{and} \qquad \int_{[-1,1]^c} |x|^{1/(1-\tau)}\, \nu(\mathrm{d}x) < \infty. \tag{5.6}$$

Thus it is necessary that (5.6) and (I)–(III) on page 81 are satisfied for $(X_t)_{t\geq 0}$ to be well-defined, and when $(Z_t)_{t\in\mathbb{R}}$ is symmetric these conditions are also sufficient. [25] studies processes of the form (5.4) under the assumptions that $\sigma^2 = 0$, $\int_{[-1,1]^c} |x|^2\, \nu(\mathrm{d}x) < \infty$, $\gamma = -\int_{[-1,1]^c} x\, \nu(\mathrm{d}x)$ and $0 < \tau < 1/2$. See also [4] for a study of the well-balanced case.

To avoid trivialities assume $\tau \neq 0$. As an application of Proposition 5.3 and Corollary 3.5 we have the following.

Corollary 5.4. *Assume $(Z_t)_{t\in\mathbb{R}}$ has no Brownian component and let $(X_t)_{t\geq 0}$ be an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale given by (5.4). Then $\int_{[-1,1]} |x|^{1/(1-\tau)}\, \nu(\mathrm{d}x) < \infty$ and $\tau \in (0, 1/2)$.*

In particular let $(X_t)_{t\geq 0}$ denote a linear fractional stable motion with indexes $\alpha \in (0,2]$ and $H \in (0,1)$, that is

$$X_t = \int_{-\infty}^t \left((t-s)^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right) \mathrm{d}Z_s, \qquad t \geq 0, \tag{5.7}$$

where $(Z_t)_{t\in\mathbb{R}}$ is a symmetric α-stable Lévy process (see Samorodnitsky and Taqqu [37, Definition 7.4.1]). For $\alpha = 2$, $(X_t)_{t\geq 0}$ is a fractional Brownian motion (fBm) with Hurst parameter H (up to a scaling constant). From Corollary 5.4 it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if $H = 1/\alpha$.

* * *

Let $(X_t)_{t\geq 0}$ be given by (5.1) and assume $(Z_t)_{t\in\mathbb{R}}$ is a symmetric α-stable Lévy process with $\alpha \in (1,2]$. If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale it follows by Proposition 5.3 and (1) on page 79 that ϕ is absolutely continuous on $\mathbb{R}_+$ with a density having locally α-moment. The next result shows that this condition is actually necessary and sufficient for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if we delete "locall". Thus, extending Knight [19, Theorem 6.5] from $\alpha = 2$ to $\alpha \in (1,2]$ we have the following.

Proposition 5.5. *Let $(X_t)_{t\geq 0}$ be given by (5.1) and assume $(Z_t)_{t\in\mathbb{R}}$ is a symmetric α-stable Lévy process with $\alpha \in (1,2]$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a density in $L^\alpha(\mathbb{R}_+, \lambda)$.*

Let B denote a Banach space (not necessarily separable) and assume there exists a countable subset D of the unit ball of B' (the topological dual space of B) such that

$$\|x\| = \sup_{F\in D} |F(x)|, \qquad \forall x \in B. \tag{5.8}$$

Following Ledoux and Talagrand [21, page 133], a B-valued random element X is called α-stable if $\sum_{i=1}^n a_i F_i(X)$ is a real-valued α-stable random variable for all $n \geq 1$, $F_1, \ldots, F_n \in D$ and $a_1, \ldots, a_n \in \mathbb{R}$.

Let T denote an interval in $\mathbb{R}_+$ and let B denote the subspace of $\mathbb{R}^T$ containing all functions which are càdlàg and of bounded variation. Then B is a Banach space in the total variation norm (but not separable) and since the unit ball of B' consists of F of the form

$$F(f) = \sum_{i=1}^{n} a_i(f(t_i) - f(t_{i-1})), \qquad f \in B, \tag{5.9}$$

where $(a_i)_{i=1}^n \subseteq [-1,1]$ and $(t_i)_{i=0}^n$ is an increasing sequence in T, it follows that B satisfies (5.8).

Proof of Proposition 5.5. For $\alpha = 2$ the result follows by Cherny [10, Theorem 3.1]; thus let us assume $\alpha \in (1,2)$.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. According to Lemma 5.2 $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation. Consider $(X_t^2)_{t\geq 0}$ as an α-stable random element with values in the Banach space consisting of functions which are càdlàg and of bounded variation equipped with the total variation norm. Hence from Ledoux and Talagrand [21, Proposition 5.6] it follows that $(X_t^2)_{t\geq 0}$ is of integrable variation on each compact interval. Moreover, by Marcus and Rosiński [23, Corollary 1.1] we have

$$\mathrm{E}[|n(X^2_{i/n} - X^2_{(i-1)/n})|] \geq \tfrac{1}{4}\left(a_{i,n} \wedge \sqrt{a_{i,n}}\right), \qquad i,n \geq 1, \tag{5.10}$$

where

$$a_{i,n} := \int_{(i-1)/n}^{\infty} \tilde{\xi}(|\Delta_n \phi(s)|)\,\mathrm{d}s, \quad \text{and} \quad \tilde{\xi}(x) := \int (|xs|^2 \wedge |xs|)\,\nu(\mathrm{d}s). \tag{5.11}$$

Since $i \mapsto a_{i,n}$ is decreasing it follows that

$$\mathrm{E}[\mathrm{Var}_1(X^2)] \geq \sup_{n\geq 1} \sum_{i=1}^{n} \mathrm{E}[|X^2_{i/n} - X^2_{(i-1)/n}|] \geq \sup_{n\geq 1} \frac{1}{4}\left(a_{n,n} \wedge \sqrt{a_{n,n}}\right). \tag{5.12}$$

By (5.12) we conclude that $(a_{n,n})_{n\geq 1}$ is bounded and hence $(\tilde{\xi}(|\Delta_n\phi|))_{n\geq 1}$ is bounded in $L^1([1,\infty),\lambda)$. A straightforward calculation shows $\tilde{\xi}(x) = c_1 x^\alpha$ for all $x \geq 0$ for some constant $c_1 > 0$, which implies that $(\Delta_n\phi)_{n\geq 1}$ is bounded in $L^\alpha([1,\infty),\lambda)$. Since $\alpha > 1$, a sequence in $L^\alpha([1,\infty),\lambda)$ is bounded if and only if it is weakly sequentially compact (see Dunford and Schwartz [14, Chapter IV.8, Corollary 4]). Thus, by arguing as in Lemma 4.1 it follows that ϕ is absolutely continuous with a density in $L^\alpha([1,\infty),\lambda)$. Furthermore, since $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale it follows by Corollary 3.4 that ϕ is absolutely continuous on $\mathbb{R}_+$ with a density locally in $L^\alpha(\mathbb{R}_+,\lambda)$. This shows the *only if*-part.

Assume conversely ϕ is absolutely continuous on $\mathbb{R}_+$ with a density in $L^\alpha(\mathbb{R}_+,\lambda)$. By Corollary 3.4 $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}^Z)_{t\geq 0}$-semimartingale. Thus it is enough to show that $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation. Since ϕ is absolutely continuous on $\mathbb{R}_+$ with a density in $L^\alpha(\mathbb{R}_+,\lambda)$ it follows by arguing as in Lemma 4.1 that $\|\phi(t-\cdot) - \phi(u-\cdot)\|_{L^\alpha((-\infty,0),\lambda)} \leq c(t-u)$ for some $c > 0$ and all $0 \leq u \leq t$. For all $p \in [1,\alpha)$ and all $u,t \geq 0$ we have

$$\|X_t^2 - X_u^2\|_{L^p(\mathrm{P})} = K_{p,\alpha}\|\phi(t-\cdot) - \phi(u-\cdot)\|_{L^\alpha((-\infty,0),\lambda)} \leq K_{p,\alpha} c|t-u|, \tag{5.13}$$

for some constant $K_{p,\alpha} > 0$ only depending on p and α. By letting $p \in (1,\alpha)$, (5.13) and the Kolmogorov-Čentsov Theorem show that $(X_t^2)_{t\geq 0}$ has a continuous modification. Moreover, by letting $p = 1$ (5.13) shows that this modification is of integrable variation on each compact interval. This completes the proof. □

Motivated by Lemma 5.2 we study in the following proposition infinitely divisible processes $(X_t)_{t\geq 0}$ of bounded variation, where $(X_t)_{t\geq 0}$ is on the form $X_t = \int_{\mathbb{R}} f(t,s)\,\mathrm{d}Z_s$. Assume $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation. Rosiński [33, Theorem 4] shows that $t \mapsto f(t,s)$ is of bounded variation for λ-a.a. $s \in \mathbb{R}$. Extending this we show that the total variation of $f(\cdot,s)$ must satisfy an integrability condition which is equivalent to the existence of $\int_{\mathbb{R}} \mathrm{Var}_t(f(\cdot,s))\,\mathrm{d}Z_s$ for all $t > 0$ when $(Z_t)_{t\in\mathbb{R}}$ is symmetric and has no Brownian component.

Proposition 5.6. *Let $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ denote a measurable function such that $X_t = \int_{\mathbb{R}} f(t,s)\,\mathrm{d}Z_s$ is well-defined for all $t \geq 0$. If $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation then*

$$\int\!\!\int \Big(1 \wedge |x\mathrm{Var}_t(f(\cdot,s))|^2\Big)\,\nu(\mathrm{d}x)\,\mathrm{d}s < \infty, \qquad \forall t > 0. \tag{5.14}$$

Let $(\epsilon_i)_{i\geq 1}$ denote a Rademacher sequence, i.e. $(\epsilon_i)_{i\geq 1}$ is an i.i.d. sequence such that $\mathrm{P}(\epsilon_1 = -1) = \mathrm{P}(\epsilon_1 = 1) = 1/2$. It is well-known that if $(\alpha_i)_{i\geq 1} \subseteq \mathbb{R}$ then $\sum_{i=1}^\infty \epsilon_i\alpha_i$ converges P-a.s. if and only if $\sum_{i=1}^\infty \alpha_i^2 < \infty$. Let B denote a Banach space satisfying (5.8). Following Ledoux and Talagrand [21, page 99], a B-valued random element X is called a vector-valued Rademacher series if there exists a sequence $(x_i)_{i\geq 1}$ in B such that $\sum_{i=1}^\infty F^2(x_i) < \infty$ for all $F \in D$, and for all $n \geq 1$ and all $F_1,\dots,F_n \in D$ $(F_1(X),\dots,F_n(X))$, and $(\sum_{i=1}^\infty \epsilon_i F_1(x_i),\dots,\sum_{i=1}^\infty \epsilon_i F_n(x_i))$ has the same distribution.

Proof of Proposition 5.6. By a symmetrization argument we may and do assume that $\sigma^2 = 0$ and $(Z_t)_{t\in\mathbb{R}}$ is symmetric. Define

$$Y_t = \sum_{j=1}^\infty \epsilon_j C_j f(t,U_j), \qquad t \geq 0, \tag{5.15}$$

where $(\epsilon_j)_{j\geq 1}$ is a Rademacher sequence, $(\tau_j)_{j\geq 1}$ are the partial sums of i.i.d. standard exponential random variables and $(U_j)_{j\geq 1}$ are i.i.d. standard normal random variables with density ρ, and $(\epsilon_j)_{j\geq 1}$, $(\tau_j)_{j\geq 1}$ and $(U_j)_{j\geq 1}$ are independent. Let $\nu^{\leftarrow}\colon \mathbb{R}_+ \to \mathbb{R}_+$ denote the right-continuous inverse of the mapping $x \mapsto \nu((x,\infty))$, that is, $\nu^{\leftarrow}(s) = \inf\{x > 0 : \nu((x,\infty)) \leq s\}$, and let $C_j := \nu^{\leftarrow}(\tau_j\rho(U_j))$ for all $j \geq 1$. By Rosiński [33, Proposition 2], the series (5.15) converges P-a.s. and $(Y_t)_{t\geq 0}$ has the same finite dimensional distributions as $(X_t)_{t\geq 0}$. Thus, $(Y_t)_{t\geq 0}$ has a càdlàg modification of locally bounded variation. Hence we may and do assume $(X_t)_{t\geq 0}$ is given by (5.15). Moreover we may define $(\epsilon_j)_{j\geq 1}$ on a probability space $(\Omega',\mathcal{F}',\mathrm{P}')$, $(\tau_j)_{j\geq 1}$ and $(U_j)_{j\geq 1}$ on a probability space $(\Omega'',\mathcal{F}'',\mathrm{P}'')$ and $(X_t)_{t\geq 0}$ on the product space. Let $T = [0,t]$ denote a compact interval in $\mathbb{R}_+$ and let B denote the subspace of $\mathbb{R}^T$ consisting of functions which are càdlàg and of bounded variation. Inspired by [24] let us fix $\omega'' \in \Omega''$ and consider $\underline{X} = (X_t)_{t\in T}$ as a B-valued Rademacher series under P'. From Ledoux and Talagrand [21, Theorem 4.8] it follows that $\mathrm{E}'[e^{\alpha\|\underline{X}\|^2}] < \infty$ for all $\alpha > 0$, which in particular shows that $(X_t)_{t\in T}$ is of P'-integrable variation. By Khinchine's inequality there exists a constant $c > 0$ such that $\mathrm{E}'[|X_t - X_u|] \geq c\|X_t - X_u\|_{L^2(\mathrm{P}')}$ for all $u,t \geq 0$. Together with the triangle inequality in l^2 this shows that

$$\mathrm{E}'\Big[\sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}|\Big] \geq c\sum_{i=1}^n \Big(\sum_{j=1}^\infty C_j^2(f(t_i,U_j) - f(t_{i-1},U_j))^2\Big)^{1/2} \tag{5.16}$$

$$\geq c\Big(\sum_{j=1}^\infty \Big(\sum_{i=1}^n |C_j(f(t_i,U_j) - f(t_{i-1},U_j))|\Big)^2\Big)^{1/2} \tag{5.17}$$

$$= c\Big(\sum_{j=1}^{\infty}\Big(|C_j|\sum_{i=1}^{n}|f(t_i,U_j)-f(t_{i-1},U_j)|\Big)^2\Big)^{1/2}. \tag{5.18}$$

Thus, by monotone convergence we conclude

$$\mathrm{E}'[\mathrm{Var}_t(X)] \geq c\Big(\sum_{j=1}^{\infty}\big(C_j\mathrm{Var}_t(f(\cdot,U_j))\big)^2\Big)^{1/2}, \tag{5.19}$$

and in particular $(C_j\mathrm{Var}_t(f(\cdot,U_j))_{j\geq 1} \in l^2$. Hence, $\sum_{j=1}^{\infty}\epsilon_j C_j\mathrm{Var}_t(f(\cdot,U_j))$ converges P-a.s. and from Theorem 2.4 and Proposition 2.7 in [34] it follows that

$$\int_0^{\infty}\int\big(1\wedge H(u,v)^2\big)\,\rho(v)\,\mathrm{d}v\,\mathrm{d}u < \infty, \tag{5.20}$$

where $H(u,v) = \nu^{\leftarrow}(u\rho(v))\mathrm{Var}_t(f(\cdot,v))$. Furthermore, (5.20) equals

$$\int\int\big(1\wedge(\nu^{\leftarrow}(u)\mathrm{Var}_t(f(\cdot,v)))^2\big)\frac{1}{\rho(v)}\,\mathrm{d}u\,\rho(v)\,\mathrm{d}v \tag{5.21}$$

$$= \int\int\big(1\wedge(u\mathrm{Var}_t(f(\cdot,v))^2\big)\,\nu(\mathrm{d}u)\,\mathrm{d}v, \tag{5.22}$$

which shows (5.14). $\square$

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PAPER E

Path and semimartingale properties of chaos processes

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Abstract

The present paper characterizes various properties of chaos processes which in particular includes processes where all time variables admit a Wiener chaos expansion of a fixed finite order. The main focus is on the semimartingale property, p-variation and continuity. The general results obtained are finally used to characterize when a moving average is a semimartingale.

Keywords: semimartingales; p-variation; moving averages; chaos processes; absolutely continuity

AMS Subject Classification: 60G48; 60G51; 60G17; 60G15; 60G10

1 Introduction

The present paper is concerned with various properties of chaos processes. Chaos processes includes processes for which all coordinates belongs to a Wiener chaos of a fixed finite order, infinitely divisible processes, Rademacher processes, linear processes and more general processes which are limits of tetrahedral polynomials; see Section 2 for more details. In Rosiński et al. [29] continuity and zero-one laws are derived for some classes of chaos processes. Houdré and Pérez-Abreu [11] and Janson [16] provides good surveys on various aspects of chaos processes.

In the first part we extend important results for Gaussian to chaos processes. In particular that of Jain and Monrad [15] saying that if a separable Gaussian process is of bounded variation then the L^2-expansion converge in total variation norm to the process. Together with the observation by Jeulin [17] that the process in this case is absolutely continuous with respect to a deterministic measure. Likewise the characterization of a stationary Gaussian processes of bounded variation, Ibragimov [12], and the canonical decomposition of a Gaussian quasimartingale, Jain and Monrad [15], together with the extension to Gaussian semimartingales, Stricker [30], are generalized. Extensions of the result on Gaussian Dirichlet processes obtained by Stricker [31] are also given. Furthermore we prove that chaos processes admitting a p-variation for some $p \geq 1$ are almost surely continuous except on an at most countable set, generalizing a result of Itô and Nisio [13].

In the second part we study moving averages $X = \phi * Y$ also known as stochastic convolutions. When Y is a Brownian motion, Knight [19] has characterized those kernels ϕ for which X is an $\mathcal{F}^Y$-semimartingale, and Jeulin and Yor [18] and Basse [2] those ϕ for which X is an $\mathcal{F}^X$-semimartingale. Basse and Pedersen [4] have characterized those ϕ for which X is an $\mathcal{F}^Y$-semimartingale in the case where Y is Lévy process. Moreover, Basse [1] extends Knight's result to the spectral representation of general Gaussian processes. Using the obtained decomposition results we provide necessary and sufficient conditions on ϕ for X to be an $\mathcal{F}^Y$-semimartingale. This result covers in particular the case where $\mathrm{d}Y_t = \sigma_t \,\mathrm{d}W_t$ and σ is Gaussian chaos process associated with the Brownian motion W.

2 Preliminaries

Let $(\Omega, \mathcal{B}, \mathrm{P})$ denote a complete probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. $T > 0$ is here a fixed positive number. A càdlàg $\mathcal{F}$-adapted process $X = (X_t)_{t\in[0,T]}$ is called an $\mathcal{F}$-semimartingale if it admits a representation

$$X_t = X_0 + A_t + M_t, \qquad t \in [0, T], \tag{2.1}$$

where M is a càdlàg $\mathcal{F}$-local martingale starting at 0 and A is a càdlàg process of bounded variation starting at 0. Furthermore, X is called a special $\mathcal{F}$-semimartingale if A in (2.1) can be chosen predictable and in this case the decomposition is unique. A special $\mathcal{F}$-semimartingale X with canonical decomposition $X = X_0 + M + A$, is said to belong to H^p for $p \geq 1$ if $\mathrm{E}[[M]_T^{p/2} + V_A(T)^p] < \infty$. $V_A(t)$ denotes the total variation of $s \mapsto A_s$ on $[0, t]$ and $[M]_t$ the quadratic variation of M on $[0, t]$. For each càdlàg process X set $D_X = \{t \in [0, T] : \mathrm{P}(X_t = X_{t-}) < 1\}$. Then as it is well-known D_X is at most countable and D_X is empty if and only if X is continuous in probability.

Variation of processes will be important. To simplify the notation we set for each $p \geq 1$, $X = (X_t)_{t\in[0,T]}$ and $\tau = \{0 \leq t_0 < \cdots < t_n \leq T\}$

$$|\tau| = \max_{1\leq i\leq n} |t_i - t_{i-1}| \qquad \text{and} \qquad V_X^{p,\tau} = \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}|^p. \tag{2.2}$$

We say that X admits a p-variation if there exists a right-continuous process $[X]^{(p)}$ such that for all $t \in [0,T]$ $V_X^{p,\tau} \to [X]_t^{(p)}$ in probability as $|\tau| \to 0$, where τ runs through all subdivisions of $[0,t]$. Furthermore, X is said to be of bounded p-variation if $\{V_X^{p,\tau} : \tau \text{ subdivision of } [0,T]\}$ is bounded in L^0. If $p = 2$ we use the short-hand notation $[X]$ for the quadratic variation of X, that is $[X] = [X]^{(2)}$. Observe that $V_X(t) = [X]_t^{(1)}$, if $V_X(T) < \infty$ a.s.

If X admits a p-variation then it is also of bounded p-variation. Likewise if X is of bounded p-variation it is also of bounded q-th variation for all $q \geq p$ since $p \mapsto (\sum_{i=1}^n |a_i|^p)^{1/p}$ is decreasing. If X is càdlàg and τ_n are subdivisions of $[0,T]$ such that $|\tau_n| \to 0$ then

$$\liminf_{n\to\infty} V_X^{p,\tau_n} \geq \sum_{0<s\leq T} |\Delta X_s|^p, \qquad \text{a.s.} \tag{2.3}$$

Thus using

$$\mathrm{P}(\liminf_{n\to\infty} V_X^{p,\tau_n} > x) \leq \sup_{n\geq 1} \mathrm{P}(V_X^{p,\tau_n} > x), \qquad \text{for all } x > 0, \tag{2.4}$$

we have that $\sum_{0<s\leq T} |\Delta X_s|^p < \infty$ a.s. if X is of bounded p-variation.

Throughout the following I denotes a set and for all $i \in I$, $\mathcal{H}_i$ is a family of independent random variables. Set $\mathcal{H} = \{\mathcal{H}_i\}_{i\in I}$. For each Banach space F and $i \in I$ let $\mathcal{P}^d_{\mathcal{H}_i}(F)$ denote the set of variables $p(Z_1, \ldots, Z_n)$ where $n \geq 1$, $Z_1, \ldots, Z_n$ different elements in $\mathcal{H}_i$ and p is an F-valued tetrahedral polynomial of order d. Recall that $p\colon \mathbb{R}^n \to F$ is called an F-valued tetrahedral polynomial of order d if there exist $x_0, x_{i_1,\ldots,i_k} \in F$ such that

$$p(z_1, \ldots, z_n) = x_0 + \sum_{k=1}^d \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1,\ldots,i_k} \prod_{j=1}^k z_{i_j}. \tag{2.5}$$

Let $\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ denote the closure in distribution of $\cup_{i\in I} \mathcal{P}^d_{\mathcal{H}_i}(F)$, that is, $\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ is the set of all F-valued random elements X for which there exists a sequence $(X_k)_{k\geq 1} \subseteq \cup_{i\in I} \mathcal{P}^d_{\mathcal{H}_i}(F)$ converging weakly to X.

The following two conditions on $\mathcal{H}$ will be important:

(a) For $q \in (0,\infty)$ there exists $\beta_1, \beta_2 > 0$ such that for all $Z \in \cup_{i\in I} \mathcal{H}_i$ there exists $c_Z > 0$ satisfying

$$\mathrm{P}(|Z| \geq c_Z) \geq \beta_1 \quad \text{and} \quad \mathrm{E}[|Z|^q, |Z| > s] \leq \beta_2 s^q \mathrm{P}(|Z| > s) \quad s \geq c_Z. \tag{2.6}$$

(b) $$\bigcup_{i\in I} \mathcal{H}_i \subseteq L^1 \quad \text{and} \quad \sup_{i\in I} \sup_{Z\in\mathcal{H}_i} \Big(\frac{\|Z - \mathrm{E}[Z]\|_\infty}{\|Z - \mathrm{E}[Z]\|_2}\Big) = \beta_3 < \infty. \tag{2.7}$$

Notation, chaos processes. *A real-valued stochastic process $X = (X_t)_{t\in U}$ is said to be a chaos process of order d if $(X_{t_1}, \ldots, X_{t_n}) \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R}^n)$ for all $n \geq 1$ and $t_1, \ldots, t_n \in U$. Furthermore X is said to be a chaos process if it is a chaos process of order d for some $d \geq 1$. A chaos process X is said to satisfy C_q for $0 < q < \infty$, if the associated $\mathcal{H}$ satisfies* (a) *for the given q and if $d \geq 2$ all $Z \in \cup_{i\in I} \mathcal{H}_i$ are symmetric. Moreover, X is said to satisfy C_∞ if $\mathcal{H}$ satisfies* (b).

Following Fernique [9] a mapping N, from a vector space V into $[0,\infty]$, is called a pseudo-seminorm if for all $\theta \in \mathbb{R}$ and $x, y \in V$ we have

$$N(\theta x) = |\theta| N(x) \qquad \text{and} \qquad N(x+y) \leq N(x) + N(y). \tag{2.8}$$

The following result, which is taken from Basse [3, Theorem 2.7], is crucial for this paper. Here $d \geq 1$ and $q > 0$ are given numbers.

Theorem 2.1. *Let U denote a countable set, $X = (X_t)_{t\in U}$ a chaos process of order d satisfying C_q and N a lower semi-continuous pseudo-seminorm on $\mathbb{R}^U$ equipped with the product topology such that $N(X) < \infty$ a.s. Then for all finite $p \leq q$ there exists a real constant $k_{p,q,d,\beta}$, only depending on p, q, d and the β's from* (a) *and* (b)*, such that*

$$\|N(X)\|_q \leq k_{p,q,d,\beta} \|N(X)\|_p < \infty. \tag{2.9}$$

Three important examples of chaos processes satisfying C_q are given as follows:

(1): Let $\mathcal{G}$ denote a vector space of Gaussian random variables, and for $d \geq 1$ $\overline{\mathcal{P}}^d_{\mathcal{G}}$ be the closure in probability of all random variables of the form $p(Z_1,\dots,Z_n)$, where $n \geq 1$, $Z_1,\dots,Z_n \in \mathcal{G}$ and $p\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree at most d (not necessarily tetrahedral). $X = (X_t)_{t\in U}$ satisfying $\{X_t : t \in U\} \subseteq \overline{\mathcal{P}}^d_{\mathcal{G}}$ is then called a Gaussian chaos process of order d, and it is in particular a chaos process satisfying C_∞ (see Basse [3]); in fact we may chose $I = \{0\}$ and $\mathcal{H}_0$ to be a Rademacher sequence. Recall that a Rademacher sequence is an independent, identically distributed sequence $(Z_n)_{n\geq 1}$ such that $\mathrm{P}(Z_1 = \pm 1) = \frac{1}{2}$. The key example of a Gaussian vector space $\mathcal{G}$ is

$$\mathcal{G} = \left\{ \int_0^\infty h(s)\,\mathrm{d}W_s : h \in L^2(\mathbb{R}_+, \lambda) \right\}, \tag{2.10}$$

where W is a Brownian motion and λ is the Lebesgue measure. In this case X is a Gaussian chaos process of order d if and only if it has the following representation in terms of multiple Wiener-Itô integrals

$$X_t = \sum_{k=0}^{d} \int_{\mathbb{R}_+^k} f_{k,t}(s_1,\dots,s_k)\,\mathrm{d}W_{s_1}\cdots\mathrm{d}W_{s_k}, \qquad t \in U, \tag{2.11}$$

where $f_{k,t} \in L^2(\mathbb{R}_+^k)$. Processes of the form (2.11) appear as weak limits of U-statistics, see Janson [16, Chapter 11] and de la Peña and Giné [7]. For a detailed survey on Gaussian chaos processes and expansions, see Janson [16], Nualart [25] and Houdré and Pérez-Abreu [11].

(2): Let $X = (X_t)_{t\in U}$ be given by

$$X_t = \int_S f(t,s)\,\Lambda(\mathrm{d}s), \qquad t \in U, \tag{2.12}$$

where Λ is an independently scattered infinitely divisible random measure (or random measure for short) on some non-empty space S equipped with a δ-ring $\mathcal{S}$, and $s \mapsto f(t,s)$ are Λ-integrable deterministic functions in the sense of Rajput and Rosiński [28]. The associated $\mathcal{H} = \{\mathcal{H}_i\}_{i\in I}$ is here described by

$$\mathcal{H}_i = \{\Lambda(A_1),\dots,\Lambda(A_n)\}, \qquad i \in I, \tag{2.13}$$

for I denoting the set of all finite collections $\{A_1,\dots,A_n\}$ where $A_1,\dots,A_n$ are disjoint sets in $\mathcal{S}$. In this case X is a chaos process of order 1. For example if X is a symmetric

α-stable process separable in L^0, then X has a representation of the form (2.12) and hence it follows that it is a chaos process of order 1 satisfying C_q for all $q < \alpha$. For further examples of random measures Λ for which X given by (2.12) satisfies C_q see Basse [3].

(3): Assume that $(Z_n)_{n\geq 1}$ is a sequence of independent, identically distributed random variables and $x(t), x_{i_1,\dots,i_k}(t) \in \mathbb{R}$ are real numbers such that

$$X_t = x(t) + \sum_{k=1}^{d} \sum_{1\leq i_1<\cdots<i_k<\infty} x_{i_1,\dots,i_k}(t) \prod_{j=1}^{k} Z_{i_j}, \tag{2.14}$$

exists in probability for all $t \in U$, then $X = (X_t)_{t\in U}$ is a chaos process of order d associated to $I = \{0\}$ and $\mathcal{H}_0 = \{Z_n : n \geq 1\}$. If for some $\alpha > 0$, $x \mapsto \mathrm{P}(|Z_1| > x)$ is regularly varying with index $-\alpha$ then $\mathcal{H}$ satisfies (a) for all $q \in (0, \alpha)$; see Bingham et al. [5, Theorem 1.5.11]. In particular, if Z_1 follows a symmetric α-stable distribution for some $\alpha \in (0, 2)$ then $\mathcal{H}$ satisfies (a) for all $q \in (0, \alpha)$. If the common distribution is Poisson, exponential, gamma or Gaussian then $\mathcal{H}$ satisfies (a) for all $q > 0$. Finally, $\mathcal{H}$ satisfies (b) if and only if Z_1 is a.s. bounded.

3 Path properties

For all $p \geq 0$ and all subset A of L^p denote by $\overline{\mathrm{span}}_{L^p} A$ the L^p-closure of the linear span of A. Let $X = (X_t)_{t\in[0,T]}$ be a square-integrable process for which $\overline{\mathrm{span}}_{L^2}\{X_t : t \in [0, T]\}$ is a separable Hilbert space with orthonormal basis $(U_i)_{i\geq 1}$. Let $X_t^{(n)}$ denote the n-th order L^2-expansion of X_t given by

$$X_t^{(n)} = \sum_{j=1}^{n} f_j(t) U_j, \tag{3.1}$$

where $f_j(t) = \mathrm{E}[U_j X_t]$ for $j \geq 1$. Note that for $t \in [0, T]$, $\lim_n X_t^{(n)} = X_t$ in L^2. The above separability assumption is always satisfied if X is a càdlàg process satisfying C_q for some $q \in [2, \infty]$.

If X is càdlàg and of integrable variation μ_X denotes the Lebesgue–Stieltjes measure on $[0, T]$ induced by $t \mapsto \mathrm{E}[V_X(t)]$. In this context we have the following extension of Jain and Monrad [15, Theorem 1.2] and Jeulin [17] in the Gaussian case. Here $BV([0, T])$ denotes the Banach space $\{f \in \mathbb{R}^{[0,T]} : f \text{ càdlàg and } V_f(T) < \infty\}$ equipped with the norm $\|f\|_{BV} = V_f(T) + |f(0)|$.

Theorem 3.1. *Let $X = (X_t)_{t\in[0,T]}$ denote a càdlàg process of bounded variation satisfying C_q for some $q \in [2, \infty]$. Then there exists a subsequence $(n_k)_{k\geq 1}$ such that $X^{(n_k)}$ converges a.s. to X in $BV([0, T])$ and X is a.s. absolutely continuous with respect to μ_X.*

For an α-stable process X of the form (2.12) with $1 < \alpha < 2$, it is shown in Pérez-Abreu and Rocha-Arteaga [26, Theorem 4(b)] that if X is of bounded variation and satisfies some additional assumption then it is absolutely continuous with respect to μ_X. This situation is not covered by Theorem 3.1 since for such processes only C_q for $q \in (0, \alpha)$ is satisfied. If the sample paths of X are contained in a separable subspace of $BV([0, T])$ Theorem 3.1 follows by Basse [3, Corollary 2.11]. On the other hand, Theorem 3.1 insures that almost all sample paths of X do belong to a separable subspace of $BV([0, T])$, more precisely to the space of functions which are absolutely continuous with respect to μ_X.

Theorem 3.1 is a direct consequence of Theorem 2.1 and the following lemma, in which X, $X^{(n)}$ and f_j are as above.

Lemma 3.2. *Assume that $X = (X_t)_{t\in[0,T]}$ is a càdlàg process of integrable variation such that $\|X_s - X_u\|_2 \le c\|X_s - X_u\|_1$ for all $0 \le s < u \le T$ and some $c > 0$. Then each f_j is absolutely continuous with respect to μ_X and $\lim_n \mathrm{E}[V_{X-X^{(n)}}(T)] = 0$.*

Proof. For $j \ge 1$ and $0 \le s < u \le T$ we have

$$|f_j(s) - f_j(u)| \le \|U_j\|_2 \|X_s - X_u\|_2 \le c\|X_s - X_u\|_1, \tag{3.2}$$

which shows that each f_j is absolutely continuous with respect to μ_X. Let ψ_j denote the density of f_j with respect to μ_X. We have

$$\mathrm{E}[V_{X-X^{(n)}}(T)] \le \sup_{k\ge1} \sum_{i=1}^{a_k} \Big(\sum_{j=n+1}^{\infty} (f_j(t_i^k) - f_j(t_{i-1}^k))^2 \Big)^{1/2}, \tag{3.3}$$

where $\tau_k = \{0 = t_0^k < \cdots < t_{a_k}^k = T\}$ are nested subdivisions of $[0,T]$ satisfying $|\tau_k| \to 0$. By Jeulin [17, Lemme 3] the right-hand side of (3.3) equals

$$\int_0^T \Big(\sum_{j=n+1}^{\infty} \psi_j(s)^2 \Big)^{1/2} \mu_X(\mathrm{d}s). \tag{3.4}$$

Another application of Jeulin [17, Lemme 3] yields

$$\int_0^T \Big(\sum_{j=1}^{\infty} \psi_j(s)^2 \Big)^{1/2} \mu_X(\mathrm{d}s) \tag{3.5}$$

$$= \sup_{k\ge1} \sum_{i=1}^{a_k} \Big(\sum_{j=1}^{\infty} (f_j(t_i^k) - f_j(t_{i-1}^k))^2 \Big)^{1/2} \le c\mathrm{E}[V_X(T)] < \infty. \tag{3.6}$$

Thus by Lebesgue's dominated convergence theorem, $\lim_n \mathrm{E}[V_{X-X^{(n)}}(T)] = 0$. This completes the proof. □

The equivalence of the L^1- and L^2-norms of the increments of X is crucial for Lemma 3.2 to be true. For example if X is a Poisson process with parameter $\lambda > 0$ then μ_X is proportional to the Lebesgue measure but all paths are step functions.

Corollary 3.3. *Let $X = (X_t)_{t\in[0,T]}$ be as in Theorem 3.1. Then for every Radon measure μ on $[0,T]$ there exists a unique decomposition $X_t = Y_t + A_t$ of X, where Y and A are càdlàg processes of bounded variation such that Y is absolutely continuous with respect to μ and A is singular to μ and $\{Y_t, A_t : t \in [0,T]\} \subseteq \overline{\mathrm{span}}_{L^0}\{X_t : t \in [0,T]\}$.*

Proof. Let $\mathcal{S}_0 = \overline{\mathrm{span}}_{L^0}\{X_t : t \in [0,T]\}$. Since $\mathcal{S}_0$ is L^2-closed the U_n's in (3.1) belong to $\mathcal{S}_0$. For each $j \ge 1$, decompose f_j in (3.1) as $f_j = g_j + h_j$, where g_j, h_j are càdlàg functions of bounded variation, g_j being absolutely continuous with respect to μ and h_j singular to μ. Set

$$Y_t^{(n)} = \sum_{j=1}^{n} g_j(t) U_j \quad \text{and} \quad A_t^{(n)} = \sum_{j=1}^{n} h_j(t) U_j, \qquad t \in [0,T]. \tag{3.7}$$

For all $n, k \ge 1$,

$$V_{X^{(n)}-X^{(k)}}(T) = V_{Y^{(n)}-Y^{(k)}}(T) + V_{A^{(n)}-A^{(k)}}(T). \tag{3.8}$$

By Theorem 3.1 there exists a subsequence $(n_k)_{k\geq 1}$ such that $\lim_k X^{(n_k)} = X$ in the total variation norm on $[0,T]$ and so by completeness (3.8) implies that $\lim_k Y^{(n_k)}$ and $\lim_k A^{(n_k)}$ exist in total variation norm a.s. Calling these limit processes Y and A we have for all $t \in [0,T]$

$$\lim_{k\to\infty} Y_t^{(n_k)} = Y_t \qquad \text{and} \qquad \lim_{k\to\infty} A_t^{(n_k)} = A_t, \qquad \text{a.s.,} \tag{3.9}$$

showing that $Y_t, A_t \in \mathcal{S}_0$. Moreover since the sets of functions which are absolutely continuous with respect to μ respectively singular to μ are closed in $BV([0,T])$ the proof of the corollary is complete. $\square$

Lemma 3.4. *Let X denote a càdlàg process process of bounded p-th variation. Then X admits an q-variation for all $q > p$ and*

$$[X]_t^{(q)} = \sum_{0<s\leq t} |\Delta X_s|^q < \infty, \qquad 0 \leq t \leq T. \tag{3.10}$$

Proof. Fixed $q > p$ and set for $0 \leq t \leq T$ and $n \geq 1$

$$X_t^n = \sum_{0<s\leq t} \Delta X_s 1_{\{|\Delta X_s|>1/n\}}, \qquad S_t = \sum_{0<s\leq t} |\Delta X_s|^q. \tag{3.11}$$

Recall that $S_t < \infty$ a.s. since X is of bounded q-variation. For all $n \geq 1$ X^n has piecewise constant sample paths and so X^n admits a q-variation and

$$[X^n]_t^{(q)} = \sum_{0<s\leq t} |\Delta X_s|^q 1_{\{|\Delta X_s|>1/n\}} \xrightarrow[n\to\infty]{} S_t \qquad \text{a.s.,}\ \ t \in [0,T]. \tag{3.12}$$

Therefore it reduces to show

$$\lim_{n\to\infty} \limsup_{|\tau|\to 0} \mathrm{P}\big(\big|V_X^{q,\tau} - V_{X^n}^{q,\tau}\big| > \epsilon\big) = 0 \qquad \text{for all } \epsilon > 0. \tag{3.13}$$

Writing $\tilde{X}_t^n$ for $X_t - X_t^n$ we have for all $n \geq 1, t \in [0,T]$ and subdivisions $\tau = \{0 = t_0 < \cdots < t_k = t\}$

$$\big|V_X^{q,\tau} - V_{X^n}^{q,\tau}\big| \leq \sum_{i=1}^k \big||X_{t_i} - X_{t_{i-1}}|^q - |X_{t_i}^n - X_{t_{i-1}}^n|^q\big| \leq q\sum_{i=1}^k C_i^{q-1} |\tilde{X}_{t_i}^n - \tilde{X}_{t_{i-1}}^n|, \tag{3.14}$$

for some C_i's between $|X_{t_i}^n - X_{t_{i-1}}|$ and $|\tilde{X}_{t_i}^n - \tilde{X}_{t_{i-1}}|$, and hence by Hölder's inequality

$$\big|V_X^{q,\tau} - V_{X^n}^{q,\tau}\big| \leq q\Big(\sum_{i=1}^k C_i^q\Big)^{(q-1)/q} \Big(\sum_{i=1}^k |\tilde{X}_{t_i}^n - \tilde{X}_{t_{i-1}}^n|^q\Big)^{1/q} \tag{3.15}$$

$$\leq q\Big(V_X^{q,\tau} + V_{X^n}^{q,\tau}\Big)^{(q-1)/q} \Big(\max_{1\leq i\leq k} |\tilde{X}_{t_i}^n - \tilde{X}_{t_{i-1}}^n|^{q-p} V_{\tilde{X}^n}^{p,\tau}\Big)^{1/q} \tag{3.16}$$

$$\leq q2^{p/q}\Big(V_X^{q,\tau} + V_{X^n}^{q,\tau}\Big)^{(q-1)/q} \Big(V_X^{p,\tau} + V_{X^n}^{p,\tau}\Big)^{1/q} \max_{1\leq i\leq k} |\tilde{X}_{t_i}^n - \tilde{X}_{t_{i-1}}^n|^{(q-p)/q}. \tag{3.17}$$

Using that $\max_{1\leq i\leq k}|\tilde{X}_{t_i}^n - \tilde{X}_{t_{i-1}}^n| < 2n^{-1}$ for $|\tau|$ sufficiently small we have

$$\limsup_{|\tau|\to 0} \mathrm{P}\big(\big|V_X^{q,\tau} - V_{X^n}^{q,\tau}\big| > \epsilon\big) \tag{3.18}$$

$$\leq \limsup_{|\tau|\to 0} \mathrm{P}\left(q2^{p/q}(V_X^{q,\tau} + S_t)^{(q-1)/q}(V_X^{p,\tau} + S_t^{p/q})^{1/q} 2n^{-1} > \frac{\epsilon}{2}\right), \tag{3.19}$$

which implies (3.13) since $\{V_X^{p,\tau} : \tau\}$ is bounded in L^0. $\square$

Proposition 3.5. *Let X denote a càdlàg process. Assume that it admits a p-variation and satisfies C_q for some $q \in [2p, \infty]$ or that it is of bounded p-variation and satisfies C_q for some $q \in (2p, \infty]$. Then a.s. X is discontinuous only on D_X, and hence X is a.s. continuous if and only if it is continuous in probability.*

In the proof we need the following two remarks concerning any càdlàg process X:

(i) If X is of integrable variation then $\mu_X(\{t\}) > 0$ if and only if $t \in D_X$.

(ii) If X admits a p-variation then $\Delta[X]^{(p)} = |\Delta X|^p$.

To prove (i) let $t > 0$ and choose $(t_n)_{n\geq 1} \subseteq [0, t)$ such that $t_n \uparrow t$. By Lebesgue's dominated convergence theorem we have

$$\mu_X(\{t\}) = \lim_{n\to\infty} \mathrm{E}[V_X(t) - V_X(t_n)] = \mathrm{E}\big[\lim_{n\to\infty}(V_X(t) - V_X(t_n))\big] = \mathrm{E}[|\Delta X_t|], \tag{3.20}$$

which shows (i). For $p = 2$ (ii) follows by Jacod [14, Lemme 3.11]. The general case can be proved by imitating Jacod's proof.

Proof of Proposition 3.5. We may assume that X admits a p-variation. Indeed, if X is of bounded p-variation and satisfies C_q for some $q \in (2p, \infty]$ then according to Lemma 3.4 it admits a $\frac{q}{2}$-variation.

Assume therefore that X admits a p-variation and satisfies C_q for a $q \in [2p, \infty]$. Let $0 \leq u < t \leq T$ and choose subdivisions τ_n of $[u, t]$ such that

$$\lim_{n\to\infty} V_X^{p,\tau_n} = [X]_t^{(p)} - [X]_u^{(p)}, \qquad \text{almost surely.} \tag{3.21}$$

For $f \in \mathbb{R}^{[0,T]}$ let

$$N(f) = \limsup_{n\to\infty} (V_f^{p,\tau_n})^{1/p}. \tag{3.22}$$

Then N is a lower semicontinuous pseudo-seminorm, and since $([X]_t^{(p)} - [X]_u^{(p)})^{1/p} = N(X)$ a.s. it follows by Theorem 2.1 that

$$\|[X]_t^{(p)} - [X]_u^{(p)}\|_2 = \|N(X)\|_{2p}^p \leq k_{p,2p}^p \|N(X)\|_p^p = k_{p,2p}^p \|[X]_t^{(p)} - [X]_u^{(p)}\|_1 < \infty. \tag{3.23}$$

For $u = 0$ this gives that $[X]^{(p)}$ is integrable and since it is increasing it is also of integrable variation. Hence by Lemma 3.2 $[X]^{(p)}$ is a.s. absolutely continuous with respect to $\mu_{[X]^{(p)}}$ and so by (i) $[X]^{(p)}$ is continuous on $D^c_{[X]^{(p)}}$. Finally, by applying (ii) it follows that X is continuous on D^c_X. Therefore, X has continuous sample paths if and only if D_X is empty, that is if X is continuous in probability. $\square$

For $f\colon \mathbb{R} \to \mathbb{R}$, let $W_f\colon \mathbb{R} \to [0, \infty]$ denote its oscillation function given by

$$W_f(t) = \lim_{n\to\infty} \sup_{u,s\in[t-1/n,t+1/n]} |f(s) - f(u)|, \qquad t \in \mathbb{R}. \tag{3.24}$$

Itô and Nisio [13, Theorem 1] show that each separable Gaussian process which is continuous in probability has a deterministic oscillation function. By Marcus and Rosen [22, Theorem 5.3.7] this is also true for Rademacher processes. Furthermore, Cambanis et al. [6] show that a very large class of infinitely divisible processes also have this property. Thus for such processes Proposition 3.5 holds even without the assumption of being of bounded p-variation. On the other hand the following example shows that Gaussian chaos processes do not in general have deterministic oscillation functions. Let

$(Y_t)_{t\geq 0}$ denote a Gaussian process which is continuous in probability and has oscillation function $t\mapsto \alpha(t)\in(0,\infty)$ and such that Y_0 is non-deterministic. Then X, given by $X_t=Y_0Y_t$, is a separable second-order Gaussian chaos process continuous in probability with oscillation function $t\mapsto |Y_0|\alpha(t)$.

3.1 The stationary increment case

According to e.g. Doob [8], a centered and L^2-continuous process $X=(X_t)_{t\in\mathbb{R}}$ with stationary increments has a spectral measure m_X, which is the unique symmetric measure integrating $s\mapsto(1+s^2)^{-1}$ and satisfying

$$\Gamma_X(t,u):=\mathrm{E}[(X_t-X_0)(X_u-X_0)]=\int_{\mathbb{R}}\frac{(e^{its}-1)(e^{-ius}-1)}{s^2}\,m_X(\mathrm{d}s). \tag{3.25}$$

Furthermore set $v_X(t)=\Gamma_X(t,t)$, and if X is stationary denote by R_X its auto covariance function, and by n_X the unique finite and symmetric measure satisfying

$$R_X(t)=\mathrm{E}[X_tX_0]=\int_{\mathbb{R}}e^{its}\,n_X(\mathrm{d}s),\qquad t\in\mathbb{R}. \tag{3.26}$$

Proposition 3.6. *Assume that X is an L^2-continuous process with stationary increments satisfying condition C_q for some $q\in[2,\infty]$. Then the following five conditions are equivalent:*

(i) X has a.s. càdlàg paths of bounded variation,

(ii) X has a.s. absolutely continuous paths,

(iii) $m_X(\mathbb{R})<\infty$, (iv) $\Gamma_X\in C^2(\mathbb{R}^2;\mathbb{R})$, (v) $v_X\in C^2(\mathbb{R};\mathbb{R})$.

If X is stationary then (i)-(v) are also equivalent to $\int_{\mathbb{R}}t^2\,n_X(\mathrm{d}t)<\infty$ or $R_X\in C^2(\mathbb{R};\mathbb{R})$.

The Gaussian case is covered by Ibragimov [12, Theorem 12]. See also Doob [8, page 536] for general results about mean-square differentiability. A Hermite process X with parameter $(d,H)\in\mathbb{N}\times(\frac{1}{2},1)$ is a Gaussian chaos process of order d with stationary increments and the same covariance function as the fractional Brownian motion with Hurst parameter H; see Maejima and Tudor [21] for a precise definition. The corresponding spectral measure is $m_X(\mathrm{d}s)=c_H|s|^{1-2H}\,\mathrm{d}s$, that is a non-finite measure, and so by Proposition 3.6 X is not of bounded variation.

Proof. Assume (i), that is X has càdlàg paths of bounded variation. The stationary increments implies that μ_X equals the Lebesgue measure up to a scaling constant. Thus (i)⇒(ii) since by Theorem 3.1 X is absolutely continuous with respect to μ_X. (ii)⇒(i) is obvious. Furthermore if X is càdlàg and of bounded variation then by Proposition 3.7 below we have

$$\infty>\sup_{n\geq 1}\Big(n^2v_X(1/n)\Big)\geq\sup_{n\geq 1}\int_{\mathbb{R}}\Big(\frac{\sin(s/n)}{s/n}\Big)^2\,m_X(\mathrm{d}s). \tag{3.27}$$

Hence by Fatou's lemma $m_X(\mathbb{R})<\infty$ and so (i)⇒(iii). (iii)⇒(iv)⇒(v) are easy. To see that (v) implies (i) assume $v_X\in C^2(\mathbb{R};\mathbb{R})$. Since v_X is symmetric and $v_X(0)=0$ we have $v_X'(0)=0$. Thus $v_X(t)=O(t^2)$ as $t\to 0$ and hence by Proposition 3.7 X is of bounded 1-variation. To show that a.a. sample paths of X are càdlàg and of bounded variation let τ_n be nested subdivisions of $[a,b]$ such that $|\tau_n|\to 0$. Using that an increasing sequence which is bounded in L^0 is a.s. bounded, $\sup_{n\geq 1} V_X^{1,\tau_n} < \infty$ a.s. Since X has sample paths of bounded variation through $\cup_{n\geq 1}\tau_n$ and is L^2-continuous we may choose a right-continuous modification of X. This modification will then have càdlàg paths of bounded variation, showing (i). The stationary case follows by similarly arguments. □

Proposition 3.7. *Let $p \geq 1$ and assume that X is an L^2-continuous process with stationary increments and satisfies C_q for some $q \in [p,\infty]$. Then X is of bounded p-variation if and only if $v_X(t) = O(t^{2/p})$ as $t \to 0$. Furthermore, X admits a p-variation zero, i.e. $[X]_t^{(p)} \equiv 0$, if and only if $v_X(t) = o(t^{2/p})$ as $t \to 0$.*

Proof. Assume that X is of bounded p-variation. For all $r \leq v \leq q$ there exists, according to Theorem 2.1, a constant $k_{r,v}$ such that for all subdivisions τ

$$\|(V_X^{p,\tau})^{1/p}\|_v \leq k_{r,q}\|(V_X^{p,\tau})^{1/p}\|_r < \infty. \tag{3.28}$$

Since $\{(V_X^{p,\tau})^{1/p} : \tau\}$ is bounded in L^0, (3.28) and Krakowiak and Szulga [20, Corollary 1.4] shows that $\sup_\tau \|(V_X^{p,\tau})^{1/p}\|_v < \infty$. In particular for $v = p$

$$\infty > \sup_\tau \mathrm{E}[V_X^{p,\tau}] = \sup_\tau \sum_{i=1}^k \mathrm{E}\Big[|X_{t_i} - X_{t_{i-1}}|^p\Big], \tag{3.29}$$

where $\tau = \{0 = t_0 < \cdots < t_k = T\}$. Using the equivalence of moments of X, see Theorem 2.1, it now follows that X is of bounded p-variation if and only if

$$\sup_\tau \sum_{i=1}^k v_X(t_i - t_{i-1})^{p/2} < \infty. \tag{3.30}$$

This proves the first part of the statement since (3.30) is equivalent to $v_X(t) = O(t^{2/p})$. Similar arguments show that X admits a p-variation zero if and only if

$$\lim_{|\tau|\to 0} \sum_{i=1}^k v_X(t_i - t_{i-1})^{p/2} = 0. \tag{3.31}$$

Thus by observing that (3.31) is satisfied if and only if $v_X(t) = o(t^{2/p})$ the proof is complete. □

By definition $v_X(t) = t^{2H}$ for a Hermite process X with parameters (d, H). Thus by Proposition 3.7 X is of bounded p-variation if and only if $p \geq \frac{1}{H}$. Moreover, X has p-variation zero if and only if $p > \frac{1}{H}$. If X is Gaussian such that v_X is concave and $\alpha := \lim_{t\to 0} v_X(t)/t^{2/p}$ exists in $\mathbb{R}$ for some $p \geq 2$ it is possible to show that X admits a p-variation; see Marcus and Rosen [22, Theorem 10.2.3]. The special case $\alpha = 0$ is included in the above Proposition 3.7, however a generalization to $\alpha > 0$ is not straightforward since the proof here relies on Borell's isoperimetric inequality in which the Gaussian assumption is crucial.

4 Semimartingales

In this section we characterize the canonical decomposition of chaos semimartingales, and in the next section this characterization is used to study when a moving average is a semimartingale.

The canonical decomposition of Gaussian quasimartingales are characterized in Jain and Monrad [15] and their result is extended to Gaussian semimartingales in Stricker [30]. Theorem 2.1 allows us to generalize this to a much larger setting. The proof by Stricker [30] relies on the fact that a càdlàg Gaussian process X, and in particular Gaussian semimartingales, only has jumps on D_X. If X is a chaos process satisfying C_q for some $q \in [4,\infty]$ admitting a quadratic variation we know by Proposition 3.5 that X has only jumps on D_X, allowing us to proceed as in Stricker [30]. However, in the case $q \in [1,4)$ we need a result by Meyer [23].

We shall need the following notation: Given a filtration $\mathcal{F}$, a process X is said to be $(\mathcal{F},q)$-stable if $(\mathrm{E}[X_t|\mathcal{F}_s])_{s,t\in[0,T]}$ is a chaos process satisfying C_q. In this case set $\mathcal{PC} = \overline{\operatorname{span}}_{L^0}\{\mathrm{E}[X_t|\mathcal{F}_s] : s,t \in [0,T]\}$.

Theorem 4.1. *Let $X = (X_t)_{t\in[0,T]}$ denote an $(\mathcal{F},q)$-stable chaos process for some $q \in [1,\infty]$. If X is an $\mathcal{F}$-semimartingale then $X \in H^p$ for all finite $p \in [1,q]$ and $\{A_t, M_t : t \in [0,T]\} \subseteq \mathcal{PC}$, where $X = X_0 + M + A$ is the $\mathcal{F}$-canonical decomposition of X. In particular A and M are chaos processes satisfying C_q.*

Let M^d and M^c denote, respectively, the purely discontinuous and continuous martingale component of M and A^c, A^{sc} and A^d the absolutely continuous, singular continuous respectively discrete component of A. If $q \in [4,\infty]$ then X has a.s. only jumps on D_X and has therefore a.s. continuous paths if and only if it is continuous in probability. Moreover, $\{M^c_t, M^d, A^c_t, A^{sc}, A^d_t : t \in [0,T]\} \subseteq \mathcal{PC}$, and for each $t \in [0,T]$ we have

$$M^d_t = \sum_{s\in(0,t]\cap D_X} \Delta M_s \quad \text{and} \quad A^d_t = \sum_{s\in(0,t]\cap D_X} \Delta A_s, \tag{4.1}$$

where both sums converge in L^p for all finite $p \le q$ and the second converges also absolutely a.s.

Proof. Consider subdivisions $\tau_n = \{0 = t^n_0 < \cdots < t^n_{2^n} = T\}$ where $t^n_i = Ti2^{-n}$ for $i = 0,\dots,2^n$. By passing to a subsequence we may assume that $\lim_{n\to\infty} V^{2,\tau_n}_X$ exists a.s. For $f\colon [0,T]\cap\mathbb{Q} \to \mathbb{R}$ define

$$\Phi(f) := \sup_{n\ge1} \sqrt{V^{2,\tau_n}_f}. \tag{4.2}$$

Then Φ is a lower semicontinuous pseudo-seminorm on $\mathbb{R}^{[0,T]\cap\mathbb{Q}}$ and $\Phi(X) < \infty$ a.s. Since X is a chaos process satisfying C_q Theorem 2.1 shows that $\mathrm{E}[\Phi(X)^p] < \infty$ for all finite $p \le q$. In particular $\Phi(X)$ is integrable and hence by Meyer [23] X is a special $\mathcal{F}$-semimartingale. Denoting by A its bounded variation component Meyer [23] shows moreover that

$$S^X_n := \sum_{i=1}^{2^n} \mathrm{E}[X_{t_i} - X_{t_{i-1}}|\mathcal{F}_{t_{i-1}}] \xrightarrow[n\to\infty]{} A_T \qquad \text{in the weak } L^1\text{-topology.} \tag{4.3}$$

Since $\mathcal{PC}$ is L^1-closed, (4.3) shows that $A_T \in \mathcal{PC}$. Similar arguments show that $\{A_s : s \in [0,T]\} \subseteq \mathcal{PC}$ and hence also $\{M_s : s \in [0,T]\} \subseteq \mathcal{PC}$. Since X is $(\mathcal{F},q)$-stable this shows that A and M are chaos processes satisfying C_q. Thus by arguing as above we have $\mathrm{E}[[M]^{p/2}_T] < \infty$ for all finite $p \le q$. Moreover define for $f\colon [0,T]\cap\mathbb{Q} \to \mathbb{R}$

$$\Psi(f) := \sup_{n\ge1} V^{1,\tau_n}_f. \tag{4.4}$$

Then Ψ is a lower semicontinuous pseudo-seminorm on $\mathbb{R}^{[0,T]\cap\mathbb{Q}}$ and $\Psi(A) < \infty$ a.s.. Hence by Theorem 2.1, $\mathrm{E}[V_A(T)^p] < \infty$ for all finite $p \le q$ implying that $X \in H^p$ for all finite $p \le q$.

To prove the second part assume $q \geq 4$. By Corollary 3.3, $A^c, A^{sc}, A^d \subseteq \mathcal{PC}$, since $A \subseteq \mathcal{PC}$. We claim that $D_A \subseteq D_X$. Assume on the contrary there exists a number $t \in D_A \setminus D_X$. Then

$$\Delta A_t = \mathrm{E}[\Delta A_t | \mathcal{F}_{t-}] = -\mathrm{E}[\Delta M_t | \mathcal{F}_{t-}] = 0, \qquad \text{a.s.} \tag{4.5}$$

contradicting the assumption that $t \in D_A$. Hence D_A and therefore also D_M are contained in D_X. By Proposition 3.5, A and M are continuous on D_A^c respectively D_M^c, implying that they are continuous on D_X^c. This shows that A^d is of the form (4.1). Set

$$(Y_t)_{t\in[0,T]} = \left(\int_0^t 1_{D_X^c}(s)\,\mathrm{d}M_s\right)_{t\in[0,T]} \quad \text{and} \quad (U_t)_{t\in[0,T]} = \left(\int_0^t 1_{D_X}(s)\,\mathrm{d}M_s\right)_{t\in[0,T]}. \tag{4.6}$$

Since $(\Delta Y_t)_{t\in[0,T]} = (1_{D_X^c}(t)\Delta M_t)_{t\in[0,T]}$ and M is continuous on D_X^c, Y is a continuous martingale. On the other hand for every continuous bounded martingale N we have

$$\langle U, N\rangle_t = \int_0^t 1_{D_X}(s)\,\mathrm{d}\langle M, N\rangle_s = 0, \tag{4.7}$$

since $\langle M, N\rangle$ is continuous and D_X is countable. Thus U is a purely discontinuous martingale, and so U and Y are the purely discontinuous respectively the continuous martingale component of M. Finally, since D_X is countable,

$$U_t = \sum_{s\in(0,t]\cap D_X} \Delta M_s, \tag{4.8}$$

where the sum converges in probability and therefore also in L^p for all finite $p \leq q$ according to Theorem 2.1. $\square$

Essentially due to Föllmer [10] a process X is called an $\mathcal{F}$-Dirichlet processes if it can be decomposed as

$$X = Y + A, \tag{4.9}$$

where Y is an $\mathcal{F}$-semimartingale and A is $\mathcal{F}$-adapted, continuous and has quadratic variation zero. A Dirichlet process X is said to be special if it has a decomposition $X = Y + A$ where Y is a special semimartingale. In this case X has a unique decomposition

$$X = X_0 + M + A^c + A^d, \tag{4.10}$$

where M is a local martingale, A^d is a predictable pure jump process of bounded variation and A^c is a continuous process of quadratic variation zero. We have the following extension of Stricker [31, Theorem 1]:

Proposition 4.2. *Let X denote an $(\mathcal{F}, q)$-stable chaos process for some $q \in [4, \infty]$. If X is an $\mathcal{F}$-Dirichlet process then it is special, has almost surely only jumps on D_X and $M_t, A_t^d, A_t^c \in \mathcal{PC}$ for all $t \in [0,T]$. Furthermore, M is a true martingale belonging to H^p for all finite $p \leq q$ and A^d is a pure jump process of integrable variation having almost surely only jumps on D_X. Finally, A^c is of zero energy, that is $\lim_{|\tau|\to 0} \mathrm{E}\big[V_{A^c}^{2,\tau}\big] = 0$.*

Proof. Let Φ be given as in (4.2). Arguing as in Theorem 4.1 it follows that $\mathrm{E}[\Phi(X)^p] < \infty$ for all finite $p \leq q$. Hence by Stricker [31, Theorem 1] X is special and $S_n^X \to A_T$ in the weak L^1-topology, where $A_t = A_t^d + A_t^c$. Since $\mathcal{PC}$ is L^1-closed we have $A_T \in \mathcal{PC}$ and similar $M_t, A_t \in \mathcal{PC}$ for all $t \in [0,T]$. Assume there exists $t \in D_A \setminus D_X$. Due to the fact that A is $\mathcal{F}$-predictable we have

$$\Delta A_t = \mathrm{E}[\Delta A_t|\mathcal{F}_{t-}] = -\mathrm{E}[\Delta M_t|\mathcal{F}_{t-}] = 0, \qquad \text{a.s.} \tag{4.11}$$

which contradicts $t \in D_A$ and so $D_A \subseteq D_X$. Furthermore, since A admits a quadratic variation, Proposition 3.5 implies that A has a.s. only jumps on the countable set $D_A \subseteq D_X$. Using moreover that A^d is a pure jump process of bounded variation and A^c is continuous we have that

$$A_t^d = \sum_{0<s\leq t} \Delta A_s^d = \sum_{0<s\leq t} \Delta A_s = \sum_{s\in D_X\cap(0,t]} \Delta A_s, \tag{4.12}$$

and we conclude that $A_t^d \in \mathcal{PC}$. The rest of the proof is now a straightforward consequence of Theorem 2.1. □

Remark 4.3.

(i) X is $(\mathcal{F},q)$-stable if

$$X_t = \int_0^T f(t,s)\,\mathrm{d}M_s, \qquad t \in [0,T], \tag{4.13}$$

where M is a càdlàg $\mathcal{F}$-martingale being also a chaos process satisfying C_q for some $q \in [1,\infty]$, and $f(t,\cdot)$ are deterministic functions for which the integrals exist. The $(\mathcal{F},q)$-stability follows easily since for $u,t \in [0,T]$

$$\mathrm{E}[X_t|\mathcal{F}_u] = \int_0^u f(t,s)\,\mathrm{d}M_s \in \overline{\mathrm{span}}_{L^0}\left\{M_s : s \in [0,T]\right\}. \tag{4.14}$$

(ii) The $(\mathcal{F},q)$-stability of X is not automatic even when X is a Gaussian chaos process of order d. However, if $\mathcal{G}$ is given by (2.10) then X is $(\mathcal{F}^W,\infty)$-stable and more generally this is true if each $\mathcal{F}_s$ is generated by elements in $\mathcal{G}$; see Nualart et al. [24] for related results. Thus for $d=1$ X is always $(\mathcal{F}^X,\infty)$-stable, but when $d \geq 2$ this may fail as the following example shows.

Example 4.4. Assume $\mathcal{G}$ is given by (2.10) for some Wiener process $(W_t)_{t\in[0,3]}$. Let $X = (X_t)_{t\in[0,3]}$ be the second-order Gaussian chaos process

$$X_t = \big(W_1^2 + W_1\big)1_{[1,2)}(t) + W_2 1_{[2,3]}(t), \qquad t \in [0,3]. \tag{4.15}$$

Then $(\mathrm{E}[X_t|\mathcal{F}_s^X])_{s,t\in[0,3]}$ is not a Gaussian chaos process. In fact, X is a special $\mathcal{F}^X$-semimartingale but the $\mathcal{F}^X$-bounded variation component of X is not a Gaussian chaos process.

To see this, note that X is a special $\mathcal{F}^X$-semimartingale since it is of integrable variation. Moreover, the $\mathcal{F}^X$-bounded variation component of X is

$$A_t = \mathrm{E}[\Delta X_1|\mathcal{F}_{1-}^X]1_{[1,3]}(t) + \mathrm{E}[\Delta X_2|\mathcal{F}_{2-}^X]1_{[2,3]}(t) \tag{4.16}$$
$$= 1_{[1,3]}(t) + \Big(W_1^2 + W_1 - \mathrm{E}[W_1|W_1^2 + W_1]\Big)1_{[2,3]}(t). \tag{4.17}$$

So to show that A is not a Gaussian chaos process it is enough to show $Y := \mathrm{E}[W_1|W_1^2 + W_1] \notin \cup_{d=1}^\infty \overline{\mathcal{P}}_{\mathcal{G}}^d$. For each integrable random variable U, which is absolutely continuous with density $f>0$, we have

$$\mathrm{E}\big[U\big||U|\big] = |U|\frac{f(|U|) - f(-|U|)}{f(|U|) + f(-|U|)}. \tag{4.18}$$

Applying (4.18) with $U = W_1 + 1/2$, we get

$$Y = -1/2 + \mathrm{E}[W_1 + 1/2 \big| |W_1 + 1/2|] \tag{4.19}$$

$$= -1/2 + |W_1 + 1/2| \tanh\Big(|W_1 + 1/2|/2\Big), \tag{4.20}$$

where $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$. Since $x \mapsto e^{x^2/4}$ is convex we have

$$\mathrm{E}[e^{Y^2/4}] \leq \mathrm{E}[\mathrm{E}[e^{W_1^2/4} | W_1^2 + W_1]] = \mathrm{E}[e^{W_1^2/4}] < \infty. \tag{4.21}$$

For contradiction assume $Y \in \cup_{d=1}^{\infty} \overline{\mathcal{P}}_{\mathcal{G}}^d$. By (4.21) and Janson [16, Theorem 6.12] this implies $Y \in \overline{\mathcal{P}}_{\mathcal{G}}^1 = \mathcal{G} + \mathbb{R}$. Moreover, (4.20) shows that $Y \geq -1/2$ and hence Y is constant. This contradict (4.20) and gives $Y \notin \cup_{d=1}^{\infty} \overline{\mathcal{P}}_{\mathcal{G}}^d$. $\Diamond$

5 The semimartingale property of moving averages

This section is concerned with the semimartingale property of moving averages (also known as stochastic convolutions). In Subsection 5.1 we treat the one-sided case and in Subsection 5.2 the two-sided case is considered.

5.1 The one-sided case

In this subsection $(\mathcal{F}_t)_{t\geq 0}$ denotes a filtration and $(M_t)_{t\geq 0}$ a square-integrable càdlàg $(\mathcal{F}_t)_{t\geq 0}$-martingale. Set $\gamma_M(t) = \mathrm{E}[M_t^2]$ for $t \geq 0$ and note that γ_M is càdlàg and increasing and hence γ_M' exists Lebesgue a.s. Let $X = (X_t)_{t\geq 0}$ be given by

$$X_t = \int_0^t \phi(t-s)\,\mathrm{d}M_s, \qquad t \geq 0, \tag{5.1}$$

where ϕ is a measurable deterministic function for which all the integrals exist, i.e. $\phi(t - \cdot) \in L^2(\gamma_M)$ for all $t \geq 0$. In this set up we have the following theorem where all locally integrability conditions are with respect to the Lebesgue measure λ.

Theorem 5.1. *Assume that M is a chaos process satisfying C_q for some $q \in [2, \infty]$ such that γ_M' is bounded away from zero on some non-empty open interval. Then X defined by (5.1) is an $\mathcal{F}$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density.*

Extensions to $q < 2$ is not possible. To see this let M denote an α-stable motion with $\alpha \in (1, 2)$. Then M is an $\mathcal{F}^M$-martingale satisfying C_q for all $q < \alpha$, but Basse and Pedersen [4, Theorem 3.1] yields that X given by (5.1) is an $\mathcal{F}^M$-semimartingale if and only if ϕ is absolutely continuous with an α-integrable density.

The proof of Theorem 5.1 relies on two lemmas. Here for each $f : \mathbb{R} \to \mathbb{R}$ and $h > 0$ $\Delta_h f$ denotes the function $t \mapsto (f(t+h) - f(t))/h$.

Lemma 5.2 (Hardy and Littlewood)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ denote a locally integrable function. Then $(\Delta_{\frac{1}{n}} f)_{n\geq 1}$ is bounded in $L^2([a,b], \lambda)$ for all $0 \leq a < b$ if and only if f is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density.*
For every $a \geq 0$ $(\Delta_{\frac{1}{n}} f)_{n\geq 1}$ is bounded in $L^2([a, \infty), \lambda)$ if and only if f is absolutely continuous on $[a, \infty)$ with a square-integrable density.

Lemma 5.3. *Let $\mathcal{F}$ denote a filtration, Y an $\mathcal{F}$-semimartingale and X be given by*

$$X_t = \int_0^t \phi(t-s)\,\mathrm{d}Y_s, \qquad t \geq 0, \tag{5.2}$$

where ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density. Then X is an $\mathcal{F}$-semimartingale.

Proof. For fixed $t > 0$ we have

$$\begin{aligned} X_t &= \phi(0)Y_t + \int_0^t \Big(\int_0^{t-s} \phi'(u)\,\mathrm{d}u \Big)\,\mathrm{d}Y_s & (5.3)\\ &= \phi(0)Y_t + \int_0^t \Big(\int_0^t 1_{[s,t]}(u)\phi'(u-s)\,\mathrm{d}u \Big)\,\mathrm{d}Y_s. & (5.4) \end{aligned}$$

Since

$$\mathbb{R}_+ \ni s \mapsto \sqrt{\int_s^t |\phi'(u-s)|^2\,\mathrm{d}u} = \sqrt{\int_0^{t-s} |\phi'(u)|^2\,\mathrm{d}u} \tag{5.5}$$

is locally bounded, Protter [27, Chapter IV, Theorem 65] shows that

$$\begin{aligned} X_t &= \phi(0)Y_t + \int_0^t \Big(\int_0^t 1_{[s,t]}(u)\phi'(u-s)\,\mathrm{d}Y_s \Big)\,\mathrm{d}u & (5.6)\\ &= \phi(0)Y_t + \int_0^t \Big(\int_0^u \phi'(u-s)\,\mathrm{d}Y_s \Big)\,\mathrm{d}u, \qquad \text{a.s.} & (5.7) \end{aligned}$$

Thus X has a modification which is an $\mathcal{F}$-semimartingale. □

Proof of Theorem 5.1. Assume X is an $\mathcal{F}$-semimartingale. By assumption there exists an interval $(a,b) \subseteq \mathbb{R}_+$ and an $\epsilon > 0$ such that $\gamma_M' \geq \epsilon$ λ-a.s. on (a,b). By Remark 4.3(i) X is $(\mathcal{F},q)$-stable and since $q \geq 1$ it follows by Theorem 4.1 that X is an $\mathcal{F}$-quasimartingale on each compact interval and in particular

$$\sup_{n\geq 1} \sum_{i=1}^{Nn} \mathrm{E}[|\mathrm{E}[X_{i/n} - X_{(i-1)/n}|\mathcal{F}_{(i-1)/n}]|] < \infty, \qquad \text{for all } N \geq 1. \tag{5.8}$$

By Theorem 2.1 there exists a constant $C > 0$ such that $C\|U\|_2 \leq \|U\|_1 < \infty$ for all $U \in \mathcal{PC}$. Moreover, for all $a < u \leq t$ we have

$$\begin{aligned} \mathrm{E}[|\mathrm{E}[X_t - X_u|\mathcal{F}_u]|] &= \mathrm{E}\Big[\Big| \int_0^u (\phi(t-s) - \phi(u-s))\,\mathrm{d}M_s \Big|\Big] & (5.9)\\ &\geq C \Big\| \int_0^u (\phi(t-s) - \phi(u-s))\,\mathrm{d}M_s \Big\|_2 & (5.10)\\ &= C \int_0^u \big(\phi(t-s) - \phi(u-s)\big)^2\,\gamma_M(\mathrm{d}s) & (5.11)\\ &\geq C \int_0^u \big(\phi(t-s) - \phi(u-s)\big)^2\,\gamma_M'(s)\,\mathrm{d}s & (5.12)\\ &= C \int_0^u \big(\phi(t-u+s) - \phi(s)\big)^2\,\gamma_M'(u-s)\,\mathrm{d}s & (5.13)\\ &\geq C\epsilon \int_{(u-b)\vee 0}^{u-a} \big(\phi(t-u+s) - \phi(s)\big)^2\,\mathrm{d}s. & (5.14) \end{aligned}$$

Put $\delta = (b-a)/4$ and set $l_x = x + (b+3a)/4$ and $r_x = x + (5b-a)/4$ for $x > 0$. By (5.8) and (5.14) we have

$$\sup_{n\geq 1} \sum_{i=[l_x n]+2}^{[r_x n]+1} \sqrt{\int_{(x-\delta)\vee 0}^{x+\delta} (\phi(1/n+s)-\phi(s))^2 \, \mathrm{d}s} < \infty, \tag{5.15}$$

showing that

$$\sup_{n\geq 1} n \sqrt{\int_{(x-\delta)\vee 0}^{x+\delta} (\phi(1/n+s)-\phi(s))^2 \, \mathrm{d}s} < \infty. \tag{5.16}$$

Thus $\{\Delta_{\frac{1}{n}}\phi : n \geq 1\}$ is bounded in $L^2([(x-\delta)\vee 0, x+\delta], \lambda)$ and so by Lemma 5.2 we need only show that ϕ is locally integrable. But this follows immediately from $\phi(t-\cdot) \in L^2([0,t], \gamma_M)$ for all $t \geq 0$ and $\gamma_M' \geq \epsilon$ λ-a.s. on (a,b). The reverse implication follows by Lemma 5.3. □

Let us rewrite Theorem 5.1 in the Gaussian chaos case. Define $\mathcal{G}$ by

$$\mathcal{G} = \left\{ \int_0^\infty h(s) \, \mathrm{d}W_s : h \in L^2(\mathbb{R}_+, \lambda) \right\}, \tag{5.17}$$

for some Wiener process W and let X be given by

$$X_t = \int_0^t \phi(t-s)\sigma_s \, \mathrm{d}W_s, \qquad t \geq 0, \tag{5.18}$$

where σ is $\mathcal{F}^W$-progressively measurable and not the zero-process, and ϕ is a measurable deterministic function such that all the integrals exist. We have the following corollary to Theorem 5.1:

Corollary 5.4. *Let X be given by* (5.18)*, where σ is a Gaussian chaos process which is right- or left-continuous in probability. Then X is an $\mathcal{F}^W$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density.*

5.2 Two-sided case

Let now $M = (M_t)_{t\in\mathbb{R}}$ denote a two-sided square-integrable $\mathcal{F}$-martingale, in the sense that $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ is an increasing family of σ-algebras, M is a square-integrable càdlàg process such that for all $-\infty < u \leq t$ we have $\mathrm{E}[M_t - M_u|\mathcal{F}_u] = 0$ and $M_t - M_u$ is $\mathcal{F}_t$-measurable. Let $\gamma_M(t) = \mathrm{sign}(t)\mathrm{E}[(M_t - M_0)^2]$ for all $t \in \mathbb{R}$ and note that γ_M is increasing and càdlàg. Let X be given by

$$X_t = \int_{-\infty}^t \big(\phi(t-s) - \psi(-s)\big) \, \mathrm{d}M_s, \qquad t \geq 0, \tag{5.19}$$

where ϕ and ψ are deterministic functions for which all the integrals are well-defined, that is $\phi(t-\cdot) - \psi(-\cdot)$ is square-integrable with respect to the measure γ_M. Assume there exists an interval $(-\infty, c)$ on which γ_M is absolutely continuous with

$$0 < \liminf_{t\to-\infty} \gamma_M'(t) \leq \limsup_{t\to-\infty} \gamma_M'(t) < \infty \quad \text{and} \quad \inf_{t\in(a,b)} \gamma_M'(t) > 0, \tag{5.20}$$

for some $0 \leq a < b$. Note that when M has stationary increments, and therefore $\gamma_M(t) = \kappa t$ for some $\kappa > 0$, the conditions are trivially satisfied.

Theorem 5.5. *Let the setting be as just described and assume that M is a chaos process satisfying C_q for some $q \in [2, \infty]$. Then X given by (5.19) is an $\mathcal{F}$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a square-integrable density.*

Proof. Assume that X is an $\mathcal{F}$-semimartingale. Since γ_M' is bounded away from 0 on some interval of $\mathbb{R}_+$, it follows (just as in the proof of Theorem 5.1) that ϕ is absolutely continuous on $\mathbb{R}_+$ with a locally square-integrable density. Choose $\epsilon > 0$ and $\tilde{c} < 0$ such that $\epsilon \leq \gamma_M'$ on $(-\infty, \tilde{c}]$. As in the proof of Theorem 5.1 $\{\Delta_{\frac{1}{n}}\phi : n \geq 1\}$ is bounded in $L^2([-\tilde{c}+1, \infty), \lambda)$ which by Lemma 5.2 implies that ϕ is absolutely continuous on $[-\tilde{c}+1, \infty)$ with a square-integrable density. This completes the proof of the *only if*-implication.

Assume now ϕ is absolutely continuous on $\mathbb{R}_+$ with a square-integrable density and choose $C > 0$ and $\tilde{c} < 0$ such that $\gamma_M' \leq C$ on $(-\infty, \tilde{c}]$. Let

$$Y_t = \int_{\tilde{c}}^{t} (\phi(t-s) - \psi(-s))\, \mathrm{d}M_s, \qquad t \geq 0. \tag{5.21}$$

By the same argument as in Lemma 5.3 it follows that Y is an $\mathcal{F}$-semimartingale. Thus it is enough to show that

$$U_t = \int_{-\infty}^{\tilde{c}} (\phi(t-s) - \psi(-s))\, \mathrm{d}M_s, \qquad t \geq 0, \tag{5.22}$$

is of bounded variation. For $0 \leq u \leq t$ we have

$$\mathrm{E}[|U_t - U_u|] \leq \|U_t - U_u\|_2 = \Big(\int_{-\infty}^{\tilde{c}} (\phi(t-s) - \phi(u-s))^2\, \gamma_M(\mathrm{d}s)\Big)^{1/2} \tag{5.23}$$

$$\leq C\Big(\int_{-\infty}^{\tilde{c}} (\phi(t-s) - \phi(u-s))^2\, \mathrm{d}s\Big)^{1/2} = C\Big(\int_{-\tilde{c}+u}^{\infty} (\phi(t-u+s) - \phi(s))^2\, \mathrm{d}s\Big)^{1/2}. \tag{5.24}$$

According to Lemma 5.2 this shows that U is of integrable variation on each compact interval and the proof is complete. □

Again we rewrite the result in a Gaussian the setting. More precisely consider the following: Let $\mathcal{G} = \{\int_{\mathbb{R}} h(s)\, \mathrm{d}W_s : h \in L^2(\mathbb{R}, \lambda)\}$, where $W = (W_t)_{t\in\mathbb{R}}$ is a two-sided Wiener process with $W_0 = 0$. Let

$$\mathcal{F}_t^W = \begin{cases} \sigma(W_s : s \in (-\infty, t]) & t \geq 0 \\ \sigma(W_t - W_s : s \in (-\infty, t]) & t < 0. \end{cases} \tag{5.25}$$

Consider a process X of the form

$$X_t = \int_{-\infty}^{t} \big(\phi(t-s) - \psi(-s)\big)\sigma_s\, \mathrm{d}W_s, \qquad t \geq 0, \tag{5.26}$$

where σ is $(\mathcal{F}_t)_{t\in\mathbb{R}}$-progressively measurable Gaussian chaos process satisfying

$$0 < \liminf_{t\to-\infty} \mathrm{E}[\sigma_t^2] \leq \limsup_{t\to-\infty} \mathrm{E}[\sigma_t^2] < \infty \quad \text{and} \quad \inf_{t\in(a,b)} \mathrm{E}[\sigma_t^2] > 0, \tag{5.27}$$

for some $0 \leq a < b$. Theorem 5.5 now gives the following corollary:

Theorem 5.6. *X is an $\mathcal{F}^W$-semimartingale if and only if ϕ is absolutely continuous on $\mathbb{R}_+$ with a square-integrable density.*

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PAPER F

Integrability of seminorms

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Abstract

We study integrability and equivalence of L^p-norms of polynomial chaos elements. Relying on known results for Banach space valued polynomials, a simple technique is presented to obtain integrability results for random elements that are not necessarily limits of Banach space valued polynomials. This enables us to prove integrability results for a large class of seminorms of stochastic processes and to answer, partially, a question raised by C. Borell (1979, *Séminaire de Probabilités*, *XIII*, 1–3).

Keywords: integrability; chaos processes; seminorms; regularly varying distributions

AMS Subject Classification: 60G17; 60B11; 60B12; 60E15

1 Introduction

Let T denote a countable set, $X = (X_t)_{t\in T}$ a stochastic process and N a seminorm on $\mathbb{R}^T$. This paper focuses on integrability and equivalence of L^p-norms of $N(X)$ in the case where X is a weak chaos process; see Definition 1.1. Of particular interest is the supremum and the p-variation norm given by

$$N(f) = \sup_{t\in T}|f(t)| \quad \text{and} \quad N(f) = \sup_{n\geq 1}\Big(\sum_{i=1}^{k_n}|f(t_i^n) - f(t_{i-1}^n)|^p\Big)^{1/p},\ p \geq 1, \tag{1.1}$$

for $f \in \mathbb{R}^T$. In the p-th variation case we assume moreover $T = [0,1]\cap\mathbb{Q}$ and $\pi_n = \{0 = t_0^n < \cdots < t_{k_n}^n = 1\}$ are nested subdivisions of T satisfying $\cup_{n=1}^\infty \pi_n = T$. Note that if N is given by (1.1), $B = \{x \in \mathbb{R}^T : N(x) < \infty\}$ and $\|x\| = N(x)$ for $x \in B$, then $(B, \|\cdot\|)$ is a non-separable Banach space when T is infinite.

Our results partly unify and partly extend known results in this area. For relations to the literature see Subsection 1.2. We note, however, that in the setting of the present paper we are able to treat Rademacher chaos processes of arbitrary order as well as infinitely divisible integral processes as in (1.4) below.

1.1 Chaos Processes and Condition C_q

Let $(\Omega, \mathcal{F}, \mathrm{P})$ denote a probability space. When F is a topological space, a Borel measurable mapping $X\colon \Omega \to F$ is called an F-valued random element, however when $F = \mathbb{R}$, X is, as usual, called a random variable. For each $p > 0$ and random variable X we let $\|X\|_p := \mathrm{E}[|X|^p]^{1/p}$, which defines a norm when $p \geq 1$; moreover, let $\|X\|_\infty := \inf\{t \geq 0 : \mathrm{P}(|X| \leq t) = 1\}$. When F is a Banach space, $L^p(\mathrm{P};F)$ denotes the space of all F-valued random elements, X, satisfying $\|X\|_{L^p(\mathrm{P};F)} = \mathrm{E}[\|X\|^p]^{1/p} < \infty$. Throughout the paper I denotes a set and for all $\xi \in I$, $\mathcal{H}_\xi$ is a family of independent random variables. Set $\mathcal{H} = \{\mathcal{H}_\xi : \xi \in I\}$. Furthermore, $d \geq 1$ is a natural number and F is a locally convex Hausdorff topological vector space (l.c.TVS) with dual space F^*. Following Fernique [11], a map N from F into $[0,\infty]$ is called a pseudo-seminorm if for all $x, y \in F$ and $\lambda \in \mathbb{R}$, we have

$$N(\lambda x) = |\lambda| N(x) \qquad \text{and} \qquad N(x+y) \leq N(x) + N(y). \tag{1.2}$$

For $\xi \in I$ let $\mathcal{P}^d_{\mathcal{H}_\xi}(F)$ denote the set of $p(Z_1,\ldots,Z_n)$ where $n \geq 1$, $Z_1,\ldots,Z_n$ are different elements in $\mathcal{H}_\xi$ and p is an F-valued tetrahedral polynomial of order d. Recall that $p\colon \mathbb{R}^n \to F$ is called an F-valued tetrahedral polynomial of order d if there exist $x_0, x_{i_1,\ldots,i_k} \in F$ such that

$$p(z_1,\ldots,z_n) = x_0 + \sum_{k=1}^d \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1,\ldots,i_k} \prod_{j=1}^k z_{i_j}. \tag{1.3}$$

Moreover, let $\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ denote the closure in distribution of $\cup_{\xi\in I}\mathcal{P}^d_{\mathcal{H}_\xi}(F)$, that is, $\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ is the set of all F-valued random elements X for which there exists a sequence $(X_k)_{k\geq 1} \subseteq \cup_{\xi\in I}\mathcal{P}^d_{\mathcal{H}_\xi}(F)$ converging weakly to X. Inspired by Ledoux and Talagrand [19] we introduce the following definition:

Definition 1.1. An F-valued random element X is said to be a weak chaos element of order d associated with $\mathcal{H}$ if for all $n \geq 1$ and all $(x_i^*)_{i=1}^n \subseteq F^*$ we have

$(x_1^*(X),\dots,x_n^*(X)) \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R}^n)$, and in this case we write $X \in$ *weak*-$\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$. Similarly, a real-valued stochastic process $(X_t)_{t\in T}$ is said to be a weak chaos process of order d associated with $\mathcal{H}$ if for all $n \geq 1$ and $(t_i)_{i=1}^n \subseteq T$ we have $(X_{t_1},\dots,X_{t_n}) \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R}^n)$.

An important example of a weak chaos process of order one is $(X_t)_{t\in T}$ of the form

$$X_t = \int_S f(t,s)\,\Lambda(\mathrm{d}s), \qquad t \in T, \tag{1.4}$$

where Λ is an independently scattered infinitely divisible random measure (or random measure for short) on some non-empty space S equipped with a δ-ring $\mathcal{PC}$, and $s \mapsto f(t,s)$ are Λ-integrable deterministic functions in the sense of [23]. To obtain the associated $\mathcal{H}$ let I be the set of all ξ given by $\xi = \{A_1,\dots,A_n\}$ for some $n \geq 1$ and disjoint sets $A_1,\dots,A_n$ in $\mathcal{PC}$, and let

$$\mathcal{H}_\xi = \{\Lambda(A_1),\dots,\Lambda(A_n)\} \qquad \text{and} \qquad \mathcal{H} = \{\mathcal{H}_\xi\}_{\xi\in I}. \tag{1.5}$$

Then, by definition of the stochastic integral (1.4) as the limit of integrals of simple functions, $(X_t)_{t\in T}$ is a weak chaos process of order one associated with $\mathcal{H}$.

Another example is where $(Z_n)_{n\geq 1}$ is sequence of independent random variables and $x(t), x_{i_1,\dots,i_k}(t) \in \mathbb{R}$ are real numbers for which

$$X_t = x(t) + \sum_{k=1}^{d} \sum_{1\leq i_1<\cdots<i_k<\infty} x_{i_1,\dots,i_k}(t) \prod_{j=1}^{k} Z_{i_j}, \tag{1.6}$$

exists in probability for all $t \in T$; then $X = (X_t)_{t\in T}$ is a weak chaos process of order d associated with $I = \{0\}$, $\mathcal{H}_0 = \{Z_n : n \geq 1\}$ and $\mathcal{H} = \{\mathcal{H}_0\}$.

In what follows we shall need the next conditions:

Notation, Condition C_q

- For $q \in (0,\infty)$, $\mathcal{H}$ is said to satisfy C_q if there exists $\beta_1, \beta_2 > 0$ such that for all $Z \in \cup_{\xi\in I}\mathcal{H}_\xi$ there exists $c_Z > 0$ with $\mathrm{P}(|Z| \geq c_Z) \geq \beta_1$ and

$$\mathrm{E}[|Z|^q, |Z| > s] \leq \beta_2 s^q \mathrm{P}(|Z| > s), \qquad s \geq c_Z. \tag{1.7}$$

- $\mathcal{H}$ is said to satisfy C_∞ if $\cup_{\xi\in I}\mathcal{H}_\xi \subseteq L^1$ and

$$\sup_{\xi\in I} \sup_{Z\in\mathcal{H}_\xi} \left(\frac{\|Z - \mathrm{E}[Z]\|_\infty}{\|Z - \mathrm{E}[Z]\|_2} \right) = \beta_3 < \infty. \tag{1.8}$$

Remark 1.2. If $\mathcal{H}$ satisfies C_q for some $q < \infty$ then for all $p \in (0,q)$ we have

$$\sup_{\xi\in I} \sup_{Z\in\mathcal{H}_\xi} \frac{\|Z\|_q}{\|Z\|_p} \leq (\beta_2 \vee 1)^{1/q} \beta_1^{-1/p} < \infty. \tag{1.9}$$

This follows by the next two estimates:

$$\begin{aligned} \mathrm{E}[|Z|^q] &= \mathrm{E}[|Z|^q, |Z| > c_Z] + \mathrm{E}[|Z|^q, |Z| \leq c_Z] & (1.10)\\ &\leq \beta_2 c_Z^q \mathrm{P}(|Z| > c_Z) + c_Z^q \mathrm{P}(|Z| \leq c_Z) \leq (\beta_2 \vee 1) c_Z^q & (1.11) \end{aligned}$$

and

$$c_Z^p \beta_1 \leq c_Z^p \mathrm{P}(|Z| \geq c_Z) \leq \mathrm{E}[|Z|^p]. \tag{1.12}$$

For example, when all $Z \in \cup_{\xi \in I} \mathcal{H}_\xi$ have the same distribution, $\mathcal{H}$ satisfies C_q for all $q \in (0,\alpha)$ for $\alpha > 0$ if $x \mapsto \mathrm{P}(|Z| > x)$ is regularly varying with index $-\alpha$, by Karamata's Theorem; see Bingham et al. [4, Theorem 1.5.11]. In particular, if the common distribution is symmetric α-stable for some $\alpha \in (0,2)$ then $\mathcal{H}$ satisfies C_q for all $q \in (0,\alpha)$. If the common distribution is Poisson, exponential, Gamma or Gaussian then C_q is satisfied for all $q > 0$. Finally $\mathcal{H}$ satisfies C_∞ if and only if the common distribution has compact support.

As we shall see in Section 2, C_q is crucial in order to obtain integrability results and equivalence of L^p-norms, so let us consider some cases where the important example (1.4) does or does not satisfy C_q. For this purpose let us introduce the following distributions: The inverse Gaussian distribution $\mathrm{IG}(\mu,\lambda)$ with $\mu,\lambda > 0$ is the distribution on $\mathbb{R}_+$ with density

$$f(x;\mu,\lambda) = \left[\frac{\lambda}{2\pi x^3}\right]^{1/2} e^{-\lambda(x-\mu)^2/(2\mu^2 x)}, \qquad x > 0. \tag{1.13}$$

Moreover, the normal inverse Gaussian distribution $\mathrm{NIG}(\alpha,\beta,\mu,\delta)$ with $\mu \in \mathbb{R}$, $\delta \geq 0$, and $0 \leq \beta \leq \alpha$, is symmetric if and only if $\beta = \mu = 0$, and in this case it has the following density

$$f(x;\alpha,\delta) = \frac{\alpha e^{\delta\alpha}}{\pi\sqrt{1+x^2\delta^{-2}}} K_1\left(\delta\alpha(1+x^2\delta^{-2})^{1/2}\right), \qquad x \in \mathbb{R}, \tag{1.14}$$

where K_1 is the modified Bessel function of the third kind and index 1 given by $K_1(z) = \frac{1}{2}\int_0^\infty e^{-z(y+y^{-1})/2}\,\mathrm{d}y$ for $z > 0$.

For each finite number $t_0 > 0$, a random measure Λ is said to be induced by a Lévy process $Y = (Y_t)_{t\in[0,t_0]}$ if $S = [0,t_0]$, $\mathcal{PC} = \mathcal{B}([0,t_0])$ and $\Lambda(A) = \int_A \mathrm{d}Y_s$ for all $A \in \mathcal{PC}$.

Proposition 1.3. *Let $t_0 \geq 1$ be a finite number, Λ a random measure induced by a Lévy process $Y = (Y_t)_{t\in[0,t_0]}$ and $\mathcal{H}$ be given by* (1.5)*.*

(i) If Y_1 has an IG-distribution, then $\mathcal{H}$ satisfies C_q if and only if $q \in (0,\frac{1}{2})$.

(ii) If Y_1 has a symmetric NIG-distribution, then $\mathcal{H}$ satisfies C_q if and only if $q \in (0,1)$.

(iii) If Y is non-deterministic and has no Gaussian component, then $\mathcal{H}$ does not satisfy C_q for any $q \geq 2$. In fact, for all square-integrable non-deterministic Lévy processes Y with no Gaussian component we have that $\lim_{t\to 0}\|Y_t\|_2/\|Y_t\|_1 = \infty$.

By the scaling property it is not difficult to show that if Λ is a symmetric α-stable random measure with $\alpha \in (0,2]$, then $\mathcal{H}$ satisfies C_q for all $q > 0$ when $\alpha = 2$ and for all $q < \alpha$ when $\alpha < 2$. For $\alpha < 2$ we have the following minor extension: Assume Λ is induced by a Lévy process Y with Lévy measure $\nu(\mathrm{d}x) = f(x)\,\mathrm{d}x$ where f is a symmetric function satisfying $c_1|x|^{-1-\alpha} \leq f(x) \leq c_2|x|^{-1-\alpha}$ for some $c_1, c_2 > 0$, then $\mathcal{H}$ satisfies C_q if and only if $q < \alpha$. Proposition 1.3 gives some insight about when C_q is satisfied; however, it would be interesting to develop more general conditions. We postpone the proof of Proposition 1.3 to Section 3.

1.2 Results on Integrability of Seminorms

Let T denote a countable set, $X = (X_t)_{t\in T}$ a real-valued stochastic process and N a measurable pseudo-seminorm on $\mathbb{R}^T$ such that $N(X) < \infty$ a.s. For X Gaussian [10] shows that $e^{\epsilon N(X)^2}$ is integrable for some $\epsilon > 0$. This result is extended to Gaussian chaos processes by Borell [5, Theorem 4.1]. Moreover, if X is α-stable for some $\alpha \in (0,2)$, de Acosta [8, Theorem 3.2] shows that $N(X)^p$ is integrable for all $p < \alpha$. When X is infinitely divisible [25] provide conditions on the Lévy measure ensuring integrability of $N(X)$. See also Hoffmann-Jørgensen [12] for further results.

Given a sequence $(Z_n)_{n\geq 1}$ of independent random variables, Borell [7] studies, under the condition

$$\sup_{n\geq 1} \frac{\|Z_n - \mathrm{E}[Z_n]\|_q}{\|Z_n - \mathrm{E}[Z_n]\|_2} < \infty, \qquad q \in (2, \infty], \tag{1.15}$$

integrability of Banach space valued random elements which are limits in probability of tetrahedral polynomials associated with $(Z_n)_{n\geq 1}$. For $q = \infty$, (1.15) is C_∞ but when $q < \infty$ (1.15) is weaker than C_∞, at least when $(Z_n)_{n\geq 1}$ are centered random variables. As shown in Borell [7], (1.15) implies equivalence of L^p-norms for Hilbert space valued tetrahedral polynomials for $p \leq q$, but not for Banach space valued tetrahedral polynomials except in the case $q = \infty$. Under the assumption that $(Z_n)_{n\geq 1}$ are symmetric random variables satisfying C_q, Kwapień and Woyczyński [18, Theorem 6.6.2] show that we have equivalence of L^p-norms in the above setting. Contrary to Borell [7], [18] and others, we consider random elements which are not necessarily limits of tetrahedral polynomials, and also more general spaces are considered. This enables us to obtain our integrability results for seminorms of stochastic processes.

Weak chaos processes appear in the context of multiple integral processes; see e.g. Krakowiak and Szulga [17] for the α-stable case. Rademacher chaos processes are applied repeatedly when studying U-statistics; see de la Peña and Giné [9]. They are also used to study infinitely divisible chaos processes; see Marcus and Rosiński [20], Rosiński and Samorodnitsky [26], Basse and Pedersen [3] and others. Using the results of the present paper, [2] extend some results on Gaussian semimartingales (e.g. Jain and Monrad [14] and Stricker [28]) to a large class of chaos processes.

2 Main results

The next lemma, which is a combination of several results, is crucial for this paper.

Lemma 2.1. *Let F denote a Banach space and X an F-valued tetrahedral polynomial of order d in the independent random variables $Z_1, \ldots, Z_n$. Assume that $\mathcal{H} = \{\mathcal{H}_0\}$ satisfies C_q for some $q \in (0, \infty]$, where $\mathcal{H}_0 = \{Z_1, \ldots, Z_n\}$; if $d \geq 2$ and $q < \infty$ assume moreover that $Z_1, \ldots, Z_n$ are symmetric. Then for all $0 < p < r \leq q$ with $r < \infty$ we have that*

$$\|X\|_{L^r(\mathrm{P};F)} \leq k_{p,r,d,\beta} \|X\|_{L^p(\mathrm{P};F)} < \infty, \tag{2.1}$$

where $k_{p,r,d,\beta}$ depends only on p, q, d and the β's from C_q. If $q = \infty$ and $p \geq 2$ we may choose $k_{p,r,d,\beta} = A_d \beta^{2d} r^{d/2}$ with $A_d = 2^{d^2/2+2d}$.

For $q < \infty$ and $d = 1$, Lemma 2.1 is a consequence of Kwapień and Woyczyński [18, Corollary 2.2.4]. Furthermore, for $q \in (1, \infty)$ and $d \geq 2$ it is taken from the proof of [18, Theorem 6.6.2] and using [18, Remark 6.9.1] the result is seen to hold also for $q \in (0, 1]$. For $q = \infty$, Lemma 2.1 is a consequence of Borell [7, Theorem 4.1]. In [7] the result is only stated for $2 \leq p < r$, however, a standard application of Hölder's inequality shows that it is valid for all $0 < p < r$; see e.g. Pisier [21, Lemme 1.1]. Finally, in [7] there are no explicit expression for A_d; this can, however, be obtained by applying the next Lemma 2.2 in the proof of [7, Theorem 4.1].

Lemma 2.2. *Let V denote a vector space, N a seminorm on V, $\epsilon \in (0,1)$ and $x_0, \dots, x_d \in V$.*

$$\text{If } N\Big(\sum_{k=0}^{d} \lambda^k x_k\Big) \leq 1 \text{ for all } \lambda \in [-\epsilon, \epsilon] \quad \text{then} \quad N\Big(\sum_{k=0}^{d} x_k\Big) \leq 2^{d^2/2+d}\epsilon^{-d}. \tag{2.2}$$

The proof of Lemma 2.2 is postponed to Section 3.

An F-valued random element X is said to be a.s. separably valued if $\mathrm{P}(X \in A) = 1$ for some separable closed subset A of F. We have the following result:

Theorem 2.3. *Let F denote a metrizable l.c.TVS, $X \in$ weak-$\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ an a.s. separably valued random element and N a lower semicontinuous pseudo-seminorm on F such that $N(X) < \infty$ a.s. Assume that $\mathcal{H}$ satisfies C_q for some $q \in (0, \infty]$ and if $q < \infty$ and $d \geq 2$ that all elements in $\cup_{\xi \in I} \mathcal{H}_\xi$ are symmetric. Then for all finite $0 < p < r \leq q$ we have*

$$\|N(X)\|_r \leq k_{p,r,d,\beta} \|N(X)\|_p < \infty, \tag{2.3}$$

where $k_{p,r,d,\beta}$ depends only on p, q, d and the β's from C_q. Furthermore, in the case $q = \infty$ we have that $\mathrm{E}[e^{\epsilon N(X)^{2/d}}] < \infty$ for all $\epsilon < d/(e 2^{d+5} \beta_3^4 \|N(X)\|_2^{2/d})$.

For $q = \infty$, Theorem 2.3 answers in the case where the pseudo-seminorm is lower semicontinuous a question raised by Borell [6] concerning integrability of pseudo-seminorms of Rademacher chaos elements. This additional assumption is satisfied in most examples, in particular in the examples in (1.1). Using the equivalence of norms in Theorem 2.3 we have by Krakowiak and Szulga [16, Corollary 1.4] the following corollary:

Corollary 2.4. *Let F and $\mathcal{H}$ be as in Theorem 2.3 and N be a continuous seminorm on F. Then given $(X_n)_{n \geq 1} \subseteq$ weak-$\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ all a.s. separably valued such that $\lim_n X_n = 0$ in probability we have $\|N(X_n)\|_p \to 0$ for all finite $p \in (0, q]$.*

Theorem 2.3 relies on the following two lemmas together with an application of Lemma 2.1 on the Banach space l^n_∞, that is $\mathbb{R}^n$ equipped with the sup norm. First, arguing as in Fernique [11, Lemme 1.2.2] we have:

Lemma 2.5. *Assume F is a strongly Lindelöf l.c.TVS. Then a pseudo-seminorm N on F is lower semicontinuous if and only if there exists $(x_n^*)_{n \geq 1} \subseteq F^*$ such that $N(x) = \sup_{n \geq 1} |x_n^*(x)|$ for all $x \in F$.*

Proof. The *if*-implication is trivial. To show the *only if*-implication let $A := \{x \in F : N(x) \leq 1\}$. Then A is convex and balanced since N is a pseudo-seminorm and closed since N is lower semicontinuous. Thus by the Hahn-Banach theorem, see Rudin [27, Theorem 3.7], for all $x \notin A$ there exists $x^* \in F^*$ such that $|x^*(y)| \leq 1$ for all $y \in A$ and $x^*(y) > 1$, showing that

$$A^c = \bigcup_{x \in A^c} \{y \in F : |x^*(y)| > 1\}. \tag{2.4}$$

Since F is strongly Lindelöf, there exists $(x_n)_{n \geq 1} \subseteq A^c$ such that

$$A^c = \bigcup_{n=1}^{\infty} \{y \in F : |x_n^*(y)| > 1\}, \tag{2.5}$$

implying that $A = \{y \in F : \sup_{n \geq 1} |x_n^*(y)| \leq 1\}$. Thus by homogeneity we have $N(y) = \sup_{n \geq 1} |x_n^*(y)|$ for all $y \in F$. $\square$

Lemma 2.6. *Let $n \geq 1$, $0 < p < q$ and $C > 0$ be given such that*

$$\|X\|_{L^q(\mathrm{P};l_\infty^n)} \leq C\|X\|_{L^p(\mathrm{P};l_\infty^n)} < \infty, \qquad X \in \mathcal{P}^d_{\mathcal{H}_\xi},\ \xi \in I. \tag{2.6}$$

Then, for all $(X_1, \ldots, X_n) \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R}^n)$ we have that

$$\|\max_{1\leq k\leq n}|X_k|\|_q \leq C\|\max_{1\leq k\leq n}|X_k|\|_p < \infty. \tag{2.7}$$

Proof. Let $X \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R}^n)$ and choose $(\xi_k)_{k\geq 1} \subseteq I$ and $X_k \in \mathcal{P}^d_{\mathcal{H}_{\xi_k}}(\mathbb{R}^n)$ for $k \geq 1$ such that $X_k \xrightarrow{\mathscr{D}} X$. Moreover, let $U_k = \|X_k\|_{l_\infty^n}$ and $U = \|X\|_{l_\infty^n}$. Then, $U_k \xrightarrow{\mathscr{D}} U$ showing that $(U_k)_{k\geq 1}$ is bounded in L^0, and by (2.6) and Krakowiak and Szulga [16, Corollary 1.4], $\{U_k^p : k \geq 1\}$ is uniformly integrable. This shows that

$$\|U\|_q \leq \liminf_{k\to\infty}\|U_k\|_q \leq C\liminf_{k\to\infty}\|U_k\|_p = C\|U\|_p < \infty, \tag{2.8}$$

and the proof is complete. □

Proof of Theorem 2.3. Since X is a.s. separably valued we may and will assume that F is separable. Hence according to Lemma 2.5 there exists $(x_n^*)_{n\geq 1} \subseteq F^*$ such that $N(x) = \sup_{n\geq 1}|x_n^*(x)|$ for all $x \in F$. For $n \geq 1$, let $X_n := x_n^*(X)$ and $U_n = \sup_{1\leq k\leq n}|X_k|$. Then $(U_n)_{n\geq 1}$ converges almost surely to $N(X)$. For finite $0 < p < r \leq q$ let $C = k_{p,r,d,\beta}$. Combining Lemmas 2.1 and 2.6 show $\|U_n\|_q \leq C\|U_n\|_p < \infty$ for all $n \geq 1$. This implies that $\{U_n^p : n \geq 1\}$ is uniformly integrable and hence we have that

$$\|N(X)\|_r \leq \liminf_{n\to\infty}\|U_n\|_r \leq C\liminf_{n\to\infty}\|U_n\|_p = C\|N(X)\|_p < \infty. \tag{2.9}$$

Finally, the exponential integrability under C_∞ follows by the last part of Lemma 2.1 since

$$\mathrm{E}[e^{\epsilon N(X)^{2/d}}] \leq 1 + \sum_{k=1}^{d}\|N(X)\|_{2k/d}^{2k/d} + \sum_{k=d+1}^{\infty}\big(\epsilon 2^{d+5}\beta_3^4\|N(X)\|_2^{2/d}/d\big)^k\frac{k^k}{k!}. \tag{2.10}$$

This completes the proof. □

Let T denote a countable set and $F = \mathbb{R}^T$ equipped with the product topology. F is then a separable and locally convex Fréchet space and all $x^* \in F^*$ are of the form $x \mapsto \sum_{i=1}^n \alpha_i x(t_i)$, for some $n \geq 1$, $t_1, \ldots, t_n \in T$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Thus for $X = (X_t)_{t\in T}$ we have that $X \in \mathit{weak}\text{-}\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ if and only if X is a weak chaos process of order d. Rewriting Theorem 2.3 in the case $F = \mathbb{R}^T$ we obtain the following result:

Theorem 2.7. *Assume $\mathcal{H}$ satisfies C_q for some $q \in (0,\infty]$ and if $q < \infty$ and $d \geq 2$ that all elements in $\cup_{\xi\in I}\mathcal{H}_\xi$ are symmetric. Let T denote a countable set, $(X_t)_{t\in T}$ a weak chaos process of order d and N a lower semicontinuous pseudo-seminorm on $\mathbb{R}^T$ such that $N(X) < \infty$ a.s. Then for all finite $0 < p < r \leq q$ we have*

$$\|N(X)\|_r \leq k_{p,r,d,\beta}\|N(X)\|_p < \infty, \tag{2.11}$$

and in the case $q = \infty$ that $\mathrm{E}[e^{\epsilon N(X)^{2/d}}] < \infty$ for all $\epsilon < d/(e2^{d+5}\beta_3^4\|N(X)\|_2^{2/d})$.

For example, let $T = [0,1] \cap \mathbb{Q}$, $(X_t)_{t\in T}$ be of the form $X_t = \int_0^1 f(t,s)\,\mathrm{d}Y_s$ where Y is a symmetric normal inverse Gaussian Lévy process, and $N\colon \mathbb{R}^T \to [0,\infty]$ is given by (1.1). Then, N is a lower semicontinuous pseudo-seminorm and X is weak chaos process of order one satisfying C_q for all $q < 1$ according to Proposition 1.3. Thus, if $N(X) < \infty$ a.s. then $\mathrm{E}[N(X)^p] < \infty$ for all $p < 1$, according to Theorem 2.7.

Let $\mathbb{G}$ denote a vector space of Gaussian random variables and $\overline{\Pi}^d_{\mathbb{G}}(\mathbb{R})$ be the closure in probability of the random variables $p(Z_1,\dots,Z_n)$, where $n \geq 1$, $Z_1,\dots,Z_n \in \mathbb{G}$ and $p\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree at most d (not necessary tetrahedral).

Lemma 2.8. *Let F be a l.c.TVS and X an F-valued random element such that $x^*(X) \in \overline{\Pi}^d_{\mathbb{G}}(\mathbb{R})$ for all $x^* \in F^*$; then $X \in$ weak-$\overline{\mathcal{P}}^d_{\mathcal{H}}(F)$ where $\mathcal{H} = \{\mathcal{H}_0\}$ and $\mathcal{H}_0$ is a Rademacher sequence.*

Recall that a sequence of independent, identically distributed random variables $(Z_n)_{n\geq 1}$ such that $\mathrm{P}(Z_1 = \pm 1) = 1/2$ is called a Rademacher sequence.

Proof. Let $n \geq 1$, $x_1^*,\dots,x_n^* \in F^*$ and $W = (x_1^*(X),\dots,x_n^*(X))$. We need to show that $W \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R}^n)$. For all $k \geq 1$ we may choose polynomials $p_k\colon \mathbb{R}^k \to \mathbb{R}^n$ of degree at most d and $Y_{1,k},\dots,Y_{k,k}$ independent standard normal random variables such that with $Y_k = (Y_{1,k},\dots,Y_{k,k})$ we have $\lim_k p_k(Y_k) = W$ in probability. Hence it suffices to show $p_k(Y_k) \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R}^n)$ for all $k \geq 1$. Fix $k \geq 1$ and let us write p and Y for p_k and Y_k. Reenumerate $\mathcal{H}_0$ as k independent Rademacher sequences $(Z_{i,m})_{i\geq 1}$ $m = 1,\dots,k$ and set

$$U_j = \frac{1}{\sqrt{j}} \sum_{i=1}^{j} (Z_{1,i},\dots,Z_{k,i}), \qquad j \geq 1. \tag{2.12}$$

Then, by the central limit theorem $U_j \xrightarrow{\mathscr{D}} Y$ and hence $p(U_j) \xrightarrow{\mathscr{D}} p(Y)$. Due to the fact that all $Z_{i,m}$ only takes on the values ± 1, $p(U_j) \in \mathcal{P}^d_{\mathcal{H}_0}(\mathbb{R}^n)$ for all $j \geq 1$, showing that $p(Y) \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R}^n)$. ☐

The $\mathcal{H}$ in Lemma 2.8 trivially satisfies C_∞ with $\beta_3 = 1$ and hence a combination of Theorem 2.3 and Lemma 2.8 shows:

Proposition 2.9. *Let F be a l.c.TVS and X an a.s. separably valued random element in F such that $x^*(X) \in \overline{\Pi}^d_{\mathbb{G}}(\mathbb{R})$ for all $x^* \in F^*$. Then, for all lower semicontinuous pseudo-seminorms N on F satisfying $N(X) < \infty$ a.s. we have*

$$\|N(X)\|_r \leq 2^{d^2/2+d} \left(\frac{r-1}{p-1}\right)^{d/2} \|N(X)\|_p < \infty, \tag{2.13}$$

and $\mathrm{E}[e^{\epsilon N(X)^{2/d}}] < \infty$ *for all* $\epsilon < d/(e2^{d+5}\|N(X)\|_2^{2/d})$.

The integrability of $e^{\epsilon N(X)^{2/d}}$ for some $\epsilon > 0$ is a consequence of the seminal work Borell [5, Theorem 4.1]. However, the above provides a very simple proof of this result and gives also equivalence of L^p-norms and explicit constants. When $F = \mathbb{R}^T$ for some countable set T, Proposition 2.9 covers processes $X = (X_t)_{t\in T}$, where all time variables have the following representation in terms of multiple Wiener-Itô integrals with respect to a Brownian motion W,

$$X_t = \sum_{k=0}^{d} \int_{\mathbb{R}_+^k} f(t,k;s_1,\dots,s_k)\,\mathrm{d}W_{s_1}\cdots\mathrm{d}W_{s_k}, \qquad t \in T. \tag{2.14}$$

The next result is known from Arcones and Giné [1, Theorem 3.1] for general Gaussian polynomials.

Proposition 2.10. *Assume that $\mathcal{H} = \{\mathcal{H}_0\}$ satisfies C_q for some $q \in [2,\infty]$ and $\mathcal{H}_0$ consists of symmetric random variables. Let F denote a Banach space and X an a.s. separably valued random element in F with $x^*(X) \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R})$ for all $x^* \in F^*$. Then there exists $x_0, x_{i_1,\dots,i_k} \in F$ and $\{Z_n : n \geq 1\} \subseteq \mathcal{H}_0$ such that for all finite $p \leq q$*

$$X = \lim_{n\to\infty} \Big(x_0 + \sum_{k=1}^{d} \sum_{1\leq i_1<\dots<i_k\leq n} x_{i_1,\dots,i_k} \prod_{j=1}^{k} Z_{i_j} \Big) \qquad \text{a.s. and in } L^p(\mathrm{P};F). \tag{2.15}$$

Proof. We follow Arcones and Giné [1, Lemma 3.4]. Since X is a.s. separably valued we may and do assume F is separable, which implies that $F_1^* := \{x^* \in F^* : \|x^*\| \leq 1\}$ is metrizable and compact in the weak*-topology by the Banach-Alaoglu theorem; see Rudin [27, Theorem 3.15+3.16]. Moreover, the map $x^* \mapsto x^*(X)$ from F_1^* into L^0 is trivially weak*-continuous and thus a weak*-continuous map into L^2 by Corollary 2.4. This shows that $\{x^*(X) : x^* \in F_1^*\}$ is compact in L^2 and hence separable. By definition of $\overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R})$, this implies that there exists a countable set $\{Z_n : n \geq 1\} \subseteq \mathcal{H}_0$ such that

$$x^*(X) = \sum_{A\in N_d} a(A,x^*) Z_A, \qquad \text{in } L^2, \tag{2.16}$$

for some $a(A,x^*) \in \mathbb{R}$, where $N_d = \{A \subseteq \mathbb{N} : |A| \leq d\}$ and $Z_A = \prod_{i\in A} Z_i$ for $A \in N_d$. For $A \in N_d$, the map $x^* \mapsto a(A,x^*)$ from F^* into $\mathbb{R}$ is linear and weak*-continuous and hence there exists $x_A \in F$ such that $a(A,x^*) = x^*(x_A)$, showing that

$$x^*(X) = \lim_{n\to\infty} x^*\Big(\sum_{A\in N_d^n} x_A Z_A \Big), \qquad \text{in } L^2, \tag{2.17}$$

where $N_d^n = \{A \in N_d : A \subseteq \{1,\dots,n\}\}$. Since F is separable, (2.17) and Kwapień and Woyczyński [18, Theorem 6.6.1] show that

$$\lim_{n\to\infty} \sum_{A\in N_d^n} x_A Z_A = X \qquad \text{a.s.} \tag{2.18}$$

By Corollary 2.4 the convergence also takes place in $L^p(\mathrm{P};F)$ for all finite $p \leq q$, which completes the proof. ☐

The above proposition gives rise to the following corollary:

Corollary 2.11. *Assume that $\mathcal{H} = \{\mathcal{H}_0\}$ satisfies C_q for some $q \in [2,\infty]$ and $\mathcal{H}_0$ consists of symmetric random variables. Let T denote a set, $V(T) \subseteq \mathbb{R}^T$ a separable Banach space where the map $f \mapsto f(t)$ from $V(T)$ into $\mathbb{R}$ is continuous for all $t \in T$, and $X = (X_t)_{t\in T}$ a stochastic process with sample paths in $V(T)$ satisfying $X_t \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R})$ for all $t \in T$. Then there exists $x_0, x_{i_1,\dots,i_k} \in V(T)$ and $\{Z_n : n \geq 1\} \subseteq \mathcal{H}_0$ such that*

$$X = \lim_{n\to\infty} \Big(x_0 + \sum_{k=1}^{d} \sum_{1\leq i_1<\dots<i_k\leq n} x_{i_1,\dots,i_k} \prod_{j=1}^{k} Z_{i_j} \Big) \tag{2.19}$$

a.s. in $V(T)$ and in $L^p(\mathrm{P};V(T))$ for all finite $p \leq q$.

Proof. For $t \in T$, let $\delta_t \colon V(T) \to \mathbb{R}$ denote the map $f \mapsto f(t)$. Since $V(T)$ is a separable Banach space and $\{\delta_t : t \in T\} \subseteq V(T)^*$ separate points in $V(T)$ we have

(i) the Borel σ-field on $V(T)$ equals the cylindrical σ-field $\sigma(\delta_t : t \in T)$,

(ii) $\{\sum_{i=1}^n \alpha_i \delta_{t_i} : \alpha_i \in \mathbb{R},\ t_i \in T,\ n \geq 1\}$ is sequentially weak*-dense in $V(T)^*$,

see e.g. Rosiński [24, page 287]. By (i) we may regard X as a random element in $V(T)$ and by (ii) it follows that $x^*(X) \in \overline{\mathcal{P}}^d_{\mathcal{H}}(\mathbb{R})$ for all $x^* \in V(T)^*$. Hence the result is a consequence of Proposition 2.10. □

Borell [7, Theorem 5.1] shows Corollary 2.11 assuming (1.15), T is a compact metric space, $V(T) = C(T)$ and $X \in L^q(\mathrm{P}; V(T))$. By assuming C_q instead of the weaker condition (1.15) we can omit the assumption $X \in L^q(\mathrm{P}; V(T))$. Note also that by Theorem 2.7 the last assumption is satisfied under C_q. When $\mathcal{H}_0$ consists of symmetric α-stable random variables and $d = 1$, Corollary 2.11 is known from Rosiński [24, Corollary 5.2]. The separability assumption on $V(T)$ in Corollary 2.11 is crucial. Indeed, for all $p > 1$, Jain and Monrad [15, Proposition 4.5] construct a separable centered Gaussian process $X = (X_t)_{t\in[0,1]}$ with sample paths in the non-separable Banach space B_p of functions of finite p-variation on $[0, 1]$ such that the range of X is a non-separable subset of B_p and hence the conclusion in Corollary 2.11 can not be true. However, for the non-separable Banach space B_1 a result similar to Corollary 2.11 is shown in [14] for Gaussian processes, and extended to weak chaos processes in [2].

3 Two proofs

Let us start by proving Proposition 1.3.

Proof of Proposition 1.3. Assume that Λ is a random measure induced by a Lévy process $Y = (Y_t)_{t\in[0,T]}$. For arbitrary $A \in \mathcal{PC}$ let $Z = \Lambda(A)$.

To prove the *if*-implication of (i) let $q \in (0, \frac{1}{2})$ and assume that $Y_1 \overset{\mathscr{D}}{=} \mathrm{IG}(\mu, \lambda)$. Then $Z \overset{\mathscr{D}}{=} \mathrm{IG}(m(A)\mu, m(A)^2\lambda)$, where m is the Lebesgue measure, and hence with $c_Z = m(A)^2\lambda$ we have that $Z/c_Z \overset{\mathscr{D}}{=} \mathrm{IG}(\mu/(\lambda m(A)), 1)$, which has a density which on $[1, \infty)$ is bounded from below and above by constants (not depending on x) times $g_Z(x)$, where

$$g_Z \colon \mathbb{R}_+ \to \mathbb{R}_+, \qquad x \mapsto x^{-3/2} \exp[-x(\lambda m(A))^2/(2\mu^2)]. \tag{3.1}$$

Thus there exists a constant $c > 0$, not depending on A or s, such that

$$\frac{\mathrm{E}[|Z/c_Z|^q, |Z/c_Z| > s]}{s^q \mathrm{P}(|Z/c_Z| > s)} \leq c \sup_{u>0} \left(\frac{\int_u^\infty x^{q-3/2} e^{-x}\,\mathrm{d}x}{u^q \int_u^\infty x^{-3/2} e^{-x}\,\mathrm{d}x} \right) \qquad s \geq 1. \tag{3.2}$$

Using e.g. l'Hôpital's rule it is easily seen that (3.2) is finite, showing (1.7). Therefore C_q follows by the inequality

$$\mathrm{P}(Z/c_Z \geq 1) \geq \frac{e^{-1/2}}{\sqrt{2\pi}} \int_1^\infty x^{-3/2} \exp[-x(\lambda T)^2/(2\mu^2)]\,\mathrm{d}x. \tag{3.3}$$

To show the *only if*-implication of (i) note that $n^2 Y_{1/n} \overset{\mathscr{D}}{\to} X$ as $n \to \infty$, where X follows a $\frac{1}{2}$-stable distribution on $\mathbb{R}_+$. Assume that $\mathcal{H}$ satisfies C_q for some $q \geq 1/2$. Then, by

Remark 1.2 there exists $c > 0$ such that $\|Y_t\|_{1/2} \leq c\|Y_t\|_{1/4}$ for all $t \in [0,1]$, and since $\{n^2 Y_{1/n} : n \geq 1\}$ is bounded in L^0 it is also bounded in $L^{1/2}$. But this contradicts

$$\infty = \|X\|_{1/2} \leq \liminf_{n\to\infty} \|n^2 Y_{1/n}\|_{1/2}, \tag{3.4}$$

and shows that $\mathcal{H}$ does not satisfy C_q.

To show the *if*-implication of (ii) assume that $Y_1 \stackrel{\mathscr{D}}{=} \mathrm{NIG}(\alpha,0,0,\delta)$. Then, $Z \stackrel{\mathscr{D}}{=} \mathrm{NIG}(\alpha,0,0,m(A)\delta)$ and with $c_Z = m(A)\delta$ we have that $Z/c_Z \stackrel{\mathscr{D}}{=} \epsilon U_Z^{1/2}$, where U_Z and ϵ are independent, $U_Z \stackrel{\mathscr{D}}{=} \mathrm{IG}(1/(m(A)\delta\alpha),1)$ and $\epsilon \stackrel{\mathscr{D}}{=} \mathrm{N}(0,1)$. For $q \in (0,1)$,

$$\begin{aligned} \mathrm{E}[|Z/c_Z|^q, |Z/c_Z| > s] = \sqrt{2\pi^{-1}}\Big(\int_0^s \mathrm{E}[|xU_Z^{1/2}|^q, |xU_Z^{1/2}| > s] e^{-x^2/2}\,\mathrm{d}x \qquad (3.5)\\ + \int_s^\infty \mathrm{E}[|xU_Z^{1/2}|^q, |xU_Z^{1/2}| > s] e^{-x^2/2}\,\mathrm{d}x\Big). \qquad (3.6)\end{aligned}$$

Using the above (i) on U_Z and $q/2$, there exists a constant $c_1 > 0$ such that

$$\begin{aligned} &\int_0^s \mathrm{E}[|xU_Z^{1/2}|^q, |xU_Z^{1/2}| > s] e^{-x^2/2}\,\mathrm{d}x \leq c_1 s^q \int_0^s \mathrm{P}\big(U_Z > (s/x)^2\big) e^{-x^2/2}\,\mathrm{d}x \qquad (3.7)\\ &\leq c_1 s^q \int_0^\infty \mathrm{P}\big(xU_Z^{1/2} > s\big) e^{-x^2/2}\,\mathrm{d}x = c_1\sqrt{\pi 2^{-1}} s^q \mathrm{P}(|Z/c_Z| > s). \qquad (3.8)\end{aligned}$$

Furthermore, it well known that there exists a constant $c_2 > 0$ such that for all $s \geq 1$

$$\begin{aligned} &\int_s^\infty \mathrm{E}[|xU_Z^{1/2}|^q, |xU_Z^{1/2}| > s] e^{-x^2/2}\,\mathrm{d}x \qquad (3.9)\\ &\leq \mathrm{E}[U_Z^{q/2}] \int_s^\infty x^q e^{-x^2/2}\,\mathrm{d}x \leq c_2 s^q \mathrm{E}[U_Z^{q/2}] \int_s^\infty e^{-x^2/2}\,\mathrm{d}x. \qquad (3.10)\end{aligned}$$

Since U_Z has a density given by (1.13) it is easily seen that

$$\mathrm{E}[U_Z^{q/2}] \leq 1 + \frac{1}{\sqrt{2\pi}} \int_1^\infty x^{q/2-3/2}\,\mathrm{d}x. \tag{3.11}$$

Moreover, using that $Z/c_Z \stackrel{\mathscr{D}}{=} \mathrm{NIG}(m(A)\alpha\delta,0,0,1)$ and that $K_1(z) \geq e^{-z}/z$ for all $z > 0$, it is not difficult to show that there exists a constant c_3, not depending on s and A, such that

$$\int_s^\infty e^{-x^2/2}\,\mathrm{d}x \leq c_3 \mathrm{P}(|Z/c_Z| > s), \qquad \text{for all } s \geq 1. \tag{3.12}$$

By combining the above we obtain (1.7) and by (3.12) applied on $s = 1$, C_q follows. The *only if*-implication of (ii) follows similar to the one of (i), now using that $(n^{-1}Y_{1/n})_{n\geq 1}$ converge weakly to a symmetric 1-stable distribution. (iii) is a consequence of the next lemma. □

The following lemma is concerned with the dynamics of the first and second moments of Lévy processes, and it has Proposition 1.3 (iii) as a direct consequence.

Lemma 3.1. *Let Y denote a non-deterministic and square-integrable Lévy process with no Gaussian component. Then $\|Y_t\|_1 = o(t^{1/2})$ and $\|Y_t\|_2 \sim t^{1/2}\sqrt{\mathrm{E}[(Y_1 - \mathrm{E}[Y_1])^2]}$ as $t \to 0$.*

Proof. We have

$$\mathrm{E}[Y_t^2] = Var(Y_t) + \mathrm{E}[Y_t]^2 = Var(Y_1)t + \mathrm{E}[Y_1]^2 t^2, \tag{3.13}$$

which shows that $\|Y_t\|_2 \sim t^{1/2} Var(Y_1)^{1/2}$ as $t \to 0$.

To show that $\|Y_t\|_1 = o(t^{1/2})$ as $t \to 0$ we may assume that Y is symmetric. Indeed let $\mu = \mathrm{E}[Y_1]$, Y' an independent copy of Y and $\tilde{Y}_t = Y_t - Y'_t$. Then $\tilde{Y}$ is a symmetric square-integrable Lévy process and

$$\|Y_t\|_1 \leq \|Y_t - \mu t\|_1 + |\mu| \leq \|Y_t - \mu t - (Y'_t - \mu t)\|_1 + |\mu| t = \|\tilde{Y}_t\|_1 + |\mu| t. \tag{3.14}$$

Hence assume that Y is symmetric. Recall, e.g. from Hoffmann-Jørgensen [13, Exercise 5.7], that for any random variable U we have

$$\|U\|_1 = \frac{1}{\pi} \int \frac{1 - \Re \phi_U(s)}{s^2} \,\mathrm{d}s, \tag{3.15}$$

where ϕ_U denotes the characteristic function of U. Using the inequalities $1 - e^{-x} \leq 1 \wedge x$ and $1 - \cos(x) \leq 4(1 \wedge x^2)$ for all $x \geq 0$ it follows that with $\psi(s) := 4 \int (1 \wedge |sx|^2)\, \nu(\mathrm{d}x)$ we have

$$\|Y_t\|_1 \leq \frac{1}{\pi} \int \frac{1 - e^{-t\psi(s)}}{s^2} \,\mathrm{d}s \leq \frac{1}{\pi} \int \frac{|t\psi(s)| \wedge 1}{s^2} \,\mathrm{d}s. \tag{3.16}$$

Note that $\psi(s) < \infty$ since Y is square-integrable. By substitution we get

$$\int \frac{|t\psi(s)| \wedge 1}{s^2} \,\mathrm{d}s \leq 2t^{1/2} \int_0^\infty \frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2} \,\mathrm{d}s. \tag{3.17}$$

Hence to complete the proof we need only to show that

$$\lim_{t \to 0} \int_0^\infty \frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2} \,\mathrm{d}s = 0. \tag{3.18}$$

Setting $c = 4 \int x^2 \,\nu(\mathrm{d}x)$ we have for all $\epsilon > 0$

$$\limsup_{t \to 0} \int_0^\infty \frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2} \,\mathrm{d}s \tag{3.19}$$

$$\leq \limsup_{t \to 0} \int_0^{\epsilon/c} \frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2} \,\mathrm{d}s + \limsup_{t \to 0} \int_{\epsilon/c}^\infty \frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2} \,\mathrm{d}s. \tag{3.20}$$

Using that $\psi(x) \leq cx^2$ for $x \geq 0$ we get

$$\limsup_{t \to 0} \int_0^{\epsilon/c} \frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2} \,\mathrm{d}s \leq \epsilon. \tag{3.21}$$

On the other hand, Lebesgue's dominated convergence theorem shows that

$$\psi(x) x^{-2} = 4 \int (x^{-2} \wedge s^2) \,\nu(\mathrm{d}x) \xrightarrow[x \to \infty]{} 0, \tag{3.22}$$

implying that $t\psi(t^{-1/2}s) \to 0$ as $t \to 0$ for all $s \geq 0$. Thus another application of Lebesgue's dominated convergence theorem yields

$$\limsup_{t \to 0} \int_{\epsilon/c}^\infty \frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2} \,\mathrm{d}s = 0, \tag{3.23}$$

which by (3.20) and (3.21) shows (3.18). $\square$

Let us proceed with the proof of Lemma 2.2.

Proof of Lemma 2.2. Assume first that $x_0, \dots, x_d \in \mathbb{R}$. By induction in d, let us show:

$$\text{If } \Big|\sum_{k=0}^{d} \lambda^k x_k\Big| \le 1 \text{ for all } \lambda \in [-\epsilon, \epsilon] \quad \text{then} \quad \Big|\sum_{k=0}^{d} x_k\Big| \le 2^{d^2/2+d}\epsilon^{-d}. \tag{3.24}$$

For $d = 1, 2$ (3.24) follows by a straightforward argument, so assume $d \ge 3$, (3.24) holds for $d-1$ and that the left-hand side of (3.24) holds for d. We have

$$\Big|\sum_{k=0}^{d} \lambda^k (\epsilon^k x_k)\Big| \le 1, \qquad \text{for all } \lambda \in [-1, 1], \tag{3.25}$$

which by Pólya and Szegö [22, Aufgabe 77] shows that $|x_d \epsilon^d| \le 2^d$ and hence $|x_d| \le 2^d \epsilon^{-d}$. For $\lambda \in [-\epsilon, \epsilon]$, the triangle inequality yields

$$\Big|\sum_{k=0}^{d-1} \lambda^k x_k\Big| \le 1 + 2^d, \quad \text{and hence} \quad \Big|\sum_{k=0}^{d-1} \lambda^k \frac{x_k}{1+2^d}\Big| \le 1. \tag{3.26}$$

The induction hypothesis implies

$$\Big|\sum_{k=0}^{d-1} x_k\Big| \le \epsilon^{-(d-1)} 2^{(d-1)^2+(d-1)}(1+2^d), \tag{3.27}$$

and hence another application of the triangle inequality shows that

$$\Big|\sum_{k=0}^{d} x_k\Big| \le \epsilon^{-d}2^d + \epsilon^{-(d-1)}2^{(d-1)^2/2+(d-1)}(1+2^d) \tag{3.28}$$

$$\le \epsilon^{-d}2^{d^2/2+d}\Big(2^{-d^2/2} + 2^{-1/2-d} + 2^{-1/2}\Big), \tag{3.29}$$

which is less than or equal to $\epsilon^{-d}2^{d^2/2+d}$ since $d \ge 3$. This completes the proof of (3.24).

Now let $x_0, \dots, x_d \in V$. Since N is a seminorm, Hahn-Banach theorem (see Rudin [27, Theorem 3.2]) shows that there exists a family Λ of linear functionals on V such that

$$N(x) = \sup_{F \in \Lambda} |F(x)|, \qquad \text{for all } x \in V. \tag{3.30}$$

Assuming that the left-hand side of (2.2) is satisfied we have

$$\Big|\sum_{k=0}^{d} \lambda^k F(x_k)\Big| \le 1, \qquad \text{for all } \lambda \in [-\epsilon, \epsilon] \text{ and all } F \in \Lambda, \tag{3.31}$$

which by (3.24) shows

$$\Big|F\Big(\sum_{k=0}^{d} x_k\Big)\Big| = \Big|\sum_{k=0}^{d} F(x_k)\Big| \le 2^{d(d-1)}\epsilon^{-d}, \qquad \text{for all } F \in \Lambda. \tag{3.32}$$

This completes the proof. □

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PAPER G

Martingale-type processes indexed by $\mathbb{R}$

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Abstract

Some classes of increment martingales, and the corresponding localised classes, are studied. An increment martingale is indexed by $\mathbb{R}$ and its increment processes are martingales. We focus primarily on the behavior as time goes to $-\infty$ in relation to the quadratic variation or the predictable quadratic variation, and we relate the limiting behaviour to the martingale property. Finally, integration with respect to an increment martingale is studied.

Keywords: Martingales; increments; integration; compensators.

AMS Subject Classification: 60G44; 60G48; 60H05

1 Introduction

Stationary processes are widely used in many areas, and the key example is a moving average, that is, a process X of the form

$$X_t = \int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}M_s, \qquad t \in \mathbb{R}, \tag{1.1}$$

where $M = (M_t)_{t\in\mathbb{R}}$ is a process with stationary increments. A particular example is a stationary Ornstein-Uhlenbeck process which corresponds to the case $\psi(t) = e^{-\lambda t} 1_{[0,\infty)}(t)$ and M is a Brownian motion indexed by $\mathbb{R}$. See [6] for second order properties of moving averages and [1] for applications of them in turbulence.

Integration with respect to a local martingale indexed by $\mathbb{R}_+$ is well-developed and in this case one can even allow the integrand to be random. However, when trying to define a stochastic integral from $-\infty$ as in (1.1) with random integrands, the class of local martingales indexed by $\mathbb{R}$ does not provide the right framework for $M = (M_t)_{t\in\mathbb{R}}$; indeed, in simple cases, such as when M is a Brownian motion, M is not a martingale in any filtration. Rather, it seems better to think of M as a process for which the increment $(M_{t+s} - M_s)_{t\geq 0}$ is a martingale for all $s \in \mathbb{R}$. It is natural to call such a process an *increment martingale.* Another interesting example within this framework is a diffusion on natural scale started in ∞ (cf. Example 3.17); indeed, if ∞ is an entrance boundary then all increments are local martingales but the diffusion itself is not. Thus, the class of increment (local) martingales indexed by $\mathbb{R}$ is strictly larger than the class of (local) martingales indexed by $\mathbb{R}$ and it contains several interesting examples. We refer to Subsection 1.1 for a discussion of the relations to other kinds of martingale-type processes indexed by $\mathbb{R}$.

In the present paper we introduce and study basic properties of some classes of increment martingales $M = (M_t)_{t\in\mathbb{R}}$ and the corresponding localised classes. Some of the problems studied are the following. Necessary and sufficient conditions for M to be a local martingale up to addition of a random variable will be given when M is either an increment martingale or an increment square integrable martingale. In addition, we give various necessary and sufficient conditions for $M_{-\infty} = \lim_{t\to-\infty} M_t$ to exist P-a.s. and $M - M_{-\infty}$ to be a local martingale expressed in terms of either the predictable quadratic variation $\langle M\rangle$ or the quadratic variation $[M]$ for M, where the latter two quantities will be defined below for increment martingales. These conditions rely on a convenient decomposition of increment martingales, and are particularly simple when M is continuous. We define two kinds of integrals with respect to M; the first of these is an increment integral $\phi \overset{\text{in}}{\bullet} M$, which we can think of as process satisfying $\phi \overset{\text{in}}{\bullet} M_t - \phi \overset{\text{in}}{\bullet} M_s = \int_{(s,t]} \phi_u \,\mathrm{d}M_u$; i.e. increments in $\phi \overset{\text{in}}{\bullet} M$ correspond to integrals over finite intervals. The second integral, $\phi \bullet M$, is a usual stochastic integral with respect to M which we can think of as an integral from $-\infty$. The integral $\phi \bullet M$ exists if and only if the increment integral $\phi \overset{\text{in}}{\bullet} M$ has an a.s. limit, $\phi \overset{\text{in}}{\bullet} M_{-\infty}$, at $-\infty$ and $\phi \overset{\text{in}}{\bullet} M - \phi \overset{\text{in}}{\bullet} M_{-\infty}$ is a local martingale. Thus, $\phi \overset{\text{in}}{\bullet} M_{-\infty}$ may exists without $\phi \bullet M$ being defined and in this case we may think of $\phi \overset{\text{in}}{\bullet} M_{-\infty}$ as an improper integral. In special cases we give necessary and sufficient conditions for $\phi \overset{\text{in}}{\bullet} M_{-\infty}$ to exist.

The present paper relies only on standard martingale results and martingale integration as developed in many textbooks, see e.g. [8] and [7]. While we focus primarily on the behaviour at $-\infty$, it is also of interest to consider the behaviour at ∞; we refer to [5], and references therein, for a study of this case for semimartingales, and to [12], and references therein, for a study of improper integrals with respect to Lévy processes when the integrand is deterministic.

1.1 Relations to other martingale-type processes

Let us briefly discuss how to define processes with some kind of martingale structure when processes are indexed by $\mathbb{R}$. There are at least three natural definitions:

(i) $\mathrm{E}[M_t|\mathcal{F}_s^M] = M_s$ for all $s \leq t$, where $\mathcal{F}_s^M = \sigma(M_u : u \in (-\infty, s])$.

(ii) $\mathrm{E}[M_t - M_u|\mathcal{F}_{v,s}^{IM}] = M_s - M_v$ for all $u \leq v \leq s \leq t$, where $\mathcal{F}_{v,s}^M = \sigma(M_s - M_u : v \leq u \leq t \leq s)$.

(iii) $\mathrm{E}[M_t - M_s|\mathcal{F}_s^{IM}] = 0$ for all $s \leq t$, where $\mathcal{F}_s^{\mathcal{I}M} = \sigma(M_t - M_u : u \leq t \leq s)$.

(The first definition is the usual martingale definition and the third one corresponds to increment martingales). Both (i) and (iii) generalise the usual notion of martingales indexed by $\mathbb{R}_+$, in the sense that if $(M_t)_{t\in\mathbb{R}}$ is a process with $M_t = 0$ for $t \in (-\infty, 0]$, then $(M_t)_{t\geq 0}$ is a martingale (in the usually sense) if and only if $(M_t)_{t\in\mathbb{R}}$ is a martingale in the sense of (i), or equivalently in the sense of (iii). Definition (ii) does not generalise martingales indexed by $\mathbb{R}_+$ in this manner. Note moreover that a centered Lévy process indexed by $\mathbb{R}$ (cf. Example 3.3) is a martingale in the sense of (ii) and (iii) but not in the sense of (i). Thus, (iii) is the only one of the above definitions which generalise the usual notion of martingales on $\mathbb{R}_+$ and is general enough to allow centered Lévy processes to be martingales. Note also that both (i) and (ii) imply (iii).

The general theory of martingales indexed by partially ordered sets (for short, posets) does not seem to give us much insight about increment martingales since the research in this field mainly has a different focus; indeed, one of the main problems has been to study martingales $M = (M_t)_{t\in I}$ in the case where $I = [0, 1]^2$; see e.g. [4, 3]. However, below we recall some of the basic definitions and relate them to the above (i)–(iii).

Consider a poset $(I, \leq)$ and a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in I}$, that is, for all $s, t \in I$ with $s \leq t$ we have that $\mathcal{F}_s \subseteq \mathcal{F}_t$. Then, $(M_t)_{t\in I}$ is called a martingale with respect to $\leq$ and $\mathcal{F}$, if for all $s, t \in I$ with $s \leq t$ we have that $\mathrm{E}[M_t|\mathcal{F}_s] = M_s$. Let $M = (M_t)_{t\in\mathbb{R}}$ denote a stochastic process. Then, definition (i) corresponds to $I = \mathbb{R}$ with the usually order. To cover (ii) and (iii) let $I = \{(a_1, a_2] : a_1, a_2, \in \mathbb{R},\ a_1 < a_2\}$, and for $A = (a_1, a_2] \in I$ let $M_A = M_{a_2} - M_{a_1}$, $\mathcal{F}_A^M = \sigma(M_B : B \in I,\ B \subseteq A)$. Furthermore, for all $A = (a_1, a_2], B = (b_1, b_2] \in I$ we will write $A \leq_2 B$ if $A \subseteq B$, and $A \leq_3 B$ if $a_1 = b_1$ and $a_2 \leq b_2$. Clearly, $\leq_2$ and $\leq_3$ are two partial orders on I. Moreover, it is easily seen that $(M_t)_{t\in\mathbb{R}}$ satisfies (ii)/(iii) if and only if $(M_A)_{A\in I}$ is a martingale with respect to $\leq_2/\leq_3$ and $\mathcal{F}^M$. Recall that a poset $(I, \leq)$ is called directed if for all $s, t \in I$ there exists an element $u \in I$ such that $s \leq u$ and $t \leq u$. Note that $(I, \leq_2)$ is directed, but $(I, \leq_3)$ is not; and in particular $(I, \leq_3)$ is not a lattice. We refer to [9] for some nice considerations about martingales indexed by directed posets.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathrm{P})$ denote a complete probability space on which all random variables appearing in the following are defined. Let $\mathcal{F}_\cdot = (\mathcal{F}_t)_{t\in\mathbb{R}}$ denote a filtration in $\mathcal{F}$, i.e. a right-continuous increasing family of sub σ-algebras in $\mathcal{F}$ satisfying $\mathcal{N} \subseteq \mathcal{F}_t$ for all t, where $\mathcal{N}$ is the collection of all P-null sets. Set $\mathcal{F}_{-\infty} := \cap_{t\in\mathbb{R}}\mathcal{F}_t$ and $\mathcal{F}_\infty := \cup_{t\in\mathbb{R}}\mathcal{F}_t$. The notation $\stackrel{\mathscr{D}}{=}$ will be used to denote identity in distribution. Similarly, $\stackrel{\mathrm{P}}{=}$ will denote equality up to P-indistinguishability of stochastic processes. When $X = (X_t)_{t\in\mathbb{R}}$ is a real-valued stochastic process we say that $\lim_{s\to-\infty} X_s$ exists P-a.s. if X_s converges almost surely as $s \to -\infty$, to a finite limit.

Definition 2.1. A *stopping time* σ is a mapping $\sigma : \Omega \to (-\infty, \infty]$ satisfying $\{\sigma \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}$. A *localising sequence* $(\sigma_n)_{n \geq 1}$ is a sequence of stopping times satisfying $\sigma_1(\omega) \leq \sigma_2(\omega) \leq \cdots$ for all ω, and $\sigma_n \to \infty$ P-a.s.

Let $\mathcal{P}(\mathcal{F}_\cdot)$ denote the *predictable σ-algebra* on $\mathbb{R} \times \Omega$. That is, the σ-algebra generated by the set of *simple predictable sets*, where a subset of $\mathbb{R} \times \Omega$ is said to be simple predictable if it is of the form $B \times C$ where, for some $t \in \mathbb{R}$, C is in $\mathcal{F}_t$ and B is a bounded Borel set in $]t, \infty[$. Note that the set of simple predictable sets is closed under finite intersections.

Any left-continuous and adapted process is predictable. Moreover, the set of predictable processes is stable under stopping in the sense that whenever $\alpha = (\alpha_t)_{t \in \mathbb{R}}$ is predictable and σ is a stopping time, the stopped process $\alpha^\sigma := (\alpha_{t \wedge \sigma})_{t \in \mathbb{R}}$ is also predictable.

By an *increasing process* we mean a process $V = (V_t)_{t \in \mathbb{R}}$ (not necessarily adapted) for which $t \mapsto V_t(\omega)$ is nondecreasing for all $\omega \in \Omega$. Similarly, a process V is said to be càdlàg if $t \mapsto V_t(\omega)$ is right-continuous and has left limits in $\mathbb{R}$ for all $\omega \in \Omega$.

In what follows increments of processes play an important role. Whenever $X = (X_t)_{t \in \mathbb{R}}$ is a process and $s, t \in \mathbb{R}$ define the *increment of X over the interval* $(s, t]$, to be denoted ${}^sX_t$, as

$$ {}^sX_t := X_t - X_{t \wedge s} = \begin{cases} 0 & \text{if } t \leq s \\ X_t - X_s & \text{if } t \geq s. \end{cases} \tag{2.1} $$

Set furthermore ${}^sX = ({}^sX_t)_{t \in \mathbb{R}}$. Note that

$$ ({}^sX)^\sigma = {}^s(X^\sigma) \quad \text{for } s \in \mathbb{R} \text{ and } \sigma \text{ a stopping time.} \tag{2.2} $$

Moreover, for $s \leq t \leq u$ we have

$$ {}^t({}^sX)_u = {}^tX_u. \tag{2.3} $$

Definition 2.2. Let $\mathcal{A}(\mathcal{F}_\cdot)$ denote the class of increasing adapted càdlàg processes.

Let $\mathcal{A}^1(\mathcal{F}_\cdot)$ denote the subclass of $\mathcal{A}(\mathcal{F}_\cdot)$ consisting of integrable increasing càdlàg adapted processes; $\mathcal{LA}^1(\mathcal{F}_\cdot)$ denotes the subclass of $\mathcal{A}(\mathcal{F}_\cdot)$ consisting of càdlàg increasing adapted processes $V = (V_t)_{t \in \mathbb{R}}$ for which there exists a localising sequence $(\sigma_n)_{n \geq 1}$ such that $V^{\sigma_n} \in \mathcal{A}^1(\mathcal{F}_\cdot)$ for all n.

Let $\mathcal{A}_0(\mathcal{F}_\cdot)$ denote the subclass of $\mathcal{A}(\mathcal{F}_\cdot)$ consisting of increasing càdlàg adapted processes $V = (V_t)_{t \in \mathbb{R}}$ for which $\lim_{t \to -\infty} V_t = 0$ P-a.s. Set $\mathcal{A}_0^1(\mathcal{F}_\cdot) := \mathcal{A}_0(\mathcal{F}_\cdot) \cap \mathcal{A}^1(\mathcal{F}_\cdot)$ and $\mathcal{LA}_0^1(\mathcal{F}_\cdot) := \mathcal{A}_0(\mathcal{F}_\cdot) \cap \mathcal{LA}^1(\mathcal{F}_\cdot)$.

Let $\mathcal{IA}(\mathcal{F}_\cdot)$ (resp. $\mathcal{IA}^1(\mathcal{F}_\cdot)$, $\mathcal{ILA}^1(\mathcal{F}_\cdot)$) denote the class of càdlàg increasing processes V for which ${}^sV \in \mathcal{A}(\mathcal{F}_\cdot)$ (resp. ${}^sV \in \mathcal{A}^1(\mathcal{F}_\cdot)$, ${}^sV \in \mathcal{LA}^1(\mathcal{F}_\cdot)$) for all $s \in \mathbb{R}$. We emphasize that V is not assumed adapted.

Motivated by our interest in increments we say that two càdlàg processes $X = (X_t)_{t \in \mathbb{R}}$ and $Y = (Y_t)_{t \in \mathbb{R}}$ have *identical increments*, and write $X \overset{\text{in}}{=} Y$, if ${}^sX \overset{\text{P}}{=} {}^sY$ for all $s \in \mathbb{R}$. In this case also $X^\sigma \overset{\text{in}}{=} Y^\sigma$ whenever σ is a stopping time.

Remark 2.3. Assume X and Y are càdlàg processes with $X \overset{\text{in}}{=} Y$. Then by definition $X_t - X_s = Y_t - Y_s$ for all $s \leq t$ P-a.s. for all t and so by the càdlàg property $X_t - X_s = Y_t - Y_s$ for all $s, t \in \mathbb{R}$ P-a.s. This shows that there exists a random variable Z such that $X_t = Y_t + Z$ for all $t \in \mathbb{R}$ P-a.s., and thus ${}^sX_t = {}^sY_t$ for all $s, t \in \mathbb{R}$ P-a.s.

For any stochastic process $X = (X_t)_{t \in \mathbb{R}}$ we have

$$ {}^sX_t + {}^tX_u = {}^sX_u \quad \text{for } s \leq t \leq u. \tag{2.4} $$

This leads us to consider increment processes, defined as follows. Let $I = \{{}^sI\}_{s\in\mathbb{R}}$ with ${}^sI = ({}^sI_t)_{t\in\mathbb{R}}$ be a family of stochastic processes. We say that I is *a consistent family of increment processes* if the following three conditions are satisfied:

(1) ${}^sI$ is an adapted process for all $s \in \mathbb{R}$, and ${}^sI_t = 0$ P-a.s. for all $t \le s$.

(2) For all $s \in \mathbb{R}$ and $\omega \in \Omega$ the mapping $t \mapsto {}^sI_t(\omega)$ is càdlàg.

(3) For all $s \le t \le u$ we have ${}^sI_t + {}^tI_u = {}^sI_u$ P-a.s.

Whenever X is a càdlàg process such that ${}^sX$ is adapted for all $s \in \mathbb{R}$, the family $\{{}^sX\}_{s\in\mathbb{R}}$ of increment processes is then consistent by equation (2.4). Conversely, let I be a consistent family of increment processes. A càdlàg process $X = (X_t)_{t\in\mathbb{R}}$ is said to be *associated with* I if ${}^sX \overset{\mathrm{P}}{=} {}^sI$ for all $s \in \mathbb{R}$. It is easily seen that there exists such a process; for example, let

$$X_t = \begin{cases} {}^0I_t & \text{for } t \ge 0 \\ -{}^tI_0 & \text{for } t = -1, -2, \dots, \\ X_{-n} + {}^{-n}I_t & \text{for } t \in (-n, -n+1) \text{ and } n = 1, 2, \dots \end{cases}$$

Thus, consistent families of increment processes correspond to increments in càdlàg processes with adapted increments. If $X = (X_t)_{t\in\mathbb{R}}$ and $Y = (Y_t)_{t\in\mathbb{R}}$ are càdlàg processes associated with I then $X \overset{\text{in}}{=} Y$ and hence by Remark 2.3 there is a random variable Z such that $X_t = Y_t + Z$ for all t P-a.s.

Remark 2.4. Let I be a consistent family of increment processes, and assume X is a càdlàg process associated with I such that $X_{-\infty} := \lim_{t\to-\infty} X_t$ exists in probability. Then, $(X_t - X_{-\infty})_{t\in\mathbb{R}}$ is adapted and associated with I. Indeed, $X_t - X_{-\infty} = \lim_{s\to-\infty} {}^sX_t$ in probability for $t \in \mathbb{R}$ and since ${}^sX_t = {}^sI_t$ (P-a.s.) is $\mathcal{F}_t$-measurable, it follows that $X_t - X_{-\infty}$ is $\mathcal{F}_t$-measurable. In this case, $(X_t - X_{-\infty})_{t\in\mathbb{R}}$ is the unique (up to P-indistinguishability) càdlàg process associated with I which converges to 0 in probability as time goes to $-\infty$. If, in addition, ${}^sI$ is predictable for all $s \in \mathbb{R}$ then $(X_t - X_{-\infty})_{t\in\mathbb{R}}$ is also predictable. To see this, choose a P-null set N and a sequence $(s_n)_{n\ge1}$ decreasing to $-\infty$ such that $X_{s_n}(\omega) \to X_{-\infty}(\omega)$ as $n \to \infty$ for all $\omega \in N^c$. For $\omega \in N^c$ and $t \in \mathbb{R}$ we then have $X_t(\omega) - X_{-\infty}(\omega) = \lim_{n\to\infty} {}^{s_n}X_t(\omega)$, implying the result due to inheritance of predictability under pointwise limits.

3 Martingales and increment martingales

Let us now introduce the classes of (square integrable) martingales and the corresponding localised classes.

Definition 3.1. Let $M = (M_t)_{t\in\mathbb{R}}$ denote a càdlàg adapted process.

We call M an *$\mathcal{F}.$-martingale* if it is integrable and for all $s < t$, $\mathrm{E}[M_t|\mathcal{F}_s] = M_s$ P-a.s. If in addition M_t is square integrable for all $t \in \mathbb{R}$ then M is called a *square integrable martingale*. Let $\mathcal{M}(\mathcal{F}.)$ resp. $\mathcal{M}^2(\mathcal{F}.)$ denote the class of $\mathcal{F}.$-martingales resp. square integrable $\mathcal{F}.$-martingales. Note that these classes are both stable under stopping.

We call M a *local $\mathcal{F}.$-martingale* if there exists a localising sequence $(\sigma_n)_{n\ge1}$ such that $M^{\sigma_n} \in \mathcal{M}(\mathcal{F}.)$ for all n. The definition of a *locally square integrable martingale* is similar. Let $\mathcal{LM}(\mathcal{F}.)$ resp. $\mathcal{LM}^2(\mathcal{F}.)$ denote the class of local martingales resp. locally square integrable martingales. These classes are stable under stopping.

Remark 3.2. (1) The backward martingale convergence theorem shows that if $M \in \mathcal{M}(\mathcal{F}.)$ then M_t converges P-a.s. and in $L^1(\mathrm{P})$ to an $\mathcal{F}_{-\infty}$-measurable integrable random variable $M_{-\infty}$ as $t \to -\infty$ (cf. Chapter II, Theorem 2.3 in [6]). In this case we may consider $(M_t)_{t\in[-\infty,\infty)}$ as a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in[-\infty,\infty)}$. If $M \in \mathcal{M}^2(\mathcal{F}.)$ then M_t converges in $L^2(\mathrm{P})$ to $M_{-\infty}$.

(2) Let $M \in \mathcal{LM}(\mathcal{F}.)$ and choose a localising sequence $(\sigma_n)_{n\geq 1}$ such that $M^{\sigma_n} \in \mathcal{M}(\mathcal{F}.)$ for all n. From (1) follows that there exists an $\mathcal{F}_{-\infty}$-measurable integrable random variable $M_{-\infty}$ (which does not depend on n) such that for all n we have $M_t^{\sigma_n} \to M_{-\infty}$ P-a.s. and in $L^1(\mathrm{P})$ as $t \to -\infty$, and $M_t \to M_{-\infty}$ P-a.s. Thus, defining $M_{-\infty}^{\sigma_n} := M_{-\infty}$ it follows that for all n the process $(M_t)_{t\in[-\infty,\infty)}^{\sigma_n}$ can be considered a martingale with respect to $(\mathcal{F}_t)_{t\in[-\infty,\infty)}$, and consequently $(M_t)_{t\in[-\infty,\infty)}$ is a local martingale. (Note, though, that σ_n is not allowed to take on the value $-\infty$.) In the case $M \in \mathcal{LM}^2(\mathcal{F}.)$ assume $(\sigma_n)_{n\geq 1}$ is chosen such that $M^{\sigma_n} \in \mathcal{M}^2(\mathcal{F}.)$ for all n; then $M_t^{\sigma_n} \to M_{-\infty}$ in $L^2(\mathrm{P})$.

(3) The preceding shows that a local martingale indexed by $\mathbb{R}$ can also be regarded as a local martingale indexed by $[-\infty,\infty)$, where localising stopping times, however, are not allowed to take on the value $-\infty$. Let us argue that the latter restriction is of minor importance. Thus, call $\sigma : \Omega \to [-\infty,\infty]$ an $\bar{\mathbb{R}}$*-valued stopping time* if $\{\sigma \leq t\} \in \mathcal{F}_t$ for all $t \in [-\infty,\infty)$, and call a sequence of nondecreasing $\bar{\mathbb{R}}$-valued stopping times $\sigma_1 \leq \sigma_2 \leq \cdots$ an $\bar{\mathbb{R}}$*-valued localising sequence* if $\sigma_n \to \infty$ P-a.s. as $n \to \infty$. Then we claim that a càdlàg adapted process $M = (M_t)_{t\in\mathbb{R}}$ is a local martingale if and only if $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s and there is an $\bar{\mathbb{R}}$-valued localising sequence $(\sigma_n)_{n\geq 1}$ such that $(M_t^{\sigma_n})_{t\in[-\infty,\infty)}$ is a martingale. We emphasize that the latter characterisation is the most natural one when considering the index set $[-\infty,\infty)$, while the former is better when considering $\mathbb{R}$. Note that the *only if* part follows from (2). Conversely, assume $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s and let $(\sigma_n)_{n\geq 1}$ be an $\bar{\mathbb{R}}$-valued localising sequence such that $(M_t^{\sigma_n})_{t\in[-\infty,\infty)}$ is a martingale, and let us prove the existence of a localising sequence $(\tau_n)_{n\geq 1}$ such that M^{τ_n} is a martingale for all n. Since $M_{-\infty}$ is integrable it suffices to consider $M_t - M_{-\infty}$ instead of M_t; consequently we may and do assume $M_{-\infty} = 0$. In this case, $(\tau_n)_{n\geq 1} = (\tau \vee \sigma_n)_{n\geq 1}$ will do if τ is a stopping time such that M^τ is a martingale. To construct this τ set $Z_t^n = \mathrm{E}[|M_t^{\sigma_n}||\mathcal{F}_{-\infty}]$ for $t \in [-\infty,\infty)$. Then Z^n is $\mathcal{F}_{-\infty}$-measurable and can be chosen non-decreasing, càdlàg and 0 at $-\infty$. Therefore

$$\rho_n = \inf\{t \in \mathbb{R} : Z^n_{\frac{t}{2}} > 1\} \wedge 0$$

is real-valued, $\mathcal{F}_{-\infty}$-measurable and $Z^n_{\rho_n} \leq 1$. Define

$$\tau = \rho_n \wedge \sigma_n \text{ on } A_n = \{\sigma_1 = \cdots = \sigma_{n-1} = -\infty \text{ and } \sigma_n > -\infty\}$$

and set $\tau = 0$ on $(\cup_{n\geq 1} A_n)^c$. Then τ is a stopping time since the A_n's are disjoint and $\mathcal{F}_{-\infty}$-measurable. Furthermore, $\cup_{n\geq 1} A_n = \Omega$ P-a.s. Thus, for all $t > -\infty$,

$$\mathrm{E}[|M_{t\wedge\tau}|] = \sum_{n=1}^{\infty} \mathrm{E}[|M_{\sigma_n\wedge\rho_n\wedge t}|1_{A_n}] = \sum_{n=1}^{\infty} \mathrm{E}[|Z^n_{\rho_n\wedge\sigma_n\wedge t}|1_{A_n}] \leq 1,$$

implying

$$\mathrm{E}[M_{\tau\wedge t}|\mathcal{F}_s] = \sum_{n=1}^{\infty} \mathrm{E}[M_{\sigma_n\wedge\rho_n\wedge t}|\mathcal{F}_s]1_{A_n} = \sum_{n=1}^{\infty} M_{\sigma_n\wedge\tau_n\wedge s}1_{A_n} = M_{\tau\wedge s}$$

for all $-\infty < s < t$; thus, M^τ is a martingale.

Example 3.3. A càdlàg process $X = (X_t)_{t\in\mathbb{R}}$ is called a *Lévy process indexed by* $\mathbb{R}$ if it has stationary independent increments; that is, whenever $n \geq 1$ and $t_0 < t_1 < \cdots < t_n$, the increments ${}^{t_0}X_{t_1}, {}^{t_1}X_{t_2}, \ldots, {}^{t_{n-1}}X_{t_n}$ are independent and ${}^{s}X_t \overset{\mathscr{D}}{=} {}^{u}X_v$ whenever $s < t$ and $u < v$ satisfy $t - s = v - u$. In this case $({}^{s}X_{s+t})_{t\geq 0}$ is an ordinary Lévy process indexed by $\mathbb{R}_+$ for all $s \in \mathbb{R}$.

Let X be a Lévy process indexed by $\mathbb{R}$. There is a unique infinitely divisible distribution μ on $\mathbb{R}$ associated with X in the sense that for all $s < t$, ${}^{s}X_t \overset{\mathscr{D}}{=} \mu^{t-s}$. When $\mu = N(0, 1)$, the standard normal distribution, X is called a (standard) Brownian motion indexed by $\mathbb{R}$. If Y is a càdlàg process with $X \overset{\text{in}}{=} Y$, it is a Lévy process as well and μ is also associated with Y; that is, Lévy processes indexed by $\mathbb{R}$ are determined by the infinitely divisible μ only up to addition of a random variable.

Note that $(X_{(-t)-})_{t\in\mathbb{R}}$ (where, for $s \in \mathbb{R}$, X_{s-} denotes the left limit at s) is again a Lévy process indexed by $\mathbb{R}$ and the distribution associated with it is μ^- given by $\mu^-(B) := \mu(-B)$ for $B \in \mathcal{B}(\mathbb{R})$. Since this process appears by time reversion of X, the behaviour of X at $-\infty$ corresponds to the behaviour of $(X_{(-t)-})_{t\in\mathbb{R}}$ at ∞, which is well understood, cf. e.g. [11]; in particular, $\lim_{s\to-\infty} X_s$ does not exist (in any reasonable sense) except when X is constant. Thus, except in nontrivial cases X is not a local martingale in any filtration.

This example clearly indicates that we need to generalise the concept of a martingale.

Definition 3.4. Let $M = (M_t)_{t\in\mathbb{R}}$ denote a càdlàg process, in general not assumed adapted.

We say that M is an *increment martingale* if for all $s \in \mathbb{R}$, ${}^{s}M \in \mathcal{M}(\mathcal{F}_\cdot)$. This is equivalent to saying that for all $s < t$, ${}^{s}M_t$ is $\mathcal{F}_t$-measurable, integrable and satisfies $\mathrm{E}[{}^{s}M_t|\mathcal{F}_s] = 0$ P-a.s. If in addition all increments are square integrable, then M is called a *increment square integrable martingale.* Let $\mathcal{IM}(\mathcal{F}_\cdot)$ and $\mathcal{IM}^2(\mathcal{F}_\cdot)$ denote the corresponding classes.

M is called an *increment local martingale* if for all s, ${}^{s}M$ is an adapted process and there exists a localising sequence $(\sigma_n)_{n\geq 1}$ (which may depend on s) such that $({}^{s}M)^{\sigma_n} \in \mathcal{M}(\mathcal{F}_\cdot)$ for all n. Define an *increment locally square integrable martingale* in the obvious way. Denote the corresponding classes by $\mathcal{ILM}(\mathcal{F}_\cdot)$ and $\mathcal{ILM}^2(\mathcal{F}_\cdot)$.

Obviously the four classes of increment processes are $\overset{\text{in}}{=}$-stable and by (2.2) stable under stopping. Moreover, $\mathcal{M}(\mathcal{F}_\cdot) \subseteq \mathcal{IM}(\mathcal{F}_\cdot)$ and $\mathcal{M}^2(\mathcal{F}_\cdot) \subseteq \mathcal{IM}^2(\mathcal{F}_\cdot)$ with the following characterizations

$$\mathcal{M}(\mathcal{F}_\cdot) = \{M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_\cdot) : M \text{ is adapted and integrable}\} \tag{3.1}$$

$$\mathcal{M}^2(\mathcal{F}_\cdot) = \{M \in \mathcal{IM}^2(\mathcal{F}_\cdot) : M \text{ is adapted and square integrable}\}. \tag{3.2}$$

Likewise, $\mathcal{LM}(\mathcal{F}_\cdot) \subseteq \mathcal{ILM}(\mathcal{F}_\cdot)$ and $\mathcal{LM}^2(\mathcal{F}_\cdot) \subseteq \mathcal{ILM}^2(\mathcal{F}_\cdot)$. But no similar simple characterizations as in (3.1)–(3.2) of the localised classes seem to be valid. Note that $\mathcal{LIM}(\mathcal{F}_\cdot) \subseteq \mathcal{ILM}(\mathcal{F}_\cdot)$, where the former is the set of *local increment martingales*, i.e. the localising sequence can be chosen independent of s. A similar statement holds for $\mathcal{ILM}^2(\mathcal{F}_\cdot)$.

When τ is a stopping time, we define ${}^{\tau}M$ in the obvious way as ${}^{\tau}M_t = M_t - M_{t\wedge\tau}$ for $t \in \mathbb{R}$.

Proposition 3.5. *Let $M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_\cdot)$ and τ be a stopping time. Then ${}^{\tau}M \in \mathcal{M}(\mathcal{F}_\cdot)$ if $\left\{M_0 - M_{\tau\vee(-n)\wedge 0} : n \geq 1\right\}$ is uniformly integrable.*

If τ is bounded from below then the above set is always uniformly integrable.

Proof. Assume first that τ is bounded from below, that is, there exists an $s_0 \in (0, -\infty)$ such that $\tau \geq s_0$. Then, since $({}^{\tau}M_t)_{t\in\mathbb{R}} = ({}^{s_0}M_t - {}^{s_0}M_{\tau\wedge t})_{t\in\mathbb{R}}$, ${}^{\tau}M$ is a sum of two martingales and hence a martingale. Assume now that $\{M_0 - M_{\tau\vee(-n)\wedge 0} : n \geq 1\}$ is uniformly integrable. Then, with $\tau_n = \tau \vee (-n)$ we have

$$\{{}^{\tau_n}M_t : n \geq 1\} \text{ is uniformly integrable for all } t \in \mathbb{R}. \tag{3.3}$$

Moreover, ${}^{\tau_n}M_t \to {}^{\tau}M_t$ a.s. and hence in $L^1(\mathrm{P})$ by (3.3). For all $n \geq 1$, τ_n is bounded from below and hence ${}^{\tau_n}M$ is a martingale, implying that ${}^{\tau}M$ is an $L^1(\mathrm{P})$-limit of martingales and hence a martingale. □

Example 3.6. Let $X = (X_t)_{t\in\mathbb{R}}$ denote a Lévy process indexed by $\mathbb{R}$. *The filtration generated by the increments of* X is $\mathcal{F}^{\mathcal{I}X}_{\cdot} = (\mathcal{F}^{\mathcal{I}X}_t)_{t\in\mathbb{R}}$, where

$$\mathcal{F}^{\mathcal{I}X}_t = \sigma({}^{s}X_t : s \leq t) \vee \mathcal{N} = \sigma({}^{s}X_u : s \leq u \leq t) \vee \mathcal{N}, \quad \text{for } t \in \mathbb{R},$$

and we recall that $\mathcal{N}$ is the set of P-null sets. Using a standard technique it can be verified that $\mathcal{F}^{\mathcal{I}X}_{\cdot}$ is a filtration. Indeed, we only have to verify right-continuity of $\mathcal{F}^{\mathcal{I}X}_{\cdot}$. For this, fix $t \in \mathbb{R}$ and consider random variables Z_1 and Z_2 where Z_1 is bounded and $\mathcal{F}^{\mathcal{I}X}_t$-measurable, and Z_2 is bounded and measurable with respect to $\sigma({}^{s}X_u : t + \epsilon < s < u)$ for some $\epsilon > 0$. Then

$$\mathrm{E}[Z_1 Z_2 | \mathcal{F}^{\mathcal{I}X}_{t+}] = Z_1 \mathrm{E}[Z_2] = \mathrm{E}[Z_1 Z_2 | \mathcal{F}^{\mathcal{I}X}_t] \quad \text{P-a.s.}$$

by independence of Z_2 and $\mathcal{F}^{\mathcal{I}X}_{t+}$. Applying the monotone class lemma it follows that whenever Z is bounded and measurable with respect to $\mathcal{F}^{\mathcal{I}X}_{\infty}$ we have $\mathrm{E}[Z|\mathcal{F}^{\mathcal{I}X}_{t+}] = \mathrm{E}[Z|\mathcal{F}^{\mathcal{I}X}_t]$ P-a.s., which in turn implies right-continuity of $\mathcal{F}^{\mathcal{I}X}_{\cdot}$. It is readily seen that $X \in \mathcal{IM}(\mathcal{F}^{\mathcal{I}X}_{\cdot})$ if X has integrable centered increments.

Increment martingales are not necessarily integrable. But for $M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_{\cdot})$, $M_t \in L^1(\mathrm{P})$ for all $t \in \mathbb{R}$ if and only if $M_t \in L^1(\mathrm{P})$ for some $t \in \mathbb{R}$. Likewise $(M_s)_{s\leq t}$ is uniformly integrable for all t if and only if $(M_s)_{s\leq t}$ is uniformly integrable for some t. Similarly, for $M \in \mathcal{IM}^2(\mathcal{F}_{\cdot})$ we have $M_t \in L^2(\mathrm{P})$ for all $t \in \mathbb{R}$ if and only if $M_t \in L^2(\mathrm{P})$ for some $t \in \mathbb{R}$, and $(M_s)_{s\leq t}$ is $L^2(\mathrm{P})$-bounded for some t if and only if $(M_s)_{s\leq t}$ is $L^2(\mathrm{P})$-bounded for some t. For integrable elements of $\mathcal{IM}(\mathcal{F}_{\cdot})$ we have the following decomposition.

Proposition 3.7. *Let* $M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_{\cdot})$ *be integrable. Then* M *can be decomposed uniquely up to* P*-indistinguishability as* $M = K + N$ *where* $K = (K_t)_{t\in\mathbb{R}} \in \mathcal{M}(\mathcal{F}_{\cdot})$ *and* $N = (N_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}_{\cdot})$ *is an integrable process satisfying*

$$\mathrm{E}[N_t|\mathcal{F}_t] = 0 \textit{ for all } t \in \mathbb{R} \quad \textit{and} \quad \lim_{t\to\infty} N_t = 0 \text{ P-}\textit{a.s. and in } L^1(\mathrm{P}). \tag{3.4}$$

If M *is square integrable then so are* K *and* N*, and* $\mathrm{E}[K_t N_t] = 0$ *for all* $t \in \mathbb{R}$*. Thus* $\mathrm{E}[M_t^2] = \mathrm{E}[K_t^2] + \mathrm{E}[N_t^2]$ *for all* t *and moreover* $t \mapsto \mathrm{E}[N_t^2]$ *is decreasing.*

Proof. The uniqueness is evident. To get the existence set $K_t = \mathrm{E}[M_t|\mathcal{F}_t]$. Then K is integrable and adapted and for $s < t$ we have

$$\mathrm{E}[K_t|\mathcal{F}_s] = \mathrm{E}[M_t|\mathcal{F}_s] = \mathrm{E}[M_s|\mathcal{F}_s] + \mathrm{E}[{}^{s}M_t|\mathcal{F}_s] = K_s.$$

Thus, $K \in \mathcal{M}(\mathcal{F}_{\cdot})$ and therefore $N := M - K \in \mathcal{IM}(\mathcal{F}_{\cdot})$. Clearly, N is integrable and $\mathrm{E}[N_t|\mathcal{F}_t] = 0$ for all $t \in \mathbb{R}$. Take $s \leq t$. Then ${}^{s}N_t = \mathrm{E}[{}^{s}N_t|\mathcal{F}_t]$, giving

$${}^{s}N_t = \mathrm{E}[N_t - N_s|\mathcal{F}_t] = -\mathrm{E}[N_s|\mathcal{F}_t], \tag{3.5}$$

that is $N_t = N_s - \mathrm{E}[N_s|\mathcal{F}_t]$, proving that $\lim_{t\to\infty} N_t = 0$ P-a.s. and in $L^1(\mathrm{P})$. If M is square integrable then so are K and N and they are orthogonal. Furthermore for $s \le t$

$$\begin{aligned}\mathrm{E}[N_s(N_t - N_s)] &= \mathrm{E}[(N_t - N_s)\mathrm{E}[N_s|\mathcal{F}_t]] \\ &= \mathrm{E}[(N_t - N_s)\mathrm{E}[(N_s - N_t)|\mathcal{F}_t]] = -\mathrm{E}[(N_t - N_s)^2]\end{aligned}$$

implying

$$\mathrm{E}[N_t^2] = \mathrm{E}[N_s^2] - \mathrm{E}[(N_t - N_s)^2]. \tag{3.6}$$

□

As a corollary we may deduce the following convergence result for integrable increment martingales.

Corollary 3.8. *Let $M = (M_t)_{t\in\mathbb{R}} \in \mathcal{IM}(\mathcal{F}.)$ be integrable.*

(a) If $(M_s)_{s\le 0}$ is uniformly integrable then $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P*-a.s. and in $L^1(\mathrm{P})$ and $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ is in $\mathcal{M}(\mathcal{F}.)$.*

(b) If $(M_s)_{s\le 0}$ is bounded in $L^2(\mathrm{P})$ then $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P*-a.s. and in $L^2(\mathrm{P})$ and $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ is in $\mathcal{M}^2(\mathcal{F}.)$.*

Proof. Write $M = K + N$ as in Proposition 3.7. As noticed in Remark 3.2 the conclusion holds for K. Furthermore $(N_s)_{s\le 0}$ is uniformly integrable when this is true for M so we may and will assume $M = N$. That is, M satisfies (3.4). By uniform integrability we can find a sequence s_n decreasing to $-\infty$ and an $\tilde{M} \in L^1(\mathrm{P})$ such that $M_{s_n} \to \tilde{M}$ in $\sigma(L^1, L^\infty)$. For all t we have by (3.5)

$$M_t = M_{s_n} - \mathrm{E}[M_{s_n}|\mathcal{F}_t] \quad \text{for } s_n < t$$

and thus

$$M_t = \tilde{M} - \mathrm{E}[\tilde{M}|\mathcal{F}_t] \quad \text{for all } t,$$

proving part (a). In (b) the martingale part K again has the right behaviour at $-\infty$. Likewise, $(N_s)_{s\le 0}$ is bounded in $L^2(\mathrm{P})$ if this is true for M. Thus we may assume that M satisfies (3.4). The a.s. convergence is already proved and the $L^2(\mathrm{P})$-convergence follows from (3.6) since $t \mapsto \mathrm{E}[M_t]$ is decreasing and $\sup_{s<0} \mathrm{E}[M_s^2] < \infty$. □

Observe that $(M_t - M_{t_0})_{t\in\mathbb{R}}$ is in $\mathcal{IM}(\mathcal{F}.)$ and is integrable for every $t_0 \in \mathbb{R}$ and every $M \in \mathcal{IM}(\mathcal{F}.)$. Since a similar result holds in the square integrable case, Corollary 3.8 implies the following result relating convergence of an increment martingale to the martingale property.

Proposition 3.9. *Let $M = (M_t)_{t\in\mathbb{R}}$ be a given càdlàg process. The following are equivalent:*

(a) $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P*-a.s. and $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ is in $\mathcal{M}(\mathcal{F}.)$.*

(b) $M \in \mathcal{IM}(\mathcal{F}.)$ and $({}^sM_0)_{s<0}$ is uniformly integrable.

Likewise, the following are equivalent:

(c) $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P*-a.s. and $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ is in $\mathcal{M}^2(\mathcal{F}.)$*

(d) $M \in \mathcal{IM}^2(\mathcal{F}.)$ and $\sup_{s:s\le 0} \mathrm{E}[({}^sM_0)^2] < \infty$.

Proof. Assuming $M \in \mathcal{IM}(\mathcal{F}_\cdot)/\mathcal{IM}^2(\mathcal{F}_\cdot)$, (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) follow by using Corollary 3.8 on $(M_t - M_0)_{t\in\mathbb{R}}$. The remaining two implications follow from standard martingale theory and the identity ${}^sM_0 = (M_0 - M_{-\infty}) - (M_s - M_{-\infty})$. □

Let $M \in \mathcal{LM}(\mathcal{F}_\cdot)$ with $M_{-\infty} = 0$. It is well-known that there exists a unique (up to P-indistinguishability) process $[M]$ called *the quadratic variation for M* satisfying $[M] \in \mathcal{A}_0(\mathcal{F}_\cdot)$, $(\Delta M)_t^2 = \Delta[M]_t$ for all $t \in \mathbb{R}$ P-a.s., and $M^2 - [M] \in \mathcal{LM}(\mathcal{F}_\cdot)$. We have

$$ {}^s[M] \stackrel{\text{P}}{=} [{}^sM] \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad [M]^\sigma \stackrel{\text{P}}{=} [M^\sigma] \text{ when } \sigma \text{ is a stopping time.} \tag{3.7} $$

If, in addition, $M \in \mathcal{LM}^2(\mathcal{F}_\cdot)$, there is a unique predictable process $\langle M\rangle \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)$ satisfying $M^2 - \langle M\rangle \in \mathcal{LM}(\mathcal{F}_\cdot)$, and we shall call this process the *predictable quadratic variation for M*. In this case,

$$ {}^s\langle M\rangle \stackrel{\text{P}}{=} \langle {}^sM\rangle \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad \langle M\rangle^\sigma \stackrel{\text{P}}{=} \langle M^\sigma\rangle \text{ when } \sigma \text{ is a stopping time.} \tag{3.8} $$

Definition 3.10. Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. We say that an increasing process $V = (V_t)_{t\in\mathbb{R}}$ is a *generalised quadratic variation* for M if

$$ V \in \mathcal{IA}(\mathcal{F}_\cdot) \tag{3.9} $$
$$ (\Delta M)_t^2 = \Delta V_t \quad \text{for all } t \in \mathbb{R}, \text{ P-a.s.} \tag{3.10} $$
$$ ({}^sM)^2 - {}^sV \in \mathcal{LM}(\mathcal{F}_\cdot) \quad \text{for all } s \in \mathbb{R}. \tag{3.11} $$

We say that V is *quadratic variation for M* if, instead of (3.9), $V \in \mathcal{A}_0(\mathcal{F}_\cdot)$.

Let $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$. We say that an increasing process $V = (V_t)_{t\in\mathbb{R}}$ is a *generalised predictable quadratic variation* for M if

$$ V \in \mathcal{ILA}^1(\mathcal{F}_\cdot) \tag{3.12} $$
$$ {}^sV \text{ is predictable for all } s \in \mathbb{R} \tag{3.13} $$
$$ ({}^sM)^2 - {}^sV \in \mathcal{LM}(\mathcal{F}_\cdot) \quad \text{for all } s \in \mathbb{R}. \tag{3.14} $$

We say that V is a *predictable quadratic variation* for V if, instead of (3.12), $V \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)$.

Remark 3.11. (1) Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$ and V denote a generalised quadratic variation for M such that $V_{-\infty} := \lim_{s\to-\infty} V_s$ exists P-a.s. From Remark 2.4 it follows that $(V_t - V_{-\infty})_{t\in\mathbb{R}}$ is a quadratic variation for M.

Similarly, let $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$ and V denote a generalised predictable quadratic variation for M such that $V_{-\infty} := \lim_{s\to-\infty} V_s$ exists P-a.s. Then $(V_t - V_{-\infty})_{t\in\mathbb{R}}$ is a predictable quadratic variation for M. Indeed, by [8], Lemma I.3.10, $(V_t - V_{-\infty})_{t\in\mathbb{R}}$ is a predictable process in $\mathcal{LA}_0^1(\mathcal{F}_\cdot)$. (Strictly speaking, this lemma only ensures the existence of an $\bar{\mathbb{R}}$-value localising sequence $(\sigma_n)_{n\geq1}$ (cf. Remark 3.2 (3)) such that $(V_t - V_{-\infty})^{\sigma_n}$ is in $\mathcal{A}_0^1(\mathcal{F}_\cdot)$; this problem can, however, be dealt with as described in Remark 3.2).

(2) If $M \in \mathcal{LM}(\mathcal{F}_\cdot)$ with $M_{-\infty} = 0$ then the usual quadratic variation $[M]$ for M is, by (3.7), also a quadratic variation in the sense of Definition 3.10, and similarly, if $M \in \mathcal{LM}^2(\mathcal{F}_\cdot)$ then the usual predictable quadratic variation $\langle M\rangle$ is a predictable quadratic variation also in the sense defined above.

(3) (Existence of generalised quadratic variation). Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. Then V is a generalised quadratic variation for M if and only if we have (3.9)–(3.10) and V is associated with the family $\{[{}^sM]\}_{s\in\mathbb{R}}$. By Section 2, existence and uniqueness (up to addition of random variables) of the generalised quadratic variation is thus ensured once we have shown that the latter family is consistent. In other words, we must show for

$s \leq t \leq u$ that $[{}^sM]_u = [{}^sM]_t + [{}^tM]_u$ P-a.s. Equivalently, ${}^t([{}^sM])_u = [{}^tM]_u$ P-a.s. This follows, however, from (3.7) and (2.2).

(4) (Existence of generalised predictable quadratic variation). Similarly, let $M \in \mathcal{ILM}^2(\mathcal{F}.)$. Then V is a generalised predictable quadratic variation for M if and only if we have (3.12)–(3.13) and V is associated with $\{\langle {}^sM\rangle\}_{s\in\mathbb{R}}$. Moreover, the latter family is consistent, ensuring existence and uniqueness of the generalised predictable quadratic variation up to addition of random variables.

(5) By Remark 2.4, the quadratic variation and the predictable quadratic variation are unique up to P-indistinguishability when they exist.

(6) Generalised compensators and predictable compensators are $\stackrel{\text{in}}{=}$-invariant, i.e. if for example $M, N \in \mathcal{IM}(\mathcal{F}.)$ with $M \stackrel{\text{in}}{=} N$ then V is a generalised compensator for M if and only if it is a generalised compensator for N.

When $M \in \mathcal{ILM}(\mathcal{F}.)$ we use $[M]^{\mathrm{g}}$ to denote a generalised quadratic variation for M, and $[M]$ denotes the quadratic variation when it exists. For $M \in \mathcal{ILM}^2(\mathcal{F}.)$, $\langle M\rangle^{\mathrm{g}}$ denotes a generalised quadratic variation for M, and $\langle M\rangle$ denotes the predictable quadratic variation when it exists. Generalising (3.7)–(3.8) we have the following.

Lemma 3.12. *Let σ denote a stopping time and $s \in \mathbb{R}$. If $M \in \mathcal{ILM}(\mathcal{F}.)$ then*

$$([M]^{\mathrm{g}})^\sigma \stackrel{\text{in}}{=} [M^\sigma]^{\mathrm{g}} \quad \textit{and} \quad {}^s([M]^{\mathrm{g}}) \stackrel{\text{P}}{=} [{}^sM]. \tag{3.15}$$

If $M \in \mathcal{ILM}^2(\mathcal{F}.)$ then

$$(\langle M\rangle^{\mathrm{g}})^\sigma \stackrel{\text{in}}{=} \langle M^\sigma\rangle^{\mathrm{g}} \quad \textit{and} \quad {}^s(\langle M\rangle^{\mathrm{g}}) \stackrel{\text{P}}{=} \langle {}^sM\rangle.$$

Proof. We only prove the part concerning the quadratic variation. As seen above, $[M]^{\mathrm{g}}$ is associated with $\{[{}^sM]\}_{s\in\mathbb{R}}$, which implies the second statement in (3.15).

To prove the first statement in (3.15) it suffices to show that $([M]^{\mathrm{g}})^\sigma$ is associated with $\{[{}^sM^\sigma]\}_{s\in\mathbb{R}}$. Note that, by (2.2) and (3.7),

$${}^s(([M]^{\mathrm{g}})^\sigma) \stackrel{\text{P}}{=} ({}^s[M]^{\mathrm{g}})^\sigma \stackrel{\text{P}}{=} [{}^sM]^\sigma \stackrel{\text{P}}{=} [{}^sM^\sigma].$$

□

Example 3.13. Let τ_1 and τ_2 denote independent absolutely continuous random variables with densities f_1 and f_2 and distribution functions F_1 and F_2 satisfying $F_i(t) < 1$ for all t and $i = 1, 2$. Set

$$N_t^i = 1_{[\tau_i,\infty)}(t),\ A_t^i = \int_{-\infty}^{t\wedge\tau_i} \frac{f_i(u)}{1-F_i(u)}\,\mathrm{d}u,\ N_t = (N_t^1, N_t^1) \text{ and } \mathcal{F}_t = \sigma(N_s : s \leq t) \vee \mathcal{N}$$

for $t \in \mathbb{R}$. From [2], A2 T26, follows that $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is right-continuous and hence a filtration in the sense defined in the present paper. It is well-known that M^i defined by $M_t^i = N_t^i - A_t^i$ is a square integrable martingale with $\langle M^i\rangle_t = A_t^i$, and M^1M^2 is a martingale. Assume, in addition,

$$\int_{-\infty}^{t} \frac{uf_i(u)}{1-F_i(u)}\,\mathrm{d}u = -\infty \quad \text{for all } t \in \mathbb{R}.$$

(This is satisfied if, for example, $F_i(s)$ equals a constant times $(1 + |s|\log(|s|))^{-1}$ when s is small.) Let $B^i \in \mathcal{IA}^1(\mathcal{F}.)$ satisfy

$${}^sB_t^i = \int_{(s,t]} u\,\mathrm{d}A_u^i = \int_{s\wedge\tau_i}^{t\wedge\tau_i} \frac{uf_i(u)}{1-F_i(u)}\,\mathrm{d}u$$

for $s < t$ and set $X_t^i = \tau_i N_t^i - B_t^i$. Then

$$\lim_{s\to-\infty} X_s^i = -\lim_{s\to-\infty} B_s^i = \infty \quad \text{pointwise,}$$

implying that X^i is not a local martingale. However, since for $s < t$,

$${}^sX_t^i = \int_{(s,t]} u \, \mathrm{d}M_u^i$$

it follows that ${}^sX^i$ is a square integrable martingale. That is, $X^i \in \mathcal{ILM}^2(\mathcal{F}.)$.

The quadratic variations, $[X^i]$ resp. $[X^1 - X^2]$, of X^i resp. $X^1 - X^2$ do exist and are $[X^i]_t = (\tau_i)^2 N_t^i$ resp. $[X^1 - X^2]_t = (\tau_1)^2 N_t^1 + (\tau_2)^2 N_t^2$. Moreover, up to addition of random variables,

$$\liminf_{s\to-\infty}(X_s^1 - X_s^2) = \liminf_{s\to-\infty}(B_s^2 - B_s^1) = \liminf_{s\to-\infty} \int_s^0 u\Big(\frac{f_2(u)}{1-F_2(u)} - \frac{f_1(u)}{1-F_1(u)}\Big)\,\mathrm{d}u$$
$$\limsup_{s\to-\infty}(X_s^1 - X_s^2) = \liminf_{s\to-\infty}(B_s^2 - B_s^1) = \limsup_{s\to-\infty} \int_s^0 u\Big(\frac{f_2(u)}{1-F_2(u)} - \frac{f_1(u)}{1-F_1(u)}\Big)\,\mathrm{d}u.$$

If τ_1 and τ_2 are identically distributed then $X_s^1 - X_s^2$ converges pointwise. In other cases we may have $\limsup_{s\to-\infty}(X_s^1 - X_s^2) = -\liminf_{s\to-\infty}(X_s^1 - X_s^2) = \infty$ pointwise.

To sum up, we have seen that even if the quadratic variation exists, the process may or may not converge as time goes to $-\infty$.

The next result shows in particular that for increment local martingales with bounded jumps, a.s. convergence at $-\infty$ is closely related to the local martingale property.

Theorem 3.14. *Let $M \in \mathcal{ILM}^2(\mathcal{F}.)$. The following are equivalent.*

(a) There is a predictable quadratic variation $\langle M\rangle$ for M.

(b) $M_{-\infty} = \lim_{s\to-\infty} M_s$ exists P*-a.s. and $(M_t - M_{-\infty})_{t\in\mathbb{R}} \in \mathcal{LM}^2(\mathcal{F}.)$.*

Remark 3.15. Let M in $\mathcal{ILM}(\mathcal{F}.)$ have bounded jumps; then, $M \in \mathcal{ILM}^2(\mathcal{F}.)$ as well. In this case (b) is satisfied if and only if $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s. Indeed, if the limit exists we define

$$\sigma_n = \inf\{t \in \mathbb{R} : |M_t - M_{-\infty}| > n\}.$$

Then $(M_t^{\sigma_n} - M_{-\infty})_{t\in\mathbb{R}}$ is a bounded and adapted process in $\mathcal{ILM}(\mathcal{F}.)$ and hence in $\mathcal{IM}^2(\mathcal{F}.)$. By Proposition 3.9, $(M_t^{\sigma_n} - M_{-\infty})_{t\in\mathbb{R}}$ is in $\mathcal{M}^2(\mathcal{F}.)$.

Proof. (a) implies (b): Choose a localising sequence $(\sigma_n)_{n\geq 1}$ such that

$$\mathrm{E}[\langle M\rangle_t^{\sigma_n}] < \infty, \quad \text{for all } t \in \mathbb{R} \text{ and all } n \geq 1.$$

Since ${}^s\langle M\rangle^{\sigma_n} = \langle {}^sM^{\sigma_n}\rangle$, it follows in particular that

$$\mathrm{E}[\langle {}^sM^{\sigma_n}\rangle_t] \leq \mathrm{E}[\langle M\rangle_t^{\sigma_n}] < \infty$$

for all $s \leq t$ and n. Therefore, for all s and n we have ${}^sM^{\sigma_n} \in \mathcal{M}^2(\mathcal{F}.)$, and

$$\mathrm{E}[({}^sM_t^{\sigma_n})^2] \leq \mathrm{E}[\langle M\rangle_t^{\sigma_n}] < \infty$$

for all $s \leq t$. Using Proposition 3.9 on M^{σ_n} it follows that $M_{-\infty} := \lim_{s\to-\infty} M_s^{\sigma_n}$ exists P-a.s. (this limit does not depend on n) and $(M_t^{\sigma_n} - M_{-\infty})_{t\in\mathbb{R}}$ is a square integrable martingale.

(b) implies (a): Let $\langle M - M_{-\infty}\rangle$ denote the predictable quadratic variation for $(M_t - M_{-\infty})_{t\in\mathbb{R}}$ which exists since this process is a locally square integrable martingale. Since $M \overset{\text{in}}{=} (M_t - M_{-\infty})_{t\in\mathbb{R}}$, $\langle M - M_{-\infty}\rangle$ is a predictable quadratic variation for M as well.

□

We have seen that a continuous increment local martingale is a local martingale if it converges almost surely as time goes to $-\infty$. A main purpose of the next examples is to study the behaviour at $-\infty$ when this is not the case.

Example 3.16. In (2) below we give an example of a continuous increment local martingale which converges to zero in probability as time goes to $-\infty$ without being a local martingale. As a building block for this construction we first consider a simple example of a continuous local martingale which is nonzero only on a finite interval.

(1) Let $B = (B_t)_{t\geq 0}$ denote a standard Brownian motion and τ be the first visit to zero after a visit to k, i.e.

$$\tau = \inf\{t > 0 : B_t = 0 \text{ and there is an } s < t \text{ such that } B_s > k\}, \tag{3.16}$$

where $k > 0$ is some fixed level. Then τ is finite with probability one, the stopped process $(B_{t\wedge\tau})_{t\geq 0}$ is a square integrable martingale, and $B_{t\wedge\tau} = 0$ when $t \geq \tau$. Let $a < b$ be real numbers and $\phi : [a,b) \to [0,\infty)$ be a surjective, continuous and strictly increasing mapping and define $Y = (Y_t)_{t\in\mathbb{R}}$ as

$$Y_t = \begin{cases} 0 & \text{if } t < a \\ B_{\phi(t)\wedge\tau} & \text{if } t \in [a,b) \\ 0 & \text{if } t \geq b. \end{cases} \tag{3.17}$$

Note that $t \mapsto Y_t$ is continuous P-a.s. and that with probability one $Y_t = 0$ for $t \notin [a,b]$. Define, with $\mathcal{N}$ denoting the P-null sets,

$$\mathcal{F}_t = \sigma(B_u : u \leq \phi(t)) \vee \mathcal{N} \quad \text{for } t \in \mathbb{R}, \tag{3.18}$$

where we let $\phi(t) = 0$ for $t \leq a$ and $\phi(t) = \infty$ for $t \geq b$. Interestingly, Y is a local martingale. To see this, define the "canonical" localising sequence $(\sigma_n)_{n\geq 1}$ as $\sigma_n = \inf\{t \in \mathbb{R} : |Y_t| > n\}$. Since $(Y_t^{\sigma_n})_{t\in[a,b)}$ is a deterministic time change of $(B_{t\wedge\tau})_{t\geq 0}$ stopped at σ_n, it is a bounded, and hence uniformly integrable, martingale. By continuity of the paths and the property $Y_t^{\sigma_n} = Y_b^{\sigma_n}$ for $t \geq b$ it thus follows that $(Y_t^{\sigma_n})_{t\in\mathbb{R}}$ is a bounded martingale.

(2) For $n = 1, 2, \ldots$ let $B^n = (B_t^n)_{t\geq 0}$ denote independent standard Brownian motions, and define $Y^n = (Y_t^n)_{t\in\mathbb{R}}$ as in (3.17) with $a = -n$ and $b = -n+1$, and Y resp. B replaced by Y^n resp. B^n. Let $(\mathcal{F}_t^n)_{t\in\mathbb{R}}$ be the corresponding filtration defined as in (3.18), and $(\theta_n)_{n\geq 1}$ denote a sequence of independent Bernoulli variables that are independent of the Brownian motions as well and satisfy $\mathrm{P}(\theta_n = 1) = 1 - \mathrm{P}(\theta_n = 0) = \frac{1}{n}$ for all n. Let $X_t^n = \theta_n Y_t^n$ for $t \in \mathbb{R}$.

Define $X_t = \sum_{n=1}^\infty X_t^n$ for $t \in \mathbb{R}$, which is well-defined since $X_t^n = 0$ for $t \notin [-n, -n+1]$, and set $\mathcal{F}_t = \vee_{n=1}^\infty(\mathcal{F}_t^n \vee \sigma(\theta_n))$ for $t \in \mathbb{R}$. For $s \in [-n, -n+1]$ and $n = 1, 2, \ldots$, ${}^sX_t = \sum_{m=1}^n {}^sX_t^m$, and since it is easily seen that each $(X_t^m)_{t\in\mathbb{R}}$ is a local martingale with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}}$, it follows that ${}^sX$ is a local martingale as well; that is, X

is an increment local martingale. By Borel-Cantelli, infinitely many of the θ_n's are 1 P-a.s., implying that X_s does not converge P-a.s. as $s \to -\infty$. On the other hand, $\mathrm{P}(X_t = 0) \geq \frac{n-1}{n}$ for $t \in [-n, -n+1]$, which means that $X_s \to 0$ in probability as $s \to -\infty$.

From (3.1) it follows that if a process in $\mathcal{IM}(\mathcal{F}.)$ is adapted and integrable then it is in $\mathcal{M}(\mathcal{F}.)$. By the above there is no such result for $\mathcal{ILM}(\mathcal{F}.)$; indeed, X is both adapted and p-integrable for all $p > 0$ but it is not in $\mathcal{LM}(\mathcal{F}.)$.

Example 3.17. Let $X = (X_t)_{t \geq 0}$ denote the inverse of BES(3), the three-dimensional Bessel process. It is well-known (see e.g. [10]) that X is a diffusion on natural scale and hence for all $s > 0$ the increment process $({}^sX_t)_{t \geq 0}$ is a local martingale. That is, we may consider X as an increment martingale indexed by $[0, \infty)$. By [10], ∞ is an entrance boundary, which means that if the process is started in ∞, it immediately leaves this state and never returns. Since we can obviously stretch $(0, \infty)$ into $\mathbb{R}$, this shows that there are interesting examples of continuous increment local martingales $(X_t)_{t \in \mathbb{R}}$ for which $\lim_{t \to -\infty} X_t = \pm\infty$ almost surely.

Using the Dambis-Dubins-Schwartz theorem it follows easily that any continuous local martingale indexed by $\mathbb{R}$ is a time change of a Brownian motion indexed by $\mathbb{R}_+$. It is not clear to us whether there is some analogue of this result for continuous increment local martingales but there are indications that this it not the case; indeed, above we saw that a continuous increment local martingale may converge to ∞ as time goes to $-\infty$; in particular this limiting behaviour does not resemble that of a Brownian motion indexed by $\mathbb{R}_+$ as time goes to 0 or of a Brownian motion indexed by $\mathbb{R}$ as time goes to $-\infty$.

Let $M \in \mathcal{LM}(\mathcal{F}.)$. It is well-known that M can be decomposed uniquely up to P-indistinguishability as $M_t = M_{-\infty} + M_t^c + M_t^d$ where $M^c = (M_t^c)_{t \in \mathbb{R}}$, the *continuous part* of M, is a continuous local martingale with $M_{-\infty} = 0$, and M^d, the *purely discontinuous part* of M, is a purely discontinuous local martingale with $M_{-\infty}^d = 0$, which means that $M^d N$ is a local martingale for all continuous local martingales N. Note that for $s \in \mathbb{R}$,

$$({}^sM)^c = {}^s(M^c) \quad \text{and} \quad ({}^sM)^d = {}^s(M^d). \tag{3.19}$$

We need a further decomposition of M^d so let $\mu^M = \{\mu^M(\omega; \mathrm{d}t, \mathrm{d}x) : \omega \in \Omega\}$ denote the random measure on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ induced by the jumps of M; that is,

$$\mu^M(\omega; \mathrm{d}t, \mathrm{d}x) = \sum_{s \in \mathbb{R}} \delta_{(s, \Delta M_s(\omega))}(\mathrm{d}t, \mathrm{d}x),$$

and let $\nu^M = \{\nu^M(\omega; \mathrm{d}t, \mathrm{d}x) : \omega \in \Omega\}$ denote the compensator of μ^M in the sense of [8], II.1.8. From Proposition II.2.29 and Corollary II.2.38 in [8] it follows that $(|x| \wedge |x|^2) * \nu^M \in \mathcal{LA}_0^1(\mathcal{F}.)$ and $M^d \overset{\mathrm{P}}{=} x * (\mu^M - \nu^M)$, implying that for arbitrary $\epsilon > 0$, M can be decomposed as

$$\begin{aligned} M_t &= M_{-\infty} + M_t^c + M_t^d = M_{-\infty} + M_t^c + x * (\mu^M - \nu^M)_t \\ &= M_{-\infty} + M_t^c + (x1_{\{|x| \leq \epsilon\}}) * (\mu^M - \nu^M)_t + (x1_{\{|x| > \epsilon\}}) * \mu_t^M - (x1_{\{|x| > \epsilon\}}) * \nu_t^M. \end{aligned}$$

Recall that when M is quasi-left continuous we have

$$\nu^M(\cdot; \{t\} \times (\mathbb{R} \setminus \{0\})) = 0 \quad \text{for all } t \in \mathbb{R} \text{ P-a.s.} \tag{3.20}$$

Finally, for $s \in \mathbb{R}$, $\mu^{{}^sM}(\cdot; \mathrm{d}t, \mathrm{d}x) = 1_{(s,\infty)}(\mathrm{d}t)\mu^M(\cdot; \mathrm{d}t, \mathrm{d}x)$ and thus

$$\nu^{{}^sM}(\cdot; \mathrm{d}t, \mathrm{d}x) = 1_{(s,\infty)}(\mathrm{d}t)\nu^M(\cdot; \mathrm{d}t, \mathrm{d}x). \tag{3.21}$$

Now consider the case $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. Denote the continuous resp. purely discontinuous part of ${}^sM$ by ${}^sM^c$ resp. ${}^sM^d$. By (3.19), $\{{}^sM^c\}_{s\in\mathbb{R}}$ and $\{{}^sM^d\}_{s\in\mathbb{R}}$ are consistent families of increment processes, and M is associated with $\{{}^sM^c + {}^sM^d\}_{s\in\mathbb{R}}$. Thus, there exist two processes, which we call the *continuous* resp. *purely discontinuous* part of M, and denote M^{cg} and $M^{d\mathrm{g}}$, such that M^{cg} is associated with $\{{}^sM^c\}_{s\in\mathbb{R}}$ and $M^{d\mathrm{g}}$ is associated with $\{{}^sM^d\}_{s\in\mathbb{R}}$, and

$$M_t = M_t^{\mathrm{cg}} + M_t^{d\mathrm{g}} \quad \text{for all } t \in \mathbb{R},\ \text{P-a.s.} \tag{3.22}$$

Once again these processes are unique only up to addition of random variables. In view of (3.21) we define the compensator of μ^M, to be denoted $\{\nu^M(\omega; \mathrm{d}t, \mathrm{d}x) : \omega \in \Omega\}$, as the random measure on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ satisfying that for all $s \in \mathbb{R}$,

$$1_{(s,\infty)}(\mathrm{d}t)\nu(\omega; \mathrm{d}t, \mathrm{d}x) = \nu^{{}^sM}(\omega; \mathrm{d}t, \mathrm{d}x),$$

where, noticing that ${}^sM$ is a local martingale, the right-hand side is the compensator of $\mu^{{}^sM}$ in the sense of [8], II.1.8.

Theorem 3.18. *Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$.*

(1) The quadratic variation $[M]$ for M exists if and only if there is a continuous martingale component M^{cg} with $M^{\mathrm{cg}} \in \mathcal{LM}(\mathcal{F}_\cdot)$ and $M^{\mathrm{cg}}_{-\infty} = 0$, and for all $t \in \mathbb{R}$, $\sum_{s\le t}(\Delta M_s)^2 < \infty$ P-a.s. In this case

$$[M]_t = \langle M^{\mathrm{cg}}\rangle_t + \sum_{s\le t}(\Delta M_s)^2.$$

(2) We have that $M_{-\infty} := \lim_{s\to-\infty} M_s$ exists P-a.s. and $(M_t - M_{-\infty})_{t\in\mathbb{R}} \in \mathcal{LM}(\mathcal{F}_\cdot)$ if and only if the quadratic variation $[M]$ for M exists and $[M]^{\frac{1}{2}} \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)$.

(3) Assume (3.20) is satisfied and there is an $\epsilon > 0$ such that

$$\lim_{s\to-\infty} \int_{(s,0]} \int_{|x|>\epsilon} x\nu^M(\cdot; \mathrm{d}u, \mathrm{d}x) \tag{3.23}$$

exists P-a.s. Then, $\lim_{s\to-\infty} M_s$ exists P-a.s. if and only if $[M]$ exists.

Note that the conditions in (3) are satisfied if ν^M can be decomposed as $\nu^M(\cdot; \mathrm{d}t \times \mathrm{d}x) = F(\cdot; t, \mathrm{d}x)\,\mu(\mathrm{d}t)$ where $F(\cdot; t, \mathrm{d}x)$ is a symmetric measure for all $t \in \mathbb{R}$ and μ does not have positive point masses.

Proof. (1) For $s \le t$ we have

$$\begin{aligned}
{}^s[M]^{\mathrm{g}}_t = [{}^sM]_t &= \sum_{u:s<u\le t} (\Delta M_u)^2 + \langle {}^sM^c\rangle_t \\
&= \sum_{u:s<u\le t} (\Delta M_u)^2 + \langle {}^s(M^{\mathrm{cg}})\rangle_t \\
&= \sum_{u:s<u\le t} (\Delta M_u)^2 + {}^s\langle M^{\mathrm{cg}}\rangle^{\mathrm{g}}_t \\
&= \sum_{u:s<u\le t} (\Delta M_u)^2 + \langle M^{\mathrm{cg}}\rangle^{\mathrm{g}}_t - \langle M^{\mathrm{cg}}\rangle^{\mathrm{g}}_s,
\end{aligned} \tag{3.24}$$

where the first equality is due to the fact that $[M]^{\mathrm{g}}$ is associated with $\{[{}^sM]\}_{s\in\mathbb{R}}$, the second is a well-known decomposition of the quadratic variation of a local martingale, the third equality is due to $M^{c\mathrm{g}}$ being associated with $\{{}^sM^c\}_{s\in\mathbb{R}}$ and the fourth is due to $\langle M^{c\mathrm{g}}\rangle^{\mathrm{g}}$ being associated with $\{\langle {}^sM^{c\mathrm{g}}\rangle\}_{s\in\mathbb{R}}$. By Remark 3.11 (1), the quadratic variation $[M]$ exists if and only if $[M]^{\mathrm{g}}_s$ converges P-a.s. as $s\to-\infty$, which, by the above, is equivalent to convergence almost surely of both terms in (3.24). By Theorem 3.14, $\langle M^{c\mathrm{g}}\rangle^{\mathrm{g}}_s$ converges P-a.s. as $s\to-\infty$ if and only if $M^{c\mathrm{g}}_{-\infty}$ exists P-a.s. and $(M^{c\mathrm{g}}_t - M^{c\mathrm{g}}_{-\infty})_{t\in\mathbb{R}}$ is a continuous local martingale. If the quadratic variation exists, we may replace $M^{c\mathrm{g}}$ by $(M^{c\mathrm{g}}_t - M^{c\mathrm{g}}_{-\infty})_{t\in\mathbb{R}}$ and $M^{d\mathrm{g}}$ by $(M^{d\mathrm{g}}_t + M^{c\mathrm{g}}_{-\infty})_{t\in\mathbb{R}}$, thus obtaining a continuous part of M which starts at 0.

(2) First assume that $M_{-\infty}$ exists and $(M_t - M_{-\infty})_{t\in\mathbb{R}} \in \mathcal{LM}(\mathcal{F}_\cdot)$. Since $M \overset{\text{in}}{=} (M_t - M_{-\infty})_{t\in\mathbb{R}}$, the quadratic variation for M exists and equals the quadratic variation for $(M_t - M_{-\infty})_{t\in\mathbb{R}}$. It is well-known that since the latter is a local martingale, $[M]^{\frac{1}{2}} \in \mathcal{LA}^1_0(\mathcal{F}_\cdot)$.

Conversely assume that $[M]$ exists and $[M]^{\frac{1}{2}} \in \mathcal{LA}^1_0(\mathcal{F}_\cdot)$. Choose a localising sequence $(\sigma_n)_{n\geq 1}$ such that $[M^{\sigma_n}]^{\frac{1}{2}} \in \mathcal{A}^1_0(\mathcal{F}_\cdot)$. Since ${}^s[M^{\sigma_n}]_0 \leq [M^{\sigma_n}]_0$ if follows from Davis' inequality that for some constant $c>0$,

$$\mathrm{E}[\sup_{u:s\leq u\leq 0} |{}^sM^{\sigma_n}_0|] \leq c\mathrm{E}[[M^{\sigma_n}]_0^{\frac{1}{2}}] < \infty$$

for all $s\leq 0$, implying that $({}^sM^{\sigma_n}_0)_{s<0}$ is uniformly integrable. The result now follows from Proposition 3.9.

(3) By (3.21), the three families of increment processes $\{(x1_{\{|x|\leq\epsilon\}})*(\mu^{{}^sM} - \nu^{{}^sM}\}_{s\in\mathbb{R}}$, $\{(x1_{\{|x|>\epsilon\}})*\mu^{{}^sM}\}_{s\in\mathbb{R}}$ and $\{(x1_{\{|x|>\epsilon\}})*\nu^{{}^sM}\}_{s\in\mathbb{R}}$ are all consistent. Choose $X=(X_t)_{t\in\mathbb{R}}$, $Y=(Y_t)_{t\in\mathbb{R}}$ and $Z=(Z_t)_{t\in\mathbb{R}}$ associated with these families such that $X_t + Y_t - Z_t = M^{d\mathrm{g}}_t$; in particular we then have $M \overset{\mathrm{P}}{=} M^{c\mathrm{g}} + X + Y - Z$. Since Z is associated with $\{(x1_{\{|x|>\epsilon\}})*\nu^{{}^sM}\}_{s\in\mathbb{R}}$ we have

$$Z_0 - Z_s = \int_s^0 \int_{|x|>\epsilon} x\,\nu^M(\cdot;\mathrm{d}u,\mathrm{d}x) \quad \text{for all } s\in\mathbb{R} \text{ with probability one,}$$

implying that $s\mapsto Z_s$ is continuous by (3.20) and $\lim_{s\to-\infty} Z_s$ exists P-a.s. by (3.23). By (3.20) it also follows that $(\Delta X_s)_{s\in\mathbb{R}} \overset{\mathrm{P}}{=} (\Delta M_s 1_{\{|\Delta M_s|\leq\epsilon\}})_{s\in\mathbb{R}}$, implying that X is an increment local martingale with jumps bounded by ϵ in absolute value and

$$\sum_{s:s\leq t} (\Delta M_s)^2 = \sum_{s:s\leq t} (\Delta X_s)^2 + \sum_{s:s\leq t} (\Delta Y_s)^2 \quad \text{for all } t\in\mathbb{R} \text{ with probability one.} \tag{3.25}$$

If $[M]$ exists then by (1) $M^{c\mathrm{g}}_{-\infty}$ exists P-a.s. and (3.25) is finite for all t with probability one. Since Y is piecewise constant with jumps of magnitude at least ϵ, it follows that Y_s is constant when s is small enough almost surely. In addition, since the quadratic variation of the increment local martingale X exists and X has bounded jumps it follows from (2) that, up to addition of a random variable, X is a local martingale and thus $\lim_{s\to-\infty} X_s$ exists as well; that is, $\lim_{s\to-\infty} M_s$ exists P-a.s.

If, conversely, $\lim_{s\to-\infty} M_s$ exists P-a.s., there are no jumps of magnitude at least ϵ in M when s is small enough; thus there are no jumps in Y_s when s is sufficiently small P-a.s., implying that $\lim_{s\to-\infty}(M^{c\mathrm{g}}_s + X_s)$ exists P-a.s. Combining Theorem 3.14, (3.25) and (1) it follows that $[M]$ exists. ◻

4 Stochastic integration

In the following we define a stochastic integral with respect to an increment local martingale. Let $M \in \mathcal{LM}(\mathcal{F}_\cdot)$ and set

$$\begin{aligned}&\mathcal{L}L^1(M)\\ &\quad := \{\phi = (\phi_t)_{t\in\mathbb{R}} : \phi \text{ is predictable and } \Big(\big(\int_{(-\infty,t]} \phi_s^2 \,\mathrm{d}[M]_s\big)^{\frac{1}{2}}\Big)_{t\in\mathbb{R}} \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)\}.\end{aligned}$$

Since in this case the index set set can be taken to be $[-\infty,\infty)$, it is well-known, e.g. from [7], that the stochastic integral of $\phi \in \mathcal{L}L^1(M)$ with respect to M, which we denote $(\int_{(-\infty,t]} \phi_s \,\mathrm{d}M_s)_{t\in\mathbb{R}}$ or $\phi \bullet M = (\phi \bullet M_t)_{t\in\mathbb{R}}$, does exist. All fundamental properties of the integral are well-known so let us just explicitly mention the following two results that we are going to use in the following: For σ a stopping time, $s \in \mathbb{R}$ and $\phi \in \mathcal{L}L^1(M)$ we have

$$(\phi \bullet M)^\sigma \overset{\mathrm{P}}{=} (\phi 1_{(-\infty,\sigma]}) \bullet M \overset{\mathrm{P}}{=} \phi \bullet (M^\sigma) \tag{4.1}$$

and

$$^{s}(\phi \bullet M) \overset{\mathrm{P}}{=} \phi \bullet (^{s}M) \overset{\mathrm{P}}{=} (\phi 1_{(s,\infty)}) \bullet M. \tag{4.2}$$

Next we define and study a *stochastic increment integral* with respect an increment local martingale. For $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$ set

$$\begin{aligned}\mathcal{L}L^1(M) &:= \{\phi : \phi \text{ is predictable and } \Big(\big(\int_{(-\infty,t]} \phi_s^2 \,\mathrm{d}[M]_s^{\mathrm{g}}\big)^{\frac{1}{2}}\Big)_{t\in\mathbb{R}} \in \mathcal{LA}_0^1(\mathcal{F}_\cdot)\}\\ \mathcal{IL}L^1(M) &:= \{\phi : \phi \in \mathcal{L}L^1(^{s}M) \text{ for all } s \in \mathbb{R}\}.\end{aligned}$$

As an example, if $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$ then a predictable ϕ is in $\mathcal{L}L^1(M)$ resp. in $\mathcal{IL}L^1(M)$ if (but in general not only if) $\int_{(-\infty,t]} \phi_s^2 \,\mathrm{d}\langle M\rangle_s^{\mathrm{g}} < \infty$ for all $t \in \mathbb{R}$ P-a.s. resp. $\int_{(s,t]} \phi_u^2 \,\mathrm{d}\langle M\rangle_u^{\mathrm{g}} < \infty$ for all $s < t$ P-a.s. If $M \in \mathcal{ILM}^2(\mathcal{F}_\cdot)$ is continuous then

$$\begin{aligned}\mathcal{L}L^1(M) &= \{\phi : \phi \text{ is predictable and } \int_{(-\infty,t]} \phi_s^2 \,\mathrm{d}\langle M\rangle_s^{\mathrm{g}} < \infty \text{ P-}a.s. \text{ for all } t\}\\ \mathcal{IL}L^1(M) &= \{\phi : \phi \text{ is predictable and } \int_{(s,t]} \phi_u^2 \,\mathrm{d}\langle M\rangle_u^{\mathrm{g}} < \infty \text{ P-}a.s. \text{ for all } s < t\}.\end{aligned}$$

Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. The stochastic integral $\phi \bullet (^{s}M)$ of ϕ in $\mathcal{IL}L^1(M)$ exists for all $s \in \mathbb{R}$; in addition, $\{\phi \bullet (^{s}M)\}_{s\in\mathbb{R}}$ is a consistent family of increment processes. Indeed, for $s \le t \le u$ we must verify

$$(\phi \bullet (^{s}M))_u = (\phi \bullet (^{s}M))_t + (\phi \bullet (^{t}M))_u, \quad \text{P-a.s.}$$

or equivalently

$$^{t}(\phi \bullet (^{s}M))_u = (\phi \bullet (^{t}M))_u \quad \text{P-a.s.},$$

which follows from (2.3) and (4.2). Based on this, we define the *stochastic increment integral of ϕ with respect to M*, to be denoted $\phi \overset{\mathrm{in}}{\bullet} M$, as a càdlàg process associated with the the family $\{\phi \bullet (^{s}M)\}_{s\in\mathbb{R}}$. Note that the increment integral $\phi \overset{\mathrm{in}}{\bullet} M$ is uniquely determined only up to addition of a random variable and it is an increment local martingale. For $s < t$ and $\phi \in \mathcal{IL}L^1(M)$ we think of $\phi \overset{\mathrm{in}}{\bullet} M_t - \phi \overset{\mathrm{in}}{\bullet} M_s$ as the integral of ϕ with respect to M over the interval $(s,t]$ and hence use the notation

$$\int_{(s,t]} \phi_u \,\mathrm{d}M_u := \phi \overset{\mathrm{in}}{\bullet} M_t - \phi \overset{\mathrm{in}}{\bullet} M_s \quad \text{ for } s < t. \tag{4.3}$$

When $\phi \overset{\text{in}}{\bullet} M_{-\infty} := \lim_{s\to-\infty} \phi \overset{\text{in}}{\bullet} M_s$ exists P-a.s. we define the *improper integral of* ϕ *with respect to* M *from* $-\infty$ *to* t for $t \in \mathbb{R}$ as

$$\int_{(-\infty,t]} \phi_u \, \mathrm{d}M_u := \phi \overset{\text{in}}{\bullet} M_t - \phi \overset{\text{in}}{\bullet} M_{-\infty}. \tag{4.4}$$

Put differently, the improper integral $(\int_{(-\infty,t]} \phi_u \, \mathrm{d}M_u)_{t\in\mathbb{R}}$ is, when it exists, the unique, up to P-indistinguishability, increment integral of ϕ with respect to M which is 0 in $-\infty$. Moreover, it is an adapted process.

The following summarises some fundamental properties.

Theorem 4.1. *Let* $M \in \mathcal{ILM}(\mathcal{F}.)$.

(1) Whenever $\phi \in \mathcal{ILL}^1(M)$ *and* $s < t$ *we have* ${}^s(\phi \overset{\text{in}}{\bullet} M)_t = (\phi \bullet ({}^sM))_t$ P-*a.s.*

(2) $\phi \overset{\text{in}}{\bullet} M \in \mathcal{ILM}(\mathcal{F}.)$ *for all* $\phi \in \mathcal{ILL}^1(M)$.

(3) If $\phi, \psi \in \mathcal{ILL}^1(M)$ *and* $a, b \in \mathbb{R}$ *then* $(a\phi + b\psi) \overset{\text{in}}{\bullet} M \overset{\text{in}}{=} a(\phi \overset{\text{in}}{\bullet} M) + b(\psi \overset{\text{in}}{\bullet} M)$.

(4) For $\phi \in \mathcal{ILL}^1(M)$ *we have*

$$\Delta\phi \overset{\text{in}}{\bullet} M_t = \phi_t \Delta M_t, \quad \textit{for } t \in \mathbb{R}, \ \text{P-}a.s. \tag{4.5}$$

$${}^s[\phi \overset{\text{in}}{\bullet} M]^{\mathrm{g}}_t = \int_{(s,t]} \phi_u^2 \, \mathrm{d}[M]^{\mathrm{g}}_s \quad \textit{for } s \le t \ \text{P-}a.s. \tag{4.6}$$

In particular $[\phi \overset{\text{in}}{\bullet} M]$ *exists if and only if* $\int_{(-\infty,t]} \phi_s^2 \, \mathrm{d}[M]^{\mathrm{g}}_s < \infty$ *for all* $t \in \mathbb{R}$ P-*a.s.*

(5) If σ *a stopping time and* $\phi \in \mathcal{ILL}^1(M)$ *then*

$$(\phi \overset{\text{in}}{\bullet} M)^\sigma \overset{\text{in}}{=} (\phi 1_{(-\infty,\sigma]}) \overset{\text{in}}{\bullet} M \overset{\text{in}}{=} \phi \overset{\text{in}}{\bullet} (M^\sigma).$$

(6) Let $\phi \in \mathcal{ILL}^1(M)$ *and* $\psi = (\psi_t)_{t\in\mathbb{R}}$ *be predictable. Then* $\psi \in \mathcal{ILL}^1(\phi \overset{\text{in}}{\bullet} M)$ *if and only if* $\phi\psi \in \mathcal{ILL}^1(M)$, *and in this case* $\psi \overset{\text{in}}{\bullet} (\phi \overset{\text{in}}{\bullet} M) \overset{\text{in}}{=} (\psi\phi) \overset{\text{in}}{\bullet} M$.

(7) Let $\phi \in \mathcal{ILL}^1(M)$. *Then* $\phi \overset{\text{in}}{\bullet} M_{-\infty} := \lim_{s\to-\infty} \phi \overset{\text{in}}{\bullet} M_s$ *exists* P-*a.s. and* $(\int_{(-\infty,t]} \phi_u \, \mathrm{d}M_u)_{t\in\mathbb{R}} \in \mathcal{LM}(\mathcal{F}.)$ *if and only if* $\phi \in \mathcal{LL}^1(M)$.

Remark 4.2. (a) When M is continuous it follows from Theorem 3.14 that (7) can be simplified to the statement that $\phi \overset{\text{in}}{\bullet} M_{-\infty} = \lim_{s\to-\infty} \phi \overset{\text{in}}{\bullet} M_s$ exists P-a.s. if and only if $\phi \in \mathcal{LL}^1(M)$, and in this case $(\int_{(-\infty,t]} \phi_u \, \mathrm{d}M_u)_{t\in\mathbb{R}} \in \mathcal{LM}(\mathcal{F}.)$.

(b) Result (7) above gives a necessary and sufficient condition for the improper integral to exist and be a local martingale; however, improper integrals may exist without being a local martingale (but as noted above they are always increment local martingales). For example, assume M is purely discontinuous and that the compensator ν^M of the jump measure ν^M can be decomposed as $\nu^M(\cdot; \mathrm{d}t \times \mathrm{d}x) = F(\cdot; t, \mathrm{d}x)\mu(\mathrm{d}t)$ where $F(\cdot; t, \mathrm{d}x)$ is a symmetric measure and $\mu(\{t\}) = 0$ for all $t \in \mathbb{R}$. Then by Theorem 3.18 (3), $\phi \overset{\text{in}}{\bullet} M_{-\infty}$ exists P-a.s. if and only if the quadratic variation $[\phi \overset{\text{in}}{\bullet} M]$ exists; that is,

$$\sum_{s\le 0} \phi_s^2 (\Delta M_s)^2 < \infty \quad \text{P-}a.s.$$

Proof. Property (1) is merely by definition, and (2) is due to the fact that ${}^s(\phi \overset{\text{in}}{\bullet} M) \overset{\text{P}}{=} \phi \bullet {}^sM$, which is a local martingale.

(3) We must show that $a(\phi \overset{\text{in}}{\bullet} M)+b(\psi \overset{\text{in}}{\bullet} M)$ is associated with $\{(a\phi+b\psi)\bullet({}^sM)\}_{s\in\mathbb{R}}$, i.e. that ${}^s\big(a(\phi \overset{\text{in}}{\bullet} M) + b(\psi \overset{\text{in}}{\bullet} M)\big) \overset{\text{P}}{=} (a\phi + b\psi) \bullet ({}^sM)$. However, by definition of the stochastic increment integral and linearity of the stochastic integral we have

$$a\,{}^s\big(\phi \overset{\text{in}}{\bullet} M\big) + b\,{}^s\big(\psi \overset{\text{in}}{\bullet} M\big) \overset{\text{P}}{=} a\big(\phi \bullet ({}^sM)\big) + b\big(\psi \bullet ({}^sM)\big) \overset{\text{P}}{=} (a\phi + b\psi) \bullet ({}^sM).$$

(4) Using that ${}^s(\phi \overset{\text{in}}{\bullet} M) = \phi \bullet ({}^sM)$ and $\Delta\phi \bullet ({}^sM) \overset{\text{P}}{=} \phi\Delta({}^sM)$, the result in (4.5) follows. By definition, $[\phi \overset{\text{in}}{\bullet} M]^{\text{g}}$ is associated with $\{[{}^s(\phi \overset{\text{in}}{\bullet} M)]\}_{s\in\mathbb{R}} = \{[\phi \bullet ({}^sM)]\}_{s\in\mathbb{R}}$. That is, for $s \in \mathbb{R}$ we have, using that $[M]^{\text{g}}$ is associated with $\{[{}^sM]_s\}_{s\in\mathbb{R}}$,

$$\begin{aligned}{}^s[\phi \overset{\text{in}}{\bullet} M]^{\text{g}}_t = [\phi \bullet ({}^sM)]_t &= \int_{(s,t]} \phi_u^2 \,\mathrm{d}[{}^sM]_u \\ &= \int_{(s,t]} \phi_u^2 \,\mathrm{d}({}^s[M]^{\text{g}})_u = \int_{(s,t]} \phi_u^2 \,\mathrm{d}[M]^{\text{g}}_u \quad \text{for } s \le t \quad \text{P-a.s.,}\end{aligned}$$

which yields (4.6). The last statement in (4) follows from Remark 3.11 (1).

The proofs of (5) and (6) are left to the reader.

(7) Using (4) the result follows immediately from Theorem 3.18. □

Let us turn to the definition of a stochastic integral $\phi \bullet M$ of a predictable ϕ with respect to an increment local martingale M. Thinking of $\phi \bullet M_t$ as an integral from $-\infty$ to t it seems reasonable to say that $\phi \bullet M$ (defined for a suitable class of predictable processes ϕ) is a stochastic integral with respect to M if the following is satisfied:

(1) $\lim_{t\to-\infty} \phi \bullet M_t = 0$ P-a.s.

(2) $\phi_t \bullet M_t - \phi \bullet M_s = \int_{(s,t]} \phi_u \,\mathrm{d}M_u$ P-a.s. for all $s < t$

(3) $\phi \bullet M$ is a local martingale.

By definition of $\int_{(s,t]} \phi_u \,\mathrm{d}M_u$, (2) implies that $\phi \bullet M$ must be an increment integral of ϕ with respect to M. Moreover, since we assume $\phi \bullet M_{-\infty} = 0$, $\phi \bullet M$ is uniquely determined as $(\phi \bullet M_t)_{t\in\mathbb{R}} \overset{\text{P}}{=} (\int_{(-\infty,t]} \phi_u \,\mathrm{d}M_u)_{t\in\mathbb{R}}$, i.e. the improper integral of ϕ. Since we also insist that $\phi \bullet M$ is a local martingale, Theorem 4.1 (7) shows that $\mathcal{L}L^1(M)$ is the largest possible set on which $\phi \bullet M$ can be defined. We summarise these findings as follows.

Theorem 4.3. *Let $M \in \mathcal{ILM}(\mathcal{F}_\cdot)$. Then there exists a unique stochastic integral $\phi \bullet M$ defined for $\phi \in \mathcal{L}L^1(M)$. This integral is given by*

$$\phi \bullet M_t = \int_{(-\infty,t]} \phi_u \,\mathrm{d}M_u \quad \textit{for } t \in \mathbb{R} \tag{4.7}$$

and it satisfied the following.

(1) $\phi \bullet M \in \mathcal{LM}(\mathcal{F}_\cdot)$ and $\phi \bullet M_{-\infty} = 0$ for $\phi \in \mathcal{L}L^1(M)$.

(2) The mapping $\phi \mapsto \phi \bullet M$ is, up to P*-indistinguishability, linear in $\phi \in \mathcal{L}L^1(M)$.*

(3) For $\phi \in \mathcal{L}L^1(M)$ we have

$$\begin{aligned}\Delta\phi \bullet M_t &= \phi_t \Delta M_t, \qquad \textit{for } t \in \mathbb{R},\ \text{P-}a.s.\\ [\phi \bullet M]_t &= \int_{(-\infty,t]} \phi_s^2 \,\mathrm{d}[M]^{\text{g}}_s \quad \textit{for } t \in \mathbb{R},\ \text{P-}a.s.\end{aligned}$$

(4) For σ a stopping time, $s \in \mathbb{R}$ and $\phi \in \mathcal{L}L^1(M)$ we have

$$(\phi \bullet M)^\sigma \overset{\mathrm{P}}{=} (\phi 1_{(-\infty,\sigma]}) \bullet M \overset{\mathrm{P}}{=} \phi \bullet (M^\sigma)$$

and ${}^s(\phi \bullet M) \overset{\mathrm{P}}{=} \phi \bullet ({}^sM)$.

Example 4.4. Let $X \in \mathcal{ILM}(\mathcal{F}_\cdot)$ be continuous and assume there is a positive continuous predictable process $\sigma = (\sigma_t)_{t\in\mathbb{R}}$ such that for all $s < t$, ${}^s[X]^{\mathrm{g}}_t = \int_s^t \sigma_u^2 \,\mathrm{d}u$. Set $B = \sigma^{-1} \overset{\mathrm{in}}{\bullet} X$ and note that by Lévy's theorem B is a standard Brownian motion indexed by $\mathbb{R}$, and X is given by $X \overset{\mathrm{in}}{=} \sigma \overset{\mathrm{in}}{\bullet} B$.

Example 4.5. As a last example assume $B = (B_t)_{t\in\mathbb{R}}$ is a Brownian motion indexed by $\mathbb{R}$ and consider the filtration $\mathcal{F}^{\mathcal{I}B}_\cdot$ generated by the increments of B cf. Example 3.6. In this case a predictable ϕ is in $\mathcal{L}L^1(B)$ resp. $\mathcal{IL}L^1(B)$ if and only if $\int_{-\infty}^t \phi_u^2 \,\mathrm{d}u < \infty$ for all t P-a.s. resp. $\int_s^t \phi_u^2 \,\mathrm{d}u < \infty$ for all $s < t$ P-a.s. Moreover, if $M \in \mathcal{ILM}(\mathcal{F}^{\mathcal{I}B}_\cdot)$ then there is a $\phi \in \mathcal{IL}L^1(B)$ such that

$$M \overset{\mathrm{in}}{=} \phi \overset{\mathrm{in}}{\bullet} B \tag{4.8}$$

and if $M \in \mathcal{LM}(\mathcal{F}^{\mathcal{I}B}_\cdot)$ then there is a $\phi \in \mathcal{L}L^1(B)$ such that

$$M \overset{\mathrm{P}}{=} M_{-\infty} + \phi \bullet B. \tag{4.9}$$

That is, we have a *martingale representation result* in the filtration $\mathcal{F}^{\mathcal{I}B}_\cdot$. To see that this is the case, it suffices to prove (4.8). Let $s \in \mathbb{R}$ and set $\mathcal{H} = \mathcal{F}^{\mathcal{I}B}_s$. Since $\mathcal{F}^{\mathcal{I}B}_t = \mathcal{H} \vee \sigma(B_u - B_s : s \le u \le t)$ for $t \ge s$ it follows from [8], Theorem III.4.34, that there is a ϕ^s in $\mathcal{L}L^1({}^sB)$ such that ${}^sM \overset{\mathrm{P}}{=} \phi^s \bullet ({}^sB)$. If $u < s$ then by (2.3) and (4.2) we have ${}^sM = \phi^u \bullet ({}^sB)$; thus, there is a ϕ in $\mathcal{IL}L^1(B)$ such that ${}^sM \overset{\mathrm{P}}{=} \phi \bullet ({}^sB)$ for all s and hence $M \overset{\mathrm{in}}{=} \phi \overset{\mathrm{in}}{\bullet} B$ by definition of the increment integral.

The above generalises in an obvious way to the case where instead of a Brownian motion B we have, say, a Lévy process X with integrable centred increments. In this case, we have to add an integral with respect to $\mu^X - \nu^X$ on the right-hand sides of (4.8) and (4.9).

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PAPER H

Quasi Ornstein-Uhlenbeck processes

Ole E. Barndorff-Nielsen and Andreas Basse-O'Connor

Abstract

The question of existence and properties of stationary solutions to Langevin equations driven by noise processes with stationary increments is discussed, with particular focus on noise processes of pseudo moving average type. On account of the Wold-Karhunen decomposition theorem such solutions are in principle representable as a moving average (plus a drift like term) but the kernel in the moving average is generally not available in explicit form. A class of cases is determined where an explicit expression of the kernel can be given, and this is used to obtain information on the asymptotic behavior of the associated autocorrelation functions, both for small and large lags. Applications to Gaussian and Lévy driven fractional Ornstein-Uhlenbeck processes are presented. As an element in the derivations a Fubini theorem for Lévy bases is established.

Keywords: fractional Ornstein-Uhlenbeck processes; Fubini theorem for Lévy bases; Langevin equations; stationary processes

AMS Subject Classification: 60G22; 60G10; 60G15; 60G52; 60G57

1 Introduction

This paper studies existence and properties of stationary solutions to Langevin equations driven by a noise process with, in general, stationary dependent increments. We shall refer to such solutions as quasi Ornstein-Uhlenbeck (QOU) processes. Of particular interest are the cases where the noise process is of the pseudo moving average (PMA) type. In wide generality the stationary solutions can, in principle, be written in the form of a Wold-Karhunen type representation, but it is relatively rare that an explicit expression for the kernel of such a representation can be given. When this is possible it often provides a more direct and simpler access to the character and properties of the process, for instance concerning the autocovariance function.

The structure of the paper is as follows. Section 2 defines the concept of quasi Ornstein-Uhlenbeck processes and provides conditions for existence and uniqueness of stationary solutions to the Langevin equation. The form of the autocovariance function of the solutions is given and its asymptotic behavior for $t \to \infty$ is discussed. As a next, intermediate, step a Fubini theorem for Lévy bases is established in Section 3. In Section 4 explicit forms of Wold-Karhunen representations are derived and used to analyze the asymptotics, under more specialized assumptions, of the autocovariance functions, both for $t \to \infty$ and for $t \to 0$. The results are applied in particular to the case of Gaussian and Lévy driven fractional Ornstein-Uhlenbeck processes. Section 5 concludes.

2 Langevin equations and QOU processes

Let $N = (N_t)_{t\in\mathbb{R}}$ be a measurable process with stationary increments and let $\lambda > 0$ be a positive number. By a quasi Ornstein-Uhlenbeck (QOU) process X driven by N and with parameter λ, we mean a stationary solution to the Langevin equation $\mathrm{d}X_t = -\lambda X_t\,\mathrm{d}t + \mathrm{d}N_t$, that is, $X = (X_t)_{t\in\mathbb{R}}$ is a stationary process which satisfies

$$X_t = X_0 - \lambda \int_0^t X_s\,\mathrm{d}s + N_t, \qquad t \in \mathbb{R}, \tag{2.1}$$

where the integral is a pathwise Lebesgue integral. For all $a < b$ we use the notation $\int_b^a := -\int_a^b$. Recall that a process $Z = (Z_t)_{t\in\mathbb{R}}$ is measurable if $(t,\omega) \mapsto Z_t(\omega)$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, and that Z has stationary increments if for all $s \in \mathbb{R}$, $(Z_t - Z_0)_{t\in\mathbb{R}}$ has the same finite distributions as $(Z_{t+s} - Z_s)_{t\in\mathbb{R}}$. For $p \geq 0$ we will say that a process Z has finite p-moments if $\mathrm{E}[|Z_t|^p] < \infty$ for all $t \in \mathbb{R}$. Moreover for $t \to 0$ or ∞, we will write $f(t) \sim g(t)$, $f(t) = o(g(t))$ or $f(t) = O(g(t))$ provided that $f(t)/g(t) \to 1$, $f(t)/g(t) \to 0$ or $\limsup_t |f(t)/g(t)| < \infty$, respectively. For each process Z with finite second-moments, let $\mathrm{Var}Z(t) = \mathrm{Var}(Z_t)$ denote its variance function. When Z, in addition, is stationary, let $\mathrm{R}_Z(t) = \mathrm{Cov}(Z_t, Z_0)$ denote its autocovariance function, and $\bar{\mathrm{R}}_X(t) = \mathrm{R}_X(0) - \mathrm{R}_X(t) = \frac{1}{2}\mathrm{E}[(X_t - X_0)^2]$ its complementary autocovariance function.

Before discussing the general setting further we recall some well known cases. The stationary solution X to (2.1) where $N_t = \mu t + \sigma B_t$, and B is a Brownian motion is of particular interest in finance; here X is the Gaussian Ornstein-Uhlenbeck process, μ/λ is the mean level, λ is the speed of reversion and σ is the volatility. When N is a Lévy process the corresponding QOU process, X, exists if and only if $\mathrm{E}[\log^+|N_1|] < \infty$ or, equivalently, if $\int_{\{|x|>1\}} \log|x|\,\nu(dx) < \infty$ where ν is the Lévy measure of N; see [30]. In this case X is called an Ornstein-Uhlenbeck type process; for applications of such processes in financial economics see [5, 6].

2.1 Auxiliary continuity result

Let $(E, \mathcal{E}, \mu)$ be a σ-finite measure space, and $\phi\colon \mathbb{R} \to \mathbb{R}_+$ an even and continuous function which is non-decreasing on $\mathbb{R}_+$, with $\phi(0) = 0$. Assume there exists a constant $C > 0$ such that $\phi(2x) \leq C\phi(x)$ for all $x \in \mathbb{R}$ (that is, ϕ satisfies the Δ_2-condition). Let $L^0 = L^0(E, \mathcal{E}, \mu)$ denote the space of all measurable functions from E into $\mathbb{R}$, and let Φ denote the modular on L^0 given by

$$\Phi(g) = \int_E \phi(g)\,\mathrm{d}\mu, \qquad g \in L^0, \tag{2.2}$$

and $L^\phi = \{g \in L^0 : \Phi(g) < \infty\}$ the corresponding modular space. Furthermore, for $g \in L^0$ define

$$\rho(g) = \inf\{c > 0 : \Phi(g/c) \leq c\}, \qquad \text{and} \qquad \|g\|_\phi = \inf\{c > 0 : \Phi(g/c) \leq 1\}. \tag{2.3}$$

Then ρ is an F-norm on L^ϕ, and when ϕ is convex, the *Luxemburg norm* $\|\cdot\|_\phi$ is a norm on L^ϕ; see e.g. [20]. If not explicitly said otherwise, L^ϕ will be equipped with the metric $d_\phi(f, g) = \rho(f - g)$.

Theorem 2.1. *Let $f\colon \mathbb{R} \times E \to \mathbb{R}$ denote a measurable function satisfying that $f_t = f(t, \cdot) \in L^\phi$ for all $t \in \mathbb{R}$, and*

$$d_\phi(f_{t+u}, f_{v+u}) = d_\phi(f_t, f_v), \qquad \text{for all } t, u, v \in \mathbb{R}. \tag{2.4}$$

Then, $(t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$ is continuous. Moreover, if ϕ is convex, then there exist $\alpha, \beta > 0$ such that $\|f_t\|_\phi \leq \alpha + \beta|t|$ for all $t \in \mathbb{R}$.

To prove Theorem 2.1 we shall need the following lemma.

Lemma 2.2. *Let $f\colon \mathbb{R} \times E \to \mathbb{R}$ denote a measurable function, such that $f_t \in L^\phi$ for all $t \in \mathbb{R}$. Then, $(t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$ is Borel measurable and has a separable range.*

Recall that $f\colon E \to F$ has a separable range, if $f(E)$ is a separable subset of F.

Proof. We will use a Monotone Class Lemma argument to prove this result, so let $\mathcal{M}_2$ be the set of all functions f for which Lemma 2.2 holds, and $\mathcal{M}_1$ the set of all functions f of the form

$$f_t(s) = \sum_{i=1}^n \alpha_i 1_{A_i}(t) 1_{B_i}(s), \qquad t \in \mathbb{R},\ s \in E, \tag{2.5}$$

where for $n \geq 1$, $A_1, \ldots, A_n$ are measurable subsets of $\mathbb{R}$, $B_1, \ldots, B_n$ are measurable subsets of E of finite μ-measure, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Then, $\Psi_f : (t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$ has separable range, and since $t \mapsto d_\phi(f_t, g)$ is measurable for all $g \in L^\phi$, Ψ_f is measurable. This shows that $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Note that the set $\mathrm{b}\mathcal{M}_2$ of bounded elements from $\mathcal{M}_2$ is a vector space with $1 \in \mathrm{b}\mathcal{M}_2$, and that $(f_n)_{n\geq 1} \subseteq \mathrm{b}\mathcal{M}_2$ with $0 \leq f_n \uparrow f \leq K$ implies that $f \in \mathrm{b}\mathcal{M}_2$. Moreover, since $\mathcal{M}_1$ is stable under pointwise multiplication the Monotone Class Lemma, see e.g. Chapter II, Theorem 3.2 in [31], shows that

$$\mathrm{bM}(\mathcal{B}(\mathbb{R}) \times \mathcal{F}) = \mathrm{bM}(\sigma(\mathcal{M}_1)) \subseteq \mathrm{b}\mathcal{M}_2. \tag{2.6}$$

(For a family of functions $\mathcal{M}$, $\sigma(\mathcal{M})$ denotes the least σ-algebra for which all the functions are measurable, and for each σ-algebra $\mathcal{E}$, $\mathrm{bM}(\mathcal{E})$ denotes the space of all bounded $\mathcal{E}$-measurable functions). For a general function f define $f^{(n)}$ by $f_t^{(n)} = f_t 1_{\{|f_t| \leq n\}}$. For all $n \geq 1$, $f^{(n)}$ is a bounded measurable function and hence $\Psi_{f^{(n)}}$ is a measurable map with a separable range. Moreover, $\lim_n \Psi_{f^{(n)}} = \Psi_f$ pointwise in L^ϕ, showing that Ψ_f is measurable and has a separable range. $\square$

Proof of Theorem 2.1. Let Ψ_f denote the map $(t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$, and for fixed $\epsilon > 0$ and arbitrary $t \in \mathbb{R}$, consider the ball $B_t = \{s \in \mathbb{R} : d_\phi(f_t, f_s) < \epsilon\}$. By Lemma 2.2, Ψ_f is measurable, and hence B_t is a measurable subset of $\mathbb{R}$ for all $t \in \mathbb{R}$. According to Lemma 2.2 Ψ_f has a separable range, and therefore there exists a countable set $(t_n)_{n\geq 1} \subseteq \mathbb{R}$ such that the range of Ψ_f is included in $\cup_{n\geq 1} B(f_{t_n}, \epsilon)$, implying that $\mathbb{R} = \cup_{n\geq 1} B_{t_n}$. (Here, $B(g, r) = \{h \in L^\phi : d_\phi(g, h) < r\}$). In particular, there exists an $n \geq 1$ such that B_{t_n} has strictly positive Lebesgue measure. By the Steinhaus Lemma, see Theorem 1.1.1 in [11], there exists a $\delta > 0$ such that $(-\delta, \delta) \subseteq B_{t_n} - B_{t_n}$. Note that by (2.4) it is enough to show continuity of Ψ_f at $t = 0$. For $|t| < \delta$ there exists, by definition, $s_1, s_2 \in \mathbb{R}$ such that $d_\phi(f_{t_n}, f_{s_i}) < \epsilon$ for $i = 1, 2$, showing that

$$d_\phi(f_t, f_0) \leq d_\phi(f_t, f_{s_1}) + d_\phi(f_t, f_{s_2}) < 2\epsilon, \tag{2.7}$$

which completes the proof of the continuity part.

To show the last part of the theorem assume that ϕ is convex. For each $t > 0$ choose $n = 0, 1, 2, \ldots$ such that $n \leq t < n + 1$. Then,

$$\|f_t - f_0\|_\phi \leq \sum_{i=1}^{n} \|f_i - f_{i-1}\|_\phi + \|f_t - f_n\|_\phi \tag{2.8}$$

$$\leq n\|f_1 - f_0\|_\phi + \|f_{t-n} - f_0\|_\phi \leq t\beta + a, \tag{2.9}$$

where $\beta = \|f_1 - f_0\|_\phi$ and $a = \sup_{s\in[0,1]} \|f_s - f_0\|_\phi$. We have already shown that $t \mapsto f_t$ is continuous, and hence $a < \infty$. Since $\|f_{-t} - f_0\|_\phi = \|f_t - f_0\|_\phi$ for all $t \in \mathbb{R}$, (2.9) shows that $\|f_t - f_0\|_\phi \leq a + \beta|t|$ for all $t \in \mathbb{R}$, implying that $\|f_t\|_\phi \leq \alpha + \beta|t|$ where $\alpha = a + \|f_0\|_\phi$. □

For $(E, \mathcal{E}, \mu) = (\Omega, \mathcal{F}, \mathrm{P})$ and $\phi(t) = |t|^p$ for $p > 0$ or $\phi(t) = |t| \wedge 1$ for $p = 0$, we have the following corollary to Theorem 2.1.

Corollary 2.3. *Let $p \geq 0$ and $X = (X_t)_{t\in\mathbb{R}}$ be a measurable process with stationary increments and finite p-moments. Then, X is continuous in L^p. Moreover if $p \geq 1$, then there exist $\alpha, \beta > 0$ such that $\|X_t\|_p \leq \alpha + \beta|t|$ for all $t \in \mathbb{R}$.*

Note that in Corollary 2.3 the reversed implication is also true; in fact, all stochastic processes $X = (X_t)_{t\in\mathbb{R}}$ that are continuous in L^0 have a measurable modification according to Theorem 2 in [14].

The idea by using the Steinhaus Lemma to prove Theorem 2.1 is borrowed from [35], where Corollary 2.3 is shown for $p = 0$. Furthermore, when μ is a probability measure and $\phi(t) = |t| \wedge 1$, Lemma 2.2 is known from [14].

2.2 Existence and uniqueness of QOU processes

The next result shows existence and uniqueness for the stationary solution X to the Langevin equation $\mathrm{d}X_t = -\lambda X_t \,\mathrm{d}t + \mathrm{d}N_t$, in the case where the the noise N is integrable. That is, we show existence and uniqueness of QOU processes X, and moreover provide an explicit form of the solution which is used to calculate the mean and variance of X.

Theorem 2.4. *Let N be a measurable process with stationary increments and finite first-moments, and let $\lambda > 0$ be a positive real number. Then, $X = (X_t)_{t\in\mathbb{R}}$ given by*

$$X_t = N_t - \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} N_s \,\mathrm{d}s, \qquad t \in \mathbb{R}, \tag{2.10}$$

is a QOU process driven by N with parameter λ (the integral is a pathwise Lebesgue integral). Furthermore, any other QOU process driven by N and with parameter λ equals X in law. Finally, if N has finite p-moments, $p \geq 1$, then X has also finite p-moments and is continuous in L^p.

Remark 2.5. It is an open problem to relax the integrability of N in Theorem 2.4, e.g. is it enough that N has finite log-moments? Recall that when N is a Lévy process, finite log-moments is a necessary and sufficient condition for the existence of the corresponding QOU process.

Proof. Existence: Let $p \geq 1$ and assume that N has finite p-moments. Choose $\alpha, \beta > 0$, according to Corollary 2.3, such that $\|N_t\|_p \leq \alpha + \beta|t|$ for all $t \in \mathbb{R}$. By Jensen's inequality,

$$E\left[\left(\int_{-\infty}^{t} e^{\lambda s}|N_s|\,\mathrm{d}s\right)^p\right] \leq (e^{\lambda t}/\lambda)^{p-1}\int_{-\infty}^{t} e^{\lambda s}\mathrm{E}[|N_s|^p]\,\mathrm{d}s \tag{2.11}$$

$$\leq (e^{\lambda t}/\lambda)^{p-1}\int_{-\infty}^{t} e^{\lambda s}(\alpha+\beta|s|)^p\,\mathrm{d}s < \infty, \tag{2.12}$$

which shows that the integral in (2.10) exists almost surely as a Lebesgue integral and that X_t, given by (2.10), is p-integrable. Using substitution we obtain from (2.10),

$$X_t = \lambda\int_{-\infty}^{0} e^{\lambda u}(N_t - N_{t+u})\,\mathrm{d}u, \qquad t \in \mathbb{R}. \tag{2.13}$$

By Corollary 2.3 N is L^p-continuous and therefore it follows that the right-hand side of (2.13) exists as a limit of Riemann sums in L^p. Hence the stationarity of the increments of N implies that X is stationary. Moreover, using integration by parts on $t \mapsto \int_{-\infty}^{t} e^{\lambda s}N_s(\omega)\,\mathrm{d}s$, we get

$$\int_0^t X_s\,\mathrm{d}s = e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}N_s\,\mathrm{d}s - \int_{-\infty}^{0} e^{\lambda s}N_s\,\mathrm{d}s, \tag{2.14}$$

which shows that X satisfies (2.1), and hence X is a QOU process driven by N with parameter λ.

Since X is a measurable process with stationary increments and finite p-moments, Proposition 2.3 shows that it is continuous in L^p.

To show uniqueness in law, let $\mathcal{L}(V)$ denote law of a random vector V, and by $\lim_k \mathcal{L}(V_k) = \mathcal{L}(V)$ we mean that, $(V_k)_{k\geq 1}$ are random vectors converging in law to V. Let Y be a QOU process driven by N with parameter $\lambda > 0$, that is, Y is a stationary process which satisfies (2.1). For all $t_0 \in \mathbb{R}$ we have with $Z_t = N_t - N_{t_0} + Y_{t_0}$ that

$$Y_t = Z_t - \lambda\int_{t_0}^{t} Y_s\,\mathrm{d}s, \qquad t \geq t_0. \tag{2.15}$$

Solving (2.15) pathwise, it follows that for all $t \geq t_0$,

$$Y_t = Z_t - \lambda e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}Z_s\,\mathrm{d}s \tag{2.16}$$

$$= N_t - \lambda e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}N_s\,\mathrm{d}s + (Y_{t_0} - N_{t_0})e^{-\lambda(t-t_0)}. \tag{2.17}$$

Note that $\lim_{t\to\infty}(Y_{t_0} - N_{t_0})e^{-\lambda(t-t_0)} = 0$ a.s., thus for all $n \geq 1$ and $t_0 < t_1 < \cdots < t_n$, the stationarity of Y implies that

$$\mathcal{L}(Y_{t_1}, \ldots, Y_{t_n}) = \lim_{k\to\infty} \mathcal{L}(Y_{t_1+k}, \ldots, Y_{t_n+k}) \tag{2.18}$$

$$= \lim_{k\to\infty} \mathcal{L}\Big(N_{t_1+k} - \lambda e^{-\lambda(t_1+k)} \int_{t_0}^{t_1+k} e^{\lambda s} N_s \,\mathrm{d}s, \tag{2.19}$$

$$\ldots, N_{t_n+k} - \lambda e^{-\lambda(t_n+k)} \int_{t_0}^{t_n+k} e^{\lambda s} N_s \,\mathrm{d}s\Big). \tag{2.20}$$

This shows that the distribution of Y only depends on N and λ, and completes the proof. □

Proposition 2.1 in [35] and Proposition 2.1 in [23] provide also existence results for stationary solutions to Langevin equations. However, these results do not cover Theorem 2.4. The first result considers only Bochner type integrals and the second result requires, in particular, that the sample paths of N are Riemann integrable.

Let $B = (B_t)_{t\in\mathbb{R}}$ denote an $\mathcal{F}$-Brownian motion indexed by $\mathbb{R}$ and $\sigma = (\sigma_t)_{t\in\mathbb{R}}$ be a predictable process, that is, σ is measurable with respect to

$$\mathscr{P} = \sigma((s,t] \times A : s,t \in \mathbb{R},\ s < t,\ A \in \mathcal{F}_s). \tag{2.21}$$

Assume that for all $u \in \mathbb{R}$, $(\sigma_t, B_t)_{t\in\mathbb{R}}$ has the same finite distributions as $(\sigma_{t+u}, B_{t+u} - B_u)_{t\in\mathbb{R}}$ and that $\sigma_0 \in L^2$. Then N given by

$$N_t = \int_0^t \sigma_s \,\mathrm{d}B_s, \qquad t \in \mathbb{R}, \tag{2.22}$$

is a well-defined continuous process with stationary increments and finite second-moments. (Recall that for $t < 0$, $\int_0^t := -\int_t^0$).

Corollary 2.6. *Let N be given by* (2.22)*. Then, there exists a unique in law QOU process X driven by N with parameter $\lambda > 0$, and X is given by*

$$X_t = \int_{-\infty}^t e^{-\lambda(t-s)} \sigma_s \,\mathrm{d}B_s, \qquad t \in \mathbb{R}. \tag{2.23}$$

Proof. Since N is a measurable process with finite second-moments it follows by Theorem 2.4 that there exists a unique in law QOU process X, and it is given by

$$X_t = N_t - \lambda e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} N_s \,\mathrm{d}s = \lambda \int_{-\infty}^0 e^{\lambda s}\left(N_t - N_{t+s}\right) \mathrm{d}s \tag{2.24}$$

$$= \lambda \int_{-\infty}^0 \left(\int_{\mathbb{R}} 1_{(t+s,t]}(u) e^{\lambda s} \sigma_u \,\mathrm{d}B_u\right) \mathrm{d}s. \tag{2.25}$$

By a minor extension of Theorem 65, Chapter IV in [28] we may switch the order of integration in (2.25) and hence we obtain (2.23). □

Let us conclude this section with formulas for the mean and variance of a QOU process X. In the rest of this section let N be a measurable process with stationary increments and finite first-moments, and let X be a QOU process driven by N with parameter $\lambda > 0$ (which exists by Theorem 2.4). Since X is unique in law it makes sense to consider the mean and variance function of X. Let us assume for simplicity that $N_0 = 0$ a.s. The following proposition gives the mean and variance of X.

Proposition 2.7. *Let N and X be given as above. Then,*

$$\mathrm{E}[X_0] = \frac{\mathrm{E}[N_1]}{\lambda}, \qquad \textit{and} \qquad \mathrm{Var}(X_0) = \frac{\lambda}{2} \int_0^\infty e^{-\lambda s} \mathrm{Var} N(s) \, \mathrm{d}s. \tag{2.26}$$

In the part concerning the variance of X_0, we assume moreover that N has finite second-moments.

Note that Proposition 2.7 shows that the variance of X_0 is $\lambda/2$ times the Laplace transform of $\mathrm{Var}N$. In particular, if $N_t = \mu t + \sigma B_t^H$ where B^H is a fractional Brownian motion (fBm) of index $H \in (0,1)$, then $\mathrm{E}[N_1] = \mu$ and $\mathrm{Var}N(s) = \sigma^2 |s|^{2H}$, and hence by Proposition 2.7 we have that

$$\mathrm{E}[X_0] = \frac{\mu}{\lambda}, \qquad \text{and} \qquad \mathrm{Var}(X_0) = \frac{\sigma^2 \Gamma(1+2H)}{2\lambda^{2H}}. \tag{2.27}$$

For $H = 1/2$, (2.27) is well-known, and in this case $\mathrm{Var}(X_0) = \sigma^2/(2\lambda)$.

Before proving Proposition 2.7 let us note that $\mathrm{E}[N_t] = \mathrm{E}[N_1]t$ for all $t \in \mathbb{R}$. Indeed, this follows by the continuity of $t \mapsto \mathrm{E}[N_t]$ (see Corollary 2.3) and the stationarity of the increments of N.

Proof. Recall that by Corollary 2.3, we have that $\mathrm{E}[|N_t|] \le \alpha + \beta|t|$ for some $\alpha, \beta > 0$. Hence by (2.10) and Fubini's theorem we have that

$$\mathrm{E}[X_0] = \mathrm{E}\left[-\lambda \int_{-\infty}^0 e^{\lambda s} N_s \, \mathrm{d}s\right] = -\lambda \int_{-\infty}^0 e^{\lambda s} \mathrm{E}[N_s] \, \mathrm{d}s \tag{2.28}$$

$$= -\lambda \mathrm{E}[N_1] \int_{-\infty}^0 e^{\lambda s} s \, \mathrm{d}s = \mathrm{E}[N_1]/\lambda, \tag{2.29}$$

where in the third equality we have used that $\mathrm{E}[N_s] = \mathrm{E}[N_1]s$. This shows the part concerning the mean of X_0.

To show the last part assume that N has finite second-moments. By using $\mathrm{E}[X_0] = \mathrm{E}[N_1]/\lambda$, (2.10) shows that with $\tilde{N}_t := N_t - \mathrm{E}[N_1]t$, we have

$$\mathrm{Var}(X_0) = \mathrm{E}[(X_0 - \mathrm{E}[X_0])^2] = \mathrm{E}\left[\left(\lambda \int_{-\infty}^0 e^{\lambda s} \tilde{N}_s \, \mathrm{d}s\right)^2\right]. \tag{2.30}$$

Since $\|\tilde{N}_t\|_2 \le \alpha + \beta|t|$ for some $\alpha, \beta > 0$ by Corollary 2.3, Fubini's theorem shows

$$\mathrm{Var}(X_0) = \lambda^2 \int_{-\infty}^0 \int_{-\infty}^0 \left(e^{\lambda s} e^{\lambda u} \mathrm{E}[\tilde{N}_s \tilde{N}_u]\right) \mathrm{d}s \, \mathrm{d}u, \tag{2.31}$$

and since $\mathrm{E}[\tilde{N}_s \tilde{N}_u] = \frac{1}{2}[\mathrm{Var}N(s) + \mathrm{Var}N(u) - \mathrm{Var}N(s-u)]$ we have

$$\mathrm{Var}(X_0) = \frac{\lambda^2}{2} \int_{-\infty}^0 \int_{-\infty}^0 \left(e^{\lambda s} e^{\lambda u} (\mathrm{Var}N(s) + \mathrm{Var}N(u) - \mathrm{Var}N(s-u))\right) \mathrm{d}s \, \mathrm{d}u \tag{2.32}$$

$$= \lambda \int_{-\infty}^0 e^{\lambda s} \mathrm{Var}N(s) \, \mathrm{d}s - \frac{\lambda^2}{2} \int_{-\infty}^0 e^{\lambda u} \left(\int_{-\infty}^{-u} e^{\lambda(s+u)} \mathrm{Var}N(s) \, \mathrm{d}s\right) \mathrm{d}u. \tag{2.33}$$

Moreover,

$$\frac{\lambda^2}{2}\int_{-\infty}^{0} e^{\lambda u}\left(\int_{-\infty}^{-u} e^{\lambda(s+u)}\mathrm{Var}N(s)\,\mathrm{d}s\right)\mathrm{d}u \tag{2.34}$$

$$= \frac{\lambda^2}{2}\int_{\mathbb{R}} \mathrm{Var}N(s)e^{\lambda s}\left(\int_{-\infty}^{(-s)\wedge 0} e^{2\lambda u}\,\mathrm{d}u\right)\mathrm{d}s \tag{2.35}$$

$$= \frac{\lambda^2}{2}\left(\int_{-\infty}^{0} \mathrm{Var}N(s)e^{\lambda s}\left(\int_{-\infty}^{0} e^{2\lambda u}\,\mathrm{d}u\right)\mathrm{d}s + \int_{0}^{\infty} \mathrm{Var}N(s)e^{\lambda s}\left(\int_{-\infty}^{-s} e^{2\lambda u}\,\mathrm{d}u\right)\mathrm{d}s\right) \tag{2.36}$$

$$= \frac{\lambda}{4}\left(\int_{-\infty}^{0} \mathrm{Var}N(s)e^{\lambda s}\mathrm{d}s + \int_{0}^{\infty} \mathrm{Var}N(s)e^{\lambda s}\left(e^{-2\lambda s}\right)\mathrm{d}s\right) \tag{2.37}$$

$$= \frac{\lambda}{2}\int_{0}^{\infty} e^{-\lambda s}\mathrm{Var}N(s)\,\mathrm{d}s, \tag{2.38}$$

which by (2.33) gives the expression for the variance of X_0. □

2.3 Asymptotic behavior of the autocovariance function

The next result shows that the autocovariance function of a QOU process X driven by N with parameter λ has the same asymptotic behavior at infinity as the second derivative of the variance function of N divided by $2\lambda^2$.

Proposition 2.8. *Let N be a measurable process with stationary increments, $N_0 = 0$ a.s., and finite second-moments, and let X be a QOU process driven by N with parameter $\lambda > 0$.*

(i) Assume there exists a $\beta > 0$ such that $\mathrm{Var}N \in C^3((\beta,\infty);\mathbb{R})$, and for $t \to \infty$ we have that $\mathrm{V}_N''(t) = O(e^{(\lambda/2)t})$, $e^{-\lambda t} = o(\mathrm{V}_N''(t))$ and $\mathrm{V}_N'''(t) = o(\mathrm{V}_N''(t))$. Then, for $t \to \infty$, we have $\mathrm{R}_X(t) \sim (\frac{1}{2\lambda^2})\mathrm{V}_N''(t)$.

(ii) Assume for $t \to 0$ that $t^2 = o(\mathrm{Var}N(t))$, then for $t \to 0$ we have $\bar{\mathrm{R}}_X(t) \sim \frac{1}{2}\mathrm{Var}N(t)$. More generally, let $p \geq 1$ and assume that N has finite p-moments and $t = o(\|N_t\|_p)$ as $t \to 0$. Then, for $t \to 0$, we have $\|X_t - X_0\|_p \sim \|N_t\|_p$.

Note that by Proposition 2.8(ii) the short term asymptotic behavior of $\bar{\mathrm{R}}_X$ is not influence by λ.

Proof. (i): Let $t_0 = \beta + 1$, and let us show that $t \geq t_0$ and for $t \to \infty$,

$$\mathrm{R}_X(t) = \frac{e^{-\lambda t}}{4\lambda}\int_{t_0}^{t} e^{\lambda u}\mathrm{V}_N''(u)\,\mathrm{d}u + \frac{e^{\lambda t}}{4\lambda}\int_{t}^{\infty} e^{-\lambda u}\mathrm{V}_N''(u)\,\mathrm{d}u + O(e^{-\lambda t}). \tag{2.39}$$

If we have shown (2.39), then by using that $e^{-\lambda t} = o(\mathrm{V}_N''(t))$, $\mathrm{V}_N'''(t) = o(\mathrm{V}_N''(t))$ and l'Hôpital's rule, (i) follows.

Similar to the proof of Proposition 2.7 let $\tilde{N}_t = N_t - E[N_1]t$. To show (2.39), recall that by Corollary 2.3 we have $\|\tilde{N}_t\|_2 \leq \alpha + \beta|t|$ for some $\alpha, \beta > 0$. Hence by (2.10) and Fubini's theorem, we find that

$$\mathrm{R}_X(t) = \mathrm{E}[(X_t - \mathrm{E}[X_t])(X_0 - \mathrm{E}[X_0])] = g(t) - \lambda e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}g(s)\,\mathrm{d}s, \tag{2.40}$$

where

$$g(t) = -\lambda \int_{-\infty}^{0} e^{\lambda s} \mathrm{E}[\tilde{N}_s \tilde{N}_t] \, \mathrm{d}s, \qquad t \in \mathbb{R}. \tag{2.41}$$

Since $\mathrm{E}[\tilde{N}_s \tilde{N}_t] = \frac{1}{2}[\mathrm{Var}N(t) + \mathrm{Var}N(s) - \mathrm{Var}N(s-t)]$ we have that

$$g(t) = -\frac{\lambda}{2} \int_{-\infty}^{0} e^{\lambda s} [\mathrm{Var}N(t) + \mathrm{Var}N(s) - \mathrm{Var}N(t-s)] \, \mathrm{d}s \tag{2.42}$$

$$= -\frac{1}{2} \left(\mathrm{Var}N(t) - \lambda e^{\lambda t} \int_{t}^{\infty} e^{-\lambda s} \mathrm{Var}N(s) \, \mathrm{d}s \right) - \frac{\lambda}{2} \int_{-\infty}^{0} e^{\lambda s} \mathrm{Var}N(s) \, \mathrm{d}s. \tag{2.43}$$

From (2.43) it follows that $g \in C^1((\beta, \infty); \mathbb{R})$ and hence, using partial integration on (2.40), we have for $t \geq t_0$,

$$\mathrm{R}_X(t) = e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} g'(s) \, \mathrm{d}s + e^{-\lambda t} \left(e^{\lambda t_0} g(t_0) - \lambda \int_{-\infty}^{t_0} e^{\lambda s} g(s) \, \mathrm{d}s \right). \tag{2.44}$$

Moreover, from (2.43) and for $t \geq t_0$ we find

$$g'(t) = -\frac{1}{2} \left(\mathrm{V}_N'(t) - \lambda^2 e^{\lambda t} \int_{t}^{\infty} e^{-\lambda s} \mathrm{Var}N(s) \, \mathrm{d}s + \lambda \mathrm{Var}N(t) \right). \tag{2.45}$$

For $t \to \infty$ we have, by assumption, that $\mathrm{V}_N''(t) = O(e^{(\lambda/2)t})$, and hence also $\mathrm{V}_N'(t) = O(e^{(\lambda/2)t})$. Thus, from (2.45) and a double use of partial integration we obtain that

$$g'(t) = \frac{e^{\lambda t}}{2} \int_{t}^{\infty} e^{-\lambda s} \mathrm{V}_N''(s) \, \mathrm{d}s, \qquad t \geq t_0. \tag{2.46}$$

Using (2.46), Fubini's theorem and that $\mathrm{V}_N''(t) = O(e^{(\lambda/2)t})$ we have for $t \geq t_0$,

$$e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} g'(s) \, \mathrm{d}s = e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} \left(\frac{e^{\lambda s}}{2} \int_{s}^{\infty} e^{-\lambda u} \mathrm{V}_N''(u) \, \mathrm{d}u \right) \mathrm{d}s \tag{2.47}$$

$$= e^{-\lambda t} \int_{t_0}^{\infty} e^{-\lambda u} \mathrm{V}_N''(u) \left(\int_{t_0}^{t \wedge u} \frac{1}{2} e^{2\lambda s} \, \mathrm{d}s \right) \mathrm{d}u \tag{2.48}$$

$$= e^{-\lambda t} \int_{t_0}^{\infty} e^{-\lambda u} \mathrm{V}_N''(u) \left(\frac{1}{4\lambda} (e^{2\lambda(t \wedge u)} - e^{2\lambda t_0}) \right) \mathrm{d}u \tag{2.49}$$

$$= \frac{e^{-\lambda t}}{4\lambda} \int_{t_0}^{t} e^{\lambda u} \mathrm{V}_N''(u) \, \mathrm{d}u + \frac{e^{\lambda t}}{4\lambda} \int_{t}^{\infty} e^{-\lambda u} \mathrm{V}_N''(u) \, \mathrm{d}u - e^{-\lambda t} \left(\frac{e^{2\lambda t_0}}{4\lambda} \int_{t_0}^{\infty} e^{-\lambda u} \mathrm{V}_N''(u) \, \mathrm{d}u \right). \tag{2.50}$$

Combining this with (2.44) we obtain (2.39), and the proof of (i) is complete.

(ii): Using (2.1) we have for all for $t > 0$ that

$$\|X_t - X_0\|_p \leq \|N_t\|_p + \lambda \int_{0}^{t} \|X_s\|_p \, \mathrm{d}s = \|N_t\|_p + \lambda t \|X_0\|_p. \tag{2.51}$$

On the other hand,

$$\|X_t - X_0\|_p \geq \|N_t\|_p - \lambda \int_{0}^{t} \|X_s\|_p \, \mathrm{d}s = \|N_t\|_p - \lambda t \|X_0\|_p, \tag{2.52}$$

which shows that

$$1 - \lambda \|X_0\|_p \frac{t}{\|N_t\|_p} \leq \frac{\|X_t - X_0\|_p}{\|N_t\|_p} \leq 1 + \lambda \|X_0\|_p \frac{t}{\|N_t\|_p}. \tag{2.53}$$

A similar inequality is available when $t < 0$, and hence for $t \to 0$ we have that $\|X_t - X_0\|_p \sim \|N_t\|_p$ if $\lim_{t \to 0}(t/\|N_t\|_p) = 0$. □

When N is a fBm of index $H \in (0,1)$ then $\mathrm{Var}N(t) = |t|^{2H}$, and hence

$$\mathrm{V}_N''(t) = 2H(2H-1)t^{2H-2}, \qquad t > 0. \tag{2.54}$$

The conditions in Proposition 2.8 are clearly fulfilled and thus we have the following corollary.

Corollary 2.9. *Let N be a fBm of index $H \in (0,1)$, and let X be a QOU process driven by N with parameter $\lambda > 0$. For $H \in (0,1) \setminus \{\frac{1}{2}\}$ and $t \to \infty$, we have $\mathrm{R}_X(t) \sim (H(2H-1)/\lambda^2)t^{2H-2}$. For $H \in (0,1)$ and $t \to 0$, we have $\bar{\mathrm{R}}_X(t) \sim \frac{1}{2}|t|^{2H}$.*

The above result concerning the behavior of R_X for $t \to \infty$ when N is a fBm has been obtained previously, via a different approach, by [13], see their Theorem 2.3.

A square-integrable stationary process $Y = (Y_t)_{t\in\mathbb{R}}$ is said to have *long-range dependence* of order $\alpha \in (0,1)$ if R_Y is regulary varying at ∞ of index $-\alpha$. Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is regulary varying at ∞ of index $\beta \in \mathbb{R}$, if for $t \to \infty$, $f(t) \sim t^\beta l(t)$ where l is slowly varying, which means that for all $a > 0$, $\lim_{t\to\infty} l(at)/l(t) = 1$. Many empirical observations have shown evidence for long-range dependence in various fields, such as finance, telecommunication and hydrology; see e.g. [17]. Let X be a QOU process driven by N, then Proposition 2.8(i) shows that X has long-range dependence of order $\alpha \in (0,1)$ if and only if V_N'' is regulary varying at ∞ of order $-\alpha$.

3 A Fubini theorem for Lévy bases

Let $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ denote a centered Lévy basis on a non-empty space S equipped with a δ-ring $\mathcal{S}$. (A Lévy basis is an infinitely divisible independently scattered random measure. Recall also that a δ-ring on S is a family of subsets of S which is closed under union, countable intersection and set difference). As usual we assume that $\mathcal{S}$ is σ-finite, meaning that there exists $(S_n)_{n\geq1} \subseteq \mathcal{S}$ such that $\cup_{n\geq1} S_n = S$. All integrals $\int_S f(s)\,\Lambda(\mathrm{d}s)$ will be defined in the sense of [29]. We can now find a measurable parameterization of Lévy measures $\nu(du,s)$ on $\mathbb{R}$, a σ-finite measure m on S, and a positive measurable function $\sigma^2 : S \to \mathbb{R}_+$, such that for all $A \in \mathcal{S}$,

$$\mathrm{E}[e^{iy\Lambda(A)}] = \exp\left(\int_A \left[-\sigma^2(s)y^2/2 + \int_{\mathbb{R}} (e^{iyu} - 1 - iyu)\,\nu(\mathrm{d}u,s)\right] m(\mathrm{d}s)\right), \quad y \in \mathbb{R}, \tag{3.1}$$

see [29]. Let $\phi : \mathbb{R} \times S \mapsto \mathbb{R}$ be given by

$$\phi(y,s) = y^2\sigma^2(s) + \int_{\mathbb{R}} [(uy)^2 1_{\{|uy|\leq1\}} + (2|uy|-1)1_{\{|uy|>1\}}]\,\nu(\mathrm{d}u,s), \tag{3.2}$$

and for all measurable functions $g\colon S \to \mathbb{R}$ define

$$\|g\|_\phi = \inf\left\{c > 0 : \int_S \phi(c^{-1}g(s),s)\,m(\mathrm{d}s) \leq 1\right\} \in [0,\infty]. \tag{3.3}$$

Moreover, let $L^\phi = L^\phi(S,\sigma(\mathcal{S}),m)$ denote the *Musielak-Orlicz space* of measurable functions g with

$$\int_S \left[g(s)^2\sigma^2(s) + \int_{\mathbb{R}} \Big(|ug(s)|^2 \wedge |ug(s)|\Big)\,\nu(\mathrm{d}u,s)\right] m(\mathrm{d}s) < \infty, \tag{3.4}$$

equipped with the Luxemburg norm $\|g\|_\phi$. Note that $g \in L^\phi$ if and only if $\|g\|_\phi < \infty$, since $\phi(2x,s) \leq C\phi(x,s)$ for some $C > 0$ and all $s \in S,\ x \in \mathbb{R}$. We refer to [26] for the basic properties of Musielak-Orlicz spaces. When $\sigma^2 \equiv 0$ and $g \in L^\phi$, Theorem 2.1 in [24] shows that $\int_S g(s)\,\Lambda(\mathrm{d}s)$ is well-defined, integrable and centered and

$$c_1\|g\|_\phi \leq E\left[\left|\int_S g(s)\,\Lambda(\mathrm{d}s)\right|\right] \leq c_2\|g\|_\phi, \tag{3.5}$$

and we may choose $c_1 = 1/8$ and $c_2 = 17/8$. Hence for general σ^2 it is easily seen that for all $g \in L^\phi$, $\int_S g(s)\,\Lambda(\mathrm{d}s)$ is well-defined, integrable and centered and

$$E\left[\left|\int_S g(s)\,\Lambda(\mathrm{d}s)\right|\right] \leq 2c_2\|g\|_\phi. \tag{3.6}$$

Let T denote a complete separable metric space, and $Y = (Y_t)_{t\in T}$ be given by

$$Y_t = \int_S f(t,s)\,\Lambda(\mathrm{d}s), \qquad t \in T, \tag{3.7}$$

for some measurable function $f(\cdot,\cdot)$ for which the integrals are well-defined. Then we can and will choose a measurable modification of Y. Indeed, the existence of a measurable modification of Y is equivalent to measurability of $(t \in T) \mapsto (Y_t \in L^0)$ according to Theorem 3 and the Remark in [14]. Hence, since f is measurable, the maps $(t \in T) \mapsto (\|f(t,\cdot) - g(\cdot)\|_\phi \in \mathbb{R})$ for all $g \in L^\phi$, are measurable. This shows that $(t \in \mathbb{R}) \mapsto (f(t,\cdot) \in L^\phi)$ is measurable since L^ϕ is a separable Banach space. Hence by continuity of $(f(t,\cdot) \in L^\phi) \mapsto (Y_t \in L^0)$, see [29], it follows that $(t \in T) \mapsto (Y_t \in L^0)$ is measurable.

Assume that μ is a σ-finite measure on a complete and separable metric space T, then we have the following stochastic Fubini result extending Rosiński [33, Lemma 7.1], Pérez-Abreu and Rocha-Arteaga [27, Lemma 5] and Basse and Pedersen [10, Lemma 4.9]. Stochastic Fubini type results for semimartingales can be founded in [28] and [18], however the assumptions in these results are too strong for our purpose.

Theorem 3.1 (Fubini). *Let $f : T \times S \mapsto \mathbb{R}$ be an $\mathcal{B}(T) \otimes \sigma(\mathcal{S})$-measurable function such that*

$$f_x = f(x,\cdot) \in L^\phi, \text{ for } x \in T, \qquad \text{and} \qquad \int_E \|f_x\|_\phi\,\mu(\mathrm{d}x) < \infty. \tag{3.8}$$

Then $f(\cdot,s) \in L^1(\mu)$ for m-a.a. $s \in S$ and $s \mapsto \int_T f(x,s)\,\mu(\mathrm{d}x)$ belongs to L^ϕ, all of the below integrals exist and

$$\int_T \Big(\int_S f(x,s)\,\Lambda(\mathrm{d}s)\Big)\,\mu(\mathrm{d}x) = \int_S \Big(\int_T f(x,s)\,\mu(\mathrm{d}x)\Big)\,\Lambda(\mathrm{d}s) \qquad a.s. \tag{3.9}$$

Remark 3.2. If μ is a finite measure then the last condition in (3.8) is equivalent to

$$\int_T \Big[\int_S f(x,s)^2\sigma^2(s) + \int_{\mathbb{R}} \Big(|uf(x,s)|^2 \wedge |uf(x,s)|\Big)\,\nu(\mathrm{d}u,s)\Big] m(\mathrm{d}s)\,\mu(\mathrm{d}x) < \infty. \tag{3.10}$$

Before proving Theorem 3.1 we will need the following observation:

Lemma 3.3. *For all measurable functions $f\colon T \times S \to \mathbb{R}$ we have*

$$\left\|\int_T |f(x,\cdot)|\,\mu(\mathrm{d}x)\right\|_\phi \leq \int_T \|f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x). \tag{3.11}$$

Moreover, if $f\colon T \times S \to \mathbb{R}$ is a measurable function such that $\int_T \|f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x) < \infty$, then for m-a.a. $s \in S$, $f(\cdot,s) \in L^1(\mu)$ and $s \mapsto \int_T f(x,s)\,\mu(\mathrm{d}x)$ is a well-defined function which belongs to L^ϕ.

Proof. Let us sketch the proof of (3.11). For f of the form

$$f(x,s) = \sum_{i=1}^{k} g_i(s) 1_{A_i}(x), \tag{3.12}$$

where $k \geq 1$, $g_1, \ldots, g_k \in L^\phi$ and $A_1, \ldots, A_k$ are disjoint measurable subsets of T of finite μ-measure, (3.11) easily follows. Hence by a Monotone Class Lemma argument it is possible to show (3.11) for all measurable f. The second statement is a consequence of (3.11). $\square$

Recall that if $(F, \|\cdot\|)$ is a separable Banach space, μ is a measure on T, and $f\colon T \to F$ is a measurable map such that $\int_T \|f(x)\|\,\mu(\mathrm{d}x) < \infty$, then the Bochner integral $\mathrm{B}\!\int_T f(x)\,\mu(\mathrm{d}x)$ exists in F and $\|\mathrm{B}\!\int_T f(x)\,\mu(\mathrm{d}x)\| \leq \int_T \|f(x)\|\,\mu(\mathrm{d}x)$. Even though $(L^\phi, \|\cdot\|_\phi)$ is a Banach space, this result does not cover Lemma 3.3.

Proof of Theorem 3.1. For f of the form

$$f(x,s) = \sum_{i=1}^{n} \alpha_i 1_{A_i}(x) 1_{B_i}(s), \qquad x \in T,\ s \in S, \tag{3.13}$$

where $n \geq 1$, $A_1, \ldots, A_n$ are measurable subsets of T of finite μ-measure, $B_1, \ldots, B_n \in \mathcal{S}$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, the theorem is trivially true. Thus for a general f, as in the theorem, choose f_n for $n \geq 1$ of the form (3.13) such that $\int_T \|f_n(x,\cdot) - f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x) \to 0$. Indeed, the existence of such a sequence follows by an application of the Monotone Class Lemma. Let

$$X_n = \int_E \Big(\int_S f_n(x,s)\,\Lambda(\mathrm{d}s) \Big)\,\mu(\mathrm{d}x), \qquad X = \int_E \Big(\int_S f(x,s)\,\Lambda(\mathrm{d}s) \Big)\,\mu(\mathrm{d}x), \tag{3.14}$$

and let us show that X is well-defined and $X_n \to X$ in L^1. This follows since

$$E\left[\int_E \left| \int_S f(x,s)\,\Lambda(\mathrm{d}s) \right| \mu(\mathrm{d}x) \right] \leq 2c_2 \int_E \|f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x) < \infty, \tag{3.15}$$

and

$$\mathrm{E}[|X_n - X|] \leq 2c_2 \int_E \|f_n(x,\cdot) - f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x). \tag{3.16}$$

Similarly, let

$$Y_n = \int_S \Big(\int_E f_n(x,s)\,\mu(\mathrm{d}x) \Big)\,\Lambda(\mathrm{d}s), \qquad Y = \int_S \Big(\int_E f(x,s)\,\mu(\mathrm{d}x) \Big)\,\Lambda(\mathrm{d}s), \tag{3.17}$$

and let us show that Y is well-defined and $Y_n \to Y$ in L^1. By Remark 3.3, $s \mapsto \int_E f(x,s)\,\mu(\mathrm{d}x)$ is a well-defined function which belongs to L^ϕ, which shows that Y is well-defined. By (3.6) and (3.11) we have

$$\mathrm{E}[|Y_n - Y|] \leq 2c_2 \int_E \|f_n(x,\cdot) - f(x,\cdot)\|_\phi\,\mu(\mathrm{d}x), \tag{3.18}$$

which shows that $Y_n \to Y$ in L^1. We have therefore proved (3.9), since $Y_n = X_n$ a.s., $X_n \to X$ and $Y_n \to Y$ in L^1. $\square$

Let $Z = (Z_t)_{t\in\mathbb{R}}$ denote an integrable and centered Lévy process with Lévy measure ν and Gaussian component σ^2. Then Z induces a Lévy basis Λ on $S = \mathbb{R}$ and $\mathcal{S} = \mathcal{B}_b(\mathbb{R})$, the bounded Borel sets, which is uniquely determined by $\Lambda((a,b]) = Z_b - Z_a$ for all $a, b \in \mathbb{R}$ with $a < b$. In this case m is the Lebesgue measure on $\mathbb{R}$ and

$$\phi(y,s) = \phi(y) = \sigma^2 + \int_{\mathbb{R}} \left(|uy|^2 1_{\{|uy|\leq 1\}} + (2|uy| - 1)1_{\{|uy|>1\}}\right) \nu(\mathrm{d}u). \tag{3.19}$$

We will often write $\int f(s)\,\mathrm{d}Z_s$ instead of $\int f(s)\,\Lambda(\mathrm{d}s)$. Note that, $\int_{\mathbb{R}} f(s)\,\mathrm{d}Z_s$ exists and is integrable if and only if $f \in L^\phi$, i.e.,

$$\int_{\mathbb{R}} \left(f(s)^2\sigma^2 + \int_{\mathbb{R}} \left(|uf(s)|^2 \wedge |uf(s)|\right) \nu(\mathrm{d}x)\right) \mathrm{d}s < \infty. \tag{3.20}$$

Moreover, if Z is a symmetric α-stable Lévy process, $\alpha \in (0, 2]$, then $L^\phi = L^\alpha(\mathbb{R}, \lambda)$, where $L^\alpha(\mathbb{R}, \lambda)$ is the space of α-integrable functions with respect to the Lebesgue measure λ.

4 Moving average representations

In wide generality, if X is a continuous time stationary processes then it is representable, in principle, as a moving average (MA), i.e.

$$X_t = \int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}\Xi_s \tag{4.1}$$

where ϕ is a deterministic function and Ξ has stationary and orthogonal increments, at least in the second order sense. (For a precise statements, see the beginning of Subsection 4.1 below). However, an explicit expression for ϕ is seldom available.

We show in Subsection 4.2 below that an expression can be found in cases where the process X is the stationary solution to a Langevin equation for which the driving noise process N is a pseudo moving average (PMA), i.e.

$$N_t = \int_{\mathbb{R}} (f(t-s) - f(-s))\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}, \tag{4.2}$$

where $Z = (Z_t)_{t\in\mathbb{R}}$ is a suitable process specified later on and $f\colon \mathbb{R} \to \mathbb{R}$ a deterministic function for which the integrals exist.

In Subsection 4.3, continuing the discussion from Subsection 2.3, we use the MA representation to study the asymptotic behavior of the associated autocovariance functions. Subsection 4.4 comments on a notable cancellation effect. But first, in Subsection 4.1 we summarize known results concerning Wold-Karhunen type representations of stationary continuous time processes.

4.1 Wold-Karhunen type decompositions

Let $X = (X_t)_{t\in\mathbb{R}}$ be a second-order stationary process of mean zero and continuous in quadratic-mean. Let F_X denote the spectral measure of X, i.e., F_X is a finite and symmetric measure on $\mathbb{R}$ satisfying

$$\mathrm{E}[X_t X_u] = \int_{\mathbb{R}} e^{i(t-u)x}\,F_X(\mathrm{d}x), \qquad t, u \in \mathbb{R}, \tag{4.3}$$

and let F_X' denote the density of the absolutely continuous part of F_X. For each $t \in \mathbb{R}$ let $\mathcal{X}_t = \overline{\mathrm{span}}\{X_s : s \leq t\}$, $\mathcal{X}_{-\infty} = \cap_{t\in\mathbb{R}}\mathcal{X}_t$ and $\mathcal{X}_\infty = \overline{\mathrm{span}}\{X_s : s \in \mathbb{R}\}$ ($\overline{\mathrm{span}}$ denotes the L^2-closure of the linear span). Then X is called deterministic if $\mathcal{X}_{-\infty} = \mathcal{X}_\infty$ and purely non-deterministic if $\mathcal{X}_{-\infty} = \{0\}$. The following result, which is due to Satz 5–6 in [19] (cf. also [16]), provides a decomposition of stationary processes as a sum of a deterministic process and a purely non-deterministic process.

Theorem 4.1 (Karhunen). *Let X and F_X be given as above. If*

$$\int_{\mathbb{R}} \frac{|\log F_X'(x)|}{1+x^2}\,\mathrm{d}x < \infty \tag{4.4}$$

then there exists a unique decomposition of X as

$$X_t = \int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}\Xi_s + V_t, \qquad t \in \mathbb{R}, \tag{4.5}$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is a Lebesgue square-integrable deterministic function, and Ξ is a process with second-order stationary and orthogonal increments, $\mathrm{E}[|\Xi_u - \Xi_s|^2] = |u-s|$ and for all $t \in \mathbb{R}$ $\mathcal{X}_t = \overline{\mathrm{span}}\{\Xi_s - \Xi_u : -\infty < u < s \leq t\}$, and V is a deterministic second-order stationary process.

Moreover, if F_X is absolutely continuous and (4.4) is satisfied then $V \equiv 0$ and hence X is a backward moving average. Finally, the integral in (4.4) is infinite if and only if X is deterministic.

The results in [19] are formulated for complex-valued processes, however if X is real-valued (as it is in our case) then one can show that all the above processes and functions are real-valued as well. Note also that if X is Gaussian then the process Ξ in (4.5) is a standard Brownian motion. If σ is a stationary process with $\mathrm{E}[\sigma_0^2] = 1$ and B is a Brownian motion, then $\mathrm{d}\Xi_s = \sigma_s \mathrm{d}B_s$ is of the above type.

A generalization of the classical Wold-Karhunen result to a broad range of non-Gaussian infinitely divisible processes was given in [32].

4.2 Explicit MA solutions of Langevin equations

Assume initially that Z is an integrable and centered Lévy process, and recall that L^ϕ is the space of all measurable functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying (3.20). Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f(t-\cdot) - f(-\cdot) \in L^\phi$ for all $t \in \mathbb{R}$, and let N be given by

$$N_t = \int_{\mathbb{R}} \big(f(t-s) - f(-s)\big)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}. \tag{4.6}$$

Proposition 4.2. *Let N be given as above. Then there exists an unique in law QOU process X driven by N with parameter $\lambda > 0$, and X is a MA of the form*

$$X_t = \int_{\mathbb{R}} \psi_f(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}, \tag{4.7}$$

where $\psi_f\colon \mathbb{R} \to \mathbb{R}$ belongs to L^ϕ, and is given by

$$\psi_f(t) = \left(f(t) - \lambda e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s} f(s)\,\mathrm{d}s\right), \qquad t \in \mathbb{R}. \tag{4.8}$$

Proof. Since $(t,s)\mapsto f(t-s)-f(-s)$ is measurable we may choose a measurable modification of N, see Section 3, and hence, by Theorem 2.4, there exists a unique in law QOU process X driven by N with parameter λ. For fixed $t\in\mathbb{R}$, we have by (2.10) and with $h_u(s)=f(t-s)-f(t+u-s)$ for all $u,s\in\mathbb{R}$ and $\mu(\mathrm{d}u)=1_{\{u\leq 0\}}e^{\lambda u}\,\mathrm{d}u$ that

$$X_t=\lambda\int_{-\infty}^0 e^{\lambda u}(N_t-N_{t+u})\,\mathrm{d}u=\int_{-\infty}^0\left(\int_{\mathbb{R}} h_u(s)\,\mathrm{d}Z_s\right)\mu(\mathrm{d}u). \tag{4.9}$$

By Theorem 2.1 there exist $\alpha,\beta>0$ such that $\|h_u\|_\phi\leq\alpha+\beta|t|$ for all $u\in\mathbb{R}$, implying that $\int_{\mathbb{R}}\|h_u\|_\phi\,\mu(\mathrm{d}u)<\infty$. By Theorem 3.1, $(u\mapsto h_u(s))\in L^1(\mu)$ for Lebesgue almost all $s\in\mathbb{R}$, which implies that $\int_{-\infty}^t|f(u)|e^{\lambda u}\,\mathrm{d}u<\infty$ for all $t>0$, and hence ψ_f, defined in (4.8), is a well-defined function. Moreover by Theorem 3.1, $\psi_f\in L^\phi(\mathbb{R},\lambda)$ and

$$X_t=\int_{\mathbb{R}}\left(\int_{-\infty}^0 h(u,s)\,\mu(\mathrm{d}u)\right)\mathrm{d}Z_s=\int_{\mathbb{R}}\psi_f(t-s)\,\mathrm{d}Z_s,\qquad t\in\mathbb{R}, \tag{4.10}$$

which completes the proof. □

Note that for $f=1_{\mathbb{R}_+}$, we have $N_t=Z_t$ and $\psi_f(t)=e^{-\lambda t}1_{\mathbb{R}_+}(t)$. Thus, in this case we recover the well-known result that the QOU process X driven by Z with parameter $\lambda>0$ is a MA of the form $X_t=\int_{-\infty}^t e^{-\lambda(t-s)}\,\mathrm{d}Z_s$.

Let us use the notation $x_+:=x1_{\{x\geq 0\}}$, and let c_H be given by

$$c_H=\frac{\sqrt{2H\sin(\pi H)\Gamma(2H)}}{\Gamma(H+1/2)}. \tag{4.11}$$

A PMA N of the form (4.2), where Z is an α-stable Lévy process with $\alpha\in(0,2]$ and f is given by $t\mapsto c_H t_+^{H-1/\alpha}$ is called a *linear fractional α-stable motion* of index $H\in(0,1)$; see [34]. Moreover, PMAs with $f(t)=t^\alpha$ for $\alpha\in(0,\frac{1}{2})$ and where Z is a square-integrable and centered Lévy process is called *fractional Lévy processes* in [25].

A QOU process driven by a linear fractional α-stable motion is called a fractional Ornstein-Uhlenbeck process. For previous work on such processes see [23], where $\alpha\in(1,2)$, and [13], where $\alpha=2$.

Corollary 4.3. *Let $\alpha\in(1,2]$ and N be a linear fractional α-stable motion of index $H\in(0,1)$. Then there exists a unique in law QOU process X driven by N with parameter $\lambda>0$, and X is a MA of the form*

$$X_t=\int_{-\infty}^t\psi_{\alpha,H}(t-s)\,\mathrm{d}Z_s,\qquad t\in\mathbb{R}, \tag{4.12}$$

where $\psi_{\alpha,H}:\mathbb{R}_+\to\mathbb{R}$ is given by

$$\psi_{\alpha,H}(t)=c_H\left(t^{H-1/\alpha}-\lambda e^{-\lambda t}\int_0^t e^{\lambda u}u^{H-1/\alpha}\,\mathrm{d}u\right),\qquad t\geq 0. \tag{4.13}$$

For $t\to\infty$, we have $\psi_{\alpha,H}(t)\sim(c_H(H-1/\alpha)/\lambda)t^{H-1/\alpha-1}$, and for $t\to 0$, $\psi_{\alpha,H}(t)\sim c_H t^{H-1/\alpha}$.

Remark 4.4. For $H\in(0,1/\alpha)$ the existence of the stationary solution to the Langevin equation is somewhat unexpected due to the fact that the sample paths of the linear fractional α-stable motion are unbounded on each compact interval, cf. page 4 in [23] where nonexistence is surmised.

In the next lemma we will show a special property of ψ_f, given by (4.8); namely that $\int_0^\infty \psi_f(s)\,\mathrm{d}s = 0$ whenever f tends to zero fast enough. This property has a great impact on the behavior of the autocovariance function of QOU processes. We will return to this point in Section 4.4.

Lemma 4.5. *Let $\alpha \in (-\infty, 0)$, $c \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable function which is zero on $(-\infty, 0)$ and satisfies that $f(t) \sim ct^\alpha$ for $t \to \infty$. Then, $\int_0^\infty \psi_f(s)\,\mathrm{d}s = 0$.*

Proof. For $t > 0$,

$$\int_0^t \left(\lambda e^{-\lambda s} \int_0^s e^{\lambda u} f(u)\,\mathrm{d}u\right) \mathrm{d}s \tag{4.14}$$

$$= \int_0^t \left(\int_u^t \lambda e^{-\lambda s}\,\mathrm{d}s\right) e^{\lambda u} f(u)\,\mathrm{d}u = \int_0^t f(u)\,du - e^{-\lambda t}\int_0^t e^{\lambda u} f(u)\,\mathrm{d}u, \tag{4.15}$$

and hence by l'Hôpital's rule we have that

$$\int_0^\infty \psi_f(s)\,\mathrm{d}s = \lim_{t\to\infty}\int_0^t \psi_f(s)\,\mathrm{d}s = \lim_{t\to\infty}\left(e^{-\lambda t}\int_0^t e^{\lambda u} f(u)\,\mathrm{d}u\right) = 0. \tag{4.16}$$

$\square$

Proposition 4.2 carries over to a much more general setting. E.g. if N is of the form

$$N_t = \int_{\mathbb{R}\times V} \left[f(t-s,x) - f(-s,x)\right] \Lambda\left(\mathrm{d}s, \mathrm{d}x\right), \qquad t \in \mathbb{R}, \tag{4.17}$$

where Λ is a centered Lévy basis on $\mathbb{R}\times V$ (V is a non-empty space) with control measure $m(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s\, n(\mathrm{d}x)$ and $a(s,x), \sigma^2(s,x)$ and $\nu(\mathrm{d}u, (s,x))$, from (3.1), do not depend on $s \in \mathbb{R}$, and $f(t-\cdot,\cdot) - f(-\cdot,\cdot) \in L^\phi$ for all $t \in \mathbb{R}$, then using Theorem 2.1, 2.4 and 3.1 the arguments from Proposition 4.2 show that there exists a unique in law QOU process X driven by N with parameter $\lambda > 0$, and X is given by

$$X_t = \int_{\mathbb{R}\times V} \psi_f(t-s,x)\,\Lambda(\mathrm{d}s,\mathrm{d}x), \qquad t \in \mathbb{R}, \tag{4.18}$$

where

$$\psi_f(s,x) = f(s,x) - \lambda e^{-\lambda s}\int_{-\infty}^s f(u,x) e^{\lambda u}\,\mathrm{d}u, \qquad s \in \mathbb{R},\ x \in V. \tag{4.19}$$

We recover Proposition 4.2 when $V = \{0\}$ and $n = \delta_0$ is the Dirac delta measure at 0.

4.3 Asymptotic behavior of the autocovariance function

The representation, from the previous section, of QOU processes as moving averages enables us to calculate the autocovariance function in case it exists. In Section 4.3.1 we calculate the autocovariance function for general MAs. By use of these results Section 4.3.2 relates the asymptotic behavior of the kernel of the noise N to the asymptotic behavior of the autocovariance function of the QOU process X driven by N.

4.3.1 Autocovariance function of general MAs

Let ψ be a Lebesgue square-integrable function and Z a centered process with stationary and orthogonal increments, and assume for simplicity that $Z_0 = 0$ a.s. and $\mathrm{Var} Z(t) = t$. Let $X = \psi * Z = (\int_{-\infty}^t \psi(t-s)\,\mathrm{d}Z_s)_{t\in\mathbb{R}}$ be a backward moving average and R_X its autocovariance function, i.e.

$$\mathrm{R}_X(t) = \mathrm{E}[X_t X_0] = \int_0^\infty \psi(t+s)\psi(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}, \tag{4.20}$$

and let $\bar{\mathrm{R}}_X(t) = \mathrm{R}_X(0) - \mathrm{R}_X(t) = \frac{1}{2}\mathrm{E}[(X_t - X_0)^2]$. The behavior of R_X at 0 or ∞ corresponds in large extent to the behavior of the kernel ψ at 0 or ∞, respectively.

Indeed, we have the following result, in which k_α and j_α are constants given by

$$k_\alpha = \Gamma(1+\alpha)\Gamma(-1-2\alpha)\Gamma(-\alpha)^{-1}, \qquad \alpha \in (-1,-1/2), \tag{4.21}$$

$$j_\alpha = (2\alpha+1)\sin(\pi(\alpha+1/2))\Gamma(2\alpha+1)\Gamma(\alpha+1)^{-2}, \qquad \alpha \in (-1/2,1/2). \tag{4.22}$$

Proposition 4.6. *Let the setting be as described above.*

(i) For $t \to \infty$ and $\alpha \in (-1,-\frac{1}{2})$, $\psi(t) \sim ct^\alpha$ implies $\mathrm{R}_X(t) \sim (c^2 k_\alpha)t^{2\alpha+1}$ provided $|\psi(t)| \le c_1 t^\alpha$ for all $t > 0$ and some $c_1 > 0$.

(ii) For $t \to \infty$ and $\alpha \in (-\infty,-1)$, $\psi(t) \sim ct^\alpha$ implies $\mathrm{R}_X(t)/t^\alpha \to c\int_0^\infty \psi(s)\,\mathrm{d}s$, and hence $\mathrm{R}_X(t) \sim (c\int_0^\infty \psi(s)\,\mathrm{d}s)t^\alpha$ provided $\int_0^\infty \psi(s)\,\mathrm{d}s \neq 0$.

(iii) For $t \to 0$ and $\alpha \in (-\frac{1}{2},\frac{1}{2})$, $\psi(t) \sim ct^\alpha$ implies $\bar{\mathrm{R}}_X(t) \sim (c^2 j_\alpha/2)|t|^{2\alpha+1}$ provided ψ is absolutely continuous on $(0,\infty)$ with density ψ' satisfying $|\psi'(t)| \le c_2 t^{\alpha-1}$ for all $t > 0$ and some $c_2 > 0$.

Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be absolutely continuous on $(0,\infty)$ if there exists a locally integrable function f' such that for all $0 < u < t$

$$f(t) - f(u) = \int_u^t f'(s)\,\mathrm{d}s. \tag{4.23}$$

Proof. (i): Let $\alpha \in (-1,-\frac{1}{2})$ and assume that $\psi(t) \sim ct^\alpha$ as $t \to \infty$ and $|\psi(t)| \le c_1 t^\alpha$ for $t > 0$, then

$$\mathrm{R}_X(t) = \int_0^\infty \psi(t+s)\psi(s)\,\mathrm{d}s = t\int_0^\infty \psi(t(s+1))\psi(ts)\,\mathrm{d}s \tag{4.24}$$

$$= t^{2\alpha+1}\int_0^\infty \frac{\psi(t(1+s))\psi(ts)}{(t(1+s))^\alpha (ts)^\alpha}(1+s)^\alpha s^\alpha\,\mathrm{d}s \tag{4.25}$$

$$\sim t^{2\alpha+1}c^2\int_0^\infty (1+s)^\alpha s^\alpha\,\mathrm{d}s \qquad \text{as } t \to \infty. \tag{4.26}$$

Since

$$\int_0^\infty (1+s)^\alpha s^\alpha\,\mathrm{d}s = \frac{\Gamma(1+\alpha)\Gamma(-1-2\alpha)}{\Gamma(-\alpha)} = k_\alpha, \tag{4.27}$$

(4.26) shows that $\mathrm{R}_X(t) \sim (c^2 k_\alpha)t^{2\alpha+1}$ for $t \to \infty$.

(ii): Let $\alpha \in (-\infty,-1)$ and assume that $\psi(t) \sim ct^\alpha$ for $t \to \infty$. Note that $\psi \in L^1(\mathbb{R}_+,\lambda)$ and for some $K > 0$ we have for all $t \ge K$ and $s > 0$ that $|\psi(t+s)|/t^\alpha \le$

$2|c|(t+s)^\alpha/t^\alpha \le 2|c|$. Hence by applying Lebesgue's dominated convergence theorem we obtain,

$$\mathrm{R}_X(t) = t^\alpha \int_0^\infty \left(\frac{\psi(t+s)}{t^\alpha}\psi(s)\right) \mathrm{d}s \sim t^\alpha c \int_0^\infty \psi(s)\,\mathrm{d}s \qquad \text{for } t \to \infty. \tag{4.28}$$

(iii): By letting

$$f_t(s) := \frac{\psi(t(s+1)) - \psi(ts)}{t^\alpha} \qquad t > 0,\ s \in \mathbb{R}, \tag{4.29}$$

we have

$$\mathrm{E}[(X_t - X_0)^2] = t\int [(\psi(t(s+1)) - \psi(ts)]^2\,\mathrm{d}s = t^{2\alpha+1}\int |f_t(s)|^2\,\mathrm{d}s. \tag{4.30}$$

As $t \to 0$, we find

$$f_t(s) = \frac{\psi(t(s+1))}{(t(s+1))^\alpha}(s+1)^\alpha - \frac{\psi(ts)}{(ts)^\alpha}s^\alpha \to c((s+1)_+^\alpha - s_+^\alpha). \tag{4.31}$$

Choose $\delta > 0$ such that $|\psi(x)| \le 2x^\alpha$ for $x \in (0,\delta)$. By our assumptions we have for all $s \ge \delta$ that

$$|f_t(s)| = t^{-\alpha}\left|\int_{ts}^{t(1+s)} \psi'(u)\,\mathrm{d}u\right| \le t^{-\alpha+1} \sup_{u\in[st,t(s+1)]} |\psi'(u)| \tag{4.32}$$

$$\le c_2 t^{-\alpha+1} \sup_{u\in[st,t(s+1)]} |u|^{\alpha-1} = c_2 t^{-\alpha+1}|ts|^{\alpha-1} = c_2 s^{\alpha-1}, \tag{4.33}$$

and for $s \in [-1,\delta)$, $|f_t(s)| \le 2c[(1+s)^\alpha + s_+^\alpha]$. This shows that there exists a function $g \in L^2(\mathbb{R}_+,\lambda)$ such that $|f_t| \le g$ for all $t > 0$, and thus, by Lebesgue's dominated converging theorem, we have

$$\int |f_t(s)|^2\,\mathrm{d}s \xrightarrow[t\to 0]{} c^2 \int \left((s+1)_+^\alpha - s_+^\alpha\right)^2\,\mathrm{d}s = c^2 j_\alpha. \tag{4.34}$$

Together with (4.30), (4.34) shows that $\bar{\mathrm{R}}_X(t) \sim (c^2 j_\alpha/2)t^{2\alpha+1}$ for $t \to 0$. □

Remark 4.7. It would be of interest to obtain a general result covering Proposition 4.6(ii) in the case $\int_0^\infty \psi(s)\,\mathrm{d}s = 0$. Recall that for ψ_f given by (4.8) we often have that $\int_0^\infty \psi_f(s)\,\mathrm{d}s = 0$, according to Lemma 4.5.

Example 4.8. Consider the case where $\psi(t) = t^\alpha e^{-\lambda t}$ for $\alpha \in (-\frac{1}{2},\infty)$ and $\lambda > 0$. For $t \to 0$, $\psi(t) \sim t^\alpha$, and hence $\bar{\mathrm{R}}_X(t) \sim (j_\alpha/2)t^{2\alpha+1}$ for $t \to 0$ and $\alpha \in (-\frac{1}{2},\frac{1}{2})$, by Proposition 4.6(iii) (compare with [3]).

Note that if $X = \psi * Z$ is a moving average, as above, then by Proposition 4.6(i) and for $t \to \infty$, $\mathrm{R}_X(t) \sim c_1 t^{-\alpha}$ with $\alpha \in (0,1)$, provided that $\psi(t) \sim c_2 t^{-(\alpha+1)/2}$ and $|\psi(t)| \le c_3 t^{-(\alpha+1)/2}$. This shows that X has long-range dependence of order α.

Let us conclude this subsection with a short discussion of when a MA $X = \psi * Z$ is a semimartingale. It is often very important that the process of interest is a semimartingale, especially in finance, where the semimartingale property the asset price is equivalent to that the capital process depends continuously on the chosen strategy, see e.g. Section 8.1.1 in [15]. In the case where Z is a Brownian motion, Theorem 6.5 in [21] shows that X is an $\mathcal{F}^Z$-semimartingale if and only if ψ is absolutely continuous on $[0,\infty)$ with a square-integrable density. (Here $\mathcal{F}_t^Z := \sigma(Z_s : s \in (-\infty,t]))$. For a further study to the semimartingale property of pseudo moving averages and more general processes see [7, 8, 9] in the Gaussian case, and [10] for the infinitely divisible case.

4.3.2 QOU processes with PMA noise

Let us return to the case of a QOU process driven by a PMA, so let Z be a centered Lévy process, $f\colon \mathbb{R} \to \mathbb{R}$ be a measurable function which is 0 on $(-\infty, 0)$ and satisfies that $f(t - \cdot) - f(-\cdot) \in L^\phi$ for all $t \in \mathbb{R}$, and let N be given by

$$N_t = \int_{\mathbb{R}} [f(t-s) - f(-s)]\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}. \tag{4.35}$$

First we will consider the relationship between the behavior of the kernel of the noise N and that of the kernel ψ_f of the corresponding moving average X.

Proposition 4.9. *Let N be given by* (4.35)*, and X be a QOU process driven by N with parameter $\lambda > 0$.*

(i) Let $\alpha \in (-1, -\frac{1}{2})$ and assume that for some $\beta \geq 0$ and $c \neq 0$, $f \in C^1((\beta, \infty); \mathbb{R})$ with $f'(t) \sim ct^\alpha$ for $t \to \infty$. Then, for $t \to \infty$, we have $\mathrm{R}_X(t) \sim (\frac{c^2 k_\alpha}{\lambda^2}) t^{2\alpha+1}$, provided $|f(t)| \leq rt^\alpha$ for all $t > 0$ and some $r > 0$.

(ii) Let $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ and $f(t) \sim ct^\alpha$ for $t \to 0$. Then, for $t \to 0$, we have $\bar{\mathrm{R}}_X(t) \sim (c^2 j_\alpha/2)|t|^{2\alpha+1}$, provided there exists a $\beta \geq 0$ such that $f \in C^2((\beta, \infty); \mathbb{R})$ with $f''(t) = O(t^{\alpha-1})$ for $t \to \infty$, and that f is absolutely continuous on $(0, \infty)$ with density f' satisfying $\sup_{t \in (0,t_o)} |f'(t)| t^{1-\alpha} < \infty$ for all $t_0 > 0$.

Proof. (i): By partial integration, we have for $t \geq \beta$,

$$\psi_f(t) = e^{-\lambda t}\left(e^{\lambda a} f(a) - \lambda \int_{-\infty}^a e^{\lambda s} f(s)\,\mathrm{d}s\right) + e^{-\lambda t} \int_a^t e^{\lambda s} f'(s)\,\mathrm{d}s, \tag{4.36}$$

showing that $\psi_f(t) \sim (\frac{c}{\lambda}) t^\alpha$ for $t \to \infty$. Choose $k > 0$ such that $|\psi_f(t)| \leq (2c/\lambda) t^\alpha$ for all $t \geq k$. By (4.8) we have that $\sup_{t \in [0,k]} |\psi_f(t) t^{-\alpha}| < \infty$ since $\sup_{t \in [0,k]} |f(t) t^{-\alpha}| < \infty$, and hence there exists a constant $c_1 > 0$ such that $|\psi_f(t)| \leq c_1 t^\alpha$ for all $t > 0$. Therefore, (i) follows by Proposition 4.6(i).

(ii): Since $f \in C^2((\beta, \infty); \mathbb{R})$, it follows by (4.36) and partial integration that for $t > \beta$ and $t \to \infty$,

$$\psi_f'(t) = f'(t) - \lambda \psi_f(t) = f'(t) - \lambda e^{-\lambda t} \int_\beta^t e^{\lambda s} f'(s)\,\mathrm{d}s + O(e^{-\lambda t}) \tag{4.37}$$

$$= e^{-\lambda t} \int_\beta^t e^{\lambda s} f''(s)\,\mathrm{d}s + O(e^{-\lambda t}) = O(t^{\alpha-1}), \tag{4.38}$$

where we in the last equality have used that $f''(t) = O(t^{\alpha-1})$ for $t \to \infty$. Using that $|\psi_f'(t)| \leq |f'(t)| + \lambda |\psi_f(t)|$ and $\sup_{t \in (0,t_0)} |f'(t) t^{1-\alpha}| < \infty$ for all $t_0 > 0$, it follows that there exists a $c_2 > 0$ such that $|\psi_f'(t)| \leq c_1 t^{\alpha-1}$ for all $t > 0$. Moreover, for $t \to 0$, we have that $\psi_f(t) \sim ct^\alpha$. Hence, (ii) follows by Proposition 4.6(iii). □

Now consider the following set-up: Let $Z = (Z_t)_{t \in \mathbb{R}}$ be a centered and square-integrable Lévy process, and for $H \in (0,1)$, $r_0 \neq 0$, $\delta \geq 0$, let

$$f(t) = r_0 (\delta \vee t)^{H-1/2}, \qquad \text{and} \qquad N_t^{H,\delta} = \int_{\mathbb{R}} [f(t-s) - f(-s)]\,\mathrm{d}Z_s. \tag{4.39}$$

Note that when $\delta = 0$ and Z is a Brownian motion then $N^{H,\delta}$ is a constant times the fBm of index H, and when $\delta > 0$ then $N^{H,\delta}$ is a semimartingale. We have the following corollary to Proposition 4.9:

Corollary 4.10. *Let $N^{H,\delta}$ be given by (4.39), and let $X^{H,\delta}$ be a QOU process driven by $N^{H,\delta}$ with parameter $\lambda > 0$. Then, for $H \in (\frac{1}{2}, 1)$ and $t \to \infty$,*

$$\mathrm{R}_{X^{H,\delta}}(t) \sim (r_0^2 k_{H-3/2}(H-1/2)/\lambda^2)t^{2H-2}, \qquad \delta \geq 0, \tag{4.40}$$

and for $H \in (0,1)$ and $t \to 0$,

$$\bar{\mathrm{R}}_{X^{H,\delta}}(t) \sim \begin{cases} (r_0^2 \delta^{2-1}/2)|t|, & \delta > 0, \\ (r_0^2 j_{H-1/2}/2)|t|^{2H}, & \delta = 0. \end{cases} \tag{4.41}$$

Proof. For $H \in (\frac{1}{2}, 1)$, let $\beta = \delta$. Then, $f \in C^1((\beta,\infty);\mathbb{R})$ and for $t > \beta$, $f'(t) = ct^\alpha$ where $\alpha = H - 3/2 \in (-1, -\frac{1}{2})$ and $c = r(H-1/2)$. Moreover, $|f(t)| \leq r\delta t^\alpha$. Thus, Proposition 4.9(i) shows that

$$\mathrm{R}_{X^{H,\delta}}(t) \sim (c^2 k_\alpha/\lambda^2)t^{2\alpha+1} = (r^2(H-1/2)^2 k_{H-3/2}/\lambda^2)t^{2H-2}. \tag{4.42}$$

To show (4.41) assume that $H \in (0,1)$. For $t \to 0$, we have $f(t) \sim ct^\alpha$, where $c = r_0$ and $\alpha = H - 1/2 \in (-\frac{1}{2}, \frac{1}{2})$ when $\delta = 0$, and $c = r_0\delta^{H-1/2}$ and $\alpha = 0$ when $\delta > 0$. For $\beta = \delta$, $f \in C^2((\beta,\infty);\mathbb{R})$ with $f''(t) = r_0(H-1/2)(H-3/2)t^{H-5/2}$, showing that $f''(t) = O(t^{\alpha-1})$ for $t \to \infty$ (both for $\delta > 0$ and $\delta = 0$). Moreover, f is absolutely continuous on $(0,\infty)$ with density $f'(t) = r_0(H-1/2)t^{H-3/2}1_{[\delta,\infty)}(t)$. This shows that $\sup_{t\in(0,t_0)}|f'(t)t^{1-\alpha}| < \infty$ for all $t_0 > 0$ (both for $\delta > 0$ and $\delta = 0$). Hence (4.41) follows by Proposition 4.9(ii). □

4.4 Stability of the autocovariance function

Let N be a PMA of the form (4.2), where Z is a centered square-integrable Lévy process and $f(t) = c_H t_+^{H-1/2}$ where $H \in (0,1)$. (Recall that if Z is a Brownian motion, then N is a fBm of index H). Let X be a QOU process driven by N with parameter $\lambda > 0$, and recall that by Proposition 4.2, X is a MA of the form

$$X_t = \int_{-\infty}^t \psi_H(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}, \tag{4.43}$$

where

$$\psi_H(t) = c_H\left(t^{H-2/2} - \lambda e^{-\lambda t}\int_0^t e^{\lambda u}u^{H-1/2}\,\mathrm{d}u\right), \qquad t \geq 0. \tag{4.44}$$

Below we will discus some stability properties for the autocovariance function under minor modification of the kernel function.

For all bounded measurable functions $f\colon \mathbb{R}_+ \to \mathbb{R}$ with compact support let $X_t^f = \int_{-\infty}^t (\psi_H(t-s) - f(t-s))\,\mathrm{d}Z_s$. We will think of X^f as a MA where we have made a minor change of X's kernel. Note that if we let $Y_t^f = X_t - X_t^f = \int_{-\infty}^t f(t-s)\,\mathrm{d}Z_s$, then the autocovariance function $\mathrm{R}_{Y^f}(t)$, of Y^f, is zero whenever t is large enough, due to the fact that f has compact support.

Corollary 4.11. *We have the following two situations, in which $c_1, c_2, c_3 \neq 0$ are non-zero constants.*

(i) For $H \in (0, \frac{1}{2})$ and $\int_0^\infty f(s)\,\mathrm{d}s \neq 0$, we have for $t \to \infty$,

$$\mathrm{R}_{X^f}(t) \sim c_2\mathrm{R}_X(t)t^{1/2-H} \sim c_1 t^{H-3/2}. \tag{4.45}$$

(ii) For $H \in (\frac{1}{2}, 1)$, we have for $t \to \infty$,

$$\mathrm{R}_{X^f}(t) \sim \mathrm{R}_X(t) \sim c_3 t^{2H-2}. \tag{4.46}$$

Thus for $H \in (0, \frac{1}{2})$, the above shows that the behavior of the autocovariance function at infinity is changed dramatically by making a minor change of the kernel. In particular, if f is a positive function, not the zero function, then $\mathrm{R}_{X^f}(t)$ behaves as $t^{1/2-H}\mathrm{R}_X(t)$ at infinity. On the other hand, when $H \in (\frac{1}{2}, 1)$ the behavior of the autocovariance function at infinity doesn't change if we make a minor change to the kernel. That is, in this case the autocovariance functions has a stability property, contrary to the case where $H \in (0, \frac{1}{2})$.

Remark 4.12. Note that the dramatic effect appearing from Corollary 4.11(i) is associated to the fact that $\int_0^\infty \psi_H(s)\,\mathrm{d}s = 0$, as shown in Lemma 4.5.

Proof of Corollary 4.11. By Corollary 4.3 we have for $t \to \infty$ that $\psi_H(t) \sim ct^\alpha$ where $c = c_H(H-1/2)/\lambda$ and $\alpha = H - 3/2$. To show (i) assume that $H \in (0, \frac{1}{2})$, and hence $\alpha \in (-\infty, -1)$. According to Lemma 4.5 we have that $\int_0^\infty \psi_H(s)\,\mathrm{d}s = 0$ and hence $\int_0^\infty [\psi_H(s) - f(s)]\,\mathrm{d}s \neq 0$ since $\int_0^\infty f(s)\,\mathrm{d}s \neq 0$ by assumption. From Proposition 4.6(ii) and for $t \to \infty$ we have that $\mathrm{R}_{X^f}(t)(t) \sim c_1 t^{2\alpha+1} = c_1 t^{H-3/2}$, where $c_1 = c\int_0^\infty [\psi_H(s) - f(s)]\,\mathrm{d}s$. On the other hand, by Corollary 2.9 we have that $\mathrm{R}_X(t) \sim (H(H-1/2)/\lambda^2)t^{2H-2}$ for $t \to \infty$, and hence we have shown (i) with $c_2 = c_1\lambda^2/(H(H-1/2)$. For $H \in (\frac{1}{2}, 1)$ we have that $\alpha \in (-1, -\frac{1}{2})$, and hence (ii) follows by Proposition 4.6(i). □

5 Conclusion

In recent applications of stochastics, particularly in finance and in turbulence, modifications of classic noise processes by time change or by volatility modifications are of central importance, see for instance [6] and [2] and references given there. Prominent examples of such processes are $\mathrm{d}N_t = \sigma_t\,\mathrm{d}B_t$ where B is Brownian motion and σ is a predictable stationary process (cf. [5]), and $N_t = L_{T_t}$, where L is a Lévy process and T is a time change process with stationary increments (cf. [12]). The theory discussed in the present paper applies to processes of this type (cf. Corollary 2.6). In the applications mentioned the processes are mostly semimartingales. However there is a growing interest in non-semimartingale processes, see [1], [3, 4], and the results above covers also such cases. An example in point is $N_t = \int_{\mathcal{X}} B_t^{(x)}\,m(\mathrm{d}x)$ where the processes $B_\cdot^{(x)}$ are Brownian motions in different filtrations and m is a measure on some space $\mathcal{X}$.

Moreover, extensions of the theory to wider settings would be of interest, for instance to generalized Langevin equations

$$X_t = X_0 - \lambda \int_0^t (X * k)(s)\,\mathrm{d}s + N_t \tag{5.1}$$

where k is a deterministic function and $(X * k)(s) = \int_{-\infty}^s X_u k(s-u)\,\mathrm{d}u$ denotes the convolution between k and X, as occurring in statistical mechanics and biophysics, see [22] and references given there. We hope to discuss this in future work.

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