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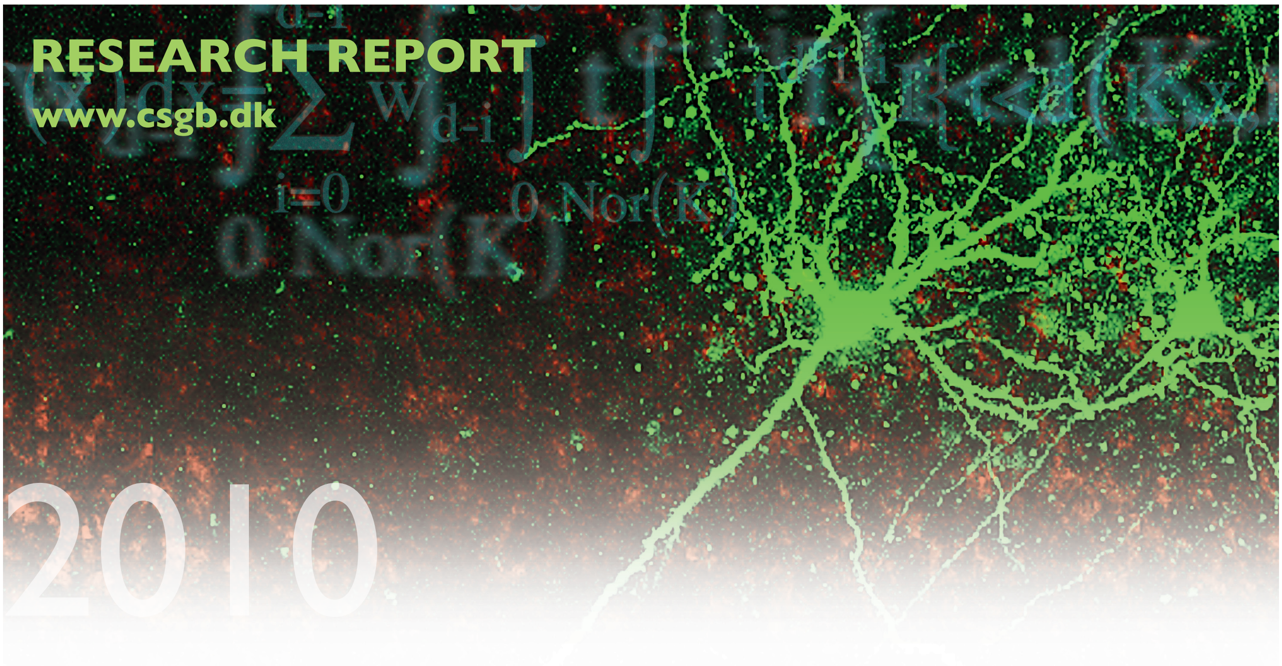

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A Rotational Crofton Formula for Flagged Intrinsic Volumes of Sets of Positive Reach

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Abstract

A rotational Crofton formula is derived relating the *flagged* intrinsic volumes of a compact set of positive reach with the flagged intrinsic volumes measured on sections passing through a fixed point. In particular cases, the flagged intrinsic volumes defined in the present paper are identical to the classical intrinsic volumes. The tight connection between our main result and other recent rotational integral formulae involving intrinsic volumes is pointed out.

Keywords: Crofton formula; Geometric measure theory; Grassmann manifold; Integral geometry; Intrinsic volume; Rotational integral; Set with positive reach; Unit normal bundle

1 Introduction

In *classical* stereology, the well-known Crofton formula relates the intrinsic volumes of a compact subset X of $\mathbb{R}^d$ with the intrinsic volumes of its affine sections

$$c_{d,r,k} V_{d-r+k}(X) = \int_{\mathcal{F}_r^d} V_k(X \cap F_r) \, \mathrm{d}F_r^d, \tag{1}$$

$r = 0, \dots, d$, $k = 0 \dots, r$. Here, $\mathcal{F}_r^d$ is the set of r-dimensional affine subspaces in $\mathbb{R}^d$ and $\mathrm{d}F_r^d$ is the element of its motion invariant measure. The kth intrinsic volume of X is denoted by $V_k(X)$, $k = 0, \dots, d$. Finally, $c_{d,r,k}$ is a known constant. In *local* stereology, the focus of interest is instead on integral geometric relations of the type

$$\beta(X) = \int_{\mathcal{L}_r^d} \alpha(X \cap L_r) \, \mathrm{d}L_r^d, \tag{2}$$

where α and β are functionals, $\mathcal{L}_r^d$ denotes the set of r-dimensional subspaces in $\mathbb{R}^d$ and $\mathrm{d}L_r^d$ is the element of its rotation invariant measure. In the special case where

α is an intrinsic volume of a compact set X of positive reach, i.e. for relations of the type

$$\beta(X) = \int_{\mathcal{L}_r^d} V_k(X \cap L_r)\, \mathrm{d}L_r^d, \tag{3}$$

Jensen and Rataj proved in [8] that β can be expressed as a certain integral over the unit normal bundle of X. In the same paper, the problem was raised of finding functionals α satisfying the integral equation (2) in the particular case where β is an intrinsic volume of X,

$$V_{d-k}(X) = \int_{\mathcal{L}_r^d} \alpha(X \cap L_r)\, \mathrm{d}L_r^d. \tag{4}$$

Recently, a solution to this problem was given independently in [3] and [6]. It was shown, for all $0 \le k < r \le d$ and for any compact set Y of positive reach contained in L_r, that the functional α given by

$$\alpha(Y) = \frac{1}{c_{d,r-1,r-k-1}} \int_{\mathcal{F}_{r-1}^r} V_{r-k-1}(Y \cap F_{r-1})\, d(F_{r-1}, O)^{d-r}\, \mathrm{d}F_{r-1}^d, \tag{5}$$

where d is the distance function, is a solution to (4). In the present paper, we shall demonstrate that solutions to (4) can be expressed as an integral over the unit normal bundle of the section $X \cap L_r$, for all $1 \le k < r$, or, as an integral over $X \cap L_r$, when $k = 0$. It appears that the functionals α and β in (3) and (4) share the same integral representation, $\alpha_{r,k}^d$, parametrized by three integers. This family of functionals *generalizes* the classical intrinsic volumes and in fact, $\alpha_{d,k}^d(X) = V_{d-k}(X)$, for any d-dimensional set X of positive reach. As a main result of the present paper, a *rotational Crofton formula* shall be derived,

$$\alpha_{j,k}^d(X) = c_{d-r,j-r} \int_{\mathcal{L}_r^d} \alpha_{j,k}^r(X \cap L_r)\, \mathrm{d}L_r^d,$$

which turns formula (3) and (4) into special cases. Here, $c_{d-r,j-r}$ is a known constant, see Section 2 below.

The paper is organized as follows. In Section 2, we present the notation and some background knowledge. Section 3 shows that the solution (5) can be expressed as an integral with respect to a q-dimensional affine subspace in L_r. In Section 4, the solution is given a more explicit expression as an integral over a unit normal bundle and the rotational Crofton formula for *flagged* intrinsic volumes is presented. Proofs are deferred to Section 5.

2 Preliminaries

In this section, we shall fix the conventions used in this paper. For any compact set $X \subseteq \mathbb{R}^d$ of positive reach, we define its kth intrinsic volume by

$$V_k(X) = \frac{1}{\sigma_{d-k}} \int_{\operatorname{nor} X} \sum_{\substack{|J|=d-k-1 \\ J \subset \{1,\dots,d-1\}}} \frac{\prod_{j\in J} \kappa_j(x,n)}{\prod_{j=1}^{d-1} \sqrt{1+\kappa_j^2(x,n)}}\, \mathcal{H}^{d-1}(\mathrm{d}(x,n)),$$

where $\kappa_j(x,n)$ denotes the jth (generalized) principal curvature at $(x,n) \in \operatorname{nor} X$ and $\sigma_{d-k} = 2\pi^{\frac{d-k}{2}}/\Gamma\left(\frac{d-k}{2}\right)$ is the surface area of the unit sphere in $\mathbb{R}^{d-k}$. An r-dimensional affine subspace $F_r \in \mathcal{F}_r^d$ can be written uniquely as $F_r = L_r + x$, where $L_r \in \mathcal{L}_r^d$ and $x \in L_r^{\perp}$. The corresponding measure decomposition is

$$\mathrm{d}F_r^d = \mathrm{d}x^{d-r}\,\mathrm{d}L_r^d. \tag{6}$$

Here, $\mathrm{d}x^{d-r}$ is a shortcut notation for the Hausdorff (Lebesgue) measure $\mathcal{H}^{d-r}(\mathrm{d}x)$. Moreover, since the superscript in $\mathrm{d}F_r^d$ and $\mathrm{d}L_r^d$ is often superfluous, we shall write $\mathrm{d}L_r$ and $\mathrm{d}F_r$, when the context allows it. The total mass of $\mathcal{L}_r^d$ is chosen to be

$$\int_{\mathcal{L}_r^d} \mathrm{d}L_r = c_{d,r},$$

where

$$c_{d,r} = \frac{\sigma_d\sigma_{d-1}\cdots\sigma_{d-r+1}}{\sigma_r\sigma_{r-1}\cdots\sigma_1}.$$

With this convention, the constant in the classical Crofton formula is given by

$$c_{d,r,k} = c_{d,r}\frac{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{d+k-r+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}.$$

The Gauss *hypergeometric series* or *hypergeometric function* is defined for $a, b, c \in \mathbb{R}$ and $z \in [-1,1]$ as

$$F(a,b;c;z) = F(b,a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

where $(x)_k$ is the *rising sequential product* or *Pochhammer symbol* defined for a non-negative integer k and $x \in \mathbb{R}$ by

$$(x)_k = \begin{cases} \frac{\Gamma(x+k)}{\Gamma(x)} & \text{if } x > 0 \\ (-1)^k \frac{\Gamma(-x+1)}{\Gamma(-x-k+1)} & \text{if } x \le 0, \end{cases}$$

cf. [1, Chapter 15]. Note that $(x)_k = 0$ whenever $x \in \{0,-1,-2,\ldots\}$ and $k > -x$. The *Gamma function* is defined on $\mathbb{R}_+$ as $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$ and it has an analytic continuation on $\mathbb{C}\setminus\{0,-1,-2,\ldots\}$. Standard formulae for the Gamma function can be found in [1, Chapter 6]. In particular, the duplication formula,

$$\Gamma(2z) = \Gamma(z)\,\Gamma\left(z+\tfrac{1}{2}\right)\,\pi^{-\frac{1}{2}}\,2^{2z-1}, \tag{7}$$

will be useful in the present paper.

3 A generalized solution

From the 1936 paper [9] by Petkantschin, a particular measure decomposition can be employed to slightly generalize the solution α, derived in [3] and [6], to the integral equation (4).

Proposition 1. *Let $Y \subseteq L_r$ be a compact subset of positive reach in some fixed linear subspace $L_r \in \mathcal{L}_r^d$. The functional*

$$\alpha_{d,k}^r(Y) = \frac{1}{c_{d,r,q,k}} \int_{\mathcal{F}_q^r} V_{q-k}(Y \cap F_q)\, d(F_q, O)^{d-r}\, \mathrm{d}F_q$$

solves the integral equation

$$V_{d-k}(X) = \int_{\mathcal{L}_r^d} \alpha_{d,k}^r(X \cap L_r)\, \mathrm{d}L_r,$$

for all $0 \le k \le q < r \le d$. Here, $c_{d,r,q,k} = c_{d-q-1,r-q-1} c_{d,q,q-k}$.

Proof. We shall use the following integral decomposition formula

$$\int_{\mathcal{F}_q^d} f(F_q)\, \mathrm{d}F_q = \frac{1}{c_{d-q-1,r-q-1}} \int_{\mathcal{L}_r^d} \int_{\mathcal{F}_q^r} f(F_q)\, d(F_q, O)^{d-r}\, \mathrm{d}F_q\, \mathrm{d}L_r, \tag{8}$$

which holds for all $0 \le q < r \le d$ and any integrable function f, cf. [9] and [11, p. 285]. Let us assume that $X \subseteq \mathbb{R}^d$ is a compact set of positive reach. The Crofton formula for sets of positive reach, cf. [10], and an application of (8) yield, for all $0 \le k \le q < r \le d$,

$$\begin{aligned} c_{d,q,q-k} V_{d-k}(X) &= \int_{\mathcal{F}_q^d} V_{q-k}(X \cap F_q)\, \mathrm{d}F_q \\ &= \frac{1}{c_{d-q-1,r-q-1}} \int_{\mathcal{L}_r^d} \int_{\mathcal{F}_q^r} V_{q-k}(X \cap F_q)\, d(F_q, O)^{d-r}\, \mathrm{d}F_q\, \mathrm{d}L_r \\ &= \frac{1}{c_{d-q-1,r-q-1}} \int_{\mathcal{L}_r^d} \int_{\mathcal{F}_q^r} V_{q-k}((X \cap L_r) \cap F_q)\, d(F_q, O)^{d-r}\, \mathrm{d}F_q\, \mathrm{d}L_r. \end{aligned}$$

The proof is complete. $\square$

Remark 1. The constant appearing on the right-hand side of formula (8) is not given explicitly in [11], but only referred to as a constant depending on d, q and r. In order to compute this constant, we set $f(F_q) = V_0(B^d \cap F_q)$, where $V_0 = \chi$ is the Euler-Poincaré characteristic and B^d is the unit ball in $\mathbb{R}^d$. By computing both side of (8) separately, we obtain

$$\int_{\mathcal{F}_q^d} f(F_q)\, \mathrm{d}F_q = \int_{\mathcal{L}_q^d} \int_{L_q^\perp \cap B^d} \mathrm{d}z^{d-q}\, \mathrm{d}L_q = \omega_{d-q}\, c_{d,q}, \tag{9}$$

where $\omega_{d-q} = (d-q)\, \sigma_{d-q}$ is the volume of the unit ball in $\mathbb{R}^{d-q}$, and

$$\int_{\mathcal{L}_r^d} \int_{\mathcal{F}_q^r} f(F_q)\, d(F_q, O)^{d-r}\, \mathrm{d}F_q\, \mathrm{d}L_r = \frac{1}{d-q}\, \sigma_{r-q}\, c_{d,r}\, c_{r,q}. \tag{10}$$

Division of (9) by (10) yields the constant appearing in formula (8). $\square$

Proposition 2. *For all $0 \le k \le q < r \le d$, the functional $\alpha_{d,k}^r$, defined in Proposition 1, does not depend on q.*

The proof of Proposition 2 is deferred to the last section. In spite of Proposition 2, the uniqueness of the solution, $\alpha^r_{d,k}$, to the integral equation (4) remains open.

Remark 2. Two representations of $\alpha^r_{d,k}$ are particularly interesting for our purposes. For $q = k$, we have

$$\alpha^r_{d,k}(Y) = \frac{1}{c_{d,r,k,k}} \int_{\mathcal{F}^r_k} \chi(Y \cap F_k)\, d(F_k, O)^{d-r}\, \mathrm{d}F_k \tag{11}$$

and, for $q = r - 1$,

$$\alpha^r_{d,k}(Y) = \frac{1}{c_{d,r-1,r-1-k}} \int_{\mathcal{F}^r_{r-1}} V_{r-k-1}(Y \cap F_{r-1})\, d(F_{r-1}, O)^{d-r}\, \mathrm{d}F_{r-1}. \tag{12}$$

Combining (11) and the identity $\mathrm{d}F^r_0 = \mathrm{d}x^r$, we obtain, for $k = 0$,

$$\alpha^r_{d,0}(Y) = \frac{1}{c_{d-1,r-1}} \int_{\mathbb{R}^r} \chi(Y \cap \{x\})|x|^{d-r}\, \mathrm{d}x^r = \frac{1}{c_{d-1,r-1}} \int_Y |x|^{d-r}\, \mathrm{d}x^r. \tag{13}$$

Furthermore, using (12), we have shown in [3] that for $k = 1$,

$$\alpha^r_{d,1}(Y) = \frac{1}{2c_{d-1,r-1}} \int_{\partial Y} |x|^{d-r} F\left(-\frac{1}{2}, -\frac{d-r}{2}; \frac{r-1}{2}; \sin^2(n(x), x)\right) \mathrm{d}x^{r-1}, \tag{14}$$

whenever Y is a C^2-manifold with boundary. Here, $n(x)$ is the vector normal to the surface at x. □

Motivated by the integral representation (14), we shall now prove that the functional $\alpha^r_{d,k}(Y)$ can be expressed as an integral over the normal bundle of Y for all $k = 1, \dots, r-1$ and $r = 2, \dots, d$.

4 Integral representation of the generalized solution

Let $X \subseteq \mathbb{R}^d$ be a compact set of positive reach and assume for now that the section $Y = X \cap L_r$, for some fixed $L_r \in \mathcal{L}^d_r$, also has positive reach. For the necessary background in multilinear algebra, current theory and sets of positive reach, the reader is referred to [4] and [5].

Let N_Y be the $(r-1)$-dimensional current on $\mathbb{R}^r \times \mathbb{R}^r$ given by

$$N_Y = (\mathcal{H}^{r-1} \llcorner \operatorname{nor} Y) \wedge a_Y, \tag{15}$$

i.e.

$$N_Y(\phi) = \int_{\operatorname{nor} Y} \langle a_Y(x,n), \phi(x,n)\rangle\, \mathcal{H}^{r-1}(\mathrm{d}(x,n))$$

for all $(r-1)$-forms ϕ on $\mathbb{R}^r \times \mathbb{R}^r$. Here, a_Y is a unit $(r-1)$-dimensional vectorfield orienting $\operatorname{nor} Y$ given explicitly by

$$a_Y(x,n) = \bigwedge_{i=1}^{r-1} \left(\frac{1}{\sqrt{1+\kappa_i(x,n)^2}}\, a_i(x,n), \frac{\kappa_i(x,n)}{\sqrt{1+\kappa_i(x,n)^2}}\, a_i(x,n) \right), \tag{16}$$

where $\kappa_i(x,n)$ is the ith principal curvature and $a_i(x,n)$ the corresponding principal direction at $(x,n) \in \operatorname{nor} Y$ for $i = 1, \dots, r-1$, cf. [8, (27)] and [12]. We apply the usual convention $\frac{\infty}{\sqrt{1+\infty^2}} = 1$ and $\frac{1}{\sqrt{1+\infty^2}} = 0$ at points where some of the principal curvatures are infinite. Assume that the principal directions are ordered in such a way that

$$a_1(x,n), \dots, a_{r-1}(x,n), n$$

constitute an orthonormal basis of $\mathbb{R}^r$. The Lipschitz-Killing curvature form ϕ_k on $\mathbb{R}^r \times \mathbb{R}^r$ of order $k = 0, \dots, r-1$ is defined by

$$\begin{aligned} &\langle (u_0^1, u_1^1) \wedge \cdots \wedge (u_0^{r-1}, u_1^{r-1}), \phi_k(x,n) \rangle \\ &\quad = \frac{1}{\sigma_{r-k}} \sum_{\substack{\epsilon_i = 0,1 \\ \epsilon_1 + \cdots + \epsilon_{r-1} = r-1-k}} \langle u_{\epsilon_1}^1 \wedge \cdots \wedge u_{\epsilon_{r-1}}^{r-1} \wedge n, \Omega_r \rangle, \end{aligned} \tag{17}$$

where Ω_r is the volume r-form in R^r. Note that the right-hand side in (17) is strictly positive whenever the number of non-zero principal curvatures at (x,n) is at least $r-1-k$ or, alternatively, when the number of infinite principal curvatures is at most $r-1-k$. For any compact set Y with positive reach, the kth intrinsic volume of Y can be expressed as

$$V_k(Y) = N_Y(\phi_k),$$

for $k = 0, \dots, r-1$, cf. [12].

Two sets Y and F with positive reach *touch*, when there exists a pair $(y,n) \in \operatorname{nor} Y$ such that $(y,-n) \in \operatorname{nor} F$, cf. [13]. In the particular case where $F = L + z$ is an affine subspace, Y and F do not touch, whenever the following condition is satisfied

$$(y,n) \in \operatorname{nor} Y \quad \wedge \quad y \in F \quad \Longrightarrow \quad n \notin L^\perp. \tag{18}$$

Remark 3. The subset of j-dimensional affine subspaces in $\mathbb{R}^r$ that do touch Y has finite $(r-1+j(r-1-j))$-dimensional measure, cf. [10, (1)]. Hence, in the special case where $j = r-1$, the set of $(r-1)$-dimensional affine subspaces touching Y has finite $(r-1)$-dimensional measure, i.e. Y is not touched by $\mathcal{H}^r$ almost all $F \in \mathcal{F}_{r-1}^r$. Whenever Y and F do not touch, their intersection $Y \cap F$ has *local* positive reach, cf. [13] or [4, Theorem 4.10]. By the compactness of $\operatorname{nor} Y$ and the continuity of the reach function, we conclude that

> if Y has positive reach, then $Y \cap F$ has positive reach for almost all $F \in \mathcal{F}_{r-1}^r$.

Furthermore, for a compact subset $X \subseteq \mathbb{R}^d$ of positive reach, it is shown in [8] that for $\mathcal{H}^d$-a.a. choices of origo, the sets X and L do not touch for almost all $L \in \mathcal{L}_r^d$. In other words, whenever X has positive reach, we may choose origo such that

$$X \cap L \text{ has positive reach for almost all } L \in \mathcal{L}_r^d. \tag{19}$$

For those reasons, the assumption, which was made at the beginning of this section on the positive reach of $Y = X \cap L_r$, is mild.

Definition 1 (Flagged intrinsic volumes). *Let $Y \in \mathbb{R}^r$ be a compact set of positive reach. Define for all $s = 1, \dots, r$, $r \geq 1$ and $j \geq s$,*

$$\alpha^r_{j,0}(Y) := \frac{1}{c_{j-1,r-1}} \int_Y |x|^{j-r} \, \mathrm{d}y^r$$

and

$$\alpha^r_{j,s}(Y) := K^r_{j,s} \int_{\operatorname{nor} Y} |x|^{j-r} \sum_{\substack{|I|=s-1 \\ I \subset \{1,\dots,r-1\}}} \frac{\prod_{i \in I} \kappa_i(x,n)}{\prod_{i=1}^{r-1} \sqrt{1+\kappa_i(x,n)^2}} Q^r_{j,s}(x,n,A_I) \, \mathcal{H}^{r-1}(\mathrm{d}(x,n)),$$

where

$$\begin{aligned} &Q^r_{j,s}(x,n,A_I) \\ &\quad := F\left(-\frac{j-r}{2}, \frac{s}{2}; \frac{r+1}{2}; \sin^2(x,n)\right) \\ &\qquad + \frac{(j-r)(r-s+1)}{r+1} \frac{\cos^2(x,A_I)}{r-s} F\left(-\frac{j-r}{2}+1, \frac{s}{2}; \frac{r+3}{2}; \sin^2(x,n)\right) \end{aligned}$$

and

$$K^r_{j,s} := \frac{1}{\sigma_s \, c_{j-1,r-1}} \frac{\Gamma(r-s+1)\,\Gamma(j)}{\Gamma(r)\,\Gamma(j-s+1)}.$$

Here, $A_I = \operatorname{span}\{a_i : i \notin I\}$ and, for the special case where we have $r = s$, we set $\frac{\cos^2(x, A_{\{1,\dots,r-1\}})}{0} := 1$. Note that $c_{j-1,r-1} := \frac{1}{c_{r-1,j-1}}$ for $j < r$. □

Remark 4. In the special case $j = r$, $K^r_{r,s} = \frac{1}{\sigma_s}$ and $Q^r_{r,s} = 1$. Consequently,

$$\alpha^r_{r,s}(Y) = \frac{1}{\sigma_s} \int_{\operatorname{nor} Y} \sum_{\substack{|J|=s-1 \\ J \subset \{1,\dots,r-1\}}} \frac{\prod_{j \in J} \kappa_j(x,n)}{\prod_{j=1}^{r-1} \sqrt{1+\kappa_j^2(x,n)}} \mathcal{H}^{r-1}(\mathrm{d}(x,n)) = V_{r-s}(Y)$$

and

$$\alpha^r_{r,0}(Y) = \int_Y \mathrm{d}\mathcal{H}^r = V_r(Y),$$

for any compact set $Y \subseteq \mathbb{R}^r$ of positive reach. □

The functionals defined above are identical to those given in Proposition 1. This result is formulated in the proposition below. The proof is deferred to the next section.

Proposition 3. *The flagged intrinsic volumes presented in Definition 1 are identical to the functional $\alpha^r_{d,s}$ given in Proposition 1 for all $s = 0, \dots, r-1$, $r = 1, \dots, d$. As a consequence, when origo is chosen such that condition (19) is satisfied, the functional $\alpha^r_{d,s}$ from Definition 1 satisfies the integral equation,*

$$V_{d-s}(X) = \int_{\mathcal{L}^d_r} \alpha^r_{d,s}(X \cap L_r) \, \mathrm{d}L_r,$$

for all $0 \leq s < r \leq d$.

Remark 5. Since $\cos^2(x, A_\emptyset) = \sin^2(x, n)$, the hypergeometric identity (34) implies

$$Q^r_{d,1}(x, n, A_\emptyset) = F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r-1}{2}; \sin^2(x, n)\right)$$

and with

$$K^r_{d,1} = \frac{1}{\sigma_1\, c_{d-1,r-1}} \frac{\Gamma(r)\Gamma(d)}{\Gamma(r)\Gamma(d)} = \frac{1}{2c_{d-1,r-1}},$$

we conclude that

$$\alpha^r_{d,1}(Y) = \frac{1}{2c_{d-1,r-1}} \int_{\operatorname{nor} Y} |x|^{d-r} F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r-1}{2}; \sin^2(x, n)\right) \mathcal{H}^{r-1}(\mathrm{d}(x, n)),$$

i.e. a generalization of (14) to sets of positive reach. □

Remark 6. Applying the hypergeometric identity (35) with $z = \sin^2(x, n) > 0$, we obtain

$$\begin{aligned}
&\frac{(j-d)(d-j+k+1)}{d+1} F\left(\frac{d-j}{2} + 1, \frac{j-k}{2}; \frac{d+3}{2}; \sin^2(x, n)\right) \\
&\quad = (j - d - (d-1)\cot^2(x, n)) F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d+1}{2}; \sin^2(x, n)\right) \\
&\qquad + (d-1)\cot^2(x, n) F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d-1}{2}; \sin^2(x, n)\right),
\end{aligned}$$

or, in other terms,

$$(d-j+k)\, Q^d_{j,j-k}(x, n) = f_1(\angle(x, n)) + f_2(\angle(x, n)) \cos^2 \alpha_I(x, n),$$

where the right-hand side is written in the notation of [2, Theorem 3.1]. Since $\frac{1}{d-j+k} K^d_{j,j-k}$ is equal to the constant $C_{d,j,k}$ defined in [2], (recall that $c_{j-1,d-1} := \frac{1}{c_{d-1,j-1}}$), we conclude that

$$|x|^{j-d}\, K^d_{j,j-k}\, Q^d_{j,j-k}(x, n) = \omega_{I,j,k}(x, n),$$

where the functional $\omega_{I,j,k}$ given in [2, Theorem 3.1] satisfies the integral equation

$$\int_{\mathcal{L}^d_j} V_k(X \cap L_j)\, \mathrm{d}L_j = \int_{\operatorname{nor} X} \sum_{\substack{|I|=j-1-k \\ I\subset\{1,\ldots,d-1\}}} \omega_{I,j,k} \frac{\prod_{i\in I} \kappa_i}{\prod_{i=1}^{d-1} \sqrt{1+\kappa_i^2}}\, \mathrm{d}\mathcal{H}^{r-1} = \alpha^d_{j,j-k}(X), \tag{20}$$

for all $0 \le k < j \le d$, whenever $X \subseteq \mathbb{R}^d$ is a set of positive reach and the origin is chosen such that condition (19) is satisfied, cf. Remark 3. □

Having made these two important remarks, the following Theorem can be proven easily.

Theorem 1 (Rotational Crofton Formula)**.** *Let $X \subset \mathbb{R}^d$ be a compact subset of positive reach and assume origo is chosen such that condition (19) is satisfied. Then,*

$$\alpha^d_{j,k}(X) = c_{d-r,j-r} \int_{\mathcal{L}^d_r} \alpha^r_{j,k}(X \cap L_r)\, \mathrm{d}L_r,$$

for all $0 \le k < r \le j \le d$.

Remark 7. With Remark 4 in mind, we notice that [8, Theorem] and Proposition 1 are special cases of Theorem 1 for $r = j$ and $d = j$, respectively. The case $r = k$ is not covered by Theorem 1. Nevertheless, for $r = j = k$, formula (20) implies

$$\alpha^d_{r,r}(X) = \int_{\mathcal{L}^d_r} V_0(X \cap L_r)\, \mathrm{d}L_r = c_{d-r,r-r} \int_{\mathcal{L}^d_r} \alpha^r_{r,r}(X \cap L_r)\, \mathrm{d}L_r.$$

Proof of Theorem 1. On the one hand, we have, according to Remark 6 and under the mild assumption on the choice of origo,

$$\alpha^d_{j,k}(X) = \int_{\mathcal{L}^d_j} V_{j-k}(X \cap L_j)\, \mathrm{d}L_j$$

for all $0 < k \le j \le d$. The case $k = 0$ follows from the Blaschke-Petkantschin formula, cf. [7, Proposition 4.5],

$$\int_{\mathcal{L}^d_j} V_j(X \cap L_j)\, \mathrm{d}L_j = \int_{\mathcal{L}^d_j} \int_{X \cap L_j} \mathrm{d}x^j\, \mathrm{d}L_j = c_{d-1,j-1} \int_X |x|^{j-d}\, \mathrm{d}x^d = \alpha^d_{j,0}(X).$$

On the other hand, Proposition 1 and [7, (3.17)] imply

$$\begin{aligned}\int_{\mathcal{L}^d_j} V_{j-k}(X \cap L_j)\, \mathrm{d}L_j &= \int_{\mathcal{L}^d_j} \int_{\mathcal{L}^j_r} \alpha^r_{j,k}(X \cap L_j \cap L_r)\, \mathrm{d}L_r\, \mathrm{d}L_j \\ &= c_{d-r,j-r} \int_{\mathcal{L}_r} \alpha^r_{j,k}(X \cap L_r)\, \mathrm{d}L_r,\end{aligned}$$

for all $0 \le k < r \le j \le d$. □

5 Proofs

Proof of Proposition 2. If $k = r-1$, there is only one possible choice for q satisfying $r > q \geq k$ and the proof is complete. Assume that $r > q > k \geq 0$. Crofton's formula for sets of positive reach and the measure decomposition (6) imply

$$\begin{aligned}
&c_{q,q-1,q-k-1} \int_{\mathcal{F}_q^r} V_{q-k}(Y \cap F_q)\, \mathrm{d}(F_q, O)^{d-r}\, \mathrm{d}F_q \\
&\quad = c_{q,q-1,q-k-1} \int_{\mathcal{L}_q^r} \int_{L_q^\perp} V_{q-k}(Y \cap (L_q + x))|x|^{d-r}\, \mathrm{d}x^{r-q}\, \mathrm{d}L_q \\
&\quad = \int_{\mathcal{L}_q^r} \int_{L_q^\perp} |x|^{d-r} \int_{\mathcal{F}_{q-1}^q} V_{q-k-1}((Y-x) \cap L_q \cap F_{q-1})\, \mathrm{d}F_{q-1}\, \mathrm{d}x^{r-q}\, \mathrm{d}L_q \\
&\quad = \int_{\mathcal{L}_q^r} \int_{L_q^\perp} |x|^{d-r} \int_{\mathcal{L}_{q-1}^q} \int_{L_{q-1}^\perp \cap L_q} V_{q-k-1}(Y \cap (L_{q-1} + x + y))\, \mathrm{d}y^1\, \mathrm{d}L_{q-1}\, \mathrm{d}x^{r-q}\, \mathrm{d}L_q.
\end{aligned}$$

An application of the measure transformation $\mathrm{d}L_{q-1}^q\, \mathrm{d}L_q^r = \mathrm{d}L_{q(q-1)}^r\, \mathrm{d}L_{q-1}^r$, cf. [7, (3.15)], and of the orthogonal decomposition $L_{q-1}^\perp = L_q^\perp \oplus (L_{q-1}^\perp \cap L_q)$, turns the last expression into

$$\begin{aligned}
&\int_{\mathcal{L}_{q-1}^r} \int_{\mathcal{L}_{q(q-1)}^r} \int_{L_q^\perp} |x|^{d-r} \int_{L_{q-1}^\perp \cap L_q} V_{q-k-1}(Y \cap (L_{q-1} + x + y))\, \mathrm{d}y^1\, \mathrm{d}x^{r-q}\, \mathrm{d}L_{q(q-1)}\, \mathrm{d}L_{q-1} \\
&\quad = \int_{\mathcal{L}_{q-1}^r} \int_{L_{q-1}^\perp} V_{q-k-1}(Y \cap (L_{q-1} + z)) \int_{\mathcal{L}_{q(q-1)}^r} |p(z|L_q^\perp)|^{d-r}\, \mathrm{d}L_{q(q-1)}\, \mathrm{d}z^{d-q+1}\, \mathrm{d}L_{q-1}.
\end{aligned}$$

It can be shown, as we shall see in Remark 11, that

$$\int_{\mathcal{L}_{q(q-1)}^r} |p(z|L_q^\perp)|^{d-r} \mathrm{d}L_{q(q-1)} = |z|^{d-r} \frac{\sigma_{r-q}}{2} B\left(\frac{1}{2}, \frac{d-q}{2}\right), \tag{21}$$

whenever $z \in L_{q-1}^\perp$. Thus,

$$\begin{aligned}
&\int_{\mathcal{L}_{q-1}^r} \int_{L_{q-1}^\perp} V_{q-k-1}(Y \cap (L_{q-1} + z)) \int_{\mathcal{L}_{q(q-1)}^r} |p(z|L_q^\perp)|^{d-r}\, \mathrm{d}L_{q(q-1)}\, \mathrm{d}z^{d-q+1}\, \mathrm{d}L_{q-1} \\
&\quad = \frac{\sigma_{r-q}}{2} B\left(\frac{1}{2}, \frac{d-q}{2}\right) \int_{\mathcal{F}_{q-1}^r} V_{q-k-1}(Y \cap F_{q-1})\, d(F_{q-1}, O)^{d-r}\, \mathrm{d}F_{q-1}.
\end{aligned}$$

Since $c_{q,q-1,q-k-1} = \frac{\sigma_{q+1}}{2B(\frac{q-k}{2}, \frac{1}{2})}$, we conclude that

$$\begin{aligned}
&\int_{\mathcal{F}_q^r} V_{q-k}(Y \cap F_q)\, d(F_q, O)^{d-r}\, \mathrm{d}F_q \\
&\quad = \frac{\sigma_{r-q} B\left(\frac{1}{2}, \frac{d-q}{2}\right) B\left(\frac{q-k}{2}, \frac{1}{2}\right)}{\sigma_{q+1}} \int_{\mathcal{F}_{q-1}^r} V_{q-k-1}(Y \cap F_{q-1})\, d(F_{q-1}, O)^{d-r}\, \mathrm{d}F_{q-1},
\end{aligned}$$

for all $0 \leq k < q < r$. A routine calculation shows that

$$\frac{\sigma_{r-q}\, B\left(\frac{1}{2}, \frac{d-q}{2}\right)\, B\left(\frac{q-k}{2}, \frac{1}{2}\right)}{\sigma_{q+1}\, c_{d-q-1,r-q-1}\, c_{d,q,q-k}} = \frac{1}{c_{d-(q-1)-1,r-(q-1)-1}\, c_{d,q-1,q-1-k}},$$

therefore,

$$\alpha_{r,k}(Y) = \frac{1}{c_{d-(q-1)-1,j-(q-1)-1}\, c_{d,q-1,q-1-k}} \int_{\mathcal{F}^r_{q-1}} V_{q-1-k}(Y \cap F_{q-1})\, d(F_{q-1}, O)^{d-r}\, \mathrm{d}F_{q-1}.$$

Hence, in the definition of $\alpha^r_{d,k}$, the variable q can be replaced by $q-1$. A recursive argument implies that $\alpha^r_{d,k}$ is independent of q. □

Proof of Proposition 3. Without loss of generality, we will use the representation (12) of $\alpha^r_{d,s}$ (i.e. $q = r-1$). Let $L_r \in \mathcal{L}^d_r$ be a fixed r-dimensional subset of $\mathbb{R}^d$ and let $Y \subset L_r$ be a compact set of positive reach. The case $s = 0$ holds by definition, cf. formula (13). Assume that $0 < s < r-1$. According to (12),

$$\begin{aligned} c_{d,r,r-1,s}\, \alpha^r_{d,s}(Y) &= \int_{\mathcal{F}^r_{r-1}} V_{r-s-1}(Y \cap F_{r-1})\, d(F_{r-1}, O)^{d-r}\, \mathrm{d}F_{r-1} \\ &= \int_{\mathcal{L}^r_{r-1}} \mathcal{I}(L_{r-1})\, \mathrm{d}L_{r-1}, \end{aligned}$$

where

$$\mathcal{I}(L_{r-1}) = \int_{L^\perp_{r-1}} V_{r-s-1}(Y \cap (L_{r-1} + x))\, |x|^{d-r} \mathrm{d}x^1.$$

Note that $c_{d,r,r-1,s} = c_{d-r,0}\, c_{d,r-1,r-s-1} = c_{d,r-1,r-s-1}$. Let $L_{r-1} \in \mathcal{L}^r_{r-1}$ be a fixed $(r-1)$-dimensional subspace of L_r and let ω_1 be a unit vector st. $\mathrm{span}\{\omega_1\} = L^\perp_{r-1}$. Let ω_{r-1} be a simple unit $(r-1)$-vector orienting L_{r-1} such that $\langle \omega_1 \wedge \omega_{r-1}, \Omega_r \rangle = 1$. We define the two volume forms, Ω_1 and Ω_{r-1}, to be the dual vectors of ω_1 and ω_{r-1}, respectively. Define

$$\begin{gathered} f\colon \operatorname{nor} Y \setminus \{(x,n) \mid n \perp L_{r-1}\} \to \mathbb{R}^r \times S^{r-2}(L_{r-1}) \\ f(x,n) = (x, \pi(n|L_{r-1})) \end{gathered}$$

and

$$\begin{gathered} g\colon \operatorname{nor} Y \to L^\perp_{r-1} \\ g(x,n) = \langle x, \omega_1 \rangle \omega_1 = p(x|L^\perp_{r-1}). \end{gathered}$$

Since the differential of the spherical projection $\pi_L\colon n \mapsto \pi(n|L)$ is

$$D\pi_L(n)v = \frac{p(v|L \cap n^\perp)}{|p(n|L)|},$$

cf. [8, Lemma 2], we have

$$D_{(x,n)} f(u,v) = \left(u, \frac{p(v|L_{r-1} \cap n^\perp)}{|p(n|L_{r-1})|} \right)$$

and the linearity of g implies

$$D_{(x,n)} g(u,v) = g(u,v) = \langle u, \mathbf{e} \rangle \mathbf{e} = p(u|L^\perp_{r-1})$$

for all $(u,v) \in \operatorname{Tan}(\operatorname{nor} Y, (x,n))$. Next, we show that for almost all $z \in L^\perp_{r-1}$, the point $(x,n) \in \operatorname{nor} Y$ is uniquely determined by the projection $f(x,n) = (x, n_0)$ for $\mathcal{H}^{r-2}$-a.a. $(x, n_0) \in f(g^{-1}(z))$.

Lemma 1. *For almost all $L_{r-1} \in \mathcal{L}^r_{r-1}$ and $\mathcal{H}^1$-almost all $z \in L^\perp_{r-1}$,*

$$\mathcal{H}^{r-2}\left(\{(x, n_0) \in f(g^{-1}(z)) \,:\, \operatorname{card} f^{-1}\{(x, n_0)\} > 1\}\right) = 0.$$

Remark 8. Let $f^{(z)}$ be the restriction of f to $g^{-1}(z)$. Note that f is well defined on the set $g^{-1}(z)$ only when $L_{r-1} + z$ and Y do not touch, which is the case for almost all pairs (L_{r-1}, z). When Y and $L_{r-1} + z$ do not touch, there is no point $(y, n) \in \operatorname{nor} Y$ such that $y \in L_{r-1} + z$ and $n \in L^\perp_{r-1}$, i.e f is well-defined at all points $(y, n) \in \operatorname{nor} Y$ with $y \in L_{r-1} + z$. Since the normal bundle, $\operatorname{nor} Y$, is compact, the function f can be extended to a locally Lipschitz (differentiable!) function on an open set containing $g^{-1}(z) \setminus \{(y, n) \,|\, n \perp L_{r-1}\}$. Thus, the assumptions for the area and coarea formulae are satisfied, cf. [5].

Proof. Assume that $L_{r-1} + z$ and $\operatorname{nor} Y$ are do not touch, i.e. that (18) is satisfied. Then, $f(x, n)$ is well-defined for all $(x, n) \in \operatorname{nor} Y \cap (L_{r-1} + z) \times \mathbb{R}^d$. Let N be the subset of $\operatorname{nor} Y$ where f is well-defined but not injective. More precisely N is the set of all $(x, n) \in \operatorname{nor} Y$ with $n \notin L^\perp_{r-1}$ such that there exist $n' \neq n$ with $(x, n') \in \operatorname{nor} Y$ and $n' \notin L^\perp_{r-1}$ and $f(x, n) = f(x, n')$. It is enough to show that

$$\mathcal{H}^{r-2}(f(N \cap g^{-1}(z))) \leq \int_{N \cap g^{-1}(z)} J_{r-2} f^{(z)} \, \mathrm{d}\mathcal{H}^{r-2} = 0,$$

for almost all $z \in L^\perp_{r-1}$, cf. [5, (3.2.3)]. Using the coarea formula, [5, (3.2.22)], we obtain

$$\int_{L^\perp_{r-1}} \int_{N \cap g^{-1}(z)} J_{r-2} f^{(z)} \, \mathrm{d}\mathcal{H}^{r-2} \, \mathrm{d}z = \int_N J_{r-2} f^{(z)}(x, n) J_1 g(x, n) \, \mathcal{H}^{r-1}(\mathrm{d}(x, n)).$$

Without loss of generality, assume that $\operatorname{Tan}(Y, (x, n))$ is the $(r-1)$-dimensional subspace given by (16), cf. [5, (3.2.19)]. Note that

$$\ker D_{(x,n)} g = (L_{r-1} \times \mathbb{R}^r) \cap \operatorname{Tan}(Y, (x, n)) = \operatorname{Tan}(g^{-1}(z), (x, n)).$$

If $\dim \ker D_{(x,n)} g \geq r - 1$, then $J_1 g(x, n)$ must be equal to zero. Let us assume that $\dim \ker D_{(x,n)} g \leq r - 2$. Since the domain of $D_{(x,n)} f^{(z)}$ is equal to $\ker D_{(x,n)} g$, the $(r-2)$-dimensional Jacobian of $f^{(z)}$ is zero if there exists a point $(u, v) \in \operatorname{Tan}(g^{-1}(z), (x, n))$ such that $D_{(x,n)} f^{(z)}(u, v) = 0$. Note that $(x, n_t) := \left(x, \frac{\sin(1-t)\theta}{\sin\theta} n + \frac{\sin t\theta}{\sin\theta} n'\right) \in g^{-1}(z)$ for all $t \in [0, 1]$, where $\theta = \angle(n, n')$. By definition of the tangent cone, we have

$$(0, \pi(n - n' | n^\perp)) = \lim_{t \downarrow 0} \frac{(x, n_t) - (x, n)}{|(x, n_t) - (x, n)|} \in \operatorname{Tan}(g^{-1}(z), (x, n)).$$

Since, $p(n' | L_{r-1} \cap n^\perp) = p(p(n' | L_{r-1}) | L_{r-1} \cap n^\perp) = p(n | L_{r-1} \cap n^\perp) = 0$, we deduce that $D_{(x,n)} f^{(z)}(0, \pi(n - n' | n^\perp)) = 0$ and therefore, $J f^{(z)}(x, n) = 0$. $\square$

Given a subspace $L_j \in \mathcal{L}^r_j$ and $y \in L^\perp_j$ such that $L_j + y$ and Y do not touch, the restriction of the normal bundle of $Y \cap (L_j + y)$ to $L_j + y$ is given by

$$\operatorname{nor}^{(j)}(Y \cap (L_j + y)) = \{(x, \pi(n, L_j)) \,|\, x \in Y \cap (L_j + y) \text{ and } (x, n) \in \operatorname{nor} Y\}, \quad (22)$$

i.e. the intersection of $\operatorname{nor} Y + \operatorname{nor}(L_j + y)$ with $(L_j + y) \times S^{r-1}$, see [4, Theorem 4.10]. The corresponding orienting unit vectorfield $a_{Y \cap (L_j + x)}$ will be computed later.

Lemma 2. *Let $Y \subseteq \mathbb{R}^r$ be a compact set with positive reach and let $L_{r-1} \in \mathcal{L}^r_{r-1}$. Then,*

$$N_{Y\cap(L_{r-1}+z)} = f_\sharp\langle N_Y, g, z\rangle$$

for almost all $z \in L^\perp_{r-1}$, whenever Y and $L_j + z$ do not touch.

Proof. Applying [5, Section 4.3.8 and 4.3.13] (with $n = 1$ and $m = r-1$) to the integral current (15), we get

$$\langle N_Y, g, z\rangle = (\mathcal{H}^{r-1-1} \llcorner g^{-1}(z)) \wedge \zeta \tag{23}$$

for almost all $z \in L^\perp_{j-1}$. Here, ζ is the $(r-2)$-vectorfield such that

$$\zeta(x,n) = \frac{a_Y(x,n) \llcorner \langle \Omega_1, \bigwedge^1 Dg(x,n)\rangle}{J_1 g(x,n)} = a_Y(x,n) \llcorner \left\langle \Omega_1, \bigwedge^1 Dg(x,n)\right\rangle \tag{24}$$

(because $J_1 g(x,n) = \sqrt{\det(Dg(x,n)Dg(x,n)^t)} = 1$). Applying the area formula for currents [5, Section 4.1.30] to (23), we obtain

$$f_\sharp\langle N_Y, g, z\rangle = \left(\mathcal{H}^{r-2} \llcorner f(g^{-1}\{z\})\right) \wedge \eta,$$

with unit vector field

$$\eta(x,v) = \frac{\left(\bigwedge_{r-2} Df(f^{-1}(x,v))\right) \zeta(f^{-1}(x,v))}{J_{r-2} f(f^{-1}(x,v))}. \tag{25}$$

for $\mathcal{H}^{2r-3}$-almost all $(x,v) \in g^{-1}(z) \times S^{r-2}(L_{r-1})$. Whenever nor Y and $L_{r-1}+z$ do not touch, then $f(g^{-1}(z))$ is equal to the normal bundle $\operatorname{nor}^{(r-1)}(Y \cap (L_{r-1}+z))$. Hence, in order to prove the lemma, it is enough to check that the orientation of the respective orienting vectorfields, $a_{X\cap(L_{r-1}+z)}$ and η, have the same orientation. By convention, the orientation of $a_{X\cap(L_{r-1}+z)}$ is chosen such that $\big\langle a_{X\cap(L_{r-1}+z)}(x,n), \varphi^{(r-1)}_{k(x,n)}(n)\big\rangle > 0$, where $k(x,n)$ is the number of principal curvatures at (x,n) that are 0. Hence, we have to check that $\big\langle \eta(x,n), \varphi^{(r-1)}_{k(x,n)}(n)\big\rangle > 0$. By combining (25) and (24), we see that η is proportional to $\bigwedge_{r-2} Df\left(a_{Y\cap(L_{r-1}+z)} \llcorner g^\# \Omega_1\right)$ with some strictly positive proportionality constant λ. Thus,

$$\lambda\left\langle \eta(x,n), \varphi^{(r-1)}_{k(x,n)}(x,n)\right\rangle = \lambda\left\langle a_{Y\cap(L_{r-1}+z)}(x,n), g^\#\Omega_1 \wedge f^\#\varphi^{(r-1)}_{k(x,n)}(x,n)\right\rangle \geq 0$$

and the inequality is strict when $k(x,n)$ is the number of zero principal curvatures at (x,n), cf. (29) below. □

Using the normal current of $Y \cap (L_{r-1} + |x|)$, the integral $\mathcal{I}(L_{r-1})$ can be written as

$$\begin{aligned}\mathcal{I}(L_{j-1}) &= \int_{L^\perp_{r-1}} V_{r-s-1}(Y \cap (L_{r-1}+x))|x|^{d-r}\mathrm{d}x^1 \\ &= \int_{L^\perp_{r-1}} N_{Y\cap(L_{r-1}+x)}(\phi^{(r-1)}_{r-s-1})|x|^{d-r}\mathrm{d}x^1.\end{aligned}$$

By applying Lemma 2 and the coarea formula for currents, [5, Section 4.3.13], we obtain

$$\begin{aligned}
\mathcal{I}(L_{r-1}) &= \int_{L_{r-1}^\perp} |x|^{d-r} N_{Y\cap(L_{r-1}+x)}(\phi_{r-s-1}^{(r-1)})\,\mathrm{d}x^1 \\
&= \int_{L_{r-1}^\perp} |x|^{d-r} f_\sharp\langle N_Y, g, x\rangle(\phi_{r-s-1}^{(r-1)})\,\mathrm{d}x^1 \\
&= \int_{L_{r-1}^\perp} |x|^{d-r} \langle N_Y, g, x\rangle(f^\sharp\phi_{r-s-1}^{(r-1)})\,\mathrm{d}x^1 \\
&= N_Y \llcorner g^\sharp(q\cdot\Omega_1)(f^\sharp\phi_{r-s-1}^{(r-1)}) \\
&= \int_{\operatorname{nor} Y} \langle a_Y, g^\sharp(q\cdot\Omega_1)\wedge f^\sharp\phi_{r-s-1}^{(r-1)}\rangle\,\mathrm{d}\mathcal{H}^{r-1},
\end{aligned}$$

where $q(x) = |x|^{d-r}$, $x \in \mathbb{R}^r$. Using the shuffle formula, [5, Section 1.4.2], the integrand can be written

$$\begin{aligned}
&\langle a_Y, g^\sharp(q\cdot\Omega_1)\wedge f^\sharp\phi_{r-s-1}^{(r-1)}\rangle \\
&\quad = \sum_{i=1}^{r-1}(-1)^{i+1}\left\langle (u_i,v_i), g^\sharp(q\cdot\Omega_1)\right\rangle \\
&\qquad\qquad \cdot\left\langle (u_1,v_1)\wedge\cdots\wedge\widehat{(u_i,v_i)}\wedge\cdots\wedge(u_{r-1},v_{r-1}), f^\sharp\phi_{r-s-1}^{(r-1)}\right\rangle. \qquad (26)
\end{aligned}$$

The second term can we expressed more explicitly (for the definition of the push-forward, the reader is referred to [5, Section 4.1.6]),

$$\begin{aligned}
&\left\langle (u_1,v_1)\wedge\cdots\wedge\widehat{(u_i,v_i)}\wedge\cdots\wedge(u_{r-1},v_{r-1}), f^\sharp\phi_{r-s-1}^{(r-1)}(x,n)\right\rangle \\
&\quad = \left\langle \left[\textstyle\bigwedge_{r-2} Df(x,n)\right](u_1,v_1)\wedge\cdots\wedge\widehat{(u_i,v_i)}\wedge\cdots\wedge(u_{r-1},v_{r-1}), \phi_{r-s-1}^{(r-1)}\circ f(x,n)\right\rangle \\
&\quad = \left\langle \textstyle\bigwedge_{j\in\{1,\dots,r-1\}\setminus\{i\}}\left(u_j, \dfrac{p(v_j|L_{r-1}\cap n^\perp)}{|p(n|L_{r-1})|}\right), \phi_{r-s-1}^{(r-1)}(\pi(n|L_{r-1}))\right\rangle \\
&\quad = \frac{1}{\sigma_s}\sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}}(\operatorname{sgn} J)\left\langle \bigwedge_{j\in J} u_j\wedge\bigwedge_{j\in J^c}\left(\frac{p(v_j|L_{r-1}\cap n^\perp)}{|p(n|L_{r-1})|}\right)\wedge\pi(n|L_{r-1}), \Omega_{r-1}\right\rangle \\
&\quad = \frac{1}{\sigma_s}\frac{1}{|p(n|L_{r-1})|^s} \\
&\qquad \times\sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}}(\operatorname{sgn} J)\left\langle \bigwedge_{j\in J} u_j\wedge\bigwedge_{j\in J^c} p(v_j|L_{r-1}\cap n^\perp)\wedge p(n|L_{r-1}), \Omega_{r-1}\right\rangle,
\end{aligned} \qquad (27)$$

for all $(x,n)\in\operatorname{nor} Y$. Here, $\operatorname{sgn} J$ is the number of permutations needed to map $J\cup J^c$ into $(1,\dots,r-1)\setminus\{i\}$, where J and J^c are sorted in increasing order. The first term in (26) can be expressed as

$$\begin{aligned}
\langle (u_i,v_i), g^\sharp(q\cdot\Omega_1(x,n))\rangle &= \langle Dg(x,n)(u_i,v_i), q\circ g(x,n)\cdot\Omega_1\circ g(x,n)\rangle \\
&= |p(x|L_{r-1}^\perp)|^{d-r}\left\langle p(u_i|L_{r-1}^\perp), \Omega_1\right\rangle. \qquad (28)
\end{aligned}$$

By inserting into (27) and (28) the explicit representation of a_Y given in (16), we can write (26) as

$$
\begin{aligned}
&\langle a_Y, g^\sharp(q\cdot\Omega_1)\wedge f^\sharp\phi^{(r-1)}_{r-s-1}\rangle \\
&\quad= \sum_{i=1}^{r-1} \sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}} \frac{(-1)^{i+1}(\operatorname{sgn} J)|p(x|L_{r-1}^\perp)|^{d-r}}{\sigma_s|p(n|L_{r-1})|^s} \frac{\prod_{l\in J^c}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}} \\
&\qquad\times \left\langle p(a_i|L_{r-1}^\perp),\Omega_1\right\rangle \left\langle \bigwedge_{j\in J} a_j \wedge \bigwedge_{j\in J^c} p(a_j|L_{r-1}\cap n^\perp)\wedge p(n|L_{r-1}),\Omega_{r-1}\right\rangle.
\end{aligned}
$$

Note that $(-1)^{i+1}(\operatorname{sgn} J)$ is the sign of the permutation necessary to order $\{i\}\cup J\cup J^c$ increasingly. Moreover, orthogonal projections are orientation preserving (eigenvalues are either 0 or 1), therefore,

$$
\begin{aligned}
&(-1)^{i+1}(\operatorname{sgn} J)\left\langle p(a_i|L_{r-1}^\perp),\Omega_1\right\rangle \left\langle \bigwedge_{j\in J} a_j \wedge \bigwedge_{j\in J^c} p(a_j|L_{r-1}\cap n^\perp)\wedge p(n|L_{r-1}),\Omega_{r-1}\right\rangle \\
&\quad= (-1)^{i+1}(\operatorname{sgn} J)\left\langle p(a_i|L_{r-1}^\perp),\Omega_1\right\rangle \left\langle \bigwedge_{j\in J\cup J^c} p(a_j|L_{r-1}\cap n^\perp)\wedge p(n|L_{r-1}),\Omega_{r-1}\right\rangle \\
&\quad= \left\langle \bigwedge_{j=1}^{i-1} p(a_j|L_{r-1}\cap n^\perp)\wedge p(a_i|L_{r-1}^\perp)\wedge \bigwedge_{j=i+1}^{r-1} p(a_j|L_{r-1}\cap n^\perp)\wedge p(n|L_{r-1}),\Omega_r\right\rangle \\
&\quad> 0,
\end{aligned}
$$

where we used the decomposition

$$
a_j = p(a_j|L_{r-1}\cap n^\perp) + p(a_j|\pi(n|L_{r-1})) + p(a_j|L_{r-1}^\perp)
$$

for the first equality. Thus,

$$
\begin{aligned}
&\langle a_Y, g^\sharp(q\cdot\Omega_1)\wedge f^\sharp\phi^{(r-1)}_{r-s-1}\rangle \\
&\quad= \frac{1}{\sigma_s}\sum_{i=1}^{r-1} \sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}} \frac{|p(x|L_{r-1}^\perp)|^{d-r}}{|p(n|L_{r-1})|^s} \frac{\prod_{l\in J^c}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}} \\
&\qquad\times \left|\left\langle p(a_i|L_{r-1}^\perp),\Omega_1\right\rangle\right| \left|\left\langle \bigwedge_{j\in J\cup J^c} p(a_j|L_{r-1}\cap n^\perp)\wedge p(n|L_{r-1}),\Omega_{r-1}\right\rangle\right| \\
&\quad= \frac{1}{\sigma_s}\sum_{i=1}^{r-1} \sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}} \frac{|p(x|L_{r-1}^\perp)|^{d-r}|p(a_i|L_{r-1}^\perp)|^2}{|p(n|L_{r-1})|^s} \frac{\prod_{l\in J^c}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}},
\end{aligned}
$$

where the last equality follows from [7, Proposition 5.2]. Applying the re-indexing identity

$$\sum_{i=1}^{r-1} \sum_{\substack{|J|=r-s-1 \\ J\subset\{1,\dots,r-1\}\setminus\{i\}}} \frac{\prod_{l\in J^c}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}} |p(a_i|L_{r-1}^\perp)|^2 = \sum_{\substack{|J|=s-1 \\ J\subset\{1,\dots,r-1\}}} \frac{\prod_{l\in J}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}} \sum_{i\in J^c} |p(a_i|L_{r-1}^\perp)|^2,$$

we conclude that

$$\begin{aligned} c_{d,r-1,r-s-1}\alpha_{d,s}^r(Y) &= \int_{\mathcal{L}_{r-1}^r} \mathcal{I}(L_{r-1})\,\mathrm{d}L_{r-1} \\ &= \int_{\mathcal{L}_{r-1}^r}\int_{\operatorname{nor} Y} \langle a_Y, g^\sharp(q\cdot\Omega_1)\wedge f^\sharp\phi_{r-s-1}^{(r-1)}\rangle\,\mathrm{d}\mathcal{H}^{r-1}\,\mathrm{d}L_{r-1} \\ &= \int_{\operatorname{nor} Y}\int_{\mathcal{L}_{r-1}^r} \langle a_Y, g^\sharp(q\cdot\Omega_1)\wedge f^\sharp\phi_{r-s-1}^{(r-1)}\rangle\,\mathrm{d}L_{r-1}\,\mathrm{d}\mathcal{H}^{r-1} \\ &= \frac{1}{\sigma_s}\int_{\operatorname{nor} Y} |x|^{d-r} \sum_{\substack{|J|=s-1 \\ J\subset\{1,\dots,r-1\}}} \frac{\prod_{j\in J}\kappa_j}{\prod_{j=1}^{r-1}\sqrt{1+\kappa_j^2}}\,\widetilde{Q}_{r,s}^d(x,n,A_J)\,\mathrm{d}\mathcal{H}^{r-1}, \end{aligned} \tag{29}$$

where

$$\widetilde{Q}_{d,s}^r(x,n,A_J) = \int_{\mathcal{L}_{r-1}^r} \frac{|p(x/|x||L_{r-1}^\perp)|^{d-r}}{|p(n|L_{r-1})|^s} \sum_{i\in J^c} |p(a_i|L_{r-1}^\perp)|^2\,\mathrm{d}L_{r-1}.$$

In the remainder of this section, we shall prove that the above expression for $\widetilde{Q}_{d,s}^r$ can be written in terms of two hypergeometric series.

The hypergeometric series and the Gamma function have been introduced earlier. The *Beta function*, $B(a,b)=B(b,a)$, is defined for all $a,b>0$ as

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$$

and is often expressed in terms of the Gamma function as

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Lemma 3. *Let $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$. Then, for all $k>0$, $a\in\mathbb{R}$, $b>-\frac{1}{k}$ and $c>-1$,*

$$\begin{aligned} &\int_0^1 \left(\alpha+\beta t^k\right)^{-a} (t^k)^b(1-t^k)^c\,\mathrm{d}t \\ &\quad = \tfrac{1}{k}B\left(b+\tfrac{1}{k}, c+1\right) F\left(a, c+1; c+b+\tfrac{1}{k}+1; \beta\right). \end{aligned} \tag{30}$$

When $\beta=1$, the extra assumption $b+\frac{1}{k}>a$ is necessary.

Proof. According to [1, 15.3.1], the analytic continuation of the hypergeometric function is given by

$$F(a,b;c;\beta) = \frac{1}{B\left(b, c-b\right)} \int_0^1 (1-\beta s)^{-a}(1-s)^{c-b-1} s^{b-1} \, \mathrm{d}s$$

whenever $c > b > 0$. Hence, a substitution by $s = 1 - r^k$ proves the Lemma. $\square$

Remark 9. In the special case where $a = 0$ and $k = 2$, Lemma 3 yields the identity

$$\int_{-1}^{1} (t^2)^b (1-t^2)^c \, \mathrm{d}t = B\left(b + \frac{1}{2}, c+1\right).$$

Lemma 4. *Let $x, z \in S^{d-1}$, $m, n \in \mathbb{N}$. Then,*

$$\begin{aligned} &\int_{S^{d-1}} |x \cdot \omega|^m |z \cdot \omega|^n \, \mathrm{d}\omega^{d-1} \\ &\quad = \sigma_{d-2} B\left(\frac{n+m+2}{2}, \frac{d-2}{2}\right) B\left(\frac{m+1}{2}, \frac{n+1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{1}{2}; \cos^2(x,z)\right). \end{aligned} \tag{31}$$

Proof. This Lemma was proven in [3] for any *even* natural number n. The exact same procedure as the one in the proof of Proposition 4 below, in particular the use of Lemma 6, shows that (31) also holds when n is odd. The details are omited here. In order to prove the main results of the present paper, the special case where $n = 2$ is sufficient, though. $\square$

Remark 10. Let L_{d-1} be a $(d-1)$-dimensional subspace of $\mathbb{R}^d$, with $z \in L_{d-1}$ and $x \notin L_{d-1}^{\perp}$. Then, for all $m, n \in \mathbb{N}$,

$$\begin{aligned} &\int_{S^{d-2}(L_{d-1})} |x \cdot \omega|^m |z \cdot \omega|^n \, \mathrm{d}\omega^{d-1} \\ &\quad = \sigma_{d-3} B\left(\frac{n+m+2}{2}, \frac{d-3}{2}\right) B\left(\frac{m+1}{2}, \frac{n+1}{2}\right) \\ &\quad\quad \times \cos^m(x, L_{d-1}) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{1}{2}; \cos^2(\pi(x|L_{d-1}), z)\right). \end{aligned}$$

Recall that the binomial coefficient $\binom{a}{k}$ is defined for all $a \in \mathbb{R}$ and all $k \in \mathbb{N}$ by

$$\binom{a}{k} = \frac{(-a)_k (-1)^k}{k!} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-k+1)\Gamma(k+1)} & \text{for } a > 0, \\ \frac{\Gamma(a+k)(-1)^k}{\Gamma(a)\Gamma(k+1)} & \text{for } a < 0, \\ 0 & \text{for } a = 0. \end{cases}$$

Lemma 5. *For all $a \in \mathbb{R}$ and all $s \in \mathbb{N}$,*

$$\binom{a}{2s} = \frac{\binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}}{\binom{2s}{s}} 2^{2s}.$$

Proof. A routine calculation yields

$$\binom{a}{2s} = \frac{\Gamma(a+1)}{\Gamma(2s+1)\Gamma(a-2s+1)} = \binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(s+\frac{1}{2})} = \frac{\binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}}{\binom{2s}{s}}2^{2s}.$$

For the second equality, we applied the duplication formula on each Gamma function appearing in the second term, and for the third equality, we applied the duplication formula to $\Gamma(2s+1)$. $\square$

Lemma 6. *For all $a \in \mathbb{R}$ and any function f, the following identity holds,*

$$\sum_{k=0}^{\infty}\binom{\frac{a}{2}}{k}\sum_{l=0}^{k}\binom{k}{l}2^{k-l}f(k+l) = \sum_{s=0}^{\frac{a}{2}}\binom{a}{2s}f(2s),$$

where the double sum on the l.h.s. is over k and l ***with the same parity*** *only.*

Proof. Substition of $k+l$ by $2s$ yields

$$\sum_{k=0}^{\infty}\binom{\frac{a}{2}}{k}\sum_{l=0}^{k}\binom{k}{l}2^{k-l}f(k+l) = \sum_{s=0}^{\infty}\sum_{l=0}^{s}\binom{\frac{a}{2}}{2s-l}\binom{2s-l}{l}2^{2s-2l}f(2s).$$

Applying the duplication formula to $\Gamma(2s-2l+1)$, we get

$$\begin{aligned}
\binom{\frac{a}{2}}{2s-l}\binom{2s-l}{l}2^{2s-2l} &= \frac{\Gamma(\frac{a}{2}+1)}{\Gamma(\frac{a}{2}-2s+l+1)}\frac{2^{2s-2l}}{\Gamma(l+1)\Gamma(2s-2l+1)} \\
&= \frac{\Gamma(\frac{a}{2}+1)}{\Gamma(\frac{a}{2}-2s+l+1)}\frac{\Gamma(\frac{1}{2})}{\Gamma(l+1)\Gamma(s-l+1)\Gamma(s-l+\frac{1}{2})} \\
&= \binom{\frac{a}{2}}{s}\binom{s-\frac{1}{2}}{l}\binom{\frac{a-1}{2}-(s-\frac{1}{2})}{s-l}\frac{\Gamma(\frac{1}{2})\Gamma(s+1)}{\Gamma(s+\frac{1}{2})}.
\end{aligned}$$

Then, the well-known identity, $\sum_{l=0}^{k}\binom{m}{l}\binom{n-m}{k-l} = \binom{n}{k}$, valid for any complex numbers m and n, and the duplication formula applied to $\Gamma(2s+1)$ imply

$$\sum_{l=0}^{s}\binom{\frac{a}{2}}{2s-l}\binom{2s-l}{l}2^{2s-2l} = \frac{\binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}}{\binom{2s}{s}}2^{2s}.$$

Thanks to Lemma 5, the proof is complete. $\square$

Proposition 4. *Let x, y and z be unit vectors in $\mathbb{R}^d$ with $y \perp z$ and let $a, b, c \in \mathbb{Z}$. Then, if $x \neq y$ and $x \notin y^{\perp}$, the following identity holds,*

$$\begin{aligned}
&\int_{S^{d-1}} |x\cdot\omega|^a\sqrt{1-(y\cdot\omega)^2}^{\,b}|z\cdot\omega|^c\,\mathrm{d}\omega^{d-1} \\
&\quad = \sigma_{d-2}\,|x\cdot y|^a B\left(\frac{a}{2}+\frac{1}{2},\frac{b+c+d-1}{2}\right)B\left(\frac{c}{2}+\frac{1}{2},\frac{d-2}{2}\right) \\
&\qquad \times\sum_{s=0}^{\infty}\frac{\left(-\frac{a}{2}\right)_s\left(\frac{b+c+d-1}{2}\right)_s}{\left(\frac{c+d-1}{2}\right)_s}\frac{(-1)^s}{s!}\tan^{2s}(x,y)F\left(-s,-\frac{c}{2};\frac{1}{2};\frac{\cos^2(x,z)}{\sin^2(x,y)}\right),
\end{aligned}$$

whenever both sides of the equation converge. Moreover, for $x=\pm y$,

$$\int_{S^{d-1}} |y\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^{d-1}$$
$$= \sigma_{d-2}\, B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+c+d-1}{2}\right) B\left(\frac{c}{2}+\frac{1}{2}, \frac{d-2}{2}\right),$$

and, for $x\perp y$,

$$\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^{d-1}$$
$$= \sigma_{d-3}\, B\left(\frac{a+b+c+d-1}{2}, \frac{1}{2}\right) B\left(\frac{a+c+2}{2}, \frac{d-3}{2}\right)$$
$$\times B\left(\frac{a+1}{2}, \frac{c+1}{2}\right) F\left(-\frac{a}{2}, -\frac{c}{2}; \frac{1}{2}; \cos^2(x,z)\right).$$

Proof. The mapping

$$f\colon S^{d-1} \to [-1,1]$$
$$\omega \mapsto \omega\cdot y = t$$

has 1-dimensional Jacobian $J_1 f(\omega) = \sqrt{1-(\omega\cdot y)^2}$ for all $\omega \in S^{d-1}\setminus\{y\}$. The coarea formula implies

$$\int_{S^{d-1}} h(\omega)\,\mathrm{d}\omega^{d-1} = \int_{-1}^{1}\int_{S^{d-1}\cap f^{-1}(t)} \frac{1}{\sqrt{1-t^2}} h(\omega)\,\mathrm{d}\omega^{d-2}\,\mathrm{d}t,$$

for any positive, $\mathcal{H}^{d-1}$-measurable function $h\colon S^{d-1}\to\mathbb{R}$. Then, an application of the area formula with the injective mapping (whenever $t\in(-1,1)$)

$$g\colon S^{d-1}\cap f^{-1}(t) \to S^{d-2}(y^\perp)$$
$$\omega \mapsto \pi(\omega|y^\perp),$$

with $(d-2)$-dimensional Jacobian $J_{d-2}g(\omega) = \sqrt{1-(\omega\cdot y)^2}^{2-d}$, yields

$$\int_{S^{d-1}} h(\omega)\,\mathrm{d}\omega^{d-1} = \int_{-1}^{1}\int_{S^{d-2}(y^\perp)} \sqrt{1-t^2}^{d-3} h(ty+\sqrt{1-t^2}\omega)\,\mathrm{d}\omega^{d-2}\,\mathrm{d}t.$$

Therefore,

$$\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^{d-1}$$
$$= \int_{-1}^{1}\int_{S^{d-2}(y^\perp)} \sqrt{1-t^2}^{d+b+c-3} |x\cdot(ty+\sqrt{1-t^2}\omega)|^a |z\cdot\omega|^c \,\mathrm{d}\omega^{d-2}\,\mathrm{d}t,$$

and by a double application of the binomial formula, the last expression becomes

$$\sum_{k=0}^{\infty}\binom{\frac{a}{2}}{k}\int_{-1}^{1}\sqrt{1-t^2}^{d+b+c-3}\int_{S^{d-2}(y^\perp)}(t^2(x\cdot y)^2)^{\frac{a}{2}-k}$$
$$\times\left((1-t^2)(x\cdot\omega)^2+2t\sqrt{1-t^2}(x\cdot y)(x\cdot\omega)\right)^k\,\mathrm{d}\omega^{d-2}\,\mathrm{d}t$$
$$=\sum_{k=0}^{\infty}\binom{\frac{a}{2}}{k}\sum_{l=0}^{k}\binom{k}{l}2^{k-l}|x\cdot y|^{a-2k}(x\cdot y)^{k-l}\int_{-1}^{1}|t|^{a-2k}t^{k-l}\sqrt{1-t^2}^{d+k+l+b+c-3}\,\mathrm{d}t$$
$$\times\int_{S^{d-2}(y^\perp)}(x\cdot\omega)^{2l}(x\cdot\omega)^{k-l}|z\cdot\omega|^c\,\mathrm{d}\omega^{d-2}.$$

Notice that both integrals are non-zero only if k an l have the same parity. Then, using Remark 9 and Lemma 4, we get

$$\int_{S^{d-1}}|x\cdot\omega|^a\sqrt{1-(y\cdot\omega)^2}^b|z\cdot\omega|^c\,\mathrm{d}\omega^{d-1}$$
$$=\sigma_{d-3}|x\cdot y|^a\sum_{k=0}^{\infty}\binom{\frac{a}{2}}{k}\sum_{l=0}^{k}\binom{k}{l}2^{k-l}\tan^{k+l}(x,y)$$
$$\times B\left(\frac{a}{2}-\frac{k+l}{2}+\frac{1}{2},\frac{d}{2}+\frac{k+l}{2}+\frac{b+c}{2}-\frac{1}{2}\right)B\left(\frac{c}{2}+\frac{k+l}{2}+1,\frac{d-3}{2}\right)$$
$$\times B\left(\frac{k+l}{2}+\frac{1}{2},\frac{c}{2}+\frac{1}{2}\right)F\left(-\frac{k+l}{2},-\frac{c}{2};\frac{1}{2};\cos^2(x_1,z)\right),$$

where $x_1=\pi(x|y^\perp)$ and the double sum is over k and l with the same parity only. Finally, an application of Lemma 6 yields

$$\int_{S^{d-1}}|x\cdot\omega|^a\sqrt{1-(y\cdot\omega)^2}^b|z\cdot\omega|^c\,\mathrm{d}\omega^{d-1}$$
$$=2\sigma_{d-2}|x\cdot y|^a\sum_{s=0}^{\infty}\binom{a}{2s}B\left(\frac{a}{2}-s+\frac{1}{2},\frac{d}{2}+s+\frac{b+c}{2}-\frac{1}{2}\right)$$
$$\times\tan^{2s}(x,y)B\left(\frac{c}{2}+s+1,\frac{d-3}{2}\right)$$
$$\times B\left(s+\frac{1}{2},\frac{c}{2}+\frac{1}{2}\right)F\left(-s,-\frac{c}{2};\frac{1}{2};\cos^2(x_1,z)\right).$$

Using the duplication formula for the Gamma function, the following identity can be derived,

$$\sigma_{d-3}\binom{a}{2s}B\left(\frac{a}{2}-s+\frac{1}{2},\frac{d}{2}+s+\frac{b+c}{2}-\frac{1}{2}\right)$$
$$\times B\left(\frac{c}{2}+s+1,\frac{d-3}{2}\right)B\left(s+\frac{1}{2},\frac{c}{2}+\frac{1}{2}\right)$$
$$=\sigma_{d-2}B\left(\frac{a}{2}+\frac{1}{2},\frac{b+c+d-1}{2}\right)B\left(\frac{c}{2}+\frac{1}{2},\frac{d-2}{2}\right)\frac{\left(-\frac{a}{2}\right)_s\left(\frac{b+c+d-1}{2}\right)_s}{\left(\frac{c+d-1}{2}\right)_s}\frac{(-1)^s}{s!}.$$

The proof of the first identity is complete. The two remaining identities are easily proven by a slight modification of the above argument. □

Remark 11. Alternatively, the three identities in Proposition 4 can be written as

$$
\begin{aligned}
&\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \, \mathrm{d}\omega^{d-1} \\
&\quad = \sigma_{d-2} B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+c+d-1}{2}\right) B\left(\frac{c}{2}+\frac{1}{2}, \frac{d-2}{2}\right) \\
&\quad\quad \times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{c}{2}\right)_s \left(\frac{b+c+d-1}{2}\right)_s}{\left(\frac{1}{2}\right)_s \left(\frac{c+d-1}{2}\right)_s} \frac{\cos^{2s}(x,z)}{s!} \\
&\quad\quad\quad \times F\left(-\frac{a}{2}+s, -\frac{b}{2}; \frac{c+d-1}{2}+s; \sin^2(x,y)\right).
\end{aligned}
$$

Then, for $b=0$, it is easy to check that Proposition 4 is Lemma 4, exactly. For $a=0$ or $c=0$, Proposition 4 is equivalent to [3, Proposition 4], i.e.

$$
\begin{aligned}
&\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b \, \mathrm{d}\omega^{d-1} \\
&\quad = \sigma_{d-1} B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+d-1}{2}\right) F\left(-\frac{a}{2}, -\frac{b}{2}; \frac{d-1}{2}; \sin^2(x,y)\right).
\end{aligned} \tag{32}
$$

Recall the identity (21) in the proof of Proposition 2. Using [7, Proposition 3.5], we notice that

$$
\begin{aligned}
2\int_{\mathcal{L}^j_{q(q-1)}} |p(z|L_q^\perp)|^{d-j} \, \mathrm{d}L_{q(q-1)} &= \int_{S^{j-q}(L_{q-1}^\perp)} |p(z|(L_{q-1}\oplus\omega)^\perp)|^{d-j} \, \mathrm{d}\omega^{j-q} \\
&= |z|^{d-j} \int_{S^{j-q}(L_{q-1}^\perp)} \sqrt{1-\left(\frac{z}{|z|}\cdot\omega\right)^2}^{d-j} \mathrm{d}\omega^{j-q},
\end{aligned}
$$

where we used $z\in L_{q-1}^\perp$ for the second equality. An application of (32) with $a=0$ yields (21). □

In the special case where $c=2$, it is easily seen using Remark 11, that

$$
\begin{aligned}
&\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^2 \, \mathrm{d}\omega^{d-1} \\
&\quad = \frac{\sigma_{d-1}}{d-1} B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+d+1}{2}\right) \left[F\left(-\frac{a}{2}, -\frac{b}{2}; \frac{d+1}{2}; \sin^2(x,y)\right)\right. \\
&\quad\quad \left. + \frac{a(b+d+1)}{d+1} \cos^2(x,z) F\left(-\frac{a}{2}+1, -\frac{b}{2}; \frac{d+3}{2}; \sin^2(x,y)\right)\right].
\end{aligned} \tag{33}
$$

When $(x,n)\in \operatorname{nor} Y$ and $a_i(x,n)$ is the i'th principal direction at (x,n), we have, thanks to formula (33),

$$
\begin{aligned}
\widetilde{Q}^r_{d,s}(x,n,A_J) &= \int_{\mathcal{L}^r_{r-1}} \frac{|p(x/|x||L^{\perp}_{r-1})|^{d-r}}{|p(n|L_{r-1})|^s} \sum_{i\in J^c} |p(a_i|L^{\perp}_{r-1})|^2 \mathrm{d}L_{r-1} \\
&= \frac{1}{2}\int_{S^{r-1}} |x\cdot\omega|^{d-r}\sqrt{1-(n\cdot\omega)^2}^{\,-s} \sum_{i\in J^c} |a_i\cdot\omega|^2 \,\mathrm{d}\omega^{r-1} \\
&= \frac{(r-s)\,\sigma_{r-1}}{2(r-1)} B\left(\frac{d-r}{2}+\frac{1}{2}, \frac{r-s+1}{2}\right) Q^r_{d,s},
\end{aligned}
$$

where

$$
\begin{aligned}
Q^r_{d,s} := F&\left(-\frac{d-r}{2}, \frac{s}{2}; \frac{r+1}{2}; \sin^2(x,n)\right) \\
&+ \frac{(d-r)(r-s+1)}{r+1}\frac{\cos^2(x,A_J)}{r-s} F\left(-\frac{d-r}{2}+1, \frac{s}{2}; \frac{r+3}{2}; \sin^2(x,n)\right).
\end{aligned}
$$

Combining with (29), we obtain

$$
\alpha^r_{d,s}(X) = K^r_{d,s}\int_{\operatorname{nor} Y} |x|^{d-r} \sum_{\substack{|J|=s-1 \\ J\subset\{1,\dots,r-1\}}} \frac{\prod_{j\in J}\kappa_j}{\prod_{j=1}^{r-1}\sqrt{1+\kappa_j^2}} Q^r_{d,s}(x,n,A_J)\mathrm{d}\mathcal{H}^{r-1},
$$

with

$$
\begin{aligned}
K^r_{d,s} &:= \frac{(r-s)\sigma_{r-1} B\left(\frac{d-r}{2}+\frac{1}{2}, \frac{r-s+1}{2}\right)}{2(r-1)c_{d,r-1,r-s-1}\sigma_s} \\
&= \frac{(r-s)\sigma_{r-1}\,\sigma_{d-r+1}}{2(r-1)\,\sigma_s\,\sigma_d\, c_{d-1,r-1}} \frac{\Gamma\left(\frac{d-r+1}{2}\right)\Gamma\left(\frac{r-s+1}{2}\right)\Gamma\left(\frac{r-s}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{d-s+1}{2}\right)\Gamma\left(\frac{d-s+2}{2}\right)} \\
&= \frac{r-s}{(r-1)\,\sigma_s\, c_{d-1,r-1}} \frac{\Gamma\left(\frac{r-s}{2}\right)\Gamma\left(\frac{r-s+1}{2}\right)}{\Gamma\left(\frac{r-1}{2}\right)\Gamma\left(\frac{r}{2}\right)} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d-s+1}{2}\right)\Gamma\left(\frac{d-s+2}{2}\right)} \\
&= \frac{1}{\sigma_s\, c_{d-1,r-1}} \frac{\Gamma(r-s+1)}{\Gamma(r)} \frac{\Gamma(d)}{\Gamma(d-s+1)}.
\end{aligned}
$$

The proof of Proposition 3 is now complete. □

Lemma 7 (Two hypergeometric identities)**.** *For all $d>0$, $r\in\mathbb{R}$ and $z\in[-1,1]$,*

$$
\begin{aligned}
(r+1)(r-1)&F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r-1}{2}; z\right) \\
&= (r+1)(r-1)F\left(-\frac{d-r}{2}, \frac{1}{2}; \frac{r+1}{2}; z\right) \\
&\quad - r(d-r)zF\left(-\frac{d-r}{2}+1, \frac{1}{2}; \frac{r+3}{2}; z\right).
\end{aligned} \tag{34}
$$

If $z \neq 0, 1, -1$, then, for all $j, k \in \mathbb{R}$,

$$
\begin{aligned}
-\frac{(d-j)(d-j+k+1)}{d+1} & F\left(\frac{d-j}{2}+1, \frac{j-k}{2}; \frac{d+3}{2}; z\right) \\
&= (d-1)\frac{1-z}{z} F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d-1}{2}; z\right) \\
&\quad \times \left[(j-d) - (d-1)\frac{1-z}{z}\right] F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d+1}{2}; z\right). \qquad (35)
\end{aligned}
$$

Proof. First, apply [1, (15.2.17)] with $a = -\frac{1}{2}$, $b = -\frac{d-r}{2}$ and $c = \frac{r+1}{2}$ to obtain

$$
\begin{aligned}
\frac{r-1}{2} & F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r-1}{2}; z\right) \\
&= \frac{r}{2} F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r+1}{2}; z\right) - \frac{1}{2} F\left(-\frac{d-r}{2}, \frac{1}{2}; \frac{r-1}{2}; z\right).
\end{aligned}
$$

Then, an application of [1, (15.2.15)] to the first term of the r.h.s. with $a = -\frac{d-r}{2}$, $b = \frac{1}{2}$ and $c = \frac{r+1}{2}$ yields

$$
\begin{aligned}
\frac{r}{2} & F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r+1}{2}; z\right) \\
&= \frac{d}{2} F\left(-\frac{d-r}{2}, \frac{1}{2}; \frac{r+1}{2}; z\right) - \frac{d-r}{2}(1-z) F\left(-\frac{d-r}{2}+1, \frac{1}{2}; \frac{r+1}{2}; z\right).
\end{aligned}
$$

Furthermore, we can transform the second term of the r.h.s. of the last expression using [1, (15.2.20)] with $a = -\frac{d-r}{2}+1$, $b = \frac{1}{2}$ and $c = \frac{r+1}{2}$,

$$
\begin{aligned}
\frac{r+1}{2} & (1-z) F\left(-\frac{d-r}{2}+1, \frac{1}{2}; \frac{r+1}{2}; z\right) \\
&= \frac{r+1}{2} F\left(-\frac{d-r}{2}, \frac{1}{2}; \frac{r+1}{2}; z\right) - \frac{r}{2} z F\left(-\frac{d-r}{2}+1, \frac{1}{2}; \frac{r+3}{2}; z\right).
\end{aligned}
$$

Hence, combining the three identities above, we obtain (34). Note that in (34), the three hypergeometric series are absolute convergent on the circle of convergence, $|z| = 1$, whenever $d > 0$, cf. [1, (15.1.1)].

According to [1, (15.2.20)] with $a = \frac{d-j}{2}+1$, $b = \frac{j-k}{2}$ and $c = \frac{d+1}{2}$, we have

$$
\begin{aligned}
-\frac{d-j+k+1}{2} & z F\left(\frac{d-j}{2}+1, \frac{j-k}{2}; \frac{d+3}{2}; z\right) \\
&= \frac{d+1}{2}(1-z) F\left(\frac{d-j}{2}+1, \frac{j-k}{2}; \frac{d+1}{2}; z\right) \\
&\quad - \frac{d+1}{2} F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d+1}{2}; z\right).
\end{aligned}
$$

The first term on the r.h.s. can be re-written using [1, (15.2.17)] with $a = \frac{d-j}{2}$, $b = \frac{j-k}{2}$ and $c = \frac{d+1}{2}$,

$$
\begin{aligned}
\frac{d-j}{2} & F\left(\frac{d-j}{2}+1, \frac{j-k}{2}; \frac{d+1}{2}; z\right) \\
&= \frac{d-1}{2} F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d-1}{2}; z\right) - \frac{j-1}{2} F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d+1}{2}; z\right).
\end{aligned}
$$

Combining the last two identities, we obtain identity (35). Note that in (35), the absolute convergence of $F(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d-1}{2}; z)$ on the circle $|z| = 1$ requires that $k > 1$, cf. [1, (15.1.1)]. $\square$

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